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الفصل الدراسي الأول

CHAPTER (1)

ERRORS

1. Introduction

In numerical analysis solving a problem is only a part of the process. Another part is to know how far the results are accurate. This is a very important part and is often more difficult than achieving the results themselves. In this part, we take in consideration the errors that arise whether from rounding errors in arithmetic operations or from some other source. Throughout this book, as we look at the numerical solution of various problems, we will simultaneously consider the errors involved in whatever computational procedure is being used.

2. Absolute error and relative error

The error in a computed quantity is defined as

$$\text{Error} = \text{true value} - \text{approximate value}$$

The relative error is a measure of the error in relation to the size of the true value:

$$\text{Relative error} = \frac{\text{error}}{\text{true value}}$$

To simplify the notation when working with these numbers, we will usually denote the true and approximate values of a number x by x_T and x_A, respectively. Then we write

$$\text{Error}(x_A) = x_T - x_A$$

$$\text{Rel}(x_A) = \frac{x_T - x_A}{x_T}$$

As an illustration, consider the well-known approximation

$$\pi = \frac{22}{7}$$

Here $x_T = \pi = 3.14159265\ldots$ and $x_A = 22/7 = 3.1428571\ldots$,

$$\text{Error}\left(\frac{22}{7}\right)=\pi-\frac{22}{7}=-0.00126$$

$$\text{Rel}\left(\frac{22}{7}\right)=\frac{\pi-(22/7)}{\pi}=-0.000402$$

An idea related to relative error is that of significant digits.

ملحوظة

عندما نقرب عدد الى k من الارقام العشرية ان الخطأ الدائري يقع بين النهايتين $\pm\frac{1}{2}\times10^{-k}$ على سبيل المثال عندما نقرب الى ثلاثة ارقام عشريه فإن الخطأ يقع بين $-\frac{1}{2}\times10^{-3}\ ,\ \frac{1}{2}\times10^{-3}$ اي يقع بين $-0.0005\ ,\ 0.0005$

In other words, we say that x_A has m significant digits with respect to x_T if

$$\left|\frac{x_T-x_A}{x_T}\right|\le 0.5\times10^{-m} \qquad (1.1)$$

Example (1.1)

(a) $x_A=0.222$ has three digits of accuracy relative to $x_T=2/9$.

(b) $x_A=23.496$ has four digits of accuracy relative to $x_T=23.494$.

(c) $x_A=0.02138$ has just two digits of accuracy relative to $x_T=0.02144$.

(d) $x_A=22/7$ has three digits of accuracy relative to $x_T=\pi$.

Most people find it easier to measure relative error than significant digits; and in some textbooks, satisfaction of (1.1) is used as the definition of x_A having m significant digits of accuracy.

3. Functional error

If e is the error in the approximated value x_A to the true value x_T so that $x_T=x_A+e$. If e_f denote the error when a function f is evaluated at x_A instead of at x_T, we have

$$f(x_T)=f(x_A)+e_f$$

Therefore

$$e_f = f(x_T) - f(x_A)$$
$$= f(x_A + e) - f(x_A)$$

Expanding $f(x_A + e)$ in a Taylor series, we have

$$e_f = f(x_A + e) - f(x_A)$$
$$= f(x_A) + ef'(x_A) + \frac{1}{2}e^2 f''(x_A) + \ldots - f(x_A)$$

Therefore,

$$e_f = ef'(x_A) + \frac{1}{2}e^2 f''(x_A) + \cdots$$

Hence if e is small (and the second and higher derivatives of f evaluated at x_A are not excessively large) we see that

$$e_f \simeq ef'(x_A)$$

Thus

$$|e_f| \approx |e||f'(x_A)|$$

and if x_A has m significant digits of accuracy, then

$$|e_f| \le 0.5 \times 10^{-m} |f'(x_A)|$$

4. Sources of errors

Imagine solving a scientific-mathematical problem, and suppose this involves a computational procedure. Errors will usually be involved in this process, often of several different kinds. We will give a simple classification of the kinds of error that might occur.

A. Round-off error

When carrying out numerical calculations, digital computers have precision limit on their ability to represent numbers.

The difference between the result produced by a given algorithm using exact arithmetic and the result produced by the same algorithm using finite-precision, rounded arithmetic is called the round-off error. For example,

$$\text{Exact number } \frac{4}{3} = 1.333\ldots$$

$$\text{Rounded number to four significant digits } \frac{4}{3} = 1.333$$

$$\text{Hence Round-off error} = 1.333\ldots - 1.333 = 0.00033\ldots$$

$$\text{Exact number } \frac{5}{3} = 1.666\ldots$$

$$\text{Rounded number to four significant digits } \frac{5}{3} = 1.667$$

$$\text{Hence, Round-off error} = 1.666\ldots - 1.667 = -0.000333\ldots$$

The following table shows the result of rounding exact numbers to N significant digits:

Number	N	Round number	Round-of error
23.764462	5	23.764	0.000462
0.0092746	3	0.00927	0.0000046
1.650045	5	1.6500	0.000045
0.0003786	3	0.000379	−0.0000004
0.57386	3	0.574	−0.00014

The following table shows the result of rounding exact numbers to N decimal places:

Number	N	Round number	Round-of error
23.764462	5	23.76446	0.000002
0.0092746	3	0.009	0.0002746
0.0003786	3	0.000	0.0003786
1.650045	5	1.65004	0.000005
0.57386	3	0.574	−0.00014

B. Imprecision of the given data

If the data are obtained experimentally, then they are known within the limits of experimental error (which can normally be estimated), and this will limit the accuracy of the results of any subsequent calculations. This is obvious fact that the accuracy of results is limited by the accuracy of any initial data.

C. Mistakes

Mistakes are errors which are created by the person performing the calculations. A common mistake is to invert the order of two digits occurring in a number. For example, it is very easy to use the number 62381 instead of the number 63281.

When doing calculations, as many checks as possible should be incorporated in the method itself so that any mistakes come quickly to light.

D. Mathematical approximation error (Truncation error)

Mathematical approximation errors are due to replacing an exact quantity by an approximation one. For example, we introduce an error if we use only a finite number of terms from an infinite series expansion. This error is called a truncation error, that is, the error due to truncating the series somewhere. For example $\sin x$ can be expressed as the infinite series expansion

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots$$

and when x is small the sum of the first three terms, namely

$$x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

will give a good approximation to $\sin x$. The truncation error is then the sum of the remaining terms of the infinite series expansion namely

$$-\frac{x^7}{7!} + \frac{x^5}{9!} - \frac{x^{11}}{11!} + \ldots$$

In general if $f(x)$ is approximated using Taylor series about x_0, where $x = x_0 + h$, then

$$f(x) = f(x_0) + \frac{h}{1!} f'(x_0) + \frac{h^2}{2!} f''(x_0) + \ldots + \frac{h^n}{h!} f^{(n)}(x_0) + R_n$$

where R_n is the truncation error. This error can be calculated as

$$R_n = \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(x_0 + \theta h), \; 0 \le \theta \le 1$$

CHAPTER (2)

FINITE DIFFERENCES

1. Introduction

The calculus of finite differences plays an important role in Numerical methods. It deals with the variations in a function when the independent variable changes by finite jumps which may be equal or unequal. In this chapter, we shall study the variations in a function due to the changes in the independent variable by equal intervals.

2. Finite differences

Let $y = f(x)$ be a discrete function. If $x_0, x_0 + h, x_0 + 2h, \cdots x_0 + nh$ are the successive values of x, where two consecutive values differ by a quantity h, then the corresponding values of y are $y_0, y_1, y_2, \ldots, y_n$. The value of the independent variable x is usually called the arguments and the corresponding functional value is known as the entry. The arguments and entries can be shown in a tabular form as follows:

Argument x	x_0	$x_1 = x_0 + h$	$x_2 = x_0 + 2h$	$\ldots$	$x_n = x_0 + nh$
Entry $y = f(x)$	$y_0 = f(x_0)$	$y_1 = f(x_0 + h)$	$y_2 = f(x_0 + 2h)$	$\ldots$	$y_n = f(x_0 + nh)$

To determine the values of $f(x)$ or $f'(x)$ etc., for some intermediate arguments, the following three types of differences are found useful:

(i) Forward differences
(ii) Backward differences
(iii) Central differences

3. Forward differences

If we subtract from each value of y (except y_0) the preceding value of y we get $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ respectively, known as the first differences of y. These results which may be denoted $\Delta y_0, \Delta y_1, \ldots, \Delta y_n$

i.e.

$$\Delta y_0 = y_1 - y_0, \Delta y_1 = y_2 - y_1, \ldots, \Delta y_{n-1} = y_n - y_{n-1}$$

where Δ is a symbol representing an operation of forward difference, are called first forward differences. Thus, the first forward differences are given by

$$\Delta y_i = y_{i+1} - y_i \ , i = 0,1,2,\ldots,n.$$

Now, the second forward differences are defined as the differences of the first differences, that is,

$$\begin{aligned}
\Delta^2 y_0 &= \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0 \\
&= (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0 \\
\Delta^2 y_1 &= \Delta(\Delta y_1) = \Delta y_2 - \Delta y_1 = y_3 - 2y_2 + y_1 \\
\ldots & \qquad\qquad \ldots \\
\Delta^2 y_n &= \Delta y_{n+1} - \Delta y_n = y_{n+2} - 2y_{n+1} + y_n
\end{aligned}$$

Here, Δ^2 is called second forward difference operator.

Similarly, the third forward differences are:

$$\begin{aligned}
\Delta^3 y_0 &= \Delta(\Delta^2 y_0) = \Delta^2 y_1 - \Delta^2 y_0 = \Delta(\Delta y_1) - \Delta(\Delta y_0) \\
&= \Delta(y_2 - y_1) - \Delta(y_1 - y_0) = \Delta y_2 - 2\Delta y_1 + \Delta y_0 \\
&= (y_3 - y_2) - 2(y_2 - y_2) + y_1 - y_0 \\
&= y_3 - 3y_2 + 3y_1 - y_0 \\
\Delta^3 y_1 &= \Delta^2 y_2 - \Delta^2 y_1 = y_4 - 3y_3 + 3y_2 - y_1 \\
\ldots & \qquad \ldots \qquad \ldots \\
\Delta^3 y_n &= \Delta^2 y_{n+1} - \Delta^2 y_n = y_{n+1} - 3y_{n+2} + 3y_{n+1} - y_n
\end{aligned}$$

In general, the n th forward differences are defined as

$$\boxed{\Delta^n y_k = \Delta^{n-1} y_{k+1} - \Delta^{n-1} y_k}$$

In function notation, the forward differences are as written below:

$$\begin{aligned}
\Delta f(x) &= f(x+h) - f(x) \\
\Delta^2 f(x) &= f(x+2h) - 2f(x+h) + f(x) \\
\Delta^3 f(x) &= f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)
\end{aligned}$$

and so on, where h is step size.

The forward differences are usually arranged in a tabular form in the following manner:

x argument	$y=f(x)$ entry	1st difference	2nd difference	3rd difference	4th difference	5th difference
x_0	$y_0=f(x_0)$					
		Δy_0				
$x_1=x_0+h$	$y_1=f(x_1)$		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
$x_2=x_0+2h$	$y_2=f(x_2)$		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
$x_3=x_0+3h$	$y_3=f(x_3)$		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
$x_4=x_0+4h$	$y_4=f(x_4)$		$\Delta^2 y_3$			
		Δy_4				
$x_5=x_0+5h$	$y_5=f(x_5)$					

The first term in the table y_0 is called the leading term and the differences $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \ldots$ are called leading differences. It can be seen that the differences $\Delta^k y_i$ with a subscript 'i' lie along the diagonal sloping downwards, that is, forward with respect to the direction of x. The above difference table is known as Forward difference table or Diagonal difference table.

- **Properties of Δ**

  The operator "Δ" satisfies the following properties:

  (i) $\Delta[f(x) \pm g(x)] = \Delta f(x) \pm \Delta g(x)$, i.e. Δ is linear.

  (ii) $\Delta[\alpha f(x)] = \alpha \Delta f(x)$, α is a constant.

  (iii) $\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x) = \Delta^n \Delta^m f(x)$, where m and n are positive integers.

  (iv) $\Delta[f(x) \cdot g(x)] \neq f(x) \cdot \Delta g(x)$.

- **Observation 1**

  We can express any higher order forward difference of y_0 in terms of the entries $y_0, y_1, y_2, \ldots, y_n$. From

$$\Delta y_0 = y_1 - y_0$$
$$\Delta^2 y_0 = y_2 - 2y_1 + y_0$$
$$\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$$

  and so on, we can see that the coefficients of the entries on the RHS are binomial coefficients. Therefore, in general,

$$\Delta^n y_0 = y_n - C_1^n y_{n-1} + C_2^n y_{n-2} - \cdots + (-1)^n y_0$$

Very important relation for the Higher order diff

Observation 2

We can express any value of y in terms of leading entry y_0

We know that $\Delta y_0 = y_1 - y_0$

$$\therefore \quad y_1 = y_0 + \Delta y_0 = (1+\Delta) y_0$$

Now,

$$y_2 = y_1 + \Delta y_1 = (1+\Delta) y_1 = (1+\Delta)^2 y_0$$

Similarly, $y_3 = (1+\Delta)^3 y_0$ and so on.

In general,

$$y_n = (1+\Delta)^n y_0 = y_0 + C_1^n \Delta y_0 + C_2^n \Delta^2 y_0 + \cdots + \Delta^n y_0$$

4. Backward differences

The differences $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ when denoted by $\nabla y_1, \nabla y_2, \ldots, \nabla y_n$ respectively, are called the first backward differences, where ∇ is the backward difference operator called nabla operator.

$$\therefore \quad \nabla y_1 = y_1 - y_0, \quad \nabla y_2 = y_2 - y_1, \ldots, \nabla y_n = y_n - y_{n-1}$$

Now the second backward differences are defined as the differences of the first backward differences, i.e.

$$\nabla^2 y_2 = \nabla(\nabla y_2) = \nabla(y_2 - y_1) = \nabla y_2 - \nabla y_1 = (y_2 - y_1) - (y_1 - y_0)$$

$$= y_2 - 2y_1 + y_0$$

$\nabla^2 y_3 = \nabla y_3 - \nabla y_2 = y_3 - 2y_2 + y_1$ and so on.

In general,

$$\nabla^n y_k = \nabla^{n-1} y_k - \nabla^{n-1} y_{k-1}$$

In function notation, these differences are written as

$$\nabla f(x) = f(x) - f(x-h)$$

$$\nabla f(x+h) = f(x+h) - f(x)$$

$$\nabla^2 f(x+2h) = f(x+2h) - 2f(x+h) + f(x)$$

$$\nabla^3 f(x+3h) = f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)$$

and so on, where h is step size.

These backward differences are arranged in a tabular form in the following manner. In this table, the difference $\nabla^k y_i$ with a fixed subscript 'i' lies along the diagonal sloping upwards; that is, backwards with respect to the direction of increasing argument x.

x argument	$y=f(x)$ entry	1st difference	2nd difference	3rd difference	4th difference	5th difference
x_0	$y_0=f(x_0)$					
		∇y_1				
$x_1=x_0+h$	$y_1=f(x_1)$		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
$x_2=x_0+2h$	$y_2=f(x_2)$		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
$x_3=x_0+3h$	$y_3=f(x_3)$		$\nabla^2 y_4$		$\nabla^4 y_5$	
		∇y_4		$\nabla^3 y_5$		
$x_4=x_0+4h$	$y_4=f(x_4)$		$\nabla^2 y_5$			
		∇y_5				
$x_5=x_0+5h$	$y_5=f(x_5)$					

- **Properties of ∇**
  (i) $\nabla[f(x)\pm g(x)]=\nabla f(x)\pm\nabla g(x)$, i.e. ∇ is a linear operator.
  (ii) $\nabla[\alpha f(x)]=\alpha\nabla f(x)$, α is a constant.
  (iii) $\nabla^m\nabla^n f(x)=\nabla^{m+n} f(x)$, m and n are positive integers.
  (iv) $\nabla[f(x)g(x)]\neq[\nabla f(x)].g(x)$.

- **Observation 1**

We can express any higher order backward difference of y_n in terms of the entries $y_0, y_1, y_2, \ldots, y_n$. From

$$\nabla y_n = y_n - y_{n-1}$$
$$\nabla^2 y_n = y_n - 2y_{n-1} + y_{n-2}$$
$$\nabla^3 y_n = y_n - 3y_{n-1} + 3y_{n-2} - y_{n-3}$$

and so on, we can see that the coefficients of the entries on the RHS are binomial coefficients. Therefore, in general,

$$\nabla^n y_n = y_n - C_1^n y_{n-1} + C_2^n y_{n-2} - \cdots + (-1)^n y_0$$

- **Observation 2**

We can express any value of y in terms of y_n and the backward differences ∇y_n, $\nabla^2 y_n$, etc. By definition,

$$\nabla y_n = y_n - y_{n-1}$$

or

$$y_{n-1} = y_n - \nabla y_n = (1-\nabla) y_n$$

Now,

$$y_{n-2} = y_{n-1} - \nabla y_{n-1} = (1-\nabla) y_{n-1} = (1-\nabla)^2 y_n$$

Similarly,

$$y_{n-3} = (1-\nabla)^3 y_n$$

and so on. In general,

$$y_{n-k} = (1-\nabla)^k y_n$$

$$\therefore \quad y_{n-k} = y_n - C_1^k \nabla y_n + C_2^k \nabla^2 y_n - \cdots + (-1)^k \nabla^k y_n$$

5. Central differences

Sometimes, it is more convenient to employ another system of differences known as central differences. In this system the symbol δ is used instead of Δ and is known as central difference operator. The subscript of δy for any difference is the average of the subscripts of the two entries.

$$\therefore \quad \delta y_{1/2} = y_1 - y_0,\ \delta y_{3/2} = y_2 - y_1,\ \delta y_{5/2} = y_3 - y_2, \ldots$$

For higher order differences, we have

$$\delta^2 y_1 = \delta y_{3/2} - \delta y_{1/2},\ \ \delta^2 y_2 = \delta y_{3/2} - \delta y_{12}, \ \ldots,\ \delta^2 y_{3/2} = \delta^2 y_2 - \delta^2 y_1,$$

and so on. The central differences are tabulated below.

x argument	$y=f(x)$ entry	1st difference	2nd difference	3rd difference	4th difference	5th difference
x_0	$y_0=f(x_0)$					
		$\delta y_{1/2}$				
$x_1=x_0+h$	$y_1=f(x_1)$		$\delta^2 y_1$			
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$		
$x_2=x_0+2h$	$y_2=f(x_2)$		$\delta^2 y_2$		$\delta^4 y_2$	
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$
$x_3=x_0+3h$	$y_3=f(x_3)$		$\delta^2 y_3$		$\delta^4 y_3$	
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		
$x_4=x_0+4h$	$y_4=f(x_4)$		$\delta^2 y_4$			
		$\delta y_{9/2}$				
$x_5=x_0+5h$	$y_5=f(x_5)$					

We can see from the table that central differences on the same horizontal line have the same subscript. Also, all odd differences have a fractional subscript, and the even differences have integer subscript.

- **Note**

  From all the three tables, we can see that only the notation changes, not the differences. For examples,

$$y_1-y_0=\Delta y_0=\nabla y_1=\delta y_{1/2}$$

6. Other differences operators

So far we have studied the operators Δ, ∇ and δ. Now we shall introduce other operators like E, μ, D etc. which also play a vital role in numerical methods.

- **Shift operator** E

  If h is step size for the argument x then the operator E is defined as

$$E\,f(x)=f(x+h).$$

It is also called translation operator due to the reason that it results the next value of the function. The higher orders of shift operator are defined as

$$E^2 f(x)=E[Ef(x)]=Ef(x+h)=f(x+2h)$$

Similarly,

$$E^3 f(x)=f(x+3h),$$

$$E^4 f(x) = f(x+4h)$$

In general

$$\boxed{E^n f(x) = f(x+nh) \text{ for any real } n}$$

The inverse shift operator E^{-1} is defined as

$$E^{-1} f(x) = f(x-h)$$

Similarly

$$E^{-n} f(x) = f(x-nh) \text{ for any real } n$$

If y_k, is the function $f(x)$ then $Ey_k = y_{k+1}$ and

$$E^n y_k = y_{k+n}$$

- **Average operator** μ

The average operator μ is defined by

$$\mu f(x) = \frac{1}{2}[f(x+h/2) + f(x-h/2)]$$

i.e. $\mu y(x) = \frac{1}{2}\left[y(x+h/2) + y(x-h/2)\right]$

- **Differential operator** D

The differential operator D is defined as $Df(x) = \frac{d}{dx} f(x)$

In general,

$$D^n f(x) = \frac{d^n}{dx^n} f(x)$$

- **Note**

All the above operators are linear and obey index laws.

7. Relation between different differences operators

- **Relation between** Δ **and** E

$$\Delta f(x) = f(x+h) - f(x)$$
$$= Ef(x) - f(x)$$
$$= (E-1) f(x)$$

Thus $\Delta = E - 1$ or $E = 1 + \Delta$

- **Relation between** E **and** ∇

$$\nabla f(x) = f(x) - f(x-h)$$
$$= f(x) - E^{-1}f(x) = \left(1 - E^{-1}\right)f(x)$$
$$\therefore \nabla = 1 - E^{-1} \text{ or } E^{-1} = 1 - \nabla$$
$$\therefore E = (1-\nabla)^{-1} \quad \left[\because \left(E^{-1}\right)^{-1} = E\right]$$

- **Relation between E and δ**

$$\delta f(x) = f(x+h/2) - f(x-h/2)$$
$$= E^{1/2}f(x) - E^{-1/2}f(x)$$
$$= \left(E^{1/2} - E^{-1/2}\right)f(x)$$
$$\therefore \delta = E^{1/2} - E^{-1/2}$$

Also,

$$\delta = E^{1/2}\left(1 - E^{-1}\right) = E^{1/2}\nabla$$
$$\delta = E^{-1/2}(E-1) = E^{-1/2}\Delta$$

Hence

$$\delta = E^{1/2}\nabla = E^{-1/2}\Delta$$

- **Relation between E and μ**

$$\mu f(x) = \frac{1}{2}[f(x+h/2) + f(x-h/2)]$$
$$= \frac{1}{2}\left[E^{1/2}f(x) + E^{-1/2}f(x)\right]$$
$$= \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)f(x)$$
$$\therefore \mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$$

- **Relation of D with other Operators**

We know that $Df(x) = \dfrac{d}{dx}f(x) = f'(x)$ etc.

By Taylor's series

$$f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots$$

Or

$$Ef(x) = f(x) + hDf(x) + \frac{h^2}{2!}D^2 f(x) + \frac{h^3}{3!}D^3 f(x) + \dots$$

$$= \left[1 + hD + \frac{h^2 D^2}{2!} + \frac{h^3 D^3}{3!} + \dots\right] f(x) = e^{hD} f(x)$$

Thus

$$E = e^{hD}$$

Taking logarithms on both sides, we get

$$hD = \ln E = \ln(1 + \Delta)$$

$$D = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \dots\right]$$

Also,

$$\nabla = 1 - E^{-1}$$

Thus

$$E^{-1} = 1 - \nabla = e^{-hD}$$

Taking logarithm on both sides,

$$-hD = \ln(1 - \nabla)$$

$$D = -\frac{1}{h}\left[-\nabla - \frac{\nabla^2}{2} - \frac{\nabla^3}{3} - \frac{\nabla^4}{4} - \dots\right]$$

$$= \frac{1}{h}\left[\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \dots\right]$$

$$\therefore \quad \sinh(hD) = \frac{e^{hD} - e^{-hD}}{2} = \frac{E - E^{-1}}{2}$$

$$= \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]\left[E^{1/2} - E^{-1/2}\right] = \mu\delta$$

$$\therefore hD = \sinh^{-1}(\mu\delta)$$

Example (2.1)

Construct the forward difference table from the following data:

x	0	1	2	3	4
y	1	1.5	2.2	3.1	4.6

.

Then evaluate $\Delta^3 y_1, y_n$ and y_5.

Solution

The forward differences table is as given below:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		0.5			
1	1.5		0.2		
		0.7		0	
2	2.2		0.2		0.4
		0.9		0.4	
3	3.1		0.6		
		1.5			
4	4.6				

Now

$$\Delta^3 y_1 = (E-1)^3 y_1 = \left(E^3 - 3E^2 + 3E - 1\right) y_1$$
$$= y_4 - 3y_3 + 3y_2 - y_1 = 4.6 - 3(3.1) + 3(2.2) - 1.5 = 0.4$$

Again from observation 2 of section, we have

$$y_n = y_0 + C_1^n \Delta y_0 + C_2^n \Delta^2 y_0 + C_3^n \Delta^3 y_0 + C_4^n \Delta^4 y_0$$
$$= 1 + n(0.5) + \frac{1}{2} n(n-1)(0.2) + \frac{1}{3!} n(n-1)(n-2)(0)$$
$$+ \frac{1}{4!} n(n-1)(n-2)(n-3)(0.4)$$
$$= 1 + \frac{1}{2} n + \frac{1}{10}\left(n^2 - n\right) + \frac{1}{60}\left(n^4 - 6n^3 + 11n^2 - 6n\right)$$
$$\therefore y_5 = \frac{1}{60}\left[5^4 - 6(5)^3 + 17(5)^2 + 18(5) + 60\right] = 7.5$$

Example (2.2)

Evaluate

(i) $\Delta \cos x$ (ii) $\Delta \ln f(x)$ (iii) $\Delta^2 \sin(px+q)$ (1v) $\Delta \tan^{-1} x$ (v) $\Delta^n e^{ax+b}$

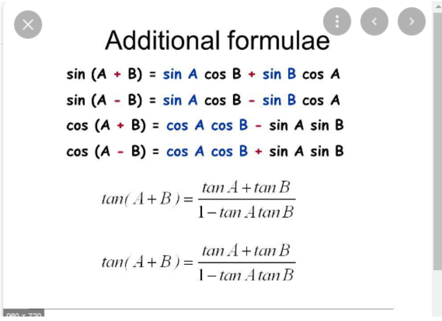
Solution

Let h be the interval step size.

(i)

$$\Delta \cos x = \cos(x+h) - \cos x = -2\sin\left(x + \frac{h}{2}\right)\sin\frac{h}{2}$$

(ii)



$$\Delta \ln f(x) = \ln f(x+h) - \ln f(x)$$
$$= \ln\left[\frac{f(x+h)}{f(x)}\right] = \ln\left[\frac{f(x)+\Delta f(x)}{f(x)}\right]$$
$$= \ln\left[1 + \frac{\Delta f(x)}{f(x)}\right]$$

(iii)

$$\Delta \sin(px+q) = \sin[p(x+h)+q] - \sin(px+q)$$
$$= 2\cos\left(px + q + \frac{ph}{2}\right)\sin\frac{ph}{2}$$
$$= 2\sin\frac{ph}{2}\sin\left(\frac{\pi}{2} + px + q + \frac{ph}{2}\right)$$
$$= 2\sin\frac{ph}{2}\sin\left(px + q + \frac{1}{2}(\pi + ph)\right)$$

Hence

$$\Delta^2 \sin(px+q) = 2\sin\frac{ph}{2}\Delta\left[\sin\left(px + q + \frac{1}{2}(\pi + ph)\right)\right]$$
$$= \left(2\sin\frac{ph}{2}\right)^2 \sin\left(px + q + 2\cdot\frac{1}{2}(\pi + ph)\right)$$

(iv)

$$\Delta \tan^{-1} x = \tan^{-1}(x+h) - \tan^{-1} x$$
$$= \tan^{-1}\left[\frac{x+h-x}{1+x(x+h)}\right]$$
$$= \tan^{-1}\frac{h}{1+x(x+h)}$$

(v)

$$\begin{aligned}\Delta e^{ax+b} &= e^{a(x+h)+b} - e^{ax+b} \\ &= e^{ax+b}(e^{ah}-1) \\ \Delta^2 e^{ax+b} &= \Delta\left[\Delta e^{ax+b}\right] = \Delta\left[(e^{ah}-1)e^{ax+b}\right] \\ &= (e^{ah}-1)^2 e^{ax+b}, \qquad \left[(e^{ah}-1) \text{ is constant}\right]\end{aligned}$$

Proceeding on, we get,

$$\Delta^n\left(e^{ax+b}\right) = \left(e^{ah}-1\right)^n e^{ax+b}$$

Example (2.3)

Prove the following results:

(i) $\Delta\nabla = \nabla\Delta = \Delta - \nabla = \delta^2$

(ii) $\Delta + \nabla = \dfrac{\Delta}{\nabla} - \dfrac{\nabla}{\Delta}$

(iii) $\left(E^{1/2} + E^{-1/2}\right)(1+\Delta)^{1/2} = 2 + \Delta$

(iv) $1 + \mu^2\delta^2 = \left(1 + \dfrac{\delta^2}{2}\right)^2$

(v) $\Delta = \dfrac{\delta^2}{2} + \delta\sqrt{\left(1 + \dfrac{\delta^2}{4}\right)}$

(vi) $\mu^{-1} = 1 - \dfrac{1}{8}\delta^2 + \dfrac{3}{128}\delta^4 - \dfrac{5}{1024}\delta^6 + \dots$

Solution

(i) We have,

$$\begin{aligned}\Delta\nabla f(x) &= \Delta[\nabla f(x)] = \Delta[f(x) - f(x-h)] \\ &= \Delta f(x) - \Delta f(x-h) \\ &= [f(x+h) - f(x)] - [f(x) - f(x-h)] \\ &= \Delta f(x) - \nabla f(x) = (\Delta - \nabla) f(x)\end{aligned}$$

$$\therefore \Delta\nabla = \Delta - \nabla$$

Similarly,

$$\begin{aligned}\nabla\Delta f(x) &= \nabla[\Delta f(x)] = \nabla[f(x+h) - f(x)] \\ &= \nabla f(x+h) - \nabla f(x) \\ &= [f(x+h) - f(x)] - [f(x) - f(x-h)] \\ &= \Delta f(x) - \nabla f(x) = (\Delta - \nabla) f(x)\end{aligned}$$

$$\therefore \nabla\Delta = \Delta - \nabla$$

Again

$$\begin{aligned}\delta^2 f(x) &= \left[E^{1/2} - E^{-1/2}\right]^2 f(x) \\ &= \left(E + E^{-1} - 2\right) f(x) \\ &= f(x+h) + f(x-h) - 2f(x) \\ &= f(x+h) - f(x)] - [f(x) - f(x-h)] \\ &= \Delta f(x) - \nabla f(x) = (\Delta - \nabla) f(x)\end{aligned}$$

$$\delta^2 = \Delta - \nabla$$

Hence

$$\Delta\nabla = \nabla\Delta = \Delta - \nabla = \delta^2$$

(ii)

$$\begin{aligned}\text{R.H.S.} &= \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} = \frac{\Delta^2 - \nabla^2}{\nabla\Delta} \\ &= \frac{(\Delta + \nabla)(\Delta - \nabla)}{(\Delta - \nabla)} \\ &= \Delta + \nabla = \text{ L.H.S.}\end{aligned}$$

(iii)

$$\left(E^{1/2}+E^{-1/2}\right)(1+\Delta)^{1/2}=\left(E^{1/2}+E^{-1/2}\right)E^{1/2}$$
$$=E+1=1+\Delta+1=2+\Delta$$

(iv)

$$1+\mu^2\delta^2=1+\left[\frac{E^{1/2}+E^{-1/2}}{2}\right]^2\left[E^{1/2}-E^{-1/2}\right]^2$$
$$=1+\left[\frac{E-E^{-1}}{2}\right]^2=\frac{4+\left(E-E^{-1}\right)^2}{4}$$
$$=\left[\frac{E+E^{-1}}{2}\right]^2 \qquad (2.1)$$

Now,

$$\left[1+\frac{1}{2}\delta^2\right]^2=\left[1+\frac{1}{2}\left[E^{1/2}-E^{-1/2}\right]^2\right]^2$$
$$=\left[1+\frac{1}{2}\left[E+E^{-1}-2\right]\right]^2=\left[\frac{E+E^{-1}}{2}\right]^2 \qquad (2.2)$$

Hence, from Eqns. (2.1) and (2.2), we have

$$1+\mu^2\delta^2=\left[1+\frac{1}{2}\delta^2\right]^2$$

(v)

$$\text{RHS}=\frac{1}{2}\delta^2+\delta\sqrt{1+\frac{\delta^2}{4}}=\frac{1}{2}\delta\left[\delta+\sqrt{4+\delta^2}\right]$$
$$=\frac{1}{2}\delta\left[\left(E^{1/2}-E^{-1/2}\right)+\sqrt{4+\left(E^{1/2}-E^{-1/2}\right)^2}\right]$$
$$=\frac{1}{2}\delta\left[\left(E^{1/2}-E^{-1/2}\right)+\left(E^{1/2}+E^{-1/2}\right)\right]$$
$$=\frac{1}{2}\left(E^{1/2}-E^{-1/2}\right)\left(2E^{1/2}\right)=E-1=\Delta=\text{LHS}$$

(vi) By definition, we have

$$\mu^2 = \left[\frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)\right]^2$$

$$= \frac{1}{4}\left[\left(E^{1/2} - E^{-1/2}\right)^2 + 4\right]$$

$$= \frac{1}{4}\left(\delta^2 + 4\right) = \frac{\delta^2}{4} + 1$$

$$\therefore \mu = \left[1 + \frac{\delta^2}{4}\right]^{1/2}$$

or

$$\mu^{-1} = \left[1 + \frac{\delta^2}{4}\right]^{-1/2}$$

$$= 1 - \frac{1}{1!}\left(\frac{1}{2}\right)\frac{\delta^2}{4} + \frac{1}{2!}\left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right)\left[\frac{\delta^2}{4}\right]^2 - \frac{1}{3!}\left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)\left[\frac{\delta^2}{4}\right]^3 + \cdots$$

$$= 1 - \frac{1}{8}\delta^2 + \frac{3}{128}\delta^4 - \frac{5}{1024}\delta^6 + \cdots$$

CHAPTER (3)

INTERPOLATION

1. Introduction

Interpolation is a technique of obtaining the value of a function for any intermediate values of the independent variable, i.e. argument within an interval, when the values of the arguments are given. Suppose that the following values of $y = f(x)$ for a set of values of x are given:

x (argument)	x_0	x_1	x_2	$\cdots$	x_n
$y(x)$	y_0	y_1	y_2	$\cdots$	y_n

Then the process of finding the value of y corresponding to any value of $x = x_i$ between x_0 and x_n is called in interpolation.

The process of finding the value of a function outside the given range of arguments is called extrapolation

If the form of the function $f(x)$ is known we can find $f(x)$ for any value of x by simple substitution. But in most practical problems that occur in engineering and science the form of the function $f(x)$ is unknown and it is very difficult to determine its exact form which is the help of tabulated set of values in such cases we replace $f(x)$ by simple function $\varphi(x)$ is called interpolating function which assumes the same values as those of $f(x)$ and from which others are values may be computed to the desired degree of accuracy.

If $\varphi(x)$ is a polynomial then it is called interpolating polynomial and the process is known as polynomial interpolation. If $\varphi(x)$ is a finite trigonometric series the process is called trigonometric interpolation. Usually, polynomial interpolation is preferred due to the reason that they are free from singularities is and the easy to differentiate and integrate. Even though there are other methods like graphical method and method of curve fitting, in this chapter we will study polynomial interpolation using the calculus of finite differences by driving two important interpolation formulae which are used often in all fields by means of forward and the backward differences of a function.

2. Newton forward interpolation formula

Let $y = f(x)$ be a function which takes the values $y_0, y_1, \cdots, y_n$ for $(n+1)$ values $x_0, x_1, \cdots, x_n$ of the dependent variable. Let these values be equidistant $x_i = x_0 + ih$, $i = 0,1,2,\cdots,n$ and let $P(x)$ be a polynomial of n degree such as

$$P(x_i) = f(x_i) = y_i \ , i = 0,1,2,\cdots,n .$$

$$\begin{aligned} P(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) \\ &\quad + a_3(x - x_0)(x - x_1)(x - x_2) + \cdots \\ &\quad + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}) \end{aligned} \tag{3.1}$$

Putting $x = x_0, x_1, \cdots, x_n$ successfully in equation (3.1), we get

$$y_0 = a_0 \ , \ y_1 = a_0 + a_1(x_1 - x_0) ,$$

$$y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\begin{aligned} y_n &= a_0 + a_1(x_n - x_0) + a_2(x_n - x_0)(x_n - x_1) + \cdots \\ &\quad + a_n(x_n - x_0)(x_n - x_1)\cdots(x_n - x_{n-1}) \end{aligned}$$

from these

$$a_0 = y_0 \ , \ a_1 = \frac{y_1 - a_0}{x_1 - x_0} = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h} \ ,$$

$$a_2 = \frac{y_2 - a_0 - a_1(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} = \frac{y_2 - y_0 - 2y_1 + 2y_0}{2h^2}$$

$$= \frac{y_2 - 2y_1 + y_0}{2!h^2} = \frac{\Delta^2 y_0}{2!h^2},$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a_n = \frac{\Delta^n y_0}{n!h^n}$$

Putting these values in equation (3.1) we get

$$\begin{aligned} P(x) &= y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)(x - x_1) \\ &\quad + \frac{\Delta^3 y_0}{3!h^3}(x - x_0)(x - x_1)(x - x_2) + \cdots \\ &\quad + \frac{\Delta^n y_0}{n!h^n}(x - x_0)(x - x_1)\cdots(x - x_{n-1}) \end{aligned} \tag{3.2}$$

Putting

$$\frac{x - x_0}{h} = q$$

$$x = x_0 + qh$$

$$\Downarrow \quad \Downarrow \quad \Downarrow$$

$$x - x_i = x - x_0 + x_0 - x_i = qh - ih = (q - i)h \text{ , } i = 1, 2, \cdots, n$$

where $0 < q < 1$ is real number. Eq. (3.2) takes the form

$$P(x) = y_0 + q\Delta y_0 + \frac{q(q-1)}{2!}\Delta^2 y_0 + \frac{q(q-1)(q-2)}{3!}\Delta^3 y_0 + \cdots + \frac{q(q-1)(q-2)\cdots(q-n+1)}{n!}\Delta^n y_0 \tag{3.3}$$

Equation (3.3) is known as Newton forward interpolation formula

- **Note**

  Formula (3.3) is called Newton forward interpolation formula due the fact that this formula contains values of the tabulated function from y_0 onward to right and none to the left of this value. This formula is used mainly to interpolating the values of y near the beginning of a set of tabulated values and to extrapolating y a little to the left of y_0. The first two terms of the equation will give a linear interpolation while the first three terms a quadratic interpolation and so on.

3. Newton backward interpolation formula

Let $y = f(x)$ be a function which takes the values $y_0, y_1, \cdots, y_n$ for $(n+1)$ values $x_0, x_1, \cdots, x_n$ of the dependent variable. Let these values be equidistant $x_i = x_0 + ih$, $i = 0, 1, 2, \cdots, n$ and let $P(x)$ be a polynomial of n degree such as

$$P(x_i) = f(x_i) = y_i \text{ , } i = 0, 1, 2, \cdots, n.$$

Suppose that it is required to evaluate $y(x)$ near the end of the table values then we can assume that

$$\begin{aligned} P(x) = a_0 &+ a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) \\ &+ a_3(x - x_n)(x - x_{n-1})(x - x_{n-2}) + \cdots \\ &+ a_n(x - x_n)(x - x_{n-1})\cdots(x - x_1) \end{aligned} \tag{3.4}$$

Putting $x = x_0, x_1, \cdots, x_n$ successfully in Eq. (3.4), we get

$$
\begin{aligned}
&a_0 = y_n,\\
&y_{n-1} = a_0 + a_1(x_{n-1} - x_n)\\
&y_{n-2} = a_0 + a_1(x_{n-2} - x_n) + a_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1})\\
&\vdots \qquad\quad \vdots \qquad\qquad \vdots\\
&y_0 = a_0 + a_1(x_0 - x_n) + a_2(x_0 - x_n)(x_0 - x_{n-1})\\
&\qquad + a_3(x_0 - x_n)(x_0 - x_{n-1})(x_0 - x_{n-2}) + \cdots\\
&\qquad + a_n(x_0 - x_n)(x_0 - x_{n-1})\cdots(x_0 - x_1)
\end{aligned}
$$

These equations give

$$
a_0 = y_n\ ,\ \ a_1 = \frac{y_{n-1} - a_0}{x_{n-1} - x_n} = \frac{y_{n-1} - y_n}{x_{n-1} - x_n} = \frac{y_n - y_{n-1}}{x_n - x_{n-1}} = \frac{\nabla y_n}{h}\ ,
$$

$$
a_2 = \frac{y_{n-2} - a_0 - a_1(x_{n-2} - x_n)}{(x_{n-2} - x_n)(x_{n-2} - x_{n-1})} = \frac{y_{n-2} - y_n - 2y_{n-1} + 2y_n}{-2h^2}
$$

$$
= \frac{y_n - 2y_{n-1} + y_{n-2}}{2!h^2} = \frac{\nabla^2 y_n}{2!h^2},
$$

$$
\vdots \qquad\quad \vdots \qquad\qquad \vdots
$$

$$
a_n = \frac{\nabla^n y_n}{n!h^n}
$$

Putting these values in Eq.(3.4) we get

$$
\begin{aligned}
P(x) = y_n &+ \frac{\nabla y_n}{h}(x - x_n) + \frac{\nabla^2 y_n}{2!h^2}(x - x_n)(x - x_{n-1})\\
&+ \frac{\nabla^3 y_n}{3!h^3}(x - x_n)(x - x_{n-1})(x - x_{n-2}) + \cdots\\
&+ \frac{\nabla^n y_n}{n!h^n}(x - x_n)(x - x_{n-1})\cdots(x - x_1) \qquad (3.5)
\end{aligned}
$$

Let

$$
\frac{x - x_n}{h} = q
$$

$$
x = x_n + qh
$$

$$
\Downarrow \quad \Downarrow \quad \Downarrow
$$

$$
x - x_i = x - x_n + x_n - x_i = qh + \left(n - i\right)h = (q + n - i)h\ ,\ i = 1, 2, \cdots, n
$$

Where q is real number. Then Eq.(3.5) takes the form

$$P(x) = y_n + q\nabla y_n + \frac{q(q+1)}{2!}\nabla^2 y_n + \frac{q(q+1)(q+2)}{3!}\nabla^3 y_n + \cdots + \frac{q(q+1)(q+2)\cdots(q+n-1)}{n!}\nabla^n y_n \tag{3.6}$$

Eq.(36) is known as Newton backward interpolation formula

- **Note**

  Since the formula (3.6) involves the backward differences it is called backward interpolation formula and it is used to interpolate the values of y near to the end of a set of tabular values. This may also be used to extrapolate the values of y a little to the right of y_n

Example (3.1)

Find a polynomial which takes the following values

x	0	1	2	3	4	5
$y(x)$	5.2	8.0	10.4	12.4	14.0	15.2

Solution

We take

$$x_0 = 0\,,\ h = x_1 - x_0 = 1\,,\ q = \frac{x - x_0}{h} = \frac{x - 0}{1} = x$$

The forward differences table is as follows:

x	$y(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	5.2				
		2.8			
1	8.0		–0.4		
		2.4		0	
2	10.4		–0.4		0
		2.0		0	
3	12.4		–0.4		0
		1.6		0	
4	14.0		–0.4		
		1.2			
5	15.2				

Using Newton forward interpolation formula, we get

$$P(x) = y_0 + q\Delta y_0 + \frac{q(q-1)}{2!}\Delta^2 y_0 + \frac{q(q-1)(q-2)}{3!}\Delta^3 y$$

$$= 5.2 + 2.8x - \frac{0.4}{2}(x)(x-1)$$

$$= 5.2 + 2.6x - 0.2x^2$$

Example (3.2)

Find a polynomial which takes the following values:

x	1	1.5	2.0	2.5
$y(x)$	4.0	18.25	44.0	84.25

and hence compute $y(1.25)$.

Solution

Take

$$x_0 = 1.0\ ,\ h = x_1 - x_0 \ \Rightarrow\ h = 1.5 - 1.0 = 0.5$$

$$x = x_0 + qh \ \Rightarrow\ q = \frac{x - x_0}{0.5} = \frac{x - 1.0}{0.5} = 2(x-1)$$

The forward differences table is as follows:

x	$y(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
1.0	4.0			
		14.25		
1.5	18.25		11.5	
		25.75		3.0
2.0	44.0		14.5	
		40.25		
2.5	84.25			

Thus

$$y_0 = 4.0\ ,\ \Delta y_0 = 14.25\ \ ,\ \Delta^2 y_0 = 11.5\ ,\ \Delta^3 y_0 = 3.0$$

Using Newton forward interpolation formula, we get

$$P(x) = y_0 + q\Delta y_0 + \frac{q(q-1)}{2!}\Delta^2 y_0 + \frac{q(q-1)(q-2)}{3!}\Delta^3 y$$

$$= 4 + 2(x-1)(14.25) + \frac{1}{2!}[2(x-1)][2(x-1)-1](11.5)$$

$$+ \frac{1}{3!}[2(x-1)][2(x-1)-1][2(x-1)-2](3)$$

Now

$$y(1.25) = 4.0 + (0.5)(14.25) + \frac{(0.5)(-0.5)}{2!}(11.5)$$

$$+ \frac{(0.5)(-0.5)(-1.5)}{3!}(3) = 9.875$$

Example (3.3)

Find a polynomial which takes the following values

x	1	3	5	7	9
y	3	14	19	21	23

and hence compute $y(2), y(10)$.

Solution

The differences table as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	3				
		11			
3	14		-6		
		5		3	
5	19		-3		0
		2		3	
7	21		0		
		2			
9	23				

Take $x_0 = 1, y_0 = 3, h = 2, q = \dfrac{x-1}{2}$

Using Newton forward interpolation formula, we get

$$y(x) = y_0 + q\Delta y_0 + \frac{q(q-1)}{2!}\Delta^2 y_0 + \frac{q(q-1)(q-2)}{3!}\Delta^3 y_0$$

$$= 3 + \frac{x-1}{2}(11) + \frac{1}{2!}\frac{x-1}{2}\left[\frac{x-1}{2} - 1\right](-6)$$

$$+ \frac{1}{3!}\frac{x-1}{2}\left[\frac{x-1}{2} - 1\right]\left[\frac{x-1}{2} - 2\right](3)$$

$$= \frac{1}{16}\left(x^3 - 21x^2 + 159x - 91\right).$$

Again $x_n = 9, y_n = 23, h = 2, q = \dfrac{x-9}{2}$

Using Newton backward interpolation formula, we get

$$y(x) = y_n + q\nabla y_n + \frac{q(q+1)}{2!}\nabla^2 y_n + \frac{q(q+1)(q+2)}{3!}\nabla^3 y_n$$

$$= 23 + \frac{x-9}{2}(2) + \frac{1}{2!}\frac{x-9}{2}\left[\frac{x-9}{2}+1\right](0)$$

$$+ \frac{1}{3!}\frac{x-9}{2}\left[\frac{x-9}{2}+1\right]\left[\frac{x-9}{2}+2\right](3)$$

$$= 23 + (x-9) + \frac{1}{16}(x-9)(x-7)(x-5).$$

Then

$$y(2) = \frac{1}{16}\left(2^3 - 21(2)^2 + 159(2) - 91\right) = 9.4375$$

and

$$y(10) = 23 + (10-9) + \frac{1}{16}(10-9)(10-7)(10-5)$$

Example (3.4)

The amount A of a substance remaining in a reacting system after a time t in a certain chemical experiment is tabulated below

t	2	5	8	11
A	94.8	87.9	81.3	75.1

Obtain the value of A when $t = 9$ using Newton backward interpolation formula.

Solution

Since the value $t = 9$ is near the end of the table, to get the corresponding value of t we use Newton backward interpolation formula.

The backward differences are calculated and tabulated below:

t	A	∇A	$\nabla^2 A$	$\nabla^3 A$
2.0	94.8			
		−6.9		
5.0	87.9		0.3	
		−6.6		0.1
8.0	81.3		0.4	
		−6.2		
11.0	75.1			

Here

$$h = t_1 - t_0 \;\Rightarrow\; h = 5 - 2 = 3 \, , \; t_n = 11.0$$

Hence the interpolation polynomial is

$$A(t) = A_n + q\nabla A_n + \frac{q(q+1)}{2!}\nabla^2 A_n + \frac{q(q+1)(q+2)}{3!}\nabla^3 A_n .$$

If $t = 9$, we have

$$t = t_n + qh \;\Rightarrow\; q = \frac{t - t_n}{h} = \frac{9 - 11.0}{3} = -\frac{2}{3}$$

Therefore

$$A(9) = 75.1 + \left(-\frac{2}{3}\right)(-6.2) + \frac{1}{2!}\left(-\frac{2}{3}\right)\left(-\frac{2}{3}+1\right)(0.4)$$
$$+ \frac{1}{3!}\left(-\frac{2}{3}\right)\left(-\frac{2}{3}+1\right)\left(-\frac{2}{3}+2\right)(0.1) = 79.183951$$

Example (3.5)

Find the missing value in the following table

x	16	18	20	22	24	26
y	43	89		155	268	388

Solution

Since five values are given, it is possible to express y as a polynomial of fourth degree. Hence the fifth differences of y are zeros. Taking the origin for x at 16 , from the given data we have:

$$y_0 = 43, y_1 = 89, y_3 = 155, y_4 = 268, y_5 = 388,$$

and we have to find y_2 . We know that $\Delta^5 y_0 = 0$

$$\Delta^5 y_0 = (E-1)^5 y_0 = 0$$

i.e.

$$\left(E^5 - C_1^5 E^4 + C_2^5 E^3 - C_3^5 E^2 + C_4^5 E - 1\right) y_0 = 0$$

$$\left(E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1\right) y_0 = 0,$$
$$E^5 y_0 - 5E^4 y_0 + 10E^3 y_0 - 10E^2 y_0 + 5Ey_0 - y_0 = 0,$$
$$y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 = 0$$

Substituting the given values, we have

$$388 - 5(268) + 10(155) - 10y_2 + 5(89) - 43 = 0$$
$$\Downarrow$$
$$y_2 = 100$$

Lagrange Interpolation

CHAPTER (4)

NUMERICAL DIFFERENTIATION

1. Introduction

This chapter deals with numerical approximations of derivatives. The first question that comes up to mind is: why do we need to approximate derivatives at all? After all, we know how to analytically differentiate every function. Nevertheless, there are several reasons as of why we still need to approximate derivatives:

- Even if there exists an underlying function that we need to differentiate, we might know its values only at a sampled data set without knowing the function itself.
- There are some cases where it may not be obvious that an underlying function exists and all that we have is a discrete data set. We may still be interested in studying changes in the data, which are related, of course, to derivatives.
- There are times in which exact formulas are available but they are very complicated to the point that an exact computation of the derivative requires a lot of function evaluations. It might be significantly simpler to approximate the derivative numerically instead of computing its exact value.
- When approximating solutions to ordinary (or partial) differential equations, we typically represent the solution as a discrete approximation that is defined on a grid. Since we then have to evaluate derivatives at the grid points, we need to be able to come up with methods for approximating the derivatives at these points, and again, this will typically be done using only values that are defined on a lattice. The underlying function itself (which in this case is the solution of the equation) is unknown.

Consider a set of values (x_i, y_i) of a function $y = f(x)$. The process of computing the derivative or a derivative of the function at some value x from the given set of values is called numerical differentiation. This may be done by first approximating the function by a suitable interpolation formula and then differentiating it as many times as desired.

2. Derivatives using Newton forward interpolation formula

If the values of x are equispaced and the derivative is required near the beginning of the table, we employ Newton forward interpolation formula.

Newton forward interpolation formula is

$$y(x) = y_0 + q\Delta y_0 + \frac{q(q-1)}{2!}\Delta^2 y_0 + \frac{q(q-1)(q-2)}{3!}\Delta^3 y_0 + \cdots + \frac{q(q-1)(q-2)\cdots(q-n+1)}{n!}\Delta^n y_0, \quad (4.1)$$

where $q = \dfrac{x - x_0}{h}$.

Differentiating both sides of equation (4.1) with respect to q, we have

$$\frac{dy}{dq} = \Delta y_0 + \frac{2q-1}{2!}\Delta^2 y_0 + \frac{3q^2-6q+2}{3!}\Delta^3 y_0 + \frac{4q^3-18q^2+22q-6}{4!}\Delta^4 y_0 + \cdots$$

Now

$$\frac{dy}{dx} = \frac{dy}{dq}\cdot\frac{dq}{dx} = \frac{1}{h}\frac{dy}{dq}, \quad \left(\frac{dq}{dx} = \frac{1}{h}\right)$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2q-1}{2!}\Delta^2 y_0 + \frac{3q^2-6q+2}{3!}\Delta^3 y_0 + \frac{4q^3-18q^2+22q-6}{4!}\Delta^4 y_0 + \cdots\right] \quad (4.2)$$

At $x = x_0 \Rightarrow q = 0$. Hence putting $q = 0$ in equation, we get

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

Differentiating Eq.(4.2) with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dq}\left(\frac{dy}{dx}\right)\frac{dq}{dx} = \frac{1}{h}\cdot\frac{d}{dq}\left(\frac{dy}{dx}\right)$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 + (q-1)\Delta^3 y_0 + \frac{6q^2-18q+11}{12}\Delta^4 y_0 + \cdots\right] \quad (4.3)$$

Putting $q = 0$ in equation, we get

$$\left.\frac{d^2y}{dx^2}\right|_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots\right]$$

Similarly

$$\left.\frac{d^3y}{dx^3}\right|_{x=x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{2}{3}\Delta^4 y_0 + \cdots\right]$$

And so on.

3. Derivatives using Newton backward interpolation formula

If the derivative is required near the end of the table, we use the backward interpolation formula.

Newton backward interpolation formula

$$y(x) = y_n + q\nabla y_n + \frac{q(q+1)}{2!}\nabla^2 y_n + \frac{q(q+1)(q+2)}{3!}\nabla^3 y_n + \cdots + \frac{q(q+1)(q+2)\cdots(q+n-1)}{n!}\nabla^n y_n, \quad (4.4)$$

where $q = \dfrac{x - x_n}{h}$.

Differentiating both sides of Eq. (4.4) with respect to q , we have

$$\frac{dy}{dq} = \nabla y_n + \frac{2q+1}{2!}\nabla^2 y_n + \frac{3q^2+6q+2}{3!}\nabla^3 y_n + \frac{4q^3+18q^2+22q+6}{4!}\nabla^4 y_n + \cdots$$

Now

$$\frac{dy}{dx} = \frac{dy}{dq}\cdot\frac{dq}{dx} = \frac{1}{h}\frac{dy}{dq}, \quad \left(\frac{dq}{dx} = \frac{1}{h}\right)$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2q+1}{2!}\nabla^2 y_n + \frac{3q^2+6q+2}{3!}\nabla^3 y_n + \frac{4q^3+18q^2+22q+6}{4!}\nabla^4 y_n + \cdots\right] \quad (4.5)$$

At $x = x_n \Rightarrow q = 0$. Hence, putting $q = 0$ in equation, we get

$$\left.\frac{dy}{dx}\right|_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right]$$

Again differentiating Eq. (4.5) with respect to x we get

$$\frac{d^2y}{dx^2} = \frac{d}{dq}\left(\frac{dy}{dx}\right)\frac{dq}{dx} = \frac{1}{h}\cdot\frac{d}{dq}\left(\frac{dy}{dx}\right)$$

$$= \frac{1}{h^2}\left[\nabla^2 y_n + (q+1)\nabla^3 y_n + \frac{6q^2+18q+11}{12}\nabla^4 y_n + \cdots\right] \quad (4.6)$$

Putting $q = 0$ in Eq. (4.6) , we get

$$\left.\frac{d^2y}{dx^2}\right|_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n - \cdots\right]$$

Similarly

$$\left.\frac{d^3y}{dx^3}\right|_{x=x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{2}{3}\nabla^4 y_n + \cdots\right]$$

and so on.

Example (4.1)

Find the first, second and third derivatives of $y(x)$ at $x = 1.5$ if

x	1.5	2.0	2.5	3.0	3.5	4.0
$y(x)$	3.375	7.000	13.625	24.000	38.875	59.000

Solution

We have to find the derivative at the point $x = 1.5$ which is at the beginning of the given data. Therefore we use here the derivative of Newton forward interpolation formula. The forward differences table as follows

x	$y(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.5	3.375				
		3.625			
2.0	7.000		3.000		
		6.625		0.750	
2.5	13.625		3.750		0
		10.375		0.750	
3.0	24.000		4.500		0
		14.875		0.750	
3.5	38.875		5.250		
		20.125			
4.0	59.000				

Here $x_0 = 1.5$, $h = x_1 - x_0 = 0.5$, from Eq. (4.2) we have

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

Thus

$$y'(1.5) = \frac{1}{0.5}\left[3.625 - \frac{1}{2}(3) + \frac{1}{3}(0.75)\right] = 4.75$$

from Eq.(4.3) we have

$$\left.\frac{d^2 y}{dx^2}\right|_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots\right]$$

Hence

$$y''(1.5) = \frac{1}{(0.5)^2}\left[3 - 0.75\right] = 9$$

Again from Eq.(4.4) we have

$$\left.\frac{d^3y}{dx^3}\right|_{x=x_0}=\frac{1}{h^3}\left[\Delta^3 y_0-\frac{2}{3}\Delta^4 y_0+\cdots\right]$$

Thus

$$y'''(1.5)=\frac{1}{(0.5)^3}[0.75]=6$$

Example (4.2)

The population of a certain town is shown in the following table

x	1951	1961	1971	1981	1991
y	19.96	36.65	58.81	77.21	94.61

Find the rate of growth of the population in the year 1981.

Solution

Here we have to find the derivative at 1981 which is near the end of the table. Hence we use derivative of Newton backward difference formula. The table of differences is as follows

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1951	19.96				
		16.69			
1961	36.65		5.47		
		22.16		–9.23	
1971	58.81		–3.76		11.9
		18.40		2.76	
1981	77.21		–1		
		17.40			
1991	94.61				

Hence

$$h=10,\ x_n=1991,\ q=\frac{x-x_n}{h}=\frac{1981-1991}{10}=-1$$

we know from Eq.(4.5) that

$$\left.\frac{dy}{dx}\right|_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{2q+1}{2!}\nabla^2 y_n + \frac{3q^2+6q+2}{3!}\nabla^3 y_n + \frac{4q^3+18q^2+22q+6}{4!}\nabla^4 y_n + \cdots\right]$$

Now we have to find out the rate of growth of the population in the year 1981

$$y'(1981) = \frac{1}{10}\left[17.4 + \frac{2(-1)+1}{2!}(-1) + \frac{3(-1)^2+6(-1)+2}{3!}(2.76) + \frac{4(-1)^3+18(-1)^2+22(-1)+6}{4!}(11.99)\right] = 1.6440833$$

The rate of growth of the population in year 1981 is 1.6440833

Example (4.3)

Find the first and second derivative of the function tabulated below at the point $x = 1.9$.

x	1.0	1.2	1.4	1.6	1.8	2.0
$y(x)$	0.000	0.128	0.544	1.296	2.432	4.00

Solution

We have to find the derivative at the point $x = 1.9$ which is near the end of the given data. Therefore we use the derivative of Newton backward interpolation formula. The backward differences table as follows

x	$y(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1.0	0.000				
		0.128			
1.2	0.128		0.288		
		0.416		0.048	
1.4	0.544		0.336		0
		0.752		0.048	
1.6	1.296		0.384		0
		1.136		0.048	
1.8	2.432		0.432		
		1.568			
2.0	4.000				

Here

$$x_n = 2, \; h = x_1 - x_0 = 0.2 \; , \; q = \frac{x - x_n}{h} = \frac{1.9 - 2.0}{0.2} = -0.5$$

we know from Eq.(4.5) that

$$\left.\frac{dy}{dx}\right|_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{2q+1}{2!}\nabla^2 y_n + \frac{3q^2+6q+2}{3!}\nabla^3 y_n + \frac{4q^3+18q^2+22q+6}{4!}\nabla^4 y_n + \cdots\right]$$

Thus

$$y'(1.9) = \frac{1}{0.2}\left[1.568 + \frac{2(-0.5)+1}{2!}(0.432) + \frac{3(-0.5)^2+6(-0.5)+2}{3!}(0.048)\right] = 7.83$$

we know from Eq.(4.6) that

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (q+1)\nabla^3 y_n + \frac{6q^2+18q+11}{12}\nabla^4 y_n + \cdots\right]$$

Hence

$$y''(1.9) = \frac{1}{(0.2)^2}\left[0.432 + (-0.5+1)(0.048)\right] = 11.4.$$

4. Two points first derivative approximation

I. First derivative forward differences approximation

The Taylor expansion of $f(x_i + h)$ about x_i is given by:

$$f(x_i + h) = f(x_i) + \frac{h}{1!}f'(x_i) + \frac{h^2}{2!}f''(x_i) + \frac{h^3}{3!}f'''(x_i) + \cdots$$
$$+ \frac{h^n}{n!}f^{(n)}(x_i) + \frac{h^{n+1}}{(n+1)!}f^{(n+1)}(\xi), \quad \xi \in (x_i, x_i + h)$$

For such expansion to be valid, we assume that $f(x)$ has $(n+1)$th continuous derivatives at the point $x = x_i$. Neglecting terms of degree higher than two, we obtain

$$f(x_i + h) = f(x_i) + \frac{h}{1!}f'(x_i) + \frac{h^2}{2!}f''(\xi), \quad \xi \in (x_i, x_i + h)$$

which turns into

$$f'(x_i) = \frac{f(x_i + h) - f(x_i)}{h} - \frac{h}{2!}f''(\xi), \quad \xi \in (x_i, x_i + h) \tag{4.7}$$

Eq. (4.7) can be written as

$$f'(x_i) = F + E_F,$$

where

$$F = \frac{f(x_i + h) - f(x_i)}{h}, \quad E_F = -\frac{h}{2!}f''(\xi), \quad \xi \in (x_i, x_i + h)$$

F is called forward differences formula for approximating $f'(x_i)$ and E_F is the error.

II. First derivative backward differences approximation

The Taylor expansion of $f(x_i - h)$ about x_i is given by:

$$f(x_i - h) = f(x_i) - \frac{h}{1!}f'(x_i) + \frac{h^2}{2!}f''(x_i) - \frac{h^3}{3!}f'''(x_i) + \cdots$$
$$+ (-1)^n\frac{h^n}{n!}f^{(n)}(x_i) + (-1)^{n+1}\frac{h^{n+1}}{(n+1)!}f^{(n+1)}(\xi), \quad \xi \in (x_i - h, x_i)$$

For such expansion to be valid, we assume that $f(x)$ has (n+1)*th* continuous derivatives at the point $x = x_i$. Neglecting terms of degree higher than two, we obtain

$$f(x_i - h) = f(x_i) - \frac{h}{1!} f'(x_i) + \frac{h^2}{2!} f''(\xi), \quad \xi \in (x_i - h, x_i)$$

which turns into

$$f'(x_i) = \frac{f(x_i) - f(x_i - h)}{h} + \frac{h}{2!} f''(\xi), \quad \xi \in (x_i - h, x_i) \qquad (4.8)$$

Eq. (4.8) can be written as

$$f'(x_i) = B + E_B,$$

where

$$B = \frac{f(x_i) - f(x_i - h)}{h}, \quad E_B = \frac{h}{2!} f''(\xi), \quad \xi \in (x_i - h, x_i)$$

B is called backward differences formula for approximating $f'(x_i)$ and E_B is the error.

Example (4.4)

Find the first derivative approximation of the function $f(x) = \cos(\pi x)$ at $x = \frac{\pi}{4}$ using forward differences approximation formula (take $h = 0.01$)

Solution

The forward differences approximation formula of the first derivative defined as

$$f'(x_i) = \frac{f(x_i + h) - f(x_i)}{h},$$

then we have

$$f'(\pi/4) = \frac{f(\pi/4 + 0.01) - f(\pi/4)}{0.01}$$

$$= \frac{0.700000476 - 0.707106781}{0.01} = 0.71063051$$

Example (4.5)

Find the first derivative of the function tabulated below at the point $x = 0.2$ using both forward differences and backward differences approximation formulae

x	0.1	0.2	0.3	0.4	0.5
y	0.0001	0.0016	0.0081	0.0256	0.0625

Solution

The forward differences approximation formula of the first derivative defined as

$$f'(x_i) = \frac{f(x_i + h) - f(x_i)}{h},$$

then we have

$$f'(0.2) = \frac{f(0.3) - f(0.2)}{0.1} = \frac{0.0081 - 0.0016}{0.1} = 0.065$$

The backward differences approximation formula of the first derivative defined as

$$f'(x_i) = \frac{f(x_i) - f(x_i - h)}{h},$$

then we have

$$f'(0.2) = \frac{f(0.2) - f(0.1)}{0.1} = \frac{0.0016 - 0.0001}{0.1} = 0.015$$

5. **Three points first derivative approximation**

The Taylor expansion of $f(x_i + h)$ about x_i is given by:

$$f(x_i + h) = f(x_i) + \frac{h}{1!}f'(x_i) + \frac{h^2}{2!}f''(x_i) + \frac{h^3}{3!}f'''(x_i) + \cdots + \frac{h^n}{n!}f^{(n)}(x_i) + \frac{h^{n+1}}{(n+1)!}f^{(n+1)}(\xi), \quad \xi \in (x_i, x_i + h). \tag{4.9}$$

While, the Taylor expansion of $f(x_i - h)$ about x_i is given by:

$$f(x_i - h) = f(x_i) - \frac{h}{1!} f'(x_i) + \frac{h^2}{2!} f''(x_i) - \frac{h^3}{3!} f'''(x_i) + \cdots$$
$$+ (-1)^n \frac{h^n}{n!} f^{(n)}(x_i) + (-1)^{n+1} \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(\xi), \quad \xi \in (x_i - h, x_i). \tag{4.10}$$

Subtracting Eq. (4.10) from Eq. (4.9) and neglecting terms of degree higher than three, we obtain

$$f(x_i + h) - f(x_i - h) = 2h f'(x_i) + \frac{h^3}{3!}\left[f'''(\xi_1) + f'''(\xi_2)\right]$$

If the third-order derivative $f'''(x)$ is a continuous function in the interval $\left[x_i - h, x_i + h\right]$, then the intermediate value theorem implies that there exists a point $\xi \in \left(x_i - h, x_i + h\right)$ such that

$$f'''(\xi) = \frac{1}{2}\left[f'''(\xi_1) + f'''(\xi_2)\right]$$

Hence

$$f'(x_i) = \frac{f(x_i + h) - f(x_i - h)}{2h} - \frac{h^2}{6} f'''(\xi) \tag{4.11}$$

Eq. (4.11) can be written as

$f'(x_i) = C + E_C$,

where

$$C = \frac{f(x_i + h) - f(x_i - h)}{h} \quad , \; E_C = \frac{h}{2!} f''(\xi), \quad \xi \in (x_i - h, x_i + h)$$

C is called central differences formula for approximating $f'(x_i)$ and E_C is the error.

Example (4.6)

Find the first derivative of the function tabulated below at the point $x = 0.2$ using central differences approximation formula

x	0.1	0.2	0.3	0.4	0.5
y	0.0001	0.0016	0.0081	0.0256	0.0625

Solution

The central differences approximation formula of the first derivative defined as:

$$f'(x_i)=\frac{f(x_i+h)-f(x_i-h)}{2h},$$

so we have

$$f'(0.2)=\frac{f(0.3)-f(0.1)}{2(0.1)}=\frac{0.0081-0.0001}{2(0.1)}=0.04$$

6. **Three points second derivative approximation**

For the second derivative approximation, we add Eq. (4.9) and Eq. (4.10) and neglecting terms of degree higher than four to obtain

$$f(x_i+h)+f(x_i-h)=2f(x_i)+h^2f''(x_i)+\frac{2h^4}{4!}f''''(\xi),\ \xi\in\left(x_i-h,x_i+h\right)$$

So, we have

$$f''(x_i)=\frac{f(x_i+h)-2f(x_i)+f(x_i-h)}{h^2}-\frac{h^2}{12}f''''(\xi),\ \xi\in\left(x_i-h,x_i+h\right) \qquad (4.12)$$

Eq. (4.12) can be written as

$$f''(x_i)=S+E_S,$$

where

$$S=\frac{f(x_i+h)-2f(x_i)+f(x_i-h)}{h^2},\quad E_S=-\frac{h^2}{12}f''''(\xi),\quad \xi\in(x_i-h,x_i+h)$$

S is called differences approximation formula of $f''(x_i)$ and E_S is the error.

Example (4.7)

Find the second derivative of the function tabulated below at the point $x=0.2$ using differences approximation formula

x	0.1	0.2	0.3	0.4	0.5
y	0.0001	0.0016	0.0081	0.0256	0.0625

Solution

The differences approximation formula of the second derivative defined as

$$f''(x_i) = \frac{f(x_i + h) - 2f(x_i) + f(x_i - h)}{h^2}.$$

So, we have

$$f''(0.2) = \frac{f(0.3) - 2f(0.2) + f(0.1)}{(0.1)^2}$$

$$= \frac{0.0081 - 2\left(0.0016\right) + 0.0001}{(0.1)^2} = 0.5$$

CHAPTER (5)

NUMERICAL INTEGRATION

1. Introduction

The process of computing $\int_a^b y(x)dx$ where $y = f(x)$ is given by a set of tabulated values $[x_i, y_i]$, $i = 0,1,2,\cdots,n$, $a = x_0$, $b = x_n$ is called numerical integration. Like that of numerical differentiation, here we also replace $y = f(x)$ by an interpolation formula and integrate it between the given limits. In this way we can derive a quadrature formula for approximate integration of a function defined by a set of numerical values.

2. General quadrature formula

In this section we will derive a general quadrature formula for equidistant mesh points.

Let

$$I = \int_a^b y\, dx \text{ , where } y = f(x),$$

takes the values $y_0, y_1, \ldots, y_n$ for $x_0, x_1, \ldots, x_n$. Let us divide the interval (a,b) into n equal parts of width h, so that

$$a = x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, \ldots, x_n = x_0 + nh = b .$$

Then,

$$I = \int_{x_0}^{x_0+nh} f(x)dx$$

Putting, $x = x_0 + qh$, so that $dx = hdq$ in above, we get,

$$I = h\int_0^n f(x_0 + qh)dq = h\int_0^n y(x)dq .$$

Now replacing $y(x)$ by Newton forward interpolation formula we get,

$$I = h\int_0^n \Big[y_0 + q\Delta y_0 + \frac{q(q-1)}{2!}\Delta^2 y_0 + \frac{q(q-1)(q-2)}{3!}\Delta^3 y_0$$
$$+ \frac{q(q-1)(q-2)(q-3)}{4!}\Delta^4 y_0$$
$$+ \frac{q(q-1)(q-2)(q-3)(q-4)}{5!}\Delta^5 y_0$$
$$+ \frac{q(q-1)(q-2)(q-3)(q-4)(q-5)}{6!}\Delta^6 y_0 + \cdots \Big] dq$$

Now integrating a term by term we get after substituting the limits as

$$I = h\Bigg[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left\{\frac{n^3}{3} - \frac{n^2}{2}\right\}\Delta^2 y_0 + \frac{1}{3!}\left\{\frac{n^4}{4} - n^3 + n^2\right\}\Delta^3 y_0$$
$$+ \frac{1}{4!}\left\{\frac{n^5}{5} - \frac{3n^4}{2} + \frac{11n^3}{3} - 3n^2\right\}\Delta^4 y_0$$
$$+ \frac{1}{5!}\left\{\frac{n^6}{6} - 2n^5 + \frac{35n^4}{4} - \frac{50n^3}{3} + 12n^2\right\}\Delta^5 y_0$$
$$+ \frac{1}{6!}\left\{\frac{n^7}{7} - \frac{15n^6}{6} + 17n^5 - \frac{225n^4}{4} + \frac{274n^3}{3} - 60n^2\right\}\Delta^6 y_0 \Bigg] \qquad (5.1)$$

Eq.(5.1) is known as Newton-Cote's quadrature formula which is general quadratic formula for equidistant mesh points. In the following sections we deduce important quadrature formula for this equation taking $n = 1,2,3$.

3. Trapezoidal rule

Putting $n = 1$ in Eq. (5.1) and neglecting second and higher order differences we get

$$\int_{x_0}^{x_0+h} y(x)dx = h\int_0^1 y(x)dq = h\left[y_0 + \frac{1}{2}\Delta y_0 \right]$$
$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0) \right] = \frac{h}{2}[y_0 + y_1]$$

Similarly

$$\int_{x_0+h}^{x_0+2h} y(x)dx = \frac{h}{2}[y_1 + y_2]$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\int_{x_0+(n-1)h}^{x_0+nh} y(x)dx = \frac{h}{2}[y_{n-1} + y_n]$$

Adding these n integrals, we get,

$$I = \int_{x_0}^{x_0+nh} y(x)dx = \frac{h}{2}\left[\left(y_0 + y_n\right) + 2\left(y_1 + y_2 + \ldots + y_{n-1}\right)\right] \tag{5.2}$$

Eq.(5.2) is known as trapezoidal rule.

4. Simpson's $1/3$ rule

Here, taking $n = 2$ in Eq.(5.1) and neglecting third and higher-order differences, we get

$$\int_{x_0}^{x_0+2h} y(x)dx = h\int_0^2 y(x)dq = h\left[2y_0 + 2\Delta y_0 + \frac{1}{2}\left(\frac{8}{3} - 2\right)\Delta^2 y_0\right]$$

$$= h\left[2y_0 + 2(y_1 - y_0) + \frac{1}{3}(y_2 - 2y_1 + y_0)\right]$$

$$= \frac{h}{3}[y_0 + 4y_1 + y_2]$$

Similarly

$$\int_{x_0+2h}^{x_0+4h} y(x)dx = \frac{h}{3}[y_2 + 4y_3 + y_4]$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\int_{x_0+(n-2)h}^{x_0+nh} y(x)dx = \frac{h}{3}[y_{n-2} + 4y_{n-1} + y_n],$$

where n is even. Adding all these integrals, we get

$$I = \int_{x_0}^{x_0+nh} y(x)dx = \frac{h}{3}\Big[\left(y_0 + y_n\right) + 4\left(y_1 + y_3 + \ldots + y_{n-1}\right)$$

$$+2\left(y_2 + y_4 + \ldots + y_{n-2}\right)\Big] \tag{5.3}$$

Eq.(5.3) is known as Simpson's $1/3$ rule.

5. Simpson's $3/8$ rule

Putting $n=3$ in Eq.(5.1) and neglecting all differences above the third order, we get

$$\int_{x_0}^{x_0+3h} y(x)dx = h\int_0^3 y(x)dq$$

$$= h\left[3y_0 + \frac{9}{2}\Delta y_0 + \frac{1}{2}\left(\frac{27}{3}-\frac{9}{2}\right)\Delta^2 y_0 + \frac{1}{3!}\left(\frac{81}{4}-27+9\right)\Delta^3 y_0\right]$$

$$= h\left[3y_0 + \frac{9}{2}(y_1 - y_0) + \frac{9}{4}(y_2 - 2y_1 + y_0) + \frac{3}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

Similarly

$$\int_{x_0+3h}^{x_0+6h} y(x)dx = \frac{3h}{8}\left[y_3 + 3y_4 + 3y_5 + y_6\right]$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\int_{x_0+(n-3)h}^{x_0+nh} y(x)dx = \frac{3h}{8}\left[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right]$$

Adding all these integrals, where n is a multiple of 3, we get

$$I = \int_{x_0}^{x_0+nh} y(x)dx = \frac{3h}{8}\Big[(y_0 + y_n)$$
$$+3(y_1 + y_2 + y_4 + y_5 + y_7 + y_8 + \ldots + y_{n-2} + y_{n-1})$$
$$+2(y_3 + y_6 + \ldots + y_{n-3})\Big] \qquad (5.4)$$

Eq. (5.4) known as Simpson's $3/8$ rule.

Note

- The trapezoidal rule $f(x)$ is linear function of x i.e. of the form $f(x)=ax+b$. It is the simplest rule but least accurate.
- In Simpson's $1/3$ rule, $f(x)$ is a polynomial of second degree, i.e. $f(x)=ax^2+bx+c$. To apply this rule, the number of intervals n must be even.
- In Simpson's $3/8$ rule $f(x)$ is a polynomial of third degree, i.e. $f(x)=ax^3+bx^2+cx+d$. To apply this rule the number of intervals n must be a multiple of 3.

Example (5.1)

Evaluate

$$I = \int_0^{10} \frac{dx}{1+x^2},$$

by using

1. Trapezoidal rule
2. Simpson's $1/3$ rule. Compare the results with the actual value.

Solution

Taking $n = 10$, divide the whole range of the integration into ten equal parts. The value of the integrand function for each point of sub-division are given below:

x	y	y_n
0	1.00000	y_0
1	0.50000	y_1
2	0.200000	y_2
3	0.100000	y_3
4	0.0588235	y_4
5	0.0384615	y_5
6	0.027027	y_6
7	0.0200000	y_7
8	0.0153846	y_8
9	0.0121951	y_9
10	9.9009901×10^{-3}	y_{10}
$\sum$		

1. By Trapezoidal rule

$$I = \int_0^{10} \frac{dx}{1+x^2} = \frac{h}{2}\left[\left(y_0 + y_{10}\right) + 2\left(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9\right)\right]$$

$$= \frac{1}{2}\left[(1 + 9.9009901\times 10^{-3}) + 2(0.5 + 0.2 + 0.1 + 0.0588235 + 0.0384615\right.$$

$$\left. + 0.027027 + 0.02 + 0.0153846 + 0.0121951)\right] = 1.4768422$$

2. By Simpson's $1/3$ rule

$$I = \int_0^{10} \frac{dx}{1+x^2} = \frac{h}{3}\left[\left(y_0 + y_{10}\right) + 4\left(y_1 + y_3 + y_5 + y_7 + y_9\right) + 2(y_2 + y_4 + y_6 + y_8)\right]$$

$$= \frac{1}{3}\left[(1 + 9.9009901\times 10^{-3}) + 4(0.5 + 0.1 + 0.0384615 + 0.02 + 0.0121951)\right.$$

$$\left. + 2(0.2 + 0.0588235 + 0.027027 + 0.0153846)\right] = 1.4316659$$

Example (5.2)

The velocity v of a particle at a distance x from a point on its path is given by the following table:

$x\ (ft)$	0	10	20	30	40	50	60
$v\ (ft\ /s)$	47	58	64	65	61	52	38

Estimate the time taken to travel to $60ft$ using Simpson's $1/3$ rule. Compare the result with Simpson's $3/8$ rule.

Solution

We know that the rate of displacement is velocity , i.e. $v = \dfrac{dx}{dt}$. Therefore the time taken to travel $60ft$ is given by

$$t = \int_0^{60} \frac{1}{v} dx = \int_0^{60} y\, dx$$

where $y = 1/v$. The table is as given below.

x	$y=1/v$	y_n
0	0.0212765	y_0
10	0.0172413	y_1
20	0.015625	y_2
30	0.0153846	y_3
40	0.0163934	y_4
50	0.0192307	y_5
60	0.0263157	y_6

By Simpson's $1/3$ rule

$$I=\int_0^{60} y\,dx=\frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$=\frac{10}{3}\left[(0.0212765+0.0263157)+4(0.0172413+0.0153846+0.0192307)\right.$$

$$\left.+2(0.015625+0.0163934)\right]=1.063518$$

Hence the time taken to travel $60ft$ is $1.064s$.

By Simpson's $3/8$ rule

$$I=\int_0^{60} y\,dx=\frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3\right]$$

$$=\frac{30}{8}\left[(0.0212765+0.0263157)+3(0.0172413+0.015625+0.0163934+0.0192307)\right.$$

$$\left.+2(0.0153846)\right]=1.0643723$$

By this method also the time taken to travel $60ft$ is $1.064s$.

Example (5.3)

Find the following integral by

(i) Trapezoidal rule (ii) Simpson's $1/3$ rule (iii) Simpson's $3/8$ rule

$$I=\int_4^{5.2}\ln x\,dx$$

Solution

Taking $n=6$, divide the whole range of the integration into six equal parts. The value of the integrand function for each point of sub-division are given below:

x	4	4.2	4.4	4.6	4.8	5	5.2
$f(x)=\ln x$	1.386	1.435	1.482	1.526	1.569	1.609	1.649

1. By Trapezoidal rule

$$I=\int_4^{5.2}\ln x\,dx=\frac{h}{2}\left[\left(y_0+y_6\right)+2\left(y_1+y_2+y_3+y_4+y_5\right)\right]$$

$$=\frac{0.2}{2}\left[(1.386+1.649)+2(1.435+1.482+1.526+1.569+1.609)\right]=1.8277$$

2. By Simpson's $1/3$ rule

$$I=\int_4^{5.2}\ln x\,dx=\frac{h}{3}\left[\left(y_0+y_6\right)+4(y_1+y_3+y_5)+2\left(y_2+y_4\right)\right]$$

$$=\frac{0.2}{3}\left[(1.386+1.649)+4(1.435+1.526+1.609)+2(1.482+1.569)\right]=1.8278$$

3. By Simpson's $3/8$ rule

$$I=\int_4^{5.2}\ln x\,dx=\frac{3h}{8}\left[\left(y_0+y_6\right)+3\left(y_1+y_2+y_4+y_5\right)+2y_3\right]$$

$$=\frac{0.6}{8}\left[(1.386+1.649)+3(1.435+1.482+1.569+1.609)+2(1.526)\right]=1.8279$$

Example (5.4)

A rocket is launched from the ground . Its acceleration is registered during the 90 seconds and are given in the table below. Using Simpson's $3/8$ rule, find the velocity of the rocket at $t=90$.

t (s)	0	10	20	30	40	50	60	70	80	90
a (m/ s^2)	30	31.63	33.64	35.47	37.75	40.33	43.25	46.69	50.67	54.87

Solution

We know that the rate of velocity is acceleration , i.e. $a = \frac{dv}{dt}$ Therefore the velocity of the rocket at $t = 90$ is given by

$$v = \int_0^{90} a\, dt\ .$$

By Simpson's $3/8$ rule

$$I = \int_0^{90} a\,dt = \frac{3h}{8}\left[\left(y_0 + y_9\right) + 3\left(y_1 + y_2 + y_4 + y_5 + y_7 + y_8\right) + 2(y_3 + y_6)\right]$$

$$= \frac{30}{8}\left[(30 + 54.87) + 3(31.63 + 33.64 + 37.75 + 40.33 + 46.69 + 50.67) + 2(35.47 + 43.25)\right]$$

$$= 3616.65$$

CHAPTER (6)

SOLUTIONS OF ALGEBRIAC

AND TRANSCENDENTAL EQUATIONS

1. Introduction

We have seen that an expression of the form

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n,$$

where $a's$ are constants $(a_0 \neq 0)$ and n is positive integer, is called a polynomial in x of degree n and the equation $f(x) = 0$ is called an algebraic equation of degree n. If $f(x)$ contains some other functions like exponential, trigonometric, logarithmic , then $f(x) = 0$ is called transcendental equation. For example

$x^3 - 3x + 6 = 0$, $x^5 - 7x^4 + 3x^2 + 36x - 7 = 0$

are algebraic equations. Whereas

$x^2 - 3\cos x + 1 = 0$, $xe^x - 2 = 0, x \log x = 1.2$

are transcendental equations.

In this chapter we will solve algebraic and the transcendental equations. For equation s of degree two or three or four, methods are available to solve them. But the need often arises to solve higher degree or transcendental equation for which no direct method exists. Such equations can be solved by approximate methods. Before we proceed to solve such equations let us recall the fundamental theorem on roots of $f(x) = 0$ in $a \le x \le b$.

Theorem 6.1

If $f(x) = 0$ is continuous function in a closed interval $[a, b]$ and $f(a), f(b)$ are of opposite signs, then the equation $f(x) = 0$ will have at least one real root between a and b.

2. Bisection method

Let the function $f(x)$ be continuous between a and b. For definiteness let $f(a)$ be negative and $f(b)$ be positive, then there is a root of $f(x) = 0$ lying between a and b. Let the first the approximation be $x_1 = \dfrac{a+b}{2}$ (the average of the ends of the range).

Now if $f(x_1) = 0$, then x_1 is a root of $f(x) = 0$. Otherwise, the root will lie between a and x_1 or x_1 and b depending upon whether $f(x_1)$ is positive or negative.
Then, as before we bisect the interval and continue the process till the root is found to the desired accuracy. If $f(x_1)$ is positive, therefore the root lies between a and x_1. The second approximation to the root now is $x_2 = \dfrac{a + x_1}{2}$. If $f(x_2)$ is negative, then the root lies between x_1 and x_2 then, the third approximation to the root is $x_3 = \dfrac{x_1 + x_2}{2}$ and so on. This method is simple but slowly convergent.

Example (6.1)

Find a root of the equation

$$x^3 - x - 11 = 0,$$

correct to four decimal places using bisection method.

Solution

Let

$$f(x) = x^3 - x - 11.$$

Since $f(2) = -5 < 0$ and $f(3) = 13 > 0$, then there exist a real root lies between 2 and 3. Hence, the first approximation to the root is

$$x_1 = \frac{2+3}{2} = 2.5.$$

Now

$$f(2.5) = (2.5)^3 - 2.5 - 11 = 2.125 > 0.$$

Therefore the second approximation lies between 2 and 2.5. Thus the second approximation to the root is

$$x_2 = \frac{2+2.5}{2} = 2.25.$$

Now

$$f\,(2.25) = (2.25)^3 - 2.25 - 11 = -1.859375 < 0.$$

Therefore the third approximation lies between 2.5 and 2.25. Thus the third approximation to the root is

$$x_3 = \frac{x_1 + x_2}{2} = \frac{2.5 + 2.25}{2} = 2.375.$$

Now

$$f\,(2.375) = (2.375)^3 - 2.375 - 11 = 0.0214843 > 0.$$

Therefore the fourth approximation lies between 2.25 and 2.375. Thus the fourth approximation to the root is

$$x_4 = \frac{x_2 + x_3}{2} = \frac{2.25 + 2.375}{2} = 2.3125.$$

Now

$$f\,(2.3125) = (2.3125)^3 - 2.3125 - 11 = -0.9460449 < 0.$$

Therefore the fifth approximation lies between 2.375 and 2.3125. Thus the fifth approximation to the root is

$$x_5 = \frac{x_3 + x_4}{2} = \frac{2.375 + 2.3125}{2} = 2.34375.$$

Now

$$f\,(2.34375) = (2.34375)^3 - 2.34375 - 11 = -0.4691467 < 0.$$

Therefore the sixth approximation lies between 2.375 and 2.34375. Thus the sixth approximation to the root is

$$x_6 = \frac{x_3 + x_5}{2} = \frac{2.375 + 2.34375}{2} = 2.359375.$$

Now

$$f\ (2.359375) = (2.359375)^3 - 2.359375 - 11 = -0.2255592 < 0\,.$$

Therefore the seventh approximation lies between 2.375 and 2.359375. Thus the seventh approximation to the root is

$$x_7 = \frac{x_3 + x_6}{2} = \frac{2.375 + 2.359375}{2} = 2.3671875\,.$$

Now

$$f\ (2.3671875) = (2.3671875)^3 - 2.3671875 - 11 = -0.1024708 < 0\,.$$

Which means that the eighth approximation lies between 2.375 and 2.3671875. Thus the eighth approximation to the root is

$$x_8 = \frac{x_3 + x_7}{2} = \frac{2.375 + 2.3671875}{2} = 2.3710938\,.$$

Now

$$f\ (2.3710938) = (2.3710938)^3 - 2.3710938 - 11 = -0.040601 < 0\,.$$

Which means that the ninth approximation lies between 2.375 and 2.3710938. Thus the ninth approximation to the root is

$$x_9 = \frac{x_3 + x_8}{2} = \frac{2.375 + 2.3710938}{2} = 2.3730469\,.$$

Now

$$f\ (2.3730469) = (2.3730469)^3 - 2.3730469 - 11 = -9.585864 \times 10^{-3} < 0\,.$$

Therefore the tenth approximation lies between 2.375 and 2.3730469. Thus the tenth approximation to the root is

$$x_{10} = \frac{x_3 + x_9}{2} = \frac{2.375 + 2.3730469}{2} = 2.3740235\,.$$

Now

$$f\ (2.3740235) = (2.3740235)^3 - 2.3740235 - 11 = 5.942463 \times 10^{-3} > 0\,.$$

Therefore the eleventh approximation lies between 2.3730469 and 2.3740235. Thus the eleventh approximation to the root is

$$x_{11} = \frac{x_9 + x_{10}}{2} = \frac{2.3730469 + 2.3740235}{2} = 2.3735352 .$$

Now

$$f\,(2.3135352) = (2.3135352)^3 - 2.3135352 - 11 = -1.823398 \times 10^{-3} < 0$$

Therefore the twelfth approximation lies between 2.3740235 and 2.3735352. Thus the twelfth approximation to the root is

$$x_{12} = \frac{x_{10} + x_{11}}{2} = \frac{2.3740235 + 2.3735352}{2} = 2.3737793 .$$

Now

$$f\,(2.3737793) = (2.3737793)^3 - 2.3737793 - 11 = 2.059107 \times 10^{-3} > 0 .$$

Therefore the thirteenth approximation lies between 2.3735352 and 2.3737793. Thus the thirteenth approximation to the root is

$$x_{13} = \frac{x_{11} + x_{12}}{2} = \frac{2.3735352 + 2.3737793}{2} = 2.3736572 .$$

Now

$$f\,(2.3736572) = (2.3736572)^3 - 2.3736572 - 11 = 1.17748 \times 10^{-4} > 0 .$$

Therefore the fourteenth approximation lies between 2.3735352 and 2.3736572. Thus the fourteenth approximation to the root is

$$x_{14} = \frac{x_{11} + x_{13}}{2} = \frac{2.3735352 + 2.3736572}{2} = 2.3735962 .$$

Now

$$f\,(2.3735962) = (2.3735962)^3 - 2.3735962 - 11 = -8.52851 \times 10^{-4} < 0 .$$

Therefore the fifteenth approximation lies between 2.3736572 and 2.3735962. Thus the fifteenth approximation to the root is

$$x_{15} = \frac{x_{13} + x_{14}}{2} = \frac{2.3736572 + 2.3735962}{2} = 2.3736267 .$$

Now

$f(2.3736267) = (2.3736267)^3 - 2.3736267 - 11 = -3.67558 \times 10^{-4} < 0$.

Therefore from x_{14} and x_{15} we can see that $f(x_{14})$ and $f(x_{15})$ are nearly equal to 0. Hence the root is correct to 4 decimal places is 2.37362.

Example (6.2)

Using bisection method, find the negative root of

$$x^3 - x + 11 = 0$$

Solution

Let

$$f(x) = x^3 - x + 11.$$

Hence

$$f(-x) = -x^3 + x + 11.$$

The negative root of $f(x) = 0$ is the positive root of $f(-x) = 0$. Therefore we will find the positive root of $f(-x) = 0$,

i.e.

$$x^3 - x - 11 = 0.$$

Proceeding as explained in example (1), we get $x = 2.37362$ and hence the negative root is $x = -2.37362$.

3. **Iteration method**

   Let $f(x) = 0$ by the given equation whose roots are to be determined this equation can be written in the form

$$x = \phi(x). \qquad (6.1)$$

   Let $x = x_0$ an initial approximation to the actual root say α of Eq. (6.1). Then the first approximation is $x_1 = \phi(x_0)$ and successive approximations are $x_2 = \phi(x_1)$, $x_3 = \phi(x_2)$, $x_4 = \phi(x_3)$, … , $x_n = \phi(x_{n-1})$. If the sequence of approximate roots $x_0, x_1, x_2, \ldots, x_n$ converges to α, then the value x_n it is taking as the root of the equation

$f(x)=0$. For the convergence purpose the function $\phi(x)$ have to be chosen carefully. The choice of $\phi(x)$ is determined according to the following theorem.

Theorem 6.2

If α is a root of $f(x)=0$ which is equivalent to $x=\phi(x)$. Let I be an interval contains the point $x=\alpha$. Then the sequence of approximations $x_0, x_1, x_2, \ldots, x_n$ will converge to the root α, if

$$|\phi'(x)|<1 \;\forall\; x \in I .$$

Note

The smaller values of $\phi'(x)$ the more rapid convergence

Example (6.3)

Find a real root of the equation

$$x^3+x^2-1=0.$$

By iteration method.

Solution

Let $f(x)=x^3+x^2-1$. Now $f(0)=-1$ and $f(1)=1$. Hence a real root lies between 0 and 1. Rewrite $x^3+x^2-1=0$ as

$$x=\frac{1}{\sqrt{1+x}}=\phi(x).$$

Now

$$\phi'(x)=-\frac{1}{2(1+x)^{3/2}}.$$

It is clear that

$$|\phi'(x)|<1 \;\forall\; x \in [0,1].$$

Hence the iteration method can be applied. Let $x_0=0.65$ be the initial approximation to the desired root, then

$$x_0 = 0.65,$$

$$x_1 = \phi(x_0) = \frac{1}{\sqrt{1+x_0}} = \frac{1}{\sqrt{1.65}} = 0.7784989,$$

$$x_2 = \frac{1}{\sqrt{1+x_1}} = \frac{1}{\sqrt{1.7784989}} = 0.7498479,$$

$$x_3 = \frac{1}{\sqrt{1+x_2}} = \frac{1}{\sqrt{1.7498479}} = 0.7559617,$$

$$x_4 = \frac{1}{\sqrt{1+x_3}} = \frac{1}{\sqrt{1.7559617}} = 0.7546446,$$

$$x_5 = \frac{1}{\sqrt{1+x_4}} = \frac{1}{\sqrt{1.7546446}} = 0.7549278,$$

$$x_6 = \frac{1}{\sqrt{1+x_5}} = \frac{1}{\sqrt{1.7549278}} = 0.7548668,$$

$$x_7 = \frac{1}{\sqrt{1+x_6}} = \frac{1}{\sqrt{1.7548668}} = 0.7548799,$$

$$x_8 = \frac{1}{\sqrt{1+x_7}} = \frac{1}{\sqrt{1.7548799}} = 0.7548771,$$

$$x_9 = \frac{1}{\sqrt{1+x_8}} = \frac{1}{\sqrt{1.7548771}} = 0.7548777,$$

$$x_{10} = \frac{1}{\sqrt{1+x_9}} = \frac{1}{\sqrt{1.7548777}} = 0.7548776,$$

$$x_{11} = \frac{1}{\sqrt{1+x_{10}}} = \frac{1}{\sqrt{1.7548776}} = 0.7548776,$$

Hence the root is 0.7548776.

Example (6.4)

Find a real root of the equation $\cos x - 3x + 1 = 0$ correct to seven decimal places.

Solution

Let $f(x) = \cos x - 3x + 1$. Now $f(0) = 2 > 0$ and $f(\pi/2) = -\frac{3\pi}{2} + 1 < 0$. Therefore there exist a real root lies between 0 and $\pi/2$. Rewrite $\cos x - 3x + 1 = 0$ as

$$x = \frac{1}{3}(\cos x + 1) = \phi(x).$$

Now

$$\phi'(x) = -\frac{\sin x}{3}.$$

It is clear that

$$|\phi'(x)| = \left|-\frac{\sin x}{3}\right| < \frac{1}{3} \;\forall\; x .$$

Hence the iteration method can be applied. Let $x_0 = 0.5$ be the initial approximation to the desired root, then

$$x_1 = \phi(x_0) = \frac{1}{3}(\cos x + 1) = 0.6258608,$$

$$x_2 = \frac{1}{3}(\cos(0.6258608) + 1) = 0.6034863,$$

$$x_3 = \frac{1}{3}(\cos(0.6034863) + 1) = 0.6077873,$$

$$x_4 = \frac{1}{3}(\cos(0.6077873) + 1) = 0.6069711,$$

$$x_5 = \frac{1}{3}(\cos(0.6069711) + 1) = 0.6071264,$$

$$x_6 = \frac{1}{3}(\cos(0.6071264) + 1) = 0.6070969,$$

$$x_7 = \frac{1}{3}(\cos(0.6070969) + 1) = 0.6071025,$$

$$x_8 = \frac{1}{3}(\cos(0.6071025) + 1) = 0.6071014,$$

$$x_9 = \frac{1}{3}(\cos(0.6071014) + 1) = 0.6071016,$$

$$x_{10} = \frac{1}{3}\left(\cos(0.6071016)+1\right) = 0.6071016,$$

Hence the root is 0.6071016.

4. Newton-Raphson method

This method, is a particular form of the iteration method discussed in section 3. When an approximate value of a root of an equation is given, a better and closer approximation to the root can be found using this method. It can be derived as follows:

Let x_0 be an approximation of a root of the given equation $f(x)=0$, which may be algebraic or transcendental. Let x_0+h be the exact value or the better approximation of the corresponding root, h being a small quantity. Then $f(x_0+h)=0$. Expanding $f(x_0+h)=0$ by Taylor's theorem, we get

$$f(x_0+h)=f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0)+\ldots=0.$$

Since h is small, we can neglect second, third and higher degree terms in h and thus we get,

$$f(x_0)+hf'(x_0)=0.$$

Or

$$h=-\frac{f(x_0)}{f'(x_0)}; \quad f'(x_0)\neq 0.$$

Hence,

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Now substituting x_1 for x_0 and x_2 for x_1, then the next better approximations are given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)},$$

and

$$x_3 = x_2 - \frac{f(x_n)}{f'(x_n)}.$$

Proceeding in the same way n times, we get the general formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{for } n = 0,1,2,\ldots, \qquad (6.2)$$

which is known as Newton-Raphson formula.

Example (6.5)

Find an iterative formula to find $\sqrt{N}$, where N is a positive number and hence, find $\sqrt{12}$ correct to four decimal places.

Solution

Let

$$x = \sqrt{N} \Rightarrow x^2 - N = 0.$$

Assume

$$f(x) = x^2 - N .$$

Then,

$$f'(x) = 2x$$

Now, from Newton-Raphson formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - N}{2x_n}$$

$$= \frac{1}{2}\left[x_n + \left(\frac{N}{x_n}\right)\right] \qquad (6.3)$$

Eq. (6.3) is the required iterative formula. Putting $N = 12$ in $f(x)$, we have $f(x) = x^2 - 12$.

Now, $f(3) < 0$ and $f(4) > 0$. Therefore, the root lies in between 3 and 4. Let the initial approximation x_0 be 3.1. Then, from Eq. (6.3) the first approximation to the root

$$x_1 = \frac{1}{2}\left[x_0 + \frac{12}{x_0}\right] = \frac{1}{2}\left[3.1 + \frac{12}{3.1}\right] = 3.4854839.$$

The second approximation is

$$x_2 = \frac{1}{2}\left[x_1 + \frac{12}{x_1}\right] = \frac{1}{2}\left[3.4854839 + \frac{12}{3.4854839}\right] = 3.4641672.$$

The third approximation is

$$x_3 = \frac{1}{2}\left[3.4641672 + \frac{12}{3.4641672}\right] = 3.4641016.$$

The fourth approximation is

$$x_4 = \frac{1}{2}\left[3.4641016 + \frac{12}{3.4641016}\right] = 3.4641016.$$

Thus, the value of $\sqrt{12}$ correct to four decimals is 3.4641.

Example (6.6)

Solve $x^3 + 2x^2 + 10x - 20 = 0$ by Newton-Raphson method.

Solution

Let

$$f(x) = x^3 + 2x^2 + 10x - 20.$$

Therefore

$$f'(x) = 3x^2 + 4x + 10.$$

From Eq. (6.2)

$$
\begin{aligned}
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\
&= x_n - \left[\frac{x_n^3 + 2x_n^2 + 10x_n - 20}{3x_n^2 + 4x_n + 10}\right] \\
&= \frac{2\left(x_n^3 + x_n^2 + 10\right)}{3x_n^2 + 4x_n + 10}. \qquad (6.4)
\end{aligned}
$$

Now we can see that $f(1) = -7 < 0$ and $f(2) = 16 > 0$. Therefore, the root lies in between 1 and 2. Let $x_0 = 1.2$ be the initial approximation $(\because f(1.2) < 0)$.

Putting $n = 0$ in Eq. (6.4), first approximation x_1 is given by

$$
\begin{aligned}
x_1 &= \frac{2\left(x_0^3 + x_0^7 + 10\right)}{3x_0^2 + 4x_0 + 10} = \frac{2\left[(1.2)^2 + (1.2)^2 + 10\right]}{3(1.2)^2 + 4(1.2) + 10} \\
&= \frac{26.336}{19.12} = 1.3774059.
\end{aligned}
$$

The second approximation x_2 is

$$
\begin{aligned}
x_2 &= \frac{2\left(x_1^3 + x_1^2 + 10\right)}{3x_1^2 + 4x_1 + 10} = \frac{2\left[(1.3774059)^3 + (1.3774059)^2 + 10\right]}{3(1.3774059)^2 + 4(1.3774059) + 10} \\
&= \frac{29.021052}{21.201364} = 1.3688295.
\end{aligned}
$$

The third approximation x_3 is given by

$$
\begin{aligned}
x_3 &= \frac{2\left(x_2^3 + x_2^2 + 10\right)}{3x_2^2 + 4x_2 + 10} = \frac{2\left[(1.3688295)^3 + (1.3688295)^2 + 10\right]}{3(1.3688295)^2 + 4(1.3688295) + 10} \\
&= \frac{28.876924}{210064} = 1.3688081.
\end{aligned}
$$

The fourth approximation x_4 (to the root) is given by

$$x_4 = \frac{2\left(x_3^3 + x_3^2 + 10\right)}{3x_3^2 + 4x_3 + 10} = \frac{2\left[(1.3688081)^3 + (1.3688081)^2 + 10\right]}{3(1.3688081)^2 + 4(13688081) + 10}$$

$$= \frac{28.876567}{21.09614} = 1.3688081.$$

Hence the root is 1.3688081.

Example (6.7)

Using Newton-Raphson method, find the root of the equation

$$x \ln x = 1.2 \ .$$

Solution

Let

$$f(x) = x \ln x - 1.2 \ \Rightarrow \ f'(x) = \ln x + 1.$$

From Newton-Raphson formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n \ln x_n - 1.2}{\ln x_n + 1}.$$

Therefore

$$x_{n+1} = \frac{x_n + 1.2}{\ln x_n + 1}. \qquad (6.5)$$

Now $f(2.5) = -0.2051499 < 0$ and $f(3) = 0.2313637 > 0$. Therefore, the real root of $f(x)$ lies in $(2.5, 3)$. Let $x_0 = 2.7$ be the initial approximation. Putting $n = 0$ in Eq. (6.5), the first approximation x_1 is given by

$$x_1 = \frac{x_0 + 1.2}{\ln x_0 + 1} = \frac{2.7 + 1.2}{\ln 2.7 + 1} = 1.9566.$$

The second approximation x_2 is

$$x_2 = \frac{x_1 + 1.2}{\ln x_1 + 1} = \frac{1.9566 + 1.2}{\ln(1.9566) + 1} = 1.8888.$$

Similarly, the third approximation is

$$x_3 = \frac{x_2 + 1.2}{\ln x_2 + 1} = \frac{1.8888 + 1.2}{\ln(1.8888) + 1} = 1.88809.$$

Hence, the root is 1.88809.

Example (6.8)

Solve $\sin x = 1 + x^3$ using Newton-Raphson method.

Solution

Let

$$f(x) = \sin x - 1 - x^3 \qquad \Rightarrow \; f'(x) = \cos x - 3x^2.$$

Then, from Newton-Raphson formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\sin x_n - 1 - x_n^3}{\cos x_n - 3x_n^2} \quad .$$

Hence

$$x_{n+1} = \frac{x_n \cos x_n - \sin x_n - 2x_n^3 + 1}{\cos x_n - 3x_n^2}. \qquad (6.6)$$

Now

$$f(-1) = \sin(-1) - 1 - (-1)^3 = -0.8414709 < 0,$$

and

$$f(-2) = \sin(-2) - 1 - (-2)^3 = 6.0907026 > 0,$$

which means that the root lies in between -1 and -2. Let $x_0 = -1.1$ be the initial approximation. Then, by putting

$n = 0, 1, 2, \ldots$ in Eq. (6.6), we obtain the successive approximations as

$$x_1 = \frac{x_0 \cos x_0 - \sin x_0 - 2x_0^3 + 1}{\cos x_0 - 3x_0^2} = \frac{4.0542516}{-3.1764039} = -1.2763653$$

$$x_2 = \frac{5.7452469}{-4.5971297} = -1.2497465$$

$$x_3 = \frac{5.4584049}{-4.370036} = -1.2490526$$

$$x_4 = \frac{5.4510835}{-4.364176} = -1.2490522$$

$$x_5 = \frac{5.4510786}{-4.3641722} = -1.2490521$$

$$x_6 = \frac{5.4510785}{-4.3641721} = -1.2490522$$

Hence the approximated root is x_6, i.e. -1.2490522 .