جامعة جنوب الوادى

كلية التربية بالغردقة

الفرقة الثانية عام physics & chemistry

المادة : Mathematics(part of static)

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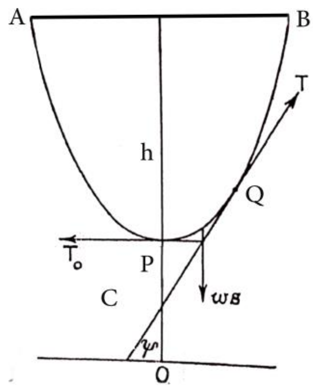
Chapter (1)

The Catenary

Definition(The common Catenary):

The catenary is the curve in which a uniform chain or string hangs when freely suspended from two points$A$&B



Denote the tension at the lowest point

y T_0 ,this will be horizontal. Let$s$

be the length of chain measured from P

to any point Q .Let the tension at Qp be T

and let its inclination to the horizontal be ψ.

Let the weight per unit length of the chain be ω.

The part of the chain PQ will be in

equilibrium under the action of three forces, its weight ωs ,T_0,and T , the tensions at P and Q .

The intrinsic Equation of the catenary:

Resolving vertically and horizontally weget,

$$T \sin\psi = \omega s \quad , \quad T\cos\psi = T_0$$

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For convenience we introduce another constant c ,which is such that $T_{0=}\omega c$. Then

$$T\ \sin\psi = \omega s \qquad , \qquad T\cos\psi = \omega c$$

Dividing

$$s = c\ \tan\psi \qquad\qquad (i)$$

This is the intrinsic equation of the curve, (c is called the parameter of the catenary).

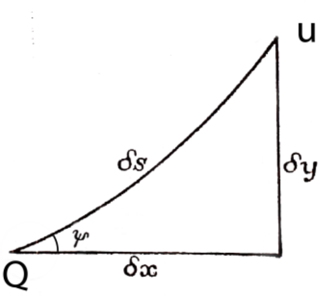
The cartesian Equation of the catenary:

To find the Cartesian equation of the curve we flow:

Since $\tan\psi = \frac{dy}{dx}$, then from $(i)\ \frac{dy}{dx} = \frac{s}{c}$

Consider a small element δs of a curve joining two points Q and U on the curve. Let the coordinates of Q and U be (x, y)&$(x + \delta x, y + \delta y)$respectively. Then

$$(\delta s)^2 \cong (\delta x)^2 + (\delta y)^2$$



Dividing by $(\delta x)^2$ then $(\delta y)^2$ respectively we get:

$$\left(\frac{\delta s}{\delta x}\right)^2 \cong 1 + \left(\frac{\delta y}{\delta x}\right)^2$$

and

$$\left(\frac{\delta s}{\delta y}\right)^2 \cong \left(\frac{\delta x}{\delta y}\right)^2 + 1$$

When $\delta s, \delta x, \delta y \to 0$, the above equations becomes

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 \quad (ii)$$

and

$$\left(\frac{ds}{dy}\right)^2 = \left(\frac{dx}{dy}\right)^2 + 1 \qquad (iii)$$

(ii) gives

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{s}{c}\right)^2$$

$$\therefore \quad \frac{ds}{dx} = \frac{\sqrt{(c^2 + s^2)}}{c}$$

$$\therefore \quad dx = \frac{c\ ds}{\sqrt{(c^2 + s^2)}}$$

$$\therefore \quad x = c\ sinh^{-1}\ \frac{s}{c} \qquad (iv)$$

or $\quad s = c \sinh\frac{x}{c} \qquad (v)$

provided $x = 0$ when $s = 0$.

(iii) gives

$$\left(\frac{ds}{dy}\right)^2 = 1 + \left(\frac{c}{s}\right)^2$$

$$\therefore \quad \frac{ds}{dy} = \frac{\sqrt{(c^2 + s^2)}}{s}$$

$$\therefore \quad dy = \frac{s\, ds}{\sqrt{(c^2 + s^2)}}$$

$$\therefore \quad y = \sqrt{(c^2 + s^2)}$$

i.e. $y^2 = s^2 + c^2 \qquad (vi)$

provided $y = c$ when $s = 0 \& x = 0$

Substituting from (v) in (vi)

$$. \qquad y^2 = c^2\left(1 + sinh^2 \frac{x}{c}\right)$$

$$= c^2 cosh^2 (\frac{x}{c})$$

$$\therefore \quad y = c \cosh\left(\frac{x}{c}\right) \qquad (viii)$$

This is the Cartesian equation of the catenary.

<u>The tension at any point</u>:

Since

$$T \sin\psi = \omega s \quad , \quad T\cos\psi = \omega c$$

then $T^2 = \omega^2 (s^2 + c^2)$

which from (vi) gives

$$T^2 = \omega^2 y^2$$

$$\therefore \quad T = \omega y$$

Thus,the tension at any point of the catenary is proportional to the height of the point above the x −axis which is usually called the (directric).

The Lightning and telephone wires:

When c is large, from equation (vii),

$$\therefore \quad y = c\cosh\left(\frac{x}{c}\right) = \frac{C}{2}\left(e^{x/c} + e^{-x/c}\right)$$

$$= s = \frac{c}{2}\left\{1 + \frac{x}{c} + \frac{x^2}{2c^2} + \ + \cdots + (1 - \frac{x}{c} + \frac{x^2}{2c^2} - \cdots)\right\}$$

$$= c + \frac{x^2}{2c} + \cdots$$

i.e. $y - c \cong \frac{x^2}{2c}$ (X) provided c is large.

In this case the curve is approximately a parabola of latus rectum $2c$

Definition (the span): The span is distance AB ,ie. the distancebetween the two hangs points $A\&B$.

If $k$is half the span, half the length of the chain is given by:

$$s = \frac{c}{2}\left\{1 + \frac{k}{c} + \frac{k^2}{2c^2} + \frac{k^3}{6c^3} + \cdots - (1 - \frac{k}{c} + \frac{k^2}{2c^2} - \frac{k^3}{6c^3} \ldots)\right\}$$

$$= \frac{c}{2}\left\{\frac{2k}{c} + \frac{k^3}{3c^3} + \cdots\right\}$$

$= k + \frac{k^3}{6c^2}$ provided c is large.

$$\therefore \quad s - k = \frac{k^3}{6c^2} \qquad (Xi)$$

<u>Definition (the sag)</u>: The sag is the difference between the coordinates of y at values of$x$for the two point$P\&B$.Or the normal distance from the lowest point P to the span line AB.

<u>The Relation between the span and sag</u>:

If h is the sag, then for $x = 0$,$y = c$ and $x = k$,$y \cong c + \frac{k^2}{2c}$ those come from (X) .Her we can get:

$$h = \frac{k^2}{2c} \qquad (*)$$

this leads to $1/c^2 = 4h^2/k^4$

then from (Xi) we have:

$$s - k = k^3/6c^2 = (k^3/6).(1/c^2) = (k^3/6).(4h^2/k^4) = 4h^2/6k$$

$$\therefore \quad 2(s - k) = (8/3)\,.(h^2/2k)$$

this means that the difference between the length of the chain $2s$ and

the span $2k$ is equal to $2(s-k) = (8/3)\,.((sag)^2/span).$ $(**)$

The equations (*)&(**) clarify two relations between the span and sag for the catenary.

Note: when c is large as mentioned above the chain or wire represents

the Lightning and telephone wires. In this case the length of the wire

$2s$ is little bigger than the span AB .So also the sag$h$ will be small.

Examples

Many problems involving catenary cables can be solved using the following formulas:

$s = c\sinh\left(\frac{x}{c}\right)$ (i) $x = c\ sinh^{-1}\left(\frac{s}{c}\right)$ (ii)

$y^2 - s^2 = c^2$ (iii) $y = c\ cosh\left(\frac{x}{c}\right)$ (iv)

$T_0 = \omega c$ $(v\ \ )$ $T = \omega y$ (vi)

$W = \omega s$ (vii)

All the parameters in the above equations have been defined before.

Example (1):

an electric power of line length $140\ m$and mass per unit length of $3\ kg/m$ is to be suspended between two towers $120\ m$ apart and of the same height. Determine the sag and maximum tension in the power line.

The solution

The sag, h, can be found from Eq(iii),provided that wecan determinethe distance, c

$y_B^2 - s_B^2 = \mathrm{c}^2$ (Eq(iii)evaluated at point B)

or

$$(h + c)^2 - (70\ m)^2 = c^2 \qquad (1)$$

The distance c ,can be determined from Eq.(i) :

$s_B = c\ \sinh(\frac{x_B}{c})$ (Eq(i)evaluated at point B)

or $70\ m = c \sinh\left(\frac{60\ m}{c}\right) \qquad (2)$

This equation must be solved numerically for c. An initialestimate for

c, whenthesolver on acalculator isto beused, could be

$$c = s_B = 70m$$

The solution to Eq.(2) is

$$c = 61.45\ m$$

Another possible solution is $c = -61.45\ m$,but this has no physical meaning. You can get the same result directly by using a modern calculator like($casio\ \ f_x - 991ES\ PLUS$).

$$(h + 61.45\ m)^2 - (70\ m)^2 = (61.45\ m)^2$$

Solving gives the sag:

$$h = 31.70\ m$$

The other negative root has no physical meaning.

The maximum tension, T_{max}, occurs where the cable has its steepest slope, point B (or point A). This can be calculated from Eq.(vi) :

$T_{max} = \omega y_B$(Eq(vi)evaluated at point B)

ω is given, then:

$$T_{max} = [(3\text{kg/m})(9.81\ \ \text{m/s}^2)][31.70\text{m} + 61.45\text{m}]$$

$$= 2740\ N = 2.74\ KN$$

Example (2):

A cable is supported at two points 400 ft apart and at the same elevation. If the sag is 40 ft and the weight per unit length of the cable is 4 lb/ft, determine the length of the cable and the tension at the low point, C.

The solution

The length of cable, s_B, from the low point to point B can be found from Eq. (i) provided that we can determine the distance c:

s_B = c sinh (x_B/c) (Eq. (i) evaluated at point B)

$= \text{c}\sinh(200/\text{c})$ (1)

The distance c can be determined from Eq.(iv)

$y_B = c\cosh(xB/c)$(Eq. (iv) evaluated at point B)

or,

c + 40 ft = c cosh (200 ft/c) (2)

This equation must be solved numerically for c. An initial estimate for c, when the solver on a calculator is to be used, could be

$$c = sag = 40\ ft$$

The solution to Eq.(2) is

$$c = 506.53\ ft$$

Using this value of c in Eq. (1) gives

$s_B = \text{c}\sinh(\frac{x_B}{c})$ (Eq. (1) repeated)

= (506.53 ft) sinh (200 ft/506.53 ft)

= 205.237 ft

Because the tension at the low point of the cable is horizontal, it can be found from Eq.(v):

$$T_0 = \omega c$$

$$= (4\ lb/ft)(506.53)$$

$$= 2.025\ lb.$$

Example (3) :

A 20-m chain is suspended between two points at the same elevation and with a sag of 6 m as shown. If the total mass of the chain 45 kg, determine the distance between the supports. Also determine the maximum tension.

The solution

The distance between the supports is $2x_B$, and x_B can befound from

Eq.(i), provided that we can determine the distance c;

$s_B = c\ \sinh(\frac{x_B}{c})$ (Eq(i)evaluated at point B)

since $s_B = 10\text{m}$, then:

$$10\text{m} = c\ \sinh(\frac{x_B}{c})$$

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This equation can be solved explicitly for x_B by rearranging it as

$$\sinh\left(\frac{x_B}{c}\right) = \frac{10\mathrm{m}}{c}$$

Which implies:

$$\frac{x_B}{c} = sinh^{-1}\left(\frac{10\ m}{c}\right)$$

So $\qquad x_B = c\ sinh^{-1}\left(\frac{10\ m}{c}\right) \qquad (1)$

The distance c can be determined from Eq.(ii):

$y_B^2 - s_B^2 = c^2 \qquad$ (Eq(iii)evaluated at point B)

$$(6\ m + c)^2 - (10\ m)^2 = (c)^2$$

or $36 + 12c + c^2 - 100 = c^2$

The c^2 terms cancel and resulting linear equation has the solution:

$$c = 5.333\ m$$

Substituting this value of c into Eq.(1)gives:

$$x_B = 5.333\ m\ sinh^{-1}(10\ m/5.333\ m) = 7{,}393\ m$$

Thus, the distance between supports $2x_B$ can be found:

$2x_B = 2(70393\ m) = 14.786\ m.$

The maximum tension, T_{max}, occurs where the slope of the cable is a maximum, at point B (or point A). This can be calculated from Eq.(vi):

$T_{max} = \omega y_B$(Eq(vi)evaluated at point B)

$$= \left(\frac{Tatol\ weight\ of\ the\ cable}{Tatol\ length\ of\ the\ cable}\right) y_B$$

$$= \left(\frac{(45.Kg)(9{,}81\ m/s^2)}{20\ m}\right)(6\ m5.333\ m) = 250\ N\ .$$

<u>Example (4):</u>

A certain cable will break if the maximum tension exceeds 500 N. If the cable is 50-mlong and has a mass of 50 kg, determine the greatest span possible. Also determine the sag.

<u>The solution</u>

The maximum tension has been specified (500 N) ,so a good place to start our solution is to see how we can use the fact that $T_{max} = 500$ N .Eq.(vii)relates the tension,T ,to they , coordinate of a point on the curve:

$$T = \omega y \text{ (Eq. (vi) repeated)}$$

The maximum tension, T_{max}, occurs where the cable has its steepest slope, point B (or point A). This can be calculated from Eq.(vi) :

$T_{max} = \omega y_B$(Eq(vi)evaluated at point B)

Thus, because we know the maximum tension, we can compute y_B :

$$y_B = \frac{T_{max}}{\omega} = \frac{T_{max}}{\left(\frac{Tatol\ weight\ of\ the\ \text{cable}}{Tatol\ length\ of\ the\ cable}\right)}$$

$$= \frac{500\ N}{\left(\frac{(50\ Kg)(9{,}81\ m/s^2)}{50\ m}\right)} = 50.97\ m$$

The distance between supports is $2x_B$, so we need to use the value of

y_B to determine x_B.this can be done by using Eq.(vi).provided that we can determine c :

$$y_B = c\cosh(x_B/c) \text{(Eq. (iv) evaluated at point B)}$$

We can solve this equation explicitly for x_B by rewriting it as:

$$\cosh(x_B/c) = y_B/c$$

So

$$x_B = c\ cosh^{-1}(y_B/c) \qquad (2)$$

The distance c , can be calculated from Eq.(iii) :

$y_B^2 - s_B^2 = \text{c}^2$ (Eq(iii)evaluated at point B)

$$(50.97\ m)^2 - (25\ m)^2 = (c)^2$$

The solution is

$$c = \pm 44.42\ m$$

The negative root has no physical meaning.

Substituting the value of $c = 44.42\, m$ and $y_B = 50.97$ into Eq.(2)gives:

$$x_B = 44{,}42\, m\, cochh^{-1}(50.97\, m/44.42\, m) = 23.836\, m$$

So, the distance between supports $2x_B$isknown:

$2x_B = 2(23.836\, m) = 47.7\, m.$

Since c and y_B are known, the sag can be computed:

$$h = y_B - c$$

$= (50.97m) - (44.42\, m) = 6.55\, m$.

<u>Example (5):</u>

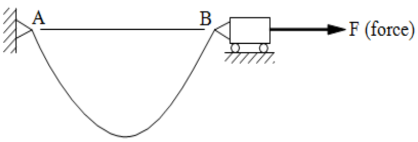
The cable is attached to a fixed support at A and a moveable support

at B. If the cable is80-ft long, weighs 0.3 lb/ft, and spans 50 ft, determine the force F holding the moveable support in place. Also determine the sag.

<u>The solution</u>

The force F acting on the moveable

support at$B$ equals the horizontal component,T_0, of tension in the cable,



$F = T_0$. Eq.(v) can be used to calculate T_0 , provided that we can

determine the distance c :

$T_0 = \omega c$ (Eq. (v) repeated)

$$= (0.3\ lb/ft)\ c = F \qquad (1)$$

The distance c ,can be calculated from Eq.(i) :

$s_B = c\ \sinh(\frac{x_B}{c})$ (Eq(i)evaluated at point B)

since $s_B = 40$ ft ,

then:

$$40\text{ft} = c\ \sinh(25\ \text{ft}/c) \qquad (2)$$

This equation must be solved numerically for c. An initial estimate for c, when the solver on a calculator is to be used, could be

$$c = x_B = 25\ ft$$

The solution to Eq.(2) is

$$c = \pm 14.229\ ft$$

The negative root has no physical meaning.

Using $c = 14.229\ ft$ in Eq.(1)gives:

$T_0 = \omega c$ (Eq. (v) repeated)

$$= (0.3\ lb/ft)(14.229\ ft)$$

$$= 4.27\ ft$$

The sag,h , can be calculated from Eq.(iv)and the known value of c :

$$h = y_B - c \quad = c\ \cosh(x_B/c) - c$$

$$= (14.229\ ft)\cosh(25.ft/14.229ft) - 14.229ft$$

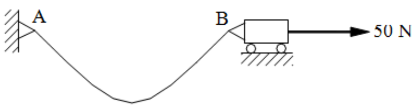
$= 28.2\ ft$.

Example (6):

. The cable is attached to a fixed support at A and a moveable support at B. If the cable is 40 m long, and has mass of 0.4Kg/m. If the force F holding the moveable support at the B is equal to $50\ N$ in the horizontal direction, determine the span and the sag.

The solution

The span is $2x_B$, and x_B can be found from Eq.(i), provided that we can determine The distance c;



$s_B = c\ \sinh(\frac{x_B}{c})$ (Eq(i)evaluated at point B)

This equation can be solved explicitly for x_B by rearranging it as

$$\sinh\left(\frac{x_B}{c}\right) = \frac{s_B}{c}$$

Which implies: $\frac{x_B}{c} = sinh^{-1}\left(\frac{s_B}{c}\right)$

So $x_B = c\ sinh^{-1}\left(\frac{s_B}{c}\right)$

Then $x_B = c\ sinh^{-1}\left(\frac{20\ \mathrm{m}}{c}\right)$ (1)

Because the 50 N force acting on the moveable support equals the horizontal component, T_0 , of the tension in the cable, Eq.(v) with $T_0 = 50\ N$ can be used to solve for c :

$\mathrm{T_0} = \omega\mathrm{c}$ (Eq. (v) repeated)

or $$50\ N = [(0.4\ Kg/m)(9.81 m/s^2)]c$$

solving gives:

$$c = 12.742\ m$$

Using this value of c in Eq. (1)gives:

$x_B = c\ sinh^{-1}(20\ \mathrm{m}/c)$(Eq. (1) repeated)

$$= (12.742\ m) sinh^{-1}(20\ \mathrm{m}/12.742\ m) = 15.708\ m$$

so, the span is

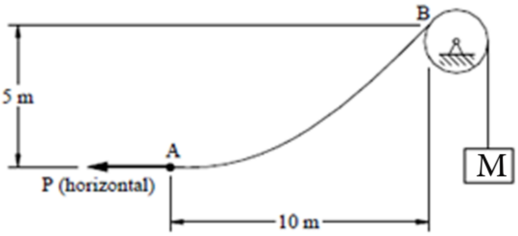
$$span = 2x_B = 2(15.708\ m) = 31.4\ m$$

The sag,h , can be calculated from Eq.(iv) and the known value of c :$h = y_B - c = c\ \cosh(x_B/c) - c$

$$= (12.742\ m)\cosh(15.708. ft/12.742\ m) - 12.742\ m = 28.2 m$$

Example (7):

A cable goes over a frictionless pulley at B and supports a block of mass M. The other end of the cable is pulled by a horizontal force P.



If the cable has a mass per length of 0.3 kg/m, determine values of P and M that will maintain the cable in the position shown.

The solution

The force P equals T_0 ,the horizontal component of the cable tension

given $T_0 = \omega c$ (Eq. (v) repeated)

so, with $T_0 = P$ then:

$$P = \omega c \qquad (1)$$

Here:

$$\omega = (0.3\ Kg/m)(9.81\ m/s^2)$$

$$= 2.943\ N/m \qquad (2)$$

The value of c in Eq.(1) can be found from Eq.(iv):

$$y_B = c\cosh(x_B/c) \text{(Eq. (iv) evaluated at point}$$

or

$$5\ m + c = c\cosh(10\ m/c)$$

Solving numerically gives:

$$c = 10.743\ m$$

Using this value of c in Eq.(1)gives:

$$\mathrm{P} = \omega \mathrm{c}$$

$$= (2.943\ N/m)(10.743\ m)$$

$$= 31.617\ N$$

The cable tension at B must equals the weight,mg :

$$T_B = Mg$$

thus, the mass is

$$M = T_B/g$$

By Eq.(vi)

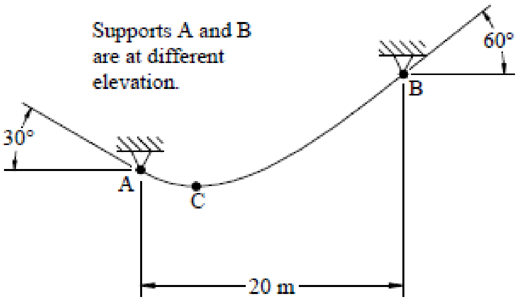
$$M = \omega y_B/g$$

By Eq. (2)

$$M = [(2.943\ \ N/m)(5\ m + 10.743\ m)]/(9.81\ N/m^2)$$

$$= 4.72\ Kg$$

Example (8):



A chain makes angles of 30° and 60° at its supports as shown.

Determine the location of the low point C of the chain relative to A. Also determine the tension at support A, if the cable has a mass per length of 0.6 kg/m.

The solution

The geometric data are shown in the figure. To determine the location of the low point C relative to A, we need to determine the coordinates xA and yA. We can get an equation

for xA by using the fact that the slope is known at A:

$$-\tan 30^\circ = \left[\frac{dy}{dx}\right]_{atA} = \left[\frac{d(c\cosh(x/c))}{dx}\right]_{atA} \qquad \text{by Eq.}(iv)$$

$$= \sinh(x_A/c)$$

Solving for x_A gives:

$$x_A = c\sinh^{-1}(-\tan 30^\circ) \qquad (1)$$

Similarly at point B , we have

$$x_B = c\sinh^{-1}(\tan 60^\circ) \qquad (2)$$

The coordinates x_A and x_B are related to the 20-m span through the equation:

$$x_A - x_B = 20\text{ m}$$

By substituting from Eqs.(1)&(2) we get:

$$c\sinh^{-1}(-\tan 30^\circ) - c\sinh^{-1}(\tan 60^\circ) = 20$$

Since this equation is linear in c, it is easily solved to give c = 10.717 m. Eq. (1) then gives

$x_A = c\sinh^{-1}(-\tan 30^\circ)$(Eq (1) repeated)

$$= (10.717\ m)\sinh^{-1}(-\tan 30^\circ)$$

$$= -5.887\ m$$

The y coordinate of point A can now be calculated fromEq. (iv):

$y_A = c\ \cosh(x_A/c)$ (Eq.(iv) evaluated at point A)

$$= (10.717\ m)\cosh(-5.887\ m/10.717m)$$

$$= 12.375\ m \qquad (3)$$

The vertical distance between support A and the low point C is given by

$$d = y_A - c$$

$$= 12.375\ m - 10.717\ m$$

$= 1.658\ m$ (by Eq. (3))

The tension at A is given by Eq. (vi):

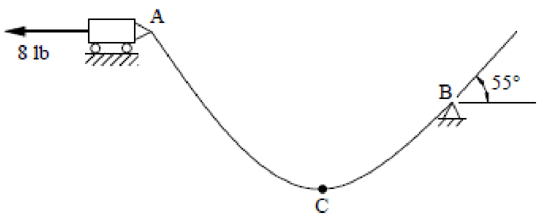
$T_A = \omega y_A$ (Eq.(vi) evaluated at point A)

$= [(0.6\ Kg/m)(9.81\ m/s^2)](12.375\ m) = 72.8\ N.$

Example (9):

A wire weighting 0.2 lb/ft is attached to a moveable support at A and makes an angle of 55° at a fixed support at B. Supports A and B are at different elevations. Determine the location of the low point C of the wire relative to support B. Also, determine the tension in the wire at C.

The solution



To determine the location of the low point, C , relative to the support at B, we need to determine the coordinates x_B and y_B . We can get an equation for x_B by using the fact that the slop is known at B.

$$\tan 55^\circ = \left[\frac{dy}{dx}\right]_{atB}$$

$= \left[\frac{d(c\cosh(x/c))}{dx}\right]_{at\boldsymbol{B}}$ by Eq.(iv)

$$= \sinh(x_B/c)$$

Thus

$$x_B = c\sinh^{-1}(\tan 55^{\circ}) \qquad (1)$$

The value of c occurring in Eq. (1) can be found by observing that the 8-lb force acting at support A equals T_0 the horizontal component of tension at A, so Eq. (v) gives

$\mathrm{T_0 = \omega c}$ (Eq. (v) repeated)

$$\therefore 8\ Ib = (0.2\ \omega\ Ib/ft)c$$

Solving gives:

$$\mathrm{C = 40\ ft} \qquad (2)$$

Using this result,$C = 40\ ft$ in Eq.(1)gives:

$x_B = c\sinh^{-1}(\tan 55^{\circ})$(Eq. (1) repeated)

$$= (40\ ft)\sinh^{-1}(\tan 55^{\circ})$$

$$= 46.169\ ft$$

The vertical distance between B and C is:

$$d = y_B - c$$

$$= c\cosh(x_B/c) - c$$

$$= (40\ ft)\cosh(46{,}169\ ft/40\ ft) - 40\ ft$$

$$= 29.7\ ft$$

Since point C is the low point of the cable, the tension there is horizontal and so must equal the horizontal component of tension at A

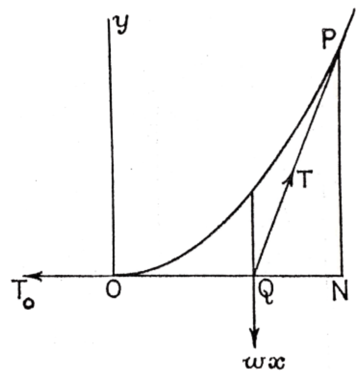
which is known to be 8 Ib that is:

$T_c = 8\ Ib$.

Worked examples

Example (1): (The suspension bridge)

If a chain supports a continuous load, uniformly distributed, the chain hangs in the form of a parabola. O is the lowest point of the chain and P any point of the chain whose coordinates referred to horizontal and



vertical through O are (x, y) The weight carried by the portion OP will be proportional to ON and acts through Q the midpoint of ON. We may call it ωx .

The other forces acting on the portion OP are T_0 the horizontal tension At O and the tension$T$ at P, three of them must therefore meet at Q and PNQ is a triangle of forces.

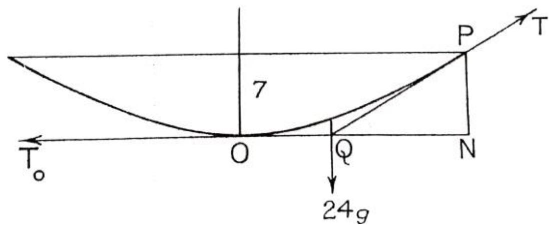
$$\therefore \quad \frac{\omega x}{PN} = \frac{T_0}{NQ} \qquad \therefore \quad T_0 y = \frac{1}{2}\omega x^2$$

Hence, if we denote T_0 by ωc , then we can get $y = \frac{x^2}{2\,c}$

this means that the curve of the chain is a parabola.

Now if the span of a suspension bridge is $96\ m$ and the sag in the chain is $7\ m$ .TheTwo branches of the chain support a load of $1000\ kg$per horizontal meter. Find the tension at the lowest and highest points. The load carried by OP is $24gkN$ .The triangle QPN is a triangle of forces.

<u>The solution</u>



$$QN = 24\,m \ \ , \ \ PN = 7\,m \qquad \therefore Qp = 25\,m$$

$$\frac{T_0}{24} = \frac{T}{25} = \frac{24g}{7}$$

$T = 840\ kN \quad , \quad T_0 \cong 810\ kN$.

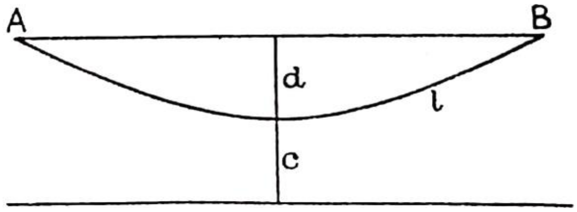
==

Example (2):

A uniform chain of length $2l$ and weight ω per unit length is suspended between two points at the same level and has a maximum depth$d$.Prove the tension at the lowest is $\omega\ (l^2 - d^2)/2d$.If $l = 50\ m$ and $d = 20\ m$ find the distance between the points of suspension.

The solution

For the catenary $y^2 = c^2 + s^2$,



At $By = c + d \quad , \quad s = l$

$$\therefore \quad (c + d)^2 = c^2 + l^2$$

$$2cd = l^2 - d^2$$

$$\therefore \quad c = l^2 - d^2/2d$$

$\therefore$ the tension at the lowest is $= \omega c = \omega\ (l^2 - d^2)/2d$.

If $l = 50\ m$ and $d = 20\ m$, then $c = 2500 - 400/40 = 105/2$

Now $s = c \sinh x/c$

Hence if $AB = 2x$,

$$\therefore \quad x = (105/2)\sinh^{-1}(20/21)$$

$$= (105/2) lin\left[(20/21) + \sqrt{\{1 + (20/21)^2\}}\right]$$

$$= (105/2) lin(49/21)$$

$\therefore \quad AB = 105 \times 2.303\ log_{10}(49/21) \cong 89\ m$.

==

EXERCISES:

(1)A rope has an effective length of $20\ m$and mass5 kg per miter. One end of the rope is $4\ m$ higher than the other. Find the maximum tension in the rope when the tangent at the lower end is horizontal.

(2)A uniform chain of length $2l$ has its ends fixed at two points at the same level. The sag at the middle is h .prove that the span is $[(l^2 - h)/h] lin[(l + h)/(l - h)]$.

(3)A uniform wire hangs freely from tow points at the same level $200\ m$apart. The sag is $15\ m$. Show the greatest tension is approximately $348\ \omega$ and the length of wire is approximately $203\ m$.

(4)Find approximately the greatest tension in a wire which has mass $100\ g$ per miter when it hangs with a sag of $25\ cm$ when stretched between two points at the same level $40\ m$apart.

(5)A uniform heavy chain of length $31\ m$ is suspended from tow points at the same level and$30\ m$apart. Show that the tension at the lowest point is about 1.08times the weight of the chain.

Chapter (2)

Direct Stress and Strain

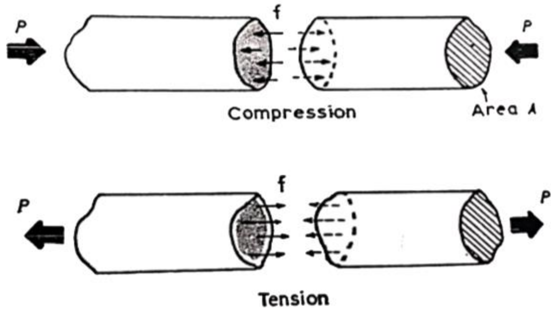
----

(1) Stress:

The ability of a structural member to withstand load or transmit force, as in a machine, depends upon its dimensions. In particular, the cross-sectional area over which the load is distributed determines the intensity or average stress in the member. If the intensity of loading is uniform the direct stress, , f is defined as the ratio of load, P ,to cross-sectional area,A , normal to the load as shown in the Fig. Thus:

$$\text{stress} = \frac{\text{load}}{\text{area}}$$

or
$$\mathrm{f} = \frac{\mathrm{P}}{\mathrm{A}}$$



If the load is in pounds and the area in square inches the units of stress are pounds per square inch $(Ib/in.^2)$.There are another unit:

If,P , is expressed in Newton (N) , and A, original area, in square meters(m^2) ,the stress,f , will be expresses in N/m^2 , this unit is called Pascal (Pa) .

As Pascal is a small quantity in practice, multiples of this unit is used.

$$1\ KPa = 10^3 Pa = 10^3\ N/m^2\ (KPa = Kilo\ Pascal)$$

$$1\ MPa = 10^6 Pa = 10^6\ N/m^2$$

$$= 1\ N/mm^2\ (MPa = Mega\ Pascal)$$

$$1\ GPa = 10^9 Pa = 10^9\ N/m^2\ (GPa = Giga\ Pascal)$$

The direct stress may be tensile or compressive according as the load is a pull (tension), or push (compression). It is often convenient to consider tensile stresses and loads as positive and compressive stresses and loads as negative.

(2) Strain:

A member under any loading experiences a change in shape or size inthe case of a bar loading in tension the extension of the bar depends upon its total length. The bar is said to be strained and the strain is defined as the extension per unit of original length of the bar. Strain may produced in two ways:

1- By application of a load.

2- By a change in temperature, unaccompanied by load or stress.

If l is the original length of the bar, x the extension or contraction in length under load or temperature change, and $e$the strain, then:

$$\text{strain} = \frac{\text{change in length}}{\text{original lenth}}$$

or $$e = \frac{x}{l}$$

Strain is a ratio and has therefore no units.

Strain due to an extension is considered positive, that associated with a contraction is negative.

<u>(3) Relation between Stress and Strain</u>:

If the extension or compression in a member due to a load disappears on removal of the load, then the material is said to be elastic. Most metals are elastic over a limited range of stress known as the elastic range. Elastic materials, with some exception, obey Hooke's, which states that: the strain is directly proportional to the applied stress Thus

$$\frac{stress}{strain} = constant\ (E)$$

i.e.$\frac{f}{e} = E$or $e = \frac{f}{E}$

where E is the constant of proportionality, known as the modulus of elasticity or Young's modulus?

Since strain is a ratio, the units of $E$are those of stress, i.e. pounds per square inch.

Examples

Example (1):

A rubber pad for a machine mounting is to carry a load of $1000\ Ib$ and to compress$0.2in$. If the stress in the rubber is not exceed $40\ \ Ib/in.^2$, determine the diameter and thickness of a pad of circular cross-section.

Take E for rubber as $150\ \ Ib/in.^2$.

The solution

$$\text{stress} = \frac{\text{load}}{\text{area}}$$

i.e $$\text{f} = \frac{\text{P}}{\text{A}}$$

$$40 = \frac{1000}{\pi \text{d}^2/4}$$

hence $d^2 = 31.83\ in.^2$ and $d = 5.64\ in$

i.e. diameter of pad=5.64 in.

The increase in area due to compression has been neglected.

Also $stress = \frac{reduction\ in\ length}{original\ length}$

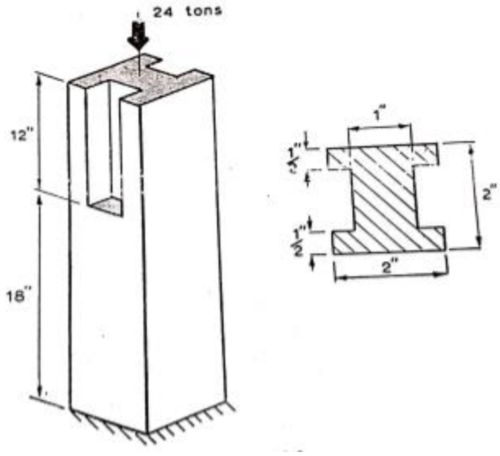
then $\frac{f}{E} = \frac{x}{l}$ this leads to $\frac{40}{150} = \frac{0.2}{l}$

therefore, thickness of pad is given by

$$l = 0.75\ in.$$

Example (2):

The Fig shows a steel strut with tow grooves cut out along part of its length. Calculate the total compression of the strut due to a load of $24\ tons. E = 12500\ \ ton/in.^2$



The solution

Suffices 1 and 2 denote solid and grooved portions, respectively. the load at every section is the same, $24\ ton$.

For the solid length of $18\ in.$

compression $x_1 = e_1 l = e_1 \times 18$

$$\text{stress}, f_1 = \frac{P}{A_1} = \frac{24}{2 \times 2} = 6\ tons/in.^2$$

$$\text{strain}, e_1 = \frac{f_1}{E} = \frac{6}{E}$$

For the grooved length 0f $12\ in.$

compression $x_2 = e_2 l = e_2 \times 12$

$$\text{stress}, f_2 = \frac{P}{A_2} = \frac{24}{(4-1)(1)} = 8\ tons/in.^2$$

$$\text{strain}, e_2 = \frac{f_2}{E} = \frac{8}{E}$$

The total compression of the strut is equal to the sum of the compressions of the solid and grooved portions. Therefore

$$x = x_1 + x_2$$

$$= (e_1 \times 18) + (e_2 \times 12)$$

$$= \left(\frac{6}{E} \times 18\right) + \left(\frac{8}{E} \times 12\right)$$

$$= \frac{204}{E}$$

$$= \frac{204}{12500}$$

$$= 0.0163\ in.$$

Note: It has been assumed here that the stress distribution is uniform over all sections, but at the change in cross-section the stress distribution is actually very complex. The assumption produces little error in the calculated compression.

Example (3):

A rod $10\ mm \times 10\ mm$ cross-section is carrying an axial tensile load $10\ KN$. In this rod the tensile stress developed is given by:

$$f = \frac{P}{A} = \frac{10\ KN}{(10\ mm \times 10\ mm)} = \frac{10 \times 10^3\ N}{100\ mm^2} = 100\ MPa$$

Example (4):

A rod $100\ mm$ in original length. When we apply an axial tensile loa $10\ KN$ .The final length of the rod after application the tensile is $100.1\ mm$ .So in this rod tensile strain is developed and is given by;

$$e = \frac{x}{l} = \frac{100.1mm - 100\ mm}{100\ mm} = \frac{0.1mm}{100\ mm} = 0.001(\text{Dimenionless})\text{Tensile.}$$

Example (5):

A rod $100\ mm$ in original length. When we apply an axial compressive load $10\ KN.$ .The final length of the rod after application compressive is $99\ mm$ .So, in this rod compressive strain is developed and is given by;

$$e = \frac{x}{l} = \frac{99\ mm - 100\ mm}{100\ mm} = \frac{-0.1mm}{100\ mm} = -0.001(\text{Dimenionless})\text{Tensile.}$$

===

Exercises

(1) A bar of 1 $in.$ diameter is subjected to a tensile load of 10000 Ib.Calculate the extensionon a 1 $ft.$ length. $E = 30 \times 10^4\ Ib/in.^2$.

(2)A light alloy bar is observed to increase in length by 0.35 per centwhen subjected to a tensile stress of 18 $ton/in.^2$ .Calculate Young' modulus for the material.

(3)A duralumnin tie, 2 ft long 1.5 $in.$ diameter, has a hole drilled out along its length .The hole is of 1 $in.$diameter and 4 $in.$ long.Calculate the total extension of the tiedue to a load of 18 $tons$. $E = 12 \times 10^4\ Ib/in.^2$.

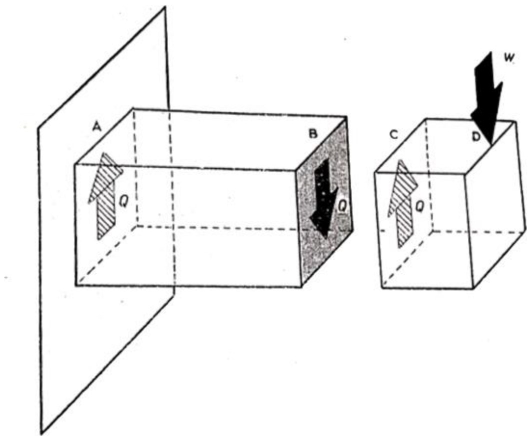
(4) A steel strut of rectangular section is made up of two lengths. The first 6 $in.$ long, has breadth 2 $in.$and depth 1.5 $in.$; the second,4 $in.$long is 1 $in.$ square. If $E = 14000\ tons/in.^2$, calculate the compression of the strut under a load of 10 $tons$.

Chapter (3)

Shear force and Bending Moment

(1) Shear force (SF):

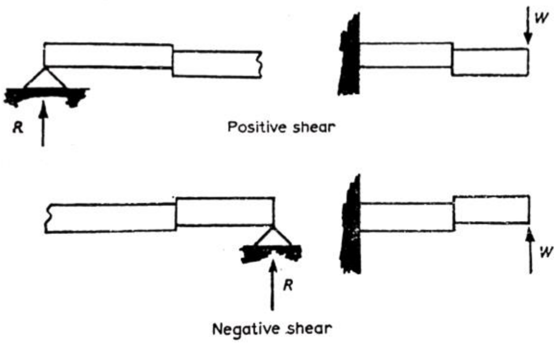
The shear force in a beam at any section is the force transverse to the beam tending to cause it to shear across the section. Fig.(3.1) shows abeam under a transverse load W at the end D ; the other end A is built in to the wall. Such a beam is called a cantilever and the load W , which is assumed to act at a point, is called a concentrated or point load.



Consider the equilibrium of any portion of beam CD . At section C for balance of forces there must be an upward force Q equal and opposite to the load W at D . This force Q is provided by the resistance of the beam to shear at the plane B ; this plane being coincident with the plane section at C . Q is the shear force at B and in this cace has the same magnitude for any section in AD . Consider now the equilibrium of the portion of beam AB . There is a downward force $Q = W$, exerted on plane B , so for balance there must be an upward force Q at A . This latter force being exerted on the beam by the wall.

Sign Convention

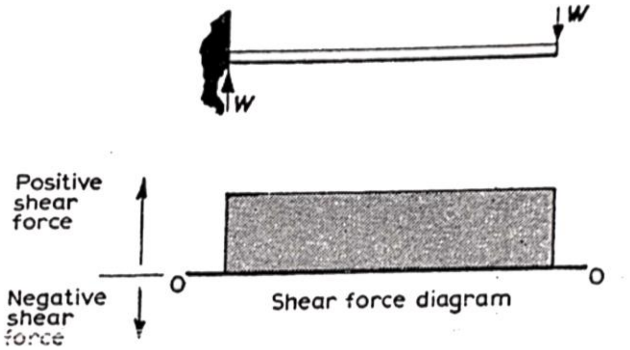
The shear force at any section is taken positive if the right-hand side tends to slide downwards relative to the lift-hand portion, fig.(3.2).A negative shear force tends to cause the right-hand portion to slide upward relative to the lift. (In some books flowed totally opposite sign convention).



If several loads act on the beam to the right-hand side of section C the shear force a C is the resultant of these loads. Thus, the shear force at any section of a loaded beam is the algebraic sum of the loads to one side of the section. It does not matter which side of the section is considered provided all loads on that side are taken into account-including the forces exerted by fixings and props.

(2) Shear Force Diagram (SFD):

The graph showing the variation of shear force along a beam is known as the shear force diagram. for the beam of Fig 3.1 the shear force was $+W$, uniform along the beam. Fig (3.3) shows the shear force diagram for this beam, $0 - 0$ being the axisof zero shear force.

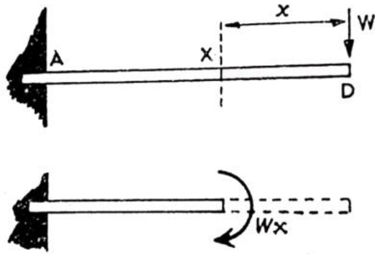


(3) Bending Moment (BM):

The bending effect at any section X of a concentrated load W at D ,

Fig.(3.4) , is measured by the applied moment Wx , where x is the perpendicular distance of the line of section of$W$ from section X .

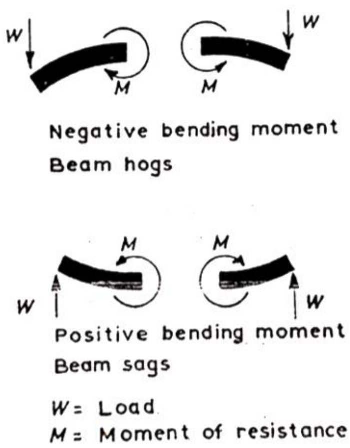
This moment is called the bending moment and is balanced by an equal and opposite moment M exered by the material of the beam at$X$, called the moment of resistance.



Sign Convention

A bending moment is taken as positive if its effect is to tend to make the beam sag at the section considered, Fig.(3.5).If the moment tends make the beam bend upward or hog at the section it is negative.

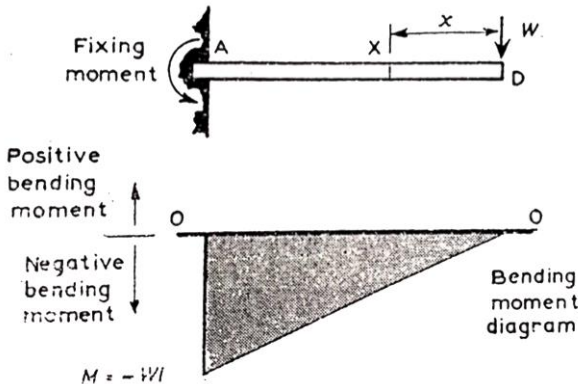
When more than one load act on a beam the bending moment at any section is the algebraic sum of the moments due to all the loads on one side of the beam. It does not matter which side of the section is considered but all loads on that side must be taken into account, including any moments exerted by fixings.



(4) Bending Moment Diagram (BMD):

The variation of bending moment along the beam is shown in a bending moment diagram. for the cantilever beam of Fig 3.1 the bending moment at any section X is given by:

bending moment$= -Wx$ (negative, since the beam hogs at X)



Since there is no other load on the beam this expression for the bending moment applies for the whole length of beam from $x = 0$ to $x = l$ .The moment is proportional to$x$ and hence the bending moment diagram is a straight line. Hence the diagram can be drawn by calculating the moment at two points and joining two corresponding points on the graph by a straight line.

At D, $x = 0$ and bending moment $= 0$

At A, $x = l$ and bending moment $= -Wl$

Since the bending moment is everywhere negative the graph plotted is below the line $0 - 0$ of zero bending moment, Fig.(3.6). At the fixed end A the wall exerts a moment Wl anticlockwise onthe beam; this is calleda fixing moment.

(5) Calculation of Beam Reactions:

When a beam is fixed at some point, or supported by props the fixings and props exert reaction forces on beam. To calculate these reactions the procedure is:

(a) equate the net vertical force to zero;

(b) equate the total moment about any convenient point to zero.

Note (1): Distinguish carefully between "taking moments" and calculating a "bending moment":

(1) The Principle of Moments states that the algebraic sum of the moments of all the forces about any point is zero, i.e. when forces on both sides of a beam section are considered.

(2) The bending Moment is the algebraic sum of the moments of forces on one side of the section about that section.

Note (2): What are the benefits of drawing shear force (SF) and bending

Moment (BM)diagram?

The benefits of drawing a variation of (SF) and (BM)in a beam as a function of $'x'$ measuredfrom one end of the beam is that it becomes easier to determine the maximum absolute value of (SF) and (BM).The (SF) and (BM) diagram gives a clear picture in our mind about the variation of (SF) and (BM) throughout the entire section of the beam.

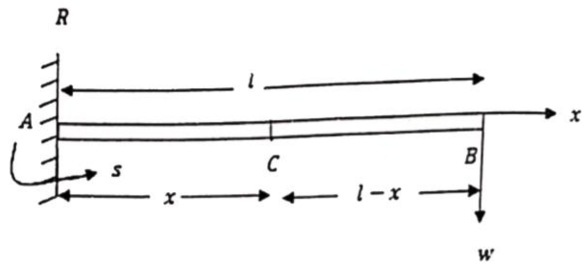
Further, the determination of value of <u>deflection</u> of beam subjected to a given loading where we will use the formula $EL\ \frac{d^2y}{d\,x^2} = M_x$.

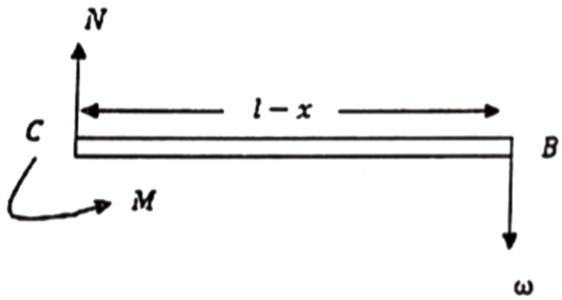
<u>Examples</u>

<u>Example (1):</u>

Draw the (SF) & (BM) diagrams at any section for a light horizontal beam AB ,its length is L .The end A of the beam is fixed at a vertical wall, while the free end $B$is loaded by a weight W .

<u>The Solution</u>





We take a section for the beam at C , where:

$$AC = x \qquad \& \qquad CB = L - x$$

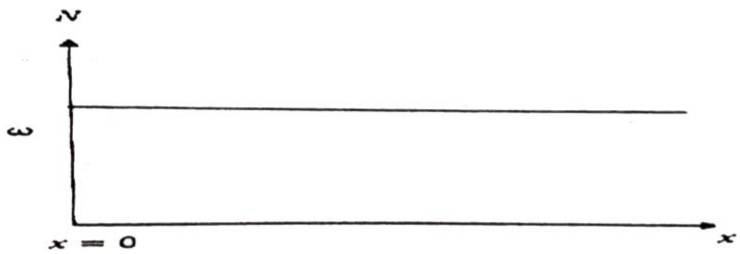
We study the equilibrium of the part CB or the part AC .

The study of the right part CB is easier than the lift part AC , because the existence of the reaction R and the couple S .

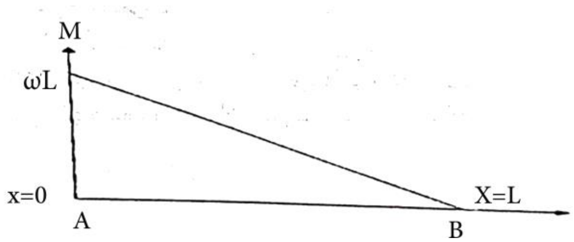
The shear force (SF) is $N = W$ (1)

and the bending moment (BM) is $M = W(L - x)(2)$

From EQ.(1) the (SF) N is constant at any section and so it is a straight line parallel to x −axis As shown in the next Fig..



But from EQ.(2) the (BN) M is depending on x , and its diagram isshown in the next Fig.



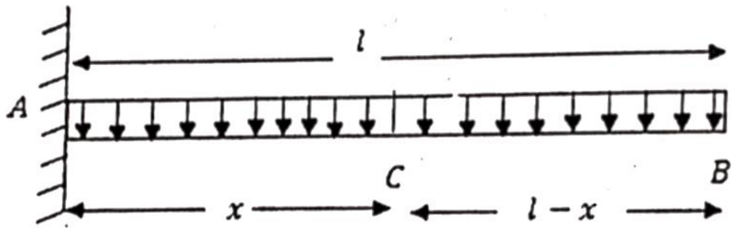
For the equilibrium of the beam AB we find that:

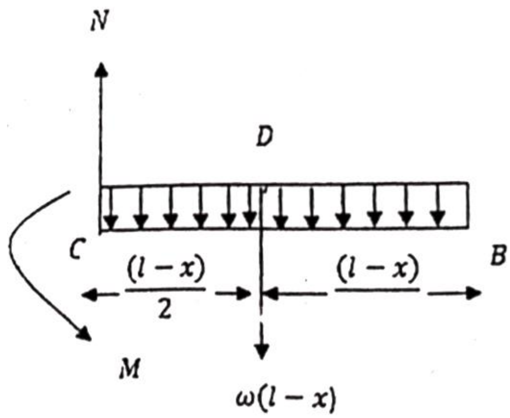
$R = W$, $S = WL.$

Example (2):

Draw the (SF) & (BM) diagrams for a light horizontal beam AB ,its length is L .The end A of the beam is fixed at a vertical wall, while the free end B is free. The beam is loaded uniformly by a weightω per unit length.

The Solution



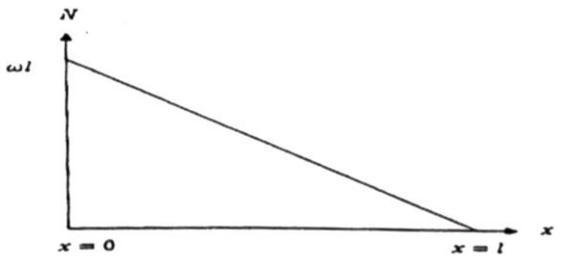


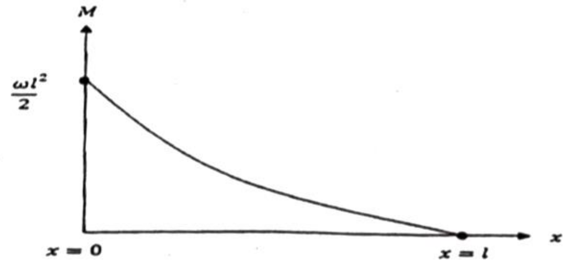
We note that the weight of the part CB is $\omega(L-x)$ and acts at its middle point. From the equilibrium of this part we find that:

The (SF) is $N = \omega(L-x) \quad (1)$,

and the (BM) is $M = \omega(L-x)\frac{(L-x)}{2} = \frac{\omega}{2}(L-x)^2 \quad (2)$.

Then the (SFD) &(BMD) will be shown in the following Figs.

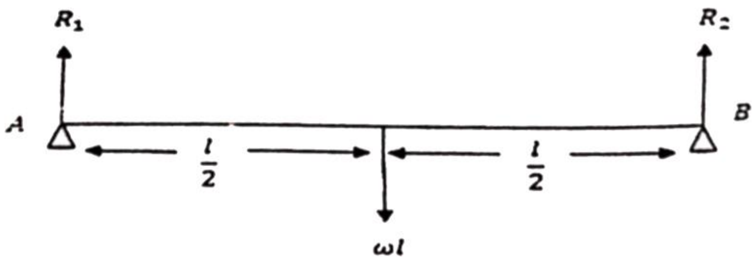




Example (3):

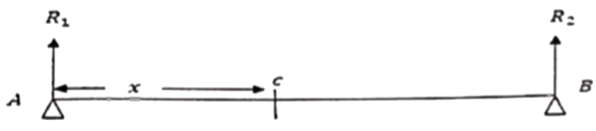
Draw the (SF) & (BM) diagrams for a heavy horizontal beam AB ,its length is L , and ω is its weigh per length. The beam is standing on two weidges in the same horizontal plane atits ends.

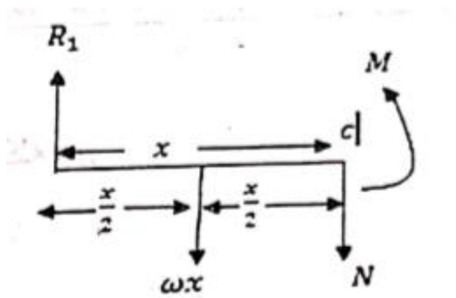
The Solution



From the symmetry we find that:

$R_1 = R_2 = \omega L/2$ Where $R_1 + R_2 = \omega L.$





By considering the equilibrium of the part AC , we find that:

$$R_1 = \omega x + N$$

$$\therefore \ \frac{1}{2}\,\omega L = \omega x + N$$

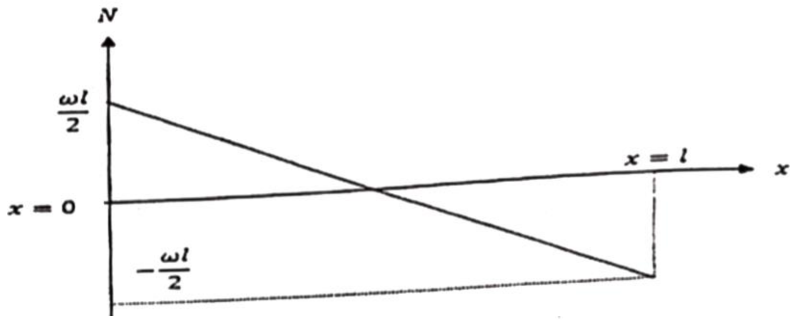
$$\therefore \ N = \ \omega\left(\frac{L}{2} - x\right)$$

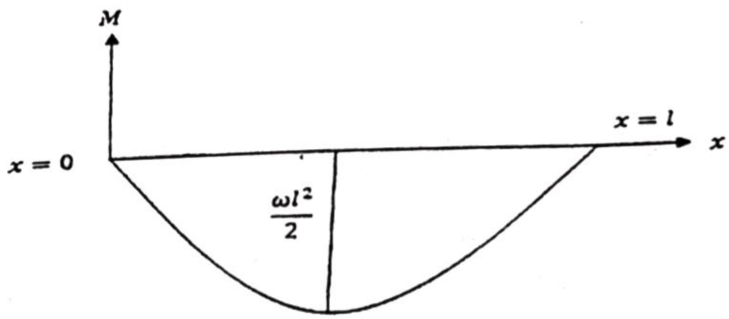
And by taking the moment about the point C , we get:

$$M + \omega x\left(\frac{x}{2}\right) = R_1 x$$

$\therefore \ M = \frac{\omega}{2}\,Lx - \frac{\omega}{2}x^2 = -\frac{\omega}{2}(x^2 - Lx).$

The (SFD) &(BMD) will be shown in the following Figs.

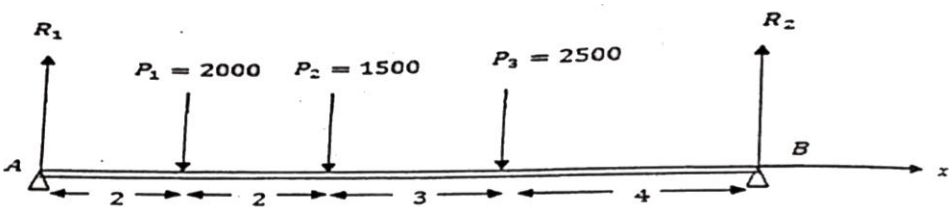




Example (4):

Draw the (SF) & (BM) diagrams for a light horizontal beam AB ,its length is 11 ft , and stands on two weidges in the same plane at its ends. The beam carries $p_1 = 2000, p_2 = 1500, p_3 = 2500\ lb$ at the three pointes a, b, c such that $Aa = ab = 2\ ft$ &$bc = 3\ ft$.

The Solution



From the equilibrium of the beam we get:

$$R_1 + R_2 = 2000 + 1500 + 2500 = 6000\ lb \qquad (1)$$

By taking the moment about the point B we get:

$$11\ R_1 = 2000 \times 9 + 1500 \times 7 + 2500 \times 4$$

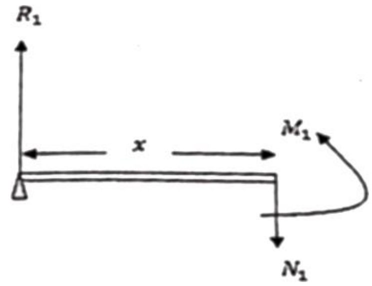
$$\therefore 11\ R_1 = 38500 \qquad \rightarrow \qquad R_1 = 3500\ lb \qquad (2)$$

By substituting from (1) in (2) we get:

$$R_2 = 2500\ lb \qquad (3)(\quad)$$

For determination the (SF) &(BM) at any point we consider the sections where:

$(i) 0 < x < 2$



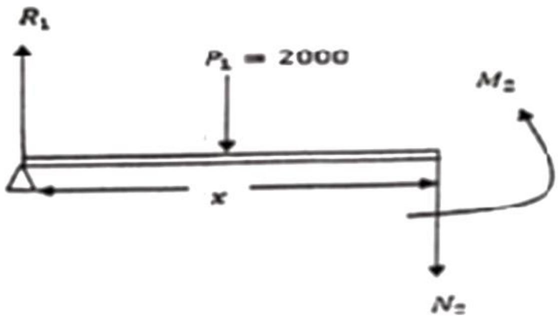
$N_1 = R_1 = 3500\ lb$

$M_1 = R_1\, x = 3500x\ lb$

$(ii) 2 < x < 4$

$N_2 = R_1 - p_1 = 3500 - 2000 = 1500\ lb$

$$M_2 = R_1\, x - 2000(x-2) = 3500\, x - 2000(x-2)\ lb$$



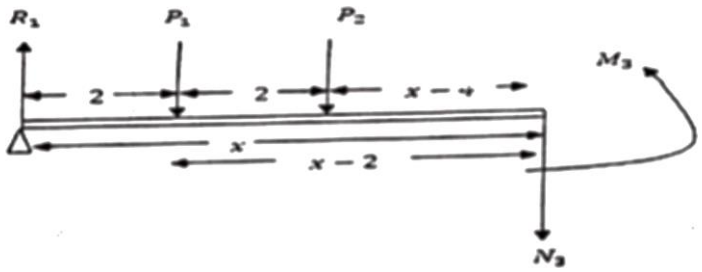
$(iii) 4 < x < 7$

$N_3 = R_1 - p_1 - p_2$

$= 3500 - 2000 - 1500 = 0\ lb$

$M_3 = R_1\, x - 2000(x-2) - 1500(x-4)$

$$= 3500\, x - 2000(x-2) - 1500(x-4)\ lb$$



$(v) 7 < x < 11$

$N_4 = R_1 - p_1 - p_2 - p_3$

$= 3500 - 2000 - 1500 - 2500 = -2500\, lb$

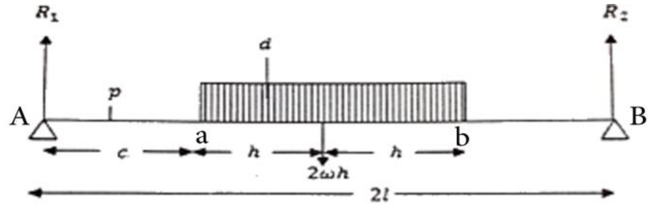
$M_4 = R_1\, x - 2000(x-2) - 1500(x-4) - 2500(x-7) -$

$= 3500\, x - 2000(x-2) - 1500(x-4) -\ lb2500(x-7)$

<u>Example (5):</u>

Find the (SF) and defined the maximum (BM) at a point d for a light horizontal beam AB ,its length is $2L$ and stands on two weidges in the same plane at its ends. The beam carries a movable weight$ab = 2h\omega$ where $2h(h < L)$is its length. Then draw (SFD) &(BMD), and prove that $\frac{ad}{ab} = \frac{Ad}{dB}$.

<u>The Solution</u>



By taking a position for the beam ABas shown in the figure such that $Aa = c$, and by finding the value of c , which makes the (BM) at d is maximum.

In case, the equilibrium of AB , we get:

$$R_1 + R_2 = 2\omega h$$

By taking the moment about the point B we get:

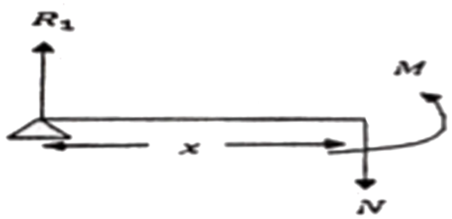
$$R_1 \times 2L = 2\omega h(2L - c - h)$$

$$\therefore\ R_1 = \frac{\omega h}{L}(2L - c - h)$$

By taking a section at $p$where,$AP = x$ & $x < c$, we get:

$$N = R_1 = \frac{\omega h}{L}(2L - c - h)$$

$M = R_1 x = \frac{\omega h}{L}(2L - c - h)x$

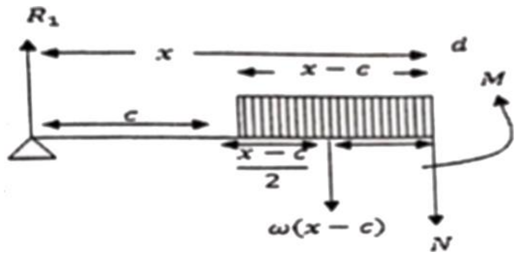


And, by taking a section at d where ,$Ad = x$ & $x > c$, we get:

$$N = R_1 - \omega(x - c)$$

$$= \frac{\omega h}{L}(2L - c - h) - \omega(x - c)$$

$$M = \frac{\omega h}{L}(2L - c - h)x - \frac{1}{2}\omega(x - c)$$



The maximum value of M will be when

$\frac{dM}{dc} = 0$, i.e.$-\frac{x\omega h}{L} + \omega(x - c) = 0$

$$\therefore \quad c = \left(1 - \frac{b}{L}\right)x$$

By substituting in M , we have:

$$M_{max} = \frac{\omega h}{L}\left(2L - h - x\left(1 - \frac{h}{L}\right)\right)x - \frac{1}{2}\omega\left(x - x\left(1 - \frac{h}{L}\right)\right)x$$

In this case, we find that :

$$\frac{ad}{db} = \frac{x-c}{2h-(x-c)} = \frac{\frac{h}{L}x}{2h-\frac{h}{L}x} = \frac{hL(x)}{\frac{h}{L}(2L-x)} = \frac{x}{2L-x} = \frac{Ad}{dB} .$$

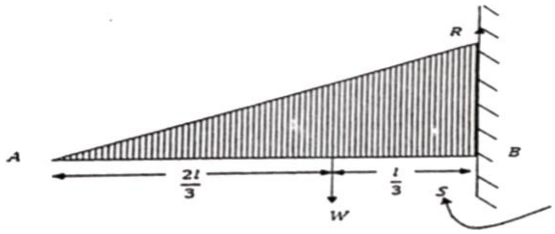
Example (6):

AB is a beam, its length is L , and the end $B$is fixed at a vertical wall. The beam is loaded by a weight Wdistributedlinearly, by uniformly increasing, starting from zero at the free end A.Find the (SF) & (BM) then draw its diagrams.

The Solution

The density of loading is $\omega = \omega(x)$ at the section C ,where $AC = x$,

Then $\omega = \gamma x$ (linearly distribution)

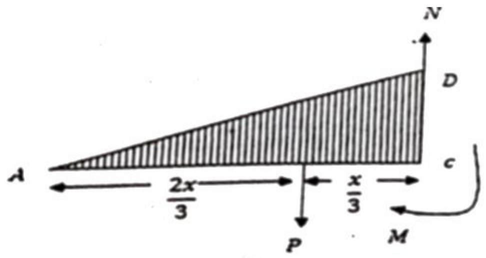


$W = \int_0^L \omega\, dx = \int_0^L \gamma x\, dx$

$\therefore W = \gamma L^2/2 \therefore \gamma = 2W/L^2$

$\therefore \omega = \gamma x = (2W/L^2)x$

For the section AC , we get



$$N = P = \int_0^x \omega\, dx = \int_0^x (2Wx/L^2)\, dx$$

$\therefore N = (Wx^2/L^2)$

We note that the weight P dived AC by the ratio

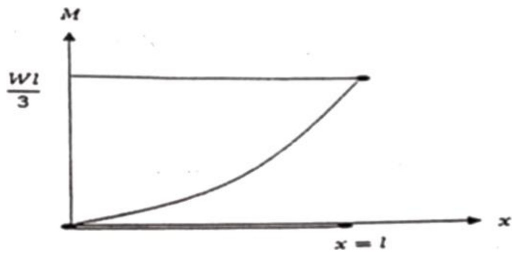
$AE = 2EC = 2x/3$

By taking the moment about C we find that

$$M = P(x/3) = (Wx^2/L^2)(x/3) = (Wx^3/3L^3)$$

We note that $R = W$, $S = WL/3$ and

$AF = 2FB = (2/3)L$.



Exercises

(1)Draw the (SF) & (BM) diagrams at any section for a light horizontal beam AB ,its length is L .The end A of the beam is fixed at a vertical, while the free end B is loaded by a weight ωL .

(2)Draw the (SF) & (BM) diagrams for a light horizontal beam AB ,its length is L , and stands on two weidges in the same plane at its ends. The beam carries two equal weights$p_1 = p_2 = \omega$ at the two points $C\&D$ such that $AC = DB = a,\ (a < L/2)$.

(3)Find and draw the (SF)&(BM) for a light horizontal beam AB ,its length is$10\ ft$and stands on two weidges in the same plane at its ends. The beam is loaded by a uniformally distributed weight, where $\omega = 10\ lb$ per unit length.