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المادة : (Pure 1) (Calculus I)

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الفصل الدراسي الأول

Lectures in Calculus (I)

Prepared

By

Dr. Ahmed Mohamed Yousef

Associate Professor of Pure Mathematics

Chapter 1

Real valued Functions

One of the important themes in calculus is the analysis of relationships between physical or mathematical quantities. Such relationships can be described in terms of graphs, formulas, numerical data, or words. In this chapter we will develop the concept of a function, which is the basic idea that underlies almost all mathematical and physical relationships, regardless of the form in which they are expressed. We will study properties of some of the most basic functions that occur in calculus.

Let us begin with some illustrative examples.

- The area A of a circle depends on its radius r by the equation $A = \pi r^2$, so we say that A is a function of r.
- Volume of a sphere depends on its radius by the equation $V = \frac{4}{3}\pi r^3$.
- Surface area of a cube depends on the length of its side by the equation $S = 6x^2$
- The velocity A of a ball falling freely in the Earth's gravitational field increases with time A until it hits the ground, so we say that A is a function of A.

This idea is captured in the following definition:

Definition 1.

If a variable y depends on a variable x in such a way that each value of x determines exactly one value of y, then we say that y **is a function of** x.

In the mid-eighteenth century the mathematician Euler conceived the idea of denoting functions by letters of the alphabet, thereby making it possible to describe functions without stating specific formulas, graphs, or tables.

This suggests the following definition:

Definition 2.

A function f is a rule that associates a unique output with each input. If the input is denoted by x, then the output is denoted by $f(x)$ (read "f of x").

This output is sometimes called the value of f at x or the image of x under f. Sometimes we will want to denote the output by a single letter, say y, and write

$$y = f(x)$$

This equation expresses y as a function of x. The variable x is called the independent variable of f, and the variable y is called the dependent variable of f. This terminology is intended to suggest that x is free to vary, but that once x has a specific value a corresponding value of y is determined. For now, we will only consider functions in which the independent and dependent variables are real numbers, in which case we say that f is a real-valued function of a real variable.

In the previous definition the term unique means "exactly one". Thus, a function cannot assign two different outputs to the same input.

For example, the following equation

$$y = x\sqrt{x^2 - 9}$$

describes y as a function of x because each input x in the interval $-3 \leq x \leq 3$ produces exactly one output $y = x\sqrt{x^2 - 9}$.

Definition 3.

A function f from set A to set B (written as $f : A \to B$) is a rule of correspondence that associates to each element of A, one and only one element of B. (A function is also called a mapping from A to B.)

We observe that

- Each element of B need not be in the association, but every element of A must be involved in it. Hence, a function is a one way pairing process. (Every element of A pairs off with some element of B but not conversely.) يقترن
- One element of A cannot be associated to more than one element of B, but one element of B may correspond to two or more elements of A.

The correspondence from the elements of set A to set B, shown in Figs 1.1-1.3 represents function(s) whereas that shown in Figs 1.4 and 1.5 does not represent functions.

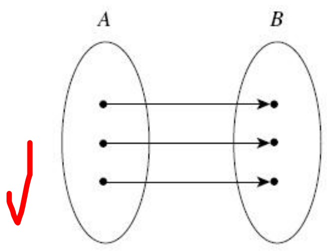


Fig 1.1

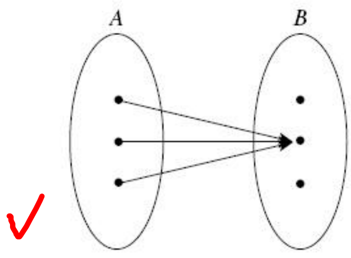


Fig 1.2

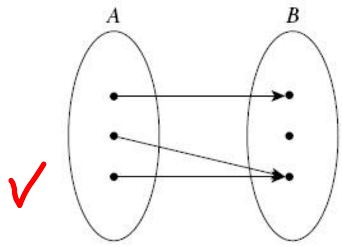


Fig 1.3

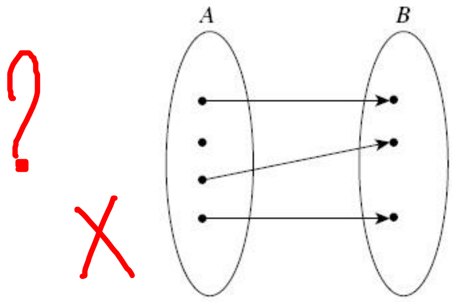


Fig 1.4

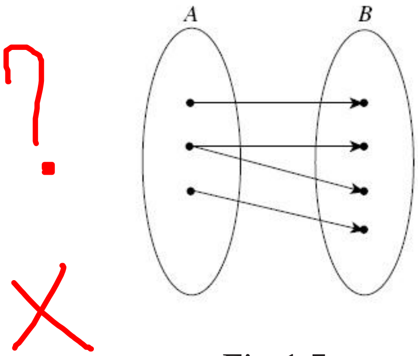


Fig 1.5

Example (1)

For $f(x) = x^2 - 2x$, find and simplify

(a) $f(4)$, (b) $f(4+h)$, (c) $f(4+h) - f(4)$

(d) $[f(4+h) - f(4)]\ /\ h$, where $h \neq 0$.

Solution

$$f(4) = 4^2 - 2(4) = 16 - 8 = 8$$

$$\begin{aligned} f(4+h) &= (4+h)^2 - 2(4+h) \\ &= (16 + 8h + h^2) - (8 + 2h) \\ &= 8 + 6h + h^2 \end{aligned}$$

$$\begin{aligned} f(4+h) - f(4) &= 8 + 6h + h^2 - 8 \\ &= 6h + h^2 \end{aligned}$$

$$[f(4+h) - f(4)]\ /\ h = (6h + h^2)\ /\ h = 6 + h$$

Domain and Range of a Function

Definition 4.

Let f be a function from set A to set B ($f : A \to B$), then

- The (entire) set A is called the domain of f.
- The (entire) set B is called the codomain of f.
- An element y of B that corresponds to some element x of A is denoted by $f(x)$, and it is called the image of x under f.
- The set of all images تشكل constitute the range of f. The range of f is denoted by $f(A)$ and it is a subset of set B. In other words $f(A) \subseteq B$.

Definition 5.

If $y = f(x)$ then the set of all possible inputs (x-values) is called the domain of f, and the set of outputs (y-values) that result when x varies over the domain is called the range of f.

For example, consider the equations Ex1 D_f, R_f

$$y = x^2$$

and Ex2

$$y = x^2, \quad x \geq 2$$

In the first equation there is no restriction on x, so we may assume that any real value of x is an allowable input. Thus, the equation defines a function $f(x) = x^2$ with domain $-\infty \leq x \leq \infty$. In the second equation, the inequality $x \geq 2$ restricts the allowable inputs to be greater than or equal to 2, so the equation defines a function $g(x) = x^2$, $x \geq 2$ with domain $2 \leq x \leq \infty$.

As x varies over the domain of the function $f(x) = x^2$, the values of $y = x^2$ vary over the interval $0 \leq y \leq \infty$, so this is the range of f. By comparison, as x varies over the domain of the function $g(x) = x^2, x \geq 2$, the values of $y = x^2$ vary over the interval $4 \leq y \leq \infty$, so this is the range of g. It is important to understand here that even though $f(x) = x^2$ and $g(x) = x^2, x \geq 2$ involve the same formula, we regard them to be different functions because they have different domains. In short, to fully describe a function you must not only specify the rule that relates the inputs and outputs, but you must also specify the domain, that is, the set of allowable inputs.

Example (2)

Find the domain of :

$$\text{(a)}\ \ f(x) = x^3 \qquad \text{(b)}\ \ f(x) = \frac{1}{(x-1)(x-3)}$$

$$\text{(c)}\ \ f(x) = \tan x \quad \text{(d)}\ \ f(x) = \sqrt{x^2 - 5x + 6}$$

Solution

(a) The function f has real values for all real x, so its domain is the interval $(-\infty, \infty)$.

(b) The function f has real values for all real x, except $x = 1$ and $x = 3$, where divisions by zero occur. Thus, the domain is $\{x : x \in R, x \neq 1 \text{ and } x \neq 3\} = (-\infty, 1) \cup (1, 3) \cup (3, \infty)$.

(c) Since $f(x) = \tan x = \dfrac{\sin x}{\cos x}$, the function f has real values except where $\cos x = 0$, and this occurs when x is an odd integer multiple of $\dfrac{\pi}{2}$. Thus, the domain consists of all real numbers except $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \ldots$

(d) The function f has real values, except when the expression inside the radical is negative. Thus the domain consists of all real numbers x such that $x^2 - 5x + 6 = (x-3)(x-2) \geq 0$. This inequality is satisfied if $x \leq 2$ or $x \geq 3$, so the natural domain of f is $(-\infty, 2] \cup [3, \infty)$.

Example (3)

Find the domain and range of

$$\text{(a)}\ \ f(x) = 2 + \sqrt{x-1} \quad \text{(b)}\ \ f(x) = \frac{x+1}{x-1}$$

Solution

(a) The domain of $f(x)$ is $[1, \infty)$. As x varies over the interval $[1, \infty)$, the value of $\sqrt{x-1}$ varies over the interval $[0, \infty)$, so the value of $f(x) = 2 + \sqrt{x-1}$ varies over the interval $[2,\infty)$, which is the range of $f(x)$.

(b) The given function $f(x)$ is defined for all real $x \neq 1$, so the domain of $f(x)$ is $(-\infty,1)\cup(1,\infty)$. To determine the range it will be convenient to introduce a dependent variable

$$y = \frac{x+1}{x-1} \qquad (*)$$

we solve (*) for x in terms of

$$(x-1)y = x+1$$
$$xy - y = x+1$$
$$xy - x = y+1$$
$$x(y-1) = y+1$$
$$x = \frac{y+1}{y-1}$$

So, the range of the function $f(x)$ is $(-\infty,1)\cup(1,\infty)$.

Example (4)

Find the domain for $\phi(\text{t}) = \sqrt{9-t^2}$.

Solution

Here, we must restrict t so that $9-t^2 \geq 0$, in order to avoid nonreal values for $\sqrt{9-t^2}$. This is achieved by requiring that $t^2 \leq 9$ or $-3 \leq t \leq 3$. Thus, the domain of $\phi(t)$ is $\{t \in R : -3 \leq t \leq 3\}$. In interval notation, we can write the domain as $[-3,3]$.

Example (5)

Determine the domains of the functions

$$\text{(a) } y = \sqrt{4-x^2} \quad \text{(b) } y = \sqrt{x^2-16} \quad \text{(c) } y = \frac{1}{x-2}$$

$$\text{(d) } y = \frac{1}{x^2-9} \qquad \text{(e) } y = \frac{x}{x^2+4}$$

Solution

a. Since y must be real, $4-x^2 \geq 0$ or $x^2 \leq 4$. The domain is the interval

$-2 \le x \le 2$.

b. Here, $x^2 - 16 \ge 0$, or $x^2 \ge 16$. The domain consists of the intervals $x \ge 4$ and $x \le -4$.

c. The function is defined for every value of x except 2.

d. The function is defined for $x \ne \pm 3$.

e. Since $x^2 + 4 \ne 0$ for all x, the domain is the set of all real numbers.

Example (6)

Determine the domain of each of the following functions:

(a) $y = x^2 + 4$ (b) $y = \sqrt{x^2 + 4}$ (c) $y = \sqrt{x^2 - 4}$ (d) $y = \dfrac{x}{x+3}$

(e) $y = \dfrac{2x}{(x-2)(x+1)}$ (f) $y = \dfrac{1}{\sqrt{9 - x^2}}$ (g) $y = \dfrac{x^2 - 1}{x^2 + 1}$ (h) $y = \sqrt{\dfrac{x}{2-x}}$

Solution

(a), (b), (g) all values of x (c) $|x| \ge 2$ (d) $x \ne 3$ (e) $x \ne -1, 2$ (f) $-3 < x < 3$ (h) $0 \le x < 2$.

Example (7)

Find the domains and ranges of the following functions:

(a) $f(x) = -x^2 + 1$ (b) $f(x) = \begin{cases} x - 1 & \text{if } 0 < x < 1 \\ 2x & \text{if } x \ge 1 \end{cases}$

(c) $f(x) = [x]$ = the greatest integer less than or equal to x

(d) $y = \dfrac{x^2 - 4}{x - 2}$ (e) $f(x) = 5 - x^2$ (f) $f(x) = -4\sqrt{x}$

(g) $f(x) = |x - 3|$ (h) $f(x) = 4 / x$ (i) $f(x) = |x| / x$

(j) $f(x) = x - |x|$ (k) $f(x) = \begin{cases} x & \text{if } x \ge 0 \\ 2 & \text{if } x < 0 \end{cases}$

Solution

(a) domain, all numbers; range, $y \le 1$

(b) domain, $x > 0$; range, $-1 < y < 0$ or $y \ge 2$

(c) domain, all numbers; range, all integers

(d) domain, $x \neq 2$; range $y \neq 4$

(e) domain, all numbers; range, $y \leq 5$

(f) domain, $x \geq 0$; range, $y \leq 0$

(g) domain, all numbers; range, $y \leq 0$

(h) domain, $x \neq 0$; range, $y \neq 0$

(i) domain, $x \neq 0$; range, $y = \ -1, 1$

(j) domain, all numbers; range, $y \leq 0$

(k) domain, all numbers; range, $y \geq 0$

Example (8)

Find the domains and ranges of the following functions:

(a) $f(x) = \begin{cases} x+2 & \text{if} -1 < x < 0 \\ x & \text{if}\ 0 \leq x < 1 \end{cases}$ (b) $f(x) = \begin{cases} 2-x & \text{if}\ 0 < x < 2 \\ x-1 & \text{if}\ 3 \leq x < 4 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if}\ x \neq 2 \\ 4 & \text{if}\ x = 2 \end{cases}$

Solution

(a) domain $= (-1, 1]$, range $= [0, 2)$

(b) domain $= (0, 2) \cup [3, 4]$, range $= (0, 3)$

(c) domain and range $=$ set of all real numbers

Types of Functions

(A) **One-One Function**

A function is one-one provided distinct elements of the domain are related to distinct element of the range. In other words, a function $f : A \rightarrow B$ is defined to be one-one if the images of distinct element of A under f are distinct, that is, for every $a_1, a_2 \in A$, $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$. [It also means that, $f(a_1) \neq f(a_2) \Rightarrow a_1 \neq a_2$]. A one–one function is also called injective function (Figure 1.6 and 1.7).

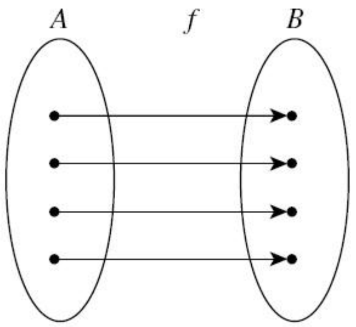


Fig. 1.6

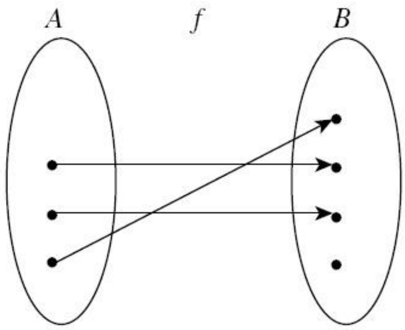


Fig. 1.7

(B) **Many-One Function**

If the range of the function has at least one element, which is the image for two or more elements of the domain, then the function is said to be many-one function (Figure 2.8a and b). It means that there is at least one pair of distinct elements, $a_1, a_2 \in A$, such that $f(a_1) = f(a_2)$ though $a_1 \neq a_2$. A constant function is a special case of many-one function (Figures 1.8 and 1.9).

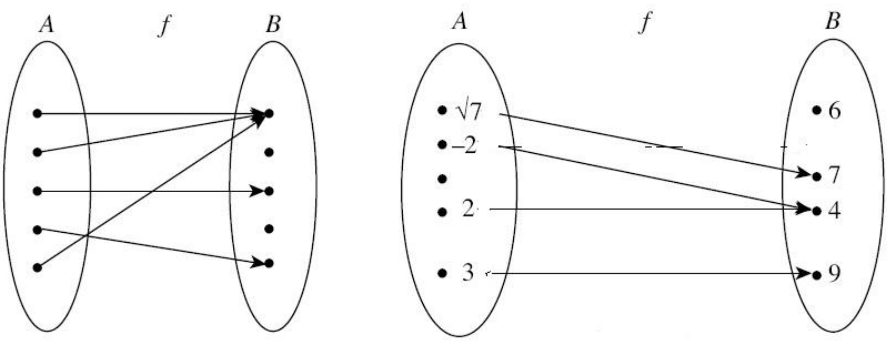


Fig. 1.8

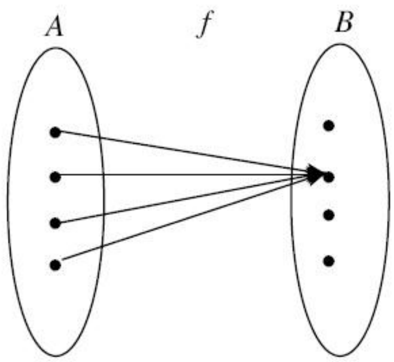


Fig. 1.9 Constant function

(C) **Onto Function**

A function $f : A \to B$ is called an onto function if each element of the codomain is involved in the relation. (Here, range of f = codomain B.) In other words, a function $f : A \to B$ is said to be onto if every element of B is the image of some element of A, under f, that is, for every $b \in B$, there exist an element $a \in A$ such that $f(a) = b$ (Figure 1.10 and 1.11). Onto function is also called surjective function.

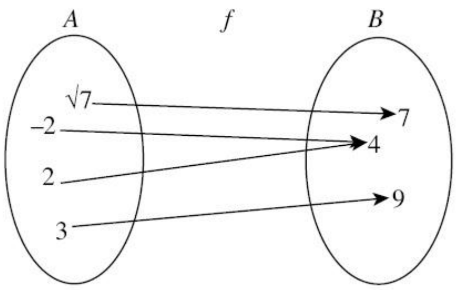


Fig. 1.10

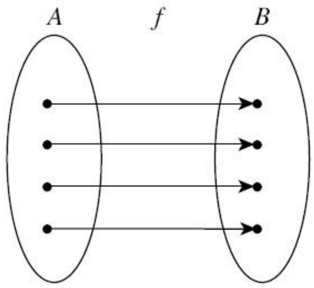
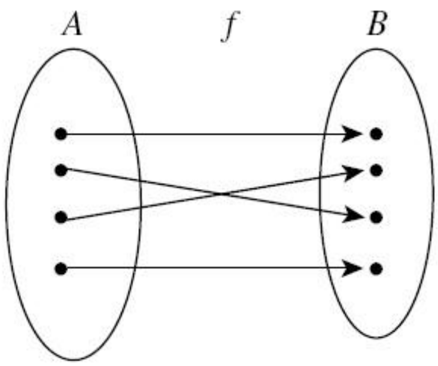


Fig. 1.11

(D) **Bijective Function** (or One-to-One Correspondence)

The most important functions are those which are both one-one and onto. In a function that is one-one and onto, each image corresponds to exactly one element of the domain and each element of codomain is involved in the relation as shown in Figure 1,12. Such a function is also called one-to-one correspondence or a bijective function.



One–one and onto function

Fig. 12

Example (9)

Consider the function $y = f(x) = x^3$. Here, for every value of $x \in R$, there corresponds a single value of y, and, conversely, to each $y \in R$, there corresponds a single value of x given by $x = \sqrt[3]{y}$. Therefore, f specifies a one-to-one mapping, from R onto R.

Example (10)

Consider the function $y = g(x) = x^2$. Here, for every value of $x \in R$, there corresponds a single value of $y \in (0, \infty)$. However, to every $y > 0$, there correspond two values of $x : x = \pm\sqrt{y}$. Therefore, "g" is not one-to-one correspondence.

Example (11)

Consider the exponential function y $y = f(x) = e^x$. It can be shown that the function $y = f(x) = e^x$ is one-to-one mapping from $(-\infty, \infty)$ onto $(0, \infty)$. Note that for $x_1 \neq x_2$, we have $e^{x_1} \neq e^{x_2}$, where $x_1, x_2 \in R$x1, , and $e^{x_1}, e^{x_2} \in R^+$.

Consider $e^{x_1} / e^{x_2} \neq 1 \Rightarrow e^{x_1 - x_2} \neq 1$or $e^{x_1 - x_2} \neq e^0$ (since $e^0 = 1$)

$x_1 - x_2 \neq 0 \Rightarrow x_1 \neq x_2$. In other words, $e^{x_1} \neq e^{x_2} \Rightarrow x_1 \neq x_2$. Thus,

$x_1 \neq x_2 \Leftrightarrow e^{x_1} \neq e^{x_2}$. Therefore, "$f$" defines a one-to-one correspondence from $(-\infty, \infty)$ onto $(0, \infty)$.

Classification of Functions

Even and Odd Functions

(i) A function is an even function if for every x in the domain of f $f(-x) = f(x)$.

(ii) A function is an odd function if for every x in the domain of f $f(-x) = -f(x)$.

Example (12)

I. A polynomial function of the following form is an even function:

$$f(x) = a_0 + a_1 x^2 + a_2 x^4 + \cdots + a_n x^{2n}$$

Observe that the power of x in each term is an even integer.

II. We have , that $\cos(-x) = \cos x$ for all x . Thus, the cosine function is an even function.

III. A constant function is always even (why?).

Example (13)

I. It can be easily verified that the functions $f(x) = x$ and $f(x) = x^3$ are odd functions. In fact, any polynomial function in which the power of each term is an odd integer is an odd function.

II. We have for all x , $\sin(-x) = -\sin x$ and $\tan(-x) = -\tan x$. Thus, the sine and the tangent functions are odd functions.

Note

The property of functions whether even or odd is very useful. In particular, it helps in drawing graph of such functions.

Definition 6.

A function $f:\mathbb{R}\to\mathbb{R}$ is said to be periodic, if there exists a real number $p(p\neq 0)$ such that $f(x+p)=f(x)$ for all $x\in\mathbb{R}$.

Period of a Periodic Function

If a function f is periodic, then the smallest $p>0$, if it exists such that $f(x+p)=f(x)$ for all x, is called the period of the function.

Obviously, the period of the sine and cosine functions is 2π. It can be shown that the period of the tangent function (and that of the cotangent function) is π.

Remark

Aperiodic function may not have a period. Note that a constant function f is periodic as $f(x+p)=f(x)=$constant for all $p>0$, however, there is no smallest $p>0$ for which the relation holds. Hence, there is no period of this function, though it is periodic by definition.

Algebraic operation on functions

Functions are not numbers. But, just as two numbers a and b can be added to produce a new number $(a+b)$, two functions f and g can be added to produce a new function $(f+g)$. This is just one of the several operations on functions.

(a) Sums, Differences, Products and Quotients of Functions

Let f and g be functions. We define the sum $f+g$, the difference $f-g$, and the product $f.g$ to be the functions whose domains consist of all those numbers that are common in the domains of both f and g and whose rules are given by

$$(f+g)(x)=f(x)+g(x)$$

$$(f-g)(x)=f(x)-g(x)$$

$$(f.g)(x)=f(x).g(x).$$

In each case, the domain is consisting of those values of x for which both $f(x)$ and $g(x)$ are defined.

Next, because division by 0 is excluded, we give the definition of quotient of two functions separately as follows: The quotient $\frac{f}{g}$ is the function whose domain consists of all numbers x in the domains of both $f(x)$ and $g(x)$ for which $g(x) \neq 0$, and whose rule is given by

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)}, \ \ g(x) \neq 0$$

Example (14)

Let $f(x) = \frac{1}{x}$ and $g(x) = \sqrt{x}$. Find the domain and rule of $f+g$.

Solution

The domain of f is $x \in R : x \neq 0$ and the domain of $g(x)$ $x \in R : x \geq 0$.

The only numbers in both domains are the positive numbers, which constitute the domain of $f+g$.

For the rule, we have

$$(f+g)(x) = f(x) + g(x) = \frac{1}{x} + \sqrt{x}, \ x > 0.$$

Example (15)

Let $f(x) = \sqrt{4-x^2}$ and $g(x) = \sqrt{x-1}$. Find the domain and rule of $f \cdot g$.

Solution:

The domain of $f(x)$ is the interval $[-2,2]$ and the domain of $g(x)$ is the interval $[1,\infty)$. The domain of $f \cdot g = [-2,2] \cap [1,\infty) = [1,2]$. The rule of $f \cdot g$ is given by

$$\begin{aligned}(f.g)(x) = f(x).g(x) &= \sqrt{4-x^2}\sqrt{x-1} \\ &= \sqrt{(4-x^2)(x-1)} \quad \text{for} \quad 1 \leq x \leq 2\end{aligned}$$

Caution

This example illustrates a surprising fact about the domain of functions combination . We found that the domain of $f(x) \cdot g(x)$ is the interval $[1,2]$. Now observe that the expression $\sqrt{(4-x^2)(x-1)}$ is also meaningful for x in $(-\infty,-2]$.

This is true because $(4-x^2)(x-1) \geq 0 \Rightarrow x \leq -2$. However, $(-\infty,-2]$ cannot be considered a part of the domain of $f(x).g(x)$. By definition, the domain of the resulting function $f(x) \cdot g(x)$ consists of those values of x common to domains of $f(x)$ and $g(x)$. It is not to be determined from the expression (or the rule) for $f(x).g(x)$.

Similar comments hold for the domains of $f(x)+g(x)$ and $f(x)-g(x)$.

For the domain of $f(x)\ /\ g(x)$, there is an additional requirement that the values of x, for which $g(x) \neq 0$, are excluded.

Example (16)

Let $f(x)=x+3$ and $g(x)=(x-3)(x+2)$. Let us find the domain and rule of $f(x)\ /\ g(x)$.

Solution

Observe that the domains of $f(x)$ and $g(x)$ are all real numbers, but $g(x)=0$, for $x=3$ and $x=-2$. It follows that the domain of $f(x)\ /\ g(x)$ consists of all real numbers except $x=-2$ and $x=3$. The

rule of $f(x)\ /\ g(x)$ is given by

$$(\frac{f}{g})x = \frac{f(x)}{g(x)} = \frac{x+3}{(x-3)(x+2)} \quad \text{for} \quad x \neq -2 \text{ and } x \neq 3$$

Note

We can add or multiply more than two functions. For example, if f, g, and h are functions, then for all x common to the domains of f, g, and h, we have ($(f+g+h)x = f(x)+g(x)+h(x)$ and $(f.g.h)x = f(x).g(x).h(x)$.

(b) Composition of Functions

Given the two function f and g, the composite function denoted by ($g \circ f$) is defined by

$$(g \circ f)(x) = g(f(x))$$

and the domain of $g(f(x))$ is the set of all numbers x in the domain of f such that $f(x)$ is in the domain of $g(x)$. The definition indicates that when computing $(f \circ g)(x)$, we first apply g to x and then the function f to $g(x)$. We write

$$(f \circ g)(x) = f(g(x))$$

Example (17)

Let $f(x) = \dfrac{x-3}{2}$ and $g(x) = \sqrt{x}$. We may composite them as follows:

I. $(g \circ f)(x) = g(f(x)) = g(\dfrac{x-3}{2}) = \sqrt{\dfrac{x-3}{2}}$

II. $(f \circ g)(x) = f(\sqrt{x}) = \dfrac{\sqrt{x}-2}{2}$

Remark

Note that $(g \circ f)(x) \neq (f \circ g)(x)$). Thus, composition of functions is not commutative, $(g \circ f)(x)$ and $(f \circ g)(x)$ are usually different.

Domain of a Composite Function

We must be more careful in describing the domain of a composite function. Let $f(x)$ and $g(x)$ be defined for certain values of x. Then, the domain of $(g \circ f)(x)$ is that part of the domain of $f(x)$ (i.e., those values of x x) for which g can accept $f(x)$ as input. In the above example, the domain of $(g \circ f)(x)$ is $[3, \infty$, since x must be greater than or equal to 3 in order to give a nonnegative number $\dfrac{x-3}{2}$ for g to work on.

Example (18)

Consider the function $\phi(x) = \sqrt{x^3 + 7}$.

We can express $\phi(x)$ as the composition of the two functions $g(x)$ and $f(x)$, given by $f(x) = x^3 + 7$ and $g(x) = \sqrt{x}$.

Now, we have $\phi(x) = (g \circ f)(x) = g(f(x)) = g(x^3 + 7) = \sqrt{x^3 + 7}$

Next, we can also express $\phi(x)$ as the composition of another pair of functions g and f given by $f(x) = x^3$ and $g(x) = \sqrt{x + 7}$.

Consider $\phi(x) = (g \circ f)(x) = g(f(x)) = g(x^3) = g(\sqrt{x^3 + 7})$.

Example (19)

Given $\phi(x) = \dfrac{1}{\sqrt{x^2 + 3}}$.

Express $\phi(x)$ as the composition of two function f and g in two ways:

(i) The function f containing the radical.

(ii) The function g containing the radical.

Solution

To solve such problems, it is necessary to develop the ability of decomposing the given function into composite pieces.

I. We choose $f(x) = \dfrac{1}{\sqrt{x + 3}}$ and $g(x) = x^2$.

Now, $\phi(x) = (f \circ g)(x) = f(g(x)) = f(x^2) = \dfrac{1}{\sqrt{x^2 + 3}}$

(Observe that to express $f(g(x))$ first we insert the expression for $g(x)$ and obtain $f(t)$, where t stands for $g(x)$. Next, we write the expression for $f(t)$ and replace t by $g(x)$.)

II. Now, we choose $f(x) = \dfrac{1}{x}$ and $g(x) = \sqrt{x^2 + 3}$. Then,

$$\phi(x) = (f \circ g)(x) = f(g(x)) = f(\sqrt{x^2 + 3}) = \frac{1}{\sqrt{x^2 + 3}}$$

(Here again, to express $f(g(x))$, first we insert the expression for $g(x)$ and obtain $f(t)$, where $f(t)$ stands for $g(x)$. Now we look at the expression for $f(t)$, which suggests that we must take the reciprocal of t.)

Example (20)

Let $f(x) = \sqrt{x - 1}$ and $g(x) = \dfrac{1}{x}$. We shall determine the functions

$g \circ f$ and $f \circ g$, and then find $g(f(5))$ and f $f(\text{g}(\frac{1}{4}))$

Solution

The function is $(g \circ f)(x)$ given by

$(g \circ f)(x) = g(f(x)) = g(\sqrt{x-1}) = \dfrac{1}{\sqrt{x-1}}$. The domain of $f(x)$ is $[1,\infty)$. Therefore, the domain of $g \circ f$ consists of those numbers x in $[1,\infty)$ for which g can accept $f(x)$ as input. This demands that

$g\left(\dfrac{1}{\sqrt{x-1}}\right) = \dfrac{1}{\sqrt{x-1}}$ must be defined, which requires that $x \neq 1$.

Therefore, the domain of $g \circ f$ is $(1,\infty)$.

The rule for $f \circ g$ is given by

$$(f \circ g)(x) = f(g(x)) = f(\frac{1}{x}) = \sqrt{\frac{1}{x} - 1}$$

The domain of $g(x)$ is the set of nonzero numbers, that is $(-\infty,0) \cup (0,\infty)$ Therefore, the domain of $f \circ g$ consists of those numbers x in the above domain for which f can accept $g(x)$ as input. This demands that

$f(\dfrac{1}{x}) = \sqrt{\dfrac{1}{x} - 1}$ must be defined.

It requires that $\dfrac{1}{x} - 1 \geq 0 \Rightarrow \dfrac{1}{x} \geq 1 (x \text{ must be positive with } \dfrac{1}{x} \geq 1)$.

So, The domain is $(0,1]$.

Inverse Function f^{-1}

If a function "f" is one-to-one and onto, then the correspondence associating the same pairs of elements in the reverse order is also a function. This reverse function is denoted by f^{-1}, and we call it the inverse of the function f. Note that, f^{-1} is also one-to one and onto. See figure 1.13

Remark

A function f has an inverse provided that there exists a function, f^{-1} such that

I. the domain of f^{-1} is the range of f

II. $f(x) = y$ if and only if $f^{-1}(y) = x$ for all x in the domain of "f" and for all y in the range of "f ".

Note

Not every function has an inverse. If a function $f: A \to B$ has an inverse, then $f^{-1}: B \to A$ is defined, such that, the domain of f^{-1} is the range of f , and the range of f^{-1} is the domain of f , associating the same pairs of elements.

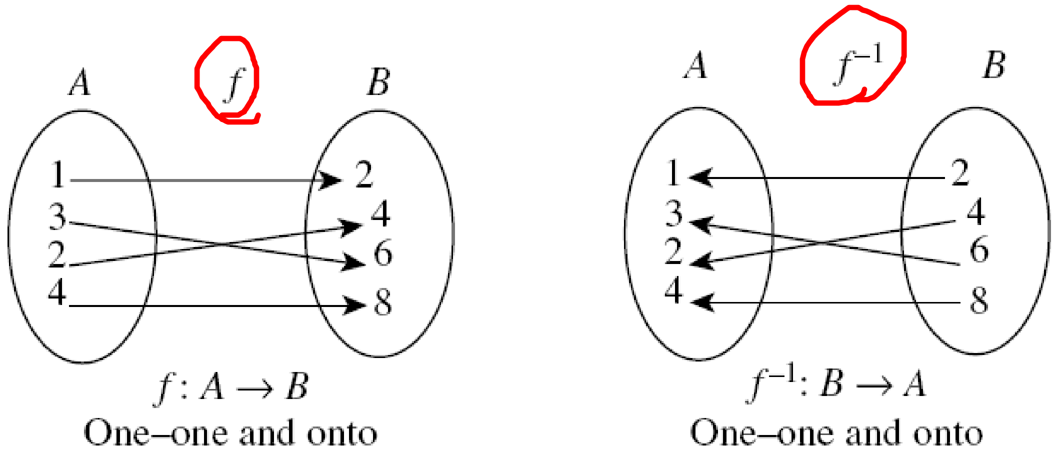


Fig. 1.13

It can be shown that if f has an inverse, then the inverse function is uniquely determined. Sometimes, we can give a formula for f^{-1}. For example $y = f(x) = 2x$, then $x = f^{-1}(y) = \frac{1}{2}y$. Similarly, if $y = f(x) = x^3 - 1$, then $x = f^{-1}(y) = \sqrt{y^3 + 1}$. In each case, we simply solve the equation that determines x in terms of y . The formula in y expresses the (new) function f^{-1}.

We cannot always give the formula for f^{-1}. For example, consider the function $y = f(x) = x^5 + 2x + 1$. It is beyond our capabilities to solve this equation for x.

Note that, in such cases, we cannot decide whether a given function has an inverse or not.

Fortunately, there are criteria that tell whether a given function $y = f(x)$ has an inverse, irrespective of whether we can solve it for x .

In this notation, the letter x stands for the independent variable and the letter y the dependent variable for both the mutually inverse functions. Thus the func-

tions $y = x^3$ and $y = \sqrt[3]{x}$, represent a pair of mutually inverse functions. Also $y = 10^x$ and $y = \log_{10} x$ are mutually inverse functions.
There is a simple relationship between the graphs of two mutually inverse functions $y = f(x)$ and $y = f^{-1}(x)$. They are symmetric with respect to the line $y = x$ (see Figure 1.14 and 1.15).

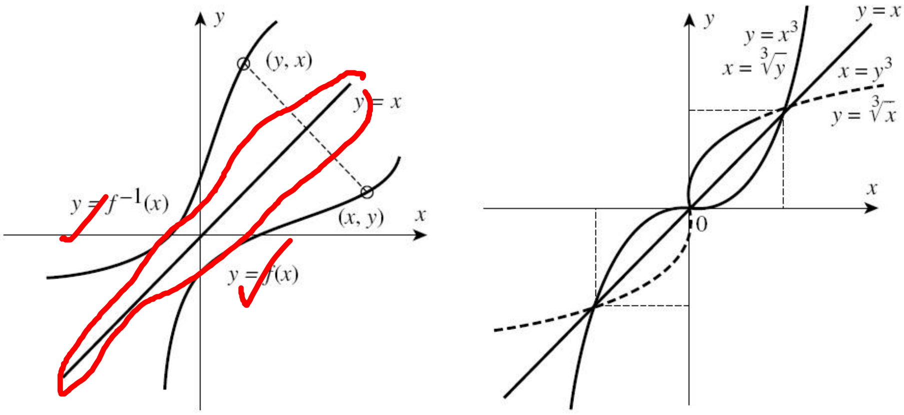


Fig. 1.14

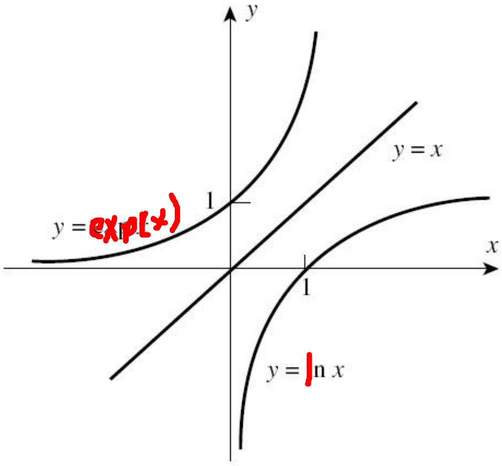


Fig. 1.15

In the case of simple functions (like linear functions, etc.) there is a three-step process that gives a formula for the inverse.
Step (1): Solve the equation $y = f(x)$ for x, in terms of y .

Step (2): Use the symbol f^{-1} to name the resulting expression in y .

Step (3): Replace y by x to get the formula for $f^{-1}(x)$.

Example (21)
Consider the function $y = f(x) = 3x - 2,\ x \in R$, and let us find its inverse function.

Solution

Step (1): $y = f(x) = 3x - 2 \quad \Rightarrow x = \dfrac{y+2}{3}$

Step (2): $f^{-1}(y) = \dfrac{y+2}{3}$

Step (3): $f^{-1}(x) = \dfrac{x+2}{3}$

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Example (22)

Let us find the formula for $f^{-1}(x)$ if $y = f(x) = \dfrac{x}{1-x}$

Step (1): $y = f(x) = \dfrac{x}{1-x} \ \Rightarrow \ x = \dfrac{y}{1+y}$

Step (2): $f^{-1}(y) = \dfrac{y}{1+y}\ (y \neq -1)$

Step (3): $f^{-1}(x) = \dfrac{x}{1+x}(x \neq -1)$

Algebraic Functions and Their Combinations

(a) **Constant Function**:

A function of the form $f(x) = a$, where "a" is a nonzero real number (i.e., $a \neq 0$), is called a constant function. The range of a constant function consists of only one nonzero number.

(b) **Identity Function**:

The function $f(x) = x$ is called the identity function . The range of identity function is all real number. From the functions at (a) and (b) above, we can build many important functions of calculus: polynomials, rational functions, power functions, root functions, and so on.

(c) **Polynomial Function**:

Any function, that can be obtained from the constant functions and the identity function by using the operations of addition, subtraction, and multiplication, is called a polynomial function. This amounts to say that "$f(x)$" is a polynomial function, if it is of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$$

where $a_n, a_{n-1}, ..., a_2, a_1, a_0$ are real numbers ($a_n \neq 0$) and n is a nonnegative integer. If the coefficient $a_n \neq 0$, then "n" (in x^n), the nonnegative integral exponent of x, is called the degree of the polynomial. Obviously, the degree of constant functions is zero.

I. Linear Function: Polynomials of degree 1 are called linear functions. They are of the form $f(x) = a_1 x + a_0$, with $a_1 \neq 0$. Note that, the identity function [$f(x) = x$] is a particular linear function.

II. $f(x) = a_2 x^2 + a_1 x + a_0$ is a second degree polynomial, called a quadratic function. If the degree of the polynomial is 3, the function is called a cubic function.

III. Rational Functions: Quotients of polynomials are called rational functions. Examples are as follows:

$$f(x) = \frac{1}{x^2}, \quad f(x) = x^3 + \sqrt{5}\,x; \; f(x) = \frac{x^3 - 2x + \pi}{x - \sqrt{2}}$$

$$f(x) = \frac{x^2 + x - 2}{x^2 + 5x - 6}.$$

Example (23)

Let $f(x)=\dfrac{x^2+x-2}{x^2+5x-6}$. Find the domain of f.

Solution

We have $x^2+5x-6=(x-1)(x+6)$. Therefore, the denominator is 0 for $x=1$ and $x=-6$. Thus, the domain of f consists of all numbers except 1 and -6.

Remark

Sometimes, it may happen that both the numerator and the denominator have a common factor. For example, we have $x^2+x-2=(x-1)(x+2)$, and $x^2+5x-6=(x-1)(x+6)$. So, we have

$$f(x)=\frac{x^2+x-2}{x^2+5x-6}=\frac{(x-1)(x+2)}{(x-1)(x+6)}$$

which may be simplified to read $\dfrac{x+2}{x+6}$, provided $x\neq 1$. Note that, while the expression $\dfrac{x+2}{x+6}$ is meaningful for $x=1$, the number 1 is not in the domain of function f. (This again suggests that the domain of a combination of functions must be determined from the original description of the function(s), and not from their simplified form.)

(d) **Power Functions**

These are functions of the form $f(x)=x^\alpha$, where α is real number. Examples are $x^4, x^{\sqrt{2}}, x^{-3}, x^{-\sqrt{5}}, x^0, x^{-3}$

(e) **Root Functions**

I. Square root function

Consider the relation $y^2=x$. We write it as $y=\sqrt{x}$ or $y=x^{1/2}$ and call it the square root function of x. We know that there is no real number whose square is a negative number. Hence, we define square root function $f(x)=\sqrt{x}$ that assigns to each nonnegative number x the nonnegative number $f(x)$. We emphasize that

$f(x) = \sqrt{x}$ is defined only for $x \geq 0$and that $f(x) \geq 0$, for all $x \geq 0$. Accordingly, it is meaningful to write $\sqrt{8}, \sqrt{1/3}$, and $\sqrt{0}$, and so on, but $\sqrt{-5}$ has no meaning. Furthermore, while $\sqrt{4} = \pm 2$, we write $\sqrt{4} = 2$ and we never write $\sqrt{4} = -2$.

II. Cube Root Function

Consider the relation $y^3 = x$. We write it as $y = \sqrt[3]{x}$ or $y = x^{1/3}$, and call it the cube root function. It assigns to any number x, the unique number y such that $y^3 = x$. Of course, our interest lies only in real roots. In contrast to the square root function, the cube root function has in its domain all real numbers, including negative numbers. For example, $\sqrt[3]{-8} = -2$, $\sqrt[3]{-1} = -1$and $\sqrt[3]{-27/64} = -3/4$. Similarly $\sqrt[3]{8} = 2$; $\sqrt[3]{-125} = -5$, and $\sqrt[3]{125} = 5$. Thus cube root of any negative number is a negative number and that of any positive number is a positive number.

III. nth Root Function

We note that cube root function "$f(x) = \sqrt[3]{x}$"is defined for all real numbers x, whereas square root function "$f(x) = \sqrt{x}$" is defined only for $x \geq 0$ with the understanding that $\sqrt{x} \geq 0$ (i.e., only nonnegative square roots are accepted). By extending these concepts to the roots of higher order, we get that if n is odd, then nth root function "$\sqrt[n]{x}$" is defined for all real numbers, and on the other hand, if n is even, then "$\sqrt[n]{x}$" is defined only for $x \geq 0$

Note

In view of the above, the expressions $\sqrt[3]{-1}$; $\sqrt[5]{-32}$ and $\sqrt[7]{-128}$ are meaningful, whereas the expressions $\sqrt[4]{-1}$; $\sqrt[6]{-64}$; and $\sqrt[3]{-9/4}$ are meaningless. For every positive integer n, we also have $\sqrt[n]{1} = 1$, $\sqrt[n]{0} = 0$.

Non-algebraic Functions and Their Combinations

I. Trigonometric functions

Let a point $p(x, y)$ moves along a circle perimeter with radius $r = 1$ and θ is the angle that the revolving line OP makes with the x-axis (see figure 1.16). Then, we can define the **sine** and **cosine** functions of θ by:

$$\sin\theta = \frac{x}{r},\ \cos\theta = \frac{y}{r}$$

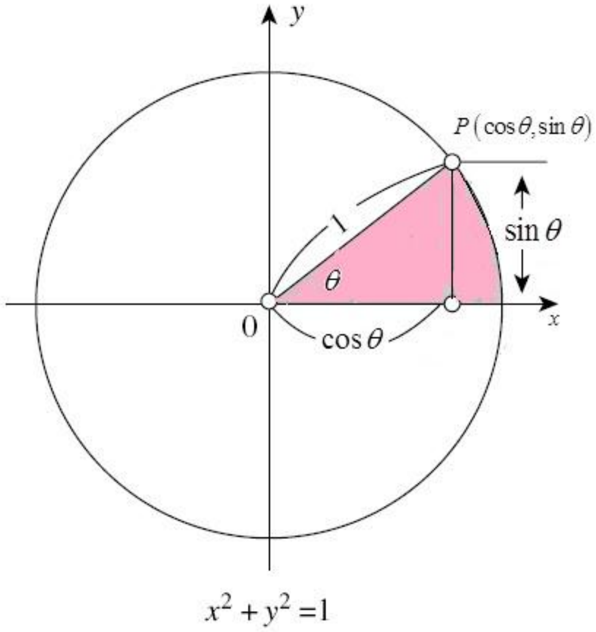


Fig. 1.16

Here , it is important to keep in mind that the angle θ can be of any magnitude and sign. Therefore, the terminal side OP can be in any quadrant. Thus, the angle θ that the revolving line makes with the x-axis need not be acute. However, we define the trigonometric function of the angle θ with reference to the right-angled triangle in which the revolving line (as hypotenuse) makes the angle θ with the x-axis. Obviously, θ may be acute or obtuse or negative.

There are four other basic trigonometric functions that are defined in terms of $\sin\theta$ and $\cos\theta$, we define

$$\tan\theta = \frac{\sin\theta}{\cos\theta},\quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta},\quad \operatorname{co}\sec\theta = \frac{1}{\sin\theta}$$

The values of these functions can be quickly computed from the corresponding values of $\sin\theta$ and $\cos\theta$.

Properties of trigonometric functions

1. **Sine function**

   Sine function has the following properties(Fig. 1.17)

   a. $\sin : R \to R$

   b. Its domain is R and its range is $[-1,1]$

   c. It is periodic function with period 2π, that is $\sin(\theta + 2\pi) = \sin\theta$.

   d. It is odd function, that is, $\sin(-x) = -\sin x$.

   e. Sine function is not one-to-one function.

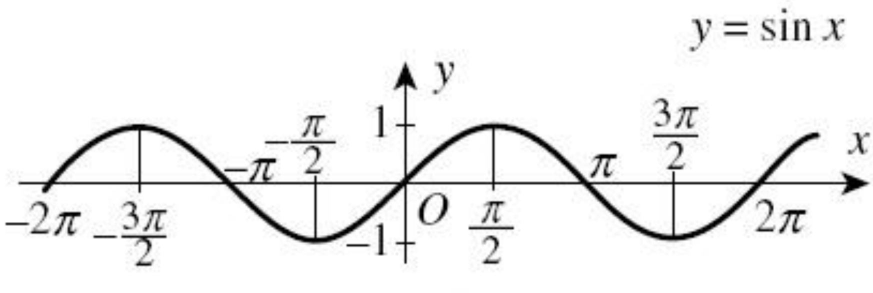


Fig. 1.17

2. **Cosine function**

   Cosine function has the following properties (see Fig. 1.18)

   a. $\cos : R \to R$

   b. Its domain is R and its range is $[-1,1]$

   c. It is periodic function with period 2π, that is $\cos(x + 2\pi) = \cos x$.

   d. It is even function, that is, $\cos(-x) = \cos x$.

   e. Cosine function is not one-to-one function.

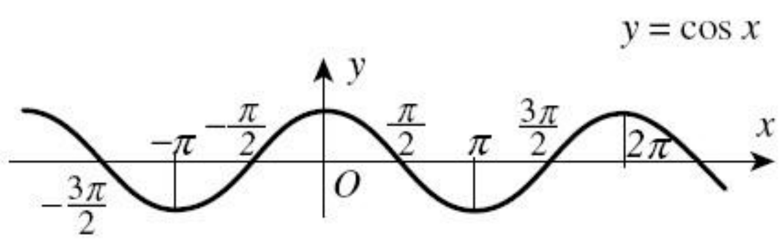


Fig. 1.18

3. **Tangent function**

Tangent function has the following properties (see Fig. 1.19)

a. $\tan : R - \{k\pi + \frac{\pi}{2}\} \to R,\ k \in Z$.

b. Its domain is $R - \{k\pi + \frac{\pi}{2}\},\ k \in Z$ and its range is R.

c. It is periodic function with period π, that is $\tan(x + \pi) = \tan x$.

d. It is odd function, that is, $\tan(-x) = -\tan x$.

e. It is not one-to-one function.

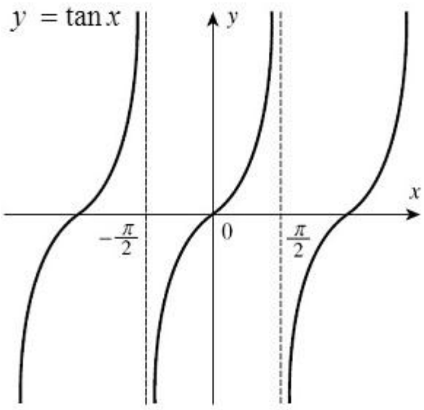


Fig. 1.19

4. **Secant function**

Secant function has the following properties (see Fig. 1.20).

a. $\sec : R - \{k\pi + \frac{\pi}{2}\} \to R,\ k \in Z$.

b. Its domain is $R - \{k\pi + \frac{\pi}{2}\},\ k \in Z$ and its range is $(-\infty, -1] \cup [1, \infty)$.

c. It is periodic function with period 2π, that is $\sec(x + 2\pi) = \sec x$.

d. It is even function, that is, $\sec(-x) = \sec x$.

e. It is not one-to-one function.

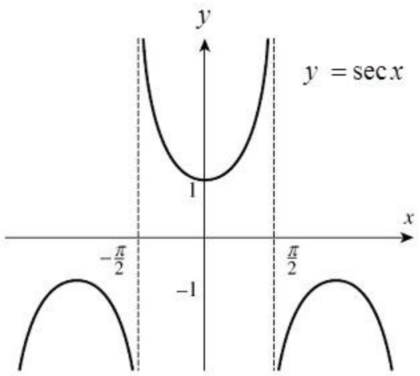


Fig. 1.20

5. **Cosecant function**

Cosecant function has the following properties (see Fig. 1.21)

a. $\text{cosec} : R - \{k\pi\} \rightarrow R,\ k \in Z$

b. Its domain is $R - \{k\pi\},\ k \in Z$ and its range is $(-\infty, -1] \cup [1, \infty)$

c. It is periodic function with period 2π, that is $\text{cosec}(x + 2\pi) = \text{cosec}\, x$.

d. It is odd function, that is, $\text{cosec}(-x) = -\text{cosec}\, x$.

e. It is not one-to-one function.

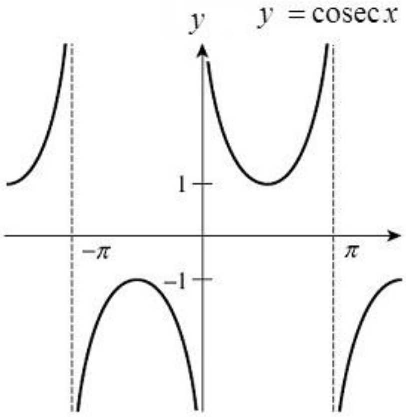


Fig. 1.21

6. **Cotangent function**

Cotangent function has the following properties (see Fig. 1.22).

a. $\cot : R - \{k\pi\} \rightarrow R,\ k \in Z$.

b. Its domain is $R - \{k\pi\},\ k \in Z$ and its range is R.

c. It is periodic function with period π, that is $\cot(x+\pi) = \cot x$.
d. It is odd function, that is, $\cot(-x) = -\cot x$.
e. It is not one-to-one function.

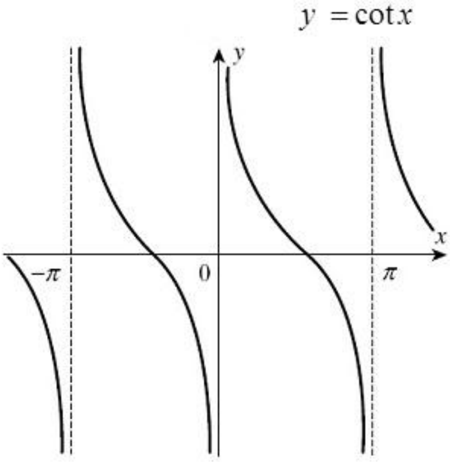


Fig. 1.22

Some Values of Trigonometric Functions

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

$\sin(x+\pi) = -\sin x$

$\cos(x+\pi) = -\cos x$

Trigonometric Identities

1. $\sin^2 x + \cos^2 x = 1$
2. $1 + \tan^2 x = \sec^2 x$
3. $1 + \cot^2 x = \operatorname{cosec}^2 x$
4. $\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x$
5. $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
6. $\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

7. $\sin 2x = 2\sin x \cos x$

8. $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$

9. $\sin^2 x = \dfrac{1-\cos 2x}{2}$

10. $\cos^2 x = \dfrac{1+\cos 2x}{2}$

11. $\sin x \cos y = \dfrac{1}{2}[\sin(x+y) + \sin(x-y)]$

12. $\sin x \sin y = \dfrac{1}{2}[\cos(x-y) - \cos(x+y)]$

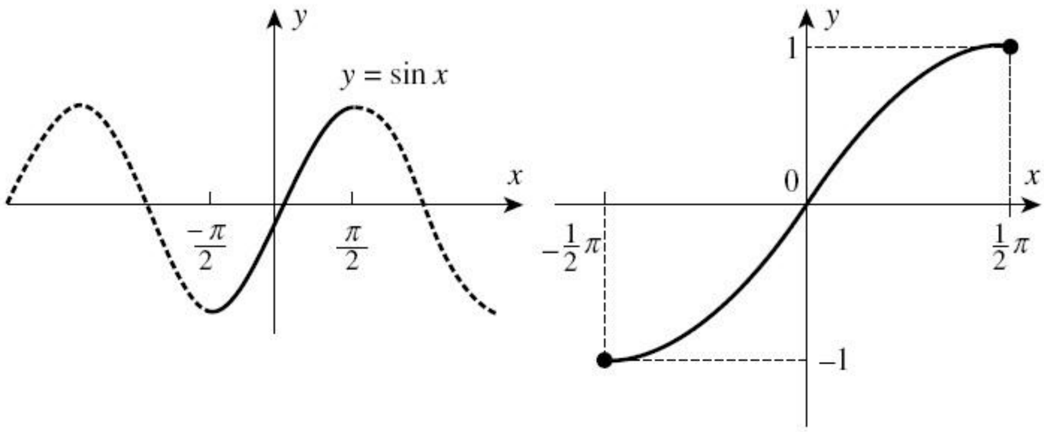
13. $\cos x \cos y = \dfrac{1}{2}[\cos(x+y) + \cos(x-y)]$

II. Trigonometric Functions (With Restricted Domains) and Their Inverses

We begin with the sine function $y = \sin x$, whose graph appears in Figure 1.17. Observe from the figure that the sine function is strictly increasing on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Consequently, the function $f(x) = \sin x$, for which

$$f(x) = \sin x, \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

is one-to-one, and hence it does have an inverse in this interval. The graph of is sketched in figure 1.23. Its domain is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and its range is $[-1,1]$. The inverse of this function is called the inverse sine function.



Domain restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $\qquad$ $f(x) = \sin x,\ x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Fig. 1.23

1 . **Inverse Sine Function**

The inverse sine function, denoted by $\sin^{-1}$ is defined by

$y = \sin^{-1} x$, if and only if $x = \sin y$ and $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

The domain of $\sin^{-1} x$ is the closed interval $[-1,1]$ and the range is the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (see Fig. 1.24).

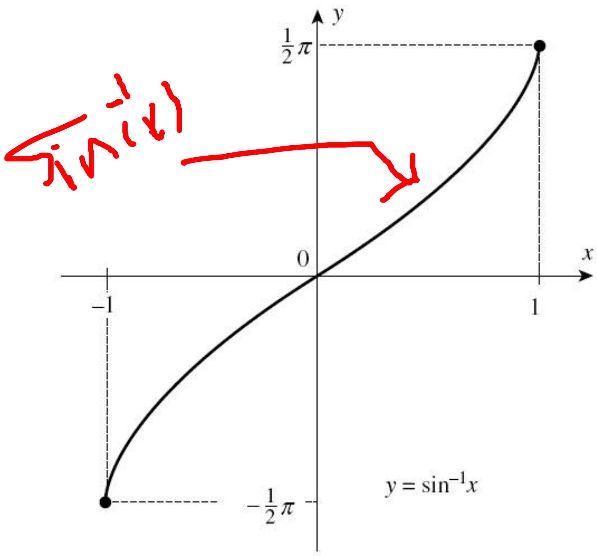


Fig. 1.24

Remarks

$\sin^{-1}(-1) = -\frac{\pi}{2}$ as $\sin(-\frac{\pi}{2}) = -1$.

$\sin^{-1}(0) = 0$ as $\sin(0) = 0$.

$\sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$ as $\sin\frac{\pi}{6} = \frac{1}{2}$.

$\sin^{-1}(\pm\frac{1}{\sqrt{2}}) = \pm\frac{\pi}{4}$ as $\sin(\pm\frac{\pi}{4}) = \pm\frac{1}{\sqrt{2}}$.

$\sin^{-1}(1) = \frac{\pi}{2}$ as $\sin(\frac{\pi}{2}) = 1$.

The use of the symbol "-1" to represent the inverse sine function makes it necessary to denote the reciprocal of $\sin x$ by $(\sin x)^{-1}$, to avoid confusion.
A similar convention is applied when using any negative exponent with a trigonometric function. For instance, $\frac{1}{\tan x} = (\tan x)^{-1}$

$\frac{1}{\cos x} = (\cos x)^{-1}$ and so on.
The terminology arc sine is sometimes used in place of inverse sine, and the notation arc sine is then used instead of $\sin^{-1} x$.

2 . Inverse Cosine Function

The graph of cosine function $y = \cos x$, appears in Figure 1.18. Observe from the figure that the cosine function is strictly decreasing on the interval $[0, \pi]$. Consequently, the function
$f(x) = \cos x$, for which

$$f(x) = \cos x , \quad x \in [0, \pi]$$

is one-to-one, and hence it does have an inverse in this interval. The graph of is sketched in figure 1.25. Its domain is $[0, \pi]$ and its range is $[-1, 1]$. The inverse of this function is called the inverse cosine function.

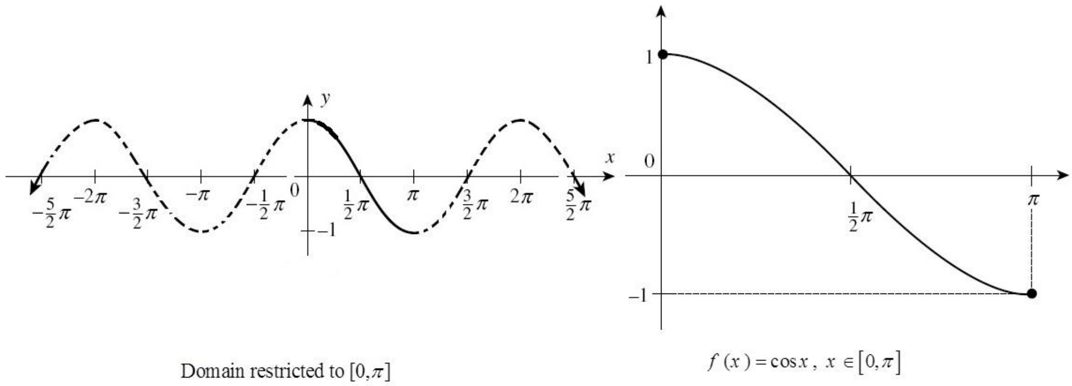


Fig. 1.25

The inverse cosine function, denoted by $\cos^{-1}x$, is defined by $y=\cos^{-1}x$, if and only if $x=\cos y$ and $y\in[0,\pi]$. The domain of $\cos^{-1}x$ is the closed interval $[-1,1]$ and the range is the closed interval $[0,\pi]$(see Fig. 1.26).

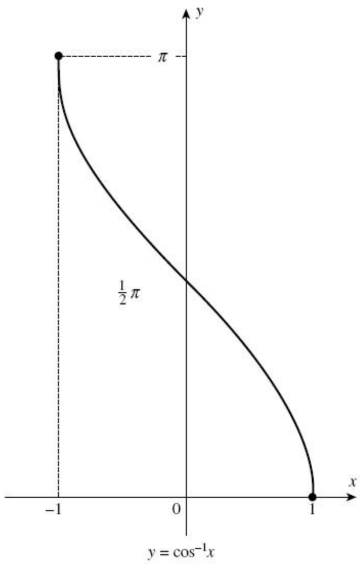


Fig. 1.26

$\cos^{-1}(-1)=\pi$ as $\cos(\pi)=-1$.

$\cos^{-1}(0)=\dfrac{\pi}{2}$ as $\cos(\dfrac{\pi}{2})=0$.

$\cos^{-1}(\dfrac{1}{2})=\dfrac{\pi}{3}$ as $\cos\dfrac{\pi}{3}=\dfrac{1}{2}$.

$\cos^{-1}(\pm\dfrac{1}{\sqrt{2}})=\pm\dfrac{\pi}{4}$ as $\cos(\pm\dfrac{\pi}{4})=\pm\dfrac{1}{\sqrt{2}}$.

$\cos^{-1}(1) = 0$ as $\cos(0) = 1$.

3 . **Inverse Tangent Function**

The inverse tangent function, denoted by $\tan^{-1}$, is defined by

$y = \tan^{-1} x$, if and only if, $x = \tan y$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$. The domain of $\tan^{-1} x$ is the set $\mathbb{R}$ of real numbers and the range is the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. The graph of the inverse tangent function is shown in Figure 1.27.

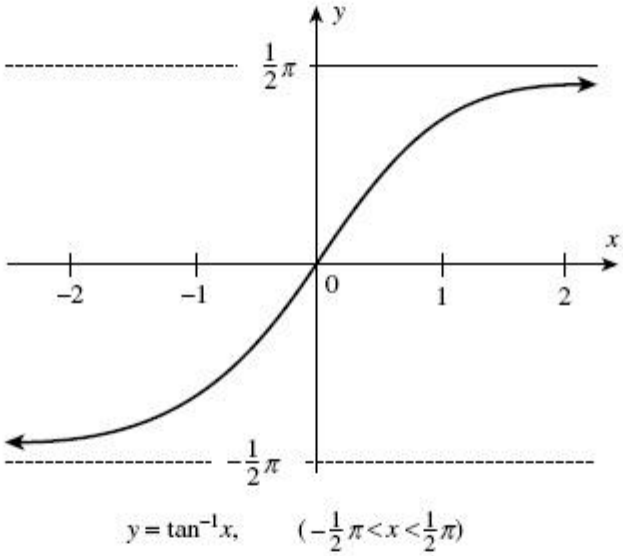


Fig. 1.27

4 . **Inverse Cotangent Function**

To define the inverse cotangent function, we use the identity

$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$, where x is any real number.

The inverse cotangent function, denoted by $\cot^{-1}$, is defined by

$y = \cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$ where x is any real number.

The domain of $\cot^{-1} x$ is the set $\mathbb{R}$ of real numbers. To obtain the range, we write the equation in the definition as

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x \qquad (**)$$

We know that;

$$-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2} \qquad (***)$$

Using (**) in (***), we get

$$-\frac{\pi}{2} < \frac{\pi}{2} - \cot^{-1} x < \frac{\pi}{2}$$

Subtracting $\frac{\pi}{2}$ from each member, we get

$$-\pi < -\cot^{-1} x < 0$$

Now, multiplying each member by -1 , we get

$$0 < \cot^{-1} x < \pi$$

The range of the inverse cotangent function is therefore the open interval $(0, \pi)$ (see Fig. 1.28).

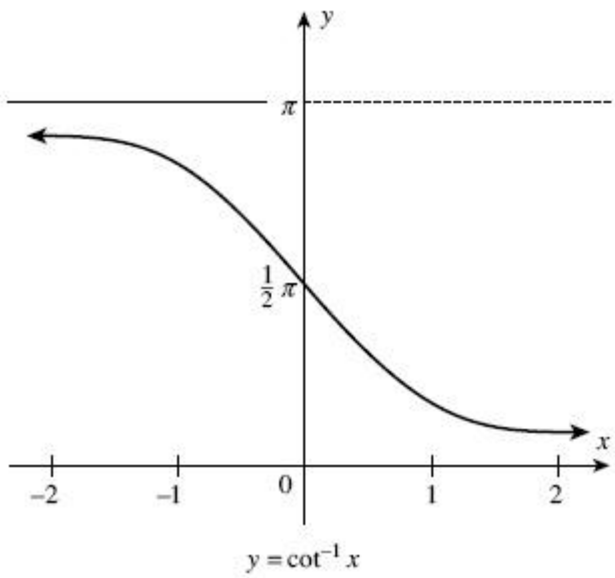


Fig. 1.28

Illustration

(a) $\tan^{-1}(1) = \frac{\pi}{4}$

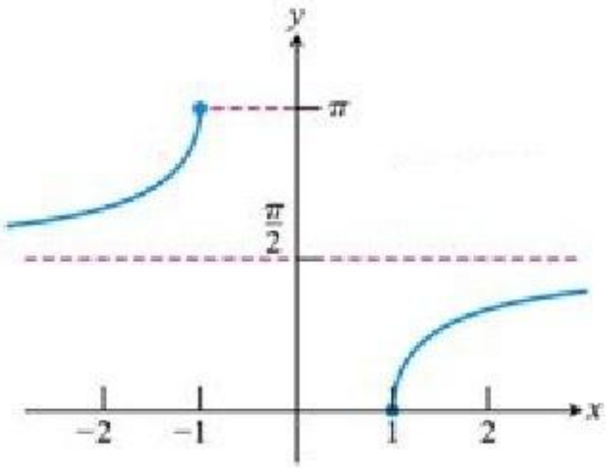
(b) $\tan^{-1}(-1) = -\frac{\pi}{4}$

(c) $\cot^{-1}(1) = \frac{\pi}{2} - \tan^{-1}(1) = \frac{\pi}{4}$

(d) $\cot^{-1}(-1) = \frac{\pi}{2} - \tan^{-1}(-1) = \frac{3\pi}{4}$

5 . **Inverse secant Function**

The inverse secant function, denoted by $\sec^{-1}$, is defined by $y = \sec^{-1} x$, if and only if, $x = \sec y$ and $y \in [0, \pi] - \{\frac{\pi}{2}\}$. The domain of $\sec^{-1} x$ is the set $\mathbb{R} - (-1,1)$ of real numbers and the range is $[0, \pi] - \{\frac{\pi}{2}\}$. The graph of the inverse secant function is shown in Figure 1.29.



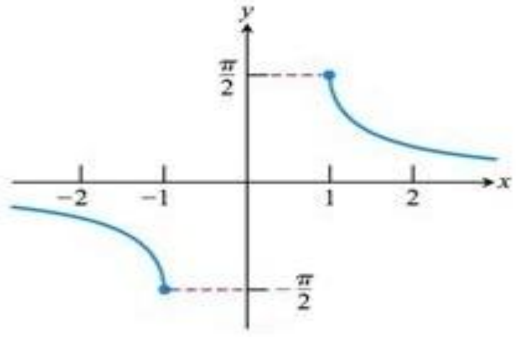
$y = \sec^{-1} x$

Fig. 1.29

6 . **Definition of the Inverse cosecant Function**

The inverse secant function, denoted by $\operatorname{cosec}^{-1}$, is defined by $y = \operatorname{cosec}^{-1} x$, if and only if, $x = \operatorname{cosec} y$ and $y \in [-\frac{\pi}{2}, \frac{\pi}{2}] - \{0\}$. The domain of $\operatorname{cosec}^{-1} x$ is the set $\mathbb{R} - (-1,1)$ of real numbers and the range is $[-\frac{\pi}{2}, \frac{\pi}{2}] - \{0\}$.

The graph of the inverse cosecant function is shown in Figure 1.30.



$y = \text{cosec}^{-1} x$

Fig. 1.30

III. Exponential Function

The product $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$, is conveniently written in the form $2^6 = 64$, to mean that the number is multiplied by itself, six times. In the expression 2^6 , the number "2 "is called the base and "6 " is called the exponent. We say that the number 64 is expressed in the exponential form as 2^6 . Similarly, we can write $4^3 = 64$ and $64^1 = 64$, which are two other exponential forms for 64.

In fact, any positive number can be expressed in any number of exponential form(s), by choosing a positive base and an appropriate exponent.

Definition

The exponential function is defined as

$$y = f(x) = a^x,\ a > 0,\ a \neq 1$$

The domain of exponential function is the set of all real numbers $\mathbb{R}$ and its range is the set of positive numbers. This function monotonically increases, if the base is $a > 1$ and monotonically decreases if $0 < a < 1$ (see Fig. 1.31).

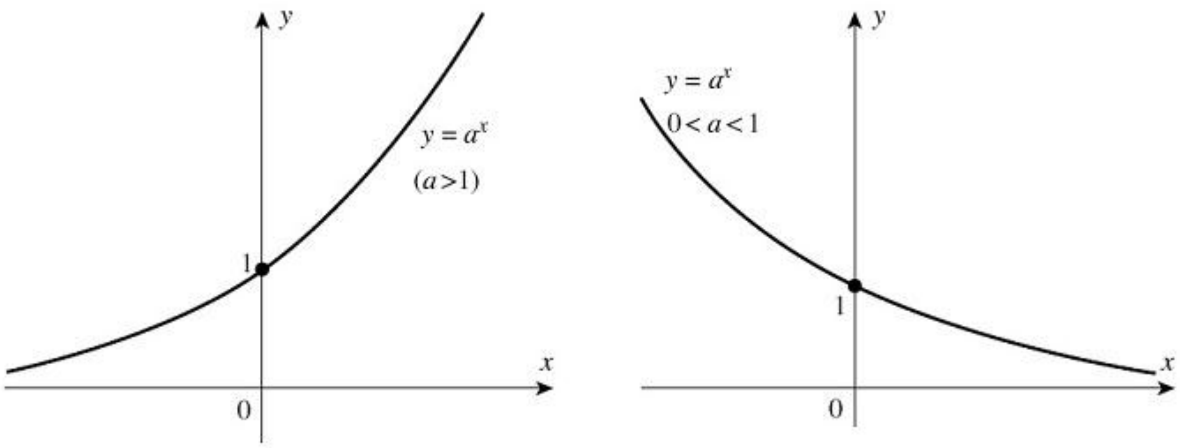


Fig. 1.31

The Natural Exponential Function
The exponential function to the base e is called the natural exponential and is usually denoted by $y = f(x) = e^x$ (see Fig. 1.32).

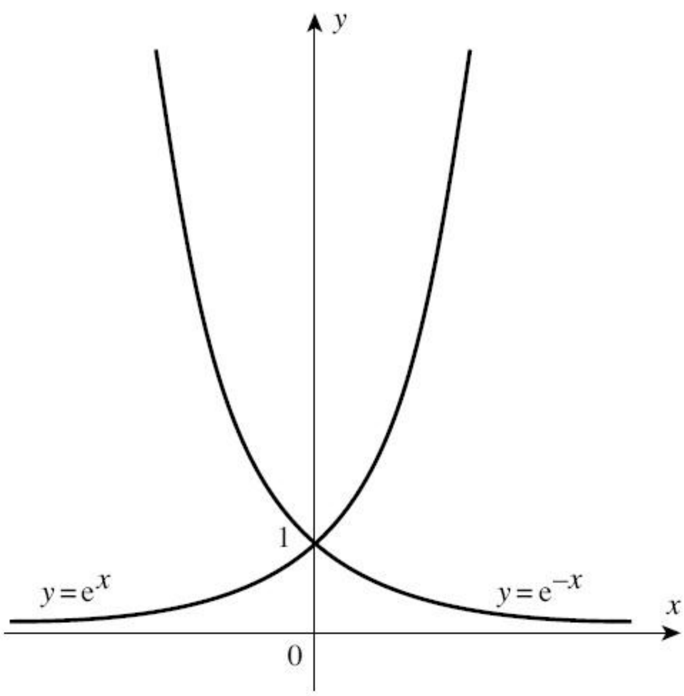


Fig. 1.32

Laws of Exponents (or Laws of Indices) for real exponents
For any positive real numbers $a \neq 1, b \neq 1$, m, n natural numbers and real variables x, y , the following laws are valid:

I. $a^x \cdot a^y = a^{x+y}$

II. $\dfrac{a^x}{a^y} = a^{x-y}\,,\ a \neq 0$

III. $\left(a^x\right)^y = a^{xy}$

IV. $\left(ab\right)^x = a^x . b^x$

V. $a^0 = 1$

VI. $\sqrt[n]{a^m} = a^{m/n}$

The Exponential Series

Now, we will show that,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

Proof.

Consider the expression $\left(1+\dfrac{1}{n}\right)^{nx}$, by making use of the binomial theorem, we can expand this expression and get

$$\left(1+\frac{1}{n}\right)^{nx} = 1 + \frac{nx}{1!}.\frac{1}{n} + \frac{nx(nx-1)}{2!}\frac{1}{n^2} + \frac{nx(nx-1)(nx-2)}{3!}\frac{1}{n^3} + \ldots$$

$$= 1 + \frac{x}{1!} + \frac{n^2x(x-1/n)}{2!}\frac{1}{n^2} + \frac{n^3x(x-1/n)(x-2/n)}{3!}\frac{1}{n^3} + \ldots$$

$$= 1 + \frac{x}{1!} + \frac{x(x-1/n)}{2!} + \frac{x(x-1/n)(x-2/n)}{3!} + \ldots$$

But, as $n \to \infty$, the terms $1/n,\ 2/n$, and so on approach 0. Therefore, the right-hand side simplifies to the following:

$$\text{R.H.S.} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

Moreover, the number of terms (being $n+1$) becomes infinitely large as $n \to \infty$, whatever x may be. Hence, the series continues to infinity.

Also,

$$\lim_{n\to\infty} \text{L.H.S.} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx} = \left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\right)^{x} = e^{x}$$

We get,

$$e^{x} = 1+\frac{x}{1!}+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\frac{x^{4}}{4!}+...$$

IV. The Logarithmic Function

Firstly, we introduce the concept of logarithm of a positive real number. If three numbers a,b , and c are so related that

$$a^{b} = c$$

then the exponent "b " is called the logarithm of "c " to the base "a "

We write

$$\log_{a} c = b$$

It may be noted that the logarithm of a number can be different for different bases. In the system of logarithms, which we use in our day-to-day calculations (such as those in the field of engineering, etc.), the base 10 is found to be most useful. Logarithms to the base 10 are called common logarithms. Once the base "10 " is chosen, it has to be raised with a suitable real number "b "(positive, zero, or negative) so that, it represents the given (positive) number c , exactly or very close to it.

Thus, we write,

$10^{b} = c$ or $10^{b} \approx c$ where the symbol "$\approx$" stands for "very close to".

For example,

$$\log_{10} 100 = 2,\ \log_{10} 1000 = 3\ .$$

These values of logarithms are exact, since $10^{2} = 100$ and$10^{3} = 1000$.

On other hand,

$$\log_{10} 5 = 0.669\ ,\ \log_{10} 27.8 = 1.4453$$

These values of logarithms are not exact, but they are very close to the numbers in equations, since $(10)^{0.699} \approx 5$, $(10)^{1.4453} \approx 27.8$.

In common logarithms, the base is always10 , so that, if no base is mentioned, the base 10 is always understood. However, it is useful only while dealing with arithmetical calculations.

Important in calculus are logarithms to the base "e ", called natural logarithms . The number"e ", (which is the base for natural logarithms) is a typical irrational number, lying between 2 and 3 ($e = 2.71828...$).

The notation for "natural logarithm" is "ln".

Definition of the logarithm

The logarithm of a given number to a given base, is equal to the power to which, the base should be raised to get the given number.

We know that	Therefore we say that	we write
$2^6=64$	log of 64 to the base 2=6	$\log_2 64=6$
$4^3=64$	log of 64 to the base 4=3	$\log_4 64=3$
$64^1=64$	log of 64 to the base 64=1	$\log_{64} 64=1$
$5^2=25$	log of 25 to the base 5=2	$\log_5 25=2$
$5^{-3}=1/125$	log of 1/125 to the base 5=-3	$\log_5(1/125)=-3$
$a^0=1, (a \neq 0)$	log of 1 to the base a=0	$\log_a 1=0$
$a^1=a$	log of a to the base a=1	$\log_a a=1$

Note

I. From the first three illustrations, we observe that the logarithm of a (positive) number is different for different bases.

II. The logarithm of 1 to any base is zero.

III. The logarithm of any number to the same base (as the number itself) is 1 (i.e. $\log_a a=1, \log_{10} 10=1, \log_e e=1$.)

Definition

the general logarithmic function is defined as

$$y=f(x)=\log_a x, \quad a>0, \ a\neq 1$$

and defined by the condition

$$y=\log_a x \quad \Leftrightarrow \quad a^y=x$$

The domain of the logarithmic function $y=\log_a x$ is the set of all positive real numbers $(0,\infty)$, and its range is the open interval $(-\infty,\infty)$.

This function monotonically increases if $a>1$, and monotonically decreases if $0<a<1$ (see Fig. 1.33).

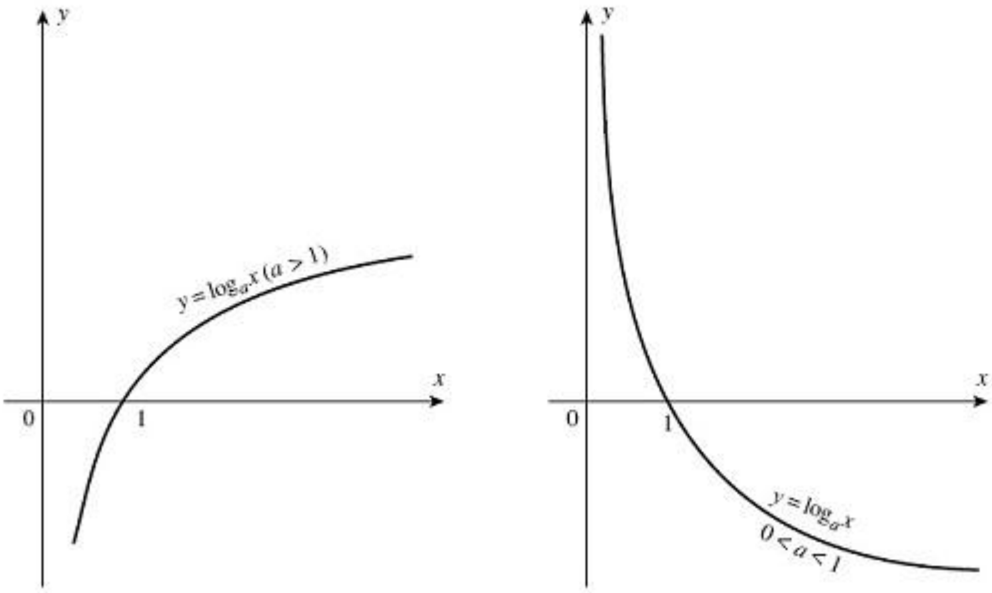


Fig. 1.33

The logarithmic function, $y = \log_a x$ is the inverse of the exponential function $y = a^x$.

The Natural Logarithm

The logarithmic function to the base e is called the natural logarithmic function and is usually denoted by $\ln x$ (or $\log_e x$) see Fig. 1.34.

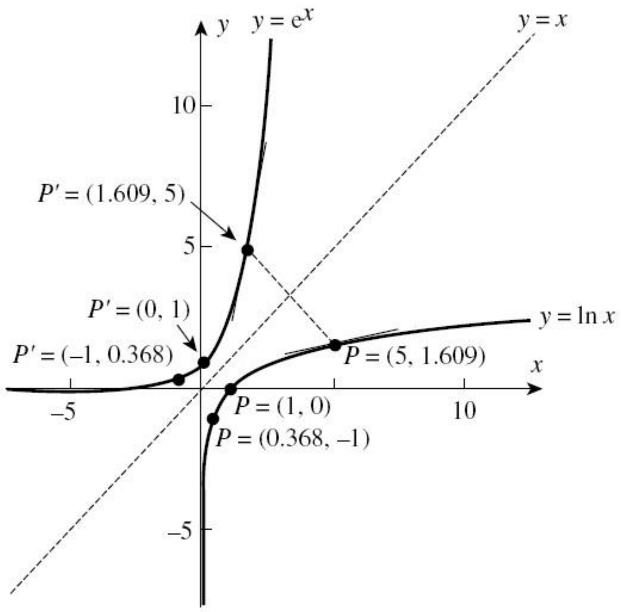


Fig. 1.34

The Common Logarithm
The logarithmic function to the base 10 is called the common logarithmic function and sometimes denoted by $\log x$.

The fundamental Laws of Logarithms

(i) $\log_a b^x = x \log_a b$

Proof.

Let $b = a^u \Rightarrow \log_a b = u$

$$\therefore \text{ L.H.S.} = \log_a \left(a^u\right)^x = \log_a \left(a^{ux}\right)$$
$$= u\,x = x \log_a b = \text{R.H.S.}$$

(ii) $\log_a (x\,y) = \log_a x + \log_a y$

(iii) $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$

Change of Base
We will now show that, if we are given the logarithm of a number, to any base, then we can easily compute the logarithm of that number to any other base. The following relation states the rule.

$$\log_a x = \frac{\log_b x}{\log_b a} \qquad (1)$$

Proof.
Let

$$x = b^y,\ a = b^c \Rightarrow x = a^{y/c}$$

The left hand side of (1)

$$\text{L.H.S.} = \log_a x = \log_a a^{y/c} = \frac{y}{c} \qquad (2)$$

The right hand side of (1)

$$\text{R.H.S.} = \frac{\log_b x}{\log_b a} = \frac{\log_b b^y}{\log_b b^c} = \frac{y}{c} \qquad (3)$$

Comparing (2) and (3) we have the result.

Relation Between Exponential Function and Logarithmic Function

Now, it is easy to show that

$$a^{\log_a x} = x$$

Proof.

Let

$$a^{\log_a x} = t \qquad (1)$$

Taking the logarithm to base a for both sides of (1), we have

$$\log_a a^{\log_a x} = \log_a t \Rightarrow \log_a x = \log_a t$$

So, we have

$$t = x$$

Corollaries

I. $y = \ln x \Leftrightarrow x = e^y$ **.**

II. $y = a^x \quad \Rightarrow \ln y = x \ln a$.

III. $\log_a x = \dfrac{\ln x}{\ln a}$.

IV. $\ln e^x = x$.

V. $e^{\ln x} = x$.

V.Hyperbolic Functions and Their Properties

Certain special combinations of e^x and e^{-x} appear so often in both mathematics and science that they are given special names.

Definitions

The functions

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2} \qquad (1)$$

are respectively, called the hyperbolic sine and hyperbolic cosine.

the parametric equations $x = \cosh t$, $y = \sinh t$ describe the right branch of the unit hyperbola $x^2 - y^2 = 1$ [which is the special case of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$](Figure 1.35). Moreover, the parameter t is related to the shaded area S by $t = 2S$.

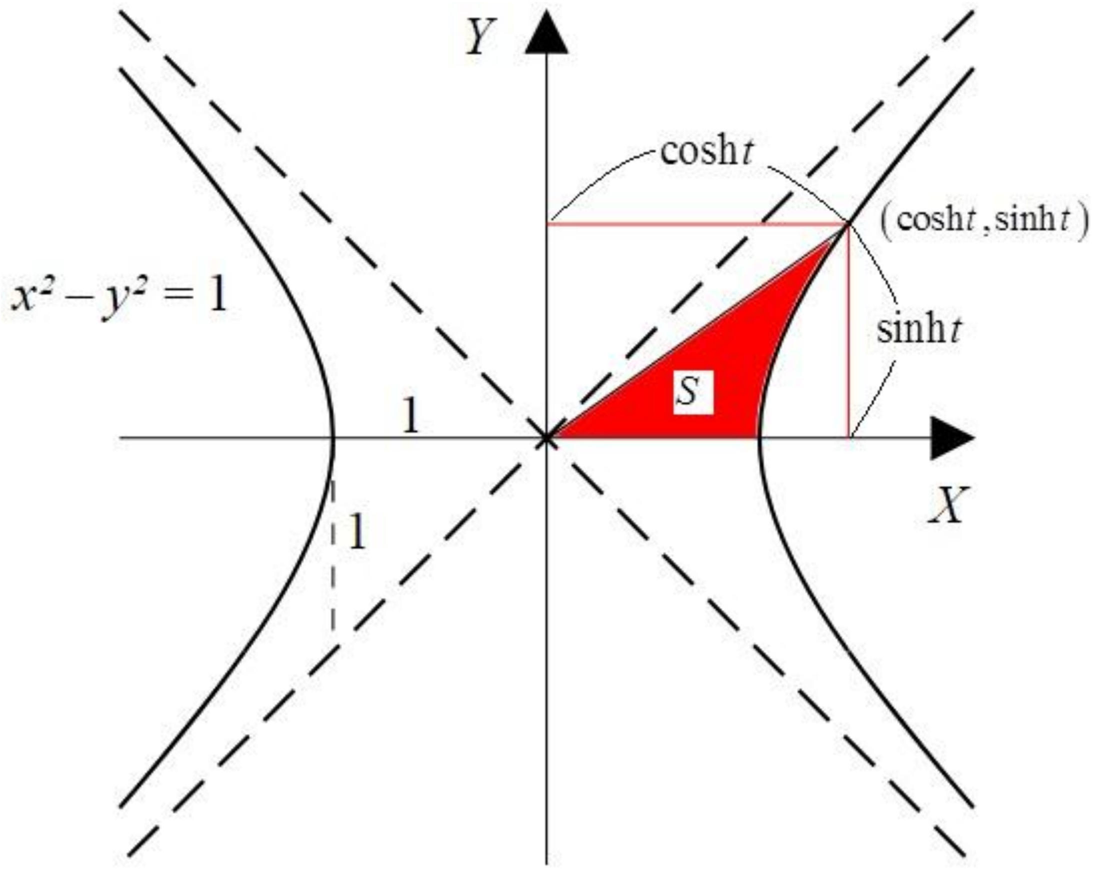


Fig. 1.35

There are six basic hyperbolic functions. The other four hyperbolic functions are defined in the terms of the hyperbolic sine and hyperbolic cosine.

Definitions

The functions

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

$$\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

are respectively called the hyperbolic tangent, the hyperbolic cotangent, the hyperbolic secant, and the hyperbolic cosecant.

Hyperbolic functions are connected by a number of algebraic relations similar to those connecting trigonometric functions. In particular, the fundamental identity for the hyperbolic functions is

$$\cosh^2 x - \sinh^2 x = 1$$
$$1 - \tanh^2 x = \text{sech}^2 x$$
$$1 - \coth^2 x = -\text{cosech}^2 x$$
$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$
$$\sinh(x \pm y) = \sinh x \cosh y \pm \sinh y \cosh x$$

If y is replaced by x in these identities we obtain,

$$\cosh(2x) = \cosh^2 x + \sinh^2 x$$
$$\sinh(2x) = 2\sinh x \cosh x$$

Note

From the definitions (1) , we can obtain

$$\sinh x + \cosh x = e^x$$
$$\cosh x - \sinh x = e^{-x}$$

It is, therefore, apparent that any combination of the exponentials e^x and e^{-x} can be replaced by a combination of $\sinh x$ and $\cosh x$ and conversely.

The important hyperbolic identities

$$\cosh^2 x - \sinh^2 x = 1$$
$$\sinh 2x = 2\sinh x \cosh x$$
$$\cosh 2x = \cosh^2 x - \sinh^2 x$$
$$\text{sech}^2 x = 1 - \tanh^2 x$$
$$\text{cosech}^2 x = \coth^2 x - 1$$
$$\sinh(x \pm y) = \sinh x \cosh y \pm \sinh y \cosh x$$
$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

Note

Hyperbolic functions are defined in terms of exponential functions. This is very different from the way we defined trigonometric functions. However, if you study complex analysis, you will discover that trigonometric functions can also be defined in terms of exponential functions of a complex variable.

The Properties of Hyperbolic Functions

The graphs of hyperbolic cosine and hyperbolic sine are shown in Figs. 1.36 and 1.37.

At $x = 0$, $\cosh x = 1$ and $\sinh x = 0$. Note that these value are same as in the case of corresponding trigonometric functions at $x = 0$. Therefore, all the hyperbolic functions have the same values at $x = 0$ that the corresponding trigonometric functions have.

Further, note that

$$\sinh(-x) = \frac{e^{-x} - e^{x}}{2} = -\frac{e^{x} - e^{-x}}{2} = -\sinh x$$

$$\cosh(-x) = \frac{e^{-x} + e^{x}}{2} = \frac{e^{x} + e^{-x}}{2} = \cosh x$$

Thus, hyperbolic sine is an odd function and the hyperbolic cosine is an even function. So the graph of $\sinh x$ is symmetric with respect to the origin and that of $\cosh x$ is symmetric about the $y-$axis .

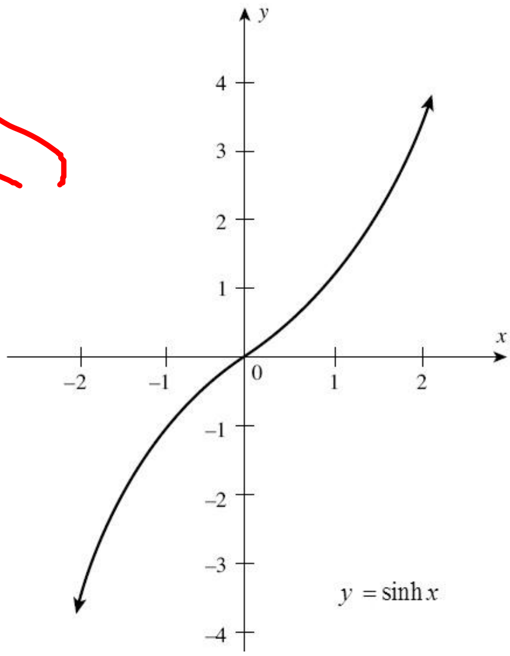


Fig. 1.36

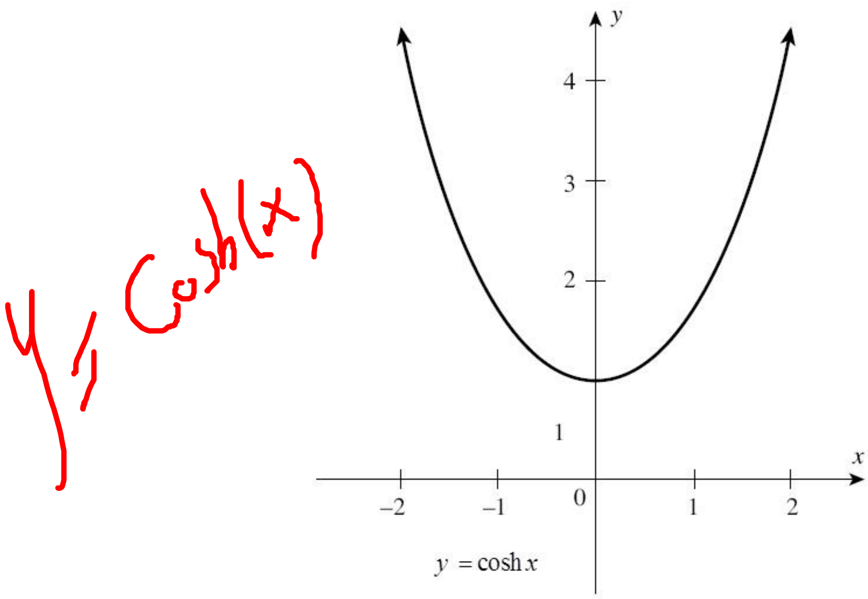


Fig. 1.37

1. The domain of the function $\sinh x$ is the set of all real numbers $\mathbb{R}$ and its range is $(-\infty,\infty)$ (Fig. 1.36).
2. The domain of the function $\cosh x$ is the set of all real numbers $\mathbb{R}$ and its range is $[1,\infty)$ (Fig. 1.37).
3. The domain of the function $\tanh x$ is the set of all real numbers $\mathbb{R}$ and its range is $(-1,1)$ (Fig. 1.38).
4. The domain of the function $\coth x$ is the set of all real numbers $\mathbb{R}$ except at $x = 0$ $\left(\mathbb{R}-\{0\}\right)$ and its range is $\mathbb{R}-\left[-1,1\right]=(-\infty,-1)\cup(1,\infty)$ (Fig. 1.39).
5. The domain of the function $\operatorname{sech} x$ is the set of all real numbers $\mathbb{R}$ and its range is $(0,1]$ (Fig. 1.40).
6. The domain of the function $\coth x$ is the set of all real numbers $\mathbb{R}$ except at $x = 0$ $\left(\mathbb{R}-\{0\}\right)$ and its range is $\left(\mathbb{R}-\{0\}\right)$ (Fig. 1.41).

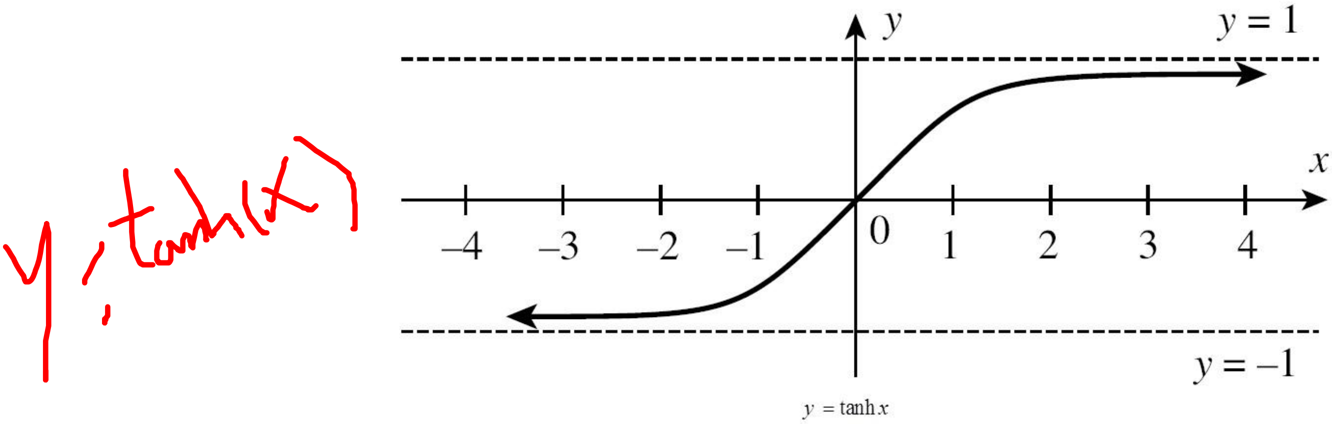


Fig. 1.38

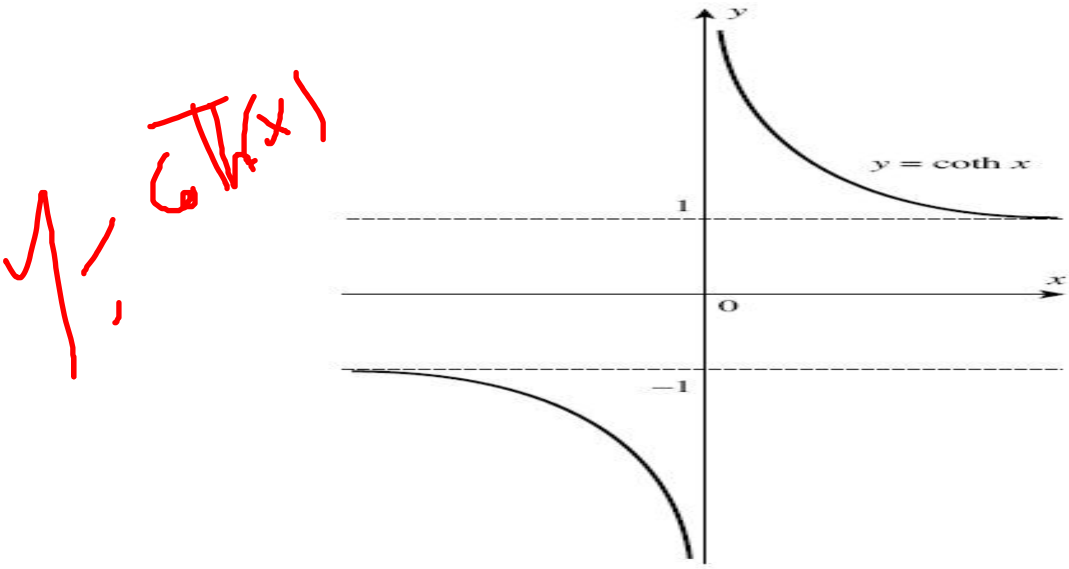


Fig. 1.39

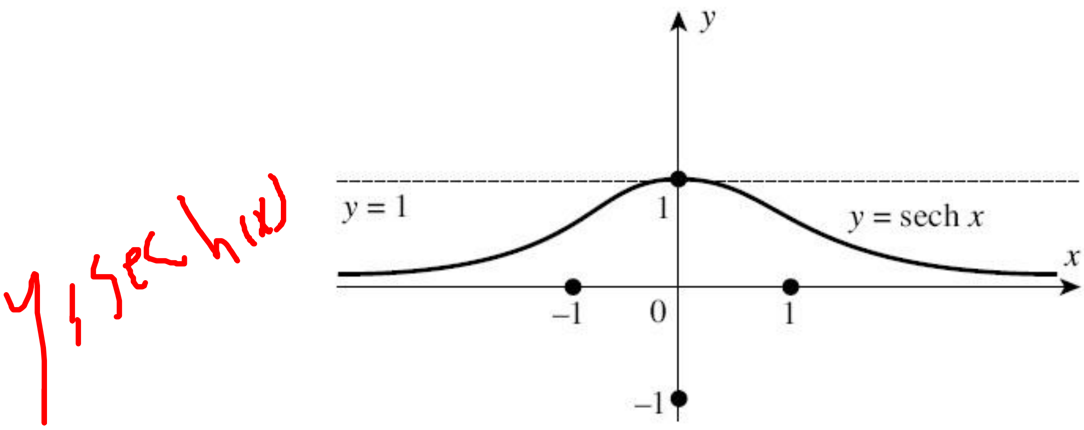


Fig. 1.40

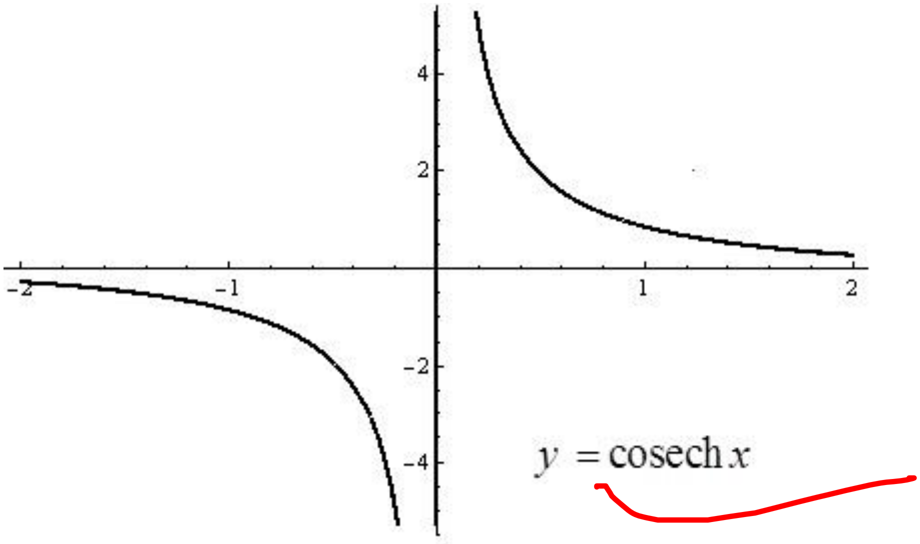


Fig. 1.41

VI. Inverse Hyperbolic Functions

1. **Inverse Hyperbolic Sine Function**.

From the graph of the hyperbolic sine in Figure 1.36, observe that the hyperbolic sine is one-to-one. Furthermore, the hyperbolic sine is continuous and increasing on its domain. Thus, this function has an inverse that we now define.

Definition (A): The inverse hyperbolic sine function denoted by $\sinh^{-1} x$, is defined as follows:

$y = \sinh^{-1} x$, if and only if, $x = \sinh y$, where y is any real number (Figure 1.42).

Both, the domain and range of $\sinh^{-1} x$, are the set $\mathbb{R}$ of real numbers. From the definition (A),

$$\sinh\left(\sinh^{-1} x\right) = x \text{ and } \sinh^{-1}\left(\sinh y\right) = y$$

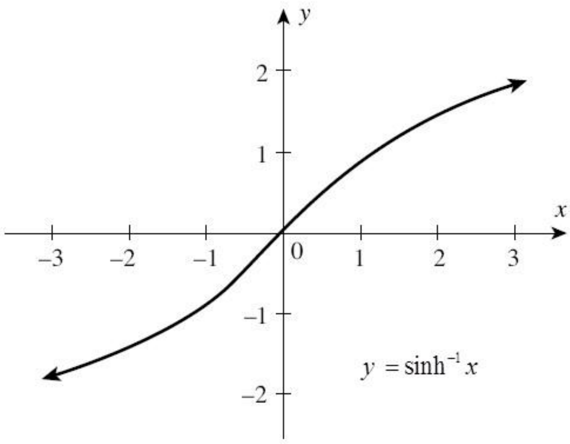


Fig.1.42

2. **Inverse Hyperbolic Cosine Function**

As in the case of inverse trigonometric functions, we restrict the domain and define a new function $f(x) = \cosh x$, $x \geq 0$ as follows: The domain of this function is the interval $[0,\infty)$ and the range is the interval $[1,\infty)$. Because $f(x)$ is continuous and increasing on its domain, it has an inverse, called the inverse hyperbolic cosine function.

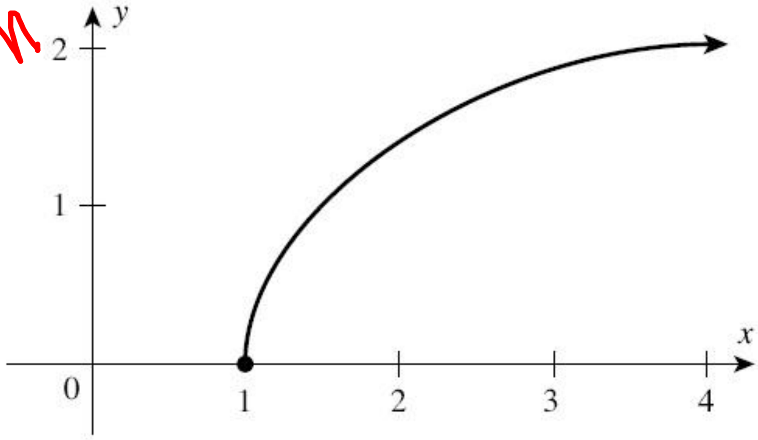
Definition (B): The inverse hyperbolic cosine function denoted by $\cosh^{-1} x$, is defined as follows:

$$y = \cosh^{-1} x \text{, if and only if } x = \cosh y\text{, } y \geq 0$$

The domain of $\cosh^{-1} x$ is in the interval $[1,\infty)$ and the range is in the interval $[0,\infty)$ (See Fig. 1.43) . From the definition (B),

$$\cosh\left(\cosh^{-1} x\right) = x \text{ if } x \geq 1,$$

$$\text{and } \cosh^{-1}\left(\cosh y\right) = y \text{ if } y \geq 0$$



$y = \cosh^{-1} x$

Fig. 1.43

3. **Inverse Hyperbolic Tangent Function**

   The hyperbolic tangent function is one-to-one and has an inverse.

   Definition (C): The inverse hyperbolic tangent function denoted by $\tanh^{-1} x$ is defined as follows:

   $$y = \tanh^{-1} x \text{ if and only if } x = \tanh y,$$

   where y is any real number.

   The domain of the inverse hyperbolic tangent function is the interval $(-\infty, \infty)$ and the range is the set $\mathbb{R}$ of real numbers. The graph of $\tanh^{-1} x$ appears in Figure 1.44.

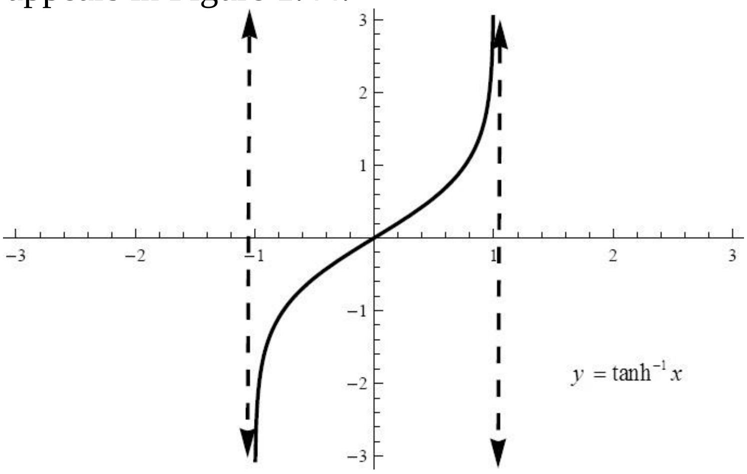


Fig.1.44

4. **Inverse Hyperbolic Cotangent Function**.

   The hyperbolic cotangent function is one-to-one and has an inverse. The graphs of $y = \coth^{-1} x$ is given in Figures 1.45.

The domain of the inverse hyperbolic cotangent function is $(-\infty,1)\cup(1,\infty)$ and the range is $(-\infty,0)\cup(0,\infty)$.

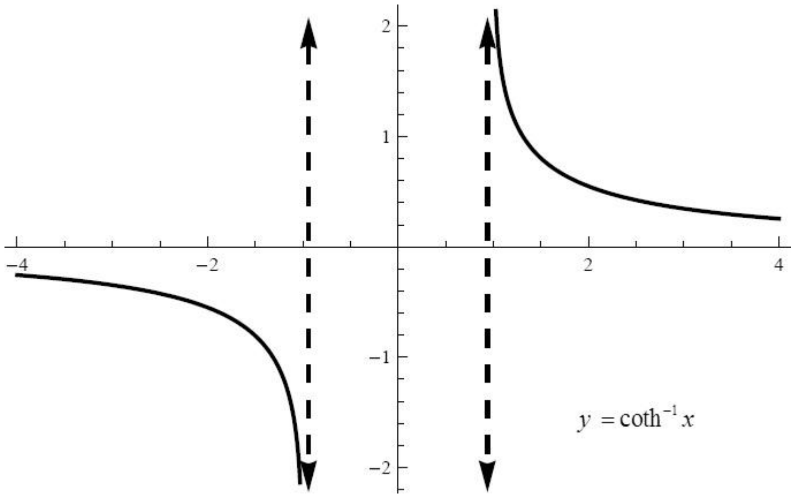


Fig. 1.45

5. **Inverse Hyperbolic Secant Function.**

We restrict the domain of hyperbolic secant function and define a new function $f(x) = \operatorname{sech} x$, $x \geq 0$ as follows:

The domain of this function is the interval $[0,\infty)$ and the range is the interval $(0,1]$. Because $f(x)$ is continuous and increasing on its domain, it has an inverse, called the inverse hyperbolic secant function.

Definition (D): The inverse hyperbolic secant function denoted by $\operatorname{sech}^{-1} x$, is defined as follows:

$$y = \operatorname{sech}^{-1} x, \text{ if and only if } x = \cosh y,\ y \geq 0$$

The domain of $\operatorname{sech}^{-1} x$ is the interval $(0,1]$ and the range is the interval $[0,\infty)$ (see Fig. 1.46).

From the definition (D),

$$\operatorname{sech}\left(\operatorname{sech}^{-1} x\right) = x \text{ if } 0 < x \leq 1,$$

$$\text{and } \operatorname{sech}^{-1}\left(\operatorname{sech} y\right) = y \text{ if } y \geq 0$$

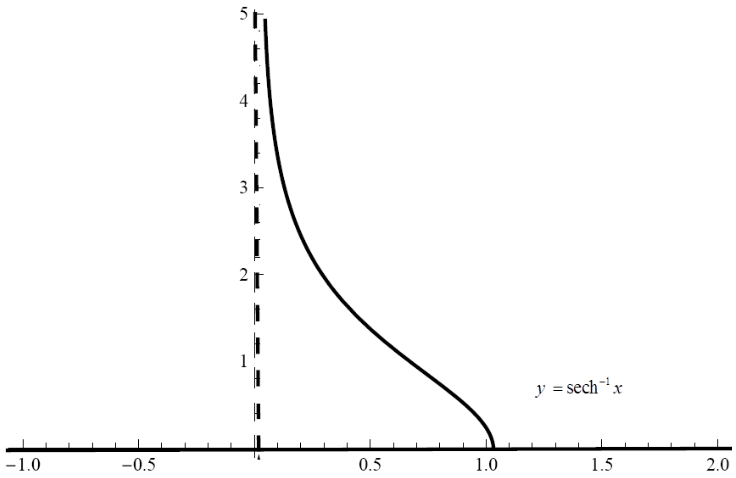


Fig. 1.46

6. **Inverse Hyperbolic Cosecant Function**.
   The hyperbolic cosecant function is one-to-one and has an inverse.
   The graphs of $y = \text{cosech}^{-1} x$ is given in Figures 1.47.
   The domain of the inverse hyperbolic cotangent function is $(-\infty, 0) \cup (0, \infty)$ and the range is $(-\infty, 0) \cup (0, \infty)$.

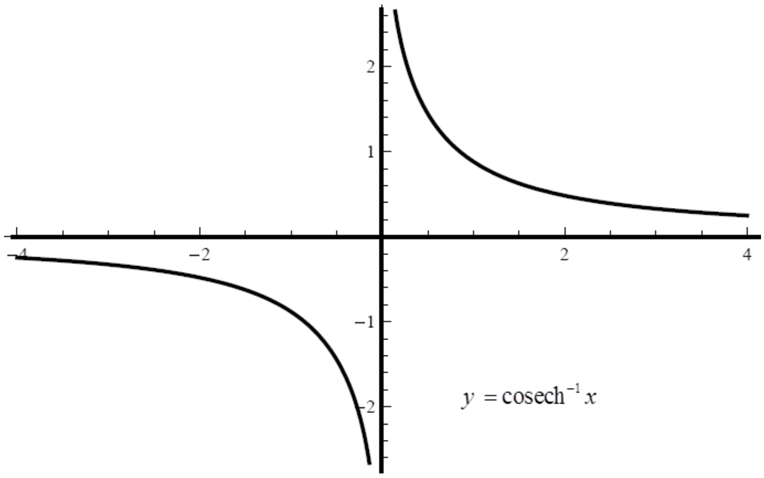


Fig. 1.47

Logarithm Equivalents of the Inverse Hyperbolic Functions

Since the hyperbolic functions are defined in terms of e^x and e^{-x} , it is not too surprising that the inverse hyperbolic functions can be expressed in terms of the natural logarithm. Following are these expressions for the six inverse hyperbolic functions we have discussed.

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right),\ x \in \mathbb{R}$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right),\ x \geq 1$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),\ |x| < 1$$

$$\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right),\ |x| > 1$$

$$\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right),\ 0 < x \leq 1$$

$$\operatorname{cosech}^{-1} x = \ln\left(\frac{1+\sqrt{1+x^2}}{x}\right),\ |x| > 0$$

To prove

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right),\ x \in \mathbb{R}$$

Let $y = \sinh^{-1} x$

From definition (A)

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

$$\sqrt{1+x^2} = \sqrt{1+\sinh^2 y} = \cosh y = \frac{e^y + e^{-y}}{2}$$

$$\therefore x + \sqrt{1+x^2} = e^y$$

$$\Downarrow$$

$$y = \sinh^{-1} x = \ln\left(x + \sqrt{1+x^2}\right)$$

To prove

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right),\ |x| \geq 1$$

Let $y = \cosh^{-1} x$

From definition (B)

$$x = \cosh y = \frac{e^{y} + e^{-y}}{2}$$

$$\sqrt{x^{2} - 1} = \sqrt{\cosh^{2} y - 1} = \sinh y = \frac{e^{y} - e^{-y}}{2}$$

$$\therefore x + \sqrt{x^{2} - 1} = e^{y}$$

$$\Downarrow$$

$$y = \cosh^{-1} x = \ln\left(x + \sqrt{x^{2} - 1}\right)$$

To prove

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \left|x\right| < 1$$

Let $y = \tanh^{-1} x$

From definition (C)

$$x = \tanh y = \frac{e^{y} - e^{-y}}{e^{y} + e^{-y}} = \frac{e^{2y} - 1}{e^{2y} + 1}$$

$$x\left(e^{2y} + 1\right) = e^{2y} - 1 \Rightarrow e^{2y}\left(x - 1\right) = -x - 1$$

$$e^{2y} = \frac{1+x}{1-x} \Rightarrow e^{y} = \pm\sqrt{\frac{1+x}{1-x}}$$

But $e^{y} \geq 0$, we have

$$e^{y} = \sqrt{\frac{1+x}{1-x}}$$

$$y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

The other relations can be proved in similar way.

Chapter 2

Limits of Real Functions

Introduction

Addition, subtraction, multiplication, division, raising to a power, extracting a root, taking a logarithm, or a modulus are operations of elementary mathematics. In order to pass from elementary mathematics to higher mathematics, we must add to this list one more mathematical operation, namely, "finding the limit of a function".

The notion of limit is an important new idea that lies at the foundation of Calculus. In fact, we might define Calculus as the study of limits. It is, therefore, important that we have a deep understanding of this concept. Although the topic of limit is rather theoretical in nature, we shall try to represent it in a very simple and concrete way.

Useful Notations

- Meaning of the notation $x \to a$ Let x be a variable and "a" be a constant. If x assumes values nearer and nearer to "a" (without assuming the value "a" itself), then we say x tends to a (or x approaches a) and we write $x \to a$. In other words, the procedure of giving values to x (from the domain of "f") nearer and nearer to "a", but not permitting x to assume the value "a", is denoted by the symbol "$x \to a$". Thus, $x \to 1$ means, we assign values to x which are nearer and nearer to 1 (but not permitting x to assume the value 1), which means that x comes closer and closer to "1" reducing the distance between "x" and "1", in the process. Thus, by the statement "x" tends to "a", we mean that:
  - $x \neq a$,
  - (x assumes values nearer and nearer to a, and

- The way in which x should approach a is not specified. (Different ways of approaching "a" are given below.)

- Meaning of $x \to a^-$
  If we consider x to be approaching closer and closer to "a" from the left side (i.e., through the values less than "a"), then we denote this procedure by writing $x \to a^-$ and read it as "x " tends to "a minus".
- Meaning of $x \to a^+$
  If we consider x approaching closer and closer to "a" through the values greater than "a " (i.e., x approaching "a" from the right side), then this procedure is denoted by writing $x \to a^+$ and we read it as "x " tends to "a plus".

Example (1)

Consider the function

$$f(x) = 3x + 5, \; x \in (2,3) \cup (3,5]$$

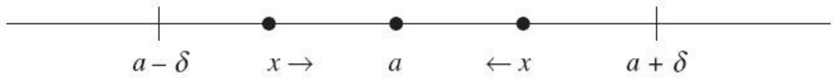
Note the following points

1. "4" is in the domain of f , and it can be approached from both the sides. Therefore, we can write $x \to 4$.
2. "5 " is in the domain of f , but x can approach 5 , only from the left of 5 (i.e., through values of $x < 5$). Thus, in this case, it is meaningful to write $x \to 5^-$, but we cannot write $x \to 5$.
3. "2" is not in the domain of f , but x can approach "2", from the right of "2" (i.e., through values of $x > 2$). Thus, in this case, it is meaningful to write $x \to 2^+$, but we cannot write $x \to 2^-$ or $x \to 2$.
4. "3 " is not in the domain of f , but x can approach " 3 " from both the sides of "3 ". Thus, we can write $x \to 3^+$ and $x \to 3^-$ or $x \to 3$

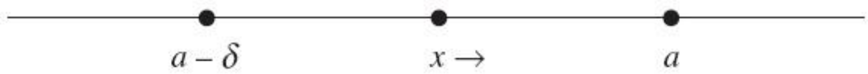
Notes

1. If x can approach "a " from both sides, then for an

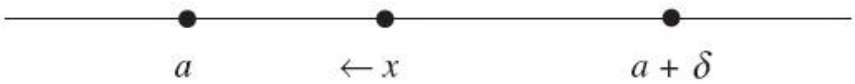
arbitrary small $\delta > 0$, x always belongs to the δ-neighborhood of "a", that is, $x \in (a-\delta, a+\delta)$ with $x \neq a$. This is equivalent to assigning values to "x", closer and closer to "a"from both sides of "a". (This procedure is useful for studying the values of a function in the neighborhood of the given point "a".)



2. If $x \to a^-$ (i.e., if approaches "a"from the left) then, for an arbitrary small $\delta > 0$, x always belongs $(a-\delta, a)$



3. If $x \to a^+$ (i.e., if x approaches “a” from the right) then, for an arbitrary small $\delta > 0$, x always belongs to $(\mathrm{a}, \mathrm{a}+\delta)$



Definition of the limit

Let f(x) be a function. If x assumes values nearer and nearer to the number "a"except possibly the value "a" and f(x) assumes the values nearer and nearer to l , which is a finite real number, then we say that f(x) tends to the limit l as x tends to a, and we write

$$\lim_{x \to a} f(x) = l$$

Notice that the function f need not even be defined at "a". If $f(x)$ assumes the values nearer and nearer to l as x approaches closer and closer to "a" from the left side, then the number "l" is the limit of $f(x)$ as x approaches "a"from the left and we write

$$\lim_{x \to a^-} f(x) = l$$

If $f(x)$ assumes the values nearer and nearer to l as x approaches closer and closer to "a" from the right side, then the number "l" is the limit of $f(x)$ as x approaches "a"from the right and we write

$$\lim_{x \to a^+} f(x) = l$$

Since "a" may be approached from both the sides of a (i.e., left side and right side of a) when we say that

$$\lim_{x \to a} f(x) = l$$

we really mean to say that

$$\lim_{x \to a^-} f(x) = l = \lim_{x \to a^+} f(x)$$

If these conditions are not satisfied simultaneously, we say that $\lim_{x \to a} f(x)$ does not exist.

Example (2)

Consider

$$f(x) = \frac{x^2 - 4}{x - 2}, \ x \neq 2$$

Find $\lim_{x \to 2} f(x)$.

Solution

We prepare the following calculations, by choosing successive values of x from a small neighborhood of 2 (say $\delta = 0.1$ is neighborhood of 2) and compute corresponding values $f(x)$. From the calculations, we get the data of our interest, which is given in Table 2.1.

x	$f(x)$	x	$f(x)$
1.91	3.91	2.1	4.1
1.92	3.92	2.01	4.01
1.96	3.96	2.001	4.001
1.99	3.99	2.0001	4.0001
1.997	3.997	2.00001	4.00001
1.9998	3.9998	2.000001	4.000001
1.999998	3.999998	2.0000001	4.0000001
1.99999999	3.99999999	2.00000001	4.00000001
2	Not defined	2	Not defined

Table 2.1

From the table, we observe that as x approaches 2, $f(x)$ takes up values closer and closer to 4.We, therefore, say that the limit of $f(x)$ as x approaches 2, is 4. In symbols, we write

$$\lim_{x \to 2} f(x) = 4$$

Note that the preparation of Table 2.1 is time consuming and tedious. On the other hand, we have

$$f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x-2)(x+2)}{(x-2)}, x \neq 2 \quad (1)$$

Note that, if $(x-2) \neq 0$, (i.e., if $x \neq 2$) then we can cancel the factor $(x-2)$ from the numerator and the denominator of the above expression on the right-hand side of Equation (1), and get,

$$f(x) = x + 2, x \neq 2 \quad (2)$$

Thus, we have two Equations (1) and (2), both representing the same function $f(x)$, when $x \neq 2$. We may choose any of them for computing the limit of the function in question. Obviously, the Equation (2) is simpler to handle in view of the difficulty observed in connection with the expression $\frac{x^2-4}{x-2}, x \neq 2$, in listing the values of $f(x)$ in the neighborhood of 2. Hence, we choose the expression ($f(x)=x+2$) for computing the limit in question. We get

$$\begin{aligned}\lim_{x\to 2} f(x) &= \lim_{x\to 2}\frac{x^2-4}{x+2}, x \neq 2\\ &= \lim_{x\to 2}(x+2), x \neq 2\\ &= 2+2=4\end{aligned}$$

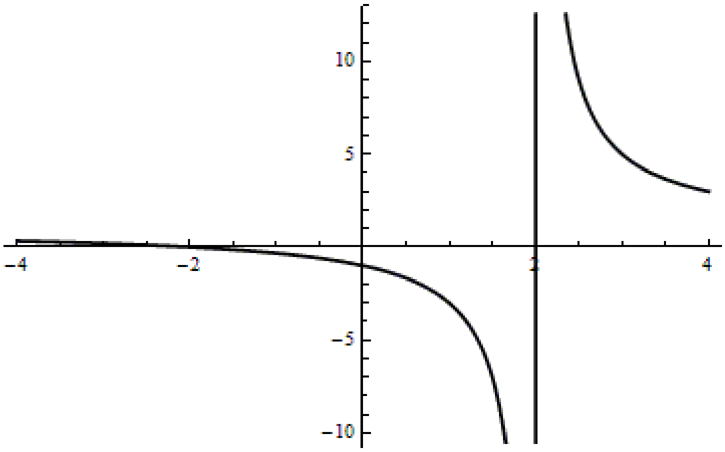
Note that whereas $f(2)$ does not exist (since 2 is not in the domain of "f"), $\lim_{x\to 2} f(x)$ exists, and it is given by the number 4. This shows that the existence or nonexistence of the limit of a function at a point does not depend on the existence or nonexistence of the value of the function at that point.

Example (3)

Consider

$$G(x)=\frac{x+2}{x-2}, x \neq 2$$

Note that this function is defined for all real values of x , except $x=2$. However, the limit $\lim_{x\to 2}\frac{x+2}{x-2}, x \neq 2$ does not exist (see Fig. 2.1).



$$G(x)=\frac{x+2}{x-2},\ x\neq 2$$

Fig. 2.1

This is because, as $x\to 2^{+}$, the numerator $(x+2)$ approaches the number 4 whereas the denominator approaches the number "0" from right, so that $G(x)$ approaches positive large values. On the other hand, as $x\to 2^{-}$, the numerator $(x+2)$ approaches the number 4 whereas the denominator approaches the number "0"from left, so that $G(x)$ approaches negative large values. Whenever such a situation arises, we say that the limit of the function does not exist. Later, we shall introduce infinity as limit of a function.

Example (4)

Let

$$f(x)=\begin{cases}x+5, & x>0\\ x+2, & x<0\end{cases}$$

Find $\lim\limits_{x\to 0} f(x)$.

Solution

Observe that $f(0)$ is not defined. Let us study the values $f(x)$ as of $x\to 0$. We note that as

$$x \to 0^- \Rightarrow f(x) \to 2.$$

On the other hand, as

$$x \to 0^+ \Rightarrow f(x) \to 5.$$

Thus

$$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x).$$

When this happens, we say that the limit of the function does not exist.

Example (5)

$$f(x) = \begin{cases} 2x - 1, & 1 \le x < 2 \\ 4x - 5, & 2 < x \le 3 \end{cases}$$

Observe that $f(2)$ is not defined. Let us study the values of $f(x)$ as $x \to 2$. We prepare Table 2.2.

x	$f(x)$	x	$f(x)$
1.9	2.8	2.1	3.4
1.99	2.98	2.01	3.04
1.999	2.998	2.001	3.004
1.9999	2.9998	2.0001	3.0004
1.9999	2.99998	2.00001	3.00004
$As\ x \to 2^-$	$f(x) \to 3$	$As\ x \to 2^+$	$f(x) \to 3$

Table 2.2

From Table 2.2, we observe that

$$\lim_{x \to 2^-} f(x) = 3$$

And

$$\lim_{x \to 2^+} f(x) = 3$$

Thus, the left-hand limit of $f(x)$ at $x = 2$ is equal to its right-hand limit at $x = 2$. In this case, we say that the limit of $f(x)$ as $x = 2$ exists, and we write

$$\lim_{x \to 2} f(x) = 3$$

Example (6)

Let

$$f(x) = \begin{cases} x, & x < 1 \\ 2, & x = 1 \\ x + 2, & x > 1 \end{cases}$$

Find $\lim_{x \to 1} f(x)$

Solution

We have the following observations:

(a) $\lim_{x \to 1^-} f(x) = 1$ (left-hand limit)

(b) $\lim_{x \to 1^+} f(x) = 3$ (right-hand limit)

(c) $f(1) = 2$

Thus

$$\lim_{x \to 1^-} f(x) = 1 \neq \lim_{x \to 1^+} f(x) = 3$$

Obviously, $\lim_{x \to 1} f(x)$ does not exist.

Example (7)

Let

$$f(x) = \frac{1}{x - 1}, x \neq 1$$

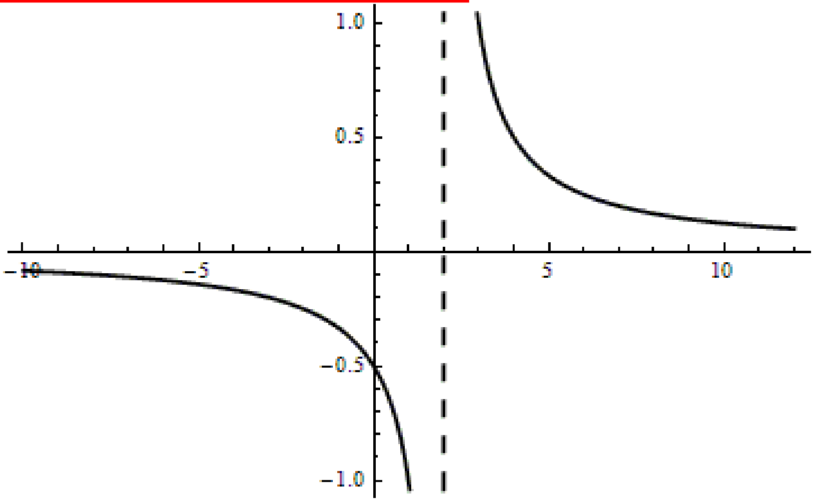
Find $\lim_{x \to 1} f(x)$

Solution

Observe that as $x \to 1^+$ (as x assumes values closer and closer to 1 from the right hand side) $f(x)$ gets larger and larger positive values. On the other hand, when $x \to 1^-$ (as

x assumes values closer and closer to 1 from the left hand side), $f(x)$ gets larger and larger negative values (see Fig. 2.2).

Thus, $\lim_{x \to 1} f(x)$ does not exist.



$$f(x) = \frac{1}{x-1}, x \neq 1$$

Fig. 2.2

Example (8)

Evaluate the following limit

$$\lim_{x \to 0} \frac{\sin x}{x}, \qquad (x \text{ in radians})$$

Solution

Here, there is no way of canceling terms in the numerator and denominator. Since $\sin x \to 0$ as $x \to 0$, the quotient $\frac{\sin x}{x}$ might appear to approach $\frac{0}{0}$. But, we know that $\frac{0}{0}$ is undefined, so if the above limit exists, then we must find it by a different technique. Since we do not have any other simpler way of rewriting $\frac{\sin x}{x}$ to obtain the limit, we use a calculator to find the values of $\frac{\sin x}{x}$ for values of x close to 0 and angles x (in sin x) in radians. (Other methods of finding this limit will be discussed later.)

x	$\sin x$	$\frac{\sin x}{x}$
-0.10	0.0998333	0.99833
-0.09	0.0898785	0.99865
-0.05	0.0499792	0.99958
-0.03	0.0299955	0.99985
-0.02	0.0199987	0.99993
-0.01	0.00999983	0.999983
0.00	0.00000	Not defined
0.01	0.00999983	0.999983
0.02	0.0199987	0.99993
0.03	0.0299955	0.99985

Table 2.3

From Table 2.3, it is obvious that, as $x \to 0$, either from the right or from the left, the value of $\frac{\sin x}{x}$ approaches closer and closer to the number 1. We, therefore, agree to write

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

This limit is used very often to find the limits of many trig-onometric functions (including various functions involving trigonometric functions), and plays a very important role in deriving many useful results.

Simpler and Powerful Rules for Finding Limits (Algebra of Limits)

Limits are extremely important throughout Calculus. A general method, we can prepare a table listing values of x, closer and closer to “a”, and the corresponding values $f(x)$. Such a table may help us guess a number to which $f(x)$ approaches, suggesting the limit of $f(x)$, as $x \to a$. However, such a process of finding the values of “f” as $x \to a$ is both time consuming and generally very tedious.

Let n be a positive integer, k be a constant, and $f(x)$, $g(x)$ and $h(x)$ be functions, such that $\lim_{x \to a} f(x)$, $\lim_{x \to a} g(x)$ and $\lim_{x \to a} h(x)$ exist. Then

1. $\lim_{x \to a} k = k$
2. $\lim_{x \to a} x = a$
3. $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
4. $\lim_{x \to a} k.f(x) = k.\lim_{x \to a} f(x)$
5. $\lim_{x \to a} [f(x).g(x)] = \lim_{x \to a} f(x).\lim_{x \to a} g(x)$
6. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \quad \lim_{x \to a} g(x) \neq 0$
7. $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$
8. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ provided $\lim_{x \to a} f(x) > 0$ when n is even.
9. $\lim_{x \to 0} (f \circ g)(x) = \lim_{x \to 0} f(g(x)) = f\left(\lim_{x \to 0} g(x)\right)$

10. If $f(x) \le g(x) \le h(x)$ for all x near a, except possibly at a. If $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = l$, then $\lim_{x \to a} g(x) = l$.

Example (9)

Find the following limit

$$\lim_{x \to 1} \frac{x^{1/4} - 1}{x^{1/3} - 1}$$

Solution

Here, we observe that the indices of x are fractions. Hence, it is not possible to factorize both numerator and denominator. We substitute $x = y^{12}$. Required limit is

$$\lim_{x \to 1} \frac{x^{1/4} - 1}{x^{1/3} - 1} = \lim_{y \to 1} \frac{y^3 - 1}{y^4 - 1}$$

$$= \lim_{y \to 1} \frac{(y-1)(y^2 + y + 1)}{(y-1)(y^3 + y^2 + y + 1)}$$

$$= \lim_{y \to 1} \frac{y^2 + y + 1}{y^3 + y^2 + y + 1} = \frac{3}{4}$$

Example (10)

Determine the following limit

$$\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x}$$

Solution

Put $y = 1 + x$, then as $x \to 0 \Rightarrow y \to 1$. Hence, the limit reduces to the form $\lim_{y \to 1} \frac{y^{1/2} - 1}{y - 1}$.

Example (11)

One can show that $\lim\limits_{x \to 0^+} \sqrt{x} = 0$ but it must be clear that neither $\lim\limits_{x \to 0} \sqrt{x}$ nor $\lim\limits_{x \to 0^-} \sqrt{x}$ exists (because $\sqrt{x}$ is not defined to the left of 0).

Methods for Evaluating Limits of Various Algebraic Functions

1. Direct Method [or Method of Direct Substitution]

This method is applicable in the case of very simple functions, in which the value of the function and the limit of the function both are the same.

Example (12)

$$\lim_{x \to 2}[x^2 + 3] = \lim_{x \to 2} x^2 + \lim_{x \to 2} 3 = 4 + 3 = 7$$

Example (13)

$$\lim_{x \to 5}\left[\frac{\sqrt{x-1}+2}{\sqrt{x+31}}\right] = \frac{\lim\limits_{x \to 5}\sqrt{x-1} + \lim\limits_{x \to 5} 2}{\lim\limits_{x \to 5}\sqrt{x+31}} = \frac{4}{6} = \frac{2}{3}$$

Example (14)

$$\lim_{x \to 1} \frac{x^2 - 9}{x - 3}, x \neq 3$$

$$= \frac{\lim\limits_{x \to 1}(x^2 - 9)}{\lim\limits_{x \to 1}(x - 3)} = 4$$

2. Factorization Method

For computing limit(s) of the type, $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$, where $f(a) = 0$ and $g(a) = 0$, the direct substitution method fails. In such cases, we search for a common factor $(x - a)$ in f(x) and g(x) by factorizing them and canceling this factor to reduce the quotient to the simplest form and then apply the direct method to obtain the limit. [Remember that $x \to a$ means that $x \neq a$, at any stage. In other words $(x - a) \neq 0$, at any stage. This permits us to cancel the common factor (x – a) from both numerator and denominator.

Example (15)

Evaluate

$$\lim_{x \to 1} \frac{x^2 - 4x + 3}{x^2 + 2x - 3}$$

Solution

$$\lim_{x \to 1} \frac{x^2 - 4x + 3}{x^2 + 2x - 3} = \lim_{x \to 1} \frac{(x-3)(x-1)}{(x+3)(x-1)}$$

$$= \lim_{x \to 1} \frac{(x-3)}{(x+3)} = -\frac{1}{2}, \quad [(x-1) \neq 0]$$

Note: For evaluating $\lim_{x \to a} \frac{f(x)}{g(x)}$, we may also follow the following steps:

I. Put $x = a + h \quad (\therefore x \to a \ as \ h \to 0)$

II. Simplify numerator and denominator and cancel the common factor h .

III. Put $h = 0$, in the remaining expression in h and obtain the limit.

Example (16)

Evaluate

$$\lim_{x \to 4} \frac{x^3 - 8x^2 + 16x}{x^3 - x - 60}$$

Solution

$$\lim_{x \to 4} \frac{x^3 - 8x^2 + 16x}{x^3 - x - 60}$$

$$= \lim_{x \to 4} \frac{x(x^2 - 8x + 16)}{x^3 - 4x^2 + 4x^2 - 16x + 15x - 60}$$

$$= \lim_{x \to 4} \frac{x(x^2 - 4x - 4x + 16)}{(x-4)[(x-4) + 4x + 15]}$$

$$= \lim_{x \to 4} \frac{x(x-4)(x-4)}{(x-4)[(x-4) + 4x + 15]}$$

$$= \lim_{x \to 4} \frac{x(x-4)}{[(x-4) + 4x + 15]} = 0$$

An Important Standard Limit

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}, \quad \text{n is natural number} \qquad (*)$$

Example (17)

Evaluate

$$\lim_{x \to 1} \frac{x^n + x^{n-1} + x^{n-2} + \ldots + x^3 + x^2 + x - n}{x - 1}, \quad \text{n is natural number}$$

Solution

$$\lim_{x \to 1} \frac{x^n + x^{n-1} + x^{n-2} + \ldots + x^3 + x^2 + x - n}{x - 1}$$

$$= \lim_{x \to 1} \frac{x^n + x^{n-1} + x^{n-2} + \ldots + x^3 + x^2 + x - (1 + 1 + 1 + \ldots\text{n times})}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x^n - 1) + (x^{n-1} - 1) + \ldots + (x^3 - 1) + (x^2 - 1) + (x - 1)}{x - 1}$$

$$= \lim_{x\to 0}\frac{(x^n-1)}{x-1}+\lim_{x\to 0}\frac{(x^{n-1}-1)}{x-1}+\lim_{x\to 0}\frac{(x^{n-2}-1)}{x-1}$$

$$+\ldots+\lim_{x\to 0}\frac{(x^3-1)}{x-1}+\lim_{x\to 0}\frac{(x^2-1)}{x-1}+\lim_{x\to 0}\frac{(x-1)}{x-1}$$

$$= n+n-1+n-2+\ldots+1=\frac{n}{n(n+1)}$$

The above formula can be used to evaluate limits of the

$$\lim_{x\to a}\frac{x^n-a^n}{x^m-a^n}$$

For this purpose, we write

$$\lim_{x\to a}\frac{x^n-a^n}{x^m-a^n}=\lim_{x\to a}\frac{x^n-a^n}{x-a}\div\lim_{x\to 0}\frac{x^m-a^m}{x-a}$$

and apply the standard limit to obtain

$$\lim_{x\to a}\frac{x^n-a^n}{x^m-a^n}=\frac{n}{m}a^{n-m} \qquad (**)$$

Example (18)
Evaluate

$$\lim_{x\to a}\frac{x^5-a^5}{x^3-a^3}$$

Solution

$$\lim_{x\to a}\frac{x^5-a^5}{x^3-a^3}=\frac{5}{3}a^{5-3}$$

Remark
Formula (*) has been proved for natural numbers n and m. However, the result is true for rational values of n and m. The following examples tell how this is justified.

Example (19)
Evaluate

$$\lim_{x\to1}\frac{x^{1/4}-1^{1/4}}{x^{1/3}-1^{1/3}}$$

Note : In such cases the important point is that the given limit can be converted in the form (*) by substitution as follows.
Here, the indices of x are fractions and hence we cannot factorize. The denominators of these indices are 4 and 3. Their L.C.M. is 12. Therefore, we use the substitution $x=t^{12}$, for our purpose.
Solution
Put $x=t^{12}$ $(t\to1 \text{ as } x\to1)$

$$\lim_{x\to1}\frac{x^{1/4}-1^{1/4}}{x^{1/3}-1^{1/3}}=\lim_{t\to1}\frac{t^{3}-1^{3}}{t^{4}-1^{4}}=\frac{3}{4}$$

Note
We can also apply Corollary (**) directly and obtain the limit as follows:

$$\lim_{x\to1}\frac{x^{1/4}-1^{1/4}}{x^{1/3}-1^{1/3}}=\frac{1/4}{1/3}.1^{1/4-1/3}=\frac{3}{4}$$

Example (20)
Find

$$\lim_{x\to3}\frac{x^{2/5}-3^{2/5}}{x^{1/2}-3^{1/2}}$$

Solution

$$\lim_{x\to3}\frac{x^{2/5}-3^{2/5}}{x^{1/2}-3^{1/2}}=\frac{2/5}{1/2}.3^{2/5-1/2}=\frac{4}{5}3^{-1/10}$$

Example (21)
Evaluate

$$\lim_{x \to 2} \frac{x^{-3} - 2^{-3}}{x - 2}$$

Solution

$$\lim_{x \to 2} \frac{x^{-3} - 2^{-3}}{x - 2} = \frac{-3}{1} . 2^{-3-1} = -3.2^{-4} = -\frac{3}{16}$$

Note : To evaluate limits of this type, it is always useful to convert the given limit to the standard form as follows:

$$\lim_{x \to 2} \frac{x^{-3} - 2^{-3}}{x - 2} = \lim_{x \to 2} \frac{1/x^{3} - 1/2^{3}}{x - 2}$$

$$= \lim_{x \to 2} \frac{1}{-8x^{3}} \frac{x^{3} - 2^{3}}{x - 2} = -\frac{1}{64} 3.2^{3-1} = -\frac{3}{16}$$

Example (22)

Evaluate

$$\lim_{x \to a} \frac{(x+2)^{5/3} - (a+2)^{5/3}}{x - a}$$

Solution

$$\lim_{x \to a} \frac{(x+2)^{5/3} - (a+2)^{5/3}}{x - a}$$

$$= \lim_{x+2 \to a+2} \frac{(x+2)^{5/3} - (a+2)^{5/3}}{(x+2) - (a+2)}$$

$$= \frac{5}{3} (\mathrm{a}+2)^{5/3-1}$$

Example (23)

Evaluate

$$\lim_{x \to 1} \frac{1 - x^{-1/3}}{1 - x^{-2/3}}$$

Solution

$$\lim_{x\to 1}\frac{1-x^{-1/3}}{1-x^{-2/3}}=\lim_{x\to 1}\frac{(x^{1/3}-1)/x^{1/3}}{(x^{2/3}-1)/x^{2/3}}$$

$$=\lim_{x\to 1}\frac{\mathrm{x}^{1/3}(x^{1/3}-1)}{(x^{2/3}-1)}=1^{1/3}.\lim_{x\to 1}\frac{x^{1/3}-1}{x^{2/3}-1}$$

$$=\frac{1}{2}$$

Method of Simplification

Sometimes it is required to simplify the given function and then evaluate the limit.

Example (24)

Evaluate

$$\lim_{x\to 5}\left(\frac{1}{x-5}-\frac{5}{x^2-5x}\right)$$

Solution

$$\lim_{x\to 5}\left(\frac{1}{x-5}-\frac{5}{x^2-5x}\right)=\lim_{x\to 5}\left(\frac{x-5}{x^2-5x}\right)$$

$$=\lim_{x\to 0}\frac{x-5}{x(x-5)}=\lim_{x\to 5}\frac{1}{x}=\frac{1}{5}$$

Example (25)

Evaluate

$$\lim_{x\to -2}\left(\frac{1}{x^2+5x+6}+\frac{1}{x^2+3x+2}\right)$$

Solution

We have

$$\lim_{x \to -2}\left(\frac{1}{x^2+5x+6}+\frac{1}{x^2+3x+2}\right)$$

$$= \lim_{x \to -2}\left[\frac{1}{(x+2)(x+3)}+\frac{1}{(x+2)(x+1)}\right]$$

$$= \lim_{x \to -2}\frac{(x+1)+(x+3)}{(x+1)(x+2)(x+3)}$$

$$= \lim_{x \to -2}\frac{2(x+2)}{(x+1)(x+2)(x+3)}$$

$$= \lim_{x \to -2}\frac{2}{(x+1)(x+3)} = -2$$

Method of Rationalization

If the numerator or the denominator or both contain functions of the type $[\sqrt{f(x)} - g(x)]$ or $[\sqrt{f(x)} - \sqrt{g(x)}]$ and the direct method fails to give the limit, we rationalize the given function by multiplying and dividing by $[\sqrt{f(x)} + g(x)]$ or $[\sqrt{f(x)} + \sqrt{g(x)}]$, as the case may be. After simplification of the function, we evaluate the limit by the earlier methods.

Example (26)

Evaluate

$$\lim_{x \to 0}\frac{x}{\sqrt{1+x}-1}$$

Solution
Consider

$$\lim_{x\to 0}\frac{x}{\sqrt{1+x}-1}=\lim_{x\to 0}\frac{x}{\sqrt{1+x}-1}\times\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}$$

$$=\lim_{x\to 0}\frac{x(\sqrt{1+x}+1)}{(1+x)-1}=\lim_{x\to 0}(\sqrt{1+x}+1)=2$$

Example (27)

$$\lim_{x\to 3}\frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}$$

Solution
Consider

$$\lim_{x\to 3}\frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}$$

$$=\lim_{x\to 3}\frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}\times\frac{\sqrt{x-2}+\sqrt{4-x}}{\sqrt{x-2}+\sqrt{4-x}}$$

$$=\lim_{x\to 3}\frac{(x-3)(\sqrt{x-2}+\sqrt{4-x})}{(x-2)-(4-x)}$$

$$=\lim_{x\to 3}\frac{(x-3)(\sqrt{x-2}+\sqrt{4-x})}{2(x-3)}$$

$$=\lim_{x\to 3}\frac{(\sqrt{x-2}+\sqrt{4-x})}{2}=1$$

Example (28)
Evaluate

$$\lim_{x\to 0}\frac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{b+x}-\sqrt{b-x}}$$

Solution

$$\lim_{x \to 0} \frac{\sqrt{a+x} - \sqrt{a-x}}{\sqrt{b+x} - \sqrt{b-x}}$$

$$= \lim_{x \to 0} \left[\left(\sqrt{a+x} - \sqrt{a-x} \right) \times \frac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} + \sqrt{a-x}} \right]$$

$$\div \lim_{x \to 0} \left[\left(\sqrt{b+x} - \sqrt{b-x} \right) \times \frac{\sqrt{b+x} + \sqrt{b-x}}{\sqrt{b+x} + \sqrt{b+x}} \right]$$

$$= \lim_{x \to 0} \left[\frac{2x}{\sqrt{a+x} + \sqrt{a-x}} \div \frac{2x}{\sqrt{b+x} + \sqrt{b-x}} \right]$$

$$= \frac{2\sqrt{b}}{2\sqrt{a}} = \sqrt{\frac{b}{a}}$$

Infinite Limits

So far we have considered the cases where as $x \to a$ (a finite number), $f(x) \to l$, (a finite number).

But, it may happen that as $x \to a$, $f(x)$ increases (or decreases) endlessly. Symbolically, we express these statements as follows:

$$\lim_{x \to a^-} f(x) = \infty, \lim_{x \to a^+} f(x) = \infty$$

Or

$$\lim_{x \to a^-} f(x) = -\infty, \lim_{x \to a^+} f(x) = -\infty$$

Consider the graph of $f(x) = \dfrac{1}{x-2}$, as shown in Figure 2.3. Note that it makes no sense to ask $\lim_{x \to 2} \dfrac{1}{x-2}$ (why?), but we think it is reasonable to write $\lim_{x \to 2^-} \dfrac{1}{x-2} = -\infty$ and $\lim_{x \to 2^+} \dfrac{1}{x-2} = \infty$. The following definition relates to this situation.

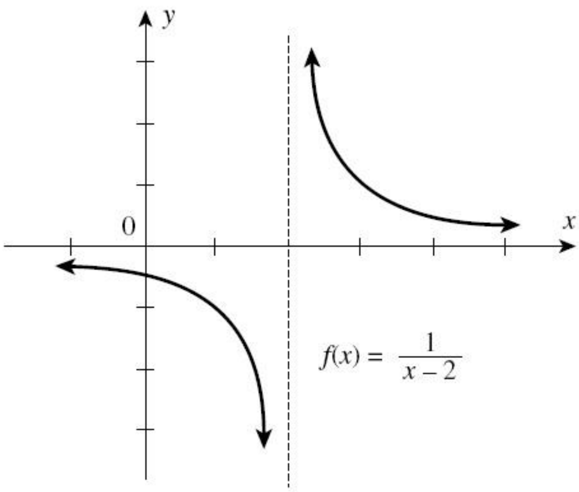


Fig 2.3

Definition (Infinite Limits)

We say that $\lim_{x \to a} f(x) = \infty$, if $f(x)$ gets larger and larger without bound, when x assumes values nearer and nearer to "a". On other hand, we say that $\lim_{x \to a} f(x) = -\infty$, if $f(x)$ is permitted to assume smaller and smaller values endlessly, when x assumes values nearer and nearer to "a".

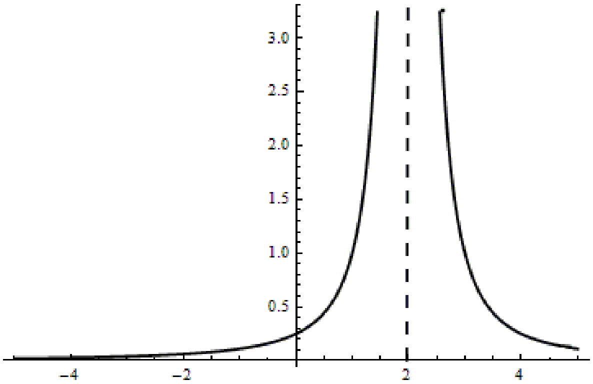
Example (29)

Find

$$\lim_{x \to 2^-} \frac{1}{(x-2)^2} \quad \text{and} \quad \lim_{x \to 2^+} \frac{1}{(x-2)^2}$$

Solution

The graph of $f(x) = \frac{1}{(x-2)^2}$ is shown in Figure 2.4.



$$f(x) = \frac{1}{(x-2)^2}$$

Fig. 2.4

We think it is quite clear that

$$\lim_{x \to 2^-} \frac{1}{(x-2)^2} = \infty$$

And

$$\lim_{x \to 2^+} \frac{1}{(x-2)^2} = \infty$$

Since both limits are ∞ , we could also write

$$\lim_{x \to 2} \frac{1}{(x-2)^2} = \infty$$

Example (30)

Find

$$\lim_{x \to 2^+} \frac{x+1}{x^2+5x+6}$$

Solution

$$\lim_{x \to 2^+} \frac{x+1}{x^2-5x+6} = \lim_{x \to 2^+} \frac{x+1}{(x-2)(x-3)}$$

As $x \to 2^+$, we see that $x+1 \to 3$, $x-3 \to -1$, and $x-2 \to 0$. Thus, the numerator is approaching 3, but the denominator is negative and approaching 0. We conclude that

$$\lim_{x \to 2^+} \frac{x+1}{(x-2)(x-3)} = -\infty$$

Asymptotes

Definition: An asymptote to a curve is defined as a straight line, which has the property that the distance from a point on the curve to the line tends to zero as the distance of this point to the origin increases without bound. There are vertical, horizontal asymptotes.

Vertical Asymptotes

The graph of the function $y = f(x)$ has a vertical asymptote for $x \to a$, if $\lim_{x \to a} f(x) = \infty$ or $\lim_{x \to a} f(x) = -\infty$ (see Figure 3.3a and b). The equation of the vertical asymptote has the form $x = a$. (In Figure 2.5a, it is $x = 0$, and in Figure 2.5b it is $x = a$.)

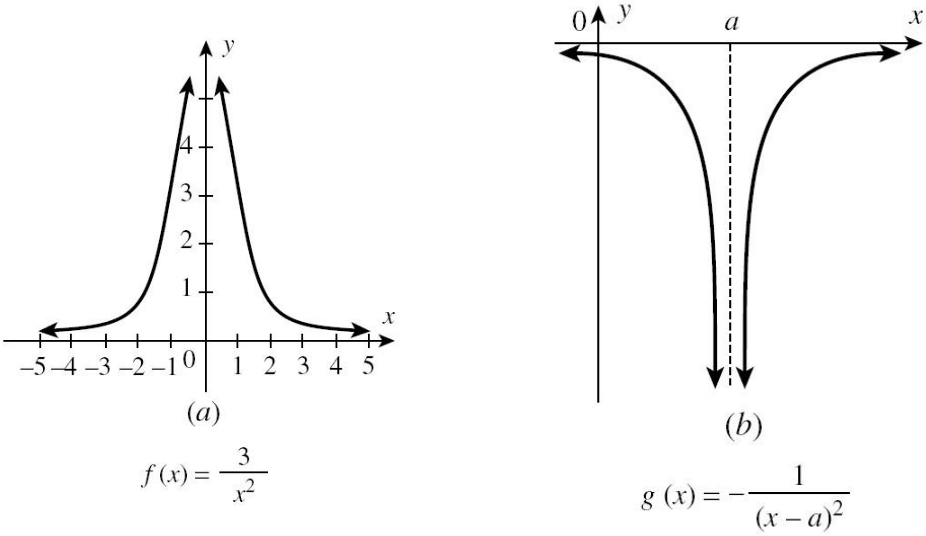


Fig. 2.5

Horizontal Asymptotes

The graph of the function $y = f(x)$ for $x \to \infty$ or for $x \to -\infty$, has a horizontal asymptote, if $\lim_{x\to\infty} f(x) = b$ Or $\lim_{x\to-\infty} f(x) = b$, where b is a finite number. It may happen that either only one or none of these limits is finite. Then, the graph has either one or no horizontal asymptote. Of course, the graph of a function may have two horizontal asymptotes. The equation of the horizontal asymptote has the form $y = a$. (In Figure 2.6a, it is $y = b$, and in Figure 2.6b the two asymptotes are $y = \pm 1$.)

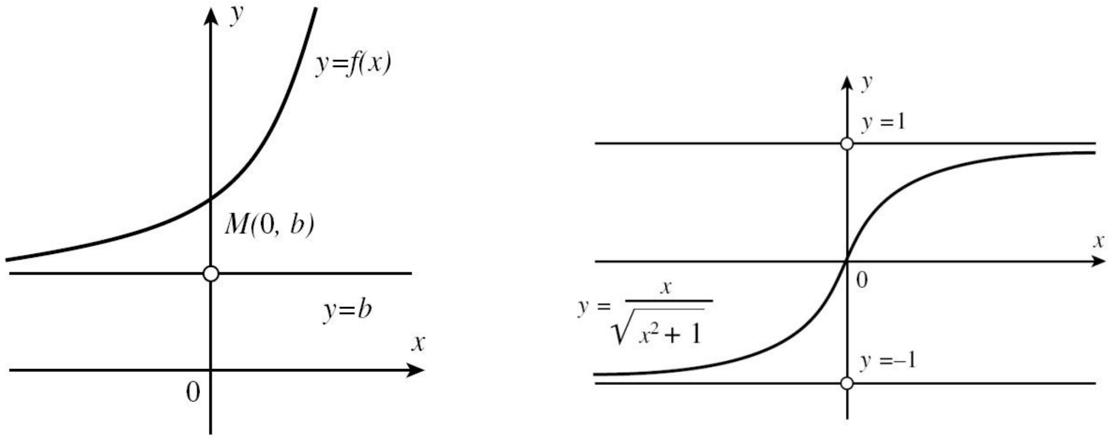


Fig. 2.6

Example (31)
Find the asymptotes to the curve

$$y = \frac{1}{x - 3}$$

Solution:
We have

$$\lim_{x \to \pm\infty} \frac{1}{x - 3} = 0$$

Therefore, the curve has a horizontal asymptote at $y = 0$
Further, we observe that

$$\lim_{x \to 3^-} \frac{1}{x - 3} = -\infty$$

and

$$\lim_{x \to 3^+} \frac{1}{x - 3} = \infty$$

Hence, the curve has a vertical asymptote at $x = 3$ (see Figure 2.7).

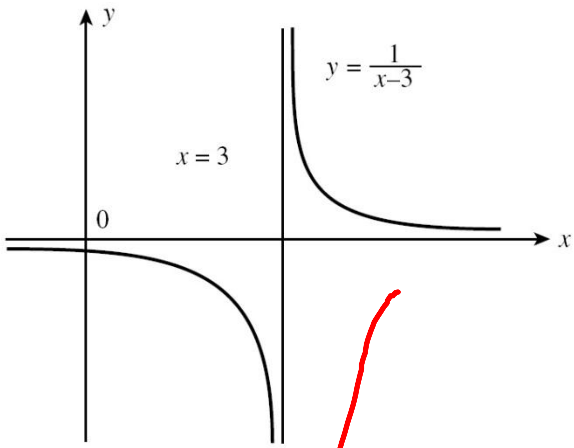


Fig. 2.7

Limit at Infinity

The symbol for infinity is "∞ ". In modern mathematics, the symbol "∞ " is not a number, and not all algebraic operations are defined for this symbol.

Often we shall have to study the behavior of functions of x, as x becomes infinitely large, that is, when x is permitted to assume larger and larger values exceeding any bound K, no matter how big K is chosen.

For example, take

$$f(n) = \frac{1}{n}.$$

Then if n takes the values 1, 2, 3, . . ., 100, the class, or set, consisting of the values of $f(n)$, for various values of n consisting of the fractions(1, 1/2, 1/3, . . . 1/100).

We wish to discuss the behavior of this function for very large values of n. It is immediately obvious that $f(n) = \frac{1}{n}$ becomes very small when n is very large.

Note: It is wrong to say that $\frac{1}{n} = 0$ when $n = \infty$. Remember that ∞ is not a number, so it cannot be equated to any number, howsoever large. Further, $\frac{1}{n}$ can never be equated to zero, however big n is chosen. However, it makes sense to say that the function $f(n) = \frac{1}{n}$ tends to zero for values of n that tend to infinity.

If we now consider the function

$$f(n) = n^2,$$

it is clear that this function can be made as large as we please by taking sufficiently large values of n. We may therefore, say that the function

$$f(n) = n^2$$

tends to infinity when n tends to infinity.

Now, let us consider the function

$f(n) = -n^2$

In this case, we say that f(n) tends to $-\infty$ when n tends to ∞ . We would usually write these statements briefly as given below:

$$n^2 \to \infty \quad \text{as} \quad n \to \infty$$

$$-n^2 \to -\infty \quad \text{as} \quad n \to \infty$$

Consider the function

$$f(x) = \frac{x}{1 + x^2}$$

We ask the question:

What happens to $f(x)$ as x gets larger and larger? In symbols, we ask for the value $\lim_{x \to \infty} f(x)$

We use the symbol $x \to \infty$ as a shorthand way of saying that x gets larger and larger without bound.

(When we write $x \to \infty$, we are not implying that somewhere far, far to the right on the x-axis, there is a number bigger than all other numbers to which x is approaching. Rather, we use $x \to \infty$ to say that x is permitted to assume larger and larger values endlessly.)

In Table 2.4, we have listed values of $f(x)$, for larger and larger values of x, for several values of x .

x	$f(x)=\frac{x}{x^2+1}$
10	0.099
100	0.010
1000	0.001
10,000	0.0001
$\downarrow$	$\downarrow$
∞	0

Table 2.4

It appears that $f(x)$ gets smaller and smaller as x gets larger and larger. Therefore, we

$$\lim_{x\to\infty}\frac{x}{x^2+1}=0$$

Experimenting with large negative values of x, would again lead us to write

$$\lim_{x\to-\infty}\frac{x}{x^2+1}=0$$

Definitions of Limits $x\to+\infty$

If $f(x)$ gets closer and closer to the value l as x is permitted to assume larger and larger values endlessly(without bound). In symbols, we write

$$\lim_{x\to\infty}f(x)=l$$

Definitions of Limits $x\to-\infty$

If $f(x)$ gets closer and closer to the value l as x is permitted to assume larger and larger negative values endlessly(without bound). In symbols, we write

$$\lim_{x\to-\infty}f(x)=l$$

Simpler and Powerful Rules for Finding Limits
$x \to \pm\infty$

1. $\lim_{x \to \infty} x^n = \infty$
2. $\lim_{x \to \pm\infty} x^n = \infty, \text{ (n is even)}$
3. $\lim_{x \to -\infty} x^n = -\infty, \text{ (n is odd)}$
4. $\lim_{x \to \pm\infty} \frac{1}{x^n} = 0$
5. If $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n$, then
   $\lim_{x \to \pm\infty} f(x) = a_0 \lim_{x \to \pm\infty} x^n$
6. If $f(x) = \dfrac{a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n}{b_0 x^m + b_1 x^{m-1} + b_2 x^{m-2} + \ldots + b_m}$, then
   $\lim_{x \to \pm\infty} f(x) = \dfrac{a_0 \lim_{x \to \pm\infty} x^n}{b_0 \lim_{x \to \pm\infty} x^m}$

Example (31)
Find

$$\lim_{x \to -\infty} \frac{2x^3}{1 + x^3}$$

Solution
Here we use a standard trick: dividing numerator and denominator by the highest power of x that appears in the denominator.

$$\lim_{x \to -\infty} \frac{2x^3}{1 + x^3} = \lim_{x \to -\infty} \frac{2x^3 / x^3}{1/x^3 + x^3/x^3} = \lim_{x \to -\infty} \frac{2}{0 + 1} = 2$$

Exercise

Evaluate the following limits

(i) $\lim_{x \to -\infty} \dfrac{2x^2 - 4x + 5}{3x^3 - x + 7}$

(ii) $\lim_{x \to \infty} \dfrac{(2x-1)^{20}(3x-1)^{30}}{(2x+1)^{50}}$

(iii) $\lim_{x \to \infty} \left(\sqrt{x+1} - \sqrt{x} \right)$

Limits of Trigonometric Functions

We shall be using the following basic trigonometric limits:

$$\text{(i)}\lim_{x\to 0}\sin x = 0 \qquad \text{(ii)}\lim_{x\to 0}\cos x = 1$$

$$\text{(iii)}\lim_{x\to 0}\frac{\sin x}{x} = 1 \qquad \text{(iv)}\lim_{x\to 0}\frac{\cos x - 1}{x} = 0$$

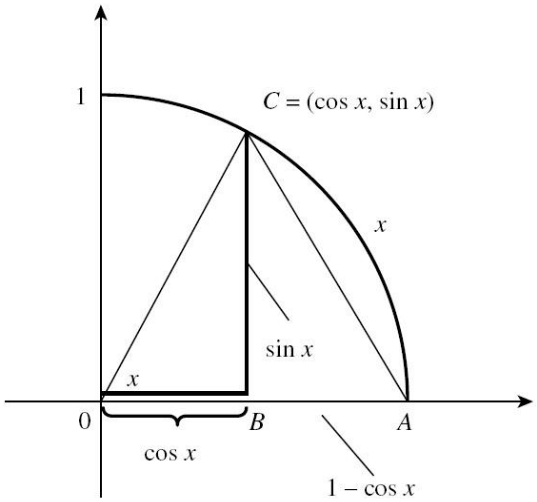


Fig.2.8

In Figure 2.8, let C be any point on the unit circle (placed in the standard position) such that it is at the end of the arc length x. Since this arc length subtends an angle of x radians at the center, we identify the point C as a function of the angle x . We recall the definitions of the sine and cosine functions as follows:

$$\sin x = y - \text{coordinate of } C$$

$$\cos x = x - \text{coordinate of } C$$

Since C ($\cos x, \sin x$) can move endlessly around the unit circle (with positive or negative arc length), the domain of both sine and cosine functions is $(-\infty, \infty)$. The largest value either function may have is 1 and the smallest value is -1 . Also, observe that both these functions assume all values between -1 and 1. Hence, the range of both the functions is $\left[-1,1\right]$.

Note that as $x \to 0$, the point P($\cos x$,$\sin x$) moves toward (1, 0) so that we get

$$\lim_{x \to 0} \cos x = 1, \ \lim_{x \to 0} \sin x = 0$$

Thus, we have shown the correctness of the results (i) and (ii). Now, onward, we shall be using results (i) and (ii) freely in solving problems and obtaining other results.

Now, our next goal is to show that for any real number "a",

$$\lim_{x \to a} \sin x = \sin a$$

and

$$\lim_{x \to a} \cos x = \cos a$$

We know that, if "a" is a fixed number and $x = a + h$, then

$\lim_{x \to a} f(x) = l$ if and only if $\lim_{h \to 0} f(a+h) = l$

Therefore, in order to prove the result(s) at (1) above, we can instead show that

$\lim_{h \to 0} \sin(a+h) = \sin a$ and $\lim_{h \to 0} \cos(a+h) = \cos a$

So,

$$\begin{aligned}\lim_{h \to 0} \sin(a+h) &= \lim_{h \to 0} [\sin a \cos h + \cos a \sin h] \\ &= \sin a \lim_{h \to 0} \cos h + \cos a \lim_{h \to 0} \sin h \\ &= \sin a\end{aligned}$$

And

$$\begin{aligned}\lim_{h \to 0} \cos(a+h) &= \lim_{h \to 0} [\cos a \cos h - \sin a \sin h] \\ &= \cos a \lim_{h \to 0} \cos h + \sin a \lim_{h \to 0} \sin h \\ &= \cos a\end{aligned}$$

To prove (iii) , ($\lim_{x \to 0} \frac{\sin x}{x} = 1$) consider a unit circle with center “O”, placed at the origin, and let the radian measure of angle AOC be x radians (Figure 2.9).
Using Figure 3a.4, we obtain the following equations, which are valid for $0 < x < \frac{\pi}{2}$.

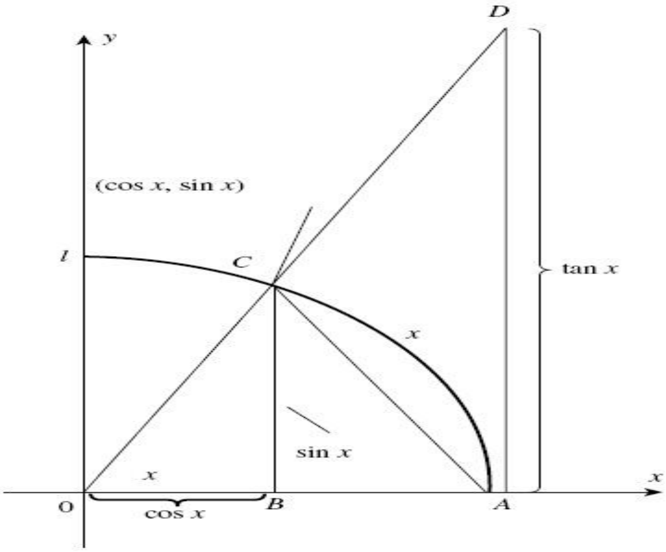


Fig. 2.9

Area of triangle OAC

$$= \frac{1}{2}|OA|.|BC| = \frac{1}{2}.1.\sin x = \frac{\sin x}{2}$$

Area of sector

$$= \frac{1}{2}x.r^2 = \frac{x}{2}$$

Area of triangle OAD

$$= \frac{1}{2}|OA|.|AD| = \frac{1}{2}.1.\tan x = \frac{\tan x}{2}$$

It is geometrically clear that

Area of $\Delta OAC \le$ area of sector $OAC \le$ area of ΔOAD

So that,

$$\frac{\sin x}{2} \le \frac{x}{2} \le \frac{\tan x}{2}$$

So, we have

$$\cos x \le \frac{\sin x}{x} \le 1$$

But $\lim_{x \to 0} \cos x = 1$ and $\lim_{x \to 0} 1 = 1$, it follows from the squeezing theorem

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0} \frac{\cos x - 1}{x} = \lim_{x \to 0} \frac{\cos x - 1}{x} . \frac{\cos x + 1}{\cos x + 1}$$

$$= \lim_{x \to 0} \frac{\cos^2 x - 1}{x(\cos x + 1)} = \lim_{x \to 0} \frac{-\sin^2 x}{x(\cos x + 1)}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} . \lim_{x \to 0} \frac{-\sin x}{\cos x + 1} = 1 . \frac{0}{1+1} = 0$$

Corollaries

(i) $\lim_{x \to 0} \frac{x}{\sin x} = 1$ (ii) $\lim_{x \to 0} \frac{\tan x}{x} = 1$ (iii) $\lim_{x \to 0} \frac{x}{\tan x} = 1$

(iv) $\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$ (v) $\lim_{x \to 0} \frac{x^2}{1 - \cos x} = 2$

Ex

Proposition:

If $f(x)$ is a bounded function, and if $\lim_{x \to a} g(x) = 0$

Then,

$$\lim_{x \to a} f(x).g(x) = 0$$

Example (32)
Evaluate

$$\lim_{x \to \infty} \frac{\sin x}{x}$$

Solution
Note that $-1 \le \sin x \le 1$ for all x

$\therefore \sin x$ is a bounded function. Also $\lim_{x \to \infty} \frac{1}{x} = 0$

$$\lim_{x \to \infty} \frac{\sin x}{x} = \lim_{x \to \infty} \frac{1}{x} . \lim_{x \to \infty} \sin x = 0$$

Example (33)
Evaluate

$$\lim_{x \to 0} \frac{\sin(1/x)}{1/x}$$

Solution

$$\lim_{x \to 0} \frac{\sin(1/x)}{1/x} = \lim_{x \to 0} x . \sin \frac{1}{x}$$

We know that
$-1 \le \sin x \le 1$ for all x

$$\therefore \sin \frac{1}{x} \text{ is a bounded function.}$$

Next,

$$\lim_{x \to 0} x = 0$$

$$\therefore \lim_{x \to 0} x \sin(1/x) = 0$$

$$\therefore \lim_{x \to 0} \frac{\sin(1/x)}{1/x} = 0$$

Example (34)
Evaluate

$$\lim_{x \to 0} \frac{\sin 3x}{x}$$

Solution

$$\lim_{x \to 0} \frac{\sin 3x}{x} = \lim_{x \to 0} \frac{\sin 3x}{3x}.3$$

Note that as $x \to 0$, $3x \to 0$. If we put $3x = t$, we get the given limit as

$$\lim_{t \to 0} \frac{\sin t}{t}.3 = \lim_{t \to 0} \frac{\sin t}{t}.\lim_{t \to 0} 3 = 1.3 = 3$$

Example (35)
Evaluate

$$\lim_{x \to 0} \frac{x \cos x + \sin x}{x + \tan x}$$

Solution

$$\lim_{x \to 0} \frac{x \cos x + \sin x}{x + \tan x} = \lim_{x \to 0} \frac{\cos x + \dfrac{\sin x}{x}}{1 + \dfrac{\tan x}{x}}$$

$$= \frac{1+1}{1+1} = 1$$

Example (36)
Evaluate

$$\lim_{x \to 0} \frac{\operatorname{cosec} 2x - \cot 2x}{\sin x}$$

Solution

$$\lim_{x \to 0} \frac{\operatorname{cosec} 2x - \cot 2x}{\sin x} = \lim_{x \to 0} \frac{\dfrac{1}{\sin 2x} - \dfrac{\cos 2x}{\sin 2x}}{\sin x}$$

$$= \lim_{x \to 0} \frac{1 - \cos 2x}{\sin 2x \ \sin x} = \lim_{x \to 0} \frac{2 \sin^2 x}{2 \sin^2 x \ \cos x}$$

$$= \lim_{x \to 0} \frac{1}{\cos x} = 1$$

Example (37)

Evaluate

$$\lim_{x \to 0} \frac{\sqrt{2} - \sqrt{1 + \cos 2x}}{\sin^2 x}$$

Solution

$$\lim_{x \to 0} \frac{\sqrt{2} - \sqrt{1 + \cos 2x}}{\sin^2 x}$$

$$= \lim_{x \to 0} \frac{\sqrt{2} - \sqrt{1 + \cos 2x}}{\sin^2 x} \cdot \frac{\sqrt{2} + \sqrt{1 + \cos 2x}}{\sqrt{2} + \sqrt{1 + \cos 2x}}$$

$$= \lim_{x \to 0} \frac{2 - 1 - \cos 2x}{\sin^2 x \ (\sqrt{2} + \sqrt{1 + \cos 2x}\ )}$$

$$= \lim_{x \to 0} \frac{2 \sin^2 x}{\sin^2 x \ (\sqrt{2} + \sqrt{1 + \cos 2x}\ )}$$

$$= \lim_{x \to 0} \frac{2}{\sqrt{2} + \sqrt{1 + \cos 2x}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Example (38)

Evaluate

$$\lim_{x \to 0} \frac{1 - \cos 4x}{x^2}$$

Solution

$$\lim_{x \to 0} \frac{1 - \cos 4x}{x^2} = \lim_{x \to 0} \frac{1 - \cos 4x}{x^2} \cdot \frac{1 + \cos 4x}{1 + \cos 4x}$$

$$=\lim_{x\to 0}\frac{1-\cos^2 4x}{x^2(1+\cos 4x)}=\lim_{x\to 0}\frac{\sin^2 4x}{x^2(1+\cos 4x)}$$

$$=\lim_{x\to 0}\frac{\sin^2 4x}{(4x)^2}.16.\lim_{x\to 0}\frac{1}{1+\cos 4x}=1.16.\frac{1}{2}=8$$

Example (39)
Evaluate

$$\lim_{x\to 0}\frac{3\sin x-\sin 3x}{x^3}$$

Solution

Since $\sin 3x=3\text{sinx}-4\sin^3 x$

$$\therefore \lim_{x\to 0}\frac{3\sin x-\sin 3x}{x^3}=\lim_{x\to 0}\frac{4\sin^3 x}{x^3}=4$$

Example (40)
Evaluate

$$\lim_{x\to 0}\frac{\cos ax-\text{cosbx}}{\cos cx-\cos dx}$$

Solution
Since

$$\cos x-\cos y=-2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

$$\lim_{x\to 0}\frac{\cos ax-\cos bx}{\cos cx-\cos dx}$$

$$=\lim_{x\to 0}\frac{-2\sin\frac{(a+b)x}{2}\sin\frac{(a-b)x}{2}}{-2\sin\frac{(c+d)x}{2}\sin\frac{(c-d)x}{2}}$$

$$=\left[\lim_{x\to 0}\frac{\sin\frac{(a+b)x}{2}\sin\frac{(a-b)x}{2}}{\frac{(a+b)x}{2}\,\frac{(a-b)x}{2}}\right]$$

$$\times\left[\lim_{x\to 0}\frac{\frac{(c+d)x}{2}\,\frac{(c-d)x}{2}}{\sin\frac{(c+d)x}{2}\sin\frac{(c-d)x}{2}}\right]$$

$$\times\lim_{x\to 0}\left[\frac{\frac{(a+b)x}{2}\,\frac{(a-b)x}{2}}{\frac{(c+d)x}{2}\,\frac{(c-d)x}{2}}\right]$$

$$=\frac{a^2-b^2}{c^2-d^2}$$

Example (41)
Evaluate

$$\lim_{x\to a}\frac{\sin x-\sin a}{\sqrt{x}-\sqrt{a}}$$

Solution

$$\lim_{x\to a}\frac{\sin x-\sin a}{\sqrt{x}-\sqrt{a}}=\lim_{x\to a}\frac{\sin x-\sin a}{\sqrt{x}-\sqrt{a}}\cdot\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}$$

$$=\lim_{x\to a}\frac{(\sin x-\sin a)(\sqrt{x}+\sqrt{a})}{x-a}$$

Let $x-a=t \quad \Rightarrow x=t+a$

As $x\to a,\ t\to 0$

$$\lim_{x \to a} \frac{(\sin x - \sin a)(\sqrt{x} + \sqrt{a})}{x - a}$$

$$= \lim_{t \to 0} \frac{\left[\sin(t + a) - \sin a\right](\sqrt{t + a} + \sqrt{a})}{t}$$

But

$$\sin x - \sin y = 2\cos(\frac{x + y}{2})\sin(\frac{x - y}{2})$$

So,

$$\lim_{t \to 0} \frac{\left[2\cos(\frac{t}{2} + a)\sin\frac{t}{2}\right](\sqrt{t + a} + \sqrt{a})}{t}$$

$$= \lim_{t \to 0} \cos(\frac{t}{2} + a).\frac{\sin t / 2}{t / 2}.(\sqrt{t + a} + \sqrt{a})$$

$$= \cos(\mathrm{a} + 0).1.(2\sqrt{a}) = 2\sqrt{a}\cos a$$

Example (42)
Evaluate

$$\lim_{x \to \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2}$$

Solution
Put $x - \pi = t$
Note that $x \to \pi, t \to 0$

$$\lim_{x \to \pi} \frac{\sqrt{2+\cos x}-1}{(\pi - x)^2} = \lim_{t \to 0} \frac{\sqrt{2+\cos(t+\pi)}-1}{t^2}$$

$$= \lim_{t \to 0} \frac{\sqrt{2-\cos t}-1}{t^2} = \lim_{t \to 0} \frac{\sqrt{2-\cos t}-1}{t^2} . \frac{\sqrt{2-\cos t}+1}{\sqrt{2-\cos t}+1}$$

$$= \lim_{t \to 0} \frac{1-\cos t}{t^2 \sqrt{2-\cos t}+1} = \lim_{t \to 0} \frac{1-\cos t}{t^2} . \lim_{t \to 0} \frac{1}{\sqrt{2-\cos t}+1}$$

$$= \frac{1}{2} . \frac{1}{2} = \frac{1}{4}$$

Example (43)

Evaluate

$$\lim_{x \to \frac{\pi}{4}} \frac{1-\tan x}{1-\sqrt{2}\sin x}$$

Solution

$$\lim_{x \to \frac{\pi}{4}} \frac{1-\tan x}{1-\sqrt{2}\sin x} = \lim_{x \to \frac{\pi}{4}} \frac{\cos x - \sin x}{\cos x} . \frac{1}{1-\sqrt{2}\sin x}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\cos x - \sin x}{\cos x} . \frac{1}{1-\sqrt{2}\sin x} . \frac{1+\sqrt{2}\sin x}{1+\sqrt{2}\sin x}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\cos x - \sin x}{\cos x} . \frac{1+\sqrt{2}\sin x}{\cos^2 x - \sin^2 x}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{1+\sqrt{2}\sin x}{\cos x \left(\cos x + \sin x\right)} = \frac{2}{1} = 2$$

Example (44)

Evaluate

$$\lim_{x \to 1} \frac{x^2 - 3x + 2}{x^2 - x + \sin(x-1)}$$

Solution

$$\lim_{x \to 1} \frac{x^2 - 3x + 2}{x^2 - x + \sin(x-1)}$$
$$= \lim_{x \to 1} \frac{(x-1)(x-2)}{x(x-1) + \sin(x-1)}$$

Put $x - 1 = t \Rightarrow \text{as} \, x \to 1, \ t \to 0$

$$\lim_{x \to 1} \frac{(x-1)(x-2)}{x(x-1) + \sin(x-1)}$$
$$= \lim_{t \to 0} \frac{t(t-1)}{(t+1)(t) + \sin t}$$
$$= \lim_{t \to 0} \frac{t-1}{t+1+\sin t / t} = \frac{-1}{0+1+1} = -\frac{1}{2}$$

Limits of exponential and logarithmic functions

(i) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

(ii) $\lim_{x \to 0} (1+x)^{1/x} = e$

(iii) $\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^{x/a} = e$

If $f(x) \to 0$, as $x \to 0$, then

(iv) $\lim_{x \to 0} \left(1 + kf(x)\right)^{\frac{1}{kf(x)}} = e, \ k \neq 0$

If $f(x) \to \infty$, as $x \to \infty$, then

(v) $\lim_{x \to \infty} \left(1 + \frac{1}{kf(x)}\right)^{kf(x)} = e, \ k \neq 0$

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e \quad and \quad \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

It follows that

$$\text{(vi)} \lim_{x\to 0}\frac{a^x - 1}{x} = \ln a$$

By replacing a with e in (vi), we get

$$\text{(vii)} \lim_{x\to 0}\frac{e^x - 1}{x} = \ln e = 1$$

Let $f(x) \to 0$ as $x \to 0$. If $k \neq 0$, then any number $t = k.f(x) \to 0$ as $x \to 0$.

We have

$$\text{(viii)} \lim_{x\to 0}\frac{a^{k.f(x)} - 1}{k.f(x)} = \lim_{t\to 0}\frac{a^t - 1}{t} = \ln a$$

Example (45)

$$\lim_{x\to 0}\left(\frac{3+2x}{3-2x}\right)^{1/x}$$

Solution

$$\lim_{x\to 0}\left(\frac{3+2x}{3-2x}\right)^{1/x} = \lim_{x\to 0}\left(\frac{1+\frac{2x}{3}}{1-\frac{2x}{3}}\right)^{1/x}$$

$$= \frac{\lim_{x\to 0}\left(1+\frac{2x}{3}\right)^{1/x}}{\lim_{x\to 0}\left(1-\frac{2x}{3}\right)^{1/x}} = \lim_{x\to 0}\left(1+\frac{2x}{3}\right)^{1/x} . \lim_{x\to 0}\left(1-\frac{2x}{3}\right)^{-1/x}$$

First consider

$$\lim_{x\to 0}\left(1+\frac{2x}{3}\right)^{1/x}$$

(If we put $\frac{2x}{3}=t$, then $\frac{3}{2x}=\frac{1}{t}$. Furthermore, note that as $x\to 0, t\to 0$ and $\frac{1}{t}\to\infty$.)

$$\lim_{x\to 0}\left(1+\frac{2x}{3}\right)^{1/x} = \left[\lim_{x\to 0}\left(1+\frac{2x}{3}\right)^{3/2x}\right]^{2/3}$$

$$=\left[\lim_{t\to 0}(1+t)^{1/t}\right]^{2/3} = e^{2/3}$$

Next, consider,

$$\lim_{x\to 0}\left(1-\frac{2x}{3}\right)^{-1/x}$$

$$=\left[\lim_{x\to 0}\left(1-\frac{2x}{3}\right)^{-3/2x}\right]^{2/3} = \left[\lim_{t\to 0}(1-t)^{-1/t}\right]^{2/3} = e^{2/3}$$

Thus,

$$\lim_{x\to 0}\left(\frac{3+2x}{3-2x}\right)^{1/x} = e^{4/3}$$

Example (46)

$$\lim_{x\to\infty}\left(\frac{2x+3}{2x-1}\right)^{x+1}$$

Solution

$$\lim_{x\to\infty}\left(\frac{2x+3}{2x-1}\right)^{x+1} = \lim_{x\to\infty}\left(\frac{2x+3}{2x-1}\right)^{x} . \lim_{x\to\infty}\left(\frac{2x+3}{2x-1}\right)$$

$$= \lim_{x \to \infty} \left(\frac{1 + \frac{3}{2x}}{1 - \frac{1}{2x}} \right)^x . \lim_{x \to \infty} \left(\frac{2 + \frac{3}{x}}{2 - \frac{1}{x}} \right)$$

$$= \lim_{x \to \infty} \left(\frac{1 + \frac{3}{2x}}{1 - \frac{1}{2x}} \right)^x = \frac{\lim_{x \to \infty} \left(1 + \frac{3}{2x} \right)^x}{\lim_{x \to \infty} \left(1 - \frac{1}{2x} \right)^x}$$

$$= \lim_{x \to \infty} \left(1 + \frac{3}{2x} \right)^x . \lim_{x \to \infty} \left(1 - \frac{1}{2x} \right)^{-x}$$

First consider

$$\lim_{x \to \infty} \left(1 + \frac{3}{2x} \right)^x$$

(If we put $\frac{3}{2x} = t$, then $\frac{2x}{3} = \frac{1}{t}$. Furthermore, note that as $x \to \infty, t \to 0$.)

$$\lim_{x \to \infty} \left(1 + \frac{3}{2x} \right)^x = \left[\lim_{x \to \infty} \left(1 + \frac{3}{2x} \right)^{2x/3} \right]^{3/2}$$

$$= \left[\lim_{t \to \infty} (1 + t)^{1/t} \right]^{3/2} = e^{3/2}$$

Next, consider

$$\lim_{x \to \infty} \left(1 - \frac{1}{2x} \right)^{-x} = \left[\lim_{x \to \infty} \left(1 - \frac{1}{2x} \right)^{-2x} \right]^{1/2}$$

$$= e^{1/2}$$

Thus

$$\lim_{x \to \infty} \left(\frac{2x+3}{2x-1} \right)^{x+1} = e^{\frac{3}{2}+\frac{1}{2}} = e^{2}$$

Example (47)

Evaluate

$$\lim_{x \to 1} x^{1/(x-1)}$$

Solution

Put $x-1=t$. Therefore, $x=1+t$. Note that, as $x \to 1, t \to 0$,

$$\lim_{x \to 1} x^{1/(x-1)} = \lim_{t \to 0} (1+t)^{1/t} = e$$

Example (48)

Evaluate;

$$\lim_{x \to 4} (x-3)^{1/(x-4)}$$

Solution

Put $x-4=t$. Therefore, $x=t+4$. Note that, as $x \to 4, t \to 0$,

$$\lim_{x \to 4} (x-3)^{1/(x-4)} = \lim_{t \to 0} (1+t)^{1/t} = e$$

Example (49)

Evaluate

$$\lim_{x \to 3} \frac{\ln x - \ln 3}{x-3}$$

Solution

Put $x-3=t$. Therefore, $x=t+3$. Note that as $x \to 3, t \to 0$ 0.

Thus,

$$\lim_{x \to 3} \frac{\ln x - \ln 3}{x - 3} = \lim_{t \to 0} \frac{\ln(t+3) - \ln 3}{t}$$

$$= \lim_{t \to 0} \frac{1}{t} \ln\left(\frac{t+3}{3}\right) = \lim_{t \to 0} \ln\left(1 + \frac{t}{3}\right)^{1/t}$$

$$= \ln\left[\lim_{t \to 0}\left(1 + \frac{t}{3}\right)^{3/t}\right]^{1/3} = \ln e^{1/3} = \frac{1}{3}$$

Example (50)
Evaluate

$$\lim_{x \to e} \frac{\ln x - 1}{x - e}$$

Solution
Let
Put $x - e = t$. Therefore, $x = t + e$. Also, note that as $x \to e, t \to 0$

$$\lim_{x \to e} \frac{\ln x - 1}{x - e} = \lim_{t \to 0} \frac{\ln(t + e) - \ln e}{t}$$

$$= \lim_{t \to 0} \frac{1}{t} \ln\left(1 + \frac{t}{e}\right) = \lim_{t \to 0} \ln\left(1 + \frac{t}{e}\right)^{1/t}$$

$$= \ln\left[\lim_{t \to 0}\left(1 + \frac{t}{e}\right)^{e/t}\right]^{1/e} = \ln e^{1/e} = \frac{1}{e}$$

Example (51)
Evaluate

$$\lim_{x \to 0} \frac{\ln 10 + \ln(x + 0.1)}{x}$$

Solution
Consider

$$\ln 10 + \ln(x + 0.1) = \ln 10 + \ln\left(\frac{10x + 1}{10}\right)$$

$$= \ln 10 + \ln\left(10x + 1\right) - \ln 10 = \ln\left(10x + 1\right)$$

Therefore, the given limit can be expressed in the form

$$\lim_{x \to 0} \frac{\ln 10 + \ln(x + 0.1)}{x} = \lim_{x \to 0} \frac{1}{x} \ln\left(10x + 1\right)$$

$$= \lim_{x \to 0} \ln\left(10x + 1\right)^{1/x} = \ln\left[\lim_{x \to 0}\left(10x + 1\right)^{1/(10x)}\right]^{10}$$

$$= \ln e^{10} = 10$$

Example (52)

Evaluate

$$\lim_{x \to 0} \frac{a^x - b^x}{x}$$

Solution

$$\lim_{x \to 0} \frac{a^x - b^x}{x} = \lim_{x \to 0} \frac{(a^x - 1) - (b^x - 1)}{x}$$

$$= \lim_{x \to 0} \frac{(a^x - 1)}{x} - \lim_{x \to 0} \frac{(b^x - 1)}{x} = \ln a - \ln b = \ln \frac{a}{b}$$

Example (53)

Evaluate

$$\lim_{x \to 0} \frac{3^{8x} - 1}{x}$$

Solution

Put $8x = t$. Then, $x \to 0, \quad t \to 0$.

$$\lim_{x \to 0} \frac{3^{8x} - 1}{x} = \lim_{t \to 0} \frac{3^t - 1}{t / 8}$$

$$= 8.\lim_{t \to 0} \frac{3^t - 1}{t} = 8 \ln 3$$

Example (54)
Evaluate

$$\lim_{x\to 0}\frac{e^{x}-e^{-x}}{\sin x}$$

Solution

$$\lim_{x\to 0}\frac{e^{x}-e^{-x}}{\sin x}=\lim_{x\to 0}\frac{e^{2x}-1}{e^{x}\sin x}$$

$$=\lim_{x\to 0}\frac{e^{2x}-1}{2x}.\frac{x}{\sin x}.\frac{2}{e^{x}}=\ln e\ .1.2=2$$

Example (55)
Evaluate

$$\lim_{x\to 0}\frac{(ab)^{x}-a^{x}-b^{x}+1}{x^{2}}$$

Solution
Consider

$$(ab)^{x}-a^{x}-b^{x}+1$$

$$=a^{x}b^{x}-a^{x}-b^{x}+1$$

$$=a^{x}\left(b^{x}-1\right)-\left(b^{x}-1\right)$$

$$=\left(a^{x}-1\right).\left(b^{x}-1\right)$$

The required limit

$$\lim_{x\to 0}\frac{(ab)^{x}-a^{x}-b^{x}+1}{x^{2}}=\lim_{x\to 0}\frac{\left(a^{x}-1\right).\left(b^{x}-1\right)}{x^{2}}$$

$$=\lim_{x\to 0}\frac{\left(a^{x}-1\right)}{x}.\lim_{x\to 0}\frac{\left(b^{x}-1\right)}{x}=\ln a.\ln b$$

Example (56)
Evaluate

$$\lim_{x\to 0}\frac{a^{x}+a^{-x}-2}{x^{2}}$$

Solution

Consider

$$a^x + a^{-x} - 2 = \frac{a^{2x} + 1 - 2a^x}{a^x}$$

$$= \frac{a^{2x} - 2a^x + 1}{a^x} = \frac{\left(a^x - 1\right)^2}{a^x}$$

The required limit

$$\lim_{x \to 0} \frac{a^x + a^{-x} - 2}{x^2} = \lim_{x \to 0} \frac{\left(a^x - 1\right)^2}{a^x . x^2}$$

$$= \lim_{x \to 0} \left(\frac{a^x - 1}{x} \right)^2 . \lim_{x \to 0} \frac{1}{a^x} = \left(\ln a\right)^2 .1 = \left(\ln a\right)^2$$

Example (57)

Evaluate

$$\lim_{x \to 0} \frac{3^{5x} - 1}{\tan 3x}$$

Solution

$$\lim_{x \to 0} \frac{3^{5x} - 1}{\tan 3x} = \lim_{x \to 0} \frac{3^{5x} - 1}{5x} . \frac{5x}{\tan 3x}$$

$$= \frac{5}{3} \lim_{x \to 0} \frac{3^{5x} - 1}{5x} . \frac{3x}{\tan 3x} = \frac{5}{3} \ln 3.1 = \frac{5}{3} \ln 3$$

Example (58)

Evaluate

$$\lim_{x \to 0} \frac{12^x + 4^x - 3^x - 1}{x}$$

Solution

$$\lim_{x \to 0} \frac{12^x + 4^x - 3^x - 1}{x}$$

$$= \lim_{x \to 0} \frac{(12^x - 1) + (4^x - 1) - (3^x - 1)}{x}$$

$$= \lim_{x \to 0} \frac{(12^x - 1)}{x} + \lim_{x \to 0} \frac{(4^x - 1)}{x} - \lim_{x \to 0} \frac{(3^x - 1)}{x}$$

$$= \ln 12 + \ln 4 - \ln 3 = \ln 16$$

Example (59)

Evaluate

$$\lim_{x \to 0} \frac{12^x - 4^x - 3^x + 1}{x \sin x}$$

Solution

$$\lim_{x \to 0} \frac{12^x - 4^x - 3^x + 1}{x \sin x} = \lim_{x \to 0} \frac{4^x . 3^x - 4^x - 3^x + 1}{x \sin x}$$

$$= \lim_{x \to 0} \frac{4^x . 3^x - 4^x - 3^x + 1}{x \sin x} = \lim_{x \to 0} \frac{4^x \left(3^x - 1\right) - \left(3^x - 1\right)}{x \sin x}$$

$$= \lim_{x \to 0} \frac{\left(4^x - 1\right) . \left(3^x - 1\right)}{x \sin x} . \frac{x}{x}$$

$$= \lim_{x \to 0} \frac{4^x - 1}{x} . \lim_{x \to 0} \frac{3^x - 1}{x} . \lim_{x \to 0} \frac{x}{\sin x}$$

$$= \ln 4 . \ln 3 . 1 = \ln 4 . \ln 3$$

Chapter 3
Continuity of Real Functions

Introduction

We can introduce the concept of continuity proceeding from a graphic representation of a function.

A function is continuous if its graph is unbroken, i.e., free from sudden jumps or gaps.

Suppose a function is defined on an interval I . We say that the function is continuous on the interval I , if its graph consists of one continuous curve, so that it can be drawn without lifting the pencil. There is no break in any of the graphs of continuous functions (Figure 3.1a-b).

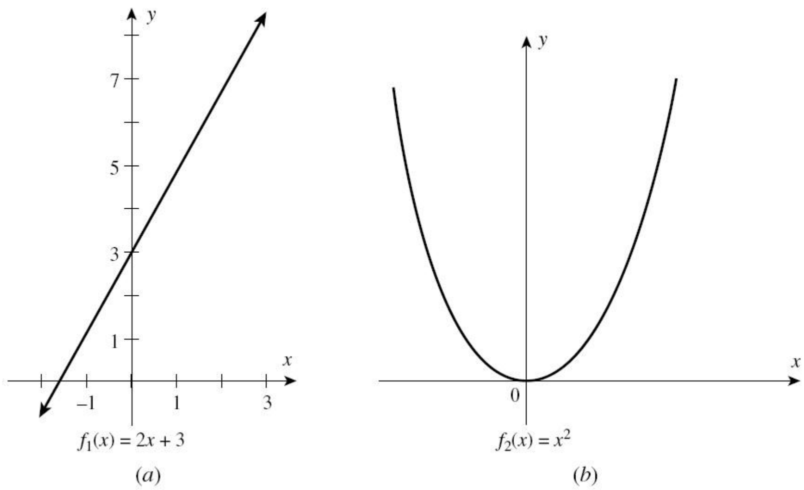


Fig. 3.1

If the graph of a function is broken at any point "a" of an interval, we say that the function is not continuous (or that it is discontinuous) at “a”.

The Natural Domain

If the domain of the given function is not specified, we take the domain as the largest set of real numbers for which the rule of the function makes sense and gives real-number values. This is called the natural domain of the function.

To understand the concept of continuity better, it is useful to study the following graphs of functions, which represent discontinuous functions.

The graph of the function $f_1(x)$ appears in Figure 3.2a. It consist of all points on the line $y = 2x + 3$, except$(2,5)$. The graph has a break at the point (1, 5). Here $f_1(x)$ is not continuous at $x = 1$ since “1” is not in the domain of $f_1(x)$. We say that $f_1(x)$ is not defined at $x = 1$ .We can

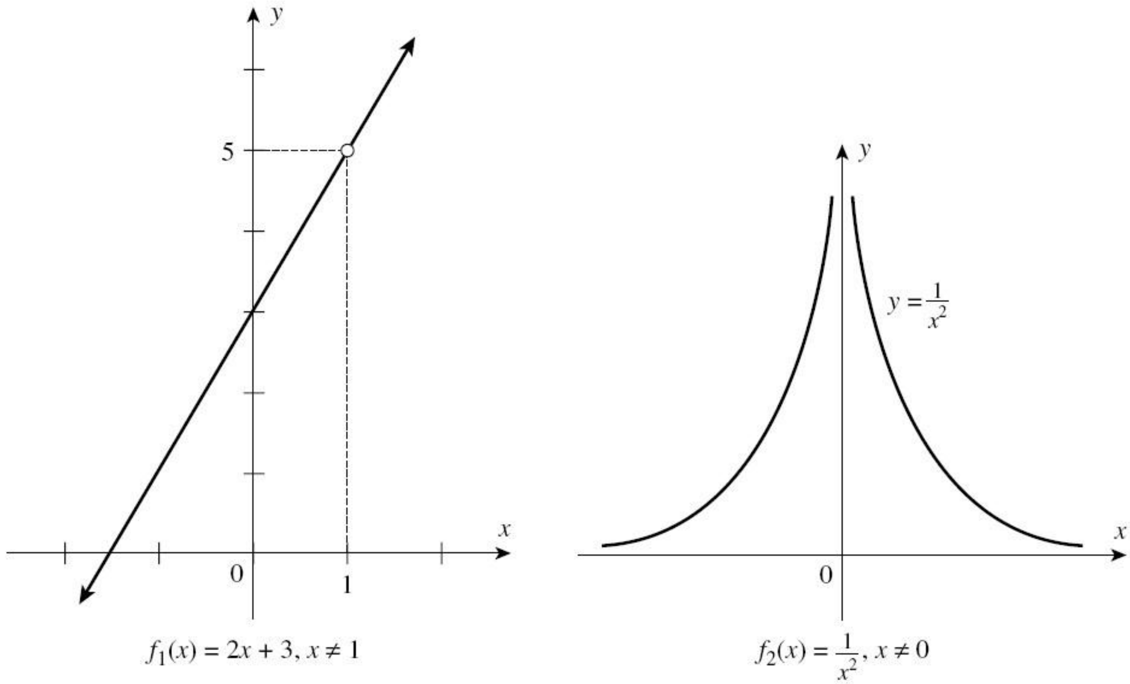


Fig. 3.2

also say that $f_1(x)$ is continuous for all x, except for $x = 1$. It is also correct to say that $f_1(x)$ is discontinuous at $x = 1$(or that it is discontinuous in any interval containing "1").

Now consider the function $f_2(x) = \frac{1}{x^2}$, $x \neq 0$. Its graph appears in the Figure 3.2b. Observe that as $x \to 0$, $\frac{1}{x^2} \to \infty$, which means that $f_2(x)$ does not exist at $x = 0$ or that $f_2(x) = \frac{1}{x^2}$ is not defined at $x = 0$ .We say that in any interval containing "0", the function $f_2(x)$ is discontinuous at the point $x = 0$.

Note

We say that a function $f(x)$ is not defined at $x = a$ if either "a" is not in the domain of $f(x)$ or $f(x) \to \infty$ as $x \to a$.

We give below some more situations when a function may be discontinuous "at a point", in the interval of its definition. The functions $f_3(x)$ is defined for all x . Note that the point (1, 5) is torn out from the graph of $f_3(x)$ and shifted to the location (1, 2). Here, the point (1, 5) of the graph jumps out

from the height 5 to 2, creating a break in the graph at $x = 1$ (Figures 3.3 and 3.4).
The graph of the function $f_4(x)$, shows a break at the point $x = 1$. Here, a portion of the graph has a finite vertical jump at $x = 1$ making the graph discontinuous at $x = 1$.

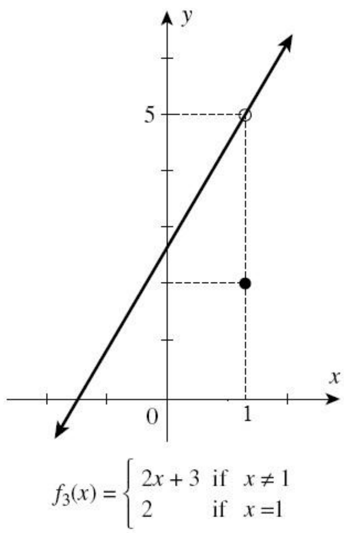


Fig.3.3

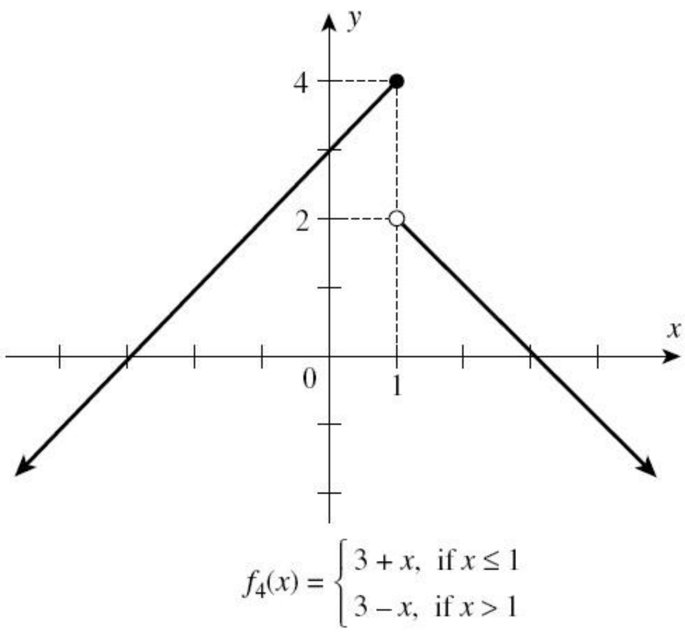


Fig.3.4

Next, consider the graph of the function $f_5(x)$ (Fig. 3.5). The function $f_5(x)$ is not defined at $x = 0$ but it is defined for all other values of x. We observe that as $x \to 0^+$, $\frac{1}{x} \to \infty$, and as $x \to 0^-$, $\frac{1}{x} \to -\infty$. Thus, $f_5(x)$ is discontinuous at the point $x = 0$.

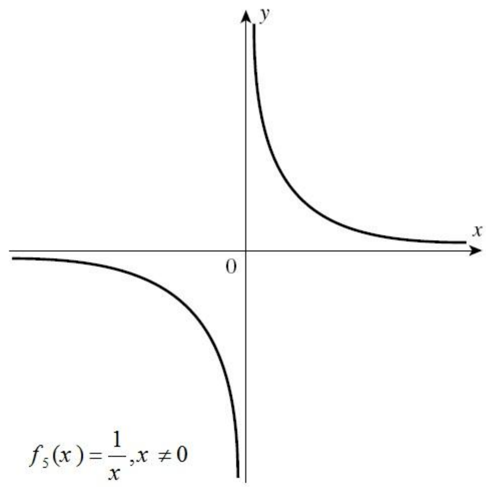


Fig. 3.5

From the above discussion (and the graphs), it is clear that the question of continuity must be considered only for those points, which are in the domain of the function. However, a point of discontinuity may or may not be in the domain of the function.

Definition

Let a function "$f(x)$" be defined in an interval I, and let "a" be any point in I. The function "f" is said to be continuous at the point "a", if and only if the following three conditions are met:

(i) $f(x)$ is defined at $x = a$

(ii) $\lim_{x \to a} f(x)$ exists

(iii) $\lim_{x \to 0} f(x) = f(a)$

In fact, these three conditions of continuity "at a point", are summed up in the following short definition.

A function f(x) is said to be continuous at a point $x = a$, if the limit of the function as $x \to a$, is equal to the value of the function for $x = a$, which we express by the statement,

$$\lim_{x \to a} f(x) = f(a) \qquad (*)$$

There is another way to express continuity of a function at a point "a". In the statement (*), if we replace x by $a + h$, then as $x \to a$, we have $h \to 0$.

Thus, the statement

$$\lim_{h \to 0} f(a+h) = f(a)$$

defines continuity of the function "f " at "a".

Remark

I. f(x) is defined at $x = a$ means, the value f(a) is a finite number.

II. f(x) is not defined at $x = a$ means, either the point $(a, f(a))$ is missing from the graph (which also means that "a" is not in the domain of "f ") or f(a) is not finite [i.e., as $x \to a$, $f(x) \to \pm\infty$].

III. $\lim_{x \to a} f(x)$ exists means $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ and both being finite

Note

It is important to remember that the value f(a) and $\lim_{x \to a} f(x)$ are two different concepts and hence even when both the numbers exist, they may be different. The concept of continuity of the function (at any point $x = a$, in its domain) is based on the existence and equality of these two values, at "a".

Definition [Discontinuity]

We can say that, a function defined on an interval I is discontinuous at a point $a \in I$, if at least one of the following conditions occur at the point $x = a$.

I. The function f(x) is not defined at $x = a$,

II. $\lim_{x \to a} f(x)$ does not exist [which means that $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$ or at least one of the one-sided limits is infinite],

III. $\lim_{x \to a} f(x) \neq f(a)$, in the arbitrary approach of $x \to a$ (which means that the expressions on the right and the left both exist but they are unequal).

One-Sided Continuity

In Chapter 2, the concept of limit of a function was extended to include one-sided limits (and limits involving ∞). The importance of one-sided limits has since been seen in testing the continuity of a function at any point and in identifying the type of discontinuity at that point.

Now, we extend the concept of limit to define the concept of one-sided continuity, which is useful in defining continuity in a closed interval.

Example(1)

Consider the function $f(x)=\sqrt{x}$. We know that the domain of the square root function $f(x)=\sqrt{x}$ is $[0,\infty)$ Therefore, the $\lim_{x\to 0} f(x)$ does not exist. As a consequence, under the definition of continuity, the square root function $f(x)=\sqrt{x}$ is not continuous at $x=0$ (Why?).

However, it has a right-hand limit at 0 . We express this fact by saying that the square root function $f(x)=\sqrt{x}$ is continuous from the right of "0".We can give the following definitions of one-sided continuity.

Definition [Continuity from the Right]

A function f(x) is continuous from the right at a point "a" in its domain, if

$$\lim_{x\to a^+} f(x)=f(a)$$

Definition [Continuity from the Left]

A function f(x) is continuous from the left at a point "a" in its domain, if

$$\lim_{x\to a^-} f(x)=f(a)$$

In view of the above definitions a function whose domain is a singleton is considered continuous at that point.

Continuity on An Interval

We say that a function is continuous on an open interval if it is continuous at each point there. It must be clear that each point in the interval has to satisfy all the three conditions of continuity at a point as stated in the definition (1).

When we consider a closed interval $[a,b]$ we face a problem as we have seen in the case of the square root function $f(x)=\sqrt{x}$.

We overcome this situation by agreeing as follows: we say that "f " is continuous on closed interval $[a,b]$, if it is continuous at each point of (a,b) and if the following limits exist:

$$\lim_{x\to a^+} f(x)=f(a) \text{ and } \lim_{x\to b^-} f(x)=f(b)$$

Example (2)

Given

$$f(x)=\frac{x}{x-2} .$$

Test the continuity of the function in the intervals (1, 2), [1, 2], and (1, 3).

Solution

Note that, f(x) is not defined for $x = 2$. Accordingly, f(x) is continuous in any interval which does not contain 2. Thus, "f " is continuous on (1, 2), but it is discontinuous on [1, 2] and on (1, 3).

Some Theorems on Continuity (Without Proof)

I. If $f(x)$ and $g(x)$ are two functions continuous at the number "a", then $f(x) \pm g(x)$, $f(x).g(x)$ are continuous at "a" and $\frac{f(x)}{g(x)}$ is continuous at "a", provided that $g(a) \neq 0$.

II. Continuity of a Composite Function: If the function $g(x)$ is continuous at "a" and the function $f(x)$ is continuous at $g(a)$, then the composite function $(f \circ g)(x)$ is continuous at "a".

Continuity of Some Elementary Functions

It can be shown that

I. A constant function is continuous for all x.

II. A polynomial function $f(x) = a_0 x^n + a_1 x^{n-1} + ... + a_n$ is continuous for all values of x on $(-\infty, \infty)$.

III. $x^n, n > 0$ is continuous for all values of x.

IV. A rational function is continuous at every point in its domain.

V. $\frac{1}{x^n}, n > 0$ is continuous for all values of x, except $x = 0$.

VI. Trigonometric functions: $f(x) = \sin x$ and $g(x) = \cos x$ are continuous on $(-\infty, \infty)$. Other trigonometric functions (i.e., tan x, cot x, sec x, cosec x) are continuous for all values of x for which they are defined.

VII. Inverse trigonometric functions are continuous for all values of x for which they are defined.

VIII. The exponential function: f$f(x) = a^x$ is continuous on $(-\infty, \infty)$. (In particular, $f(x) = e^x$ is continuous for all x.)

IX. The logarithmic function: $f(x) = \log_a x$, (a>0) is continuous o $(0, \infty)$.

Example (3)

Discus the continuity of the function

$$f(x) = \frac{|x|}{x} \text{ at } x = 0.$$

Solution

The arrows at the ends of the rectilinear portions of the graph mean that for $x = 0$, the function is not defined but for the values of x less than zero the value of the function is "-1", and for the values of x exceeding zero, it is equal to "1". Hence, the function has no limit as $x = 0$. Thus, the function $f(x)$ discontinuous at $x = 0$.

Example (4)

The greatest integer function of x denoted by $f(x) = [x]$ is defined as: $[x] =$ the greatest integer less than or equal to x. Thus, for all numbers x less than 2 but near 2, $[x] = 1$, and for all numbers greater than 2 but near 2, $[x] = 2$.

The graph of $[x]$ takes a jump at each integer as clear from the graph (Fig. 3.6).

Now, for any integer number k, we have

$$\lim_{x \to k^-} [x] = k - 1 \text{, but when } \lim_{x \to k^+} [x] = k.$$

Thus, $\lim_{x \to k} [x]$ does not exist. Thus, $[x]$ is not continuous for any integer x.

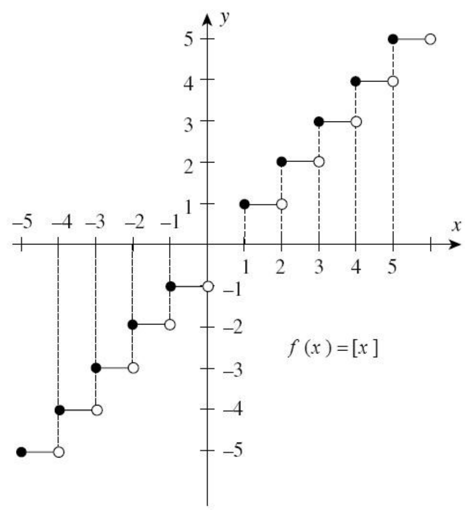


Fig. 3.6

Example (5)

Find any points of discontinuity for the function $f(x)$ given by

$$f(x)=\frac{x^4-3x^2+2x-1}{x^2-1}$$

Solution

The denominator is zero when $x=\pm 2$. Hence "$f(x)$" is not defined at ± 2 and accordingly it is discontinuous at these points. Otherwise, the function is "well behaved". In fact, any rational function (i.e., any quotient of polynomials) is discontinuous at points where the denominator becomes 0, but it is continuous at all other points.

Example (6)

Check whether the function

$$f(x)=\frac{2^{1/x}+1}{2^{1/x}+2}$$

is continuous at $x=0$.

Solution

Note that the function $f(x)$ is not defined at $x=0$. To check whether this function is continuous at $x=0$, we compute its one-sided limits.

As $x\to 0^-$, $\frac{1}{x}\to -\infty$, so that $2^{1/x}\to 0$.

$$\therefore \lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{2^{1/x}+2}{2^{1/x}+1}=\frac{0+1}{0+2}=\frac{1}{2}$$

However, as $x\to 0^+$, $\frac{1}{x}\to \infty$, so that $2^{1/x}\to \infty$.

$$\therefore \lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{2^{1/x}+2}{2^{1/x}+1}=\lim_{x\to 0^+}\frac{2^{1/x}(1+2.2^{-1/x})}{2^{1/x}(1+2^{-1/x})}$$

$$=\lim_{x\to 0^+}\frac{1+2.2^{-1/x}}{1+2^{-1/x}}=\frac{1+0}{1+0}=1$$

Therefore, the $f(x)$ is discontinuous at $x=0$.

Example (7)

Prove that the function defined by

$$f(x)=\begin{cases} x\sin\frac{1}{x}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$$

is continuous at $x=0$.

Solution

We shall compute the left-hand limit and right-hand limit of this function, at $x=0$.

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-} x\sin\frac{1}{x}=(\lim_{x\to 0^-} x)(\lim_{x\to 0^-}\sin\frac{1}{x})=0$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+} x\sin\frac{1}{x}=(\lim_{x\to 0^+} x)(\lim_{x\to 0^+}\sin\frac{1}{x})=0$$

(Since $\sin\frac{1}{x}$ is a bounded function, which lies between -1 and 1.)

As $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0)$, $f(x)$ is continuous at $x=0$.

Example (8)

$$f(x)=\begin{cases} \sin\frac{1}{x}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$$

Test the continuity of f(x) at $x=0$.

Solution

Note that f(x) is defined for all x. $\lim_{x\to 0}\sin\frac{1}{x}$ does not exist. [Indeed, the $\lim_{x\to 0}\sin\frac{1}{x}$ oscillates between -1 and 1]. Hence, the given function f(x) is not continuous at $x=0$.

Note

The function $\sin\frac{1}{x}$ is defined for all values of x except for $x=0$. It does not approach either a finite limit or infinity as $x\to 0$. The graph of this function is shown below (Fig. 3.7).

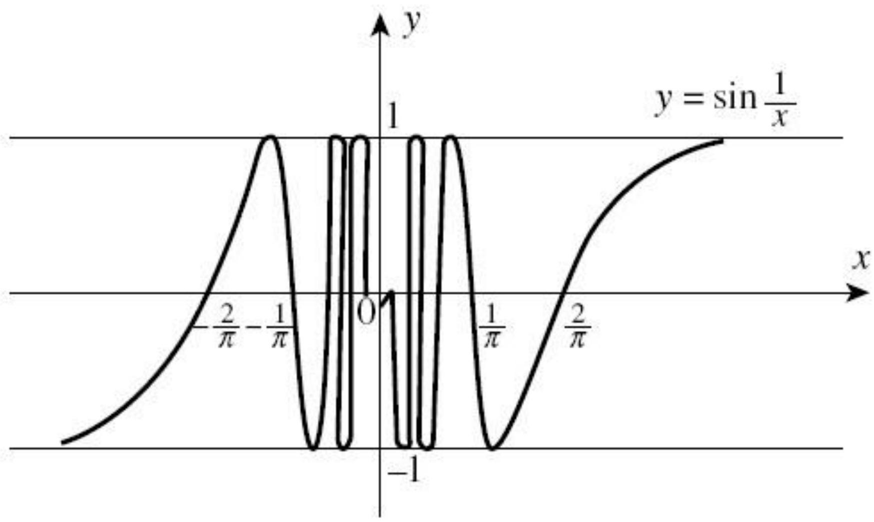


Fig. 3.7

Example (9)

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & if \ x \neq 0 \\ 0, & if \ x = 0 \end{cases}$$

Test the continuity of $f(x)$ at $x = 0$.

Solution

Note that $f(x)$ is defined for all x. We have

I. $f(0) = 0$

II. $\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \sin\frac{1}{x} = 0$

III. $\lim_{x \to 0} f(x) = f(0) = 0$

Thus, $f(x)$ is continuous at $x = 0$.

Example (10)

Test the continuity/discontinuity of the following function at $x = 0$.

$$f(x) = \begin{cases} \frac{e^{1/x}}{1+e^{1/x}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Solution

Observe that,

I. $f(0) = 0$

II. $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \dfrac{e^{1/x}}{1+e^{1/x}} = \dfrac{0}{1+0} = 0$ and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{e^{1/x}}{1+e^{1/x}} = \lim_{x\to 0^+} \frac{e^{1/x}}{e^{1/x}(\mathrm{e}^{-1/x}+1)}$$

$$= \lim_{x\to 0^+} \frac{1}{(\mathrm{e}^{-1/x}+1)} = \frac{1}{0+1} = 1$$

Thus, $\lim\limits_{x\to 0} f(x)$ does not exist. We conclude that f(x) is discontinuous at $x = 0$.

Example (11)

$$f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Is f (x) continuous at $x = 0$?

Solution

Note that the function is defined for all x. To find whether f(x) is continuous at $x = 0$ or not, we check the left-hand and the right-hand limits at $x = 0$.

I. $f(0) = 1$

II. $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \dfrac{\sin 2x}{x} = 2$ and $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \dfrac{\sin 2x}{x} = 2$.

Thus, $\lim\limits_{x\to 0} f(x) = 2$

III. $\lim\limits_{x\to 0} f(x) \neq f(0)$

We conclude that f(x) is discontinuous at $x = 0$.

Example (12)

Let

$$f(x) = \frac{\sin x}{x}.$$

Define a function $g(x)$ which is continuous, and $g(x) = f(x)$ for all $x \neq 0$.

Solution

We have $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$

Let

$$g(x)=\begin{cases}\dfrac{\sin x}{x}, & x\neq 0\\ 1\end{cases}$$

Then, g(x) is continuous at "0". Since $\lim\limits_{x\to 0} g(x)=g(0)=1$. Furthermore, $g(x)=f(x)$ for all $x\neq 0$, as was desired.

Note

The graph (Fig. 3.8)of the function $\dfrac{\sin x}{x}$ is given below. It gives a feel of how it becomes continuous when we redefine it at $x=0$ as 1.

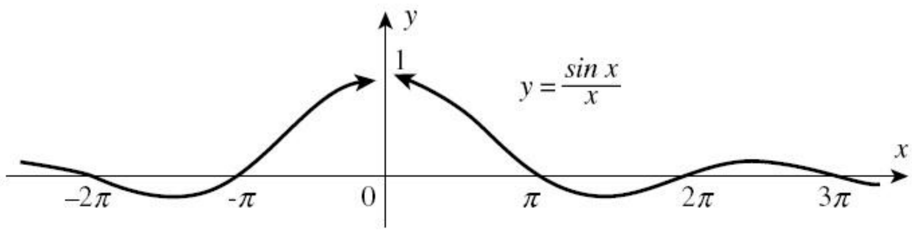


Fig. 3.8

Example (13)

Discuss the continuity of the function

$$f(x)=\begin{cases}\dfrac{(3^x-1)^2}{\sin x\ \ln(1+x)}, & x\neq 0\\ 2\ln 3, & x=0\end{cases}$$

Solution:

Given $f(0)=2\ln 3$

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{(3^x-1)^2}{\sin x\ \ln(1+x)}$$

$$=\lim_{x\to 0}\frac{\left(\dfrac{3^x-1}{x}\right)^2}{\dfrac{\sin x}{x}\ \dfrac{\ln(1+x)}{x}}=\frac{(\ln 3)^2}{1.1}=(\ln 3)^2$$

Thus, we have $\lim_{x\to 0} f(x) \neq f(0)$. Hence, $f(x)$ is discontinuous at $x = 0$.

Example (14)

Find the value of k, if

$$f(x) = \begin{cases} \dfrac{1-\cos kx}{x\sin x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$$

is continuous.

Solution

Since $f(x)$ is continuous at $x = 0$,

$$\lim_{x\to 0} f(x) = f(0) = 2$$

Hence our problem reduces to computing the limit of f(x) as $x \to 0$.
Consider,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1-\cos kx}{x\sin x} = \lim_{x\to 0} \frac{2\sin^2 \frac{kx}{2}}{x^2 \frac{\sin x}{x}}$$

Thus,

$$\frac{k^2}{2} = 2 \Rightarrow k = \pm 2$$

Example (15)

If $f(x) = \dfrac{\left(5^x - 2^x\right).x}{\cos 5x - \cos 3x}$, for $x \neq 0$, is continuous at $x = 0$, find $f(0)$.

Solution

It is given that $f(x)$ is continuous at $x = 0$. Therefore, by definition, we have,

$$\lim_{x\to 0} f(x) = f(0)$$

Thus, our problem is reduced to computing the $\lim_{x\to 0} f(x)$.

Now,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\left(5^x - 2^x\right).x}{\cos 5x - \cos 3x}$$

$$= \lim_{x\to 0} \frac{\left(5^x - 2^x\right).x}{-2\sin 4x.\sin x} \quad (\text{since } \cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2})$$

$$= \lim_{x\to 0} \frac{\left(\frac{5^x - 1}{x} - \frac{2^x - 1}{x}\right)}{-8\frac{\sin 4x}{4x}.\frac{\sin x}{x}} = \frac{\ln 5 - \ln 2}{-8} = -\frac{1}{8}\ln\frac{5}{2}$$

Example (16)

The function $f(x)$ is defined by

$$f(x) = \begin{cases} \frac{e^x - 1 - x}{x^2}, & x \neq 0 \\ \frac{1}{2}, & x = 0 \end{cases}$$

is continuous at $x = 0$. What is $\lim_{x\to 0} f(x)$?

Solution

If the problem is read carefully, it must be clear that we do not have to compute $\lim_{x\to 0} f(x)$]. Since, $f(x)$ is continuous at $x = 0$,

$$\lim_{x\to 0} f(x) = f(0) = \frac{1}{2}$$

Example (17)

Discus the continuity of the function

$$f(x) = \frac{1}{x-2} \text{ at } x = 2$$

Solution

Since $f(x)$ is not defined at $x = 2$. Hence, $f(x)$ is discontinuous at 2.

Again, $\lim_{x \to 2} f(x)$ does not exist (see Fig. 3.9) (Why?).

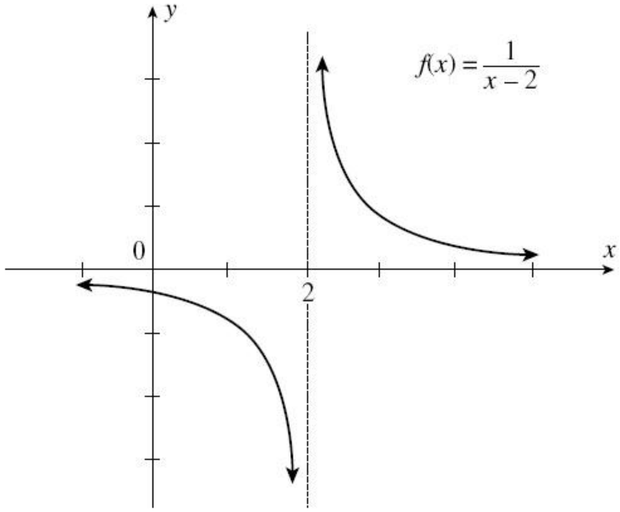


Fig. 3.9

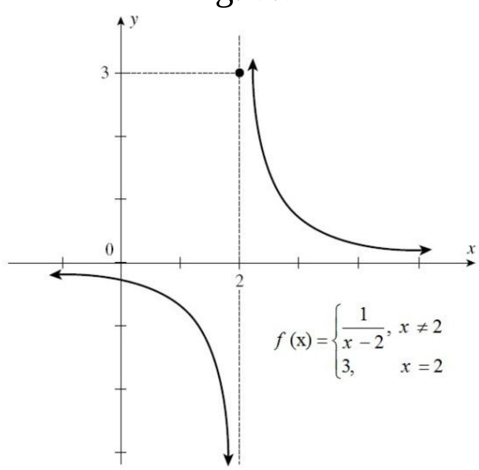


Fig. 3.10

Example (18)

Discus the continuity of the function

$$f(x) = \begin{cases} \dfrac{1}{x-2}, & x \neq 2 \\ 3, & x = 2 \end{cases}$$

at $x = 2$

Solution

Here , the graph of $f(x)$ has a break at 2 (see Fig.3.10).We check the conditions of $f(x)$, at $x = 2$. Observe that

I. $f(2) = 3$

II. $\lim_{x \to 2^-} \frac{1}{x-2} = -\infty$, and $\lim_{x \to 2^+} \frac{1}{x-2} = \infty$, Thus, $\lim_{x \to 2} \frac{1}{x-2}$ does not exist.

Obviously, $f(x)$ is discontinuous at $x = 2$.

Example (19)

Discus the continuity of the function

$$f(x) = \begin{cases} |x-3|, & x \neq 3 \\ 2, & x = 3 \end{cases}$$

Solution

We check the three conditions of continuity at $x = 3$

I. $f(3) = 2$

II. $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (3-x) = 0$, and $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x-3) = 0$.

Thus, $\lim_{x \to 0} |x-3|$ exists and equals 0 (see Fig. 3.11) .

III. $\lim_{x \to 3} f(x) \neq f(3)$

Thus, $f(x)$ is discontinuous at 3.

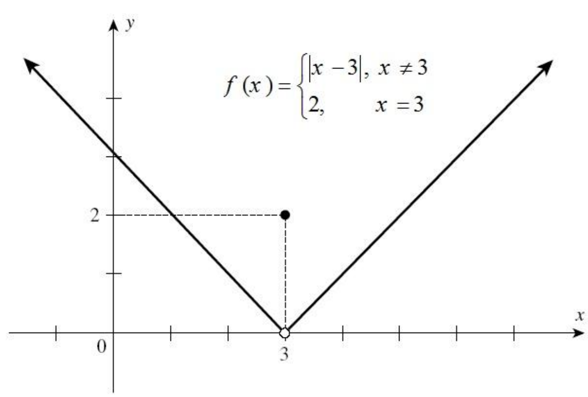


Fig. 3.11

Example (20)

Discus the continuity of the function

$$f(x) = \begin{cases} x^2 + 2, & x > 1 \\ 5x - 1, & x \leq 1 \end{cases}$$

Solution

The functions having values $x^2 + 2$ and $5x - 1$ are polynomials and are therefore continuous everywhere. Thus, the only number at which continuity is questionable is 1. We check the three conditions for continuity at “1”.

I. $f(1)=4$. Thus, $f(1)$ exists.

II. $\lim_{x\to1^-} f(x)=\lim_{x\to1^-}(x^2+2)=3$ and $\lim_{x\to1^+} f(x)=\lim_{x\to1^+}(5x-1)=4$

Thus, $\lim_{x\to1^-} f(x)\neq\lim_{x\to1^+} f(x)$. Therefore, $\lim_{x\to1} f(x)$ does not exist, and so "$f(x)$" is discontinuous at $x=1$.

Example (21)

Discus the continuity of the function

$$f(x)=\begin{cases} x+6, & x\geq3 \\ x^2, & x<3 \end{cases}$$

Solution

We observe that,

I. $f(3)=9$.

II. $\lim_{x\to3^-} f(x)=\lim_{x\to3^-} x^2=9$, and $\lim_{x\to3^+} f(x)=\lim_{x\to3^+}(x+3)=9$,

Thus, $\lim_{x\to3} f(x)=f(3)$ and $f(x)$ is continuous at $x=3$

Example (22)

Discus the continuity of the function

$$f(x)=\begin{cases} x+2, & x>2 \\ x^2, & x<2 \end{cases}$$

Solution

Since "$f(x)$" is not defined at $x=2$, it is discontinuous there. (It is continuous for all other x.). Note that

$$\lim_{x\to2^-} f(x)=\lim_{x\to2^-}(x^2)=4 \text{ and } \lim_{x\to2^+} f(x)=\lim_{x\to2^+}(x+2)=4$$

Thus $\lim_{x\to2} f(x)=4$ exists.

Example (23)

Discus the continuity of the function

$$f(x)=\begin{cases} x^2, & x\leq1 \\ x, & x>1 \end{cases}$$

Solution

Note that

I. $f(1)=1$

II. $\lim_{x\to1^-} f(x) = \lim_{x\to1^-}(x^2) = 1$ and $\lim_{x\to1^+} f(x) = \lim_{x\to1^+}(x) = 1$. Thus $\lim_{x\to1} f(x) = 1$ exists (see Fig. 3.12).

III. $\lim_{x\to1} f(x) = f(1) = 1$

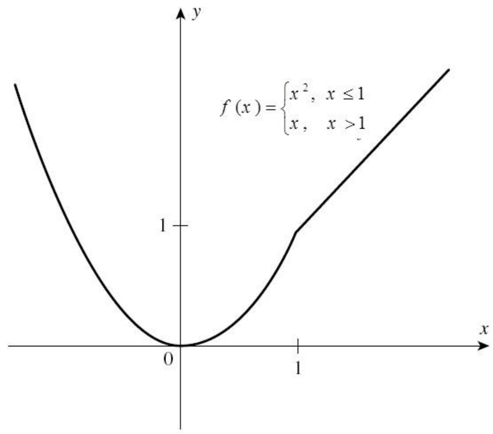


Fig. 3.12

Example (24)

Discus the continuity of the function

$$f(x) = \frac{x^2}{1+x^2}$$

Solution

Here again "$f(x)$" is a rational function, but its denominator ($1+x^2$) is never 0. Thus, "$f(x)$" is defined for all x and therefore "f " is continuous for every real value of x.

Example (25)

Show that the function $f(x) = 5$ is continuous for every value of x .

Solution

We must verify that the conditions for continuity at arbitrary point $x = a$ are satisfied.

I. $f(a) = 5$

II. $\lim_{x\to a^-} f(x) = 5$ and $\lim_{x\to a^+} f(x) = 5$. Thus, $\lim_{x\to a} f(x) = 5$

III. $\lim_{x\to a} f(x) = f(a)$

Therefore, $f(x)$ is continuous at $x = a$.

Example (26)
Let

$$f(x) = \operatorname{sgn} x = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

Discus the continuity of $f(x)$.

Solution

The function f(x) is called signum function (or sign function) denoted by $\operatorname{sgn} x$ and read "signum of x" (Figure 3.13). (It gives the sign of x .) Note that the function $\operatorname{sgn} x$ is defined for all x .

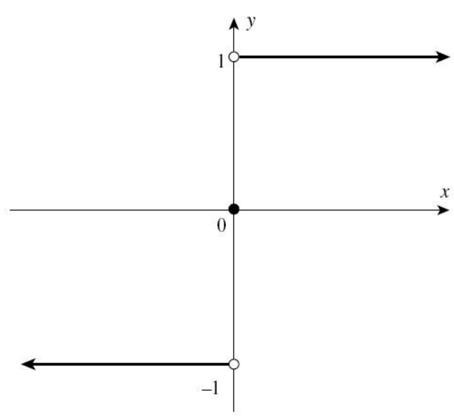


Fig. 3.13

Because

$\operatorname{sgn} x = -1,$ If $x < 0$, $\operatorname{sgn} x = 0,$ If $x = 0$ and $\operatorname{sgn} x = 1,$ If $x > 0$,

we have

$$\lim_{x\to 0^-} \operatorname{sgn} x = \lim_{x\to 0^-} (-1) = -1, \; \lim_{x\to 0^+} \operatorname{sgn} x = \lim_{x\to 0^+} (1) = 1$$

Thus, the left-hand limit and the right-hand limit are not equal, which means that $\lim_{x\to 0} \operatorname{sgn} x$ does not exist. Accordingly, f(x) is discontinuous at $x = 0$.

Chapter 4
Differentiation of Real Functions

Let $y = f(x)$ be a given function defined in an open interval (a, b). Let the points x and $(x + \Delta x)$ both belong to the domain of function $f(x)$ where Δx is an arbitrary nonzero number. From the function f(x), we form a new function

$$\phi(x) = \frac{\Delta f}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The limit of this ratio, as $\Delta x \to 0$, may or may not exist. If

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

exists, then we call it the derivative of $f(x)$ with respect to x. It is denoted by $f'(x)$ or $\frac{dy}{dx} = \frac{df}{dx}$.

Derivative of a Function at a Particular Point

The derivative of a function $y = f(x)$ at a particular point $x = x_1$ in the domain of $f(x)$ is given by the limit

$$\lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

if this limit exists. It is denoted by $f'(x_1)$.

If we replace $(x_1 + \Delta x)$ by x, and accordingly $\Delta x = x - x_1$, then the derivative of $f(x)$ at x_1 is given by

$$f'(x_1) = \lim_{x \to x_1} \frac{f(x) - f(x_1)}{x - x_1}$$

if this limit exists.

In all cases, the number x_1 at which f' is evaluated is held fixed during the limit operation. Here, x is the variable and x_1 is regarded as a constant.

Note

Observe that if $f'(x_1)$ exists, then the letter x in (C) can be replaced by any other letter. For example, we can write

$$f'(a)=\lim_{t\to a}\frac{f(t)-f(a)}{t-a} \qquad (*)$$

Example (1)
Let

$$f(x)=\frac{x^2}{4}+1 .$$

Find $f'(-1)$ and $f'(3)$

Solution
Using (*), we obtain

$$f'(-1)=\lim_{x\to -1}\frac{(x^2/4)+1-\frac{5}{4}}{x-(-1)}$$

$$=\lim_{x\to -1}\frac{\frac{x^2}{4}-\frac{1}{4}}{x+1}=\lim_{x\to -1}\frac{(1/4)\left(x^2-1\right)}{x+1}$$

$$=\lim_{x\to -1}\frac{\frac{1}{4}\left(x^2-1\right)}{x+1}=\lim_{x\to -1}\frac{(1/4)(x+1)(x-1)}{x+1}$$

$$=\lim_{x\to -1}(1/4)(x-1)=-\frac{1}{2}$$

$$f'(3)=\lim_{x\to 3}\frac{(x^2/4)+1-\frac{13}{4}}{x-3}$$

$$=\lim_{x\to 3}\frac{(1/4)x^2-9/4}{x-3}=\lim_{x\to 3}\frac{(1/4)\left(x^2-9\right)}{x-3}$$

$$=\lim_{x\to 3}\frac{(1/4)\left(x^2-1\right)}{x-3}=\lim_{x\to 3}\frac{(1/4)(x+3)(x-3)}{x-3}$$

$$=\lim_{x\to 3}(1/4)(x+3)=-\frac{1}{2}$$

Next, we give the following formal definitions.

Differentiability of Functions

I. Functions differentiable at a point

If a function has a derivative at x_1 of its domain, then it is said to be differentiable at x_1.

II. Functions differentiable in an open interval

A function is differentiable in an open interval (a,b) if it is differentiable at every number in the open interval.

III. Functions differentiable in a closed interval

If f(x) is defined in a closed interval [a, b], then the definitions of the derivatives at the end points are modified so that the point ($x+\Delta x$) lies in the interval $[a,b]$. Hence, we define the one side derivative at the end points as follows:

The right-hand derivative

$$f_+'(a)=\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}$$

The left-hand derivative

$$f_-'(b)=\lim_{x\to a^-}\frac{f(x)-f(b)}{x-b}$$

IV. Differentiable Function

If a function is differentiable at every number in its domain, it is called a differentiable function.

Note

The above definition appears to be quite simple, but certain situations might create confusion. Hence, to get a clear idea of a differentiable function, it is useful to consider the following example:

Example (2)

Check the differentiability of the function $f(x)=\sqrt{x}$ at $x=0$

Solution

The right-hand derivative

$$f_+'(0)=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{\sqrt{x}-0}{x-0}$$

$$=\lim_{x\to 0^+}\frac{1}{\sqrt{x}}=\infty$$

The left-hand derivative

$$f'_-(0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{\sqrt{x} - 0}{x - 0}$$

$$= \lim_{x \to 0^-} \frac{1}{\sqrt{x}} \text{ does not exist}$$

Here, the domain of $f(x)$ is $[0, \infty)$ but $f'(x)$ does not exist at $x = 0$. Thus, $f(x)$ is not differentiable at "0", which is in the domain of $f(x)$. Therefore, we will say that $f(x)$ is not a differentiable function.

However, if we define the function $f(x) = \sqrt{x}$ in the open interval $(0, \infty)$, then it becomes a differentiable function.

In view of the above, we agree to say that if the domain of $f'(x)$ is the same as that of $f(x)$, then $f(x)$ is a differentiable function.

Nearly every function we will encounter is differentiable at all numbers or all but finitely many numbers in its domain.

Note

To obtain the derivative of a function, by using the definition of the derivative, is known as the method of finding the derivative from the first principle.

Notation for Derivative

We know that differentiation of $y = f(x)$ by the first principle involves two steps:

I. First, the formation of the difference quotient $\frac{f(x + \Delta x) - f(x)}{\Delta x}$

II. Second, the evaluation of the limit $\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$

III. If the limit, $\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ exists, then we denote it by the symbol $f'(x)$ or $\frac{dy}{dx}$ and call it the derivative of the function $f(x)$.

Note

We can look at the process of differentiation as an operation. The operation of obtaining $f'(x)$, from $f(x)$, is called differentiation of $f(x)$. The symbol $\frac{d}{dx}$ is assigned for this operation. We call it the operator of differentiation.

The Operator of Differentiation $\frac{d}{dx}$

In view of the above discussion, we can say that the symbol $\frac{d}{dx}$ stands for the operation of computing the derivative of a given function by the first principle. In other words, we agree to say that $\frac{d}{dx}$ constructs from the difference quotient $\frac{f(x+\Delta x)-f(x)}{\Delta x}$, and determines its limit as $\Delta x \to 0$ (treating the difference quotient as a function of variable Δx)

Note

The notation $\frac{d}{dx}$ should be interpreted as a single entity and not as a ratio. (It reads "d over dx"). It is also used in a formula to stand for the phrase "the derivative of ". Thus, the symbol $\frac{d}{dx}$ is used to define the derivatives of combinations of functions.

Derivatives of Simple Algebraic Functions

Now, we proceed to evaluate the derivatives of some simple algebraic functions by definition.

Example (3)

Let $y = f(x) = x^n, n \in \mathbb{N}$. Then, we have

$$f'(x) = \frac{dy}{dx} = \frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} = nx^{n-1}$$

Example (4)

Let $y = f(x) = x^\alpha, \alpha \in \mathbb{R}$. Then, we have

$$f'(x)=\frac{dy}{dx}=\frac{df(x)}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\left(x+\Delta x\right)^{\alpha}-x^{\alpha}}{\Delta x}=\alpha x^{\alpha-1}$$

Remark

To obtain, the limit $\lim_{\Delta x\to 0}\frac{\left(x+\Delta x\right)^{\alpha}-x^{\alpha}}{\Delta x}$, by making use of binomial theorem , we can expand the amount $\frac{\left(x+\Delta x\right)^{\alpha}-x^{\alpha}}{\Delta x}$ as follows:

$$\frac{\left(x+\Delta x\right)^{\alpha}-x^{\alpha}}{\Delta x}=\frac{x^{\alpha}\left(1+\Delta x/x\right)^{\alpha}-x^{\alpha}}{\Delta x}\ (\text{since } \frac{\Delta x}{x}<1)$$

$$=\frac{x^{\alpha}\left[1+\frac{\alpha}{1!}\frac{\Delta x}{x}+\frac{\alpha(\alpha-1)}{2!}\left(\frac{\Delta x}{x}\right)^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}\left(\frac{\Delta x}{x}\right)^3+...\right]-x^{\alpha}}{\Delta x}$$

$$=\frac{\frac{\alpha x^{\alpha-1}}{1!}(\Delta x)+\frac{\alpha(\alpha-1)x^{\alpha-2}}{2!}\left(\Delta x\right)^2+\frac{\alpha(\alpha-1)(\alpha-2)x^{\alpha-3}}{3!}\left(\Delta x\right)^3+...}{\Delta x}$$

$$=\frac{\alpha x^{\alpha-1}}{1!}+\frac{\alpha(\alpha-1)x^{\alpha-2}}{2!}\left(\Delta x\right)+\frac{\alpha(\alpha-1)(\alpha-2)x^{\alpha-3}}{3!}\left(\Delta x\right)^2+...$$

So, we have

$$\lim_{\Delta x\to 0}\frac{\left(x+\Delta x\right)^{\alpha}-x^{\alpha}}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\left(\frac{\alpha x^{\alpha-1}}{1!}+\frac{\alpha(\alpha-1)x^{\alpha-2}}{2!}\left(\Delta x\right)+\frac{\alpha(\alpha-1)(\alpha-2)x^{\alpha-3}}{3!}\left(\Delta x\right)^2+...\right)$$

$$=\alpha x^{\alpha-1}$$

Note

Later, where the method of logarithmic differentiation is discussed, we shall show prove the above formula by using logarithmic differentiation .

Example (5)

Find the derivative of $y=\sqrt{x}$

Solution

$$\frac{dy}{dx}=\frac{d}{dx}(\sqrt{x})=\frac{d}{dx}x^{1/2}=\frac{1}{2}x^{(1/2)-1}$$

$$=\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}}$$

Now, Let Us Consider the Derivative of a Constant, $y=f(x)=C$.

$$\frac{dy}{dx}=\lim_{x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=\lim_{x\to 0}\frac{C-C}{\Delta x}=0$$

Example (6)

Find the derivative of

$$f(x)=\sqrt{3x+7}$$

Solution

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{3(x+h)+7}-\sqrt{3x+7}}{h}$$

By rationalizing the numerator, we get

$$f'(x)=\lim_{h\to 0}\frac{\sqrt{3(x+h)+7}-\sqrt{3x+7}}{h}\cdot\frac{\sqrt{3(x+h)+7}+\sqrt{3x+7}}{\sqrt{3(x+h)+7}+\sqrt{3x+7}}$$

$$=\lim_{h\to 0}\frac{3(x+h)+7-(3x+7)}{h\sqrt{3(x+h)+7}+\sqrt{3x+7}}$$

$$=\lim_{h\to 0}\frac{3h}{h\sqrt{3(x+h)+7}+\sqrt{3x+7}}$$

$$=\lim_{h\to 0}\frac{3}{\sqrt{3(x+h)+7}+\sqrt{3x+7}}$$

$$=\frac{3}{2\sqrt{3x+7}}$$

Example (7)

Find the derivative of $f(x)=\frac{1}{\sqrt{x}}$.

Solution

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1/(\sqrt{x+h})-1/\sqrt{x}}{h}$$

By rationalizing the numerator, we get

$$f'(x)=\lim_{h\to 0}\frac{1/\sqrt{x+h}-1/\sqrt{x}}{h}.\frac{1/\sqrt{x+h}+1/\sqrt{x}}{1/\sqrt{x+h}+1/\sqrt{x}}$$

$$=\lim_{h\to 0}\frac{1/(x+h)-1/x}{h\left(1/\sqrt{x+h}+1/\sqrt{x}\right)}$$

$$=\lim_{h\to 0}\frac{\frac{x-x-h}{x(x+h)}}{h\left(1/\sqrt{x+h}+1/\sqrt{x}\right)}$$

$$=\lim_{h\to 0}\frac{\frac{-1}{x(x+h)}}{\left(1/\sqrt{x+h}+1/\sqrt{x}\right)}$$

$$=\frac{-1}{x^2}.\frac{1}{2/\sqrt{x}}=-\frac{1}{2}x^{-3/2}$$

Rules of Differentiation of Functions

We find the result of applying the operator $\frac{d}{dx}$ to certain combinations of differentiable functions, namely, sums, products, and ratios. (It turns out that the rules for differentiating such combinations of functions are easily established in terms of the derivatives of the constituent functions).

I. Derivative of a sum (or difference) of functions

Let $f_1(x)$ and $f_2(x)$ be differentiable functions of x, with the same domain, then

$$\frac{d}{dx}\left[f_1(x) \pm f_2(x)\right] = \frac{d}{dx}f_1(x) \pm \frac{d}{dx}f_2(x)$$

This rule can be extended to the derivative of the sum (or difference) of any finite number of differentiable functions, with the same domain. Thus,

$$\frac{d}{dx}\left[f_1(x) \pm f_2(x) \pm \ldots \pm f_n(x)\right]$$
$$= \frac{d}{dx}f_1(x) \pm \frac{d}{dx}f_2(x) \pm \ldots \pm \frac{d}{dx}f_n(x)$$

II. The Constant Rule for Derivatives

If k is any constant, $f(x)$ is any differentiable function, then

$$\frac{d}{dx}\left[k.f(x)\right] = k\frac{d}{dx}f(x)$$

III. The derivative of product of two functions

Let $f_1(x)$ and $f_2(x)$ be differentiable functions of x, then

$$\frac{d}{dx}\left[f_1(x).f_2(x)\right] = f_1(x)\frac{d}{dx}f_2(x) + f_2(x)\frac{d}{dx}f_1(x)$$

This rule can be extended to the product of more than two functions (and in general for a product of finite number of differentiable functions). Thus,

$$\frac{d}{dx}\left[f_1(x).f_2(x).f_3(x)\right] = \frac{d}{dx}\left[\left(f_1(x).f_2(x)\right).f_3(x)\right]$$
$$= \left(f_1(x).f_2(x)\right)\frac{d}{dx}f_3(x) + f_3(x)\frac{d}{dx}\left(f_1(x).f_2(x)\right)$$
$$= \left(f_1(x).f_2(x)\right)\frac{d}{dx}f_3(x) + f_3(x).\left[f_1(x)\frac{d}{dx}f_2(x) + f_2(x)\frac{d}{dx}f_1(x)\right]$$

IV. The derivative of quotient of two functions

Let $f_1(x)$ and $f_2(x)$ be differentiable functions of x, then

$$\frac{d}{dx}\left(\frac{f_1(x)}{f_2(x)}\right) = \frac{f_2(x)\frac{d}{dx}f_1(x) - f_1(x)\frac{d}{dx}f_2(x)}{\left[f_2(x)\right]^2}$$

Example (8)

If $y = \dfrac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}$, find $\dfrac{dy}{dx}$.

Solution

$$y = \frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}.\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}$$

$$= \frac{(x+1)+(x-1)+2\sqrt{x+1}\sqrt{x-1}}{(x+1)-(x-1)}$$

$$= x+\sqrt{x+1}\sqrt{x-1}$$

$$\therefore \frac{dy}{dx} = 1+\frac{\sqrt{x+1}}{2\sqrt{x-1}}+\frac{\sqrt{x-1}}{2\sqrt{x+1}}$$

$$= 1+\frac{x}{\sqrt{x^2-1}}$$

Example (9)

If $y = \dfrac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$, find $\dfrac{dy}{dx}$.

Solution

$$\frac{dy}{dx} = \frac{\left(\frac{1}{2\sqrt{x}}\right)\left(\sqrt{a}-\sqrt{x}\right)+\left(\frac{1}{2\sqrt{x}}\right)\left(\sqrt{a}+\sqrt{x}\right)}{\left[\sqrt{a}-\sqrt{x}\right]^2}$$

$$= \frac{\sqrt{a}/\sqrt{x}}{\left[\sqrt{a}-\sqrt{x}\right]^2} = \frac{\sqrt{a}}{\sqrt{x}\left[\sqrt{a}-\sqrt{x}\right]^2}$$

The Derivative of a Composite Function

We have already introduced the concept of composite functions in Chapter 1. Many of the functions we encounter in mathematics and in applications are composite functions. Consider the following examples:

I. $y = (x^3+1)^{10}$ is a function of x^3+1, and x^3+1 is a function of x.

So, $y = \left(x^3+1\right)^{10}$ can be considered as a composition of two functions as follows

$$y = u^{10},\ u = x^3+1 \Rightarrow (y \circ u)(x)$$

$$= y(u(x)) = y(x^3+1) = \left(x^3+1\right)^{10}$$

II. $y = \sqrt[3]{x^4+1}$ is a function of x^4+1, and x^4+1 is a function of x. So, $y = \sqrt[3]{x^4+1}$ can be considered as a composition of two functions as follows

$$y = \sqrt[3]{u},\ u = x^4+1 \Rightarrow (y \circ u)(x)$$

$$= y(u(x)) = y(x^4+1) = \sqrt[3]{x^4+1}$$

III. $y = \sqrt[7]{\left(\frac{x}{2}+1\right)^{10}}$ is a function of $\left(\frac{x}{2}+1\right)^{10}$, $\left(\frac{x}{2}+1\right)^{10}$ is a function of $\frac{x}{2}+1$, and $\frac{x}{2}+1$ is a function of x.

Thus, $y = \left(x^3+1\right)^{10}$, $y = \sqrt[3]{x^4+1}$, $y = \sqrt[7]{\left(\frac{x}{2}+1\right)^{10}}$ and so on are examples of composite functions of x. If we could discover a general rule for the derivative of a composite function in terms of the component functions, then we would be able to find its derivative without resorting to the definition of the derivative.

To find the derivative of a composite function, we apply the chain rule, which is one of the important computational theorems in calculus. It assumes a very suggestive form in the Leibniz notation .

The Chain Rule

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable functions of x, such that the composite function

$y = (f \circ g)(x) = f(g(x))$ is defined, then $\frac{dy}{dx}$ is given by

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

If y is a function of u, defined by $y = f(u)$ and $\frac{dy}{du}$ exists, and if u is a function of x defined by $u = g(x)$ and $\frac{du}{dx}$ exists, then y

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad (*)$$

Note

Here, it is important to note that in the product of derivatives on RHS, there are two separate operators of differentiation, namely, $\frac{d}{du}$ and $\frac{d}{dx}$. Hence, $\frac{dy}{dx}$ is not obtained by canceling du from the numerator and the denominator.

Extension of Chain Rule (i.e. The Compound Chain Rule)

In general, if $y = f(t)$, $t = g(u)$, and $u = h(x)$, where $\frac{dy}{dt}$, $\frac{dt}{du}$ and $\frac{du}{dx}$ exist, then y is a function of x and $\frac{dy}{dx}$ exists, given by

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{du} \cdot \frac{du}{dx}$$

Thus, the derivative of y is obtained in a chain-like fashion. In practice, it is convenient to identify the functions t, u, and so on at different stages of differentiation.

Remark

In formula (*), y is represented in two different ways: once as a function of x and once as a function of u. The expression $\frac{dy}{dx}$ is the derivative of y, when y is regarded as a function of x. In the same way, $\frac{dy}{du}$ is the derivative of y, when y is regarded as a function of u. Formula (*) is especially useful when y is not given explicitly in terms of x, but is given in terms of an intermediate variable .

Example (10)

If $y=\sqrt{\dfrac{x-2}{x+2}}$, find $\dfrac{dy}{dx}$.

Solution

Let $u=\dfrac{x-1}{x+1} \Rightarrow y=\sqrt{u}$.

Then,

$$\frac{dy}{du}=\frac{1}{2\sqrt{u}},\ \frac{du}{dx}=\frac{4}{(x+2)^2}$$

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=\frac{1}{2\sqrt{\dfrac{x-1}{x+1}}}\cdot\frac{4}{(x+1)^2}$$

Example (11)

If $y=\left(x^3+3\right)^5$, find $\dfrac{dy}{dx}$.

Solution

Let $u=x^3+3 \Rightarrow y=u^5$.

Then,

$$\frac{dy}{du}=5u^4,\ \frac{du}{dx}=3x^2$$

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=5\left(x^3+3\right)^4\cdot 3x^2=15x^2\left(x^3+3\right)^4$$

Derivatives of Trigonometric Functions

By using the basic trigonometric limits and applying the definition of the derivative, we can compute the derivatives of all basic trigonometric functions.

The Derivatives of $\sin x$ and $\cos x$ (From the First Principle)

To find the derivative of $f(x)=\sin x$, using the definition of the derivative. We have,

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

provided the limit on the RHS exists.

$$\frac{d}{dx}(\sin x) = \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x}$$

$$(\because \sin(x + y) = \sin x \cos y + \cos x \sin y)$$

$$\frac{d}{dx}(\sin x) = \lim_{\Delta x \to 0} \frac{\sin x (\cos \Delta x - 1) + \sin \Delta x \cos x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin x (\cos \Delta x - 1)}{\Delta x} + \cos x \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} \quad \left[\because \lim_{\Delta x \to 0} \frac{(\cos \Delta x - 1)}{\Delta x} = 0\right]$$

$$= 0 + \cos x = \cos x$$

Similarly we can find the derivative of $f(x) = \cos x$, using the definition of the derivative. We have,

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

provided the limit on the RHS exists.

$$\frac{d}{dx}(\cos x) = \lim_{\Delta x \to 0} \cos \frac{\cos(x + \Delta x) - \cos x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cos x \cos \Delta x - \sin x \sin \Delta x - \cos x}{\Delta x}$$

$$(\because \cos(x + y) = \cos x \cos y - \sin x \sin y)$$

$$\frac{d}{dx}(\cos x) = \lim_{\Delta x \to 0} \frac{\cos x (\cos \Delta x - 1) - \sin x \sin \Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cos x (\cos \Delta x - 1)}{\Delta x} - \sin x \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} \quad \left[\because \lim_{\Delta x \to 0} \frac{(\cos \Delta x - 1)}{\Delta x} = 0\right]$$

$$= 0 - \sin x = -\sin x.$$

Theorem

If $f(x)$ is a differentiable function of x ,

$$\frac{d}{dx}\left[\sin\left(f(x)\right)\right]=\cos\left[f(x)\right]\cdot\frac{d}{dx}f(x) \quad \text{[by chain rule]}$$

$$=f'(x)\left[\cos\left(f(x)\right)\right]$$

$$\frac{d}{dx}\left[\cos\left(f(x)\right)\right]=-\sin\left[f(x)\right]\cdot\frac{d}{dx}f(x) \quad \text{[by chain rule]}$$

$$=-f'(x)\left[\sin\left(f(x)\right)\right]$$

The Derivative of $\tan x$

$$\frac{d}{dx}\tan x=\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)=\frac{(\cos x)(\cos x)-(-\sin x)(\sin x)}{\cos^2 x}$$

$$=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}=\frac{1}{\cos^2 x}=\sec^2 x$$

The Derivative of $\cot x$

$$\frac{d}{dx}\cot x=\frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)=\frac{(-\sin x)(\sin x)-(\cos x)(\cos x)}{\sin^2 x}$$

$$=-\frac{\cos^2 x+\sin^2 x}{\sin^2 x}=-\frac{1}{\sin^2 x}=-\text{cosec}^2 x$$

The Derivative of $\sec x$

$$\frac{d}{dx}\sec x=\frac{d}{dx}\left(\frac{1}{\cos x}\right)=\frac{0.(\cos x)-(-\sin x).1}{\cos^2 x}$$

$$=\frac{\sin x}{\cos^2 x}=\frac{1}{\cos x}\frac{\sin x}{\cos x}=\sec x\tan x$$

The Derivative of $\text{cosec} x$

$$\frac{d}{dx}\text{cosec} x=\frac{d}{dx}\left(\frac{1}{\sin x}\right)=\frac{0.(\sin x)-(\cos x).1}{\sin^2 x}$$

$$=-\frac{\cos x}{\sin^2 x}=-\frac{1}{\sin x}\frac{\cos x}{\sin x}=-\text{cosec} x\cot x$$

Theorem

If $f(x)$ is a differentiable function of x,

$$\frac{d}{dx}\left[\tan\left(f(x)\right)\right]=\sec^2\left[f(x)\right]\cdot\frac{d}{dx}f(x) \text{ [by chain rule]}$$

$$=f'(x)\left[\sec^2\left(f(x)\right)\right]$$

$$\frac{d}{dx}\left[\cot\left(f(x)\right)\right]=-\operatorname{cosec}^2\left[f(x)\right]\cdot\frac{d}{dx}f(x) \text{ [by chain rule]}$$

$$=-f'(x)\left[\operatorname{cosec}^2\left(f(x)\right)\right]$$

$$\frac{d}{dx}\left[\sec\left(f(x)\right)\right]=\sec\left[f(x)\right]\cdot\tan\left[f(x)\right]\cdot\frac{d}{dx}f(x) \text{ [by chain rule]}$$

$$=f'(x)\left[\sec\left(f(x)\right)\cdot\tan\left(f(x)\right)\right]$$

$$\frac{d}{dx}\left[\operatorname{cosec}\left(f(x)\right)\right]=-\operatorname{cosec}\left[f(x)\right]\cdot\cot\left[f(x)\right]\cdot\frac{d}{dx}f(x) \text{ [by chain rule]}$$

$$=-f'(x)\left[\operatorname{cosec}\left(f(x)\right)\cdot\cot\left(f(x)\right)\right]$$

Example (12)

Differentiate

$$y=\left(x^3+5x^2\right)\sin x\,.$$

Solution

$$\frac{dy}{dx}=\left(x^3+5x^2\right).\frac{d}{dx}\sin x+\sin x\,.\frac{d}{dx}\left(x^3+5x^2\right)$$

$$=\left(x^3+5x^2\right).\cos x+\sin x\,.\left(3x^2+10x\right)$$

Example (13)

If $y=\sqrt{\dfrac{1-\sin x}{1+\sin x}}$, find $\dfrac{dy}{dx}$

Solution

Let $u=\dfrac{1-\sin x}{1+\sin x}$, then $y=\sqrt{u}$ and

$$\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}$$

$$\frac{dy}{du} = \frac{1}{2\sqrt{u}},$$

$$\frac{du}{dx} = \frac{(-\cos x)(1+\sin x) - (\cos x)(1-\sin x)}{(1+\sin x)^2}$$

$$= \frac{-2\cos x}{(1+\sin x)^2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{u}}\left[\frac{-2\cos x}{(1+\sin x)^2}\right]$$

$$= -\sqrt{\frac{1+\sin x}{1-\sin x}}\left[\frac{\cos x}{(1+\sin x)^2}\right]$$

Example (14)

If $y = \dfrac{\tan x + \sec x}{\tan x - \sec x}$, find $\dfrac{dy}{dx}$.

Solution:

$$\frac{dy}{dx} = \frac{(\sec^2 x + \sec x \tan x)(\tan x - \sec x)}{(\tan x - \sec x)^2}$$

$$- \frac{(\sec^2 x - \sec x \tan x)(\tan x + \sec x)}{(\tan x - \sec x)^2}$$

$$= \frac{2\sec x \tan^2 x - 2\sec^3 x}{(\tan x - \sec x)^2}$$

$$= \frac{2\sec x (\tan x + \sec x)}{(\tan x - \sec x)}$$

Derivative of Exponential Function

To find the derivative of the exponential function $y = f(x) = a^x$, we use the principal definition

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x} = \lim_{\Delta x \to 0} \frac{a^x a^{\Delta x} - a^x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{a^x \left(a^{\Delta x} - 1\right)}{\Delta x} = a^x \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$$

$$= a^x \ln a$$

So, we have

$$\frac{d}{dx} a^x = a^x \ln a$$

Also, we have

$$\frac{d}{dx} e^x = e^x \ln e = e^x$$

Derivatives of Logarithmic Function

To find the derivative of the natural logarithmic function $y = f(x) = \ln x$, we use the principal definition

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\ln(x+\Delta x) - \ln x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\ln\left(1 + \frac{\Delta x}{x}\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \ln\left(1 + \frac{\Delta x}{x}\right) = \lim_{\Delta x \to 0} \ln\left(1 + \frac{\Delta x}{x}\right)^{1/\Delta x}$$

$$= \ln \lim_{\Delta x \to 0} \left[\left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x}\right]^{1/x} = \ln\left[\lim_{\Delta x \to 0} \left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x}\right]^{1/x}$$

$$= \ln e^{1/x} = \frac{1}{x}$$

So, we have

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

Also, we have

$$\frac{d}{dx}\log_a x = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right)$$

$$= \frac{1}{\ln a}\frac{d}{dx}\ln x = \frac{1}{\ln a}.\frac{1}{x}$$

Example (15)

$$\frac{d}{dx}(\sin x - \cos x) = \cos x + \sin x$$

$$\frac{d}{dx}\left(x^3 + 7x - 5\right) = 3x^2 + 7$$

$$\frac{d}{dx}\left(a^x - \tan x + \ln x\right) = a^x \ln a - \sec^2 x + \frac{1}{x}$$

$$\frac{d}{dx}\left(x^5 + e^x - \sec x\right) = 5x^4 + e^x - \sec x \tan x$$

Exercise

Find the derivative of the following functions with respect to x	Answer
$\dfrac{e^x}{\sin x}$	$\dfrac{e^x(\sin x - \cos x)}{\sin^2 x}$
$\dfrac{a^x}{x^n}$	$\dfrac{a^x}{x^n}\left(\ln a - \dfrac{n}{x}\right)$
$\dfrac{x \cos x}{\ln x}$	$\dfrac{(\cos x - x \sin x)\ln x - \cos x}{(\ln x)^2}$
$\dfrac{\ln x}{\cos x}$	$\dfrac{\cos x + x \sin x \ln x}{x \cos^2 x}$
$\dfrac{e^x + e^{-x}}{e^x - e^{-x}}$	$-\dfrac{4}{(e^x - e^{-x})^2}$
$\dfrac{\sqrt{1+x}}{\sqrt{1-x}}$	$\dfrac{1}{(1-x)\sqrt{1-x^2}}$
$\ln\left(\sqrt{\dfrac{a+x}{a-x}}\right)$	$\dfrac{a}{a^2 - x^2}$
$\dfrac{1}{\log_{10} x}$	$-\dfrac{\ln 10}{x(\ln x)^2}$

Example (16)

Differentiate

$$y = 3^x \log_5 x$$

Solution

$$\frac{dy}{dx}=3^x\ \frac{d}{dx}\log_5 x\ +\log_5 x\ \frac{d}{dx}3^x$$

$$=3^x\left(\frac{1}{\ln 5}\frac{1}{x}\right)+\log_5 x\left(3^x\ \ln 3\right)$$

Example (17)

Differentiate with respect to x , the function

$$y\ =\log_x a$$

Solution

We have,

$$y\ =\log_x\ a=\frac{\ln a}{\ln x}$$

$$\frac{dy}{dx}=\ln a\frac{d}{dx}\frac{1}{\ln x}=\ln a\left\{\frac{-1/x}{\left[\ln x\right]^2}\right\}$$

$$=\frac{-\ln a}{x\left[\ln x\right]^2}$$

Exercise	Answer
(1) Differentiate $x\ \ln x$	$1+\ln x$
(2) If $y\ =\left(x^{\,2}+2x\right)3^x$, find $\frac{dy}{dx}$ at $x\ =2$	$36\left(1+2\ln 3\right)$
(3) If $y\ =6x\ \tan x$, find $\frac{dy}{dx}$ at $x\ =0$	0

Theorem

If $f\ (x)$ is a differentiable function of x ,

$$\frac{d}{dx}\left[a^{f\,(x)}\right]=a^{f\,(x)}\cdot\ln a\cdot\frac{d}{dx}f\ (x)\ \ \text{[by chain rule]}$$

$$=a^{f\,(x)}\cdot\ln a\cdot f\ '(x)$$

$$\frac{d}{dx}\left[e^{f(x)}\right]=e^{f(x)}\cdot\frac{d}{dx}f(x) \text{ [by chain rule]}$$

$$=f'(x)\cdot a^{f(x)}$$

$$\frac{d}{dx}\left[\log_a\left[f(x)\right]\right]=\frac{1}{\ln a}\cdot\frac{1}{f(x)}\cdot\frac{d}{dx}f(x) \text{ [by chain rule]}$$

$$=\frac{1}{\ln a}\cdot\frac{f'(x)}{f(x)}$$

$$\frac{d}{dx}\left[\ln\left[f(x)\right]\right]=\frac{1}{f(x)}\cdot\frac{d}{dx}f(x) \text{ [by chain rule]}$$

$$=\frac{f'(x)}{f(x)}$$

Example (18)

If $y=\ln\left(\ln\left(\sin x\right)\right)$ find $\dfrac{dy}{dx}$.

Solution

Let $t=\sin x$, $u=\ln\left(\sin x\right)$. Then, $y=\ln u$ and $u=\ln t$.

So, we have

$$\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dt}.\frac{dt}{dx}$$

$$\frac{dy}{du}=\frac{1}{u},\ \frac{du}{dt}=\frac{1}{t},\ \frac{dt}{dx}=\cos x$$

$$\frac{dy}{dx}=\frac{1}{u}.\frac{1}{t}.\cos x=\frac{1}{\ln\left(\sin x\right)}.\frac{1}{\sin x}.\cos x$$

$$=\frac{\cot x}{\ln\left(\sin x\right)}$$

Example (19)

If $y=\sqrt{\sec\sqrt{x}}$; find $\dfrac{dy}{dx}$

Solution:

Let $t=\sqrt{x}$, $u=\sec\sqrt{x}$. Then $y=\sqrt{u}$, $u=\sec t$

$$\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dt}.\frac{dt}{dx}=\frac{1}{2\sqrt{u}}.\sec t\ \tan t.\frac{1}{2\sqrt{x}}$$

$$=\frac{1}{2\sqrt{\sec\sqrt{x}}}.\sec\sqrt{x}\ \tan\sqrt{x}.\frac{1}{2\sqrt{x}}$$

$$=\frac{\sec\sqrt{x}\ \tan\sqrt{x}}{4\sqrt{x}\sqrt{\sec\sqrt{x}}}$$

Example (20)

If $y=\ln\left(\sqrt{\frac{1+\sin mx}{1-\sin mx}}\right)$; find $\frac{dy}{dx}$.

Solution

Let $t=\frac{1+\sin mx}{1-\sin mx}$ $u=\sqrt{\frac{1+\sin mx}{1-\sin mx}}$. Then $y=\ln u,\ u=\sqrt{t}$

$$\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dt}.\frac{dt}{dx}$$

$$=\frac{1}{u}.\frac{1}{2\sqrt{t}}.\frac{m\cos mx\,(1-\sin mx)+m\cos mx\,(1+\sin mx)}{(1-\sin mx)^2}$$

$$=\sqrt{\frac{1-\sin mx}{1+\sin mx}}.\frac{1}{2}\sqrt{\frac{1-\sin mx}{1+\sin mx}}.\frac{2m\cos mx}{(1-\sin mx)^2}$$

$$=\frac{m\cos mx}{1-\sin^2 mx}=\frac{m\cos mx}{\cos^2 mx}=\frac{m}{\cos mx}=m\sec mx$$

Simpler method for other similar problems:

When computing derivatives by the chain rule, we do not actually write the function t, u and so on, but bear them in mind, and keep on obtaining the derivatives of the component functions, stepwise, as shown in the following solved examples.

Example (21)

If $y=\ln\left(\sin x^2\right)$ find $\frac{dy}{dx}$.

Solution

$$\frac{dy}{dx}=\frac{d}{dx}\left[\ln\left(\sin x^{2}\right)\right]$$

$$=\frac{1}{\sin x^{2}}.\frac{d}{dx}\sin x^{2}$$

$$=\frac{1}{\sin x^{2}}.\cos x^{2}.\frac{d}{dx}\left(x^{2}\right)$$

$$=\frac{1}{\sin x^{2}}.\cos x^{2}.2x=2x\cot x^{2}$$

Note

Observe that when we differentiate a function by using the chain rule, we differentiate from the outside inward. Thus, to differentiate $\sin(3x+5)$, we first differentiate the outer function $\sin x$ (at $3x+5$) and then differentiate the inner function $3x+5$ at x . Similarly, to differentiate $\cos x^{7}$, we first differentiate the outer function $\cos x$ (at x^{7}) and then differentiate the inner function x^{7} , at x . The chain rule can be applied to even longer composites. The procedure is always the same:

Differentiate from outside inward and multiply the resulting derivatives (evaluated at the appropriate numbers).

For example,

$$\frac{d}{dx}\left[\sin\left(\cos\left(\tan^{5}x\right)\right)\right]$$

$$=\left[\cos\left(\cos\left(\tan^{5}x\right)\right)\right]\left[-\sin\left(\tan^{5}x\right)\right]\left(5\tan^{4}x\right)\sec^{2}x$$

Example (22)

If $y=\ln\left(\ln\left(\ln x\right)\right)$ find $\frac{dy}{dx}$.

Solution
We have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\ln\left(\ln\left(\ln x\right)\right)\right]$$

$$= \frac{1}{\ln\left(\ln x\right)} \frac{d}{dx}\left(\ln\left(\ln x\right)\right)$$

$$= \frac{1}{\ln\left(\ln x\right)} . \frac{1}{\ln x} . \frac{d}{dx} \ln x$$

$$= \frac{1}{\ln\left(\ln x\right)} . \frac{1}{\ln x} . \frac{1}{x}$$

$$= \frac{1}{x\left(\ln x\right)\left[\ln\left(\ln x\right)\right]}$$

Example (23)

If $y = \ln\left(\ln\left(\ln x^3\right)\right)$ find $\frac{dy}{dx}$.

Solution
We have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\ln\left(\ln\left(\ln x^3\right)\right)\right]$$

$$= \frac{1}{\ln\left(\ln x^3\right)} \frac{d}{dx}\left(\ln\left(\ln x^3\right)\right)$$

$$= \frac{1}{\ln\left(\ln x^3\right)} . \frac{1}{\ln x^3} . \frac{d}{dx} \ln x^3$$

$$= \frac{1}{\ln\left(\ln x\right)} . \frac{1}{\ln x^3} . \frac{1}{x^3} \frac{d}{dx} x^3$$

$$= \frac{1}{\ln\left(\ln x^3\right)} . \frac{1}{\ln x^3} . \frac{1}{x^3} . 3x^2$$

$$= \frac{3}{x\left(\ln x^3\right)\left[\ln\left(\ln x^3\right)\right]}$$

Example (24)

If $y = e^{x^3}$, find $\frac{dy}{dx}$

Solution

We have,

$$\frac{dy}{dx} = \frac{d}{dx}e^{x^3} = e^{x^3}\frac{d}{dx}x^3$$
$$= e^{x^3}\left(3x^2\right) = 3x^2 e^{x^3}$$

Example (25)

If $y = \sqrt{\cos\sqrt{x}}$, find $\frac{dy}{dx}$.

Solution

We have,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{\cos\sqrt{x}}\right)$$
$$= \frac{1}{2\sqrt{\cos\sqrt{x}}}.\frac{d}{dx}\cos\sqrt{x}$$
$$= \frac{1}{2\sqrt{\cos\sqrt{x}}}.\left(-\sin\sqrt{x}\right).\frac{d}{dx}\sqrt{x}$$
$$= \frac{1}{2\sqrt{\cos\sqrt{x}}}.\left(-\sin\sqrt{x}\right).\frac{1}{2\sqrt{x}}$$
$$= -\frac{\sin\sqrt{x}}{4\sqrt{x\cos\sqrt{x}}}$$

Example (26)

If $y = \sin\left(\log_{10} x\right)$; find $\frac{dy}{dx}$.

We have,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sin\left(\log_{10} x\right)\right)$$

$$= \cos\left(\log_{10} x\right)\frac{d}{dx}\left(\log_{10} x\right)$$

$$= \cos\left(\log_{10} x\right)\left(\frac{1}{x \ln 10}\right)$$

$$= \frac{\cos\left(\log_{10} x\right)}{x \ln 10}$$

Example (27)

If $y = \ln\left[\sin x + \cos x\right]$; find $\frac{dy}{dx}$.

Solution

We have

$$\frac{dy}{dx} = \frac{d}{dx}\left(\ln\left[\sin x + \cos x\right]\right)$$

$$= \frac{1}{\sin x + \cos x}\frac{d}{dx}\left(\sin x + \cos x\right)$$

$$= \frac{1}{\sin x + \cos x}\left(\cos x - \sin x\right)$$

$$= \frac{\cos x - \sin x}{\sin x + \cos x}$$

Example (28)

If $y = 2^x \cos\left(3x - 2\right)$; find $\frac{dy}{dx}$.

Solution

We have

$$\frac{dy}{dx} = \frac{d}{dx}\left[2^x \cos\left(3x - 2\right)\right]$$

$$= 2^x \frac{d}{dx}\left[\cos\left(3x - 2\right)\right] + \left[\cos\left(3x - 2\right)\right]\frac{d}{dx}2^x$$

$$= 2^x \left[-\sin\left(3x - 2\right)\right]\frac{d}{dx}\left(3x - 2\right) + \left[\cos\left(3x - 2\right)\right]\left(2^x \ln 2\right)$$

$$= 2^x \left[-\sin\left(3x - 2\right)\right]\left(3\right) + \left[\cos\left(3x - 2\right)\right]\left(2^x \ln 2\right)$$

$$= 2^x \left[\ln 2\cos\left(3x - 2\right) - 3\sin\left(3x - 2\right)\right]$$

Example (29)

If $y = \frac{1}{x \ln x}$; find $\frac{dy}{dx}$.

Solution

We have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{1}{x \ln x}\right]$$

$$= \frac{(0)\left(x \ln x\right) - (1)\frac{d}{dx}\left(x \ln x\right)}{\left(x \ln x\right)^2}$$

$$= -\frac{x\left(\frac{1}{x}\right) + \left(\ln x\right)(1)}{\left(x \ln x\right)^2} = -\frac{1 + \ln x}{\left(x \ln x\right)^2}$$

Summary of Differentiation Rules

Derivative of a sum (difference) of functions)

$$\frac{d}{dx}\left[f_1(x) \pm f_2(x)\right] = \frac{d}{dx}f_1(x) \pm \frac{d}{dx}f_2(x)$$

Derivative of a constant multiple of a function

$$\frac{d}{dx}\left[k\, f(x)\right] = k\, \frac{d}{dx}f(x)$$

Derivative of a product of functions

$$\frac{d}{dx}\left[f_1(x) f_2(x)\right] = f_1(x)\frac{d}{dx}f_2(x) + f_2(x)\frac{d}{dx}f_1(x)$$

Derivative of ratio of functions

$$\frac{d}{dx}\left[\frac{f_1(x)}{f_2(x)}\right]=\frac{f_2(x)\frac{d}{dx}f_1(x)-f_1(x)\frac{d}{dx}f_2(x)}{\left[f_2(x)\right]^2}$$

Derivative of composite functions (the chain rule)

$$\frac{d}{dx}\left[f\left(g(x)\right)\right]=\frac{d}{dg}\left[f(g)\right]\frac{d}{dx}\left[g(x)\right]$$

Summary of basic functions derivatives .

y	$\frac{dy}{dx}$
$\left[f(x)\right]^{\alpha},\ \alpha\in\mathbb{R}$	$\alpha\left[f(x)\right]^{\alpha-1}\left[f'(x)\right]$
$\sqrt{f(x)}$	$\frac{f'(x)}{2\sqrt{f(x)}}$
$\sin\left[f(x)\right]$	$\cos\left[f(x)\right]\left(f'(x)\right)$
$\cos\left[f(x)\right]$	$-\sin\left[f(x)\right]\left(f'(x)\right)$
$\tan\left[f'(x)\right]$	$\sec^2\left[f(x)\right]\left(f'(x)\right)$
$\cot\left[f(x)\right]$	$-\operatorname{cosec}^2\left[f(x)\right]\left(f'(x)\right)$
$\sec\left[f(x)\right]$	$\sec\left[f(x)\right]\tan\left[f(x)\right]\left(f'(x)\right)$
$\operatorname{cosec}\left[f(x)\right]$	$-\operatorname{cosec}\left[f(x)\right]\cot\left[f(x)\right]\left(f'(x)\right)$
y	$\frac{dy}{dx}$
$a^{f(x)}$	$\left[a^{f(x)}\right]\left[f'(x)\right]\left[\ln a\right]$
$e^{f(x)}$	$\left[e^{f(x)}\right]\left(f'(x)\right)$
$\ln\left[f(x)\right]$	$\frac{f'(x)}{f(x)}$
$\log_a\left[f(x)\right]$	$\frac{1}{\ln a}\frac{f'(x)}{f(x)}$

Exercise : Differentiate the following functions w.r.t. x :

(1) $y=\ln\left(\ln\left(\sin x\right)\right)$ (2) $y=\left[\ln\left(\ln\left(\ln x\right)\right)\right]^4$ (3) $y=\sqrt{\sin\sqrt{x}}$

(4) $y=\cos\left(x^3e^x\right)$ (5) $y=\dfrac{\sin\sqrt{x}}{\sqrt{x}}$ (6) $y=e^{e^{e^x}}$

(7) $y=2^{2^x}$ (8) $y=\log_7\left(\log_7 x\right)$ (9) $y=\ln\sqrt{\dfrac{1+\cos x}{1-\cos 3x}}$

Implicit Functions and Their Differentiation

First, let us distinguish between explicit and implicit functions. Functions of the form, $y=f(x)$ in which y (alone) is directly expressed in terms of the function(s) of x, are called explicit functions.

Example (30)

$y=x^2+3x-2$, $y=\sin x+2e^x$, $y=\dfrac{x+3}{1+x^2}$

$y=\cos x+\ln(1+x^2)$ and so on.

Not all functions, however, can be defined by equations of this type. For example, we cannot solve the following equations for y (alone) in terms of the functions of x .

Examples (31)

$x^3+y^3=2xy$, $y^5+3y^2-2x^2+2=0$, $x^2+y^2=36$

$\sin y=x\sin(a+y)$, $y^3+7y=x^3$ and so on.

Such relations connecting x and y are called implicit relations. An implicit relation (in x and y) may represent jointly two or more functions x .

As an example, the relation $x^2+y^2=36$ jointly represents two functions: $y=\sqrt{36-x^2}$ and $y=-\sqrt{36-x^2}$.

Remark

Every explicit function $y=f(x)$ can also be expressed as an implicit function. For example, we may write the above equation in the form $y-f(x)=0$ and call it an implicit function of x . Thus, the term explicit function and implicit function do not characterize the nature of a function but merely the way a function is defined.

The Differentiation of implicit Functions

The technique of implicit differentiation is based on the chain rule.

For example, consider the equation

$$y^3 + 7y = x^3$$

Differentiating both the sides with respect to x, treating y as a function of x, we get (via the rule for differentiating a composite function)

$$3y^2 \frac{dy}{dx} + 7\frac{dy}{dx} = 3x^2 \qquad (*)$$

Now solving (*) for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx}\left(3y^2 + 7\right) = 3x^2 \quad \therefore \frac{dy}{dx} = \frac{3x^2}{3y^2 + 7}$$

Note that, the above expression for $\frac{dy}{dx}$ involves both x and y. If it is required to find the value of the derivative of an implicit function for a given value of x, then we have to first find the corresponding value of y, using the given relation . This will help in computing the value of $\frac{dy}{dx}$ at those points.

Example (32)

Find $\frac{dy}{dx}$, if $y^5 + 3y^2 - 2x^2 = -4$.

Solution

Differentiating both sides of the given equation "with respect to x" (using the chain rule), we obtain

$$5y^4 \frac{dy}{dx} + 6y\frac{dy}{dx} - 4x = 0$$

We now solve for $\frac{dy}{dx}$, obtaining

$$\left(5y^4 + 6y\right)\frac{dy}{dx} = 4x \quad \therefore \frac{dy}{dx} = \frac{4x}{5y^4 + 6y}$$

Derivatives of the Inverse Trigonometric Functions

I. Derivative of the Inverse Sine Function

Let $y = \sin^{-1} x$, which is equivalent to

$$x = \sin y \quad \text{and} \quad y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Differentiating both the sides of this equation with respect to x , we obtain

$$1 = [\cos y]\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$$

If $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\cos y$ is non-negative.

Here, we have to write the right-hand side in terms of x .

Since, $\sin y = x$, we have

$$\cos y = \pm\sqrt{1-\sin^2 y} = \pm\sqrt{1-x^2}$$

Of these two values for $\cos y$, we should take $\cos y = \sqrt{1-x^2}$, since $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

So, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin^{-1} x\right] = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left[\sin^{-1} x\right] = \frac{1}{\sqrt{1-x^2}}$$

Theorem (A): If $f(x)$ is a differentiable function of x ,

$$\frac{d}{dx}\left[\sin^{-1}\left(f(x)\right)\right] = \frac{1}{\sqrt{1-[f(x)]^2}} \cdot \frac{d}{dx} f(x) \quad [\text{by chain rule}]$$

$$= \frac{f'(x)}{\sqrt{1-[f(x)]^2}}$$

Example (33)

Find $\frac{dy}{dx}$, if $y = \sin^{-1} x^2$

Solution

$$\frac{dy}{dx} = \frac{d}{dx}\sin^{-1} x^2 = \frac{2x}{\sqrt{1-x^4}}$$

II. Derivative of the Inverse Cosine Function

Let $y = \cos^{-1} x$, which is equivalent to

$$x = \cos y \text{ and } y \in [0,\pi]$$

Differentiating both the sides of this equation with respect to x , we obtain

$$1 = [-\sin y]\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y}$$

If $y \in [0,\pi]$, $\sin y$ is non-negative.

Here, we have to write the right-hand side in terms of x .

Since, $\cos y = x$, we have

$$\sin y = \pm\sqrt{1-\cos^2 y} = \pm\sqrt{1-x^2}$$

Of these two values for $\sin y$, we should take $\sin y = \sqrt{1-x^2}$, since $y \in [0,\pi]$.

So, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\cos^{-1} x\right] = -\frac{1}{\sin y} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left[\cos^{-1} x\right] = -\frac{1}{\sqrt{1-x^2}}$$

Theorem (B): If $f(x)$ is a differentiable function of x ,

$$\frac{d}{dx}\left[\cos^{-1}\left(f(x)\right)\right] = -\frac{1}{\sqrt{1-[f(x)]^2}} \cdot \frac{d}{dx} f(x) \quad \text{[by chain rule]}$$

$$= -\frac{f'(x)}{\sqrt{1-[f(x)]^2}}$$

Example (34)

Find $\dfrac{dy}{dx}$, if $y = \cos^{-1} e^{2x}$

Solution

$$\frac{dy}{dx} = \frac{d}{dx}\cos^{-1} e^{2x} = -\frac{2e^{2x}}{\sqrt{1-e^{4x}}}$$

III. Derivative of the Inverse Tangent Function

Let $y = \tan^{-1} x$, which is equivalent to

$$x = \tan y \text{ and } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Differentiating both the sides of this equation with respect to x , we obtain

$$1 = \left[\sec^2 y\right]\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y}$$

Here, we have to write the right-hand side in terms of x .
Since, $x = \tan y$, , we have

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2$$

So, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\tan^{-1} x\right] = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left[\tan^{-1} x\right] = \frac{1}{1+x^2}$$

Theorem (C): If $f(x)$ is a differentiable function of x ,

$$\frac{d}{dx}\left[\tan^{-1}(f(x))\right] = \frac{1}{1+[f(x)]^2}\cdot\frac{d}{dx}f(x) \quad \text{[by chain rule]}$$

$$= \frac{f'(x)}{1+[f(x)]^2}$$

Example (35)

Find $\dfrac{dy}{dx}$, if $y = \tan^{-1}\dfrac{1}{1+x}$

Solution

$$\frac{dy}{dx}=\frac{d}{dx}\tan^{-1}\left(\frac{1}{1+x}\right)=\frac{-\frac{1}{(1+x)^2}}{1+\left(\frac{1}{1+x}\right)^2}=-\frac{1}{(x+1)^2+1}$$

IV. The Derivative of Inverse Cotangent function

From the definition of inverse cotangent function, we have

$$y=\cot^{-1}x=\frac{\pi}{2}-\tan^{-1}x$$

Differentiating both sides with respect to x , we get

$$\frac{dy}{dx}=\frac{d}{dx}\cot^{-1}x=\frac{d}{dx}\left(\frac{\pi}{2}\right)-\frac{d}{dx}\tan^{-1}x$$

1

$$=0-\frac{1}{1+x^2}=-\frac{1}{1+x^2}$$

Theorem (D): If $f(x)$ is a differentiable function of x ,

$$\frac{d}{dx}\cot^{-1}[f(x)]=-\frac{1}{1+[f(x)]^2}\cdot\frac{d}{dx}f(x)$$

$$=-\frac{f'(x)}{1+[f(x)]^2}$$

V. Derivative of the Inverse Secant Function

Let $y=\sec^{-1}x$, which is equivalent to

$$x=\sec y \text{ and } y\in[0,\pi]-\left\{\frac{\pi}{2}\right\}$$

Differentiating both the sides of this equation with respect to x , we obtain

$$1=[\sec y\tan y]\frac{dy}{dx}\Rightarrow\frac{dy}{dx}=\frac{1}{\sec y\tan y}$$

If $y\in[0,\pi]-\left\{\frac{\pi}{2}\right\}$, $\sec y\tan y$ is non-negative.

Here, we have to write the right-hand side in terms of x .

Since, $x = \sec y$, we have

$$\sec y \ \tan y = \sec y \sqrt{\sec^2 y - 1} = x\sqrt{x^2 - 1}$$

So, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sec^{-1} x\right] = \frac{1}{\sec y \ \tan y} = \frac{1}{\sec y \sqrt{\sec^2 y - 1}}$$

$$= \frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}\left[\sec^{-1} x\right] = \frac{1}{x\sqrt{x^2 - 1}}$$

Theorem (E): If $f(x)$ is a differentiable function of x ,

$$\frac{d}{dx}\sec^{-1}\left[f(x)\right] = \frac{1}{f(x)\sqrt{\left[f(x)\right]^2 - 1}} \cdot \frac{d}{dx} f(x)$$

$$= \frac{f'(x)}{f(x)\sqrt{\left[f(x)\right]^2 - 1}}$$

VI. The Derivative of Inverse Cosecant function

From the definition of inverse cosecant function, we have

$$y = \operatorname{cosec}^{-1} x = \frac{\pi}{2} - \sec^{-1} x$$

Differentiating both sides with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}\operatorname{cosec}^{-1} x = \frac{d}{dx}\left(\frac{\pi}{2}\right) - \frac{d}{dx}\sec^{-1} x$$

1

$$= 0 - \frac{1}{x\sqrt{x^2 - 1}} = -\frac{1}{x\sqrt{x^2 - 1}}$$

Theorem (D): If $f(x)$ is a differentiable function of x ,

$$\frac{d}{dx}\operatorname{cosec}^{-1}\left[f(x)\right] = -\frac{1}{f(x)\sqrt{\left[f(x)\right]^2 - 1}} \cdot \frac{d}{dx} f(x)$$

$$= -\frac{f'(x)}{f(x)\sqrt{\left[f(x)\right]^2 - 1}}$$

Example (36)

If $y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$, find $\frac{dy}{dx}$

Solution

$$\frac{dy}{dx} = \frac{1}{1+\left(\frac{1+x}{1-x}\right)^2} \frac{d}{dx}\left[\frac{1+x}{1-x}\right]$$

$$= \frac{1}{1+\left(\frac{1+x}{1-x}\right)^2}\left[\frac{1\cdot(1-x)-(-1)(1+x)}{(1-x)^2}\right]$$

$$= \frac{1}{1+x^2}$$

Example (37)

If $y = \cos^{-1}\left[\frac{1-e^{2x}}{1+e^{2x}}\right]$; find $\frac{dy}{dx}$

Solution

We have,

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-\left[\frac{1-e^{2x}}{1+e^{2x}}\right]^2}} \cdot \frac{d}{dx}\left[\frac{1-e^{2x}}{1+e^{2x}}\right]$$

$$=-\frac{1}{\sqrt{1-\left[\frac{1-e^{2x}}{1+e^{2x}}\right]^2}}\cdot\left[\frac{\left(-2e^{2x}\right)\left(1+e^{2x}\right)-\left(2e^{2x}\right)\left(1-e^{2x}\right)}{\left(1+e^{2x}\right)^2}\right]$$

$$=\frac{2e^{x}}{1+e^{2x}}$$

Example (38)

Differentiate $y=\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to x .

Solution

We have,

$$\frac{dy}{dx}=\frac{1}{1+\left(\frac{\sqrt{1+x^2}-1}{x}\right)^2}\frac{d}{dx}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$$

$$=\frac{1}{1+\left(\frac{\sqrt{1+x^2}-1}{x}\right)^2}\left[\frac{\left(\frac{x}{\sqrt{1+x^2}}\right)(x)-(1)\left(\sqrt{1+x^2}-1\right)}{x^2}\right]$$

$$=\frac{1}{2\left(1+x^2\right)}$$

Example (39)

Differentiate

$$y=\sin^{-1}\left(x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right)$$

Solution

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-\left(x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right)^2}}\frac{d}{dx}\left[x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right]$$

$$=\frac{1}{\sqrt{1-\left(x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right)^2}}$$

$$\times\left[(x)\left(\frac{-1}{2\sqrt{1-x}}\right)+\left(\sqrt{1-x}\right)(1)-\left(\sqrt{x}\right)\left(\frac{-x}{\sqrt{1-x^2}}\right)-\left(\frac{1}{2\sqrt{x}}\right)\left(\sqrt{1-x^2}\right)\right]$$

$$=\frac{1}{\sqrt{1-\left(x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right)^2}}\left[\frac{2-3x}{2\sqrt{1-x}}+\frac{3x^2-1}{2\sqrt{x}\sqrt{1-x^2}}\right]$$

Example (40)

If $y=\sec^{-1}\left(\frac{1+4^x}{1-4^x}\right)$, find $\frac{dy}{dx}$

Solution

$$\frac{dy}{dx}=\frac{1}{\left(\frac{1+4^x}{1-4^x}\right)\sqrt{\left(\frac{1+4^x}{1-4^x}\right)^2-1}}\frac{d}{dx}\left(\frac{1+4^x}{1-4^x}\right)$$

$$=\frac{1}{\left(\frac{1+4^x}{1-4^x}\right)\sqrt{\left(\frac{1+4^x}{1-4^x}\right)^2-1}}$$

$$\times\frac{\left(4^x\ln 4\right)\left(1-4^x\right)-\left(-4^x\ln 4\right)\left(1+4^x\right)}{\left(1-4^x\right)^2}$$

$$=\frac{2^{x+1}\ln 2}{1+4^x}$$

Derivatives of Hyperbolic Functions

The formulas for the derivatives of the hyperbolic sine and hyperbolic cosine functions are obtained by considering their definitions, and differentiating the expressions involving exponential functions. Thus,

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \left(\frac{e^x + e^{-x}}{2}\right) = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \left(\frac{e^x - e^{-x}}{2}\right) = \sinh x$$

From these formulas and the chain rule we have the following theorem.

Theorem (A): If $f(x)$ is a differentiable function of x,

$$\frac{d}{dx}\left[\sinh\left(f(x)\right)\right] = \left[\cosh\left(f(x)\right)\right] f'(x)$$

$$\frac{d}{dx}\left[\cosh\left(f(x)\right)\right] = \left[\sinh\left(f(x)\right)\right] f'(x)$$

The derivative of $\tanh x$ may be found from the exponential definition or we may use the above result(s) (i.e., the derivatives of $\sinh x$ and $\cosh x$). Since

$$\tanh x = \frac{\sinh x}{\cosh x}$$

Then,

$$\frac{d}{dx}[\tanh x] = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

The formulas for the derivatives of the remaining three hyperbolic functions are

$$\frac{d}{dx}[\coth x] = -\operatorname{cosech}^2 x,$$

$$\frac{d}{dx}[\operatorname{sech} x] = -\operatorname{sech} x \tanh x,$$

$$\frac{d}{dx}[\operatorname{cosech} x] = -\operatorname{cosech} x \coth x.$$

From these formulas and the chain rule, we have the following theorem.
Theorem (B): If $f(x)$ is a differentiable function of x,

$$\frac{d}{dx}\left[\tanh f(x)\right]=\left[\operatorname{sech}^2 f(x)\right]f'(x)$$

$$\frac{d}{dx}\left[\coth f(x)\right]=\left[-\operatorname{cosech}^2 f(x)\right]f'(x)$$

$$\frac{d}{dx}\left[\operatorname{sech} f(x)\right]=\left[-\operatorname{sech} f(x)\tanh f(x)\right]f'(x)$$

$$\frac{d}{dx}\left[\operatorname{cosech} f(x)\right]=\left[-\operatorname{cosech} f(x)\coth f(x)\right]f'(x)$$

Differentiation of Inverse Hyperbolic Functions
Inverse hyperbolic functions correspond to inverse circular functions, and their derivatives are found by similar methods.

I. **Derivative of** $y=\sinh^{-1}x$

Let $y=\sinh^{-1}x$. Then $x=\sinh y$
Differentiating both sides w.r.t. x

$$1=\left[\cosh y\right]\frac{dy}{dx}$$

$$\Rightarrow\ \frac{dy}{dx}=\frac{1}{\cosh y}=\frac{1}{\sqrt{1+\sinh^2 y}}=\frac{1}{\sqrt{1+x^2}}$$

II. **Derivative of** $y=\cosh^{-1}x$

Let $y=\cosh^{-1}x$. Then $x=\cosh y$
Differentiating both sides w.r.t. x

$$1=\left[\sinh y\right]\frac{dy}{dx}$$

$$\Rightarrow\ \frac{dy}{dx}=\frac{1}{\sinh y}=\frac{1}{\sqrt{\cosh^2 y-1}}=\frac{1}{\sqrt{x^2-1}}$$

III. **Derivative of** $y=\tanh^{-1}x$

Let $y=\tanh^{-1}x$. Then $x=\tanh y$
Differentiating both sides w.r.t. x

$$1 = \left[\operatorname{sech}^2 y\right]\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\operatorname{sech}^2 y} = \frac{1}{1-\tanh^2 x} = \frac{1}{1-x^2}$$

The differential coefficient of the reciprocals of the above can be found by the same methods.

They are,

$$y = \operatorname{sech}^{-1} x \qquad \frac{dy}{dx} = -\frac{1}{x\sqrt{1-x^2}}$$

$$y = \operatorname{cosech}^{-1} x \qquad \frac{dy}{dx} = -\frac{1}{x\sqrt{1+x^2}}$$

$$y = \coth^{-1} x \qquad \frac{dy}{dx} = -\frac{1}{x^2-1}$$

From these formulas and the chain rule, we can obtain the following results.

If $f(x)$ is a differentiable function of x

$$\frac{d}{dx}\left[\sinh^{-1}(f(x))\right] = \frac{f'(x)}{\sqrt{[f(x)]^2+1}}$$

$$\frac{d}{dx}\left[\cosh^{-1}(f(x))\right] = \frac{f'(x)}{\sqrt{[f(x)]^2-1}}, \; f(x) > 1$$

$$\frac{d}{dx}\left[\tanh^{-1}(f(x))\right] = \frac{f'(x)}{1-[f(x)]^2}, \; |f(x)| < 1$$

$$\frac{d}{dx}\left[\operatorname{sech}^{-1}(f(x))\right] = -\frac{f'(x)}{f(x)\sqrt{1-[f(x)]^2}}$$

$$\frac{d}{dx}\left[\operatorname{cosech}^{-1}(f(x))\right] = -\frac{f'(x)}{f(x)\sqrt{1+[f(x)]^2}}$$

$$\frac{d}{dx}\left[\coth^{-1}(f(x))\right] = -\frac{f'(x)}{[f(x)]^2-1}, \; |f(x)| > 1$$

Example (41)

Find $\frac{dy}{dx}$ if $y = \tanh^{-1}(\cos 2x)$.

Solution

We have,

$$\frac{dy}{dx} = \frac{d}{dx}\left[\tanh^{-1}(\cos 2x)\right]$$

$$= \frac{1}{1-(\cos 2x)^2}\cdot(-2\sin 2x)$$

$$= \frac{-2\sin 2x}{\sin^2 2x} = -\frac{2}{\sin 2x} = -2\operatorname{cosec} 2x$$

Example (42)

Find $\frac{dy}{dx}$, if $y = \sinh^{-1}(\tan x)$.

Solution

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sinh^{-1}(\tan x)\right]$$

$$= \frac{1}{\sqrt{1+\tan^2 x}}\cdot\sec^2 x = \frac{\sec^2 x}{|\sec x|} = |\sec x|$$

Derivatives Higher Orders

We have studied several methods of finding derivatives of differentiable functions. If $y = f(x)$ is a differentiable function of x, then its derivative is denoted by

$$\frac{dy}{dx} \text{ or } f'(x) \text{ or } y'$$

The notation $f'(x)$ suggests that the derivative of $f(x)$ is also a function of x. If the function $f'(x)$ is in turn differentiable, its derivative is called the second derivative (or the derivative of the second order) of the original function $f(x)$ and is denoted by $f''(x)$. This leads us to the concept of the derivatives of higher orders.

$$f''(x) = [f'(x)]' = \lim_{\Delta x \to 0} \frac{f'(x+\Delta x) - f'(x)}{\Delta x}$$

We write,

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \text{ or } \left[\frac{d\left(f(x)\right)}{dx} = f''(x) \text{ or } y''\right]$$

Similarly, we can find the derivative of $\frac{d^2y}{dx^2}$ provided it exists, and is denoted by $\frac{d^3y}{dx^3}$ [or $f'''(x)$ or y'''], called the third derivative of $y = f(x)$ and so on.

Notations for Derivatives of $y = f(x)$

Order of Derivative	Prime Notation $(')$	Leibniz Notation
1st	y' or $f'(x)$	$\frac{dy}{dx}$
2nd	y'' or $f''(x)$	$\frac{d^2y}{dx^2}$
3rd	y''' or $f'''(x)$	$\frac{d^3y}{dx^3}$
4th	y^{iv} or $f^{\text{iv}}(x)$	$\frac{d^4y}{dx^4}$
$\vdots$		
nth	$y^{(n)}$ or $f^{(n)}(x)$	$\frac{d^ny}{dx^n}$

Example (43)

If $y = 2x^5 - x^2 + 3$, then

$$\frac{dy}{dx} = 10x^4 - 2x, \quad \frac{d^2y}{dx^2} = 40x^3 - 2$$

$$\frac{d^3y}{dx^3} = 120x^2, \quad \frac{d^4y}{dx^4} = 240x$$

$$\frac{d^5y}{dx^5} = 240, \quad \frac{d^6y}{dx^6} = 0, \cdots, \frac{d^ny}{dx^n} = 0$$

Note that, for a polynomial function $f(x)$ of degree 5, $f^{(n)}(x) = 0$ for $n \geq 6$. More generally, the $(n+1)^{\text{th}}$ and all higher derivatives of any polynomial of degree n are equal to 0.

However, there are functions [like $\sin x$, $\cos x$, e^x, $\ln x$, and their extended forms, [that is $\sin(ax+b)$, $\cos(ax+b)$, e^{ax}, $\ln(ax+b)$ or more general ones like $\sin\left(f(x)\right)$, $e^{f(x)}$ and $\log_a\left(f(x)\right)$] that can be differentiated any number of times and $f^{(n)}(x)$ is never 0.

Example (44)

Let us find the nth derivatives of the following:

$$\text{(i)}\,x^n \qquad \text{(ii)}\,e^x$$

$$\text{(iii)}\,a^x \qquad \text{(iv)}\,\sin x$$

Solution

I. Let $y = x^n$

$$\therefore \frac{dy}{dx} = nx^{n-1}, \quad \frac{d^2y}{dx^2} = n(n-1)x^{n-2}, \quad \frac{d^3y}{dx^3} = n(n-1)(n-2)x^{n-3}$$

$$\vdots$$

$$\frac{d^ny}{dx^n} = n(n-1)(n-2)\cdots(n-(n-2))(n-(n-1))x^{n-n}$$

$$= n(n-1)(n-2)\cdots(2)(1) = n!$$

II. Let $y = e^x$

$$\therefore \frac{dy}{dx} = e^x, \quad \frac{d^2y}{dx^2} = e^x, \quad \frac{d^3y}{dx^3} = e^x$$

$$\vdots$$

$$\frac{d^ny}{dx^n} = e^x$$

III. Let $y = a^x$

$$\therefore \frac{dy}{dx} = a^x \ln a, \quad \frac{d^2y}{dx^2} = (a^x)(\ln a)^2, \quad \frac{d^3y}{dx^3} = (a^x)(\ln a)^3$$

$$\vdots$$

$$\frac{d^ny}{dx^n} = (a^x)(\ln a)^n$$

IV. Let $y = \sin x$

$$\therefore \frac{dy}{dx} = \cos x = \sin\left(x + \frac{\pi}{2}\right),$$

$$\frac{d^2y}{dx^2} = \cos\left(x + \frac{\pi}{2}\right) = \sin\left(\left(x + \frac{\pi}{2}\right) + \frac{\pi}{2}\right) = \sin\left(x + 2\cdot\frac{\pi}{2}\right)$$

$$\frac{d^3y}{dx^3} = \cos\left(x + 2\cdot\frac{\pi}{2}\right) = \sin\left(\left(x + 2\cdot\frac{\pi}{2}\right) + \frac{\pi}{2}\right) = \sin\left(x + 3\cdot\frac{\pi}{2}\right)$$

$$\vdots$$

$$\frac{d^ny}{dx^n} = \cos\left(x + (n-1)\cdot\frac{\pi}{2}\right) = \sin\left(\left(x + (n-1)\cdot\frac{\pi}{2}\right) + \frac{\pi}{2}\right) = \sin\left(x + n\cdot\frac{\pi}{2}\right)$$

Exercise

find the nth derivatives of the following:

(1) $\cos x$ (2) $\frac{1}{x}$

(3) $\ln x$

Derivatives of Higher Orders: Product of Two Functions (Leibniz Formula)

It helps us to find the nth derivative of the product of two functions.
Let $f(x)$ and $g(x)$ be functions of x and $y = f(x)\cdot g(x)$.
Then, the nth derivative of y is

$$y^{(n)} = C_0^n f^{(n)}(x)\cdot g(x) + C_1^n f^{(n-1)}(x)\cdot g'(x)$$
$$+ C_2^n f^{(n-2)}(x)\cdot g''(x) + C_3^n f^{(n-3)}(x)\cdot g'''(x)$$
$$+ \cdots + C_n^n f(x)\cdot g^{(n)}(x).$$

Where,

$$C_k^n = \frac{n!}{k!(n-k)!}$$

Note

When one of the functions in the above theorem is of the form x^m, $m \in \mathbb{N}$, then we should choose it as (the second function) $g(x)$, and the other as (the first function) $f(x)$, because x^m, $m \in \mathbb{N}$ shall have only m derivatives (and not more).

Example (45)

If $y = e^{ax}x^2$, find $y^{(n)}$.

Solution

$f(x) = e^{ax}$ $\qquad g(x) = x^2$

$f'(x) = ae^{ax}$ $\qquad g'(x) = 2x$

$f''(x) = a^2e^{ax}$ $\qquad g''(x) = 2$

$f^{(n)}(x) = a^ne^{ax}$ $\qquad g'''(x) = 0 = g^{(4)}(x) = \cdots = g^{(n)}(x)$

$$y^{(n)} = a^ne^{ax}x^2 + 2na^{n-1}e^{ax}x + (n)(n-1)a^{n-2}e^{ax}$$

Example (46)

Let us compute the $100th$ derivative of the function $y = x^2 \sin x$.

Solution

We have

$$y^{(100)} = (\sin x)^{(100)}x^2 + 200(\sin x)^{(99)}x + (100)(99)(\sin x)^{(98)}$$

All the subsequent terms are omitted here since they are identically equal to zero. Consequently,

$$y^{(100)} = x^2\sin\left(x + 100\cdot\frac{\pi}{2}\right) + 200x\sin\left(x + 99\cdot\frac{\pi}{2}\right) + 9900\sin\left(x + 98\cdot\frac{\pi}{2}\right)$$

$$= x^2\sin x - 200x\cos x - 9900\sin x$$

The Method of Logarithmic Differentiation

For (complicated) functions such as general exponential functions and other expressions involving products, quotients, and powers of functions.)

Recall that to find the derivative $\dfrac{d(x^n)}{dx}$, we use the power rule

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Also, we get

$$\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}f'(x)$$

using power rule and the chain rule.

But, we cannot use the power rule to find $\frac{d}{dx}\left(e^{x}\right)$. Thus,

$$\frac{d}{dx}\left(e^{x}\right) \neq x \cdot e^{x-1}$$

Recall that, $\frac{d}{dx}\left(a^{x}\right) = a^{x} \ln a$, which is the differentiation formula for the exponential function.
Thus, we get,

$$\frac{d}{dx}e^{x} = e^{x} \ln e = e^{x}$$

and

$$\frac{d}{dx}\left[a^{f(x)}\right] = a^{f(x)} \cdot f'(x) \cdot \ln a$$

using differentiation formula for exponential function and the chain rule.

Now, we ask the question; what can we write for $\frac{d}{dx}\left(x^{x}\right)$?

Of course, it would be sheer nonsense to write $\frac{d}{dx}\left(x^{x}\right) = x \cdot x^{x-1}$.

It is for these types of functions, and more generally for functions of the type

$$y = \left[f(x)\right]^{g(x)}$$

where both $f(x)$ and $g(x)$ are differentiable functions of x, that we can use the technique of logarithmic differentiation for computing their derivatives. This technique is also used to simplify differentiation of many (complicated) functions involving products, quotients, and powers of different functions. We list below the right technique for differentiating each of the following forms of functions:

$$\left[f(x)\right]^{n} \rightarrow \text{ Power rule}$$

$$a^{f(x)} \rightarrow \text{ Differentiation formula for exponential functions}$$

$$\left[f(x)\right]^{g(x)} \rightarrow \text{ Logarithmic differentiation}$$

Remark

The technique of logarithmic differentiation is so powerful that it can be used for each of these forms.

Procedure of Logarithmic Differentiation

The procedure of logarithmic differentiation involves taking natural logarithm of each side of the given equation. After simplifying (by using properties of logarithms), we differentiate both sides w.r.t. x .

The usefulness of the process is due to the fact that the differentiation of the product of functions is reduced to that of a sum; of their quotients to that of a difference; and of the general exponential to that of the product of simpler functions.

The following solved examples will illustrate the process of logarithmic differentiation.

First, we start with the differentiation of certain (complicated) function involving products, quotients, and powers of functions.

Example (47)

If $y = e^{5x} \sin 2x \cos x$, find $\dfrac{dy}{dx}$.

Solution

Taking the natural logarithm of both sides, we get

$$\ln y = 5\ln e^{x} + \ln\sin 2x + \ln\cos x$$

Differentiating w.r.t. x , we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{5}{e^{x}}\cdot e^{x} + \frac{1}{\sin 2x}\cdot\left(2\cos 2x\right) - \frac{\sin x}{\cos x}$$

$$= 5 + 2\cot 2x - \tan x$$

$$\frac{dy}{dx} = y\left[5 + 2\cot 2x - \tan x\right]$$

$$= e^{5x}\sin 2x\cos x\left[5 + 2\cot 2x - \tan x\right]$$

Example (48)

If $y = e^{4x} \sin^{2} x \tan^{3} x$, find $\dfrac{dy}{dx}$.

Solution

Taking the natural logarithms of both sides, we get

$$\ln y = 4\ln e^{x} + 2\ln\sin x + 3\ln\tan x$$

Differentiating w.r.t. x , we get

$$\frac{1}{y}\frac{dy}{dx}=4\frac{e^{x}}{e^{x}}+\frac{2\cos x}{\sin x}+\frac{3\sec^{2}x}{\tan x}$$

$$=4+2\cot x+\frac{3}{\sin x\cos x}$$

$$\frac{dy}{dx}=y\left[4+2\cot x+\frac{3}{\sin x\cos x}\right]$$

$$=e^{4x}\sin^{2}x\tan^{3}x\left[4+2\cot x+\frac{3}{\sin x\cos x}\right]$$

Example (49)

If $y=\sqrt{\frac{(1+x)(2+x)}{(1-x)(2-x)}}$, find $\frac{dy}{dx}$

Solution

Taking natural logarithm of both sides, we get

$$\ln y=\frac{1}{2}\left[\ln(1+x)+\ln(2+x)-\ln(1-x)-\ln(2-x)\right]$$

Differentiating w.r.t. x , we get

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{2}\left[\frac{1}{1+x}+\frac{1}{2+x}+\frac{1}{1-x}+\frac{1}{2-x}\right]$$

$$=\frac{1}{2}\left[\frac{2}{1-x^{2}}+\frac{4}{4-x^{2}}\right]$$

$$\frac{dy}{dx}=\frac{y}{2}\left[\frac{2}{1-x^2}+\frac{4}{4-x^2}\right]$$

$$=y\left[\frac{1}{1-x^2}+\frac{2}{4-x^2}\right]$$

$$=y\left[\frac{6-3x^2}{\left(1-x^2\right)\left(4-x^2\right)}\right]$$

$$=\sqrt{\frac{(1+x)(2+x)}{(1-x)(2-x)}}\left[\frac{6-3x^2}{\left(1-x^2\right)\left(4-x^2\right)}\right]$$

Now, we consider functions of the type $[f(x)]^{g(x)}$.

Example (50)

If $y=5^{\tan x}$, find $\frac{dy}{dx}$.

Solution

Taking natural logarithm of each side, we get

$$\ln y=\tan x \ln 5$$

Differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx}=\sec^2 x \ln 5$$

$$\frac{dy}{dx}=y\left[\sec^2 x \ln 5\right]$$

$$=5^{\tan x}\left[\sec^2 x \ln 5\right]$$

Example (51)

If x^x, find $\frac{dy}{dx}$.

Solution

Taking the natural logarithm of each side, we obtain

$\ln y=x \ln x$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + 1 \cdot \ln x = 1 + \ln x$$

$$\frac{dy}{dx} = y\left[1 + \ln x\right] = x^{x}\left[1 + \ln x\right]$$

Example (52)

If $y = x^{x^x}$, find $\frac{dy}{dx}$.

Solution

Taking the natural logarithm of each side, we get

$$\ln y = x^{x} \ln x$$

Differentiating both sides w.r.t. x , we get

$$\frac{1}{y}\frac{dy}{dx} = \left(x^{x}\right)\left(\frac{1}{x}\right) + \left(\ln x\right)\left[x^{x}\left(1 + \ln x\right)\right]$$

$$\frac{dy}{dx} = y\left\{x^{x-1} + \left(\ln x\right)\left[x^{x}\left(1 + \ln x\right)\right]\right\}$$

$$= x^{x^x}\left\{x^{x-1} + \left(\ln x\right)\left[x^{x}\left(1 + \ln x\right)\right]\right\}$$

Example (53)

If $y = \left(x^{x}\right)^{x}$, then find $\frac{dy}{dx}$.

Solution

We have $y = \left(x^{x}\right)^{x} = x^{x^2}$

Taking natural logarithm of both sides, we get

$\ln y = x^{2} \ln x$

Differentiating w.r.t. x , we get

$$\frac{1}{y}\frac{dy}{dx} = x^{2}\left(\frac{1}{x}\right) + \left(x^{2}\right)\ln x$$

$$\frac{dy}{dx} = y\left[x + x^{2}\ln x\right]$$

$$= x^{x^2}\left[x + x^{2}\ln x\right]$$

Example (54)

If $y = (\ln x)^x$ find $\dfrac{dy}{dx}$.

Solution

Taking natural logarithm of both the sides, we get

$$\ln y = x \ln[\ln x]$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = x\left(\frac{1}{\ln x}\cdot\frac{1}{x}\right) + 1\cdot\ln(\ln x)$$

$$\frac{dy}{dx} = y\left[\frac{1}{\ln x} + \ln(\ln x)\right]$$

$$= (\ln x)^x\left[\frac{1}{\ln x} + \ln(\ln x)\right]$$

Example (55)

If $y = (\cos x)^{\sin x}$, find $\dfrac{dy}{dx}$.

Solution

Taking natural logarithm of both sides, we get

$\ln y = \sin x \ln \cos x$

Differentiating both sides w.r.t. x , we get

$$\frac{1}{y}\frac{dy}{dx} = \sin x\left[-\frac{\sin x}{\cos x}\right] + \cos x\,(\ln\cos x)$$

$$\frac{dy}{dx} = y\left\{\sin x\left[-\frac{\sin x}{\cos x}\right] + \cos x\,(\ln\cos x)\right\}$$

$$= (\cos x)^{\sin x}\left\{\cos x\,(\ln\cos x) - \frac{\sin^2 x}{\cos x}\right\}$$

Example (56)

If $y = (\tan x)^{\ln x}$, find $\dfrac{dy}{dx}$.

Solution

Taking natural logarithm of each side, we get

$$\ln y = \ln x \cdot \tan x$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \ln x \cdot \frac{\sec^2 x}{\tan x} + \frac{1}{x}\ln(\tan x)$$

$$\frac{dy}{dx} = y\left[\frac{\ln x}{\sin x \cos x} + \frac{\ln(\tan x)}{x}\right]$$

$$= (\tan x)^{\ln x}\left[\frac{\ln x}{\sin x \cos x} + \frac{\ln(\tan x)}{x}\right]$$

Example (57)

If $y = (\sin x)^{\tan x}$, find $\frac{dy}{dx}$.

Solution

Taking the natural logarithm of each side, we get

$$\ln y = \tan x \cdot \ln(\sin x)$$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{y}\frac{dy}{dx} = \tan x \cdot \frac{\cos x}{\sin x} + \sec^2 x \ln \sin x$$

$$\frac{dy}{dx} = y\left[1 + \sec^2 x \cdot \ln \sin x\right]$$

$$= (\sin x)^{\tan x}\left[1 + \sec^2 x \cdot \ln \sin x\right]$$

Example (58)

If $y = (\cos x)^{\ln x}$, find $\frac{dy}{dx}$

Solution

Taking the natural logarithm of each side, we get

$$y = \ln x \cdot \ln \cos x$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \ln x \cdot \frac{-\sin x}{\cos x} + \frac{1}{x}\ln\cos x$$

$$\frac{dy}{dx} = y\left[\frac{1}{x}\ln\cos x - \tan x \cdot \ln x\right]$$

$$= (\cos x)^{\ln x}\left[\frac{1}{x}\ln\cos x - \tan x \cdot \ln x\right]$$

Example (59)

If $x^{y} \cdot y^{x} = 1$, find $\frac{dy}{dx}$

Solution

Taking natural logarithm of both sides, we get

$$\ln x^{y} + \ln y^{x} = 0$$

$$y\ \ln x + x\ \ln y = 0$$

Differentiating w.r.t. x , we get

$$y \cdot \frac{1}{x} + (\ln x)\frac{dy}{dx} + x \cdot \frac{1}{y}\frac{dy}{dx} + \ln y = 0$$

$$\frac{dy}{dx}\left(\ln x + \frac{x}{y}\right) = -\frac{y}{x} - \ln y$$

$$\frac{dy}{dx} = -\frac{\frac{y}{x} + \ln y}{\ln x + \frac{x}{y}}$$

$$= -\frac{y + x \cdot \ln y}{x\left(\ln x + \frac{x}{y}\right)} = -\frac{y\,(y + x \cdot \ln y)}{x\,(x + y\ \ln x)}$$

Example (60)

$$x^{y} + y^{x} = a^{b}\ ,$$

find $\frac{dy}{dx}$

Solution

Putting $u = x^y$ and $v = y^x$, we get

$$u + v = a^b$$

Differentiating w.r.t. x , we have

$$\frac{du}{dx} + \frac{dv}{dx} = 0 \qquad (*)$$

Now, consider $u = x^y$

Taking natural logarithm of both sides, we get

$$\ln u = y \ln x$$

Differentiating both sides w.r.t x , we get

$$\frac{1}{u}\frac{du}{dx} = y \cdot \frac{1}{x} + \frac{dy}{dx} \cdot \ln x$$

$$\frac{du}{dx} = u\left[\frac{y}{x} + \frac{dy}{dx} \cdot \ln x\right]$$

$$= x^y\left[\frac{y}{x} + \frac{dy}{dx} \cdot \ln x\right] \qquad (**)$$

Now, consider $v = y^x$

Taking natural logarithm of both sides, we get

$$\ln v = x \ln y$$

Differentiating both sides w.r.t . x , we get

$$\frac{1}{v}\frac{dv}{dx} = x \cdot \frac{1}{y}\frac{dy}{dx} + \ln y$$

$$\frac{dv}{dx} = v\left[\frac{x}{y}\frac{dy}{dx} + \ln y\right]$$

$$= y^x\left[\frac{x}{y}\frac{dy}{dx} + \ln y\right] \qquad (***)$$

Using (**) and (***) in (*), we get

$$x^y\left[\frac{y}{x} + \frac{dy}{dx} \cdot \ln x\right] + y^x\left[\frac{x}{y}\frac{dy}{dx} + \ln y\right] = 0$$

$$\therefore \frac{dy}{dx}\left[x^{y}\ln x + y^{x}\frac{x}{y}\right] = -\left[x^{y}\frac{y}{x} + y^{x}\ln y\right]$$

$$\therefore \frac{dy}{dx} = -\frac{x^{y}\left(y/x\right) + y^{x}\ln y}{x^{y}\ln x + y^{x}\left(x/y\right)}$$