تطبيقية 1 (جزء الاستاتيكا)

طلاب الفرقة الأولى عام – شعبة انجليزي

كلية التربية

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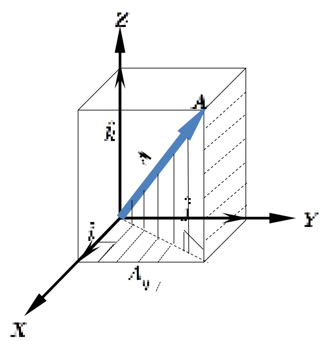
Statics

VECTORS WITH APPLICATIONS

The physical quantities or measurable objects of reasoning in Applied Mathematics or Mechanics are of two classes. The one class, called Vectors, consists of all measurable objects of reasoning which possess directional properties, such as displacement, velocity, acceleration, momentum, force, etc. The other class, called Scalars, comprises measurable objects of reasoning which possess no directional properties, such as mass, work, energy, temperature, etc.

♦ Rectangular Components of a Vector



A vector $\underline{A}$ may have one, two, or three rectangular components along the **X,Y,Z** coordinate axes, depending on how the vector is oriented relative to the axes. In general, though, when $\underline{A}$ is directed within an octant of the **X,Y,Z** frame, Figure behind, then by two successive applications of the parallelogram law, we may resolve the vector into components as $\underline{A} = \underline{A}' + A_z\hat{k}$ and then $\underline{A}' = A_x\hat{i} + A_{\hat{j}}$. Combining these equations, to eliminate $\underline{A}, \underline{A}'$ is represented by the vector sum of its three rectangular components, $\underline{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

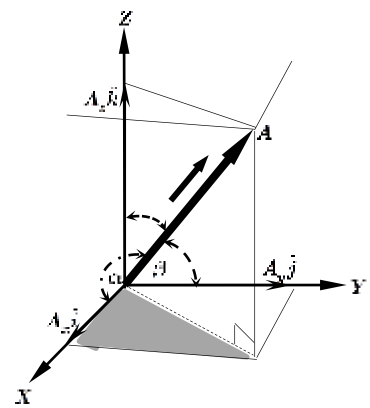
♦ **Magnitude of a Cartesian Vector** It is always possible to obtain the magnitude of **A** provided it is expressed in Cartesian vector form. As shown

$$A = \left|\underline{A}\right| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Since the magnitude of a vector is equal to the positive square root of the sum of the squares of the magnitudes of its components, and $\underline{A}$ has a magnitude of

♦ Coordinate Direction Angles

We will define the direction of $\underline{A}$ by the coordinate direction angles α (alpha), β (beta), and γ (gamma), measured between the tail of $\underline{A}$ and the positive **X,Y,Z** axes provided they are located at the tail of $\underline{A}$, Figure. Note that regardless of where $\underline{A}$ is directed, each of these angles will be between 0° and 180°.



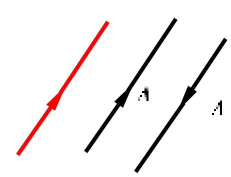
$$\cos\alpha = \frac{A_x}{\sqrt{A_x^2 + A_y^2 + A_z^2}} = \frac{A_x}{A}$$
$$\cos\beta = \frac{A_y}{\sqrt{A_x^2 + A_y^2 + A_z^2}} = \frac{A_y}{A}$$
$$\cos\gamma = \frac{A_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}} = \frac{A_z}{A}$$

A is the magnitude of $\underline{A}$. It is obvious that from previous relation, an important relation among the direction cosines can be formulated as, by squaring and adding

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Here we can see that if only two of the coordinate angles are known, the third angle can be found using this equation.

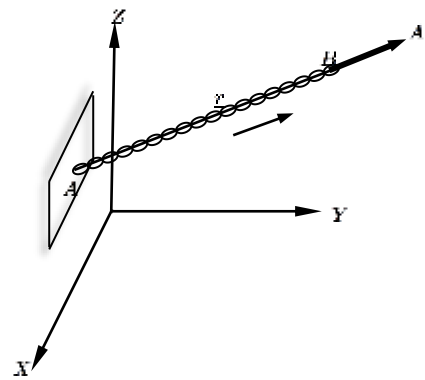
◈ The two vectors $\underline{A}, \underline{B}$ is said to be equal if they have the same magnitude and point in the same direction, while $-\underline{A}$ (negative of a vector $\underline{A}$) has the same magnitude and opposite direction.



♦ **Unit vector of a vector.** A vector is said to be a unit vector if its magnitude equals unity, A unit vector may, therefore, be chosen in any direction. In particular the unit vector along a vector $\underline{A}$ or in direction of the vector $\underline{A}$ is defined by $\hat{A} = \frac{\underline{A}}{A} = \left(\frac{A_x}{A}, \frac{A_y}{A}, \frac{A_z}{A}\right) \equiv \cos\alpha, \cos\beta, \cos\gamma$

♦ Vector Directed Along a Line

Quite often in three-dimensional statics problems, the direction of a force is specified by two points through which its line of action passes. Such a situation is shown in Figure behind, where the vector $\underline{A}$ is directed along the cord AB. We can formulate $\underline{A}$ as a Cartesian vector by realizing that it has the same direction and sense as the position vector $\underline{r}$ directed from point A to point B on the cord. This common direction is specified by the unit vector $\hat{u} = \underline{r} / r$. Hence,



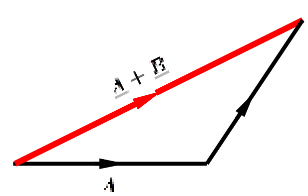
$$\underline{A} = A\hat{u} = A\left(\frac{\underline{r}}{r}\right) = A\left(\frac{(x_B - x_A)\hat{i} + (y_B - y_A)\hat{j} + (z_B - z_A)\hat{k}}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}\right)$$

♦ Addition of Cartesian Vectors

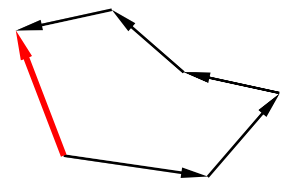
The addition (or subtraction) of two or more vectors is greatly simplified if the vectors are expressed in terms of their Cartesian components. For example, let $\underline{A}, \underline{B}$ be two vectors of components $\underline{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\underline{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$ then the addition or subtraction is given by

$$\begin{aligned}\underline{A} \pm \underline{B} &= (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \pm (B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) \\ &= (A_x \pm B_x)\hat{i} + (A_y \pm B_y)\hat{j} + (A_z \pm B_z)\hat{k}\end{aligned}$$

◈ **Law of Triangle**, states that if a body is acted upon by two vectors represented by two sides of a triangle taken in order, the resultant vector is represented by the third side of the triangle.



♦ Polygon of Vectors



If any number of vectors, acting on a particle be represented, in magnitude and direction, by the sides of a polygon, taken in order, the resultant vector is represented by the last side that will closed the polygon, as shown in red color.

♦ Scalar Product

Occasionally in statics one has to find the angle between two lines or the components of a force parallel and perpendicular to a line. In two dimensions, these problems can readily be solved by trigonometry since the geometry is easy to visualize. In three dimensions, however, this is often difficult, and consequently vector methods should be employed for the solution. The dot product, which defines a particular method for "multiplying" two vectors, can be used to solve the above-mentioned problems.

Let $\underline{A}, \underline{B}$ be two vectors of components $\underline{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\underline{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$ then the scalar product, notation $\underline{A}\bullet\underline{B}$, is expressed in equation form $\underline{A}\bullet\underline{B} = AB\cos\theta$ Or may be given by the Cartesian vector formulation

$$\begin{aligned} \underline{A}\bullet\underline{B} &= (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\bullet(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) \\ &= A_xB_x + A_yB_y + A_zB_z \end{aligned}$$

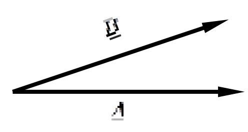
In which A, B represent the magnitude of $\underline{A}, \underline{B}$ and θ is the angle between them. Note that the scalar product is a scalar quantity. It is easy to deduce that

$\underline{A}\bullet\underline{A} = A^2, \qquad (\theta = 0^0)$

$\underline{A}\bullet\underline{B} = \underline{B}\bullet\underline{A}, \qquad$ (Commutatitve law)

$\underline{A}\bullet(\underline{B} + \underline{C}) = \underline{A}\bullet\underline{B} + \underline{A}\bullet\underline{C}, \qquad$ (Associative law)

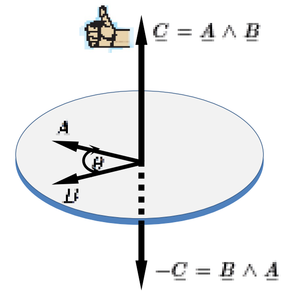
$(\lambda\underline{A})\bullet\underline{B} = \underline{A}\bullet(\lambda\underline{B}) = \lambda(\underline{A}\bullet\underline{B})$



The dot product can be applied to determine the angle formed between two vectors or intersecting lines where $\qquad \theta = \cos^{-1}(\underline{A}\bullet\underline{B} \,/\, AB)$

In particular, notice that if $\underline{A}\bullet\underline{B} = 0 \quad \Rightarrow \theta = \cos^{-1} 0 = \frac{\pi}{2}$, so that $\underline{A}$ will be perpendicular to $\underline{B}$. On the other hand the scalar product gives the work done by a force.

♦ Cross- product



Let $\underline{A}, \underline{B}$ be two vectors of components $\underline{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\underline{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$ then the cross product $\underline{A} \wedge \underline{B}$ or $\underline{A} \times \underline{B}$ is defined by

$$\underline{A} \wedge \underline{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$
$$= (A_yB_z - A_zB_y)\hat{i} - (A_xB_z - A_zB_x)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$$

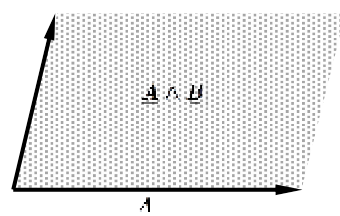
Or $\qquad \underline{A} \wedge \underline{B} = AB\sin\theta \ \hat{n}$

In which $\hat{n}$ is a unit vector normal to the plane that contains the vectors $\underline{A}, \underline{B}$ and can be determined by using the right-hand rule, as shown.

Besides, it is easy to deduce that

(i) $\underline{A} \wedge \underline{A} = \underline{0}$, (ii) $\underline{A} \wedge (\underline{B} + \underline{C}) = \underline{A} \wedge \underline{B} + \underline{A} \wedge \underline{C}$,
(iii) $\underline{A} \wedge \underline{B} = -(\underline{B} \wedge \underline{A})$, (iv) $(\lambda\underline{A}) \wedge \underline{B} = \underline{A} \wedge (\lambda\underline{B}) = \lambda(\underline{A} \wedge \underline{B})$

One of important application of cross product is to evaluate the area of parallelogram in which $\underline{A}, \underline{B}$ represents the sides of the parallelogram which is equal $\left|\underline{A} \wedge \underline{B}\right| = AB\sin\theta$



♦ Triple-Dot product

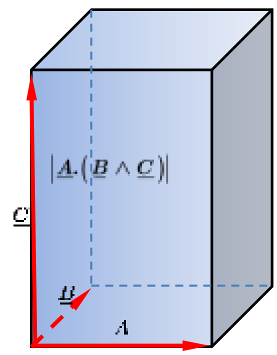
If $\underline{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$, $\underline{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$ and $\underline{C} = C_x\hat{i} + C_y\hat{j} + C_z\hat{k}$ are three vectors then the triple scalar product is defined by $\underline{A}.\ \underline{B} \wedge \underline{C}$

$$\underline{A}\bullet(\underline{B}\wedge\underline{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$
$$= A_x(B_yC_z - B_zC_y) - A_y(B_xC_z - B_zC_x) + A_z(B_xC_y - B_yC_x)$$

It is easy to proof that (properties of determinants)

$$\underline{A}\bullet(\underline{B}\wedge\underline{C}) = \underline{C}\bullet(\underline{A}\wedge\underline{B})$$
$$= \underline{B}\bullet(\underline{C}\wedge\underline{A})$$

In addition, the absolute value of triple scalar product $|\underline{A}\bullet(\underline{B}\wedge\underline{C})|$ gives the volume of parallelepiped in which $\underline{A},\underline{B},\underline{C}$ are three vectors at the corner of the parallelepiped. In particular case as $\underline{A}\bullet(\underline{B}\wedge\underline{C}) = 0$ then the three vectors lie in a plane.



♦ Triple-Cross product

Triple-cross product $\underline{A}\wedge(\underline{B}\wedge\underline{C})$ for any three vectors $\underline{A},\underline{B},\underline{C}$ is defined by

$$\underline{A}\wedge(\underline{B}\wedge\underline{C}) = (\underline{A}\bullet\underline{C})\underline{B} - (\underline{A}\bullet\underline{B})\underline{C}$$

Note that $$\underline{A}\wedge(\underline{B}\wedge\underline{C}) \neq (\underline{A}\wedge\underline{B})\wedge\underline{C}$$

If the triple vector product $\underline{A}\wedge(\underline{B}\wedge\underline{C}) = 0$ then either $\underline{A}$ or $\underline{B}$ or $\underline{C}$ is zero singly or in combination, or $\underline{A}$ is in the plane containing $\underline{B}$ and $\underline{C}$.

♦ λ-μ Theorem

If ABO is a triangle and the point C divides the line AB such that $\lambda : \mu = CB : CA$ then $\lambda\underline{OA} + \mu\underline{OB} = (\lambda+\mu)\underline{OC}$.

Proof.

Let the point C divide the line AB such that $\lambda CA = \mu CB = \mu BC$ then

$\lambda\underline{CA} = \mu\underline{BC}$ (1) (since $\underline{CA}$ and $\underline{BC}$ are in the same direction)

Now in ΔOAC $\quad \underline{OA} = \underline{OC} + \underline{CA} \quad \Rightarrow \lambda\underline{OA} = \lambda\underline{OC} + \lambda\underline{CA}$ (2)

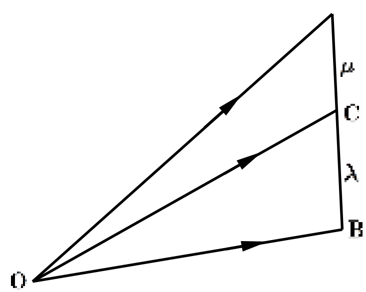
again in ΔOBC $\quad \underline{OB} = \underline{OC} + \underline{CB} \quad \Rightarrow \mu\underline{OB} = \mu\underline{OC} + \mu\underline{CB}$ (3)

Adding equations (2) and (3), we get

$$\begin{aligned}\lambda\underline{OA} + \mu\underline{OB} &= (\lambda + \mu)\underline{OC} + \lambda\underline{CA} + \mu\underline{CB} \\ &= (\lambda + \mu)\underline{OC} + \lambda\underline{CA} - \mu\underline{BC} \quad (\underline{CB} = -\underline{BC}) \\ &= (\lambda + \mu)\underline{OC} \qquad (\lambda\underline{CA} = \mu\underline{BC} \quad \text{from (1)})\end{aligned}$$

◈ **Cor.** If $\lambda = \mu$, then we have

$$\underline{OA} + \underline{OB} = 2\underline{OC}$$



■ Illustrative Examples ■

❒ EXAMPLE 1

Determine a unit vector that parallel to resultant of the vectors $\underline{A} = 2\hat{i} - 7\hat{j} + 3\hat{k}$ and $\underline{B} = -4\hat{i} + 8\hat{j} - \hat{k}$

❒ SOLUTION

The resultant of the two vectors $\underline{A}, \underline{B}$ is

$$\begin{aligned}\underline{R} = \underline{A} + \underline{B} &= (2\hat{i} - 7\hat{j} + 3\hat{k}) + (-4\hat{i} + 8\hat{j} - \hat{k}) \\ &= -2\hat{i} + \hat{j} + 2\hat{k}\end{aligned}$$

Therefore the unit vector $\hat{R}$ parallel to the resultant $\underline{R}$ is given by

$$\hat{R} = \frac{\underline{R}}{R} = \frac{-2\hat{i} + \hat{j} + 2\hat{k}}{3}$$

❒ EXAMPLE 2

Determine the constant λ so that the vector $\underline{A} = 2\lambda\hat{i} + \lambda\hat{j} + \hat{k}$ be perpendicular to the vector $\underline{B} = 4\hat{i} - 3\lambda\hat{j} + \lambda\hat{k}$

❒ SOLUTION

Since the vectors $\underline{A}, \underline{B}$ will be orthogonal if $\underline{A} \bullet \underline{B} = 0$ therefore,

$$\begin{aligned}\therefore\ \underline{A} \bullet \underline{B} &= (2\lambda\hat{i} + \lambda\hat{j} + \hat{k}) \bullet (4\hat{i} - 3\lambda\hat{j} + \lambda\hat{k}) \\ &= 8\lambda - 3\lambda^2 + \lambda = 0 \qquad \Rightarrow \lambda = 0 \quad \text{and} \quad \lambda = 3\end{aligned}$$

❒ EXAMPLE 3

Find a unit vector normal to the plane that contains the vectors $\underline{a} = 2\hat{i} - 6\hat{j} - 3\hat{k}$ and $\underline{b} = 4\hat{i} + 3\hat{j} - \hat{k}$

❒ SOLUTION

Since $\underline{a} \wedge \underline{b}$ is a vector normal to the plane that contains $\underline{a}, \underline{b}$ hence,

$$\underline{a} \wedge \underline{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix} = 15\hat{i} - 10\hat{j} + 30\hat{k}$$

Then the unit vector

$$\hat{n} = \frac{\underline{a} \wedge \underline{b}}{|\underline{a} \wedge \underline{b}|} = \cancel{5}\frac{3\hat{i} - 2\hat{j} + 6\hat{k}}{\cancel{5}\sqrt{49}} = \frac{1}{7}(3\hat{i} - 2\hat{j} + 6\hat{k})$$

❐ EXAMPLE 4

If $\underline{A} \wedge \underline{B} = 8\hat{i} - 14\hat{j} + \hat{k}$ and $\underline{A} + \underline{B} = 5\hat{i} + 3\hat{j} + 2\hat{k}$. Find the vectors $\underline{A}, \underline{B}$

❐ SOLUTION

Let the components of the vector $\underline{A}$ be A_x, A_y, A_z and

$$\because \underline{A} + \underline{B} = 5\hat{i} + 3\hat{j} + 2\hat{k} \qquad \text{then}$$

$$\Rightarrow \underline{A} \wedge (\underline{A} + \underline{B}) = \overbrace{\cancel{\underline{A} \wedge \underline{A}}}^{0} + \underline{A} \wedge \underline{B} = \underline{A} \wedge \underline{B}$$

$$\therefore \underline{A} \wedge (\underline{A} + \underline{B}) = \underline{A} \wedge \underline{B}$$

$$\Rightarrow (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \wedge (5\hat{i} + 3\hat{j} + 2\hat{k}) = 8\hat{i} - 14\hat{j} + \hat{k}$$

$$\therefore \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ 5 & 3 & 2 \end{vmatrix} = (2A_y - 3A_z)\hat{i} - (2A_x - 5A_z)\hat{j} + (3A_x - 5A_y)\hat{k}$$

By equating the components

$$\because (2A_y - 3A_z)\hat{i} - (2A_x - 5A_z)\hat{j} + (3A_x - 5A_y)\hat{k} = 8\hat{i} - 14\hat{j} + \hat{k}$$

$$\therefore 2A_y - 3A_z = 8, \qquad 2A_x - 5A_z = 14, \qquad 3A_x - 5A_y = 1$$

Solving these three equations we get,

$$A_x = 2, \quad A_y = 1, \quad A_z = -2 \qquad \therefore \underline{A} = 2\hat{i} + \hat{j} - 2\hat{k}$$

But it is given that $\underline{A} + \underline{B} = 5\hat{i} + 3\hat{j} + 2\hat{k}$ so $\underline{B} = 3\hat{i} + 2\hat{j} + 4\hat{k}$

Note there are an infinite numbers of vectors

$\underline{A} = 7\hat{i} + 4\hat{j}$ and $\underline{B} = -2\hat{i} - \hat{j} + 2\hat{k}$ etc. (How?)

❒ EXAMPLE 5

Find the vector $\underline{x}$ that satisfies the equations $\underline{a} \wedge \underline{x} = \underline{b} + \underline{a}$, if $\underline{a}.\underline{x} = b$

❒ SOLUTION

Multiply the equation $\underline{a} \wedge \underline{x} = \underline{b} + \underline{a}$, by vector $\underline{a}$ using cross-product so

$$\underline{a} \wedge (\underline{a} \wedge \underline{x}) = \underline{a} \wedge (\underline{b} + \underline{a}) \qquad \textbf{using triple cross-product}$$

$$\therefore \underbrace{(\underline{a}\bullet\underline{x})}_{b}\underline{a} - \underbrace{(\underline{a}\bullet\underline{a})}_{a^2}\underline{x} = \underline{a} \wedge \underline{b} + \cancel{\underline{a} \wedge \underline{a}}$$

$$\therefore b\underline{a} - a^2\underline{x} = \underline{a} \wedge \underline{b} \qquad \Rightarrow \underline{x} = \frac{b\underline{a} - \underline{a} \wedge \underline{b}}{a^2}$$

❒ EXAMPLE 6

Obtain the vector that satisfies the equation $(\underline{a} \wedge \underline{x}) + \underline{x} + m\underline{a} = \underline{0}$ where m is a scalar

❒ SOLUTION

Multiply the equation $(\underline{a} \wedge \underline{x}) + \underline{x} + m\underline{a} = \underline{0}$ by vector $\underline{a} \wedge \underline{x}$ using scalar-product so

$$(\underline{a} \wedge \underline{x}) \wedge ((\underline{a} \wedge \underline{x}) + \underline{x} + m\underline{a}) = \underline{0} \qquad \textbf{from associating law}$$

$$(\underline{a} \wedge \underline{x})\bullet(\underline{a} \wedge \underline{x}) + \underbrace{\underline{x}\bullet(\underline{a} \wedge \underline{x})}_{0} + m\,\underbrace{\underline{a}\bullet(\underline{a} \wedge \underline{x})}_{0} = \underline{0}$$

$$\therefore \left|\underline{a} \wedge \underline{x}\right|^2 = 0 \quad \Rightarrow \quad \underline{a} \wedge \underline{x} = \underline{0}$$

Using this formula and substitute it in equation $\underline{a} \wedge \underline{x} + \underline{x} + m\underline{a} = \underline{0}$ we get

$$\therefore \underline{x} + m\underline{a} = \underline{0} \qquad \Rightarrow \underline{x} = -m\underline{a}$$

❒ EXAMPLE 7

Solve for vector $\underline{x}$ the equation $k\underline{x} + \underline{a} \wedge \underline{x} = \underline{b}$ where k is a scalar.

❒ SOLUTION

Multiply the equation $k\underline{x} + \underline{a} \wedge \underline{x} = \underline{b}$ by vector $\underline{a}$ using scalar-product so

$$\underline{a}\bullet(k\underline{x} + \underline{a} \wedge \underline{x}) = \underline{a}\bullet\underline{b}$$

$$k(\underline{a}\bullet\underline{x}) + \underbrace{\underline{a}\bullet(\underline{a}\wedge\underline{x})}_{0} = \underline{a}.\underline{b} \qquad \Rightarrow k(\underline{a}\bullet\underline{x}) = \underline{a}\bullet\underline{b} \qquad \therefore\ \underline{a}\bullet\underline{x} = \frac{\underline{a}\bullet\underline{b}}{k}$$

Once again, multiply the equation $k\underline{x} + \underline{a}\wedge\underline{x} = \underline{b}$ by vector $\underline{a}$ using cross-product so

$$k(\underline{a}\wedge\underline{x}) + \underbrace{\underline{a}\wedge(\underline{a}\wedge\underline{x})}_{(\underline{a}\bullet\underline{x})\underline{a}-(\underline{a}\bullet\underline{a})\underline{x}} = \underline{a}\wedge\underline{b}$$

$$\Rightarrow k(\underline{a}\wedge\underline{x}) + (\underline{a}\bullet\underline{x})\underline{a} - a^2\underline{x} = \underline{a}\wedge\underline{b}$$

Equation $k\underline{x} + \underline{a}\wedge\underline{x} = \underline{b}$ gives $\underline{a}\wedge\underline{x} = \underline{b} - k\underline{x}$ and then substituting in previous equation we have

$$\Rightarrow k(\underline{b} - k\underline{x}) + \left(\frac{\underline{a}\bullet\underline{b}}{k}\right)\underline{a} - a^2\underline{x} = \underline{a}\wedge\underline{b}$$

$$\Rightarrow (a^2 + k^2)\underline{x} = k\underline{b} + \left(\frac{\underline{a}\bullet\underline{b}}{k}\right)\underline{a} - \underline{a}\wedge\underline{b}$$

Or $$\underline{x} = \frac{1}{k(a^2 + k^2)}\{k^2\underline{b} + (\underline{a}\bullet\underline{b})\underline{a} - k\underline{a}\wedge\underline{b}\}$$

❐ EXAMPLE 8

For any vectors three $\underline{A},\underline{B}$ and $\underline{C}$ show that

(i) $\underline{A}\wedge\ \ \underline{B}\wedge\underline{C}\ \ +\underline{B}\wedge\ \ \underline{C}\wedge\underline{A}\ \ +\underline{C}\wedge\ \ \underline{A}\wedge\underline{B}\ \ = \underline{0}$

(ii) $(\underline{A}+\underline{B})\wedge(\underline{B}-\underline{A}) = 2\underline{A}\wedge\underline{B}$

(iii) $\underline{A}.\ \ \underline{A}\wedge\underline{B}\ \ = 0$

❐ SOLUTION

(i) By applying the triple cross product principle, we have

$$\underline{A}\wedge\ \ \underline{B}\wedge\underline{C}\ \ = \ \ \underline{A}\bullet\underline{C}\ \ \underline{B} -\ \ \underline{A}\bullet\underline{B}\ \ \underline{C}$$
$$\underline{B}\wedge\ \ \underline{C}\wedge\underline{A}\ \ = \ \ \underline{B}\bullet\underline{A}\ \ \underline{C} -\ \ \underline{B}\bullet\underline{C}\ \ \underline{A}$$
$$\underline{C}\wedge\ \ \underline{A}\wedge\underline{B}\ \ = \ \ \underline{C}\bullet\underline{B}\ \ \underline{A} -\ \ \underline{C}\bullet\underline{A}\ \ \underline{B}$$

Adding the three equations we obtain

$$\underline{A}\wedge\ \ \underline{B}\wedge\underline{C}\ \ +\underline{B}\wedge\ \ \underline{C}\wedge\underline{A}\ \ +\underline{C}\wedge\ \ \underline{A}\wedge\underline{B}\ \ = \underline{0}$$

(ii) $(\underline{A}+\underline{B})\wedge(\underline{B}-\underline{A}) = \underline{A}\wedge\underline{B} - \overbrace{\underline{A}\wedge\underline{A}}^{0} + \overbrace{\underline{B}\wedge\underline{B}}^{0} - \underline{B}\wedge\underline{A}$

$$= \underline{A}\wedge\underline{B} + \underline{A}\wedge\underline{B} = 2\ \ \underline{A}\wedge\underline{B}$$

(iii) From properties of triple-scalar product $\underline{A}.\ (\underline{A}\wedge\underline{B}) = \underline{B}.\ \overbrace{(\underline{A}\wedge\underline{A})}^{0} = 0$ we have

$$\underline{A}.\ (\underline{A}\wedge\underline{B}) = \underline{B}\bullet\ \overbrace{(\underline{A}\wedge\underline{A})}^{0} = 0$$

Another technique from the properties of determinants (two equal rows)

$$\underline{A}.\ (\underline{A}\wedge\underline{B}) = \begin{vmatrix} A_x & A_y & A_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = 0$$

❐ EXAMPLE 9

For any four vectors $\underline{A}, \underline{B}, \underline{C}$ and $\underline{D}$ prove that

$$\underline{D}\bullet\ (\underline{A}\wedge\ (\underline{B}\wedge\ (\underline{C}\wedge\underline{D}))) = (\underline{B}\bullet\underline{D})\ ((\underline{C}\wedge\underline{D})\bullet\underline{A})$$

❐ SOLUTION

$$\text{L.H.S.} = \underline{D}\bullet\left\{\underline{A}\wedge\ \underbrace{(\underline{B}\wedge\ (\underline{C}\wedge\underline{D}))}_{\downarrow}\right\}$$

$$= \underline{D}\bullet\left\{\underline{A}\wedge\ \overbrace{((\underline{B}\bullet\underline{D})\ \underline{C} - (\underline{B}\bullet\underline{C})\ \underline{D})}\right\} \quad \textbf{Triple-cross product}$$

$$= \underline{D}\bullet\ ((\underline{A}\wedge\underline{C})\ (\underline{B}\bullet\underline{D}) - (\underline{B}\bullet\underline{C})\ (\underline{A}\wedge\underline{D}))$$

$$= (\underline{B}\bullet\underline{D})\{\underline{D}\bullet(\underline{A}\wedge\underline{C})\} - (\underline{B}\bullet\underline{C})\underbrace{\{\underline{D}\bullet(\underline{A}\wedge\underline{D})\}}_{0}$$

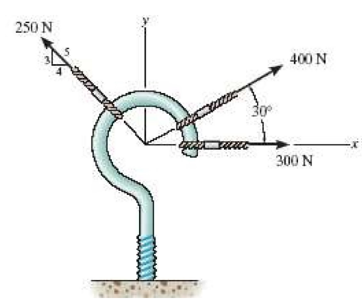
$$= (\underline{B}\bullet\underline{D})\{\underline{D}\bullet(\underline{A}\wedge\underline{C})\}$$

$$= (\underline{B}\bullet\underline{D})\{\underline{A}\bullet(\underline{C}\wedge\underline{D})\} = \text{R.H.S.}$$

L.H.S. means Left hand side, **R.H.S. means Right hand side**

❐ EXAMPLE 10

Determine the magnitude and direction of the resultant force for the forces acting on the hook.



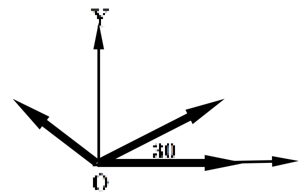
❐ SOLUTION

The forces can be in written Cartesian coordinates as

$\underline{F}_1 = 300\,\hat{i},$

$\underline{F}_2 = 400\cos 30\,\hat{i} + 400\sin 30\,\hat{j}, = 200\sqrt{3}\hat{i} + 200\hat{j}$

$\underline{F}_3 = -250 \;\; 0.8 \;\; \hat{i} + 250(0.6)\hat{j} = -200\hat{i} + 150\hat{j}$



Therefore the resultant is

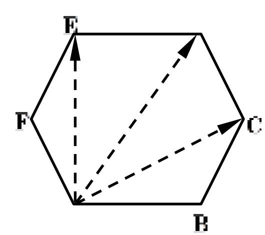
$\underline{F} = \;\; 100 + 200\sqrt{3} \;\; \hat{i} + 350\hat{j}$

❐ EXAMPLE 11

ABCDEF is a regular hexagon, prove that $\underline{AB}+\underline{AC}+\underline{AE}+\underline{AF} = 2\underline{AD}$

❐ SOLUTION

According to the triangle law, we have



$\because\ \underline{AD} = \underline{AC}+\underline{CD}, \qquad \text{and} \qquad \underline{AD} = \underline{AE}+\underline{ED}$

$\therefore\ 2\underline{AD} = \underline{AC}+\underline{AE}+\underbrace{\underline{CD}}_{\underline{AF}} + \underbrace{\underline{ED}}_{\underline{AB}}$

but $\qquad \underline{AB} = \underline{ED}, \quad \text{and} \quad \underline{AF} = \underline{CD}$

Therefore, $\qquad \underline{AB}+\underline{AC}+\underline{AE}+\underline{AF} = 2\underline{AD}$

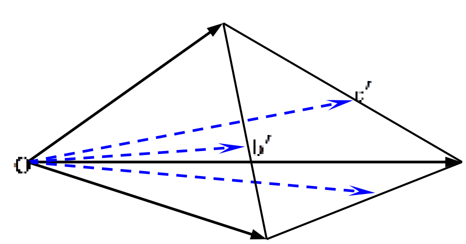
❐ EXAMPLE 12

Let a', b', c' be the middle points of the sides of the triangle **abc** prove that

$$\underline{Oa'} + \underline{Ob'} + \underline{Oc'} = \underline{Oa} + \underline{Ob} + \underline{Oc}$$

For any arbitrary point b'.

❐ SOLUTION



By applying the λ-μ theorem in (a' divides **bc** by a ratio **1:1**, etc.)

$\Delta Obc \qquad \Rightarrow 2\underline{Oa'} = \underline{Ob} + \underline{Oc}$

$\Delta Oac \qquad \Rightarrow 2\underline{Ob'} = \underline{Oa} + \underline{Oc}$

$\Delta Oab \qquad \Rightarrow 2\underline{Oc'} = \underline{Oa} + \underline{Ob}$

Adding these three equations we get

$$\begin{aligned} 2\ \underline{Oa'} + \underline{Ob'} + \underline{Oc'} &= \underline{Oa} + \underline{Ob} + \underline{Oa} + \underline{Oc} + \underline{Ob} + \underline{Oc} \\ &= 2\ \ \underline{Oa} + \underline{Ob} + \underline{Oc} \end{aligned}$$

Dividing by 2, we have

$$\therefore \underline{Oa'} + \underline{Ob'} + \underline{Oc'} = \underline{Oa} + \underline{Ob} + \underline{Oc}$$

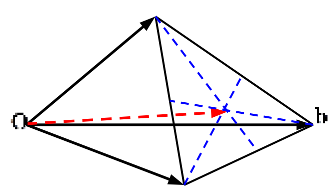
❒ EXAMPLE 13

Let S be a median point of a triangle abc, show that for any arbitrary point O

$$\underline{Oa} + \underline{Ob} + \underline{Oc} = 3\,\underline{OS}$$

❒ SOLUTION

By applying the λ-μ theorem in (b divides bc by a ratio $1{:}1$)



$\Delta OaS \qquad \Rightarrow \underline{Oa} = \underline{OS} + \underline{Sa}$

$\Delta ObS \qquad \Rightarrow \underline{Ob} = \underline{OS} + \underline{Sb}$

$\Delta OcS \qquad \Rightarrow \underline{Oc} = \underline{OS} + \underline{Sc}$

By adding these three equations

$$\underline{Oa} + \underline{Ob} + \underline{Oc} = \underline{OS} + \underline{Sc} + \underline{OS} + \underline{Sb} + \underline{OS} + \underline{Sa}$$

$$= 3\underline{OS} + \underline{Sa} + \underbrace{\underline{Sb} + \underline{Sc}}_{2\underline{Sa'}} = 3\underline{OS} + \underbrace{\underline{Sa} + 2\underline{Sa'}}_{\underline{0}} = 3\underline{OS}$$

Since S divides any median of the triangle by a ratio $2{:}1$.

PROBLEMS

❒ Determine the components of a vector whose its magnitude is 18 and acts along the line passing through the point $(2,3,-1)$ to point $(-2,12,7)$.

❒ Obtain a unit vector of the nonzero vector $8\hat{i}+7\hat{j}-12\hat{k}$.

❒ Calculate the angle between the two vectors $\underline{A}=2\hat{i}-5\hat{j}+6\hat{k}$, $\underline{B}=4\hat{i}-2\hat{j}-3\hat{k}$.

❒ For any two vectors $\underline{A},\underline{B}$, show that $\left|\underline{A}\wedge\underline{B}\right|^2+(\underline{A}\bullet\underline{B})^2=A^2B^2$.

❒ Evaluate the constant λ so that the three vectors $\underline{A}=2\hat{i}+\hat{j}-2\hat{k}$ ، $\underline{B}=\hat{i}+\hat{j}+3\hat{k}$ ، $\underline{C}=\hat{i}+\lambda\hat{j}$ be coplanar.

❒ Determine the vector $\underline{x}$ that satisfy the equation $\underline{a}\wedge\underline{x}=\underline{a}\wedge\underline{b}$ and $\underline{a}\bullet\underline{x}=0$.

❒ Determine the vector $\underline{x}$ that satisfies the equation $(\underline{x}\wedge\underline{a})+(\underline{x}.\underline{b})\underline{c}=\underline{d}$ in terms of the known vectors $\underline{a},\underline{b},\underline{c},\underline{d}$.

❒ Prove that $(\underline{a}\wedge\underline{b})\wedge\underline{c}=\{(\underline{a}\wedge\underline{b}).\hat{n}\}(\hat{n}\wedge\underline{c})$, where $\hat{n}$ is a unit vector perpendicular to the plane that contains the vectors $\underline{a},\underline{b}$.

❒ Solve, for vector $\underline{x}$, the equation $k\underline{x} + \underline{a} \wedge \underline{x} = \underline{b}$ where k is a scalar.

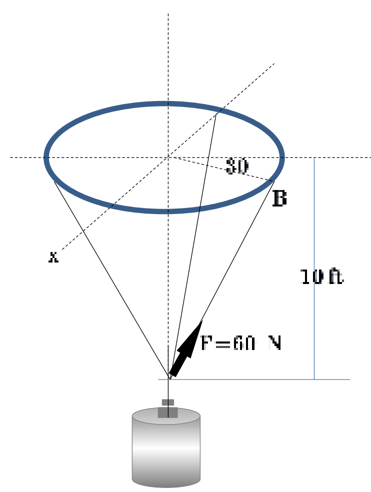
❒ For any three vectors $\underline{a}, \underline{b}, \underline{c}$, deduce that

(i) $(\underline{a} \wedge \underline{b}) \bullet \{(\underline{b} \wedge \underline{c}) \wedge (\underline{c} \wedge \underline{a})\} = \{(\underline{a} \wedge \underline{b}) \bullet \underline{c}\}^2$

(ii) $\{\underline{a} \wedge (\underline{b} \wedge \underline{c})\} \wedge \underline{c} = (\underline{a} \bullet \underline{c})(\underline{b} \wedge \underline{c})$

❒ **ABCD** is a quadrilateral, the points **P**, **M** are bisected the sides **AC**, **BD** respectively, prove that $\underline{\mathbf{AB}} + \underline{\mathbf{CD}} + \underline{\mathbf{AD}} + \underline{\mathbf{CB}} = 4\underline{\mathbf{PM}}$.

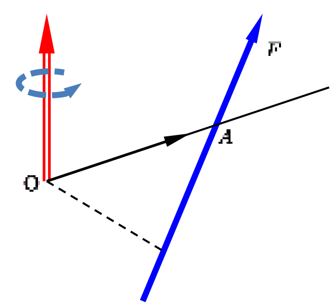
❒ The load at A creates a force of 60 N in wire AB. Express this force as a Cartesian vector acting on A and directed toward B as shown.



MOMENTS AND COUPLES

In this chapter we will obtain the moment of a force about a point or about an axis, reduction the forces at a point.

♦ The Moment



The moment of a force is the tendency of some forces to cause rotation. The moment of a force about a point is defined to be the product of the force and the perpendicular distance of its line of action from the point. On the other hand The moment of a force $\underline{F}$ about point $\mathbf{O}$, or actually about the moment axis passing through $\mathbf{O}$ and perpendicular to the plane containing $\mathbf{O}$ and $\underline{F}$, as shown, can be expressed using the vector cross product, namely,

$$\underline{M}_{\mathbf{O}} = \underline{r} \wedge \underline{F}$$

Here $\underline{r}$ represents a position vector directed from $\mathbf{O}$ to any point on the line of action of $\underline{F}$. Note that

$$\left|\underline{M}_{\mathbf{O}}\right| = \left|\underline{r} \wedge \underline{F}\right| = rF\sin\theta = h$$

So if the force $\underline{F}$ in Cartesian coordinates is $\underline{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$ and the vector $\underline{r}$ is given by $\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then

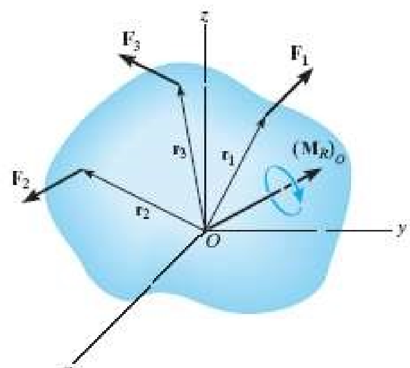
$$\underline{M}_{\mathbf{O}} = \underline{r} \wedge \underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= (yF_z - zF_y)\hat{i} - (xF_z - zF_x)\hat{j} + (xF_y - yF_x)\hat{k}$$

◈ **Theorem** If a number of coplanar forces acting at a point of a rigid body have a resultant, then the vector sum of the moments of the all forces about any arbitrary point is equal to the moments of the resultant about the same point.

Proof.

Let the coplanar forces $\underline{F}_1, \underline{F}_2, ..., \underline{F}_n$ acting at a a rigid body have the resultant $\underline{F}$.



$$\underline{F} = \underline{F}_1 + \underline{F}_2 + ... + \underline{F}_n = \sum \underline{F}_i$$

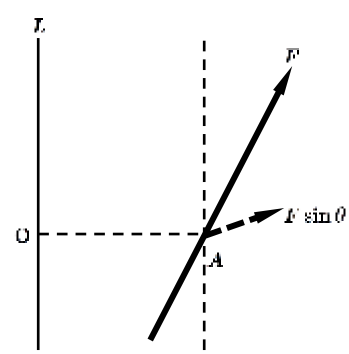
Let **O** be an arbitrary point and $\underline{r}_i$ be the position vector directed from **O** to any point on the line of action of $\underline{F}$. The sum of the moment of the forces $\underline{F}_1, \underline{F}_2, ..., \underline{F}_n$ about **O** is

$$\begin{aligned}\sum \underline{r} \wedge \underline{F}_i &= \underline{r} \wedge \underline{F}_1 + \underline{r} \wedge \underline{F}_2 + ... + \underline{r} \wedge \underline{F}_n \\ &= \underline{r} \wedge \left(\underline{F}_1 + \underline{F}_2 + ... + \underline{F}_n\right) \\ &= \underline{r} \wedge \underline{F}\end{aligned}$$

which is equal to the moment of the resultant about O. Any system of forces, acting in one plane upon a rigid body, can be reduced to either a single force or a single couple.

◈Three forces represented in magnitude, direction and position by the sides of a triangle taken the same way round are equivalent to a couple.

♦ Moment of a force about an axis



Thus if $\underline{F}$ be a force and $\underline{L}$ be a line which does not intersect $\underline{F}$, $\mathbf{OA} = h$ the shortest distance between $\underline{F}$ and $\underline{L}$, and θ the angle between $\underline{F}$ and a line through A parallel to $\underline{L}$, then $F \sin\theta$ is the resolved part of $\underline{F}$ at right angles to $\underline{L}$ and $Fh\sin\theta$ is the moment of $\underline{F}$ about $\underline{L}$ notation by $\underline{M}_L$. If $\underline{F}$ intersects the line $\underline{L}$

or is parallel to $\underline{L}$, then the moment of $\underline{F}$ about $\underline{L}$ is zero, because in the one case $h = 0$ and in the other $\sin\theta = 0$.

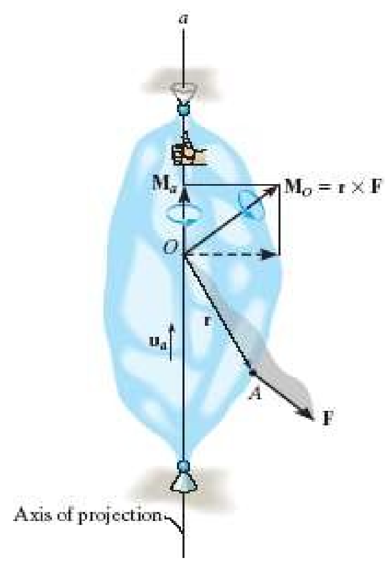
Or on the other hand $\underline{M}_{\underline{L}} = (\underline{M}_o \bullet \hat{n})\hat{n}$ where $\hat{n}$ is a unit vector of axis $\underline{L}$

and $\underline{M}_o$ represents the moment of the force $\underline{F}$ about a point **O** (say) lies on the axis $\underline{L}$, here

$$\left|\underline{M}_{\underline{L}}\right| = \hat{n}\bullet(\underline{r} \wedge \underline{F}) = \begin{vmatrix} \ell & m & n \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \ell(yF_z - zF_y) - m(xF_z - zF_x) + n(xF_y - yF_x)$$

◈ When two forces act at a point the algebraical sum of their moments about any line is equal to the moment of their resultant about this line.



♦ In brief to calculate the moment of a force about an axis, one does the following three steps

(i) Obtain a unit vector of the axis (say $\hat{n}$)

(ii) Determine the moment $\underline{M}_o$ of the force $\underline{F}$ about a point lies on the axis, say **O** .

(iii) The moment of a force about an axis is $\underline{M}_{\underline{L}} = (\underline{M}_o \bullet \hat{n})\hat{n}$

◈ <u>Particular cases</u>

The moment of a force $\underline{F}$ about **X** axis is $\underline{M}_{OX} = (\underline{M}_o \bullet \hat{i})\hat{i}$

The moment of a force $\underline{F}$ about **Y** axis is $\underline{M}_{OX} = (\underline{M}_o \bullet \hat{j})\hat{j}$

The moment of a force $\underline{F}$ about **Z** axis is $\underline{M}_{OX} = (\underline{M}_o \bullet \hat{k})\hat{k}$

♦ Couples

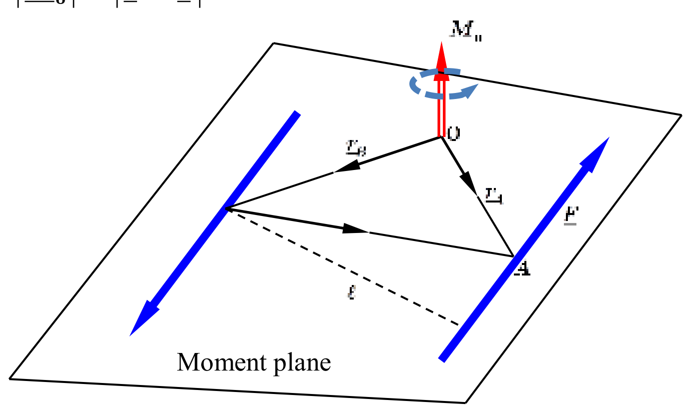
Couples play an important part in the general theory of systems of forces and we shall now establish some of their principal properties. Since a couple consists of two equal and opposite parallel forces (unlike forces), the algebraical sum of the resolved parts of the forces in every direction is zero, so that there is no tendency for the couple to produce in any direction a displacement of translation of the body upon which it acts; and the couple cannot be replaced by a single force. The effect of a couple must therefore be measured in some other way, and, since it has no tendency to produce translation, we next consider what tendency it has to produce rotation.

Let the couple consist of two forces of magnitude F. It is of course assumed that they are both acting upon the same rigid body. Let us take the algebraical sum of the moments of the forces about any point **O** in their plane as the measure of their tendency to turn the body upon which they act about the point **O**.

$$\underline{M}_{\text{o}} = \underline{r}_A \wedge \underline{F} + \underline{r}_B \wedge (-\underline{F})$$

$$\underline{M}_{\text{o}} = (\underline{r}_A - \underline{r}_B) \wedge \underline{F} = \underline{r} \wedge \underline{F}$$

Where its magnitude is $\left|\underline{M}_{\text{o}}\right| = \left|\underline{r} \wedge \underline{F}\right| = rF\sin\theta = F\ell$



◈ Forces completely represented by the sides of a plane polygon taken the same way round are equivalent to a couple whose moment is represented by twice the area of the polygon.

♦ Reduction a system of forces

Suppose a system of forces $\underline{F}_1, \underline{F}_2, ..., \underline{F}_i, ..., \underline{F}_n$ is reduced at a chosen point $\mathbf{O}$ to a single force $\underline{F}$ and a single couple $\underline{M}$ viz. the obtaining result is $(\underline{M}_{\text{o}}, \underline{F})$ where

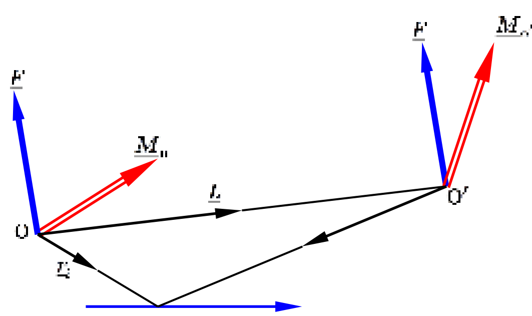
$$\underline{M}_{\text{o}} = \sum_{i=1}^{n} \underline{r}_i \wedge \underline{F}_i, \qquad \underline{F} = \sum_{i=1}^{n} \underline{F}_i$$

Once again if the system of these forces reduced at another point $\mathbf{O}'$ where the obtaining results is

$$\underline{M}_{\text{o}'} = \sum_{i=1}^{n} \underline{r}_i' \wedge \underline{F}_i, \qquad \underline{F} = \sum_{i=1}^{n} \underline{F}_i$$

That is when the point of reduction changed from $\mathbf{O}$ to $\mathbf{O}'$, the resultant of the forces does not change while the moment altered, such that

$$\begin{aligned} \therefore \underline{M}_{\text{o}'} &= \sum_{i=1}^{n} \underline{r}_i' \wedge \underline{F}_i \\ &= \sum_{i=1}^{n} (\underline{r}_i - \underline{L}) \wedge \underline{F}_i \\ &= \sum_{i=1}^{n} \underline{r}_i \wedge \underline{F}_i - \sum_{i=1}^{n} \underline{L} \wedge \underline{F}_i \\ &= \underline{M}_{\text{o}} - \sum_{i=1}^{n} \underline{L} \wedge \underline{F}_i \\ &= \underline{M}_{\text{o}} - \underline{L} \wedge \sum_{i=1}^{n} \underline{F}_i \end{aligned}$$



$$\therefore \underline{M}_{\text{o}'} = \underline{M}_{\text{o}} - \underline{L} \wedge \underline{F}$$

Also it is obvious

$$\therefore \underline{F}.\underline{M}_{\text{o}'} = \underline{F}.(\underline{M}_{\text{o}} - \underline{L} \wedge \underline{F}) = \underline{F}.\underline{M}_{\text{o}} - \overbrace{\underline{F}.(\underline{L} \wedge \underline{F})}^{0} = \underline{F}.\underline{M}_{\text{o}} = \text{const.}$$

The quantity $\underline{F}.\underline{M}_{\text{o}}$ is called invariant quantity

◈Wrench

Suppose a system of forces is reduced to a single force $\underline{F}$ and a single couple $\underline{M}$ such that the axis of the couple is coincides with the line of action of the force $\underline{F}$, then that line is called central axis. In addition, $\underline{F}$ and $\underline{M}$ taken together are called wrench of the system and are written as $(\underline{F},\underline{M})$. The single force $\underline{F}$ is called the intensity of the wrench and the ratio $\underline{M} / \underline{F}$ is called the pitch of the system and is denoted by λ. Since $\underline{F}$ and $\underline{M}_{o'}$ have the same direction so

$\underline{M}_{o'} = \underline{M}_o - \underline{r} \wedge \underline{F} = \lambda \underline{F}$ multiply by $\underline{F}$ using scalar product

$\Rightarrow \underline{F} \bullet \{\underline{M}_o - \underline{r} \wedge \underline{F}\} = \lambda F^2$ $\therefore \lambda = \dfrac{\underline{F} \bullet \underline{M}_o}{F^2} = \dfrac{FM_O}{F^2} = \dfrac{M_O}{F}$

Where λ is known as the pitch of equivalent wrench

Also since $\underline{F} \wedge \underline{M}_{o'} = \underline{0}$ multiply by $\underline{F}$ using cross product we have,

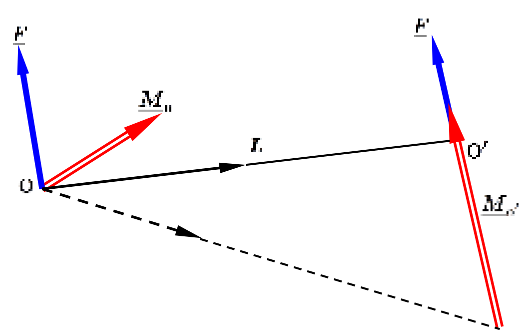
$$\therefore \underline{F} \wedge \underbrace{(\underline{M}_o - \underline{r} \wedge \underline{F})}_{\underline{M}_{o'}} = \underline{F} \wedge \underline{M}_o - \underline{F} \wedge (\underline{r} \wedge \underline{F}) = \underline{0}$$

According to the properties of triple vector product

$$\underline{F} \wedge (\underline{r} \wedge \underline{F}) = (\underline{F}.\underline{F})\underline{r} - (\underline{F}.\underline{r})\underline{F} = F^2 \underline{r} - (\underline{F}.\underline{r})\underline{F}$$

$$\therefore \underline{F} \wedge \underline{M}_o - \{F^2 \underline{r} - (\underline{F}.\underline{r})\underline{F}\} = \underline{0}$$

$$\therefore \underline{r} = \underbrace{\frac{\underline{F} \wedge \underline{M}_o}{F^2}}_{\underline{r}_1} + \underbrace{\frac{(\underline{r}.\underline{F})}{F^2}}_{\mu} \underline{F} \qquad \textbf{Or} \qquad \underline{r} = \underline{r}_1 + \mu \underline{F}$$



The previous equation represents the equation of the central axis or axis of equivalent wrench in vector form and to get the Cartesian form let

$$\underline{r} = (x, y, z), \quad \underline{r}_1 = (a, b, c), \quad \underline{F} = (F_x, F_y, F_z)$$

Therefore, the Cartesian form of central axis is

$$\frac{x-a}{F_x} = \frac{y-b}{F_y} = \frac{z-c}{F_z}$$

◈Special cases

(i) $\underline{F}.\underline{M}_o = 0$ **and** $\underline{F} \neq 0,\ \underline{M}_o = 0$

The system reduced to a single force that acts along the line $\underline{r} = \lambda \underline{F}$

(ii) $\underline{F}.\underline{M}_o = 0$ **and** $\underline{F} = 0,\ \underline{M}_o \neq 0$

The system reduced to a single moment

(iii) $\underline{F}.\underline{M}_o = 0$ **and** $\underline{F} \neq 0,\ \underline{M}_o \neq 0$

In this case $\underline{M}_o$ will be perpendicular to $\underline{F}$ and the system can be reduced to wrench in which the central axis is

$$\therefore \underline{r} = \frac{\underline{F} \wedge \underline{M}_o}{F^2} + \mu \underline{F}$$

(iv) $\underline{F} = 0$ **and** $\underline{M}_o = 0$

The system of forces will be in equilibrium or it is a balanced system of forces.

■ Illustrative Examples ■

❒ EXAMPLE 1

Determine the moment of the force $\underline{F} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ acting at the point $A(3,2,0)$ about the origin and the point $B(2,1,-1)$.

❒ SOLUTION

Since the moment is given by $\underline{M}_o = \underline{r} \wedge \underline{F}$ where

$\underline{r} = \underline{OA} = \underline{A} - \underline{O} = (3,2,0) - (0,0,0) = 3\hat{i} + 2\hat{j}$

Therefore the moment of the given force about the origin is

$$\underline{M}_o = \underline{r} \wedge \underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 0 \\ 2 & 3 & 4 \end{vmatrix} = 8\hat{i} - 12\hat{j} + 5\hat{k}$$

Again, $\underline{r}' = \underline{BA} = \underline{A} - \underline{B} = (3,2,0) - (2,1,-1) = \hat{i} + \hat{j} + \hat{k}$

Hence, the moment f the given force about the point $B(2,1,-1)$ is

$$\underline{M}_B = \underline{r}' \wedge \underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3-2 & 2-1 & 0+1 \\ 2 & 3 & 4 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{vmatrix} = \hat{i} - 2\hat{j} + \hat{k}$$

❒ EXAMPLE 2

Calculate the moment of the force of magnitude $10\sqrt{3}$ and passing through the point $A(5,3,-3)$ to $B(4,4,-4)$ about the origin.

❒ SOLUTION

We have to write the force in vector form, to do this the unit vector in the direction of the force $\hat{F}$, viz. from point $A(5,3,-3)$ to $B(4,4,-4)$ so

$$\because \underline{AB} = \underline{B} - \underline{A} = (4,4,-4) - (5,3,-3) = -\hat{i} + \hat{j} - \hat{k}$$

$$\Rightarrow \hat{F} = \frac{\underline{AB}}{AB} = \frac{-\hat{i} + \hat{j} - \hat{k}}{\sqrt{3}} \equiv \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$$

Therefore the force be

$$\therefore \underline{F} = F\hat{F} = 10\sqrt{3}\left\{\frac{-\hat{\imath} + \hat{\jmath} - \hat{k}}{\sqrt{3}}\right\} \equiv -10\hat{\imath} + 10\hat{\jmath} - 10\hat{k}$$

Choosing any point as an acting point of the force, then the moment of the force about the origin O (consider $A(5,3,-3)$ as an acting point)

$$\because \underline{r} = (5,3,-3) - (0,0,0) = (5,3,-3)$$

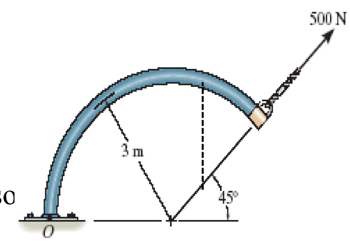
$$\Rightarrow \underline{M}_o = \underline{r} \wedge \underline{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 5 & 3 & -3 \\ -10 & 10 & -10 \end{vmatrix} = 80\hat{\jmath} + 80\hat{k}$$

Also if we choose the point $B(4,4,-4)$ as an acting point

$$\Rightarrow \underline{M}_o = \underline{r'} \wedge \underline{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 4 & -4 \\ -10 & 10 & -10 \end{vmatrix} = 40\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & -1 \\ -1 & 1 & -1 \end{vmatrix} = 80\hat{\jmath} + 80\hat{k}$$

❒ **EXAMPLE 3**

Determine the moment of the force as shown about point **O**.



❒ **SOLUTION**

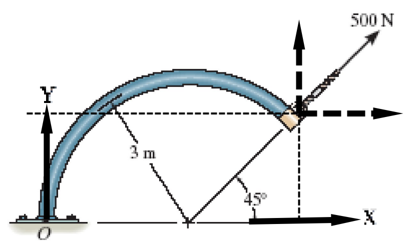
Taking horizontal axis $\mathbf{X}$ as shown, the force 500 can be reso

$$500\cos 45^0\,\hat{\imath} + 500\sin 45^0\,\hat{\jmath} = 250\sqrt{2}(\hat{\imath} + \hat{\jmath})$$

Therefore, the moment is given by,

$$M_o = 250\sqrt{2}\left(3 + \frac{3}{\sqrt{2}}\right) - 250\sqrt{2}\left(\frac{3}{\sqrt{2}}\right) = 750\sqrt{2}$$

Or by cross product where



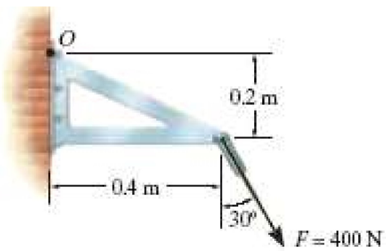
$$\underline{r} = \left(3 + \frac{3}{\sqrt{2}}\right)\hat{\imath} + \frac{3}{\sqrt{2}}\hat{\jmath}$$

$$\underline{F} = 500\cos 45^0\,\hat{\imath} + 500\sin 45^0\,\hat{\jmath} = 250\sqrt{2}(\hat{\imath} + \hat{\jmath})$$

$$\Rightarrow \underline{M}_o = \underline{r} \wedge \underline{F} = 250\sqrt{2}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 + \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} & 0 \\ 1 & 1 & 0 \end{vmatrix} = 750\sqrt{2}\,\hat{k}$$

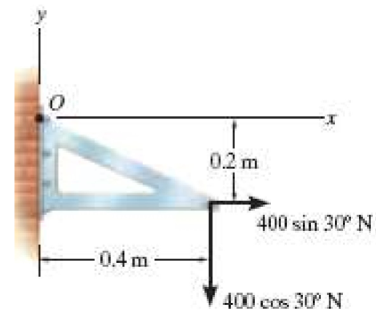
❒ EXAMPLE 4

Force **F** acts at the end of the angle bracket as shown. Determine the moment of the force about point **O**.



❒ SOLUTION

Using a Cartesian vector approach, the force and position vectors are



$$\underline{\boldsymbol{r}} = 0.4\hat{\imath} - 0.2\hat{\jmath}$$

$$\underline{\boldsymbol{F}} = 400\sin 30^0\,\hat{\imath} - 400\cos 30^0\,\hat{\jmath} = 200\hat{\imath} - 346.4\hat{\jmath}$$

The moment is therefore,

$$\Rightarrow \underline{\boldsymbol{M}}_o = \underline{\boldsymbol{r}} \wedge \underline{\boldsymbol{F}} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0.4 & -0.2 & 0 \\ 200 & -346.4 & 0 \end{vmatrix} = -98.6\hat{k}$$

❒ EXAMPLE 5

Find the sum of moment of the forces, $\underline{\boldsymbol{F}} = 2\hat{\imath}$ acts at the origin, the force $-\frac{1}{2}\underline{\boldsymbol{F}}$ acts at $\underline{\boldsymbol{r}}_2 = 3\hat{\jmath}$ and the force $-\frac{1}{2}\underline{\boldsymbol{F}}$ acts at $\underline{\boldsymbol{r}}_3 = 5\hat{k}$ about the origin.

❒ SOLUTION

As clear the resultant of these three forces is zero but the moment about the origin is given by

$$\Rightarrow \underline{\boldsymbol{M}}_o = \sum_{i=1}^{3} \underline{\boldsymbol{r}}_i \wedge \underline{\boldsymbol{F}}_i = \underline{\boldsymbol{r}}_1 \wedge \underline{\boldsymbol{F}}_1 + \underline{\boldsymbol{r}}_2 \wedge \underline{\boldsymbol{F}}_2 + \underline{\boldsymbol{r}}_3 \wedge \underline{\boldsymbol{F}}_3$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{vmatrix} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 3 & 0 \\ -1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & 5 \\ -1 & 0 & 0 \end{vmatrix}$$

$$\therefore \underline{\boldsymbol{M}}_o = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 3 & 5 \\ -1 & 0 & 0 \end{vmatrix} = -5\hat{\jmath} + 3\hat{k} \qquad \text{and} \qquad \left|\underline{\boldsymbol{M}}_o\right| = \sqrt{34}$$

❒ EXAMPLE 6

The force $2\hat{i} - \hat{j}$ acts along the line that passing through the point $(4, 4, 5)$ and the force $3\hat{k}$ acting at the origin. Find the pitch and axis of equivalent wrench.

❒ SOLUTION

The two forces reduced at the origin to a resultant force $\underline{F}$ and a moment $\underline{M}_O$ so that

$$\underline{F} = \underline{F}_1 + \underline{F}_2 = 2\hat{i} - \hat{j} + 3\hat{k}, \qquad \therefore F^2 = 14$$

$$\underline{M}_O = \underline{r}_1 \wedge \underline{F}_1 + \underline{r}_2 \wedge \underline{F}_2 = \underline{0} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 5 \\ 2 & -1 & 0 \end{vmatrix} = 5\hat{i} + 10\hat{j} - 12\hat{k}$$

Thus the pitch of equivalent wrench is given by $\lambda = \dfrac{\underline{F} \bullet \underline{M}}{F^2}$ that is

$$\lambda = \frac{\underline{F} \bullet \underline{M}}{F^2} = \frac{(2\hat{i} - \hat{j} + 3\hat{k}) \bullet (5\hat{i} + 10\hat{j} - 12\hat{k})}{14} = -\frac{36}{14} = -\frac{18}{7}$$

In addition the equation of axis of wrench $\underline{r} = \underline{r}_1 + \mu \underline{F}$

$$\underline{r}_1 = \frac{\underline{F} \wedge \underline{M}}{F^2} = \frac{1}{14} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 3 \\ 5 & 10 & -12 \end{vmatrix} = \frac{1}{14}(-18\hat{i} + 39\hat{j} + 25\hat{k})$$

Then the vector form of the axis becomes

$$\underline{r} = \frac{1}{14}\ -18\hat{i} + 39\hat{j} + 25\hat{k}\ + \mu(2\hat{i} - \hat{j} + 3\hat{k})$$

And Cartesian form is

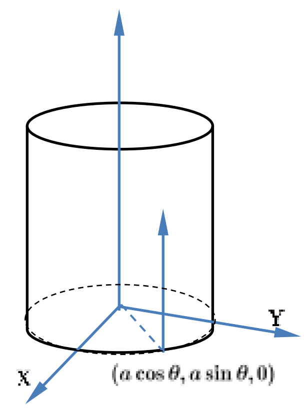
$$\frac{x + \frac{18}{14}}{2} = \frac{y - \frac{39}{14}}{-1} = \frac{z - \frac{25}{14}}{3} \qquad \text{Or} \qquad \frac{14x + 18}{2} = \frac{14y - 39}{-1} = \frac{14z - 25}{3}$$

❒ EXAMPLE 7

A force P acts along the axis of **OX** and another force nP acts along a generator of the cylinder $x^2 + y^2 = a^2$ at the point $(a\cos\theta, a\sin\theta, 0)$; show that the central axis lies on the cylinder $n^2(nx - z)^2 + (1 + n^2)^2 y^2 = n^4 a^2$

❒ SOLUTION

Generators of the cylinder are parallel to the axis of . Let one generator of it pass through the point and its unit vector is and the force acts along this line. Also the force acts along axis then



$$\underline{F}_1 = P\hat{i}, \qquad \text{acts at } (0,0,0)$$
$$\underline{F}_2 = nP\hat{k}, \qquad \text{acts at } (a\cos\theta, a\sin\theta, 0)$$
$$\underline{F} = P(\hat{i} + n\hat{k}), \qquad (F^2 = (1+n^2)P^2)$$

The system reduces to a single force and a moment so that

$$\because \underline{M}_o = \underline{r}_1 \wedge \underline{F}_1 + \underline{r}_2 \wedge \underline{F}_2$$

$$\therefore \underline{M}_o = P\left\{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\theta & a\sin\theta & 0 \\ 0 & 0 & n \end{vmatrix}\right\}$$

$$= anP(\sin\theta\hat{i} - \cos\theta\hat{j})$$

The pitch of equivalent wrench is given by $\lambda = \dfrac{\underline{F}\bullet\underline{M}}{F^2}$ that is

$$\lambda = \frac{\underline{F}\bullet\underline{M}}{F^2} = \frac{P(\hat{i} + n\hat{k})\bullet anP(\sin\theta\hat{i} - \cos\theta\hat{j})}{(1+n^2)P^2} = \frac{an\sin\theta}{1+n^2}$$

In addition the equation of axis of wrench $\underline{r} = \underline{r}_1 + \mu\underline{F}$

$$\underline{r}_1 = \frac{\underline{F} \wedge \underline{M}}{F^2} = \frac{anP^2}{(1+n^2)P^2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & n \\ \sin\theta & -\cos\theta & 0 \end{vmatrix}$$
$$= \frac{an}{1+n^2}(n\cos\theta\hat{i} + n\sin\theta\hat{j} - \cos\theta\hat{k})$$

Then the vector form of the axis becomes

$$\underline{r} = \frac{an}{1+n^2}(n\cos\theta\hat{i} + n\sin\theta\hat{j} - \cos\theta\hat{k}) + \mu(\hat{i} + \hat{k})$$

And Cartesian form is given by

$$\frac{x - \dfrac{an^2\cos\theta}{1+n^2}}{1} = \frac{y - \dfrac{an^2\sin\theta}{1+n^2}}{0} = \frac{z + \dfrac{an\cos\theta}{1+n^2}}{n} \qquad \text{Or}$$

$$y - \frac{an^2\sin\theta}{1+n^2} = 0, \qquad n\left(x - \frac{an^2\cos\theta}{1+n^2}\right) = z + \frac{an\cos\theta}{1+n^2}$$

$$y = \frac{an^2}{1+n^2}\sin\theta, \qquad nx - z = \frac{an(1+n^2)\cos\theta}{1+n^2}$$

Squaring these equations

$$y^2 = n^2\left(\frac{an}{1+n^2}\right)^2\sin^2\theta, \qquad (nx-z)^2 = (1+n^2)^2\left(\frac{an}{1+n^2}\right)^2\cos^2\theta$$

then multiply first equation by $(1+n^2)^2$ and the second by n^2 then adding the result we get

$$(1+n^2)^2y^2 + n^2(nx-z)^2 = n^2(1+n^2)^2\left(\frac{an}{1+n^2}\right)^2 = a^2n^4$$

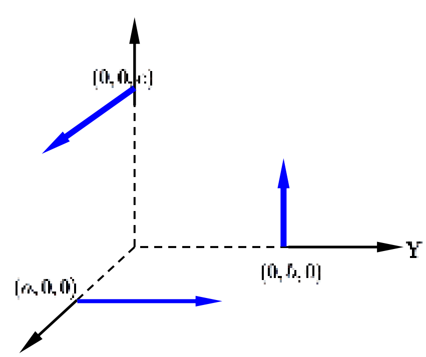
❒ EXAMPLE 8

Three forces each equal to P act on a body, one at point $(a,0,0)$ parallel to **OY**, the second at the point $(0,b,0)$ parallel to **OZ** and the third at the point $(0,0,c)$ parallel to **OX**, the axes being rectangular. Find the resultant wrench.

❒ SOLUTION

As given we see

$$\underline{F}_1 = P\hat{i}, \qquad \text{acts at } (0,0,c)$$
$$\underline{F}_2 = P\hat{j}, \qquad \text{acts at } (a,0,0)$$
$$\underline{F}_3 = P\hat{k}, \qquad \text{acts at } (0,b,0)$$
$$\underline{F} = P(\hat{i} + \hat{j} + \hat{k}), \qquad (F^2 = 3P^2)$$



The system reduces to a single force and a moment so that

$$\because \underline{M}_o = \underline{r}_1 \wedge \underline{F}_1 + \underline{r}_2 \wedge \underline{F}_2 + \underline{r}_3 \wedge \underline{F}_3$$

$$\therefore \underline{M}_o = P\left\{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & c \\ 1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & b & 0 \\ 0 & 0 & 1 \end{vmatrix}\right\}$$

$$= P(b\hat{i} + c\hat{j} + a\hat{k})$$

The pitch of equivalent wrench is given by $\lambda = \dfrac{\underline{F}\bullet\underline{M}}{F^2}$ that is

$$\lambda = \frac{\underline{F}\bullet\underline{M}}{F^2} = \frac{P(\hat{i} + \hat{j} + \hat{k})\bullet P(b\hat{i} + c\hat{j} + a\hat{k})}{3P^2} = \frac{a + b + c}{3}$$

In addition the equation of axis of wrench $\underline{r} = \underline{r}_1 + \mu\underline{F}$

$$\underline{r}_1 = \frac{\underline{F} \wedge \underline{M}}{F^2} = \frac{P^2}{3P^2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ b & c & a \end{vmatrix}$$

$$= \frac{1}{3}((a - c)\hat{i} + (b - a)\hat{j} + (c - b)\hat{k})$$

Then the vector form of the axis becomes

$$\underline{r} = \frac{1}{3}\ (a - c)\hat{i} + (b - a)\hat{j} + (c - b)\hat{k}\ \ + \mu(\hat{i} + \hat{j} + \hat{k})$$

and Cartesian form is

$$\frac{x - \frac{1}{3}(a - c)}{1} = \frac{y - \frac{1}{3}(b - a)}{1} = \frac{z - \frac{1}{3}(c - b)}{1} \qquad \text{Or}$$

$$3\ \ y - x\ \ = b + c - 2a \qquad \text{and} \qquad 3\ \ z - y\ \ = a + c - 2b$$

❒ EXAMPLE 9

Forces X, Y, Z act along three lines given by the equations

$$y = 0, z = c; \qquad z = 0, x = a; \qquad x = 0, y = b$$

Prove that the pitch of the equivalent wrench is

$$(aYZ + bZX + cXY)\ /\ (X^2 + Y^2 + Z^2)$$

If the wrench reduces to a single force, show that the line of the action of the force lies on the hyperboloid $(x-a)(y-b)(z-c) = xyz$

❐ **SOLUTION**

As given

$$\begin{aligned}
\underline{F}_1 &= X\hat{i}, && \text{acts at } (0,0,c)\\
\underline{F}_2 &= Y\hat{j}, && \text{acts at } (a,0,0)\\
\underline{F}_3 &= Z\hat{k} && \text{acts at } (0,b,0)\\
\underline{F} &= X\hat{i} + Y\hat{j} + Z\hat{k}, && F^2 = X^2 + Y^2 + Z^2
\end{aligned}$$

The system reduces to a single force and a moment so that

$$\because \underline{M}_o = \underline{r}_1 \wedge \underline{F}_1 + \underline{r}_2 \wedge \underline{F}_2 + \underline{r}_3 \wedge \underline{F}_3$$

$$\therefore \underline{M}_o = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & c \\ X & 0 & 0 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & 0 & 0 \\ 0 & Y & 0 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & b & 0 \\ 0 & 0 & Z \end{vmatrix}$$

$$= bZ\hat{i} + cX\hat{j} + aY\hat{k}$$

The pitch of equivalent wrench is given by $\lambda = \dfrac{\underline{F} \bullet \underline{M}}{F^2}$ that is

$$\lambda = \frac{\underline{F} \bullet \underline{M}}{F^2} = \frac{(X\hat{i} + Y\hat{j} + Z\hat{k}) \bullet (bZ\hat{i} + cX\hat{j} + aY\hat{k})}{X^2 + Y^2 + Z^2}$$

$$= \frac{bXZ + cXY + aYZ}{X^2 + Y^2 + Z^2}$$

Besides, the equation of axis of wrench $\underline{r} = \underline{r}_1 + \mu \underline{F}$

$$\underline{r}_1 = \frac{\underline{F} \wedge \underline{M}}{F^2} = \frac{1}{(X^2 + Y^2 + Z^2)} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ X & Y & Z \\ bZ & cX & aY \end{vmatrix}$$

$$= \frac{1}{(X^2 + Y^2 + Z^2)} ((aY^2 - cXZ)\hat{i} + (bZ^2 - aXY)\hat{j} + (cX^2 - bYZ)\hat{k})$$

Then the vector form of the axis becomes

$$\underline{r} = \frac{((aY^2 - cXZ)\hat{i} + (bZ^2 - aXY)\hat{j} + (cX^2 - bYZ)\hat{k})}{(X^2 + Y^2 + Z^2)} + \mu(X\hat{i} + Y\hat{j} + Z\hat{k})$$

And Cartesian form is

$$\frac{x - \dfrac{aY^2 - cXZ}{X^2 + Y^2 + Z^2}}{X} = \frac{y - \dfrac{bZ^2 - aXY}{X^2 + Y^2 + Z^2}}{Y} = \frac{z - \dfrac{cX^2 - bYZ}{X^2 + Y^2 + Z^2}}{Z} \qquad \textbf{Or}$$

$$Y\left(x - \frac{aY^2 - cXZ}{X^2 + Y^2 + Z^2}\right) = X\left(y - \frac{bZ^2 - aXY}{X^2 + Y^2 + Z^2}\right) \qquad \textbf{and}$$

$$Y\left(z - \frac{cX^2 - bYZ}{X^2 + Y^2 + Z^2}\right) = Z\left(y - \frac{bZ^2 - aXY}{X^2 + Y^2 + Z^2}\right)$$

<u>Complete</u>

❒ EXAMPLE 10

Two forces each equal to P act along the lines $\dfrac{x \mp a\cos\theta}{a\sin\theta} = \dfrac{y - b\sin\theta}{\mp b\cos\theta} = \dfrac{z}{c}$ show that the axis of equivalent wrench lays on the surface $y\left(\dfrac{x}{z} + \dfrac{z}{x}\right) = b\left(\dfrac{a}{c} + \dfrac{c}{a}\right)$

❒ SOLUTION

First line is $\dfrac{x - a\cos\theta}{a\sin\theta} = \dfrac{y - b\sin\theta}{-b\cos\theta} = \dfrac{z}{c}$ passing through $(a\cos\theta, b\sin\theta, 0)$

the second line is $\dfrac{x + a\cos\theta}{a\sin\theta} = \dfrac{y - b\sin\theta}{b\cos\theta} = \dfrac{z}{c}$ passing $(-a\cos\theta, b\sin\theta, 0)$

The unit vector of first line is

$$\begin{aligned}\hat{n}_1 &= \frac{1}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta + c^2}}(a\sin\theta, -b\cos\theta, c) \\ &= \frac{1}{\mu}\ a\sin\theta\hat{i} - b\cos\theta\hat{j} + c\hat{k}\end{aligned}$$

The unit vector of second line is

$$\hat{n}_2 = \frac{1}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta + c^2}}(a\sin\theta, b\cos\theta, c)$$
$$= \frac{1}{\mu}\ a\sin\theta\hat{i} + b\cos\theta\hat{j} + c\hat{k} \qquad (\mu = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta + c^2})$$

Therefore,

$$\underline{F}_1 = P\hat{n}_1 = \frac{P}{\mu}(a\sin\theta\hat{i} - b\cos\theta\hat{j} + c\hat{k})$$

$$\underline{F}_2 = P\hat{n}_2 = \frac{P}{\mu}(a\sin\theta\hat{i} + b\cos\theta\hat{j} + c\hat{k})$$

The system reduces to a single force and a moment so that

$$\underline{F} = \underline{F}_1 + \underline{F}_2$$
$$= \frac{P}{\mu}(a\sin\theta\hat{i} - b\cos\theta\hat{j} + c\hat{k}) + \frac{P}{\mu}(a\sin\theta\hat{i} + b\cos\theta\hat{j} + c\hat{k})$$
$$= \frac{2P}{\mu}(a\sin\theta\hat{i} + c\hat{k}) \qquad \text{and} \quad F^2 = \frac{4P^2}{\mu^2}(a^2\sin^2\theta + c^2)$$

$$\underline{M} = \underline{r}_1 \wedge \underline{F}_1 + \underline{r}_2 \wedge \underline{F}_2$$
$$= \frac{P}{\mu}\left\{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\theta & b\sin\theta & 0 \\ a\sin\theta & -b\cos\theta & c \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\cos\theta & b\sin\theta & 0 \\ a\sin\theta & b\cos\theta & c \end{vmatrix}\right\}$$

$$\therefore \underline{M} = \frac{2P}{\mu}(cb\sin\theta\hat{i} - ab\hat{k})$$

since the equation of axis of equivalent wrench is $\underline{r} = \underline{r}_1 + \mu\underline{F}$

$$\underline{r}_1 = \frac{\underline{F} \wedge \underline{M}}{F^2} = \frac{1}{a^2\sin^2\theta + c^2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\sin\theta & 0 & c \\ cb\sin\theta & 0 & -ab \end{vmatrix} = \frac{(c^2 + a^2)b\sin\theta}{a^2\sin^2\theta + c^2}\hat{j}$$

Then the vector form of the axis becomes

$$\underline{r} = \frac{(c^2 + a^2)b\sin\theta}{a^2\sin^2\theta + c^2}\hat{j} + \mu(a\sin\theta\hat{i} + c\hat{k})$$

While the Cartesian form is

$$\frac{x - 0}{a\sin\theta} = \frac{y - \dfrac{(c^2 + a^2)b\sin\theta}{a^2\sin^2\theta + c^2}}{0} = \frac{z - 0}{c}$$

Thus we can deduce from these equation

$$y = \frac{(c^2 + a^2)b\sin\theta}{a^2\sin^2\theta + c^2} \qquad \text{and} \qquad \frac{x}{z} = \frac{a\sin\theta}{c}$$

$$y(a^2\sin^2\theta + c^2) = (c^2 + a^2)b\sin\theta$$

$$\Rightarrow y\left(a^2\sin\theta + \frac{c^2}{\sin\theta}\right) = b(c^2 + a^2)$$

Dividing by ac and substituting $\frac{x}{z} = \frac{a\sin\theta}{c}$ we get

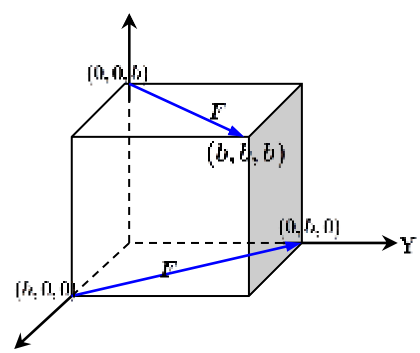
$$y\left(\frac{x}{z} + \frac{z}{x}\right) = b\left(\frac{c}{a} + \frac{a}{c}\right)$$

❒ EXAMPLE 11

Two forces each equal to F act along the sides of a cube of length b as shown, Fin the axis of equivalent wrench.

❒ SOLUTION

By calculating the unit vectors of the forces we get,



$$\hat{n}_1 = (b,b,b) - (0,0,b) = \frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$$

$$\therefore \underline{F}_1 = F\hat{n}_1 = \frac{F}{\sqrt{2}}(\hat{i} + \hat{j})$$

And for the second force

$$\hat{n}_2 = (0,b,0) - (b,0,0) = \frac{1}{\sqrt{2}}(-\hat{i} + \hat{j})$$

$$\therefore \underline{F}_2 = F\hat{n}_2 = \frac{F}{\sqrt{2}}(-\hat{i} + \hat{j})$$

The system reduces to a single force and a moment at the origin so that

$$\underline{R} = \underline{F}_1 + \underline{F}_2 = \sqrt{2}F\hat{j} \qquad \therefore R^2 = 2F^2$$

$$\underline{M} = \underline{r}_1 \wedge \underline{F}_1 + \underline{r}_2 \wedge \underline{F}_2 = \frac{Fb}{\sqrt{2}}\left\{\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 0\\ 0 & 0 & 1\end{vmatrix} + \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 1 & 0\\ 1 & 0 & 0\end{vmatrix}\right\} = \frac{Fb}{\sqrt{2}}(\hat{i} - \hat{j} - \hat{k})$$

Here we choose the point $(0,0,b)$ as an acting point of first force and the point $(b,0,0)$ of the second force. The pitch of equivalent wrench is given by $\lambda = \dfrac{\underline{F} \bullet \underline{\underline{M}}}{F^2}$ that is

$$\lambda = \frac{\underline{R} \bullet \underline{\underline{M}}}{R^2} = \frac{(\sqrt{2}F\hat{j}) \bullet \frac{Fb}{\sqrt{2}}(\hat{i} - \hat{j} - \hat{k})}{2F^2} = -\frac{b}{2}$$

since the equation of axis of equivalent wrench is $\underline{r} = \underline{r_1} + \mu \underline{F}$ so

$$\underline{r_1} = \frac{\underline{R} \wedge \underline{\underline{M}}}{R^2} = \frac{\cancel{F^2} b}{2\cancel{F^2}} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{vmatrix} = -\frac{b}{2}(\hat{i} + \hat{k})$$

Then the vector form of the axis becomes

$$\underline{r} = -\frac{b}{2}\ \hat{i} + \hat{k}\ + \mu \hat{j}$$

While the Cartesian form is given by

$$\frac{x + \frac{b}{2}}{0} = \frac{y - 0}{1} = \frac{z + \frac{b}{2}}{0} \qquad \text{Or} \qquad z = -\frac{b}{2} \text{ and } x = -\frac{b}{2}$$

PROBLEMS

❒ If the force $\underline{F} = 3\hat{i} - \hat{j} + 7\hat{k}$ acts at the origin, determine its moment about the point $(4, 4, 6)$.

❒ A force of magnitude 100 acts along the line passing through the point $(0, 1, 0)$ to $(1, 0, 0)$. Obtain its moment about the origin point and about the axes.

❒ The three forces $(2\hat{i} + 2\hat{j})$, $(\hat{j} - 2\hat{k})$, $(-\hat{i} + 2\hat{j} + \hat{k})$ act at the points $(0, 1, 0)$, $(1, 0, 0)$, $(0, 0, 1)$ respectively, Find the pitch of the equivalent wrench.

❒ Two forces each equal to $3F$ act along the lines $\frac{x-1}{2} = \frac{y+1}{2} = \frac{z-2}{1}$ and $\frac{x-2}{1} = \frac{y+1}{-2} = \frac{z-1}{2}$. Find the equivalent wrench.

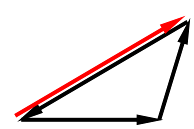
❒ The magnitude of two forces is F_2, F_1 act along the lines $(z = -c,\ y = -x\tan\alpha)$ and $(z = c,\ y = x\tan\alpha)$. Determine the central axis of equivalent wrench.

EQUILIBRIUM OF FORCES

Study of Statics and the whole study of Mechanics is actually the study about the actions of forces or force systems and the effect of these actions on bodies. So it is important to understand the action of forces, characteristics of force systems, and particular methods to analyze them. A particle is said to be in equilibrium if it remains at rest if originally at rest, or has a constant velocity if originally in motion. Most often, however, the term “equilibrium” or, more specifically, “static equilibrium” is used to describe an object at rest.

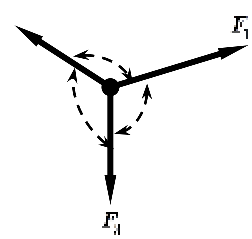
♦ Triangle of Forces

If three forces, acting at a point, be represented in magnitude and direction by the sides of a triangle, taken in order, they will be in equilibrium.



♦ Lami's Theorem

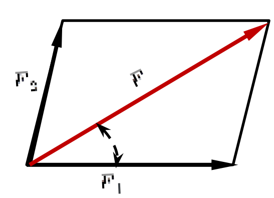
If three forces acting at a point are in equilibrium, then each force is proportional to the sine of the angle between the other two that is



$$\frac{F_1}{\sin\alpha} = \frac{F_2}{\sin\beta} = \frac{F_3}{\sin\gamma}.$$

♦ Theorem

◈ If three forces, acting in one plane upon a rigid body, keep it in equilibrium, they must either meet in a point or be parallel.



◈ If two forces acting at a point are represented in magnitude and direction by the sides of a parallelogram drawn from that point, then their resultant is represented by the diagonal of the parallelogram drawn from that point. In addition the magnitude of the resultant can be obtain by

$\because \underline{F} = \underline{F}_1 + \underline{F}_2 \qquad \Rightarrow \underline{F} \bullet \underline{F} = (\underline{F}_1 + \underline{F}_2) \bullet (\underline{F}_1 + \underline{F}_2)$

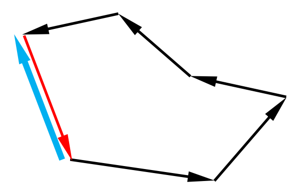
$$F^2 = F_1^2 + F_2^2 + 2F_1F_2\cos\alpha$$

where α is the angle between the two forces. The resultant $\underline{F}$ makes an angle θ to the force $\underline{F}_1$ determined by

since $\quad F\cos\theta = F_1 + F_2\cos\alpha, \quad$ **and** $\quad F\sin\theta = F_2\sin\alpha$

Therefore by dividing these two relations, $\quad \tan\theta = \dfrac{F_2\sin\alpha}{F_1 + F_2\cos\alpha}$

♦ Polygon of forces



If any number of forces, acting on a particle be represented, in magnitude and direction, by the sides of a polygon, taken in order, then the forces are in equilibrium.

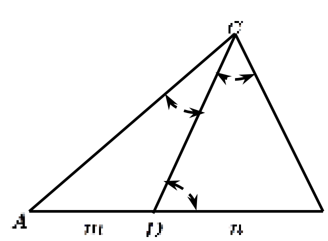
♦ Theorem

If a system of forces act in one plane upon a rigid body, and if the algebraic sum, of their moments about each of three points in the plane (not lying in the same straight line) vanish separately, the system of forces is in equilibrium.

♦ Theorem

A system of forces, acting in one plane upon a rigid body, is in equilibrium, if the sum of their components parallel to each of two lines in their plane be zero, and if the algebraic sum of their moments about any point be zero also.

♦ Two important trigonometric theorems



There are two trigonometrical theorems which are useful in There are two Statical Problems. If a line CD be drawn through the vertex C of a triangle ABC meeting the opposite side AB in point D and dividing it into two parts m and n and the angle C into two parts α and β, and if $\angle CDB = \theta$ then

(i) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

(ii) $(m+n)\cot\theta = n\cot A - m\cot B$

Proof

$$\because \frac{m}{n} = \frac{AD}{DB} = \frac{AD}{DC}\times\frac{DC}{DB} = \frac{\sin\alpha}{\sin\angle A}\times\frac{\sin\angle B}{\sin\beta}$$

$$= \frac{\sin\alpha}{\sin(\theta-\alpha)}\times\frac{\sin(\theta+\beta)}{\sin\beta}, \quad \angle DBC = 180^0 - (\beta+\theta)$$

$$= \frac{\sin\alpha(\sin\theta\cos\beta + \cos\theta\sin\beta)}{\sin\beta(\sin\theta\cos\alpha - \cos\theta\sin\alpha)} = \frac{\cot\beta + \cot\theta}{\cot\alpha - \cot\theta}$$

$$\Rightarrow m(\cot\alpha - \cot\theta) = n(\cot\beta + \cot\theta) \qquad \text{or}$$

$$(m+n)\cot\theta = m\cot\alpha - n\cot\beta$$

Again

$$\because \frac{m}{n} = \frac{\sin\angle ACD}{\sin\angle DAC}\times\frac{\sin\angle B}{\sin\beta}$$

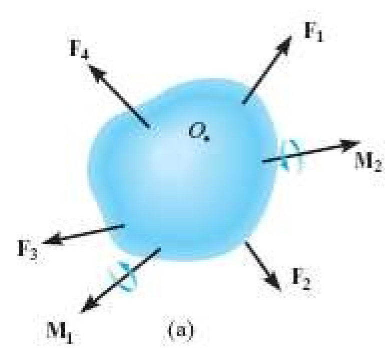
$$= \frac{\sin(\theta-A)}{\sin A}\times\frac{\sin B}{\sin(\theta+B)},$$

$$= \frac{\sin B(\sin\theta\cos A - \cos\theta\sin A)}{\sin A(\sin\theta\cos B + \cos\theta\sin B)} = \frac{\cot A - \cot\theta}{\cot B + \cot\theta}$$

$$\Rightarrow m(\cot B + \cot\theta) = n(\cot A - \cot\theta) \qquad \text{or}$$

$$(m+n)\cot\theta = n\cot A - m\cot B$$

♦ Conditions for rigid-body Equilibrium



In this section, we will develop both the necessary and sufficient conditions for the equilibrium of the rigid body. As shown, this body is subjected to an external force and couple moment system that is the result of the effects of gravitational, electrical, magnetic, or contact forces caused by adjacent bodies. The internal forces caused by interactions between particles within the body are not shown in this figure because these forces occur in equal but opposite collinear pairs and hence will cancel out, a consequence of Newton's third law.

Using the methods of the previous chapter, the force and couple moment system acting on a body can be reduced to an equivalent resultant force and resultant couple moment at any arbitrary point **O** on or off the body. If this resultant force and couple moment are both equal to zero, then the body is said to be in equilibrium. Mathematically, the equilibrium of a body is expressed as

$$\underline{\boldsymbol{R}} = \sum_{i=1}^{n} \underline{\boldsymbol{F}}_i = \underline{\boldsymbol{0}}, \qquad \underline{\boldsymbol{M}}_{\mathrm{o}} = \sum_{i=1}^{n} \underline{\boldsymbol{M}}_i = \underline{\boldsymbol{0}}$$

These relations can be rewritten in Cartesian form as

$$\sum_{i=1}^{n} \boldsymbol{F}_{ix} = \mathbf{0}, \qquad \sum_{i=1}^{n} \boldsymbol{F}_{iy} = \mathbf{0}, \qquad \sum_{i=1}^{n} \boldsymbol{F}_{iz} = \mathbf{0},$$

$$\sum_{i=1}^{n} \boldsymbol{M}_{ix} = \mathbf{0}, \qquad \sum_{i=1}^{n} \boldsymbol{M}_{iy} = \mathbf{0}, \qquad \sum_{i=1}^{n} \boldsymbol{M}_{iz} = \mathbf{0}$$

♦ Particular cases

◈ Forces act along the same line

In this case the equation of equilibrium tends to $\sum_{i=1}^{n} \boldsymbol{F}_i = \mathbf{0}$ since there is no rotation.

◈ Parallel forces system

If the acting forces are parallel then the rigid body may be in equilibrium if the resultant of acting forces is zero and the sum of moment of acting forces about a chosen point is zero too so that the two following equations are satisfying

$$\sum_{i=1}^{n} \boldsymbol{F}_i = \mathbf{0}\,, \qquad \sum_{i=1}^{n} \boldsymbol{M}_i = \mathbf{0}$$

◈ Coplanar forces system

If the acting forces are coplanar then the rigid body may be in equilibrium if the three following equations are satisfied (the forces considered to be in **XY** plane)

$$\sum_{i=1}^{n} F_{ix} = 0, \qquad \sum_{i=1}^{n} F_{iy} = 0, \qquad \sum_{i=1}^{n} M_{iz} = 0$$

Note the moment will be in a direction normal to the $\mathbf{XY}$ plane i.e. $\mathbf{Z}$-axis

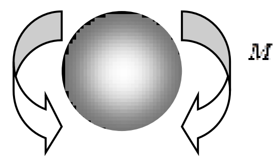
◈Spatial forces system

If the acting forces are in space then the rigid body may be in equilibrium if the following equations are satisfied

$$\sum_{i=1}^{n} F_{ix} = 0, \quad \sum_{i=1}^{n} F_{iy} = 0, \quad \sum_{i=1}^{n} F_{iz} = 0$$

$$\sum_{i=1}^{n} M_{ix} = 0 \quad \sum_{i=1}^{n} M_{iy} = 0 \quad \sum_{i=1}^{n} M_{iz} = 0$$

If two equal and inverse moments are acting on a body then the body will be in equilibrium



♦ Reactions at Joints

There are a large number of problems in which two bodies are described as smoothly hinged' at a point. In such a case the hinge may be regarded as a pin passing through cylindrical holes in the bodies, closely fitting and so smooth that each body can turn about the pin without friction. When the hinge or joint is smooth the reaction of the pin on either body reduces to a single force, because, no matter how many points of contact there may be between the pin and the cylindrical hole in the body, the reaction at each of these points acts along the common normal and therefore passes through the center of the pin (considering only forces in one plane) and all such forces can be combined into a single force through the center of the pin. When the pin connects two bodies *A* and *B* only, then the pin is subject to two forces only, namely the reactions of *A* and *B* upon it, and in equilibrium these must be equal and opposite. But the reactions of the pin on the bodies are equal and opposite to the former forces, so that the result of the smooth joint is to set up equal and opposite forces on the bodies *A* and *B* and it is unnecessary to consider the precise form of the joint, because it is sufficient to know that, as the result of the smooth joint, there is a pair of equal and opposite forces between the bodies at a certain point and that the bodies are so constrained that the only possible relative motion is one of turning about this point.

■ Illustrative Examples ■

❒ EXAMPLE 1

If the resultant of the forces $F, 2F$ perpendicular to F. Determine the angle between the two forces.

❒ SOLUTION

Let α be the angle between the two forces $F, 2F$ then from the law

$$\tan\theta = \frac{2F\sin\alpha}{F + 2F\cos\alpha} \qquad \Rightarrow \tan 90 = \frac{2\sin\alpha}{1 + 2\cos\alpha}$$

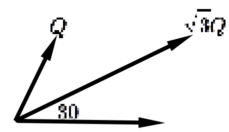
$$\Rightarrow 1 + 2\cos\alpha = 0$$

$$\Rightarrow \alpha = \cos^{-1}\left\{-\frac{1}{2}\right\} \qquad \text{Or} \qquad \alpha = 120^{\circ}$$

❒ EXAMPLE 2

The resultant of two forces P and Q is equal to $\sqrt{3}Q$ and makes an angle of 30^0 with the direction of P; show that P is either equal to, or is double of Q.

❒ SOLUTION



$$\sqrt{3}Q\cos 30 = P + Q\cos\alpha \qquad (1)$$

$$\sqrt{3}Q\sin 30 = Q\sin\alpha \qquad (2)$$

Equation (2) leads to $\alpha = 60^0$ or $\alpha = \pi - 60^0$ therefore, from equation (1) we get

$$P = Q(\sqrt{3}\cos 30 - \cos\alpha) \quad \text{when} \quad \alpha = 60^0 \quad \Rightarrow P = Q$$

$$\text{when} \quad \alpha = 120^0 \quad \Rightarrow P = 2Q$$

❒ EXAMPLE 3

The greatest resultant which two forces can have is P and the least is P'. Show that if they act an angle θ the resultant is of magnitude

$$\sqrt{P^2\cos^2\frac{1}{2}\theta + P'^2\sin^2\frac{1}{2}\theta}$$

❒ SOLUTION

Let the magnitude of the two forces be F and F' the resultant of the forces is greatest when they act in the same direction and is equal $F + F'$. Also the resultant is least when they act in opposite directions and is equal $F - F'$, consider $F > F'$ therefore,

$$P = F + F', \qquad P' = F - F'$$

Solving for F, F' we get $F = \frac{1}{2}\ P + P'\ , \qquad F' = \frac{1}{2}\ P - P'$

Then the magnitude of the resultant of the forces F and F' when they act at an angle θ is given by

$$\begin{aligned}
& R^2 = F^2 + F'^2 + 2FF'\cos\theta \\
\Rightarrow\ & R^2 = \frac{1}{4}(P + P')^2 + \frac{1}{4}(P - P')^2 + \frac{1}{2}(P + P')(P - P')\cos\theta \\
& \quad = \frac{1}{2}P^2(1 + \cos\theta) + \frac{1}{2}P'^2(1 - \cos\theta) \\
& \quad = P^2\cos^2\frac{1}{2}\theta + P'^2\sin^2\frac{1}{2}\theta \\
\Rightarrow\ & R = \sqrt{P^2\cos^2\frac{1}{2}\theta + P'^2\sin^2\frac{1}{2}\theta}
\end{aligned}$$

❒ EXAMPLE 4

Two forces P, Q act at a point along two straight lines making an angle α. with each other and R is their resultant: two other forces P', Q' acting along the same two lines have a resultant R'. Find the angle between the lines of action of the resultants.

❒ SOLUTION

Let the resultants R, R' make angles θ, θ' with the line of action of P and P'. By resolving along and perpendicular to this line, we get

$$R\cos\theta = P + Q\cos\alpha, \qquad R\sin\theta = Q\sin\alpha$$
$$R'\cos\theta' = P' + Q'\cos\alpha, \qquad R'\sin\theta' = Q'\sin\alpha$$

Multiplying two equations, we have

$$RR' \cos\theta \cos\theta' = (P + Q\cos\alpha)(P' + Q'\cos\alpha)$$

$$RR' \sin\theta \sin\theta' = QQ' \sin^2\alpha$$

By adding these two equations we get

$$RR'(\cos\theta\cos\theta' + \sin\theta\sin\theta') = (P + Q\cos\alpha)(P' + Q'\cos\alpha) + QQ'\sin^2\alpha$$

Or $$RR'\cos(\theta - \theta') = (P + Q\cos\alpha)(P' + Q'\cos\alpha) + QQ'\sin^2\alpha$$

Therefore,

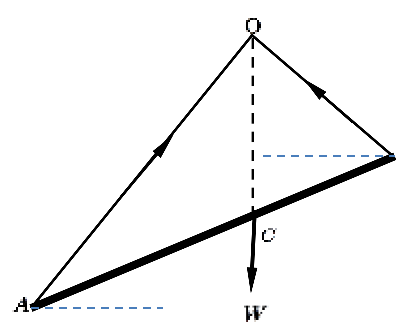
$$\begin{aligned} \cos(\theta - \theta') &= \frac{(P + Q\cos\alpha)(P' + Q'\cos\alpha) + QQ'\sin^2\alpha}{RR'} \\ \theta - \theta' &= \cos^{-1}\left(\frac{(P + Q\cos\alpha)(P' + Q'\cos\alpha) + QQ'\sin^2\alpha}{RR'}\right) \\ &= \cos^{-1}\left(\frac{PP' + QQ' + \cos\alpha(PQ' + P'Q)}{RR'}\right) \end{aligned}$$

❒ EXAMPLE 5

A rod whose center of gravity divides it into two portions, whose lengths are *a* and b, has a string, of length ℓ, tied to its two ends and the string is slung over a small smooth peg ; find the position of equilibrium of the rod, in which it is not vertical.

❒ SOLUTION

Since there are only three forces acting on the body they must meet in a point. And the two tensions pass through $\mathbf{O}$; hence the line of action of the weight W must pass through $\mathbf{O}$. The tension of the string is not altered, since the string passes round a smooth peg; that is the weight W balances the resultant of two equal forces, so it must bisected the angle between them.



$$\angle AOC = \angle BOC = \alpha \text{ (say)}$$

Hence $\frac{x}{y} = \frac{AC}{CB} = \frac{a}{b}$

Also $x + y = \ell$

Solving these equations we obtain

$$\frac{x}{a} = \frac{y}{b} = \frac{\ell}{a+b}$$

Again from the triangle AOB, we have

$$(a+b)^2 = x^2 + y^2 - 2xy\cos 2\alpha = (x+y)^2 - 2xy(1+\cos 2\alpha)$$

$$= (x+y)^2 - 4xy\cos^2\alpha = \ell^2 - \frac{4\ell^2 ab}{(a+b)^2}\cos^2\alpha$$

$$\Rightarrow \cos^2\alpha = \frac{\ell^2 - (a+b)^2}{4\ell^2}\frac{(a+b)^2}{ab}$$

Let be the inclination of the rod to the horizontal, so that

$$\angle OCA = 90^0 + \theta$$

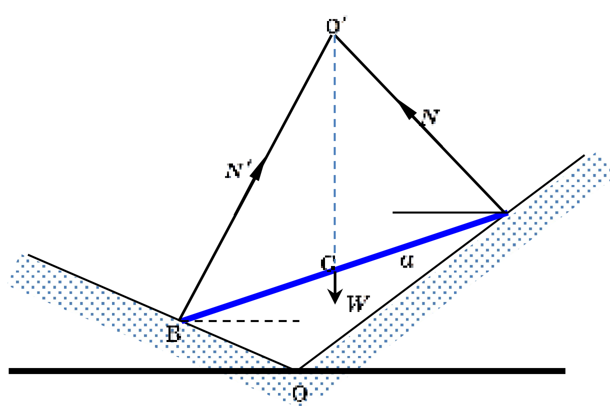
From the triangle ACO we have

$$\frac{\sin(90+\theta)}{\sin\alpha} = \frac{AO}{AC} = \frac{x}{a} = \frac{\ell}{a+b} \qquad \text{Since } \frac{x}{a} = \frac{y}{b} = \frac{\ell}{a+b}$$

$$\Rightarrow \cos\theta = \frac{\ell\sin\alpha}{a+b}$$

❐ EXAMPLE 6

A beam whose center of gravity divides it into two portions of lengths a and b respectively, rests in equilibrium with its ends resting on two smooth planes inclined at angles α, β respectively to the horizon, the planes intersecting in a horizontal line; find the inclination of the beam to the horizon and the reactions of the planes.



❐ SOLUTION

Let N and N' be the reactions at A and B perpendicular to the inclined planes, let θ be the inclination of the beam to the horizon.

Resolving vertically and horizontally, we have

$$N\cos\alpha + N'\cos\beta = W \qquad (1)$$

$$N\sin\alpha = N'\sin\beta \qquad (2)$$

Also, by taking moments about G, we get

$$N.GA\sin GAO' = N'.GB\sin GBO'$$

$$\text{Now} \quad \angle GAO' = 90^{\circ} - \angle BAO = 90^{\circ} - (\alpha - \theta)$$

$$\text{and} \quad \angle GBO' = 90^{\circ} - \angle ABO = 90^{\circ} - (\beta + \theta)$$

Hence the equation of moments becomes

$$Na\cos(\alpha - \theta) = N'b\cos(\beta + \theta) \qquad (3)$$

From equation (2) we have

$$\frac{N}{\sin\beta} = \frac{N'}{\sin\alpha} = \frac{N\cos\alpha + N'\cos\beta}{\sin\beta\cos\alpha + \sin\alpha\cos\beta} = \frac{W}{\sin(\alpha+\beta)} \qquad \text{from Eq. (1)}$$

These equations gives N and N'; also substituting for N and N' in Eq. (3) we obtain

$$a\sin\beta\cos(\alpha - \theta) = b\sin\alpha\cos(\beta + \theta);$$

$$\Rightarrow a\sin\beta \;\; \cos\alpha\cos\theta + \sin\alpha\sin\theta \;\; = b\sin\alpha \;\; \cos\beta\cos\theta - \sin\beta\sin\theta \;\; ;$$

$$\Rightarrow (a+b)\sin\alpha\sin\beta\sin\theta = \cos\theta(b\sin\alpha\cos\beta - a\cos\alpha\sin\beta);$$

$$\Rightarrow (a+b)\tan\theta = b\cot\beta - a\cot\alpha$$

❐ EXAMPLE 7

A heavy uniform rod, of length $2a$, rests partly within and partly without a fixed smooth hemispherical bowl, of radius r ; the rim of the bowl is horizontal, and one point of the rod is in contact with the rim; if θ be the inclination of the rod to the horizon, show that $2r\cos 2\theta = a\cos\theta$.

❒ SOLUTION

Since OC and AE are parallel,

$\angle OCA = \angle CAE = \theta$

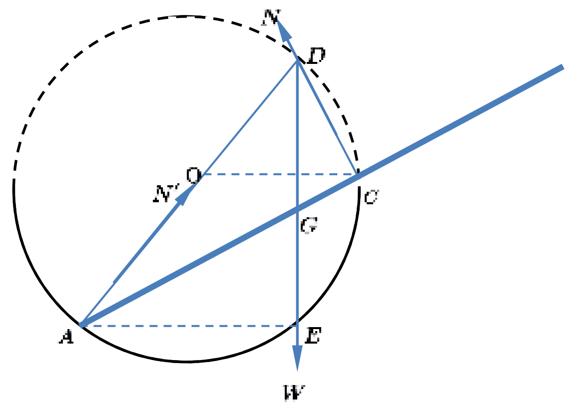
Since OC=OA, $\angle OAC = \angle OCA = \theta$

Also $\angle GDC = 90^0 - \angle DGC = \theta$

Now . $AE = AG\cos\theta = a\cos\theta,$,

and $AE = AD\cos 2\theta = 2r\cos 2\theta,$

$\Rightarrow 2r\cos 2\theta = a\cos\theta$



Hence, if N and N' are the reactions, by Lami's Theorem

$$\frac{N}{\sin 2\theta} = \frac{N'}{\sin\theta} = \frac{W}{\sin\theta}$$

$$\Rightarrow N' = W, \qquad N = 2W\cos\theta$$

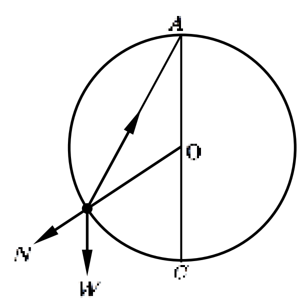
❒ EXAMPLE 8

A bead of weight W can slide on a smooth circular wire in a vertical plane. The bead is attached by a light thread to the highest point of the wire, and in equilibrium the thread is taut and makes an angle θ Find the tension of the thread and the reaction of the wire on the bead.

❒ SOLUTION

Let **B** be the bead, AB the thread, AOC the vertical diameter of the circle, and O the center. Then the angle $\angle OBA = \angle OAB = \theta$ and $\angle BOC = 2\theta$

Hence, if T denotes the tension and N the reaction, by Lami's Theorem



$$\frac{T}{\sin 2\theta} = \frac{N}{\sin\theta} = \frac{W}{\sin\theta}$$

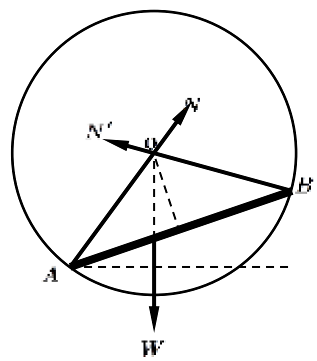
Therefore, we get $T = 2W\cos\theta, \quad N = W$

❒ EXAMPLE 9

A beam whose center of gravity divides it into two portions, is placed inside a smooth spherical bowl, show that if θ be its inclination to the horizontal in the position of equilibrium and 2α be the angle subtended by the beam at the center of the sphere, then

$$\tan\theta = \frac{b-a}{b+a}\tan\alpha$$

❒ SOLUTION



Let a beam AB of weight W be in equilibrium inside a smooth sphere of center O and radius r (say) . If G is the center of gravity of the beam then $AG = a$ and $BG = b$. As clear, the beam AB is in equilibrium under the action of the following forces

- N, the reaction at point A along the normal to the sphere at A and so passing through the center,
- W, the weight of the beam vertically downwards and
- N' the reaction at point B along the normal BO to the sphere at B

It is given that $\angle AOB = 2\alpha$ and according to the trigonometric theorem in triangle ΔOAB we get

$$(a+b)\cot(90-\theta) = b\cot(90-\alpha) - a\cot(90-\alpha)$$

$$\Rightarrow \tan\theta = \frac{b-a}{b+a}\tan\alpha$$

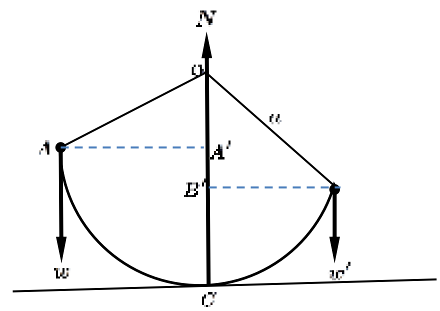
❒ EXAMPLE 10

A rigid wire, without weight, in the form of the arc of a circle subtending an angle α at its center and having two weights w and w' at its extremities, rests with its concavity downwards, upon a smooth horizontal plane. Show that, if θ be the inclination to the vertical of the radius to the end at which w is suspended, then

$$\tan\theta = \frac{w'\sin\alpha}{w + w'\cos\alpha}$$

❒ SOLUTION

The wire is in equilibrium under the action of following three forces



w, the weight at A and weight w' at B vertically downwards and the reaction N at the point of contact C, acting at right angle to the horizontal plane where the line of action of the reaction will pass through the center of the circle,

Given that $\angle AOC = \theta$ then $\angle BOC = \alpha - \theta$. To avoid the reaction , taking moments of all the forces about , we have

$$\sum M_o = 0 \qquad \Rightarrow w(AA') - w'(BB') = 0$$
$$\Rightarrow wa\sin\theta = w'a\sin(\alpha - \theta)$$
$$\Rightarrow w\sin\theta = w'(\sin\alpha\cos\theta - \sin\theta\cos\alpha)$$

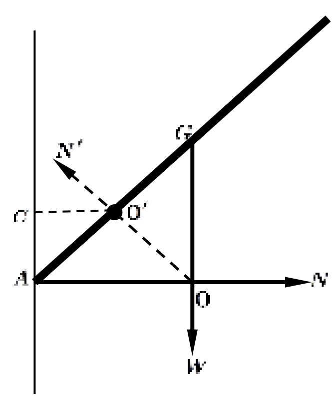
Dividing by $\cos\theta$ then we obtain

$$w\tan\theta = w'\sin\alpha - w'\tan\theta\cos\alpha$$
$$\Rightarrow \tan\theta = \frac{w'\sin\alpha}{w + w'\cos\alpha}$$

❒ EXAMPLE 11

A uniform beam, of length $2a$, rests in equilibrium against a smooth vertical wall and upon a peg as a distance b from the wall, prove that the inclination of the beam to the vertical is

$$\sin^{-1}\left(\frac{b}{a}\right)^{1/3}$$



❒ SOLUTION

The beam is in equilibrium under the action of the forces namely, N, the reaction at point A along the normal to the vertical, W, the weight of the beam vertically downwards and N' the reaction at peg along the normal AB. Let θ be the inclination of the beam to the vertical and since in

$$\Delta ACO' \quad \sin\theta = \frac{CO'}{AO'},$$
$$\Delta AOO' \quad \sin\theta = \frac{AO'}{AO},$$
$$\Delta AGO \quad \sin\theta = \frac{AO}{AG}$$

Multiplying these three formulas we get

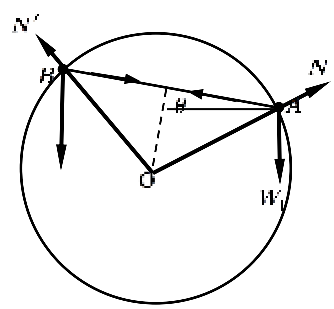
$$\sin^3\theta = \frac{CO'}{AO'} \times \frac{AO'}{AO} \times \frac{AO}{AG} = \frac{CO'}{AG} = \frac{b}{a}$$

$$\Rightarrow \sin\theta = \left(\frac{b}{a}\right)^{1/3} \qquad \text{or} \qquad \theta = \sin^{-1}\left(\frac{b}{a}\right)^{1/3}$$

❒ EXAMPLE 12

Two small rings of weights W_1 and W_2 each capable of sliding freely on a smooth circular hoop fixed in the vertical plane are connected by a light string, show that in the position of equilibrium in which the string be straight and inclined at angle θ to the horizontal $(W_1 + W_2)\tan\theta = (W_1 - W_2)\tan\frac{1}{2}\alpha$ where α is the angle subtended by the string at the center.

❒ SOLUTION



The ring at A is in equilibrium under the following forces, weight of the ring W_1 acting vertically downwards, Tension T in the string along AB and the reaction N along the normal OA passing through the center of the circle. The string is inclined at an angle θ to the horizontal and $\angle AOB = \alpha$, therefore by Lami's theorem at point A, we have

$$\frac{T}{\sin\angle\mathrm{NAW_1}} = \frac{W_1}{\sin\angle\mathrm{BAN}}$$
$$\Rightarrow \frac{T}{\sin(\pi - (\theta + \alpha / 2))} = \frac{W_1}{\sin(\pi / 2 + \alpha / 2)}$$
$$\Rightarrow T = \frac{\sin(\theta + \alpha / 2)}{\cos(\alpha / 2)} W_1 \quad (1)$$

in the same manner for ring at B we have

$$\frac{T}{\sin\angle\mathrm{N'BW_2}} = \frac{W_2}{\sin\angle\mathrm{ABN'}}$$
$$\Rightarrow \frac{T}{\sin(\pi - (\alpha / 2 - \theta))} = \frac{W_2}{\sin(\pi / 2 + \alpha / 2)}$$
$$\Rightarrow T = \frac{\sin(\alpha / 2 - \theta)}{\cos(\alpha / 2)} W_2 \quad (2)$$

From Eqs. (1) and (2)

$$\frac{\sin(\alpha / 2 - \theta)}{\cos(\alpha / 2)} W_2 = \frac{\sin(\theta + \alpha / 2)}{\cos(\alpha / 2)} W_1$$

$$W_2 \left(\sin\theta \cos\frac{1}{2}\alpha + \cos\theta \sin\frac{1}{2}\alpha \right) = W_1 \left(\sin\frac{1}{2}\alpha \cos\theta - \cos\frac{1}{2}\alpha \sin\theta \right)$$

Dividing by $\cos\theta \cos(\alpha / 2)$, we get

$$W_2 \left(\tan\theta + \tan\frac{1}{2}\alpha \right) = W_1 \left(\tan\frac{1}{2}\alpha - \tan\theta \right)$$

$$\text{Or } (W_1 + W_2)\tan\theta = (W_1 - W_2)\tan\frac{1}{2}\alpha$$

❒ EXAMPLE 13

Two equal rods, each of length 2ℓ and weight w, are freely jointed at and the others ends of the rods are suspended from a fixed point, If the lengths of each string is 2ℓ and the angle between the rods is 2θ, a disk of weight $3w$ and radius a is putted between the rods in equilibrium in a vertical plane, show that $a = 6\ell \sin^2\theta \tan\theta$

❒ SOLUTION

With respect to the disk, $3w = 2N\sin\theta \qquad \Rightarrow N = \dfrac{3w}{2\sin\theta}$

The equations of equilibrium for the whole figure (i) in vertical direction

$$5w = 2T\cos\theta \qquad \Rightarrow T = \frac{5w}{2\cos\theta}$$

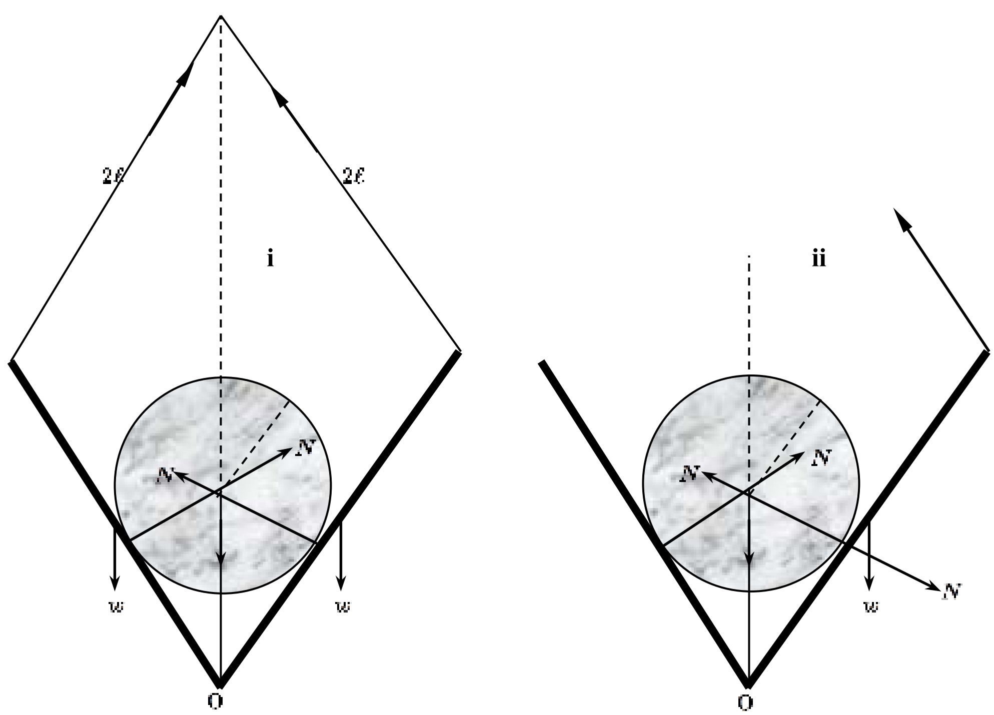
Considering one of the rods (right one say) and taking moment about the point **O**, reaction is reaction of the disk on the rod which equals the reaction of the rod on the disk but in opposite direction, we get

$$Na\cot\theta + w\ell\sin\theta = T\cos\theta(2\ell\sin\theta) + T\sin\theta(2\ell\cos\theta)$$

$$\left(\frac{3w}{2\sin\theta}\right)a\cot\theta + w\ell\sin\theta = \left\{\frac{5w}{\cos\theta}\right\}\ 2\ell\cos\theta\sin\theta$$

$$\therefore \frac{3a\cos\theta}{2\sin^2\theta} + \ell\sin\theta = 10\ell\sin\theta$$

$$\therefore 3a = 18\ell\tan\theta\sin^2\theta \qquad \therefore a = 6\ell\tan\theta\sin^2\theta$$

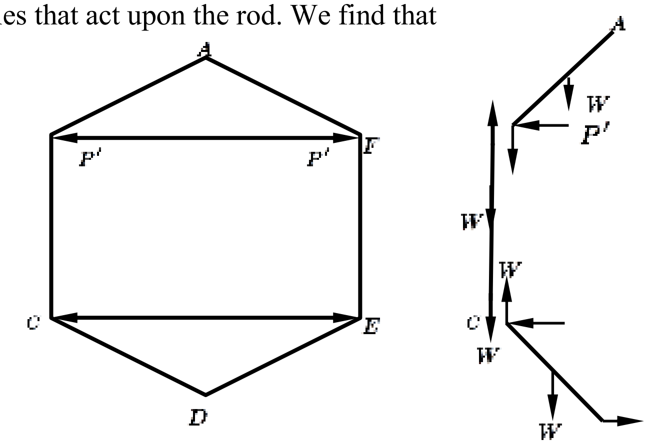


❒ EXAMPLE 14

A hexagon **ABCDEF** is formed of six equal rods of the same weight $\boldsymbol{W}$ smoothly jointed at their extremities. It is suspended from the point **A** and the regular form is maintained by light rods **bf** and **ce**. Prove that the thrust in the former **BF** is five times that in the latter **CE**.

❒ SOLUTION

Suppose that the rod BF is attached to the two upper rods and the rod GE to the two lower rods. Let $\boldsymbol{P'}$ and $\boldsymbol{P}$ denote the thrusts in BF and CE. Then since the only effect of these rods is to produce thrusts at their ends, we may ignore these rods if instead of them we suppose horizontal forces $\boldsymbol{P'}$ to act outwards on AB at B and on AF at F, and horizontal forces $\boldsymbol{P}$ to act outwards on CD at C and on DE at E. Begin by inserting these forces in the figure. Then consider the equilibrium of the rod CD. The reaction at D is horizontal because there is symmetry about the vertical through D. But the only horizontal forces on CD are the force $\boldsymbol{P}$ at C and the reaction at D, so that this reaction at D must be equal and opposite to $\boldsymbol{P}$. Then as regards vertical forces: the weight W acts vertically downwards through the middle point of CD and the only other vertical force can be at C, therefore there is a reaction at C which acts vertically upwards and is equal to W. Insert this in the figure; and, since it is produced by the rod CB, also insert an equal and opposite force W downwards acting at C on CB We can now express $\boldsymbol{P}$ in terms of W by taking moments for the rod CD about C or about D, or, what is the same thing, equating the moments of the two couples that act upon the rod. We find that



$$P\,CD\sin 30^{o} = W\left(\frac{1}{2}CD\cos 30^{o}\right) \qquad \Rightarrow P = \frac{\sqrt{3}}{2}W$$

Then, returning to the figure, consider the rod BC. It is in equilibrium under the action of its weight W, a downward force W at C and the reaction at B. This latter force must therefore act vertically upwards and be equal to 2W. Insert this force in the figure and also the equal and opposite reaction 2W at B on AB. Then consider the rod AB. It is in equilibrium under the action of its weight W, the horizontal and vertical forces P' and 2W at B and the reaction at A. It is not necessary to specify the latter because we can take moments about A; by so doing we find that

$$P'\,AB\sin 30^{o} = \frac{5}{2}W\ \ AB\cos 30^{o} \qquad \Rightarrow P' = 5\left(W\frac{\sqrt{3}}{2}\right) = 5P$$

❐ EXAMPLE 15

A regular pentagon ABODE formed of five uniform rods, each of weight W, freely hinged to each other at their ends is placed in a vertical plane with CD resting on a horizontal plane and the regular pentagonal form is maintained by means of a string joining the middle points of the rods BC and DE. Prove that the tension in the string is

$$\left(\cot\frac{\pi}{5} + 3\cot\frac{2\pi}{5}\right)W$$

❐ SOLUTION

It is only necessary to consider the reactions at the corners A and B. By symmetry that at A is horizontal and equal say to X. The rod AB is also acted on by its weight W and the reaction at B. The latter must therefore have a horizontal component X and a vertical component W upwards. Insert in the diagram forces at B acting upon BC in the opposite senses. Then by taking moments about B for the rod AB, since the rod AB makes an angle \n with the horizontal, we get

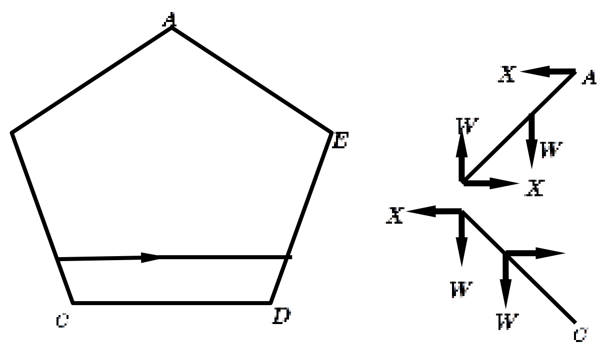
$$X \sin\frac{\pi}{5} = \frac{1}{2} W \cos\frac{\pi}{5} \qquad (1)$$

after dividing by the length of the rod. Again if T denotes the tension in the string which joins the middle points of BC and DE, by taking moments about C for the rod BC, which makes an angle £n with the horizontal, we get

$$\frac{1}{2} T \sin\frac{2\pi}{5} = \frac{1}{2} W \cos\frac{2\pi}{5} + W \cos\frac{2\pi}{5} + X \sin\frac{2\pi}{5} \qquad (2)$$

after dividing by the length of the rod. On substituting for X in terms of W from (1), we find that

$$T = W\left(\cot\frac{\pi}{5} + 3\cot\frac{2\pi}{5}\right)$$



PROBLEMS

❐ If two forces P and Q act at such an angle that R=P, show that, if P be doubled, the new resultant is at right angles to Q.

❐ The resultant of two forces P and Q acting at an angle θ is equal to $(2m+1)\sqrt{P^2+Q^2}$; when they act at an angle $90-\theta$, the resultant is $(2m-1)\sqrt{P^2+Q^2}$; prove that $\tan\theta = \frac{m-1}{m+1}$

❐ The resultant of forces P and Q is R; if Q be doubled R is doubled, whilst, if Q be reversed, R is again doubled; show that $P:Q:R=\sqrt{2}:\sqrt{3}:\sqrt{2}$

❐ The sides BC and DA of a quadrilateral ABCD are bisected in F and H respectively; show that if two forces parallel and equal to AB and DC act on a particle, then the resultant is parallel to HF and equal to 2HF.

❐ A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface of the hemisphere is in contact. If θ, φ are the inclinations of the string and the plane base of the hemisphere to the vertical, prove that

$$\tan\varphi = \frac{3}{8} + \tan\theta$$

❐ The sides AB, BC, CD, and DA of a quadrilateral ABCD are bisected at E, F, G, and H respectively. Show that the resultant of the forces acting at a point which are represented in magnitude and direction by EG ana HF is represented in magnitude and direction by AC.

❒ Two uniform rods AB, BC, rigidly jointed at B so that ABC is a right angle, hang freely in equilibrium from a fixed point at A. The lengths of the rods are a, b and their weights are wa and wb. Prove that, if AB makes an angle θ with the vertical, then $a^2 + 2ab\ \tan\theta = b^2$

❒ Two equal rods, AB and AC, each of length $2b$, are freely jointed at A and rest on a smooth vertical circle of radius a, show that if 2θ be the angle between the rods then $b\sin^3\theta = a\cos\theta$

❒ Weights W_1, W_2 are fastened to alight inextensible string ABC at the points B, C the end A being fixed. Prove that, if a horizontal force P is applied at C and in equilibrium AB, BC are inclined at angles θ, α to the vertical then

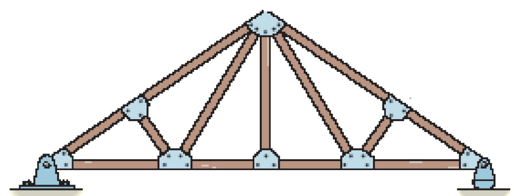
$P = (W_1 + W_2)\tan\theta = W_2 \tan\alpha$

❒ A sphere, of given weight W, rests between two smooth planes, one vertical and the other inclined at a given angle α to the vertical; find the reactions of the planes.

❒ A picture frame, rectangular in shape, rests against a smooth vertical wall, from two points in which it is suspended by parallel strings attached to two points in the upper edge of the back of the frame, the length of each string being equal to the height of the frame. Show that, if the center of gravity of the frame coincide with its center of figure, the picture will hang against the wall at an angle $\theta = \tan^{-1}\dfrac{b}{3a}$ to the vertical, where a is the height and b the thickness of the picture.

FRAMEWORK

A framework is an assembly of bars connected by hinged or pinned joints and intended to carry loads at the joints only. Each hinge joint is assumed to rotate freely without friction; hence all the bars in the frame exert direct forces only and are therefore in tension or compression. A tensile load is taken as positive and a member carrying tension is called a tie. A compressive load is negative and a member in compression is called a strut. The bars are usually assumed to be light compared with the applied loads. In practice the joints of a framework may be riveted or welded but the direct forces are often calculated assuming pin-joints. This assumption gives values of tension or compression which are on the safe side. In order that the framework shall be stiff and capable of carrying a load, each portion forms a triangle, the whole frame being built up of triangles. Note that the wall ad forms the third side of the triangle. The forces in the members of a pin-jointed stiff frame can be obtained by the methods of statics, i.e. using triangle and polygon of forces, resolution of forces and principle of moments. The system of forces in such a frame is said to be statically determinate.



♦ Free-Body Diagrams

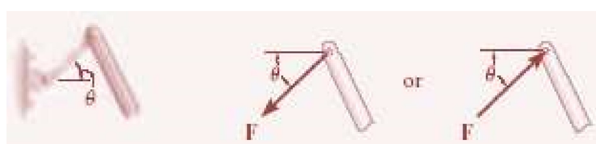
Successful application of the equations of equilibrium requires a complete specification of all the known and unknown external forces that act on the body. The best way to account for these forces is to draw a free-body diagram. This diagram is a sketch of the outlined shape of the body, which represents it as being isolated or "free" from its surroundings, i.e., a "free body." On this sketch it is necessary to show all the forces and couple moments that the surroundings exert on the body so that these effects can be accounted for when the equations of equilibrium are applied. A thorough understanding of how to draw a free-body diagram is of primary importance for solving problems in mechanics.

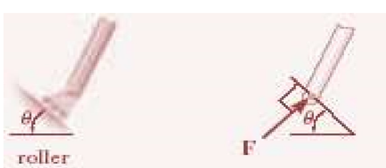
◈ Support Reactions

Before presenting a formal procedure as to how to draw a free-body diagram, we will first consider the various types of reactions that occur at supports and points of contact between bodies subjected to coplanar force systems. As a general rule. A support prevents the translation of a body in a given direction by exerting a force on the body in the opposite direction. A support prevents the rotation of a body in a given direction by exerting a couple moment on the body in the opposite direction. Here, other common types of supports for bodies subjected to coplanar force systems. (In all cases the angle θ is assumed to be known.)

♦ One unknown. The reaction is a tension force which acts away from the member in the direction of the cable.

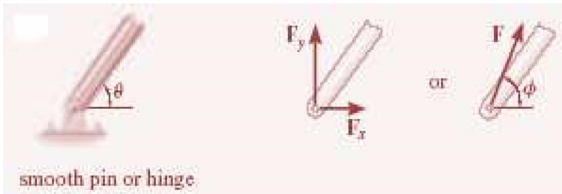
♦ One unknown. The reaction is a force which acts along the axis of the link.





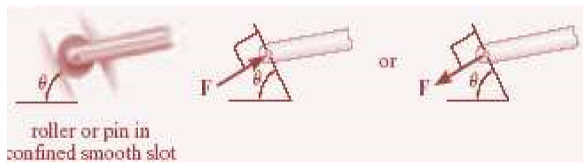
♦ One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.

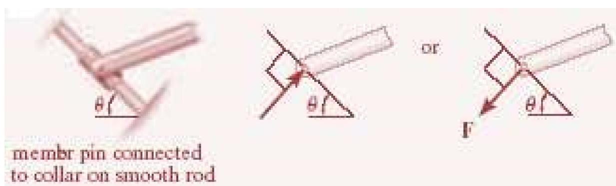
♦ One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.



♦ Two unknowns. The reactions are two components of force, or the magnitude and direction of the resultant force. Note that and are not necessarily equal [usually not, unless the rod shown is a link as in (2)].

♦ One unknown. The reaction is a force which acts perpendicular to the slot.





♦ One unknown. The reaction is a force which acts perpendicular to the rod.

♦ One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.

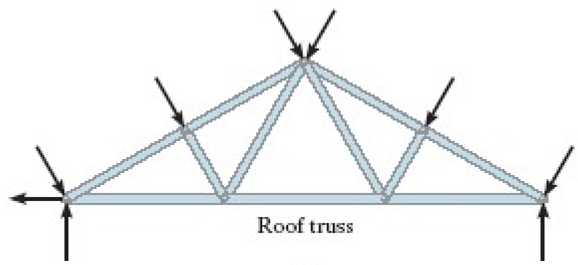
♦ Three unknowns. The reactions are the couple moment and the two force components, or the couple moment and the magnitude and direction of the resultant force.

♦ Two unknowns. The reactions are the couple moment and the force which acts perpendicular to the rod.

■ A TRUSS

A truss is a structure composed of slender members joined together at their end points. The members commonly used in construction consist of wooden struts or metal bars. In particular, planar trusses lie in a single plane and are often used to support roofs and bridges. The truss shown in the figure is an example of a typical roof-supporting truss. In this figure, the roof load is transmitted to the truss at the joints by means of a series of purlins. Since this loading acts in the same plane as the truss, the analysis of the forces developed in the truss members will be two-dimensional

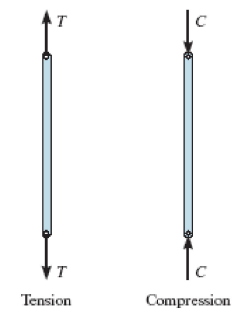
To design both the members and the connections of a truss, it is necessary first to determine the force developed in each member when the truss is subjected to a given loading. To do this we will make two important assumptions:



◈ **All loadings are applied at the joints.** In most situations, such as for bridge and roof trusses, this assumption is true. Frequently the weight of the members is neglected because the force supported by each member is usually much larger than its weight. However, if the weight is to be included in the analysis, it is generally satisfactory to apply it as a vertical force, with half of its magnitude applied at each end of the member.

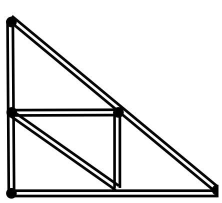
◈**The members are joined together by smooth pins.** The joint connections are usually formed by bolting or welding the ends of the members to a common plate, called a gusset plate, as shown in the figure, or by simply passing a large bolt or pin through each of the members, as shown. We can assume these connections act as pins provided the center lines of the joining members are concurrent, as shown.

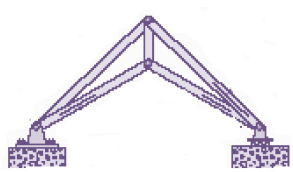
Because of these two assumptions, each truss member will act as a two force member, and therefore the force acting at each end of the member will be directed along the axis of the member. If the force tends to elongate the member, it is a tensile force (***T***), as in the figure; whereas if it tends to shorten the member, it is a compressive force (***C***), Fig. 6–4b. In the actual design of a truss it is important to state whether the nature of the force is tensile or compressive. Often, compression members must be made thicker than tension members because of the buckling or column effect that occurs when a member is in compression.

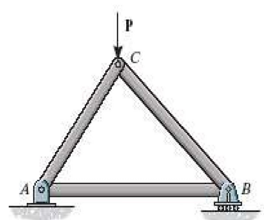


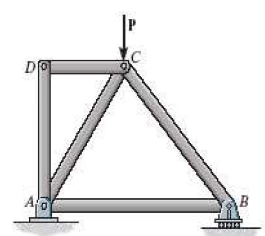
♦ Simple Truss

If three members are pin connected at their ends, they form a triangular truss that will be rigid, as shown. Attaching two more members and connecting these members to a new joint D forms a larger truss, as shown. This procedure can be repeated as many times as desired to form an even larger truss. If a truss can be constructed by expanding the basic triangular truss in this way, it is called a simple truss. The basic equation between numbers of members of a truss m and numbers of joints n so that $\boldsymbol{m = 2n - 3}$









♦ The Method of Joints

In order to analyze or design a truss, it is necessary to determine the force in each of its members. One way to do this is to use the method of joints. This method is based on the fact that if the entire truss is in equilibrium, then each of its joints is also in equilibrium. Therefore, if the free-body diagram of each joint is drawn, the force equilibrium equations can then be used to obtain the member forces acting on each joint. Since the members of a plane truss are straight two-force members lying in a single plane, each joint is subjected to a force system that is coplanar and concurrent. As a result, only $\sum \boldsymbol{F_x} = \mathbf{0}$ and $\sum \boldsymbol{F_y} = \mathbf{0}$ are need to be satisfied for equilibrium.

When using the method of joints, always start at a joint having at least one known force and at most two unknown forces. In this way, application of $\sum \boldsymbol{F_x} = \mathbf{0}$ and $\sum \boldsymbol{F_y} = \mathbf{0}$ yields two algebraic equations which can be solved for the two unknowns. When applying these equations, the correct sense of an unknown member force can be determined using one of two possible methods.

◈ The correct sense of direction of an unknown member force can, in many cases, be determined "by inspection." In more complicated cases, the sense of an unknown member force can be assumed; then, after applying the equilibrium equations, the assumed sense can be verified from the numerical results. A positive answer indicates that the sense is correct, whereas a negative answer indicates that the sense shown on the free-body diagram must be reversed.

◈ Always assume the unknown member forces acting on the joint's free-body diagram to be in tension; i.e., the forces "pull" on the pin. If this is done, then numerical solution of the equilibrium equations will yield positive scalars for members in tension and negative scalars for members in compression. Once an unknown member force is found, use its correct magnitude and sense (T or C) on subsequent joint free-body diagrams.

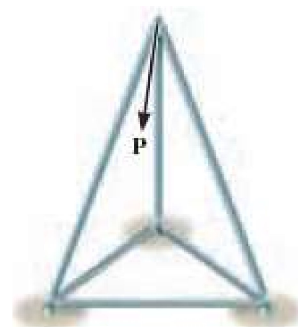
♦ Zero-Force Members

Truss analysis using the method of joints is greatly simplified if we can first identify those members which support no loading. These zero-force members are used to increase the stability of the truss during construction and to provide added support if the loading is changed. The zero-force members of a truss can generally be found by inspection of each of the joints.

♦ The Method of Sections

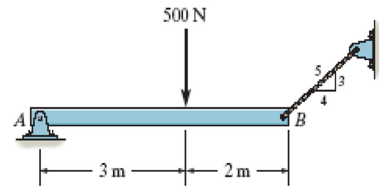
When we need to find the force in only a few members of a truss, we can analyze the truss using the method of sections. It is based on the principle that if the truss is in equilibrium then any segment of the truss is also in equilibrium. When applying the equilibrium equations, we should carefully consider ways of writing the equations so as to yield a direct solution for each of the unknowns, rather than having to solve simultaneous equations. This ability to determine directly the force in a particular truss member is one of the main advantages of using the method of sections

♦ Space Trusses



A space truss consists of members joined together at their ends to form a stable three-dimensional structure. The simplest form of a space truss is a tetrahedron, formed by connecting six members together, as shown. Any additional members added to this basic element would be redundant in supporting the force P. A simple space truss can be built from this basic tetrahedral element by adding three additional members and a joint, and continuing in this manner to form a system of multi connected tetrahedrons. Assumptions for Design. The members of a space truss may be treated as two-force members provided the external loading is applied at the joints and the joints consist of ball-and-socket connections. These assumptions are justified if the welded or bolted connections of the joined members intersect at a common point and the weight of the members can be neglected. In cases where the weight of a member is to be included in the analysis, it is generally satisfactory to apply it as a vertical force, half of its magnitude applied at each end of the member.

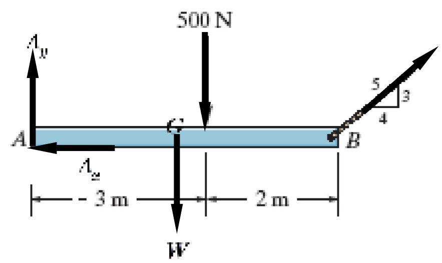
■ Illustrative Examples ■



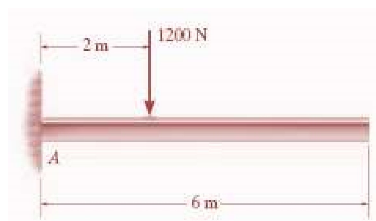
❒ EXAMPLE 1

Draw the free-body diagram of the object as shown.

❒ SOLUTION



Free-Body Diagram. The supports are removed, and the free-body diagram of the beam is shown in the figure below. Since the support at A is pin, the pin exerts two reactions on the beam, denoted as A_x and A_y. The magnitudes of these reactions are unknown, and their sense has been assumed. The weight of the beam, W N, acts through the beam's center of gravity G, which is 2.5 m from A since the beam is uniform. The tension in the string as illustrated.

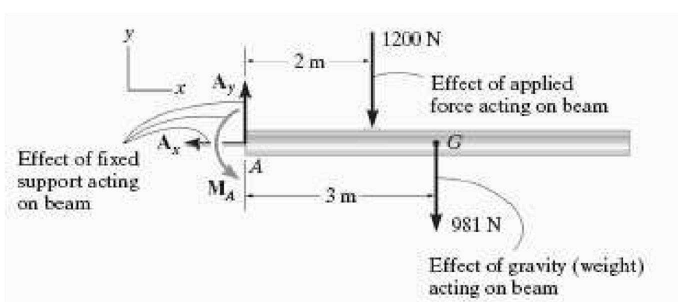


❒ EXAMPLE 2

Draw the free-body diagram of the uniform beam shown in the figure. The beam has a mass of 100 kg.

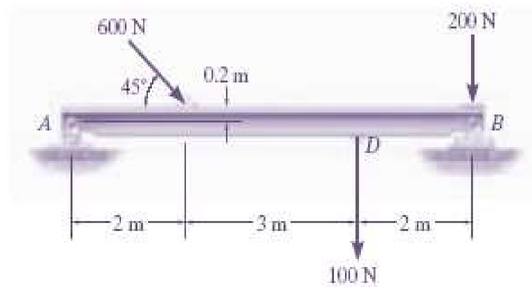
❒ SOLUTION

The free-body diagram of the beam is shown in figure behind. Since the support at A is fixed, the wall exerts three reactions on the beam, denoted as A_x, A_y and M_A. The magnitudes of these reactions are unknown, and their sense has been assumed. The weight of the beam, W = 100(9.81) N = 981 N, acts through the beam's center of gravity G, which is 3 m from A since the beam is uniform.



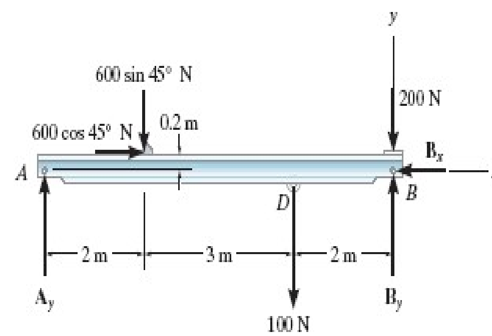
❒ EXAMPLE 3

Determine the horizontal and vertical components of reaction on the beam caused by the pin at *B* and the rocker at *A* as shown. Neglect the weight of the beam.



❒ SOLUTION

The supports are removed, and the free-body diagram of the beam is shown in figure besides. For simplicity, the 600-N force is represented by its x and y components as shown.



Equations of Equilibrium.

Summing forces in the x direction yields

$$\overset{+}{\rightarrow} \quad \sum F_x = 0, \qquad 600\cos 45 - B_x = 0, \qquad \Rightarrow B_x = 424 \text{ N}$$

A direct solution for A_y can be obtained by applying the moment equation $\sum M_B = 0$ about point B.

$$\sum M_B = 0, \qquad 100(2) + 600\sin 45(5) - 600\cos 45(0.2) - A_y(7) = 0$$

$$\Rightarrow A_y = 319 \text{ N}$$

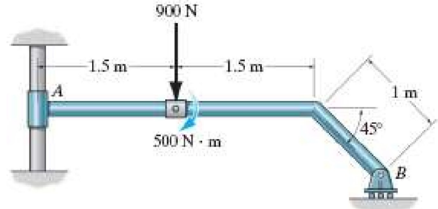
Summing forces in the y direction, using this result, gives

$$+\uparrow \sum F_y = 0, \qquad 319 - 600\sin 45 - 100 - 200 + B_y = 0, \qquad \Rightarrow B_y = 405$$

NOTE: Remember, the support forces in the figure are the result of pins that act on the beam. The opposite forces act on the pins

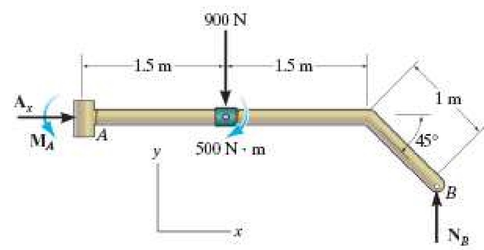
❒ EXAMPLE 3

Determine the support reactions on the member in the figure. The collar at *A* is fixed to the member and can slide vertically along the vertical shaft.



❒ SOLUTION

Free-Body Diagram. Removing the supports, the free-body diagram of the member is shown. The collar exerts a horizontal force A_x and moment M_A on the member. The reaction N_B of the roller on the member is vertical.



Equations of Equilibrium. The forces A_x and N_B can be determined directly from the force equations of equilibrium.

$$\overset{+}{\rightarrow} \sum F_x = 0, \qquad A_x = 0$$
$$+\uparrow \sum F_y = 0, \qquad N_B - 900 = 0 \qquad \Rightarrow N_B = 900 \text{ N}$$

The moment M_A can be determined by summing moments either about point A or point B.

$$\sum M_A = 0, \qquad M_A - 500 + 900((1.5) + (1)\cos 45) = 0$$
$$\Rightarrow M_A = -1486 \text{ N}$$

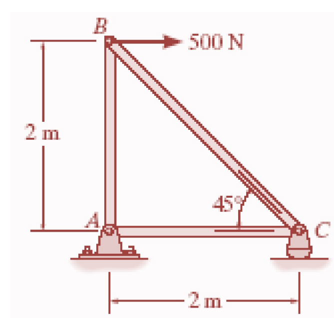
or B

$$\sum M_B = 0, \qquad M_A + 900((1.5) + (1)\cos 45) - 500 = 0$$
$$\Rightarrow M_A = -1486 \text{ N}$$

The negative sign indicates that M_A has the opposite sense of rotation to that shown on the free-body diagram.

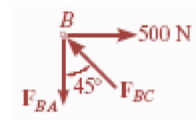
❒ EXAMPLE 5

Determine the force in each member of the truss as shown and indicate whether the members are in tension or compression.



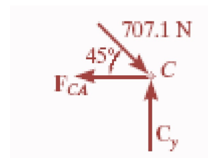
❒ SOLUTION

Since we should have no more than two unknown forces at the joint and at least one known force acting there, we will begin our analysis at joint *B*.



◈Joint *B*. The free-body diagram of the joint at *B* is shown. Applying the equations of equilibrium, we have

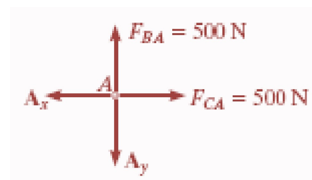
$$\overset{+}{\rightarrow} \sum F_x = 0, \qquad 500 - F_{BC}\sin 45 = 0 \qquad \Rightarrow F_{BC} = 707.1 \text{ N}$$
$$+\uparrow \sum F_y = 0, \qquad F_{BC}\cos 45 - F_{BA} = 0 \qquad \Rightarrow F_{BA} = 500 \text{ N}$$



Since the force in member *BC* has been calculated, we can proceed to analyze joint *C* to determine the force in member *CA* and the support reaction at the rocker.

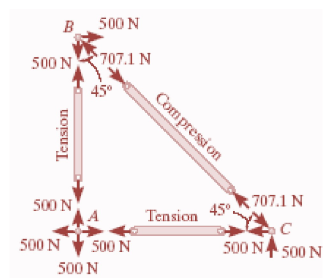
◈Joint *C*. From the free-body diagram of joint *C*, as shown, we have

$$\overset{+}{\rightarrow} \sum F_x = 0, \qquad -F_{CA} + 707.1\cos 45 = 0 \qquad \Rightarrow F_{CA} = 500 \text{ N}$$
$$+\uparrow \sum F_y = 0, \qquad C_y - 707.1\sin 45 = 0 \qquad \Rightarrow C_y = 500 \text{ N}$$



◈Joint *A*. Although it is not necessary, we can determine the components of the support reactions at joint *A* using the results of F_{CA} and F_{BA}. From the free-body diagram, we have

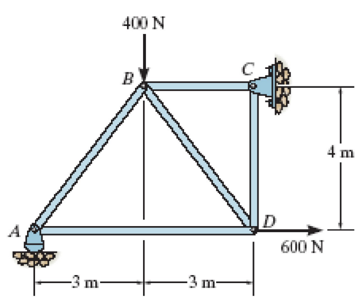
$$\overset{+}{\rightarrow} \sum F_x = 0, \qquad 500 - A_x = 0 \qquad \Rightarrow A_x = 500 \text{ N}$$
$$+\uparrow \sum F_y = 0, \qquad 500 - A_y = 0 \qquad \Rightarrow A_y = 500 \text{ N}$$



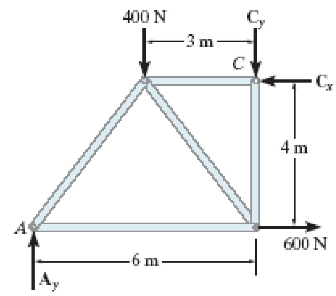
NOTE: The results of the analysis are summarized in last figure. Note that the free-body diagram of each joint (or pin) shows the effects of all the connected members and external forces applied to the joint, whereas the free-body diagram of each member shows only the effects of the end joints on the member.

❒ EXAMPLE 5

Determine the force in each member of the truss shown in the figure. Indicate whether the members are in tension or compression



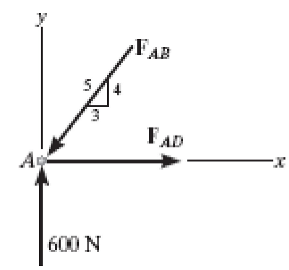
❒ SOLUTION



Support Reactions. No joint can be analyzed until the support reactions are determined, because each joint has at least three unknown forces acting on it. A free-body diagram of the entire truss is given in the figure. Applying the equations of equilibrium, we have

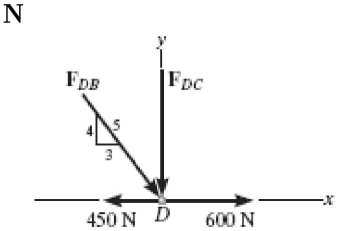
$$\overset{+}{\rightarrow} \sum F_x = 0, \quad 600 - C_x = 0 \quad \Rightarrow C_x = 600 \text{ N}$$
$$+\uparrow \sum F_y = 0, \quad 600 - 400 - C_y = 0 \quad \Rightarrow C_y = 200 \text{ N}$$
$$\sum M_C = 0, \quad -A_y(6) + 400(3) + 600(4) = 0 \quad \Rightarrow A_y = 600 \text{ N}$$

The analysis can now start at either joint *A* or *C*. The choice is arbitrary since there are one known and two unknown member forces acting on the pin at each of these joints.



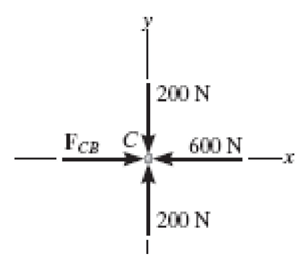
◈ Joint *A*. As shown on the free-body diagram, F_{AB} is assumed to be compressive and F_{AD} is tensile. Applying the equations of equilibrium, we have

$$+\uparrow \sum F_y = 0, \quad 600 - \frac{4}{5} F_{AB} = 0 \quad \Rightarrow F_{AB} = 750 \text{ N}$$
$$\overset{+}{\rightarrow} \sum F_x = 0, \quad F_{AD} - \frac{3}{5}(750) = 0 \quad \Rightarrow F_{AD} = 730 \text{ N}$$



◈ Joint *D*. Using the result for F_{AD} and summing forces in the horizontal direction, we have

$$\overset{+}{\rightarrow} \sum F_x = 0, \quad -450 + \frac{3}{5} F_{DB} + 600 = 0 \quad \Rightarrow F_{DB} = -250 \text{ N}$$

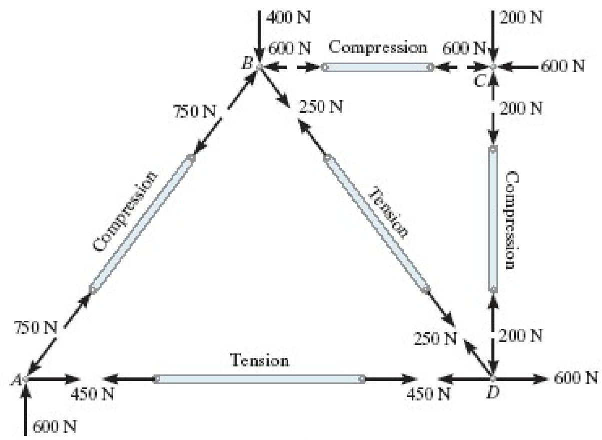


The negative sign indicates that F_{DB} acts in the opposite sense to that supposed. To determine F_{DC}, we can either correct the sense of F_{DB} on the free body diagram, and then apply $F_y = 0$, or apply this equation and retain the negative sign for F_{DB}, i.e.,

$$+\uparrow \sum F_y = 0, \qquad -F_{DC} - \frac{4}{5}(-250) = 0 \qquad \Rightarrow F_{DC} = 200 \text{ N}$$

◈ **Joint *C*.**

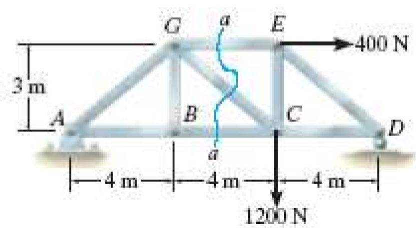
$$\underset{\rightarrow}{+} \quad \sum F_x = 0, \qquad F_{CB} - 600 = 0$$
$$\Rightarrow F_{CB} = 600 \text{ N}$$
$$+\uparrow \sum F_y = 0, \qquad 200 - 200 = 0$$



NOTE: The analysis is summarized in last figure, which shows the free body diagram for each joint and member.

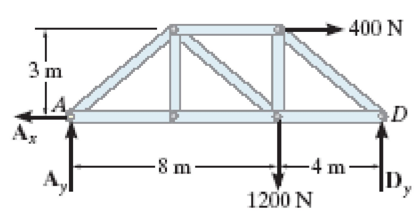
❒ **EXAMPLE 6**

Determine the force in members GE, GC, and BC of the truss shown in the figure. Indicate whether the members are in tension or compression.



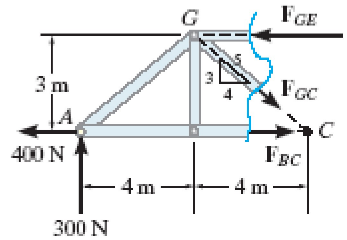
❒ **SOLUTION**

Section aa in the figure has been chosen since it cuts through the three members whose forces are to be determined. In order to use the method of sections, however, it is first necessary to determine the external reactions at *A* or *D*. Why? A free-body diagram of the entire truss is shown in second figure. Applying the equations of equilibrium, we have



$$\underset{\rightarrow}{+} \quad \sum F_x = 0, \qquad 400 - A_x = 0 \qquad \Rightarrow A_x = 400 \text{ N}$$
$$\sum M_A = 0, \qquad -1200(8) - 400(3) + D_y(12) = 0 \quad \Rightarrow D_y = 900 \text{ N}$$
$$+\uparrow \sum F_y = 0, \qquad A_y - 1200 + 900 = 0 \qquad \Rightarrow A_y = 300 \text{ N}$$

For the analysis the free-body diagram of the left portion of the sectioned truss will be used, since it involves the least number of forces. Summing moments about point G eliminates F_{GE} and F_{GC} yields a direct solution for F_{BC}.



$$\sum M_G = 0, \qquad -300(4) - 400(3) + F_{BC}(3) = 0 \ \Rightarrow F_{BC}$$

In the same manner, by summing moments about point C we obtain a direct solution for F_{GE}.

$$\sum M_C = 0, \qquad -300(8) - F_{GE}(3) = 0 \ \Rightarrow F_{GE} = 800 \text{ N}$$

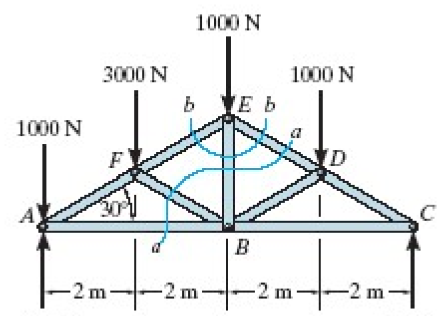
Since F_{BC} and F_{GE} have no vertical components, summing forces in the y direction directly yields F_{GC}, i.e.,

$$+\uparrow \sum F_y = 0, \qquad 300 - \frac{3}{5} F_{GC} = 0 \qquad \Rightarrow F_{GC} = 500 \text{ N}$$

NOTE: Here it is possible to tell, by inspection, the proper direction for each unknown member force. For example, $\sum M_C = 0$ requires F_{GE} to be compressive because it must balance the moment of the 300-N force about C.

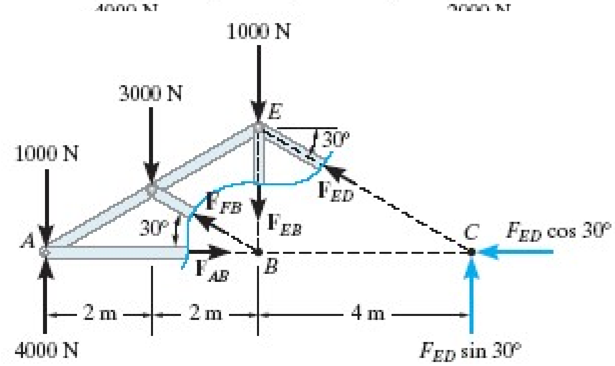
❒ EXAMPLE 7

Determine the force in member EB of the roof truss shown in the figure. Indicate whether the member is in tension or compression.

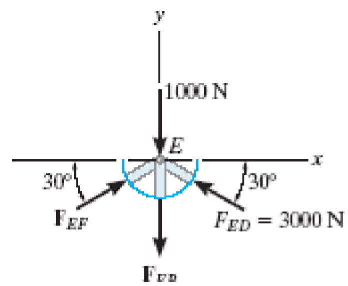


❒ SOLUTION

Free-Body Diagrams. By the method of sections, any imaginary section that cuts through EB, as shown, will also have to cut through three other members for which the forces are unknown. For example, section aa cuts



through *ED, EB, FB*, and *AB*. If a free-body diagram of the left side of this section is considered, it is possible to obtain F_{ED} by summing moments about *B* to eliminate the other three unknowns; however, F_{EB} cannot be determined from the remaining two equilibrium equations. One possible way of obtaining F_{EB} is first to determine F_{ED} from section *aa*, then use this result on section *bb*, which is shown in the figure. Here the force system is concurrent and our sectioned free-body diagram is the same as the free-body diagram for the joint at *E*. In order to determine the moment of F_{ED} about point *B*, we will use the principle of transmissibility and slide the force to point *C* and then resolve it into its rectangular components as shown. Therefore,



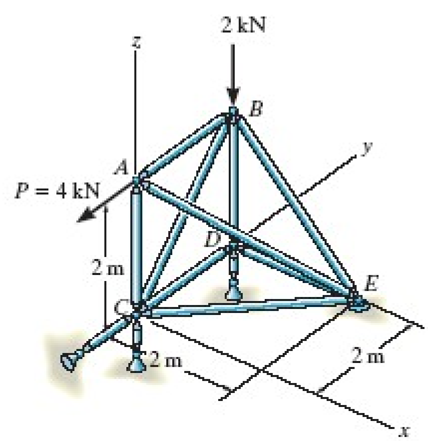
$$\sum M_B = 0, \quad 1000(4) + 3000(2) - 4000(4) + F_{ED} \sin 30 (4) = 0$$
$$\Rightarrow F_{ED} = 3000 \text{ N}$$

Considering now the free-body diagram of section *bb*, we have

$$\overset{+}{\rightarrow} \sum F_x = 0, \quad F_{EF} \cos 30 - 3000 \cos 30 = 0 \quad \Rightarrow F_{EF} = 3000 \text{ N}$$
$$+\uparrow \sum F_y = 0, \quad 2(3000 \sin 30) - 1000 - F_{EB} = 0 \quad \Rightarrow F_{EB} = 2000 \text{ N}$$

❒ EXAMPLE 7

Determine the forces acting in the members of the space truss shown in the figure. Indicate whether the members are in tension or compression.



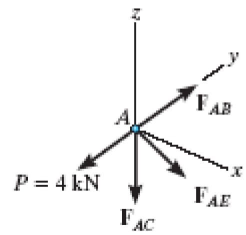
❒ SOLUTION

Since there are one known force and three unknown forces acting at joint *A*, the force analysis of the truss will begin at this joint.

◈ ***Joint A***. Expressing each force acting on the free-body diagram of joint *A* as a Cartesian vector, we have

$$\boldsymbol{P} = -4000\hat{j}, \qquad \underline{\boldsymbol{F}}_{AB} = F_{AB}\hat{j}, \qquad \underline{\boldsymbol{F}}_{AC} = -F_{AC}\hat{k}$$

$$\underline{\boldsymbol{F}}_{AE} = F_{AE}\left(\frac{\underline{r}_{AE}}{r_{AE}}\right) = F_{AE}(0.577\hat{i} + 0.577\hat{j} - 0.577\hat{k})$$



For equilibrium,

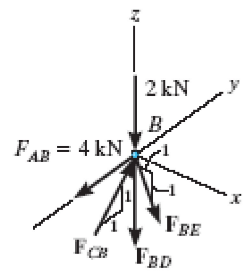
$$\sum F = 0, \qquad P + \underline{F}_{AB} + \underline{F}_{AC} + \underline{F}_{AE} = 0$$

$$\Rightarrow \quad -4000\hat{j} + F_{AB}\,\hat{j} - F_{AC}\hat{k} + 0.577F_{AE}\hat{i} + 0.577F_{AE}\hat{j} - 0.577F_{AE}\hat{k}$$

$$\sum F_x = 0, \qquad 0.577F_{AE} = 0 \qquad \Rightarrow F_{AE} = 0$$

$$\sum F_y = 0, \qquad -4000 + F_{AB} + 0.577F_{AE} = 0 \qquad \Rightarrow F_{AB} = 4000 \text{ N}$$

$$\sum F_z = 0, \qquad -F_{AC} - 0.577F_{AE} = 0 \qquad \Rightarrow F_{AC} = 0$$



Since F_{AB} is known, joint *B* can be analyzed next.

◈ ***Joint B.***

$$\sum F_x = 0, \qquad \frac{1}{\sqrt{2}}F_{BE} = 0 \qquad \Rightarrow F_{BE} = 0$$

$$\sum F_y = 0, \qquad -4000 + \frac{1}{\sqrt{2}}F_{CB} = 0 \qquad \Rightarrow F_{CB} = 5650 \text{ N}$$

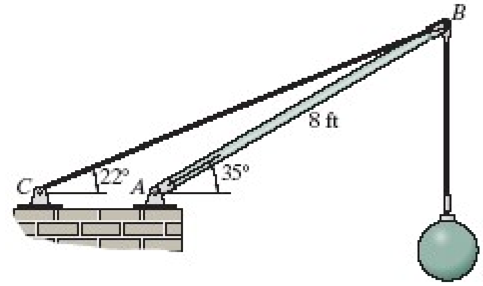
$$\sum F_z = 0, \qquad -2000 + F_{BD} - \frac{1}{\sqrt{2}}F_{BE} + \frac{1}{\sqrt{2}}F_{CB} = 0 \qquad \Rightarrow F_{BD} = 2000 \text{ N}$$

The scalar equations of equilibrium can now be applied to the forces acting on the free-body diagrams of joints D and C. Show that

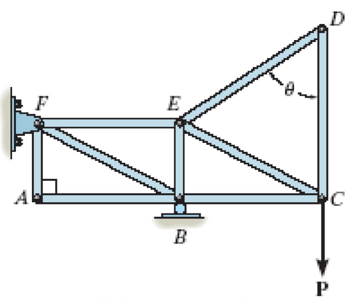
$$F_{DE} = F_{DC} = F_{CE} = 0$$

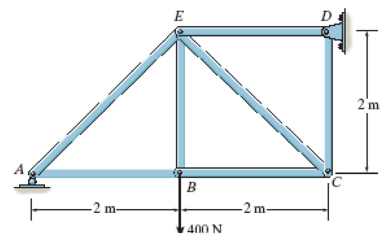
PROBLEMS

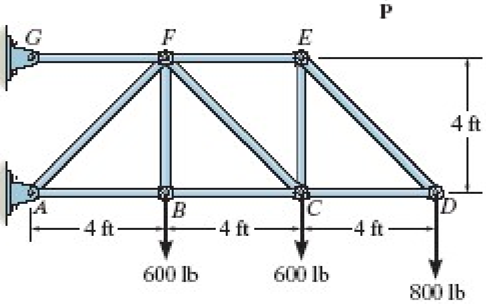
❒ Determine the magnitude of force at the pin *A* and in the cable *BC* needed to support the 500-lb load. Neglect the weight of the boom *AB*.

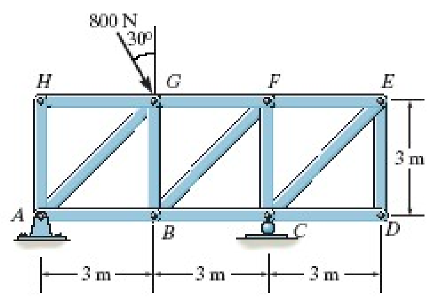


❒ In each case, calculate the support reactions and then draw the free-body diagrams of joints A, B, and C of the truss.

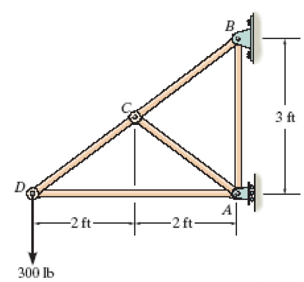




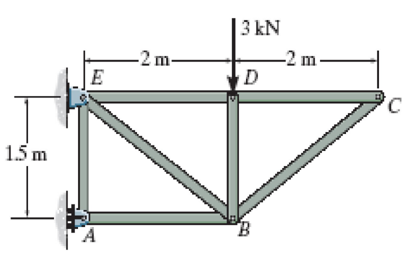


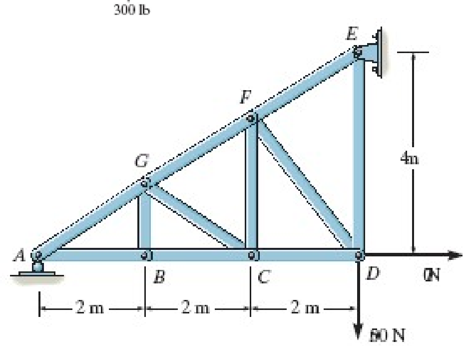


❒ Determine the force in each member of the truss. State if the members are in tension or compression.



❒ Identify the zero-force members in the truss.





المــــــراجع

١- أسس علم الميكانيكا ٩. احمد بدر الدين خليل ، عبدالشافي فهمي عبادة ، على محمد أبوستة ، عبدالرحمن أحمد السمان ، دار الفكر العربي ٢٠٠٥.

٢- امجد ابراهيم شحاته- الاستاتيكا- دار الفجر للنشر والتوزيع ٢٠٠١

3- Arthur Stanley Ramsey, Statics A Text-Book, Cambridge University Press.

4- R. C. Hibbeler, Engineering Mechanics, Statics, 14Edition.

5- S. L. Loney, The elements of Statics and Dynamics, Part I, Cambridge University Press.