Faculty of Science

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Mathematics Department

Dynamics: Lecture Notes

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Introduction

Kinematics is the branch of dynamics which describes the motion of bodies without reference to the forces which either cause the motion or are generated as a result of the motion.

Kinematics is often described as the "geometry of motion." Some engineering applications of kinematics include the design of cams, gears, linkages, and other machine elements to control or produce certain desired motions, and the calculation of

flight trajectories for aircraft, rockets, and spacecraft. A thorough working knowledge of kinematics is a prerequisite to kinetics, which is the study of the relationships between motion and the corresponding forces which cause or accompany the motion.

3.1.1 Definition of a particle

A `Particle' is a point mass at some position in space. It can move about, but has no characteristic orientation or rotational inertia. It is characterized by its mass.

Examples of applications where you might choose to idealize part of a system as a particle include:

1. Calculating the orbit of a satellite – for this application, you don't need to know the orientation of the satellite, and you know that the satellite is very small compared with the dimensions of its orbit.

2. A molecular dynamic simulation, where you wish to calculate the motion of individual atoms in a material. Most of the mass of an atom is usually concentrated in a very small region (the nucleus) in comparison to inter-atomic spacing. It has negligible rotational inertia. This approach is also sometimes

used to model entire molecules, but rotational inertia can be important in this case.

Obviously, if you choose to idealize an object as a particle, you will only be able to calculate its position. Its orientation or rotation cannot be computed.

Newton's Laws

Newton published his three laws in 1687 in his Principia Mathematica. The laws are fairly intuitive, although it seems a bit strange to attach the adjective "intuitive" to a set of statements that took millennia for humans to write down. The laws may be stated as follows.

First Law: A body moves with constant velocity (which may be zero) unless acted on by a force.

Second Law: The time rate of change of the momentum of a body equals the force acting on the body.

Third Law: The forces two bodies apply to each other are equal in magnitude and opposite in direction.

Position: The straight-line path of a particle will be defined using a

single coordinate axis x

Displacement: The displacement of the particle is defined as the change in its position. For example, if the particle moves from one point to another point.

Velocity. If the particle moves through a displacement s during the time interval t, the average velocity of the particle during this time

Acceleration. Provided the velocity of the particle is known at two points, the average acceleration of the particle during the time interval t is (a).

Motion in one dimension with constant acceleration

Constant acceleration

Motion with constant acceleration is a motion for which instantaneous acceleration is a constant function

If A particle start for the origin point with velocity υ_0 and constant acceleration proof that. $\upsilon = \upsilon_o + a_c\, t, \quad x = \upsilon_o\, t + \frac{1}{2} a_c\, t^2, \quad \upsilon^2 = \upsilon_o^2 + 2\, a_c\, x$

<u>Proof</u>

We start with

$$a = \frac{d\upsilon}{dt} \qquad \rightarrow \qquad a_c = \frac{d\upsilon}{dt}$$

$$d\upsilon = a_c\, dt \rightarrow \int d\upsilon = a_c \int dt$$

(1) $$\upsilon = a_c\, t + c_1$$

From the initial condition, we have $c_1 = \upsilon_0$ .Then

$$\upsilon = \upsilon_0 + a_c\, t \qquad (2)$$

For Eq. (2) and where $\upsilon = \frac{dx}{dt}$ then

$$\frac{dx}{dt} = \upsilon_0 + a_c\, t \rightarrow dx = \left\{ \upsilon_0 + a_c\, t \right\} dt \rightarrow$$

$$\int dx = \int \upsilon_0\, dt + \int a_c\, t\, dt \rightarrow \int dx = \upsilon_0 \int dt + a_c \int t\, dt$$

$$x = \upsilon_0\, t + \frac{1}{2} a_c\, t^2 + c_2 \qquad (3)$$

From the initial condition , we find that $c_2 = 0$ and in Eq. (3)

$$x = \upsilon_0\, t + \frac{1}{2} a_c\, t^2 \qquad (4)$$

Form Eq. (2) $t = \frac{\upsilon - \upsilon_0}{a_c}$ and into Eq. (4), we have

$$x=\upsilon_0\left\{\frac{\upsilon-\upsilon_0}{a_c}\right\}+\frac{1}{2}a_c\left\{\frac{\upsilon-\upsilon_0}{a_c}\right\}^2$$

$$x=\left\{\frac{\upsilon\upsilon_0-\upsilon_0^2}{a_c}\right\}+\frac{1}{2}a_c\left\{\frac{\upsilon^2-2\upsilon_0\upsilon+\upsilon_0^2}{a_c^2}\right\}$$

$$2x=\frac{2\upsilon\upsilon_0-2\upsilon_0^2}{a_c}+\frac{\upsilon^2-2\upsilon_0\upsilon+\upsilon_0^2}{a_c}$$

$$2xa_c=\underbrace{2\upsilon\upsilon_0}_{1+4=0}\underbrace{-2\upsilon_0^2}_{2+5=-\upsilon_0^2}+\upsilon^2-\underbrace{2\upsilon_0\upsilon}_{1+4=0}+\underbrace{\upsilon_0^2}_{2+5=-\upsilon_0}$$

$$2xa_c=-\upsilon_0^2+\upsilon^2$$

$$v^2=\upsilon_0^2+2a_c\,x \qquad (5)$$

Example:1 A Particle moves in a straight line such that for a short time its distance is defined by $x=\frac{v_0}{k}(1-e^{-kt})$, where v_0,k are constant. Determine $(v,t),(a,t),(v,x),(a,x),(a,v)$.

Solution

$$x=\frac{v_0}{k}(1-e^{-kt})$$

$$v=\frac{dx}{dt}=\frac{v_0}{k}(0-(-k)e^{-kt})=\frac{v_0}{k}k\,e^{-kt}=v_0\,e^{-kt}$$

$$v=v_0\,e^{-kt}\ \ldots\ldots\ldots\ldots(v,t) \qquad (1)$$

$$a=\frac{dv}{at}=-k\,v_0\,e^{-kt}\ldots\ldots\ldots\ldots(a,t) \qquad (2)$$

$$\frac{k}{v_0}x=1-e^{-kt}\rightarrow\quad e^{-kt}=1-\frac{k}{v_0}x\ldots\ldots\ldots\ldots v_0e^{-kt}=v_0-k\,x \qquad (3)$$

From Eq. (1) into Eq. (3)

$$v=v_0-k\,x\ \ldots\ldots\ldots\ldots(v,x) \qquad (4)$$

From Eq. (2) into Eq. (3)

$$a = -k\,(v_0 - k\,x) = k(k\,x - v_0)\text{..........}(a,x) \qquad (5)$$

From Eq. (4) into Eq. (5)

$$a = -k\,(v_0 - k\,x) = -k\,v\text{..........}(a,v) \qquad (6)$$

Example 2: A particle moves along a horizontal path with an acceleration of $a = -0.4v$, where v is velocity. If the particle start with velocity $0.4\sec$ at $1km$ from the origin point . Determine the velocity and distance

Solution

$a = -0.4v$

$$v\frac{dv}{dx} = -0.4v \quad \to \quad \frac{dv}{dx} = -0.4 \quad \to$$

$$dv = -0.4\;dx \to \; v = -0.4x + c_1$$

Form the initial condition $0.8 = c_1$, then

$$v = -0.4x + 0.8$$

$$\frac{dx}{dt} = -0.4x + 0.8 \to \qquad \frac{dx}{-0.4x + 0.8} = dt \to \frac{-1}{0.4}\int \frac{-dx}{-x+2} = \int dt$$

$$\frac{-1}{0.4}Ln(-x+2) = t + c_2$$

Form the initial condition $\frac{-1}{0.4}Ln(1) = 0 + c_2 \to \quad c_2 = 0$, then

$$Ln\,(-x+2) = -0.4t \;\to\; -x+2 = e^{-0.4t} \;\to\; x = 2 - e^{-0.4t}$$

Example: 3 A particle travels along a straight line with an acceleration of $a = (10 - 0.2x)$, where x is measured in meters. Determine the velocity of the particle when $x = 10m$, if $v = 5m/s$ at $x = 0$.

Solution

$$a = \frac{dv}{dt} = v\frac{dv}{dx} = (10 - 0.2x) \;\to\; \int v dv = \int (10 - 0.2x)\,dx$$

$$\frac{v^2}{2} = 10x - \frac{0.2}{2}x^2 + c_1$$

Form the initial condition $\frac{25}{2} = 10(0) - \frac{0.2}{2}(0) + c_1 \rightarrow c_1 = \frac{25}{2}$, then

$$\frac{v^2}{2} = 10x - \frac{0.2}{2}x^2 + \frac{25}{2} \qquad \rightarrow \qquad v = \sqrt{20x - 0.2x^2 + 25}$$

At $x = 10m \rightarrow v(10) = \sqrt{20(10) - 0.2(10)^2 + 25} = \sqrt{200 - 20 + 25} = \sqrt{155}$

Example 4: The velocity of a particle moving on the x-axis is given by $v = x^2 + x$, where x is in m and v is in m/s. What is its position when its acceleration is $30 m/s^2$.

Solution

It is well known $a = \frac{dv}{dt} = v\frac{dv}{dx}$, then

$$a = v\frac{dv}{dx} = \left(x^2 + x\right)\frac{d}{dx}(x^2 + x) = (x^2 + x)(2x + 1) = 2x^3 + 3x^2 + x \qquad (1)$$

Now at $a = 30$ Eq. (1) become

$$30 = 2x^3 + 3x^2 + x \qquad \rightarrow \qquad 2x^3 + 3x^2 + x + 30 = 0 \qquad (2)$$

Eq. (1) satisfy at $x = 2$

Applications

Example:1 The velocity of a particle traveling along a straight line is $v = v_0 - kx$, where k is constant. If $x = 0$ when $t = 0$, determine the position, velocity and acceleration of the particle as a function of time. Also the acceleration of the particle as a function of position. When the particle stop and the time of necessary?

Solution

Where

$$v = v_0 - kx, \qquad (1)$$

Then

$$\frac{dx}{dt} = v_0 - kx \rightarrow \frac{dx}{v_0 - kx} = dt \rightarrow \frac{1}{(-k)}\int \frac{(-k)\,dx}{v_0 - kx} = \int dt \rightarrow \frac{1}{(-k)} Ln(v_0 - kx) = t + c_1$$

Form the initial condition $t = 0$ at $x = 0$, then

$$\frac{1}{(-k)} Ln(v_0 - 0) = 0 + c_1 \rightarrow c_1 = \frac{1}{(-k)} Ln(v_0)$$

Then in Eq. (1)

$$\frac{1}{(-k)} Ln(v_0 - kx) = t + \frac{1}{(-k)} Ln(v_0) \rightarrow Ln(v_0 - kx) - Ln(v_0) = -kt \rightarrow$$

$$Ln\left(\frac{v_0 - kx}{v_0}\right) = -kt \rightarrow \frac{v_0 - kx}{v_0} = e^{-kt}$$

$$x = \frac{v_0}{k}\left(1 - e^{-kt}\right) \qquad (x, t) \qquad (2)$$

If we derive Eq. (2) with respect to t

$$v = \frac{v_0}{k}\left(0 - (-k)e^{-kt}\right) = v_0 e^{-kt} \qquad (v, t) \qquad (3)$$

Again, if we derive Eq. (2) with respect to t

$$a = -v_0 k e^{-kt} \qquad (a,t) \qquad (4)$$

From Eq. (2), $v_0 e^{-kt} = v_0 - xk$ and into Eq. (4)

$$f = -k(v_0 - xk) = k(xk - v_0) \qquad (a,x) \qquad (5)$$

when the particle stop, then $v = v_0 - kx = 0$. So the transversed distance given by

$x = \frac{v_0}{k}$ and into Eq. (3), necessary time given by

$$x = \frac{v_0}{k}\left(1 - e^{-kt}\right) = \frac{v_0}{k} \to 1 - e^{-kt} = 1 \to e^{-kt} = 0 \to t = \infty$$

Example:2 A particle is moving with a velocity of v_0 when $x = 0$ and $t = 0$. If it is subjected to a deceleration of $a = -kv^3$, where k is a constant, determine its velocity and position as functions of time .

Solution

Where $a = -kv^3$ and $a = \frac{dv}{dt}$, then

$$a = \frac{dv}{dt} = -kv^3 \to \frac{dv}{v^3} = -k\,dt \to \int \frac{dv}{v^3} = -k\int dt \to \frac{-1}{2v^2} = -kt + c_1 .$$

Form the initial condition $\frac{-1}{2v_0^2} = -k(0) + c_1 \to c_1 = \frac{-1}{2v_0^2}$, then

$$\frac{-1}{2v^2} = -kt - \frac{1}{2v_0^2} \to \frac{1}{2v^2} = \frac{2v_0^2 kt + 1}{2v_0^2}$$

$$v^2 = \frac{v_0^2}{2v_0^2 kt + 1} \to v = \frac{v_0}{\left(2v_0^2 kt + 1\right)^{\frac{1}{2}}} \qquad (v,\ t)$$

$$v = \frac{dx}{dt} = \frac{v_0}{\left(2v_0^2 kt + 1\right)^{\frac{1}{2}}} \to \int dx = \int \frac{v_0}{\left(2v_0^2 kt + 1\right)^{\frac{1}{2}}} dt \to$$

$$\int dx = \int v_0 \left(2v_0^2 kt + 1\right)^{-\frac{1}{2}} dt$$

$$x=\frac{v_0\left(2v_0^2kt+1\right)^{\frac{1}{2}}}{\frac{1}{2}(2v_0^2k)}=\frac{\left(2v_0^2kt+1\right)^{\frac{1}{2}}}{v_0k}$$

$$x=\frac{\left(2v_0^2kt+1\right)^{\frac{1}{2}}}{v_0k} \qquad (x,t\ )$$

Example:3 A particle is moving with a velocity of v_0 when $x=0$ and $t=0$. If it is subjected to a deceleration of $a=-k\,v^3$, where k is a constant. Prove that $t=\frac{x}{v_0}+\frac{1}{2}k\,x^2$, $\quad ?\ v=\frac{v_0}{v_0\,k\,x+1}$

Solution

Where $a=-k\,v^3$ and $a=v\frac{dv}{dx}$, then

$$a=v\frac{dv}{dx}=-k\,v^3 \rightarrow \frac{dv}{dx}=-k\,v^2 \rightarrow \frac{dv}{v^2}=-k\,dx \rightarrow \int\frac{dv}{v^2}=-k\int dx \rightarrow \frac{-1}{v}=-k\,x+c_1$$

Form the initial condition $\frac{-1}{v_0}=-k\,(0)+c_1 \rightarrow c_1=\frac{-1}{v_0}$, then

$$\frac{-1}{v}=-k\,x-\frac{1}{v_0} \rightarrow \frac{1}{v}=k\,x+\frac{1}{v_0}=\frac{v_0k\,x+1}{v_0}$$

$$v=\frac{v_0}{v_0\,k\,x+1} \qquad (1)$$

But $v=\frac{dx}{dt}$, then into Eq. (1)

$$v=\frac{dx}{dt}=\frac{v_0}{v_0\,k\,x+1} \rightarrow \int v_0dt=\int\left(v_0\,k\,x+1\right)dx \rightarrow v_0\,t=\left(\frac{1}{2}v_0\,k\,x^2+x\right)+c_2$$

Again, form the initial condition $v_0\,(0)=\left(\frac{1}{2}v_0\,k\,(0)^2+(0)\right)+c_2 \rightarrow c_2=0$.

Then $v_0\,t=\left(\frac{1}{2}v_0\,k\,x^2+x\right)+0 \rightarrow t=\frac{1}{2\,v_0}v_0\,k\,x^2+\frac{x}{v_0}=\frac{x}{v_0}+\frac{1}{2}k\,x^2$

$$t=\frac{x}{v_0}+\frac{1}{2}k\,x^2 \qquad (2)$$

Example:4 A particle moves along a straight line with an acceleration given by $a = k e^{-nt} \quad m/s^2$, where n are constant. If it starts from the rest, prove k, that $nx = (nt-1)\mathrm{v} + at.$, where v is its velocity?

Solution

Where $a = \frac{d\mathrm{v}}{dt}$, then

$$a = \frac{d\mathrm{v}}{dt} = k e^{-nt} \rightarrow d\mathrm{v} = k e^{-nt}\, dt \rightarrow \int d\mathrm{v} = \int k e^{-nt} dt \rightarrow \mathrm{v} = -\frac{k}{n} e^{-nt} + c_1$$

Where the particle rest i. e. at $t = 0$, $\mathrm{v} = 0, x = 0$, then

$$(0) = -\frac{k}{n} k e^{-n(0)} + c_1 \rightarrow c_1 = \frac{k}{n}$$ and then

$$\mathrm{v} = -\frac{k}{n} e^{-nt} + \frac{k}{n} = \frac{k}{n}\left(1 - e^{-nt}\right) \qquad (1)$$

But $\mathrm{v} = \frac{dx}{dt}$, so

$$\mathrm{v} = \frac{dx}{dt} = \frac{k}{n}(1 - e^{-nt}) \rightarrow dx = \frac{k}{n}(1 - e^{-nt})\, dt \rightarrow \int dx = \int \frac{k}{n}(1 - e^{-nt})\, dt \rightarrow$$

$$x = \frac{k}{n}\left(t - \frac{1}{-n} e^{-nt}\right) + c_2$$

Again, at $t = 0$, $\mathrm{v} = 0, x = 0$, then

$$(0) = \frac{k}{n}\left((0) - \frac{1}{-n} e^{-n(0)}\right) + c_2 \rightarrow c_2 = -\frac{k}{n^2}$$, then

$$x = \frac{k}{n}\left(t + \frac{1}{n} e^{-nt}\right) - \frac{k}{n^2}$$

$$nx = k\, t + \frac{k}{n} e^{-nt} - \frac{k}{n} \qquad (2)$$

From the basic equation, we find that

$$e^{-nt} = \frac{a}{k} \qquad (3)$$

Substitution from Eq. (3) into Eq. (1)

$$v = \frac{k}{n}\left(1-e^{-nt}\right)=\frac{k}{n}\left(1-\frac{a}{k}\right)=\frac{k}{n}-\frac{a}{n} \rightarrow \quad k = nv + a \qquad (4)$$

Again Substitution from Eq. (3) and Eq. (4) into Eq. (2)

$$n\,x=k\,t+\frac{k}{n}e^{-nt}-\frac{k}{n}= t\,(nv+a)+\frac{a}{n}-\frac{nv+a}{n}=t\,(nv+a)+\frac{a}{n}-v-\frac{a}{n}=t\,(nv+a)-v$$

Then $n\,x = v(nt-1)+a\,t$

Exercises

Exercise:1 A particle is moving along a straight line such that its acceleration is defined as $a=-2v \;\; m/s^2$, where v is in meters per second. If $v=20m/s$ v = 20 when x = 0 and $t=0$, determine the particle's position, velocity, and acceleration as functions of time.

Exercise:2 The acceleration of a particle traveling along a straight line is $a=\frac{1}{4}x^{\frac{1}{2}} \;\; m/s^2$, where x is in meters. If $v=0$,, determine the $t=0$at $x=1\,m$ particle's velocity at $x=2\,m$

Exercise:3 The velocity of a particle traveling along a straight line is $v=8-8x$, where k is constant. If $x=0$ when $t=0$, determine the position, velocity and acceleration of the particle as a function of time. Also the acceleration of the particle as a function of position. When the particle stop and the time of necessary?

Exercise:4 A freight train travels at $v=60\left(1-e^{-t}\right) \; ft/s$, where t is the elapsed time in seconds. Determine the distance traveled in three seconds, and the acceleration at this time.

Exercise:4 A particle moves along a straight line with an acceleration of $a=\frac{5}{\left(3x^{\frac{1}{3}}+x^{\frac{5}{2}}\right)} \;\; m/s^2$, where x is in meters. Determine the particle's velocity when $x=2\,m$, if it starts from rest when $x=1\,m$.

Motion of a particle in -xy plane

Introduction

In the plane $x - y$ Coordinates, the particle is located by the distances x and y from a fixed point in $x-$axis and $y-$ axis, respectively. The unit vectors $\vec{i}$, $\vec{j}$ are established in the positive $x-$ and $y-$directions, respectively.

Thus, the location of the particle at any point is

$$\vec{r} = x\ \vec{i} + y\ \vec{j} \qquad (1)$$

$$x = x(t) \quad , \quad y = y(t) \qquad (2)$$

Eqs. (2) are called the parametric equation of path

The Velocity

The speed of a particle whose motion is described by a parametric equation is given in terms of the time derivatives of the $x-$coordinate, and $y-$coordinate

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} \qquad \therefore \vec{v} = \dot{\vec{r}} = \dot{x}\,\vec{i} + \dot{y}\,\vec{j}$$

The magnitude of the velocity is $v = \sqrt{\dot{x}^2 + \dot{y}^2}$

The direction of the velocity is $\theta = \tan^{-1}\dfrac{\dot{y}}{\dot{x}}$.

The acceleration

The magnitude of the acceleration of a particle whose motion is described by a parametric function is given in terms of the second time derivatives of the $x-$coordinate, and $y-$coordinate

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \ddot{\vec{r}} = \dot{\vec{v}} = \frac{d\dot{x}}{dt}\vec{i} + \frac{d\dot{y}}{dt}\vec{j} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j}$$

$$\vec{a} = \ddot{x}\,\vec{i} + \ddot{y}\,\vec{j} = \dot{x}\,\frac{d\dot{x}}{dx}\,\vec{i} + \dot{y}\,\frac{d\dot{y}}{dy}\,\vec{j}$$

The magnitude of the acceleration $a = \sqrt{\ddot{x}^2 + \ddot{y}^2}$,

The magnitude of the acceleration $\varphi = \tan^{-1}(\frac{\ddot{y}}{\ddot{x}})$.

Example:1 A particle has a position given by $x = t^2, \quad y = 3t^2 + 5$. Determine the equation of path and what is the particle's speed and acceleration at? $t = 0$

Solution

$x = t^2$ (1)

$y = 3t^2 + 5$ (2)

If we delete the time between Eqs. (1) and (2), we get

$y = 3x + 5$ (The equation of path) (3)

From Eqs. (1) and (2), we get

$x^{\bullet} = 2t, \qquad y^{\bullet} = 6t,$

$x^{\bullet\bullet} = 2, \qquad y^{\bullet\bullet} = 6.$

Then the particle's speed and acceleration given by

$$v = \sqrt{x^{\bullet 2} + y^{\bullet 2}} = \sqrt{(2t)^2 + (6t)^2} = 2t\sqrt{10}\ m/\sec$$

$$a = \sqrt{x^{\bullet\bullet 2} + y^{\bullet\bullet 2}} = \sqrt{(2)^2 + (6)^2} = 2\sqrt{10}\ m/\sec^2$$

At $t = 0$, $v = 0$ and $a = \sqrt{10}\ m/\sec^2$

Example:2 A particle has a position given by $x = b\cos\omega t, \quad y = b\sin\omega t$. Determine the equation of path and what is the particle's speed and acceleration as a function of? t

Solution

$x = b\cos\omega t,$ (1)

$y = b\sin\omega t$ (2)

We delete the time between Eqs. (1) and (2) as follows

$$x^2 = (b\cos\omega t)^2, \quad y^2 = (b\sin\omega t)^2.$$

$x^2 + y^2 = b^2\left\{(\cos\omega t)^2 + (\sin\omega t)^2\right\} = b^2$, then

(3) (The equation of path) $x^2 + y^2 = b^2$

From Eqs. (1) and (2), we get

$$x = b\cos\omega t, \qquad y = b\sin\omega t$$
$$\dot{x} = -b\omega\sin\omega t, \qquad \dot{y} = b\omega\cos\omega t,$$
$$\ddot{x} = -b\omega^2\cos\omega t, \qquad \ddot{y} = -b\omega^2\sin\omega t.$$

Then the particle's speed and acceleration given by

$$v = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{(-b\omega\sin\omega t)^2 + (b\omega\cos\omega t)^2} = b\omega \ m/\sec.$$

$$a = \sqrt{\ddot{x}^2 + \ddot{y}^2} = \sqrt{(-b\omega^2\cos\omega t)^2 + (-b\omega^2\sin\omega t)^2} = b\omega^2 \ m/\sec^2.$$

Example:3 A particle has a position given by $x = 5\cos t + 3, \quad y = 5\sin t + 4.$ Determine the equation of path and what is the particle's speed and acceleration as a function of t?

Solution

$x = 5\cos t + 3,$ (1)

$y = 5\sin t + 4$ (2)

We delete the time between Eqs. (1) and (2) as follows

$$x-3=5\cos t, \qquad y-4=5\sin t.$$

$$(x-3)^2=25(\cos t)^2, \qquad (y-4)^2=25(\sin t)^2.$$

$$(x-3)^2+(y-4)^2=25\left\{(\cos t)^2+(\sin t)^2\right\}=25$$

$$(x-3)^2+(y-4)^2=25$$
$$x^2-6x+9+y^2-8y+16=25$$
$$x^2-6x+y^2-8y=0$$

(The equation of path) (3) $x^2+y^2=b^2$

From Eqs. (1) and (2), we get

$$\dot{x}=-5\sin t, \qquad \dot{y}=5\cos t,$$
$$\ddot{x}=-5\cos t, \qquad \ddot{y}=-5\sin t.$$

Then the particle's speed and acceleration given by

$$v=\sqrt{\dot{x}^2+\dot{y}^2}=\sqrt{(-5\sin t)^2+(5\cos t)^2}=\sqrt{25(\sin^2 t+\cos^2 t)}=5\ m/\sec$$
$$a=\sqrt{\ddot{x}^2+\ddot{y}^2}=\sqrt{(-5\cos t)^2+(5\sin t)^2}=\sqrt{25(\sin^2 t+\cos^2 t)}=5\ m/\sec^2$$

Example:4 A particle has a position given by $x=a\cos\omega t, \quad y=b\sin\omega t.$ Determine the equation of path and what is the particle's speed and acceleration as a function of? $x=0,\ \frac{a}{2}$ at and t

Solution

$$x=a\cos\omega t, \qquad (1)$$
$$y=b\sin\omega t \qquad (2)$$

We delete the time between Eqs. (1) and (2) as follows

$$x^2 = (a\cos\omega t)^2, \quad y^2 = (b\sin\omega t)^2.$$

$$x^2 = a^2\cos^2\omega t = a^2(1-\sin^2\omega t) = a^2\left(1-\frac{y^2}{b^2}\right)$$

(The equation of path) (3) $x^2 = a^2\left(1-\frac{y^2}{b^2}\right)$

From Eqs. (1) and (2), we get

$$x = a\cos\omega t, \qquad y = b\sin\omega t$$

$$.\dot{x} = -a\omega\sin\omega t, \qquad \dot{y} = b\omega\cos\omega t,$$

$$\ddot{x} = -a\omega^2\cos\omega t, \qquad \ddot{y} = -b\omega^2\sin\omega t.$$

Then the particle's speed and acceleration given by

$$v = \sqrt{\dot{x}^2+\dot{y}^2} = \sqrt{(-a\omega\sin\omega t)^2+(b\omega\cos\omega t)^2}\ m/\sec$$

$$a = \sqrt{\ddot{x}^2+\ddot{y}^2} = \sqrt{(-a\omega^2\cos\omega t)^2+(-b\omega^2\sin\omega t)^2}\ m/\sec^2$$

At, then $x = a\cos\omega t = 0 \rightarrow \omega t = 90^O = \frac{\pi}{2}$, we find that $x=0$

$$v = \sqrt{\dot{x}^2+\dot{y}^2} = \sqrt{(-a\omega\sin\frac{\pi}{2})^2+(b\omega\cos\frac{\pi}{2})^2} = a\omega\, m/\sec.$$

$$a = \sqrt{\ddot{x}^2+\ddot{y}^2}\ \sqrt{(-a\omega^2\cos\frac{\pi}{2})^2+(-b\omega^2\sin\frac{\pi}{2})^2} = b\omega^2\, m/\sec^2$$

At $x=a/2$, we find that $x = a\cos\omega t = a/2 \rightarrow \cos\omega t = 1/2 = \omega t = 60^O = \frac{\pi}{3}$, then

$$v = \sqrt{\dot{x}^2+\dot{y}^2} = \sqrt{(-a\omega\sin\frac{\pi}{3})^2+(b\omega\cos\frac{\pi}{3})^2} = \sqrt{(-a\omega\frac{\sqrt{3}}{2})^2+(b\omega\frac{1}{2})^2}$$

$$v = \frac{\omega}{2}\sqrt{3a^2 + b^2}\ m/\sec$$

$$a = \sqrt{\ddot{x}^2 + \ddot{y}^2} = \sqrt{(-a\omega^2 \cos\frac{\pi}{3})^2 + (-b\omega^2 \sin\frac{\pi}{3})^2} = \sqrt{(-a\omega^2 \frac{1}{2})^2 + (-b\omega^2 \frac{\sqrt{3}}{2})^2}$$

$$a = \frac{\omega^2}{2}\sqrt{a^2 + 3b^2}\ m/\sec^2$$

$$\vec{a} = -a\omega^2 \cos\omega t\,\vec{i} - b\omega^2 \sin\omega t\,\vec{j} = -\omega^2\left(a \cos\omega t\,\vec{i} + b \sin\omega t\,\vec{j}\right) = -\omega^2\,\vec{r}$$

Exercises:

Exercise:1 A particle has a position given by $x = 5t, \quad y = 20 - 5t^2$. Determine the equation of path and what is the particle's speed and acceleration at? $t = 0$

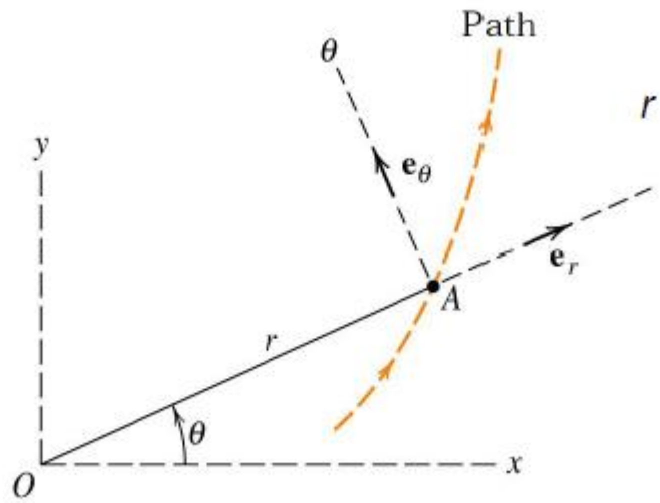
Exercise:2 A particle has a position given by $x = 5t, \quad y = 20 - 5t^2$. Determine the equation of path and what is the particle's speed and acceleration at? $t = 0$

Exercise:3 A particle has a position given by $x = a\left\{5t + \sin 2t\right\}, \quad y = a\left\{1 - \cos 2t\right\}$. Determine the equation of path and what is the particle's speed and acceleration at? $t = 0$

Motion in polar Plane

<u>Polar coordinates</u> $(r-\theta)$.

In Polar Coordinates, the particle is located by the radial distance (r) from a fixed point and by an angular measurement (θ) to the radial line.



$+r$ direction (positive radial direction) is measured from the pole to the point the particle.

$+\theta$ (positive angular direction) is in the direction of increasing θ and is orthogonal to r.

<u>Position Vector</u>

In Polar Coordinates the position vector define as

$$\vec{r} = r\,\vec{e}_r \quad , \quad (1)$$

Where $\vec{e}_r$ is the unit vector in the direction of $\vec{r}$ and given as

$$\vec{e}_r = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}\,, \quad (2)$$

Also, we define $\vec{e}_\theta$ is the unit vector in the θ direction and define as

$$\vec{e}_\theta = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}\,. \qquad (3)$$

Time derivative of unit vectors

$$\vec{e}_r = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$$

$$\frac{d\vec{e}_r}{dt} = \frac{d}{dt}\left(\cos\theta\,\vec{i} + \sin\theta\,\vec{j}\right) = \left(-\sin\theta\,\vec{i} + \cos\theta\,\vec{j}\right)\frac{d\theta}{dt} = \theta^\bullet\,\vec{e}_\theta \qquad (4)$$

$$\vec{e}_\theta = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}$$

$$\frac{d\vec{e}_\theta}{dt} = \frac{d}{dt}\left(-\sin\theta\,\vec{i} + \cos\theta\,\vec{j}\right) = -\left(\cos\theta\,\vec{i} + \sin\theta\,\vec{j}\right)\frac{d\theta}{dt} = -\theta^\bullet\,\vec{e}_r \qquad (5)$$

Note that $\frac{d\vec{i}}{dt} = 0,\ \frac{d\vec{j}}{dt} = 0$

The velocity in polar coordinate

The velocity can be determined by taking a time derivative of position,

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\,(r\vec{e}_r)}{dt} = \frac{dr}{dt}\vec{e}_r + r\,\frac{d\vec{e}_r}{dt}$$

Substituting the above relationships into the velocity equation gives

$\vec{v} = r^\bullet\vec{e}_r + r\,\theta^\bullet\,\vec{e}_\theta$ (radian velocity, transfer velocity

)Then $\vec{v} = (v_r,\ v_\theta) = (r^\bullet,\ r\,\theta^\bullet)$, $\quad v = \sqrt{v_r^2 + v_\theta^2}$

The Acceleration in polar coordinate

A time derivative of the velocity will give an expression for the acceleration,

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(r^\bullet\vec{e}_r + r\,\theta^\bullet\,\vec{e}_\theta\right) = \frac{dr^\bullet}{dt}\vec{e}_r + r^\bullet\frac{d\vec{e}_r}{dt} + \left(\theta^\bullet\,\vec{e}_\theta\right)\frac{dr}{dt} + \left(r\ \vec{e}_\theta\right)\frac{d\theta^\bullet}{dt} + \left(r\,\theta^\bullet\right)\frac{d\vec{e}_\theta}{dt}$$

By substituting this into the previous equation, and rearranging, gives the acceleration in terms of radial and transverse components,

$$\vec{a} = \ddot{r}\vec{e}_r + \dot{r}\dot{\theta}\,\vec{e}_\theta + \dot{r}\dot{\theta}\,\vec{e}_\theta + r\,\ddot{\theta}\,\vec{e}_\theta - (r\,\dot{\theta})\dot{\theta}\,\vec{e}_r$$

$$\vec{a} = \left(\ddot{r} - r\,\dot{\theta}^2\right)\vec{e}_r + \left(r\,\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\vec{e}_\theta$$

Then $\vec{a} = (a_r, a_\theta) = \left(\ddot{r} - r\,\dot{\theta}^2,\; r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$ $\qquad a = \sqrt{a_r^2 + a_\theta^2}$

<u>Example: 1</u> A particle moves in the $r-\theta$-plane in such a way that its position at time t is $r = 2t^3 - 3t^2$ and $\theta = t^2 + t$. Find the values of its velocity and acceleration at $t = 2$ second ?

Solution

In polar coordinates the velocity and acceleration are given by

$$\vec{v} = (\dot{r},\; r\dot{\theta}),\quad \vec{a} = (\ddot{r} - r\dot{\theta}^2, r\ddot{\theta} + 2\dot{r}\;\dot{\theta})$$

$$r = 2t^3 - 3t^2, \qquad \theta = t^2 + t,$$
$$\dot{r} = 6t^2 - 6t, \qquad \dot{\theta} = 2t + 1 \;, \qquad \text{At } t = 2$$
$$\ddot{r} = 12t - 6, \qquad \ddot{\theta} = 2.$$

$$\dot{r} = 6(2)^2 - 6(2) = 12, \qquad \dot{\theta} = 2(2) + 1 = 5$$
$$\ddot{r} = 12(2) - 6 = 18, \qquad \ddot{\theta} = 2$$
,Then

$\vec{v} = (\dot{r},\; r\dot{\theta}) = (12,\; a + b\,\sin\theta)$, Then

$$v = \sqrt{b^2\;\cos^2\theta + (a + b\,\sin\theta)^2} = \sqrt{b^2\cos^2\theta + a^2 + 2ab\,\sin\theta + b^2\;\sin^2\theta}\,,$$
$$v = \sqrt{b^2(\cos^2\theta + \sin^2\theta) + a^2 + 2ab\,\sin\theta} = \sqrt{b^2 + a^2 + 2ab\,\sin\theta}$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2, r\ddot{\theta} + 2\dot{r}\;\dot{\theta}) = \left(-b\,\omega^2\sin\theta - \omega^2(a + b\sin\theta),\; 0 + 2b\,\omega^2\cos\theta\right),$$
$$\vec{a} = \;\omega^2\left(-2b\sin\theta - a,\; 2b\cos\theta\right)$$

Then

$$a=\sqrt{(-2b\sin\theta-a)^2+(2b\cos\theta)^2}=\omega^2\sqrt{4b^2\sin^2\theta+4ab\sin\theta+a^2+4b^2\cos^2\theta},$$
$$a=\omega^2\sqrt{4b^2(\sin^2\theta+\cos^2\theta)+4ab\sin\theta+a^2}=\omega^2\sqrt{4b^2+4ab\sin\theta+a^2}$$

<u>Example: 2</u> A particle moves in the $(r-\theta)$ plane in such a way that its position is $r=b\theta$ and move with a constant angular velocity, b is constant. Find the values of its velocity and acceleration ?

Solution

In polar coordinates the velocity and acceleration are given by

$$\vec{v}=(\dot{r},\ r\dot{\theta}),\quad \vec{a}=(\ddot{r}-r\dot{\theta}^2, r\ddot{\theta}+2\dot{r}\,\dot{\theta})$$

$$\dot{\theta}=\omega=constant\ \ddot{\theta}=0,$$

$$\dot{\theta}=\frac{d\theta}{dt}=\omega\rightarrow\int d\theta=\omega\int dt\rightarrow\theta=\omega t+c$$

At $t=0,\ \theta=0$, then , $\theta=\omega t$. Thus $c=0$

$$r=b\omega t,\quad \dot{r}=b\omega,\quad \ddot{r}=0,\quad \theta=\omega t,\quad \dot{\theta}=\omega,\quad \ddot{\theta}=0$$

Therefore

$$v=(\dot{r},\ r\dot{\theta})=(b\omega, b\omega^2 t)$$
$$v=\sqrt{(b\omega)^2+(b\omega^2 t)^2}=b\omega\sqrt{1+t^2},$$

$$\vec{a}=(\ddot{r}-r\dot{\theta}^2,\ r\ddot{\theta}+2\dot{r}\,\dot{\theta})=(-b\omega^3 t,\ 2b\omega^2)$$

$$a=b\omega^2\sqrt{4+\omega^2 t^2}$$

Example: 3 A particle moves in the ($r-\theta$) plane in such a way that its position is given by $r=c+b\sin\theta$ with a constant angular velocity, b, c are constants. Find the values of its velocity and acceleration ?

Solution

In polar coordinates the velocity and acceleration are given by

$$\vec{v}=(\dot{r},\ r\dot{\theta}),\quad \vec{a}=(\ddot{r}-r\dot{\theta}^{2},\ r\ddot{\theta}+2\dot{r}\ \dot{\theta})$$

where

$$\dot{\theta}=\omega,\ \ddot{\theta}=0,$$
$$r=c+b\sin\theta,$$
$$\dot{r}=\dot{\theta}\,b\cos\theta=b\,\omega\cos\theta,$$
$$\ddot{r}=-\dot{\theta}\,\omega b\sin\theta=-b\,\omega^2\sin\theta.$$

$$\vec{v}=(\dot{r},\ r\dot{\theta})=\omega\left(b\ \cos\theta,\ \ c+b\ \sin\theta\right)$$

$$v=\sqrt{b^2\ \cos^2\theta+(c+b\ \sin\theta)^2}=\sqrt{b^2\cos^2\theta+c^2+2cb\ \sin\theta+b^2\ \sin^2\theta}$$
$$v=\sqrt{b^2(\cos^2\theta+\sin^2\theta)+c^2+2cb\ \sin\theta}=\sqrt{b^2+c^2+2cb\ \sin\theta}$$
$$\vec{a}=(\ddot{r}-r\dot{\theta}^{2},r\ddot{\theta}+2\dot{r}\ \dot{\theta})=\left(-b\omega^2\sin\theta-\omega^2(c+b\sin\theta),\ 0+2b\omega^2\cos\theta\right),$$
$$\vec{a}=\ \omega^2\left(-2b\sin\theta-c,\ 2b\cos\theta\right),$$

$$a=\sqrt{\left(-2b\sin\theta-c\right)^2+\left(2b\cos\theta\right)^2}=\omega^2\sqrt{4b^2\sin^2\theta+4cb\sin\theta+a^2+4b^2\cos^2\theta},$$
$$a=\omega^2\sqrt{4b^2(\sin^2\theta+\cos^2\theta)+4cb\sin\theta+c^2}=\omega^2\sqrt{4b^2+4cb\sin\theta+c^2}$$

Exercise

A particle moves in the ($r-\theta$) plane at the two paths (i) $r=c+b\theta$ (ii) $r=be^{\theta}$. with a constant angular velocity, b, c are constants. Find the values of its velocity and acceleration in the two paths?