
بيانات الكتاب

Specialist : Mathematics

Subject : (pure 5)Algebraic equation

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كلية التربية بالغردقة – جامعة جنوب الوادى

رؤية الكلية

كلية التربية بالغردقة مؤسسة رائدة محلياً ودولياً فى مجالات التعليم ،والبحث العلمى ،وخدمة المجتمع ، بما يؤهلها للمنافسة على المستوى : المحلى ، والإقليمى ، والعالمى .

رسالة الكلية

تقديم تعليم مميز في مجالات العلوم الأساسية و إنتاج بحوث علمية تطبيقية للمساهمة فى التنمية المستدامة من خلال إعداد خريجين متميزين طبقا للمعايير الأكاديمية القومية، و تطوير مهارات و قدرات الموارد البشرية، و توفير خدمات مجتمعية وبيئية تلبي طموحات مجتمع جنوب الوادي، و بناء الشراكات المجتمعية الفاعلة.

THEORY OF EQUATIONS

1.0 Introduction

In this module, we will study about polynomial functions and various methods to find out the roots of polynomial equations. 'Solving equations' was an important problem from the beginning of study of Mathematics itself. The notion of complex numbers was first introduced because equations like $x^2 + 1 = 0$ has no solution in the set of real numbers. The "fundamental theorem of algebra" which states that every polynomial of degree ≥ 1 has at least one zero was first proved by the famous German Mathematician Karl Fredrich Gauss. We shall look at polynomials in detail and will discuss various methods for solving polynomial equations.

1.1. Polynomial Functions

Definition:

A function defined by

$$f(x) = a_0x^n + a_1x^{n-1} + \ldots.. + a_n, \quad where \quad a_o \neq 0, \text{ n is a non negative}$$

integer and a_i (i = 0, 1….,n) are fixed complex numbers is called a **polynomial** of **degree** n in x. Then numbers $a_o, a_1, \ldots.., a_n$ are called the **coefficients** of f.

If α is a complex number such that $f(\alpha) = 0$, then α is called **zero** of the polynomial.

1.1.1 Theorem (Fundamental Theorem of Algebra)

Every polynomial function of degree $n \geq 1$ has at least one zero.

Remark:

Fundamental theorem of algebra says that, if $f(x) = a_0x^n + a_1x^{n-1} + \ldots.. + a_n$, where $a_0 \neq 0$ is the given polynomial of degree $n \geq 1$, then there exists a complex number α such that $a_0\alpha^n + a_1\alpha^{n-1} + ... + a_n = 0$.

We use the Fundamental Theorem of Algebra, to prove the following result.

1.1.2 Theorem

Every polynomial of degree n has n and only n zeroes.

Proof:

Let $f(x) = a_0x^n + a_1x^{n-1} + ... + a_n,\ \ where\ \ a_o \neq 0,$ be a polynomial of degree $n \geq 1$.

By fundamental theorem of algebra, f(x) has at least one zero, let α_1 be that zero.

Then $(x - \alpha_1)$ is a factor of f(x).

Therefore, we can write:

f(x) = $(x - \alpha_1)Q_1(x)$, where $Q_1(x)$ is a polynomial function of degree $n - 1$.

If $n - 1 \geq 1$, again by Fundamental Theorem of Algebra, $Q_1(x)$ has at least one zero, say α_2.

Therefore, $f(x) = (x - \alpha_1)(x - \alpha_2)Q_2(x)$ *where* $Q_2(x)$ is a polynomial function of degree $n - 2$.

Repeating the above arguments, we get

$f(x) = (x - \alpha_1)(x - \alpha_2).....(x - \alpha_n)Q_n(x)$, where $Q_n(x)$ is a polynomial function of degree $n - n = 0$, i.e., $Q_n(x)$ is a constant.

Equating the coefficient of x^n on both sides of the above equation, we get

$Q_n(x) = a_o$.

Therefore, $f(x) = a_o(x - \alpha_1)(x - \alpha_2)....(x - \alpha_n)$.

If α is any number other than $\alpha_1, \alpha_2, ...\alpha_n$, then $f(x) \neq 0 \Rightarrow \alpha$ is not a zero of f(x).

Hence f(x) has n and only n zeros, namely $\alpha_1, \alpha_2 ...., \alpha_n$.

Note:

Let $f(x) = a_ox^n + a_1x^{n-1} + ... + a_n; a_o \neq 0$ be an n^{th} degree polynomial in x.

Then, $a_ox^n + a_1x^{n-1} + ... + a_n = 0$ --------- (1)

is called a **polynomial equation** in x of degree n.

A number α is called a **root** of the equation (1) if α is a zero of the polynomial f(x).

So theorem (1.1.2)can also be stated as : "Every polynomial equation of degree n has n and only n roots".

1.1.3 Theorem

If the equation $a_o x^n + a_1 x^{n-1} + ... + a_n = 0$, where $a_o, a_1, ...a_n$ are real numbers $(a_o \neq 0)$, has a complex root $\alpha + i\beta$, then it also has a complex root $\alpha - i\beta$. (i.e., complex roots occur in conjugate pairs for a polynomial equation with real coefficients).

Proof:

Let f(x) = $a_o x^n + a_1 x^{n-1} + ... + a_n, a_o \neq 0$

Given that $\alpha + i\beta$ is a root of f(x) = 0.

Consider $(x - (\alpha + i\beta)(x - (\alpha - i\beta) = (x - \alpha)^2 + \beta^2$.

Divide f(x) by $(x - \alpha)^2 + \beta^2$.

Let Q(x) be the quotient and Ax + B be the remainder.

Then, $f(x) = \left[(x - \alpha)^2 + \beta^2\right] Q(x) + Ax + B$

$$= \left[(x - (\alpha + i\beta))(x - (\alpha - i\beta))\right] Q(x) + Ax + B$$

$$\Rightarrow f(\alpha + i\beta) = 0 + A(\alpha + i\beta) + B = A(\alpha + i\beta) + B = (A\alpha + B) + iA\beta$$

But $f(\alpha + i\beta) = 0$.

Equating real and imaginary parts, we see that $A\alpha + B = 0$ *and* $A\beta = 0$

But $\beta \neq 0 \Rightarrow A = 0$ *and so* $B = 0$

$\Rightarrow$ The remainder Ax + B is zero. i.e., $\left[(x - (\alpha + i\beta))(x - (\alpha - i\beta))\right]$ is a factor of f(x)

i.e., $\alpha - i\beta$ is a root of f(x) = 0.

1.1.4. Theorem

In an equation with rational coefficients, the roots which are quadratic surds occur in conjugate pairs.

Proof:

Let $f(x) = a_o x^n + a_1 x^{n-1} + ... + a_n, a_o \neq 0$, be an n^{th} degree polynomial with rational coefficients.

Let $\alpha + \sqrt{\beta}$ is a root of f(x) = 0.

Divide f(x) by $\left[(x - (\alpha + \sqrt{\beta}))(x - (\alpha - \sqrt{\beta}))\right] = (x - \alpha)^2 - \beta$.

Let $Q(x)$ be the quotient and $Ax + B$ be the remainder.

Proceeding exactly as in the above theorem, we get $Ax + B = 0$.

Thus we conclude that $\alpha - \sqrt{\beta}$ is also a root of $f(x) = 0$.

Solved Problems

1. Solve $x^4 - 4x^2 + 8x + 35 = 0$, given $2 + i\sqrt{3}$ is a root.

Solution :

Given that $2 + i\sqrt{3}$ is a root of $x^4 - 4x^2 + 8x + 35 = 0$; since complex roots occurs in conjugate pairs $2 - i\sqrt{3}$ is also a root of it.

$\Rightarrow [x - (2 + i\sqrt{3})][x - (2 - i\sqrt{3})] = (x - 2)^2 + 3 = x^2 - 4x + 7$ is a factor of the given polynomial.

Dividing the given polynomial by this factor, we obtain the other factor as $x^2 + 4x + 5$.

The roots of $x^2 + 4x + 5 = 0$ are given by $\dfrac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i$.

Hence the roots of the given polynomial are $2 + i\sqrt{3},\ 2 - i\sqrt{3},\ -2 + i$ *and* $-2 - i$.

2. Solve $x^4 - 5x^3 + 4x^2 + 8x - 8 = 0$, given that one of the roots is $1 - \sqrt{5}$.

Solution:

Since quadratic surds occur in conjugate pairs as roots of a polynomial equation, $1 + \sqrt{5}$ is also a root of the given polynomial.

$\Rightarrow [x - (1 - \sqrt{5})][x - (1 + \sqrt{5})] = (x - 1)^2 - 5 = x^2 - 2x - 4$ is a factor.

Dividing the given polynomial by this factor, we obtain the other factor as $x^2 - 3x + 2$.

Also, $x^2 - 3x + 2 = (x - 2)(x - 1)$

Thus the roots of the given polynomial equation are $1 + \sqrt{5}, 1 - \sqrt{5}, 1, 2$.

3. Find a polynomial equation of the lowest degree with rational coefficients having $\sqrt{3}$ and $1 - 2i$ as two of its roots.

Solution:

Since quadratic surds occur in pairs as roots, $-\sqrt{3}$ is also a root.

Since complex roots occur in conjugate pairs, $1 + 2i$ is also a root of the required polynomial equation. Therefore the desired equation is given by

$$(x - \sqrt{3})(x + \sqrt{3})(x - (1 - 2i)(x - (1 + 2i)) = 0$$

$$\text{i.e., } x^4 - 2x^3 + 2x^2 + 6x - 15 = 0$$

ALGEBRA AND CALCULUS

Unit – I
Theory of Equations

Let us consider

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$$

This a polynomial in 'x' of degree 'n' provided $a_0 \neq 0$.

The equation is obtained by putting f(x) = 0 is called an **algebraic equation** of degree n.

RELATIONS BETWEEN THE ROOTS AND COEFFICIENTS OF EQUATIONS

Let the given equation be $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0$

Let $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ be its roots.

$\sum \alpha_1$ = sum of the roots taken one at a time = $-\frac{a_1}{a_0}$

$\sum \alpha_1 \alpha_2$ = sum of the product of the roots taken two at a time = $\frac{a_2}{a_0}$

$\sum \alpha_1 \alpha_2 \alpha_3$ = sum of the product of the roots taken three at a time = $-\frac{a_3}{a_0}$

finally we get $\alpha_1.\alpha_2\ldots\ldots\alpha_n = (-1)^n\frac{a_n}{a_0}$.

Problem:

If α and β are the roots of $2x^2 + 3x + 5 = 0$, find α + β, αβ.

Solution:

Here $a_0 = 2, a_1 = 3, a_3 = 5.$

$\sum \alpha = \alpha + \beta = -\frac{a_1}{a_0} = -\frac{3}{2}$

$\alpha\beta = \frac{a_2}{a_0} = \frac{5}{2}$.

Problem:

Solve the equation $x^3 + 6x + 20 = 0$, one root being 1 + 3i.

Solution:

Given equation is cubic. Hence we have 3 roots. One root is (1+3i) = α (say) complex roots occur in pairs.

∴ β = 1 – 3i is another root.

To find third root γ (say)

Sum of the roots taken one at a time

$$\alpha + \beta + \gamma = \frac{0}{1} = 0.$$

i.e., $1 + 3i + 1 - 3i + \gamma = 0$

$$\gamma = -2$$

∴ The roots of the given equation are $1 + 3i$, $1 - 3i$, -2.

Problem:

Solve the equation $3x^3 - 23x^2 + 72x - 20 = 0$ having given that $3 + \sqrt{-5}$ is a root.

Solution:

Given equation is cubic. Hence we have three roots.

One root is $3 + i\sqrt{5} = \alpha$

Since complex roots occur in pairs, $3 - i\sqrt{5} = \beta$ is another root.

Sum of the roots is $\alpha + \beta + \gamma = \frac{23}{3}$

i.e., $3 + i\sqrt{5} + 3 - i\sqrt{5} + \gamma = \frac{23}{3}$

$$6 + \gamma = \frac{23}{3}$$

$$\gamma = \frac{23}{3} - 6$$

$$\gamma = \frac{5}{3}$$

Hence the roots of the given equation are $3 + i\sqrt{5}$, $3 - i\sqrt{5}$, $\frac{5}{3}$.

Problem:

Solve the equation $x^4 + 2x^3 - 16x^2 - 22x + 7 = 0$ which has a root $2 + \sqrt{3}$.

Solution:

Given $x^4 + 2x^3 - 16x^2 - 22x + 7 = 0$. ----- (1)

This equation is biquadratic, i.e., fourth degree equation.

∴ It has 4 roots. Given $2 + \sqrt{3}$ is a root which is clearly irrational. Since irrational roots occur in pairs, $2 - \sqrt{3}$ is also a root of the given equation.

$\therefore [x-(2+\sqrt{3})]\,[x-(2-\sqrt{3})]$ is a factor of (1)

i.e., $x^2-4x+1=0$ is a factor.

Dividing (1) by $x^2-4x+1=0$, we get

	x^2+6x+7
x^2-4x+1	$x^4+2x^3-16x^2-22x+7$
	$x^4-4x^3+x^2$
	(-) $6x^3-17x^2-22x+7$
	$6x^3-24x^2+6x$
	(-) $7x^2-28x+7$
	$7x^2-28x+7$
	0

Hence the quotient is $x^2+6x+7=0$. Solving this quadratic equation, we get $= -3\pm\sqrt{2}$.

Hence the roots of the given equation are $2+\sqrt{3}$, $2-\sqrt{3}$, $-3+\sqrt{2}$, $-3-\sqrt{2}$.

Problem:
Form the equation, with rational coefficients one root of whose roots is $\sqrt{2}+\sqrt{3}$.
Solution:

One root is $\sqrt{2}+\sqrt{3}$

i.e., $x=\sqrt{2}+\sqrt{3}$

i.e., $x-\sqrt{2}=\sqrt{3}$

Squaring on both sides we get

$$(x-\sqrt{2})^2=3$$

$$x^2-2\sqrt{2}x+2=3$$

$$x^2-1=2\sqrt{2}x$$

Again squaring, we get

$(x^2-1)^2 = \left(2\sqrt{2}x\right)^2$

$x^4 - 2x^2 + 1 = 4.2.x^2$

$x^4 - 10x^2 + 1 = 0$, which is the required equation.

Problem:

Form the equation with rational coefficients having $1 + \sqrt{5}$ and $1 + \sqrt{-5}$ as two of its roots.

Solution:

Given x = $1 + \sqrt{5}$ and x = $1 + i\sqrt{5}$

i.e., $[x - (1 + \sqrt{5})]\ [x - (1 + i\sqrt{5})]$ are the factors of the required equation.

Since complex and irrational roots occur in pairs, we have x = $1 - \sqrt{5}$, x = $1 - i\sqrt{5}$ are also the roots of the required equation.

i.e., $x - (1 - \sqrt{5}\,)$ and $x - (1 - i\sqrt{5})$ are also factors of the required equation.

Hence the required equation is,

$[x - (1 + \sqrt{5})]\ [x - (1 + i\sqrt{5})]\ [x - (1 - \sqrt{5})]\ [x - (1 - i\sqrt{5})] = 0$

i.e., $[(x-1)^2 - 5][(x-1)^2 + 5] = 0$

$(x^2 - 2x - 4)(x^2 - 2x + 6) = 0$

Simplifying we get

$x^4 - 4x^3 + 6x^2 - 4x - 24 = 0$ which is the required equation.

Problem:

Solve the equation $32x^3 - 48x^2 + 22x - 3 = 0$ whose roots are in A.P.

Solution:

Let the roots be $\alpha - d, \alpha, \alpha + d$.

Sum of the roots taken one at a time is,

$\alpha - d + \alpha + \alpha + d = \frac{48}{32}$

$3\alpha = \frac{48}{32}$

$\alpha = \frac{1}{2}$

$\therefore \frac{1}{2}$ is a root of the given equation. By division we have,

$\frac{1}{2}$	32	-48	22	-3
	0	16	-16	3
	32	-32	6	0

The reduced equation is $32x^2 - 32x + 6 = 0$

Solving this quadratic equation we get the remaining two roots $\frac{1}{4}, \frac{3}{4}$.

Hence the roots of the given equation are $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$.

Problem:

Find the value of k for which the roots of the equation $2x^3 + 6x^2 + 5x + k = 0$ are in A.P.

Solution:

Given $2x^3 + 6x^2 + 5x + k = 0$ --------- (1)

Let the roots be $\alpha - d, \alpha, \alpha + d$.

Sum of the roots taken one at a time is,

$\alpha - d + \alpha + \alpha + d = \frac{-6}{2}$

$3\alpha = -3$

i.e., $\alpha = -1$

i.e., $\alpha = -1$ is a root of (1).

$\therefore$ put x = - 1 in (1), we get k = 1.

Problem:

Solve the equation $27x^3 + 42x^2 - 28x - 8 = 0$ whose roots are in G.P.

Solution:

Given $27x^3 + 42x^2 - 28x - 8 = 0$ --------- (1)

Let the roots be $\frac{\alpha}{r}$, α, αr

Product of the roots taken three at a time is $\frac{\alpha}{r} . \alpha . \alpha r = \frac{8}{27}$

i.e., $\alpha^3 = \frac{8}{27}$

i.e., $\alpha = \frac{2}{3}$.

i.e., $\alpha = \frac{2}{3}$ is a root of the given equation (1)

i.e., $x = \frac{2}{3}$ is a root of the given equation (1)

i.e., $(x - \frac{2}{3})$ is a factor of (1).

$$
\begin{array}{r|l}
 & 27x^2 + 60x + 12 \\
\hline
x - \frac{2}{3} & 27x^3 + 42x^2 - 28x - 8 \\
 & 27x^3 - 18x^2 \\
\hline
 & (-) \quad 60x^2 - 28x - 8 \\
 & \quad 60x^2 - 40x \\
\hline
 & (-) \quad 12x - 8 \\
 & \quad 12x - 8 \\
\hline
 & 0
\end{array}
$$

Hence the quotient is $27x^2 + 60x + 12 = 0$

i.e., $9x^2 + 20x + 4 = 0$

Solving this quadratic equation we get $x = -2 \text{ or } -\frac{2}{9}$

Hence the roots of the given equation are $-2, -\frac{2}{9}, \frac{2}{3}$.

Problem:

Find the condition that the roots of the equation $x^3 - px^2 + qx - r = 0$ may be in G.P.

Solution:

Given $x^3 - px^2 + qx - r = 0$ --------- (1)

Let the roots be $\frac{\alpha}{r}$, α, αr

Product of the roots taken three at a time $\frac{\alpha}{r} . \alpha . \alpha r = r$

i.e., $\alpha^3 = r$ -------- (2)

But α is a root of the equation (1). Put x = α in (1), we get,

$$\alpha^3 - p\alpha^2 + q\alpha - r = 0 \quad \text{-------- (3)}$$

Substituting (2) in (3) we get

$$r - p\alpha^2 + q\alpha - r = 0$$

$$p\alpha^2 - q\alpha = 0$$

$$\alpha(p\alpha - q) = 0$$

$\alpha \neq 0$ $\therefore$ $p\alpha - q = 0$

i.e., $p\alpha = q$

i.e., $\alpha = \frac{q}{p}$

$$\therefore \alpha^3 = \frac{\alpha^3}{\beta^3}$$

$$r = \frac{q^3}{p^3}$$

Hence the required condition is $p^3 r = q^3$.

Transformation of Equations:

Problem:

If the roots of $x^3 - 12x^2 + 23x + 36 = 0$ are -1, 4, 9, find the equation whose roots are

1, -4, -9.

$\alpha = \frac{1}{2}$

$\therefore \frac{1}{2}$ is a root of the given equation. By division we have,

$\frac{1}{2}$	32	-48	22	-3
	0	16	-16	3
	32	-32	6	0

The reduced equation is $32x^2 - 32x + 6 = 0$

Solving this quadratic equation we get the remaining two roots $\frac{1}{4}, \frac{3}{4}$.

Hence the roots of the given equation are $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$.

Problem:

Find the value of k for which the roots of the equation $2x^3 + 6x^2 + 5x + k = 0$ are in A.P.

Solution:

Given $2x^3 + 6x^2 + 5x + k = 0$ --------- (1)

Let the roots be $\alpha - d, \alpha, \alpha + d$.

Sum of the roots taken one at a time is,

$\alpha - d + \alpha + \alpha + d = \frac{-6}{2}$

$3\alpha = -3$

i.e., $\alpha = -1$

i.e., $\alpha = -1$ is a root of (1).

$\therefore$ put x = - 1 in (1), we get k = 1.

Problem:

Solve the equation $27x^3 + 42x^2 - 28x - 8 = 0$ whose roots are in G.P.

Solution:

Given $x^3 - 12x^2 + 23x + 36 = 0$ -------- (1)

The roots are -1, 4, 9.

Now we find an equation whose roots are 1, -4, -9 ie., to find an equation whose roots are the roots of (1) but the signs are changed. Hence in (1) we have to change the sign of odd powers of x.

Hence the required equation is

$$-x^3 - 12x^2 - 23x + 36 = 0$$

i.e., $x^3 + 12x^2 + 23x - 36 = 0$

This gives the required equation.

Problem:

Multiply the roots of the equation $x^4 + 2x^3 + 4x^2 + 6x + 8 = 0$ by $\frac{1}{2}$.

Solution:

Given $x^4 + 2x^3 + 4x^2 + 6x + 8 = 0$ -------- (1)

To multiply the roots of (1) by $\frac{1}{2}$, we have to multiply the successive coefficients beginning with the second by $\frac{1}{2}, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \left(\frac{1}{2}\right)^4$

i.e., $$x^4 + \frac{1}{2}\,2x^3 + \left(\frac{1}{2}\right)^2 4x^2 + \left(\frac{1}{2}\right)^3 6x + \left(\frac{1}{2}\right)^4 8 = 0$$

$$x^4 + x^3 + x^2 + \frac{3}{4}x + \frac{1}{2} = 0$$

i.e., $$4x^4 + 4x^3 + 4x^2 + 3x + 2 = 0$$

which is the required equation.

Problem:

Remove the fractional coefficients from the equation $x^3 - \frac{1}{4}x^2 + \frac{1}{3}x - 1 = 0$.

Solution:

Given $x^3 - \frac{1}{4}x^2 + \frac{1}{3}x - 1 = 0$ --------- (1)

Multiply by the roots of (1) by m, we get

$$x^3 - \frac{m}{4}x^2 + \frac{m^2}{3}x - m^3 = 0 \quad \text{--------- (2)}$$

If m = 12 (L.C.M. of 4 and 3), the fractions will be removed. Put m = 12 in (2), we get

i.e., $x^3 - 3x^2 + 48x - 1728 = 0.$

Problem:

Solve the equation $6x^3 - 11x^2 - 3x + 2 = 0$ given that its roots are in H.P.

Solution:

Given $6x^3 - 11x^2 - 3x + 2 = 0$ ---------- (1)

Its roots are in H.P. x to $\frac{1}{x}$ in (1), we get

$$6\left(\frac{1}{x}\right)^3 - 11\left(\frac{1}{x}\right)^2 - 3\left(\frac{1}{x}\right) + 2 = 0$$

$$\Rightarrow 2x^3 - 3x^2 - 11x + 6 = 0 \quad \text{------ (2)}$$

Now the roots of (2) are in A.P. (Since H.P. is a reciprocal of A.P.). Let the roots of (2) be $\alpha - d,\ \alpha,\ \alpha + d$.

Sum of the roots

$$\alpha - d + \alpha + \alpha + d = \frac{3}{2}$$

$$\Rightarrow 3\alpha = \frac{3}{2}$$

$$\alpha = \frac{1}{2} \quad \text{------ (3)}$$

Product of the roots taken 3 at the time is $(\alpha - d) \times \alpha \times (\alpha + d) = \frac{-11}{2}$

d = $\pm \frac{5}{2}$.

Case(i) :

When $d=\frac{5}{2}$ *and* $\alpha=\frac{1}{2}$, *the roots of* ( ) *are* $\frac{1}{2}-\frac{5}{2}, \frac{1}{2}, \frac{1}{2}+\frac{5}{2}$

i.e., $-2, \frac{1}{2}, 3.$

$\therefore$ *The roots of the given equation are the reciprocal of the roots of* ( )

i.e., $-\frac{1}{2}, 2, \frac{1}{3}.$ *are roots of* ( )

Case (ii) :

When $d=\frac{-5}{2}$ *and* $\alpha=\frac{1}{2}$, *the roots of* ( ) *are* $\frac{1}{2}+\frac{5}{2}, \frac{1}{2}, \frac{1}{2}-\frac{5}{2}$

i.e., $3, \frac{1}{2}, -2.$

$\therefore$ *The roots of the given equation are the reciprocal of the roots of* ( )

i.e., $\frac{1}{3}, 2, -\frac{1}{2}.$ *are roots of* ( )

Problem:

Diminish the roots of $x^4-5x^3+7x^2-4x+5=0$ *by* 2 and find the transformed equation.

Solution :

Diminishing the roots by 2, we get

2	1	-5	7	-4	5	
	0	2	-6	2	-4	
2	1	-3	1	-2	1	(constant term of the transformed equation)
	0	2	-2	-2		
2	1	-1	-1	-4	(coefficient of x)	
	0	2	2			
2	1	1	1	(coefficient of x^2)		
	0	2				
2	1	3	(coefficient of x^3)			
	0					
	1	(coefficient of x^4 in the transformed equation)				

The transformed equation whose roots are less by 2 of the given equation is

$x^4+3x^3+x^2-4x+1=0$

Problem:

Increase by 7 the roots of the equation $3x^4 + 7x^3 - 15x^2 + x - 2 = 0$ and find the transformed equation.

Solution :

Increasing by 7 the roots of the given equation is the same as diminishing the roots by -7.

-7	3	7	-15	1	-2	
	0	-21	98	-581	4060	
-7	3	-14	83	-580	4058	(constant term of the transformed equation)
	0	-21	245	-2296		
-7	3	-35	328	-2876	(coefficient of x)	
	0	-21	392			
-7	3	-56	720	(coefficient of x^2)		
	0	-21				
-7	3	-77	(coefficient of x^3)			
	0					
	3	(coefficient of x^4 in the transformed equation)				

The transformed equation is $3x^4 - 77x^3 + 720x^2 - 2876x + 4058 = 0$.

Problem:

Find the equation whose roots are the roots of $x^4 - x^3 - 10x^2 + 4x + 24 = 0$ increased by 2.

Solution :

-2	1	-1	-10	4	24	
	0	-2	6	8	24	
-2	1	-3	-4	12	0	(constant term of the transformed equation)
	0	-2	10	-12		
-2	1	-5	6	0	(coefficient of x)	
	0	-2	14			
-2	1	-7	20	(coefficient of x^2)		
	0	-2				
-2	1	-9	(coefficient of x^3)			
	0					
	1	(coefficient of x^4 in the transformed equation)				

The transformed equation is $x^4 - 9x^3 + 20x = 0$.

Problem:

If α, β, γ are the roots of the equation $x^3 - 6x^2 + 12x - 8 = 0$, find an equation whose roots are $\alpha - 2, \beta - 2, \gamma - 2$.

Solution :

2	1	-6	12	-8
	0	2	-8	8
2	1	-4	4	0
	0	2	-4	
2	1	-2	0	
	0	2		
2	1	0		
	0			
	1			

The transformed equation is $x^3 = 0$.

i.e., the roots are = 0, 0, 0.

i.e., $\alpha - 2 = 0,\ \beta - 2 = 0,\ \gamma - 2 = 0$

i.e., $\alpha = 2,\ \beta = 2,\ \gamma = 2$.

Problem:

Find the transformed equation with sign changed $x^5 + 6x^4 + 6x^3 - 7x^2 + 2x - 1 = 0$.

Solution:

Given that $x^5 + 6x^4 + 6x^3 - 7x^2 + 2x - 1 = 0$

Given sign =	+	+	+	-	+	-
	+	-	+	-	+	-
	+	-	+	+	+	+

Now the transformed equation $x^5 - 6x^4 + 6x^3 + 7x^2 + 2x + 1 = 0$ which is the required equation.

Nature of the Roots:

Problem:
Determine completely the nature of the roots of the equation $x^5 - 6x^2 - 4x + 5 = 0$.
Solution:

Given that $f(x) = x^5 - 6x^2 - 4x + 5$

There are 2 times sign changed.

∴ There exist 2 positive roots.

Put x = -x

$$f(-x) = (-x)^5 - 6(-x)^2 - 4(-x) + 5$$

$$= -x^5 - 6x^2 + 4x + 5$$

There is 1 time sign changed.

∴ There is only one positive root.

∴ There are 3 real roots.

The degree of the equation is 5.

Number of imaginary roots = degree of equation – number of real roots

= 5 – 3

= 2

∴ The number of imaginary roots = 2.

1.6 Cardan's Method of Solving a Standard Cubic Equation

Knowledge of the quadratic formula is older than the Pythagorean Theorem. Solving a cubic equation, on the other hand, was done by Renaissance mathematicians in Italy. In this section we describe some methods to find one root of the cubic equation

$$ax^3 + bx^2 + cx + d = 0. \qquad \text{----------} \quad (1)$$

so that other two roots (real or complex) can then be found by polynomial division and the quadratic formula. The solution proceeds in two steps. First, the cubic equation is *depressed*; then one solves the depressed cubic.

1.6.1 Depressing the cubic equation

This trick, which transforms the general cubic equation into a new cubic equation with missing x^2-term is due to Italian mathematician **Nicolo Fontana Tartaglia** (1500-1557). To remove the second term of Eq. (1), we diminish the roots of (1) by $h = -\frac{a_1}{na_0}$ with $n = 3$ (i.e., the degree of the polynomial equation), $a_0 = a$, $a_1 = b$; so that $h = -\frac{b}{3a}$. Set $x - h = y$ or $x = y + h = y - \frac{b}{3a}$.

We apply the substitution $x = y - \frac{b}{3a}$ to the cubic equation (1), and obtain

$$a\left(y - \frac{b}{3a}\right)^3 + b\left(y - \frac{b}{3a}\right)^2 + c\left(y - \frac{b}{3a}\right) + d = 0.$$

Multiplying out and simplifying, we obtain $ay^3 + \left(c - \frac{b^2}{3a}\right)y + \left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right) = 0,$

a cubic equation in which y^2-term is absent.

1.6.2 Solving the Depressed Cubic

Method to solve a depressed cubic equation of the form

$$y^3 + Ay = B \qquad \ldots(2)$$

had been discovered earlier by Italian mathematician **Scipione dal Ferro** (1465-1526). The procedure is as follows:

First find s and t so that $3st = A$ ……… (3)

and $s^3 - t^3 = B$ ……….(4)

Then $y = s - t$ will be a solution of the depressed cubic. This can be verified as follows: Substituting for *A*, *B* and *y*, equation (2) gives

$$(s-t)^3 + 3st(s-t) = s^3 - t^3.$$

This is true since we can simplify the left side using the binomial formula to obtain $s^3 - t^3.$

Now to find *s* and *t* satisfying (3) and (4), we proceed as follows: From Eq.(3), we have $s = \frac{A}{3t}$ and substituting this into Eq.(4), we obtain, $\left(\frac{A}{3t}\right)^3 - t^3 = B$

Simplifying, this turns into the *tri-quadratic* equation, $t^6 + Bt^3 - \frac{A^3}{27} = 0,$

which using the substitution $u = t^3$ becomes the quadratic equation, $u^2 + Bu - \frac{A^3}{27} = 0.$

From this, we can find a value for *u* by the quadratic formula, then obtain *t*, afterwards *s*. Hence the root $s - t$ can be obtained.

Illustrative Examples:

1. Using the discussion above, find a root of the cubic equation

$$2x^3 - 30x^2 + 162x - 350 = 0.$$

Solution:

Comparing with $ax^3 + bx^2 + cx + d = 0,$ ....(5)

we have $a = 2, b = -30, c = 162,$ and $d = -350.$

Hence substituting $x = y - \frac{b}{3a} = y + 5$ in (5), expanding and simplifying, we obtain the depressed cubic equation $y^3 + 6y - 20 = 0.$

Now to find the solution of depressed equation $y^3 + 6y = 20,$ we proceed as follows:

We need *s* and *t* to satisfy $3st = 6$ ...(6)

and $s^3 - t^3 = 20.$ ...(7)

Solving for *s* in (6) and substituting the result into (7) yields: $\frac{8}{t^3} - t^3 = 20,$

which multiplied by t^3 becomes, $t^6 + 20t^3 - 8 = 0.$

Using the substitution $u = t^3$ the above becomes the quadratic equation

$u^2 + 20u - 8 = 0.$

Using the quadratic formula, we obtain that $u = -10 \pm \sqrt{108}$.

We take the cube root of the positive value of u and obtain $t = \sqrt[3]{-10+\sqrt{108}}$.

By Equation (7), $s^3 = 20 + t^3 = 20 + \left(-10+\sqrt{108}\right) = 10 + \sqrt{108}$

and hence $s = \sqrt[3]{10+\sqrt{108}}$.

Hence a root y for the depressed cubic equation is

$$y = s - t = \sqrt[3]{10+\sqrt{108}} - \sqrt[3]{-10+\sqrt{108}}.$$

Hence a root of the original cubic equation

$$2x^3 - 30x^2 + 162x - 350 = 0$$

is given by $x = y + 5 = \sqrt[3]{10+\sqrt{108}} - \sqrt[3]{-10+\sqrt{108}} + 5.$

2. Find one real root of the cubic equation $x^3 - 2x - 5 = 0$.

Solution.

Since the term x^2 is absent, the given equation is in the *depressed form*.

Here $x^3 - 2x = 5$, and hence $3st = -2$ and $s^3 - t^3 = 5$.

Now substituting $s = -\frac{2}{3t}$ in $s^3 - t^3 = 5$, we obtain $\left(-\frac{2}{3t}\right)^3 - t^3 = 5$

or $$-\frac{8}{27t^3} - t^3 = 5 \quad \text{or} \quad t^6 + 5t^3 + \frac{8}{27} = 0.$$

Take $u = t^3$. Then the above becomes the quadratic equation $u^2 + 5u + \frac{8}{27} = 0$ with

$$u = \frac{-5 \pm \sqrt{25 - \frac{4\times 8}{27}}}{2} = \frac{-5 \pm 4.88}{2} = -4.94 \text{ or } -0.06.$$

We take the cube of root of the value with largest absolute value, and obtain

$$t = u^{1/3} = -1.7031.$$

Putting this value in $s = -\frac{2}{3t}$, we obtain $s = 0.3914$.

Hence one of the roots of the given equation is $s - t = 2.0945$.

1.6.3 Cardan's Solution of the Standard cubic

Italian Renaissance mathematician **Girolamo Cardano** (1501 –1576) published the solution to a cubic equation in his Algebra book ***Ars Magna***.

Usually we take the cubic equation as $a_0x^3 + a_1x^2 + a_2x + a_3 = 0$.

But it has been found it is more advantageous to take the general cubic as

$$ax^3 + 3bx^2 + 3cx + d = 0 \qquad \dots (8)$$

This method of writing is referred to as the ***cubic with binomial coefficients***.

Taking the form (8) and putting $y = ax + b$ or $x = (y-b)/a$ and multiplying throughout by a^2, we obtain : $(y-b)^3 + 3b(y-b)^2 + 3ac(y-b) + a^2d = 0$.

i.e., $$y^3 + 3(ac-b^2)y + (a^2d - 3abc + 2b^3) = 0$$

i.e., $$y^3 + 3Hy + G = 0, \qquad \dots (9)$$

where $H = ac - b^2$ and $G = a^2d - 3abc + 2b^3$. The equation (9), where the term in y^2 absent, is the **"standard form" of the cubic.**

Now to solve (9) using Cardano's method, assume that the roots are of the form $p^{1/3} + q^{1/3}$; where p and q are to be determined.

Putting $y = p^{1/3} + q^{1/3}$, we get $y^3 = p + q + 3p^{1/3}q^{1/3}(p^{1/3} + q^{1/3})$

$$= p + q + 3p^{1/3}q^{1/3}y.$$

Hence, $$y^3 - 3p^{1/3}q^{1/3}y - (p+q) = 0. \dots (10)$$

Comparing the coefficients in (9) and (10), we have $p + q = -G$, and

$p^{1/3}q^{1/3} = -H$. i.e., $p + q = -G \qquad \dots (11)$

and $pq = -H^3$.

Now, $p - q = [(p+q)^2 - 4pq]^{1/2}$

i.e., $p - q = \sqrt{G^2 + 4H^3} \qquad \dots (12)$

Using (11) and (12), solving for p and q, we get

$$p = \frac{-G + \sqrt{G^2 + 4H^3}}{2}, \quad q = \frac{-G - \sqrt{G^2 + 4H^3}}{2}$$

Then the solution is given by $y = p^{1/3}q^{1/3}$.

Remark

We notice that $p^{1/3}$ has three values, viz., m, $m\omega$ and $m\omega^2$ where m is a cube root of p (i.e., $m = \sqrt[3]{\frac{-G + \sqrt{G^2 + 4H^3}}{2}}$) and ω is one of the imaginary cube roots of unity. But we cannot take the three values of $q^{1/3}$ independently, for we have the

relation $p^{1/3}q^{1/3} = -H$. Thus if $n, n\omega,\ n\omega^2$ are the three values of $q^{1/3}$ where n is a cube root of q and ω is one of the imaginary cube roots of unity, we have to choose those pairs of cube roots of p and q such that the product of each pair is rational. Hence the three admissible roots of equation (9) are

$$m+n,\ \ m\omega+n\omega^2,\ \ m\omega^2+n\omega$$

Illustrative Examples

1. Solve the cubic $x^3 - 9x + 28 = 0$ by Cardan's method.

Solution.

Let $x = p^{1/3} + q^{1/3}$ be a solution. Then

$$x^3 = p + q + 3p^{1/3}q^{1/3}(p^{1/3} + q^{1/3})$$

Hence $x^2 - 3p^{1/3}q^{1/3}x - (p+q) = 0$

Comparing this with the given cubic equation, we get

$$p + q = -28 \qquad \ldots (13)$$

$$p^{1/3}q^{1/3} = 3$$

Hence $pq = 27$. Now, $(p-q)^2 = (p+q)^2 - 4pq = 784 - 108 = 676$

Hence $p - q = 26$ … (14)

From (13) and (14), we get $p = -1$ and $q = -27$.

Hence $p^{1/3} = -1, -\omega, -\omega^2$ and $q^{1/3} = -3, -3\omega, -3\omega^2$; where ω is one of the imaginary cube roots of unity.

Hence the roots of the given cubic are $-1-3,\ -\omega-3\omega^2$ and $-\omega^2-3\omega$.

i.e., $-4,\ -\omega-3\omega^2,\ -\omega^2-3\omega$.

Another Method

Since -4 is a root of the given cubic, $x+4$ is a factor of the polynomial in the given cubic equation. Removing the factor $x+4$, the cubic equation yields the quadratic equation $x^2 - 4x + 7 = 0$

Hence $x = \dfrac{4 \pm \sqrt{-12}}{2} = 2 \pm i\sqrt{3}$

Hence the roots of the given cubic are $-4,\ 2+i\sqrt{3},\ 2-i\sqrt{3}$

2. Solve $x^3 - 6x^2 + 3x - 2 = 0$

Solution.

To reduce to standard form [noting that $h = -b/3a = -6/3 = -2$], put $x - 2 = y$. i.e, $x = y + 2$ and obtain $(y+2)^3 - 6(y+2)^2 + 3(y+2) - 2 = 0$

i.e., $y^3 - 9y - 2 = 0$, which is the *standard form* of the cubic.

Putting $y = p^{1/3} + q^{1/3}$, taking the cube and a rearrangement yields

$$y^3 - 3p^{1/3}q^{1/3}y - (p+q) = 0.$$

Comparing this with the standard form of the cubic, we obtain

$p + q = 12$ … (15) and $p^{1/3}q^{1/3} = 3$

Hence $pq = 27$ and $p - q = \sqrt{(p+q)^2 - 4pq} = 6$ … (16)

From (15) and (16), $p = 9$ and $q = 3$.

Hence $y = 3^{1/3} + 9^{1/3}$ or $3^{1/3}\omega + 9^{1/3}\omega^2$ or $3^{1/3}\omega^2 + 9^{1/3}\omega$

Hence the roots of the given equation are

$2 + 3^{1/3} + 9^{1/3}$, $2 + 3^{1/3}\omega + 9^{1/3}\omega^2$, $2 + 3^{1/3}\omega^2 + 9^{1/3}\omega$.

1.6.4 Nature of the roots of a cubic

Let α, β, γ be the roots of the cubic

$$y^3 + 3Hy + G = 0 \qquad \text{… (17)}$$

Then the equation whose roots are $(\beta-\gamma)^2, (\gamma-\alpha)^2$ and $(\alpha-\beta)^2$ is

$$z^2 + 18Hz^2 + 81H^2z + 27(G^2 + 4H^3) = 0.$$

Hence $(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2 = -27(G^2 + 4H^3)$. … (18)

Then nature of the roots α, β, γ of Eq. (17) can be obtained by a consideration of the product in Eq. (18). Since imaginary roots occur in pairs, equation (17) will have either all real roots, or one real and two imaginary roots. The following cases can occur.

Case 1: The roots α, β, γ are all real and different. In this case

$(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2$ is positive. Therefore by Eq. (18), G^2+4H^3 is negative.

Case 2: One root, say α, is real and the other two imaginary. Let β and γ be $m \pm in$.

Then $(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2 = (2in)^2(m-in-\alpha)^2(m+in-\alpha)^2$

$$= -4n^2\{(m-\alpha)^2+n^2\}^2,$$

which is negative, whatever α, m, n may be. Therefore by Eq.(18), G^2+4H^3 is positive in this case.

Case 3: Two of the roots, say β, γ are equal. Then $(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2$, and therefore G^2+4H^3, is zero.

Case 4: α, β, γ are all equal. In this case all the three roots of equation (17) are zero. This will be so if $H = G = 0$.

Conversely, it is easy to see that

(i) when $G^2+4H^3<0$, the roots of the cubic in Eq. (17) are all real;

(ii) when $G^2+4H^3>0$, the cubic in Eq. (17) has two imaginary roots;

(iii) when $G^2+4H^3=0$, the cubic in Eq. (17) has two equal roots; and

(iv) when $G=H=0$, all the roots of the cubic in Eq. (17) are equal.

Remark

On substituting the values of G and H, it can be seen that

$$G^2+4H^3 \equiv a^2\{a^2d^2-6abcd+4ac^3+4b^3d-3b^2c^2\}.$$

The expression in brackets is called the **discriminant** of the general cubic in Eq. (7), and is denoted by Δ.

quartic equation. In this method the solution of the quartic depends on the solution of a cubic. We now describe the **Ferrari's method.**

Writing the quartic equation $u = ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$ ... (19)

We assume that $au = (ax^2 + 2bx + s)^2 - (2mx + n)^2$.

Equating coefficients of like powers of x and then eliminating m and n we will obtain a cubic equation in s. Then corresponding values of m and n will be obtained. Using these, roots of the quartic equation will be obtained.

The method is illustrated in the following problems.

Illustrative Examples

1. Solve $x^4 - 2x^3 - 12x^2 + 10x + 3 = 0$.

Solution:

Let $u = x^4 - 2x^3 - 12x^2 + 10x + 3 = 0$.

and $u = (x^2 + px + s)^2 - (mx + n)^2$ ...(20)

Then, $x^4 - 2x^3 - 12x^2 + 10x + 3 = (x^2 + px + s)^2 - (mx + n)^2$

Equating coefficients of like powers of x, we obtain

$$-2 = 2p, \quad -12 = 2s + p^2 - m^2, \quad 10 = 2ps - 2mn, \; 3 = s^2 - n^2.$$

Hence, $p = -1, \quad -12 = 2s + 1 - m^2, \; 10 = -2s - 2mn, \quad 3 = s^2 - n^2$.

Thus $p = -1, \quad m^2 = 2s + 13, \quad mn = -s - 5, \quad n^2 = s^2 - 3$.

To eliminate m and n, we note that $(mn)^2 = m^2 n^2$

which gives $(-s - 5)^2 = (2s + 13)(s^2 - 3)$ or $s^2 + 10s + 25 = 2s^3 - 6s + 13s^2 - 39$

which is the cubic equation $2s^3 - 16s + 12s^2 - 64 = 0$

or $s^3 + 6s^2 - 8s - 32 = 0$. ...(21)

(Remarks :

- Equation (21), which is a cubic in s, is known as the *reducing cubic.*
- The reducing cubic gives three values of s. These **do not** however lead to three different sets of roots for the quartic equation. They only give three different methods of factorizing the left hand side of the quartic. *Hence it is enough to find any one root of the reducing cubic.*)

By inspection, $s = -2$ is a root of the cubic equation (21).

1.7. Quartic (or Biquadratic) Equation

A **quartic function** is a polynomial of degree four and is of the form

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e, \text{ where } a \text{ is nonzero.}$$

Such a function is sometimes called a **biquadratic function,** but the latter term can occasionally also refer to a quadratic function of a square, having the form

$ax^4 + bx^2 + c,$ or a product of two quadratic factors, having the form $(ax^2 + bx + c)(dy^2 + ey + f)$.

Setting $f(x) = 0$ results in a **quartic equation** (or **biquadratic equation)** of the form

$$ax^4 + bx^3 + cx^2 + dx + e = 0 \quad \text{where } a \neq 0.$$

Quartic equation is some times called **biquadratic equation.**

Solution of Quartic Equations (Ferrari's method)

Shortly after the discovery of a method to solve the cubic equation, **Lodovico Ferrari** (1522-1565), a student of Cardano, found a similar method to solve the

Illustrative Examples

1. Solve $x^4 - 2x^3 - 12x^2 + 10x + 3 = 0$.

Solution:

Let $u = x^4 - 2x^3 - 12x^2 + 10x + 3 = 0$.

and $u = (x^2 + px + s)^2 - (mx + n)^2$ …(20)

Then, $x^4 - 2x^3 - 12x^2 + 10x + 3 = (x^2 + px + s)^2 - (mx + n)^2$

Equating coefficients of like powers of x, we obtain

$$-2 = 2p, \quad -12 = 2s + p^2 - m^2, \quad 10 = 2ps - 2mn, \; 3 = s^2 - n^2.$$

Hence, $p = -1, \quad -12 = 2s + 1 - m^2, \; 10 = -2s - 2mn, \quad 3 = s^2 - n^2.$

Thus $p = -1, \quad m^2 = 2s + 13, \; mn = -s - 5, \quad n^2 = s^2 - 3.$

To eliminate m and n, we note that $(mn)^2 = m^2 n^2$

which gives $(-s-5)^2 = (2s+13)(s^2 - 3)$ or $s^2 + 10s + 25 = 2s^3 - 6s + 13s^2 - 39$

which is the cubic equation $2s^3 - 16s + 12s^2 - 64 = 0$

or $s^3 + 6s^2 - 8s - 32 = 0.$ …(21)

(Remarks :

- Equation (21), which is a cubic in s , is known as the *reducing cubic.*
- The reducing cubic gives three values of s . These **do not** however lead to three different sets of roots for the quartic equation. They only give three different methods of factorizing the left hand side of the quartic. *Hence it is enough to find any one root of the reducing cubic.*)

By inspection, $s = -2$ is a root of the cubic equation (21).

Hence, $m^2 = 2s + 13 = 2(-2) + 13 = 9$

$n^2 = s^2 - 3 = (-2)^2 - 3 = 1.$

$mn = -s - 5 = 2 - 5 = -3.$

We take $m = 3$. Then, as $mn = -3$ and $n^2 = 1$, we take $n = -1$.

We take (Referring Remark above), $s = -2, m = 3, n = -1.$

Noting [by Eq.(20)] that $u = \{x^2 + (p+m)x + s + n\}\{x^2 + (p-m)x + s - n\}$

we have $u = \{x^2 + (-1+3)x - 2 - 1\}\{x^2 + (-1-3)x - 2 + 1\}$

or $u = \{x^2 + 2x - 3\}\{x^2 - 4x - 1\}$, or $u = (x-1)(x+3)\{x^2 - 4x - 3\}$ ...(22)

As the quadratic equation $x^2 - 4x - 3$ has roots $1+\sqrt{5}$ and $1-\sqrt{5}$, by (22), we have the roots of the given quartic equation are $x = 1, -3, 1+\sqrt{5}, 1-\sqrt{5}.$

2. Solve $x^4 - 3x^2 - 6x - 2 = 0$

Solution.

Let $u = x^4 - 3x^2 - 6x - 2 = 0.$ and $u = (x^2 + px + s)^2 - (mx + n)^2$ ...(23)

Then, $x^4 - 3x^2 - 6x - 2 = (x^2 + px + s)^2 - (mx + n)^2$

Equating coefficients of like powers of x, we obtain

$$p = 0, \quad m^2 = 2s + 3, \quad mn = 3, \quad n^2 = s^2 + 2.$$

To eliminate m and n, we note that $(mn)^2 = m^2 n^2$ which gives $9 = (2s+3)(s^2+2)$

i.e., we obtain the cubic equation, $2s^3 + 3s^2 + 4s - 3 = 0$. ... (24)

In order to find a root of (24) we proceed as follows: If α, β, γ are the roots of (24), then the equation with roots $2\alpha, 2\beta, 2\gamma$ is given by

$$2\left(\frac{y}{2}\right)^3+3\left(\frac{y}{2}\right)^2+4\left(\frac{y}{2}\right)-3=0$$

Or $\quad y^3+3y^2+8y-12=0 \qquad \dots (25)$

$y=1$ is evidently a root of equation (25). If we let $2\alpha=1$ then $\alpha=\frac{1}{2}$ is a root of (24).

i.e., $s=\frac{1}{2}$ is a solution of the reducing cubic (24). Hence

$$m^2=2\times\frac{1}{2}+3=4, \;\; n^2=\left(\frac{1}{2}\right)^2+2=\frac{9}{4}.$$

Take $m=2$. Hence, as $mn=3$, and $n^2=\frac{9}{4}$, we have to take $n=\frac{3}{2}$.

Putting $p=0, \;\; s=\frac{1}{2}, \; m=2, \; n=\frac{3}{2}$, in

$u=\{x^2+(p+m)x+s+n\}\{x^2+(p-m)x+s-n\}$, we get $u=\{x^2+2x+2\}\{x^2-2x-1\}$.

As the quadratic equation x^2+2x+2 have roots $-1+i, -1-i$ and x^2-2x-1 have roots $1+\sqrt{2}, 1-\sqrt{2}$; hence the roots of the given quartic equation are

$$1+\sqrt{2}, 1-\sqrt{2}, -1+i, -1-i.$$

3. Solve $x^4+3x^3+x^2-2=0$.

Solution.

Let $\quad u=x^4+3x^3+x^2-2=0$. and $u=(x^2+px+s)^2-(mx+n)^2 \quad \dots(26)$

Then, $x^4+3x^3+x^2-2=(x^2+px+s)^2-(mx+n)^2$

Equating coefficients of like powers of x, we obtain $\quad p=\frac{3}{2}, \quad m^2=2s+\frac{5}{4}$,

$mn=\frac{3}{2}s, \;\; n^2=s^2+2$. To eliminate m and n, we note that $(mn)^2=m^2n^2$, which gives

$\left(\frac{3}{2}s\right)^2=\left(2s+\frac{5}{4}\right)\left(s^2+2\right)$ and is the cubic equation

$4s^3-2s^2+8s+5=0. \qquad \dots(27)$

In order to find a root of (27) we proceed as follows: If α, β, γ are the roots of (24), then the equation with roots $4\alpha, 4\beta, 4\gamma$ is given by

$$4\left(\frac{y}{4}\right)^3 - 2\left(\frac{y}{4}\right)^2 + 8\left(\frac{y}{4}\right) + 5 = 0.$$

Or $$y^3 - 2y^2 + 32y + 80 = 0. \qquad \dots (28)$$

$y = -2$ is evidently a root of equation (28).

If we let $4\alpha = -2$ then $\alpha = -\frac{1}{2}$ is a root of (27). i.e., $s = -\frac{1}{2}$ is a solution of the reducing cubic (27). Hence

$$m^2 = 2s + \frac{5}{4} = -1 + \frac{5}{4} = \frac{1}{4}$$

$$n^2 = s^2 + 2 = \frac{1}{4} + 2 = \frac{9}{4}.$$

We take $m = \frac{1}{2}$. Since $mn = \frac{3}{2}s = -\frac{3}{4}$, and since $n^2 = \frac{9}{4}$, we are forced to take $n = -\frac{3}{2}$.

Hence $u = \{x^2 + (p+m)x + s + n\}\{x^2 + (p-m)x + s - n\}$ gives $u = \{x^2 + 2x - 2\}\{x^2 + x + 1\}$

Hence the roots of the given quartic equation are $x = -1 + \sqrt{3}, 1 + \sqrt{3}, \omega, \omega^2$,

where ω is an imaginary cube root of 1.

1.6. Insolvability of the Quintic.

A **quintic function** is a polynomial function of the form

$$g(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f,$$

where a, b, c, d, e and f are rational numbers, real numbers or complex numbers, and a is nonzero. In other words, a quintic function is defined by a polynomial of degree five.

If a is zero but one of the coefficients b, c, d, or e is non-zero, the function is classified as either a quartic function, cubic function, quadratic function or linear function.

As we noted above, solving linear, quadratic, cubic and quartic equations by factorization into radicals is fairly straightforward, no matter whether the roots are rational or irrational, real or complex; there are also formulae that yield the required solutions. However, ***there is no formula for general quintic equations over the rationals in terms of radicals***; this is known as the **Abel–Ruffini** theorem, first published in 1824, which was one of the first applications of group theory in algebra. This result also holds for equations of higher degrees.

This means that unlike quadratic, cubic, and quartic polynomials, the *general* quintic or all polynomials of degree greater than 5 cannot be solved algebraically in terms of a finite number of additions, subtractions, multiplications, divisions, and root extractions.

An example quintic whose roots cannot be expressed by radicals is $x^5 - x + 1 = 0$. Some fifth-degree equations can be solved by factorizing into radicals; for example, $x^5 - x^4 - x + 1 = 0$, which can be written as $(x^2 + 1)(x + 1)(x - 1)^2 = 0$, or, as another example, $x^5 - 2 = 0$, which has $\sqrt[5]{2}$ as solution.

Évariste Galois developed techniques for determining whether a given equation could be solved by radicals which gave rise to ***Galois theory***.

1.7. Descarte's Rule of Signs and Sturm's Theorem

Nature of Roots - Descarte's Rule of Signs

To determine the nature of some of the roots of a polynomial equation it is not always necessary to solve it; for instance, the truth of the following statements will be readily admitted.

1. If the coefficients of a polynomial equation are all positive, the equation has no positive root; for example, the equation

   $$x^4 + 3x^2 + 3 = 0$$

   cannot have a positive root.

2. If the coefficients of the even powers of x are all of one sign, and the coefficients of the odd powers are all of the opposite sign, the equation has no negative root; thus for example, the equation

   $$-x^8 + x^7 + x^5 - 2x^4 + x^3 - 3x^2 + 7x - 3 = 0$$

   cannot have a negative root.

3. If the equation contains only even powers of x and the coefficients are all of the same sign, the equation has no real root; thus for example, the equation

   $$-x^8 - 2x^4 - 3x^2 - 3 = 0$$

   cannot have a real root.

4. If the equation contains only odd powers of x, and the coefficients are all of the same sign, the equation has no real root except $x = 0$; thus the equation

   $$x^7 + x^5 + 3x^3 + 8x = 0$$

   has no real root except $x = 0$.

Suppose that the signs of the terms in a polynomial are $++--+---+-+-$; here the number of changes of sign is 7. We shall show that if this polynomial is multiplied by a binomial (corresponding to a positive root) whose signs are $+-$, there will be at least one more change of sign in the product than in the original polynomial.

Writing down only the signs of the terms in the multiplication, we have the following:

$$\begin{array}{l} ++--+---+-+- \\ +- \\ \hline ++--+---+-+- \\ \quad --++-+++-+-+ \\ \hline ++\pm-\pm+-\pm\pm+-+-+ \end{array}$$

Here in the last line the ambiguous sign $\pm$ is placed wherever there are two different signs to be added.

Here we see that in the product

(i) an ambiguity replaces each continuation of sign in the original polynomial;

(ii) the signs before and after an ambiguity or set of ambiguities are unlike;

(iii) a change of sign is introduced at the end.

Let us take the most unfavourable case (i.e., the case where the number of changes of sign is less) and suppose that all the ambiguities are replaced by continuations; then the sign of the terms become

$$++--+---+-+-+,$$

and the number of changes of sign is 8.

We conclude that if a polynomial is multiplied by a binomial (corresponding to a positive root) whose signs are $+-$, there will be at least one more change of sign in the product than in the original polynomial.

If then we suppose the factors corresponding to the negative and imaginary roots to be already multiplied together, each factor $x-a$ corresponding to a positive root introduces at least one change of sign; therefore no equation can have more positive roots than it has changes of sign.

Again, the roots of the equation $f(-x)=0$ are equal to those of $f(x)=0$ but opposite to them in sign; therefore the negative roots of $f(x)=0$ are the positive roots of $f(-x)=0$; but the number of these positive roots cannot exceed the number of changes of sign in $f(-x)$; that is, the number of negative roots of $f(x)=0$ cannot exceed the number of changes in sign in $f(-x)$.

All the above observations are included in the following result, known as **Descarte's Rule of Signs.**

In any polynomial equation $f(x) = 0$, the number of real positive roots cannot exceed the number of changes in the signs of the coefficients of the terms in $f(x)$, and the number of real negative roots cannot exceed the number of changes in the signs of the coefficients of $f(-x)$.

Example:

Consider the equation $f(x) = x^4 + 3x - 1 = 0$

This a polynomial equations of degree 4, and hence must have four roots.

The signs of the coefficients of $f(x)$ are + + –

Therefore, the number of changes in signs = 1

By Descarte's rule of signs, number of real positive roots ≤ 1.

Now $f(-x) = x^4 - 3x - 1 = 0$

The signs of the coefficients of $f(-x)$ are + – –

Therefore, the number of changes in signs = 1.

Hence the number of real negative roots of $f(x) = 0$ is ≤ 1.

Therefore, the maximum number of real roots is 2.

If the equation has two real roots, then the other two roots must be complex roots.

Since complex roots occur in conjugate pairs, the possibility of one real root and three complex roots is not admissible.

Also $f(0) < 0$, and $f(1) > 0$, so $f(x) = 0$ has a real roots between 0 and 1.

Therefore, the given equation must have two real roots and two complex roots.

Problem.

Discuss the nature of roots of the equation $x^9 + 5x^8 - x^3 + 7x + 2 = 0$.

Solution.

With $f(x) = x^9 + 5x^8 - x^3 + 7x + 2$, there are two changes of sign in $f(x) = 0$, and therefore there are at most two positive roots.

Again $f(-x) = -x^9 + 5x^8 + x^3 - 7x + 2$, and there are three changes of sign, therefore the given equation has at most three negative roots.

Obviously 0 is not a root of the given equation.

Hence the given equation has at most 2 + 3 + 0 = 5 real roots. Thus the given equation has at least four imaginary roots.

1.8. Exercises

1. Solve the equation $x^4 + x^3 - x^2 - 2x - 2 = 0$ given that one root is $\sqrt{2}$.
2. Form a rational quartic whose roots are 1, –1, $2 + \sqrt{3}$
3. Solve $x^5 - x^3 + 4x^2 - 3x + 2 = 0$ given that it has multiple roots.
4. Solve the equation $x^4 - 2x^3 - 21x^2 + 2^2x + 40 = 0$ whose roots are in A.P.
5. Solve the equation $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$ given that two of its roots are equal in magnitude and opposite in sign.
6. Find the condition that the roots of the equation $x^3 + px^2 + qx + r = 0$ may be in geometric progression.
7. Find the condition that the roots of the equation $x^3 - lx^2 + mx - n = 0$ may be in arithmetic progression.
8. If α, β, γ are the roots of $x^3 + px + 1 = 0$, prove that $\frac{1}{5}\Sigma\alpha^5 = \frac{1}{6}\Sigma\alpha^3 \cdot \Sigma\alpha^2$.
9. If α, β, γ are the roots of $x^3 + qx + r = 0$, then find the values of $\Sigma\frac{1}{\beta+\gamma}$ and $\Sigma(\beta-\gamma)^2$.
10. Prove that the sum of the ninth powers of the roots of $x^3 + 3x + 9 = 0$ is zero.
11. If α, β, γ are the roots of $x^3 - 7x + 7 = 0$, find the value of $\alpha^{-4} + \beta^{-4} + \gamma^{-4}$.
12. Find the equation whose roots are the roots of the equation

    $3x^4 + 7x^3 - 15x^2 + x - 2 = 0$, each increased by 7.
13. Remove the second term of the equation $x^3 - 6x^2 + 4x - 7 = 0$.
14. Solve the equation $x^4 - 8x^3 + 19x^2 - 12x + 2 = 0$ by removing its second term.
15. If α, β, γ are the roots of $x^3 + px + q = 0$, form the equation whose roots are $\alpha^2 + \beta\gamma, \beta^2 + \gamma\alpha, \gamma^2 + \alpha\beta$.
16. If α, β, γ are the roots of the equation $x^3 + px + q = 0$, find the equation whose roots are $\frac{\beta}{\gamma} + \frac{\gamma}{\beta}, \frac{\gamma}{\alpha} + \frac{\alpha}{\gamma}, \frac{\alpha}{\beta} + \frac{\beta}{\alpha}$.
17. Solve $6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0$.
18. Solve $x^5 - 5x^3 + 5x^2 - 1 = 0$.

20. Solve $2x^3+3x^2+3x+1=0$ using Cardan's method.

21. Solve $x^4+2x^3-7x^2-8x+12=0$ using Ferrari's method.

22. Solve $x^4+6x^3+4x^2-32=0$ using Ferrari's method.

23. Find the greatest possible number of real roots of the equation

$$x^5-6x^2-4x+5=0$$

24. Find the number of real roots of $x^7-x^5-x^4-6x^2+7=0$.

25. Show that $x^5-2x^2+7=0$ has at least two imaginary roots.

26. Determine the nature of the roots of the equation $x^4+3x^2+2x-7=0$.

1.3.2. Theorem (Newton's Theorem on the Sum of the Powers of the Roots)

If $\alpha_1, \alpha_2, ..., \alpha_n$ are the roots of the equation $x^n + P_1 x^{n-1} + P_2 x^{n-2} + ... + P_n = 0$,

and $S_r = \alpha_1^r + ... + \alpha_n^r$. Then, $S_r + S_{r-1}P_1 + ... + S_1 P_{r-1} + rP_r = 0, \quad if \quad r \le n.$

and $S_r + S_{r-1}P_1 + S_{r-2}P_2 + ... + S_{r-n}P_n = 0$ if $r > n$.

Proof:

We have $x^n + P_1 x^{n-1} + P_2 x^{n-2} + ... + P_n = (x-\alpha_1)(x-\alpha_2).....(x-\alpha_n)$

Put $x = \dfrac{1}{y}$

$$\Rightarrow \quad \frac{1}{y^n} + \frac{P_1}{y^{n-1}} + \frac{P_2}{y^{n-2}} + ... + P_n = (\frac{1}{y} - \alpha_1)(\frac{1}{y} - \alpha_2).....(\frac{1}{y} - \alpha_n),$$

and then multiplying by y^n, we obtain:

$$1 + P_1 y + P_2 y^2 + ... + P_n y^n = (1-\alpha_1 y)(1-\alpha_2 y).....(1-\alpha_n y)$$

Taking logarithm and differentiating w.r.t y, we get

$$\frac{P_1 + 2P_2 y + 3P_3 y^2 + ... + nP_n y^{n-1}}{1 + P_1 y + P_2 y^2 + ... + P_n y^n} = \frac{-\alpha_1}{1-\alpha_1 y} + \frac{-\alpha_2}{1-\alpha_2 y} + ... + \frac{-\alpha_n}{1-\alpha_n y}$$

$$= -\alpha_1(1-\alpha_1 y)^{-1} - \alpha_2(1-\alpha_2 y)^{-1} - ...... - \alpha_n(1-\alpha_n y)^{-1}$$

$$= -\alpha_1(1+\alpha_1 y + \alpha^2 y^2 + ...) - \alpha_2(1+\alpha_2 y + \alpha_2^2 y^2 + ...) - ...... - \alpha_n(1+\alpha_n y + \alpha_n^2 y^2 + ...)$$

$$= -S_1 - S_2 y - S_3 y^2 - ..... - S_{r+1} y^r - ...$$

Cross – multiplying, we get

$$P_1 + 2P_2 y + 3P_3 y^2 + ... + nP_n y^{n-1} = -(1 + P_1 y + P_2 y^2 + ... + P_n y^n)$$

$$[S_1 + S_2 y + ... + S_{r+1} y^r + ...]$$

Equating coefficients of like powers of y, we see that

$$P_1 = -S_1 \quad \Rightarrow \quad S_1 + 1.P_1 = 0$$

$$2P_2 = -S_2 - S_1 P_1 \quad \Rightarrow \quad S_2 + S_1 P_1 + 2P_2 = 0$$

$$3p_3 = -S_3 - S_2 P_1 - S_1 P_2 \quad \Rightarrow \quad S_3 + S_2 P_1 + S_1 P_2 + 3P_3 = 0, \text{ and so on .}$$

If $r \le n$, equating coefficients of y^{r-1} on both sides,

$$rP_r = -S_r - S_{r-1}P_1 - S_{r-2}P_2 - ...... - S_1 P_{r-1}$$

$$\Rightarrow \quad S_r + S_{r-1}P_1 + S_{r-2}P_2 + \ldots\ldots + S_1P_{r-1} + rP_r = 0$$

If $r > n$, then $r-1 > n-1$.

Equating coefficients of y^{r-1} on both sides,

$$0 = -S_r - S_{r-1}P_1 - S_{r-2}P_2 - \ldots\ldots - S_{r-n}P_n$$

$$\text{i.e.,} \quad S_r + S_{r-1}P_1 + S_{r-2}P_2 + \ldots\ldots + S_{r-n}P_n = 0$$

Remark:

To find the sum of the negative powers of the roots of $f(x) = 0$, put $x =$

and find the sums of the corresponding positive powers of the roots of the new equation.

Illustrative Examples

1. If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, find the value of the following in terms of the coefficients.

$$(i) \Sigma \frac{1}{\beta\gamma} (ii) \Sigma \frac{1}{\alpha} (iii) \Sigma \alpha^2\beta$$

Solution:

Here $\alpha + \beta + \gamma = -p$, $\alpha\beta + \beta\gamma + \alpha\gamma = q$, $\alpha\beta\gamma = -r$

(i) $$\sum \frac{1}{\beta\gamma} = \frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\alpha\gamma} = \frac{\alpha+\beta+\gamma}{\alpha\beta\gamma} = \frac{-p}{-r} = \frac{p}{r}$$

(ii) $$\sum \frac{1}{\alpha} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta+\beta\gamma+\alpha\gamma}{\alpha\beta\gamma} = \frac{q}{-r} = -\frac{q}{r}$$

(iii) $$\Sigma \alpha^2\beta = \alpha^2\beta + \beta^2\alpha + \gamma^2\alpha + \gamma^2\beta + \alpha^2\gamma + \beta^2\gamma$$

$$= (\alpha\beta + \beta\gamma + \alpha\gamma)(\alpha + \beta + \gamma) - 3\alpha\beta\gamma = (q \,.\, -p) - 3\,(-r) = 3r - pq \;.$$

2. If α is an imaginary root of the equation $x^7 - 1 = 0$ form the equation whose roots are $\alpha + \alpha^6, \alpha^2 + \alpha^5, \alpha^3 + \alpha^4$.

Solution:

Let $a = \alpha + \alpha^6 \quad b = \alpha^2 + \alpha^5 \quad c = \alpha^3 + \alpha^4$

The required equation is $(x - a)(x - b)(x - c) = 0$

i.e., $x^3 - (a+b+c)x^2 + (ab+bc+ac)x - abc = 0$ …………… (1)

$$a + b + c = \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 = \frac{\alpha(\alpha^6 - 1)}{\alpha - 1} = \frac{\alpha^7 - \alpha}{\alpha - 1} = \frac{1 - \alpha}{\alpha - 1} = -1$$

(Since α is a root of $x^7 - 1 = 0$, we have $\alpha^7 = 1$)

Similarly we can find that $ab + bc + ac = -2$, $abc = 1$.

Thus from (1), the required equation is

$$x^3 + x^2 - 2x - 1 = 0$$

3. If α, β, γ are the roots of $x^3 + 3x^2 + 2x + 1 = 0$, find $\sum\alpha^3$ and $\sum\alpha^{-2}$.

Solution:

Here $\alpha + \beta + \gamma = -3$, $\alpha\beta + \beta\gamma + \alpha\gamma = 2$, $\alpha\beta\gamma = -1$

Using the identity $a^3+b^3+c^3 - 3abc = (a+b+c)(a^2+b^2+c^2 - ab - bc - ac)$, we find that

$$\sum\alpha^3 = (\alpha+\beta+\gamma)\left[\alpha^2+\beta^2+\gamma^2 - (\alpha\beta+\beta\gamma+\alpha\gamma)\right] + 3\alpha\beta\gamma$$

$$= (\alpha+\beta+\gamma)\left[[(\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\alpha\gamma)] - (\alpha\beta+\beta\gamma+\alpha\gamma)\right] + 3\alpha\beta\gamma$$

$$= -3[(9-4)-2] - 3$$

$$= -9 - 3 = -12$$

Also, $$\sum\alpha^{-2} = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{\beta^2\gamma^2 + \alpha^2\gamma^2 + \beta^2\alpha^2}{\alpha^2\beta^2\gamma^2}$$

$$= \frac{(\alpha\beta+\beta\gamma+\alpha\gamma)^2 - 2\sum\alpha^2\beta\gamma}{\alpha^2\beta^2\gamma^2} \quad \text{…………….. (1)}$$

We have:

$$\sum\alpha^2\beta\gamma = (\alpha+\beta+\gamma)\alpha\beta\gamma = -3 . -1 = 3$$

$$(1) \Rightarrow \sum\alpha^{-2} = \frac{4 - 2.3}{1} = -2$$

4. Find the sum of the 4^{th} powers of the roots of the equation $x^4 - 5x^3 + x - 1 = 0$.

Solution:

Let $f(x) = x^4 - 5x^3 + x - 1 = 0$

Then $f^1(x) = 4x^3 - 15x^2 + 1$

Now, $\dfrac{xf^{1}(x)}{f(x)}$ can be evaluated as follows :

$$
\begin{array}{r|l}
 & \underline{4+5+25+122} + 609 + \ldots\ldots \\
1-5+0+1-1 & 4-15+0+1+0 \\
 & \underline{4-20+0+4-4} \\
 & \quad 5+0-3+4 \\
 & \quad \underline{5-25+0+5-5} \\
 & \qquad 25-3-1+5 \\
 & \qquad \underline{25-125+0+25-25} \\
 & \qquad\quad 122-1-20+25 \\
 & \qquad\quad \underline{122-610+0+122-122} \\
 & \qquad\qquad 609-20-97+122 \\
 & \qquad\qquad \underline{609-3045+0+609-609} \\
 & \qquad\qquad\qquad \ldots\ldots\ldots\ldots\ldots\ldots
\end{array}
$$

Therefore,

$$\frac{xf^{|}(x)}{f(x)} = 4 + \frac{5}{x} + \frac{25}{x^2} + \frac{122}{x^3} + \frac{609}{x^4} + \ldots\ldots$$

Sum of the fourth powers of the roots = coefficient of x^{-4}.

= 609.

5. If $\alpha+\beta+\gamma=1$, $\alpha^2+\beta^2+\gamma^2=2$, $\alpha^3+\beta^3+\gamma^3=3$. Find $\alpha^4+\beta^4+\gamma^4$.

Solution:

Let $x^3 + P_1x^2 + P_2x + P_3 = 0$ be the equation whose roots are α, β, γ, then

$$\alpha+\beta+\gamma = -P_1 \Rightarrow P_1 = -1$$

By Newton's theorem,

$$S_2 + S_1P_1 + 2P_2 = 0$$

i.e., $2 + 1.(-1) + 2P_2 = 0 \Rightarrow P_2 = {}^{-1}/_2$

Again, by Newton's theorem

$$S_3 + S_2P_1 + S_1P_2 + 3P_3 = 0$$

i.e., $3 + 2. - 1 + 1.{}^{-1}/_2 + 3.P_3 = 0$

$\Rightarrow P_3 = {}^{-1}/_6$

Also $S_4 + S_3P_1 + S_2P_2 + S_1P_3 = 0$ (By Newton's theorem for the case $r < n$)

Substituting and simplifying, we obtain $S_4 = {}^{25}/_6$

Thus $\alpha^4 + \beta^4 + \gamma^4 = \dfrac{25}{6}$

6. Calculate the sum of the cubes of the roots of $x^4 + 2x + 3 = 0$

Solution:

Let the given equation be

$$x^4 + P_1x^3 + P_2x^2 + P_3x + P_4 = 0$$

Here $P_1 = P_2 = 0$, $P_3 = 2$ and $P_4 = 3$

By Newton's theorem, $S_3 + S_2P_1 + S_1P_2 + 3P_3 = 0$

i.e., $S_3 + 0 + 0 + 3 . 2 = 0$

$\Rightarrow S_3 = -6$

i.e., sum of the cubes of the roots of $x^4 + 2x + 3 = 0$, is -6.