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WHERE PDEs COME FROM

After thinking about the meaning of a partial differential equation, we will flex our mathematical muscles by solving a few of them. Then we will see how naturally they arise in the physical sciences. The physics will motivate the formulation of boundary conditions and initial conditions.

1.1 WHAT IS A PARTIAL DIFFERENTIAL EQUATION?

The key defining property of a partial differential equation (PDE) is that there is more than one independent variable $x, y, \ldots$. There is a dependent variable that is an unknown function of these variables $u(x, y, \ldots)$. We will often denote its derivatives by subscripts; thus $\partial u/\partial x = u_x$, and so on. A PDE is an identity that relates the independent variables, the dependent variable u, and the partial derivatives of u. It can be written as

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = F(x, y, u, u_x, u_y) = 0. \tag{1}$$

This is the most general PDE in two independent variables of *first* order. The *order* of an equation is the highest derivative that appears. The most general *second*-order PDE in two independent variables is

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0. \tag{2}$$

A *solution* of a PDE is a function $u(x, y, \ldots)$ that satisfies the equation identically, at least in some region of the $x, y, \ldots$ variables.

When solving an ordinary differential equation (ODE), one sometimes reverses the roles of the independent and the dependent variables—for instance, for the separable ODE $\dfrac{du}{dx} = u^3$. For PDEs, the distinction between the independent variables and the dependent variable (the unknown) is always maintained.

Some examples of PDEs (all of which occur in physical theory) are:

1. $u_x + u_y = 0$ (transport)
2. $u_x + yu_y = 0$ (transport)
3. $u_x + uu_y = 0$ (shock wave)
4. $u_{xx} + u_{yy} = 0$ (Laplace's equation)
5. $u_{tt} - u_{xx} + u^3 = 0$ (wave with interaction)
6. $u_t + uu_x + u_{xxx} = 0$ (dispersive wave)
7. $u_{tt} + u_{xxxx} = 0$ (vibrating bar)
8. $u_t - iu_{xx} = 0$ $(i = \sqrt{-1})$ (quantum mechanics)

Each of these has two independent variables, written either as x and y or as x and t. Examples 1 to 3 have order one; 4, 5, and 8 have order two; 6 has order three; and 7 has order four. Examples 3, 5, and 6 are distinguished from the others in that they are not "linear." We shall now explain this concept.

Linearity means the following. Write the equation in the form $\mathscr{L}u = 0$, where $\mathscr{L}$ is an *operator*. That is, if v is any function, $\mathscr{L}v$ is a new function. For instance, $\mathscr{L} = \partial/\partial x$ is the operator that takes v into its partial derivative v_x. In Example 2, the operator $\mathscr{L}$ is $\mathscr{L} = \partial/\partial x + y\partial/\partial y$. ($\mathscr{L}u = u_x + yu_y$.) The definition we want for linearity is

$$\mathscr{L}(u + v) = \mathscr{L}u + \mathscr{L}v \qquad \mathscr{L}(cu) = c\mathscr{L}u \tag{3}$$

for any functions u, v and any constant c. Whenever (3) holds (for all choices of u, v, and c), $\mathscr{L}$ is called *linear operator*. The equation

$$\mathscr{L}u = 0 \tag{4}$$

is called *linear* if $\mathscr{L}$ is a linear operator. Equation (4) is called a *homogeneous linear equation*. The equation

$$\mathscr{L}u = g, \tag{5}$$

where $g \neq 0$ is a given function of the independent variables, is called an *inhomogeneous linear equation*. For instance, the equation

$$(\cos xy^2)u_x - y^2 u_y = \tan(x^2 + y^2) \tag{6}$$

is an inhomogeneous linear equation.

As you can easily verify, five of the eight equations above are linear as well as homogeneous. Example 5, on the other hand, is not linear because although $(u + v)_{xx} = u_{xx} + v_{xx}$ and $(u + v)_{tt} = u_{tt} + v_{tt}$ satisfy property (3), the cubic term does not:

$$(u + v)^3 = u^3 + 3u^2v + 3uv^2 + v^3 \neq u^3 + v^3.$$

The advantage of linearity for the equation $\mathscr{L}u = 0$ is that if u and v are both solutions, so is $(u + v)$. If $u_1, \ldots, u_n$ are all solutions, so is any linear combination

$$c_1 u_1(x) + \cdots + c_n u_n(x) = \sum_{j=1}^{n} c_j u_j(x) \qquad (c_j = \text{ constants}).$$

(This is sometimes called the superposition principle.) Another consequence of linearity is that if you add a homogeneous solution [a solution of (4)] to an inhomogeneous solution [a solution of (5)], you get an inhomogeneous solution. (Why?) The mathematical structure that deals with linear combinations and linear operators is the vector space. Exercises 5–10 are review problems on vector spaces.

We'll study, almost exclusively, linear systems with constant coefficients. Recall that for ODEs you get linear combinations. The coefficients are the arbitrary constants. *For an ODE of order m, you get m arbitrary constants.*

Let's look at some PDEs.

Example 1.

Find all $u(x, y)$ satisfying the equation $u_{xx} = 0$. Well, we can integrate once to get $u_x =$ constant. But that's not really right since there's another variable y. What we really get is $u_x(x, y) = f(y)$, where $f(y)$ is arbitrary. Do it again to get $u(x, y) = f(y)x + g(y)$. This is the solution formula. Note that *there are two arbitrary* functions *in the solution*. We see this as well in the next two examples. □

Example 2.

Solve the PDE $u_{xx} + u = 0$. Again, it's really an ODE with an extra variable y. We know how to solve the ODE, so the solution is

$$u = f(y)\cos x + g(y)\sin x,$$

where again $f(y)$ and $g(y)$ are two arbitrary functions of y. You can easily check this formula by differentiating twice to verify that $u_{xx} = -u$. □

Example 3.

Solve the PDE $u_{xy} = 0$. This isn't too hard either. First let's integrate in x, regarding y as fixed. So we get

$$u_y(x, y) = f(y).$$

Next let's integrate in y regarding x as fixed. We get the solution

$$u(x, y) = F(y) + G(x),$$

where $F' = f$. □

Moral *A PDE has arbitrary functions in its solution.* In these examples the arbitrary functions are functions of one variable that combine to produce a function $u(x, y)$ of two variables which is only partly arbitrary.

A function of two variables contains *immensely* more information than a function of only one variable. Geometrically, it is obvious that a surface $\{u = f(x, y)\}$, the graph of a function of two variables, is a much more complicated object than a curve $\{u = f(x)\}$, the graph of a function of one variable.

To illustrate this, we can ask how a computer would record a function $u = f(x)$. Suppose that we choose 100 points to describe it using equally spaced values of x: $x_1, x_2, x_3, \ldots, x_{100}$. We could write them down in a column, and next to each x_j we could write the corresponding value $u_j = f(x_j)$. Now how about a function $u = f(x, y)$? Suppose that we choose 100 equally spaced values of x and also of y: $x_1, x_2, x_3, \ldots, x_{100}$ and $y_1, y_2, y_3, \ldots, y_{100}$. Each *pair* x_i, y_j provides a value $u_{ij} = f(x_i, y_j)$, so there will be $100^2 = 10{,}000$ lines of the form

$$x_i \qquad y_j \qquad u_{ij}$$

required to describe the function! (If we had a prearranged system, we would need to record only the values u_{ij}.) A function of three variables described discretely by 100 values in each variable would require a million numbers!

To understand this book what do you have to know from calculus? Certainly all the basic facts about partial derivatives and multiple integrals. For a brief discussion of such topics, see the Appendix. Here are a few things to keep in mind, some of which may be new to you.

1. Derivatives are *local*. For instance, to calculate the derivative $(\partial u/\partial x)(x_0, t_0)$ at a particular point, you need to know just the values of $u(x, t_0)$ for x *near* x_0, since the derivative is the limit as $x \to x_0$.
2. Mixed derivatives are equal: $u_{xy} = u_{yx}$. (We assume throughout this book, unless stated otherwise, that all derivatives exist and are continuous.)
3. The chain rule is used frequently in PDEs; for instance,
$$\frac{\partial}{\partial x}[f(g(x, t))] = f'(g(x, t)) \cdot \frac{\partial g}{\partial x}(x, t).$$
4. For the integrals of derivatives, the reader should learn or review Green's theorem and the divergence theorem. (See the end of Section A.3 in the Appendix.)
5. Derivatives of integrals like $I(t) = \int_{a(t)}^{b(t)} f(x, t)\, dx$ (see Section A.3).
6. Jacobians (change of variable in a double integral) (see Section A.1).
7. Infinite series of functions and their differentiation (see Section A.2).
8. Directional derivatives (see Section A.1).
9. We'll often reduce PDEs to ODEs, so we must know how to solve simple ODEs. But we won't need to know anything about tricky ODEs.

EXERCISES

1. Verify the linearity and nonlinearity of the eight examples of PDEs given in the text, by checking whether or not equations (3) are valid.
2. Which of the following operators are linear?
   (a) $\mathscr{L}u = u_x + xu_y$
   (b) $\mathscr{L}u = u_x + uu_y$
   (c) $\mathscr{L}u = u_x + u_y^2$
   (d) $\mathscr{L}u = u_x + u_y + 1$
   (e) $\mathscr{L}u = \sqrt{1+x^2}\,(\cos y)u_x + u_{yxy} - [\arctan(x/y)]u$
3. For each of the following equations, state the order and whether it is nonlinear, linear inhomogeneous, or linear homogeneous; provide reasons.
   (a) $u_t - u_{xx} + 1 = 0$
   (b) $u_t - u_{xx} + xu = 0$
   (c) $u_t - u_{xxt} + uu_x = 0$
   (d) $u_{tt} - u_{xx} + x^2 = 0$
   (e) $iu_t - u_{xx} + u/x = 0$
   (f) $u_x(1+u_x^2)^{-1/2} + u_y(1+u_y^2)^{-1/2} = 0$
   (g) $u_x + e^y u_y = 0$
   (h) $u_t + u_{xxxx} + \sqrt{1+u} = 0$
4. Show that the difference of two solutions of an inhomogeneous linear equation $\mathscr{L}u = g$ with the same g is a solution of the homogeneous equation $\mathscr{L}u = 0$.
5. Which of the following collections of 3-vectors $[a, b, c]$ are vector spaces? Provide reasons.
   (a) The vectors with $b = 0$.
   (b) The vectors with $b = 1$.
   (c) The vectors with $ab = 0$.
   (d) All the linear combinations of the two vectors [1, 1, 0] and [2, 0, 1].
   (e) All the vectors such that $c - a = 2b$.
6. Are the three vectors [1, 2, 3], [−2, 0, 1], and [1, 10, 17] linearly dependent or independent? Do they span all vectors or not?
7. Are the functions $1 + x$, $1 - x$, and $1 + x + x^2$ linearly dependent or independent? Why?
8. Find a vector that, together with the vectors [1, 1, 1] and [1, 2, 1], forms a basis of $\mathbb{R}^3$.
9. Show that the functions $(c_1 + c_2 \sin^2 x + c_3 \cos^2 x)$ form a vector space. Find a basis of it. What is its dimension?
10. Show that the solutions of the differential equation $u''' - 3u'' + 4u = 0$ form a vector space. Find a basis of it.
11. Verify that $u(x, y) = f(x)g(y)$ is a solution of the PDE $uu_{xy} = u_x u_y$ for all pairs of (differentiable) functions f and g of one variable.

12. Verify by direct substitution that

$$u_n(x, y) = \sin nx \sinh ny$$

is a solution of $u_{xx} + u_{yy} = 0$ for every $n > 0$.

1.2 FIRST-ORDER LINEAR EQUATIONS

We begin our study of PDEs by solving some simple ones. The solution is quite geometric in spirit.

The simplest possible PDE is $\partial u/\partial x = 0$ [where $u = u(x, y)$]. Its general solution is $u = f(y)$, where f is any function of *one* variable. For instance, $u = y^2 - y$ and $u = e^y \cos y$ are two solutions. Because the solutions don't depend on x, they are constant on the lines $y =$ constant in the xy plane.

THE CONSTANT COEFFICIENT EQUATION

Let us solve

$$\boxed{au_x + bu_y = 0,} \tag{1}$$

where a and b are constants not both zero.

Geometric Method The quantity $au_x + bu_y$ is the directional derivative of u in the direction of the vector $\mathbf{V} = (a, b) = a\mathbf{i} + b\mathbf{j}$. It must always be zero. This means that $u(x, y)$ must be constant in the direction of $\mathbf{V}$. The vector $(b, -a)$ is orthogonal to $\mathbf{V}$. The lines parallel to $\mathbf{V}$ (see Figure 1) have the equations $bx - ay =$ constant. (They are called the *characteristic lines*.) The solution is constant on each such line. Therefore, $u(x, y)$ depends on $bx - ay$ only. Thus the solution is

$$\boxed{u(x, y) = f(bx - ay),} \tag{2}$$

where f is any function of one variable. Let's explain this conclusion more explicitly. On the line $bx - ay = c$, the solution u has a constant value. Call

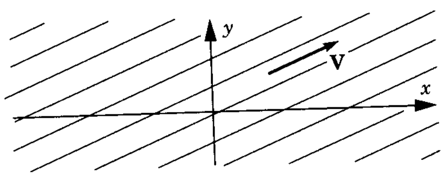

Figure 1

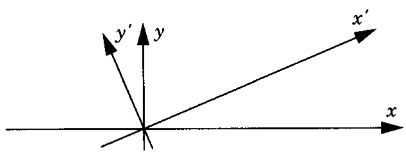

Figure 2

this value $f(c)$. Then $u(x, y) = f(c) = f(bx - ay)$. Since c is arbitrary, we have formula (2) for all values of x and y. In xyu space the solution defines a surface that is made up of parallel horizontal straight lines like a sheet of corrugated iron.

Coordinate Method Change variables (or "make a change of coordinates"; Figure 2) to

$$x' = ax + by \qquad y' = bx - ay. \tag{3}$$

Replace all x and y derivatives by x' and y' derivatives. By the chain rule,

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} = au_{x'} + bu_{y'}$$

and

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} + \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} = bu_{x'} - au_{y'}.$$

Hence $au_x + bu_y = a(au_{x'} + bu_{y'}) + b(bu_{x'} - au_{y'}) = (a^2 + b^2)u_{x'}$. So, since $a^2 + b^2 \neq 0$, the equation takes the form $u_{x'} = 0$ in the new (primed) variables. Thus the solution is $u = f(y') = f(bx - ay)$, with f an arbitrary function of *one* variable. This is exactly the same answer as before!

Example 1.

Solve the PDE $4u_x - 3u_y = 0$, together with the auxiliary condition that $u(0, y) = y^3$. By (2) we have $u(x, y) = f(-3x - 4y)$. This is the general solution of the PDE. Setting $x = 0$ yields the equation $y^3 = f(-4y)$. Letting $w = -4y$ yields $f(w) = -w^3/64$. Therefore, $u(x, y) = (3x + 4y)^3/64$.

Solutions can usually be checked much easier than they can be derived. We check this solution by simple differentiation: $u_x = 9(3x + 4y)^2/64$ and $u_y = 12(3x + 4y)^2/64$ so that $4u_x - 3u_y = 0$. Furthermore, $u(0, y) = (3 \cdot 0 + 4y)^3/64 = y^3$. □

THE VARIABLE COEFFICIENT EQUATION

The equation

$$\boxed{u_x + yu_y = 0} \tag{4}$$

is linear and homogeneous but has a variable coefficient (y). We shall illustrate for equation (4) how to use the geometric method somewhat like Example 1.

The PDE (4) itself asserts that *the directional derivative in the direction of the vector* $(1, y)$ *is zero*. The curves in the xy plane with $(1, y)$ as tangent vectors have slopes y (see Figure 3). Their equations are

$$\frac{dy}{dx} = \frac{y}{1} \tag{5}$$

This ODE has the solutions

$$y = Ce^x. \tag{6}$$

These curves are called the *characteristic curves* of the PDE (4). As C is changed, the curves fill out the xy plane perfectly without intersecting. On each of the curves $u(x, y)$ is a constant because

$$\frac{d}{dx}u(x, Ce^x) = \frac{\partial u}{\partial x} + Ce^x\frac{\partial u}{\partial y} = u_x + yu_y = 0.$$

Thus $u(x, Ce^x) = u(0, Ce^0) = u(0, C)$ is independent of x. Putting $y = Ce^x$ and $C = e^{-x}y$, we have

$$u(x, y) = u(0,\ e^{-x}y).$$

It follows that

$$\boxed{u(x, y) = f(e^{-x}y)} \tag{7}$$

is the *general solution* of this PDE, where again f is an arbitrary function of only a single variable. This is easily checked by differentiation using the chain rule (see Exercise 4). Geometrically, the "picture" of the solution $u(x, y)$ is that it is *constant on each characteristic curve* in Figure 3.

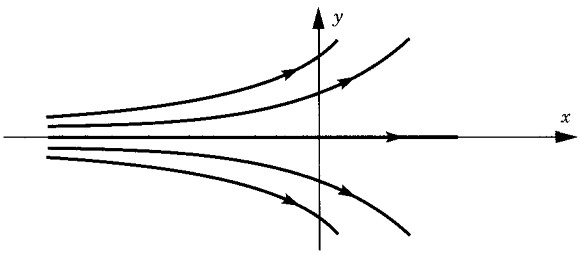

Figure 3

Example 2.

Find the solution of (4) that satisfies the auxiliary condition $u(0, y) = y^3$. Indeed, putting $x = 0$ in (7), we get $y^3 = f(e^{-0}y)$, so that $f(y) = y^3$. Therefore, $u(x, y) = (e^{-x}y)^3 = e^{-3x}y^3$. □

Example 3.

Solve the PDE

$$\boxed{u_x + 2xy^2u_y = 0.} \tag{8}$$

The characteristic curves satisfy the ODE $dy/dx = 2xy^2/1 = 2xy^2$. To solve the ODE, we separate variables: $dy/y^2 = 2x\,dx$; hence $-1/y = x^2 - C$, so that

$$y = (C - x^2)^{-1}. \tag{9}$$

These curves are the characteristics. Again, $u(x, y)$ is a constant on each such curve. (Check it by writing it out.) So $u(x, y) = f(C)$, where f is an arbitrary function. Therefore, the general solution of (8) is obtained by solving (9) for C. That is,

$$\boxed{u(x, y) = f\left(x^2 + \frac{1}{y}\right).} \tag{10}$$

Again this is easily checked by differentiation, using the chain rule: $u_x = 2x \cdot f'(x^2 + 1/y)$ and $u_y = -(1/y^2) \cdot f'(x^2 + 1/y)$, whence $u_x + 2xy^2u_y = 0$. □

In summary, the geometric method works nicely for any PDE of the form $a(x, y)u_x + b(x, y)u_y = 0$. It reduces the solution of the PDE to the solution of the ODE $dy/dx = b(x, y)/a(x, y)$. If the ODE can be solved, so can the PDE. Every solution of the PDE is constant on the solution curves of the ODE.

Moral Solutions of PDEs generally depend on arbitrary functions (instead of arbitrary constants). You need an auxiliary condition if you want to determine a unique solution. Such conditions are usually called *initial* or *boundary* conditions. We shall encounter these conditions throughout the book.

EXERCISES

1. Solve the first-order equation $2u_t + 3u_x = 0$ with the auxiliary condition $u = \sin x$ when $t = 0$.
2. Solve the equation $3u_y + u_{xy} = 0$. (*Hint*: Let $v = u_y$.)

3. Solve the equation $(1+x^2)u_x + u_y = 0$. Sketch some of the characteristic curves.
4. Check that (7) indeed solves (4).
5. Solve the equation $xu_x + yu_y = 0$.
6. Solve the equation $\sqrt{1-x^2}\,u_x + u_y = 0$ with the condition $u(0, y) = y$.
7. (a) Solve the equation $yu_x + xu_y = 0$ with $u(0, y) = e^{-y^2}$.
   (b) In which region of the xy plane is the solution uniquely determined?
8. Solve $au_x + bu_y + cu = 0$.
9. Solve the equation $u_x + u_y = 1$.
10. Solve $u_x + u_y + u = e^{x+2y}$ with $u(x, 0) = 0$.
11. Solve $au_x + bu_y = f(x, y)$, where $f(x, y)$ is a given function. If $a \neq 0$, write the solution in the form

$$u(x, y) = (a^2 + b^2)^{-1/2} \int_L f\, ds + g(bx - ay),$$

where g is an arbitrary function of one variable, L is the characteristic line segment from the y axis to the point (x, y), and the integral is a line integral. (*Hint:* Use the coordinate method.)
12. Show that the new coordinate axes defined by (3) are orthogonal.
13. Use the coordinate method to solve the equation

$$u_x + 2u_y + (2x - y)u = 2x^2 + 3xy - 2y^2.$$

1.3 FLOWS, VIBRATIONS, AND DIFFUSIONS

The subject of PDEs was practically a branch of physics until the twentieth century. In this section we present a series of examples of PDEs as they occur in physics. They provide the basic motivation for all the PDE problems we study in the rest of the book. We shall see that most often in physical problems the independent variables are those of space x, y, z, and time t.

Example 1. Simple Transport

Consider a fluid, water, say, flowing at a constant rate c along a horizontal pipe of fixed cross section in the positive x direction. A substance, say a pollutant, is suspended in the water. Let $u(x, t)$ be its concentration in grams/centimeter at time t. Then

$$\boxed{u_t + cu_x = 0.} \qquad (1)$$

(That is, the rate of change u_t of concentration is proportional to the gradient u_x. Diffusion is assumed to be negligible.) Solving this equation as in Section 1.2, we find that the concentration is a function of $(x - ct)$

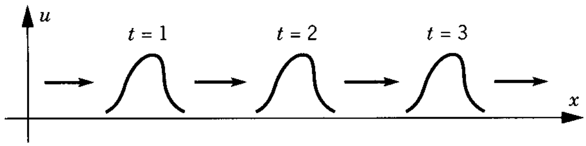

Figure 1

only. This means that the substance is transported to the right at a fixed speed c. Each individual particle moves to the right at speed c; that is, in the xt plane, it moves precisely along a characteristic line (see Figure 1). □

Derivation of Equation (1). The amount of pollutant in the interval $[0, b]$ at the time t is $M = \int_0^b u(x,t)\,dx$, in grams, say. At the later time $t+h$, the same molecules of pollutant have moved to the right by $c \cdot h$ centimeters. Hence

$$M = \int_0^b u(x,t)dx = \int_{ch}^{b+ch} u(x, t+h)\,dx.$$

Differentiating with respect to b, we get

$$u(b,t) = u(b+ch, t+h).$$

Differentiating with respect to h and putting $h = 0$, we get

$$0 = cu_x(b,t) + u_t(b,t),$$

which is equation (1). □

Example 2. Vibrating String

Consider a flexible, elastic homogenous string or thread of length l, which undergoes relatively small transverse vibrations. For instance, it could be a guitar string or a plucked violin string. At a given instant t, the string might look as shown in Figure 2. Assume that it remains in a plane. Let $u(x,t)$ be its displacement from equilibrium position at time t and position x. Because the string is perfectly flexible, the tension (force) is directed tangentially along the string (Figure 3). Let $T(x,t)$ be the magnitude of this tension vector. Let ρ be the density (mass per unit length) of the string. It is a constant because the string is homogeneous. We shall write down Newton's law for the part of the string between any two points at $x = x_0$ and $x = x_1$. The slope of the string at x_1 is

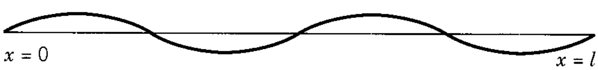

Figure 2

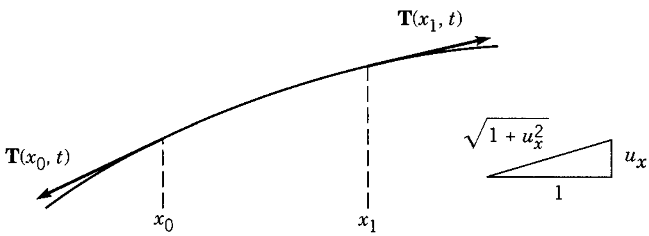

Figure 3

$u_x(x_1, t)$. Newton's law $\mathbf{F} = m\mathbf{a}$ in its longitudinal (x) and transverse (u) components is

$$\left. \frac{T}{\sqrt{1+u_x^2}} \right|_{x_0}^{x_1} = 0 \qquad \text{(longitudinal)}$$

$$\left. \frac{T u_x}{\sqrt{1+u_x^2}} \right|_{x_0}^{x_1} = \int_{x_0}^{x_1} \rho u_{tt}\, dx \qquad \text{(transverse)}$$

The right sides are the components of the mass times the acceleration integrated over the piece of string. Since we have assumed that the motion is purely transverse, there is no longitudinal motion.

Now we also assume that the motion is small—more specifically, that $|u_x|$ is quite small. Then $\sqrt{1+u_x^2}$ may be replaced by 1. This is justified by the Taylor expansion, actually the binomial expansion,

$$\sqrt{1+u_x^2} = 1 + \tfrac{1}{2}u_x^2 + \cdots$$

where the dots represent higher powers of u_x. If u_x is small, it makes sense to drop the even smaller quantity u_x^2 and its higher powers. With the square roots replaced by 1, the first equation then says that T is constant along the string. Let us assume that T is independent of t as well as x. The second equation, differentiated, says that

$$(Tu_x)_x = \rho u_{tt}.$$

That is,

$$\boxed{u_{tt} = c^2 u_{xx} \quad \text{where } c = \sqrt{\frac{T}{\rho}}.} \qquad (2)$$

This is the *wave equation*. At this point it is not clear why c is defined in this manner, but shortly we'll see that c is the *wave speed*. □

There are many *variations* of this equation:

(i) If significant air resistance r is present, we have an extra term proportional to the speed u_t, thus:

$$u_{tt} - c^2 u_{xx} + r u_t = 0 \quad \text{where } r > 0. \tag{3}$$

(ii) If there is a transverse elastic force, we have an extra term proportional to the displacement u, as in a coiled spring, thus:

$$u_{tt} - c^2 u_{xx} + ku = 0 \quad \text{where } k > 0. \tag{4}$$

(iii) If there is an externally applied force, it appears as an extra term, thus:

$$u_{tt} - c^2 u_{xx} = f(x, t), \tag{5}$$

which makes the equation inhomogeneous.

Our derivation of the wave equation has been quick but not too precise. A much more careful derivation can be made, which makes precise the physical and mathematical assumptions [We, Chap. 1].

The same wave equation or a variation of it describes many other wavelike phenomena, such as the vibrations of an elastic bar, the sound waves in a pipe, and the long water waves in a straight canal. Another example is the equation for the electrical current in a transmission line,

$$u_{xx} = CLu_{tt} + (CR + GL)u_t + GRu,$$

where C is the capacitance per unit length, G the leakage resistance per unit length, R the resistance per unit length, and L the self-inductance per unit length.

Example 3. Vibrating Drumhead

The two-dimensional version of a string is an elastic, flexible, homogeneous drumhead, that is, a membrane stretched over a frame. Say the frame lies in the xy plane (see Figure 4), $u(x, y, t)$ is the vertical

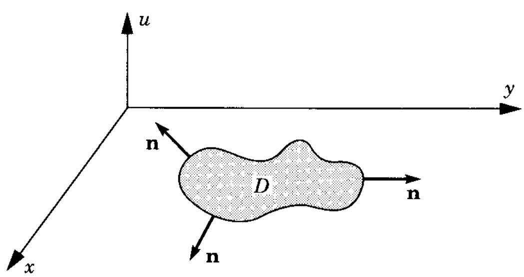

Figure 4

displacement, and there is no horizontal motion. The horizontal components of Newton's law again give constant tension T. Let D be any domain in the xy plane, say a circle or a rectangle. Let bdy D be its boundary curve. We use reasoning similar to the one-dimensional case. The vertical component gives (approximately)

$$F = \int_{\text{bdy } D} T \frac{\partial u}{\partial n}\, ds = \iint_D \rho u_{tt}\, dx\, dy = ma,$$

where the left side is the total force acting on the piece D of the membrane, and where $\partial u/\partial n = \mathbf{n} \cdot \nabla u$ is the directional derivative in the outward normal direction, $\mathbf{n}$ being the unit outward normal vector on bdy D. By Green's theorem (see Section A.3 in the Appendix), this can be rewritten as

$$\iint_D \nabla \cdot (T \nabla u)\, dx\, dy = \iint_D \rho u_{tt}\, dx\, dy.$$

Since D is arbitrary, we deduce from the second vanishing theorem in Section A.1 that $\rho u_{tt} = \nabla \cdot (T \nabla u)$. Since T is constant, we get

$$\boxed{u_{tt} = c^2 \nabla \cdot (\nabla u) \equiv c^2 (u_{xx} + u_{yy}),} \qquad (6)$$

where $c = \sqrt{T/\rho}$ as before and $\nabla \cdot (\nabla u) = \operatorname{div} \operatorname{grad} u = u_{xx} + u_{yy}$ is known as the *two-dimensional laplacian*. Equation (6) is the two-dimensional wave equation. □

The pattern is now clear. Simple three-dimensional vibrations obey the equation

$$\boxed{u_{tt} = c^2 (u_{xx} + u_{yy} + u_{zz}).} \qquad (7)$$

The operator $\mathscr{L} = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial/\partial z^2$ is called the *three-dimensional laplacian* operator, usually denoted by Δ or ∇^2. Physical examples described by the three-dimensional wave equation or a variation of it include the vibrations of an elastic solid, sound waves in air, electromagnetic waves (light, radar, etc.), linearized supersonic airflow, free mesons in nuclear physics, and seismic waves propagating through the earth.

Example 4. Diffusion

Let us imagine a motionless liquid filling a straight tube or pipe and a chemical substance, say a dye, which is diffusing through the liquid. Simple diffusion is characterized by the following law. [It is not to

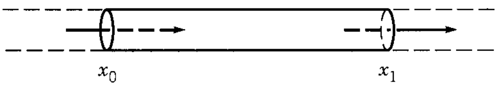

Figure 5

be confused with convection (transport), which refers to currents in the liquid.] The dye moves from regions of higher concentration to regions of lower concentration. The rate of motion is proportional to the concentration gradient. (This is known as Fick's law of diffusion.) Let $u(x, t)$ be the concentration (mass per unit length) of the dye at position x of the pipe at time t.

In the section of pipe from x_0 to x_1 (see Figure 5), the mass of dye is

$$M(t) = \int_{x_0}^{x_1} u(x,t)\,dx, \quad \text{so} \quad \frac{dM}{dt} = \int_{x_0}^{x_1} u_t(x,t)\,dx.$$

The mass in this section of pipe cannot change except by flowing in or out of its ends. By Fick's law,

$$\frac{dM}{dt} = \text{flow in} - \text{flow out} = ku_x(x_1,t) - ku_x(x_0,t),$$

where k is a proportionality constant. Therefore, those two expressions are equal:

$$\int_{x_0}^{x_1} u_t(x,t)\,dx = ku_x(x_1,t) - ku_x(x_0,t).$$

Differentiating with respect to x_1, we get

$$\boxed{u_t = ku_{xx}.} \tag{8}$$

This is the *diffusion equation*.

In three dimensions we have

$$\iiint_D u_t\,dx\,dy\,dz = \iint_{\text{bdy } D} k(\mathbf{n}\cdot\nabla u)\,dS,$$

where D is any solid domain and bdy D is its bounding surface. By the divergence theorem (using the arbitrariness of D as in Example 3), we get the *three-dimensional diffusion equation*

$$\boxed{u_t = k(u_{xx} + u_{yy} + u_{zz}) = k\,\Delta u.} \tag{9}$$

If there is an external source (or a "sink") of the dye, and if the rate k of diffusion is a variable, we get the more general inhomogeneous equation

$$u_t = \nabla \cdot (k \nabla u) + f(x, t).$$

The same equation describes the conduction of heat, brownian motion, diffusion models of population dynamics, and many other phenomena.

□

Example 5. Heat Flow

We let $u(x, y, z, t)$ be the temperature and let $H(t)$ be the amount of heat (in calories, say) contained in a region D. Then

$$H(t) = \iiint_D c\rho u \, dx\, dy\, dz,$$

where c is the "specific heat" of the material and ρ is its density (mass per unit volume). The change in heat is

$$\frac{dH}{dt} = \iiint_D c\rho u_t \, dx\, dy\, dz.$$

Fourier's law says that heat flows from hot to cold regions proportionately to the temperature gradient. But heat cannot be lost from D except by leaving it through the boundary. This is the law of conservation of energy. Therefore, the change of heat energy in D also equals the heat flux across the boundary,

$$\frac{dH}{dt} = \iint_{\text{bdy } D} \kappa(\mathbf{n} \cdot \nabla u)\, dS,$$

where κ is a proportionality factor (the "heat conductivity"). By the divergence theorem,

$$\iiint_D c\rho \frac{\partial u}{\partial t}\, dx\, dy\, dz = \iiint_D \nabla \cdot (\kappa \nabla u)\, dx\, dy\, dz$$

and we get the *heat equation*

$$\boxed{c\rho \frac{\partial u}{\partial t} = \nabla \cdot (\kappa \nabla u).} \tag{10}$$

If c, ρ, and κ are constants, it is exactly the same as the diffusion equation!

□

Example 6. Stationary Waves and Diffusions

Consider any of the four preceding examples in a situation where the physical state does not change with time. Then $u_t = u_{tt} = 0$. So *both* the wave *and* the diffusion equations reduce to

$$\boxed{\Delta u = u_{xx} + u_{yy} + u_{zz} = 0.} \tag{11}$$

This is called the *Laplace equation*. Its solutions are called *harmonic functions*. For example, consider a hot object that is constantly heated in an oven. The heat is not expected to be evenly distributed throughout the oven. The temperature of the object eventually reaches a steady (or equilibrium) state. This is a harmonic function $u(x, y, z)$. (Of course, if the heat were being supplied evenly in all directions, the steady state would be $u \equiv$ constant.) In the *one*-dimensional case (e.g., a laterally insulated thin rod that exchanges heat with its environment only through its ends), we would have u a function of x only. So the Laplace equation would reduce simply to $u_{xx} = 0$. Hence $u = c_1 x + c_2$. The two- and three-dimensional cases are *much* more interesting (see Chapter 6 for the solutions). □

Example 7. The Hydrogen Atom

This is an electron moving around a proton. Let m be the mass of the electron, e its charge, and h Planck's constant divided by 2π. Let the origin of coordinates (x, y, z) be at the proton and let $r = (x^2 + y^2 + z^2)^{1/2}$ be the spherical coordinate. Then the motion of the electron is given by a "wave function" $u(x, y, z, t)$ which satisfies Schrödinger's equation

$$\boxed{-ihu_t = \frac{h^2}{2m}\Delta u + \frac{e^2}{r}u} \tag{12}$$

in all of space $-\infty < x, y, z < +\infty$. Furthermore, we are supposed to have $\iiint |u|^2 dx\, dy\, dz = 1$ (integral over all space). Note that $i = \sqrt{-1}$ and u is complex-valued. The coefficient function e^2/r is called the potential. For any other atom with a single electron, such as a helium ion, e^2 is replaced by Ze^2, where Z is the atomic number. □

What does this mean physically? In quantum mechanics quantities cannot be measured exactly but only with a certain probability. The *wave function* $u(x, y, z, t)$ represents a possible *state* of the electron. If D is *any* region in xyz space, then

$$\iiint_D |u|^2\, dx\, dy\, dz$$

is the probability of finding the electron in the region D at the time t. The *expected z coordinate of the position* of the electron at the time t is the value of the integral

$$\iiint z|u(x, y, z, t)|^2 \, dx \, dy \, dz;$$

similarly for the x and y coordinates. The *expected z coordinate of the momentum* is

$$\iiint -ih\frac{\partial u}{\partial z}(x, y, z, t) \cdot \bar{u}(x, y, z, t) \, dx \, dy \, dz,$$

where $\bar{u}$ is the complex conjugate of u. All other observable quantities are given by operators A, which act on functions. The expected value of the observable A equals

$$\iiint Au(x, y, z, t) \cdot \bar{u}(x, y, z, t) \, dx \, dy \, dz.$$

Thus the position is given by the operator $Au = \mathbf{x}u$, where $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and the momentum is given by the operator $Au = -ih\nabla u$.

Schrödinger's equation is most easily regarded simply as an axiom that leads to the correct physical conclusions, rather than as an equation that can be derived from simpler principles. It explains why atoms are stable and don't collapse. It explains the energy levels of the electron in the hydrogen atom observed by Bohr. In principle, elaborations of it explain the structure of *all* atoms and molecules and so all of chemistry! With many particles, the wave function u depends on time t and all the coordinates of all the particles and so is a function of a large number of variables. The Schrödinger equation then becomes

$$-ihu_t = \sum_{i=1}^{n} \frac{h^2}{2m_i}(u_{x_i x_i} + u_{y_i y_i} + u_{z_i z_i}) + V(x_1, \ldots, z_n)u$$

for n particles, where the potential function V depends on all the $3n$ coordinates. Except for the hydrogen and helium atoms (the latter having two electrons), the mathematical analysis is impossible to carry out completely and cannot be calculated even with the help of the modern computer. Nevertheless, with the use of various approximations, many of the facts about more complicated atoms and the chemical binding of molecules can be understood. □

This has been a brief introduction to the sources of PDEs in physical problems. Many realistic situations lead to much more complicated PDEs. See Chapter 13 for some additional examples.

EXERCISES

1. Carefully derive the equation of a string in a medium in which the resistance is proportional to the velocity.
2. A flexible chain of length l is hanging from one end $x = 0$ but oscillates horizontally. Let the x axis point downward and the u axis point to the right. Assume that the force of gravity at each point of the chain equals the weight of the part of the chain below the point and is directed tangentially along the chain. Assume that the oscillations are small. Find the PDE satisfied by the chain.
3. On the sides of a thin rod, heat exchange takes place (obeying Newton's law of cooling—flux proportional to temperature difference) with a medium of constant temperature T_0. What is the equation satisfied by the temperature $u(x, t)$, neglecting its variation across the rod?
4. Suppose that some particles which are suspended in a liquid medium would be pulled down at the constant velocity $V > 0$ by gravity in the absence of diffusion. Taking account of the diffusion, find the equation for the concentration of particles. Assume homogeneity in the horizontal directions x and y. Let the z axis point upwards.
5. Derive the equation of one-dimensional diffusion in a medium that is moving along the x axis to the right at constant speed V.
6. Consider heat flow in a long circular cylinder where the temperature depends only on t and on the distance r to the axis of the cylinder. Here $r = \sqrt{x^2 + y^2}$ is the cylindrical coordinate. From the three-dimensional heat equation derive the equation $u_t = k(u_{rr} + u_r/r)$.
7. Solve Exercise 6 in a ball except that the temperature depends only on the spherical coordinate $\sqrt{x^2 + y^2 + z^2}$. Derive the equation $u_t = k(u_{rr} + 2u_r/r)$.
8. For the hydrogen atom, if $\int |u|^2\, d\mathbf{x} = 1$ at $t = 0$, show that the same is true at all later times. (*Hint:* Differentiate the integral with respect to t, taking care about the solution being complex valued. Assume that u and $\nabla u \to 0$ fast enough as $|\mathbf{x}| \to \infty$.)
9. This is an exercise on the divergence theorem

$$\iiint_D \nabla \cdot \mathbf{F}\, d\mathbf{x} = \iint_{\text{bdy } D} \mathbf{F} \cdot \mathbf{n}\, dS,$$

valid for any bounded domain D in space with boundary surface bdy D and unit outward normal vector $\mathbf{n}$. If you never learned it, see Section A.3. It is crucial that D be bounded As an exercise, verify it in the following case by calculating both sides separately: $\mathbf{F} = r^2\mathbf{x}$, $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $r^2 = x^2 + y^2 + z^2$, and $D =$ the ball of radius a and center at the origin.

10. If $\mathbf{f}(\mathbf{x})$ is continuous and $|\mathbf{f}(\mathbf{x})| \le 1/(|\mathbf{x}|^3 + 1)$ for all $\mathbf{x}$, show that

$$\iiint_{\text{all space}} \nabla \cdot \mathbf{f}\, d\mathbf{x} = 0.$$

(*Hint:* Take D to be a large ball, apply the divergence theorem, and let its radius tend to infinity.)

11. If curl $\mathbf{v} = \mathbf{0}$ in all of three-dimensional space, show that there exists a scalar function $\phi(x, y, z)$ such that $\mathbf{v} = \text{grad}\, \phi$.

1.4 INITIAL AND BOUNDARY CONDITIONS

Because PDEs typically have so many solutions, as we saw in Section 1.2, we single out one solution by imposing auxiliary conditions. We attempt to formulate the conditions so as to specify a unique solution. These conditions are motivated by the physics and they come in two varieties, initial conditions and boundary conditions.

An *initial condition* specifies the physical state at a particular time t_0. For the diffusion equation the initial condition is

$$u(\mathbf{x}, t_0) = \phi(\mathbf{x}), \tag{1}$$

where $\phi(\mathbf{x}) = \phi(x, y, z)$ is a given function. For a diffusing substance, $\phi(\mathbf{x})$ is the initial concentration. For heat flow, $\phi(\mathbf{x})$ is the initial temperature. For the Schrödinger equation, too, (1) is the usual initial condition.

For the wave equation there is a *pair* of initial conditions

$$u(\mathbf{x},\ t_0) = \phi(\mathbf{x}) \quad \text{and} \quad \frac{\partial u}{\partial t}(\mathbf{x},\ t_0) = \psi(\mathbf{x}), \tag{2}$$

where $\phi(\mathbf{x})$ is the initial position and $\psi(\mathbf{x})$ is the initial velocity. It is clear on physical grounds that both of them must be specified in order to determine the position $u(\mathbf{x}, t)$ at later times. (We shall also prove this mathematically.)

□

In each physical problem we have seen that there is a *domain D* in which the PDE is valid. For the vibrating string, D is the interval $0 < x < l$, so the boundary of D consists only of the two points $x = 0$ and $x = l$. For the drumhead, the domain is a plane region and its boundary is a closed curve. For the diffusing chemical substance, D is the container holding the liquid, so its boundary is a surface $S = \text{bdy}\, D$. For the hydrogen atom, the domain is all of space, so it has no boundary.

It is clear, again from our physical intuition, that it is necessary to specify some *boundary condition* if the solution is to be determined. The three most important kinds of boundary conditions are:

(D) u is specified ("*Dirichlet* condition")

(N) the normal derivative $\partial u/\partial n$ is specified ("*Neumann* condition")

(R) $\partial u/\partial n + au$ is specified ("*Robin* condition")

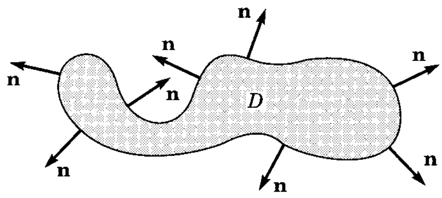

Figure 1

where a is a given function of x, y, z, and t. Each is to hold for all t and for $\mathbf{x} = (x, y, z)$ belonging to bdy D. Usually, we write (D), (N), and (R) as equations. For instance, (N) is written as the equation

$$\frac{\partial u}{\partial n} = g(\mathbf{x}, t) \tag{3}$$

where g is a given function that could be called the boundary datum. Any of these boundary conditions is called *homogeneous* if the specified function $g(\mathbf{x}, t)$ vanishes (equals zero). Otherwise, it is called inhomogenous. As usual, $\mathbf{n} = (n_1, n_2, n_3)$ denotes the unit normal vector on bdy D, which points outward from D (see Figure 1). Also, $\partial u/\partial n \equiv \mathbf{n} \cdot \nabla u$ denotes the directional derivative of u in the outward normal direction.

In *one-dimensional* problems where D is just an interval $0 < x < l$, the boundary consists of just the two endpoints, and these boundary conditions take the simple form

$$\text{(D)} \quad u(0, t) = g(t) \quad \text{and} \quad u(l, t) = h(t)$$

$$\text{(N)} \quad \frac{\partial u}{\partial x}(0, t) = g(t) \quad \text{and} \quad \frac{\partial u}{\partial x}(l, t) = h(t)$$

and similarly for the Robin condition. □

Following are some illustrations of physical problems corresponding to these boundary conditions.

THE VIBRATING STRING

If the string is held *fixed* at both ends, as for a violin string, we have the homogeneous Dirichlet conditions $u(0, t) = u(l, t) = 0$.

Imagine, on the other hand, that one end of the string is *free* to move transversally without any resistance (say, along a frictionless track); then there is no tension T at that end, so $u_x = 0$. This is a Neumann condition.

Third, the Robin condition would be the correct one if one were to imagine that an end of the string were free to move along a track but were attached to a coiled spring or rubber band (obeying Hooke's law) which tended to pull it back to equilibrium position. In that case the string would exchange some of its energy with the coiled spring.

Finally, if an end of the string were simply moved in a specified way, we would have an inhomogeneous Dirichlet condition at that end.

DIFFUSION

If the diffusing substance is enclosed in a container D so that none can escape or enter, then the concentration gradient in the normal direction must vanish, by Fick's law (see Exercise 2). Thus $\partial u/\partial n = 0$ on $S = \text{bdy } D$, which is the Neumann condition.

If, on the other hand, the container is permeable and is so constructed that any substance that escapes to the boundary of the container is immediately washed away, then we have $u = 0$ on S.

HEAT

Heat conduction is described by the diffusion equation with $u(\mathbf{x}, t) =$ temperature. If the object D through which the heat is flowing is perfectly *insulated*, then no heat flows across the boundary and we have the Neumann condition $\partial u/\partial n = 0$ (see Exercise 2).

On the other hand, if the object were immersed in a large *reservoir* of specified temperature $g(t)$ and there were perfect thermal conduction, then we'd have the Dirichlet condition $u = g(t)$ on bdy D.

Suppose that we had a uniform rod insulated along its length $0 \le x \le l$, whose end at $x = l$ were immersed in the reservoir of temperature $g(t)$. If heat were exchanged between the end and the reservoir so as to obey Newton's law of cooling, then

$$\frac{\partial u}{\partial x}(l, t) = -a[u(l, t) - g(t)],$$

where $a > 0$. Heat from the hot rod radiates into the cool reservoir. This is an inhomogeneous Robin condition.

LIGHT

Light is an electromagnetic field and as such is described by Maxwell's equations (see Chapter 13). Each component of the electric and magnetic field satisfies the wave equation. It is through the boundary conditions that the various components are related to each other. (They are "coupled.") Imagine, for example, light reflecting off a ball with a mirrored surface. This is a scattering problem. The domain D where the light is propagating is the exterior of the ball. Certain boundary conditions then are satisfied by the electromagnetic field components. When polarization effects are not being studied, some scientists use the wave equation with homogeneous Dirichlet or Neumann conditions as a considerably simplified model of such a situation.

SOUND

Our ears detect small disturbances in the air. The disturbances are described by the equations of gas dynamics, which form a system of nonlinear equations with velocity $\mathbf{v}$ and density ρ as the unknowns. But *small* disturbances are described quite well by the so-called linearized equations, which are a lot simpler; namely,

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{c_0^2}{\rho_0} \operatorname{grad} \rho = 0 \tag{4}$$

$$\frac{\partial \rho}{\partial t} + \rho_0 \operatorname{div} \mathbf{v} = 0 \tag{5}$$

(four scalar equations altogether). Here ρ_0 is the density and c_0 is the speed of sound in still air.

Assume now that the curl of $\mathbf{v}$ is zero; this means that there are no sound "eddies" and the velocity $\mathbf{v}$ is irrotational. It follows that ρ and all three components of $\mathbf{v}$ satisfy the wave equation:

$$\frac{\partial^2 \mathbf{v}}{\partial t^2} = c_0^2 \, \Delta \mathbf{v} \quad \text{and} \quad \frac{\partial^2 \rho}{\partial t^2} = c_0^2 \, \Delta \rho. \tag{6}$$

The interested reader will find a derivation of these equations in Section 13.2.

Now if we are describing sound propagation in a closed, sound-insulated room D with *rigid* walls, say a concert hall, then the air molecules at the wall can only move parallel to the boundary, so that no sound can travel in a normal direction to the boundary. So $\mathbf{v} \cdot \mathbf{n} = 0$ on bdy D. Since curl $\mathbf{v} = 0$, there is a standard fact in vector calculus (Exercise 1.3.11) which says that there is a "potential" function ψ such that $\mathbf{v} = -\operatorname{grad} \psi$. The potential also satisfies the wave equation $\partial^2\psi/\partial t^2 = c_0^2 \, \Delta\psi$, and the boundary condition for it is $-\mathbf{v} \cdot \mathbf{n} = \mathbf{n} \cdot \operatorname{grad} \psi = 0$ or Neumann's condition for ψ.

At an *open* window of the room D, the atmospheric pressure is a constant and there is no difference of pressure across the window. The pressure p is proportional to the density ρ, for small disturbances of the air. Thus ρ is a constant at the window, which means that ρ satisfies the Dirichlet boundary condition $\rho = \rho_0$.

At a *soft* wall, such as an elastic membrane covering an open window, the pressure difference $p - p_0$ across the membrane is proportional to the normal velocity $\mathbf{v} \cdot \mathbf{n}$, namely

$$p - p_0 = Z \, \mathbf{v} \cdot \mathbf{n},$$

where Z is called the acoustic impedance of the wall. (A rigid wall has a very large impedance and an open window has zero impedance.) Now $p - p_0$ is in turn proportional to $\rho - \rho_0$ for small disturbances. Thus the system of four equations (4),(5) satisfies the boundary condition

$$\mathbf{v} \cdot \mathbf{n} = a(\rho - \rho_0),$$

where a is a constant proportional to $1/Z$. (See [MI] for further discussion.) □

A different kind of boundary condition in the case of the wave equation is

$$\frac{\partial u}{\partial n} + b\frac{\partial u}{\partial t} = 0. \tag{7}$$

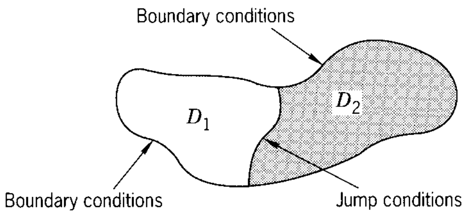

Figure 2

This condition means that energy is *radiated to* ($b > 0$) or *absorbed from* ($b < 0$) the exterior through the boundary. For instance, a vibrating string whose ends are immersed in a viscous liquid would satisfy (7) with $b > 0$ since energy is radiated to the liquid.

CONDITIONS AT INFINITY

In case the domain D is unbounded, the physics usually provides conditions at infinity. These can be tricky. An example is Schrödinger's equation, where the domain D is all of space, and we require that $\int |u|^2\, d\mathbf{x} = 1$. The finiteness of this integral means, in effect, that u "vanishes at infinity."

A second example is afforded by the scattering of acoustic or electromagnetic waves. If we want to study sound or light waves that are radiating outward (to infinity), the appropriate condition at infinity is "Sommerfeld's outgoing radiation condition"

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - \frac{\partial u}{\partial t}\right) = 0, \tag{8}$$

where $r = |\mathbf{x}|$ is the spherical coordinate. (In a given mathematical context this limit would be made more precise.) (See Section 13.3.)

JUMP CONDITIONS

These occur when the domain D has two parts, $D = D_1 \cup D_2$ (see Figure 2), with different physical properties. An example is heat conduction, where D_1 and D_2 consist of two different materials (see Exercise 6).

EXERCISES

1. By trial and error, find a solution of the diffusion equation $u_t = u_{xx}$ with the initial condition $u(x, 0) = x^2$.
2. (a) Show that the temperature of a metal rod, insulated at the end $x = 0$, satisfies the boundary condition $\partial u/\partial x = 0$. (Use Fourier's law.)
   (b) Do the same for the diffusion of gas along a tube that is closed off at the end $x = 0$. (Use Fick's law.)

(c) Show that the three-dimensional version of (a) (insulated solid) or (b) (impermeable container) leads to the boundary condition $\partial u/\partial n = 0$.

3. A homogeneous body occupying the solid region D is completely insulated. Its initial temperature is $f(\mathbf{x})$. Find the steady-state temperature that it reaches after a long time. (*Hint:* No heat is gained or lost.)

4. A rod occupying the interval $0 \le x \le l$ is subject to the heat source $f(x) = 0$ for $0 < x < \frac{l}{2}$, and $f(x) = H$ for $\frac{l}{2} < x < l$ where $H > 0$. The rod has physical constants $c = \rho = \kappa = 1$, and its ends are kept at zero temperature.
   (a) Find the steady-state temperature of the rod.
   (b) Which point is the hottest, and what is the temperature there?

5. In Exercise 1.3.4, find the boundary condition if the particles lie above an impermeable horizontal plane $z = a$.

6. Two homogeneous rods have the same cross section, specific heat c, and density ρ but different heat conductivities κ_1 and κ_2 and lengths L_1 and L_2. Let $k_j = \kappa_j/c\rho$ be their diffusion constants. They are welded together so that the temperature u and the heat flux κu_x at the weld are continuous. The left-hand rod has its left end maintained at temperature zero. The right-hand rod has its right end maintained at temperature T degrees.
   (a) Find the *equilibrium* temperature distribution in the composite rod.
   (b) Sketch it as a function of x in case $k_1 = 2,\ k_2 = 1,\ L_1 = 3,\ L_2 = 2$, and $T = 10$. (This exercise requires a lot of elementary algebra, but it's worth it.)

7. In linearized gas dynamics (sound), verify the following.
   (a) If curl $\mathbf{v} = \mathbf{0}$ at $t = 0$, then curl $\mathbf{v} = \mathbf{0}$ at all later times.
   (b) Each component of $\mathbf{v}$ and ρ satifies the wave equation.

1.5 WELL-POSED PROBLEMS

Well-posed problems consist of a PDE in a domain together with a set of initial and/or boundary conditions (or other auxiliary conditions) that enjoy the following fundamental properties:

(i) *Existence:* There exists at least one solution $u(x, t)$ satisfying all these conditions.

(ii) *Uniqueness:* There is at most one solution.

(iii) *Stability:* The unique solution $u(x, t)$ depends in a stable manner on the data of the problem. This means that if the data are changed a little, the corresponding solution changes only a little.

For a physical problem modeled by a PDE, the scientist normally tries to formulate physically realistic auxiliary conditions which all together make a well-posed problem. The mathematician tries to prove that a given problem is or is not well-posed. If too few auxiliary conditions are imposed, then there may be more than one solution (*nonuniqueness*) and the problem is called underdetermined. If, on the other hand, there are too many auxiliary conditions, there may be no solution at all (*nonexistence*) and the problem is called overdetermined.

The stability property (iii) is normally required in models of physical problems. This is because you could never measure the data with mathematical precision but only up to some number of decimal places. You cannot distinguish a set of data from a tiny perturbation of it. The solution ought not be significantly affected by such tiny perturbations, so it should change very little.

Let us take an example. We know that a vibrating string with an external force, whose ends are moved in a specified way, satisfies the problem

$$\begin{gathered} Tu_{tt} - \rho u_{xx} = f(x,t) \\ u(x,0) = \phi(x) \qquad u_t(x,0) = \psi(x) \\ u(0,t) = g(t) \qquad u(L,t) = h(t) \end{gathered} \tag{1}$$

for $0 < x < L$. The *data* for this problem consist of the five functions $f(x,t)$, $\phi(x)$, $\psi(x)$, $g(t)$, and $h(t)$. Existence and uniqueness would mean that there is exactly one solution $u(x, t)$ for arbitrary (differentiable) functions f, ϕ, ψ, g, h. Stability would mean that if any of these five functions are slightly perturbed, then u is also changed only slightly. To make this precise requires a definition of the "*nearness*" of functions. Mathematically, this requires the concept of a "distance", "metric", "norm", or "topology" in function space and will be discussed in the context of specific examples (see Sections 2.3, 3.4, or 5.5). Problem (1) is indeed well-posed if we make the appropriate choice of "nearness."

As a second example, consider the diffusion equation. Given an initial condition $u(x, 0) = f(x)$, we expect a unique solution, in fact, well-posedness, for $t > 0$. But consider the backwards problem! Given $f(x)$, find $u(x, t)$ for $t < 0$. What past behavior could have led up to the concentration $f(x)$ at time 0? Any chemist knows that diffusion is a smoothing process since the concentration of a substance tends to flatten out. Going backward ("*antidiffusion*"), the situation becomes more and more chaotic. Therefore, you would *not* expect well-posedness of the backward-in-time problem for the diffusion equation.

As a third example, consider solving a matrix equation instead of a PDE: namely, $Au = b$, where A is an $m \times n$ matrix and b is a given m-vector. The "data" of this problem comprise the vector b. If $m > n$, there are more rows than columns and the system is overdetermined. This means that no solution can exist for certain vectors b; that is, you don't necessarily have existence. If, on the other hand, $n > m$, there are more columns than rows and the system is underdetermined. This means that there are lots of solutions for certain vectors b; that is, you can't have uniqueness.

Now suppose that $m = n$ but A is a singular matrix; that is, $\det A = 0$ or A has no inverse. Then the problem is still ill-posed (neither existence nor uniqueness). It is also unstable. To illustrate the instability further, consider a nonsingular matrix A with one very small eigenvalue. The solution is unique but if b is slightly perturbed, then the error will be greatly magnified in the solution u. Such a matrix, in the context of scientific computation, is called ill-conditioned. The ill-conditioning comes from the instability of the matrix equation with a singular matrix.

As a fourth example, consider Laplace's equation $u_{xx} + u_{yy} = 0$ in the region $D = \{-\infty < x < \infty, 0 < y < \infty\}$. It is *not* a well-posed problem to specify both u and u_y on the boundary of D, for the following reason. It has the solutions

$$u_n(x, y) = \frac{1}{n} e^{-\sqrt{n}} \sin nx \sinh ny. \tag{2}$$

Notice that they have boundary data $u_n(x, 0) = 0$ and $\partial u_n/\partial y(x, 0) = e^{-\sqrt{n}} \sin nx$, which tends to zero as $n \to \infty$. But for $y \neq 0$ the solutions $u_n(x, y)$ do not tend to zero as $n \to \infty$. Thus the stability condition (iii) is violated.

EXERCISES

1. Consider the problem

$$\frac{d^2u}{dx^2} + u = 0$$
$$u(0) = 0 \quad \text{and} \quad u(L) = 0,$$

consisting of an ODE and a pair of boundary conditions. Clearly, the function $u(x) \equiv 0$ is a solution. Is this solution *unique, or not*? Does the answer depend on L?

2. Consider the problem

$$u''(x) + u'(x) = f(x)$$
$$u'(0) = u(0) = \tfrac{1}{2}[u'(l) + u(l)],$$

with $f(x)$ a given function.
   (a) Is the solution *unique*? Explain.
   (b) Does a solution necessarily *exist*, or is there a condition that $f(x)$ must satisfy for existence? Explain.

3. Solve the boundary problem $u'' = 0$ for $0 < x < 1$ with $u'(0) + ku(0) = 0$ and $u'(1) \pm ku(1) = 0$. Do the $+$ and $-$ cases separately. What is special about the case $k = 2$?

4. Consider the Neumann problem

$$\Delta u = f(x, y, z) \quad \text{in } D$$
$$\frac{\partial u}{\partial n} = 0 \quad \text{on bdy } D.$$

(a) What can we surely add to any solution to get another solution? So we don't have uniqueness.
(b) Use the divergence theorem and the PDE to show that

$$\iiint_D f(x, y, z)\, dx\, dy\, dz = 0$$

is a necessary condition for the Neumann problem to have a solution.
(c) Can you give a physical interpretation of part (a) and/or (b) for either heat flow or diffusion?

5. Consider the equation

$$u_x + yu_y = 0$$

with the boundary condition $u(x, 0) = \phi(x)$.
(a) For $\phi(x) \equiv x$, show that no solution exists.
(b) For $\phi(x) \equiv 1$, show that there are many solutions.

6. Solve the equation $u_x + 2xy^2u_y = 0$.

1.6 TYPES OF SECOND-ORDER EQUATIONS

In this section we show how the Laplace, wave, and diffusion equations are in some sense typical among all second-order PDEs. However, these three equations are quite different from each other. It is natural that the Laplace equation $u_{xx} + u_{yy} = 0$ and the wave equation $u_{xx} - u_{yy} = 0$ should have very different properties. After all, the *algebraic* equation $x^2 + y^2 = 1$ represents a circle, whereas the equation $x^2 - y^2 = 1$ represents a hyperbola. The parabola is somehow in between.

In general, let's consider the PDE

$$\boxed{a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0.} \qquad (1)$$

This is a linear equation of order two in two variables with six real constant coefficients. (The factor 2 is introduced for convenience.)

Theorem 1. By a linear transformation of the independent variables, the equation can be reduced to one of three forms, as follows.

(i) *Elliptic case:* If $a_{12}^2 < a_{11}a_{22}$, it is reducible to

$$u_{xx} + u_{yy} + \cdots = 0$$

(where $\cdots$ denotes terms of order 1 or 0).

(ii) *Hyperbolic case:* If $a_{12}^2 > a_{11}a_{22}$, it is reducible to

$$u_{xx} - u_{yy} + \cdots = 0.$$

(iii) *Parabolic case:* If $a_{12}^2 = a_{11}a_{22}$, it is reducible to

$$u_{xx} + \cdots = 0$$

(unless $a_{11} = a_{12} = a_{22} = 0$).

The proof is easy and is just like the analysis of conic sections in analytic geometry as either ellipses, hyperbolas, or parabolas. For simplicity, let's suppose that $a_{11} = 1$ and $a_1 = a_2 = a_0 = 0$. By completing the square, we can then write (1) as

$$(\partial_x + a_{12}\partial_y)^2 u + \left(a_{22} - a_{12}^2\right)\partial_y^2 u = 0 \tag{2}$$

(where we use the operator notation $\partial_x = \partial/\partial x$, $\partial_y^2 = \partial^2/\partial y^2$, etc.). In the elliptic case, $a_{12}^2 < a_{22}$. Let $b = (a_{22} - a_{12}^2)^{1/2} > 0$. Introduce the new variables ξ and η by

$$x = \xi, \quad y = a_{12}\xi + b\eta. \tag{3}$$

Then $\partial_\xi = 1 \cdot \partial_x + a_{12}\partial_y$, $\partial_\eta = 0 \cdot \partial_x + b\partial_y$, so that the equation becomes

$$\partial_\xi^2 u + \partial_\eta^2 u = 0, \tag{4}$$

which is Laplace's. The procedure is similar in the other cases. □

Example 1.

Classify each of the equations

(a) $u_{xx} - 5u_{xy} = 0$.
(b) $4u_{xx} - 12u_{xy} + 9u_{yy} + u_y = 0$.
(c) $4u_{xx} + 6u_{xy} + 9u_{yy} = 0$.

Indeed, we check the sign of the "discriminant" $\mathscr{D} = a_{12}^2 - a_{11}a_{22}$. For (a) we have $\mathscr{D} = (-5/2)^2 - (1)(0) = 25/4 > 0$, so it is hyperbolic. For (b), we have $\mathscr{D} = (-6)^2 - (4)(9) = 36 - 36 = 0$, so it is parabolic. For (c), we have $\mathscr{D} = 3^2 - (4)(9) = 9 - 36 < 0$, so it is elliptic. □

The same analysis can be done in any number of variables, using a bit of linear algebra. Suppose that there are n variables, denoted $x_1, x_2 \ldots, x_n$, and the equation is

$$\sum_{i,j=1}^{n} a_{ij}u_{x_i x_j} + \sum_{i=1}^{n} a_i u_{x_i} + a_0 u = 0, \tag{5}$$

with real constants a_{ij}, a_i, and a_0. Since the mixed derivatives are equal, we may as well assume that $a_{ij} = a_{ji}$. Let $\mathbf{x} = (x_1, \ldots, x_n)$. Consider any linear change of independent variables:

$$(\xi_1, \ldots, \xi_n) = \boldsymbol{\xi} = B\mathbf{x},$$

where B is an $n \times n$ matrix. That is,

$$\xi_k = \sum_m b_{km}x_m. \tag{6}$$

Convert to the new variables using the chain rule:

$$\frac{\partial}{\partial x_i} = \sum_k \frac{\partial \xi_k}{\partial x_i} \frac{\partial}{\partial \boldsymbol{\xi}_k}$$

and

$$u_{x_i x_j} = \left(\sum_k b_{ki} \frac{\partial}{\partial \xi_k} \right) \left(\sum_l b_{lj} \frac{\partial}{\partial \xi_l} \right) u.$$

Therefore the PDE is converted to

$$\sum_{i,j} a_{ij} u_{x_i x_j} = \sum_{k,l} \left(\sum_{i,j} b_{ki} a_{ij} b_{lj} \right) u_{\xi_k \xi_l}. \tag{7}$$

(Watch out that on the left side u is considered as a function of $\mathbf{x}$, whereas on the right side it is considered as a function of $\boldsymbol{\xi}$.) So you get a second-order equation in the new variables $\boldsymbol{\xi}$, but with the *new coefficient matrix* given within the parentheses. That is, the new matrix is

$$BA{}^tB,$$

where $A = (a_{ij})$ is the original coefficient matrix, the matrix $B = (b_{ij})$ defines the transformation, and ${}^tB = (b_{ji})$ is its transpose.

Now a theorem of linear algebra says that for any symmetric real matrix A, there is a rotation B (an orthogonal matrix with determinant 1) such that $BA{}^tB$ is the diagonal matrix

$$BA{}^tB = D = \begin{pmatrix} d_1 & & & & & \\ & d_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & d_n \end{pmatrix}. \tag{8}$$

The real numbers $d_1, \ldots, d_n$ are the eigenvalues of A. Finally, a change of scale would convert D into a diagonal matrix with each of the d's equal to $+1$, -1, or 0. (This is what we did, in effect, early in this section for the case $n = 2$.)

Thus any PDE of the form (5) can be converted by means of a linear change of variables into a PDE with a diagonal coefficient matrix.

Definition. The PDE (5) is called *elliptic* if all the eigenvalues $d_1, \ldots, d_n$ are positive or all are negative. [This is equivalent to saying that the original coefficient matrix A (or $-A$) is positive definite.] The PDE is called *hyperbolic* if none of the $d_1, \ldots, d_n$ vanish and one of them has the opposite sign from the $(n - 1)$ others. If none vanish, but at least two of them are positive and at least two are negative, it is called *ultrahyperbolic*. If exactly one of the eigenvalues is zero and all the others have the same sign, the PDE is called *parabolic*.

Ultrahyperbolic equations occur quite rarely in physics and mathematics, so we shall not discuss them further. Just as each of the three conic sections has quite distinct properties (boundedness, shape, asymptotes), so do each of the three main types of PDEs. □

More generally, if the coefficients are variable, that is, the a_{ij} are functions of $\mathbf{x}$, the equation may be elliptic in one region and hyperbolic in another.

Example 2.

Find the regions in the xy plane where the equation

$$yu_{xx} - 2u_{xy} + xu_{yy} = 0$$

is elliptic, hyperbolic, or parabolic. Indeed, $\mathscr{D} = (-1)^2 - (y)(x) = 1 - xy$. So the equation is parabolic on the hyperbola ($xy = 1$), elliptic in the two convex regions ($xy > 1$), and hyperbolic in the connected region ($xy < 1$). □

If the equation is nonlinear, the regions of ellipticity (and so on) may depend on which solution we are considering. Sometimes nonlinear transformations, instead of linear transformations such as B above, are important. But this is a complicated subject that is poorly understood.

EXERCISES

1. What is the type of each of the following equations?
   (a) $u_{xx} - u_{xy} + 2u_y + u_{yy} - 3u_{yx} + 4u = 0$.
   (b) $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0$.
2. Find the regions in the xy plane where the equation

$$(1 + x)u_{xx} + 2xyu_{xy} - y^2u_{yy} = 0$$

   is elliptic, hyperbolic, or parabolic. Sketch them.
3. Among all the equations of the form (1), show that the only ones that are unchanged under all rotations (*rotationally invariant*) have the form $a(u_{xx} + u_{yy}) + bu = 0$.
4. What is the *type* of the equation

$$u_{xx} - 4u_{xy} + 4u_{yy} = 0?$$

   Show by direct substitution that $u(x, y) = f(y + 2x) + xg(y + 2x)$ is a solution for arbitrary functions f and g.
5. Reduce the elliptic equation

$$u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u = 0$$

to the form $v_{xx} + v_{yy} + cv = 0$ by a change of dependent variable $u = ve^{\alpha x + \beta y}$ and then a change of scale $y' = \gamma y$.

6. Consider the equation $3u_y + u_{xy} = 0$.
   (a) What is its type?
   (b) Find the general solution. (*Hint:* Substitute $v = u_y$.)
   (c) With the auxiliary conditions $u(x, 0) = e^{-3x}$ and $u_y(x, 0) = 0$, does a solution exist? Is it unique?

2

WAVES AND DIFFUSIONS

In this chapter we study the wave and diffusion equations on the whole real line $-\infty < x < +\infty$. Real physical situations are usually on finite intervals. We are justified in taking x on the whole real line for two reasons. Physically speaking, if you are sitting far away from the boundary, it will take a certain time for the boundary to have a substantial effect on you, and until that time the solutions we obtain in this chapter are valid. Mathematically speaking, the absence of a boundary is a big simplification. The most fundamental properties of the PDEs can be found most easily without the complications of boundary conditions. That is the purpose of this chapter. We begin with the wave equation.

2.1 THE WAVE EQUATION

We write the wave equation as

$$\boxed{u_{tt} = c^2 u_{xx} \qquad \text{for } -\infty < x < +\infty.} \tag{1}$$

(Physically, you can imagine a very long string.) This is the simplest second-order equation. The reason is that the operator factors nicely:

$$u_{tt} - c^2 u_{xx} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = 0. \tag{2}$$

This means that, starting from a function $u(x, t)$, you compute $u_t + cu_x$, call the result v, then you compute $v_t - cv_x$, and you ought to get the zero function. The *general solution* is

$$\boxed{u(x, t) = f(x + ct) + g(x - ct)} \tag{3}$$

where f and g are two *arbitrary* (twice differentiable) functions of a single variable.

Proof. Because of (2), if we let $v = u_t + cu_x$, we must have $v_t - cv_x = 0$. Thus we have two first-order equations

$$v_t - cv_x = 0 \tag{4a}$$

and

$$u_t + cu_x = v. \tag{4b}$$

These two first-order equations are equivalent to (1) itself. Let's solve them one at a time. As we know from Section 1.2, equation (4a) has the solution $v(x,t) = h(x+ct)$, where h is any function.

So we must solve the other equation, which now takes the form

$$u_t + cu_x = h(x+ct) \tag{4c}$$

for the unknown function $u(x, t)$. It is easy to check directly by differentiation that one solution is $u(x, t) = f(x + ct)$, where $f'(s) = h(s)/2c$. [A prime ($'$) denotes the derivative of a function of one variable.] To the solution $f(x+ct)$ we can add $g(x - ct)$ to get another solution (since the equation is linear). The most general solution of (4b) in fact turns out to be a particular solution plus any solution of the homogeneous equation; that is,

$$u(x,t) = f(x+ct) + g(x-ct),$$

as asserted by the theorem. The complete justification is left to be worked out in Exercise 4.

A different method to derive the solution formula (3) is to introduce the *characteristic coordinates*

$$\xi = x + ct \qquad \eta = x - ct.$$

By the chain rule, we have $\partial_x = \partial_\xi + \partial_\eta$ and $\partial_t = c\partial_\xi + c\partial_\eta$. Therefore, $\partial_t - c\partial_x = -2c\partial_\eta$ and $\partial_t + c\partial_x = 2c\partial_\xi$. So equation (1) takes the form

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = (-2c\partial_\xi)(2c\partial_\eta)u = 0,$$

which means that $u_{\xi\eta} = 0$ since $c \neq 0$. The solution of this transformed equation is

$$u = f(\xi) + g(\eta)$$

(see Section 1.1), which agrees exactly with the previous answer (3). □

The wave equation has a nice simple geometry. There are *two* families of characteristic lines, $x \pm ct =$ constant, as indicated in Figure 1. The most general solution is the sum of two functions. One, $g(x - ct)$, is a wave of arbitrary shape traveling to the *right* at speed c. The other, $f(x + ct)$, is another shape traveling to the *left* at speed c. A "movie" of $g(x - ct)$ is sketched in Figure 1 of Section 1.3.

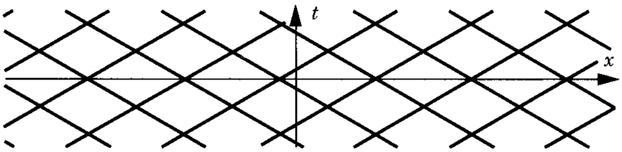

Figure 1

INITIAL VALUE PROBLEM

The initial-value problem is to solve the wave equation

$$\boxed{u_{tt} = c^2 u_{xx} \qquad \text{for } -\infty < x < +\infty} \tag{1}$$

with the initial conditions

$$\boxed{u(x, 0) = \phi(x) \qquad u_t(x, 0) = \psi(x),} \tag{5}$$

where ϕ and ψ are arbitrary functions of x. There is one, and only one, solution of this problem. For instance, if $\phi(x) = \sin x$ and $\psi(x) = 0$, then $u(x, t) = \sin x$ $\cos ct$.

The solution of (1),(5) is easily found from the general formula (3). First, setting $t = 0$ in (3), we get

$$\phi(x) = f(x) + g(x). \tag{6}$$

Then, using the chain rule, we differentiate (3) with respect to t and put $t = 0$ to get

$$\psi(x) = cf'(x) - cg'(x). \tag{7}$$

Let's regard (6) and (7) as two equations for the two unknown functions f and g. To solve them, it is convenient temporarily to change the name of the variable to some neutral name; we change the name of x to s. Now we differentiate (6) and divide (7) by c to get

$$\phi' = f' + g' \qquad \text{and} \qquad \frac{1}{c}\psi = f' - g'.$$

Adding and subtracting the last pair of equations gives us

$$f' = \frac{1}{2}\left(\phi' + \frac{\psi}{c}\right) \qquad \text{and} \qquad g' = \frac{1}{2}\left(\phi' - \frac{\psi}{c}\right).$$

Integrating, we get

$$f(s) = \frac{1}{2}\phi(s) + \frac{1}{2c}\int_0^s \psi + A$$

and

$$g(s) = \frac{1}{2}\phi(s) - \frac{1}{2c}\int_0^s \psi + B,$$

where A and B are constants. Because of (6), we have $A + B = 0$. This tells us what f and g are in the general formula (3). Substituting $s = x + ct$ into the formula for f and $s = x - ct$ into that of g, we get

$$u(x,t) = \frac{1}{2}\phi(x+ct) + \frac{1}{2c}\int_0^{x+ct} \psi + \frac{1}{2}\phi(x-ct) - \frac{1}{2c}\int_0^{x-ct} \psi.$$

This simplifies to

$$\boxed{u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds.} \qquad (8)$$

This is the solution formula for the initial-value problem, due to d'Alembert in 1746. Assuming ϕ to have a continuous second derivative (written $\phi \in C^2$) and ψ to have a continuous first derivative ($\psi \in C^1$), we see from (8) that u itself has continuous second partial derivatives in x and t ($u \in C^2$). Then (8) is a *bona fide* solution of (1) and (5). You may check this directly by differentiation and by setting $t = 0$.

Example 1.

For $\phi(x) \equiv 0$ and $\psi(x) = \cos x$, the solution is $u(x,t) = (1/2c)$ $[\sin(x+ct) - \sin(x-ct)] = (1/c)\cos x \sin ct$. Checking this result directly, we have $u_{tt} = -c\cos x \sin ct$, $u_{xx} = -(1/c)\cos x \sin ct$, so that $u_{tt} = c^2 u_{xx}$. The initial condition is easily checked. □

Example 2. The Plucked String

For a vibrating string the speed is $c = \sqrt{T/\rho}$. Consider an infinitely long string with initial position

$$\phi(x) = \begin{cases} b - \dfrac{b|x|}{a} & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases} \qquad (9)$$

and initial velocity $\psi(x) \equiv 0$ for all x. This is a "three-finger" pluck, with all three fingers removed at once. A "movie" of this solution $u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)]$ is shown in Figure 2. (Even though this solution is not twice differentiable, it can be shown to be a "weak" solution, as discussed later in Section 12.1.)

Each of these pictures is the sum of two triangle functions, one moving to the right and one to the left, as is clear graphically. To write down the formulas that correspond to the pictures requires a lot more work. The formulas depend on the relationships among the five numbers $0, \pm a, x \pm ct$. For instance, let $t = a/2c$. Then $x \pm ct = x \pm a/2$. First, if $x < -3a/2$, then $x \pm a/2 < -a$ and $u(x, t) \equiv 0$. Second, if $-3a/2 < x < -a/2$, then

$$u(x, t) = \frac{1}{2}\phi\left(x + \frac{1}{2}a\right) = \frac{1}{2}\left(b - \frac{b|x + \frac{1}{2}a|}{a}\right) = \frac{3b}{4} + \frac{bx}{2a}.$$

Third, if $|x| < a/2$, then

$$\begin{aligned} u(x, t) &= \frac{1}{2}\left[\phi\left(x + \frac{1}{2}a\right) + \phi\left(x - \frac{1}{2}a\right)\right] \\ &= \frac{1}{2}\left[b - \frac{b\left(x + \frac{1}{2}a\right)}{a} + b - \frac{b\left(\frac{1}{2}a - x\right)}{a}\right] \\ &= \frac{1}{2}b \end{aligned}$$

and so on [see Figure 2]. □

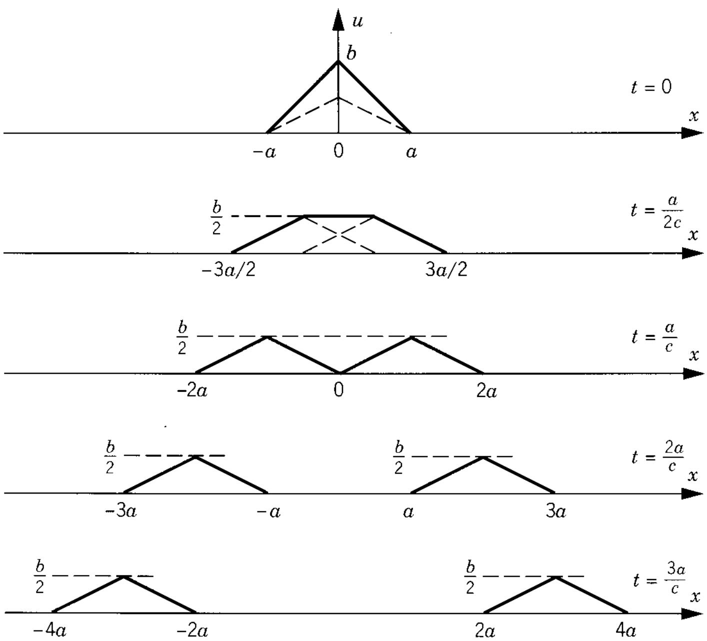

Figure 2

EXERCISES

1. Solve $u_{tt} = c^2 u_{xx},\ u(x, 0) = e^x, u_t(x, 0) = \sin x$.
2. Solve $u_{tt} = c^2 u_{xx},\ u(x, 0) = \log(1 + x^2), u_t(x, 0) = 4 + x$.
3. The midpoint of a piano string of tension T, density ρ, and length l is hit by a hammer whose head diameter is $2a$. A flea is sitting at a distance $l/4$ from one end. (Assume that $a < l/4$; otherwise, poor flea!) How long does it take for the disturbance to reach the flea?
4. Justify the conclusion at the beginning of Section 2.1 that *every* solution of the wave equation has the form $f(x + ct) + g(x - ct)$.
5. (*The hammer blow*) Let $\phi(x) \equiv 0$ and $\psi(x) = 1$ for $|x| < a$ and $\psi(x) = 0$ for $|x| \geq a$. Sketch the string profile (u versus x) at each of the successive instants $t = a/2c$, a/c, $3a/2c$, $2a/c$, and $5a/c$. [*Hint:* Calculate

$$u(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s)\, ds = \frac{1}{2c} \{\text{length of } (x - ct, x + ct) \cap (-a, a)\}.$$

Then $u(x, a/2c) = (1/2c)$ {length of $(x - a/2,\ x + a/2) \cap (-a,\ a)$}. This takes on different values for $|x| < a/2$, for $a/2 < x < 3a/2$, and for $x > 3a/2$. Continue in this manner for each case.]
6. In Exercise 5, find the greatest displacement, $\max_x u(x, t)$, as a function of t.
7. If both ϕ and ψ are odd functions of x, show that the solution $u(x, t)$ of the wave equation is also odd in x for all t.
8. A *spherical wave* is a solution of the three-dimensional wave equation of the form $u(r, t)$, where r is the distance to the origin (the spherical coordinate). The wave equation takes the form

$$u_{tt} = c^2 \left(u_{rr} + \frac{2}{r} u_r \right) \qquad \text{("spherical wave equation").}$$

   (a) Change variables $v = ru$ to get the equation for v: $v_{tt} = c^2 v_{rr}$.
   (b) Solve for v using (3) and thereby solve the spherical wave equation.
   (c) Use (8) to solve it with initial conditions $u(r, 0) = \phi(r)$, $u_t(r, 0) = \psi(r)$, taking both $\phi(r)$ and $\psi(r)$ to be even functions of r.
9. Solve $u_{xx} - 3u_{xt} - 4u_{tt} = 0$, $u(x, 0) = x^2$, $u_t(x, 0) = e^x$. (*Hint:* Factor the operator as we did for the wave equation.)
10. Solve $u_{xx} + u_{xt} - 20u_{tt} = 0, \quad u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x)$.
11. Find the general solution of $3u_{tt} + 10u_{xt} + 3u_{xx} = \sin(x + t)$.

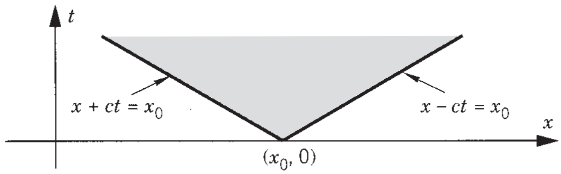

Figure 1

2.2 CAUSALITY AND ENERGY

CAUSALITY

We have just learned that the effect of an initial position $\phi(x)$ is a pair of waves traveling in either direction at speed c and at half the original amplitude. The effect of an initial velocity ψ is a wave spreading out at speed $\leq c$ in both directions (see Exercise 2.1.5 for an example). So part of the wave may lag behind (if there is an initial velocity), but *no part goes faster than speed c*. The last assertion is called the *principle of causality*. It can be visualized in the xt plane in Figure 1.

An initial condition (position or velocity or both) at the point $(x_0, 0)$ can affect the solution for $t > 0$ only in the shaded sector, which is called the *domain of influence* of the point $(x_0, 0)$. As a consequence, if ϕ and ψ vanish for $|x| > R$, then $u(x, t) = 0$ for $|x| > R + ct$. In words, the domain of influence of an interval ($|x| \leq R$) is a sector ($|x| \leq R + ct$).

An "inverse" way to express causality is the following. *Fix* a point (x, t) for $t > 0$ (see Figure 2). How is the number $u(x, t)$ synthesized from the initial data ϕ, ψ? It depends only on the values of ϕ at the two points $x \pm ct$, and it depends only on the values of ψ within the interval $[x - ct, x + ct]$. We therefore say that the interval $(x - ct, x + ct)$ is the interval of dependence of the point (x, t) on $t = 0$. Sometimes we call the entire shaded triangle Δ the *domain of dependence* or the *past history* of the point (x, t). The domain of dependence is bounded by the pair of characteristic lines that pass through (x, t).

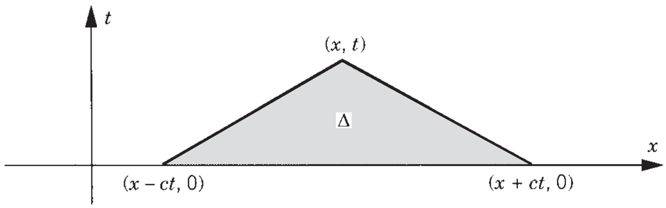

Figure 2

ENERGY

Imagine an infinite string with constants ρ and T. Then $\rho u_{tt} = T u_{xx}$ for $-\infty < x < +\infty$. From physics we know that the kinetic energy is $\frac{1}{2}mv^2$, which in our case takes the form $\text{KE} = \frac{1}{2}\rho \int u_t^2\,dx$. This integral, and the following ones, are evaluated from $-\infty$ to $+\infty$. To be sure that the integral converges, we assume that $\phi(x)$ and $\psi(x)$ vanish outside an interval $\{|x| \le R\}$. As mentioned above, $u(x, t)$ [and therefore $u_t(x, t)$] vanish for $|x| > R + ct$. Differentiating the kinetic energy, we can pass the derivative under the integral sign (see Section A.3) to get

$$\frac{dKE}{dt} = \rho \int u_t u_{tt}\,dx.$$

Then we substitute the PDE $\rho u_{tt} = T u_{xx}$ and integrate by parts to get

$$\frac{dKE}{dt} = T \int u_t u_{xx}\,dx = T u_t u_x - T \int u_{tx} u_x\,dx.$$

The term $T u_t u_x$ is evaluated at $x = \pm\infty$ and so it vanishes. But the final term is a pure derivative since $u_{tx} u_x = (\frac{1}{2}u_x^2)_t$. Therefore,

$$\frac{dKE}{dt} = -\frac{d}{dt} \int \frac{1}{2} T u_x^2\,dx.$$

Let $PE = \frac{1}{2}T \int u_x^2\,dx$ and let $E = KE + PE$. Then $dKE/dt = -dPE/dt$, or $dE/dt = 0$. Thus

$$\boxed{E = \tfrac{1}{2} \int_{-\infty}^{+\infty} \left(\rho u_t^2 + T u_x^2\right) dx} \tag{1}$$

is a constant independent of t. This is the law of *conservation of energy.*

In physics courses we learn that PE has the interpretation of the potential energy. The only thing we need mathematically is the total energy E. The conservation of energy is one of the most basic facts about the wave equation. Sometimes the definition of E is modified by a constant factor, but that does not affect its conservation. Notice that the energy is necessarily positive. The energy can also be used to derive causality (as will be done in Section 9.1).

Example 1.

The plucked string, Example 2 of Section 2.1, has the energy

$$E = \frac{1}{2} T \int \phi_x^2\,dx = \frac{1}{2} T \left(\frac{b}{a}\right)^2 2a = \frac{Tb^2}{a}. \qquad \square$$

In electromagnetic theory the equations are Maxwell's. Each component of the electric and magnetic fields satisfies the (three-dimensional) wave equation, where c is the speed of light. The principle of causality, discussed above, is the cornerstone of the theory of relativity. It means that a signal located at the position x_0 at the instant t_0 cannot move faster than the speed of light. The domain of influence of (x_0, t_0) consists of all the points that can be reached by a signal of speed c starting from the point x_0 at the time t_0. It turns out that the solutions of the *three*-dimensional wave equation always travel at speeds exactly equal to c and never slower. Therefore, the causality principle is sharper in three dimensions than in one. This sharp form is called *Huygens's principle* (see Chapter 9).

Flatland is an imaginary two-dimensional world. You can think of yourself as a waterbug confined to the surface of a pond. You wouldn't want to live there because Huygens's principle is not valid in two dimensions (see Section 9.2). Each sound you make would automatically mix with the "echoes" of your previous sounds. And each view would be mixed fuzzily with the previous views. Three is the best of all possible dimensions.

EXERCISES

1. Use the energy conservation of the wave equation to prove that the only solution with $\phi \equiv 0$ and $\psi \equiv 0$ is $u \equiv 0$. (*Hint:* Use the first vanishing theorem in Section A.1.)
2. For a solution $u(x, t)$ of the wave equation with $\rho = T = c = 1$, the energy density is defined as $e = \frac{1}{2}(u_t^2 + u_x^2)$ and the momentum density as $p = u_t u_x$.
   (a) Show that $\partial e/\partial t = \partial p/\partial x$ and $\partial p/\partial t = \partial e/\partial x$.
   (b) Show that both $e(x, t)$ and $p(x, t)$ also satisfy the wave equation.
3. Show that the wave equation has the following invariance properties.
   (a) Any translate $u(x - y, t)$, where y is fixed, is also a solution.
   (b) Any derivative, say u_x, of a solution is also a solution.
   (c) The dilated function $u(ax, at)$ is also a solution, for any constant a.
4. If $u(x, t)$ satisfies the wave equation $u_{tt} = u_{xx}$, prove the identity

   $$u(x + h,\ t + k) + u(x - h,\ t - k) = u(x + k,\ t + h) + u(x - k,\ t - h)$$

   for all x, t, h, and k. Sketch the quadrilateral Q whose vertices are the arguments in the identity.
5. For the *damped* string, equation (1.3.3), show that the energy decreases.
6. Prove that, among all possible dimensions, only in three dimensions can one have distortionless spherical wave propagation with attenuation. This means the following. A spherical wave in n-dimensional space satisfies the PDE

   $$u_{tt} = c^2 \left(u_{rr} + \frac{n-1}{r} u_r \right),$$

   where r is the spherical coordinate. Consider such a wave that has the special form $u(r, t) = \alpha(r) f(t - \beta(r))$, where $\alpha(r)$ is called the attenuation and $\beta(r)$ the delay. The question is whether such solutions exist for "arbitrary" functions f.

(a) Plug the special form into the PDE to get an ODE for f.
(b) Set the coefficients of f'', f', and f equal to zero.
(c) Solve the ODEs to see that $n = 1$ or $n = 3$ (unless $u \equiv 0$).
(d) If $n = 1$, show that $\alpha(r)$ is a constant (so that "there is no attenuation").

(T. Morley, *American Mathematical Monthly*, Vol. 27, pp. 69–71, 1985)

2.3 THE DIFFUSION EQUATION

In this section we begin a study of the one-dimensional diffusion equation

$$u_t = ku_{xx}. \tag{1}$$

Diffusions are very different from waves, and this is reflected in the mathematical properties of the equations. Because (1) is harder to solve than the wave equation, we begin this section with a general discussion of some of the properties of diffusions. We begin with the maximum principle, from which we'll deduce the uniqueness of an initial-boundary problem. We postpone until the next section the derivation of the solution formula for (1) on the whole real line.

Maximum Principle. If $u(x, t)$ satisfies the diffusion equation in a rectangle (say, $0 \le x \le l$, $0 \le t \le T$) in space-time, then the maximum value of $u(x, t)$ is assumed either initially ($t = 0$) or on the lateral sides ($x = 0$ or $x = l$) (see Figure 1).

In fact, there is a *stronger version* of the maximum principle which asserts that the maximum cannot be assumed anywhere inside the rectangle but *only on the bottom or the lateral sides* (unless u is a constant). The corners are allowed.

The minimum value has the same property; it too can be attained only on the bottom or the lateral sides. To prove the minimum principle, just apply the maximum principle to $[-u(x, t)]$.

These principles have a natural interpretation in terms of diffusion or heat flow. If you have a rod with no internal heat source, the hottest spot and the

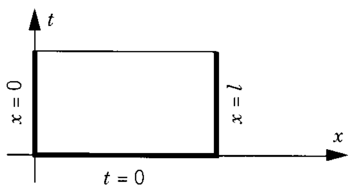

Figure 1

coldest spot can occur only initially or at one of the two ends of the rod. Thus a hot spot at time zero will cool off (unless heat is fed into the rod at an end). You can burn one of its ends but the maximum temperature will always be at the hot end, so that it will be cooler away from that end. Similarly, if you have a substance diffusing along a tube, its highest concentration can occur only initially or at one of the ends of the tube.

If we draw a "movie" of the solution, the maximum drops down while the minimum comes up. So the differential equation tends to smooth the solution out. (This is very different from the behavior of the wave equation!)

Proof of the Maximum Principle. We'll prove only the weaker version. (Surprisingly, its strong form is much more difficult to prove.) For the strong version, see [PW]. The idea of the proof is to use the fact, from calculus, that at an interior maximum the first derivatives vanish and the second derivatives satisfy inequalities such as $u_{xx} \leq 0$. If we knew that $u_{xx} \neq 0$ at the maximum (which we do not), then we'd have $u_{xx} < 0$ as well as $u_t = 0$, so that $u_t \neq ku_{xx}$. This contradiction would show that the maximum could only be somewhere on the boundary of the rectangle. However, because u_{xx} could in fact be equal to zero, we need to play a mathematical game to make the argument work.

So let M denote the maximum value of $u(x, t)$ on the three sides $t = 0$, $x = 0$, and $x = l$. (Recall that any continuous function on any bounded closed set is bounded and assumes its maximum on that set.) We must show that $u(x, t) \leq M$ throughout the rectangle R.

Let ϵ be a positive constant and let $v(x, t) = u(x, t) + \epsilon x^2$. Our goal is to show that $v(x, t) \leq M + \epsilon l^2$ throughout R. Once this is accomplished, we'll have $u(x, t) \leq M + \epsilon(l^2 - x^2)$. This conclusion is true for any $\epsilon > 0$. Therefore, $u(x, t) \leq M$ throughout R, which is what we are trying to prove.

Now from the definition of v, it is clear that $v(x, t) \leq M + \epsilon l^2$ on $t = 0$, on $x = 0$, and on $x = l$. This function v satisfies

$$v_t - kv_{xx} = u_t - k(u + \epsilon x^2)_{xx} = u_t - ku_{xx} - 2\epsilon k = -2\epsilon k < 0, \qquad (2)$$

which is the "diffusion inequality." Now suppose that $v(x, t)$ attains its maximum at an *interior* point (x_0, t_0). That is, $0 < x_0 < l$, $0 < t_0 < T$. By ordinary calculus, we know that $v_t = 0$ and $v_{xx} \leq 0$ at (x_0, t_0). This contradicts the diffusion inequality (2). So there can't be an interior maximum. Suppose now that $v(x, t)$ has a maximum (in the closed rectangle) at a point on the *top* edge $\{t_0 = T \text{ and } 0 < x < l\}$. Then $v_x(x_0, t_0) = 0$ and $v_{xx}(x_0, t_0) \leq 0$, as before. Furthermore, because $v(x_0, t_0)$ is bigger than $v(x_0, t_0 - \delta)$, we have

$$v_t(x_0, t_0) = \lim \frac{v(x_0, t_0) - v(x_0, t_0 - \delta)}{\delta} \geq 0$$

as $\delta \to 0$ through positive values. (This is not an equality because the maximum is only "one-sided" in the variable t.) We again reach a contradiction to the diffusion inequality.

But $v(x,t)$ does have a maximum *somewhere* in the closed rectangle $0 \leq x \leq l,\ 0 \leq t \leq T$. This maximum must be on the bottom or sides. Therefore $v(x,\ t) \leq M + \epsilon l^2$ throughout R. This proves the maximum principle (in its weaker version).

UNIQUENESS

The maximum principle can be used to give a proof of *uniqueness for the Dirichlet problem for the diffusion equation*. That is, there is at most one solution of

$$\begin{aligned} u_t - ku_{xx} &= f(x,t) \quad \text{for } 0 < x < l \text{ and } t > 0 \\ u(x,0) &= \phi(x) \\ u(0,t) &= g(t) \qquad u(l,t) = h(t) \end{aligned} \tag{3}$$

for four given functions f, ϕ, g, and h. Uniqueness means that any solution is determined completely by its initial and boundary conditions. Indeed, let $u_1(x,\ t)$ and $u_2(x,\ t)$ be two solutions of (3). Let $w = u_1 - u_2$ be their difference. Then $w_t - kw_{xx} = 0,\ w(x,0) = 0,\ w(0,t) = 0,\ w(l,t) = 0$. Let $T > 0$. By the maximum principle, $w(x,t)$ has its maximum for the rectangle on its bottom or sides—exactly where it vanishes. So $w(x,t) \leq 0$. The same type of argument for the minimum shows that $w(x,t) \geq 0$. Therefore, $w(x,t) \equiv 0$, so that $u_1(x,t) \equiv u_2(x,t)$ for all $t \geq 0$.

Here is a second proof of uniqueness for problem (3), by a very different technique, the *energy method*. Multiplying the equation for $w = u_1 - u_2$ by w itself, we can write

$$0 = 0 \cdot w = (w_t - kw_{xx})(w) = \left(\tfrac{1}{2}w^2\right)_t + (-kw_xw)_x + kw_x^2.$$

(Verify this by carrying out the derivatives on the right side.) Upon integrating over the interval $0 < x < l$, we get

$$0 = \int_0^l \left(\tfrac{1}{2}w^2\right)_t dx - kw_x\,w\Big|_{x=0}^{x=l} + k\int_0^l w_x^2\,dx.$$

Because of the boundary conditions ($w = 0$ at $x = 0,\ l$),

$$\frac{d}{dt}\int_0^l \frac{1}{2}[w(x,\ t)]^2\,dx = -k\int_0^l [w_x(x,\ t)]^2\,dx \leq 0,$$

where the time derivative has been pulled out of the x integral (see Section A.3). Therefore, $\int w^2\,dx$ is decreasing, so

$$\int_0^l [w(x,\ t)]^2\,dx \leq \int_0^l [w(x,\ 0)]^2\,dx \tag{4}$$

for $t \geq 0$. The right side of (4) vanishes because the initial conditions of u and v are the same, so that $\int [w(x,t)]^2\,dx = 0$ for all $t > 0$. So $w \equiv 0$ and $u_1 \equiv u_2$ for all $t \geq 0$.

STABILITY

This is the third ingredient of well-posedness (see Section 1.5). It means that the initial and boundary conditions are correctly formulated. The energy method leads to the following form of stability of problem (3), in case $h = g = f = 0$. Let $u_1(x, 0) = \phi_1(x)$ and $u_2(x, 0) = \phi_2(x)$. Then $w = u_1 - u_2$ is the solution with the initial datum $\phi_1 - \phi_2$. So from (4) we have

$$\int_0^l [u_1(x,t) - u_2(x,t)]^2\,dx \le \int_0^l [\phi_1(x) - \phi_2(x)]^2\,dx. \qquad (5)$$

On the right side is a quantity that measures the nearness of the initial data for two solutions, and on the left we measure the nearness of the solutions at any later time. Thus, if we start nearby (at $t = 0$), we stay nearby. This is exactly the meaning of stability in the "square integral" sense (see Sections 1.5 and 5.4).

The maximum principle also proves the stability, but with a different way to measure nearness. Consider two solutions of (3) in a rectangle. We then have $w \equiv u_1 - u_2 = 0$ on the lateral sides of the rectangle and $w = \phi_1 - \phi_2$ on the bottom. The maximum principle asserts that throughout the rectangle

$$u_1(x,t) - u_2(x,t) \le \max|\phi_1 - \phi_2|.$$

The "minimum" principle says that

$$u_1(x,t) - u_2(x,t) \ge -\max|\phi_1 - \phi_2|.$$

Therefore,

$$\max_{0\le x\le l} |u_1(x,t) - u_2(x,t)| \le \max_{0\le x\le l} |\phi_1(x) - \phi_2(x)|, \qquad (6)$$

valid for all $t > 0$. Equation (6) is in the same spirit as (5), but with a quite different method of measuring the nearness of functions. It is called stability in the "uniform" sense.

EXERCISES

1. Consider the solution $1 - x^2 - 2kt$ of the diffusion equation. Find the locations of its maximum and its minimum in the closed rectangle $\{0 \le x \le 1,\ 0 \le t \le T\}$.
2. Consider a solution of the diffusion equation $u_t = u_{xx}$ in $\{0 \le x \le l,\ 0 \le t < \infty\}$.
   (a) Let $M(T)$ = the maximum of $u(x, t)$ in the closed rectangle $\{0 \le x \le l, 0 \le t \le T\}$. Does $M(T)$ increase or decrease as a function of T?
   (b) Let $m(T)$ = the minimum of $u(x, t)$ in the closed rectangle $\{0 \le x \le l, 0 \le t \le T\}$. Does $m(T)$ increase or decrease as a function of T?
3. Consider the diffusion equation $u_t = u_{xx}$ in the interval (0, 1) with $u(0, t) = u(1, t) = 0$ and $u(x, 0) = 1 - x^2$. Note that this initial function does not satisfy the boundary condition at the left end, but that the solution will satisfy it for all $t > 0$.

(a) Show that $u(x, t) > 0$ at all interior points $0 < x < 1,\ 0 < t < \infty$.
(b) For each $t > 0$, let $\mu(t) =$ the maximum of $u(x, t)$ over $0 \le x \le 1$. Show that $\mu(t)$ is a decreasing (i.e., nonincreasing) function of t. (*Hint:* Let the maximum occur at the point $X(t)$, so that $\mu(t) = u(X(t), t)$. Differentiate $\mu(t)$, assuming that $X(t)$ is differentiable.)
(c) Draw a rough sketch of what you think the solution looks like (u versus x) at a few times. (If you have appropriate software available, compute it.)

4. Consider the diffusion equation $u_t = u_{xx}$ in $\{0 < x < 1,\ 0 < t < \infty\}$ with $u(0, t) = u(1, t) = 0$ and $u(x, 0) = 4x(1 - x)$.
(a) Show that $0 < u(x, t) < 1$ for all $t > 0$ and $0 < x < 1$.
(b) Show that $u(x, t) = u(1 - x, t)$ for all $t \ge 0$ and $0 \le x \le 1$.
(c) Use the energy method to show that $\int_0^1 u^2\,dx$ is a strictly decreasing function of t.

5. The purpose of this exercise is to show that the maximum principle is not true for the equation $u_t = xu_{xx}$, which has a variable coefficient.
(a) Verify that $u = -2xt - x^2$ is a solution. Find the location of its maximum in the closed rectangle $\{-2 \le x \le 2,\ 0 \le t \le 1\}$.
(b) Where precisely does our proof of the maximum principle break down for this equation?

6. Prove the *comparison principle* for the diffusion equation: If u and v are two solutions, and if $u \le v$ for $t = 0$, for $x = 0$, and for $x = l$, then $u \le v$ for $0 \le t < \infty,\ 0 \le x \le l$.

7. (a) More generally, if $u_t - ku_{xx} = f,\ v_t - kv_{xx} = g,\ f \le g$, and $u \le v$ at $x = 0$, $x = l$ and $t = 0$, prove that $u \le v$ for $0 \le x \le l$, $0 \le t < \infty$.
(b) If $v_t - v_{xx} \ge \sin x$ for $0 \le x \le \pi,\ 0 < t < \infty$, and if $v(0, t) \ge 0$, $v(\pi, t) \ge 0$ and $v(x, 0) \ge \sin x$, use part (a) to show that $v(x, t) \ge (1 - e^{-t})\sin x$.

8. Consider the diffusion equation on $(0, l)$ with the Robin boundary conditions $u_x(0, t) - a_0 u(0, t) = 0$ and $u_x(l, t) + a_l u(l, t) = 0$. If $a_0 > 0$ and $a_l > 0$, use the energy method to show that the endpoints contribute to the decrease of $\int_0^l u^2(x, t)\,dx$. (This is interpreted to mean that part of the "energy" is lost at the boundary, so we call the boundary conditions "*radiating*" or "*dissipative*.")

2.4 DIFFUSION ON THE WHOLE LINE

Our purpose in this section is to solve the problem

$$u_t = ku_{xx} \quad (-\infty < x < \infty,\ 0 < t < \infty) \tag{1}$$
$$u(x, 0) = \phi(x). \tag{2}$$

As with the wave equation, the problem on the infinite line has a certain "purity", which makes it easier to solve than the finite-interval problem. (The effects of boundaries will be discussed in the next several chapters.) Also as with the wave equation, we will end up with an explicit formula. But it will be derived by a method *very different* from the methods used before. (The characteristics for the diffusion equation are just the lines $t =$ constant and play no major role in the analysis.) Because the solution of (1) is not easy to derive, we first set the stage by making some general comments.

Our method is to solve it for a *particular* $\phi(x)$ and then build the general solution from this particular one. We'll use five basic *invariance properties* of the diffusion equation (1).

(a) The *translate* $u(x - y, t)$ of any solution $u(x, t)$ is another solution, for any fixed y.

(b) Any *derivative* (u_x or u_t or u_{xx}, etc.) of a solution is again a solution.

(c) A *linear combination* of solutions of (1) is again a solution of (1). (This is just linearity.)

(d) An *integral* of solutions is again a solution. Thus if $S(x, t)$ is a solution of (1), then so is $S(x - y, t)$ and so is

$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)g(y)\,dy$$

for any function $g(y)$, as long as this improper integral converges appropriately. (We'll worry about convergence later.) In fact, (d) is just a limiting form of (c).

(e) If $u(x,t)$ is a solution of (1), so is the *dilated* function $u(\sqrt{a}\,x, at)$, for any $a > 0$. Prove this by the chain rule: Let $v(x,t) = u(\sqrt{a}\,x, at)$. Then $v_t = [\partial(at)/\partial t]u_t = au_t$ and $v_x = [\partial(\sqrt{a}\,x)/\partial x]u_x = \sqrt{a}\,u_x$ and $v_{xx} = \sqrt{a}\cdot\sqrt{a}\,u_{xx} = a\,u_{xx}$.

Our goal is to find a particular solution of (1) and then to construct all the other solutions using property (d). The particular solution we will look for is the one, denoted $Q(x, t)$, which satisfies the *special initial condition*

$$\boxed{Q(x,0) = 1 \quad \text{for } x > 0 \quad Q(x,0) = 0 \quad \text{for } x < 0.} \tag{3}$$

The reason for this choice is that this initial condition does not change under dilation. We'll find Q in three steps.

Step 1 We'll look for $Q(x, t)$ of the special form

$$Q(x,t) = g(p) \quad \text{where } p = \frac{x}{\sqrt{4kt}} \tag{4}$$

and g is a function of only one variable (to be determined). (The $\sqrt{4k}$ factor is included only to simplify a later formula.)

Why do we expect Q to have this special form? Because property (e) says that equation (1) doesn't "see" the dilation $x \to \sqrt{a}x, t \to at$. Clearly, (3) doesn't change at all under the dilation. So $Q(x, t)$, which is defined by conditions (1) and (3), ought not see the dilation either. How could that happen? In only one way: if Q depends on x and t solely through the combination $x/\sqrt{t}$. For the dilation takes $x/\sqrt{t}$ into $\sqrt{a}x/\sqrt{at} = x/\sqrt{t}$. Thus let $p = x/\sqrt{4kt}$ and look for Q which satisfies (1) and (3) and has the form (4).

Step 2 Using (4), we convert (1) into an ODE for g by use of the chain rule:

$$Q_t = \frac{dg}{dp}\frac{\partial p}{\partial t} = -\frac{1}{2t}\frac{x}{\sqrt{4kt}}g'(p)$$

$$Q_x = \frac{dg}{dp}\frac{\partial p}{\partial x} = \frac{1}{\sqrt{4kt}}g'(p)$$

$$Q_{xx} = \frac{dQ_x}{dp}\frac{\partial p}{\partial x} = \frac{1}{4kt}g''(p)$$

$$0 = Q_t - kQ_{xx} = \frac{1}{t}\left[-\frac{1}{2}pg'(p) - \frac{1}{4}g''(p)\right].$$

Thus

$$g'' + 2pg' = 0.$$

This ODE is easily solved using the integrating factor $\exp\int 2p\,dp = \exp(p^2)$. We get $g'(p) = c_1\exp(-p^2)$ and

$$Q(x, t) = g(p) = c_1\int e^{-p^2}dp + c_2.$$

Step 3 We find a completely explicit formula for Q. We've just shown that

$$Q(x, t) = c_1\int_0^{x/\sqrt{4kt}} e^{-p^2}dp + c_2.$$

This formula is valid only for $t > 0$. Now use (3), expressed as a limit as follows.

$$\text{If } x > 0, \quad 1 = \lim_{t\searrow 0} Q = c_1\int_0^{+\infty} e^{-p^2}dp + c_2 = c_1\frac{\sqrt{\pi}}{2} + c_2.$$

$$\text{If } x < 0, \quad 0 = \lim_{t\searrow 0} Q = c_1\int_0^{-\infty} e^{-p^2}dp + c_2 = -c_1\frac{\sqrt{\pi}}{2} + c_2.$$

See Exercise 6. Here $\lim_{t \searrow 0}$ means limit from the right. This determines the coefficients $c_1 = 1/\sqrt{\pi}$ and $c_2 = \frac{1}{2}$. Therefore, Q is the function

$$\boxed{Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-p^2}\,dp} \tag{5}$$

for $t > 0$. Notice that it does indeed satisfy (1), (3), and (4).

Step 4 Having found Q, we now *define* $S = \partial Q/\partial x$. (The explicit formula for S will be written below.) By property (b), S is also a solution of (1). Given any function ϕ, we also *define*

$$\boxed{u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy \quad \text{for } t > 0.} \tag{6}$$

By property (d), u is another solution of (1). *We claim that u is the unique solution of* (1), (2). To verify the validity of (2), we write

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \frac{\partial Q}{\partial x}(x-y,t)\phi(y)\,dy \\ &= -\int_{-\infty}^{\infty} \frac{\partial}{\partial y}[Q(x-y,t)]\phi(y)\,dy \\ &= +\int_{-\infty}^{\infty} Q(x-y,t)\phi'(y)\,dy - Q(x-y,t)\phi(y)\Big|_{y=-\infty}^{y=+\infty} \end{aligned}$$

upon integrating by parts. We assume these limits vanish. In particular, let's temporarily assume that $\phi(y)$ itself equals zero for $|y|$ large. Therefore,

$$\begin{aligned} u(x,0) &= \int_{-\infty}^{\infty} Q(x-y,0)\phi'(y)\,dy \\ &= \int_{-\infty}^{x} \phi'(y)\,dy = \phi\Big|_{-\infty}^{x} = \phi(x) \end{aligned}$$

because of the initial condition for Q and the assumption that $\phi(-\infty) = 0$. This is the initial condition (2). We conclude that (6) is our solution formula, where

$$\boxed{S = \frac{\partial Q}{\partial x} = \frac{1}{2\sqrt{\pi k t}} e^{-x^2/4kt} \quad \text{for } t > 0.} \tag{7}$$

That is,

$$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)\,dy.} \tag{8}$$

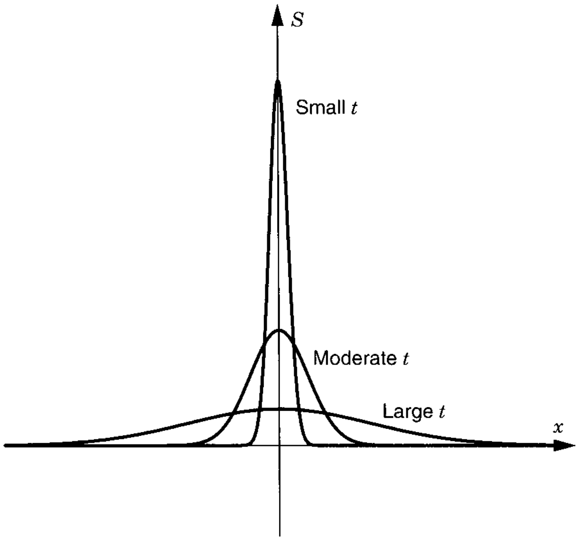

Figure 1

$S(x, t)$ is known as the *source function, Green's function, fundamental solution, gaussian*, or *propagator* of the diffusion equation, or simply the *diffusion kernel*. It gives the solution of (1),(2) with any initial datum ϕ. The formula only gives the solution for $t > 0$. When $t = 0$ it makes no sense. □

The *source function* $S(x, t)$ is defined for all real x and for all $t > 0$. $S(x, t)$ is positive and is even in x $[S(-x, t) = S(x, t)]$. It looks like Figure 1 for various values of t. For large t, it is very spread out. For small t, it is a very tall thin spike (a "*delta function*") of height $(4\pi kt)^{-1/2}$. The area under its graph is

$$\int_{-\infty}^{\infty} S(x, t)\, dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-q^2}\, dq = 1$$

by substituting $q = x/\sqrt{4kt}$, $dq = (dx)/\sqrt{4kt}$ (see Exercise 7). Now look more carefully at the sketch of $S(x, t)$ for a very small t. If we cut out the tall spike, the rest of $S(x, t)$ is very small. Thus

$$\max_{|x|>\delta} S(x, t) \to 0 \qquad \text{as} \quad t \to 0 \tag{9}$$

Notice that the value of the solution $u(x, t)$ given by (6) is a kind of weighted *average* of the initial values around the point x. Indeed, we can write

$$u(x, t) = \int_{-\infty}^{\infty} S(x - y, t)\phi(y)\, dy \simeq \sum_{t} S(x - y_i, t)\phi(y_i)\Delta y_i$$

approximately. This is the average of the solutions $S(x-y_i, t)$ with the weights $\phi(y_i)$. For very small t, the source function is a spike so that the formula exaggerates the values of ϕ near x. For *any* $t > 0$ the solution is a spread-out version of the initial values at $t = 0$.

Here's the physical interpretation. Consider diffusion. $S(x-y, t)$ represents the result of a unit mass (say, 1 gram) of substance located at time zero exactly at the position y which is diffusing (spreading out) as time advances. For any initial distribution of concentration, the amount of substance initially in the interval Δy spreads out in time and contributes approximately the term $S(x-y_i, t)\phi(y_i)\Delta y_i$. All these contributions are added up to get the whole distribution of matter. Now consider heat flow. $S(x-y, t)$ represents the result of a "hot spot" at y at time 0. The hot spot is cooling off and spreading its heat along the rod.

Another physical interpretation is brownian motion, where particles move randomly in space. For simplicity, we assume that the motion is one-dimensional; that is, the particles move along a tube. Then the probability that a particle which begins at position x ends up in the interval (a, b) at time t is precisely $\int_a^b S(x-y, t)\,dy$ for some constant k, where S is defined in (7). In other words, if we let $u(x, t)$ be the probability density (probability per unit length) and if the initial probability density is $\phi(x)$, then the probability at all later times is given by formula (6). That is, $u(x, t)$ satisfies the diffusion equation.

It is usually impossible to evaluate integral (8) completely in terms of elementary functions. Answers to particular problems, that is, to particular initial data $\phi(x)$, are sometimes expressible in terms of the *error function* of statistics,

$$\mathscr{E}\mathrm{rf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2}\,dp. \tag{10}$$

Notice that $\mathscr{E}\mathrm{rf}(0) = 0$. By Exercise 6, $\lim_{x\to+\infty} \mathscr{E}\mathrm{rf}(x) = 1$.

Example 1.

From (5) we can write $Q(x, t)$ in terms of $\mathscr{E}\mathrm{rf}$ as

$$Q(x, t) = \frac{1}{2} + \frac{1}{2}\mathscr{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right).$$ □

Example 2.

Solve the diffusion equation with the initial condition $u(x, 0) = e^{-x}$. To do so, we simply plug this into the general formula (8):

$$u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} e^{-y}\,dy.$$

This is one of the few fortunate examples that can be integrated. The exponent is

$$-\frac{x^2 - 2xy + y^2 + 4kty}{4kt}.$$

Completing the square in the y variable, it is

$$-\frac{(y + 2kt - x)^2}{4kt} + kt - x.$$

We let $p = (y + 2kt - x)/\sqrt{4kt}$ so that $dp = dy/\sqrt{4kt}$. Then

$$u(x, t) = e^{kt-x} \int_{-\infty}^{\infty} e^{-p^2} \frac{dp}{\sqrt{\pi}} = e^{kt-x}.$$

By the maximum principle, a solution in a bounded interval cannot grow in time. However, this particular solution grows, rather than decays, in time. The reason is that the left side of the rod is initially very hot [$u(x, 0) \to +\infty$ as $x \to -\infty$] and the heat gradually diffuses throughout the rod. □

EXERCISES

1. Solve the diffusion equation with the initial condition

$$\phi(x) = 1 \quad \text{for } |x| < l \qquad \text{and} \qquad \phi(x) = 0 \quad \text{for } |x| > l.$$

   Write your answer in terms of $\mathscr{E}\text{rf}(x)$.
2. Do the same for $\phi(x) = 1$ for $x > 0$ and $\phi(x) = 3$ for $x < 0$.
3. Use (8) to solve the diffusion equation if $\phi(x) = e^{3x}$. (You may also use Exercises 6 and 7 below.)
4. Solve the diffusion equation if $\phi(x) = e^{-x}$ for $x > 0$ and $\phi(x) = 0$ for $x < 0$.
5. Prove properties (a) to (e) of the diffusion equation (1).
6. Compute $\int_0^\infty e^{-x^2} dx$. (*Hint:* This is a function that *cannot* be integrated by formula. So use the following trick. Transform the double integral $\int_0^\infty e^{-x^2} dx \cdot \int_0^\infty e^{-y^2} dy$ into polar coordinates and you'll end up with a function that can be integrated easily.)
7. Use Exercise 6 to show that $\int_{-\infty}^\infty e^{-p^2} dp = \sqrt{\pi}$. Then substitute $p = x/\sqrt{4kt}$ to show that

$$\int_{-\infty}^{\infty} S(x, t)\, dx = 1.$$

8. Show that for any fixed $\delta > 0$ (no matter how small),

$$\max_{\delta \le |x| < \infty} S(x, t) \to 0 \qquad \text{as } t \to 0.$$

[This means that the tail of $S(x, t)$ is "uniformly small".]

9. Solve the diffusion equation $u_t = ku_{xx}$ with the initial condition $u(x, 0) = x^2$ by the following special method. First show that u_{xxx} satisfies the diffusion equation with *zero* initial condition. Therefore, by uniqueness, $u_{xxx} \equiv 0$. Integrating this result thrice, obtain $u(x,t) = A(t)x^2 + B(t)x + C(t)$. Finally, it's easy to solve for A, B, and C by plugging into the original problem.

10. (a) Solve Exercise 9 using the general formula discussed in the text. This expresses $u(x, t)$ as a certain integral. Substitute $p = (x - y)/\sqrt{4kt}$ in this integral.
    (b) Since the solution is unique, the resulting formula must agree with the answer to Exercise 9. Deduce the value of

$$\int_{-\infty}^{\infty} p^2 e^{-p^2}\, dp.$$

11. (a) Consider the diffusion equation on the whole line with the usual initial condition $u(x,0) = \phi(x)$. If $\phi(x)$ is an *odd* function, show that the solution $u(x, t)$ is also an *odd* function of x. (*Hint:* Consider $u(-x,t) + u(x,t)$ and use the uniqueness.)
    (b) Show that the same is true if "odd" is replaced by "even."
    (c) Show that the analogous statements are true for the wave equation.

12. The purpose of this exercise is to calculate $Q(x,t)$ approximately for large t. Recall that $Q(x,t)$ is the temperature of an infinite rod that is initially at temperature 1 for $x > 0$, and 0 for $x < 0$.
    (a) Express $Q(x, t)$ in terms of $\mathscr{E}\text{rf}$.
    (b) Find the Taylor series of $\mathscr{E}\text{rf}(x)$ around $x = 0$. (*Hint:* Expand e^z, substitute $z = -y^2$, and integrate term by term.)
    (c) Use the first two nonzero terms in this Taylor expansion to find an approximate formula for $Q(x, t)$.
    (d) *Why* is this formula a good approximation for x fixed and t large?

13. Prove from first principles that $Q(x,t)$ *must* have the form (4), as follows.
    (a) Assuming uniqueness show that $Q(x,t) = Q(\sqrt{a}\,x,\ at)$. This identity is valid for all $a > 0$, all $t > 0$, and all x.
    (b) Choose $a = 1/(4kt)$.

14. Let $\phi(x)$ be a continuous function such that $|\phi(x)| \leq Ce^{ax^2}$. Show that formula (8) for the solution of the diffusion equation makes sense for $0 < t < 1/(4ak)$, but not necessarily for larger t.

15. Prove the uniqueness of the diffusion problem with Neumann boundary conditions:

$$u_t - ku_{xx} = f(x,t) \quad \text{for } 0 < x < l, t > 0 \quad u(x,0) = \phi(x)$$
$$u_x(0,t) = g(t) \quad u_x(l,t) = h(t)$$

by the energy method.

16. Solve the diffusion equation with constant dissipation:

$$u_t - ku_{xx} + bu = 0 \qquad \text{for } -\infty < x < \infty \qquad \text{with } u(x, 0) = \phi(x),$$

where $b > 0$ is a constant. (*Hint:* Make the change of variables $u(x, t) = e^{-bt} v(x, t)$.)

17. Solve the diffusion equation with variable dissipation:

$$u_t - ku_{xx} + bt^2 u = 0 \qquad \text{for } -\infty < x < \infty \qquad \text{with } u(x, 0) = \phi(x),$$

where $b > 0$ is a constant. (*Hint:* The solutions of the ODE $w_t + bt^2 w = 0$ are $Ce^{-bt^3/3}$. So make the change of variables $u(x, t) = e^{-bt^3/3} v(x, t)$ and derive an equation for v.)

18. Solve the heat equation with convection:

$$u_t - ku_{xx} + Vu_x = 0 \qquad \text{for } -\infty < x < \infty \qquad \text{with } u(x, 0) = \phi(x),$$

where V is a constant. (*Hint:* Go to a moving frame of reference by substituting $y = x - Vt$.)

19. (a) Show that $S_2(x, y, t) = S(x, t)S(y, t)$ satisfies the diffusion equation $S_t = k(S_{xx} + S_{yy})$.
    (b) Deduce that $S_2(x, y, t)$ is the source function for two-dimensional diffusions.

2.5 COMPARISON OF WAVES AND DIFFUSIONS

We have seen that the basic property of waves is that information gets transported in both directions at a finite speed. The basic property of diffusions is that the initial disturbance gets spread out in a smooth fashion and gradually disappears. The fundamental properties of these two equations can be summarized in the following table.

Property		**Waves**	**Diffusions**
(i)	Speed of propagation?	Finite ($\leq c$)	Infinite
(ii)	Singularities for $t > 0$?	Transported along characteristics (speed $= c$)	Lost immediately
(iii)	Well-posed for $t > 0$?	Yes	Yes (at least for bounded solutions)
(iv)	Well-posed for $t < 0$?	Yes	No
(v)	Maximum principle	No	Yes
(vi)	Behavior as $t \to +\infty$?	Energy is constant so does not decay	Decays to zero (if ϕ integrable)
(vii)	Information	Transported	Lost gradually

For the wave equation we have seen most of these properties already. That there is no maximum principle is easy to see. Generally speaking, the wave equation just moves information along the characteristic lines. In more than one dimension we'll see that it spreads information in expanding circles or spheres.

For the diffusion equation we discuss property (ii), that singularities are immediately lost, in Section 3.5. The solution is differentiable to all orders even if the initial data are not. Properties (iii), (v), and (vi) have been shown already. The fact that information is gradually lost [property (vii)] is clear from the graph of a typical solution, for instance, from $S(x, t)$.

As for property (i) for the diffusion equation, notice from formula (2.4.8) that the value of $u(x, t)$ depends on the values of the initial datum $\phi(y)$ for *all* y, where $-\infty < y < \infty$. Conversely, the value of ϕ at a point x_0 has an *immediate effect everywhere* (for $t > 0$), even though most of its effect is only for a short time near x_0. Therefore, the *speed of propagation is infinite.* Exercise 2(b) shows that solutions of the diffusion equation can travel at any speed. This is in stark contrast to the wave equation (and all hyperbolic equations).

As for (iv), there are several ways to see that *the diffusion equation is not well-posed for* $t < 0$ ("backward in time"). One way is the following. Let

$$u_n(x, t) = \frac{1}{n} \sin nx \, e^{-n^2kt}. \tag{1}$$

You can check that this satisfies the diffusion equation for all x, t. Also, $u_n(x, 0) = n^{-1} \sin nx \to 0$ uniformly as $n \to \infty$. But consider any $t < 0$, say $t = -1$. Then $u_n(x, -1) = n^{-1} \sin nx \, e^{+kn^2} \to \pm\infty$ uniformly as $n \to \infty$ except for a few x. Thus u_n is close to the zero solution at time $t = 0$ but not at time $t = -1$. This violates the stability, in the uniform sense at least.

Another way is to let $u(x, t) = S(x, t + 1)$. This is a solution of the diffusion equation $u_t = ku_{xx}$ for $t > -1$, $-\infty < x < \infty$. But $u(0, t) \to \infty$ as $t \searrow -1$, as we saw above. So we cannot solve backwards in time with the perfectly nice-looking initial data $(4\pi k)^{-1} e^{-x^2/4}$.

Besides, any physicist knows that heat flow, brownian motion, and so on, are irreversible processes. Going backward leads to chaos.

EXERCISES

1. Show that there is no maximum principle for the wave equation.
2. Consider a traveling wave $u(x, t) = f(x - at)$ where f is a given function of one variable.
   (a) If it is a solution of the wave equation, show that the speed must be $a = \pm c$ (unless f is a linear function).
   (b) If it is a solution of the diffusion equation, find f and show that the speed a is arbitrary.

3. Let u satisfy the diffusion equation $u_t = \frac{1}{2}u_{xx}$. Let

$$v(x,t) = \frac{1}{\sqrt{t}} e^{x^2/2t} v\left(\frac{x}{t}, \frac{1}{t}\right).$$

Show that v satisfies the "backward" diffusion equation $v_t = -\frac{1}{2}v_{xx}$ for $t > 0$.

4. Here is a direct relationship between the wave and diffusion equations. Let $u(x, t)$ solve the wave equation on the whole line with bounded second derivatives. Let

$$v(x,t) = \frac{c}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-s^2c^2/4kt} u(x,s)\, ds.$$

(a) Show that $v(x, t)$ solves the diffusion equation!
(b) Show that $\lim_{t\to 0} v(x,t) = u(x,0)$.

(*Hint:* (a) Write the formula as $v(x,t) = \int_{-\infty}^{\infty} H(s,t)u(x,s)\, ds$, where $H(x, t)$ solves the diffusion equation with constant k/c^2 for $t > 0$. Then differentiate $v(x, t)$ using Section A.3. (b) Use the fact that $H(s, t)$ is essentially the source function of the diffusion equation with the spatial variable s.)

3

REFLECTIONS AND SOURCES

In this chapter we solve the simplest reflection problems, when there is only a single point of reflection at one end of a semi-infinite line. In Chapter 4 we shall begin a systematic study of more complicated reflection problems. In Sections 3.3 and 3.4 we solve problems with sources: that is, the inhomogeneous wave and diffusion equations. Finally, in Section 3.5 we analyze the solution of the diffusion equation more carefully.

3.1 DIFFUSION ON THE HALF-LINE

Let's take the domain to be $D =$ the half-line $(0, \infty)$ and take the *Dirichlet boundary condition* at the single endpoint $x = 0$. So the problem is

$$\boxed{\begin{array}{ll} v_t - kv_{xx} = 0 & \text{in } \{0 < x < \infty, \quad 0 < t < \infty\}, \\ v(x, 0) = \phi(x) & \text{for } t = 0 \\ v(0, t) = 0 & \text{for } x = 0 \end{array}} \tag{1}$$

The PDE is supposed to be satisfied in the open region $\{0 < x < \infty, 0 < t < \infty\}$. If it exists, we know that the solution $v(x, t)$ of this problem is unique because of our discussion in Section 2.3. It can be interpreted, for instance, as the temperature in a very long rod with one end immersed in a reservoir of temperature zero and with insulated sides.

We are looking for a solution formula analogous to (2.4.8). In fact, we shall reduce our new problem to our old one. Our method uses the idea of an *odd function*. Any function $\psi(x)$ that satisfies $\psi(-x) \equiv -\psi(+x)$ is called an odd function. Its graph $y = \psi(x)$ is symmetric with respect to the origin

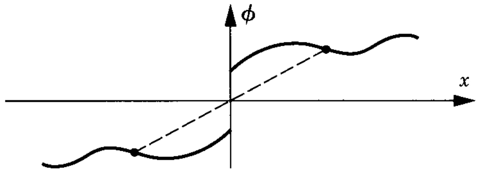

Figure 1

(see Figure 1). Automatically (by putting $x = 0$ in the definition), $\psi(0) = 0$. For a detailed discussion of odd and even functions, see Section 5.2.

Now the initial datum $\phi(x)$ of our problem is defined only for $x \geq 0$. Let ϕ_{odd} be the unique *odd extension* of ϕ to the whole line. That is,

$$\phi_{\text{odd}}(x) = \begin{cases} \phi(x) & \text{for } x > 0 \\ -\phi(-x) & \text{for } x < 0 \\ 0 & \text{for } x = 0. \end{cases} \tag{2}$$

The extension concept too is discussed in Section 5.2.

Let $u(x, t)$ be the solution of

$$\begin{aligned} u_t - ku_{xx} &= 0 \\ u(x, 0) &= \phi_{\text{odd}}(x) \end{aligned} \tag{3}$$

for the *whole line* $-\infty < x < \infty$, $0 < t < \infty$. According to Section 2.3, it is given by the formula

$$u(x, t) = \int_{-\infty}^{\infty} S(x - y,\ t)\phi_{\text{odd}}(y)\,dy. \tag{4}$$

Its "restriction,"

$$v(x, t) = u(x, t) \qquad \text{for } x > 0, \tag{5}$$

will be the unique solution of our new problem (1). There is no difference at all between v and u except that the negative values of x are not considered when discussing v.

Why is $v(x, t)$ the solution of (1)? Notice first that $u(x, t)$ must also be an odd function of x (see Exercise 2.4.11). That is, $u(-x, t) = -u(x, t)$. Putting $x = 0$, it is clear that $u(0, t) = 0$. So the boundary condition $v(0, t) = 0$ is *automatically* satisfied! Furthermore, v solves the PDE as well as the initial condition for $x > 0$, simply because it is equal to u for $x > 0$ and u satisfies the same PDE for all x and the same initial condition for $x > 0$.

The explicit formula for $v(x, t)$ is easily deduced from (4) and (5). From (4) and (2) we have

$$u(x,\ t) = \int_0^{\infty} S(x - y,\ t)\phi(y)\,dy - \int_{-\infty}^{0} S(x - y,\ t)\phi(-y)\,dy.$$

Changing the variable $-y$ to $+y$ in the second integral, we get

$$u(x, t) = \int_0^\infty [S(x - y,\ t) - S(x + y,\ t)]\, \phi(y)\, dy.$$

(Notice the change in the limits of integration.) Hence for $0 < x < \infty$, $0 < t < \infty$, we have

$$\boxed{v(x,\ t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty [e^{-(x-y)^2/4kt} - e^{-(x+y)^2/4kt}]\, \phi(y)\, dy.} \tag{6}$$

This is the complete solution formula for (1).

We have just carried out the *method of odd extensions* or *reflection method*, so called because the graph of $\phi_{\text{odd}}(x)$ is the reflection of the graph of $\phi(x)$ across the origin.

Example 1.

Solve (1) with $\phi(x) \equiv 1$. The solution is given by formula (6). This case can be simplified as follows. Let $p = (x - y)/\sqrt{4kt}$ in the first integral and $q = (x + y)/\sqrt{4kt}$ in the second integral. Then

$$\begin{aligned} u(x, t) &= \int_{-\infty}^{x/\sqrt{4kt}} e^{-p^2}\, dp/\sqrt{\pi} - \int_{x/\sqrt{4kt}}^{+\infty} e^{-q^2}\, dq/\sqrt{\pi} \\ &= \left[\frac{1}{2} + \frac{1}{2}\mathscr{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right)\right] - \left[\frac{1}{2} - \frac{1}{2}\mathscr{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right)\right] \\ &= \mathscr{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right). \end{aligned}$$

□

Now let's play the same game with the *Neumann problem*

$$\boxed{\begin{aligned} &w_t - kw_{xx} = 0 \quad \text{for } 0 < x < \infty,\ 0 < t < \infty \\ &w(x,\ 0) = \phi(x) \\ &w_x(0,\ t) = 0. \end{aligned}} \tag{7}$$

In this case the reflection method is to use *even*, rather than odd, extensions. An even function is a function ψ such that $\psi(-x) = +\psi(x)$. If ψ is an even function, then differentiation shows that its derivative is an odd function. So automatically its slope at the origin is zero: $\psi'(0) = 0$. If $\phi(x)$ is defined only on the half-line, its *even extension* is defined to be

$$\phi_{\text{even}}(x) = \begin{cases} \phi(x) & \text{for } x \geq 0 \\ +\phi(-x) & \text{for } x \leq 0 \end{cases} \tag{8}$$

By the same reasoning as we used above, we end up with an explicit formula for $w(x, t)$. It is

$$\boxed{w(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty [e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt}]\, \phi(y)\, dy.} \tag{9}$$

This is carried out in Exercise 3. Notice that the only difference between (6) and (9) is a single minus sign!

Example 2.

Solve (7) with $\phi(x) = 1$. This is the same as Example 1 except for the single sign. So we can copy from that example:

$$u(x,t) = \left[\frac{1}{2} + \frac{1}{2}\mathscr{E}\text{rf}\left(\frac{x}{4kt}\right)\right] + \left[\frac{1}{2} - \frac{1}{2}\mathscr{E}\text{rf}\left(\frac{x}{4kt}\right)\right] = 1.$$

(That was stupid: We could have guessed it!) □

EXERCISES

1. Solve $u_t = ku_{xx}$; $u(x,0) = e^{-x}$; $u(0,t) = 0$ on the half-line $0 < x < \infty$.
2. Solve $u_t = ku_{xx}$; $u(x,0) = 0$; $u(0,t) = 1$ on the half-line $0 < x < \infty$.
3. Derive the solution formula for the half-line Neumann problem $w_t - kw_{xx} = 0$ for $0 < x < \infty$, $0 < t < \infty$; $w_x(0,t) = 0$; $w(x,0) = \phi(x)$.
4. Consider the following problem with a Robin boundary condition:

$$\begin{aligned}
&\text{DE:} && u_t = ku_{xx} && \text{on the half-line } 0 < x < \infty \\
& && && \quad (\text{and } 0 < t < \infty) \\
&\text{IC:} && u(x,0) = x && \text{for } t = 0 \text{ and } 0 < x < \infty \\
&\text{BC:} && u_x(0,t) - 2u(0,t) = 0 && \text{for } x = 0.
\end{aligned} \tag{$*$}$$

The purpose of this exercise is to verify the solution formula for (*). Let $f(x) = x$ for $x > 0$, let $f(x) = x + 1 - e^{2x}$ for $x < 0$, and let

$$v(x,\ t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\, dy.$$

(a) What PDE and initial condition does $v(x,t)$ satisfy for $-\infty < x < \infty$?
(b) Let $w = v_x - 2v$. What PDE and initial condition does $w(x,t)$ satisfy for $-\infty < x < \infty$?
(c) Show that $f'(x) - 2f(x)$ is an odd function (for $x \neq 0$).
(d) Use Exercise 2.4.11 to show that w is an odd function of x.

(e) Deduce that $v(x, t)$ satisfies (*) for $x > 0$. Assuming uniqueness, deduce that the solution of (*) is given by

$$u(x,\ t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)dy.$$

5. (a) Use the method of Exercise 4 to solve the Robin problem:

DE:	$u_t = ku_{xx}$	on the half-line $0 < x < \infty$ (and $0 < t < \infty$)
IC:	$u(x, 0) = x$	for $t = 0$ and $0 < x < \infty$
BC:	$u_x(0, t) - hu(0, t) = 0$	for $x = 0$,

where h is a constant.

(b) Generalize the method to the case of general initial data $\phi(x)$.

3.2 REFLECTIONS OF WAVES

Now we try the same kind of problem for the wave equation as we did in Section 3.1 for the diffusion equation. We again begin with the *Dirichlet problem* on the half-line $(0, \infty)$. Thus the problem is

DE :	$v_{tt} - c^2 v_{xx} = 0$		for $0 < x < \infty$ and $-\infty < t < \infty$
IC :	$v(x, 0) = \phi(x)$,	$v_t(x, 0) = \psi(x)$	for $t = 0$ and $0 < x < \infty$
BC :	$v(0, t) = 0$		for $x = 0$ and $-\infty < t < \infty$.

(1)

The reflection method is carried out in the same way as in Section 3.1. Consider the *odd* extensions of both of the initial functions to the whole line, $\phi_{\text{odd}}(x)$ and $\psi_{\text{odd}}(x)$. Let $u(x, t)$ be the solution of the initial-value problem on $(-\infty, \infty)$ with the initial data ϕ_{odd} and ψ_{odd}. Then $u(x, t)$ is once again an odd function of x (see Exercise 2.1.7). Therefore, $u(0, t) = 0$, so that the boundary condition is satisfied automatically. Define $v(x, t) = u(x, t)$ for $0 < x < \infty$ [the restriction of u to the half-line]. Then $v(x, t)$ is precisely the solution we are looking for. From the formula in Section 2.1, we have for $x \geq 0$,

$$v(x, t) = u(x,\ t) = \frac{1}{2}[\phi_{\text{odd}}(x + ct) + \phi_{\text{odd}}(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{\text{odd}}(y)dy.$$

Let's "unwind" this formula, recalling the meaning of the odd extensions. First we notice that for $x > c|t|$ only positive arguments occur in the formula,

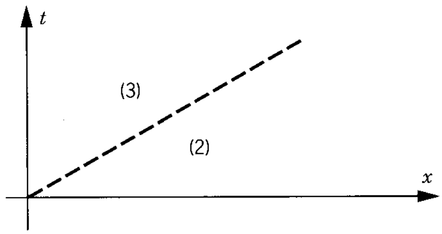

Figure 1

so that $u(x, t)$ is given by the *usual* formula:

$$v(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy \quad \text{for } x > c|t|. \tag{2}$$

But in the *other* region $0 < x < c|t|$, we have $\phi_{\text{odd}}(x - ct) = -\phi(ct - x)$, and so on, so that

$$v(x,t) = \frac{1}{2}[\phi(x+ct) - \phi(ct-x)] + \frac{1}{2c}\int_0^{x+ct} \psi(y)\,dy + \frac{1}{2c}\int_{x-ct}^{0} [-\psi(-y)]\,dy.$$

Notice the switch in signs! In the last term we change variables $y \to -y$ to get $1/2c \int_{ct-x}^{ct+x} \psi(y)\,dy$. Therefore,

$$v(x,\ t) = \frac{1}{2}[\phi(ct+x) - \phi(ct-x)] + \frac{1}{2c}\int_{ct-x}^{ct+x} \psi(y)\,dy \tag{3}$$

for $0 < x < c|t|$. The complete solution is given by the pair of formulas (2) and (3). The two regions are sketched in Figure 1 for $t > 0$.

Graphically, the result can be interpreted as follows. Draw the backward characteristics from the point (x, t). In case (x, t) is in the region $x < ct$, one of the characteristics hits the t axis ($x = 0$) before it hits the x axis, as indicated in Figure 2. The formula (3) shows that *the reflection induces a change of*

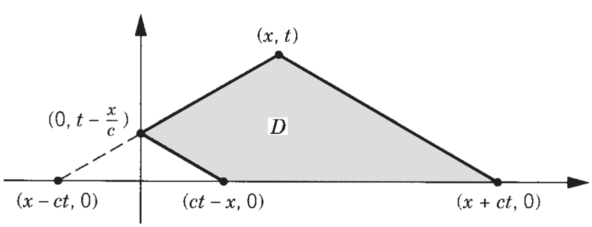

Figure 2

sign. The value of $v(x, t)$ now depends on the values of ϕ at the pair of points $ct \pm x$ and on the values of ψ in the short interval between these points. Note that the other values of ψ have canceled out. The shaded area D in Figure 2 is called the *domain of dependence of the point* (x, t).

The case of the Neumann problem is left as an exercise.

THE FINITE INTERVAL

Now let's consider the guitar string with fixed ends:

$$\boxed{\begin{array}{c} v_{tt} = c^2 v_{xx} \quad v(x, 0) = \phi(x) \quad v_t(x, 0) = \psi(x) \quad \text{for } 0 < x < l, \\ v(0, t) = v(l, t) = 0. \end{array}} \tag{4}$$

This problem is much more difficult because a typical wave will bounce back and forth an infinite number of times. Nevertheless, let's use the method of reflection. This is a bit tricky, so you are invited to skip the rest of this section if you wish.

The initial data $\phi(x)$ and $\psi(x)$ are now given only for $0 < x < l$. We extend them to the whole line to be "*odd*" with respect to *both* $x = 0$ *and* $x = l$:

$$\phi_{\text{ext}}(-x) = -\phi_{\text{ext}}(x) \quad \text{and} \quad \phi_{\text{ext}}(2l - x) = -\phi_{\text{ext}}(x).$$

The simplest way to do this is to define

$$\phi_{\text{ext}}(x) = \begin{cases} \phi(x) & \text{for} \quad 0 < x < l \\ -\phi(-x) & \text{for} \quad -l < x < 0 \\ \text{extended to be of period } 2l. \end{cases}$$

See Figure 3 for an example. And see Section 5.2 for further discussion. "Period $2l$" means that $\phi_{\text{ext}}(x + 2l) = \phi_{\text{ext}}(x)$ for all x. We do exactly the same for $\psi(x)$ (defined for $0 < x < l$) to get $\psi_{\text{ext}}(x)$ defined for $-\infty < x < \infty$.

Now let $u(x, t)$ be the solution of the infinite line problem with the extended initial data. Let v be the restriction of u to the interval $(0, l)$. Thus $v(x, t)$ is

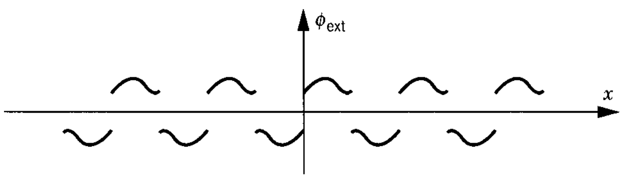

Figure 3

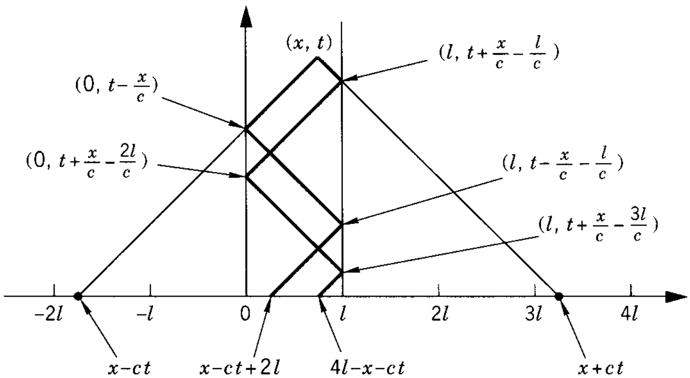

Figure 4

given by the formula

$$v(x, t) = \frac{1}{2}\phi_{\text{ext}}(x + ct) + \frac{1}{2}\phi_{\text{ext}}(x - ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\,ds \qquad (5)$$

for $0 \leq x \leq l$. This simple formula contains all the information we need. But to see it explicitly we must unwind the definitions of ϕ_{ext} and ψ_{ext}. This will give a resulting formula which appears quite complicated because it includes a precise description of *all* the reflections of the wave at both of the boundary points $x = 0$ and $x = l$.

The way to understand the explicit result we are about to get is by drawing a space-time diagram (Figure 4). From the point (x, t), we draw the two characteristic lines and reflect them each time they hit the boundary. We keep track of the change of sign at each reflection. We illustrate the result in Figure 4 for the case of a typical point (x, t). We also illustrate in Figure 5 the definition of the extended function $\phi_{\text{ext}}(x)$. (The same picture is valid for ψ_{ext}.) For instance, for the point (x, t) as drawn in Figures 4 and 5, we have

$$\phi_{\text{ext}}(x + ct) = -\phi(4l - x - ct) \quad \text{and} \quad \phi_{\text{ext}}(x - ct) = +\phi(x - ct + 2l).$$

The minus coefficient on $-\phi(-x - ct + 4l)$ comes from the odd number of reflections $(= 3)$. The plus coefficient on $\phi(x - ct + 2l)$ comes from the even

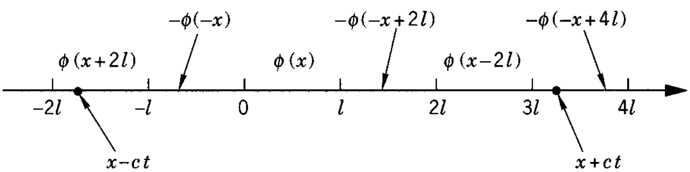

Figure 5

number of reflections (= 2). Therefore, the general formula (5) reduces to

$$\begin{aligned} v(x,t) = {} & \frac{1}{2}\phi(x-ct+2l) - \frac{1}{2}\phi(4l-x-ct) \\ & + \frac{1}{2c}\left[\int_{x-ct}^{-l} \psi(y+2l)\,dy + \int_{-l}^{0} -\psi(-y)\,dy \right. \\ & + \int_{0}^{1} \psi(y)\,dy + \int_{l}^{2l} -\psi(-y+2l)\,dy \\ & \left. + \int_{2l}^{3l} \psi(y-2l)\,dy + \int_{3l}^{x+ct} -\psi(-y+4l)\,dy \right] \end{aligned}$$

But notice that there is an exact cancellation of the four middle integrals, as we see by changing $y \to -y$ and $y - 2l \to -y + 2l$. So, changing variables in the two remaining integrals, the formula simplifies to

$$\begin{aligned} v(x,t) = {} & \frac{1}{2}\phi(x-ct+2l) - \frac{1}{2}\phi(4l-x-ct) \\ & + \frac{1}{2c}\int_{x-ct+2l}^{l} \psi(s)\,ds + \frac{1}{2c}\int_{l}^{4l-x-ct} \psi(s)\,ds. \end{aligned}$$

Therefore, we end up with the formula

$$\boxed{v(x,t) = \frac{1}{2}\phi(x-ct+2l) - \frac{1}{2}\phi(4l-x-ct) + \int_{x-ct+2l}^{4l-x-ct} \psi(s)\frac{ds}{2c}} \tag{6}$$

at the point (x, t) *illustrated*, which has three reflections on one end and two on the other. Formula (6) is valid only for such points.

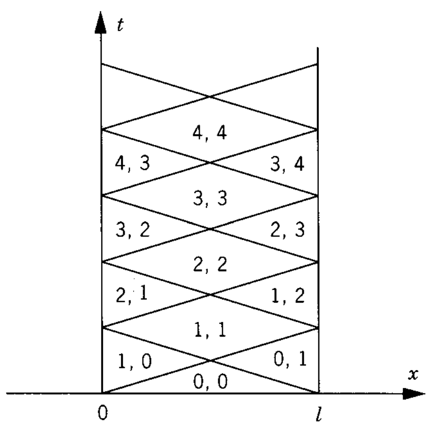

Figure 6

The solution formula at any other point (x, t) is characterized by the number of reflections at each end $(x = 0, l)$. This divides the space-time picture into diamond-shaped regions as illustrated in Figure 6. *Within each diamond the solution $v(x, t)$ is given by a different formula*. Further examples may be found in the exercises.

The formulas explain in detail how the solution looks. However, the method is impossible to generalize to two- or three-dimensional problems, nor does it work for the diffusion equation at all. Also, it is very complicated! Therefore, in Chapter 4 we shall introduce a completely different method (Fourier's) for solving problems on a finite interval.

EXERCISES

1. Solve the Neumann problem for the wave equation on the half-line $0 < x < \infty$.
2. The longitudinal vibrations of a semi-infinite flexible rod satisfy the wave equation $u_{tt} = c^2 u_{xx}$ for $x > 0$. Assume that the end $x = 0$ is free $(u_x = 0)$; it is initially at rest but has a constant initial velocity V for $a < x < 2a$ and has zero initial velocity elsewhere. Plot u versus x at the times $t = 0, a/c, 3a/2c, 2a/c$, and $3a/c$.
3. A wave $f(x + ct)$ travels along a semi-infinite string $(0 < x < \infty)$ for $t < 0$. Find the vibrations $u(x, t)$ of the string for $t > 0$ if the end $x = 0$ is fixed.
4. Repeat Exercise 3 if the end is free.
5. Solve $u_{tt} = 4u_{xx}$ for $0 < x < \infty$, $u(0, t) = 0$, $u(x, 0) \equiv 1$, $u_t(x, 0) \equiv 0$ using the reflection method. This solution has a singularity; find its location.
6. Solve $u_{tt} = c^2 u_{xx}$ in $0 < x < \infty$, $0 \le t < \infty$, $u(x, 0) = 0$, $u_t(x, 0) = V$,

$$u_t(0, t) + a u_x(0, t) = 0,$$

   where V, a, and c are positive constants and $a > c$.
7. (a) Show that $\phi_{\text{odd}}(x) = (\text{sign } x)\phi(|x|)$.
   (b) Show that $\phi_{\text{ext}}(x) = \phi_{\text{odd}}(x - 2l[x/2l])$, where $[\cdot]$ denotes the greatest integer function.
   (c) Show that

$$\phi_{\text{ext}}(x) = \begin{cases} \phi\left(x - \left[\frac{x}{l}\right] l\right) & \text{if } \left[\frac{x}{l}\right] \text{ even} \\ -\phi\left(-x - \left[\frac{x}{l}\right] l - l\right) & \text{if } \left[\frac{x}{l}\right] \text{ odd.} \end{cases}$$

8. For the wave equation in a finite interval $(0, l)$ with Dirichlet conditions, explain the solution formula within each diamond-shaped region.

9. (a) Find $u(\frac{2}{3}, 2)$ if $u_{tt} = u_{xx}$ in $0 < x < 1$, $u(x, 0) = x^2(1 - x)$, $u_t(x, 0) = (1 - x)^2$, $u(0, t) = u(1, t) = 0$.
   (b) Find $u(\frac{1}{4}, \frac{7}{2})$.
10. Solve $u_{tt} = 9u_{xx}$ in $0 < x < \pi/2$, $u(x, 0) = \cos x$, $u_t(x, 0) = 0$, $u_x(0, t) = 0$, $u(\pi/2, t) = 0$.
11. Solve $u_{tt} = c^2 u_{xx}$ in $0 < x < l$, $u(x, 0) = 0$, $u_t(x, 0) = x$, $u(0, t) = u(l, t) = 0$.

3.3 DIFFUSION WITH A SOURCE

In this section we solve the *inhomogeneous* diffusion equation on the whole line,

$$\boxed{\begin{aligned} u_t - ku_{xx} &= f(x, t) \qquad (-\infty < x < \infty, \quad 0 < t < \infty) \\ u(x, 0) &= \phi(x) \end{aligned}} \tag{1}$$

with $f(x, t)$ and $\phi(x)$ arbitrary given functions. For instance, if $u(x, t)$ represents the temperature of a rod, then $\phi(x)$ is the initial temperature distribution and $f(x, t)$ is a source (or sink) of heat provided to the rod at later times.

We will show that the solution of (1) is

$$\boxed{\begin{aligned} u(x, t) = &\int_{-\infty}^{\infty} S(x - y, t)\phi(y)\, dy \\ &+ \int_0^t \int_{-\infty}^{\infty} S(x - y, t - s) f(y, s)\, dy\, ds. \end{aligned}} \tag{2}$$

Notice that there is the usual term involving the initial data ϕ and another term involving the source f. Both terms involve the source function S.

Let's begin by explaining where (2) comes from. Later we will actually prove the validity of the formula. (If a strictly mathematical proof is satisfactory to you, this paragraph and the next two can be skipped.) Our explanation is an analogy. The simplest analogy is the ODE

$$\frac{du}{dt} + Au(t) = f(t), \qquad u(0) = \phi, \tag{3}$$

where A is a constant. Using the integrating factor e^{tA}, the solution is

$$u(t) = e^{-tA}\phi + \int_0^t e^{(s-t)A} f(s)\, ds. \tag{4}$$

A more elaborate analogy is the following. Let's suppose that ϕ is an n-vector, $u(t)$ is an n-vector function of time, and A is a fixed $n \times n$ matrix.

Then (3) is a coupled system of n linear ODEs. In case $f(t) \equiv 0$, the solution of (3) is given as $u(t) = S(t)\phi$, where $S(t)$ is the matrix $S(t) = e^{-tA}$. So in case $f(t) \neq 0$, an integrating factor for (3) is $S(-t) = e^{tA}$. Now we multiply (3) on the left by this integrating factor to get

$$\frac{d}{dt}[S(-t)u(t)] = S(-t)\frac{du}{dt} + S(-t)Au(t) = S(-t)f(t).$$

Integrating from 0 to t, we get

$$S(-t)u(t) - \phi = \int_0^t S(-s)f(s)\,ds.$$

Multiplying this by $S(t)$, we end up with the solution formula

$$u(t) = S(t)\phi + \int_0^t S(t-s)f(s)\,ds. \tag{5}$$

The first term in (5) represents the solution of the homogeneous equation, the second the effect of the source $f(t)$. For a single equation, of course, (5) reduces to (4). □

Now let's return to the original diffusion problem (1). There is an analogy between (2) and (5) which we now explain. The solution of (1) will have two terms. The first one will be the solution of the homogeneous problem, already solved in Section 2.4, namely

$$\int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy = (\mathscr{S}(t)\phi)(x). \tag{6}$$

$S(x - y, t)$ is the source function given by the formula (2.4.7). Here we are using $\mathscr{S}(t)$ to denote the *source operator*, which transforms any function ϕ to the new function given by the integral in (6). (Remember: Operators transform functions into functions.) We can now *guess* what the whole solution to (1) must be. In analogy to formula (5), we guess that the solution of (1) is

$$u(t) = \mathscr{S}(t)\phi + \int_0^t \mathscr{S}(t-s)f(s)\,ds. \tag{7}$$

Formula (7) is exactly the same as (2):

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)\,dy\,ds. \tag{2}$$

The method we have just used to find formula (2) is the operator method.

Proof of (2). All we have to do is verify that the function $u(x, t)$, which is *defined* by (2), in fact satisfies the PDE and IC (1). Since the solution of

(1) is unique, we would then know that $u(x, t)$ is that unique solution. For simplicity, we may as well let $\phi \equiv 0$, since we understand the ϕ term already.

We first verify the PDE. Differentiating (2), assuming $\phi \equiv 0$ and using the rule for differentiating integrals in Section A.3, we have

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}\int_0^t \int_{-\infty}^{\infty} S(x-y, t-s) f(y,s)\,dy\,ds \\ &= \int_0^t \int_{-\infty}^{\infty} \frac{\partial S}{\partial t}(x-y, t-s) f(y,s)\,dy\,ds \\ &\quad + \lim_{s\to t}\int_{-\infty}^{\infty} S(x-y, t-s) f(y,s)\,dy,\end{aligned}$$

taking special care due to the singularity of $S(x-y, t-s)$ at $t-s=0$. Using the fact that $S(x-y, t-s)$ satisfies the diffusion equation, we get

$$\begin{aligned}\frac{\partial u}{\partial t} &= \int_0^t \int_{-\infty}^{\infty} k\frac{\partial^2 S}{\partial x^2}(x-y,\ t-s) f(y,\ s)\,dy\,ds \\ &\quad + \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} S(x-y, \epsilon) f(y,t)\,dy.\end{aligned}$$

Pulling the spatial derivative outside the integral and using the initial condition satisfied by S, we get

$$\begin{aligned}\frac{\partial u}{\partial t} &= k\frac{\partial^2}{\partial x^2}\int_0^t \int_{-\infty}^{\infty} S(x-y,\ t-s) f(y,\ s)\,dy\,ds + f(x,t) \\ &= k\frac{\partial^2 u}{\partial x^2} + f(x,t).\end{aligned}$$

This identity is exactly the PDE (1). Second, we verify the initial condition. Letting $t \to 0$, the first term in (2) tends to $\phi(x)$ because of the initial condition of S. The second term is an integral from 0 to 0. Therefore,

$$\lim_{t\to 0} u(x,t) = \phi(x) + \int_0^0 \cdots = \phi(x).$$

This proves that (2) is the unique solution. □

Remembering that $S(x, t)$ is the gaussian distribution (2.4.7), the formula (2) takes the explicit form

$$\begin{aligned}u(x,t) &= \int_0^t \int_{-\infty}^{\infty} S(x-y, t-s) f(y,s)\,dy\,ds \\ &= \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-(x-y)^2/4k(t-s)} f(y,s)\,dy\,ds. \qquad (8)\end{aligned}$$

in the case that $\phi \equiv 0$.

SOURCE ON A HALF-LINE

For inhomogeneous diffusion on the half-line we can use the method of reflection just as in Section 3.1 (see Exercise 1).

Now consider the more complicated problem of a *boundary source* $h(t)$ on the half-line; that is,

$$\begin{aligned} v_t - kv_{xx} &= f(x,t) \quad \text{for } 0 < x < \infty, \quad 0 < t < \infty \\ \boldsymbol{v(0,t)} &= \boldsymbol{h(t)} \\ v(x,0) &= \phi(x). \end{aligned} \tag{9}$$

We may use the following subtraction device to reduce (9) to a simpler problem. Let $V(x,t) = v(x,t) - h(t)$. Then $V(x, t)$ will satisfy

$$\begin{aligned} V_t - kV_{xx} &= f(x,t) - h'(t) \qquad \text{for } 0 < x < \infty, \qquad 0 < t < \infty \\ V(0,t) &= 0 \\ V(x,0) &= \phi(x) - h(0). \end{aligned} \tag{10}$$

To verify (10), just subtract! This new problem has a homogeneous boundary condition to which we can apply the method of reflection. Once we find V, we recover v by $v(x,t) = V(x,t) + h(t)$. This simple subtraction device is often used to reduce one linear problem to another.

The domain of independent variables (x, t) in this case is a quarter-plane with specified conditions on both of its half-lines. If they do not agree at the corner [i.e., if $\phi(0) \neq h(0)$], then the solution is discontinuous there (but continuous everywhere else). This is physically sensible. Think for instance, of suddenly at $t = 0$ sticking a hot iron bar into a cold bath.

For the inhomogeneous *Neumann* problem on the half-line,

$$\begin{aligned} w_t - kw_{xx} &= f(x,t) \qquad \text{for } 0 < x < \infty, \qquad 0 < t < \infty \\ \boldsymbol{w_x(0,t)} &= \boldsymbol{h(t)} \\ w(x,0) &= \phi(x), \end{aligned} \tag{11}$$

we would subtract off the function $xh(t)$. That is, $W(x, t) = w(x, t) - xh(t)$. Differentiation implies that $W_x(0, t) = 0$. Some of these problems are worked out in the exercises.

EXERCISES

1. Solve the inhomogeneous diffusion equation on the half-line with Dirichlet boundary condition:

$$\begin{aligned} u_t - ku_{xx} &= f(x,t) \qquad (0 < x < \infty, \quad 0 < t < \infty) \\ u(0,t) &= 0 \qquad u(x,0) = \phi(x) \end{aligned}$$

using the method of reflection.

2. Solve the completely inhomogeneous diffusion problem on the half-line

$$v_t - kv_{xx} = f(x,t) \qquad \text{for } 0 < x < \infty, \quad 0 < t < \infty$$
$$v(0,t) = h(t) \qquad v(x,0) = \phi(x),$$

by carrying out the subtraction method begun in the text.

3. Solve the inhomogeneous Neumann diffusion problem on the half-line

$$w_t - kw_{xx} = 0 \qquad \text{for } 0 < x < \infty, \quad 0 < t < \infty$$
$$w_x(0,t) = h(t) \qquad w(x,0) = \phi(x),$$

by the subtraction method indicated in the text.

3.4 WAVES WITH A SOURCE

The purpose of this section is to solve

$$\boxed{u_{tt} - c^2 u_{xx} = f(x,t)} \tag{1}$$

on the whole line, together with the usual initial conditions

$$\boxed{\begin{aligned} u(x,0) &= \phi(x) \\ u_t(x,0) &= \psi(x) \end{aligned}} \tag{2}$$

where $f(x, t)$ is a given function. For instance, $f(x, t)$ could be interpreted as an external force acting on an infinitely long vibrating string.

Because $L = \partial_t^2 - c^2\partial_x^2$ is a linear operator, the solution will be the *sum of three terms*, one for ϕ, one for ψ, and one for f. The first two terms are given already in Section 2.1 and we must find the third term. We'll derive the following formula.

Theorem 1. The unique solution of (1),(2) is

$$\boxed{u(x,t) = \frac{1}{2}\,[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi + \frac{1}{2c}\iint_\Delta f} \tag{3}$$

where Δ is the characteristic triangle (see Figure 1).

The double integral in (3) is equal to the iterated integral

$$\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds.$$

We will give three different derivations of this formula! But first, let's note what the formula says. It says that the effect of a force f on $u(x, t)$ is obtained

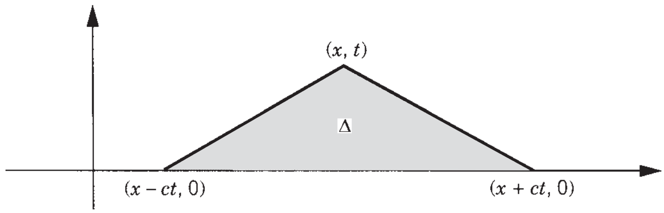

Figure 1

by simply integrating f over the past history of the point (x, t) back to the initial time $t = 0$. This is yet another example of the causality principle.

WELL-POSEDNESS

We first show that the problem (1),(2) is well-posed in the sense of Section 1.5. The well-posedness has three ingredients, as follows. Existence is clear, given that the formula (3) itself is an explicit solution. If ϕ has a continuous second derivative, ψ has a continuous first derivative, and f is continuous, then the formula (3) yields a function u with continuous second partials that satisfies the equation. Uniqueness means that there are no other solutions of (1),(2). This will follow from any one of the derivations given below.

Third, we claim that the problem (1),(2) is stable in the sense of Section 1.5. This means that if the data (ϕ, ψ, f) change a little, then u also changes only a little. To make this precise, we need a way to measure the "nearness" of functions, that is, a *metric* or *norm* on function spaces. We will illustrate this concept using the *uniform norms:*

$$\|w\| = \max_{-\infty<x<\infty} |w(x)|$$

and

$$\|w\|_T = \max_{-\infty<x<\infty,\ 0\le t\le T} |w(x,t)|.$$

Here T is fixed. Suppose that $u_1(x,t)$ is the solution with data $(\phi_1(x), \psi_1(x), f_1(x,t))$ and $u_2(x,t)$ is the solution with data $(\phi_2(x), \psi_2(x), f_2(x,t))$ (six given functions). We have the same formula (3) satisfied by u_1 and by u_2 except for the different data. We subtract the two formulas. We let $u = u_1 - u_2$. Since the area of Δ equals ct^2, we have from (3) the inequality

$$\begin{aligned}|u(x,t)| &\le \max|\phi| + \frac{1}{2c}\cdot\max|\psi|\cdot 2ct + \frac{1}{2c}\cdot\max|f|\cdot ct^2\\ &= \max|\phi| + t\cdot\max|\psi| + \frac{t^2}{2}\cdot\max|f|.\end{aligned}$$

Therefore,

$$\boxed{\|u_1 - u_2\|_T \le \|\phi_1 - \phi_2\| + T\,\|\psi_1 - \psi_2\| + \frac{T^2}{2}\|f_1 - f_2\|_T.} \qquad (4)$$

So if $\|\phi_1 - \phi_2\| < \delta$, $\|\psi_1 - \psi_2\| < \delta$, and $\|f_1 - f_2\|_T < \delta$, where δ is small, then

$$\|u_1 - u_2\|_T < \delta(1 + T + T^2) \le \epsilon$$

provided that $\delta \le \epsilon/(1 + T + T^2)$. Since ϵ is arbitrarily small, this argument proves the well-posedness of the problem (1),(2) with respect to the uniform norm.

PROOF OF THEOREM 1

Method of Characteristic Coordinates We introduce the usual characteristic coordinates $\xi = x + ct$, $\eta = x - ct$, (see Figure 2). As in Section 2.1, we have

$$Lu \equiv u_{tt} - c^2 u_{xx} = -4c^2 u_{\xi\eta} = f\left(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2c}\right).$$

We integrate this equation with respect to η, leaving ξ as a constant. Thus $u_\xi = -(1/4c^2)\int^\eta f\,d\eta$. Then we integrate with respect to ξ to get

$$u = -\frac{1}{4c^2}\int^\xi \int^\eta f\,d\eta\,d\xi \qquad (5)$$

The lower limits of integration here are arbitrary: They correspond to constants of integration. The calculation is much easier to understand if we fix a point P_0 with coordinates x_0, t_0 and

$$\xi_0 = x_0 + ct_0 \qquad \eta_0 = x_0 - ct_0.$$

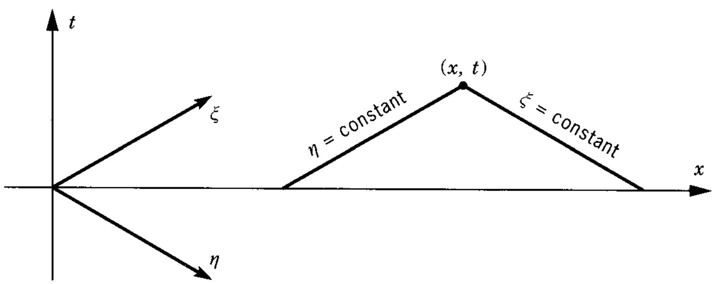

Figure 2

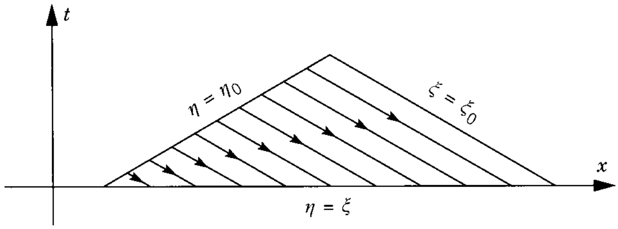

Figure 3

We evaluate (5) at P_0 *and make a particular choice of the lower limits*. Thus

$$\begin{aligned} u(P_0) &= -\frac{1}{4c^2}\int_{\eta_0}^{\xi_0}\int_{\xi}^{\eta_0} f\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}\right) d\eta\, d\xi \\ &= +\frac{1}{4c^2}\int_{\eta_0}^{\xi_0}\int_{\eta_0}^{\xi} f\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}\right) d\eta\, d\xi \end{aligned} \tag{6}$$

is a particular solution. As Figure 3 indicates, η now represents a variable going along a line segment to the base $\eta = \xi$ of the triangle Δ from the left-hand edge $\eta = \eta_0$, while ξ runs from the left-hand corner to the right-hand edge. Thus we have integrated over the whole triangle Δ.

The iterated integral, however, is not exactly the double integral over Δ because the coordinate axes are not orthogonal. The original axes (x and t) are orthogonal, so we make a change of variables back to x and t. This amounts to substituting back

$$x = \frac{\xi+\eta}{2} \qquad t = \frac{\xi-\eta}{2c}. \tag{7}$$

A little square in Figure 4 goes into a parallelogram in Figure 5. The change in its area is measured by the jacobian determinant J (see Section A.1). Since

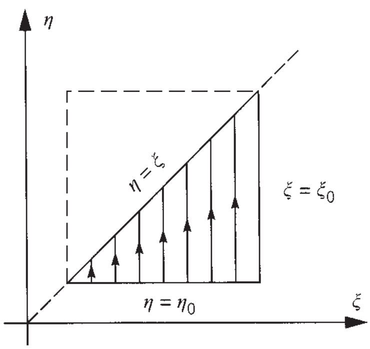

Figure 4

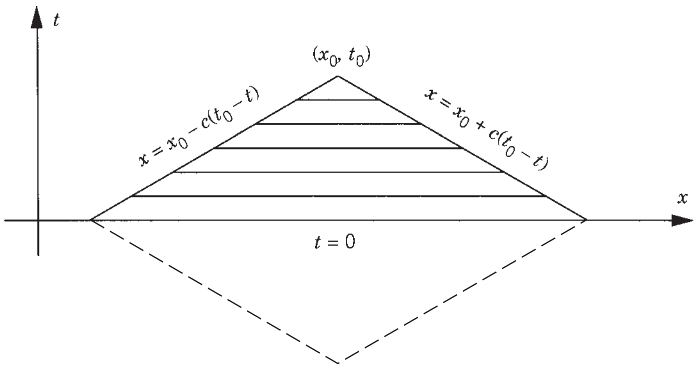

Figure 5

our change of variable is a linear transformation, the jacobian is just the determinant of its coefficient matrix:

$$J = \left|\det\begin{pmatrix} \dfrac{\partial\xi}{\partial x} & \dfrac{\partial\xi}{\partial t} \\ \dfrac{\partial\eta}{\partial x} & \dfrac{\partial\eta}{\partial t}\end{pmatrix}\right| = \left|\det\begin{pmatrix} 1 & c \\ 1 & -c\end{pmatrix}\right| = 2c.$$

Thus $d\eta\, d\xi = J\, dx\, dt = 2c\, dx\, dt$. Therefore, the rule for changing variables in a multiple integral (the jacobian theorem) then gives

$$u(P_0) = \frac{1}{4c^2}\iint_\Delta f(x,t)J\, dx\, dt. \tag{8}$$

This is precisely Theorem 1. The formula can also be written as the iterated integral in x and t:

$$u(x_0, t_0) = \frac{1}{2c}\int_0^{t_0}\int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} f(x,t)\, dx\, dt, \tag{9}$$

integrating first over the horizontal line segments in Figure 5 and then vertically.

A variant of the method of characteristic coordinates is to write (1) as the system of two equations

$$u_t + cu_x = v \quad v_t - cv_x = f,$$

the first equation being the definition of v, as in Section 2.1. If we first solve the second equation, then v is a line integral of f over a characteristic line segment $x + ct =$ constant. The first equation then gives $u(x, t)$ by sweeping out these line segments over the characteristic triangle Δ. To carry out this variant is a little tricky, however, and we leave it as an exercise.

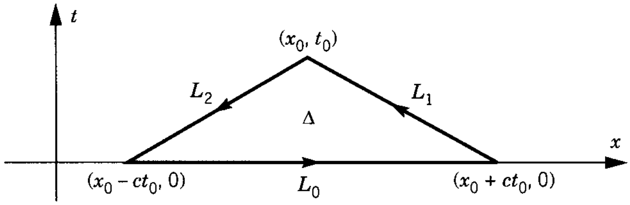

Figure 6

Method Using Green's Theorem In this method we integrate f over the past history triangle Δ. Thus

$$\iint_\Delta f\,dx\,dt = \iint_\Delta (u_{tt} - c^2 u_{xx})\,dx\,dt. \tag{10}$$

But Green's theorem says that

$$\iint_\Delta (P_x - Q_t)\,dx\,dt = \int_{\text{bdy}} P\,dt + Q\,dx$$

for any functions P and Q, where the line integral on the boundary is taken counterclockwise (see Section A.3). Thus we get

$$\iint_\Delta f\,dx\,dt = \int_{L_0+L_1+L_2} (-c^2 u_x dt - u_t\,dx). \tag{11}$$

This is the sum of three line integrals over straight line segments (see Figure 6). We evaluate each piece separately. On L_0, $dt = 0$ and $u_t(x, 0) = \psi(x)$, so that

$$\int_{L_0} = -\int_{x_0-ct_0}^{x_0+ct_0} \psi(x)\,dx.$$

On L_1, $x + ct = x_0 + ct_0$, so that $dx + c\,dt = 0$, whence $-c^2 u_x\,dt - u_t\,dx = cu_x\,dx + cu_t\,dt = c\,du$. (We're in luck!) Thus

$$\int_{L_1} = c\int_{L_1} du = cu(x_0, t_0) - c\phi(x_0 + ct_0).$$

In the same way,

$$\int_{L_2} = -c\int_{L_2} du = -c\phi(x_0 - ct_0) + cu(x_0, t_0).$$

Adding these three results, we get

$$\iint_\Delta f\,dx\,dt = 2cu(x_0, t_0) - c[\phi(x_0 + ct_0) + \phi(x_0 - ct_0)] - \int_{x_0-ct_0}^{x_0+ct_0} \psi(x)\,dx.$$

Thus

$$u(x_0, t_0) = \frac{1}{2c} \iint_\Delta f\,dx\,dt + \frac{1}{2}[\phi(x_0 + ct_0) + \phi(x_0 - ct_0)] \\ + \frac{1}{2c} \int_{x_0 - ct_0}^{x_0 + ct_0} \psi(x)\,dx, \tag{12}$$

which is the same as before.

Operator Method This is how we solved the diffusion equation with a source. Let's try it out on the wave equation. The ODE analog is the equation,

$$\frac{d^2u}{dt^2} + A^2 u(t) = f(t), \quad u(0) = \phi, \quad \frac{du}{dt}(0) = \psi. \tag{13}$$

We could think of A^2 as a positive constant (or even a positive square matrix.) The solution of (13) is

$$u(t) = S'(t)\phi + S(t)\psi + \int_0^t S(t - s) f(s)\,ds, \tag{14}$$

where

$$S(t) = A^{-1} \sin tA \quad \text{and} \quad S'(t) = \cos tA. \tag{15}$$

The key to understanding formula (14) is that $S(t)\psi$ is the solution of problem (13) in the case that $\phi = 0$ and $f = 0$.

Let's return to the PDE

$$u_{tt} - c^2 u_{xx} = f(x, t) \quad u(x, 0) = \phi(x) \quad u_t(x, 0) = \psi(x). \tag{16}$$

The basic operator ought to be given by the ψ term. That is,

$$\mathcal{S}(t)\psi = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y)\,dy = v(x, t), \tag{17}$$

where $v(x, t)$ solves $v_{tt} - c^2 v_{xx} = 0,\ v(x, 0) = 0,\ v_t(x, 0) = \psi(x)$. $\mathcal{S}(t)$ is the *source operator.* By (14) we would expect the ϕ term to be $(\partial/\partial t)\mathcal{S}(t)\phi$. In fact,

$$\frac{\partial}{\partial t}\mathcal{S}(t)\phi = \frac{\partial}{\partial t}\frac{1}{2c} \int_{x-ct}^{x+ct} \phi(y)\,dy \\ = \frac{1}{2c}\left[c\phi(x + ct) - (-c)\phi(x - ct)\right],$$

in agreement with our old formula (2.1.8)! So we must be on the right track.

Let's now take the f term; that is, $\phi = \psi = 0$. By analogy with the last term in (14), the solution *ought* to be

$$u(t) = \int_0^t \mathcal{S}(t - s) f(s)\,ds.$$

That is, using (17),

$$u(x,t) = \int_0^t \left[\frac{1}{2c} \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy \right] ds = \frac{1}{2c} \iint_\Delta f\,dx\,dt.$$

This is once again the same result!

The moral of the operator method is that if you can solve the homogeneous equation, you can also solve the inhomogeneous equation. This is sometimes known as *Duhamel's principle.*

SOURCE ON A HALF-LINE

The solution of the general inhomogeneous problem on a half-line

$$\begin{aligned} \text{DE:}\quad & v_{tt} - c^2 v_{xx} = f(x,t) \quad \text{in} \quad 0 < x < \infty \\ \text{IC:}\quad & v(x,0) = \phi(x) \quad v_t(x,0) = \psi(x) \\ \text{BC:}\quad & v(0,t) = h(t) \end{aligned} \tag{18}$$

is the sum of four terms, one for each data function ϕ, ψ, f, and h. For $x > ct > 0$, the solution has precisely the same form as in (3), with the backward triangle Δ as the domain of dependence. For $0 < x < ct$, however, it is given by

$$v(x,t) = \phi \text{ term} + \psi \text{ term} + \boldsymbol{h\left(t - \frac{x}{c}\right)} + \frac{1}{2c} \iint_D f \tag{19}$$

where $t - x/c$ is the reflection point and D is the shaded region in Figure 3.2.2. The only caveat is that the given conditions had better coincide at the origin. That is, we require that $\phi(0) = h(0)$ and $\psi(0) = h'(0)$. If this were not assumed, there would be a singularity on the characteristic line emanating from the corner.

Let's derive the boundary term $h(t - x/c)$ for $x < ct$. To accomplish this, it is convenient to assume that $\phi = \psi = f = 0$. We shall derive the solution from scratch using the fact that $v(x,t)$ must take the form $v(x,t) = j(x+ct) + g(x-ct)$. From the initial conditions ($\phi = \psi = 0$), we find that $j(s) = g(s) = 0$ for $s > 0$. From the boundary condition we have $h(t) = v(0,t) = g(-ct)$ for $t > 0$. Thus $g(s) = h(-s/c)$ for $s < 0$. Therefore, if $x < ct$, $t > 0$, we have $v(x,\ t) = 0 + h(-[x-ct]/c) = h(t - x/c)$.

FINITE INTERVAL

For a finite interval $(0, l)$ with inhomogeneous boundary conditions $v(0,t) = h(t)$, $v(l,t) = k(t)$, we get the whole series of terms

$$\begin{aligned} v(x,t) = h\left(t - \frac{x}{c}\right) - h\left(t + \frac{x-2l}{c}\right) + h\left(t - \frac{x+2l}{c}\right) + \cdots \\ + k\left(t + \frac{x-l}{c}\right) - k\left(t - \frac{x+l}{c}\right) + k\left(t + \frac{x-3l}{c}\right) + \cdots \end{aligned}$$

(see Exercise 15 and Figure 3.2.4).

EXERCISES

1. Solve $u_{tt} = c^2u_{xx} + xt, \quad u(x, 0) = 0, \quad u_t(x, 0) = 0.$
2. Solve $u_{tt} = c^2u_{xx} + e^{ax}, \quad u(x, 0) = 0, \quad u_t(x, 0) = 0.$
3. Solve $u_{tt} = c^2u_{xx} + \cos x, \quad u(x, 0) = \sin x, \quad u_t(x, 0) = 1 + x.$
4. Show that the solution of the inhomogeneous wave equation

$$u_{tt} = c^2u_{xx} + f, \quad u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x),$$

is the sum of three terms, one each for f, ϕ, and ψ.

5. Let $f(x, t)$ be any function and let $u(x, t) = (1/2c)\iint_\Delta f$, where Δ is the triangle of dependence. Verify directly by differentiation that

$$u_{tt} = c^2u_{xx} + f \quad \text{and} \quad u(x, 0) \equiv u_t(x, 0) \equiv 0.$$

(*Hint:* Begin by writing the formula as the *iterated* integral

$$u(x, t) = \frac{1}{2c}\int_0^t \int_{x-ct+cs}^{x+ct-cs} f(y, s)\, dy\, ds$$

and differentiate with care using the rule in the Appendix. This exercise is not easy.)

6. Derive the formula for the inhomogeneous wave equation in yet another way.
   (a) Write it as the system

$$u_t + cu_x = v, \quad v_t - cv_x = f.$$

   (b) Solve the first equation for u in terms of v as

$$u(x, t) = \int_0^t v(x - ct + cs, s)\, ds.$$

   (c) Similarly, solve the second equation for v in terms of f.
   (d) Substitute part (c) into part (b) and write as an iterated integral.

7. Let A be a positive-definite $n \times n$ matrix. Let

$$S(t) = \sum_{m=0}^{\infty} \frac{(-1)^m A^{2m} t^{2m+1}}{(2m + 1)!}.$$

   (a) Show that this series of matrices converges uniformly for bounded t and its sum $S(t)$ solves the problem $S''(t) + A^2S(t) = 0,\ S(0) = 0,\ S'(0) = I$, where I is the identity matrix. Therefore, it makes sense to denote $S(t)$ as $A^{-1} \sin tA$ and to denote its derivative $S'(t)$ as $\cos(t\mathrm{A})$.
   (b) Show that the solution of (13) is (14).

8. Show that the source operator for the wave equation solves the problem

$$\mathcal{S}_{tt} - c^2\mathcal{S}_{xx} = 0, \quad \mathcal{S}(0) = 0, \quad \mathcal{S}_t(0) = I,$$

where I is the identity operator.

9. Let $u(t) = \int_0^t \mathcal{S}(t-s) f(s)\,ds$. Using *only* Exercise 8, show that u solves the inhomogeneous wave equation with zero initial data.
10. Use any method to show that $u = 1/(2c) \iint_D f$ solves the inhomogeneous wave equation on the half-line with zero initial and boundary data, where D is the domain of dependence for the half-line.
11. Show by direct substitution that $u(x,\ t) = h(t - x/c)$ for $x < ct$ and $u(x,\ t) = 0$ for $x \geq ct$ solves the wave equation on the half-line $(0, \infty)$ with zero initial data and boundary condition $u(0, t) = h(t)$.
12. Derive the solution of the fully inhomogeneous wave equation on the half-line
$$v_{tt} - c^2 v_{xx} = f(x,t) \quad \text{in } 0 < x < \infty$$
$$v(x,0) = \phi(x), \qquad v_t(x,0) = \psi(x)$$
$$v(0,t) = h(t),$$
by means of the method using Green's theorem. (*Hint:* Integrate over the domain of dependence.)
13. Solve $u_{tt} = c^2 u_{xx}$ for $0 < x < \infty$,
$u(0,t) = t^2, \quad u(x,0) = x, \quad u_t(x,0) = 0.$
14. Solve the homogeneous wave equation on the half-line $(0, \infty)$ with zero initial data and with the Neumann boundary condition $u_x(0,t) = k(t)$. Use any method you wish.
15. Derive the solution of the wave equation in a finite interval with inhomogeneous boundary conditions $v(0,t) = h(t)$, $v(l,t) = k(t)$, and with $\phi = \psi = f = 0$.

3.5 DIFFUSION REVISITED

In this section we make a careful mathematical analysis of the solution of the diffusion equation that we found in Section 2.4. (On the other hand, the formula for the solution of the wave equation is so much simpler that it doesn't require a special justification.)

The solution formula for the diffusion equation is an example of a *convolution*, the convolution of ϕ with S (at a fixed t). It is

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\,\phi(y)\,dy = \int_{-\infty}^{\infty} S(z,t)\,\phi(x-z)\,dz, \qquad (1)$$

where $S(z,t) = 1/\sqrt{4\pi kt}\, e^{-z^2/4kt}$. If we introduce the variable $p = z/\sqrt{kt}$, it takes the equivalent form

$$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-p^2/4} \phi(x - p\sqrt{kt})\,dp.} \qquad (2)$$

Now we are prepared to state a precise theorem.

Theorem 1. Let $\phi(x)$ be a bounded continuous function for $-\infty < x < \infty$. Then the formula (2) defines an infinitely differentiable function $u(x, t)$ for $-\infty < x < \infty$, $0 < t < \infty$, which satisfies the equation $u_t = ku_{xx}$ and $\lim_{t\searrow 0} u(x,t) = \phi(x)$ for each x.

Proof. The integral converges easily because

$$|u(x,t)| \le \frac{1}{\sqrt{4\pi}}\,(\max|\phi|)\int_{-\infty}^{\infty} e^{-p^2/4}\,dp = \max|\phi|.$$

(This inequality is related to the maximum principle.) Thus the integral converges uniformly and absolutely. Let us show that $\partial u/\partial x$ exists. It equals $\int(\partial S/\partial x)(x-y,t)\phi(y)\,dy$ provided that this new integral also converges absolutely. Now

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{\partial S}{\partial x}(x-y,t)\phi(y)\,dy &= -\frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty}\frac{x-y}{2kt}e^{-(x-y)^2/4kt}\phi(y)\,dy\\ &= \frac{c}{\sqrt{t}}\int_{-\infty}^{\infty} pe^{-p^2/4}\phi(x-p\sqrt{kt})\,dp\\ &\le \frac{c}{\sqrt{t}}\,(\max|\phi|)\int_{-\infty}^{\infty}|p|\,e^{-p^2/4}\,dp,\end{aligned}$$

where c is a constant. The last integral is finite. So this integral also converges uniformly and absolutely. Therefore, $u_x = \partial u/\partial x$ exists and is given by this formula. All derivatives of all orders ($u_t, u_{xt}, u_{xx}, u_{tt}, \ldots$) work the same way because each differentiation brings down a power of p so that we end up with convergent integrals like $\int p^n e^{-p^2/4}\,dp$. So $u(x, t)$ is differentiable to all orders. Since $S(x, t)$ satisfies the diffusion equation for $t > 0$, so does $u(x, t)$.

It remains to prove the initial condition. It has to be understood in a limiting sense because the formula itself has meaning only for $t > 0$. Because the integral of S is 1, we have

$$\begin{aligned}u(x,t)-\phi(x) &= \int_{-\infty}^{\infty} S(x-y,t)\,[\phi(y)-\phi(x)]\,dy\\ &= \frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty} e^{-p^2/4}[\phi(x-p\sqrt{kt})-\phi(x)]\,dp.\end{aligned}$$

For fixed x we must show that this tends to zero as $t \to 0$. The idea is that for $p\sqrt{t}$ small, the continuity of ϕ makes the integral small; while for $p\sqrt{t}$ not small, p is large and the exponential factor is small.

To carry out this idea, let $\epsilon > 0$. Let $\delta > 0$ be so small that

$$\max_{|y-x|\le\delta}|\phi(y)-\phi(x)| < \frac{\epsilon}{2}.$$

This can be done because ϕ is continuous at x. We break up the integral into the part where $|p| < \delta/\sqrt{kt}$ and the part where $|p| \geq \delta/\sqrt{kt}$. The first part is

$$\left|\int_{|p|<\delta/\sqrt{kt}}\right| \leq \left(\frac{1}{\sqrt{4\pi}}\int e^{-p^2/4}dp\right)\cdot \max_{|y-x|\leq\delta}|\phi(y)-\phi(x)|$$
$$< 1\cdot\frac{\epsilon}{2} = \frac{\epsilon}{2}.$$

The second part is

$$\left|\int_{|p|\geq\delta/\sqrt{kt}}\right| \leq \frac{1}{\sqrt{4\pi}}\cdot 2(\max|\phi|)\cdot\int_{|p|\geq\delta/\sqrt{kt}} e^{-p^2/4}dp < \frac{\epsilon}{2}$$

by choosing t sufficiently small, since the integral $\int_{-\infty}^{\infty} e^{-p^2/4}\,dp$ converges and δ is fixed. (That is, the "tails" $\int_{|p|\geq N} e^{-p^2/4}\,dp$ are as small as we wish if $N = \delta/\sqrt{kt}$ is large enough.) Therefore,

$$|u(x,t)-\phi(x)| < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon$$

provided that t is small enough. This means exactly that $u(x,t)\to\phi(x)$ as $t\to 0$. □

Corollary. The solution has all derivatives of all orders for $t > 0$, even if ϕ is not differentiable. We can say therefore that all solutions become smooth as soon as diffusion takes effect. There are no singularities, in sharp contrast to the wave equation.

Proof. We use formula (1)

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy$$

together with the rule for differentiation under an integral sign, Theorem 2 in Section A.3.

Piecewise Continuous Initial Data. Notice that the continuity of $\phi(x)$ was used in only one part of the proof. With an appropriate change we can allow $\phi(x)$ to have a jump discontinuity. [Consider, for instance, the initial data for $Q(x,t)$.]

A function $\phi(x)$ is said to have a *jump* at x_0 if both the limit of $\phi(x)$ as $x\to x_0$ from the right exists [denoted $\phi(x_0+)$] and the limit from the left [denoted $\phi(x_0-)$] exists but these two limits are not equal. A function is called *piecewise continuous* if in each finite interval it has only a finite number of jumps and it is continuous at all other points. This concept is discussed in more detail in Section 5.2.

Theorem 2. Let $\phi(x)$ be a bounded function that is piecewise continuous. Then (1) is an infinitely differentiable solution for $t > 0$ and

$$\boxed{\lim_{t \searrow 0} u(x, t) = \tfrac{1}{2}[\phi(x+) + \phi(x-)]}$$

for all x. At every point of continuity this limit equals $\phi(x)$.

Proof. The idea is the same as before. The only difference is to split the integrals into $p > 0$ and $p < 0$. We need to show that

$$\frac{1}{\sqrt{4\pi}} \int_0^{\pm\infty} e^{-p^2/4} \phi(x + \sqrt{kt}\, p)\, dp \to \pm\frac{1}{2}\phi\,(x\pm).$$

The details are left as an exercise. □

EXERCISES

1. Prove that if ϕ is any piecewise continuous function, then

$$\frac{1}{\sqrt{4\pi}} \int_0^{\pm\infty} e^{-p^2/4} \phi(x + \sqrt{kt}\, p)\, dp \to \pm\frac{1}{2}\phi\,(x\pm) \quad \text{as } t \searrow 0.$$

2. Use Exercise 1 to prove Theorem 2.

4

BOUNDARY PROBLEMS

In this chapter we finally come to the physically realistic case of a finite interval $0 < x < l$. The methods we introduce will frequently be used in the rest of this book.

4.1 SEPARATION OF VARIABLES, THE DIRICHLET CONDITION

We first consider the homogeneous Dirichlet conditions for the wave equation:

$$u_{tt} = c^2 u_{xx} \quad \text{for } 0 < x < l \tag{1}$$

$$u(0, t) = 0 = u(l, t) \tag{2}$$

with some initial conditions

$$u(x, 0) = \phi(x) \quad u_t(x, 0) = \psi(x). \tag{3}$$

The method we shall use consists of building up the general solution as a linear combination of special ones that are easy to find. (Once before, in Section 2.4, we followed this program, but with different building blocks.)

A *separated solution* is a solution of (1) and (2) of the form

$$u(x, t) = X(x)T(t). \tag{4}$$

(It is important to distinguish between the independent variable written as a lowercase letter and the function written as a capital letter.) Our first goal is to look for as many separated solutions as possible.

Plugging the form (4) into the wave equation (1), we get

$$X(x)T''(t) = c^2 X''(x)T(t)$$

or, dividing by $-c^2XT$,

$$-\frac{T''}{c^2T} = -\frac{X''}{X} = \lambda.$$

This defines a quantity λ, *which must be a constant*. (*Proof:* $\partial\lambda/\partial x = 0$ and $\partial\lambda/\partial t = 0$, so λ is a constant. Alternatively, we can argue that λ doesn't depend on x because of the first expression and doesn't depend on t because of the second expression, so that it doesn't depend on any variable.) We will show at the end of this section that $\lambda > 0$. (This is the reason for introducing the minus signs the way we did.)

So let $\lambda = \beta^2$, where $\beta > 0$. Then the equations above are a pair of *separate* (!) ordinary differential equations for $X(x)$ and $T(t)$:

$$X'' + \beta^2 X = 0 \quad \text{and} \quad T'' + c^2\beta^2 T = 0. \tag{5}$$

These ODEs are easy to solve. The solutions have the form

$$X(x) = C\cos\beta x + D\sin\beta x \tag{6}$$

$$T(t) = A\cos\beta ct + B\sin\beta ct, \tag{7}$$

where A, B, C, and D are constants.

The second step is to impose the boundary conditions (2) on the separated solution. They simply require that $X(0) = 0 = X(l)$. Thus

$$0 = X(0) = C \quad \text{and} \quad 0 = X(l) = D\sin\beta l.$$

Surely we are not interested in the obvious solution $C = D = 0$. So we must have $\beta l = n\pi$, a root of the sine function. That is,

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin\frac{n\pi x}{l} \quad (n = 1, 2, 3, \ldots) \tag{8}$$

are distinct solutions. Each sine function may be multiplied by an arbitrary constant.

Therefore, there are an *infinite* (!) number of separated solutions of (1) and (2), one for each n. They are

$$u_n(x,t) = \left(A_n\cos\frac{n\pi ct}{l} + B_n\sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}$$

($n = 1, 2, 3, \ldots$), where A_n and B_n are arbitrary constants. The sum of solutions is again a solution, so *any finite sum*

$$\boxed{u(x,t) = \sum_n \left(A_n\cos\frac{n\pi ct}{l} + B_n\sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}} \tag{9}$$

is also a solution of (1) and (2).

Formula (9) solves (3) as well as (1) and (2), provided that

$$\phi(x) = \sum_n A_n \sin \frac{n\pi x}{l} \tag{10}$$

and

$$\psi(x) = \sum_n \frac{n\pi c}{l} B_n \sin \frac{n\pi x}{l}. \tag{11}$$

Thus for any initial data of this form, the problem (1), (2), and (3) has a simple explicit solution.

But such data (10) and (11) clearly are very special. So let's try (following Fourier in 1827) to take *infinite sums*. Then we ask what kind of data pairs $\phi(x)$, $\psi(x)$ can be expanded as in (10), (11) for some choice of coefficients A_n, B_n? This question was the source of great disputes for half a century around 1800, but the final result of the disputes was very simple: *Practically any* (!) *function* $\phi(x)$ *on the interval* $(0, l)$ *can be expanded in an infinite series* (10). We will show this in Chapter 5. It will have to involve technical questions of convergence and differentiability of infinite series like (9). The series in (10) is called a *Fourier sine series* on $(0, l)$. But for the time being let's not worry about these mathematical points. Let's just forge ahead to see what their implications are.

First of all, (11) is the same kind of series for $\psi(x)$ as (10) is for $\phi(x)$. What we've shown is simply that *if* (10), (11) *are true, then the infinite series* (9) *ought to be the solution of the whole problem* (1), (2), (3).

A sketch of the first few functions $\sin(\pi x/l)$, $\sin(2\pi x/l)$, ... is shown in Figure 1. The functions $\cos(n\pi ct/l)$ and $\sin(n\pi ct/l)$ which describe the

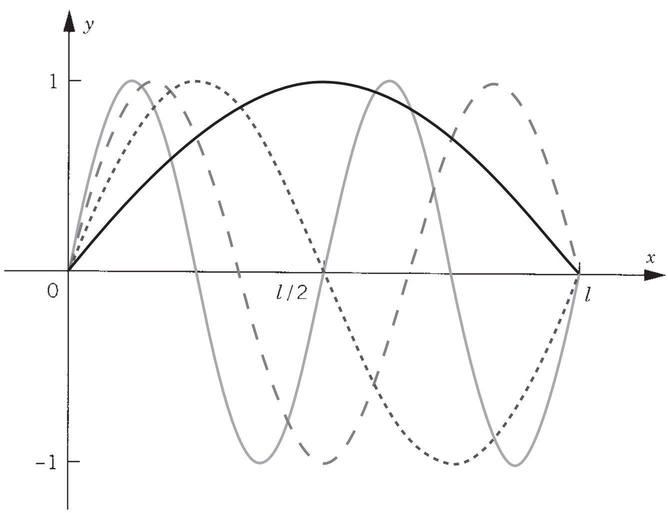

Figure 1

behavior in *time* have a similar form. The coefficients of t inside the sines and cosines, namely $n\pi c/l$, are called the *frequencies*. (In some texts, the frequency is defined as $nc/2l$.)

If we return to the violin string that originally led us to the problem (1), (2), (3), we find that the frequencies are

$$\frac{n\pi\sqrt{T}}{l\sqrt{\rho}} \quad \text{for } n = 1, 2, 3, \ldots \tag{12}$$

The "fundamental" note of the string is the smallest of these, $\pi\sqrt{T}/(l\sqrt{\rho})$. The "overtones" are *exactly* the double, the triple, and so on, of the fundamental! The discovery by Euler in 1749 that the musical notes have such a simple mathematical description created a sensation. It took over half a century to resolve the ensuing controversy over the relationship between the infinite series (9) and d'Alembert's solution in Section 2.1. □

The analogous problem for *diffusion* is

$$\text{DE:} \quad u_t = ku_{xx} \quad (0 < x < l, 0 < t < \infty) \tag{13}$$
$$\text{BC:} \quad u(0,t) = u(l,t) = 0 \tag{14}$$
$$\text{IC:} \quad u(x,0) = \phi(x). \tag{15}$$

To solve it, we separate the variables $u = T(t)X(x)$ as before. This time we get

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda = \text{constant}.$$

Therefore, $T(t)$ satisfies the equation $T' = -\lambda kT$, whose solution is $T(t) = Ae^{-\lambda kt}$. Furthermore,

$$-X'' = \lambda X \quad \text{in } 0 < x < l \quad \text{with} \quad X(0) = X(l) = 0. \tag{16}$$

This is *precisely the same problem for* $X(x)$ *as before* and so has the same solutions. Because of the form of $T(t)$,

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2 kt} \sin \frac{n\pi x}{l}} \tag{17}$$

is the solution of (13)–(15) provided that

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}. \tag{18}$$

Once again, our solution is expressible for each t as a Fourier sine series in x provided that the initial data are.

For example, consider the diffusion of a substance in a tube of length l. Each end of the tube opens up into a very large empty vessel. So the concentration $u(x, t)$ at each end is essentially zero. Given an initial concentration $\phi(x)$ in the tube, the concentration at all later times is given by formula (17). Notice that as $t \to \infty$, each term in (17) goes to zero. Thus the substance gradually empties out into the two vessels and less and less remains in the tube. □

The numbers $\lambda_n = (n\pi/l)^2$ are called *eigenvalues* and the functions $X_n(x) = \sin(n\pi x/l)$ are called *eigenfunctions*. The reason for this terminology is as follows. They satisfy the conditions

$$-\frac{d^2}{dx^2}X = \lambda X, \quad X(0) = X(l) = 0. \tag{19}$$

This is an ODE with conditions at two points. Let A denote the operator $-d^2/dx^2$, which acts on the functions that satisfy the Dirichlet boundary conditions. The differential equation has the form $AX = \lambda X$. An eigenfunction is a solution $X \not\equiv 0$ of this equation and an eigenvalue is a number λ for which there exists a solution $X \not\equiv 0$.

This situation is analogous to the more familiar case of an $N \times N$ matrix A. A vector X that satisfies $AX = \lambda X$ with $X \not\equiv 0$ is called an eigenvector and λ is called an eigenvalue. For an $N \times N$ matrix there are at most N eigenvalues. But for the differential operator that we are interested in, there are an *infinite number of eigenvalues* π^2/l^2, $4\pi^2/l^2$, $9\pi^2/l^2$, ... Thus you might say that we are dealing with *infinite-dimensional linear algebra!*

In physics and engineering the eigenfunctions are called *normal modes* because they are the natural shapes of solutions that persist for all time.

Why are all the eigenvalues of this problem positive? We assumed this in the discussion above, but now let's *prove* it. First, could $\lambda = 0$ be an eigenvalue? This would mean that $X'' = 0$, so that $X(x) = C + Dx$. But $X(0) = X(l) = 0$ implies that $C = D = 0$, so that $X(x) \equiv 0$. Therefore, zero is *not* an eigenvalue.

Next, could there be *negative* eigenvalues? If $\lambda < 0$, let's write it as $\lambda = -\gamma^2$. Then $X'' = \gamma^2 X$, so that $X(x) = C \cosh \gamma x + D \sinh \gamma x$. Then $0 = X(0) = C$ and $0 = X(l) = D \sinh \gamma l$. Hence $D = 0$ since $\sinh \gamma l \neq 0$.

Finally, let λ be any *complex* number. Let γ be either one of the two square roots of $-\lambda$; the other one is $-\gamma$. Then

$$X(x) = Ce^{\gamma x} + De^{-\gamma x},$$

where we are using the complex exponential function (see Section 5.2). The boundary conditions yield $0 = X(0) = C + D$ and $0 = Ce^{\gamma l} + De^{-\gamma l}$. Therefore $e^{2\gamma l} = 1$. By a well-known property of the complex exponential function, this implies that $\mathrm{Re}(\gamma) = 0$ and $2l\,\mathrm{Im}(\gamma) = 2\pi n$ for some integer n. Hence $\gamma = n\pi i/l$ and $\lambda = -\gamma^2 = n^2\pi^2/l^2$, which is real and positive. Thus the only eigenvalues λ of our problem (16) are positive numbers; in fact, they are $(\pi/l)^2$, $(2\pi/l)^2$, ...

EXERCISES

1. (a) Use the Fourier expansion to explain why the note produced by a violin string rises sharply by one octave when the string is clamped exactly at its midpoint.
   (b) Explain why the note rises when the string is tightened.
2. Consider a metal rod ($0 < x < l$), insulated along its sides but not at its ends, which is initially at temperature $= 1$. Suddenly both ends are plunged into a bath of temperature $= 0$. Write the differential equation, boundary conditions, and initial condition. Write the formula for the temperature $u(x, t)$ at later times. In this problem, *assume* the infinite series expansion
$$1 = \frac{4}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + \cdots\right)$$
3. A quantum-mechanical particle on the line with an infinite potential outside the interval $(0, l)$ ("particle in a box") is given by Schrödinger's equation $u_t = iu_{xx}$ on $(0, l)$ with Dirichlet conditions at the ends. Separate the variables and use (8) to find its representation as a series.
4. Consider waves in a resistant medium that satisfy the problem
$$u_{tt} = c^2u_{xx} - ru_t \quad \text{for } 0 < x < l$$
$$u = 0 \quad \text{at both ends}$$
$$u(x, 0) = \phi(x) \quad u_t(x, 0) = \psi(x),$$
where r is a constant, $0 < r < 2\pi c/l$. Write down the series expansion of the solution.
5. Do the same for $2\pi c/l < r < 4\pi c/l$.
6. Separate the variables for the equation $tu_t = u_{xx} + 2u$ with the boundary conditions $u(0, t) = u(\pi, t) = 0$. Show that there are an infinite number of solutions that satisfy the initial condition $u(x, 0) = 0$. So uniqueness is false for this equation!

4.2 THE NEUMANN CONDITION

The same method works for both the Neumann and Robin boundary conditions (BCs). In the former case, (4.1.2) is replaced by $u_x(0, t) = u_x(l, t) = 0$. Then the eigenfunctions are the solutions $X(x)$ of

$$\boxed{-X'' = \lambda X, \quad X'(0) = X'(l) = 0,} \tag{1}$$

other than the trivial solution $X(x) \equiv 0$.

As before, let's first search for the positive eigenvalues $\lambda = \beta^2 > 0$. As in (4.1.6), $X(x) = C\cos\beta x + D\sin\beta x$, so that
$$X'(x) = -C\beta\sin\beta x + D\beta\cos\beta x.$$

The boundary conditions (1) mean first that $0 = X'(0) = D\beta$, so that $D = 0$, and second that

$$0 = X'(l) = -C\beta \sin \beta l.$$

Since we don't want $C = 0$, we must have $\sin \beta l = 0$. Thus $\beta = \pi/l,\ 2\pi/l,\ 3\pi/l, \ldots$. Therefore, we have the

$$\text{Eigenvalues:} \quad \left(\frac{\pi}{l}\right)^2, \left(\frac{2\pi}{l}\right)^2, \cdots \tag{2}$$

$$\text{Eigenfunctions:} \quad X_n(x) = \cos \frac{n\pi x}{l} \quad (n = 1, 2, \ldots) \tag{3}$$

Next let's check whether zero is an eigenvalue. Set $\lambda = 0$ in the ODE (1). Then $X'' = 0$, so that $X(x) = C + Dx$ and $X'(x) \equiv D$. The Neumann boundary conditions are both satisfied if $D = 0$. C can be any number. Therefore, $\lambda = 0$ *is an eigenvalue*, and any constant function is its eigenfunction.

If $\lambda < 0$ or if λ is complex (nonreal), it can be shown directly, as in the Dirichlet case, that there is no eigenfunction. (Another proof will be given in Section 5.3.) Therefore, the list of all the eigenvalues is

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad \text{for } n = 0, 1, 2, 3, \ldots . \tag{4}$$

Note that $n = 0$ is included among them!

So, for instance, the *diffusion* equation with the Neumann BCs has the solution

$$u(x, t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2 kt} \cos \frac{n\pi x}{l}. \tag{5}$$

This solution requires the initial data to have the "Fourier cosine expansion"

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l}. \tag{6}$$

All the coefficients $A_0, A_1, A_2, \ldots$ are just constants. The first term in (5) and (6), which comes from the eigenvalue $\lambda = 0$, is written separately in the form $\frac{1}{2}A_0$ just for later convenience. (The reader is asked to bear with this ridiculous factor $\frac{1}{2}$ until Section 5.1 when its convenience will become apparent.)

What is the behavior of $u(x, t)$ as $t \to +\infty$? Since all but the first term in (5) contains an exponentially decaying factor, the solution decays quite fast to the first term $\frac{1}{2}A_0$, which is just a constant. Since these boundary conditions correspond to insulation at both ends, this agrees perfectly with our intuition of Section 2.5 that the solution "spreads out." This is the eventual behavior if we wait long enough. (To actually *prove* that the limit as $t \to \infty$ is given term by term in (5) requires the use of one of the convergence theorems in Section A.2. We omit this verification here.)

Consider now the *wave* equation with the Neumann BCs. The eigenvalue $\lambda = 0$ then leads to $X(x) =$ constant and to the differential equation $T''(t) = \lambda c^2 T(t) = 0$, which has the solution $T(t) = A + Bt$. Therefore, the wave equation with Neumann BCs has the solutions

$$\boxed{\begin{aligned} u(x,t) = {} & \frac{1}{2}A_0 + \frac{1}{2}B_0 t \\ & + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi ct}{l}\right)\cos\frac{n\pi x}{l}. \end{aligned}} \tag{7}$$

(Again, the factor $\frac{1}{2}$ will be justified later.) Then the initial data must satisfy

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} \tag{8}$$

and

$$\psi(x) = \frac{1}{2}B_0 + \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \cos\frac{n\pi x}{l}. \tag{9}$$

Equation (9) comes from first differentiating (7) with respect to t and then setting $t = 0$. □

A "mixed" boundary condition would be Dirichlet at one end and Neumann at the other. For instance, in case the BCs are $u(0,t) = u_x(l,t) = 0$, the eigenvalue problem is

$$\boxed{-X'' = \lambda X \qquad X(0) = X'(l) = 0.} \tag{10}$$

The eigenvalues then turn out to be $(n+\frac{1}{2})^2\pi^2/l^2$ and the eigenfunctions $\sin[(n+\frac{1}{2})\pi x/l]$ for $n = 0, 1, 2, \ldots$ (see Exercises 1 and 2). For a discussion of boundary conditions in the context of musical instruments, see [HJ].

For another example, consider the *Schrödinger* equation $u_t = iu_{xx}$ in $(0, l)$ with the Neumann BCs $u_x(0,t) = u_x(l,t) = 0$ and initial condition $u(x,0) = \phi(x)$. Separation of variables leads to the equation

$$\frac{T'}{iT} = \frac{X''}{X} = -\lambda = \text{constant},$$

so that $T(t) = e^{-i\lambda t}$ and $X(x)$ satisfies exactly the same problem (1) as before. Therefore, the solution is

$$\boxed{u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-i(n\pi/l)^2 t} \cos\frac{n\pi x}{l}.}$$

The initial condition requires the cosine expansion (6).

EXERCISES

1. Solve the diffusion problem $u_t = ku_{xx}$ in $0 < x < l$, with the mixed boundary conditions $u(0,t) = u_x(l,t) = 0$.
2. Consider the equation $u_{tt} = c^2 u_{xx}$ for $0 < x < l$, with the boundary conditions $u_x(0, t) = 0$, $u(l, \text{t}) = 0$ (Neumann at the left, Dirichlet at the right).
   (a) Show that the eigenfunctions are $\cos[(n + \frac{1}{2})\pi x/l]$.
   (b) Write the series expansion for a solution $u(x,t)$.
3. Solve the Schrödinger equation $u_t = iku_{xx}$ for real k in the interval $0 < x < l$ with the boundary conditions $u_x(0,t) = 0$, $u(l,t) = 0$.
4. Consider diffusion inside an enclosed circular tube. Let its length (circumference) be $2l$. Let x denote the arc length parameter where $-l \le x \le l$. Then the concentration of the diffusing substance satisfies

$$u_t = ku_{xx} \quad \text{for } -l \le x \le l$$

$$u(-l,t) = u(l,t) \quad \text{and} \quad u_x(-l,t) = u_x(l,t).$$

   These are called *periodic boundary conditions*.
   (a) Show that the eigenvalues are $\lambda = (n\pi/l)^2$ for $n = 0, 1, 2, 3, \ldots$.
   (b) Show that the concentration is

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l}\right) e^{-n^2\pi^2 kt/l^2}.$$

4.3 THE ROBIN CONDITION

We continue the method of separation of variables for the case of the Robin condition. The Robin condition means that we are solving $-X'' = \lambda X$ with the boundary conditions

$$\boxed{\begin{aligned} X' - a_0 X &= 0 \quad \text{at } x = 0 \qquad (1)\\ X' + a_l X &= 0 \quad \text{at } x = l. \qquad (2)\end{aligned}}$$

The two constants a_0 and a_l should be considered as given.

The physical reason they are written with opposite signs is that they correspond to *radiation* of energy if a_0 and a_l are positive, *absorption* of energy if a_0 and a_l are negative, and *insulation* if $a_0 = a_l = 0$. This is the interpretation for a heat problem: See the discussion in Section 1.4 or Exercise 2.3.8. For the case of the vibrating string, the interpretation is that the string shares its energy with the endpoints if a_0 and a_l are positive, whereas the string gains some energy from the endpoints if a_0 and a_l are negative: See Exercise 11.

The mathematical reason for writing the constants in this way is that the unit *outward* normal $\mathbf{n}$ for the interval $0 \le x \le l$ points to the *left* at $x = 0$ ($\mathbf{n} = -1$) and to the *right* at $x = l$ ($\mathbf{n} = +1$). Therefore, we expect that the nature of the eigenfunctions might depend on the signs of the two constants in opposite ways.

POSITIVE EIGENVALUES

Our task now is to solve the ODE $-X'' = \lambda X$ with the boundary conditions (1), (2). First let's look for the *positive eigenvalues*

$$\lambda = \beta^2 > 0.$$

As usual, the solution of the ODE is

$$X(x) = C\cos\beta x + D\sin\beta x \tag{3}$$

so that

$$X'(x) \pm aX(x) = (\beta D \pm aC)\cos\beta x + (-\beta C \pm aD)\sin\beta x.$$

At the left end $x = 0$ we require that

$$0 = X'(0) - a_0 X(0) = \beta D - a_0 C. \tag{4}$$

So we can solve for D in terms of C. At the right end $x = l$ we require that

$$0 = (\beta D + a_l C)\cos\beta l + (-\beta C + a_l D)\sin\beta l. \tag{5}$$

Messy as they may look, equations (4) and (5) are easily solved since they are equivalent to the matrix equation

$$\begin{pmatrix} -a_0 & \beta \\ a_l\cos\beta l - \beta\sin\beta l & \beta\cos\beta l + a_l\sin\beta l \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{6}$$

Therefore, substituting for D, we have

$$0 = (a_0 C + a_l C)\cos\beta l + \left(-\beta C + \frac{a_l a_0 C}{\beta}\right)\sin\beta l. \tag{7}$$

We don't want the trivial solution $C = 0$. We divide by $C\cos\beta l$ and multiply by β to get

$$\boxed{\left(\beta^2 - a_0 a_l\right)\tan\beta l = (a_0 + a_l)\beta.} \tag{8}$$

Any root $\beta > 0$ of this "algebraic" equation would give us an eigenvalue $\lambda = \beta^2$.

What would be the corresponding eigenfunction? It would be the above $X(x)$ with the required relation between C and D, namely,

$$\boxed{X(x) = C\left(\cos\beta x + \frac{a_0}{\beta}\sin\beta x\right)} \tag{9}$$

for any $C \neq 0$. By the way, because we divided by $\cos\beta l$, there is the exceptional case when $\cos\beta l = 0$; it would mean by (7) that $\beta = \sqrt{a_0 a_l}$.

Our next task is to solve (8) for β. This is not so easy, as there is no simple formula. One way is to calculate the roots numerically, say by Newton's method. Another way is by graphical analysis, which, instead of precise numerical values, will provide a lot of qualitative information. This is what we'll do. It's here where the nature of a_0 and a_l come into play. Let us rewrite the eigenvalue equation (8) as

$$\tan\beta l = \frac{(a_0 + a_l)\beta}{\beta^2 - a_0 a_l}. \tag{10}$$

Our method is to sketch the graphs of the tangent function $y = \tan\beta l$ and the rational function $y = (a_0 + a_l)\beta/(\beta^2 - a_0 a_l)$ as functions of $\beta > 0$ and to find their points of intersection. What the rational function looks like depends on the constants a_0 and a_l.

Case 1 In Figure 1 is pictured the case of *radiation at both ends:* $a_0 > 0$ and $a_l > 0$. Each of the points of intersection (for $\beta > 0$) provides an eigenvalue $\lambda_n = \beta_n^2$. The results depend very much on the a_0 and a_l. The exceptional situation mentioned above, when $\cos\beta l = 0$ and $\beta = \sqrt{a_0 a_l}$, will occur when the graphs of the tangent function and the rational function "intersect at infinity."

No matter what they are, as long as they are both positive, the graph clearly shows that

$$n^2\frac{\pi^2}{l^2} < \lambda_n < (n+1)^2\frac{\pi^2}{l^2} \qquad (n = 0, 1, 2, 3, \ldots). \tag{11}$$

Furthermore,

$$\lim_{n\to\infty} \beta_n - n\frac{\pi}{l} = 0, \tag{12}$$

which means that the larger eigenvalues get relatively closer and closer to $n^2\pi^2/l^2$ (see Exercise 19). You may compare this to the case $a_0 = a_l = 0$, the Neumann problem, where they are all *exactly* equal to $n^2\pi^2/l^2$.

Case 2 The case of absorption at $x = 0$ and radiation at $x = l$, but *more radiation than absorption*, is given by the conditions

$$a_0 < 0, \; a_l > 0, \; a_0 + a_l > 0. \tag{13}$$

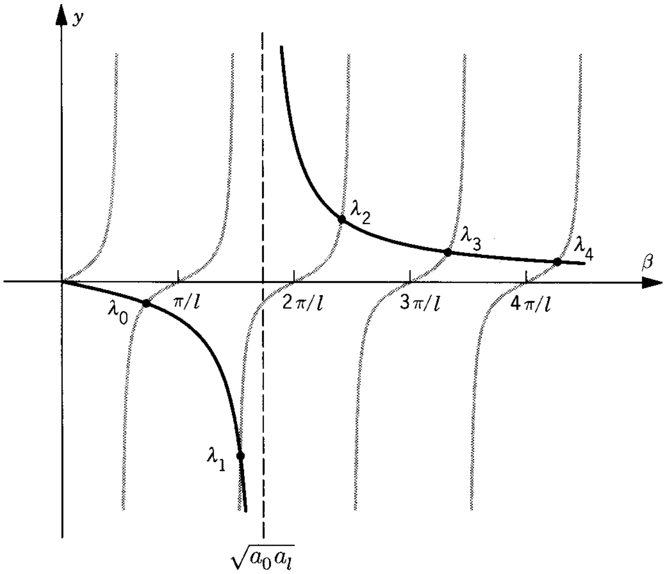

Figure 1

Then the graph looks like Figure 2 or 3, depending on the relative sizes of a_0 and a_l. Once again we see that (11) and (12) hold, except that in Figure 2 there is no eigenvalue λ_0 in the interval $(0, \pi^2/l^2)$.

There is an eigenvalue in the interval $(0, \pi^2/l^2)$ only if the rational curve crosses the *first* branch of the tangent curve. Since the rational curve has only a single maximum, this crossing can happen only if the slope of the rational curve is greater than the slope of the tangent curve at the origin. Let's

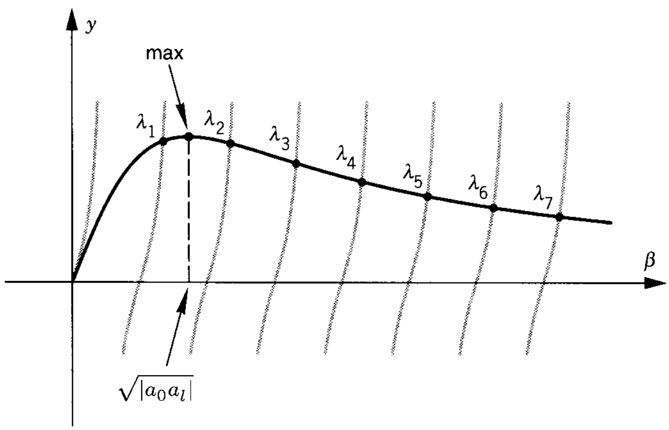

Figure 2

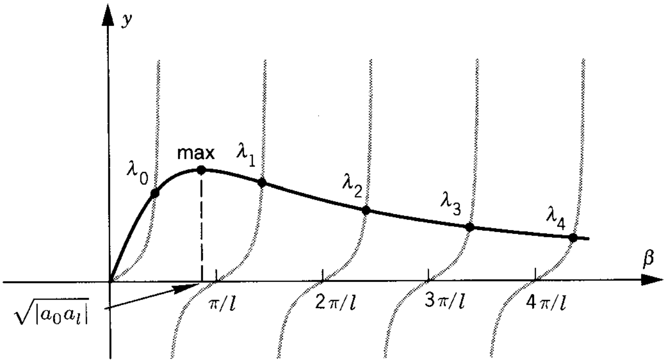

Figure 3

calculate these two slopes. A direct calculation shows that the slope $dy/d\beta$ of the rational curve at the origin is

$$\frac{a_0 + a_l}{-a_0 a_l} = \frac{a_l - |a_0|}{a_l\,|a_0|} > 0$$

because of (13). On the other hand, the slope of the tangent curve $y = \tan l\beta$ at the origin is $l \sec^2(l0) = l$. Thus we reach the following conclusion. In case

$$a_0 + a_l > -a_0 a_l l \tag{14}$$

(which means "much more radiation than absorption"), the rational curve will start out at the origin with a greater slope than the tangent curve and the two graphs must intersect at a point in the interval $(0, \pi/2l)$. Therefore, we conclude that *in Case* 2 *there is an eigenvalue* $0 < \lambda_0 < (\pi/2l)^2$ *if and only if* (14) *holds.*

Other cases, for instance absorption at both ends, may be found in the exercises, especially Exercise 8.

ZERO EIGENVALUE

In Exercise 2 it is shown that *there is a zero eigenvalue if and only if*

$$a_0 + a_l = -a_0 a_l l. \tag{15}$$

Notice that (15) can happen only if a_0 or a_l is negative and the interval has exactly a certain length or else $a_0 = a_l = 0$.

NEGATIVE EIGENVALUE

Now let's investigate the possibility of a negative eigenvalue. This is a very important question; see the discussion at the end of this section. To avoid dealing with imaginary numbers, we set

$$\lambda = -\gamma^2 < 0$$

and write the solution of the differential equation as

$$X(x) = C \cosh \gamma x + D \sinh \gamma x.$$

(An alternative form, which we used at the end of Section 4.1, is $Ae^{\gamma x} + Be^{-\gamma x}$.) The boundary conditions, much as before, lead to the eigenvalue equation

$$\boxed{\tanh \gamma l = -\frac{(a_0 + a_l)\gamma}{\gamma^2 + a_0 a_l}.} \tag{16}$$

(Verify it!) So we look for intersections of these two graphs [on the two sides of (16)] for $\gamma > 0$. Any such point of intersection would provide a negative eigenvalue $\lambda = -\gamma^2$ and a corresponding eigenfunction

$$\boxed{X(x) = \cosh \gamma x + \frac{a_0}{\gamma} \sinh \gamma x.} \tag{17}$$

Several different cases are illustrated in Figure 4. Thus in Case 1, of radiation at both ends, when a_0 and a_l are both positive, there is no intersection and so no negative eigenvalue.

Case 2, the situation with more radiation than absorption ($a_0 < 0,\ a_l > 0,\ a_0 + a_l > 0$), is illustrated by the two solid (14) and dashed (18) curves. There is either one intersection or none, depending on the slopes at the origin. The slope of the tanh curve is l, while the slope of the rational curve is

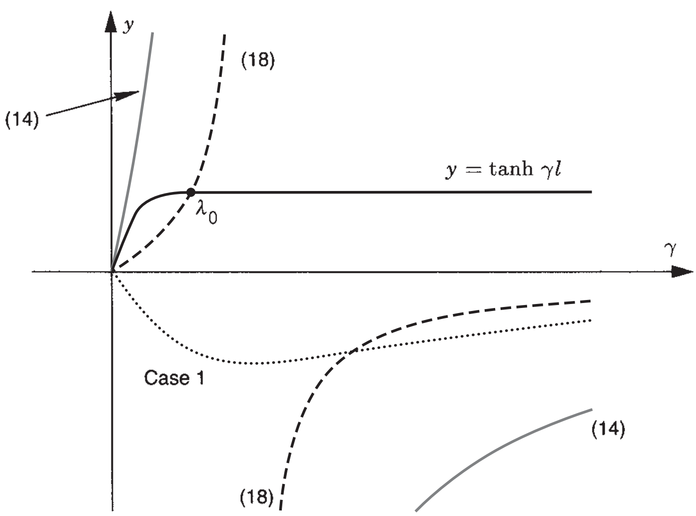

Figure 4

$-(a_0 + a_l)/(a_0 a_1) > 0$. If the last expression is smaller than l, there is an intersection; otherwise, there isn't. So our conclusion in Case 2 is as follows.

Let $a_0 < 0$ and $a_l > -a_0$. If

$$a_0 + a_l < -a_0 a_l l, \tag{18}$$

then there exists exactly one negative eigenvalue, which we'll call $\lambda_0 < 0$. If (14) *holds, then there is no negative eigenvalue.* Notice how the "missing" positive eigenvalue λ_0 in case (18) now makes its appearance as a negative eigenvalue! Furthermore, the zero eigenvalue is the borderline case (15); therefore, we use the notation $\lambda_0 = 0$ in the case of (15).

SUMMARY

We summarize the various cases as follows:

Case 1: Only positive eigenvalues.

Case 2 with (14): Only positive eigenvalues.

Case 2 with (15): Zero is an eigenvalue, all the rest are positive.

Case 2 with (18): One negative eigenvalue, all the rest are positive.

Exercise 8 provides a complete summary of all the other cases.

In any case, that is, for any values for a_0 and a_l, there are no complex, nonreal, eigenvalues. This fact can be shown directly as before but will also be shown by a general, more satisfying, argument in Section 5.3. Furthermore, there are always an infinite number of positive eigenvalues, as is clear from (10). In fact, the tangent function has an infinite number of branches. The rational function on the right side of (10) always goes from the origin to the β axis as $\beta \to \infty$ and so must cross each branch of the tangent except possibly the first one.

For all these problems it is critically important to find *all* the eigenvalues. If even one of them were missing, there would be initial data for which we could not solve the diffusion or wave equations. This will become clearer in Chapter 5. Exactly how we enumerate the eigenvalues, that is, whether we call the first one λ_0 or λ_1 or λ_5 or λ_{-2}, is not important. It is convenient, however, to number them in a consistent way. In the examples presented above we have numbered them in a way that neatly exhibits their dependence on a_0 and a_l.

What Is the Grand Conclusion for the Robin BCs? As before, we have an expansion

$$u(x, t) = \sum_n T_n(t) X_n(x), \tag{19}$$

where $X_n(x)$ are the eigenfunctions and where

$$T_n(t) = \begin{cases} A_n e^{-\lambda_n kt} & \text{for diffusions} \\ A_n \cos(\sqrt{\lambda_n}\, ct) + B_n \sin(\sqrt{\lambda_n}\, ct) & \text{for waves.} \end{cases} \tag{20}$$

Example 1.

Let $a_0 < 0 < a_0 + a_l < -a_0 a_l l$, which is Case 2 with (18). Then the grand conclusion takes the following explicit form. As we showed above, in this case there is exactly one negative eigenvalue $\lambda_0 = -\gamma_0^2 < 0$ as well as a sequence of positive ones $\lambda_n = +\beta_n^2 > 0$ for $n = 1, 2, 3, \ldots$. The complete solution of the diffusion problem

$$\begin{aligned} u_t &= k u_{xx} && \text{for } 0 < x < l, \quad 0 < t < \infty \\ u_x - a_0 u &= 0 && \text{for } x = 0, \quad u_x + a_l u = 0 \quad \text{for } x = l \\ u &= \phi && \text{for } t = 0 \end{aligned}$$

therefore is

$$\boxed{\begin{aligned} u(x,t) = {} & A_0 e^{+\gamma_0^2 kt}\left(\cosh \gamma_0 x + \frac{a_0}{\gamma_0}\sinh \gamma_0 x\right) \\ & + \sum_{n=1}^{\infty} A_n e^{-\beta_n^2 kt}\left(\cos \beta_n x + \frac{a_0}{\beta_n}\sin \beta_n x\right). \end{aligned}} \tag{21}$$

This conclusion (21) has the following physical interpretation if, say, $u(x, t)$ is the *temperature* in a rod of length l. We have taken the case when energy is supplied at $x = 0$ (absorption of energy by the rod, heat flux goes *into* the rod at its left end) and when energy is radiated from the right end (the heat flux goes *out*). For a given length l and a given radiation $a_l > 0$, there is a negative eigenvalue ($\lambda_0 = -\gamma_0^2$) if and only if the absorption is great enough [$|a_0| > a_l/(1 + a_l l)$]. Such a large absorption coefficient allows the temperature to build up to large values, as we see from the expansion (21). In fact, all the terms get smaller as time goes on, except the first one, which *grows* exponentially due to the factor $e^{+\gamma_0^2 kt}$. So the rod gets hotter and hotter (unless $A_0 = 0$, which could only happen for very special initial data).

If, on the other hand, the absorption is relatively small [that is, $|a_0| < a_l/(1 + a_l l)$], then all the eigenvalues are positive and the temperature will remain bounded and will eventually decay to zero. Other interpretations of this sort are left for the exercises. □

For the *wave equation*, a negative eigenvalue $\lambda_0 = -\gamma_0^2$ would also lead to exponential growth because the expansion for $u(x,t)$ would contain the term

$$(A_0 e^{\gamma_0 ct} + B_0 e^{-\gamma_0 ct}) X_0(x).$$

This term comes from the usual equation $-T'' = \lambda c^2 T = -(\gamma_0 c)^2 T$ for the temporal part of a separated solution (see Exercise 10).

EXERCISES

1. Find the eigenvalues graphically for the boundary conditions

$$X(0) = 0, \qquad X'(l) + aX(l) = 0.$$

Assume that $a \neq 0$.

2. Consider the eigenvalue problem with Robin BCs at both ends:

$$-X'' = \lambda X$$
$$X'(0) - a_0 X(0) = 0, \quad X'(l) + a_l X(l) = 0.$$

   (a) Show that $\lambda = 0$ is an eigenvalue if and only if $a_0 + a_l = -a_0 a_l l$.
   (b) Find the eigenfunctions corresponding to the zero eigenvalue. (*Hint:* First solve the ODE for $X(x)$. The solutions are not sines or cosines.)

3. Derive the eigenvalue equation (16) for the negative eigenvalues $\lambda = -\gamma^2$ and the formula (17) for the eigenfunctions.

4. Consider the Robin eigenvalue problem. If

$$a_0 < 0, \quad a_l < 0 \quad \text{and} \quad -a_0 - a_l < a_0 a_l l,$$

show that there are *two* negative eigenvalues. This case may be called "substantial absorption at both ends." (*Hint:* Show that the rational curve $y = -(a_0 + a_l)\gamma/(\gamma^2 + a_0 a_l)$ has a single maximum and crosses the line $y = 1$ in two places. Deduce that it crosses the tanh curve in two places.)

5. In Exercise 4 (substantial absorption at both ends) show graphically that there are an infinite number of positive eigenvalues. Show graphically that they satisfy (11) and (12).

6. If $a_0 = a_l = a$ in the Robin problem, show that:
   (a) There are *no* negative eigenvalues if $a \geq 0$, there is *one* if $-2/l < a < 0$, and there are *two* if $a < -2/l$.
   (b) Zero is an eigenvalue if and only if $a = 0$ or $a = -2/l$.

7. If $a_0 = a_l = a$, show that as $a \to +\infty$, the eigenvalues tend to the eigenvalues of the Dirichlet problem. That is,

$$\lim_{a \to \infty} \left\{ \beta_n(a) - \frac{(n+1)\pi}{l} \right\} = 0,$$

where $\lambda_n(a) = [\beta_n(a)]^2$ is the $(n+1)$st eigenvalue.

8. Consider again Robin BCs at both ends for arbitrary a_0 and a_l.
   (a) In the $a_0 a_l$ plane sketch the hyperbola $a_0 + a_l = -a_0 a_l l$. Indicate the asymptotes. For (a_0, a_l) on this hyperbola, zero is an eigenvalue, according to Exercise 2(a).
   (b) Show that the hyperbola separates the whole plane into three regions, depending on whether there are two, one, or no negative eigenvalues.
   (c) Label the directions of increasing absorption and radiation on each axis. Label the point corresponding to Neumann BCs.
   (d) Where in the plane do the Dirichlet BCs belong?
9. On the interval $0 \le x \le 1$ of length one, consider the eigenvalue problem

$$-X'' = \lambda X$$

$$X'(0) + X(0) = 0 \qquad \text{and} \qquad X(1) = 0$$

   (absorption at one end and zero at the other).
   (a) Find an eigenfunction with eigenvalue zero. Call it $X_0(x)$.
   (b) Find an equation for the positive eigenvalues $\lambda = \beta^2$.
   (c) Show graphically from part (b) that there are an infinite number of positive eigenvalues.
   (d) Is there a negative eigenvalue?
10. Solve the wave equation with Robin boundary conditions under the assumption that (18) holds.
11. (a) Prove that the (total) energy is conserved for the wave equation with Dirichlet BCs, where the energy is defined to be

$$E = \tfrac{1}{2}\int_0^l \left(c^{-2}u_t^2 + u_x^2\right) dx.$$

   (Compare this definition with Section 2.2.)
   (b) Do the same for the Neumann BCs.
   (c) For the Robin BCs, show that

$$E_R = \tfrac{1}{2}\int_0^l \left(c^{-2}u_t^2 + u_x^2\right) dx + \tfrac{1}{2}a_l[u(l,t)]^2 + \tfrac{1}{2}a_0[u(0,t)]^2$$

   is conserved. Thus, while the total energy E_R is still a constant, some of the internal energy is "lost" to the boundary if a_0 and a_l are positive and "gained" from the boundary if a_0 and a_l are negative.
12. Consider the unusual eigenvalue problem

$$-v_{xx} = \lambda v \qquad \text{for } 0 < x < l$$

$$v_x(0) = v_x(l) = \frac{v(l) - v(0)}{l}.$$

   (a) Show that $\lambda = 0$ is a double eigenvalue.
   (b) Get an equation for the positive eigenvalues $\lambda > 0$.

(c) Letting $\gamma = \frac{1}{2}l\sqrt{\lambda}$, reduce the equation in part (b) to the equation

$$\gamma \sin\gamma \cos\gamma = \sin^2\gamma.$$

(d) Use part (c) to find half of the eigenvalues explicitly and half of them graphically.

(e) Assuming that all the eigenvalues are nonnegative, make a list of all the eigenfunctions.

(f) Solve the problem $u_t = ku_{xx}$ for $0 < x < l$, with the BCs given above, and with $u(x, 0) = \phi(x)$.

(g) Show that, as $t \to \infty$, $\lim u(x, t) = A + Bx$ for some constants A, B, assuming that you can take limits term by term.

13. Consider a string that is fixed at the end $x = 0$ and is free at the end $x = l$ except that a load (weight) of given mass is attached to the right end.

(a) Show that it satisfies the problem

$$u_{tt} = c^2 u_{xx} \qquad \text{for } 0 < x < l$$

$$u(0, t) = 0 \qquad u_{tt}(l, t) = -ku_x(l, t)$$

for some constant k.

(b) What is the eigenvalue problem in this case?

(c) Find the equation for the positive eigenvalues and find the eigenfunctions.

14. Solve the eigenvalue problem $x^2u'' + 3xu' + \lambda u = 0$ for $1 < x < e$, with $u(1) = u(e) = 0$. Assume that $\lambda > 1$. (*Hint:* Look for solutions of the form $u = x^m$.)

15. Find the equation for the eigenvalues λ of the problem

$$(\kappa(x)X')' + \lambda\rho(x)X = 0 \qquad \text{for } 0 < x < l \text{ with } X(0) = X(l) = 0,$$

where $\kappa(x) = \kappa_1^2$ for $x < a$, $\kappa(x) = \kappa_2^2$ for $x > a$, $\rho(x) = \rho_1^2$ for $x < a$, and $\rho(x) = \rho_2^2$ for $x > a$. All these constants are positive and $0 < a < l$.

16. Find the positive eigenvalues and the corresponding eigenfunctions of the fourth-order operator $+d^4/dx^4$ with the four boundary conditions

$$X(0) = X(l) = X''(0) = X''(l) = 0.$$

17. Solve the fourth-order eigenvalue problem $X'''' = \lambda X$ in $0 < x < l$, with the four boundary conditions

$$X(0) = X'(0) = X(l) = X'(l) = 0,$$

where $\lambda > 0$. (*Hint:* First solve the fourth-order ODE.)

18. A tuning fork may be regarded as a pair of vibrating flexible bars with a certain degree of stiffness. Each such bar is clamped at one end and is approximately modeled by the fourth-order PDE $u_{tt} + c^2u_{xxxx} = 0$. It has initial conditions as for the wave equation. Let's say that on the end $x = 0$ it is clamped (fixed), meaning that it satisfies

$u(0, t) = u_x(0, t) = 0$. On the other end $x = l$ it is free, meaning that it satisfies $u_{xx}(l, t) = u_{xxx}(l, t) = 0$. Thus there are a total of four boundary conditions, two at each end.

(a) Separate the time and space variables to get the eigenvalue problem $X'''' = \lambda X$.

(b) Show that zero is not an eigenvalue.

(c) Assuming that all the eigenvalues are positive, write them as $\lambda = \beta^4$ and find the equation for β.

(d) Find the frequencies of vibration.

(e) Compare your answer in part (d) with the overtones of the vibrating string by looking at the ratio β_2^2/β_1^2. Explain why you hear an almost pure tone when you listen to a tuning fork.

19. Show that in Case 1 (radiation at both ends)

$$\lim_{n\to\infty}\left[\lambda_n - \frac{n^2\pi^2}{l^2}\right] = \frac{2}{l}\left(a_0 + a_l\right).$$

5

FOURIER SERIES

Our first goal in this key chapter is to find the coefficients in a Fourier series. In Section 5.3 we introduce the idea of orthogonality of functions and we show how the different varieties of Fourier series can be treated in a unified fashion. In Section 5.4 we state the basic completeness (convergence) theorems. Proofs are given in Section 5.5. The final section is devoted to the treatment of inhomogeneous boundary conditions. Joseph Fourier developed his ideas on the convergence of trigonometric series while studying heat flow. His 1807 paper was rejected by other scientists as too imprecise and was not published until 1822.

5.1 THE COEFFICIENTS

In Chapter 4 we have found Fourier series of several types. How do we find the coefficients? Luckily, there is a very beautiful, conceptual formula for them.

Let us begin with the *Fourier sine series*

$$\boxed{\phi(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}} \tag{1}$$

in the interval $(0, l)$. [It turns out that this infinite series converges to $\phi(x)$ for $0 < x < l$, but let's postpone further discussion of the delicate question of convergence for the time being.] The first problem we tackle is to try to find the coefficients A_n if $\phi(x)$ is a given function. The key observation is that the sine functions have the wonderful property that

$$\boxed{\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx = 0 \quad \text{if } m \neq n,} \tag{2}$$

m and n being positive integers. This can be verified directly by integration. [Historically, (1) was first discovered by a horrible expansion in Taylor series!]

Proof of (2). We use the trig identity

$$\sin a \sin b = \tfrac{1}{2}\cos(a-b) - \tfrac{1}{2}\cos(a+b).$$

Therefore, the integral equals

$$\frac{l}{2(m-n)\pi}\sin\frac{(m-n)\pi x}{l}\Bigg|_0^l - [\text{same with } (m+n)]$$

if $m \neq n$. This is a linear combination of $\sin(m \pm n)\pi$ and $\sin 0$, and so it vanishes. □

The far-reaching implications of this observation are astounding. Let's *fix* m, multiply (1) by $\sin(m\pi x/l)$, and integrate the series (1) term by term to get

$$\int_0^l \phi(x)\sin\frac{m\pi x}{l}\,dx = \int_0^l \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx$$

$$= \sum_{n=1}^{\infty} A_n \int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx.$$

All but one term in this sum vanishes, namely the one with $n = m$ (n just being a "dummy" index that takes on all integer values ≥ 1). Therefore, we are left with the single term

$$A_m \int_0^l \sin^2\frac{m\pi x}{l}\,dx, \tag{3}$$

which equals $\frac{1}{2}lA_m$ by explicit integration. Therefore,

$$\boxed{A_m = \frac{2}{l}\int_0^l \phi(x)\sin\frac{m\pi x}{l}\,dx.} \tag{4}$$

This is the famous *formula for the Fourier coefficients* in the series (1). That is, *if* $\phi(x)$ has an expansion (1), *then* the coefficients must be given by (4).

These are the only possible coefficients in (1). However, the basic question still remains whether (1) is in fact valid with these values of the coefficients. This is a question of convergence, and we postpone it until Section 5.4.

APPLICATION TO DIFFUSIONS AND WAVES

Going back to the diffusion equation with Dirichlet boundary conditions, the formula (4) provides the final ingredient in the solution formula for arbitrary initial data $\phi(x)$.

As for the wave equation with Dirichlet conditions, the initial data consist of a pair of functions $\phi(x)$ and $\psi(x)$ with expansions (4.1.10) and (4.1.11). The coefficients A_m in (4.1.9) are given by (4), while for the same reason the coefficients B_m are given by the similar formula

$$\frac{m\pi c}{l} B_m = \frac{2}{l} \int_0^l \psi(x) \sin \frac{m\pi x}{l}\, dx. \tag{5}$$

FOURIER COSINE SERIES

Next let's take the case of the cosine series, which corresponds to the Neumann boundary conditions on $(0, l)$. We write it as

$$\boxed{\phi(x) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l}.} \tag{6}$$

Again we can verify the magical fact that

$$\int_0^l \cos \frac{n\pi x}{l} \cos \frac{m\pi x}{l}\, dx = 0 \qquad \text{if } m \neq n$$

where m and n are nonnegative integers. (Verify it!) By exactly the same method as above, but with sines replaced by cosines, we get

$$\int_0^l \phi(x) \cos \frac{m\pi x}{l}\, dx = A_m \int_0^l \cos^2 \frac{m\pi x}{l}\, dx = \frac{1}{2} l A_m$$

if $m \neq 0$. For the case $m = 0$, we have

$$\int_0^l \phi(x) \cdot 1\, dx = \frac{1}{2} A_0 \int_0^l 1^2\, dx = \frac{1}{2} l A_0.$$

Therefore, for all *nonnegative* integers m, we have the formula for the coefficients of the cosine series

$$\boxed{A_m = \frac{2}{l} \int_0^l \phi(x) \cos \frac{m\pi x}{l}\, dx.} \tag{7}$$

[This is the reason for putting the $\frac{1}{2}$ in front of the constant term in (6).]

FULL FOURIER SERIES

The full Fourier series, or simply the Fourier series, of $\phi(x)$ on the interval $-l < x < l$, is defined as

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l}\right). \tag{8}$$

Watch out: The interval is twice as long! The eigenfunctions now are all the functions $\{1, \cos(n\pi x/l), \sin(n\pi x/l)\}$, where $n = 1, 2, 3, \dots$. Again we have the same wonderful coincidence: Multiply any two different eigenfunctions and integrate over the interval and you get zero! That is,

$$\int_{-l}^{l} \cos\frac{n\pi x}{l} \sin\frac{m\pi x}{l}\,dx = 0 \qquad \text{for all } n, m$$

$$\int_{-l}^{l} \cos\frac{n\pi x}{l} \cos\frac{m\pi x}{l}\,dx = 0 \qquad \text{for } n \neq m$$

$$\int_{-l}^{l} \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l}\,dx = 0 \qquad \text{for } n \neq m$$

$$\int_{-l}^{l} 1 \cdot \cos\frac{n\pi x}{l}\,dx = 0 = \int_{-l}^{l} 1 \cdot \sin\frac{m\pi x}{l}\,dx.$$

Therefore, the same procedure will work to find the coefficients. We also calculate the integrals of the squares

$$\int_{-l}^{l} \cos^2\frac{n\pi x}{l}\,dx = l = \int_{-l}^{l} \sin^2\frac{n\pi x}{l}\,dx \quad \text{and} \quad \int_{-l}^{l} 1^2\,dx = 2l.$$

(Verify these integrals too!) Then we end up with the formulas

$$A_n = \frac{1}{l}\int_{-l}^{l} \phi(x)\cos\frac{n\pi x}{l}\,dx \quad (n = 0, 1, 2, \dots) \tag{9}$$

$$B_n = \frac{1}{l}\int_{-l}^{l} \phi(x)\sin\frac{n\pi x}{l}\,dx \quad (n = 1, 2, 3, \dots) \tag{10}$$

for the coefficients of the full Fourier series. Note that these formulas are *not* exactly the same as (4) and (7).

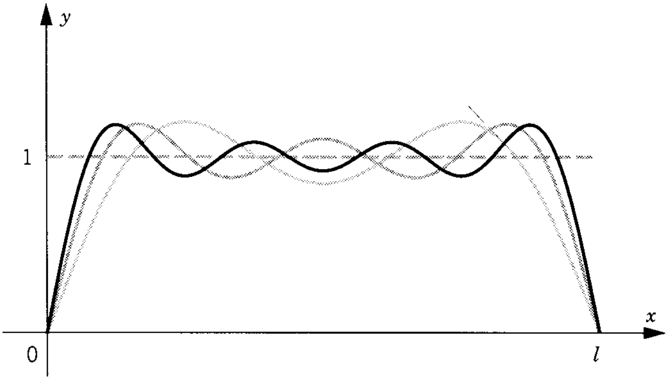

Figure 1

Example 1.

Let $\phi(x) \equiv 1$ in the interval [0, l]. It has a Fourier sine series with coefficients

$$A_m = \frac{2}{l}\int_0^l \sin\frac{m\pi x}{l}\,dx = -\frac{2}{m\pi}\cos\frac{m\pi x}{l}\bigg|_0^l$$
$$= \frac{2}{m\pi}(1-\cos m\pi) = \frac{2}{m\pi}[1-(-1)^m].$$

Thus $A_m = 4/m\pi$ if m is odd, and $A_m = 0$ if m is even. Thus

$$1 = \frac{4}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + \cdots\right) \tag{11}$$

in (0, l). (The factor $4/\pi$ is pulled out just for notational convenience.) See Figure 1 for a sketch of the first few partial sums. □

Example 2.

The same function $\phi(x) \equiv 1$ has a Fourier cosine series with coefficients

$$A_m = \frac{2}{l}\int_0^l \cos\frac{m\pi x}{l}\,dx = \frac{2}{m\pi}\sin\frac{m\pi x}{l}\bigg|_0^l$$
$$= \frac{2}{m\pi}(\sin m\pi - \sin 0) = 0 \qquad \text{for } m \neq 0.$$

So there is only one nonzero coefficient, namely, the one for $m = 0$. The Fourier cosine series is therefore trivial:

$$1 = 1 + 0\cos\frac{\pi x}{l} + 0\cos\frac{2\pi x}{l} + \cdots.$$

This is perfectly natural since the sum $1 = 1 + 0 + 0 + 0 + \cdots$ is obvious and the Fourier cosine expansion is unique. □

Example 3.

Let $\phi(x) \equiv x$ in the interval $(0, l)$. Its Fourier sine series has the coefficients

$$\begin{aligned} A_m &= \frac{2}{l}\int_0^l x \sin\frac{m\pi x}{l}\,dx \\ &= -\frac{2x}{m\pi}\cos\frac{m\pi x}{l} + \frac{2l}{m^2\pi^2}\sin\frac{m\pi x}{l}\bigg|_0^l \\ &= -\frac{2l}{m\pi}\cos m\pi + \frac{2l}{m^2\pi^2}\sin m\pi = (-1)^{m+1}\frac{2l}{m\pi}. \end{aligned}$$

Thus in $(0, l)$ we have

$$x = \frac{2l}{\pi}\left(\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} - \cdots\right). \tag{12}$$

□

Example 4.

Let $\phi(x) \equiv x$ in the interval $[0, l]$. Its Fourier cosine series has the coefficients

$$\begin{aligned} A_0 &= \frac{2}{l}\int_0^l x\,dx = l \\ A_m &= \frac{2}{l}\int_0^l x\cos\frac{m\pi x}{l}\,dx \\ &= \frac{2x}{m\pi}\sin\frac{m\pi x}{l} + \frac{2l}{m^2\pi^2}\cos\frac{m\pi x}{l}\bigg|_0^l \\ &= \frac{2l}{m\pi}\sin m\pi + \frac{2l}{m^2\pi^2}(\cos m\pi - 1) = \frac{2l}{m^2\pi^2}[(-1)^m - 1] \\ &= \frac{-4l}{m^2\pi^2} \quad \text{for } m \text{ odd,} \quad \text{and} \quad 0 \quad \text{for } m \text{ even.} \end{aligned}$$

Thus in $(0, l)$ we have

$$x = \frac{l}{2} - \frac{4l}{\pi^2}\left(\cos\frac{\pi x}{l} + \frac{1}{9}\cos\frac{3\pi x}{l} + \frac{1}{25}\cos\frac{5\pi x}{l} + \cdots\right). \tag{13}$$

□

Example 5.

Let $\phi(x) \equiv x$ in the interval $[-l, l]$. Its full Fourier series has the coefficients

$$
\begin{aligned}
A_0 &= \frac{1}{l}\int_{-l}^{l} x\,dx = 0 \\
A_m &= \frac{1}{l}\int_{-l}^{l} x\cos\frac{m\pi x}{l}\,dx \\
&= \frac{x}{m\pi}\sin\frac{m\pi x}{l} + \frac{l}{m^2\pi^2}\cos\frac{m\pi x}{l}\bigg|_{-l}^{l} \\
&= \frac{l}{m^2\pi^2}(\cos m\pi - \cos(-m\pi)) = 0 \\
B_m &= \frac{1}{l}\int_{-l}^{l} x\sin\frac{m\pi x}{l}\,dx \\
&= \frac{-x}{m\pi}\cos\frac{m\pi x}{l} + \frac{l}{m^2\pi^2}\sin\frac{m\pi x}{l}\bigg|_{-l}^{l} \\
&= \frac{-l}{m\pi}\cos m\pi + \frac{-l}{m\pi}\cos(-m\pi) = (-1)^{m+1}\frac{2l}{m\pi}.
\end{aligned}
$$

This gives us exactly the same series as (12), except that it is supposed to be valid in $(-l, l)$, which is not a surprising result because both sides of (12) are odd. □

Example 6.

Solve the problem

$$
\begin{gathered}
u_{tt} = c^2 u_{xx} \\
u(0,t) = u(l,t) = 0 \\
u(x,0) = x, \quad u_t(x,0) = 0.
\end{gathered}
$$

From Section 4.1 we know that $u(x, t)$ has an expansion

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi ct}{l} + B_n\sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}.$$

Differentiating with respect to time yields

$$u_t(x,t) = \sum_{n=1}^{\infty}\frac{n\pi c}{l}\left(-A_n\sin\frac{n\pi ct}{l} + B_n\cos\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}.$$

Setting $t = 0$, we have

$$0 = \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \sin \frac{n\pi x}{l}$$

so that all the $B_n = 0$. Setting $t = 0$ in the expansion of $u(x, t)$, we have

$$x = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}.$$

This is exactly the series of Example 3. Therefore, the complete solution is

$$u(x,t) = \frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}. \qquad \square$$

EXERCISES

1. In the expansion $1 = \sum_{n \text{ odd}} (4/n\pi) \sin n\pi$, valid for $0 < x < \pi$, put $x = \pi/4$ to calculate the sum

$$\left(1 - \tfrac{1}{5} + \tfrac{1}{9} - \tfrac{1}{13} + \cdots\right) + \left(\tfrac{1}{3} - \tfrac{1}{7} + \tfrac{1}{11} - \tfrac{1}{15} + \cdots\right)$$
$$= 1 + \tfrac{1}{3} - \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} + \cdots$$

(*Hint:* Since each of the series converges, they can be combined as indicated. However, they cannot be arbitrarily rearranged because they are only conditionally, not absolutely, convergent.)

2. Let $\phi(x) \equiv x^2$ for $0 \le x \le 1 = l$.
   (a) Calculate its Fourier sine series.
   (b) Calculate its Fourier cosine series.

3. Consider the function $\phi(x) \equiv x$ on $(0, l)$. On the same graph, *sketch* the following functions.
   (a) The sum of the first three (nonzero) terms of its Fourier sine series.
   (b) The sum of the first three (nonzero) terms of its Fourier cosine series.

4. Find the Fourier cosine series of the function $|\sin x|$ in the interval $(-\pi, \pi)$. Use it to find the sums

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}.$$

5. Given the Fourier sine series of $\phi(x) \equiv x$ on $(0, l)$. Assume that the series can be integrated term by term, a fact that will be shown later.
   (a) Find the Fourier cosine series of the function $x^2/2$. Find the constant of integration that will be the first term in the cosine series.