رياضيات (2) جزء التفاضل الجزئي

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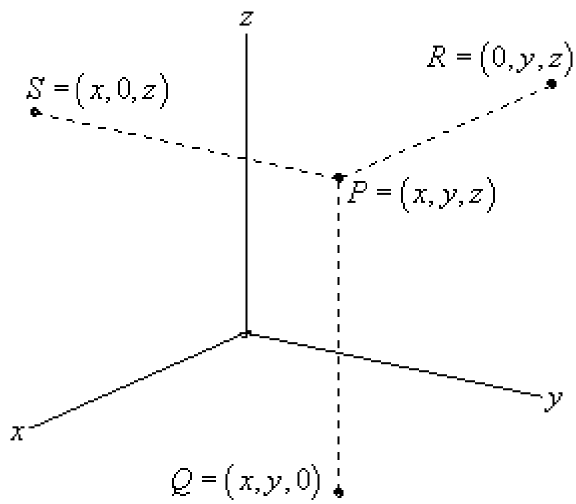
The 3-D Coordinate System

We'll start the chapter off with a fairly short discussion introducing the 3-D coordinate system and the conventions that we'll be using. We will also take a brief look at how the different coordinate systems can change the graph of an equation.

Let's first get some basic notation out of the way. The 3-D coordinate system is often denoted by $\mathbb{R}^3$. Likewise the 2-D coordinate system is often denoted by $\mathbb{R}^2$ and the 1-D coordinate system is denoted by $\mathbb{R}$. Also, as you might have guessed then a general *n* dimensional coordinate system is often denoted by $\mathbb{R}^n$.

Next, let's take a quick look at the basic coordinate system.



This is the standard placement of the axes in this class. It is assumed that only the positive directions are shown by the axes. If we need the negative axes for any reason we will put them in as needed.

Also note the various points on this sketch. The point *P* is the general point sitting out in 3-D space. If we start at *P* and drop straight down until we reach a *z*-coordinate of zero we arrive at the point *Q*. We say that *Q* sits in the *xy*-plane. The *xy*-plane corresponds to all the points which have a zero *z*-coordinate. We can also start at *P* and move in the other two directions as shown to get points in the *xz*-plane (this is *S* with a *y*-coordinate of zero) and the *yz*-plane (this is *R* with an *x*-coordinate of zero).

Collectively, the *xy*, *xz*, and *yz*-planes are sometimes called the coordinate planes. In the remainder of this class you will need to be able to deal with the various coordinate planes so make sure that you can.

Also, the point *Q* is often referred to as the projection of *P* in the *xy*-plane. Likewise, *R* is the projection of *P* in the *yz*-plane and *S* is the projection of *P* in the *xz*-plane.

Many of the formulas that you are used to working with in $\mathbb{R}^2$ have natural extensions in $\mathbb{R}^3$. For instance the distance between two points in $\mathbb{R}^2$ is given by,

$$d\left(P_1, P_2\right) = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$$

While the distance between any two points in $\mathbb{R}^3$ is given by,

$$d\left(P_1, P_2\right) = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + \left(z_2 - z_1\right)^2}$$

Likewise, the general equation for a circle with center (h, k) and radius r is given by,

$$\left(x-h\right)^2 + \left(y-k\right)^2 = r^2$$

and the general equation for a sphere with center (h, k, l) and radius r is given by,

$$\left(x-h\right)^2 + \left(y-k\right)^2 + \left(z-l\right)^2 = r^2$$

With that said we do need to be careful about just translating everything we know about $\mathbb{R}^2$ into $\mathbb{R}^3$ and assuming that it will work the same way. A good example of this is in graphing to some extent. Consider the following example.

Example 1 Graph $x = 3$ in $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$.

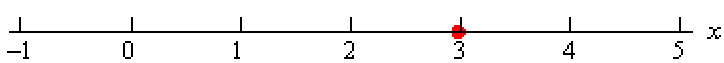
Solution

In $\mathbb{R}$ we have a single coordinate system and so $x = 3$ is a point in a 1-D coordinate system.

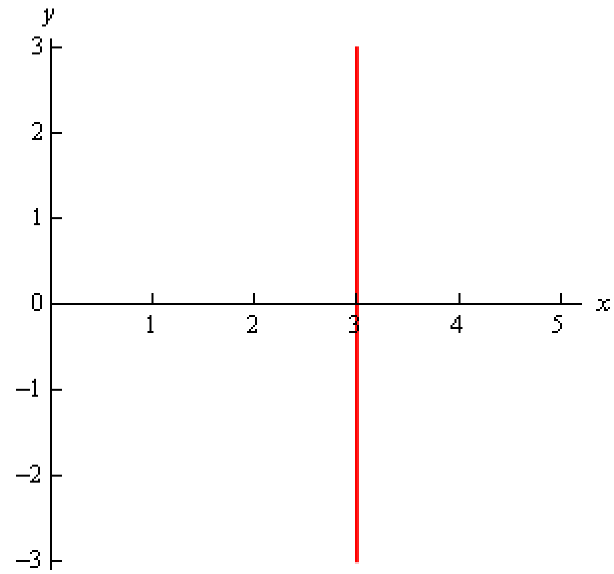
In $\mathbb{R}^2$ the equation $x = 3$ tells us to graph all the points that are in the form $(3, y)$. This is a vertical line in a 2-D coordinate system.

In $\mathbb{R}^3$ the equation $x = 3$ tells us to graph all the points that are in the form $(3, y, z)$. If you go back and look at the coordinate plane points this is very similar to the coordinates for the *yz*-plane except this time we have $x = 3$ instead of $x = 0$. So, in a 3-D coordinate system this is a plane that will be parallel to the *yz*-plane and pass through the *x*-axis at $x = 3$.

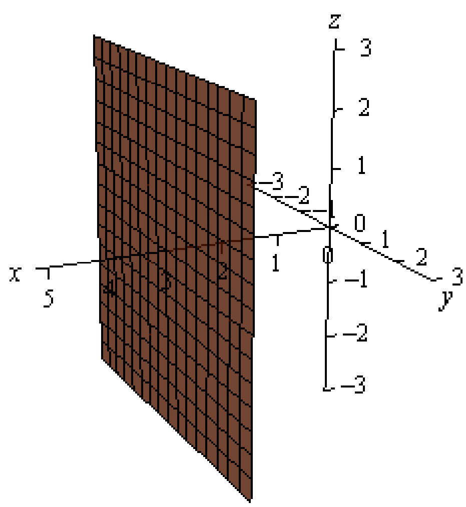
Here is the graph of $x = 3$ in $\mathbb{R}$.

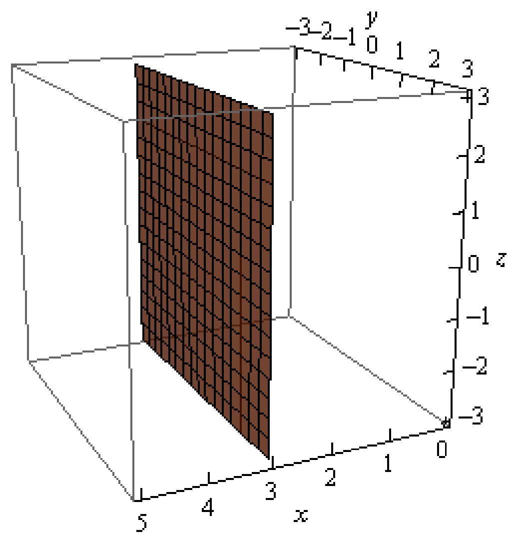


Here is the graph of $x = 3$ in $\mathbb{R}^2$.



Finally, here is the graph of $x = 3$ in $\mathbb{R}^3$. Note that we've presented this graph in two different styles. On the left we've got the traditional axis system that we're used to seeing and on the right we've put the graph in a box. Both views can be convenient on occasion to help with perspective and so we'll often do this with 3D graphs and sketches.





Note that at this point we can now write down the equations for each of the coordinate planes as well using this idea.

$$
\begin{array}{ll}
z = 0 & xy-\text{plane} \\
y = 0 & xz-\text{plane} \\
x = 0 & yz-\text{plane}
\end{array}
$$

Let's take a look at a slightly more general example.

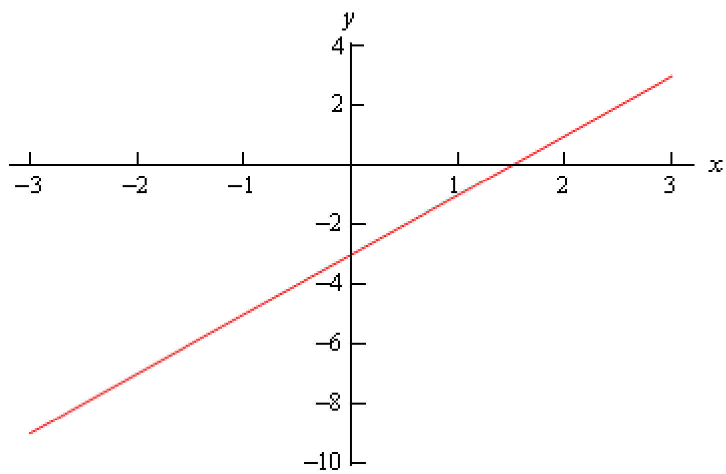
Example 2 Graph $y = 2x - 3$ in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Solution
Of course we had to throw out $\mathbb{R}$ for this example since there are two variables which means that we can't be in a 1-D space.

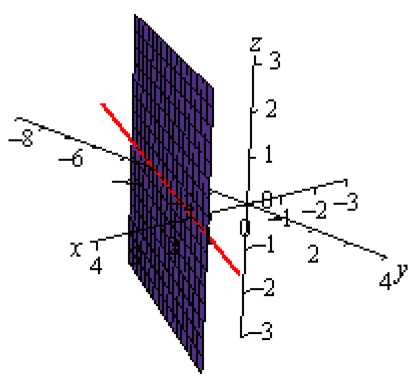
In $\mathbb{R}^2$ this is a line with slope 2 and a *y* intercept of -3.

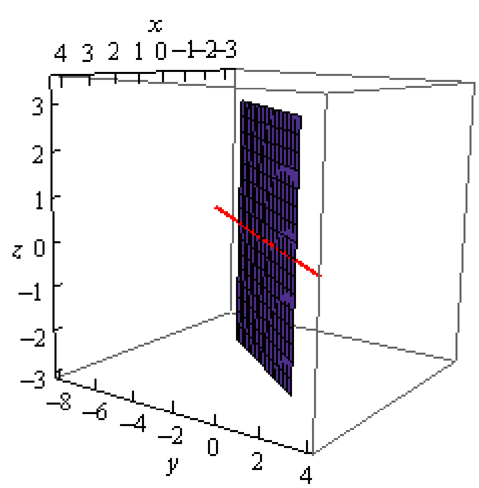
However, in $\mathbb{R}^3$ this is not necessarily a line. Because we have not specified a value of *z* we are forced to let *z* take any value. This means that at any particular value of *z* we will get a copy of this line. So, the graph is then a vertical plane that lies over the line given by $y = 2x - 3$ in the *xy*-plane.

Here is the graph in $\mathbb{R}^2$.



here is the graph in $\mathbb{R}^3$.





Notice that if we look to where the plane intersects the *xy*-plane we will get the graph of the line in $\mathbb{R}^2$ as noted in the above graph by the red line through the plane.

Let's take a look at one more example of the difference between graphs in the different coordinate systems.

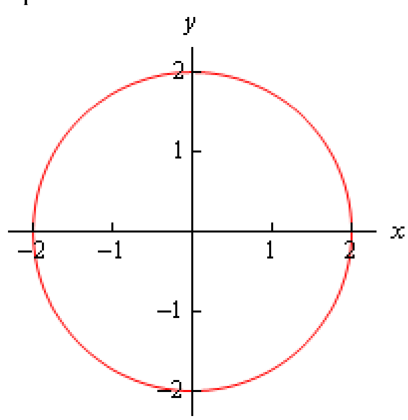
Example 3 Graph $x^2 + y^2 = 4$ in $\mathbb{R}^2$ and $\mathbb{R}^3$.

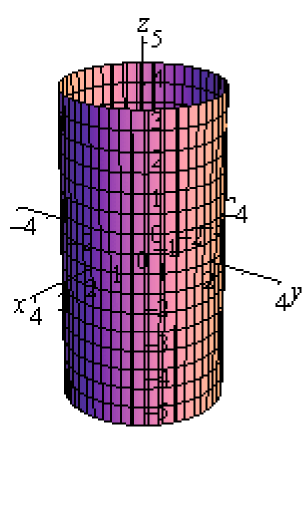
Solution
As with the previous example this won't have a 1-D graph since there are two variables.

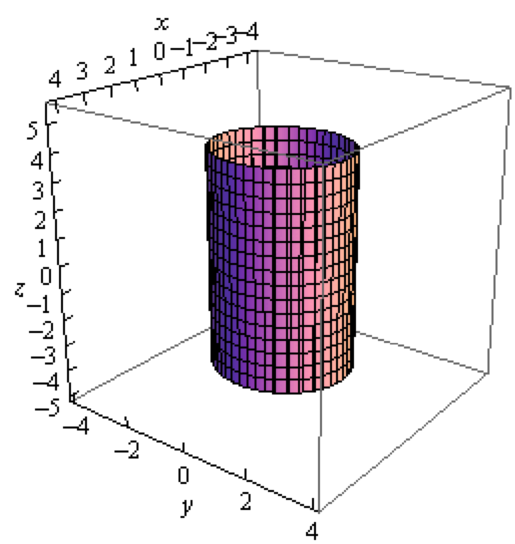
In $\mathbb{R}^2$ this is a circle centered at the origin with radius 2.

In $\mathbb{R}^3$ however, as with the previous example, this may or may not be a circle. Since we have not specified z in any way we must assume that z can take on any value. In other words, at any value of z this equation must be satisfied and so at any value z we have a circle of radius 2 centered on the z-axis. This means that we have a cylinder of radius 2 centered on the z-axis.

Here are the graphs for this example.







Notice that again, if we look to where the cylinder intersects the xy-plane we will again get the circle from $\mathbb{R}^2$.

We need to be careful with the last two examples. It would be tempting to take the results of these and say that we can't graph lines or circles in $\mathbb{R}^3$ and yet that doesn't really make sense. There is no reason for there to not be graphs of lines or circles in $\mathbb{R}^3$. Let's think about the example of the circle. To graph a circle in $\mathbb{R}^3$ we would need to do something like $x^2 + y^2 = 4$ at $z = 5$. This would be a circle of radius 2 centered on the *z*-axis at the level of $z = 5$. So, as long as we specify a *z* we will get a circle and not a cylinder. We will see an easier way to specify circles in a later section.

We could do the same thing with the line from the second example. However, we will be looking at lines in more generality in the next section and so we'll see a better way to deal with lines in $\mathbb{R}^3$ there.

The point of the examples in this section is to make sure that we are being careful with graphing equations and making sure that we always remember which coordinate system that we are in.

Another quick point to make here is that, as we've seen in the above examples, many graphs of equations in $\mathbb{R}^3$ are surfaces. That doesn't mean that we can't graph curves in $\mathbb{R}^3$. We can and will graph curves in $\mathbb{R}^3$ as well as we'll see later in this chapter.

Equations of Lines

In this section we need to take a look at the equation of a line in $\mathbb{R}^3$. As we saw in the previous section the equation $y = mx + b$ does not describe a line in $\mathbb{R}^3$, instead it describes a plane. This doesn't mean however that we can't write down an equation for a line in 3-D space. We're just going to need a new way of writing down the equation of a curve.

So, before we get into the equations of lines we first need to briefly look at vector functions. We're going to take a more in depth look at vector functions later. At this point all that we need to worry about is notational issues and how they can be used to give the equation of a curve.

The best way to get an idea of what a vector function is and what its graph looks like is to look at an example. So, consider the following vector function.

$$\vec{r}\left(t\right) = \left\langle t, 1 \right\rangle$$

A vector function is a function that takes one or more variables, one in this case, and returns a vector. Note as well that a vector function can be a function of two or more variables. However, in those cases the graph may no longer be a curve in space.

The vector that the function gives can be a vector in whatever dimension we need it to be. In the example above it returns a vector in $\mathbb{R}^2$. When we get to the real subject of this section, equations of lines, we'll be using a vector function that returns a vector in $\mathbb{R}^3$

Now, we want to determine the graph of the vector function above. In order to find the graph of our function we'll think of the vector that the vector function returns as a position vector for points on the graph. Recall that a position vector, say $\vec{v} = \left\langle a, b \right\rangle$, is a vector that starts at the origin and ends at the point $\left(a, b\right)$.

So, to get the graph of a vector function all we need to do is plug in some values of the variable and then plot the point that corresponds to each position vector we get out of the function and play connect the dots. Here are some evaluations for our example.

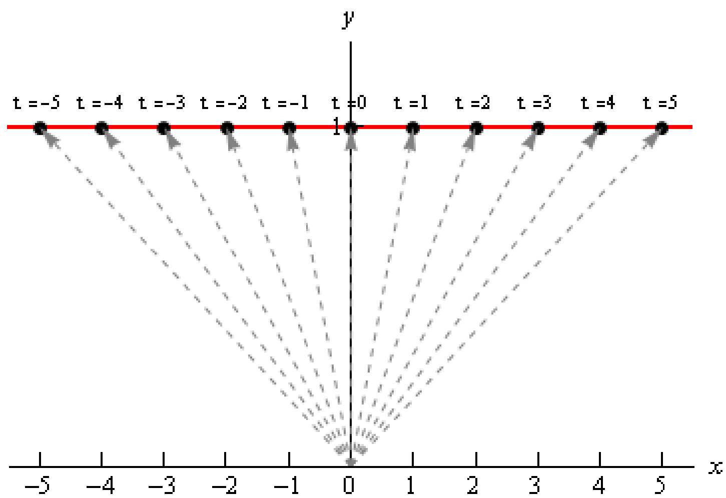
$$\vec{r}\left(-3\right) = \left\langle -3, 1 \right\rangle \qquad \vec{r}\left(-1\right) = \left\langle -1, 1 \right\rangle \qquad \vec{r}\left(2\right) = \left\langle 2, 1 \right\rangle \qquad \vec{r}\left(5\right) = \left\langle 5, 1 \right\rangle$$

So, each of these are position vectors representing points on the graph of our vector function. The points,

$$\left(-3, 1\right) \qquad \left(-1, 1\right) \qquad \left(2, 1\right) \qquad \left(5, 1\right)$$

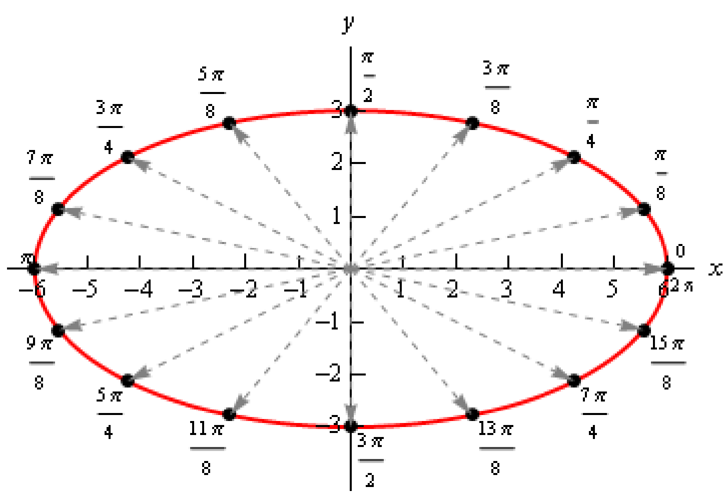
are all points that lie on the graph of our vector function.

If we do some more evaluations and plot all the points we get the following sketch.



In this sketch we've included the position vector (in gray and dashed) for several evaluations as well as the *t* (above each point) we used for each evaluation. It looks like, in this case the graph of the vector equation is in fact the line $y = 1$.

Here's another quick example. Here is the graph of $\vec{r}(t) = \langle 6\cos t, 3\sin t \rangle$.



In this case we get an ellipse. It is important to not come away from this section with the idea that vector functions only graph out lines. We'll be looking at lines in this section, but the graphs of vector functions do not have to be lines as the example above shows.

We'll leave this brief discussion of vector functions with another way to think of the graph of a vector function. Imagine that a pencil/pen is attached to the end of the position vector and as we increase the variable the resulting position vector moves and as it moves the pencil/pen on the end sketches out the curve for the vector function.

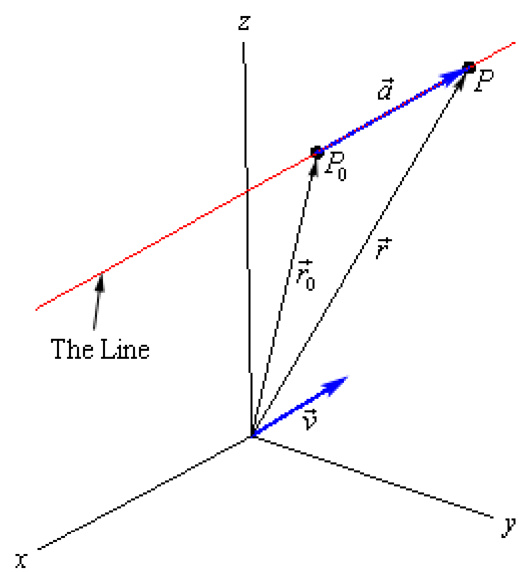
Okay, we now need to move into the actual topic of this section. We want to write down the equation of a line in $\mathbb{R}^3$ and as suggested by the work above we will need a vector function to do this. To see how we're going to do this let's think about what we need to write down the equation of a line in $\mathbb{R}^2$. In two dimensions we need the slope (*m*) and a point that was on the line in order to write down the equation.

In $\mathbb{R}^3$ that is still all that we need except in this case the "slope" won't be a simple number as it was in two dimensions. In this case we will need to acknowledge that a line can have a three dimensional slope. So, we need something that will allow us to describe a direction that is potentially in three dimensions. We already have a quantity that will do this for us. Vectors give directions and can be three dimensional objects.

So, let's start with the following information. Suppose that we know a point that is on the line, $P_0 = (x_0, y_0, z_0)$, and that $\vec{v} = \langle a, b, c \rangle$ is some vector that is parallel to the line. Note, in all likelihood, $\vec{v}$ will not be on the line itself. We only need $\vec{v}$ to be parallel to the line. Finally, let $P = (x, y, z)$ be any point on the line.

Now, since our "slope" is a vector let's also represent the two points on the line as vectors. We'll do this with position vectors. So, let $\vec{r_0}$ and $\vec{r}$ be the position vectors for P_0 and P respectively. Also, for no apparent reason, let's define $\vec{a}$ to be the vector with representation $\overrightarrow{P_0 P}$.

We now have the following sketch with all these points and vectors on it.



Now, we've shown the parallel vector, $\vec{v}$, as a position vector but it doesn't need to be a position vector. It can be anywhere, a position vector, on the line or off the line, it just needs to be parallel to the line.

Next, notice that we can write $\vec{r}$ as follows,

$$\vec{r} = \vec{r_0} + \vec{a}$$

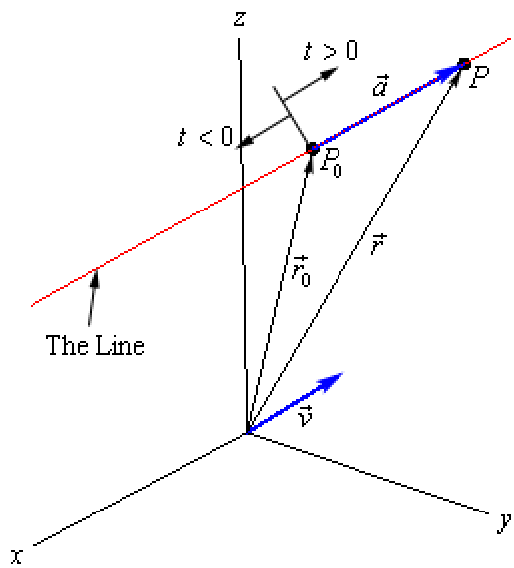
If you're not sure about this go back and check out the sketch for vector addition in the vector arithmetic section. Now, notice that the vectors $\vec{a}$ and $\vec{v}$ are parallel. Therefore there is a number, *t*, such that

$$\vec{a} = t\,\vec{v}$$

We now have,

$$\vec{r} = \overrightarrow{r_0} + t\,\vec{v} = \left\langle x_0, y_0, z_0 \right\rangle + t\left\langle a, b, c \right\rangle$$

This is called the **vector form of the equation of a line**. The only part of this equation that is not known is the t. Notice that $t\,\vec{v}$ will be a vector that lies along the line and it tells us how far from the original point that we should move. If t is positive we move away from the original point in the direction of $\vec{v}$ (right in our sketch) and if t is negative we move away from the original point in the opposite direction of $\vec{v}$ (left in our sketch). As t varies over all possible values we will completely cover the line. The following sketch shows this dependence on t of our sketch.



There are several other forms of the equation of a line. To get the first alternate form let's start with the vector form and do a slight rewrite.

$$\vec{r} = \left\langle x_0, y_0, z_0 \right\rangle + t\left\langle a, b, c \right\rangle$$
$$\left\langle x, y, z \right\rangle = \left\langle x_0 + ta, y_0 + tb, z_0 + tc \right\rangle$$

The only way for two vectors to be equal is for the components to be equal. In other words,

$$x = x_0 + ta$$
$$y = y_0 + tb$$
$$z = z_0 + tc$$

This set of equations is called the **parametric form of the equation of a line**. Notice as well that this is really nothing more than an extension of the parametric equations we've seen previously. The only difference is that we are now working in three dimensions instead of two dimensions.

To get a point on the line all we do is pick a t and plug into either form of the line. In the vector form of the line we get a position vector for the point and in the parametric form we get the actual coordinates of the point.

There is one more form of the line that we want to look at. If we assume that a, b, and c are all non-zero numbers we can solve each of the equations in the parametric form of the line for t. We can then set all of them equal to each other since t will be the same number in each. Doing this gives the following,

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

This is called the **symmetric equations of the line**.

If one of a, b, or c does happen to be zero we can still write down the symmetric equations. To see this let's suppose that $b = 0$. In this case t will not exist in the parametric equation for y and so we will only solve the parametric equations for x and z for t. We then set those equal and acknowledge the parametric equation for y as follows,

$$\frac{x - x_0}{a} = \frac{z - z_0}{c} \qquad y = y_0$$

Let's take a look at an example.

Example 1 Write down the equation of the line that passes through the points $(2, -1, 3)$ and $(1, 4, -3)$. Write down all three forms of the equation of the line.

Solution

To do this we need the vector $\vec{v}$ that will be parallel to the line. This can be any vector as long as it's parallel to the line. In general, $\vec{v}$ won't lie on the line itself. However, in this case it will. All we need to do is let $\vec{v}$ be the vector that starts at the second point and ends at the first point. Since these two points are on the line the vector between them will also lie on the line and will hence be parallel to the line. So,

$$\vec{v} = \langle 1, -5, 6 \rangle$$

Note that the order of the points was chosen to reduce the number of minus signs in the vector. We could just have easily gone the other way.

Once we've got $\vec{v}$ there really isn't anything else to do. To use the vector form we'll need a point on the line. We've got two and so we can use either one. We'll use the first point. Here is the vector form of the line.

$$\vec{r} = \langle 2, -1, 3 \rangle + t \langle 1, -5, 6 \rangle = \langle 2 + t, -1 - 5t, 3 + 6t \rangle$$

Once we have this equation the other two forms follow. Here are the parametric equations of the line.

$$x = 2 + t$$
$$y = -1 - 5t$$
$$z = 3 + 6t$$

Here is the symmetric form.

$$\frac{x-2}{1} = \frac{y+1}{-5} = \frac{z-3}{6}$$

Example 2 Determine if the line that passes through the point $(0,-3,8)$ and is parallel to the line given by $x = 10 + 3t$, $y = 12t$ and $z = -3 - t$ passes through the *xz*-plane. If it does give the coordinates of that point.

Solution

To answer this we will first need to write down the equation of the line. We know a point on the line and just need a parallel vector. We know that the new line must be parallel to the line given by the parametric equations in the problem statement. That means that any vector that is parallel to the given line must also be parallel to the new line.

Now recall that in the parametric form of the line the numbers multiplied by *t* are the components of the vector that is parallel to the line. Therefore, the vector,

$$\vec{v} = \langle 3, 12, -1 \rangle$$

is parallel to the given line and so must also be parallel to the new line.

The equation of new line is then,

$$\vec{r} = \langle 0, -3, 8 \rangle + t\langle 3, 12, -1 \rangle = \langle 3t, -3 + 12t, 8 - t \rangle$$

If this line passes through the *xz*-plane then we know that the *y*-coordinate of that point must be zero. So, let's set the *y* component of the equation equal to zero and see if we can solve for *t*. If we can, this will give the value of *t* for which the point will pass through the *xz*-plane.

$$-3 + 12t = 0 \qquad \Rightarrow \qquad t = \frac{1}{4}$$

So, the line does pass through the *xz*-plane. To get the complete coordinates of the point all we need to do is plug $t = \frac{1}{4}$ into any of the equations. We'll use the vector form.

$$\vec{r} = \left\langle 3\left(\frac{1}{4}\right), -3 + 12\left(\frac{1}{4}\right), 8 - \frac{1}{4} \right\rangle = \left\langle \frac{3}{4}, 0, \frac{31}{4} \right\rangle$$

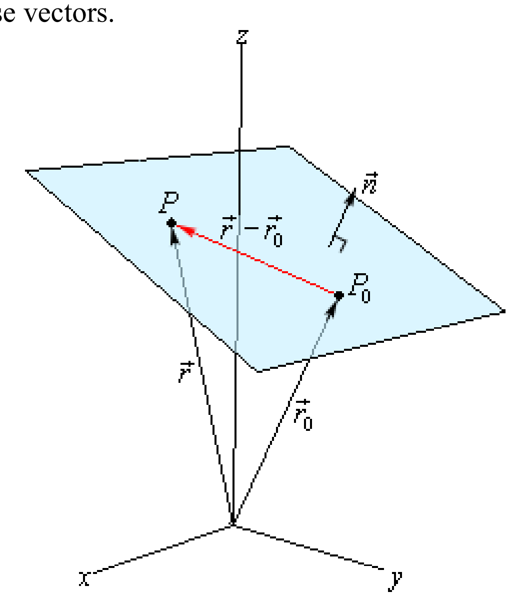
Recall that this vector is the position vector for the point on the line and so the coordinates of the point where the line will pass through the *xz*-plane are $\left(\frac{3}{4}, 0, \frac{31}{4}\right)$.

Equations of Planes

In the first section of this chapter we saw a couple of equations of planes. However, none of those equations had three variables in them and were really extensions of graphs that we could look at in two dimensions. We would like a more general equation for planes.

So, let's start by assuming that we know a point that is on the plane, $P_0 = \left(x_0, y_0, z_0\right)$. Let's also suppose that we have a vector that is orthogonal (perpendicular) to the plane, $\vec{n} = \left\langle a, b, c \right\rangle$. This vector is called the **normal vector**. Now, assume that $P = \left(x, y, z\right)$ is any point in the plane. Finally, since we are going to be working with vectors initially we'll let $\vec{r_0}$ and $\vec{r}$ be the position vectors for P_0 and P respectively.

Here is a sketch of all these vectors.



Notice that we added in the vector $\vec{r} - \vec{r_0}$ which will lie completely in the plane. Also notice that we put the normal vector on the plane, but there is actually no reason to expect this to be the case. We put it here to illustrate the point. It is completely possible that the normal vector does not touch the plane in any way.

Now, because $\vec{n}$ is orthogonal to the plane, it's also orthogonal to any vector that lies in the plane. In particular it's orthogonal to $\vec{r} - \vec{r_0}$. Recall from the Dot Product section that two orthogonal vectors will have a dot product of zero. In other words,

$$\vec{n} \bullet \left(\vec{r} - \vec{r_0}\right) = 0 \qquad \Rightarrow \qquad \vec{n} \bullet \vec{r} = \vec{n} \bullet \vec{r_0}$$

This is called the **vector equation of the plane**.

A slightly more useful form of the equations is as follows. Start with the first form of the vector equation and write down a vector for the difference.

$$\langle a,b,c\rangle \bullet \left(\langle x,y,z\rangle - \langle x_0,y_0,z_0\rangle\right) = 0$$
$$\langle a,b,c\rangle \bullet \langle x-x_0, y-y_0, z-z_0\rangle = 0$$

Now, actually compute the dot product to get,

$$a\left(x-x_0\right)+b\left(y-y_0\right)+c\left(z-z_0\right)=0$$

This is called the **scalar equation of plane**. Often this will be written as,

$$ax+by+cz=d$$

where $d = ax_0 + by_0 + cz_0$.

This second form is often how we are given equations of planes. Notice that if we are given the equation of a plane in this form we can quickly get a normal vector for the plane. A normal vector is,

$$\vec{n} = \langle a,b,c\rangle$$

Let's work a couple of examples.

Example 1 Determine the equation of the plane that contains the points $P=\left(1,-2,0\right)$, $Q=\left(3,1,4\right)$ and $R=\left(0,-1,2\right)$.

Solution
In order to write down the equation of plane we need a point (we've got three so we're cool there) and a normal vector. We need to find a normal vector. Recall however, that we saw how to do this in the Cross Product section.

We can form the following two vectors from the given points.

$$\overrightarrow{PQ} = \langle 2,3,4\rangle \qquad \overrightarrow{PR} = \langle -1,1,2\rangle$$

These two vectors will lie completely in the plane since we formed them from points that were in the plane. Notice as well that there are many possible vectors to use here, we just chose two of the possibilities.

Now, we know that the cross product of two vectors will be orthogonal to both of these vectors. Since both of these are in the plane any vector that is orthogonal to both of these will also be orthogonal to the plane. Therefore, we can use the cross product as the normal vector.

$$\vec{n} = \overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 4 \\ -1 & 1 & 2 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ 2 & 3 \\ -1 & 1 \end{matrix} = 2\vec{i} - 8\vec{j} + 5\vec{k}$$

The equation of the plane is then,

$$2(x-1)-8(y+2)+5(z-0)=0$$
$$2x-8y+5z=18$$

We used *P* for the point, but could have used any of the three points.

Example 2 Determine if the plane given by $-x+2z=10$ and the line given by $\vec{r}=\langle 5, 2-t, 10+4t\rangle$ are orthogonal, parallel or neither.

Solution
This is not as difficult a problem as it may at first appear to be. We can pick off a vector that is normal to the plane. This is $\vec{n}=\langle -1,0,2\rangle$. We can also get a vector that is parallel to the line. This is $v=\langle 0,-1,4\rangle$.

Now, if these two vectors are parallel then the line and the plane will be orthogonal. If you think about it this makes some sense. If $\vec{n}$ and $\vec{v}$ are parallel, then $\vec{v}$ is orthogonal to the plane, but $\vec{v}$ is also parallel to the line. So, if the two vectors are parallel the line and plane will be orthogonal.

Let's check this.

$$\vec{n}\times\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -1 & 0 & 2\\ 0 & -1 & 4\end{vmatrix}\begin{matrix}\vec{i} & \vec{j}\\ -1 & 0\\ 0 & -1\end{matrix}=2\vec{i}+4\vec{j}+\vec{k}\neq\vec{0}$$

So, the vectors aren't parallel and so the plane and the line are not orthogonal.

Now, let's check to see if the plane and line are parallel. If the line is parallel to the plane then any vector parallel to the line will be orthogonal to the normal vector of the plane. In other words, if $\vec{n}$ and $\vec{v}$ are orthogonal then the line and the plane will be parallel.

Let's check this.

$$\vec{n}\bullet\vec{v}=0+0+8=8\neq 0$$

The two vectors aren't orthogonal and so the line and plane aren't parallel.

So, the line and the plane are neither orthogonal nor parallel.

Quadric Surfaces

In the previous two sections we've looked at lines and planes in three dimensions (or $\mathbb{R}^3$) and while these are used quite heavily at times in a Calculus class there are many other surfaces that are also used fairly regularly and so we need to take a look at those.

In this section we are going to be looking at quadric surfaces. Quadric surfaces are the graphs of any equation that can be put into the general form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

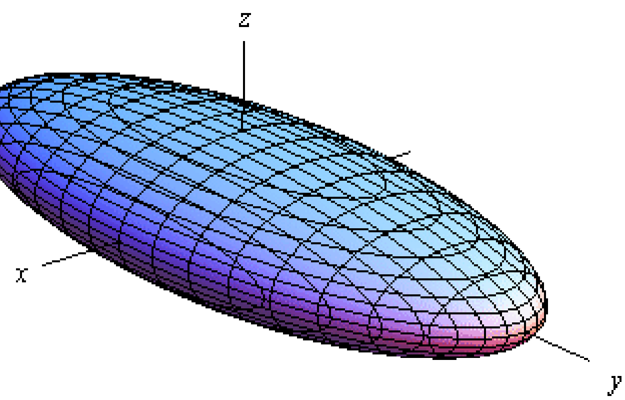
where A, … , J are constants.

There is no way that we can possibly list all of them, but there are some standard equations so here is a list of some of the more common quadric surfaces.

Ellipsoid
Here is the general equation of an ellipsoid.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Here is a sketch of a typical ellipsoid.



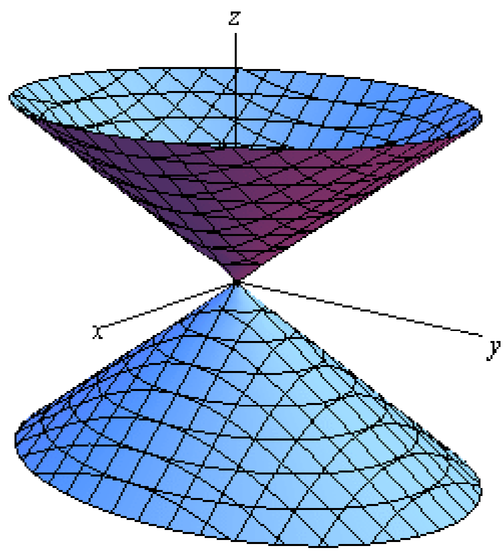
If $a = b = c$ then we will have a sphere.

Notice that we only gave the equation for the ellipsoid that has been centered on the origin. Clearly ellipsoids don't have to be centered on the origin. However, in order to make the discussion in this section a little easier we have chosen to concentrate on surfaces that are "centered" on the origin in one way or another.

Cone
Here is the general equation of a cone.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

Here is a sketch of a typical cone.



Note that this is the equation of a cone that will open along the z-axis. To get the equation of a cone that opens along one of the other axes all we need to do is make a slight modification of the equation. This will be the case for the rest of the surfaces that we'll be looking at in this section as well.

In the case of a cone the variable that sits by itself on one side of the equal sign will determine the axis that the cone opens up along. For instance, a cone that opens up along the x-axis will have the equation,

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{x^2}{a^2}$$

For most of the following surfaces we will not give the other possible formulas. We will however acknowledge how each formula needs to be changed to get a change of orientation for the surface.

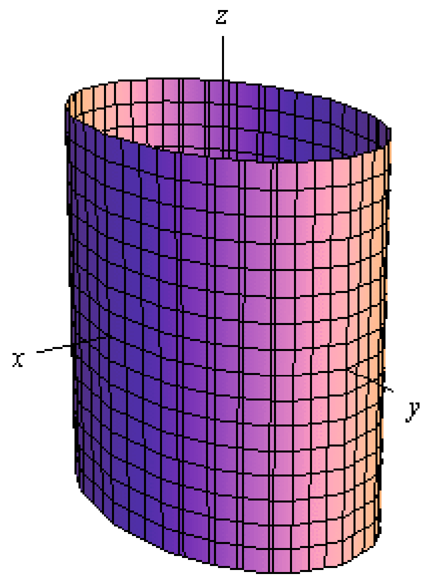
Cylinder
Here is the general equation of a cylinder.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

This is a cylinder whose cross section is an ellipse. If $a=b$ we have a cylinder whose cross section is a circle. We'll be dealing with those kinds of cylinders more than the general form so the equation of a cylinder with a circular cross section is,

$$x^2+y^2=r^2$$

Here is a sketch of typical cylinder with an ellipse cross section.



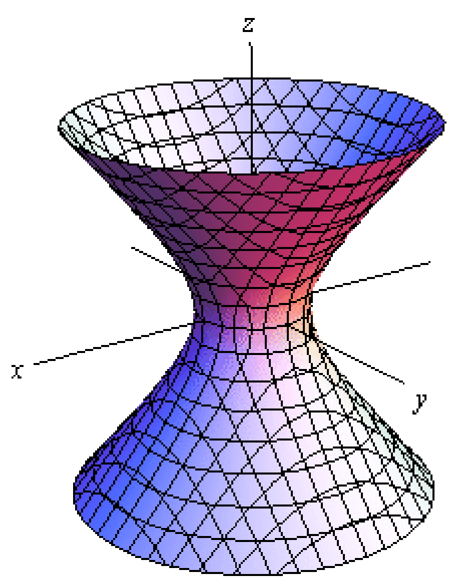
The cylinder will be centered on the axis corresponding to the variable that does not appear in the equation.

Be careful to not confuse this with a circle. In two dimensions it is a circle, but in three dimensions it is a cylinder.

Hyperboloid of One Sheet
Here is the equation of a hyperboloid of one sheet.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

Here is a sketch of a typical hyperboloid of one sheet.

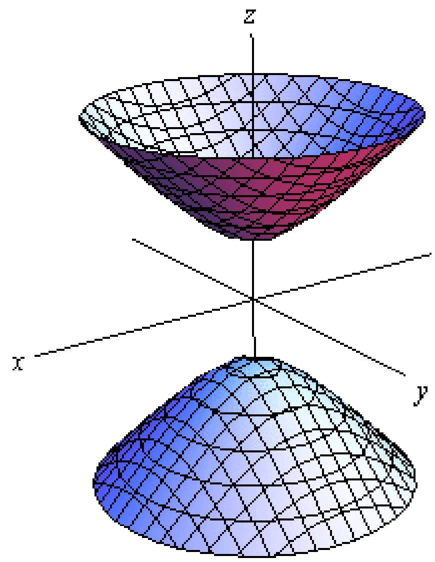


The variable with the negative in front of it will give the axis along which the graph is centered.

Hyperboloid of Two Sheets
Here is the equation of a hyperboloid of two sheets.

$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Here is a sketch of a typical hyperboloid of two sheets.



The variable with the positive in front of it will give the axis along which the graph is centered.

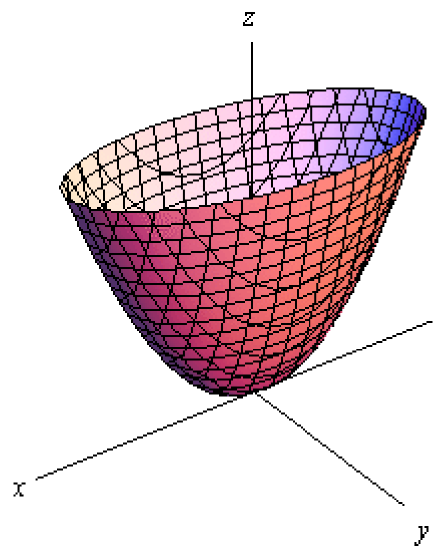
Notice that the only difference between the hyperboloid of one sheet and the hyperboloid of two sheets is the signs in front of the variables. They are exactly the opposite signs.

Elliptic Paraboloid
Here is the equation of an elliptic paraboloid.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$$

As with cylinders this has a cross section of an ellipse and if $a=b$ it will have a cross section of a circle. When we deal with these we'll generally be dealing with the kind that have a circle for a cross section.

Here is a sketch of a typical elliptic paraboloid.

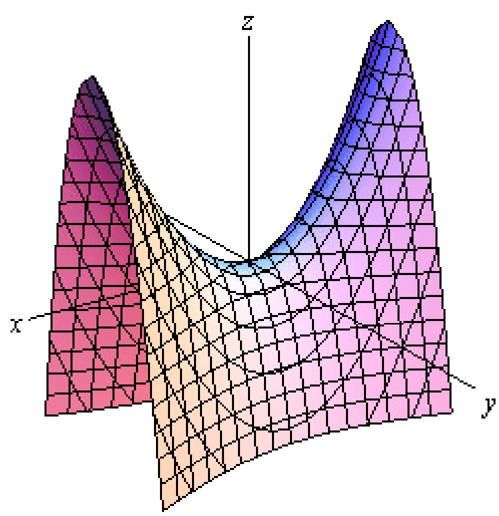


In this case the variable that isn't squared determines the axis upon which the paraboloid opens up. Also, the sign of c will determine the direction that the paraboloid opens. If c is positive then it opens up and if c is negative then it opens down.

Hyperbolic Paraboloid
Here is the equation of a hyperbolic paraboloid.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$$

Here is a sketch of a typical hyperbolic paraboloid.



These graphs are vaguely saddle shaped and as with the elliptic paraboloid the sign of c will determine the direction in which the surface "opens up". The graph above is shown for c positive.

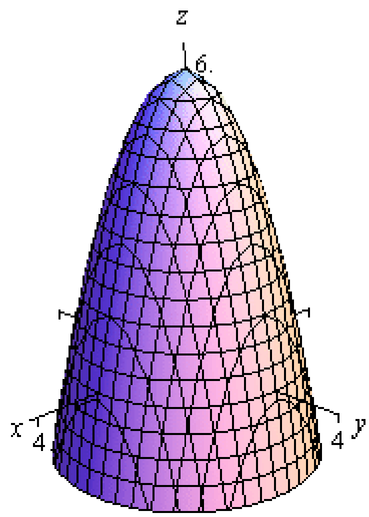
With the both of the types of paraboloids discussed above the surface can be easily moved up or down by adding/subtracting a constant from the left side.

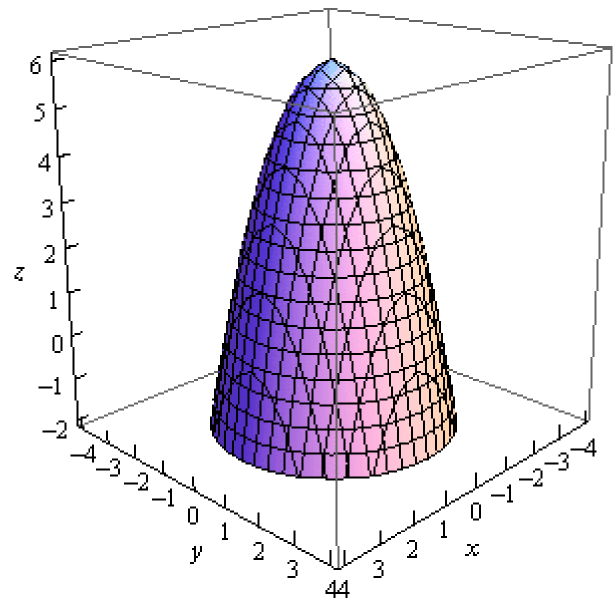
For instance

$$z = -x^2 - y^2 + 6$$

is an elliptic paraboloid that opens downward (be careful, the "-" is on the *x* and *y* instead of the *z*) and starts at $z = 6$ instead of $z = 0$.

Here are a couple of quick sketches of this surface.



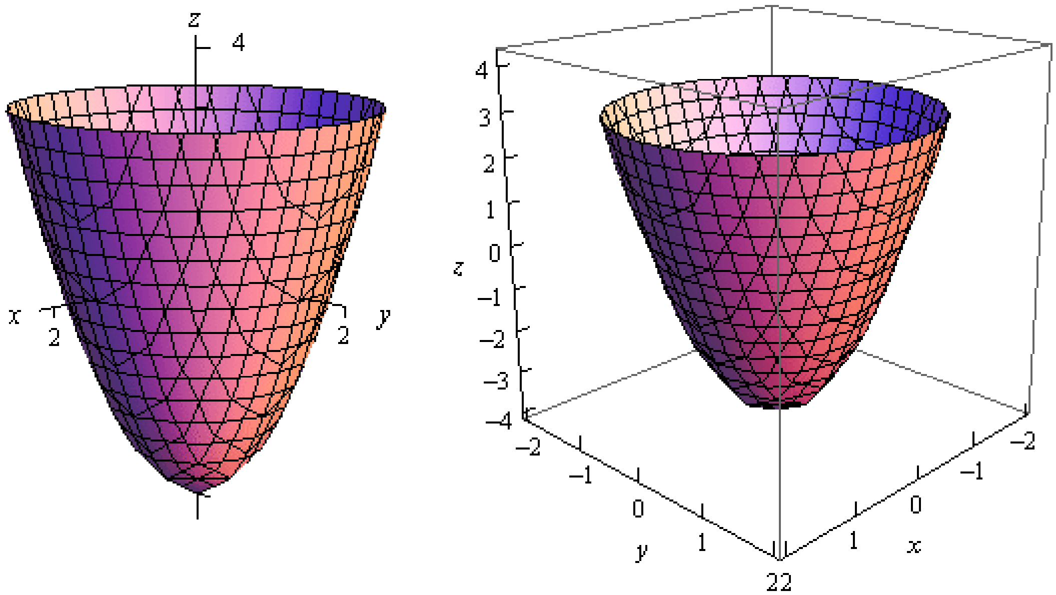


Note that we've given two forms of the sketch here. The sketch on the left has the standard set of axes but it is difficult to see the numbers on the axis. The sketch on the right has been "boxed" and this makes it easier to see the numbers to give a sense of perspective to the sketch. In most sketches that actually involve numbers on the axis system we will give both sketches to help get a feel for what the sketch looks like.

Functions of Several Variables

In this section we want to go over some of the basic ideas about functions of more than one variable.

First, remember that graphs of functions of two variables, $z = f(x, y)$ are surfaces in three dimensional space. For example here is the graph of $z = 2x^2 + 2y^2 - 4$.



This is an elliptic parabaloid and is an example of a quadric surface. We saw several of these in the previous section. We will be seeing quadric surfaces fairly regularly later on in Calculus III.

Another common graph that we'll be seeing quite a bit in this course is the graph of a plane. We have a convention for graphing planes that will make them a little easier to graph and hopefully visualize.

Recall that the equation of a plane is given by

$$ax + by + cz = d$$

or if we solve this for z we can write it in terms of function notation. This gives,

$$f(x, y) = Ax + By + D$$

To graph a plane we will generally find the intersection points with the three axes and then graph the triangle that connects those three points. This triangle will be a portion of the plane and it will give us a fairly decent idea on what the plane itself should look like. For example let's graph the plane given by,

$$f(x, y) = 12 - 3x - 4y$$

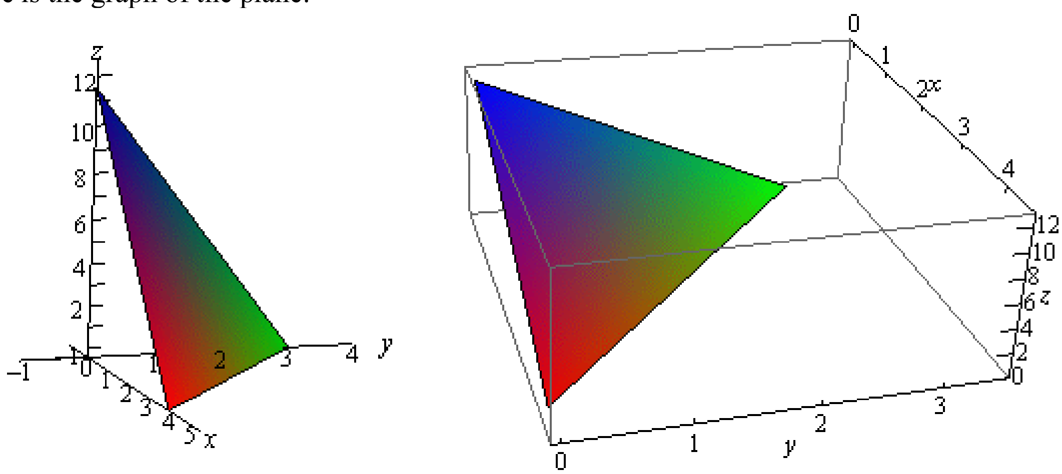
For purposes of graphing this it would probably be easier to write this as,

$$z = 12 - 3x - 4y \qquad \Rightarrow \qquad 3x + 4y + z = 12$$

Now, each of the intersection points with the three main coordinate axes is defined by the fact that two of the coordinates are zero. For instance, the intersection with the *z*-axis is defined by $x = y = 0$. So, the three intersection points are,

$$\begin{aligned} x-\text{axis}: & \ (4,0,0) \\ y-\text{axis}: & \ (0,3,0) \\ z-\text{axis}: & \ (0,0,12) \end{aligned}$$

Here is the graph of the plane.



Now, to extend this out, graphs of functions of the form $w = f(x, y, z)$ would be four dimensional surfaces. Of course we can't graph them, but it doesn't hurt to point this out.

We next want to talk about the domains of functions of more than one variable. Recall that domains of functions of a single variable, $y = f(x)$, consisted of all the values of *x* that we could plug into the function and get back a real number. Now, if we think about it, this means that the domain of a function of a single variable is an interval (or intervals) of values from the number line, or one dimensional space.

The domain of functions of two variables, $z = f(x, y)$, are regions from two dimensional space and consist of all the coordinate pairs, (x, y), that we could plug into the function and get back a real number.

Example 1 Determine the domain of each of the following.

(a) $f(x,y) = \sqrt{x+y}$ [Solution]

(b) $f(x,y) = \sqrt{x} + \sqrt{y}$ [Solution]

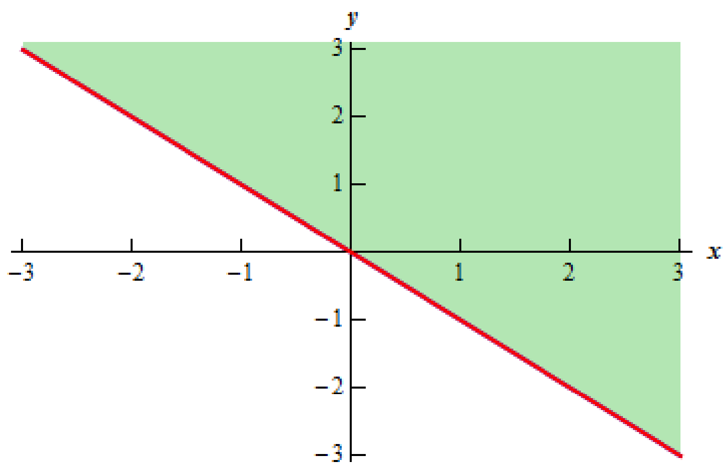
(c) $f(x,y) = \ln\left(9 - x^2 - 9y^2\right)$ [Solution]

Solution

(a) In this case we know that we can't take the square root of a negative number so this means that we must require,

$$x + y \geq 0$$

Here is a sketch of the graph of this region.



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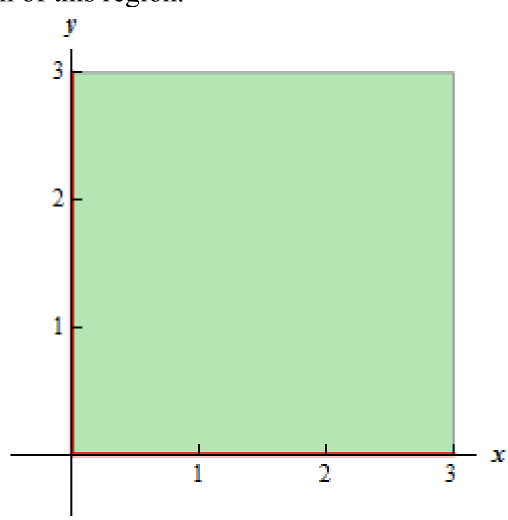
(b) This function is different from the function in the previous part. Here we must require that,

$$x \geq 0 \quad \text{and} \quad y \geq 0$$

and they really do need to be separate inequalities. There is one for each square root in the function. Here is the sketch of this region.



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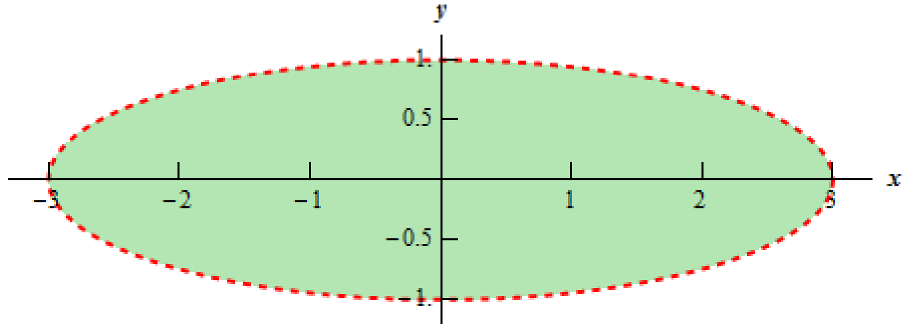
(c) In this final part we know that we can't take the logarithm of a negative number or zero. Therefore we need to require that,

$$9 - x^2 - 9y^2 > 0 \qquad \Rightarrow \qquad \frac{x^2}{9} + y^2 < 1$$

and upon rearranging we see that we need to stay interior to an ellipse for this function. Here is a sketch of this region.



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Note that domains of functions of three variables, $w = f(x, y, z)$, will be regions in three dimensional space.

Example 2 Determine the domain of the following function,

$$f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2 - 16}}$$

Solution

In this case we have to deal with the square root and division by zero issues. These will require,

$$x^2 + y^2 + z^2 - 16 > 0 \qquad \Rightarrow \qquad x^2 + y^2 + z^2 > 16$$

So, the domain for this function is the set of points that lies completely outside a sphere of radius 4 centered at the origin.

The next topic that we should look at is that of **level curves** or **contour curves**. The level curves of the function $z = f(x, y)$ are two dimensional curves we get by setting $z = k$, where k is any number. So the equations of the level curves are $f(x, y) = k$. Note that sometimes the equation will be in the form $f(x, y, z) = 0$ and in these cases the equations of the level curves are $f(x, y, k) = 0$.

You've probably seen level curves (or contour curves, whatever you want to call them) before. If you've ever seen the elevation map for a piece of land, this is nothing more than the contour curves for the function that gives the elevation of the land in that area. Of course, we probably don't have the function that gives the elevation, but we can at least graph the contour curves.

Let's do a quick example of this.

Example 3 Identify the level curves of $f\left(x,y\right)=\sqrt{x^2+y^2}$. Sketch a few of them.

Solution
First, for the sake of practice, let's identify what this surface given by $f\left(x,y\right)$ is. To do this let's rewrite it as,

$$z=\sqrt{x^2+y^2}$$

Now, this equation is not listed in the Quadric Surfaces section, but if we square both sides we get,

$$z^2=x^2+y^2$$

and this is listed in that section. So, we have a cone, or at least a portion of a cone. Since we know that square roots will only return positive numbers, it looks like we've only got the upper half of a cone.

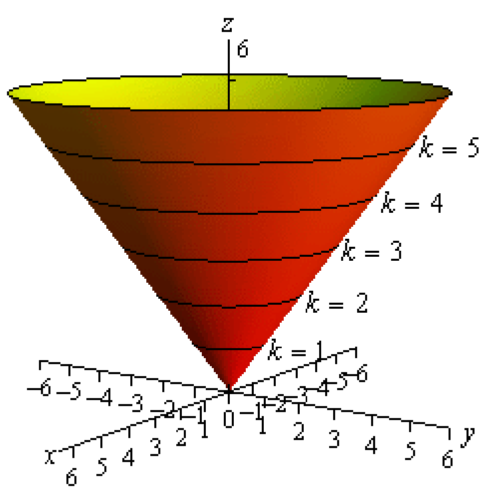
Note that this was not required for this problem. It was done for the practice of identifying the surface and this may come in handy down the road.

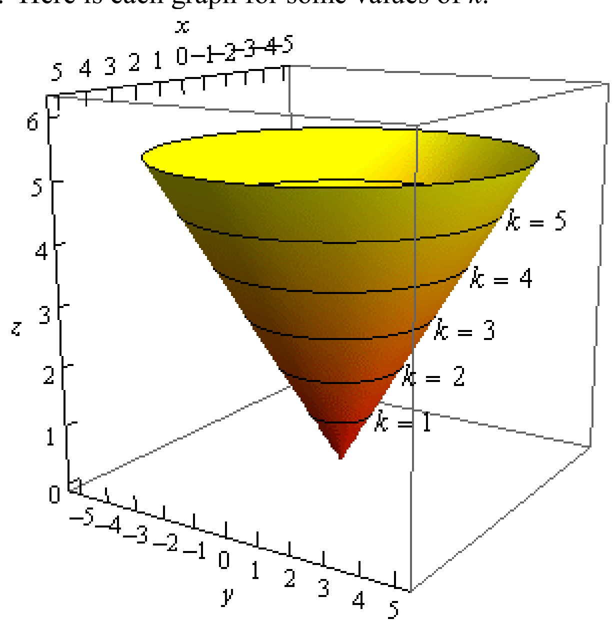
Now on to the real problem. The level curves (or contour curves) for this surface are given by the equation are found by substituting $z=k$. In the case of our example this is,

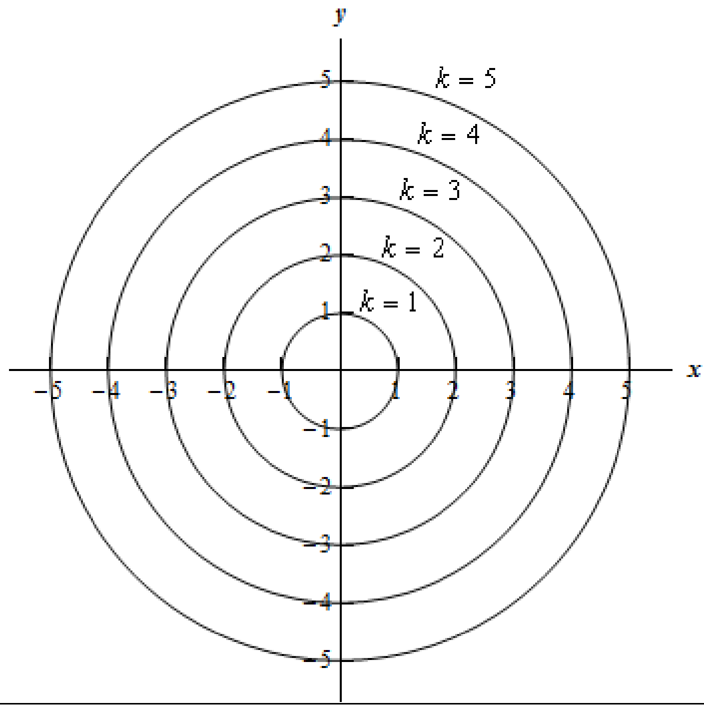
$$k=\sqrt{x^2+y^2} \qquad \Rightarrow \qquad x^2+y^2=k^2$$

where k is any number. So, in this case, the level curves are circles of radius k with center at the origin.

We can graph these in one of two ways. We can either graph them on the surface itself or we can graph them in a two dimensional axis system. Here is each graph for some values of k.







Note that we can think of contours in terms of the intersection of the surface that is given by $z = f(x, y)$ and the plane $z = k$. The contour will represent the intersection of the surface and the plane.

For functions of the form $f(x, y, z)$ we will occasionally look at **level surfaces**. The equations of level surfaces are given by $f(x, y, z) = k$ where k is any number.

The final topic in this section is that of **traces**. In some ways these are similar to contours. As noted above we can think of contours as the intersection of the surface given by $z = f(x, y)$ and the plane $z = k$. Traces of surfaces are curves that represent the intersection of the surface and the plane given by $x = a$ or $y = b$.

Let's take a quick look at an example of traces.

Example 4 Sketch the traces of $f(x, y) = 10 - 4x^2 - y^2$ for the plane $x = 1$ and $y = 2$.

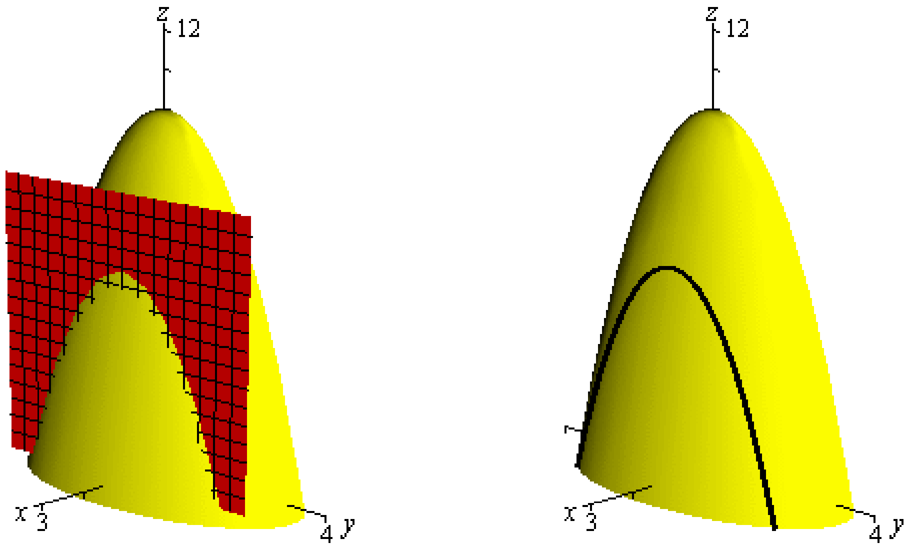
Solution

We'll start with $x = 1$. We can get an equation for the trace by plugging $x = 1$ into the equation. Doing this gives,

$$z = f(1, y) = 10 - 4(1)^2 - y^2 \qquad \Rightarrow \qquad z = 6 - y^2$$

and this will be graphed in the plane given by $x = 1$.

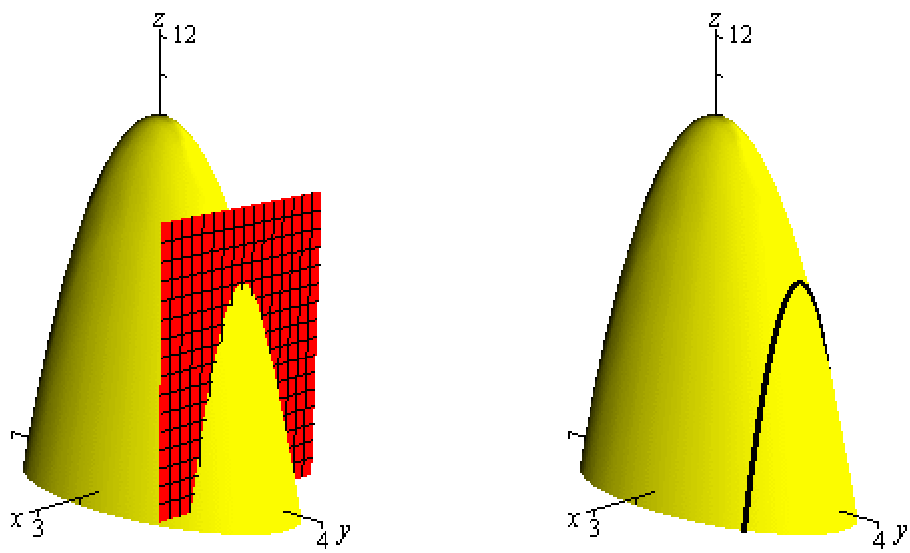
Below are two graphs. The graph on the left is a graph showing the intersection of the surface and the plane given by $x = 1$. On the right is a graph of the surface and the trace that we are after in this part.



For $y = 2$ we will do pretty much the same thing that we did with the first part. Here is the equation of the trace,

$$z = f(x, 2) = 10 - 4x^2 - (2)^2 \qquad \Rightarrow \qquad z = 6 - 4x^2$$

and here are the sketches for this case.



Vector Functions

We first saw vector functions back when we were looking at the Equation of Lines. In that section we talked about them because we wrote down the equation of a line in $\mathbb{R}^3$ in terms of a **vector function** (sometimes called a **vector-valued function**). In this section we want to look a little closer at them and we also want to look at some vector functions in $\mathbb{R}^3$ other than lines.

A vector function is a function that takes one or more variables and returns a vector. We'll spend most of this section looking at vector functions of a single variable as most of the places where vector functions show up here will be vector functions of single variables. We will however briefly look at vector functions of two variables at the end of this section.

A vector functions of a single variable in $\mathbb{R}^2$ and $\mathbb{R}^3$ have the form,

$$\vec{r}(t) = \langle f(t), g(t) \rangle \qquad\qquad \vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

respectively, where $f(t)$, $g(t)$ and $h(t)$ are called the **component functions**.

The main idea that we want to discuss in this section is that of graphing and identifying the graph given by a vector function. Before we do that however, we should talk briefly about the domain of a vector function. The **domain** of a vector function is the set of all *t*'s for which all the component functions are defined.

Example 1 Determine the domain of the following function.

$$\vec{r}(t) = \left\langle \cos t, \ln(4-t), \sqrt{t+1} \right\rangle$$

Solution

The first component is defined for all *t*'s. The second component is only defined for $t < 4$. The third component is only defined for $t \geq -1$. Putting all of these together gives the following domain.

$$[-1, 4)$$

This is the largest possible interval for which all three components are defined.

Let's now move into looking at the graph of vector functions. In order to graph a vector function all we do is think of the vector returned by the vector function as a position vector for points on the graph. Recall that a position vector, say $\vec{v} = \langle a, b, c \rangle$, is a vector that starts at the origin and ends at the point (a, b, c).

So, in order to sketch the graph of a vector function all we need to do is plug in some values of *t* and then plot points that correspond to the resulting position vector we get out of the vector function.

Because it is a little easier to visualize things we'll start off by looking at graphs of vector functions in $\mathbb{R}^2$.

Example 2 Sketch the graph of each of the following vector functions.

(a) $\vec{r}(t) = \langle t, 1 \rangle$ [Solution]

(b) $\vec{r}(t) = \langle t, t^3 - 10t + 7 \rangle$ [Solution]

Solution

(a) $\vec{r}(t) = \langle t, 1 \rangle$

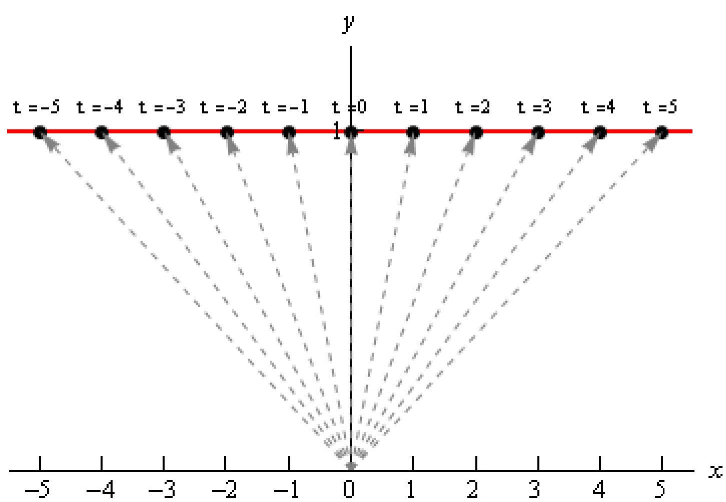
Okay, the first thing that we need to do is plug in a few values of *t* and get some position vectors. Here are a few,

$$\vec{r}(-3) = \langle -3, 1 \rangle \qquad \vec{r}(-1) = \langle -1, 1 \rangle \qquad \vec{r}(2) = \langle 2, 1 \rangle \qquad \vec{r}(5) = \langle 5, 1 \rangle$$

So, what this tells us is that the following points are all on the graph of this vector function.

$$(-3, 1) \qquad (-1, 1) \qquad (2, 1) \qquad (5, 1)$$

Here is a sketch of this vector function.



In this sketch we've included many more evaluations that just those above. Also note that we've put in the position vectors (in gray and dashed) so you can see how all this is working. Note however, that in practice the position vectors are generally not included in the sketch.

In this case it looks like we've got the graph of the line $y = 1$.

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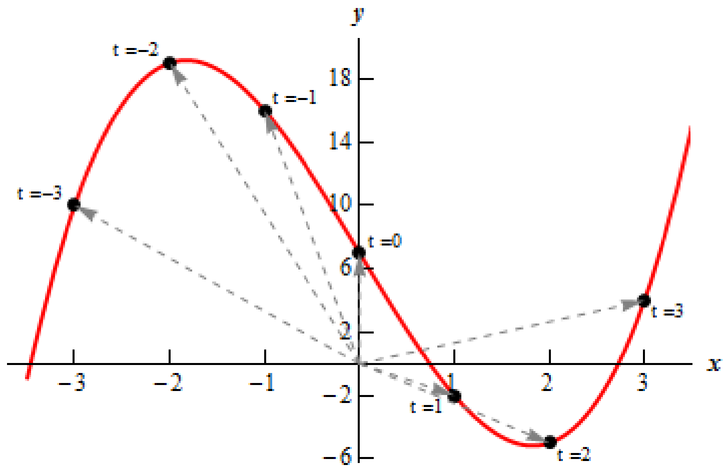
(b) $\vec{r}(t) = \langle t, t^3 - 10t + 7 \rangle$

Here are a couple of evaluations for this vector function.

$$\vec{r}(-3) = \langle -3, 10 \rangle \qquad \vec{r}(-1) = \langle -1, 16 \rangle \qquad \vec{r}(1) = \langle 1, -2 \rangle \qquad \vec{r}(3) = \langle 3, 4 \rangle$$

So, we've got a few points on the graph of this function. However, unlike the first part this isn't really going to be enough points to get a good idea of this graph. In general, it can take quite a few function evaluations to get an idea of what the graph is and it's usually easier to use a computer to do the graphing.

Here is a sketch of this graph. We've put in a few vectors/evaluations to illustrate them, but the reality is that we did have to use a computer to get a good sketch here.



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Both of the vector functions in the above example were in the form,

$$\vec{r}(t) = \langle t, g(t) \rangle$$

and what we were really sketching is the graph of $y = g(x)$ as you probably caught onto. Let's graph a couple of other vector functions that do not fall into this pattern.

Example 3 Sketch the graph of each of the following vector functions.

(a) $\vec{r}(t) = \langle 6\cos t, 3\sin t \rangle$ [Solution]

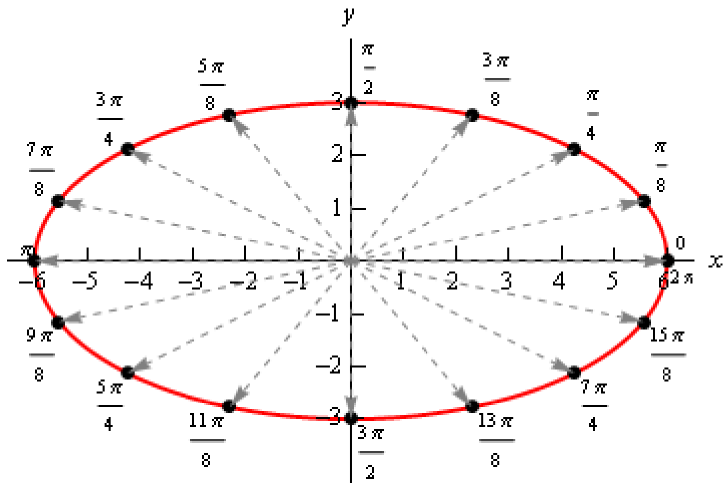
(b) $\vec{r}(t) = \langle t - 2\sin t, t^2 \rangle$ [Solution]

Solution

As we saw in the last part of the previous example it can really take quite a few function evaluations to really be able to sketch the graph of a vector function. Because of that we'll be skipping all the function evaluations here and just giving the graph. The main point behind this set of examples is to not get you too locked into the form we were looking at above. The first part will also lead to an important idea that we'll discuss after this example.

So, with that said here are the sketches of each of these.

(a) $\vec{r}(t) = \langle 6\cos t, 3\sin t \rangle$

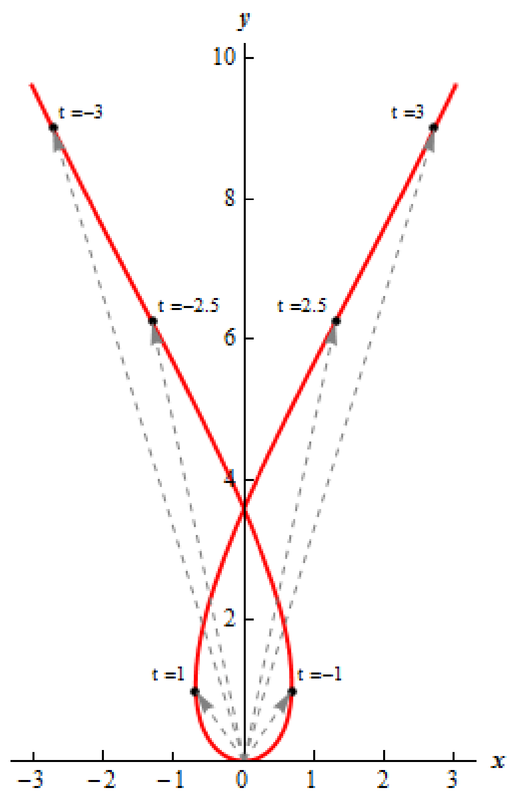


So, in this case it looks like we've got an ellipse.

[Return to Problems]

(b) $\vec{r}(t) = \langle t - 2\sin t, t^2 \rangle$

Here's the sketch for this vector function.



[Return to Problems]

Before we move on to vector functions in $\mathbb{R}^3$ let's go back and take a quick look at the first vector function we sketched in the previous example, $\vec{r}(t) = \langle 6\cos t, 3\sin t \rangle$. The fact that we got an ellipse here should not come as a surprise to you. We know that the first component function gives the x coordinate and the second component function gives the y coordinates of the point that we graph. If we strip these out to make this clear we get,

$$x = 6\cos t \qquad\qquad y = 3\sin t$$

This should look familiar to you. Back when we were looking at Parametric Equations we saw that this was nothing more than one of the sets of parametric equations that gave an ellipse.

This is an important idea in the study of vector functions. Any vector function can be broken down into a set of parametric equations that represent the same graph. In general, the two dimensional vector function, $\vec{r}(t) = \langle f(t), g(t) \rangle$, can be broken down into the parametric equations,

$$x = f(t) \qquad\qquad y = g(t)$$

Likewise, a three dimensional vector function, $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, can be broken down into the parametric equations,

$$x = f(t) \qquad y = g(t) \qquad z = h(t)$$

Do not get too excited about the fact that we're now looking at parametric equations in $\mathbb{R}^3$. They work in exactly the same manner as parametric equations in $\mathbb{R}^2$ which we're used to dealing with already. The only difference is that we now have a third component.

Let's take a look at a couple of graphs of vector functions.

Example 4 Sketch the graph of the following vector function.

$$\vec{r}(t) = \langle 2-4t, -1+5t, 3+t \rangle$$

Solution

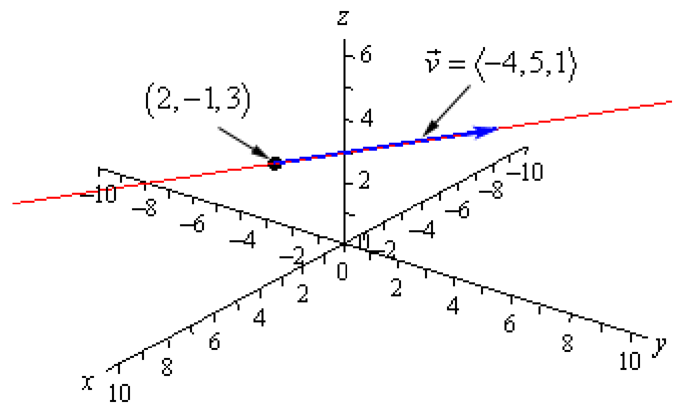
Notice that this is nothing more than a line. It might help if we rewrite it a little.

$$\vec{r}(t) = \langle 2,-1,3 \rangle + t\langle -4,5,1 \rangle$$

In this form we can see that this is the equation of a line that goes through the point $(2,-1,3)$ and is parallel to the vector $\vec{v} = \langle -4,5,1 \rangle$.

To graph this line all that we need to do is plot the point and then sketch in the parallel vector. In order to get the sketch will assume that the vector is on the line and will start at the point in the line. To sketch in the line all we do this is extend the parallel vector into a line.

Here is a sketch.



Example 5 Sketch the graph of the following vector function.

$$\vec{r}(t) = \langle 2\cos t, 2\sin t, 3 \rangle$$

Solution

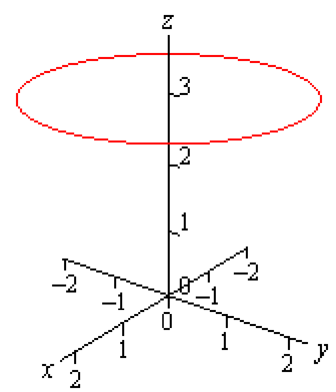
In this case to see what we've got for a graph let's get the parametric equations for the curve.

$$x = 2\cos t \qquad y = 2\sin t \qquad z = 3$$

If we ignore the *z* equation for a bit we'll recall (hopefully) that the parametric equations for *x* and *y* give a circle of radius 2 centered on the origin (or about the *z*-axis since we are in $\mathbb{R}^3$).

Now, all the parametric equations here tell us is that no matter what is going on in the graph all the *z* coordinates must be 3. So, we get a circle of radius 2 centered on the *z*-axis and at the level of $z = 3$.

Here is a sketch.



Note that it is very easy to modify the above vector function to get a circle centered on the *x* or *y*-axis as well. For instance,

$$\vec{r}(t) = \langle 10\sin t, -3, 10\cos t \rangle$$

will be a circle of radius 10 centered on the *y*-axis and at $y = -3$. In other words, as long as two of the terms are a sine and a cosine (with the same coefficient) and the other is a fixed number then we will have a circle that is centered on the axis that is given by the fixed number.

Let's take a look at a modification of this.

Example 6 Sketch the graph of the following vector function.

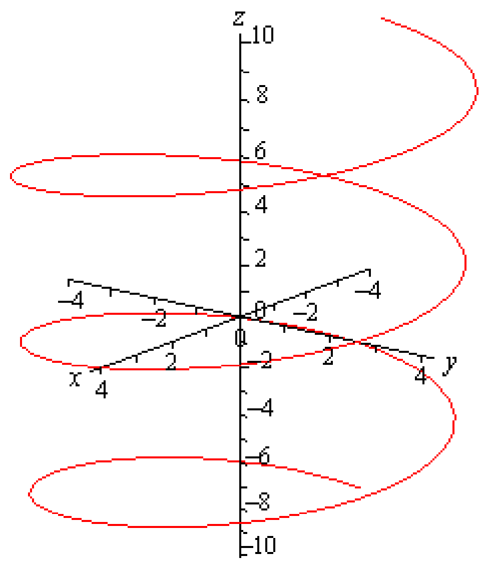
$$\vec{r}(t) = \langle 4\cos t, 4\sin t, t \rangle$$

Solution

If this one had a constant in the z component we would have another circle. However, in this case we don't have a constant. Instead we've got a t and that will change the curve. However, because the x and y component functions are still a circle in parametric equations our curve should have a circular nature to it in some way.

In fact, the only change is in the z component and as t increases the z coordinate will increase. Also, as t increases the x and y coordinates will continue to form a circle centered on the z-axis. Putting these two ideas together tells us that as we increase t the circle that is being traced out in the x and y directions should also be rising.

Here is a sketch of this curve.



So, we've got a helix (or spiral, depending on what you want to call it) here.

As with circles the component that has the t will determine the axis that the helix rotates about. For instance,

$$\vec{r}(t) = \langle t, 6\cos t, 6\sin t \rangle$$

is a helix that rotates around the x-axis.

Also note that if we allow the coefficients on the sine and cosine for both the circle and helix to be different we will get ellipses.

For example,

$$\vec{r}(t) = \langle 9\cos t, t, 2\sin t \rangle$$

will be a helix that rotates about the *y*-axis and is in the shape of an ellipse.

There is a nice formula that we should derive before moving onto vector functions of two variables.

Example 7 Determine the vector equation for the line segment starting at the point $P = (x_1, y_1, z_1)$ and ending at the point $Q = (x_2, y_2, z_2)$.

Solution

It is important to note here that we only want the equation of the line segment that starts at *P* and ends at *Q*. We don't want any other portion of the line and we do want the direction of the line segment preserved as we increase *t*. With all that said, let's not worry about that and just find the vector equation of the line that passes through the two points. Once we have this we will be able to get what we're after.

So, we need a point on the line. We've got two and we will use *P*. We need a vector that is parallel to the line and since we've got two points we can find the vector between them. This vector will lie on the line and hence be parallel to the line. Also, let's remember that we want to preserve the starting and ending point of the line segment so let's construct the vector using the same "orientation".

$$\vec{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Using this vector and the point *P* we get the following vector equation of the line.

$$\vec{r}(t) = \langle x_1, y_1, z_1 \rangle + t \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

While this is the vector equation of the line, let's rewrite the equation slightly.

$$\begin{aligned} \vec{r}(t) &= \langle x_1, y_1, z_1 \rangle + t \langle x_2, y_2, z_2 \rangle - t \langle x_1, y_1, z_1 \rangle \\ &= (1-t) \langle x_1, y_1, z_1 \rangle + t \langle x_2, y_2, z_2 \rangle \end{aligned}$$

This is the equation of the line that contains the points *P* and *Q*. We of course just want the line segment that starts at *P* and ends at *Q*. We can get this by simply restricting the values of *t*.

Notice that

$$\vec{r}(0) = \langle x_1, y_1, z_1 \rangle \qquad \vec{r}(1) = \langle x_2, y_2, z_2 \rangle$$

So, if we restrict *t* to be between zero and one we will cover the line segment and we will start and end at the correct point.

So the vector equation of the line segment that starts at $P = (x_1, y_1, z_1)$ and ends at $Q = (x_2, y_2, z_2)$ is,

$$\vec{r}(t) = (1-t) \langle x_1, y_1, z_1 \rangle + t \langle x_2, y_2, z_2 \rangle \qquad 0 \le t \le 1$$

As noted briefly at the beginning of this section we can also have vector functions of two variables. In these case the graphs of vector function of two variables are surfaces. So, to make sure that we don't forget that let's work an example with that as well.

Example 8 Identify the surface that is described by $\vec{r}\left(x,y\right)=x\,\vec{i}+y\,\vec{j}+\left(x^2+y^2\right)\vec{k}$.

Solution

First, notice that in this case the vector function will in fact be a function of two variables. This will always be the case when we are using vector functions to represent surfaces.

To identify the surface let's go back to parametric equations.

$$x=x \qquad\qquad y=y \qquad\qquad z=x^2+y^2$$

The first two are really only acknowledging that we are picking x and y for free and then determining z from our choices of these two. The last equation is the one that we want. We should recognize that function from the section on quadric surfaces. The third equation is the equation of an elliptic paraboloid and so the vector function represents an elliptic paraboloid.

As a final topic for this section let's generalize the idea from the previous example and note that given any function of one variable ($y=f\left(x\right)$ or $x=h\left(y\right)$) or any function of two variables ($z=g\left(x,y\right)$, $x=g\left(y,z\right)$, or $y=g\left(x,z\right)$) we can always write down a vector form of the equation.

For a function of one variable this will be,

$$\vec{r}\left(x\right)=x\,\vec{i}+f\left(x\right)\vec{j} \qquad\qquad \vec{r}\left(y\right)=h\left(y\right)\vec{i}+y\,\vec{j}$$

and for a function of two variables the vector form will be,

$$\vec{r}\left(x,y\right)=x\,\vec{i}+y\,\vec{j}+g\left(x,y\right)\vec{k} \qquad\qquad \vec{r}\left(y,z\right)=g\left(y,z\right)\vec{i}+y\,\vec{j}+z\,\vec{k}$$

$$\vec{r}\left(x,z\right)=x\,\vec{i}+g\left(x,z\right)\vec{j}+z\,\vec{k}$$

depending upon the original form of the function.

For example the hyperbolic paraboloid $y=2x^2-5z^2$ can be written as the following vector function.

$$\vec{r}\left(x,z\right)=x\,\vec{i}+\left(2x^2-5z^2\right)\vec{j}+z\,\vec{k}$$

This is a fairly important idea and we will be doing quite a bit of this kind of thing in Calculus III.

Calculus with Vector Functions

In this section we need to talk briefly about limits, derivatives and integrals of vector functions. As you will see, these behave in a fairly predictable manner. We will be doing all of the work in $\mathbb{R}^3$ but we can naturally extend the formulas/work in this section to $\mathbb{R}^n$ (*i.e.* n-dimensional space).

Let's start with limits. Here is the limit of a vector function.

$$\begin{aligned}\lim_{t\to a}\vec{r}(t) &= \lim_{t\to a}\left\langle f(t), g(t), h(t)\right\rangle \\ &= \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\right\rangle \\ &= \lim_{t\to a} f(t)\,\vec{i} + \lim_{t\to a} g(t)\,\vec{j} + \lim_{t\to a} h(t)\,\vec{k}\end{aligned}$$

So, all that we do is take the limit of each of the component's functions and leave it as a vector.

Example 1 Compute $\lim_{t\to 1}\vec{r}(t)$ where $\vec{r}(t) = \left\langle t^3, \dfrac{\sin(3t-3)}{t-1}, \mathbf{e}^{2t}\right\rangle$.

Solution

There really isn't all that much to do here.

$$\begin{aligned}\lim_{t\to 1}\vec{r}(t) &= \left\langle \lim_{t\to 1} t^3, \lim_{t\to 1}\frac{\sin(3t-3)}{t-1}, \lim_{t\to 1}\mathbf{e}^{2t}\right\rangle \\ &= \left\langle \lim_{t\to 1} t^3, \lim_{t\to 1}\frac{3\cos(3t-3)}{1}, \lim_{t\to 1}\mathbf{e}^{2t}\right\rangle \\ &= \left\langle 1, 3, \mathbf{e}^2\right\rangle\end{aligned}$$

Notice that we had to use L'Hospital's Rule on the y component.

Now let's take care of derivatives and after seeing how limits work it shouldn't be too surprising that we have the following for derivatives.

$$\vec{r}'(t) = \left\langle f'(t), g'(t), h'(t)\right\rangle = f'(t)\,\vec{i} + g'(t)\,\vec{j} + h'(t)\,\vec{k}$$

Example 2 Compute $\vec{r}'(t)$ for $\vec{r}(t) = t^6\,\vec{i} + \sin(2t)\,\vec{j} - \ln(t+1)\,\vec{k}$.

Solution

There really isn't too much to this problem other than taking the derivatives.

$$\vec{r}'(t) = 6t^5\,\vec{i} + 2\cos(2t)\,\vec{j} - \frac{1}{t+1}\vec{k}$$

Most of the basic facts that we know about derivatives still hold however, just to make it clear here are some facts about derivatives of vector functions.

Facts

$$\frac{d}{dt}\left(\vec{u}+\vec{v}\right)=\vec{u}'+\vec{v}'$$

$$\left(c\vec{u}\right)'=c\,\vec{u}'$$

$$\frac{d}{dt}\left(f\left(t\right)\vec{u}\left(t\right)\right)=f'\left(t\right)\vec{u}\left(t\right)+f\left(t\right)\vec{u}'$$

$$\frac{d}{dt}\left(\vec{u}\bullet\vec{v}\right)=\vec{u}'\bullet\vec{v}+\vec{u}\bullet\vec{v}'$$

$$\frac{d}{dt}\left(\vec{u}\times\vec{v}\right)=\vec{u}'\times\vec{v}+\vec{u}\times\vec{v}'$$

$$\frac{d}{dt}\left(\vec{u}\left(f\left(t\right)\right)\right)=f'\left(t\right)\vec{u}'\left(f\left(t\right)\right)$$

There is also one quick definition that we should get out of the way so that we can use it when we need to.

A **smooth curve** is any curve for which $\vec{r}'\left(t\right)$ is continuous and $\vec{r}'\left(t\right)\neq 0$ for any t except possibly at the endpoints. A helix is a smooth curve, for example.

Finally, we need to discuss integrals of vector functions. Using both limits and derivatives as a guide it shouldn't be too surprising that we also have the following for integration for indefinite integrals

$$\int\vec{r}\left(t\right)=\left\langle\int f\left(t\right)dt,\int g\left(t\right)dt,\int h\left(t\right)dt\right\rangle+\vec{c}$$

$$\int\vec{r}\left(t\right)=\int f\left(t\right)dt\;\vec{i}+\int g\left(t\right)dt\;\vec{j}+\int h\left(t\right)dt\;\vec{k}+\vec{c}$$

and the following for definite integrals.

$$\int_a^b\vec{r}\left(t\right)dt=\left\langle\int_a^b f\left(t\right)dt,\int_a^b g\left(t\right)dt,\int_a^b h\left(t\right)dt\right\rangle$$

$$\int_a^b\vec{r}\left(t\right)dt=\int_a^b f\left(t\right)dt\;\vec{i}+\int_a^b g\left(t\right)dt\;\vec{j}+\int_a^b h\left(t\right)dt\;\vec{k}$$

With the indefinite integrals we put in a constant of integration to make sure that it was clear that the constant in this case needs to be a vector instead of a regular constant.

Also, for the definite integrals we will sometimes write it as follows,

$$\int_a^b\vec{r}\left(t\right)dt=\left(\left\langle\int f\left(t\right)dt,\int g\left(t\right)dt,\int h\left(t\right)dt\right\rangle\right)\Big|_a^b$$

$$\int_a^b\vec{r}\left(t\right)dt=\left(\int f\left(t\right)dt\;\vec{i}+\int g\left(t\right)dt\;\vec{j}+\int h\left(t\right)dt\;\vec{k}\right)\Big|_a^b$$

In other words, we will do the indefinite integral and then do the evaluation of the vector as a whole instead of on a component by component basis.

Example 3 Compute $\int \vec{r}\left(t\right) dt$ for $\vec{r}\left(t\right) = \left\langle \sin\left(t\right), 6, 4t \right\rangle$.

Solution

All we need to do is integrate each of the components and be done with it.

$$\int \vec{r}\left(t\right) dt = \left\langle -\cos\left(t\right), 6t, 2t^2 \right\rangle + \vec{c}$$

Example 4 Compute $\int_0^1 \vec{r}\left(t\right) dt$ for $\vec{r}\left(t\right) = \left\langle \sin\left(t\right), 6, 4t \right\rangle$.

Solution

In this case all that we need to do is reuse the result from the previous example and then do the evaluation.

$$\begin{aligned} \int_0^1 \vec{r}\left(t\right) dt &= \left(\left\langle -\cos\left(t\right), 6t, 2t^2 \right\rangle \right)\Big|_0^1 \\ &= \left\langle -\cos\left(1\right), 6, 2 \right\rangle - \left\langle -1, 0, 0 \right\rangle \\ &= \left\langle 1 - \cos\left(1\right), 6, 2 \right\rangle \end{aligned}$$

Tangent, Normal and Binormal Vectors

In this section we want to look at an application of derivatives for vector functions. Actually, there are a couple of applications, but they all come back to needing the first one.

In the past we've used the fact that the derivative of a function was the slope of the tangent line. With vector functions we get exactly the same result, with one exception.

Given the vector function, $\vec{r}\left(t\right)$, we call $\vec{r}'\left(t\right)$ the **tangent vector** provided it exists and provided $\vec{r}'\left(t\right) \neq \vec{0}$. The tangent line to $\vec{r}\left(t\right)$ at P is then the line that passes through the point P and is parallel to the tangent vector, $\vec{r}'\left(t\right)$. Note that we really do need to require $\vec{r}'\left(t\right) \neq \vec{0}$ in order to have a tangent vector. If we had $\vec{r}'\left(t\right) = \vec{0}$ we would have a vector that had no magnitude and so couldn't give us the direction of the tangent.

Also, provided $\vec{r}'\left(t\right) \neq \vec{0}$, the **unit tangent vector** to the curve is given by,

$$\vec{T}\left(t\right) = \frac{\vec{r}'\left(t\right)}{\left\|\vec{r}'\left(t\right)\right\|}$$

While, the components of the unit tangent vector can be somewhat messy on occasion there are times when we will need to use the unit tangent vector instead of the tangent vector.

Example 1 Find the general formula for the tangent vector and unit tangent vector to the curve given by $\vec{r}\left(t\right) = t^2\,\vec{i} + 2\sin t\,\vec{j} + 2\cos t\,\vec{k}$.

Solution
First, by general formula we mean that we won't be plugging in a specific t and so we will be finding a formula that we can use at a later date if we'd like to find the tangent at any point on the curve. With that said there really isn't all that much to do at this point other than to do the work.

Here is the tangent vector to the curve.

$$\vec{r}'\left(t\right) = 2t\,\vec{i} + 2\cos t\,\vec{j} - 2\sin t\,\vec{k}$$

To get the unit tangent vector we need the length of the tangent vector.

$$\begin{aligned}\left\|\vec{r}'\left(t\right)\right\| &= \sqrt{4t^2 + 4\cos^2 t + 4\sin^2 t} \\ &= \sqrt{4t^2 + 4}\end{aligned}$$

The unit tangent vector is then,

$$\begin{aligned}\vec{T}\left(t\right) &= \frac{1}{\sqrt{4t^2+4}}\left(2t\,\vec{i} + 2\cos t\,\vec{j} - 2\sin t\,\vec{k}\right) \\ &= \frac{2t}{\sqrt{4t^2+4}}\vec{i} + \frac{2\cos t}{\sqrt{4t^2+4}}\vec{j} - \frac{2\sin t}{\sqrt{4t^2+4}}\vec{k}\end{aligned}$$

Example 2 Find the vector equation of the tangent line to the curve given by $\vec{r}(t) = t^2\,\vec{i} + 2\sin t\,\vec{j} + 2\cos t\,\vec{k}$ at $t = \frac{\pi}{3}$.

Solution
First we need the tangent vector and since this is the function we were working with in the previous example we can just reuse the tangent vector from that example and plug in $t = \frac{\pi}{3}$.

$$\vec{r}'\left(\frac{\pi}{3}\right) = \frac{2\pi}{3}\vec{i} + 2\cos\left(\frac{\pi}{3}\right)\vec{j} - 2\sin\left(\frac{\pi}{3}\right)\vec{k} = \frac{2\pi}{3}\vec{i} + \vec{j} - \sqrt{3}\,\vec{k}$$

We'll also need the point on the line at $t = \frac{\pi}{3}$ so,

$$\vec{r}\left(\frac{\pi}{3}\right) = \frac{\pi^2}{9}\vec{i} + \sqrt{3}\,\vec{j} + \vec{k}$$

The vector equation of the line is then,

$$\vec{r}(t) = \left\langle \frac{\pi^2}{9}, \sqrt{3}, 1 \right\rangle + t\left\langle \frac{2\pi}{3}, 1, -\sqrt{3} \right\rangle$$

Before moving on let's note a couple of things about the previous example. First, we could have used the unit tangent vector had we wanted to for the parallel vector. However, that would have made for a more complicated equation for the tangent line.

Second, notice that we used $\vec{r}(t)$ to represent the tangent line despite the fact that we used that as well for the function. Do not get excited about that. The $\vec{r}(t)$ here is much like y is with normal functions. With normal functions, y is the generic letter that we used to represent functions and $\vec{r}(t)$ tends to be used in the same way with vector functions.

Next we need to talk about the **unit normal** and the **binormal** vectors.

The unit normal vector is defined to be,

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\left\|\vec{T}'(t)\right\|}$$

The unit normal is orthogonal (or normal, or perpendicular) to the unit tangent vector and hence to the curve as well. We've already seen normal vectors when we were dealing with Equations of Planes. They will show up with some regularity in several Calculus III topics.

The definition of the unit normal vector always seems a little mysterious when you first see it. It follows directly from the following fact.

Fact

Suppose that $\vec{r}(t)$ is a vector such that $\left\|\vec{r}(t)\right\| = c$ for all t. Then $\vec{r}'(t)$ is orthogonal to $\vec{r}(t)$.

To prove this fact is pretty simple. From the fact statement and the relationship between the magnitude of a vector and the dot product we have the following.

$$\vec{r}(t) \bullet \vec{r}(t) = \left\| \vec{r}(t) \right\|^2 = c^2 \qquad \text{for all } t$$

Now, because this is true for all t we can see that,

$$\frac{d}{dt}\left(\vec{r}(t) \bullet \vec{r}(t) \right) = \frac{d}{dt}\left(c^2 \right) = 0$$

Also, recalling the fact from the previous section about differentiating a dot product we see that,

$$\frac{d}{dt}\left(\vec{r}(t) \bullet \vec{r}(t) \right) = \vec{r}'(t) \bullet \vec{r}(t) + \vec{r}(t) \bullet \vec{r}'(t) = 2\vec{r}'(t) \bullet \vec{r}(t)$$

Or, upon putting all this together we get,

$$2\vec{r}'(t) \bullet \vec{r}(t) = 0 \qquad \Rightarrow \qquad \vec{r}'(t) \bullet \vec{r}(t) = 0$$

Therefore $\vec{r}'(t)$ is orthogonal to $\vec{r}(t)$.

The definition of the unit normal then falls directly from this. Because $\vec{T}(t)$ is a unit vector we know that $\left\| \vec{T}(t) \right\| = 1$ for all t and hence by the Fact $\vec{T}'(t)$ is orthogonal to $\vec{T}(t)$. However, because $\vec{T}(t)$ is tangent to the curve, $\vec{T}'(t)$ must be orthogonal, or normal, to the curve as well and so be a normal vector for the curve. All we need to do then is divide by $\left\| \vec{T}'(t) \right\|$ to arrive at a unit normal vector.

Next, is the binormal vector. The binormal vector is defined to be,

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

Because the binormal vector is defined to be the cross product of the unit tangent and unit normal vector we then know that the binormal vector is orthogonal to both the tangent vector and the normal vector.

Example 3 Find the normal and binormal vectors for $\vec{r}(t) = \langle t, 3\sin t, 3\cos t \rangle$.

Solution

We first need the unit tangent vector so first get the tangent vector and its magnitude.

$$\vec{r}'(t) = \langle 1, 3\cos t, -3\sin t \rangle$$

$$\left\| \vec{r}'(t) \right\| = \sqrt{1 + 9\cos^2 t + 9\sin^2 t} = \sqrt{10}$$

The unit tangent vector is then,

$$\vec{T}(t) = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\cos t, -\frac{3}{\sqrt{10}}\sin t \right\rangle$$

The unit normal vector will now require the derivative of the unit tangent and its magnitude.

$$\vec{T}'(t) = \left\langle 0, -\frac{3}{\sqrt{10}}\sin t, -\frac{3}{\sqrt{10}}\cos t \right\rangle$$

$$\left\| \vec{T}'(t) \right\| = \sqrt{\frac{9}{10}\sin^2 t + \frac{9}{10}\cos^2 t} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}$$

The unit normal vector is then,

$$\vec{N}(t) = \frac{\sqrt{10}}{3}\left\langle 0, -\frac{3}{\sqrt{10}}\sin t, -\frac{3}{\sqrt{10}}\cos t \right\rangle = \left\langle 0, -\sin t, -\cos t \right\rangle$$

Finally, the binormal vector is,

$$\begin{aligned}
\vec{B}(t) &= \vec{T}(t) \times \vec{N}(t) \\
&= \left| \begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\cos t & -\frac{3}{\sqrt{10}}\sin t \\ 0 & -\sin t & -\cos t \end{array} \right| \begin{array}{cc} \vec{i} & \vec{j} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\cos t \\ 0 & -\sin t \end{array} \\
&= -\frac{3}{\sqrt{10}}\cos^2 t\,\vec{i} - \frac{1}{\sqrt{10}}\sin t\,\vec{k} + \frac{1}{\sqrt{10}}\cos t\;\vec{j} - \frac{3}{\sqrt{10}}\sin^2 t\,\vec{i} \\
&= -\frac{3}{\sqrt{10}}\vec{i} + \frac{1}{\sqrt{10}}\cos t\;\vec{j} - \frac{1}{\sqrt{10}}\sin t\;\vec{k}
\end{aligned}$$

Arc Length with Vector Functions

In this section we'll recast an old formula into terms of vector functions. We want to determine the length of a vector function,

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

on the interval $a \le t \le b$.

We actually already know how to do this. Recall that we can write the vector function into the parametric form,

$$x = f(t) \qquad y = g(t) \qquad z = h(t)$$

Also, recall that with two dimensional parametric curves the arc length is given by,

$$L = \int_a^b \sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2}\, dt$$

There is a natural extension of this to three dimensions. So, the length of the curve $\vec{r}(t)$ on the interval $a \le t \le b$ is,

$$L = \int_a^b \sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2 + \left[h'(t)\right]^2}\, dt$$

There is a nice simplification that we can make for this. Notice that the integrand (the function we're integrating) is nothing more than the magnitude of the tangent vector,

$$\left\| \vec{r}'(t) \right\| = \sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2 + \left[h'(t)\right]^2}$$

Therefore, the arc length can be written as,

$$L = \int_a^b \left\| \vec{r}'(t) \right\| dt$$

Let's work a quick example of this.

Example 1 Determine the length of the curve $\vec{r}(t) = \langle 2t, 3\sin(2t), 3\cos(2t) \rangle$ on the interval $0 \le t \le 2\pi$.

Solution

We will first need the tangent vector and its magnitude.

$$\vec{r}'(t) = \langle 2, 6\cos(2t), -6\sin(2t) \rangle$$

$$\left\| \vec{r}'(t) \right\| = \sqrt{4 + 36\cos^2(2t) + 36\sin^2(2t)} = \sqrt{4 + 36} = 2\sqrt{10}$$

The length is then,

$$\begin{aligned} L &= \int_a^b \left\| \vec{r}'(t) \right\| dt \\ &= \int_0^{2\pi} 2\sqrt{10}\, dt \\ &= 4\pi\sqrt{10} \end{aligned}$$

We need to take a quick look at another concept here. We define the **arc length function** as,

$$s(t) = \int_0^t \left\| \vec{r}'(u) \right\| du$$

Before we look at why this might be important let's work a quick example.

Example 2 Determine the arc length function for $\vec{r}(t) = \langle 2t, 3\sin(2t), 3\cos(2t) \rangle$.

Solution
From the previous example we know that,

$$\left\| \vec{r}'(t) \right\| = 2\sqrt{10}$$

The arc length function is then,

$$s(t) = \int_0^t 2\sqrt{10}\, du = \left(2\sqrt{10}\, u \right)\Big|_0^t = 2\sqrt{10}\, t$$

Okay, just why would we want to do this? Well let's take the result of the example above and solve it for *t*.

$$t = \frac{s}{2\sqrt{10}}$$

Now, taking this and plugging it into the original vector function and we can **reparameterize** the function into the form, $\vec{r}(t(s))$. For our function this is,

$$\vec{r}(t(s)) = \left\langle \frac{s}{\sqrt{10}}, 3\sin\left(\frac{s}{\sqrt{10}} \right), 3\cos\left(\frac{s}{\sqrt{10}} \right) \right\rangle$$

So, why would we want to do this? Well with the reparameterization we can now tell where we are on the curve after we've traveled a distance of *s* along the curve. Note as well that we will start the measurement of distance from where we are at $t = 0$.

Example 3 Where on the curve $\vec{r}(t) = \langle 2t, 3\sin(2t), 3\cos(2t) \rangle$ are we after traveling for a distance of $\frac{\pi\sqrt{10}}{3}$?

Solution
To determine this we need the reparameterization, which we have from above.

$$\vec{r}\left(t\left(s\right)\right)=\left\langle \frac{s}{\sqrt{10}},3\sin\left(\frac{s}{\sqrt{10}}\right),3\cos\left(\frac{s}{\sqrt{10}}\right)\right\rangle$$

Then, to determine where we are all that we need to do is plug in $s=\frac{\pi\sqrt{10}}{3}$ into this and we'll get our location.

$$\vec{r}\left(t\left(\frac{\pi\sqrt{10}}{3}\right)\right)=\left\langle \frac{\pi}{3},3\sin\left(\frac{\pi}{3}\right),3\cos\left(\frac{\pi}{3}\right)\right\rangle=\left\langle \frac{\pi}{3},\frac{3\sqrt{3}}{2},\frac{3}{2}\right\rangle$$

So, after traveling a distance of $\frac{\pi\sqrt{10}}{3}$ along the curve we are at the point $\left(\frac{\pi}{3},\frac{3\sqrt{3}}{2},\frac{3}{2}\right)$.

Curvature

In this section we want to briefly discuss the **curvature** of a smooth curve (recall that for a smooth curve we require $\vec{r}'(t)$ is continuous and $\vec{r}'(t) \neq 0$). The curvature measures how fast a curve is changing direction at a given point.

There are several formulas for determining the curvature for a curve. The formal definition of curvature is,

$$\kappa = \left| \frac{d\vec{T}}{ds} \right|$$

where $\vec{T}$ is the unit tangent and s is the arc length. Recall that we saw in a previous section how to reparameterize a curve to get it into terms of the arc length.

In general the formal definition of the curvature is not easy to use so there are two alternate formulas that we can use. Here they are.

$$\kappa = \frac{\left\| \vec{T}'(t) \right\|}{\left\| \vec{r}'(t) \right\|} \qquad\qquad \kappa = \frac{\left\| \vec{r}'(t) \times \vec{r}''(t) \right\|}{\left\| \vec{r}'(t) \right\|^3}$$

These may not be particularly easy to deal with either, but at least we don't need to reparameterize the unit tangent.

Example 1 Determine the curvature for $\vec{r}(t) = \langle t, 3\sin t, 3\cos t \rangle$.

Solution
Back in the section when we introduced the tangent vector we computed the tangent and unit tangent vectors for this function. These were,

$$\vec{r}'(t) = \langle 1, 3\cos t, -3\sin t \rangle$$

$$\vec{T}(t) = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\cos t, -\frac{3}{\sqrt{10}}\sin t \right\rangle$$

The derivative of the unit tangent is,

$$\vec{T}'(t) = \left\langle 0, -\frac{3}{\sqrt{10}}\sin t, -\frac{3}{\sqrt{10}}\cos t \right\rangle$$

The magnitudes of the two vectors are,

$$\left\| \vec{r}'(t) \right\| = \sqrt{1 + 9\cos^2 t + 9\sin^2 t} = \sqrt{10}$$

$$\left\| \vec{T}'(t) \right\| = \sqrt{0 + \frac{9}{10}\sin^2 t + \frac{9}{10}\cos^2 t} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}$$

The curvature is then,

$$\kappa = \frac{\left\| \vec{T}'(t) \right\|}{\left\| \vec{r}'(t) \right\|} = \frac{3/\sqrt{10}}{\sqrt{10}} = \frac{3}{10}$$

In this case the curvature is constant. This means that the curve is changing direction at the same rate at every point along it. Recalling that this curve is a helix this result makes sense.

Example 2 Determine the curvature of $\vec{r}(t) = t^2\,\vec{i} + t\,\vec{k}$.

Solution

In this case the second form of the curvature would probably be easiest. Here are the first couple of derivatives.

$$\vec{r}'(t) = 2t\,\vec{i} + \vec{k} \qquad\qquad \vec{r}''(t) = 2\,\vec{i}$$

Next, we need the cross product.

$$\begin{aligned} \vec{r}'(t) \times \vec{r}''(t) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t & 0 & 1 \\ 2 & 0 & 0 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ 2t & 0 \\ 2 & 0 \end{matrix} \\ &= 2\,\vec{j} \end{aligned}$$

The magnitudes are,

$$\left\| \vec{r}'(t) \times \vec{r}''(t) \right\| = 2 \qquad\qquad \left\| \vec{r}'(t) \right\| = \sqrt{4t^2 + 1}$$

The curvature at any value of t is then,

$$\kappa = \frac{2}{\left(4t^2 + 1\right)^{\frac{3}{2}}}$$

There is a special case that we can look at here as well. Suppose that we have a curve given by $y = f(x)$ and we want to find its curvature.

As we saw when we first looked at vector functions we can write this as follows,

$$\vec{r}(x) = x\,\vec{i} + f(x)\,\vec{j}$$

If we then use the second formula for the curvature we will arrive at the following formula for the curvature.

$$\kappa = \frac{\left| f''(x) \right|}{\left(1 + \left[f'(x) \right]^2\right)^{\frac{3}{2}}}$$

Velocity and Acceleration

In this section we need to take a look at the velocity and acceleration of a moving object.

From Calculus I we know that given the position function of an object that the velocity of the object is the first derivative of the position function and the acceleration of the object is the second derivative of the position function.

So, given this it shouldn't be too surprising that if the position function of an object is given by the vector function $\vec{r}\left(t\right)$ then the velocity and acceleration of the object is given by,

$$\vec{v}\left(t\right)=\vec{r}'\left(t\right) \qquad\qquad \vec{a}\left(t\right)=\vec{r}''\left(t\right)$$

Notice that the velocity and acceleration are also going to be vectors as well.

In the study of the motion of objects the acceleration is often broken up into a **tangential component**, a_T, and a **normal component**, a_N. The tangential component is the part of the acceleration that is tangential to the curve and the normal component is the part of the acceleration that is normal (or orthogonal) to the curve. If we do this we can write the acceleration as,

$$\vec{a}=a_T\vec{T}+a_N\vec{N}$$

where $\vec{T}$ and $\vec{N}$ are the unit tangent and unit normal for the position function.

If we define $v=\left\|\vec{v}\left(t\right)\right\|$ then the tangential and normal components of the acceleration are given by,

$$a_T=v'=\frac{\vec{r}'\left(t\right)\bullet\vec{r}''\left(t\right)}{\left\|\vec{r}'\left(t\right)\right\|} \qquad\qquad a_N=\kappa v^2=\frac{\left\|\vec{r}'\left(t\right)\times\vec{r}''\left(t\right)\right\|}{\left\|\vec{r}'\left(t\right)\right\|}$$

where κ is the curvature for the position function.

There are two formulas to use here for each component of the acceleration and while the second formula may seem overly complicated it is often the easier of the two. In the tangential component, v, may be messy and computing the derivative may be unpleasant. In the normal component we will already be computing both of these quantities in order to get the curvature and so the second formula in this case is definitely the easier of the two.

Let's take a quick look at a couple of examples.

Example 1 If the acceleration of an object is given by $\vec{a}=\vec{i}+2\vec{j}+6t\vec{k}$ find the object's velocity and position functions given that the initial velocity is $\vec{v}\left(0\right)=\vec{j}-\vec{k}$ and the initial position is $\vec{r}\left(0\right)=\vec{i}-2\vec{j}+3\vec{k}$.

Solution

We'll first get the velocity. To do this all (well almost all) we need to do is integrate the acceleration.

$$\begin{aligned}\vec{v}(t) &= \int \vec{a}(t)\,dt \\ &= \int \vec{i} + 2\vec{j} + 6t\vec{k}\,dt \\ &= t\,\vec{i} + 2t\,\vec{j} + 3t^2\,\vec{k} + \vec{c}\end{aligned}$$

To completely get the velocity we will need to determine the "constant" of integration. We can use the initial velocity to get this.

$$\vec{j} - \vec{k} = \vec{v}(0) = \vec{c}$$

The velocity of the object is then,

$$\begin{aligned}\vec{v}(t) &= t\,\vec{i} + 2t\,\vec{j} + 3t^2\,\vec{k} + \vec{j} - \vec{k} \\ &= t\,\vec{i} + (2t+1)\,\vec{j} + \left(3t^2 - 1\right)\vec{k}\end{aligned}$$

We will find the position function by integrating the velocity function.

$$\begin{aligned}\vec{r}(t) &= \int \vec{v}(t)\,dt \\ &= \int t\,\vec{i} + (2t+1)\,\vec{j} + \left(3t^2 - 1\right)\vec{k}\,dt \\ &= \frac{1}{2}t^2\,\vec{i} + \left(t^2 + t\right)\vec{j} + \left(t^3 - t\right)\vec{k} + \vec{c}\end{aligned}$$

Using the initial position gives us,

$$\vec{i} - 2\vec{j} + 3\vec{k} = \vec{r}(0) = \vec{c}$$

So, the position function is,

$$\vec{r}(t) = \left(\frac{1}{2}t^2 + 1\right)\vec{i} + \left(t^2 + t - 2\right)\vec{j} + \left(t^3 - t + 3\right)\vec{k}$$

Example 2 For the object in the previous example determine the tangential and normal components of the acceleration.

Solution

There really isn't much to do here other than plug into the formulas. To do this we'll need to notice that,

$$\begin{aligned}\vec{r}'(t) &= t\,\vec{i} + (2t+1)\,\vec{j} + \left(3t^2 - 1\right)\vec{k} \\ \vec{r}''(t) &= \vec{i} + 2\vec{j} + 6t\vec{k}\end{aligned}$$

Let's first compute the dot product and cross product that we'll need for the formulas.

$$\vec{r}'(t) \bullet \vec{r}''(t) = t + 2(2t+1) + 6t\left(3t^2 - 1\right) = 18t^3 - t + 2$$

$$\vec{r}'(t)\times\vec{r}''(t)=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ t & 2t+1 & 3t^2-1\\ 1 & 2 & 6t\end{vmatrix}\begin{matrix}\vec{i} & \vec{j}\\ t & 2t+1\\ 1 & 2\end{matrix}$$

$$=(6t)(2t+1)\vec{i}+(3t^2-1)\vec{j}+2t\vec{k}-6t^2\vec{j}-2(3t^2-1)\vec{i}-(2t+1)\vec{k}$$

$$=(6t^2+6t+2)\vec{i}-(3t^2+1)\vec{j}-\vec{k}$$

Next, we also need a couple of magnitudes.

$$\left\|\vec{r}'(t)\right\|=\sqrt{t^2+(2t+1)^2+(3t^2-1)^2}=\sqrt{9t^4-t^2+4t+2}$$

$$\left\|\vec{r}'(t)\times\vec{r}''(t)\right\|=\sqrt{(6t^2+6t+2)^2+(3t^2+1)^2+1}=\sqrt{45t^4+72t^3+66t^2+24t+6}$$

The tangential component of the acceleration is then,

$$a_T=\frac{18t^3-t+2}{\sqrt{9t^4-t^2+4t+2}}$$

The normal component of the acceleration is,

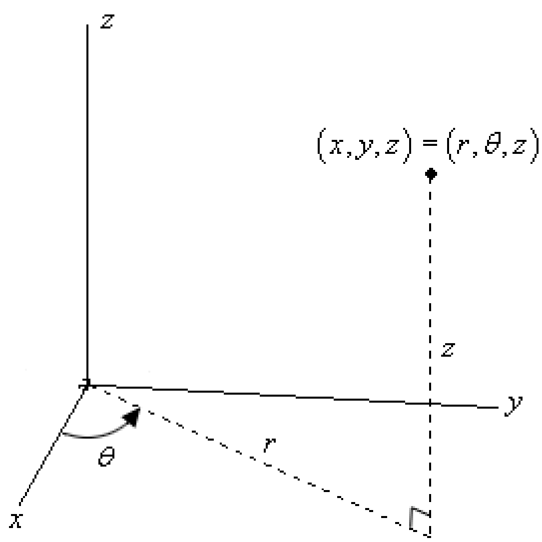
$$a_N=\frac{\sqrt{45t^4+72t^3+66t^2+24t+6}}{\sqrt{9t^4-t^2+4t+2}}=\sqrt{\frac{45t^4+72t^3+66t^2+24t+6}{9t^4-t^2+4t+2}}$$

Cylindrical Coordinates

As with two dimensional space the standard (x, y, z) coordinate system is called the Cartesian coordinate system. In the last two sections of this chapter we'll be looking at some alternate coordinate systems for three dimensional space.

We'll start off with the cylindrical coordinate system. This one is fairly simple as it is nothing more than an extension of polar coordinates into three dimensions. Not only is it an extension of polar coordinates, but we extend it into the third dimension just as we extend Cartesian coordinates into the third dimension. All that we do is add a z on as the third coordinate. The r and θ are the same as with polar coordinates.

Here is a sketch of a point in $\mathbb{R}^3$.



The conversions for x and y are the same conversions that we used back when we were looking at polar coordinates. So, if we have a point in cylindrical coordinates the Cartesian coordinates can be found by using the following conversions.

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

The third equation is just an acknowledgement that the z-coordinate of a point in Cartesian and polar coordinates is the same.

Likewise, if we have a point in Cartesian coordinates the cylindrical coordinates can be found by using the following conversions.

$$r = \sqrt{x^2 + y^2} \qquad \text{OR} \qquad r^2 = x^2 + y^2$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$z = z$$

Let's take a quick look at some surfaces in cylindrical coordinates.

Example 1 Identify the surface for each of the following equations.

(a) $r = 5$

(b) $r^2 + z^2 = 100$

(c) $z = r$

Solution

(a) In two dimensions we know that this is a circle of radius 5. Since we are now in three dimensions and there is no z in equation this means it is allowed to vary freely. So, for any given z we will have a circle of radius 5 centered on the z-axis.

In other words, we will have a cylinder of radius 5 centered on the z-axis.

(b) This equation will be easy to identify once we convert back to Cartesian coordinates.

$$r^2 + z^2 = 100$$

$$x^2 + y^2 + z^2 = 100$$

So, this is a sphere centered at the origin with radius 10.

(c) Again, this one won't be too bad if we convert back to Cartesian. For reasons that will be apparent eventually, we'll first square both sides, then convert.

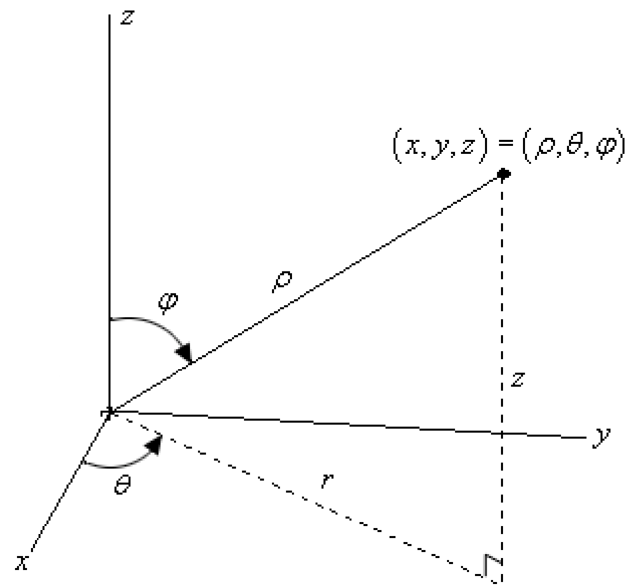
$$z^2 = r^2$$

$$z^2 = x^2 + y^2$$

From the section on quadric surfaces we know that this is the equation of a cone.

Spherical Coordinates

In this section we will introduce spherical coordinates. Spherical coordinates can take a little getting used to. It's probably easiest to start things off with a sketch.



Spherical coordinates consist of the following three quantities.

First there is ρ. This is the distance from the origin to the point and we will require $\rho \ge 0$.

Next there is θ. This is the same angle that we saw in polar/cylindrical coordinates. It is the angle between the positive *x*-axis and the line above denoted by *r* (which is also the same *r* as in polar/cylindrical coordinates). There are no restrictions on θ.

Finally there is φ. This is the angle between the positive *z*-axis and the line from the origin to the point. We will require $0 \le \varphi \le \pi$.

In summary, ρ is the distance from the origin to the point, φ is the angle that we need to rotate down from the positive *z*-axis to get to the point and θ is how much we need to rotate around the *z*-axis to get to the point.

We should first derive some conversion formulas. Let's first start with a point in spherical coordinates and ask what the cylindrical coordinates of the point are. So, we know (ρ, θ, φ) and want to find (r, θ, z). Of course we really only need to find *r* and *z* since θ is the same in both coordinate systems.

We will be able to do all of our work by looking at the right triangle shown above in our sketch. With a little geometry we see that the angle between *z* and ρ is φ and so we can see that,

$$z = \rho\cos\varphi$$
$$r = \rho\sin\varphi$$

and these are exactly the formulas that we were looking for. So, given a point in spherical coordinates the cylindrical coordinates of the point will be,

$$\begin{aligned} r &= \rho\sin\varphi \\ \theta &= \theta \\ z &= \rho\cos\varphi \end{aligned}$$

Note as well that,

$$r^2 + z^2 = \rho^2\cos^2\varphi + \rho^2\sin^2\varphi = \rho^2\left(\cos^2\varphi + \sin^2\varphi\right) = \rho^2$$

Or,

$$\rho^2 = r^2 + z^2$$

Next, let's find the Cartesian coordinates of the same point. To do this we'll start with the cylindrical conversion formulas from the previous section.

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

Now all that we need to do is use the formulas from above for r and z to get,

$$\begin{aligned} x &= \rho\sin\varphi\cos\theta \\ y &= \rho\sin\varphi\sin\theta \\ z &= \rho\cos\varphi \end{aligned}$$

Also note that since we know that $r^2 = x^2 + y^2$ we get,

$$\rho^2 = x^2 + y^2 + z^2$$

Converting points from Cartesian or cylindrical coordinates into spherical coordinates is usually done with the same conversion formulas. To see how this is done let's work an example of each.

Example 1 Perform each of the following conversions.

(a) Convert the point $\left(\sqrt{6}, \frac{\pi}{4}, \sqrt{2}\right)$ from cylindrical to spherical coordinates.

[Solution]

(b) Convert the point $\left(-1, 1, -\sqrt{2}\right)$ from Cartesian to spherical coordinates.

[Solution]

Solution

(a) Convert the point $\left(\sqrt{6},\frac{\pi}{4},\sqrt{2}\right)$ from cylindrical to spherical coordinates.

We'll start by acknowledging that θ is the same in both coordinate systems and so we don't need to do anything with that.

Next, let's find ρ.

$$\rho=\sqrt{r^2+z^2}=\sqrt{6+2}=\sqrt{8}=2\sqrt{2}$$

Finally, let's get φ. To do this we can use either the conversion for *r* or *z*. We'll use the conversion for *z*.

$$z=\rho\cos\varphi \qquad\Rightarrow\qquad \cos\varphi=\frac{z}{\rho}=\frac{\sqrt{2}}{2\sqrt{2}} \qquad\Rightarrow\qquad \varphi=\cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}$$

Notice that there are many possible values of φ that will give $\cos\varphi=\frac{1}{2}$, however, we have restricted φ to the range $0\leq\varphi\leq\pi$ and so this is the only possible value in that range.

So, the spherical coordinates of this point will are $\left(2\sqrt{2},\frac{\pi}{4},\frac{\pi}{3}\right)$.

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(b) Convert the point $\left(-1,1,-\sqrt{2}\right)$ from Cartesian to spherical coordinates.

The first thing that we'll do here is find ρ.

$$\rho=\sqrt{x^2+y^2+z^2}=\sqrt{1+1+2}=2$$

Now we'll need to find φ. We can do this using the conversion for *z*.

$$z=\rho\cos\varphi \qquad\Rightarrow\qquad \cos\varphi=\frac{z}{\rho}=\frac{-\sqrt{2}}{2} \qquad\Rightarrow\qquad \varphi=\cos^{-1}\left(\frac{-\sqrt{2}}{2}\right)=\frac{3\pi}{4}$$

As with the last parts this will be the only possible φ in the range allowed.

Finally, let's find θ. To do this we can use the conversion for *x* or *y*. We will use the conversion for *y* in this case.

$$\sin\theta=\frac{y}{\rho\sin\varphi}=\frac{1}{2\left(\frac{\sqrt{2}}{2}\right)}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2} \qquad\Rightarrow\qquad \theta=\frac{\pi}{4}\text{ or }\theta=\frac{3\pi}{4}$$

Now, we actually have more possible choices for θ but all of them will reduce down to one of the two angles above since they will just be one of these two angles with one or more complete rotations around the unit circle added on.

We will however, need to decide which one is the correct angle since only one will be. To do this let's notice that, in two dimensions, the point with coordinates $x=-1$ and $y=1$ lies in the second quadrant. This means that θ must be angle that will put the point into the second quadrant. Therefore, the second angle, $\theta=\frac{3\pi}{4}$, must be the correct one.

The spherical coordinates of this point are then $\left(2,\frac{3\pi}{4},\frac{3\pi}{4}\right)$.

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Now, let's take a look at some equations and identify the surfaces that they represent.

Example 2 Identify the surface for each of the following equations.

(a) $\rho=5$ [Solution]

(b) $\varphi=\frac{\pi}{3}$ [Solution]

(c) $\theta=\frac{2\pi}{3}$ [Solution]

(d) $\rho\sin\varphi=2$ [Solution]

Solution

(a) $\rho=5$

There are a couple of ways to think about this one.

First, think about what this equation is saying. This equation says that, no matter what θ and φ are, the distance from the origin must be 5. So, we can rotate as much as we want away from the z-axis and around the z-axis, but we must always remain at a fixed distance from the origin. This is exactly what a sphere is. So, this is a sphere of radius 5 centered at the origin.

The other way to think about it is to just convert to Cartesian coordinates.

$$\begin{aligned}\rho&=5\\ \rho^2&=25\\ x^2+y^2+z^2&=25\end{aligned}$$

Sure enough a sphere of radius 5 centered at the origin.

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(b) $\varphi=\frac{\pi}{3}$

In this case there isn't an easy way to convert to Cartesian coordinates so we'll just need to think about this one a little. This equation says that no matter how far away from the origin that we move and no matter how much we rotate around the z-axis the point must always be at an angle of $\frac{\pi}{3}$ from the z-axis.

This is exactly what happens in a cone. All of the points on a cone are a fixed angle from the z-

axis. So, we have a cone whose points are all at an angle of $\frac{\pi}{3}$ from the z-axis.

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(c) $\theta = \dfrac{2\pi}{3}$

As with the last part we won't be able to easily convert to Cartesian coordinates here. In this case no matter how far from the origin we get or how much we rotate down from the positive z-axis the points must always form an angle of $\frac{2\pi}{3}$ with the x-axis.

Points in a vertical plane will do this. So, we have a vertical plane that forms an angle of $\frac{2\pi}{3}$ with the positive x-axis.

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(d) $\rho \sin \varphi = 2$

In this case we can convert to Cartesian coordinates so let's do that. There are actually two ways to do this conversion. We will look at both since both will be used on occasion.

Solution 1

In this solution method we will convert directly to Cartesian coordinates. To do this we will first need to square both sides of the equation.

$$\rho^2 \sin^2 \varphi = 4$$

Now, for no apparent reason add $\rho^2 \cos^2 \varphi$ to both sides.

$$\begin{aligned} \rho^2 \sin^2 \varphi + \rho^2 \cos^2 \varphi &= 4 + \rho^2 \cos^2 \varphi \\ \rho^2 \left(\sin^2 \varphi + \cos^2 \varphi\right) &= 4 + \rho^2 \cos^2 \varphi \\ \rho^2 &= 4 + \left(\rho \cos \varphi\right)^2 \end{aligned}$$

Now we can convert to Cartesian coordinates.

$$\begin{aligned} x^2 + y^2 + z^2 &= 4 + z^2 \\ x^2 + y^2 &= 4 \end{aligned}$$

So, we have a cylinder of radius 2 centered on the z-axis.

This solution method wasn't too bad, but it did require some not so obvious steps to complete.

Solution 2

This method is much shorter, but also involves something that you may not see the first time around. In this case instead of going straight to Cartesian coordinates we'll first convert to cylindrical coordinates.

This won't always work, but in this case all we need to do is recognize that $r = \rho \sin \varphi$ and we will get something we can recognize. Using this we get,

$$\rho \sin \varphi = 2$$
$$r = 2$$

At this point we know this is a cylinder (remember that we're in three dimensions and so this isn't a circle!). However, let's go ahead and finish the conversion process out.

$$r^2 = 4$$
$$x^2 + y^2 = 4$$

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So, as we saw in the last part of the previous example it will *sometimes* be easier to convert equations in spherical coordinates into cylindrical coordinates before converting into Cartesian coordinates. This won't always be easier, but it can make some of the conversions quicker and easier.

The last thing that we want to do in this section is generalize the first three parts of the previous example.

$\rho = a$	sphere of radius a centered at the origin
$\varphi = \alpha$	cone that makes an angle of α with the positive $z-$axis
$\theta = \beta$	vertical plane that makes an angle of β with the positive $x-$axis

Partial Derivatives

Introduction

In Calculus I and in most of Calculus II we concentrated on functions of one variable. In Calculus III we will extend our knowledge of calculus into functions of two or more variables. Despite the fact that this chapter is about derivatives we will start out the chapter with a section on limits of functions of more than one variable. In the remainder of this chapter we will be looking at differentiating functions of more than one variable. As we will see, while there are differences with derivatives of functions of one variable, if you can do derivatives of functions of one variable you shouldn't have any problems differentiating functions of more than one variable.

Here is a list of topics in this chapter.

Limits – Taking limits of functions of several variables.

Partial Derivatives – In this section we will introduce the idea of partial derivatives as well as the standard notations and how to compute them.

Interpretations of Partial Derivatives – Here we will take a look at a couple of important interpretations of partial derivatives.

Higher Order Partial Derivatives – We will take a look at higher order partial derivatives in this section.

Differentials – In this section we extend the idea of differentials to functions of several variables.

Chain Rule – Here we will look at the chain rule for functions of several variables.

Directional Derivatives – We will introduce the concept of directional derivatives in this section. We will also see how to compute them and see a couple of nice facts pertaining to directional derivatives.

Limits

In this section we will take a look at limits involving functions of more than one variable. In fact, we will concentrate mostly on limits of functions of two variables, but the ideas can be extended out to functions with more than two variables.

Before getting into this let's briefly recall how limits of functions of one variable work. We say that,

$$\lim_{x \to a} f(x) = L$$

provided,

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$$

Also, recall that,

$$\lim_{x \to a^+} f(x)$$

is a right hand limit and requires us to only look at values of x that are greater than a. Likewise,

$$\lim_{x \to a^-} f(x)$$

is a left hand limit and requires us to only look at values of x that are less than a.

In other words, we will have $\lim_{x \to a} f(x) = L$ provided $f(x)$ approaches L as we move in towards $x = a$ (without letting $x = a$) from both sides.

Now, notice that in this case there are only two paths that we can take as we move in towards $x = a$. We can either move in from the left or we can move in from the right. Then in order for the limit of a function of one variable to exist the function must be approaching the same value as we take each of these paths in towards $x = a$.

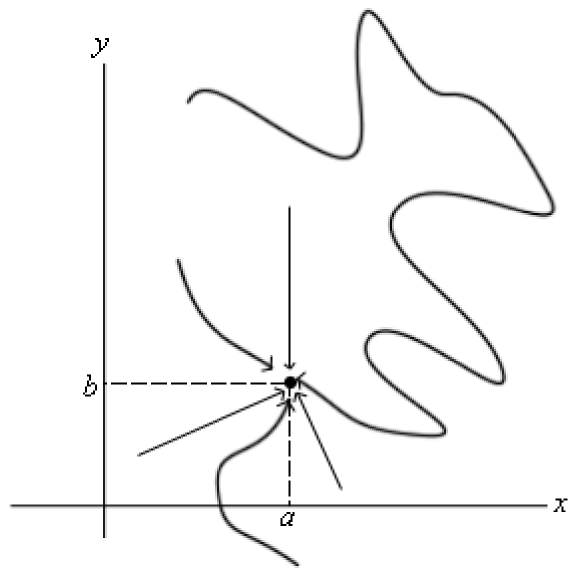
With functions of two variables we will have to do something similar, except this time there is (potentially) going to be a lot more work involved. Let's first address the notation and get a feel for just what we're going to be asking for in these kinds of limits.

We will be asking to take the limit of the function $f(x, y)$ as x approaches a and as y approaches b. This can be written in several ways. Here are a couple of the more standard notations.

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y) \qquad\qquad \lim_{(x,y) \to (a,b)} f(x, y)$$

We will use the second notation more often than not in this course. The second notation is also a little more helpful in illustrating what we are really doing here when we are taking a limit. In taking a limit of a function of two variables we are really asking what the value of $f(x, y)$ is doing as we move the point (x, y) in closer and closer to the point (a, b) without actually letting it be (a, b).

Just like with limits of functions of one variable, in order for this limit to exist, the function must be approaching the same value regardless of the path that we take as we move in towards (a,b). The problem that we are immediately faced with is that there are literally an infinite number of paths that we can take as we move in towards (a,b). Here are a few examples of paths that we could take.



We put in a couple of straight line paths as well as a couple of "stranger" paths that aren't straight line paths. Also, we only included 6 paths here and as you can see simply by varying the slope of the straight line paths there are an infinite number of these and then we would need to consider paths that aren't straight line paths.

In other words, to show that a limit exists we would technically need to check an infinite number of paths and verify that the function is approaching the same value regardless of the path we are using to approach the point.

Luckily for us however we can use one of the main ideas from Calculus I limits to help us take limits here.

Definition

A function $f(x,y)$ is **continuous** at the point (a,b) if,

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

From a graphical standpoint this definition means the same thing as it did when we first saw continuity in Calculus I. A function will be continuous at a point if the graph doesn't have any holes or breaks at that point.

How can this help us take limits? Well, just as in Calculus I, if you know that a function is continuous at (a,b) then you also know that

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

must be true. So, if we know that a function is continuous at a point then all we need to do to take the limit of the function at that point is to plug the point into the function.

All the standard functions that we know to be continuous are still continuous even if we are plugging in more than one variable now. We just need to watch out for division by zero, square roots of negative numbers, logarithms of zero or negative numbers, *etc.*

Note that the idea about paths is one that we shouldn't forget since it is a nice way to determine if a limit doesn't exist. If we can find two paths upon which the function approaches different values as we get near the point then we will know that the limit doesn't exist.

Let's take a look at a couple of examples.

Example 1 Determine if the following limits exist or not. If they do exist give the value of the limit.

(a) $\lim\limits_{(x,y,z)\to(2,1,-1)} 3x^2z + yx\cos(\pi x - \pi z)$ [Solution]

(b) $\lim\limits_{(x,y)\to(5,1)} \dfrac{xy}{x+y}$ [Solution]

(c) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^4+3y^4}$ [Solution]

(d) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3y}{x^6+y^2}$ [Solution]

Solution

(a) $\lim\limits_{(x,y,z)\to(2,1,-1)} 3x^2z + yx\cos(\pi x - \pi z)$

Okay, in this case the function is continuous at the point in question and so all we need to do is plug in the values and we're done.

$$\lim_{(x,y,z)\to(2,1,-1)} 3x^2z + yx\cos(\pi x - \pi z) = 3(2)^2(-1) + (1)(2)\cos(2\pi + \pi) = -14$$

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(b) $\lim\limits_{(x,y)\to(5,1)} \dfrac{xy}{x+y}$

In this case the function will not be continuous along the line $y = -x$ since we will get division by zero when this is true. However, for this problem that is not something that we will need to worry about since the point that we are taking the limit at isn't on this line.

Therefore, all that we need to do is plug in the point since the function is continuous at this point.

$$\lim_{(x,y)\to(5,1)} \frac{xy}{x+y} = \frac{5}{6}$$

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(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+3y^4}$

Now, in this case the function is not continuous at the point in question and so we can't just plug in the point. So, since the function is not continuous at the point there is at least a chance that the limit doesn't exist. If we could find two different paths to approach the point that gave different values for the limit then we would know that the limit didn't exist. Two of the more common paths to check are the *x* and *y*-axis so let's try those.

Before actually doing this we need to address just what exactly do we mean when we say that we are going to approach a point along a path. When we approach a point along a path we will do this by either fixing *x* or *y* or by relating *x* and *y* through some function. In this way we can reduce the limit to just a limit involving a single variable which we know how to do from Calculus I.

So, let's see what happens along the *x*-axis. If we are going to approach $(0,0)$ along the *x*-axis we can take advantage of the fact that that along the *x*-axis we know that $y=0$. This means that, along the *x*-axis, we will plug in $y=0$ into the function and then take the limit as *x* approaches zero.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+3y^4} = \lim_{(x,0)\to(0,0)} \frac{x^2(0)^2}{x^4+3(0)^4} = \lim_{(x,0)\to(0,0)} 0 = 0$$

So, along the *x*-axis the function will approach zero as we move in towards the origin.

Now, let's try the *y*-axis. Along this axis we have $x=0$ and so the limit becomes,

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+3y^4} = \lim_{(0,y)\to(0,0)} \frac{(0)^2y^2}{(0)^4+3y^4} = \lim_{(0,y)\to(0,0)} 0 = 0$$

So, the same limit along two paths. Don't misread this. This does NOT say that the limit exists and has a value of zero. This only means that the limit happens to have the same value along two paths.

Let's take a look at a third fairly common path to take a look at. In this case we'll move in towards the origin along the path $y=x$. This is what we meant previously about relating *x* and *y* through a function.

To do this we will replace all the *y*'s with *x*'s and then let *x* approach zero. Let's take a look at this limit.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+3y^4} = \lim_{(x,x)\to(0,0)} \frac{x^2x^2}{x^4+3x^4} = \lim_{(x,x)\to(0,0)} \frac{x^4}{4x^4} = \lim_{(x,x)\to(0,0)} \frac{1}{4} = \frac{1}{4}$$

So, a different value from the previous two paths and this means that the limit can't possibly exist.

Note that we can use this idea of moving in towards the origin along a line with the more general path $y = mx$ if we need to.

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(d) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6 + y^2}$

Okay, with this last one we again have continuity problems at the origin. So, again let's see if we can find a couple of paths that give different values of the limit.

First, we will use the path $y = x$. Along this path we have,

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6 + y^2} = \lim_{(x,x)\to(0,0)} \frac{x^3 x}{x^6 + x^2} = \lim_{(x,x)\to(0,0)} \frac{x^4}{x^6 + x^2} = \lim_{(x,x)\to(0,0)} \frac{x^2}{x^4 + 1} = 0$$

Now, let's try the path $y = x^3$. Along this path the limit becomes,

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6 + y^2} = \lim_{(x,x^3)\to(0,0)} \frac{x^3 x^3}{x^6 + \left(x^3\right)^2} = \lim_{(x,x^3)\to(0,0)} \frac{x^6}{2x^6} = \lim_{(x,x^3)\to(0,0)} \frac{1}{2} = \frac{1}{2}$$

We now have two paths that give different values for the limit and so the limit doesn't exist.

As this limit has shown us we can, and often need, to use paths other than lines.

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Partial Derivatives

Now that we have the brief discussion on limits out of the way we can proceed into taking derivatives of functions of more than one variable. Before we actually start taking derivatives of functions of more than one variable let's recall an important interpretation of derivatives of functions of one variable.

Recall that given a function of one variable, $f(x)$, the derivative, $f'(x)$, represents the rate of change of the function as *x* changes. This is an important interpretation of derivatives and we are not going to want to lose it with functions of more than one variable. The problem with functions of more than one variable is that there is more than one variable. In other words, what do we do if we only want one of the variables to change, or if we want more than one of them to change? In fact, if we're going to allow more than one of the variables to change there are then going to be an infinite amount of ways for them to change. For instance, one variable could be changing faster than the other variable(s) in the function. Notice as well that it will be completely possible for the function to be changing differently depending on how we allow one or more of the variables to change.

We will need to develop ways, and notations, for dealing with all of these cases. In this section we are going to concentrate exclusively on only changing one of the variables at a time, while the remaining variable(s) are held fixed. We will deal with allowing multiple variables to change in a later section.

Because we are going to only allow one of the variables to change taking the derivative will now become a fairly simple process. Let's start off this discussion with a fairly simple function.

Let's start with the function $f(x,y) = 2x^2y^3$ and let's determine the rate at which the function is changing at a point, (a,b), if we hold *y* fixed and allow *x* to vary and if we hold *x* fixed and allow *y* to vary.

We'll start by looking at the case of holding *y* fixed and allowing *x* to vary. Since we are interested in the rate of change of the function at (a,b) and are holding *y* fixed this means that we are going to always have $y = b$ (if we didn't have this then eventually *y* would have to change in order to get to the point…). Doing this will give us a function involving only *x*'s and we can define a new function as follows,

$$g(x) = f(x,b) = 2x^2b^3$$

Now, this is a function of a single variable and at this point all that we are asking is to determine the rate of change of $g(x)$ at $x = a$. In other words, we want to compute $g'(a)$ and since this is a function of a single variable we already know how to do that. Here is the rate of change of the function at (a,b) if we hold *y* fixed and allow *x* to vary.

$$g'(a) = 4ab^3$$

We will call $g'(a)$ the **partial derivative** of $f(x,y)$ with respect to x at (a,b) and we will denote it in the following way,

$$f_x(a,b) = 4ab^3$$

Now, let's do it the other way. We will now hold x fixed and allow y to vary. We can do this in a similar way. Since we are holding x fixed it must be fixed at $x = a$ and so we can define a new function of y and then differentiate this as we've always done with functions of one variable.

Here is the work for this,

$$h(y) = f(a,y) = 2a^2y^3 \qquad \Rightarrow \qquad h'(b) = 6a^2b^2$$

In this case we call $h'(b)$ the **partial derivative** of $f(x,y)$ with respect to y at (a,b) and we denote it as follows,

$$f_y(a,b) = 6a^2b^2$$

Note that these two partial derivatives are sometimes called the **first order partial derivatives**. Just as with functions of one variable we can have derivatives of all orders. We will be looking at higher order derivatives in a later section.

Note that the notation for partial derivatives is different than that for derivatives of functions of a single variable. With functions of a single variable we could denote the derivative with a single prime. However, with partial derivatives we will always need to remember the variable that we are differentiating with respect to and so we will subscript the variable that we differentiated with respect to. We will shortly be seeing some alternate notation for partial derivatives as well.

Note as well that we usually don't use the (a,b) notation for partial derivatives. The more standard notation is to just continue to use (x,y). So, the partial derivatives from above will more commonly be written as,

$$f_x(x,y) = 4xy^3 \qquad \text{and} \qquad f_y(x,y) = 6x^2y^2$$

Now, as this quick example has shown taking derivatives of functions of more than one variable is done in pretty much the same manner as taking derivatives of a single variable. To compute $f_x(x,y)$ all we need to do is treat all the y's as constants (or numbers) and then differentiate the x's as we've always done. Likewise, to compute $f_y(x,y)$ we will treat all the x's as constants and then differentiate the y's as we are used to doing.

Before we work any examples let's get the formal definition of the partial derivative out of the way as well as some alternate notation.

Since we can think of the two partial derivatives above as derivatives of single variable functions it shouldn't be too surprising that the definition of each is very similar to the definition of the derivative for single variable functions. Here are the formal definitions of the two partial derivatives we looked at above.

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \qquad\qquad f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Now let's take a quick look at some of the possible alternate notations for partial derivatives. Given the function $z = f(x,y)$ the following are all equivalent notations,

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(f(x,y)\right) = z_x = \frac{\partial z}{\partial x} = D_x f$$
$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(f(x,y)\right) = z_y = \frac{\partial z}{\partial y} = D_y f$$

For the fractional notation for the partial derivative notice the difference between the partial derivative and the ordinary derivative from single variable calculus.

$$f(x) \qquad \Rightarrow \qquad f'(x) = \frac{df}{dx}$$
$$f(x,y) \qquad \Rightarrow \qquad f_x(x,y) = \frac{\partial f}{\partial x} \;\;\&\;\; f_y(x,y) = \frac{\partial f}{\partial y}$$

Okay, now let's work some examples. When working these examples always keep in mind that we need to pay very close attention to which variable we are differentiating with respect to. This is important because we are going to treat all other variables as constants and then proceed with the derivative as if it was a function of a single variable. If you can remember this you'll find that doing partial derivatives are not much more difficult that doing derivatives of functions of a single variable as we did in Calculus I.

Example 1 Find all of the first order partial derivatives for the following functions.

(a) $f(x,y) = x^4 + 6\sqrt{y} - 10$ [Solution]

(b) $w = x^2 y - 10y^2 z^3 + 43x - 7\tan(4y)$ [Solution]

(c) $h(s,t) = t^7 \ln(s^2) + \frac{9}{t^3} - \sqrt[7]{s^4}$ [Solution]

(d) $f(x,y) = \cos\left(\frac{4}{x}\right)\mathbf{e}^{x^2 y - 5y^3}$ [Solution]

Solution

(a) $f(x,y) = x^4 + 6\sqrt{y} - 10$

Let's first take the derivative with respect to x and remember that as we do so all the y's will be treated as constants. The partial derivative with respect to x is,

$$f_x(x,y) = 4x^3$$

Notice that the second and the third term differentiate to zero in this case. It should be clear why the third term differentiated to zero. It's a constant and we know that constants always differentiate to zero. This is also the reason that the second term differentiated to zero. Remember that since we are differentiating with respect to x here we are going to treat all y's as constants. That means that terms that only involve y's will be treated as constants and hence will differentiate to zero.

Now, let's take the derivative with respect to *y*. In this case we treat all *x*'s as constants and so the first term involves only *x*'s and so will differentiate to zero, just as the third term will. Here is the partial derivative with respect to *y*.

$$f_y(x,y) = \frac{3}{\sqrt{y}}$$

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(b) $w = x^2y - 10y^2z^3 + 43x - 7\tan(4y)$

With this function we've got three first order derivatives to compute. Let's do the partial derivative with respect to *x* first. Since we are differentiating with respect to *x* we will treat all *y*'s and all *z*'s as constants. This means that the second and fourth terms will differentiate to zero since they only involve *y*'s and *z*'s.

This first term contains both *x*'s and *y*'s and so when we differentiate with respect to *x* the *y* will be thought of as a multiplicative constant and so the first term will be differentiated just as the third term will be differentiated.

Here is the partial derivative with respect to *x*.

$$\frac{\partial w}{\partial x} = 2xy + 43$$

Let's now differentiate with respect to *y*. In this case all *x*'s and *z*'s will be treated as constants. This means the third term will differentiate to zero since it contains only *x*'s while the *x*'s in the first term and the *z*'s in the second term will be treated as multiplicative constants. Here is the derivative with respect to *y*.

$$\frac{\partial w}{\partial y} = x^2 - 20yz^3 - 28\sec^2(4y)$$

Finally, let's get the derivative with respect to *z*. Since only one of the terms involve *z*'s this will be the only non-zero term in the derivative. Also, the *y*'s in that term will be treated as multiplicative constants. Here is the derivative with respect to *z*.

$$\frac{\partial w}{\partial z} = -30y^2z^2$$

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(c) $h(s,t) = t^7\ln(s^2) + \frac{9}{t^3} - \sqrt[7]{s^4}$

With this one we'll not put in the detail of the first two. Before taking the derivative let's rewrite the function a little to help us with the differentiation process.

$$h(s,t) = t^7\ln(s^2) + 9t^{-3} - s^{\frac{4}{7}}$$

Now, the fact that we're using *s* and *t* here instead of the "standard" *x* and *y* shouldn't be a problem. It will work the same way. Here are the two derivatives for this function.

$$h_s(s,t) = \frac{\partial h}{\partial s} = t^7\left(\frac{2s}{s^2}\right) - \frac{4}{7}s^{-\frac{3}{7}} = \frac{2t^7}{s} - \frac{4}{7}s^{-\frac{3}{7}}$$

$$h_t(s,t) = \frac{\partial h}{\partial t} = 7t^6 \ln\left(s^2\right) - 27t^{-4}$$

Remember how to differentiate natural logarithms.

$$\frac{d}{dx}\left(\ln g(x)\right) = \frac{g'(x)}{g(x)}$$

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(d) $f(x,y) = \cos\left(\frac{4}{x}\right)\mathbf{e}^{x^2y-5y^3}$

Now, we can't forget the product rule with derivatives. The product rule will work the same way here as it does with functions of one variable. We will just need to be careful to remember which variable we are differentiating with respect to.

Let's start out by differentiating with respect to x. In this case both the cosine and the exponential contain x's and so we've really got a product of two functions involving x's and so we'll need to product rule this up. Here is the derivative with respect to x.

$$\begin{aligned} f_x(x,y) &= -\sin\left(\frac{4}{x}\right)\left(-\frac{4}{x^2}\right)\mathbf{e}^{x^2y-5y^3} + \cos\left(\frac{4}{x}\right)\mathbf{e}^{x^2y-5y^3}\left(2xy\right) \\ &= \frac{4}{x^2}\sin\left(\frac{4}{x}\right)\mathbf{e}^{x^2y-5y^3} + 2xy\cos\left(\frac{4}{x}\right)\mathbf{e}^{x^2y-5y^3} \end{aligned}$$

Do not forget the chain rule for functions of one variable. We will be looking at the chain rule for some more complicated expressions for multivariable functions in a later section. However, at this point we're treating all the y's as constants and so the chain rule will continue to work as it did back in Calculus I.

Also, don't forget how to differentiate exponential functions,

$$\frac{d}{dx}\left(\mathbf{e}^{f(x)}\right) = f'(x)\mathbf{e}^{f(x)}$$

Now, let's differentiate with respect to y. In this case we don't have a product rule to worry about since the only place that the y shows up is in the exponential. Therefore, since x's are considered to be constants for this derivative, the cosine in the front will also be thought of as a multiplicative constant. Here is the derivative with respect to y.

$$f_y(x,y) = \left(x^2 - 15y^2\right)\cos\left(\frac{4}{x}\right)\mathbf{e}^{x^2y-5y^3}$$

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Example 2 Find all of the first order partial derivatives for the following functions.

(a) $z = \dfrac{9u}{u^2 + 5v}$ [Solution]

(b) $g(x,y,z) = \dfrac{x\sin(y)}{z^2}$ [Solution]

(c) $z = \sqrt{x^2 + \ln(5x - 3y^2)}$ [Solution]

Solution

(a) $z = \dfrac{9u}{u^2 + 5v}$

We also can't forget about the quotient rule. Since there isn't too much to this one, we will simply give the derivatives.

$$z_u = \frac{9(u^2 + 5v) - 9u(2u)}{(u^2 + 5v)^2} = \frac{-9u^2 + 45v}{(u^2 + 5v)^2}$$

$$z_v = \frac{(0)(u^2 + 5v) - 9u(5)}{(u^2 + 5v)^2} = \frac{-45u}{(u^2 + 5v)^2}$$

In the case of the derivative with respect to v recall that u's are constant and so when we differentiate the numerator we will get zero!

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(b) $g(x,y,z) = \dfrac{x\sin(y)}{z^2}$

Now, we do need to be careful however to not use the quotient rule when it doesn't need to be used. In this case we do have a quotient, however, since the x's and y's only appear in the numerator and the z's only appear in the denominator this really isn't a quotient rule problem.

Let's do the derivatives with respect to x and y first. In both these cases the z's are constants and so the denominator in this is a constant and so we don't really need to worry too much about it. Here are the derivatives for these two cases.

$$g_x(x,y,z) = \frac{\sin(y)}{z^2} \qquad\qquad g_y(x,y,z) = \frac{x\cos(y)}{z^2}$$

Now, in the case of differentiation with respect to z we can avoid the quotient rule with a quick rewrite of the function. Here is the rewrite as well as the derivative with respect to z.

$$g(x,y,z) = x\sin(y)z^{-2}$$

$$g_z(x,y,z) = -2x\sin(y)z^{-3} = -\frac{2x\sin(y)}{z^3}$$

We went ahead and put the derivative back into the "original" form just so we could say that we did. In practice you probably don't really need to do that.

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(c) $z = \sqrt{x^2 + \ln\left(5x - 3y^2\right)}$

In this last part we are just going to do a somewhat messy chain rule problem. However, if you had a good background in Calculus I chain rule this shouldn't be all that difficult of a problem. Here are the two derivatives,

$$\begin{aligned} z_x &= \frac{1}{2}\left(x^2 + \ln\left(5x - 3y^2\right)\right)^{-\frac{1}{2}} \frac{\partial}{\partial x}\left(x^2 + \ln\left(5x - 3y^2\right)\right) \\ &= \frac{1}{2}\left(x^2 + \ln\left(5x - 3y^2\right)\right)^{-\frac{1}{2}} \left(2x + \frac{5}{5x - 3y^2}\right) \\ &= \left(x + \frac{5}{2\left(5x - 3y^2\right)}\right)\left(x^2 + \ln\left(5x - 3y^2\right)\right)^{-\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} z_y &= \frac{1}{2}\left(x^2 + \ln\left(5x - 3y^2\right)\right)^{-\frac{1}{2}} \frac{\partial}{\partial y}\left(x^2 + \ln\left(5x - 3y^2\right)\right) \\ &= \frac{1}{2}\left(x^2 + \ln\left(5x - 3y^2\right)\right)^{-\frac{1}{2}} \left(\frac{-6y}{5x - 3y^2}\right) \\ &= -\frac{3y}{5x - 3y^2}\left(x^2 + \ln\left(5x - 3y^2\right)\right)^{-\frac{1}{2}} \end{aligned}$$

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So, there are some examples of partial derivatives. Hopefully you will agree that as long as we can remember to treat the other variables as constants these work in exactly the same manner that derivatives of functions of one variable do. So, if you can do Calculus I derivatives you shouldn't have too much difficulty in doing basic partial derivatives.

There is one final topic that we need to take a quick look at in this section, implicit differentiation. Before getting into implicit differentiation for multiple variable functions let's first remember how implicit differentiation works for functions of one variable.

Example 3 Find $\frac{dy}{dx}$ for $3y^4 + x^7 = 5x$.

Solution

Remember that the key to this is to always think of *y* as a function of *x*, or $y = y\left(x\right)$ and so whenever we differentiate a term involving *y*'s with respect to *x* we will really need to use the chain rule which will mean that we will add on a $\frac{dy}{dx}$ to that term.

The first step is to differentiate both sides with respect to *x*.

$$12y^3 \frac{dy}{dx} + 7x^6 = 5$$

The final step is to solve for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{5 - 7x^6}{12y^3}$$

Now, we did this problem because implicit differentiation works in exactly the same manner with functions of multiple variables. If we have a function in terms of three variables x, y, and z we will assume that z is in fact a function of x and y. In other words, $z = z(x, y)$. Then whenever we differentiate z's with respect to x we will use the chain rule and add on a $\frac{\partial z}{\partial x}$. Likewise, whenever we differentiate z's with respect to y we will add on a $\frac{\partial z}{\partial y}$.

Let's take a quick look at a couple of implicit differentiation problems.

Example 4 Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for each of the following functions.

(a) $x^3 z^2 - 5xy^5 z = x^2 + y^3$ [Solution]

(b) $x^2 \sin(2y - 5z) = 1 + y\cos(6zx)$ [Solution]

Solution

(a) $x^3 z^2 - 5xy^5 z = x^2 + y^3$

Let's start with finding $\frac{\partial z}{\partial x}$. We first will differentiate both sides with respect to x and remember to add on a $\frac{\partial z}{\partial x}$ whenever we differentiate a z.

$$3x^2 z^2 + 2x^3 z \frac{\partial z}{\partial x} - 5y^5 z - 5xy^5 \frac{\partial z}{\partial x} = 2x$$

Remember that since we are assuming $z = z(x, y)$ then any product of x's and z's will be a product and so will need the product rule!

Now, solve for $\frac{\partial z}{\partial x}$.

$$\left(2x^3 z - 5xy^5\right)\frac{\partial z}{\partial x} = 2x - 3x^2 z^2 + 5y^5 z$$

$$\frac{\partial z}{\partial x} = \frac{2x - 3x^2 z^2 + 5y^5 z}{2x^3 z - 5xy^5}$$

Now we'll do the same thing for $\frac{\partial z}{\partial y}$ except this time we'll need to remember to add on a $\frac{\partial z}{\partial y}$ whenever we differentiate a z.

$$2x^3 z\frac{\partial z}{\partial y} - 25xy^4 z - 5xy^5\frac{\partial z}{\partial y} = 3y^2$$

$$\left(2x^3 z - 5xy^5\right)\frac{\partial z}{\partial y} = 3y^2 + 25xy^4 z$$

$$\frac{\partial z}{\partial y} = \frac{3y^2 + 25xy^4 z}{2x^3 z - 5xy^5}$$

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(b) $x^2\sin\left(2y-5z\right) = 1 + y\cos\left(6zx\right)$

We'll do the same thing for this function as we did in the previous part. First let's find $\frac{\partial z}{\partial x}$.

$$2x\sin\left(2y-5z\right) + x^2\cos\left(2y-5z\right)\left(-5\frac{\partial z}{\partial x}\right) = -y\sin\left(6zx\right)\left(6z + 6x\frac{\partial z}{\partial x}\right)$$

Don't forget to do the chain rule on each of the trig functions and when we are differentiating the inside function on the cosine we will need to also use the product rule. Now let's solve for $\frac{\partial z}{\partial x}$.

$$2x\sin\left(2y-5z\right) - 5\frac{\partial z}{\partial x}x^2\cos\left(2y-5z\right) = -6zy\sin\left(6zx\right) - 6yx\sin\left(6zx\right)\frac{\partial z}{\partial x}$$

$$2x\sin\left(2y-5z\right) + 6zy\sin\left(6zx\right) = \left(5x^2\cos\left(2y-5z\right) - 6yx\sin\left(6zx\right)\right)\frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial x} = \frac{2x\sin\left(2y-5z\right) + 6zy\sin\left(6zx\right)}{5x^2\cos\left(2y-5z\right) - 6yx\sin\left(6zx\right)}$$

Now let's take care of $\frac{\partial z}{\partial y}$. This one will be slightly easier than the first one.

$$x^2\cos\left(2y-5z\right)\left(2 - 5\frac{\partial z}{\partial y}\right) = \cos\left(6zx\right) - y\sin\left(6zx\right)\left(6x\frac{\partial z}{\partial y}\right)$$

$$2x^2\cos\left(2y-5z\right) - 5x^2\cos\left(2y-5z\right)\frac{\partial z}{\partial y} = \cos\left(6zx\right) - 6xy\sin\left(6zx\right)\frac{\partial z}{\partial y}$$

$$\left(6xy\sin\left(6zx\right) - 5x^2\cos\left(2y-5z\right)\right)\frac{\partial z}{\partial y} = \cos\left(6zx\right) - 2x^2\cos\left(2y-5z\right)$$

$$\frac{\partial z}{\partial y} = \frac{\cos\left(6zx\right) - 2x^2\cos\left(2y-5z\right)}{6xy\sin\left(6zx\right) - 5x^2\cos\left(2y-5z\right)}$$

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There's quite a bit of work to these. We will see an easier way to do implicit differentiation in a later section.

Interpretations of Partial Derivatives

This is a fairly short section and is here so we can acknowledge that the two main interpretations of derivatives of functions of a single variable still hold for partial derivatives, with small modifications of course to account of the fact that we now have more than one variable.

The first interpretation we've already seen and is the more important of the two. As with functions of single variables partial derivatives represent the rates of change of the functions as the variables change. As we saw in the previous section, $f_x(x,y)$ represents the rate of change of the function $f(x,y)$ as we change x and hold y fixed while $f_y(x,y)$ represents the rate of change of $f(x,y)$ as we change y and hold x fixed.

Example 1 Determine if $f(x,y)=\frac{x^2}{y^3}$ is increasing or decreasing at $(2,5)$,

(a) if we allow x to vary and hold y fixed.
(b) if we allow y to vary and hold x fixed.

Solution
(a) If we allow *x* to vary and hold *y* fixed.

In this case we will first need $f_x(x,y)$ and its value at the point.

$$f_x(x,y)=\frac{2x}{y^3} \qquad \Rightarrow \qquad f_x(2,5)=\frac{4}{125}>0$$

So, the partial derivative with respect to x is positive and so if we hold y fixed the function is increasing at $(2,5)$ as we vary x.

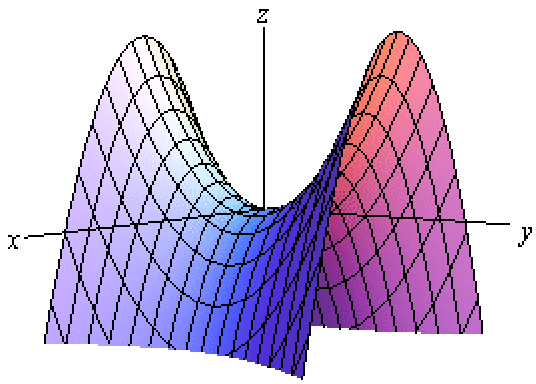
(b) If we allow *y* to vary and hold *x* fixed.

For this part we will need $f_y(x,y)$ and its value at the point.

$$f_y(x,y)=-\frac{3x^2}{y^4} \qquad \Rightarrow \qquad f_y(2,5)=-\frac{12}{625}<0$$

Here the partial derivative with respect to y is negative and so the function is decreasing at $(2,5)$ as we vary y and hold x fixed.

Note that it is completely possible for a function to be increasing for a fixed y and decreasing for a fixed x at a point as this example has shown. To see a nice example of this take a look at the following graph.



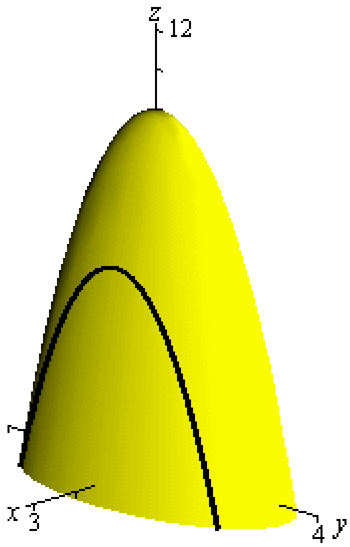
This is a graph of a hyperbolic paraboloid and at the origin we can see that if we move in along the *y*-axis the graph is increasing and if we move along the *x*-axis the graph is decreasing. So it is completely possible to have a graph both increasing and decreasing at a point depending upon the direction that we move. We should never expect that the function will behave in exactly the same way at a point as each variable changes.

The next interpretation was one of the standard interpretations in a Calculus I class. We know from a Calculus I class that $f'(a)$ represents the slope of the tangent line to $y = f(x)$ at $x = a$. Well, $f_x(a,b)$ and $f_y(a,b)$ also represent the slopes of tangent lines. The difference here is the functions that they represent tangent lines to.

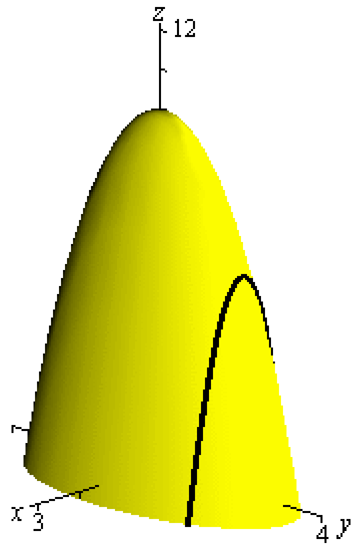
Partial derivatives are the slopes of traces. The partial derivative $f_x(a,b)$ is the slope of the trace of $f(x,y)$ for the plane $y = b$ at the point (a,b). Likewise the partial derivative $f_y(a,b)$ is the slope of the trace of $f(x,y)$ for the plane $x = a$ at the point (a,b).

Example 2 Find the slopes of the traces to $z = 10 - 4x^2 - y^2$ at the point $(1,2)$.

Solution
We sketched the traces for the planes $x = 1$ and $y = 2$ in a previous section and these are the two traces for this point. For reference purposes here are the graphs of the traces.



Trace for $x = 1$



Trace for $y = 2$

Next we'll need the two partial derivatives so we can get the slopes.

$$f_x(x,y) = -8x \qquad\qquad f_y(x,y) = -2y$$

To get the slopes all we need to do is evaluate the partial derivatives at the point in question.

$$f_x(1,2) = -8 \qquad\qquad f_y(1,2) = -4$$

So, the tangent line at $(1,2)$ for the trace to $z = 10 - 4x^2 - y^2$ for the plane $y = 2$ has a slope of -8. Also the tangent line at $(1,2)$ for the trace to $z = 10 - 4x^2 - y^2$ for the plane $x = 1$ has a slope of -4.

Finally, let's briefly talk about getting the equations of the tangent line. Recall that the equation of a line in 3-D space is given by a vector equation. Also to get the equation we need a point on the line and a vector that is parallel to the line.

The point is easy. Since we know the *x*-*y* coordinates of the point all we need to do is plug this into the equation to get the point. So, the point will be,

$$\left(a, b, f(a,b)\right)$$

The parallel (or tangent) vector is also just as easy. We can write the equation of the surface as a vector function as follows,

$$\vec{r}(x,y) = \langle x, y, z \rangle = \langle x, y, f(x,y) \rangle$$

We know that if we have a vector function of one variable we can get a tangent vector by differentiating the vector function. The same will hold true here. If we differentiate with respect to x we will get a tangent vector to traces for the plane $y = b$ (*i.e.* for fixed y) and if we differentiate with respect to y we will get a tangent vector to traces for the plane $x = a$ (or fixed x).

So, here is the tangent vector for traces with fixed y.

$$\vec{r}_x(x,y) = \left\langle 1, 0, f_x(x,y) \right\rangle$$

We differentiated each component with respect to x. Therefore the first component becomes a 1 and the second becomes a zero because we are treating y as a constant when we differentiate with respect to x. The third component is just the partial derivative of the function with respect to x.

For traces with fixed x the tangent vector is,

$$\vec{r}_y(x,y) = \left\langle 0, 1, f_y(x,y) \right\rangle$$

The equation for the tangent line to traces with fixed y is then,

$$\vec{r}(t) = \left\langle a, b, f(a,b) \right\rangle + t\left\langle 1, 0, f_x(a,b) \right\rangle$$

and the tangent line to traces with fixed x is,

$$\vec{r}(t) = \left\langle a, b, f(a,b) \right\rangle + t\left\langle 0, 1, f_y(a,b) \right\rangle$$

Example 3 Write down the vector equations of the tangent lines to the traces to $z = 10 - 4x^2 - y^2$ at the point $(1,2)$.

Solution

There really isn't all that much to do with these other than plugging the values and function into the formulas above. We've already computed the derivatives and their values at $(1,2)$ in the previous example and the point on each trace is,

$$\left(1, 2, f(1,2)\right) = (1,2,2)$$

Here is the equation of the tangent line to the trace for the plane $y = 2$.

$$\vec{r}(t) = \left\langle 1,2,2 \right\rangle + t\left\langle 1,0,-8 \right\rangle = \left\langle 1+t, 2, 2-8t \right\rangle$$

Here is the equation of the tangent line to the trace for the plane $x = 1$.

$$\vec{r}(t) = \left\langle 1,2,2 \right\rangle + t\left\langle 0,1,-4 \right\rangle = \left\langle 1, 2+t, 2-4t \right\rangle$$

Higher Order Partial Derivatives

Just as we had higher order derivatives with functions of one variable we will also have higher order derivatives of functions of more than one variable. However, this time we will have more options since we do have more than one variable..

Consider the case of a function of two variables, $f(x,y)$ since both of the first order partial derivatives are also functions of x and y we could in turn differentiate each with respect to x or y. This means that for the case of a function of two variables there will be a total of four possible second order derivatives. Here they are and the notations that we'll use to denote them.

$$\left(f_x\right)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$\left(f_x\right)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$\left(f_y\right)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$\left(f_y\right)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

The second and third second order partial derivatives are often called mixed partial derivatives since we are taking derivatives with respect to more than one variable. Note as well that the order that we take the derivatives in is given by the notation for each these. If we are using the subscripting notation, *e.g.* f_{xy}, then we will differentiate from left to right. In other words, in this case, we will differentiate first with respect to x and then with respect to y. With the fractional notation, *e.g.* $\dfrac{\partial^2 f}{\partial y \partial x}$, it is the opposite. In these cases we differentiate moving along the denominator from right to left. So, again, in this case we differentiate with respect to x first and then y.

Let's take a quick look at an example.

Example 1 Find all the second order derivatives for $f(x,y) = \cos(2x) - x^2\mathbf{e}^{5y} + 3y^2$.

Solution

We'll first need the first order derivatives so here they are.

$$f_x(x,y) = -2\sin(2x) - 2x\mathbf{e}^{5y}$$

$$f_y(x,y) = -5x^2\mathbf{e}^{5y} + 6y$$

Now, let's get the second order derivatives.

$$f_{xx} = -4\cos(2x) - 2\mathbf{e}^{5y}$$
$$f_{xy} = -10x\mathbf{e}^{5y}$$
$$f_{yx} = -10x\mathbf{e}^{5y}$$
$$f_{yy} = -25x^2\mathbf{e}^{5y} + 6$$

Notice that we dropped the (x, y) from the derivatives. This is fairly standard and we will be doing it most of the time from this point on. We will also be dropping it for the first order derivatives in most cases.

Now let's also notice that, in this case, $f_{xy} = f_{yx}$. This is not by coincidence. If the function is "nice enough" this will always be the case. So, what's "nice enough"? The following theorem tells us.

Clairaut's Theorem

Suppose that f is defined on a disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are continuous on this disk then,

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Now, do not get too excited about the disk business and the fact that we gave the theorem for a specific point. In pretty much every example in this class if the two mixed second order partial derivatives are continuous then they will be equal.

Example 2 Verify Clairaut's Theorem for $f(x,y) = x\mathbf{e}^{-x^2y^2}$.

Solution

We'll first need the two first order derivatives.

$$f_x(x,y) = \mathbf{e}^{-x^2y^2} - 2x^2y^2\mathbf{e}^{-x^2y^2}$$
$$f_y(x,y) = -2yx^3\mathbf{e}^{-x^2y^2}$$

Now, compute the two mixed second order partial derivatives.

$$f_{xy}(x,y) = -2yx^2\mathbf{e}^{-x^2y^2} - 4x^2y\mathbf{e}^{-x^2y^2} + 4x^4y^3\mathbf{e}^{-x^2y^2} = -6x^2y\mathbf{e}^{-x^2y^2} + 4x^4y^3\mathbf{e}^{-x^2y^2}$$
$$f_{yx}(x,y) = -6yx^2\mathbf{e}^{-x^2y^2} + 4y^3x^4\mathbf{e}^{-x^2y^2}$$

Sure enough they are the same.

So far we have only looked at second order derivatives. There are, of course, higher order derivatives as well. Here are a couple of the third order partial derivatives of function of two variables.

$$f_{xyx} = \left(f_{xy}\right)_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial^3 f}{\partial x \partial y \partial x}$$
$$f_{yxx} = \left(f_{yx}\right)_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial^3 f}{\partial x^2 \partial y}$$

Notice as well that for both of these we differentiate once with respect to y and twice with respect to x. There is also another third order partial derivative in which we can do this, f_{xxy}. There is an extension to Clairaut's Theorem that says if all three of these are continuous then they should all be equal,

$$f_{xxy} = f_{xyx} = f_{yxx}$$

To this point we've only looked at functions of two variables, but everything that we've done to this point will work regardless of the number of variables that we've got in the function and there are natural extensions to Clairaut's theorem to all of these cases as well. For instance,

$$f_{xz}\left(x,y,z\right) = f_{zx}\left(x,y,z\right)$$

provided both of the derivatives are continuous.

In general, we can extend Clairaut's theorem to any function and mixed partial derivatives. The only requirement is that in each derivative we differentiate with respect to each variable the same number of times. In other words, provided we meet the continuity condition, the following will be equal

$$f_{ssrtsrr} = f_{trsrssr}$$

because in each case we differentiate with respect to t once, s three times and r three times.

Let's do a couple of examples with higher (well higher order than two anyway) order derivatives and functions of more than two variables.

Example 3 Find the indicated derivative for each of the following functions.

(a) Find f_{xxyzz} for $f\left(x,y,z\right) = z^3 y^2 \ln\left(x\right)$ [Solution]

(b) Find $\dfrac{\partial^3 f}{\partial y \partial x^2}$ for $f\left(x,y\right) = \mathbf{e}^{xy}$ [Solution]

Solution

(a) Find f_{xxyzz} for $f\left(x,y,z\right) = z^3 y^2 \ln\left(x\right)$

In this case remember that we differentiate from left to right. Here are the derivatives for this part.

$$f_x = \frac{z^3 y^2}{x}$$
$$f_{xx} = -\frac{z^3 y^2}{x^2}$$
$$f_{xxy} = -\frac{2z^3 y}{x^2}$$

$$f_{xxyz} = -\frac{6z^2 y}{x^2}$$

$$f_{xxyzz} = -\frac{12zy}{x^2}$$

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(b) Find $\frac{\partial^3 f}{\partial y \partial x^2}$ **for** $f(x,y) = \mathbf{e}^{xy}$

Here we differentiate from right to left. Here are the derivatives for this function.

$$\frac{\partial f}{\partial x} = y\mathbf{e}^{xy}$$

$$\frac{\partial^2 f}{\partial x^2} = y^2\mathbf{e}^{xy}$$

$$\frac{\partial^3 f}{\partial y \partial x^2} = 2y\mathbf{e}^{xy} + xy^2\mathbf{e}^{xy}$$

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Differentials

This is a very short section and is here simply to acknowledge that just like we had differentials for functions of one variable we also have them for functions of more than one variable. Also, as we've already seen in previous sections, when we move up to more than one variable things work pretty much the same, but there are some small differences.

Given the function $z = f(x,y)$ the differential dz or df is given by,

$$dz = f_x\,dx + f_y\,dy \qquad \text{or} \qquad df = f_x\,dx + f_y\,dy$$

There is a natural extension to functions of three or more variables. For instance, given the function $w = g(x,y,z)$ the differential is given by,

$$dw = g_x\,dx + g_y\,dy + g_z\,dz$$

Let's do a couple of quick examples.

Example 1 Compute the differentials for each of the following functions.

(a) $z = \mathbf{e}^{x^2+y^2}\tan(2x)$

(b) $u = \dfrac{t^3 r^6}{s^2}$

Solution

(a) $z = \mathbf{e}^{x^2+y^2}\tan(2x)$

There really isn't a whole lot to these outside of some quick differentiation. Here is the differential for the function.

$$dz = \left(2x\mathbf{e}^{x^2+y^2}\tan(2x) + 2\mathbf{e}^{x^2+y^2}\sec^2(2x)\right)dx + 2y\mathbf{e}^{x^2+y^2}\tan(2x)\,dy$$

(b) $u = \dfrac{t^3 r^6}{s^2}$

Here is the differential for this function.

$$du = \frac{3t^2 r^6}{s^2}dt + \frac{6t^3 r^5}{s^2}dr - \frac{2t^3 r^6}{s^3}ds$$

Note that sometimes these differentials are called the **total differentials**.

Chain Rule

We've been using the standard chain rule for functions of one variable throughout the last couple of sections. It's now time to extend the chain rule out to more complicated situations. Before we actually do that let's first review the notation for the chain rule for functions of one variable.

The notation that's probably familiar to most people is the following.

$$F(x) = f(g(x)) \qquad\qquad F'(x) = f'(g(x))\,g'(x)$$

There is an alternate notation however that while probably not used much in Calculus I is more convenient at this point because it will match up with the notation that we are going to be using in this section. Here it is.

$$\text{If} \qquad y = f(x) \qquad \text{and} \qquad x = g(t) \qquad \text{then} \qquad \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Notice that the derivative $\frac{dy}{dt}$ really does make sense here since if we were to plug in for *x* then *y* really would be a function of *t*. One way to remember this form of the chain rule is to note that if we think of the two derivatives on the right side as fractions the *dx*'s will cancel to get the same derivative on both sides.

Okay, now that we've got that out of the way let's move into the more complicated chain rules that we are liable to run across in this course.

As with many topics in multivariable calculus, there are in fact many different formulas depending upon the number of variables that we're dealing with. So, let's start this discussion off with a function of two variables, $z = f(x, y)$. From this point there are still many different possibilities that we can look at. We will be looking at two distinct cases prior to generalizing the whole idea out.

<u>**Case 1 :**</u> $z = f(x, y)$, $x = g(t)$, $y = h(t)$ and compute $\frac{dz}{dt}$.

This case is analogous to the standard chain rule from Calculus I that we looked at above. In this case we are going to compute an ordinary derivative since *z* really would be a function of *t* only if we were to substitute in for *x* and *y*.

The chain rule for this case is,

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

So, basically what we're doing here is differentiating *f* with respect to each variable in it and then multiplying each of these by the derivative of that variable with respect to *t*. The final step is to then add all this up.

Let's take a look at a couple of examples.

Example 1 Compute $\dfrac{dz}{dt}$ for each of the following.

(a) $z = x\mathbf{e}^{xy}$, $x = t^2$, $y = t^{-1}$ [Solution]

(b) $z = x^2 y^3 + y\cos x$, $x = \ln\left(t^2\right)$, $y = \sin\left(4t\right)$ [Solution]

Solution

(a) $z = x\mathbf{e}^{xy}$, $x = t^2$, $y = t^{-1}$

There really isn't all that much to do here other than using the formula.

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= \left(\mathbf{e}^{xy} + yx\mathbf{e}^{xy}\right)\left(2t\right) + x^2\mathbf{e}^{xy}\left(-t^{-2}\right) \\ &= 2t\left(\mathbf{e}^{xy} + yx\mathbf{e}^{xy}\right) - t^{-2}x^2\mathbf{e}^{xy}\end{aligned}$$

So, technically we've computed the derivative. However, we should probably go ahead and substitute in for x and y as well at this point since we've already got t's in the derivative. Doing this gives,

$$\frac{dz}{dt} = 2t\left(\mathbf{e}^{t} + t\mathbf{e}^{t}\right) - t^{-2}t^{4}\mathbf{e}^{t} = 2t\mathbf{e}^{t} + t^2\mathbf{e}^{t}$$

Note that in this case it might actually have been easier to just substitute in for x and y in the original function and just compute the derivative as we normally would. For comparison's sake let's do that.

$$z = t^2\mathbf{e}^{t} \qquad \Rightarrow \qquad \frac{dz}{dt} = 2t\mathbf{e}^{t} + t^2\mathbf{e}^{t}$$

The same result for less work. Note however, that often it will actually be more work to do the substitution first.

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(b) $z = x^2 y^3 + y\cos x$, $x = \ln\left(t^2\right)$, $y = \sin\left(4t\right)$

Okay, in this case it would almost definitely be more work to do the substitution first so we'll use the chain rule first and then substitute.

$$\begin{aligned}\frac{dz}{dt} &= \left(2xy^3 - y\sin x\right)\left(\frac{2}{t}\right) + \left(3x^2y^2 + \cos x\right)\left(4\cos\left(4t\right)\right) \\ &= \frac{4\sin^3\left(4t\right)\ln t^2 - 2\sin\left(4t\right)\sin\left(\ln t^2\right)}{t} + 4\cos\left(4t\right)\left(3\sin^2\left(4t\right)\left[\ln t^2\right]^2 + \cos\left(\ln t^2\right)\right)\end{aligned}$$

Note that sometimes, because of the significant mess of the final answer, we will only simplify the first step a little and leave the answer in terms of x, y, and t. This is dependent upon the situation, class and instructor however and for this class we will pretty much always be substituting in for x and y.

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Now, there is a special case that we should take a quick look at before moving on to the next case. Let's suppose that we have the following situation,

$$z = f(x,y) \qquad\qquad y = g(x)$$

In this case the chain rule for $\frac{dz}{dx}$ becomes,

$$\frac{dz}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

In the first term we are using the fact that,

$$\frac{dx}{dx} = \frac{d}{dx}(x) = 1$$

Let's take a quick look at an example.

Example 2 Compute $\frac{dz}{dx}$ for $z = x\ln(xy) + y^3$, $y = \cos(x^2+1)$

Solution

We'll just plug into the formula.

$$\frac{dz}{dx} = \left(\ln(xy) + x\frac{y}{xy}\right) + \left(x\frac{x}{xy} + 3y^2\right)\left(-2x\sin(x^2+1)\right)$$

$$= \ln\left(x\cos(x^2+1)\right) + 1 - 2x\sin(x^2+1)\left(\frac{x}{\cos(x^2+1)} + 3\cos^2(x^2+1)\right)$$

$$= \ln\left(x\cos(x^2+1)\right) + 1 - 2x^2\tan(x^2+1) - 6x\sin(x^2+1)\cos^2(x^2+1)$$

Now let's take a look at the second case.

<u>Case 2 :</u> $z = f(x,y)$, $x = g(s,t)$, $y = h(s,t)$ and compute $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

In this case if we were to substitute in for x and y we would get that z is a function of s and t and so it makes sense that we would be computing partial derivatives here and that there would be two of them.

Here is the chain rule for both of these cases.

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \qquad\qquad \frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

So, not surprisingly, these are very similar to the first case that we looked at. Here is a quick example of this kind of chain rule.

Example 3 Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ for $z = \mathbf{e}^{2r}\sin(3\theta)$, $r = st - t^2$, $\theta = \sqrt{s^2 + t^2}$.

Solution

Here is the chain rule for $\frac{\partial z}{\partial s}$.

$$\begin{aligned}\frac{\partial z}{\partial s} &= \left(2\mathbf{e}^{2r}\sin(3\theta)\right)(t) + \left(3\mathbf{e}^{2r}\cos(3\theta)\right)\frac{s}{\sqrt{s^2+t^2}} \\ &= t\left(2\mathbf{e}^{2(st-t^2)}\sin\left(3\sqrt{s^2+t^2}\right)\right) + \frac{3s\mathbf{e}^{2(st-t^2)}\cos\left(3\sqrt{s^2+t^2}\right)}{\sqrt{s^2+t^2}}\end{aligned}$$

Now the chain rule for $\frac{\partial z}{\partial t}$.

$$\begin{aligned}\frac{\partial z}{\partial t} &= \left(2\mathbf{e}^{2r}\sin(3\theta)\right)(s-2t) + \left(3\mathbf{e}^{2r}\cos(3\theta)\right)\frac{t}{\sqrt{s^2+t^2}} \\ &= (s-2t)\left(2\mathbf{e}^{2(st-t^2)}\sin\left(3\sqrt{s^2+t^2}\right)\right) + \frac{3t\mathbf{e}^{2(st-t^2)}\cos\left(3\sqrt{s^2+t^2}\right)}{\sqrt{s^2+t^2}}\end{aligned}$$

Okay, now that we've seen a couple of cases for the chain rule let's see the general version of the chain rule.

Chain Rule

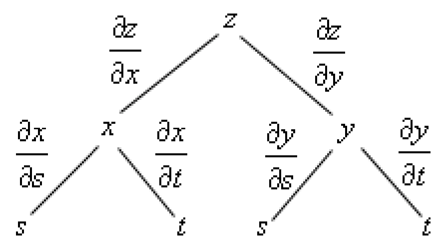
Suppose that *z* is a function of *n* variables, $x_1, x_2, \ldots, x_n$, and that each of these variables are in turn functions of *m* variables, $t_1, t_2, \ldots, t_m$. Then for any variable t_i, $i = 1, 2, \ldots, m$ we have the following,

$$\frac{\partial z}{\partial t_i} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

Wow. That's a lot to remember. There is actually an easier way to construct all the chain rules that we've discussed in the section or will look at in later examples. We can build up a **tree diagram** that will give us the chain rule for any situation. To see how these work let's go back and take a look at the chain rule for $\frac{\partial z}{\partial s}$ given that $z = f(x,y)$, $x = g(s,t)$, $y = h(s,t)$. We already know what this is, but it may help to illustrate the tree diagram if we already know the answer. For reference here is the chain rule for this case,

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

Here is the tree diagram for this case.



We start at the top with the function itself and the branch out from that point. The first set of branches is for the variables in the function. From each of these endpoints we put down a further set of branches that gives the variables that both x and y are a function of. We connect each letter with a line and each line represents a partial derivative as shown. Note that the letter in the numerator of the partial derivative is the upper "node" of the tree and the letter in the denominator of the partial derivative is the lower "node" of the tree.

To use this to get the chain rule we start at the bottom and for each branch that ends with the variable we want to take the derivative with respect to (s in this case) we move up the tree until we hit the top multiplying the derivatives that we see along that set of branches. Once we've done this for each branch that ends at s, we then add the results up to get the chain rule for that given situation.

Note that we don't usually put the derivatives in the tree. They are always an assumed part of the tree.

Let's write down some chain rules.

Example 4 Use a tree diagram to write down the chain rule for the given derivatives.

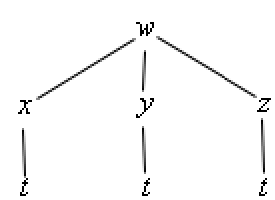
(a) $\frac{dw}{dt}$ for $w = f(x,y,z)$, $x = g_1(t)$, $y = g_2(t)$, and $z = g_3(t)$ [Solution]

(b) $\frac{\partial w}{\partial r}$ for $w = f(x,y,z)$, $x = g_1(s,t,r)$, $y = g_2(s,t,r)$, and $z = g_3(s,t,r)$ [Solution]

Solution

(a) $\frac{dw}{dt}$ for $w = f(x,y,z)$, $x = g_1(t)$, $y = g_2(t)$, and $z = g_3(t)$

So, we'll first need the tree diagram so let's get that.



From this it looks like the chain rule for this case should be,

$$\frac{dw}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}+\frac{\partial f}{\partial z}\frac{dz}{dt}$$

which is really just a natural extension to the two variable case that we saw above.

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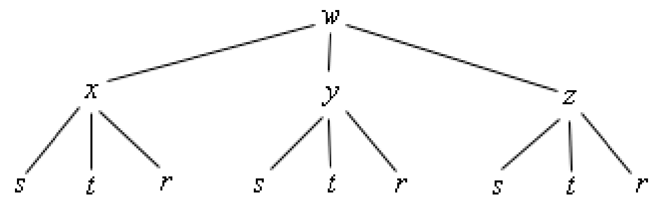
(b) $\frac{\partial w}{\partial r}$ for $w=f\left(x,y,z\right)$, $x=g_1\left(s,t,r\right)$, $y=g_2\left(s,t,r\right)$, and $z=g_3\left(s,t,r\right)$

Here is the tree diagram for this situation.



From this it looks like the derivative will be,

$$\frac{\partial w}{\partial r}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial r}$$

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So, provided we can write down the tree diagram, and these aren't usually too bad to write down, we can do the chain rule for any set up that we might run across.

We've now seen how to take first derivatives of these more complicated situations, but what about higher order derivatives? How do we do those? It's probably easiest to see how to deal with these with an example.

Example 5 Compute $\frac{\partial^2 f}{\partial \theta^2}$ for $f\left(x,y\right)$ if $x=r\cos\theta$ and $y=r\sin\theta$.

Solution

We will need the first derivative before we can even think about finding the second derivative so let's get that. This situation falls into the second case that we looked at above so we don't need a new tree diagram. Here is the first derivative.

$$\begin{aligned}\frac{\partial f}{\partial \theta}&=\frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}\\&=-r\sin\left(\theta\right)\frac{\partial f}{\partial x}+r\cos\left(\theta\right)\frac{\partial f}{\partial y}\end{aligned}$$

Okay, now we know that the second derivative is,

$$\frac{\partial^2 f}{\partial \theta^2}=\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial \theta}\right)=\frac{\partial}{\partial \theta}\left(-r\sin\left(\theta\right)\frac{\partial f}{\partial x}+r\cos\left(\theta\right)\frac{\partial f}{\partial y}\right)$$

The issue here is to correctly deal with this derivative. Since the two first order derivatives, $\frac{\partial f}{\partial x}$

and $\frac{\partial f}{\partial y}$, are both functions of *x* and *y* which are in turn functions of *r* and θ both of these terms are products. So, the using the product rule gives the following,

$$\frac{\partial^2 f}{\partial \theta^2} = -r\cos(\theta)\frac{\partial f}{\partial x} - r\sin(\theta)\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right) - r\sin(\theta)\frac{\partial f}{\partial y} + r\cos(\theta)\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial y}\right)$$

We now need to determine what $\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right)$ and $\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial y}\right)$ will be. These are both chain rule problems again since both of the derivatives are functions of *x* and *y* and we want to take the derivative with respect to θ.

Before we do these let's rewrite the first chain rule that we did above a little.

$$\frac{\partial}{\partial \theta}(f) = -r\sin(\theta)\frac{\partial}{\partial x}(f) + r\cos(\theta)\frac{\partial}{\partial y}(f) \qquad (1)$$

Note that all we've done is change the notation for the derivative a little. With the first chain rule written in this way we can think of (1) as a formula for differentiating any function of *x* and *y* with respect to θ provided we have $x = r\cos\theta$ and $y = r\sin\theta$.

This however is exactly what we need to do the two new derivatives we need above. Both of the first order partial derivatives, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, are functions of *x* and *y* and $x = r\cos\theta$ and $y = r\sin\theta$ so we can use (1) to compute these derivatives.

To do this we'll simply replace all the *f*'s in (1) with the first order partial derivative that we want to differentiate. At that point all we need to do is a little notational work and we'll get the formula that we're after.

Here is the use of (1) to compute $\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right)$.

$$\begin{aligned}\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right) &= -r\sin(\theta)\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + r\cos(\theta)\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \\ &= -r\sin(\theta)\frac{\partial^2 f}{\partial x^2} + r\cos(\theta)\frac{\partial^2 f}{\partial y \partial x}\end{aligned}$$

Here is the computation for $\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial y}\right)$.

$$\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial y}\right) = -r\sin(\theta)\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) + r\cos(\theta)\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$
$$= -r\sin(\theta)\frac{\partial^2 f}{\partial x\partial y} + r\cos(\theta)\frac{\partial^2 f}{\partial y^2}$$

The final step is to plug these back into the second derivative and do some simplifying.

$$\frac{\partial^2 f}{\partial\theta^2} = -r\cos(\theta)\frac{\partial f}{\partial x} - r\sin(\theta)\left(-r\sin(\theta)\frac{\partial^2 f}{\partial x^2} + r\cos(\theta)\frac{\partial^2 f}{\partial y\partial x}\right) -$$
$$r\sin(\theta)\frac{\partial f}{\partial y} + r\cos(\theta)\left(-r\sin(\theta)\frac{\partial^2 f}{\partial x\partial y} + r\cos(\theta)\frac{\partial^2 f}{\partial y^2}\right)$$
$$= -r\cos(\theta)\frac{\partial f}{\partial x} + r^2\sin^2(\theta)\frac{\partial^2 f}{\partial x^2} - r^2\sin(\theta)\cos(\theta)\frac{\partial^2 f}{\partial y\partial x} -$$
$$r\sin(\theta)\frac{\partial f}{\partial y} - r^2\sin(\theta)\cos(\theta)\frac{\partial^2 f}{\partial x\partial y} + r^2\cos^2(\theta)\frac{\partial^2 f}{\partial y^2}$$
$$= -r\cos(\theta)\frac{\partial f}{\partial x} - r\sin(\theta)\frac{\partial f}{\partial y} + r^2\sin^2(\theta)\frac{\partial^2 f}{\partial x^2} -$$
$$2r^2\sin(\theta)\cos(\theta)\frac{\partial^2 f}{\partial y\partial x} + r^2\cos^2(\theta)\frac{\partial^2 f}{\partial y^2}$$

It's long and fairly messy but there it is.

The final topic in this section is a revisiting of implicit differentiation. With these forms of the chain rule implicit differentiation actually becomes a fairly simple process. Let's start out with the implicit differentiation that we saw in a Calculus I course.

We will start with a function in the form $F(x,y)=0$ (if it's not in this form simply move everything to one side of the equal sign to get it into this form) where $y=y(x)$. In a Calculus I course we were then asked to compute $\frac{dy}{dx}$ and this was often a fairly messy process. Using the chain rule from this section however we can get a nice simple formula for doing this. We'll start by differentiating both sides with respect to *x*. This will mean using the chain rule on the left side and the right side will, of course, differentiate to zero. Here are the results of that.

$$F_x + F_y\frac{dy}{dx} = 0 \qquad \Rightarrow \qquad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

As shown, all we need to do next is solve for $\frac{dy}{dx}$ and we've now got a very nice formula to use for implicit differentiation. Note as well that in order to simplify the formula we switched back to using the subscript notation for the derivatives.

Let's check out a quick example.

Example 6 Find $\frac{dy}{dx}$ for $x\cos\left(3y\right)+x^3y^5=3x-\mathbf{e}^{xy}$.

Solution

The first step is to get a zero on one side of the equal sign and that's easy enough to do.

$$x\cos\left(3y\right)+x^3y^5-3x+\mathbf{e}^{xy}=0$$

Now, the function on the left is $F\left(x,y\right)$ in our formula so all we need to do is use the formula to find the derivative.

$$\frac{dy}{dx}=-\frac{\cos\left(3y\right)+3x^2y^5-3+y\mathbf{e}^{xy}}{-3x\sin\left(3y\right)+5x^3y^4+x\mathbf{e}^{xy}}$$

There we go. It would have taken much longer to do this using the old Calculus I way of doing this.

We can also do something similar to handle the types of implicit differentiation problems involving partial derivatives like those we saw when we first introduced partial derivatives. In these cases we will start off with a function in the form $F\left(x,y,z\right)=0$ and assume that $z=f\left(x,y\right)$ and we want to find $\frac{\partial z}{\partial x}$ and/or $\frac{\partial z}{\partial y}$.

Let's start by trying to find $\frac{\partial z}{\partial x}$. We will differentiate both sides with respect to x and we'll need to remember that we're going to be treating y as a constant. Also, the left side will require the chain rule. Here is this derivative.

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}=0$$

Now, we have the following,

$$\frac{\partial x}{\partial x}=1 \qquad \text{and} \qquad \frac{\partial y}{\partial x}=0$$

The first is because we are just differentiating x with respect to x and we know that is 1. The second is because we are treating the y as a constant and so it will differentiate to zero.

Plugging these in and solving for $\frac{\partial z}{\partial x}$ gives,

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}$$

A similar argument can be used to show that,

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$$

As with the one variable case we switched to the subscripting notation for derivatives to simplify the formulas. Let's take a quick look at an example of this.

Example 7 Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $x^2 \sin(2y-5z) = 1 + y\cos(6zx)$.

Solution

This was one of the functions that we used the old implicit differentiation on back in the Partial Derivatives section. You might want to go back and see the difference between the two.

First let's get everything on one side.

$$x^2 \sin(2y-5z) - 1 - y\cos(6zx) = 0$$

Now, the function on the left is $F(x,y,z)$ and so all that we need to do is use the formulas developed above to find the derivatives.

$$\frac{\partial z}{\partial x} = -\frac{2x\sin(2y-5z) + 6yz\sin(6zx)}{-5x^2\cos(2y-5z) + 6yx\sin(6zx)}$$

$$\frac{\partial z}{\partial y} = -\frac{2x^2\cos(2y-5z) - \cos(6zx)}{-5x^2\cos(2y-5z) + 6yx\sin(6zx)}$$

If you go back and compare these answers to those that we found the first time around you will notice that they might appear to be different. However, if you take into account the minus sign that sits in the front of our answers here you will see that they are in fact the same.

Directional Derivatives

To this point we've only looked at the two partial derivatives $f_x(x,y)$ and $f_y(x,y)$. Recall that these derivatives represent the rate of change of *f* as we vary *x* (holding *y* fixed) and as we vary *y* (holding *x* fixed) respectively. We now need to discuss how to find the rate of change of *f* if we allow both *x* and *y* to change simultaneously. The problem here is that there are many ways to allow both *x* and *y* to change. For instance one could be changing faster than the other and then there is also the issue of whether or not each is increasing or decreasing. So, before we get into finding the rate of change we need to get a couple of preliminary ideas taken care of first. The main idea that we need to look at is just how are we going to define the changing of *x* and/or *y*.

Let's start off by supposing that we wanted the rate of change of *f* at a particular point, say (x_0, y_0). Let's also suppose that both *x* and *y* are increasing and that, in this case, *x* is increasing twice as fast as *y* is increasing. So, as *y* increases one unit of measure *x* will increase two units of measure.

To help us see how we're going to define this change let's suppose that a particle is sitting at (x_0, y_0) and the particle will move in the direction given by the changing *x* and *y*. Therefore, the particle will move off in a direction of increasing *x* and *y* and the *x* coordinate of the point will increase twice as fast as the *y* coordinate. Now that we're thinking of this changing *x* and *y* as a direction of movement we can get a way of defining the change. We know from Calculus II that vectors can be used to define a direction and so the particle, at this point, can be said to be moving in the direction,

$$\vec{v} = \langle 2,1 \rangle$$

Since this vector can be used to define how a particle at a point is changing we can also use it describe how *x* and/or *y* is changing at a point. For our example we will say that we want the rate of change of *f* in the direction of $\vec{v} = \langle 2,1 \rangle$. In this way we will know that *x* is increasing twice as fast as *y* is. There is still a small problem with this however. There are many vectors that point in the same direction. For instance all of the following vectors point in the same direction as $\vec{v} = \langle 2,1 \rangle$.

$$\vec{v} = \left\langle \frac{1}{5}, \frac{1}{10} \right\rangle \qquad \vec{v} = \langle 6,3 \rangle \qquad \vec{v} = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

We need a way to consistently find the rate of change of a function in a given direction. We will do this by insisting that the vector that defines the direction of change be a unit vector. Recall that a unit vector is a vector with length, or magnitude, of 1. This means that for the example that we started off thinking about we would want to use

$$\vec{v} = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

since this is the unit vector that points in the direction of change.

For reference purposes recall that the magnitude or length of the vector $\vec{v} = \langle a,b,c \rangle$ is given by,

$$\|\vec{v}\| = \sqrt{a^2 + b^2 + c^2}$$

For two dimensional vectors we drop the *c* from the formula.

Sometimes we will give the direction of changing *x* and *y* as an angle. For instance, we may say that we want the rate of change of *f* in the direction of $\theta = \frac{\pi}{3}$. The unit vector that points in this direction is given by,

$$\vec{u} = \langle \cos\theta, \sin\theta \rangle$$

Okay, now that we know how to define the direction of changing *x* and *y* its time to start talking about finding the rate of change of *f* in this direction. Let's start off with the official definition.

Definition

The rate of change of $f(x, y)$ in the direction of the unit vector $\vec{u} = \langle a, b \rangle$ is called the **directional derivative** and is denoted by $D_{\vec{u}} f(x, y)$. The definition of the directional derivative is,

$$D_{\vec{u}} f(x, y) = \lim_{h \to 0} \frac{f(x + ah, y + bh) - f(x, y)}{h}$$

So, the definition of the directional derivative is very similar to the definition of partial derivatives. However, in practice this can be a very difficult limit to compute so we need an easier way of taking directional derivatives. It's actually fairly simple to derive an equivalent formula for taking directional derivatives.

To see how we can do this let's define a new function of a single variable,

$$g(z) = f(x_0 + az, y_0 + bz)$$

where x_0, y_0, *a*, and *b* are some fixed numbers. Note that this really is a function of a single variable now since *z* is the only letter that is not representing a fixed number.

Then by the definition of the derivative for functions of a single variable we have,

$$g'(z) = \lim_{h \to 0} \frac{g(z + h) - g(z)}{h}$$

and the derivative at $z = 0$ is given by,

$$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h}$$

If we now substitute in for $g(z)$ we get,

$$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h} = D_{\vec{u}} f(x_0, y_0)$$

So, it looks like we have the following relationship.

$$g'(0) = D_{\vec{u}} f(x_0, y_0) \tag{1}$$

Now, let's look at this from another perspective. Let's rewrite $g(z)$ as follows,

$$g(z) = f(x,y) \quad \text{where} \quad x = x_0 + az \quad \text{and} \quad y = y_0 + bz$$

We can now use the chain rule from the previous section to compute,

$$g'(z) = \frac{dg}{dz} = \frac{\partial f}{\partial x}\frac{dx}{dz} + \frac{\partial f}{\partial y}\frac{dy}{dz} = f_x(x,y)\,a + f_y(x,y)\,b$$

So, from the chain rule we get the following relationship.

$$g'(z) = f_x(x,y)\,a + f_y(x,y)\,b \tag{2}$$

If we now take $z = 0$ we will get that $x = x_0$ and $y = y_0$ (from how we defined x and y above) and plug these into (2) we get,

$$g'(0) = f_x(x_0,y_0)\,a + f_y(x_0,y_0)\,b \tag{3}$$

Now, simply equate (1) and (3) to get that,

$$D_{\vec{u}} f(x_0,y_0) = g'(0) = f_x(x_0,y_0)\,a + f_y(x_0,y_0)\,b$$

If we now go back to allowing x and y to be any number we get the following formula for computing directional derivatives.

$$D_{\vec{u}} f(x,y) = f_x(x,y)\,a + f_y(x,y)\,b$$

This is much simpler than the limit definition. Also note that this definition assumed that we were working with functions of two variables. There are similar formulas that can be derived by the same type of argument for functions with more than two variables. For instance, the directional derivative of $f(x,y,z)$ in the direction of the unit vector $\vec{u} = \langle a,b,c \rangle$ is given by,

$$D_{\vec{u}} f(x,y,z) = f_x(x,y,z)\,a + f_y(x,y,z)\,b + f_z(x,y,z)\,c$$

Let's work a couple of examples.

Example 1 Find each of the directional derivatives.

(a) $D_{\vec{u}} f(2,0)$ where $f(x,y) = x\mathbf{e}^{xy} + y$ and $\vec{u}$ is the unit vector in the direction of $\theta = \frac{2\pi}{3}$. [Solution]

(b) $D_{\vec{u}} f(x,y,z)$ where $f(x,y,z) = x^2 z + y^3 z^2 - xyz$ in the direction of $\vec{v} = \langle -1,0,3 \rangle$. [Solution]

Solution

(a) $D_{\vec{u}} f\left(2,0\right)$ **where** $f\left(x,y\right) = x\mathbf{e}^{xy} + y$ **and** $\vec{u}$ **is the unit vector in the direction of** $\theta = \dfrac{2\pi}{3}$**.**

We'll first find $D_{\vec{u}} f\left(x,y\right)$ and then use this a formula for finding $D_{\vec{u}} f\left(2,0\right)$. The unit vector giving the direction is,

$$\vec{u} = \left\langle \cos\left(\frac{2\pi}{3}\right), \sin\left(\frac{2\pi}{3}\right) \right\rangle = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$

So, the directional derivative is,

$$D_{\vec{u}} f\left(x,y\right) = \left(-\frac{1}{2}\right)\left(\mathbf{e}^{xy} + xy\mathbf{e}^{xy}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(x^2\mathbf{e}^{xy} + 1\right)$$

Now, plugging in the point in question gives,

$$D_{\vec{u}} f\left(2,0\right) = \left(-\frac{1}{2}\right)\left(1\right) + \left(\frac{\sqrt{3}}{2}\right)\left(5\right) = \frac{5\sqrt{3}-1}{2}$$

[Return to Problems]

(b) $D_{\vec{u}} f\left(x,y,z\right)$ **where** $f\left(x,y,z\right) = x^2 z + y^3 z^2 - xyz$ **in the direction of** $\vec{v} = \left\langle -1,0,3 \right\rangle$**.**

In this case let's first check to see if the direction vector is a unit vector or not and if it isn't convert it into one. To do this all we need to do is compute its magnitude.

$$\left\|\vec{v}\right\| = \sqrt{1+0+9} = \sqrt{10} \neq 1$$

So, it's not a unit vector. Recall that we can convert any vector into a unit vector that points in the same direction by dividing the vector by its magnitude. So, the unit vector that we need is,

$$\vec{u} = \frac{1}{\sqrt{10}}\left\langle -1,0,3 \right\rangle = \left\langle -\frac{1}{\sqrt{10}}, 0, \frac{3}{\sqrt{10}} \right\rangle$$

The directional derivative is then,

$$\begin{aligned} D_{\vec{u}} f\left(x,y,z\right) &= \left(-\frac{1}{\sqrt{10}}\right)\left(2xz - yz\right) + \left(0\right)\left(3y^2z^2 - xz\right) + \left(\frac{3}{\sqrt{10}}\right)\left(x^2 + 2y^3 z - xy\right) \\ &= \frac{1}{\sqrt{10}}\left(3x^2 + 6y^3 z - 3xy - 2xz + yz\right) \end{aligned}$$

[Return to Problems]

There is another form of the formula that we used to get the directional derivative that is a little nicer and somewhat more compact. It is also a much more general formula that will encompass both of the formulas above.

Let's start with the second one and notice that we can write it as follows,

$$\begin{aligned} D_{\vec{u}} f(x,y,z) &= f_x(x,y,z)a + f_y(x,y,z)b + f_z(x,y,z)c \\ &= \langle f_x, f_y, f_z \rangle \bullet \langle a,b,c \rangle \end{aligned}$$

In other words we can write the directional derivative as a dot product and notice that the second vector is nothing more than the unit vector $\vec{u}$ that gives the direction of change. Also, if we had used the version for functions of two variables the third component wouldn't be there, but other than that the formula would be the same.

Now let's give a name and notation to the first vector in the dot product since this vector will show up fairly regularly throughout this course (and in other courses). The **gradient of *f*** or **gradient vector of *f*** is defined to be,

$$\nabla f = \langle f_x, f_y, f_z \rangle \qquad \text{or} \qquad \nabla f = \langle f_x, f_y \rangle$$

Or, if we want to use the standard basis vectors the gradient is,

$$\nabla f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k} \qquad \text{or} \qquad \nabla f = f_x \vec{i} + f_y \vec{j}$$

The definition is only shown for functions of two or three variables, however there is a natural extension to functions of any number of variables that we'd like.

With the definition of the gradient we can now say that the directional derivative is given by,

$$D_{\vec{u}} f = \nabla f \bullet \vec{u}$$

where we will no longer show the variable and use this formula for any number of variables. Note as well that we will sometimes use the following notation,

$$D_{\vec{u}} f(\vec{x}) = \nabla f \bullet \vec{u}$$

where $\vec{x} = \langle x,y,z \rangle$ or $\vec{x} = \langle x,y \rangle$ as needed. This notation will be used when we want to note the variables in some way, but don't really want to restrict ourselves to a particular number of variables. In other words, $\vec{x}$ will be used to represent as many variables as we need in the formula and we will most often use this notation when we are already using vectors or vector notation in the problem/formula.

Let's work a couple of examples using this formula of the directional derivative.

Example 2 Find each of the directional derivative.

(a) $D_{\vec{u}} f(\vec{x})$ for $f(x,y) = x\cos(y)$ in the direction of $\vec{v} = \langle 2,1 \rangle$. [Solution]

(b) $D_{\vec{u}} f(\vec{x})$ for $f(x,y,z) = \sin(yz) + \ln(x^2)$ at $(1,1,\pi)$ in the direction of $\vec{v} = \langle 1,1,-1 \rangle$. [Solution]

Solution

(a) $D_{\vec{u}} f(\vec{x})$ **for** $f(x,y) = x\cos(y)$ **in the direction of** $\vec{v} = \langle 2,1 \rangle$**.**

Let's first compute the gradient for this function.

$$\nabla f = \langle \cos(y), -x\sin(y) \rangle$$

Also, as we saw earlier in this section the unit vector for this direction is,

$$\vec{u} = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

The directional derivative is then,

$$\begin{aligned} D_{\vec{u}} f(\vec{x}) &= \langle \cos(y), -x\sin(y) \rangle \bullet \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle \\ &= \frac{1}{\sqrt{5}}\left(2\cos(y) - x\sin(y)\right) \end{aligned}$$

[Return to Problems]

(b) $D_{\vec{u}} f(\vec{x})$ **for** $f(x,y,z) = \sin(yz) + \ln(x^2)$ **at** $(1,1,\pi)$ **in the direction of** $\vec{v} = \langle 1,1,-1 \rangle$**.**

In this case are asking for the directional derivative at a particular point. To do this we will first compute the gradient, evaluate it at the point in question and then do the dot product. So, let's get the gradient.

$$\nabla f(x,y,z) = \left\langle \frac{2}{x}, z\cos(yz), y\cos(yz) \right\rangle$$

$$\nabla f(1,1,\pi) = \left\langle \frac{2}{1}, \pi\cos(\pi), \cos(\pi) \right\rangle = \langle 2, -\pi, -1 \rangle$$

Next, we need the unit vector for the direction,

$$\|\vec{v}\| = \sqrt{3} \qquad\qquad \vec{u} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle$$

Finally, the directional derivative at the point in question is,

$$\begin{aligned} D_{\vec{u}} f(1,1,\pi) &= \langle 2, -\pi, -1 \rangle \bullet \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle \\ &= \frac{1}{\sqrt{3}}(2 - \pi + 1) \\ &= \frac{3-\pi}{\sqrt{3}} \end{aligned}$$

[Return to Problems]

Before proceeding let's note that the first order partial derivatives that we were looking at in the majority of the section can be thought of as special cases of the directional derivatives. For instance, f_x can be thought of as the directional derivative of f in the direction of $\vec{u} = \langle 1,0 \rangle$ or $\vec{u} = \langle 1,0,0 \rangle$, depending on the number of variables that we're working with. The same can be done for f_y and f_z

We will close out this section with a couple of nice facts about the gradient vector. The first tells us how to determine the maximum rate of change of a function at a point and the direction that we need to move in order to achieve that maximum rate of change.

Theorem

The maximum value of $D_{\vec{u}} f(\vec{x})$ (and hence then the maximum rate of change of the function $f(\vec{x})$) is given by $\left\| \nabla f(\vec{x}) \right\|$ and will occur in the direction given by $\nabla f(\vec{x})$.

Proof

This is a really simple proof. First, if we start with the dot product form $D_{\vec{u}} f(\vec{x})$ and use a nice fact about dot products as well as the fact that $\vec{u}$ is a unit vector we get,

$$D_{\vec{u}} f = \nabla f \bullet \vec{u} = \left\| \nabla f \right\| \left\| \vec{u} \right\| \cos\theta = \left\| \nabla f \right\| \cos\theta$$

where θ is the angle between the gradient and $\vec{u}$.

Now the largest possible value of $\cos\theta$ is 1 which occurs at $\theta = 0$. Therefore the maximum value of $D_{\vec{u}} f(\vec{x})$ is $\left\| \nabla f(\vec{x}) \right\|$ Also, the maximum value occurs when the angle between the gradient and $\vec{u}$ is zero, or in other words when $\vec{u}$ is pointing in the same direction as the gradient, $\nabla f(\vec{x})$.

■

Let's take a quick look at an example.

Example 3 Suppose that the height of a hill above sea level is given by $z = 1000 - 0.01x^2 - 0.02y^2$. If you are at the point $(60,100)$ in what direction is the elevation changing fastest? What is the maximum rate of change of the elevation at this point?

Solution

First, you will hopefully recall from the Quadric Surfaces section that this is an elliptic paraboloid that opens downward. So even though most hills aren't this symmetrical it will at least be vaguely hill shaped and so the question makes at least a little sense.

Now on to the problem. There are a couple of questions to answer here, but using the theorem makes answering them very simple. We'll first need the gradient vector.

$$\nabla f(\vec{x}) = \langle -0.02x, -0.04y \rangle$$

The maximum rate of change of the elevation will then occur in the direction of

$$\nabla f(60,100) = \langle -1.2, -4 \rangle$$

The maximum rate of change of the elevation at this point is,

$$\left\|\nabla f\left(60,100\right)\right\| = \sqrt{\left(-1.2\right)^2 + \left(4\right)^2} = \sqrt{17.44} = 4.176$$

Before leaving this example let's note that we're at the point $\left(60,100\right)$ and the direction of greatest rate of change of the elevation at this point is given by the vector $\left\langle -1.2,-4\right\rangle$. Since both of the components are negative it looks like the direction of maximum rate of change points up the hill towards the center rather than away from the hill.

The second fact about the gradient vector that we need to give in this section will be very convenient in some later sections.

Fact

The gradient vector $\nabla f\left(x_0,y_0\right)$ is orthogonal (or perpendicular) to the level curve $f\left(x,y\right)=k$ at the point $\left(x_0,y_0\right)$. Likewise, the gradient vector $\nabla f\left(x_0,y_0,z_0\right)$ is orthogonal to the level surface $f\left(x,y,z\right)=k$ at the point $\left(x_0,y_0,z_0\right)$.

Proof

We're going to do the proof for the $\mathbb{R}^3$ case. The proof for the $\mathbb{R}^2$ case is identical. We'll also need some notation out of the way to make life easier for us let's let *S* be the level surface given by $f\left(x,y,z\right)=k$ and let $P=\left(x_0,y_0,z_0\right)$. Note as well that *P* will be on *S*.

Now, let *C* be any curve on *S* that contains *P*. Let $\vec{r}\left(t\right)=\left\langle x\left(t\right),y\left(t\right),z\left(t\right)\right\rangle$ be the vector equation for *C* and suppose that t_0 be the value of *t* such that $\vec{r}\left(t_0\right)=\left\langle x_0,y_0,z_0\right\rangle$. In other words, t_0 be the value of *t* that gives *P*.

Because *C* lies on *S* we know that points on *C* must satisfy the equation for *S*. Or,

$$f\left(x\left(t\right),y\left(t\right),z\left(t\right)\right)=k$$

Next, let's use the Chain Rule on this to get,

$$\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}+\frac{\partial f}{\partial z}\frac{dz}{dt}=0 \qquad (4)$$

Notice that $\nabla f=\left\langle f_x,f_y,f_z\right\rangle$ and $\vec{r}'\left(t\right)=\left\langle x'\left(t\right),y'\left(t\right),z'\left(t\right)\right\rangle$ so (4) becomes,

$$\nabla f \bullet \vec{r}'\left(t\right)=0$$

At, $t=t_0$ this is,

$$\nabla f\left(x_0,y_0,z_0\right)\bullet\vec{r}'\left(t_0\right)=0$$

This then tells us that the gradient vector at P, $\nabla f\left(x_0,y_0,z_0\right)$, is orthogonal to the tangent vector, $\vec{r}'\left(t_0\right)$, to any curve C that passes through P and on the surface S and so must also be orthogonal to the surface S.

■

As we will be seeing in later sections we are often going to be needing vectors that are orthogonal to a surface or curve and using this fact we will know that all we need to do is compute a gradient vector and we will get the orthogonal vector that we need. We will see the first application of this in the next chapter.

Applications of Partial Derivatives

Introduction

In this section we will take a look at a couple of applications of partial derivatives. Most of the applications will be extensions to applications to ordinary derivatives that we saw back in Calculus I. For instance, we will be looking at finding the absolute and relative extrema of a function and we will also be looking at optimization. Both (all three?) of these subjects were major applications back in Calculus I. They will, however, be a little more work here because we now have more than one variable.

Here is a list of the topics in this chapter.

Tangent Planes and Linear Approximations – We'll take a look at tangent planes to surfaces in this section as well as an application of tangent planes.

Gradient Vector, Tangent Planes and Normal Lines – In this section we'll see how the gradient vector can be used to find tangent planes and normal lines to a surface.

Relative Minimums and Maximums – Here we will see how to identify relative minimums and maximums.

Absolute Minimums and Maximums – We will find absolute minimums and maximums of a function over a given region.

Lagrange Multipliers – In this section we'll see how to use Lagrange Multipliers to find the absolute extrema for a function subject to a given constraint.

Tangent Planes and Linear Approximations

Earlier we saw how the two partial derivatives f_x and f_y can be thought of as the slopes of traces. We want to extend this idea out a little in this section. The graph of a function $z = f(x,y)$ is a surface in $\mathbb{R}^3$ (three dimensional space) and so we can now start thinking of the plane that is "tangent" to the surface as a point.

Let's start out with a point (x_0, y_0) and let's let C_1 represent the trace to $f(x,y)$ for the plane $y = y_0$ (*i.e.* allowing *x* to vary with *y* held fixed) and we'll let C_2 represent the trace to $f(x,y)$ for the plane $x = x_0$ (*i.e.* allowing *y* to vary with *x* held fixed). Now, we know that $f_x(x_0, y_0)$ is the slope of the tangent line to the trace C_1 and $f_y(x_0, y_0)$ is the slope of the tangent line to the trace C_2. So, let L_1 be the tangent line to the trace C_1 and let L_2 be the tangent line to the trace C_2.

The tangent plane will then be the plane that contains the two lines L_1 and L_2. Geometrically this plane will serve the same purpose that a tangent line did in Calculus I. A tangent line to a curve was a line that just touched the curve at that point and was "parallel" to the curve at the point in question. Well tangent planes to a surface are planes that just touch the surface at the point and are "parallel" to the surface at the point. Note that this gives us a point that is on the plane. Since the tangent plane and the surface touch at (x_0, y_0) the following point will be on both the surface and the plane.

$$(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$$

What we need to do now is determine the equation of the tangent plane. We know that the general equation of a plane is given by,

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

where (x_0, y_0, z_0) is a point that is on the plane, which we know already. Let's rewrite this a little. We'll move the *x* terms and *y* terms to the other side and divide both sides by *c*. Doing this gives,

$$z - z_0 = -\frac{a}{c}(x - x_0) - \frac{b}{c}(y - y_0)$$

Now, let's rename the constants to simplify up the notation a little. Let's rename them as follows,

$$A = -\frac{a}{c} \qquad\qquad B = -\frac{b}{c}$$

With this renaming the equation of the tangent plane becomes,

$$z - z_0 = A(x - x_0) + B(y - y_0)$$

and we need to determine values for *A* and *B*.

Let's first think about what happens if we hold *y* fixed, *i.e.* if we assume that $y = y_0$. In this case the equation of the tangent plane becomes,

$$z - z_0 = A(x - x_0)$$

This is the equation of a line and this line must be tangent to the surface at (x_0, y_0) (since it's part of the tangent plane). In addition, this line assumes that $y = y_0$ (*i.e.* fixed) and *A* is the slope of this line. But if we think about it this is exactly what the tangent to C_1 is, a line tangent to the surface at (x_0, y_0) assuming that $y = y_0$. In other words,

$$z - z_0 = A(x - x_0)$$

is the equation for L_1 and we know that the slope of L_1 is given by $f_x(x_0, y_0)$. Therefore we have the following,

$$A = f_x(x_0, y_0)$$

If we hold *x* fixed at $x = x_0$ the equation of the tangent plane becomes,

$$z - z_0 = B(y - y_0)$$

However, by a similar argument to the one above we can see that this is nothing more than the equation for L_2 and that it's slope is *B* or $f_y(x_0, y_0)$. So,

$$B = f_y(x_0, y_0)$$

The equation of the tangent plane to the surface given by $z = f(x, y)$ at (x_0, y_0) is then,

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Also, if we use the fact that $z_0 = f(x_0, y_0)$ we can rewrite the equation of the tangent plane as,

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

We will see an easier derivation of this formula (actually a more general formula) in the next section so if you didn't quite follow this argument hold off until then to see a better derivation.

Example 1 Find the equation of the tangent plane to $z = \ln(2x + y)$ at $(-1, 3)$.

Solution
There really isn't too much to do here other than taking a couple of derivatives and doing some quick evaluations.

$$f(x,y) = \ln(2x+y) \qquad z_0 = f(-1,3) = \ln(1) = 0$$

$$f_x(x,y) = \frac{2}{2x+y} \qquad f_x(-1,3) = 2$$

$$f_y(x,y) = \frac{1}{2x+y} \qquad f_y(-1,3) = 1$$

The equation of the plane is then,

$$z - 0 = 2(x+1) + (1)(y-3)$$
$$z = 2x + y - 1$$

One nice use of tangent planes is they give us a way to approximate a surface near a point. As long as we are near to the point (x_0, y_0) then the tangent plane should nearly approximate the function at that point. Because of this we define the **linear approximation** to be,

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

and as long as we are "near" (x_0, y_0) then we should have that,

$$f(x,y) \approx L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

Example 2 Find the linear approximation to $z = 3 + \frac{x^2}{16} + \frac{y^2}{9}$ at $(-4,3)$.

Solution
So, we're really asking for the tangent plane so let's find that.

$$f(x,y) = 3 + \frac{x^2}{16} + \frac{y^2}{9} \qquad f(-4,3) = 3+1+1 = 5$$

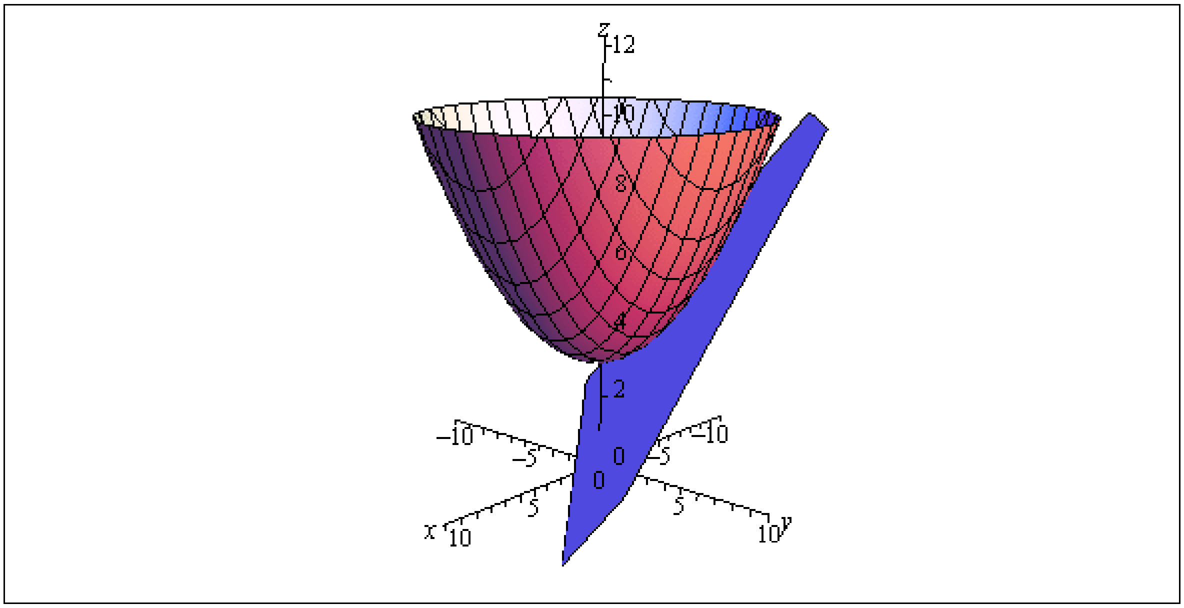
$$f_x(x,y) = \frac{x}{8} \qquad f_x(-4,3) = -\frac{1}{2}$$

$$f_y(x,y) = \frac{2y}{9} \qquad f_y(-4,3) = \frac{2}{3}$$

The tangent plane, or linear approximation, is then,

$$L(x,y) = 5 - \frac{1}{2}(x+4) + \frac{2}{3}(y-3)$$

For reference purposes here is a sketch of the surface and the tangent plane/linear approximation.


z
12
10
8
6
4
2
–10
–5
0
x
10
5
–10
–5
0
5
10
y

Gradient Vector, Tangent Planes and Normal Lines

In this section we want to revisit tangent planes only this time we'll look at them in light of the gradient vector. In the process we will also take a look at a normal line to a surface.

Let's first recall the equation of a plane that contains the point (x_0, y_0, z_0) with normal vector $\vec{n} = \langle a, b, c \rangle$ is given by,

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

When we introduced the gradient vector in the section on directional derivatives we gave the following fact.

Fact

The gradient vector $\nabla f(x_0, y_0)$ is orthogonal (or perpendicular) to the level curve $f(x, y) = k$ at the point (x_0, y_0). Likewise, the gradient vector $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface $f(x, y, z) = k$ at the point (x_0, y_0, z_0).

Actually, all we need here is the last part of this fact. This says that the gradient vector is always orthogonal, or *normal*, to the surface at a point.

Also recall that the gradient vector is,

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

So, the tangent plane to the surface given by $f(x, y, z) = k$ at (x_0, y_0, z_0) has the equation,

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$$

This is a much more general form of the equation of a tangent plane than the one that we derived in the previous section.

Note however, that we can also get the equation from the previous section using this more general formula. To see this let's start with the equation $z = f(x, y)$ and we want to find the tangent plane to the surface given by $z = f(x, y)$ at the point (x_0, y_0, z_0) where $z_0 = f(x_0, y_0)$. In order to use the formula above we need to have all the variables on one side. This is easy enough to do. All we need to do is subtract a z from both sides to get,

$$f(x, y) - z = 0$$

Now, if we define a new function

$$F(x, y, z) = f(x, y) - z$$

we can see that the surface given by $z = f(x, y)$ is identical to the surface given by $F(x, y, z) = 0$ and this new equivalent equation is in the correct form for the equation of the tangent plane that we derived in this section.

So, the first thing that we need to do is find the gradient vector for *F*.

$$\nabla F = \langle F_x, F_y, F_z \rangle = \langle f_x, f_y, -1 \rangle$$

Notice that

$$F_x = \frac{\partial}{\partial x}\left(f(x,y) - z\right) = f_x \qquad F_y = \frac{\partial}{\partial y}\left(f(x,y) - z\right) = f_y$$

$$F_z = \frac{\partial}{\partial z}\left(f(x,y) - z\right) = -1$$

The equation of the tangent plane is then,

$$f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0) - (z - z_0) = 0$$

Or, upon solving for *z*, we get,

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

which is identical to the equation that we derived in the previous section.

We can get another nice piece of information out of the gradient vector as well. We might on occasion want a line that is orthogonal to a surface at a point, sometimes called the **normal line**. This is easy enough to get if we recall that the equation of a line only requires that we have a point and a parallel vector. Since we want a line that is at the point (x_0, y_0, z_0) we know that this point must also be on the line and we know that $\nabla f(x_0, y_0, z_0)$ is a vector that is normal to the surface and hence will be parallel to the line. Therefore the equation of the normal line is,

$$\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t\,\nabla f(x_0, y_0, z_0)$$

Example 1 Find the tangent plane and normal line to $x^2 + y^2 + z^2 = 30$ at the point $(1,-2,5)$.

Solution

For this case the function that we're going to be working with is,

$$F(x,y,z) = x^2 + y^2 + z^2$$

and note that we don't have to have a zero on one side of the equal sign. All that we need is a constant. To finish this problem out we simply need the gradient evaluated at the point.

$$\nabla F = \langle 2x, 2y, 2z \rangle$$

$$\nabla F(1,-2,5) = \langle 2,-4,10 \rangle$$

The tangent plane is then,

$$2(x-1) - 4(y+2) + 10(z-5) = 0$$

The normal line is,

$$\vec{r}(t) = \langle 1,-2,5 \rangle + t\langle 2,-4,10 \rangle = \langle 1+2t, -2-4t, 5+10t \rangle$$

Relative Minimums and Maximums

In this section we are going to extend one of the more important ideas from Calculus I into functions of two variables. We are going to start looking at trying to find minimums and maximums of functions. This in fact will be the topic of the following two sections as well.

In this section we are going to be looking at identifying relative minimums and relative maximums. Recall as well that we will often use the word extrema to refer to both minimums and maximums.

The definition of relative extrema for functions of two variables is identical to that for functions of one variable we just need to remember now that we are working with functions of two variables. So, for the sake of completeness here is the definition of relative minimums and relative maximums for functions of two variables.

Definition

1. A function $f\left(x,y\right)$ has a **relative minimum** at the point $\left(a,b\right)$ if $f\left(x,y\right) \ge f\left(a,b\right)$ for all points $\left(x,y\right)$ in some region around $\left(a,b\right)$.

2. A function $f\left(x,y\right)$ has a **relative maximum** at the point $\left(a,b\right)$ if $f\left(x,y\right) \le f\left(a,b\right)$ for all points $\left(x,y\right)$ in some region around $\left(a,b\right)$.

Note that this definition does not say that a relative minimum is the smallest value that the function will ever take. It only says that in some region around the point $\left(a,b\right)$ the function will always be larger than $f\left(a,b\right)$. Outside of that region it is completely possible for the function to be smaller. Likewise, a relative maximum only says that around $\left(a,b\right)$ the function will always be smaller than $f\left(a,b\right)$. Again, outside of the region it is completely possible that the function will be larger.

Next we need to extend the idea of **critical points** up to functions of two variables. Recall that a critical point of the function $f\left(x\right)$ was a number $x = c$ so that either $f'\left(c\right) = 0$ or $f'\left(c\right)$ doesn't exist. We have a similar definition for critical points of functions of two variables.

Definition

The point $\left(a,b\right)$ is a **critical point** (or a **stationary point**) of $f\left(x,y\right)$ provided one of the following is true,

1. $\nabla f\left(a,b\right) = \vec{0}$ (this is equivalent to saying that $f_x\left(a,b\right) = 0$ and $f_y\left(a,b\right) = 0$),
2. $f_x\left(a,b\right)$ and/or $f_y\left(a,b\right)$ doesn't exist.

To see the equivalence in the first part let's start off with $\nabla f = \vec{0}$ and put in the definition of each part.

$$\nabla f(a,b) = \vec{0}$$
$$\left\langle f_x(a,b), f_y(a,b) \right\rangle = \left\langle 0,0 \right\rangle$$

The only way that these two vectors can be equal is to have $f_x(a,b) = 0$ and $f_y(a,b) = 0$. In fact, we will use this definition of the critical point more than the gradient definition since it will be easier to find the critical points if we start with the partial derivative definition.

Note as well that BOTH of the first order partial derivatives must be zero at (a,b). If only one of the first order partial derivatives are zero at the point then the point will NOT be a critical point.

We now have the following fact that, at least partially, relates critical points to relative extrema.

Fact

If the point (a,b) is a relative extrema of the function $f(x,y)$ and the first order derivatives of $f(x,y)$ exist at (a,b) then (a,b) is also a critical point of $f(x,y)$ and in fact we'll have $\nabla f(a,b) = \vec{0}$.

Proof

This is a really simple proof that relies on the single variable version that we saw in Calculus I version, often called Fermat's Theorem.

Let's start off by defining $g(x) = f(x,b)$ and suppose that $f(x,y)$ has a relative extrema at (a,b). However, this also means that $g(x)$ also has a relative extrema (of the same kind as $f(x,y)$) at $x = a$. By Fermat's Theorem we then know that $g'(a) = 0$. But we also know that $g'(a) = f_x(a,b)$ and so we have that $f_x(a,b) = 0$.

If we now define $h(y) = f(a,y)$ and going through exactly the same process as above we will see that $f_y(a,b) = 0$.

So, putting all this together means that $\nabla f(a,b) = \vec{0}$ and so $f(x,y)$ has a critical point at (a,b).

■

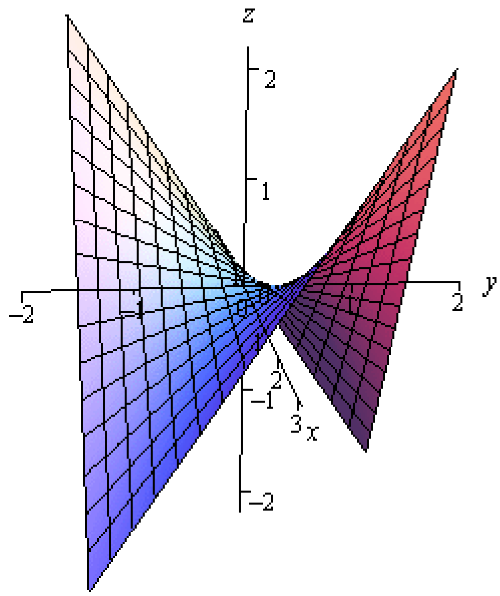
Note that this does NOT say that all critical points are relative extrema. It only says that relative extrema will be critical points of the function. To see this let's consider the function

$$f(x,y) = xy$$

The two first order partial derivatives are,

$$f_x(x,y) = y \qquad\qquad f_y(x,y) = x$$

The only point that will make both of these derivatives zero at the same time is $(0,0)$ and so $(0,0)$ is a critical point for the function. Here is a graph of the function.



Note that the axes are not in the standard orientation here so that we can see more clearly what is happening at the origin, *i.e.* at $(0,0)$. If we start at the origin and move into either of the quadrants where both x and y are the same sign the function increases. However, if we start at the origin and move into either of the quadrants where x and y have the opposite sign then the function decreases. In other words, no matter what region you take about the origin there will be points larger than $f(0,0) = 0$ and points smaller than $f(0,0) = 0$. Therefore, there is no way that $(0,0)$ can be a relative extrema.

Critical points that exhibit this kind of behavior are called **saddle points**.

While we have to be careful to not misinterpret the results of this fact it is very useful in helping us to identify relative extrema. Because of this fact we know that if we have all the critical points of a function then we also have every possible relative extrema for the function. The fact tells us that all relative extrema must be critical points so we know that if the function does have relative extrema then they must be in the collection of all the critical points. Remember however, that it will be completely possible that at least one of the critical points won't be a relative extrema.

So, once we have all the critical points in hand all we will need to do is test these points to see if they are relative extrema or not. To determine if a critical point is a relative extrema (and in fact to determine if it is a minimum or a maximum) we can use the following fact.

Fact

Suppose that (a,b) is a critical point of $f(x,y)$ and that the second order partial derivatives are continuous in some region that contains (a,b). Next define,

$$D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - \left[f_{xy}(a,b) \right]^2$$

We then have the following classifications of the critical point.

1. If $D > 0$ and $f_{xx}(a,b) > 0$ then there is a relative minimum at (a,b).
2. If $D > 0$ and $f_{xx}(a,b) < 0$ then there is a relative maximum at (a,b).
3. If $D < 0$ then the point (a,b) is a saddle point.
4. If $D = 0$ then the point (a,b) may be a relative minimum, relative maximum or a saddle point. Other techniques would need to be used to classify the critical point.

Note that if $D > 0$ then both $f_{xx}(a,b)$ and $f_{yy}(a,b)$ will have the same sign and so in the first two cases above we could just as easily replace $f_{xx}(a,b)$ with $f_{yy}(a,b)$. Also note that we aren't going to be seeing any cases in this class where $D = 0$. We will be able to classify all the critical points that we find.

Let's see a couple of examples.

Example 1 Find and classify all the critical points of $f(x,y) = 4 + x^3 + y^3 - 3xy$.

Solution

We first need all the first order (to find the critical points) and second order (to classify the critical points) partial derivatives so let's get those.

$$f_x = 3x^2 - 3y \qquad f_y = 3y^2 - 3x$$
$$f_{xx} = 6x \qquad f_{yy} = 6y \qquad f_{xy} = -3$$

Let's first find the critical points. Critical points will be solutions to the system of equations,

$$f_x = 3x^2 - 3y = 0$$
$$f_y = 3y^2 - 3x = 0$$

This is a non-linear system of equations and these can, on occasion, be difficult to solve. However, in this case it's not too bad. We can solve the first equation for *y* as follows,

$$3x^2 - 3y = 0 \qquad \Rightarrow \qquad y = x^2$$

Plugging this into the second equation gives,

$$3\left(x^2\right)^2 - 3x = 3x\left(x^3 - 1\right) = 0$$

From this we can see that we must have $x = 0$ or $x = 1$. Now use the fact that $y = x^2$ to get the critical points.

$$x = 0: \quad y = 0^2 = 0 \qquad \Rightarrow \qquad (0,0)$$
$$x = 1: \quad y = 1^2 = 1 \qquad \Rightarrow \qquad (1,1)$$

So, we get two critical points. All we need to do now is classify them. To do this we will need *D*. Here is the general formula for *D*.

$$\begin{aligned} D(x,y) &= f_{xx}(x,y)\, f_{yy}(x,y) - \left[f_{xy}(x,y) \right]^2 \\ &= (6x)(6y) - (-3)^2 \\ &= 36xy - 9 \end{aligned}$$

To classify the critical points all that we need to do is plug in the critical points and use the fact above to classify them.

$(0,0)$**:**

$$D = D(0,0) = -9 < 0$$

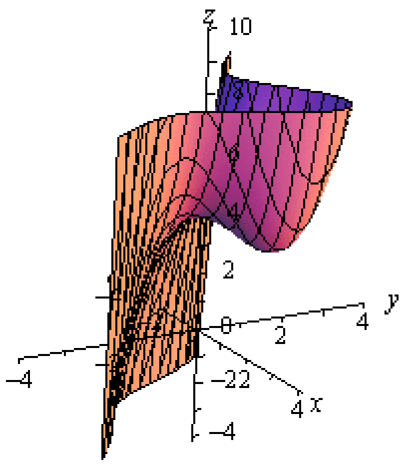
So, for $(0,0)$ *D* is negative and so this must be a saddle point.

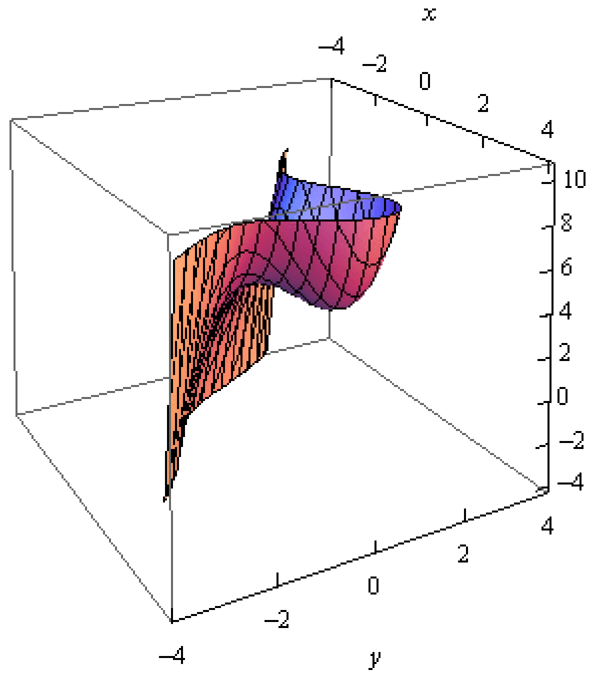
$(1,1)$**:**

$$D = D(1,1) = 36 - 9 = 27 > 0 \qquad\qquad f_{xx}(1,1) = 6 > 0$$

For $(1,1)$ *D* is positive and f_{xx} is positive and so we must have a relative minimum.

For the sake of completeness here is a graph of this function.





Notice that in order to get a better visual we used a somewhat nonstandard orientation. We can see that there is a relative minimum at $(1,1)$ and (hopefully) it's clear that at $(0,0)$ we do get a saddle point.

Example 2 Find and classify all the critical points for $f(x,y) = 3x^2 y + y^3 - 3x^2 - 3y^2 + 2$

Solution

As with the first example we will first need to get all the first and second order derivatives.

$$f_x = 6xy - 6x \qquad f_y = 3x^2 + 3y^2 - 6y$$

$$f_{xx} = 6y - 6 \qquad f_{yy} = 6y - 6 \qquad f_{xy} = 6x$$

We'll first need the critical points. The equations that we'll need to solve this time are,

$$6xy - 6x = 0$$

$$3x^2 + 3y^2 - 6y = 0$$

These equations are a little trickier to solve than the first set, but once you see what to do they really aren't terribly bad.

First, let's notice that we can factor out a $6x$ from the first equation to get,

$$6x(y-1) = 0$$

So, we can see that the first equation will be zero if $x = 0$ or $y = 1$. Be careful to not just cancel the x from both sides. If we had done that we would have missed $x = 0$.

To find the critical points we can plug these (individually) into the second equation and solve for the remaining variable.

$x = 0$:

$$3y^2 - 6y = 3y(y-2) = 0 \qquad \Rightarrow \qquad y = 0,\ y = 2$$

$y = 1$:

$$3x^2 - 3 = 3(x^2 - 1) = 0 \qquad \Rightarrow \qquad x = -1,\ x = 1$$

So, if $x = 0$ we have the following critical points,

$$(0,0) \quad (0,2)$$

and if $y = 1$ the critical points are,

$$(1,1) \quad (-1,1)$$

Now all we need to do is classify the critical points. To do this we'll need the general formula for *D*.

$$D(x,y) = (6y-6)(6y-6) - (6x)^2 = (6y-6)^2 - 36x^2$$

$(0,0)$:

$$D = D(0,0) = 36 > 0 \qquad f_{xx}(0,0) = -6 < 0$$

$(0,2)$:

$$D = D(0,2) = 36 > 0 \qquad f_{xx}(0,2) = 6 > 0$$

$(1,1)$:

$$D = D(1,1) = -36 < 0$$

$(-1,1)$:

$$D = D(-1,1) = -36 < 0$$

So, it looks like we have the following classification of each of these critical points.

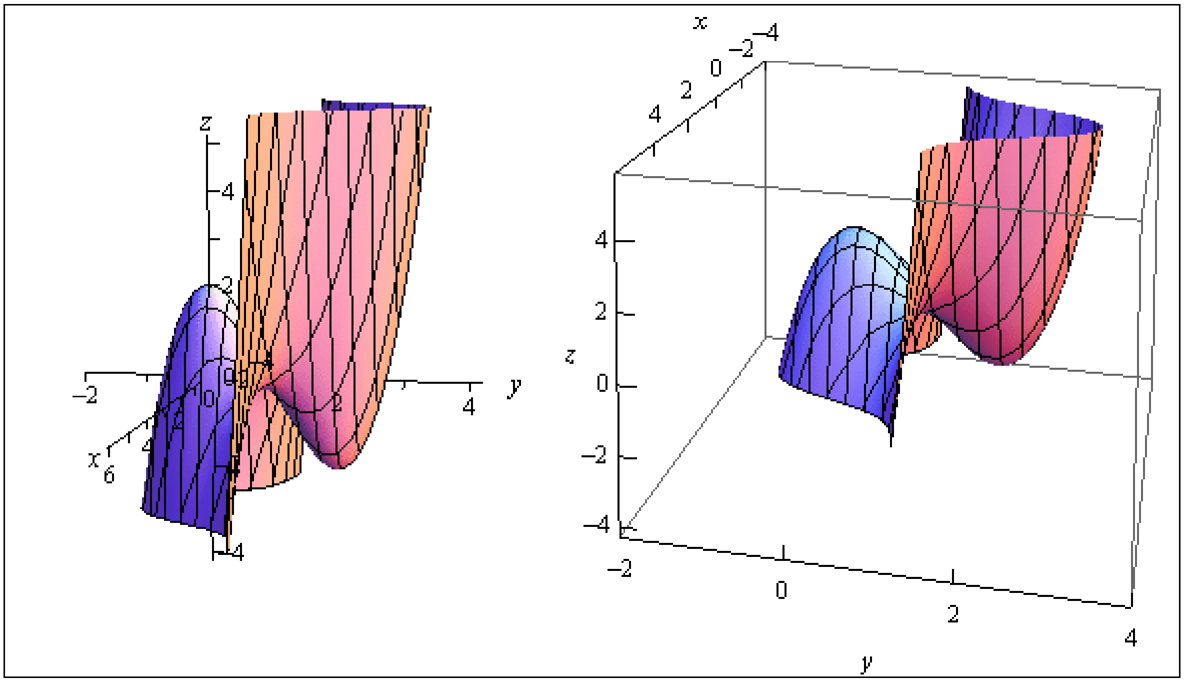
$(0,0)$: Relative Maximum

$(0,2)$: Relative Minimum

$(1,1)$: Saddle Point

$(-1,1)$: Saddle Point

Here is a graph of the surface for the sake of completeness.



Let's do one more example that is a little different from the first two.

Example 3 Determine the point on the plane $4x-2y+z=1$ that is closest to the point $(-2,-1,5)$.

Solution

Note that we are NOT asking for the critical points of the plane. In order to do this example we are going to need to first come up with the equation that we are going to have to work with.

First, let's suppose that (x,y,z) is any point on the plane. The distance between this point and the point in question, $(-2,-1,5)$, is given by the formula,

$$d=\sqrt{(x+2)^2+(y+1)^2+(z-5)^2}$$

What we are then asked to find is the minimum value of this equation. The point (x,y,z) that gives the minimum value of this equation will be the point on the plane that is closest to $(-2,-1,5)$.

There are a couple of issues with this equation. First, it is a function of x, y and z and we can only deal with functions of x and y at this point. However, this is easy to fix. We can solve the equation of the plane to see that,

$$z = 1 - 4x + 2y$$

Plugging this into the distance formula gives,

$$\begin{aligned} d &= \sqrt{(x+2)^2 + (y+1)^2 + (1-4x+2y-5)^2} \\ &= \sqrt{(x+2)^2 + (y+1)^2 + (-4-4x+2y)^2} \end{aligned}$$

Now, the next issue is that there is a square root in this formula and we know that we're going to be differentiating this eventually. So, in order to make our life a little easier let's notice that finding the minimum value of *d* will be equivalent to finding the minimum value of d^2.

So, let's instead find the minimum value of

$$f(x,y) = d^2 = (x+2)^2 + (y+1)^2 + (-4-4x+2y)^2$$

Now, we need to be a little careful here. We are being asked to find the closest point on the plane to $(-2,-1,5)$ and that is not really the same thing as what we've been doing in this section. In this section we've been finding and classifying critical points as relative minimums or maximums and what we are really asking is to find the smallest value the function will take, or the absolute minimum. Hopefully, it does make sense from a physical standpoint that there will be a closest point on the plane to $(-2,-1,5)$. Also, this point should be a relative minimum.

So, let's go through the process from the first and second example and see what we get as far as relative minimums go. If we only get a single relative minimum then we will be done since that point will also need to be the absolute minimum of the function and hence the point on the plane that is closest to $(-2,-1,5)$.

We'll need the derivatives first.

$$\begin{aligned} f_x &= 2(x+2) + 2(-4)(-4-4x+2y) = 36 + 34x - 16y \\ f_y &= 2(y+1) + 2(2)(-4-4x+2y) = -14 - 16x + 10y \\ f_{xx} &= 34 \\ f_{yy} &= 10 \\ f_{xy} &= -16 \end{aligned}$$

Now, before we get into finding the critical point let's compute *D* quickly.

$$D = 34(10) - (-16)^2 = 84 > 0$$

So, in this case *D* will always be positive and also notice that $f_{xx} = 34 > 0$ is always positive and so any critical points that we get will be guaranteed to be relative minimums.

Now let's find the critical point(s). This will mean solving the system.

$$\begin{aligned} 36 + 34x - 16y &= 0 \\ -14 - 16x + 10y &= 0 \end{aligned}$$

To do this we can solve the first equation for *x*.

$$x = \frac{1}{34}(16y - 36) = \frac{1}{17}(8y - 18)$$

Now, plug this into the second equation and solve for *y*.

$$-14 - \frac{16}{17}(8y - 18) + 10y = 0 \qquad \Rightarrow \qquad y = -\frac{25}{21}$$

Back substituting this into the equation for *x* gives $x = -\frac{34}{21}$.

So, it looks like we get a single critical point : $\left(-\frac{34}{21}, -\frac{25}{21}\right)$. Also, since we know this will be a relative minimum and it is the only critical point we know that this is also the *x* and *y* coordinates of the point on the plane that we're after. We can find the *z* coordinate by plugging into the equation of the plane as follows,

$$z = 1 - 4\left(-\frac{34}{21}\right) + 2\left(-\frac{25}{21}\right) = \frac{107}{21}$$

So, the point on the plane that is closest to $(-2, -1, 5)$ is $\left(-\frac{34}{21}, -\frac{25}{21}, \frac{107}{21}\right)$.

Absolute Minimums and Maximums

In this section we are going to extend the work from the previous section. In the previous section we were asked to find and classify all critical points as relative minimums, relative maximums and/or saddle points. In this section we want to optimize a function, that is identify the absolute minimum and/or the absolute maximum of the function, on a given region in $\mathbb{R}^2$. Note that when we say we are going to be working on a region in $\mathbb{R}^2$ we mean that we're going to be looking at some region in the *xy*-plane.

In order to optimize a function in a region we are going to need to get a couple of definitions out of the way and a fact. Let's first get the definitions out of the way.

Definitions

1. A region in $\mathbb{R}^2$ is called **closed** if it includes its boundary. A region is called **open** if it doesn't include any of its boundary points.
2. A region in $\mathbb{R}^2$ is called **bounded** if it can be completely contained in a disk. In other words, a region will be bounded if it is finite.

Let's think a little more about the definition of closed. We said a region is closed if it includes its boundary. Just what does this mean? Let's think of a rectangle. Below are two definitions of a rectangle, one is closed and the other is open.

Open	Closed
$-5 < x < 3$	$-5 \le x \le 3$
$1 < y < 6$	$1 \le y \le 6$

In this first case we don't allow the ranges to include the endpoints (*i.e.* we aren't including the edges of the rectangle) and so we aren't allowing the region to include any points on the edge of the rectangle. In other words, we aren't allowing the region to include its boundary and so it's open.

In the second case we are allowing the region to contain points on the edges and so will contain its entire boundary and hence will be closed.

This is an important idea because of the following fact.

Extreme Value Theorem

If $f(x,y)$ is continuous in some closed, bounded set *D* in $\mathbb{R}^2$ then there are points in *D*, (x_1,y_1) and (x_2,y_2) so that $f(x_1,y_1)$ is the absolute maximum and $f(x_2,y_2)$ is the absolute minimum of the function in *D*.

Note that this theorem does NOT tell us where the absolute minimum or absolute maximum will occur. It only tells us that they will exist. Note as well that the absolute minimum and/or absolute maximum may occur in the interior of the region or it may occur on the boundary of the region.

The basic process for finding absolute maximums is pretty much identical to the process that we used in Calculus I when we looked at finding absolute extrema of functions of single variables. There will however, be some procedural changes to account for the fact that we now are dealing with functions of two variables. Here is the process.

Finding Absolute Extrema

1. Find all the critical points of the function that lie in the region D and determine the function value at each of these points.

2. Find all extrema of the function on the boundary. This usually involves the Calculus I approach for this work.

3. The largest and smallest values found in the first two steps are the absolute minimum and the absolute maximum of the function.

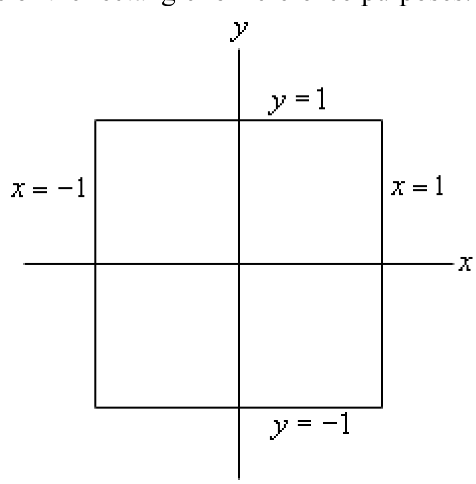
The main difference between this process and the process that we used in Calculus I is that the "boundary" in Calculus I was just two points and so there really wasn't a lot to do in the second step. For these problems the majority of the work is often in the second step as we will often end up doing a Calculus I absolute extrema problem one or more times.

Let's take a look at an example or two.

Example 1 Find the absolute minimum and absolute maximum of $f(x,y) = x^2 + 4y^2 - 2x^2y + 4$ on the rectangle given by $-1 \le x \le 1$ and $-1 \le y \le 1$.

Solution

Let's first get a quick picture of the rectangle for reference purposes.



The boundary of this rectangle is given by the following conditions.

$$\begin{aligned}&\text{right side :} && x=1,\ -1\le y\le 1\\&\text{left side :} && x=-1,\ -1\le y\le 1\\&\text{upper side :} && y=1,\ -1\le x\le 1\\&\text{lower side :} && y=-1,\ -1\le x\le 1\end{aligned}$$

These will be important in the second step of our process.

We'll start this off by finding all the critical points that lie inside the given rectangle. To do this we'll need the two first order derivatives.

$$f_x = 2x-4xy \qquad\qquad f_y = 8y-2x^2$$

Note that since we aren't going to be classifying the critical points we don't need the second order derivatives. To find the critical points we will need to solve the system,

$$\begin{aligned}2x-4xy&=0\\8y-2x^2&=0\end{aligned}$$

We can solve the second equation for *y* to get,

$$y=\frac{x^2}{4}$$

Plugging this into the first equation gives us,

$$2x-4x\left(\frac{x^2}{4}\right)=2x-x^3=x\left(2-x^2\right)=0$$

This tells us that we must have $x=0$ or $x=\pm\sqrt{2}=\pm1.414\ldots$. Now, recall that we only want critical points in the region that we're given. That means that we only want critical points for which $-1\le x\le 1$. The only value of *x* that will satisfy this is the first one so we can ignore the last two for this problem. Note however that a simple change to the boundary would include these two so don't forget to always check if the critical points are in the region (or on the boundary since that can also happen).

Plugging $x=0$ into the equation for *y* gives us,

$$y=\frac{0^2}{4}=0$$

The single critical point, in the region (and again, that's important), is $(0,0)$. We now need to get the value of the function at the critical point.

$$f(0,0)=4$$

Eventually we will compare this to values of the function found in the next step and take the largest and smallest as the absolute extrema of the function in the rectangle.

Now we have reached the long part of this problem. We need to find the absolute extrema of the function along the boundary of the rectangle. What this means is that we're going to need to look at what the function is doing along each of the sides of the rectangle listed above.

Let's first take a look at the right side. As noted above the right side is defined by

$$x = 1,\ -1 \le y \le 1$$

Notice that along the right side we know that $x = 1$. Let's take advantage of this by defining a new function as follows,

$$g(y) = f(1, y) = 1^2 + 4y^2 - 2(1^2)y + 4 = 5 + 4y^2 - 2y$$

Now, finding the absolute extrema of $f(x, y)$ along the right side will be equivalent to finding the absolute extrema of $g(y)$ in the range $-1 \le y \le 1$. Hopefully you recall how to do this from Calculus I. We find the critical points of $g(y)$ in the range $-1 \le y \le 1$ and then evaluate $g(y)$ at the critical points and the end points of the range of y's.

Let's do that for this problem.

$$g'(y) = 8y - 2 \qquad \Rightarrow \qquad y = \frac{1}{4}$$

This is in the range and so we will need the following function evaluations.

$$g(-1) = 11 \qquad g(1) = 7 \qquad g\left(\frac{1}{4}\right) = \frac{19}{4} = 4.75$$

Notice that, using the definition of $g(y)$ these are also function values for $f(x, y)$.

$$g(-1) = f(1, -1) = 11$$
$$g(1) = f(1, 1) = 7$$
$$g\left(\frac{1}{4}\right) = f\left(1, \frac{1}{4}\right) = \frac{19}{4} = 4.75$$

We can now do the left side of the rectangle which is defined by,

$$x = -1,\ -1 \le y \le 1$$

Again, we'll define a new function as follows,

$$g(y) = f(-1, y) = (-1)^2 + 4y^2 - 2(-1)^2 y + 4 = 5 + 4y^2 - 2y$$

Notice however that, for this boundary, this is the same function as we looked at for the right side. This will not always happen, but since it has let's take advantage of the fact that we've already done the work for this function. We know that the critical point is $y = \frac{1}{4}$ and we know that the function value at the critical point and the end points are,

$$g(-1) = 11 \qquad g(1) = 7 \qquad g\left(\frac{1}{4}\right) = \frac{19}{4} = 4.75$$

The only real difference here is that these will correspond to values of $f(x, y)$ at different points than for the right side. In this case these will correspond to the following function values for $f(x,y)$.

$$g(-1) = f(-1,-1) = 11$$
$$g(1) = f(-1,1) = 7$$
$$g\left(\frac{1}{4}\right) = f\left(-1,\frac{1}{4}\right) = \frac{19}{4} = 4.75$$

We can now look at the upper side defined by,

$$y = 1, \ -1 \le x \le 1$$

We'll again define a new function except this time it will be a function of x.

$$h(x) = f(x,1) = x^2 + 4(1^2) - 2x^2(1) + 4 = 8 - x^2$$

We need to find the absolute extrema of $h(x)$ on the range $-1 \le x \le 1$. First find the critical point(s).

$$h'(x) = -2x \qquad \Rightarrow \qquad x = 0$$

The value of this function at the critical point and the end points is,

$$h(-1) = 7 \qquad h(1) = 7 \qquad h(0) = 8$$

and these in turn correspond to the following function values for $f(x,y)$

$$h(-1) = f(-1,1) = 7$$
$$h(1) = f(1,1) = 7$$
$$h(0) = f(0,1) = 8$$

Note that there are several "repeats" here. The first two function values have already been computed when we looked at the right and left side. This will often happen.

Finally, we need to take care of the lower side. This side is defined by,

$$y = -1, \ -1 \le x \le 1$$

The new function we'll define in this case is,

$$h(x) = f(x,-1) = x^2 + 4(-1)^2 - 2x^2(-1) + 4 = 8 + 3x^2$$

The critical point for this function is,

$$h'(x) = 6x \qquad \Rightarrow \qquad x = 0$$

The function values at the critical point and the endpoint are,

$$h(-1) = 11 \qquad h(1) = 11 \qquad h(0) = 8$$

and the corresponding values for $f(x,y)$ are,

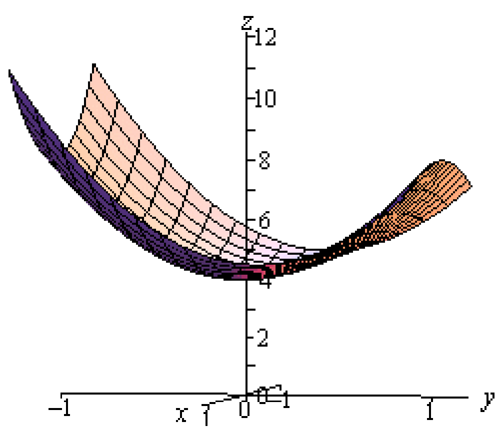
$$h(-1) = f(-1,-1) = 11$$
$$h(1) = f(1,-1) = 11$$
$$h(0) = f(0,-1) = 8$$

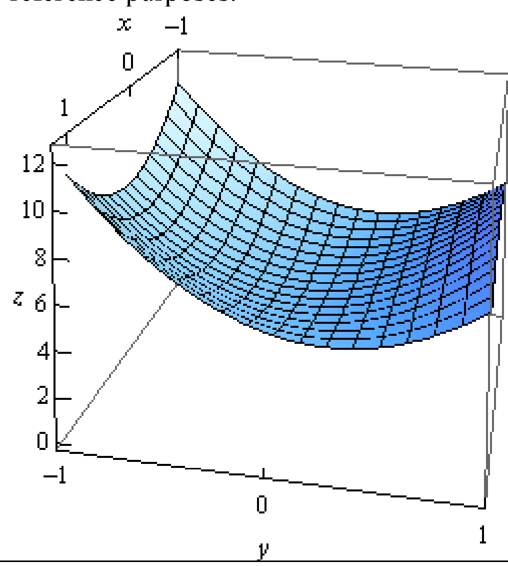
The final step to this (long…) process is to collect up all the function values for $f(x,y)$ that we've computed in this problem. Here they are,

$$f(0,0) = 4 \qquad f(1,-1) = 11 \qquad f(1,1) = 7$$
$$f\left(1,\frac{1}{4}\right) = 4.75 \qquad f(-1,1) = 7 \qquad f(-1,-1) = 11$$
$$f\left(-1,\frac{1}{4}\right) = 4.75 \qquad f(0,1) = 8 \qquad f(0,-1) = 8$$

The absolute minimum is at $(0,0)$ since gives the smallest function value and the absolute maximum occurs at $(1,-1)$ and $(-1,-1)$ since these two points give the largest function value.

Here is a sketch of the function on the rectangle for reference purposes.





As this example has shown these can be very long problems. Let's take a look at an easier problem with a different kind of boundary.

Example 2 Find the absolute minimum and absolute maximum of $f(x,y) = 2x^2 - y^2 + 6y$ on the disk of radius 4, $x^2 + y^2 \le 16$

Solution

First note that a disk of radius 4 is given by the inequality in the problem statement. The "less than" inequality is included to get the interior of the disk and the equal sign is included to get the boundary. Of course, this also means that the boundary of the disk is a circle of radius 4.

Let's first find the critical points of the function that lies inside the disk. This will require the following two first order partial derivatives.

$$f_x = 4x \qquad\qquad f_y = -2y + 6$$

To find the critical points we'll need to solve the following system.

$$4x = 0$$
$$-2y + 6 = 0$$

This is actually a fairly simple system to solve however. The first equation tells us that $x = 0$ and the second tells us that $y = 3$. So the only critical point for this function is $\left(0,3\right)$ and this is inside the disk of radius 4. The function value at this critical point is,

$$f\left(0,3\right) = 9$$

Now we need to look at the boundary. This one will be somewhat different from the previous example. In this case we don't have fixed values of *x* and *y* on the boundary. Instead we have,

$$x^2 + y^2 = 16$$

We can solve this for x^2 and plug this into the x^2 in $f\left(x,y\right)$ to get a function of *y* as follows.

$$x^2 = 16 - y^2$$
$$g\left(y\right) = 2\left(16 - y^2\right) - y^2 + 6y = 32 - 3y^2 + 6y$$

We will need to find the absolute extrema of this function on the range $-4 \le y \le 4$ (this is the range of *y*'s for the disk….). We'll first need the critical points of this function.

$$g'\left(y\right) = -6y + 6 \qquad \Rightarrow \qquad y = 1$$

The value of this function at the critical point and the endpoints are,

$$g\left(-4\right) = -40 \qquad\qquad g\left(4\right) = 8 \qquad\qquad g\left(1\right) = 35$$

Unlike the first example we will still need to find the values of *x* that correspond to these. We can do this by plugging the value of *y* into our equation for the circle and solving for *y*.

$$y = -4 \;: \qquad x^2 = 16 - 16 = 0 \qquad \Rightarrow \qquad x = 0$$
$$y = 4 \;: \qquad x^2 = 16 - 16 = 0 \qquad \Rightarrow \qquad x = 0$$
$$y = 1 \;: \qquad x^2 = 16 - 1 = 15 \Rightarrow \qquad x = \pm\sqrt{15} = \pm 3.87$$

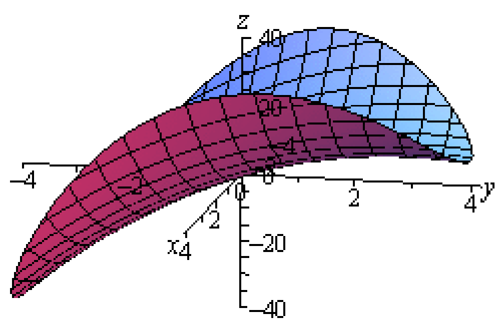
The function values for $g\left(y\right)$ then correspond to the following function values for $f\left(x,y\right)$.

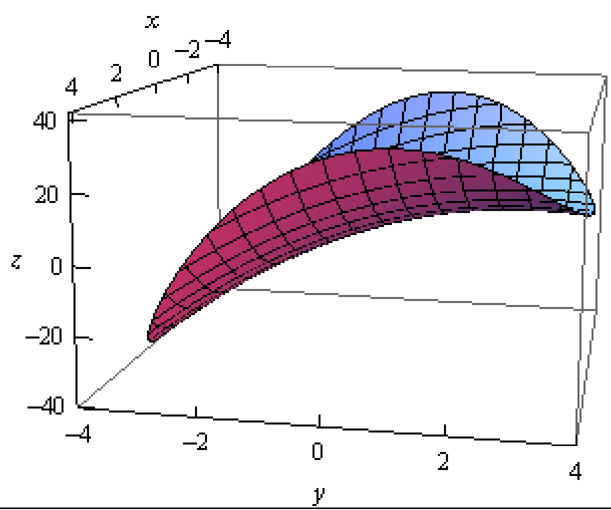
$$g\left(-4\right) = -40 \quad \Rightarrow \quad f\left(0,-4\right) = -40$$
$$g\left(4\right) = 8 \quad \Rightarrow \quad f\left(0,4\right) = 8$$
$$g\left(1\right) = 35 \quad \Rightarrow \quad f\left(-\sqrt{15},1\right) = 35 \;\text{ and }\; f\left(\sqrt{15},1\right) = 35$$

Note that the third one actually corresponded to two different values for $f(x,y)$ since that y also produced two different values of x.

So, comparing these values to the value of the function at the critical point of $f(x,y)$ that we found earlier we can see that the absolute minimum occurs at $(0,-4)$ while the absolute maximum occurs twice at $\left(-\sqrt{15},1\right)$ and $\left(\sqrt{15},1\right)$.

Here is a sketch of the region for reference purposes.





In both of these examples one of the absolute extrema actually occurred at more than one place. Sometimes this will happen and sometimes it won't so don't read too much into the fact that it happened in both examples given here.

Also note that, as we've seen, absolute extrema will often occur on the boundaries of these regions, although they don't have to occur at the boundaries. Had we given much more complicated examples with multiple critical points we would have seen examples where the absolute extrema occurred interior to the region and not on the boundary.

Lagrange Multipliers

In the previous section we optimized (*i.e.* found the absolute extrema) a function on a region that contained its boundary. Finding potential optimal points in the interior of the region isn't too bad in general, all that we needed to do was find the critical points and plug them into the function. However, as we saw in the examples finding potential optimal points on the boundary was often a fairly long and messy process.

In this section we are going to take a look at another way of optimizing a function subject to given constraint(s). The constraint(s) may be the equation(s) that describe the boundary of a region although in this section we won't concentrate on those types of problems since this method just requires a general constraint and doesn't really care where the constraint came from.

So, let's get things set up. We want to optimize (*i.e.* find the minimum and maximum value of) a function, $f(x,y,z)$, subject to the constraint $g(x,y,z)=k$. Again, the constraint may be the equation that describes the boundary of a region or it may not be. The process is actually fairly simple, although the work can still be a little overwhelming at times.

Method of Lagrange Multipliers

1. Solve the following system of equations.
$$\nabla f(x,y,z) = \lambda\ \nabla g(x,y,z)$$
$$g(x,y,z) = k$$
2. Plug in all solutions, (x,y,z), from the first step into $f(x,y,z)$ and identify the minimum and maximum values, provided they exist.

The constant, λ, is called the **Lagrange Multiplier**.

Notice that the system of equations actually has four equations, we just wrote the system in a simpler form. To see this let's take the first equation and put in the definition of the gradient vector to see what we get.

$$\langle f_x, f_y, f_z\rangle = \lambda\langle g_x, g_y, g_z\rangle = \langle \lambda g_x, \lambda g_y, \lambda g_z\rangle$$

In order for these two vectors to be equal the individual components must also be equal. So, we actually have three equations here.

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad f_z = \lambda g_z$$

These three equations along with the constraint, $g(x,y,z)=c$, give four equations with four unknowns *x*, *y*, *z*, and λ.

Note as well that if we only have functions of two variables then we won't have the third component of the gradient and so will only have three equations in three unknowns *x*, *y*, and λ.

As a final note we also need to be careful with the fact that in some cases minimums and maximums won't exist even though the method will seem to imply that they do. In every problem we'll need to go back and make sure that our answers make sense.

Let's work a couple of examples.

Example 1 Find the dimensions of the box with largest volume if the total surface area is 64 cm^2.

Solution

Before we start the process here note that we also saw a way to solve this kind of problem in Calculus I, except in those problems we required a condition that related one of the sides of the box to the other sides so that we could get down to a volume and surface area function that only involved two variables. We no longer need this condition for these problems.

Now, let's get on to solving the problem. We first need to identify the function that we're going to optimize as well as the constraint. Let's set the length of the box to be x, the width of the box to be y and the height of the box to be z. Let's also note that because we're dealing with the dimensions of a box it is safe to assume that x, y, and z are all positive quantities.

We want to find the largest volume and so the function that we want to optimize is given by,

$$f\left(x,y,z\right) = xyz$$

Next we know that the surface area of the box must be a constant 64. So this is the constraint. The surface area of a box is simply the sum of the areas of each of the sides so the constraint is given by,

$$2xy + 2xz + 2yz = 64 \qquad \Rightarrow \qquad xy + xz + yz = 32$$

Note that we divided the constraint by 2 to simplify the equation a little. Also, we get the function $g\left(x,y,z\right)$ from this.

$$g\left(x,y,z\right) = xy + xz + yz$$

Here are the four equations that we need to solve.

$$yz = \lambda\left(y+z\right) \qquad \left(f_x = \lambda g_x\right) \qquad (1)$$

$$xz = \lambda\left(x+z\right) \qquad \left(f_y = \lambda g_y\right) \qquad (2)$$

$$xy = \lambda\left(x+y\right) \qquad \left(f_z = \lambda g_z\right) \qquad (3)$$

$$xy + xz + yz = 32 \qquad \left(g\left(x,y,z\right) = 32\right) \qquad (4)$$

There are many ways to solve this system. We'll solve it in the following way. Let's multiply equation (1) by x, equation (2) by y and equation (3) by z. This gives,

$$xyz = \lambda x\left(y+z\right) \qquad (5)$$

$$xyz = \lambda y\left(x+z\right) \qquad (6)$$

$$xyz = \lambda z\left(x+y\right) \qquad (7)$$

Now notice that we can set equations (5) and (6) equal. Doing this gives,

$$\lambda x(y+z)=\lambda y(x+z)$$
$$\lambda(xy+xz)-\lambda(yx+yz)=0$$
$$\lambda(xz-yz)=0 \qquad \Rightarrow \qquad \lambda=0 \quad \text{or} \quad xz=yz$$

This gave two possibilities. The first, $\lambda=0$ is not possible since if this was the case equation (1) would reduce to

$$yz=0 \qquad \Rightarrow \qquad y=0 \text{ or } z=0$$

Since we are talking about the dimensions of a box neither of these are possible so we can discount $\lambda=0$. This leaves the second possibility.

$$xz=yz$$

Since we know that $z\neq 0$ (again since we are talking about the dimensions of a box) we can cancel the z from both sides. This gives,

$$x=y \tag{8}$$

Next, let's set equations (6) and (7) equal. Doing this gives,

$$\lambda y(x+z)=\lambda z(x+y)$$
$$\lambda(yx+yz-zx-zy)=0$$
$$\lambda(yx-zx)=0 \qquad \Rightarrow \qquad \lambda=0 \text{ or } \quad yx=zx$$

As already discussed we know that $\lambda=0$ won't work and so this leaves,

$$yx=zx$$

We can also say that $x\neq 0$ since we are dealing with the dimensions of a box so we must have,

$$z=y \tag{9}$$

Plugging equations (8) and (9) into equation (4) we get,

$$y^2+y^2+y^2=3y^2=32 \qquad\qquad y=\pm\sqrt{\frac{32}{3}}=\pm 3.266$$

However, we know that y must be positive since we are talking about the dimensions of a box. Therefore the only solution that makes physical sense here is

$$x=y=z=3.266$$

So, it looks like we've got a cube here.

We should be a little careful here. Since we've only got one solution we might be tempted to assume that these are the dimensions that will give the largest volume. The method of Lagrange Multipliers will give a set of points that will either maximize or minimize a given function subject to the constraint, provided there actually are minimums or maximums.

The function itself, $f(x,y,z)=xyz$ will clearly have neither minimums or maximums unless we put some restrictions on the variables. The only real restriction that we've got is that all the variables must be positive. This, of course, instantly means that the function does have a minimum, zero.

The function will not have a maximum if all the variables are allowed to increase without bound. That however, can't happen because of the constraint,

$$xy + xz + yz = 32$$

Here we've got the sum of three positive numbers (because x, y, and z are positive) and the sum must equal 32. So, if one of the variables gets very large, say x, then because each of the products must be less than 32 both y and z must be very small to make sure the first two terms are less than 32. So, there is no way for all the variables to increase without bound and so it should make some sense that the function, $f(x, y, z) = xyz$, will have a maximum.

This isn't a rigorous proof that the function will have a maximum, but it should help to visualize that in fact it should have a maximum and so we can say that we will get a maximum volume if the dimensions are : $x = y = z = 3.266$.

Notice that we never actually found values for λ in the above example. This is fairly standard for these kinds of problems. The value of λ isn't really important to determining if the point is a maximum or a minimum so often we will not bother with finding a value for it. On occasion we will need its value to help solve the system, but even in those cases we won't use it past finding the point.

Example 2 Find the maximum and minimum of $f(x, y) = 5x - 3y$ subject to the constraint $x^2 + y^2 = 136$.

Solution

This one is going to be a little easier than the previous one since it only has two variables. Also, note that it's clear from the constraint that region of possible solutions lies on a disk of radius $\sqrt{136}$ which is a closed and bounded region and hence by the Extreme Value Theorem we know that a minimum and maximum value must exist.

Here is the system that we need to solve.

$$\begin{aligned} 5 &= 2\lambda x \\ -3 &= 2\lambda y \\ x^2 + y^2 &= 136 \end{aligned}$$

Notice that, as with the last example, we can't have $\lambda = 0$ since that would not satisfy the first two equations. So, since we know that $\lambda \neq 0$ we can solve the first two equations for x and y respectively. This gives,

$$x = \frac{5}{2\lambda} \qquad\qquad y = -\frac{3}{2\lambda}$$

Plugging these into the constraint gives,

$$\frac{25}{4\lambda^2}+\frac{9}{4\lambda^2}=\frac{17}{2\lambda^2}=136$$

We can solve this for λ.

$$\lambda^2=\frac{1}{16} \qquad \Rightarrow \qquad \lambda=\pm\frac{1}{4}$$

Now, that we know λ we can find the points that will be potential maximums and/or minimums.

If $\lambda=-\frac{1}{4}$ we get,

$$x=-10 \qquad\qquad y=6$$

and if $\lambda=\frac{1}{4}$ we get,

$$x=10 \qquad\qquad y=-6$$

To determine if we have maximums or minimums we just need to plug these into the function. Also recall from the discussion at the start of this solution that we know these will be the minimum and maximums because the Extreme Value Theorem tells us that minimums and maximums will exist for this problem.

Here are the minimum and maximum values of the function.

$$f(-10,6)=-68 \qquad \text{Minimum at } (-10,6)$$
$$f(10,-6)=68 \qquad \text{Maximum at } (10,-6)$$

In the first two examples we've excluded $\lambda=0$ either for physical reasons or because it wouldn't solve one or more of the equations. Do not always expect this to happen. Sometimes we will be able to automatically exclude a value of λ and sometimes we won't.

Let's take a look at another example.

Example 3 Find the maximum and minimum values of $f(x,y,z)=xyz$ subject to the constraint $x+y+z=1$. Assume that $x,y,z\ge 0$.

Solution
First note that our constraint is a sum of three positive or zero number and it must be 1. Therefore it is clear that our solution will fall in the range $0\le x,y,z\le 1$. Therefore the solution must lie in a closed and bounded region and so by the Extreme Value Theorem we know that a minimum and maximum value must exist.

Here is the system of equation that we need to solve.

$$yz=\lambda \tag{10}$$
$$xz=\lambda \tag{11}$$
$$xy=\lambda \tag{12}$$
$$x+y+z=1 \tag{13}$$

Let's start this solution process off by noticing that since the first three equations all have λ they are all equal. So, let's start off by setting equations (10) and (11) equal.

$$yz = xz \qquad \Rightarrow \qquad z(y-x)=0 \qquad \Rightarrow \qquad z=0 \text{ or } y=x$$

So, we've got two possibilities here. Let's start off with by assuming that $z=0$. In this case we can see from either equation (10) or (11) that we must then have $\lambda = 0$. From equation (12) we see that this means that $xy=0$. This in turn means that either $x=0$ or $y=0$.

So, we've got two possible cases to deal with there. In each case two of the variables must be zero. Once we know this we can plug into the constraint, equation (13), to find the remaining value.

$$\begin{aligned} z=0,\ x=0: \qquad &\Rightarrow \qquad y=1 \\ z=0,\ y=0: \qquad &\Rightarrow \qquad x=1 \end{aligned}$$

So, we've got two possible solutions $(0,1,0)$ and $(1,0,0)$.

Now let's go back and take a look at the other possibility, $y=x$. We also have two possible cases to look at here as well.

This first case is $x=y=0$. In this case we can see from the constraint that we must have $z=1$ and so we now have a third solution $(0,0,1)$.

The second case is $x=y\neq 0$. Let's set equations (11) and (12) equal.

$$xz = xy \qquad \Rightarrow \qquad x(z-y)=0 \qquad \Rightarrow \qquad x=0 \text{ or } z=y$$

Now, we've already assumed that $x \neq 0$ and so the only possibility is that $z=y$. However, this also means that,

$$x=y=z$$

Using this in the constraint gives,

$$3x=1 \qquad \Rightarrow \qquad x=\frac{1}{3}$$

So, the next solution is $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$.

We got four solutions by setting the first two equations equal.

To completely finish this problem out we should probably set equations (10) and (12) equal as well as setting equations (11) and (12) equal to see what we get. Doing this gives,

$$\begin{aligned} yz = xy \qquad &\Rightarrow \qquad y(z-x)=0 \qquad \Rightarrow \qquad y=0 \text{ or } z=x \\ xz = xy \qquad &\Rightarrow \qquad x(z-y)=0 \qquad \Rightarrow \qquad x=0 \text{ or } z=y \end{aligned}$$

Both of these are very similar to the first situation that we looked at and we'll leave it up to you to show that in each of these cases we arrive back at the four solutions that we already found.

So, we have four solutions that we need to check in the function to see whether we have minimums or maximums.

$$f(0,0,1)=0 \qquad f(0,1,0)=0 \qquad f(1,0,0)=0 \qquad \text{All Minimums}$$
$$f\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)=\frac{1}{27} \qquad \text{Maximum}$$

So, in this case the maximum occurs only once while the minimum occurs three times.

Note as well that we never really used the assumption that $x,y,z \geq 0$ in this problem. This assumption is here mostly to make sure that we really do have a maximum and a minimum of the function. Without this assumption it wouldn't be too difficult to find points that give both larger and smaller values of the functions. For example.

$$x=-100,\ y=100,\ z=1:\ -100+100+1=1 \qquad f(-100,100,1)=-10000$$
$$x=-50,\ y=-50,\ z=101:\ -50-50+101=1 \qquad f(-50,-50,101)=252500$$

With these examples you can clearly see that it's not too hard to find points that will give larger and smaller function values. However, all of these examples required negative values of x, y and/or z to make sure we satisfy the constraint. By eliminating these we will know that we've got minimum and maximum values by the Extreme Value Theorem.

To this point we've only looked at constraints that were equations. We can also have constraints that are inequalities. The process for these types of problems is nearly identical to what we've been doing in this section to this point. The main difference between the two types of problems is that we will also need to find all the critical points that satisfy the inequality in the constraint and check these in the function when we check the values we found using Lagrange Multipliers.

Let's work an example to see how these kinds of problems work.

Example 4 Find the maximum and minimum values of $f(x,y)=4x^2+10y^2$ on the disk $x^2+y^2 \leq 4$.

Solution

Note that the constraint here is the inequality for the disk. Because this is a closed and bounded region the Extreme Value Theorem tells us that a minimum and maximum value must exist.

The first step is to find all the critical points that are in the disk (*i.e.* satisfy the constraint). This is easy enough to do for this problem. Here are the two first order partial derivatives.

$$f_x=8x \quad \Rightarrow \quad 8x=0 \quad \Rightarrow \quad x=0$$
$$f_y=20y \quad \Rightarrow \quad 20y=0 \quad \Rightarrow \quad y=0$$

So, the only critical point is $(0,0)$ and it does satisfy the inequality.

At this point we proceed with Lagrange Multipliers and we treat the constraint as an equality instead of the inequality. We only need to deal with the inequality when finding the critical points.

So, here is the system of equations that we need to solve.

$$8x = 2\lambda x$$
$$20y = 2\lambda y$$
$$x^2 + y^2 = 4$$

From the first equation we get,

$$2x(4-\lambda) = 0 \qquad \Rightarrow \qquad x = 0 \quad \text{or} \quad \lambda = 4$$

If we have $x = 0$ then the constraint gives us $y = \pm 2$.

If we have $\lambda = 4$ the second equation gives us,

$$20y = 8y \qquad \Rightarrow \qquad y = 0$$

The constraint then tells us that $x = \pm 2$.

If we'd performed a similar analysis on the second equation we would arrive at the same points.

So, Lagrange Multipliers gives us four points to check : $(0,2)$, $(0,-2)$, $(2,0)$, and $(-2,0)$.

To find the maximum and minimum we need to simply plug these four points along with the critical point in the function.

$$f(0,0) = 0 \qquad \text{Minimum}$$
$$f(2,0) = f(-2,0) = 16$$
$$f(0,2) = f(0,-2) = 40 \qquad \text{Maximum}$$

In this case, the minimum was interior to the disk and the maximum was on the boundary of the disk.

The final topic that we need to discuss in this section is what to do if we have more than one constraint. We will look only at two constraints, but we can naturally extend the work here to more than two constraints.

We want to optimize $f(x,y,z)$ subject to the constraints $g(x,y,z) = c$ and $h(x,y,z) = k$. The system that we need to solve in this case is,

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$$
$$g(x,y,z) = c$$
$$h(x,y,z) = k$$

So, in this case we get two Lagrange Multipliers. Also, note that the first equation really is three equations as we saw in the previous examples. Let's see an example of this kind of optimization problem.

Example 5 Find the maximum and minimum of $f\left(x,y,z\right)=4y-2z$ subject to the constraints $2x-y-z=2$ and $x^2+y^2=1$.

Solution

Verifying that we will have a minimum and maximum value here is a little trickier. Clearly, because of the second constraint we've got to have $-1\le x,y\le 1$. With this in mind there must also be a set of limits on *z* in order to make sure that the first constraint is met. If one really wanted to determine that range you could find the minimum and maximum values of $2x-y$ subject to $x^2+y^2=1$ and you could then use this to determine the minimum and maximum values of *z*. We won't do that here. The point is only to acknowledge that once again the possible solutions must lie in a closed and bounded region and so minimum and maximum values must exist by the Extreme Value Theorem.

Here is the system of equations that we need to solve.

$$0=2\lambda+2\mu x \qquad \left(f_x=\lambda g_x+\mu h_x\right) \tag{14}$$

$$4=-\lambda+2\mu y \qquad \left(f_y=\lambda g_y+\mu h_y\right) \tag{15}$$

$$-2=-\lambda \qquad \left(f_z=\lambda g_z+\mu h_z\right) \tag{16}$$

$$2x-y-z=2 \tag{17}$$

$$x^2+y^2=1 \tag{18}$$

First, let's notice that from equation (16) we get $\lambda=2$. Plugging this into equation (14) and equation (15) and solving for *x* and *y* respectively gives,

$$0=4+2\mu x \qquad \Rightarrow \qquad x=-\frac{2}{\mu}$$

$$4=-2+2\mu y \qquad \Rightarrow \qquad y=\frac{3}{\mu}$$

Now, plug these into equation (18).

$$\frac{4}{\mu^2}+\frac{9}{\mu^2}=\frac{13}{\mu^2}=1 \qquad \Rightarrow \qquad \mu=\pm\sqrt{13}$$

So, we have two cases to look at here. First, let's see what we get when $\mu=\sqrt{13}$. In this case we know that,

$$x=-\frac{2}{\sqrt{13}} \qquad\qquad y=\frac{3}{\sqrt{13}}$$

Plugging these into equation (17) gives,

$$-\frac{4}{\sqrt{13}}-\frac{3}{\sqrt{13}}-z=2 \qquad \Rightarrow \qquad z=-2-\frac{7}{\sqrt{13}}$$

So, we've got one solution.

Let's now see what we get if we take $\mu=-\sqrt{13}$. Here we have,

$$x=\frac{2}{\sqrt{13}} \qquad y=-\frac{3}{\sqrt{13}}$$

Plugging these into equation (17) gives,

$$\frac{4}{\sqrt{13}}+\frac{3}{\sqrt{13}}-z=2 \qquad \Rightarrow \qquad z=-2+\frac{7}{\sqrt{13}}$$

and there's a second solution.

Now all that we need to is check the two solutions in the function to see which is the maximum and which is the minimum.

$$f\left(-\frac{2}{\sqrt{13}},\frac{3}{\sqrt{13}},-2-\frac{7}{\sqrt{13}}\right)=4+\frac{26}{\sqrt{13}}=11.2111$$

$$f\left(\frac{2}{\sqrt{13}},-\frac{3}{\sqrt{13}},-2+\frac{7}{\sqrt{13}}\right)=4-\frac{26}{\sqrt{13}}=-3.2111$$

So, we have a maximum at $\left(-\frac{2}{\sqrt{13}},\frac{3}{\sqrt{13}},-2-\frac{7}{\sqrt{13}}\right)$ and a minimum at $\left(\frac{2}{\sqrt{13}},-\frac{3}{\sqrt{13}},-2+\frac{7}{\sqrt{13}}\right)$.