حامعة جنوب الوادى

كلية التربية بالغردقة

الفرقة الثانية عام رياضيات (Math)

المادة : (Pure 6) جزء ((Ordinary Differential Equations (ODEs))

إستاذ المادة / د. اسماعيل جاد امين

الفصل الدراسي الأول

Ordinary Differential Equations (I)

Ordinary Differential Equations

Chapter 1
Introduction to ODEs

- Objectives of Lesson

- Recall basic definitions of ODEs:
  - Order
  - Linearity
  - Initial conditions
  - Solution
- Classify ODEs based on:
  - Order, linearity, and conditions.
- Classify the solution methods.

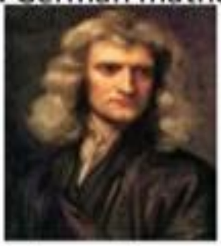
Ordinary Differential Equations (ODEs)

Chapter1

- History of differential equations:

INVENTION OF DIFFERENTIAL EQUATION:

- In mathematics, the **history of differential equations** traces the development of "differential equations" from calculus, which itself was independently invented by English physicist Isaac Newton and German mathematician Gottfried Leibniz.




- The history of the subject of differential equations, in concise form, from a synopsis of the recent article "The History of Differential Equations, **1670-1950**"

  "**Differential equations** began with Leibniz, the Bernoulli brothers, and others from the **1680s**, not long after Newton's 'fluxional equations' in the **1670s**."

- Definition of DEs and some properties:

Differential Equations

Definition A differential equation is an equation involving derivatives of an unknown function and possibly the function itself as well as the independent variable.

Example $y' = \sin(x)$, $(y')^4 - y^2 + 2xy - x^2 = 0$, $y'' + y^3 + x = 0$

1st order equations — 2nd order equation

Definition The order of a differential equation is the highest order of the derivatives of the unknown function appearing in the equation

In the simplest cases, equations may be solved by direct integration.

Examples

$$y' = \sin(x) \Rightarrow y = -\cos(x) + C$$

$$y'' = 6x + e^x \Rightarrow y' = 3x^2 + e^x + C_1 \Rightarrow y = x^3 + e^x + C_1 x + C_2$$

Observe that the set of solutions to the above 1st order equation has 1 parameter, while the solutions to the above 2nd order equation depend on two parameters.

Derivatives

Derivatives

Ordinary Derivatives	Partial Derivatives
$\frac{dv}{dt}$	$\frac{\partial u}{\partial y}$
v is a function of one independent variable	u is a function of more than one independent variable

dy/dx

ODE PDE

Differential Equations

Differential Equations

Ordinary Differential Equations	Partial Differential Equations
$\frac{d^2v}{dt^2} + 6tv = 1$	$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} = 0$
involve one or more Ordinary derivatives of unknown functions	involve one or more partial derivatives of unknown functions

Ordinary Differential Equations (ODEs) involve one or more ordinary derivatives of unknown functions with respect to one independent variable

Examples :

$$\frac{dv(t)}{dt} - v(t) = e^t$$

$$\frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt} + 2x(t) = \cos(t)$$

x(t): unknown function

t: independent variable

Example 2:

$$\frac{dv}{dt} = 9.8 - \frac{c}{M}v$$

Ordinary differential equation

$$\frac{dv}{dt} = 9.8 - \frac{c}{M}v$$

(Dependent variable) unknown function to be determined

$$\frac{dv}{dt} = 9.8 - \frac{c}{M}v$$

(independent variable) the variable with respect to which other variables are differentiated

- Order of a Differential Equation:

The **order** of an ordinary differential equation is the order of the highest order derivative

Ordinary Differential Equations (ODEs)

Examples:

$$\frac{dx(t)}{dt} - x(t) = e^t \qquad \text{First order ODE}$$

$$\frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt} + 2x(t) = \cos(t) \qquad \text{Second order ODE}$$

$$\left(\frac{d^2x(t)}{dt^2}\right)^3 - \frac{dx(t)}{dt} + 2x^4(t) = 1 \qquad \text{Second order ODE}$$

- Linear ODE:

An **ODE** is linear if the unknown function and its derivatives appear to power one**.** No product of the unknown function and/or its derivatives.

Examples:

$$\frac{dx(t)}{dt} - x(t) = e^t \qquad \text{Linear ODE}$$

$$\frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt} + 2t^2x(t) = \cos(t) \qquad \text{Linear ODE}$$

$$\left(\frac{d^2x(t)}{dt^2}\right)^3 - \frac{dx(t)}{dt} + \sqrt{x(t)} = 1 \qquad \text{Non-linear ODE}$$

- Nonlinear ODE:

Examples of nonlinear ODE :

$$\frac{dx(t)}{dt} - \cos(x(t)) = 1, \qquad \frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt}x(t) = 2$$

$$\frac{d^2x(t)}{dt^2} - \left|\frac{dx(t)}{dt}\right| + x(t) = 1$$

Ordinary Differential Equations (ODEs)

Chapter1

- Auxiliary Conditions:

Auxiliary Conditions

Initial Conditions

- All conditions are at one point of the independent variable

Boundary Conditions

- The conditions are not at one point of the independent variable

-Boundary-Value and Initial value Problems:

Initial-Value Problems

- The auxiliary conditions are at one point of the independent variable

$$\ddot{x}+2\dot{x}+x=e^{-2t}$$

$$x(0)=1, \quad \dot{x}(0)=2.5$$

same

Boundary-Value Problems

- The auxiliary conditions are not at one point of the independent variable
- More difficult to solve than initial value problems

$$\ddot{x}+2\dot{x}+x=e^{-2t}$$

$$x(0)=1, \quad x(2)=1.5$$

different

- Classification of ODEs:

ODEs can be classified in different ways:

- Order
  - First order ODE
  - Second order ODE
  - N^{th} order ODE
- Linearity
  - Linear ODE
  - Nonlinear ODE
- Auxiliary conditions
  - Initial value problems
  - Boundary value problems

- Solutions of Ordinary Differential Equations

For example, this function

$x(t) = \cos(2t)$

is a solution to the ODE

$$\frac{d^2x(t)}{dt^2} + 4x(t) = 0$$

Is it unique?

All functions of the form $x(t) = \cos(2t + c)$ (where c is a real constant) are solutions.

- Uniqueness of a Solution

In order to uniquely specify a solution to an n^{th} order differential equation we need n conditions

$$\frac{d^2x(t)}{dt^2} + 4x(t) = 0$$ Second order ODE

$x(0) = a$

$\dot{x}(0) = b$

Two conditions are needed to uniquely specify the solution

Classification of ODEs

ODEs can be classified in different ways:

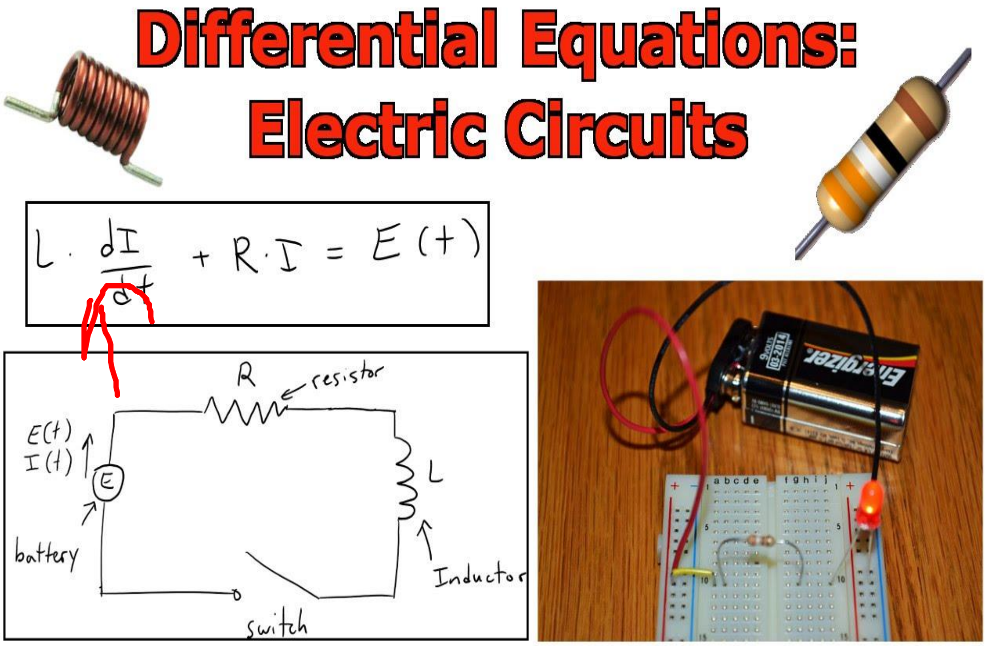
- **Order**
  - First order ODE
  - Second order ODE
  - N^{th} order ODE
- **Linearity**
  - Linear ODE
  - Nonlinear ODE
- **Auxiliary conditions**
  - Initial value problems
  - Boundary value problems

Ordinary Differential Equations (ODEs)

Chapter1

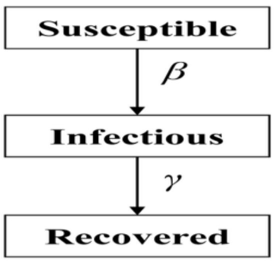
- Applications of Differential equations:

- **Electric Circuits:-**



- **Biological Systems:-**

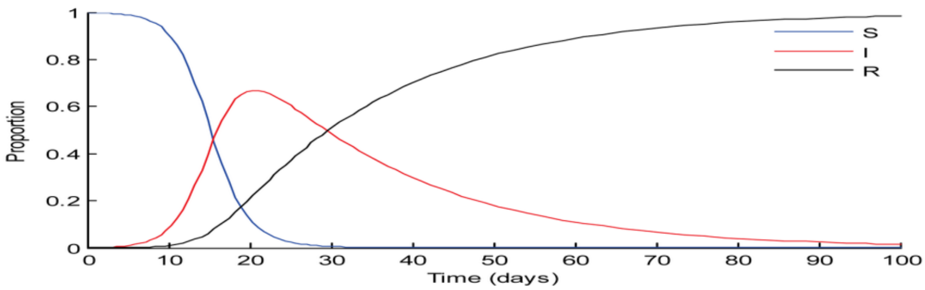
The ***SIR*** epidemic model is one of the simplest compartmental models, and many models are derivations of this basic form. The model consists of three compartments–*S* for the number susceptible, *I* for the number of infectious, and *R* for the number recovered (or immune).



$$\frac{dS}{dt} = -\beta SI$$

$$\frac{dI}{dt} = \beta SI - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$



Chapter 2

First-Order Differential Equations

First-order differential equations

Separable Differential Equations

- Objectives of Lesson

- Differential Equation of first Order and first Degree
- Method of Solution: Separation of Variables
- Equations Reducible to Variable Separable Form
- Class Exercise

Definition of DE of first order ✓

A differential equation of the first order and first degree contains independent variable x, dependent variable y and its derivative

$$\frac{dy}{dx} \text{ i.e. } \frac{dy}{dx} = f(x, y) \text{ or } f\left(x, y, \frac{dy}{dx}\right) = 0$$

where f (x, y) is the function of x and y.

For example : $xy(y+1)dy = (x^2+1)dx$, $\frac{dy}{dx} = \frac{x+y}{x-y}$,

$$\frac{dy}{dx} + y = \sin x \text{ etc.}$$

Separation of Variables ✓

A **separable** differential equation is one that can be written so that the independent variable terms (along with its differential) are collected to one side of the equal sign, and the dependent variable terms (and its differential) to the other.

Example 1: $y' = xy^2$ is separable. It is first written as $\frac{dy}{dx} = xy^2$, then "separated":

$$\frac{dy}{y^2} = x\,dx.$$

This is separated as $\frac{dy}{y^2} = x\,dx$.

$\int \frac{dy}{y^2} = \int x\,dx$ Integrate both sides.

$-\frac{1}{y} = \frac{1}{2}x^2 + C$ Don't forget the constant of integration.

$\frac{1}{y} = C - \frac{1}{2}x^2$ Negate. The C "absorbs" the negative.

$y = \frac{1}{C - \frac{1}{2}x^2} = \frac{2}{C - x^2}$ Solve for y. Note that $2C$ is written as C.

Example 2: Solve the IVP: $y' = x + xy, y(0) = 3$.

Solution

Write y' as $\frac{dy}{dx}$: $\frac{dy}{dx} = x + xy$

Factor: $\frac{dy}{dx} = x(1 + y)$

Separate: $\frac{dy}{1+y} = x\,dx$ $(y \neq -1)$

Integrate: $\int \frac{dy}{1+y} = \int x\,dx \rightarrow \ln|1 + y| = \frac{1}{2}x^2 + C$

Isolate y: $|1 + y| = e^{0.5x^2 + C} \rightarrow |1 + y| = Ce^{0.5x^2}$

$1 + y = \pm Ce^{0.5x^2} = Ce^{0.5x^2}$ $(\pm C = C)$

Thus, $y = Ce^{0.5x^2} - 1$ is the general solution of $y' = x + xy$.

- The constant of integration C is just a generic constant at this point. It absorbs all constants that come near it, so to speak. For example, $e^C = C$, $-C = C$, $2C = C$, $\frac{1}{C} = C$, and so on.
- The C can be determined with an initial condition. For example, suppose we have $y' = x + xy$ with $y(0) = 3$. The general solution is $y = C^{0.5x^2} - 1$. To find C, let $x = 0$ and $y = 3$:

$$3 = Ce^{0.5(0)^2} - 1 \quad \rightarrow \quad 3 = C - 1 \quad \rightarrow \quad C = 4.$$

Thus, the particular solution is $y = 4e^{0.5x^2} - 1$.

Example 3

- Solve the following first order nonlinear equation:

$$\frac{dy}{dx} = \frac{x^2 + 1}{y^2 - 1}$$

Solution

Separating variables, and using calculus, we obtain

$$(y^2 - 1)\,dy = (x^2 + 1)\,dx$$
$$\int (y^2 - 1)\,dy = \int (x^2 + 1)\,dx$$
$$\frac{1}{3}y^3 - y = \frac{1}{3}x^3 + x + C$$
$$y^3 - 3y = x^3 + 3x + C$$

Example 4:

- Solve the following first order nonlinear equation:

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \textit{where } y(0) = -1$$

Solution

$$2(y-1)dy = (3x^2 + 4x + 2)dx$$
$$2\int (y-1)dy = \int (3x^2 + 4x + 2)dx$$
$$y^2 - 2y = x^3 + 2x^2 + 2x + C$$

- The equation above defines the solution *y* implicitly. An explicit expression for the solution can be found in this case:

$$y^2 - 2y - (x^3 + 2x^2 + 2x + C) = 0 \quad \Rightarrow \quad y = \frac{2 \pm \sqrt{4 + 4(x^3 + 2x^2 + 2x + C)}}{2}$$
$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + C_1}$$

- Suppose we seek a solution satisfying *y*(0) = -1. Using the implicit expression of *y*, we obtain
- Thus the implicit equation defining *y* is

$$y^2 - 2y = x^3 + 2x^2 + 2x + C$$
$$(-1)^2 - 2(-1) = C \quad \Rightarrow \quad C = 3$$

- Using explicit expression of *y*,

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3$$

Example 5

- Consider the following initial value problem:

$$y' = \frac{y\cos x}{1+3y^3}, \quad y(0)=1$$

Solution

Separating variables and using calculus, we obtain

$$\frac{1+3y^3}{y}dy = \cos x dx$$

$$\int\left(\frac{1}{y}+3y^2\right)dy = \int \cos x dx$$

$$\ln|y| + y^3 = \sin x + C$$

- Using the initial condition, it follows that

$$\ln y + y^3 = \sin x + 1$$

- Thus

$$y' = \frac{y\cos x}{1+3y^3}, \quad y(0)=1 \Rightarrow \ln y + y^3 = \sin x + 1$$

Example 6

Solve the differential equation $\frac{dy}{dx} = x - 1 + xy - y$.

Solution : The given differential equation is $\frac{dy}{dx} = x - 1 + xy - y$.

$$\Rightarrow \frac{dy}{dx} = (x-1) + y(x-1)$$

$$\Rightarrow \frac{dy}{y+1} = (x-1)dx \qquad \text{[Variable separable form]}$$

Integrating both sides, we get

$$\int \frac{dy}{y+1} = \int (x-1)dx$$

$$\Rightarrow \log_e|y+1| = \frac{x^2}{2} - x + C$$

Example 7

$$\tan y \frac{dy}{dx} = \frac{x \cos^4 y}{1 - x^2}$$

→ Separation

$$\frac{\tan y}{\cos^2 y} dy = \frac{x}{1 - x^2} dx \longrightarrow \frac{\sin y}{\cos^3 y} dy = \frac{x}{1 - x^2} dx$$

$$\int \sin y \cos^{-3} y \, dy = \int \frac{x}{1 - x^2} dx$$

$$-\int -\sin y \cos^{-3} y \, dy = \frac{-1}{2} \int \frac{-2 * x}{1 - x^2} dx$$

$$\frac{-\cos^{-2} y}{-2} = \frac{-1}{2} \ln(1 - x^2) + \frac{C}{2}$$

$$\cos^{-2} y = -\ln(1 - x^2) + C$$

$$\boxed{\sec^2 y + \ln(1 - x^2) = C}$$

(i) $(e^{2y} - y) \cosh x \frac{dy}{dx} = e^{y+x} \sinh 2x$

$$\int \frac{(e^{2y} - y)}{e^y} dy = \int \frac{\sinh 2x \, e^x}{\cosh x} dx$$

$$\int (e^y - y e^{-y}) dy = \int \frac{2 \sinh x \cosh x}{\cosh x} e^x dx$$

$$e^y + y e^{-y} - \int e^{-y} dy = \int 2 e^x \frac{e^x - e^{-x}}{2} = \int e^{2x} - 1$$

$$\boxed{e^y + y e^{-y} + e^{-y} = \frac{1}{2} e^{2x} - x + C}$$

Reducible to Variable Separable Form

Differential equation of the form

$$\frac{dy}{dx} = f(ax + by + c)$$

Substitute ax + by + c = v to reducing variable separable form.

Example 8

Solve the differential equation: $(x+y)^2 \frac{dy}{dx} = 1$

Solution: We have $(x+y)^2 \frac{dy}{dx} = 1$ ...(i)

Putting $x + y = v$ and $1 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$

in (i), we get

$$v^2\left(\frac{dv}{dx} - 1\right) = 1 \Rightarrow v^2 \frac{dv}{dx} = 1 + v^2$$

$$\Rightarrow \frac{v^2}{1+v^2} dv = dx \quad \Rightarrow \frac{1 + v^2 - 1}{1+v^2} dv = dx$$

$$\Rightarrow \left(1 - \frac{1}{1+v^2}\right) dv = dx$$

$$\Rightarrow \int \left(1 - \frac{1}{1+v^2}\right) dv = \int dx \Rightarrow v - \tan^{-1} v = x + C$$

$$\Rightarrow (x+y) - \tan^{-1}(x+y) = x + C \quad \Rightarrow y - \tan^{-1}(x+y) = C$$

Homogenous Differential Equations

Homogeneous Function:-

A function $f(x,y)$ is called a homogenous function of degree n if $f(tx,ty) = t^n f(x,y)$

Examples:

$g(x, y) = x^2 - xy + y^2$ is a homogenous function of degree 2

Since $g(tx,ty) = t^2x^2 - (tx)(ty) + t^2y^2 = t^2(x^2 - xy + y^2) = t^2 g(x,y)$

$Q(x,y) = x^3 sin\left(\frac{x}{y}\right) - \sqrt{x^2 - 4xy}$ is homogenous of degree 2

Method of Solution

(1) Write the differential equation in the form $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$

(2) Substitute $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in the equation.

(3) The equation reduces to the form $v + x\frac{dv}{dx} = F(v)$

(4) Separate the variables of v and x.

(5) Integrate both sides to obtain the solution in terms of v and x.

(6) Replace v by $\frac{y}{x}$ to get the solution

Note: you could perform the same steps but with other assumption:

$x = uy$ and $\frac{dx}{dy} = u + y\frac{du}{dy}$

Example 1

Solve the differential equation $x\frac{dy}{dx} = x + y$.

Solution : The given differential equation is $x\frac{dy}{dx} = x + y$.

$$\Rightarrow \frac{dy}{dx} = \frac{x+y}{x} \quad ...(i)$$

It is a homogeneous differential equation of degree 1.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ ...(i), we get

$$v + x\frac{dv}{dx} = \frac{x+vx}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = 1 + v \quad \Rightarrow x\frac{dv}{dx} = 1 \quad \Rightarrow dv = \frac{1}{x}dx$$

Integrating both sides, we get

$$\int dv = \int \frac{1}{x}dx \quad \Rightarrow v = \log_e|x| + C$$

$$\Rightarrow \frac{y}{x} = \log_e|x| + C \quad [\because y = vx] \quad \Rightarrow y = x\log_e|x| + Cx$$

Example 2

$$\left(x^3 + y^2\sqrt{x^2+y^2}\right)dx - \left(xy\sqrt{x^2+y^2}\right)dy = 0$$

⟶ No Separation

$$\left(t^3x^3 + t^2y^2\sqrt{t^2x^2 + t^2y^2}\right)dx - \left(tx\,ty\sqrt{t^2x^2 + t^2y^2}\right)dy = 0$$

$$\left(t^3x^3 + t^2y^2\sqrt{t^2(x^2+y^2)}\right)dx - \left(t^2xy\sqrt{t^2(x^2+y^2)}\right)dy = 0$$

$$\left(t^3x^3 + t^2y^2\sqrt{t^2}\sqrt{x^2+y^2}\right)dx - \left(t^2xy\sqrt{t^2}\sqrt{x^2+y^2}\right)dy = 0$$

$$\left(t^3x^3 + t^2y^2t\sqrt{x^2+y^2}\right)dx - \left(t^2xy\,t\sqrt{x^2+y^2}\right)dy = 0$$

$$t^3\left(x^3 + y^2\sqrt{x^2+y^2}\right)dx - t^3\left(xy\sqrt{x^2+y^2}\right)dy = 0$$

⟶ Homogenous DE of 3rd order

$$\frac{dy}{dx} = \frac{x^3 + y^2\sqrt{x^2+y^2}}{xy\sqrt{x^2+y^2}}$$

Let $\quad y = u\,x \quad$ and $\quad \frac{dy}{dx} = u + x\,\frac{du}{dx}$

$$u + x\,\frac{du}{dx} = \frac{x^3 + u^2 x^2 \sqrt{x^2 + u^2 x^2}}{x^2 u \sqrt{x^2 + u^2 x^2}} = \frac{x^3 + u^2 x^2 \sqrt{x^2 (1+u^2)}}{x^2 u \sqrt{x^2 (1+u^2)}}$$

$$= \frac{x^3 + u^2 x^2 \sqrt{x^2}\sqrt{(1+u^2)}}{x^2 u \sqrt{x^2}\,\sqrt{(1+u^2)}} = \frac{x^3 + u^2 x^3 \sqrt{(1+u^2)}}{x^3 u \sqrt{(1+u^2)}}$$

$$u + x\,\frac{du}{dx} = \frac{1 + u^2 \sqrt{(1+u^2)}}{u\sqrt{(1+u^2)}} \quad \rightarrow \quad x\,\frac{du}{dx} = \frac{1 + u^2 \sqrt{(1+u^2)}}{u\sqrt{(1+u^2)}} - u$$

$$x\,\frac{du}{dx} = \frac{1 + u^2 \sqrt{1+u^2}}{u\sqrt{1+u^2}} - u * \frac{u\sqrt{1+u^2}}{u\sqrt{1+u^2}}$$

$$x\,\frac{du}{dx} = \frac{1 + u^2 \sqrt{1+u^2} - u^2 \sqrt{1+u^2}}{u\sqrt{1+u^2}}$$

$$x\,\frac{du}{dx} = \frac{1}{u\sqrt{1+u^2}} \quad \rightarrow \quad \int u\sqrt{1+u^2}\;du = \int \frac{1}{x}\,dx$$

$$\frac{1}{2}\int 2 * u\,(1+u^2)^{0.5}\,du = \ln x + \ln C$$

$$\frac{1}{2}\,\frac{(1+u^2)^{1.5}}{1.5} = \ln x + \ln C$$

$$\boxed{\frac{1}{3}\left(1 + \left(\frac{y}{x}\right)^2\right)^{1.5} = \ln C\,x}$$

Example 3

10. $(y^2 - x^2 e^{-\frac{y}{x}})\,dx - xy\,dy = 0$

$\longrightarrow$ No Separation

$$(t^2y^2 - t^2x^2 e^{-\frac{ty}{tx}})\,dx - t^2xy\,dy = 0$$

$$t^2(y^2 - x^2 e^{-\frac{y}{x}})\,dx - t^2xy\,dy = 0$$

$\longrightarrow$ Homogenous DE of 2nd order

$$\frac{dy}{dx} = \frac{y^2 - x^2 e^{-\frac{y}{x}}}{xy}$$

Let $y = u\,x$ and $\frac{dy}{dx} = u + x\frac{du}{dx}$

$$u + x\frac{du}{dx} = \frac{u^2x^2 - x^2 e^{-u}}{u\,x^2} = \frac{u^2 - e^{-u}}{u}$$

$$x\frac{du}{dx} = \frac{u^2 - e^{-u}}{u} - u * \frac{u}{u} = \frac{u^2 - e^{-u} - u^2}{u}$$

$$x\frac{du}{dx} = \frac{-e^{-u}}{u} \quad \rightarrow \quad \int u\,e^{u}\,du = \int \frac{-1}{x}\,dx$$

$$u\,e^{u} - \int e^{u}\,du = -\ln x + \ln C$$

$$u\,e^{u} - e^{u} = \ln\frac{C}{x}$$

$$\boxed{\frac{y}{x}e^{\frac{y}{x}} - e^{\frac{y}{x}} = \ln\frac{C}{x}}$$

Exact Differential Equations & Integrating Factors
Exact and Integrating factor

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

An equation is said to be solved using exact method only if:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$M_y = N_x$$

Steps to solve:

Let the solution of the differential equation, F, = C ⟶ F = C

Then :

$F = \int M(x,y)\,dx$

$= \text{----------} + g(y)$

$\frac{\partial F}{\partial y} \equiv N(x,y)$

$\text{----------} + g'(y) \equiv N(x,y)$

Then find the equivalent of $g'(y)$ in the right hand side thus,

$g(y) = \int g'(y)\,dy$

$F = \text{----------} + g(y) = C$

$F = \int N(x,y)\,dy$

$= \text{----------} + g(x)$

$\frac{\partial F}{\partial x} \equiv M(x,y)$

$\text{----------} + g'(x) \equiv M(x,y)$

Then find the equivalent of $g'(x)$ in the right hand side thus,

$g(x) = \int g'(x)\,dx$

$F = \text{----------} + g(x) =$

Solve the differential equation $(2xy - 3x^2)\,dx + (x^2 - 2y)\,dy = 0$.

Solution The given differential equation is exact because

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}[2xy - 3x^2] = 2x = \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}[x^2 - 2y].$$

The general solution, $f(x, y) = C$, is given by

$$\begin{aligned} f(x, y) &= \int M(x, y)\,dx \\ &= \int (2xy - 3x^2)\,dx = x^2y - x^3 + g(y). \end{aligned}$$

In Section 14.1, you determined $g(y)$ by integrating $N(x, y)$ with respect to y and reconciling the two expressions for $f(x, y)$. An alternative method is to partially differentiate this version of $f(x, y)$ with respect to y and compare the result with $N(x, y)$. In other words,

$$f_y(x, y) = \frac{\partial}{\partial y}[x^2y - x^3 + g(y)] = x^2 + g'(y) = \overbrace{x^2 - 2y}^{N(x,\,y)}.$$

$$\boxed{g'(y) = -2y}$$

Thus, $g'(y) = -2y$, and it follows that $g(y) = -y^2 + C_1$. Therefore,

$$f(x, y) = x^2y - x^3 - y^2 + C_1$$

and the general solution is $x^2y - x^3 - y^2 = C$. Figure 15.1 shows the solution curves that correspond to $C = 1, 10, 100,$ and 1000.

Find the particular solution of

$$(\cos x - x \sin x + y^2)\, dx + 2xy\, dy = 0$$

that satisfies the initial condition $y = 1$ when $x = \pi$.

Solution The differential equation is exact because

$$\overbrace{\frac{\partial}{\partial y}[\cos x - x \sin x + y^2]}^{\frac{\partial M}{\partial y}} = 2y = \overbrace{\frac{\partial}{\partial x}[2xy]}^{\frac{\partial N}{\partial x}}.$$

Because $N(x, y)$ is simpler than $M(x, y)$, it is better to begin by integrating $N(x, y)$.

$$f(x, y) = \int N(x, y)\, dy = \int 2xy\, dy = xy^2 + g(x)$$

$$f_x(x, y) = \frac{\partial}{\partial x}[xy^2 + g(x)] = y^2 + g'(x) = \overbrace{\cos x - x \sin x + y^2}^{M(x, y)}$$

$$g'(x) = \cos x - x \sin x$$

Thus, $g'(x) = \cos x - x \sin x$ and

$$g(x) = \int (\cos x - x \sin x)\, dx$$
$$= x \cos x + C_1$$

which implies that $f(x, y) = xy^2 + x \cos x + C_1$, and the general solution is

$$xy^2 + x \cos x = C. \qquad \text{General solution}$$

Applying the given initial condition produces

$$\pi(1)^2 + \pi \cos \pi = C$$

which implies that $C = 0$. Hence, the particular solution is

$$xy^2 + x \cos x = 0. \qquad \text{Particular solution}$$

The graph of the particular solution is shown in Figure 15.3. Notice that the graph consists of two parts: the ovals are given by $y^2 + \cos x = 0$, and the y-axis is given by $x = 0$.

- $e^{x^2}\left(2x^2y - xy^2 + y\right)dx - e^{x^2}(y-x)dy = 0$

→ No separation

→ No Homogenous

$$e^{x^2}\left(2x^2y - xy^2 + y\right)dx + e^{x^2}(x-y)dy = 0$$

$$M_y = e^{x^2}(2x^2 - 2xy + 1)$$

$$N_x = e^{x^2} + 2x^2e^{x^2} - 2xye^{x^2}$$

→ $M_y = N_x$ → *Exact*

F = C

$$F = \int N(x,y)\,dy = \int e^{x^2}(x-y)\,dy$$

$$F = xye^{x^2} - \frac{y^2}{2}e^{x^2} + g(x)$$

$$\frac{\partial F}{\partial x} \equiv M(x,y)$$

$$ye^{x^2} + 2x^2ye^{x^2} - xy^2e^{x^2} + g'(x) \equiv e^{x^2}\left(2x^2y - xy^2 + y\right)$$

$$\therefore\ g'(y) \equiv 0 \longrightarrow g(y) \equiv \int 0\,dy = C$$

$$F = xye^{x^2} - \frac{y^2}{2}e^{x^2} = C$$

$$xye^{x^2} - \frac{y^2}{2}e^{x^2} = C$$

$$\left(e^{2y} - y\cos(xy)\right)dx + (2x\,e^{2y} - x\cos(xy) + 2y)dy = 0 \quad , \; y(0) = 2$$

→ No Separation

→ No Homogenous

$M_y = 2\,e^{2y} - \cos(xy) + xy\sin(xy)$, $N_x = 2\,e^{2y} - \cos(xy) + xy\sin(xy)$

→ Exact

$F = C$

$$F = \int (e^{2y} - y\cos(xy))\,dx = x\,e^{2y} - \sin(xy) + g(y)$$

$$\frac{\partial F}{\partial y} \equiv N(x, y)$$

$$2x\,e^{2y} - x\cos(xy) + g'(y) \equiv 2x\,e^{2y} - x\cos(xy) + 2y$$

$$\therefore g'(y) = 2y \quad \longrightarrow \quad g(y) = \int 2y\,dy = \frac{y^2}{2}$$

$$F = x\,e^{2y} - \sin(xy) + \frac{y^2}{2} = C$$

By substituting $y = 2$ and $x = 0$ → $C = 2$

$$\boxed{x\,e^{2y} - \sin(xy) + \frac{y^2}{2} = 2}$$

Integrating Factors

Integrating Factor $\mu(x,y)$

If $M_y \neq N_x$ and there is a slight difference between them, you have to drive an integrating factor using either of the following ways:

If $\dfrac{M_y - N_x}{N} =$ $f(x)$ only

$$\mu(x) = e^{\int f(x)\,dx}$$

If $\dfrac{N_x - M_y}{M} =$ $f(y)$ only

$$\mu(y) = e^{\int f(y)\,dy}$$

Then multiply the original equation by μ getting a new equation in the form of :

$$\mu\, M(x, y)\, dx + \mu\, N(x, y)\, dy = 0$$

OR

$$M`(x, y)\, dx + N`(x, y)\, dy = 0$$

Then solve this new differential equation using the normal exact method.

Note: To check on your solutions, you must find that:

$$M_y` = N_x`$$

<u>Examples:-</u>

Solve the differential equation $(y^2 - x)\,dx + 2y\,dy = 0$.

Solution The given equation is not exact because $M_y(x, y) = 2y$ and $N_x(x, y) = 0$. However, because

$$\frac{M_y(x, y) - N_x(x, y)}{N(x, y)} = \frac{2y - 0}{2y} = 1 = h(x)$$

it follows that $e^{\int h(x)\,dx} = e^{\int dx} = e^x$ is an integrating factor. Multiplying the given differential equation by e^x produces the exact differential equation

$$(y^2e^x - xe^x)\,dx + 2ye^x\,dy = 0$$

whose solution is obtained as follows.

$$f(x, y) = \int N(x, y)\,dy = \int 2ye^x\,dy = y^2e^x + g(x)$$

$$f_x(x, y) = y^2e^x + g'(x) = \underbrace{y^2e^x - xe^x}_{M(x,y)}$$

$g'(x) = -xe^x$

Therefore, $g'(x) = -xe^x$ and $g(x) = -xe^x + e^x + C_1$, which implies that

$$f(x, y) = y^2e^x - xe^x + e^x + C_1.$$

The general solution is $y^2e^x - xe^x + e^x = C$, or $y^2 - x + 1 = Ce^{-x}$.

$$(2y^2x - yx^2 + x)dy + (y - x)dx = 0$$

→ No Separation

→ No Homogenous

$$(y - x)dx + (2y^2x - yx^2 + x)dy = 0$$

$M_y = 1$, $N_x = 2y^2 - 2xy + 1$ → Not Exact

Find an integrating factor:

$$\frac{N_x - M_y}{M} = \frac{2y^2 - 2xy + 1 - 1}{y - x} = \frac{2y^2 - 2xy}{y - x} = \frac{2y(y - x)}{y - x} = 2y \longrightarrow f(y)\ only$$

$$\mu(y) = e^{\int f(y)\,dy} = e^{\int 2y\,dy} = e^{y^2}$$

Therefore, the new equation will be:

$$e^{y^2}(y - x)dx + e^{y^2}(2y^2x - yx^2 + x)dy = 0$$

$$\left(ye^{y^2} - xe^{y^2}\right)dx + \left(2y^2 x e^{y^2} - y x^2 e^{y^2} + x e^{y^2}\right)dy = 0$$

F = C

$$F = \int \left(ye^{y^2} - xe^{y^2}\right)dx = x y e^{y^2} - \frac{x^2}{2} e^{y^2} + g(y)$$

$$\frac{\partial F}{\partial y} \equiv N(x, y)$$

$$x\left(y e^{y^2} * 2y + e^{y^2}\right) - x^2 y e^{x^2} + g'(y) \equiv 2y^2 x e^{y^2} - y x^2 e^{y^2} + x e^{y^2}$$

$$\therefore g'(y) = 0 \longrightarrow g(y) = \int 0 \, dy = C$$

$$F = x y e^{y^2} - \frac{x^2}{2} e^{y^2} = C$$

- $y' \tan x \sin 2y = \sin^2 x + \cos^2 y$

⟶ No separation

⟶ No Homogenous

$$\frac{dy}{dx} \tan x \sin 2y = \sin^2 x + \cos^2 y$$

$$(\sin^2 x + \cos^2 y)\, dx + (-\tan x \sin 2y)\, dy = 0$$

$$M_y = -2 \cos y \sin y$$

$$= -\sin 2y$$

$$N_x = -\sec^2 x \sin 2y$$

$$\longrightarrow M_y \neq N_x \longrightarrow \text{Not Exact}$$

So try to find an integrating factor using either of the above cases:

$$\frac{M_y - N_x}{N} = \frac{-\sin 2y + \sec^2 x \sin 2y}{-\tan x \sin 2y} = \frac{-\sin 2y + \sec^2 x \sin 2y}{-\tan x \sin 2y}$$

$$= \frac{\sin 2y\,(-1 + \sec^2 x)}{-\tan x \sin 2y} = \frac{(-1 + \sec^2 x)}{-\tan x} = \frac{\tan^2 x}{-\tan x} = -\tan x$$

Since that the result is function in x only, then

$$\mu = e^{\int -\tan x \, dx} = e^{\int \frac{-\sin x}{\cos x} dx} = e^{\ln \cos x} = \cos x$$

Therefore, the new equation will be:

$$\cos x\,(\sin^2 x + \cos^2 y)\,dx + \cos x\,(-\tan x \sin 2y)\,dy = 0$$

$$(\cos x \sin^2 x + \cos x \cos^2 y)\,dx + (-\sin x \sin 2y)\,dy = 0$$

$$(\cos x \sin^2 x + \cos x \cos^2 y)\,dx + (-2 \sin x \sin y \cos y)\,dy = 0$$

F = C

$$F = \int N(x, y)\,dy = \int (-2 \sin x \sin y \cos y)\,dy$$

$$F = \sin x \cos^2 y + g(x)$$

$$\frac{\partial F}{\partial x} \equiv M(x, y)$$

$$\cos x \cos^2 y + g'(x) \equiv \cos x \sin^2 x + \cos x \cos^2 y$$

$$\therefore \quad g'(x) \equiv \cos x \sin^2 x \longrightarrow g(x) \equiv \int \cos x \sin^2 x \, dx$$

$$g(x) \equiv \frac{\sin^3 x}{3}$$

$$F = \sin x \cos^2 y + \frac{\sin^3 x}{3} = C$$

$$\boxed{\sin x \cos^2 y + \frac{\sin^3 x}{3} = C}$$

Linear & and Bernoulli Differential Equations

To say that a 1^{st} order differential equation can be solved using Linear Method, the form of the equation must be either:

$$\frac{dy}{dx} + P(x)\,y = Q(x) \quad OR \quad \frac{dx}{dy} + P(y)\,x = Q(y)$$

Solving Criteria:

$$OR$$

$\frac{dy}{dx} + P(x)\,y = Q(x)$	$\frac{dx}{dy} + P(y)\,x = Q(y)$
Let	Let
$\mu\,(x) = e^{\int p(x)\,dx}$	$\mu\,(y) = e^{\int p(y)\,dy}$
and the solution will be	and the solution will be
$\mu\,(x) * y = \int \mu\,(x) * Q(x)\,dx$	$\mu\,(y) * x = \int \mu\,(y) * Q(y)\,dy$

Integration Factor:-

There is a process by which most first-order linear differential equations can be solved. This uses an **integration factor**, denoted $\mu(x)$ (Greek letter "mu").

The differential equation must be in the form

$$y' + f(x)y = g(x).$$

To find $\mu(x)$, we perform the following process (next slide)

Starting with $y' + f(x)y = g(x)$, multiply both sides by $\mu(x)$:

$$\mu(x)y' + \mu(x)f(x)y = \mu(x)g(x)$$

The left side is a product-rule derivative of $(\mu(x)y)$:

$$(\mu(x)y)' = \mu(x)y' + \mu'(x)y.$$

Thus, we have $\mu(x)y' + \mu'(x)y = \mu(x)y' + \mu(x)f(x)y$.

This forces $\mu'(x) = \mu(x)f(x)$. (*next slide*)

Now we find $\mu(x)$. From the last slide, we had $\mu'(x) = \mu(x)f(x)$.

This is a separable differential equation… so separate:

$$\frac{d\mu}{\mu(x)} = f(x)dx.$$

Integrating both sides, we have

$$\int \frac{d\mu}{\mu(x)} = \int f(x)dx\,.$$

(*next slide*)

After integration, we have

$$\ln \mu(x) = \int f(x)\, dx + C.$$

Here, we only need one form of the antiderivative, so we let $C = 0$. Taking base-e on both sides, we now know $\mu(x)$:

$$\mu(x) = e^{\int f(x)dx}.$$

Remark: you don't need to do all those steps each time. Just remember that if you have a differential equation of the form $y' + f(x)y = g(x)$, then find $\mu(x) = e^{\int f(x)dx}$.

Example 1

Find the general solution of the following ODE:

$$\frac{1}{x}\frac{dy}{dx} - \frac{2y}{x^2} = x\cos x$$

Solution

So $\frac{dy}{dx} + \frac{-2}{x}y = x^2\cos x$ is a linear ODE where $P(x) = -\frac{2}{x}$ and $Q(x) = x^2\cos x$

The integrating factor $\mu(x) = e^{\int P(x)dx} = e^{\int \frac{-2}{x}dx} = e^{-2\ln x} = x^{-2}$

The solution is $\mu(x)\,y = \int \mu(x)Q(x)dx$

$$x^{-2}\,y = \int \cos x\,dx = \sin x + C$$

$$y = x^2\sin x + x^2 C$$

Example 2

A rock contains two radioactive isotopes, RA_1 and RA_2, that belong to the same radioactive series; that is, RA_1 decays into RA_2, which then decays into stable atoms. Assume that the rate at which RA_1 decays into RA_2 is $50e^{-t}$ kg/sec. Because the rate of decay of RA_2 is proportional to the mass $y(t)$ of RA_2 present, the rate of change in RA_2 is

$$\frac{dy}{dt} = \text{Rate of concentration} - \text{Rate of decay}$$

$$\frac{dy}{dt} = 50e^{-t} - ky$$

Where $k > 0$ is the decay constant. If $k = 2$, $y(0) = 40\ kg$ then find the mass $y(t)$ of RA_2 for $t \geq 0$.

Solution

$\frac{dy}{dt} + 2y = 50e^{-t}$ is linear ODE where $P(t) = 2$ and $Q(t) = 50e^{-t}$

The integrating factor $\mu(t) = e^{\int P(t)dt} = e^{\int 2\,dt} = e^{2t}$

The solution is $\mu(t)\, y = \int \mu(t)Q(t)dt$

$$e^{2t}\, y = 50 \int e^{-8t}dt = \frac{-25}{4}e^{-8t} + C$$

$$y = \frac{-25}{4}e^{-8t} + e^{-2t}C$$

Since $y(0) = 40$ then $C = \frac{185}{4}$ so the solution will be

$$y = \frac{-25}{4}e^{-8t} + \frac{185}{4}e^{-2t}$$

Example: Find the general solution of
$y' + \frac{2}{x}y = x.$

(Note that $f(x) = \frac{2}{x}$ and $g(x) = x$ and that $x \neq 0$)

So $\mu(x) = e^{\int \left(\frac{2}{x}\right)dx} = e^{2\ln x} = e^{\ln x^2} = x^2.$

Now, use the formula $y = \frac{\int \mu(x)g(x)dx + C}{\mu(x)}$:

Solve $y' + \frac{2}{x}y = x$. From previous slide, we know that $\mu(x) = x^2$.

Now we use the formula $y = \frac{\int \mu(x)g(x)dx + C}{\mu(x)}$:

$$y = \frac{\int x^2 x\, dx + C}{x^2} = \frac{\int x^3 dx + C}{x^2} = \frac{\frac{1}{4}x^4 + C}{x^2} = \frac{1}{4}x^2 + Cx^{-2}.$$

Thus, $y = \frac{1}{4}x^2 + Cx^{-2}$ is the general solution of
$y' + \frac{2}{x}y = x.$

Check that $y = \frac{1}{4}x^2 + Cx^{-2}$ is the general solution of $y' + \frac{2}{x}y = x.$
First, differentiate y:

$$y' = \frac{1}{2}x - 2Cx^{-3}$$

Now insert y' and y into the differential equation and simplify:

$$\left(\frac{1}{2}x - 2Cx^{-3}\right) + \frac{2}{x}\left(\frac{1}{4}x^2 + Cx^{-2}\right) = x$$
$$\left(\frac{1}{2}x - 2Cx^{-3}\right) + \left(\frac{1}{2}x + 2Cx^{-3}\right) = x$$
$$\left(\frac{1}{2}x + \frac{1}{2}x\right) + (-2Cx^{-3} + 2Cx^{-3}) = x$$
$$x + 0 = x.$$

Ordinary Differential Equations (ODEs)

Chapter 2

Bernoulli Differential equations

Exercise 1.

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

DIVIDE by y^n: $\frac{1}{y^n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$

SET $z = y^{1-n}$: i.e. $\frac{dz}{dx} = (1-n)y^{(1-n-1)}\frac{dy}{dx}$

i.e. $\frac{1}{(1-n)}\frac{dz}{dx} = \frac{1}{y^n}\frac{dy}{dx}$

SUBSTITUTE $\frac{1}{(1-n)}\frac{dz}{dx} + P(x)z = Q(x)$

i.e. $\boxed{\frac{dz}{dx} + P_1(x)z = Q_1(x)}$ linear in z

where $P_1(x) = (1-n)P(x)$
$Q_1(x) = (1-n)Q(x)$.

OR

$\frac{dy}{dx} + P(x)\,y = Q(x) * y^n$

Divide both sides by y^n

Thus the equation will be:

$$y^{-n}\frac{dy}{dx} + P(x)\,y^{1-n} = Q(x)$$

Then substitute:

Let $u = y^{1-n}$ *and*

$$\frac{du}{dx} = (1-n)\,y^{-n}\,\frac{dy}{dx}$$

$$\therefore\ y^{-n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{du}{dx}$$

So the equation will be :

$$\frac{1}{(1-n)}\frac{du}{dx} + P(x)\,u = Q(x)$$

Then multiply both sides by (1 − n)

$$\frac{du}{dx} + (1-n)P(x)\,u = (1-n)Q(x)$$

Let $\mu(x) = e^{\int (1-n)\,p(x)\,dx}$

and the solution will be

$$\mu(x) * u = \int \mu(x) * (1-n)\,Q(x)\,dx$$

$$\mu(x) * y^{-n} = \int \mu(x) * (1-n)Q(x)dx$$

$\frac{dx}{dy} + P(y)\,x = Q(y) * x^n$

Divide both sides by x^n

Thus the equation will be:

$$x^{-n}\frac{dx}{dy} + P(y)\,x^{1-n} = Q(y)$$

Then substitute:

Let $u = x^{1-n}$ *and*

$$\frac{du}{dy} = (1-n)\ x^{-n}\,\frac{dx}{dy}$$

$$\therefore\ x^{-n}\frac{dx}{dy} = \frac{1}{(1-n)}\frac{du}{dy}$$

So the equation will be :

$$\frac{1}{(1-n)}\frac{du}{dy} + P(y)\,u = Q(y)$$

Then multiply both sides by (1 − n)

$$\frac{du}{dy} + (1-n)P(y)\,u = (1-n)Q(y)$$

Let $\mu(y) = e^{\int (1-n)\,p(y)\,dy}$

and the solution will be

$$\mu(y) * u = \int \mu(y) * (1-n)Q(y)\,dy$$

$$\mu(y) * x^{-n} = \int \mu(y) * (1-n)Q(y)dy$$

Example

Find the general solution of $2\frac{dy}{dx} + \tan x \cdot y = \frac{(4x+5)^2}{\cos x} y^3$

Solution

Divide by 2 to get standard form:

$$\frac{dy}{dx} + \frac{1}{2}\tan x \cdot y = \frac{(4x+5)^2}{2\cos x} y^3$$

This is of the form $\frac{dy}{dx} + P(x)y = Q(x)y^n$

$$\text{where} \quad P(x) = \frac{1}{2}\tan x$$

$$Q(x) = \frac{(4x+5)^2}{2\cos x}$$

$$\text{and} \quad n = 3$$

DIVIDE by y^n: i.e. $\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{2}\tan x \cdot y^{-2} = \frac{(4x+5)^2}{2\cos x}$

SET $z = y^{1-n} = y^{-2}$: i.e. $\frac{dz}{dx} = -2y^{-3}\frac{dy}{dx} = -\frac{2}{y^3}\frac{dy}{dx}$

$$\therefore \quad -\frac{1}{2}\frac{dz}{dx} + \frac{1}{2}\tan x \cdot z = \frac{(4x+5)^2}{2\cos x}$$

$$\text{i.e.} \quad \frac{dz}{dx} - \tan x \cdot z = \frac{(4x+5)^2}{\cos x}$$

Integrating factor, $\quad \text{IF} = e^{\int -\tan x \cdot dx} = e^{\int -\frac{\sin x}{\cos x}\, dx} \left[\equiv e^{\int \frac{f'(x)}{f(x)}\, dx} \right]$

$$= e^{\ln \cos x} = \cos x$$

$$\therefore \quad \cos x \frac{dz}{dx} - \cos x \tan x \cdot z = \cos x \frac{(4x+5)^2}{\cos x}$$

$$\text{i.e.} \quad \cos x \frac{dz}{dx} - \sin x \cdot z = (4x+5)^2$$

$$\text{i.e.} \quad \cos x \cdot z = \int (4x+5)^2 dx$$

$$\text{i.e.} \quad \cos x \cdot z = \left(\frac{1}{4}\right) \cdot \frac{1}{3}(4x+5)^3 + C$$

Use $z = \frac{1}{y^2}$:

$$\frac{\cos x}{y^2} = \frac{1}{12}(4x+5)^3 + C$$

$$\text{i.e.} \quad \frac{1}{y^2} = \frac{1}{12 \cos x}(4x+5)^3 + \frac{C}{\cos x} \; .$$

Example

Find the general solution of $x\frac{dy}{dx} + y = y^2 x^2 \ln x$

Solution

Standard form: $\quad \frac{dy}{dx} + \left(\frac{1}{x}\right) y = (x \ln x) y^2$

i.e. $\quad P(x) = \frac{1}{x},\ Q(x) = x \ln x \ ,\ n = 2$

DIVIDE by y^2: $\quad \frac{1}{y^2}\frac{dy}{dx} + \left(\frac{1}{x}\right) y^{-1} = x \ln x$

SET $z = y^{-1}$: $\quad \frac{dz}{dx} = -y^{-2}\frac{dy}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$

$$\therefore \quad -\frac{dz}{dx} + \left(\frac{1}{x}\right) z = x \ln x$$

$$\text{i.e.} \quad \frac{dz}{dx} - \frac{1}{x} \cdot z = -x \ln x$$

Integrating factor: $\text{IF} = e^{-\int \frac{dx}{x}} = e^{-\ln x} = e^{\ln x^{-1}} = \dfrac{1}{x}$

$$\therefore \quad \frac{1}{x}\frac{dz}{dx} - \frac{1}{x^2}z = -\ln x$$

$$\text{i.e.} \quad \frac{1}{x}z = -\int \ln x \, dx + C'$$

[Use integration by parts: $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx,$

with $u = \ln x$, $\frac{dv}{dx} = 1$]

$$\text{i.e.} \quad \frac{1}{x}z = -\left[x \ln x - \int x \cdot \frac{1}{x}dx\right] + C$$

Use $z = \frac{1}{y}$: $\quad \frac{1}{xy} = x(1 - \ln x) + C$.

Example

Find the general solution of $\dfrac{dy}{dx} = y\cot x + y^3\operatorname{cosec}x$

Solution

Standard form: $\quad \frac{dy}{dx} - (\cot x)\cdot y = (\operatorname{cosec} x)\, y^3$

DIVIDE by y^3: $\quad \frac{1}{y^3}\frac{dy}{dx} - (\cot x)\cdot y^{-2} = \operatorname{cosec} x$

SET $z = y^{-2}$: $\quad \frac{dz}{dx} = -2y^{-3}\frac{dy}{dx} = -2\cdot\frac{1}{y^3}\frac{dy}{dx}$

$$\therefore \quad -\frac{1}{2}\frac{dz}{dx} - \cot x \cdot z = \operatorname{cosec} x$$

$$\text{i.e.} \quad \frac{dz}{dx} + 2\cot x \cdot z = -2 \operatorname{cosec} x$$

Integrating factor: $\text{IF} = e^{2\int \frac{\cos x}{\sin x}dx} \equiv e^{2\int \frac{f'(x)}{f(x)}dx} = e^{2\ln(\sin x)} = \sin^2 x.$

$$\therefore \quad \sin^2 x \cdot \frac{dz}{dx} + 2\sin x \cdot \cos x \cdot z = -2\sin x$$

i.e. $\frac{d}{dx}\left[\sin^2 x \cdot z\right] = -2\sin x$

i.e. $z\sin^2 x = (-2)\cdot(-\cos x) + C$

Use $z = \frac{1}{y^2}$: $\frac{\sin^2 x}{y^2} = 2\cos x + C$

i.e. $y^2 = \frac{\sin^2 x}{2\cos x + C}$.

Example

A 30-volt battery is applied to R-L series circuit with R=50 ohm and L=0.1 henry. Find the current $i(t)$ if $i(0) = 0$. Determine the time at which $i = 0.25\, i_{ss}$.

Solution

$$L\frac{di(t)}{dt} + Ri(t) = E(t)$$

$$\frac{di(t)}{dt} + \frac{R}{L}i(t) = \frac{E(t)}{L}$$

$$\frac{di(t)}{dt} + \frac{50}{0.1}i(t) = \frac{30}{0.1}$$

$$\frac{di(t)}{dt} + 500\, i(t) = 300$$

$$\mu = e^{\int 500dt} = e^{500t}$$

$$e^{500t}\, i(t) = \int 300\, e^{500t}\, dt = \frac{3}{5}e^{500t} + C$$

$$i(t) = \frac{3}{5} + Ce^{-500t}$$

$$since\ i(0) = 0, \text{then} \quad C = \frac{-3}{5}$$

$$i(t) = \frac{3}{5} - \frac{3}{5}e^{-500t}$$

At $i(t) = 0.25\ i_{ss} = 0.25\ \left(\frac{3}{5}\right) = \frac{3}{20}$then $\frac{3}{20} = \frac{3}{5}\ (1 - e^{-500t})$

Then $t = 5.75 \times 10^{-4}\ sec$

Equations with Linear Coefficients

Equations with linear coefficients that is, equations of the form

(13) $$(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0\ ,$$

where the a_i's, b_i's, and c_i's are constants. We leave it as an exercise to show that when $a_1b_2 = a_2b_1$, equation (13) can be put in the form $dy/dx = G(ax + by)$, which we solved via the substitution $z = ax + by$.

Before considering the general case when $a_1b_2 \neq a_2b_1$, let's first look at the special situation when $c_1 = c_2 = 0$. Equation (13) then becomes

$$(a_1x + b_1y)dx + (a_2x + b_2y)dy = 0\ ,$$

which can be rewritten in the form

$$\frac{dy}{dx} = -\frac{a_1x + b_1y}{a_2x + b_2y} = -\frac{a_1 + b_1(y/x)}{a_2 + b_2(y/x)}\ .$$

This equation is homogeneous, so we can solve it using the method discussed earlier in this section.

The above discussion suggests the following procedure for solving (13). If $a_1b_2 \neq a_2b_1$, then we seek a translation of axes of the form

$$x = u + h \qquad \text{and} \qquad y = v + k\ ,$$

where h and k are constants, that will change $a_1x + b_1y + c_1$ into $a_1u + b_1v$ and change $a_2x + b_2y + c_2$ into $a_2u + b_2v$. Some elementary algebra shows that such a transformation exists if the system of equations

(14) $$\begin{aligned} a_1h + b_1k + c_1 &= 0\ , \\ a_2h + b_2k + c_2 &= 0 \end{aligned}$$

has a solution. This is ensured by the assumption $a_1b_2 \neq a_2b_1$, which is geometrically equivalent to assuming that the two lines described by the system (14) intersect. Now if (h, k) satisfies (14), then the substitutions $x = u + h$ and $y = v + k$ transform equation (13) into the homogeneous equation

(15) $$\frac{dv}{du} = -\frac{a_1u + b_1v}{a_2u + b_2v} = -\frac{a_1 + b_1(v/u)}{a_2 + b_2(v/u)}\ ,$$

which we know how to solve.

Example

Solve $(-3x + y + 6)dx + (x + y + 2)dy = 0$. (16)

Solution Since $a_1b_2 = (-3)(1) \neq (1)(1) = a_2b_1$, we will use the translation of axes $x = u + h$, $y = v + k$, where h and k satisfy the system

$$-3h + k + 6 = 0 \,,$$
$$h + k + 2 = 0 \,.$$

Solving the above system for h and k gives $h = 1, k = -3$. Hence, we let $x = u + 1$ and $y = v - 3$. Because $dy = dv$ and $dx = du$, substituting in equation (16) for x and y yields

$$(-3u + v)du + (u + v)dv = 0$$

The last equation is homogeneous, so we let $z = v/u$. Then $dv/du = z + u(dz/du)$, and, substituting for v/u, we obtain

$$z + u\frac{dz}{du} = \frac{3 - z}{1 + z} \,.$$

Separating variables gives

$$\int \frac{z + 1}{z^2 + 2z - 3}dz = -\int \frac{1}{u}du \,,$$

$$\frac{1}{2}\ln|z^2 + 2z - 3| = -\ln|u| + C_1 \,,$$

from which it follows that

$$z^2 + 2z - 3 = Cu^{-2} \,.$$

When we substitute back in for z, u, and v, we find

$$(v/u)^2 + 2(v/u) - 3 = Cu^{-2} \,,$$
$$v^2 + 2uv - 3u^2 = C \,,$$
$$(y + 3)^2 + 2(x - 1)(y + 3) - 3(x - 1)^2 = C \,.$$

This last equation gives an implicit solution to (16).

Riccati Differential Equation

Riccati Equation. An equation of the form

$$\frac{dy}{dx} = P(x)y^2 + Q(x)y + R(x) \qquad (18)$$

is called a generalized Riccati equation.

(a) If one solution—say, $u(x)$—of (18) is known, show that the substitution $y = u + 1/v$ reduces (18) to a linear equation in v.

(b) Given that $u(x) = x$ is a solution to

$$\frac{dy}{dx} = x^3(y - x)^2 + \frac{y}{x} ,$$

use the result of part (a) to find all the other solutions to this equation.

Solution:

Ordinary Differential Equations (ODEs)

Chapter 2

Chapter Summary:

In this chapter we have discussed various types of first-order differential equations. The most important were the separable, linear, and exact equations. Their principal features and method of solution are outlined below.

Separable Equations: $dy/dx = g(x)p(y)$. Separate the variables and integrate.

Linear Equations: $dy/dx + P(x)y = Q(x)$. The integrating factor $\mu = \exp\left[\int P(x)dx\right]$ reduces the equation to $d(\mu y)/dx = \mu Q$, so that $\mu y = \int \mu Q\, dx + C$.

Exact Equations: $dF(x, y) = 0$. Solutions are given implicitly by $F(x, y) = C$. If $\partial M/\partial y = \partial N/\partial x$, then $M\, dx + N\, dy = 0$ is exact and F is given by

$$F = \int M\, dx + g(y) \;, \quad \text{where} \quad g'(y) = N - \frac{\partial}{\partial y}\int M\, dx$$

or

$$F = \int N\, dy + h(x) \;, \quad \text{where} \quad h'(x) = M - \frac{\partial}{\partial x}\int N\, dy \;.$$

When an equation is not separable, linear, or exact, it may be possible to find an integrating factor or perform a substitution that will enable us to solve the equation.

Special Integrating Factors: $\mu M\, dx + \mu N\, dy = 0$ is exact. If $(\partial M/\partial y - \partial N/\partial x)/N$ depends only on x, then

$$\mu(x) = \exp\left[\int\left(\frac{\partial M/\partial y - \partial N/\partial x}{N}\right)dx\right]$$

is an integrating factor. If $(\partial N/\partial x - \partial M/\partial y)/M$ depends only on y, then

$$\mu(y) = \exp\left[\int\left(\frac{\partial N/\partial x - \partial M/\partial y}{M}\right)dy\right]$$

is an integrating factor.

Homogeneous Equations: $dy/dx = G(y/x)$. Let $v = y/x$. Then $dy/dx = v + x(dv/dx)$, and the transformed equation in the variables v and x is separable.

Equations of the Form: $dy/dx = G(ax + by)$. Let $z = ax + by$. Then $dz/dx = a + b(dy/dx)$, and the transformed equation in the variables z and x is separable.

Bernoulli Equations: $dy/dx + P(x)y = Q(x)y^n$. For $n \neq 0$ or 1, let $v = y^{1-n}$. Then $dv/dx = (1 - n)y^{-n}(dy/dx)$, and the transformed equation in the variables v and x is linear.

Linear Coefficients: $(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0$. For $a_1b_2 \neq a_2b_1$, let $x = u + h$ and $y = v + k$, where h and k satisfy

$$a_1h + b_1k + c_1 = 0 \;,$$
$$a_2h + b_2k + c_2 = 0 \;.$$

Then the transformed equation in the variables u and v is homogeneous.

Chapter 3

FIRST-ORDER DIFFERENTIAL EQUATIONS OF HIGHER DEGREE

CHAPTER 3

FIRST-ORDER DIFFERENTIAL EQUATIONS OF HIGHER DEGREE

3.1 Equations of the first-Order and not of First Degree

In this Chapter we discuss briefly basic properties of differential equations of first-order and higher degree. In general such equations may not have solutions. We confine ourselves to those cases in which solutions exist.

The most general form of a differential equation of the first order and of higher degree say of nth degree can be written as

$$\left(\frac{dy}{dx}\right)^n + a_1(x,y)\left(\frac{dy}{dx}\right)^{n-1} + a_2(x,y)\left(\frac{dy}{dx}\right)^{n-2} + \ldots \ldots$$

$$\ldots \ldots \ldots + a_{n-1}(x,y)\frac{dy}{dx} + a_n(x,y) = 0$$

or $$p^n + a_1 p^{n-1} + a_2 p^{n-2} + \ldots\ldots + a_{n-1}p + a_n = 0 \qquad (3.1)$$

where $p = \frac{dy}{dx}$ and $a_1, a_2, \ldots, a_n$ are functions of x and y.

(3.1) can be written as

$$F(x, y, p) = 0 \qquad (3.2)$$

3.2 First-Order Equations of Higher Degree Solvable for p

Let (3.2) can be solved for p and can be written as

$$(p-q_1(x,y))\ (p-q_2(x,y))\ \ldots\ldots\ldots\ (p-q_n(x,y)) = 0$$

Equating each factor to zero we get equations of the first order and first degree. One can find solutions of these equations by the methods discussed in the previous chapter. Let their solution be given as:

$$f_i(x,y,c_i)=0,\ i=1,2,3\ \ldots\ldots\ldots n \qquad (3.3)$$

Therefore the general solution of (3.1) can be expressed in the form

$$f_1(x,y,c)\ f_2(x,y,c)\ldots\ldots\ldots f_n(x,y,c) = 0 \qquad (3.4)$$

where c in any arbitrary constant.

It can be checked that the sets of solutions represented by (3.3) and (3.4) are identical because the validity of (3.4) in equivalent to the validity of (3.3) for at least one i with a suitable value of c, namely $c=c_i$

Example 3.1 Solve $xy\left(\frac{dy}{dx}\right)^2 + (x^2 + y^2)\frac{dy}{dx} + xy = 0$ (3.5)

Solution: This is first-order differential equation of degree 2. Let $p = \frac{dy}{dx}$

Equation (3.5) can be written as

$$xy\,p^2+(x^2+y^2)\,p+xy=0 \qquad (3.6)$$

$$(xp+y)(yp+x)=0$$

This implies that

$$xp+y=0,\ yp+x=0 \qquad (3.7)$$

By solving equations in (3.7) we get

$xy=c_1$ and $x^2+y^2=c_2$, respectively

$x\frac{dy}{dx}+y=0$ or $\frac{dy}{dx}+\frac{1}{x}y=0$, Integrating factor

$$I(x)=e^{\int\frac{1}{x}dx}=e^{\log x}.$$

This gives

$$y.x=\int o.x\,dx+c_1 \text{ or } xy=c_1$$

$$y\frac{dy}{dx}+x=0, \quad \text{or} \quad ydy+xdx=0$$

By integration we get $\frac{1}{2}y^2+\frac{1}{2}x^2=c$

or $x^2+y^2=c_2,\ c_2>0,\ -\sqrt{c_2}\le x\le\sqrt{c_2}$

The general solution can be written in the form

$$(x^2+y^2-c_2)(xy-c_1)=0 \qquad (3.8)$$

It can be seen that none of the nontrivial solutions belonging to $xy=c_1$ or $x^2+y^2=c_2$ is valid on the whole real line.

3.3 Equations Solvable for y

Let the differential equation given by (3.2) be solvable for y. Then y can be expressed as a function x and p, that is,

$$y = f\,(x,p) \tag{3.9}$$

Differentiating (3.9) with respect to x we get

$$\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}.\frac{dp}{dx} \tag{3.10}$$

(3.10) is a first order differential equation of first degree in x and p. It may be solved by the methods of Chapter 2. Let solution be expressed in the form

$$\varphi(x,p,c) = 0 \tag{3.11}$$

The solution of equation (3.9) is obtained by eliminating p between (3.9) and (3.11). If elimination of p is not possible then (3.9) and (3.11) together may be considered parametric equations of the solutions of (3.9) with p as a parameter.

Example 3.2: Solve $y^2-1-p^2=o$

Solution: It is clear that the equation is solvable for y, that is

$$y = \sqrt{1+p^2} \tag{3.12}$$

By differentiating (3.12) with respect to x we get

$$\frac{dy}{dx} = \frac{1}{2}\frac{1}{\sqrt{1+p^2}}.2p\frac{dp}{dx} \quad \text{or} \quad p = \frac{p}{\sqrt{1+p^2}}\frac{dp}{dx}$$

or $$p\left[1 - \frac{1}{\sqrt{1+p^2}}\frac{dp}{dx}\right] = 0 \qquad (3.13)$$

(3.13) gives p=o or $1 - \frac{p}{\sqrt{1+p^2}}\frac{dp}{dx} = 0$

By solving p=0 in (3.12) we get y=1

By $$1 - \frac{1}{\sqrt{1+p^2}}\frac{dp}{dx} = 0$$

we get a separable equation in variables p and x.

$$\frac{dp}{dx} = \sqrt{1+p^2}$$

By solving this we get

$$p = \sinh(x+c) \qquad (3.14)$$

By eliminating p from (3.12) and (3.14) we obtain

$$y = \cos h\,(x+c) \qquad (3.15)$$

(3.15) is a general solution.

Solution y=1 of the given equation is a singular solution as it cannot be obtained by giving a particular value to c in (3.15).

3.4 Equations Solvable for x

Let equation (3.2) be solvable for x, that is

$$x=f(y,p) \qquad (3.16)$$

Then as argued in the previous section for y we get a function Ψ such that

$$\Psi(y, p, c) = 0 \qquad (3.17)$$

By eliminating p from (3.16) and (3.17) we get a general solution of (3.2). If elimination of p with the help of (3.16) and (3.17) is cumbersome then these equations may be considered parametric equations of the solutions of (3.16) with p as a parameter.

Example 3.3 Solve $x\left(\frac{dy}{dx}\right)^3 - 12\frac{dy}{dx} - 8 = 0$

Solution: Let $p = \frac{dy}{dx}$, *then* $xp^3-12p-8=0$

It is solvable for x, that is,

$$x = \frac{12p+8}{p3} = \frac{12}{p^2} + \frac{8}{p^3} \qquad (3.18)$$

Differentiating (3.18) with respect to y, we get

$$\frac{dx}{dy} = -2\frac{12}{p^3}\frac{dp}{dy} - 3\frac{8}{p^4}\frac{dp}{dy} \quad \text{or} \quad \frac{1}{p} = -\frac{24}{p^3}\frac{dp}{dy} - \frac{24}{p^4}\frac{dp}{dy} \quad or \quad dy = \left(-\frac{24}{p^2} - \frac{24}{p^3}\right)dp$$

$$or \quad y = +\frac{24}{p} + \frac{12}{p^2} + c \qquad (3.19)$$

(3.18) and (3.19) constitute parametric equations of solution of the given differential equation.

3.5 Equations of the First Degree in x and y – Lagrange's and Clairaut's Equation.

Let Equation (3.2) be of the first degree in x and y, then

$$y = x\varphi_1(p) + \varphi_2(p) \qquad (3.20)$$

Equation (3.20) is known as Lagrange's equation. If $\varphi_1(p) = p$ then the equation

$$y = xp + \varphi_2(p) \qquad (3.21)$$

is known as Clairaut's equation. By differentiating (3.20) with respect to x, we get

$$\frac{dy}{dx} = \phi_1(p) + x\phi_1'(p)\frac{dp}{dx} + \phi_2'(p)\frac{dp}{dx}$$

$$\text{or} \quad p - \phi_1(p) = (x\phi_1'(p) + \phi_2'(p))\frac{dp}{dx} \qquad (3.22)$$

From (3.22) we get

$$(x + \varphi_2'(p))\frac{dp}{dx} = 0 \quad \text{for} \quad \varphi_1(p) = p$$

This gives

$$\frac{dp}{dx} = 0 \text{ or } x + \varphi_2'(p) = 0$$

$\frac{dp}{dx} = 0$ gives $p = c$ and

by putting this value in (3.21) we get

$$y = cx + \varphi_2(c)$$

This is a general solution of Clairaut's equation. The elimination of p between $x+\varphi_2'(p) = 0$ and (3.21) gives a singular solution. If $\varphi_1(p) \neq p$ for any p, then we observe from (3.22) that $\frac{dp}{dx} \neq 0$ everywhere. Division by

$[p - \varphi_1(p)]\frac{dp}{dx}$ in (3.22) gives $\frac{dx}{dp} - \frac{\varphi_1'}{p - \varphi_1(p)} x = \frac{\varphi_2'(p)}{p - \varphi_1(p)}$

which is a linear equation of first order in x and thus can be solved for x as a function of p, which together with (3.20) will form a parametric representation of the general solution of (3.20).

Example 3.4 Solve $\left(\frac{dy}{dx} - 1\right)\left(y - x\frac{dy}{dx}\right) = \frac{dy}{dx}$

Solution: Let $p = \frac{dy}{dx}$ then, $(p-1)(y-xp)=p$

This equation can be written as

$$y = xp + \frac{p}{p-1}$$

Differentiating both sides with respect to x we get

$$\frac{dp}{dx}\left[x - \frac{1}{(p-1)^2}\right] = 0$$

Thus either $\frac{dp}{dx} = 0$ or $x - \frac{1}{(p-1)^2} = 0$

$\frac{dp}{dx} = 0$ gives p=c .

Putting p=c in the equation we get

$y = cx + \frac{c}{c-1} \Rightarrow$ (y-cx)(c-1)=c

which is the required solution.

3.6 Exercises

Solve the following differential equations

1. $\left(\frac{dy}{dx}\right)^3 = \frac{dy}{dx}e^{2x}$

2. $y(y-2)p^2 - (y-2x+xy)p+x=0$

3. $-\left(\frac{dy}{dx}\right)^2 + 4y - x^2 = 0$

4. $\left(\frac{dy}{dx} + y + x\right)\left(x\frac{dy}{dx} + y + x\right)\left(\frac{dy}{dx} + 2x\right) = 0$

5. $y + x\frac{dy}{dx} - x^4\left(\frac{dy}{dx}\right)^2 = 0$

6. $\left(x\frac{dy}{dx} - y\right)\left(y\frac{dy}{dx} + x\right) = h^2\frac{dy}{dx}$

7. $y\left(\frac{dy}{dx}\right)^2 + (x-y)\frac{dy}{dx} = x$

8. $x\left(\frac{dy}{dx}\right)^2 - 2y\frac{dy}{dx} + ax = 0$