CALCULUS I

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Review

Introduction

Technically a student coming into a Calculus class is supposed to know both Algebra and Trigonometry. The reality is often much different however. Most students enter a Calculus class woefully unprepared for both the algebra and the trig that is in a Calculus class. This is very unfortunate since good algebra skills are absolutely vital to successfully completing any Calculus course and if your Calculus course includes trig (as this one does) good trig skills are also important in many sections.

The intent of this chapter is to do a very cursory review of some algebra and trig skills that are absolutely vital to a calculus course. This chapter is not inclusive in the algebra and trig skills that are needed to be successful in a Calculus course. It only includes those topics that most students are particularly deficient in. For instance factoring is also vital to completing a standard calculus class but is not included here. For a more in depth review you should visit my Algebra/Trig review or my full set of Algebra notes at http://tutorial.math.lamar.edu.

Note that even though these topics are very important to a Calculus class I rarely cover all of these in the actual class itself. We simply don't have the time to do that. I do cover certain portions of this chapter in class, but for the most part I leave it to the students to read this chapter on their own.

Here is a list of topics that are in this chapter. I've also denoted the sections that I typically cover during the first couple of days of a Calculus class.

Review : Functions – Here is a quick review of functions, function notation and a couple of fairly important ideas about functions.

Review : Inverse Functions – A quick review of inverse functions and the notation for inverse functions.

Review : Trig Functions – A review of trig functions, evaluation of trig functions and the unit circle. This section usually gets a quick review in my class.

Review : Solving Trig Equations – A reminder on how to solve trig equations. This section is always covered in my class.

Review : Solving Trig Equations with Calculators, Part I – The previous section worked problem whose answers were always the "standard" angles. In this section we work some problems whose answers are not "standard" and so a calculator is needed. This section is always covered in my class as most trig equations in the remainder will need a calculator.

Review : Solving Trig Equations with Calculators, Part II – Even more trig equations requiring a calculator to solve.

Review : Exponential Functions – A review of exponential functions. This section usually gets a quick review in my class.

Review : Logarithm Functions – A review of logarithm functions and logarithm properties. This section usually gets a quick review in my class.

Review : Exponential and Logarithm Equations – How to solve exponential and logarithm equations. This section is always covered in my class.

Review : Common Graphs – This section isn't much. It's mostly a collection of graphs of many of the common functions that are liable to be seen in a Calculus class.

Review : Functions

In this section we're going to make sure that you're familiar with functions and function notation. Both will appear in almost every section in a Calculus class and so you will need to be able to deal with them.

First, what exactly is a function? An equation will be a function if for any *x* in the domain of the equation (the domain is all the *x*'s that can be plugged into the equation) the equation will yield exactly one value of *y*.

This is usually easier to understand with an example.

Example 1 Determine if each of the following are functions.

(a) $y = x^2 + 1$

(b) $y^2 = x + 1$

Solution

(a) This first one is a function. Given an *x* there is only one way to square it and then add 1 to the result and so no matter what value of *x* you put into the equation there is only one possible value of *y*.

(b) The only difference between this equation and the first is that we moved the exponent off the *x* and onto the *y*. This small change is all that is required, in this case, to change the equation from a function to something that isn't a function.

To see that this isn't a function is fairly simple. Choose a value of *x*, say *x=3* and plug this into the equation.

$$y^2 = 3 + 1 = 4$$

Now, there are two possible values of *y* that we could use here. We could use $y = 2$ or $y = -2$. Since there are two possible values of *y* that we get from a single *x* this equation isn't a function.

Note that this only needs to be the case for a single value of *x* to make an equation not be a function. For instance we could have used *x=-1* and in this case we would get a single *y* (*y=0*). However, because of what happens at *x=3* this equation will not be a function.

Next we need to take a quick look at function notation. Function notation is nothing more than a fancy way of writing the *y* in a function that will allow us to simplify notation and some of our work a little.

Let's take a look at the following function.

$$y = 2x^2 - 5x + 3$$

Using function notation we can write this as any of the following.

$$
\begin{aligned}
f(x) &= 2x^2 - 5x + 3 & \qquad g(x) &= 2x^2 - 5x + 3 \\
h(x) &= 2x^2 - 5x + 3 & \qquad R(x) &= 2x^2 - 5x + 3 \\
w(x) &= 2x^2 - 5x + 3 & \qquad y(x) &= 2x^2 - 5x + 3 \\
& \vdots
\end{aligned}
$$

Recall that this is NOT a letter times x, this is just a fancy way of writing y.

So, why is this useful? Well let's take the function above and let's get the value of the function at $x=-3$. Using function notation we represent the value of the function at $x=-3$ as $f(-3)$. Function notation gives us a nice compact way of representing function values.

Now, how do we actually evaluate the function? That's really simple. Everywhere we see an x on the right side we will substitute whatever is in the parenthesis on the left side. For our function this gives,

$$
\begin{aligned}
f(-3) &= 2(-3)^2 - 5(-3) + 3 \\
&= 2(9) + 15 + 3 \\
&= 36
\end{aligned}
$$

Let's take a look at some more function evaluation.

Example 2 Given $f(x) = -x^2 + 6x - 11$ find each of the following.

(a) $f(2)$ [Solution]

(b) $f(-10)$ [Solution]

(c) $f(t)$ [Solution]

(d) $f(t-3)$ [Solution]

(e) $f(x-3)$ [Solution]

(f) $f(4x-1)$ [Solution]

Solution

(a) $f(2) = -(2)^2 + 6(2) - 11 = -3$

[Return to Problems]

(b) $f(-10) = -(-10)^2 + 6(-10) - 11 = -100 - 60 - 11 = -171$

Be careful when squaring negative numbers!

[Return to Problems]

(c) $f(t) = -t^2 + 6t - 11$

Remember that we substitute for the x's WHATEVER is in the parenthesis on the left. Often this will be something other than a number. So, in this case we put t's in for all the x's on the left.

[Return to Problems]

(d) $f(t-3) = -(t-3)^2 + 6(t-3) - 11 = -t^2 + 12t - 38$

Often instead of evaluating functions at numbers or single letters we will have some fairly complex evaluations so make sure that you can do these kinds of evaluations.

[Return to Problems]

(e) $f(x-3) = -(x-3)^2 + 6(x-3) - 11 = -x^2 + 12x - 38$

The only difference between this one and the previous one is that I changed the *t* to an *x*. Other than that there is absolutely no difference between the two! Don't get excited if an *x* appears inside the parenthesis on the left.

[Return to Problems]

(f) $f(4x-1) = -(4x-1)^2 + 6(4x-1) - 11 = -16x^2 + 32x - 18$

This one is not much different from the previous part. All we did was change the equation that we were plugging into function.

[Return to Problems]

All throughout a calculus course we will be finding roots of functions. A root of a function is nothing more than a number for which the function is zero. In other words, finding the roots of a function, *g(x)*, is equivalent to solving

$$g(x) = 0$$

Example 3 Determine all the roots of $f(t) = 9t^3 - 18t^2 + 6t$

Solution

So we will need to solve,

$$9t^3 - 18t^2 + 6t = 0$$

First, we should factor the equation as much as possible. Doing this gives,

$$3t(3t^2 - 6t + 2) = 0$$

Next recall that if a product of two things are zero then one (or both) of them had to be zero. This means that,

$$3t = 0 \qquad\qquad \text{OR,}$$
$$3t^2 - 6t + 2 = 0$$

From the first it's clear that one of the roots must then be *t=0*. To get the remaining roots we will need to use the quadratic formula on the second equation. Doing this gives,

$$\begin{aligned} t &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(2)}}{2(3)} \\ &= \frac{6 \pm \sqrt{12}}{6} \\ &= \frac{6 \pm \sqrt{(4)(3)}}{6} \\ &= \frac{6 \pm 2\sqrt{3}}{6} \\ &= \frac{3 \pm \sqrt{3}}{3} \\ &= 1 \pm \frac{1}{3}\sqrt{3} \\ &= 1 \pm \frac{1}{\sqrt{3}} \end{aligned}$$

In order to remind you how to simplify radicals we gave several forms of the answer.

To complete the problem, here is a complete list of all the roots of this function.

$$t = 0, \; t = \frac{3+\sqrt{3}}{3}, \;\; t = \frac{3-\sqrt{3}}{3}$$

Note we didn't use the final form for the roots from the quadratic. This is usually where we'll stop with the simplification for these kinds of roots. Also note that, for the sake of the practice, we broke up the compact form for the two roots of the quadratic. You will need to be able to do this so make sure that you can.

This example had a couple of points other than finding roots of functions.

The first was to remind you of the quadratic formula. This won't be the first time that you'll need it in this class.

The second was to get you used to seeing "messy" answers. In fact, the answers in the above list are not that messy. However, most students come out of an Algebra class very used to seeing only integers and the occasional "nice" fraction as answers.

So, here is fair warning. In this class I often will intentionally make the answers look "messy" just to get you out of the habit of always expecting "nice" answers. In "real life" (whatever that is) the answer is rarely a simple integer such as two. In most problems the answer will be a decimal that came about from a messy fraction and/or an answer that involved radicals.

The next topic that we need to discuss here is that of **function composition**. The composition of *f(x)* and *g(x)* is

$$(f \circ g)(x) = f(g(x))$$

In other words, compositions are evaluated by plugging the second function listed into the first function listed. Note as well that order is important here. Interchanging the order will usually result in a different answer.

Example 4 Given $f(x) = 3x^2 - x + 10$ and $g(x) = 1 - 20x$ find each of the following.

(a) $(f \circ g)(5)$ [Solution]

(b) $(f \circ g)(x)$ [Solution]

(c) $(g \circ f)(x)$ [Solution]

(d) $(g \circ g)(x)$ [Solution]

Solution

(a) $(f \circ g)(5)$

In this case we've got a number instead of an *x* but it works in exactly the same way.

$$\begin{aligned}(f \circ g)(5) &= f(g(5)) \\ &= f(-99) = 29512\end{aligned}$$

[Return to Problems]

(b) $(f \circ g)(x)$

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(1-20x) \\ &= 3(1-20x)^2 - (1-20x) + 10 \\ &= 3(1-40x+400x^2) - 1 + 20x + 10 \\ &= 1200x^2 - 100x + 12\end{aligned}$$

Compare this answer to the next part and notice that answers are NOT the same. The order in which the functions are listed is important!

[Return to Problems]

(c) $(g \circ f)(x)$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(3x^2 - x + 10) \\ &= 1 - 20(3x^2 - x + 10) \\ &= -60x^2 + 20x - 199\end{aligned}$$

And just to make the point. This answer is different from the previous part. Order is important in composition.

[Return to Problems]

(d) $(g \circ g)(x)$

In this case do not get excited about the fact that it's the same function. Composition still works the same way.

$$\begin{aligned}(g \circ g)(x) &= g(g(x)) \\ &= g(1-20x) \\ &= 1-20(1-20x) \\ &= 400x-19\end{aligned}$$

[Return to Problems]

Let's work one more example that will lead us into the next section.

Example 5 Given $f(x) = 3x - 2$ and $g(x) = \frac{1}{3}x + \frac{2}{3}$ find each of the following.

(a) $(f \circ g)(x)$

(b) $(g \circ f)(x)$

Solution

(a)

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f\left(\frac{1}{3}x + \frac{2}{3}\right) \\ &= 3\left(\frac{1}{3}x + \frac{2}{3}\right) - 2 \\ &= x + 2 - 2 = x\end{aligned}$$

(b)

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(3x-2) \\ &= \frac{1}{3}(3x-2) + \frac{2}{3} \\ &= x - \frac{2}{3} + \frac{2}{3} = x\end{aligned}$$

In this case the two compositions where the same and in fact the answer was very simple.

$$(f \circ g)(x) = (g \circ f)(x) = x$$

This will usually not happen. However, when the two compositions are the same, or more specifically when the two compositions are both x there is a very nice relationship between the two functions. We will take a look at that relationship in the next section.

Review : Inverse Functions

In the last example from the previous section we looked at the two functions $f(x) = 3x - 2$ and $g(x) = \frac{x}{3} + \frac{2}{3}$ and saw that

$$(f \circ g)(x) = (g \circ f)(x) = x$$

and as noted in that section this means that there is a nice relationship between these two functions. Let's see just what that relationship is. Consider the following evaluations.

$$f(-1) = 3(-1) - 2 = -5 \qquad \Rightarrow \qquad g(-5) = \frac{-5}{3} + \frac{2}{3} = \frac{-3}{3} = -1$$

$$g(2) = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \qquad \Rightarrow \qquad f\left(\frac{4}{3}\right) = 3\left(\frac{4}{3}\right) - 2 = 4 - 2 = 2$$

In the first case we plugged $x = -1$ into $f(x)$ and got a value of -5. We then turned around and plugged $x = -5$ into $g(x)$ and got a value of -1, the number that we started off with.

In the second case we did something similar. Here we plugged $x = 2$ into $g(x)$ and got a value of $\frac{4}{3}$, we turned around and plugged this into $f(x)$ and got a value of 2, which is again the number that we started with.

Note that we really are doing some function composition here. The first case is really,

$$(g \circ f)(-1) = g\left[f(-1)\right] = g\left[-5\right] = -1$$

and the second case is really,

$$(f \circ g)(2) = f\left[g(2)\right] = f\left[\frac{4}{3}\right] = 2$$

Note as well that these both agree with the formula for the compositions that we found in the previous section. We get back out of the function evaluation the number that we originally plugged into the composition.

So, just what is going on here? In some way we can think of these two functions as undoing what the other did to a number. In the first case we plugged $x = -1$ into $f(x)$ and then plugged the result from this function evaluation back into $g(x)$ and in some way $g(x)$ undid what $f(x)$ had done to $x = -1$ and gave us back the original x that we started with.

Function pairs that exhibit this behavior are called **inverse functions**. Before formally defining inverse functions and the notation that we're going to use for them we need to get a definition out of the way.

A function is called **one-to-one** if no two values of *x* produce the same *y*. Mathematically this is the same as saying,

$$f\left(x_1\right) \neq f\left(x_2\right) \qquad \text{whenever} \qquad x_1 \neq x_2$$

So, a function is one-to-one if whenever we plug different values into the function we get different function values.

Sometimes it is easier to understand this definition if we see a function that isn't one-to-one. Let's take a look at a function that isn't one-to-one. The function $f\left(x\right) = x^2$ is not one-to-one because both $f\left(-2\right) = 4$ and $f\left(2\right) = 4$. In other words there are two different values of *x* that produce the same value of *y*. Note that we can turn $f\left(x\right) = x^2$ into a one-to-one function if we restrict ourselves to $0 \leq x < \infty$. This can sometimes be done with functions.

Showing that a function is one-to-one is often tedious and/or difficult. For the most part we are going to assume that the functions that we're going to be dealing with in this course are either one-to-one or we have restricted the domain of the function to get it to be a one-to-one function.

Now, let's formally define just what inverse functions are. Given two one-to-one functions $f\left(x\right)$ and $g\left(x\right)$ if

$$\left(f \circ g\right)\left(x\right) = x \qquad \text{AND} \qquad \left(g \circ f\right)\left(x\right) = x$$

then we say that $f\left(x\right)$ and $g\left(x\right)$ are **inverses** of each other. More specifically we will say that $g\left(x\right)$ is the **inverse** of $f\left(x\right)$ and denote it by

$$g\left(x\right) = f^{-1}\left(x\right)$$

Likewise we could also say that $f\left(x\right)$ is the **inverse** of $g\left(x\right)$ and denote it by

$$f\left(x\right) = g^{-1}\left(x\right)$$

The notation that we use really depends upon the problem. In most cases either is acceptable.

For the two functions that we started off this section with we could write either of the following two sets of notation.

$$f\left(x\right) = 3x - 2 \qquad f^{-1}\left(x\right) = \frac{x}{3} + \frac{2}{3}$$

$$g\left(x\right) = \frac{x}{3} + \frac{2}{3} \qquad g^{-1}\left(x\right) = 3x - 2$$

Now, be careful with the notation for inverses. The "-1" is NOT an exponent despite the fact that is sure does look like one! When dealing with inverse functions we've got to remember that

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

This is one of the more common mistakes that students make when first studying inverse functions.

The process for finding the inverse of a function is a fairly simple one although there are a couple of steps that can on occasion be somewhat messy. Here is the process

Finding the Inverse of a Function

Given the function $f(x)$ we want to find the inverse function, $f^{-1}(x)$.

1. First, replace $f(x)$ with *y*. This is done to make the rest of the process easier.
2. Replace every *x* with a *y* and replace every *y* with an *x*.
3. Solve the equation from Step 2 for *y*. This is the step where mistakes are most often made so be careful with this step.
4. Replace *y* with $f^{-1}(x)$. In other words, we've managed to find the inverse at this point!
5. Verify your work by checking that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ are both true. This work can sometimes be messy making it easy to make mistakes so again be careful.

That's the process. Most of the steps are not all that bad but as mentioned in the process there are a couple of steps that we really need to be careful with since it is easy to make mistakes in those steps.

In the verification step we technically really do need to check that both $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ are true. For all the functions that we are going to be looking at in this course if one is true then the other will also be true. However, there are functions (they are beyond the scope of this course however) for which it is possible for only one of these to be true. This is brought up because in all the problems here we will be just checking one of them. We just need to always remember that technically we should check both.

Let's work some examples.

Example 1 Given $f(x) = 3x - 2$ find $f^{-1}(x)$.

Solution

Now, we already know what the inverse to this function is as we've already done some work with it. However, it would be nice to actually start with this since we know what we should get. This will work as a nice verification of the process.

So, let's get started. We'll first replace $f\left(x\right)$ with *y*.

$$y = 3x - 2$$

Next, replace all *x*'s with *y* and all *y's* with *x*.

$$x = 3y - 2$$

Now, solve for *y*.

$$\begin{aligned} x + 2 &= 3y \\ \frac{1}{3}\left(x+2\right) &= y \\ \frac{x}{3} + \frac{2}{3} &= y \end{aligned}$$

Finally replace *y* with $f^{-1}\left(x\right)$.

$$f^{-1}\left(x\right) = \frac{x}{3} + \frac{2}{3}$$

Now, we need to verify the results. We already took care of this in the previous section, however, we really should follow the process so we'll do that here. It doesn't matter which of the two that we check we just need to check one of them. This time we'll check that $\left(f \circ f^{-1}\right)\left(x\right) = x$ is true.

$$\begin{aligned} \left(f \circ f^{-1}\right)\left(x\right) &= f\left[f^{-1}\left(x\right)\right] \\ &= f\left[\frac{x}{3} + \frac{2}{3}\right] \\ &= 3\left(\frac{x}{3} + \frac{2}{3}\right) - 2 \\ &= x + 2 - 2 \\ &= x \end{aligned}$$

Example 2 Given $g\left(x\right) = \sqrt{x-3}$ find $g^{-1}\left(x\right)$.

Solution

The fact that we're using $g\left(x\right)$ instead of $f\left(x\right)$ doesn't change how the process works. Here are the first few steps.

$$\begin{aligned} y &= \sqrt{x-3} \\ x &= \sqrt{y-3} \end{aligned}$$

Now, to solve for *y* we will need to first square both sides and then proceed as normal.

$$x = \sqrt{y-3}$$
$$x^2 = y - 3$$
$$x^2 + 3 = y$$

This inverse is then,

$$g^{-1}(x) = x^2 + 3$$

Finally let's verify and this time we'll use the other one just so we can say that we've gotten both down somewhere in an example.

$$\begin{aligned}\left(g^{-1} \circ g\right)(x) &= g^{-1}\left[g(x)\right] \\ &= g^{-1}\left(\sqrt{x-3}\right) \\ &= \left(\sqrt{x-3}\right)^2 + 3 \\ &= x - 3 + 3 \\ &= x\end{aligned}$$

So, we did the work correctly and we do indeed have the inverse.

The next example can be a little messy so be careful with the work here.

Example 3 Given $h(x) = \dfrac{x+4}{2x-5}$ find $h^{-1}(x)$.

Solution

The first couple of steps are pretty much the same as the previous examples so here they are,

$$y = \frac{x+4}{2x-5}$$
$$x = \frac{y+4}{2y-5}$$

Now, be careful with the solution step. With this kind of problem it is very easy to make a mistake here.

$$\begin{aligned} x\left(2y-5\right) &= y+4 \\ 2xy-5x &= y+4 \\ 2xy-y &= 4+5x \\ \left(2x-1\right)y &= 4+5x \\ y &= \frac{4+5x}{2x-1} \end{aligned}$$

So, if we've done all of our work correctly the inverse should be,

$$h^{-1}\left(x\right)=\frac{4+5x}{2x-1}$$

Finally we'll need to do the verification. This is also a fairly messy process and it doesn't really matter which one we work with.

$$\begin{aligned} \left(h\circ h^{-1}\right)\left(x\right) &= h\left[h^{-1}\left(x\right)\right] \\ &= h\left[\frac{4+5x}{2x-1}\right] \\ &= \frac{\frac{4+5x}{2x-1}+4}{2\left(\frac{4+5x}{2x-1}\right)-5} \end{aligned}$$

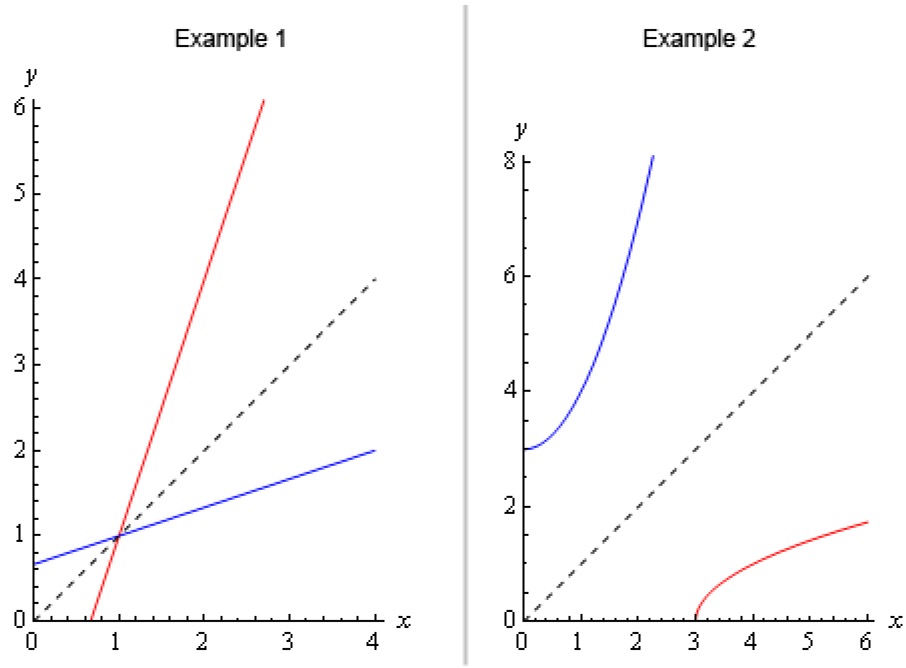
Okay, this is a mess. Let's simplify things up a little bit by multiplying the numerator and denominator by $2x-1$.

$$\begin{aligned} \left(h\circ h^{-1}\right)\left(x\right) &= \frac{2x-1}{2x-1}\,\frac{\frac{4+5x}{2x-1}+4}{2\left(\frac{4+5x}{2x-1}\right)-5} \\ &= \frac{\left(2x-1\right)\left(\frac{4+5x}{2x-1}+4\right)}{\left(2x-1\right)\left(2\left(\frac{4+5x}{2x-1}\right)-5\right)} \\ &= \frac{4+5x+4\left(2x-1\right)}{2\left(4+5x\right)-5\left(2x-1\right)} \\ &= \frac{4+5x+8x-4}{8+10x-10x+5} \\ &= \frac{13x}{13}=x \end{aligned}$$

Wow. That was a lot of work, but it all worked out in the end. We did all of our work correctly and we do in fact have the inverse.

There is one final topic that we need to address quickly before we leave this section. There is an interesting relationship between the graph of a function and the graph of its inverse.

Here is the graph of the function and inverse from the first two examples.



In both cases we can see that the graph of the inverse is a reflection of the actual function about the line $y = x$. This will always be the case with the graphs of a function and its inverse.

Limits

Introduction

The topic that we will be examining in this chapter is that of Limits. This is the first of three major topics that we will be covering in this course. While we will be spending the least amount of time on limits in comparison to the other two topics limits are very important in the study of Calculus. We will be seeing limits in a variety of places once we move out of this chapter. In particular we will see that limits are part of the formal definition of the other two major topics.

Here is a quick listing of the material that will be covered in this chapter.

Tangent Lines and Rates of Change – In this section we will take a look at two problems that we will see time and again in this course. These problems will be used to introduce the topic of limits.

The Limit – Here we will take a conceptual look at limits and try to get a grasp on just what they are and what they can tell us.

One-Sided Limits – A brief introduction to one-sided limits.

Limit Properties – Properties of limits that we'll need to use in computing limits. We will also compute some basic limits in this section

Computing Limits – Many of the limits we'll be asked to compute will not be "simple" limits. In other words, we won't be able to just apply the properties and be done. In this section we will look at several types of limits that require some work before we can use the limit properties to compute them.

Infinite Limits – Here we will take a look at limits that have a value of infinity or negative infinity. We'll also take a brief look at vertical asymptotes.

Limits At Infinity, Part I – In this section we'll look at limits at infinity. In other words, limits in which the variable gets very large in either the positive or negative sense. We'll also take a brief look at horizontal asymptotes in this section. We'll be concentrating on polynomials and rational expression involving polynomials in this section.

Limits At Infinity, Part II – We'll continue to look at limits at infinity in this section, but this time we'll be looking at exponential, logarithms and inverse tangents.

Limit Properties

The time has almost come for us to actually compute some limits. However, before we do that we will need some properties of limits that will make our life somewhat easier. So, let's take a look at those first. The proof of some of these properties can be found in the Proof of Various Limit Properties section of the Extras chapter.

Properties

First we will assume that $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist and that c is any constant. Then,

1. $\lim\limits_{x\to a}\left[cf(x)\right] = c\lim\limits_{x\to a} f(x)$

   In other words we can "factor" a multiplicative constant out of a limit.

2. $\lim\limits_{x\to a}\left[f(x)\pm g(x)\right] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$

   So to take the limit of a sum or difference all we need to do is take the limit of the individual parts and then put them back together with the appropriate sign. This is also not limited to two functions. This fact will work no matter how many functions we've got separated by "+" or "-".

3. $\lim\limits_{x\to a}\left[f(x)g(x)\right] = \lim\limits_{x\to a} f(x)\ \lim\limits_{x\to a} g(x)$

   We take the limits of products in the same way that we can take the limit of sums or differences. Just take the limit of the pieces and then put them back together. Also, as with sums or differences, this fact is not limited to just two functions.

4. $\lim\limits_{x\to a}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)},\qquad$ provided $\lim\limits_{x\to a} g(x) \neq 0$

   As noted in the statement we only need to worry about the limit in the denominator being zero when we do the limit of a quotient. If it were zero we would end up with a division by zero error and we need to avoid that.

5. $\lim\limits_{x\to a}\left[f(x)\right]^n = \left[\lim\limits_{x\to a} f(x)\right]^n,\qquad$ where n is any real number

   In this property n can be any real number (positive, negative, integer, fraction, irrational, zero, *etc.*). In the case that n is an integer this rule can be thought of as an extended case of **3**.

   For example consider the case of $n = 2$.

$$\lim_{x\to a}\left[f(x)\right]^2 = \lim_{x\to a}\left[f(x)\,f(x)\right]$$
$$= \lim_{x\to a} f(x)\lim_{x\to a} f(x) \qquad \text{using property 3}$$
$$= \left[\lim_{x\to a} f(x)\right]^2$$

The same can be done for any integer *n*.

6. $\lim_{x\to a}\left[\sqrt[n]{f(x)}\right] = \sqrt[n]{\lim_{x\to a} f(x)}$

This is just a special case of the previous example.

$$\lim_{x\to a}\left[\sqrt[n]{f(x)}\right] = \lim_{x\to a}\left[f(x)\right]^{\frac{1}{n}}$$
$$= \left[\lim_{x\to a} f(x)\right]^{\frac{1}{n}}$$
$$= \sqrt[n]{\lim_{x\to a} f(x)}$$

7. $\lim_{x\to a} c = c,\qquad c$ is any real number

In other words, the limit of a constant is just the constant. You should be able to convince yourself of this by drawing the graph of $f(x) = c$.

8. $\lim_{x\to a} x = a$

As with the last one you should be able to convince yourself of this by drawing the graph of $f(x) = x$.

9. $\lim_{x\to a} x^n = a^n$

This is really just a special case of property **5** using $f(x) = x$.

Note that all these properties also hold for the two one-sided limits as well we just didn't write them down with one sided limits to save on space.

Let's compute a limit or two using these properties. The next couple of examples will lead us to some truly useful facts about limits that we will use on a continual basis.

Example 1 Compute the value of the following limit.

$$\lim_{x\to -2}\left(3x^2 + 5x - 9\right)$$

Solution

This first time through we will use only the properties above to compute the limit.

First we will use property **2** to break up the limit into three separate limits. We will then use property **1** to bring the constants out of the first two limits. Doing this gives us,

$$\begin{aligned}\lim_{x\to-2}\left(3x^2+5x-9\right) &= \lim_{x\to-2}3x^2+\lim_{x\to-2}5x-\lim_{x\to-2}9\\ &= 3\lim_{x\to-2}x^2+5\lim_{x\to-2}x-\lim_{x\to-2}9\end{aligned}$$

We can now use properties **7** through **9** to actually compute the limit.

$$\begin{aligned}\lim_{x\to-2}\left(3x^2+5x-9\right) &= 3\lim_{x\to-2}x^2+5\lim_{x\to-2}x-\lim_{x\to-2}9\\ &= 3\left(-2\right)^2+5\left(-2\right)-9\\ &= -7\end{aligned}$$

Now, let's notice that if we had defined

$$p\left(x\right)=3x^2+5x-9$$

then the proceeding example would have been,

$$\begin{aligned}\lim_{x\to-2}p\left(x\right) &= \lim_{x\to-2}\left(3x^2+5x-9\right)\\ &= 3\left(-2\right)^2+5\left(-2\right)-9\\ &= -7\\ &= p\left(-2\right)\end{aligned}$$

In other words, in this case we were the limit is the same value that we'd get by just evaluating the function at the point in question. This seems to violate one of the main concepts about limits that we've seen to this point.

In the previous two sections we made a big deal about the fact that limits do not care about what is happening at the point in question. They only care about what is happening around the point. So how does the previous example fit into this since it appears to violate this main idea about limits?

Despite appearances the limit still doesn't care about what the function is doing at $x=-2$. In this case the function that we've got is simply "nice enough" so that what is happening around the point is exactly the same as what is happening at the point. Eventually we will formalize up just what is meant by "nice enough". At this point let's not worry too much about what "nice enough" is. Let's just take advantage of the fact that some functions will be "nice enough", whatever that means.

The function in the last example was a polynomial. It turns out that all polynomials are "nice enough" so that what is happening around the point is exactly the same as what is happening at the point. This leads to the following fact.

Fact

If *p(x)* is a polynomial then,

$$\lim_{x \to a} p(x) = p(a)$$

By the end of this section we will generalize this out considerably to most of the functions that we'll be seeing throughout this course.

Let's take a look at another example.

Example 2 Evaluate the following limit.

$$\lim_{z \to 1} \frac{6 - 3z + 10z^2}{-2z^4 + 7z^3 + 1}$$

Solution

First notice that we can use property **4)** to write the limit as,

$$\lim_{z \to 1} \frac{6 - 3z + 10z^2}{-2z^4 + 7z^3 + 1} = \frac{\lim_{z \to 1} 6 - 3z + 10z^2}{\lim_{z \to 1} -2z^4 + 7z^3 + 1}$$

Well, actually we should be a little careful. We can do that provided the limit of the denominator isn't zero. As we will see however, it isn't in this case so we're okay.

Now, both the numerator and denominator are polynomials so we can use the fact above to compute the limits of the numerator and the denominator and hence the limit itself.

$$\begin{aligned} \lim_{z \to 1} \frac{6 - 3z + 10z^2}{-2z^4 + 7z^3 + 1} &= \frac{6 - 3(1) + 10(1)^2}{-2(1)^4 + 7(1)^3 + 1} \\ &= \frac{13}{6} \end{aligned}$$

Notice that the limit of the denominator wasn't zero and so our use of property **4** was legitimate.

Notice in this last example that again all we really did was evaluate the function at the point in question. So it appears that there is a fairly large class of functions for which this can be done. Let's generalize the fact from above a little.

Fact

Provided *f(x)* is "nice enough" we have,

$$\lim_{x \to a} f(x) = f(a) \qquad \lim_{x \to a^-} f(x) = f(a) \qquad \lim_{x \to a^+} f(x) = f(a)$$

Again, we will formalize up just what we mean by "nice enough" eventually. At this point all we want to do is worry about which functions are "nice enough". Some functions are "nice enough" for all *x* while others will only be "nice enough" for certain values of *x*. It will all depend on the function.

As noted in the statement, this fact also holds for the two one-sided limits as well as the normal limit.

Here is a list of some of the more common functions that are "nice enough".

- Polynomials are nice enough for all *x*'s.
- If $f(x)=\dfrac{p(x)}{q(x)}$ then *f(x)* will be nice enough provided both *p(x)* and *q(x)* are nice enough and if we don't get division by zero at the point we're evaluating at.
- $\cos(x)$, $\sin(x)$ are nice enough for all *x*'s
- $\sec(x)$, $\tan(x)$ are nice enough provided $x \neq \ldots,-\frac{5\pi}{2},-\frac{3\pi}{2},\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\ldots$ In other words secant and tangent are nice enough everywhere cosine isn't zero. To see why recall that these are both really rational functions and that cosine is in the denominator of both then go back up and look at the second bullet above.
- $\csc(x)$, $\cot(x)$ are nice enough provided $x \neq \ldots,-3\pi,-\pi,0,\pi,3\pi,\ldots$ In other words cosecant and cotangent are nice enough everywhere sine isn't zero.
- $\sqrt[n]{x}$ is nice enough for all *x* if *n* is odd.
- $\sqrt[n]{x}$ is nice enough for $x \geq 0$ if *n* is even. Here we require $x \geq 0$ to avoid having to deal with complex values.
- a^x, $\mathbf{e}^x$ are nice enough for all *x*'s.
- $\log_b x$, $\ln x$ are nice enough for *x>0*. Remember we can only plug positive numbers into logarithms and not zero or negative numbers.
- Any sum, difference or product of the above functions will also be nice enough. Quotients will be nice enough provided we don't get division by zero upon evaluating the limit.

The last bullet is important. This means that for any combination of these functions all we need to do is evaluate the function at the point in question, making sure that none of the restrictions are violated. This means that we can now do a large number of limits.

Example 3 Evaluate the following limit.

$$\lim_{x\to 3}\left(-\sqrt[5]{x}+\frac{\mathbf{e}^x}{1+\ln(x)}+\sin(x)\cos(x)\right)$$

Solution

This is a combination of several of the functions listed above and none of the restrictions are violated so all we need to do is plug in $x=3$ into the function to get the limit.

$$\lim_{x\to 3}\left(-\sqrt[5]{x}+\frac{\mathbf{e}^{x}}{1+\ln\left(x\right)}+\sin\left(x\right)\cos\left(x\right)\right)=-\sqrt[5]{3}+\frac{\mathbf{e}^{3}}{1+\ln\left(3\right)}+\sin\left(3\right)\cos\left(3\right)$$
$$=8.185427271$$

Not a very pretty answer, but we can now do the limit.

Computing Limits

In the previous section we saw that there is a large class of function that allows us to use

$$\lim_{x\to a} f(x) = f(a)$$

to compute limits. However, there are also many limits for which this won't work easily. The purpose of this section is to develop techniques for dealing with some of these limits that will not allow us to just use this fact.

Let's first got back and take a look at one of the first limits that we looked at and compute its exact value and verify our guess for the limit.

Example 1 Evaluate the following limit.

$$\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x}$$

Solution

First let's notice that if we try to plug in $x=2$ we get,

$$\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x}=\frac{0}{0}$$

So, we can't just plug in $x=2$ to evaluate the limit. So, we're going to have to do something else.

The first thing that we should always do when evaluating limits is to simplify the function as much as possible. In this case that means factoring both the numerator and denominator. Doing this gives,

$$\begin{aligned}\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x}&=\lim_{x\to 2}\frac{(x-2)(x+6)}{x(x-2)}\\&=\lim_{x\to 2}\frac{x+6}{x}\end{aligned}$$

So, upon factoring we saw that we could cancel an $x-2$ from both the numerator and the denominator. Upon doing this we now have a new rational expression that we can plug $x=2$ into because we lost the division by zero problem. Therefore, the limit is,

$$\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x}=\lim_{x\to 2}\frac{x+6}{x}=\frac{8}{2}=4$$

Note that this is in fact what we guessed the limit to be.

On a side note, the 0/0 we initially got in the previous example is called an **indeterminate form**. This means that we don't really know what it will be until we do some more work. Typically zero in the denominator means it's undefined. However that will only be true if the numerator isn't also zero. Also, zero in the numerator usually means that the fraction is zero, unless the denominator is also zero. Likewise anything divided by itself is 1, unless we're talking about zero.

So, there are really three competing "rules" here and it's not clear which one will win out. It's also possible that none of them will win out and we will get something totally different from undefined, zero, or one. We might, for instance, get a value of 4 out of this, to pick a number completely at random.

There are many more kinds of indeterminate forms and we will be discussing indeterminate forms at length in the next chapter.

Let's take a look at a couple of more examples.

Example 2 Evaluate the following limit.

$$\lim_{h\to 0}\frac{2\left(-3+h\right)^2-18}{h}$$

Solution

In this case we also get 0/0 and factoring is not really an option. However, there is still some simplification that we can do.

$$\begin{aligned}\lim_{h\to 0}\frac{2\left(-3+h\right)^2-18}{h} &= \lim_{h\to 0}\frac{2\left(9-6h+h^2\right)-18}{h}\\ &= \lim_{h\to 0}\frac{18-12h+2h^2-18}{h}\\ &= \lim_{h\to 0}\frac{-12h+2h^2}{h}\end{aligned}$$

So, upon multiplying out the first term we get a little cancellation and now notice that we can factor an *h* out of both terms in the numerator which will cancel against the *h* in the denominator and the division by zero problem goes away and we can then evaluate the limit.

$$\begin{aligned}\lim_{h\to 0}\frac{2\left(-3+h\right)^2-18}{h} &= \lim_{h\to 0}\frac{-12h+2h^2}{h}\\ &= \lim_{h\to 0}\frac{h\left(-12+2h\right)}{h}\\ &= \lim_{h\to 0}\, -12+2h=-12\end{aligned}$$

Example 3 Evaluate the following limit.

$$\lim_{t\to 4}\frac{t-\sqrt{3t+4}}{4-t}$$

Solution

This limit is going to be a little more work than the previous two. Once again however note that we get the indeterminate form 0/0 if we try to just evaluate the limit. Also note that neither of the two examples will be of any help here, at least initially. We can't factor and we can't just multiply something out to get the function to simplify.

When there is a square root in the numerator or denominator we can try to rationalize and see if that helps. Recall that rationalizing makes use of the fact that

$$(a+b)(a-b)=a^2-b^2$$

So, if either the first and/or the second term have a square root in them the rationalizing will eliminate the root(s). This *might* help in evaluating the limit.

Let's try rationalizing the numerator in this case.

$$\lim_{t\to 4}\frac{t-\sqrt{3t+4}}{4-t}=\lim_{t\to 4}\frac{\left(t-\sqrt{3t+4}\right)}{(4-t)}\frac{\left(t+\sqrt{3t+4}\right)}{\left(t+\sqrt{3t+4}\right)}$$

Remember that to rationalize we just take the numerator (since that's what we're rationalizing), change the sign on the second term and multiply the numerator and denominator by this new term.

Next, we multiply the numerator out being careful to watch minus signs.

$$\begin{aligned}\lim_{t\to 4}\frac{t-\sqrt{3t+4}}{4-t}&=\lim_{t\to 4}\frac{t^2-(3t+4)}{(4-t)\left(t+\sqrt{3t+4}\right)}\\&=\lim_{t\to 4}\frac{t^2-3t-4}{(4-t)\left(t+\sqrt{3t+4}\right)}\end{aligned}$$

Notice that we didn't multiply the denominator out as well. Most students come out of an Algebra class having it beaten into their heads to always multiply this stuff out. However, in this case multiplying out will make the problem very difficult and in the end you'll just end up factoring it back out anyway.

At this stage we are almost done. Notice that we can factor the numerator so let's do that.

$$\lim_{t\to 4}\frac{t-\sqrt{3t+4}}{4-t}=\lim_{t\to 4}\frac{(t-4)(t+1)}{(4-t)\left(t+\sqrt{3t+4}\right)}$$

Now all we need to do is notice that if we factor a "-1"out of the first term in the denominator we can do some canceling. At that point the division by zero problem will go away and we can evaluate the limit.

$$\begin{aligned}\lim_{t\to4}\frac{t-\sqrt{3t+4}}{4-t} &= \lim_{t\to4}\frac{(t-4)(t+1)}{-(t-4)\left(t+\sqrt{3t+4}\right)}\\ &= \lim_{t\to4}\frac{t+1}{-\left(t+\sqrt{3t+4}\right)}\\ &= -\frac{5}{8}\end{aligned}$$

Note that if we had multiplied the denominator out we would not have been able to do this canceling and in all likelihood would not have even seen that some canceling could have been done.

So, we've taken a look at a couple of limits in which evaluation gave the indeterminate form 0/0 and we now have a couple of things to try in these cases.

Let's take a look at another kind of problem that can arise in computing some limits involving piecewise functions.

Example 4 Given the function,

$$g(y)=\begin{cases} y^2+5 & \text{if } y<-2\\ 1-3y & \text{if } y\ge -2\end{cases}$$

Compute the following limits.

(a) $\lim\limits_{y\to6} g(y)$ [Solution]

(b) $\lim\limits_{y\to-2} g(y)$ [Solution]

Solution

(a) $\lim\limits_{y\to6} g(y)$

In this case there really isn't a whole lot to do. In doing limits recall that we must always look at what's happening on both sides of the point in question as we move in towards it. In this case $y=6$ is completely inside the second interval for the function and so there are values of y on both sides of $y=6$ that are also inside this interval. This means that we can just use the fact to evaluate this limit.

$$\begin{aligned}\lim_{y\to6} g(y) &= \lim_{y\to6} 1-3y\\ &= -17\end{aligned}$$

[Return to Problems]

(b) $\lim\limits_{y\to-2} g(y)$

This part is the real point to this problem. In this case the point that we want to take the limit for is the cutoff point for the two intervals. In other words we can't just plug $y=-2$ into the second portion because this interval does not contain values of y to the left of $y=-2$ and we need to know what is happening on both sides of the point.

To do this part we are going to have to remember the fact from the section on one-sided limits that says that if the two one-sided limits exist and are the same then the normal limit will also exist and have the same value.

Notice that both of the one sided limits can be done here since we are only going to be looking at one side of the point in question. So let's do the two one-sided limits and see what we get.

$$\lim_{y\to -2^-} g(y) = \lim_{y\to -2^-} y^2+5 \qquad \text{since } y\to 2^- \text{ implies } y<-2$$
$$= 9$$

$$\lim_{y\to -2^+} g(y) = \lim_{y\to -2^+} 1-3y \qquad \text{since } y\to 2^+ \text{ implies } y>-2$$
$$= 7$$

So, in this case we can see that,

$$\lim_{y\to -2^-} g(y) = 9 \neq 7 = \lim_{y\to -2^+} g(y)$$

and so since the two one sided limits aren't the same

$$\lim_{y\to -2} g(y)$$

doesn't exist.

[Return to Problems]

Note that a very simple change to the function will make the limit at $y=-2$ exist so don't get in into your head that limits at these cutoff points in piecewise function don't ever exist.

Example 5 Evaluate the following limit.

$$\lim_{y\to -2} g(y) \qquad \text{where, } g(y) = \begin{cases} y^2+5 & \text{if } y<-2 \\ 3-3y & \text{if } y\geq -2 \end{cases}$$

Solution

The two one-sided limits this time are,

$$\lim_{y\to -2^-} g(y) = \lim_{y\to -2^-} y^2+5 \qquad \text{since } y\to 2^- \text{ implies } y<-2$$
$$= 9$$

$$\lim_{y\to -2^+} g(y) = \lim_{y\to -2^-} 3-3y \qquad \text{since } y\to 2^+ \text{ implies } y>-2$$
$$= 9$$

The one-sided limits are the same so we get,

$$\lim_{y\to -2} g(y) = 9$$

fact that the normal limit will exist only if the two one-sided limits exist and have the same value.

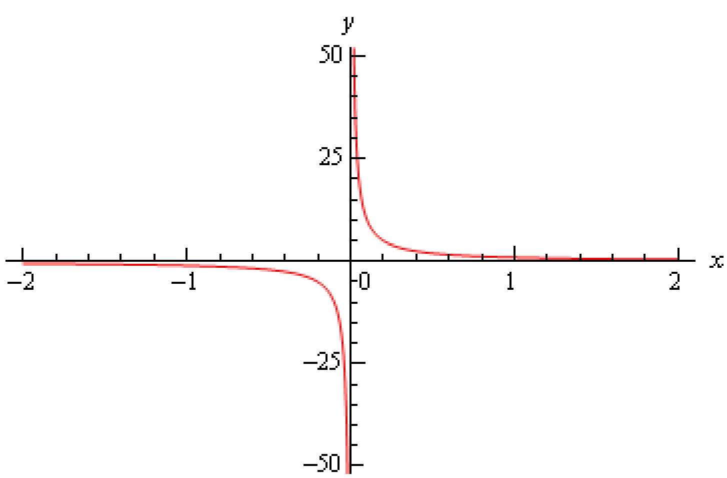
x	$\frac{1}{x}$	x	$\frac{1}{x}$
-0.1	-10	0.1	10
-0.01	-100	0.01	100
-0.001	-1000	0.001	1000
-0.0001	-10000	0.0001	1000

From this table we can see that as we make x smaller and smaller the function $\frac{1}{x}$ gets larger and larger and will retain the same sign that x originally had. It should make sense that this trend will continue for any smaller value of x that we chose to use. The function is a constant (one in this case) divided by an increasingly small number. The resulting fraction should be an increasingly large number and as noted above the fraction will retain the same sign as x.

We can make the function as large and positive as we want for all x's sufficiently close to zero while staying positive (*i.e.* on the right). Likewise, we can make the function as large and negative as we want for all x's sufficiently close to zero while staying negative (*i.e.* on the left). So, from our definition above it looks like we should have the following values for the two one sided limits.

$$\lim_{x\to 0^+}\frac{1}{x} = -\infty \qquad\qquad \lim_{x\to 0^-}\frac{1}{x} = \infty$$

Another way to see the values of the two one sided limits here is to graph the function. Again, in the previous section we mentioned that we won't do this too often as most functions are not something we can just quickly sketch out as well as the problems with accuracy in reading values off the graph. In this case however, it's not too hard to sketch a graph of the function and, in this case as we'll see accuracy is not really going to be an issue. So, here is a quick sketch of the graph.



So, we can see from this graph that the function does behave much as we predicted that it would from our table values. The closer x gets to zero from the right the larger (in the positive sense) the function gets, while the closer x gets to zero from the left the larger (in the negative sense) the function gets.

Finally, the normal limit, in this case, will not exist since the two one-sided have different values.

So, in summary here are the values of the three limits for this example.

$$\lim_{x \to 0^+} \frac{1}{x} = -\infty \qquad \lim_{x \to 0^-} \frac{1}{x} = \infty \qquad \lim_{x \to 0} \frac{1}{x} \text{ doesn't exist}$$

For most of the remaining examples in this section we'll attempt to "talk our way through" each limit. This means that we'll see if we can analyze what should happen to the function as we get very close to the point in question without actually plugging in any values into the function. For most of the following examples this kind of analysis shouldn't be all that difficult to do. We'll also verify our analysis with a quick graph.

So, let's do a couple more examples.

Example 2 Evaluate each of the following limits.

$$\lim_{x \to 0^+} \frac{6}{x^2} \qquad \lim_{x \to 0^-} \frac{6}{x^2} \qquad \lim_{x \to 0} \frac{6}{x^2}$$

Solution

As with the previous example let's start off by looking at the two one-sided limits. Once we have those we'll be able to determine a value for the normal limit.

So, let's take a look at the right-hand limit first and as noted above let's see if we can see if we can figure out what each limit will be doing without actually plugging in any values of x into the function. As we take smaller and smaller values of x, while staying positive, squaring them will only make them smaller (recall squaring a number between zero and one will make it smaller) and of course it will stay positive. So we have a positive constant divided by an increasingly small positive number. The result should then be an increasingly large positive number. It looks like we should have the following value for the right-hand limit in this case,

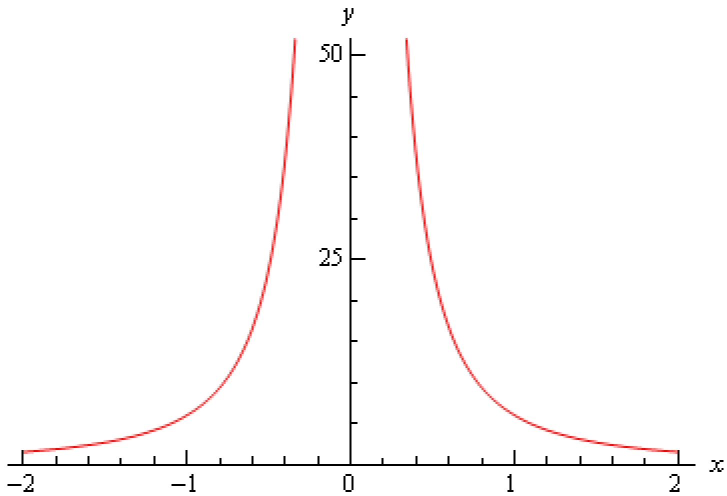
$$\lim_{x \to 0^+} \frac{6}{x^2} = \infty$$

Now, let's take a look at the left hand limit. In this case we're going to take smaller and smaller values of x, while staying negative this time. When we square them we'll get smaller, but upon squaring the result is now positive. So, we have a positive constant divided by an increasingly small positive number. The result, as with the right hand limit, will be an increasingly large positive number and so the left-hand limit will be,

$$\lim_{x \to 0^-} \frac{6}{x^2} = \infty$$

Now, in this example, unlike the first one, the normal limit will exist and be infinity since the two one-sided limits both exist and have the same value. So, in summary here are all the limits for this example as well as a quick graph verifying the limits.

$$\lim_{x \to 0^+} \frac{6}{x^2} = \infty \qquad \lim_{x \to 0^-} \frac{6}{x^2} = \infty \qquad \lim_{x \to 0} \frac{6}{x^2} = \infty$$



With this next example we'll move away from just an x in the denominator, but as we'll see in the next couple of examples they work pretty much the same way.

Example 3 Evaluate each of the following limits.

$$\lim_{x \to -2^+} \frac{-4}{x+2} \qquad \lim_{x \to -2^-} \frac{-4}{x+2} \qquad \lim_{x \to -2} \frac{-4}{x+2}$$

Solution

Let's again start with the right-hand limit. With the right hand limit we know that we have,

$$x > -2 \qquad \Rightarrow \qquad x+2 > 0$$

Also, as x gets closer and closer to -2 then $x+2$ will be getting closer and closer to zero, while staying positive as noted above. So, for the right-hand limit, we'll have a negative constant divided by an increasingly small positive number. The result will be an increasingly large and negative number. So, it looks like the right-hand limit will be negative infinity.

For the left hand limit we have,

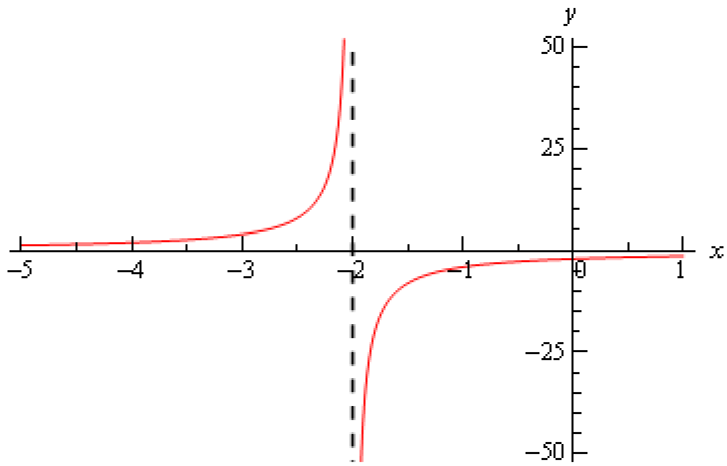
$$x < -2 \qquad \Rightarrow \qquad x+2 < 0$$

and $x+2$ will get closer and closer to zero (and be negative) as x gets closer and closer to -2. In this case then we'll have a negative constant divided by an increasingly small negative number. The result will then be an increasingly large positive number and so it looks like the left-hand limit will be positive infinity.

Finally, since two one sided limits are not the same the normal limit won't exist.

Here are the official answers for this example as well as a quick graph of the function for verification purposes.

$$\lim_{x \to -2^+} \frac{-4}{x+2} = -\infty \qquad \lim_{x \to -2^-} \frac{-4}{x+2} = \infty \qquad \lim_{x \to -2} \frac{-4}{x+2} \text{ doesn't exist}$$



At this point we should briefly acknowledge the idea of vertical asymptotes. Each of the three previous graphs have had one. Recall from an Algebra class that a vertical asymptote is a vertical line (the dashed line at $x = -2$ in the previous example) in which the graph will go towards infinity and/or minus infinity on one or both sides of the line.

In an Algebra class they are a little difficult to define other than to say pretty much what we just said. Now that we have infinite limits under our belt we can easily define a vertical asymptote as follows,

Definition

The function *f(x)* will have a vertical asymptote at $x = a$ if we have any of the following limits at $x = a$.

$$\lim_{x \to a^-} f(x) = \pm\infty \qquad \lim_{x \to a^+} f(x) = \pm\infty \qquad \lim_{x \to a} f(x) = \pm\infty$$

Note that it only requires one of the above limits for a function to have a vertical asymptote at $x = a$.

Using this definition we can see that the first two examples had vertical asymptotes at $x = 0$ while the third example had a vertical asymptote at $x = -2$.

We aren't really going to do a lot with vertical asymptotes here, but wanted to mention them at this since we'd reached a good point to do that.

Let's now take a look at a couple more examples of infinite limits that can cause some problems on occasion.

Example 4 Evaluate each of the following limits.

$$\lim_{x \to 4^+} \frac{3}{(4-x)^3} \qquad \lim_{x \to 4^-} \frac{3}{(4-x)^3} \qquad \lim_{x \to 4} \frac{3}{(4-x)^3}$$

Solution

Let's start with the right-hand limit. For this limit we have,

$$x > 4 \qquad \Rightarrow \qquad 4 - x < 0 \qquad \Rightarrow \qquad (4-x)^3 < 0$$

also, $4 - x \to 0$ as $x \to 4$. So, we have a positive constant divided by an increasingly small negative number. The results will be an increasingly large negative number and so it looks like the right-hand limit will be negative infinity.

For the left-handed limit we have,

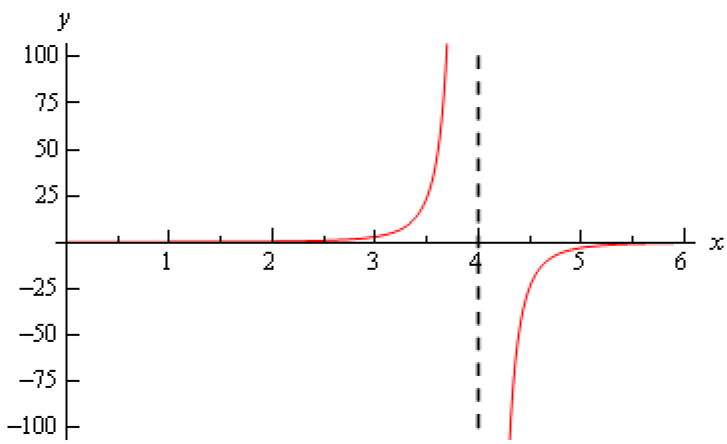
$$x < 4 \qquad \Rightarrow \qquad 4 - x > 0 \qquad \Rightarrow \qquad (4-x)^3 > 0$$

and we still have, $4 - x \to 0$ as $x \to 4$. In this case we have a positive constant divided by an increasingly small positive number. The results will be an increasingly large positive number and so it looks like the right-hand limit will be positive infinity.

The normal limit will not exist since the two one-sided limits are not the same. The official answers to this example are then,

$$\lim_{x \to 4^+} \frac{3}{(4-x)^3} = -\infty \qquad \lim_{x \to 4^-} \frac{3}{(4-x)^3} = \infty \qquad \lim_{x \to 4} \frac{3}{(4-x)^3} \text{ doesn't exist}$$

Here is a quick sketch to verify our limits.



All the examples to this point have had a constant in the numerator and we should probably take a quick look at an example that doesn't have a constant in the numerator.

Example 5 Evaluate each of the following limits.

$$\lim_{x\to 3^+} \frac{2x}{x-3} \qquad \lim_{x\to 3^-} \frac{2x}{x-3} \qquad \lim_{x\to 3} \frac{2x}{x-3}$$

Solution

Let's take a look at the right-handed limit first. For this limit we'll have,

$$x > 3 \qquad \Rightarrow \qquad x-3 > 0$$

The main difference here with this example is the behavior of the numerator as we let *x* get closer and closer to 3. In this case we have the following behavior for both the numerator and denominator.

$$x-3\to 0 \;\text{ and }\; 2x\to 6 \;\text{ as }\; x\to 3$$

So, as we let *x* get closer and closer to 3 (always staying on the right of course) the numerator, while not a constant, is getting closer and closer to a positive constant while the denominator is getting closer and closer to zero, and will be positive since we are on the right side.

This means that we'll have a numerator that is getting closer and closer to a non-zero and positive constant divided by an increasingly smaller positive number and so the result should be an increasingly larger positive number. The right-hand limit should then be positive infinity.

For the left-hand limit we'll have,

$$x < 3 \qquad \Rightarrow \qquad x-3 < 0$$

As with the right-hand limit we'll have the following behaviors for the numerator and the denominator,

$$x-3\to 0 \;\text{ and }\; 2x\to 6 \;\text{ as }\; x\to 3$$

The main difference in this case is that the denominator will now be negative. So, we'll have a numerator that is approaching a positive, non-zero constant divided by an increasingly small negative number. The result will be an increasingly large and negative number.

The formal answers for this example are then,

$$\lim_{x\to 3^+} \frac{2x}{x-3} = \infty \qquad \lim_{x\to 3^-} \frac{2x}{x-3} = -\infty \qquad \lim_{x\to 3} \frac{2x}{x-3} \text{ doesn't exist}$$

As with most of the examples in this section the normal limit does not exist since the two one-sided limits are not the same.

Here's a quick graph to verify our limits.

Derivatives

Introduction

In this chapter we will start looking at the next major topic in a calculus class. We will be looking at derivatives in this chapter (as well as the next chapter). This chapter is devoted almost exclusively to finding derivatives. We will be looking at one application of them in this chapter. We will be leaving most of the applications of derivatives to the next chapter.

Here is a listing of the topics covered in this chapter.

The Definition of the Derivative **–** In this section we will be looking at the definition of the derivative.

Interpretation of the Derivative – Here we will take a quick look at some interpretations of the derivative.

Differentiation Formulas – Here we will start introducing some of the differentiation formulas used in a calculus course.

Product and Quotient Rule – In this section we will took at differentiating products and quotients of functions.

Derivatives of Trig Functions – We'll give the derivatives of the trig functions in this section.

Derivatives of Exponential and Logarithm Functions – In this section we will get the derivatives of the exponential and logarithm functions.

Derivatives of Inverse Trig Functions – Here we will look at the derivatives of inverse trig functions.

Derivatives of Hyperbolic Functions – Here we will look at the derivatives of hyperbolic functions.

Chain Rule – The Chain Rule is one of the more important differentiation rules and will allow us to differentiate a wider variety of functions. In this section we will take a look at it.

Implicit Differentiation – In this section we will be looking at implicit differentiation. Without this we won't be able to work some of the applications of derivatives.

Related Rates – In this section we will look at the lone application to derivatives in this chapter. This topic is here rather than the next chapter because it will help to cement in our minds one of the more important concepts about derivatives and because it requires implicit differentiation.

Higher Order Derivatives – Here we will introduce the idea of higher order derivatives.

Logarithmic Differentiation – The topic of logarithmic differentiation is not always presented in a standard calculus course. It is presented here for those how are interested in seeing how it is done and the types of functions on which it can be used.

The Definition of the Derivative

In the first section of the last chapter we saw that the computation of the slope of a tangent line, the instantaneous rate of change of a function, and the instantaneous velocity of an object at $x = a$ all required us to compute the following limit.

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

We also saw that with a small change of notation this limit could also be written as,

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \qquad (3)$$

This is such an important limit and it arises in so many places that we give it a name. We call it a **derivative**. Here is the official definition of the derivative.

Definition

The **derivative of** $f(x)$ **with respect to *x*** is the function $f'(x)$ and is defined as,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad (4)$$

Note that we replaced all the *a*'s in (1) with *x*'s to acknowledge the fact that the derivative is really a function as well. We often "read" $f'(x)$ as "*f* prime of *x*".

Let's compute a couple of derivatives using the definition.

Example 1 Find the derivative of the following function using the definition of the derivative.

$$f(x) = 2x^2 - 16x + 35$$

Solution

So, all we really need to do is to plug this function into the definition of the derivative, (1), and do some algebra. While, admittedly, the algebra will get somewhat unpleasant at times, but it's just algebra so don't get excited about the fact that we're now computing derivatives.

First plug the function into the definition of the derivative.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{2(x+h)^2 - 16(x+h) + 35 - (2x^2 - 16x + 35)}{h} \end{aligned}$$

Be careful and make sure that you properly deal with parenthesis when doing the subtracting.

Now, we know from the previous chapter that we can't just plug in $h = 0$ since this will give us a division by zero error. So we are going to have to do some work. In this case that means multiplying everything out and distributing the minus sign through on the second term. Doing this gives,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{2x^2+4xh+2h^2-16x-16h+35-2x^2+16x-35}{h} \\ &= \lim_{h\to 0} \frac{4xh+2h^2-16h}{h} \end{aligned}$$

Notice that every term in the numerator that didn't have an *h* in it canceled out and we can now factor an *h* out of the numerator which will cancel against the *h* in the denominator. After that we can compute the limit.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{h(4x+2h-16)}{h} \\ &= \lim_{h\to 0} 4x+2h-16 \\ &= 4x-16 \end{aligned}$$

So, the derivative is,

$$f'(x) = 4x-16$$

Example 2 Find the derivative of the following function using the definition of the derivative.

$$g(t) = \frac{t}{t+1}$$

Solution

This one is going to be a little messier as far as the algebra goes. However, outside of that it will work in exactly the same manner as the previous examples. First, we plug the function into the definition of the derivative,

$$\begin{aligned} g'(t) &= \lim_{h\to 0} \frac{g(t+h)-g(t)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{t+h}{t+h+1}-\frac{t}{t+1}\right) \end{aligned}$$

Note that we changed all the letters in the definition to match up with the given function. Also note that we wrote the fraction a much more compact manner to help us with the work.

As with the first problem we can't just plug in $h=0$. So we will need to simplify things a little. In this case we will need to combine the two terms in the numerator into a single rational expression as follows.

$$g'(t) = \lim_{h \to 0} \frac{1}{h}\left(\frac{(t+h)(t+1) - t(t+h+1)}{(t+h+1)(t+1)}\right)$$
$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{t^2 + t + th + h - (t^2 + th + t)}{(t+h+1)(t+1)}\right)$$
$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{h}{(t+h+1)(t+1)}\right)$$

Before finishing this let's note a couple of things. First, we didn't multiply out the denominator. Multiplying out the denominator will just overly complicate things so let's keep it simple. Next, as with the first example, after the simplification we only have terms with *h*'s in them left in the numerator and so we can now cancel an *h* out.

So, upon canceling the *h* we can evaluate the limit and get the derivative.

$$g'(t) = \lim_{h \to 0} \frac{1}{(t+h+1)(t+1)}$$
$$= \frac{1}{(t+1)(t+1)}$$
$$= \frac{1}{(t+1)^2}$$

The derivative is then,

$$g'(t) = \frac{1}{(t+1)^2}$$

Example 3 Find the derivative of the following function using the derivative.

$$R(z) = \sqrt{5z-8}$$

Solution

First plug into the definition of the derivative as we've done with the previous two examples.

$$R'(z) = \lim_{h \to 0} \frac{R(z+h) - R(z)}{h}$$
$$= \lim_{h \to 0} \frac{\sqrt{5(z+h)-8} - \sqrt{5z-8}}{h}$$

In this problem we're going to have to rationalize the numerator. You do remember rationalization from an Algebra class right? In an Algebra class you probably only rationalized the denominator, but you can also rationalize numerators. Remember that in rationalizing the numerator (in this case) we multiply both the numerator and denominator by the numerator except we change the sign between the two terms. Here's the rationalizing work for this problem,

$$R'(z) = \lim_{h\to 0} \frac{\left(\sqrt{5(z+h)-8} - \sqrt{5z-8}\right)}{h} \frac{\left(\sqrt{5(z+h)-8} + \sqrt{5z-8}\right)}{\left(\sqrt{5(z+h)-8} + \sqrt{5z-8}\right)}$$

$$= \lim_{h\to 0} \frac{5z+5h-8-(5z-8)}{h\left(\sqrt{5(z+h)-8} + \sqrt{5z-8}\right)}$$

$$= \lim_{h\to 0} \frac{5h}{h\left(\sqrt{5(z+h)-8} + \sqrt{5z-8}\right)}$$

Again, after the simplification we have only *h*'s left in the numerator. So, cancel the *h* and evaluate the limit.

$$R'(z) = \lim_{h\to 0} \frac{5}{\sqrt{5(z+h)-8} + \sqrt{5z-8}}$$

$$= \frac{5}{\sqrt{5z-8} + \sqrt{5z-8}}$$

$$= \frac{5}{2\sqrt{5z-8}}$$

And so we get a derivative of,

$$R'(z) = \frac{5}{2\sqrt{5z-8}}$$

Let's work one more example. This one will be a little different, but it's got a point that needs to be made.

Example 4 Determine $f'(0)$ for $f(x) = |x|$

Solution

Since this problem is asking for the derivative at a specific point we'll go ahead and use that in our work. It will make our life easier and that's always a good thing.

So, plug into the definition and simplify.

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{|0+h| - |0|}{h}$$

$$= \lim_{h\to 0} \frac{|h|}{h}$$

We saw a situation like this back when we were looking at limits at infinity. As in that section we can't just cancel the h's. We will have to look at the two one sided limits and recall that

$$|h| = \begin{cases} h & \text{if } h \geq 0 \\ -h & \text{if } h < 0 \end{cases}$$

$$\begin{aligned} \lim_{h \to 0^-} \frac{|h|}{h} &= \lim_{h \to 0^-} \frac{-h}{h} && \text{because } h < 0 \text{ in a left-hand limit.} \\ &= \lim_{h \to 0^-} (-1) \\ &= -1 \end{aligned}$$

$$\begin{aligned} \lim_{h \to 0^+} \frac{|h|}{h} &= \lim_{h \to 0^+} \frac{h}{h} && \text{because } h > 0 \text{ in a right-hand limit.} \\ &= \lim_{h \to 0^+} 1 \\ &= 1 \end{aligned}$$

The two one-sided limits are different and so

$$\lim_{h \to 0} \frac{|h|}{h}$$

doesn't exist. However, this is the limit that gives us the derivative that we're after.

If the limit doesn't exist then the derivative doesn't exist either.

In this example we have finally seen a function for which the derivative doesn't exist at a point. This is a fact of life that we've got to be aware of. Derivatives will not always exist. Note as well that this doesn't say anything about whether or not the derivative exists anywhere else. In fact, the derivative of the absolute value function exists at every point except the one we just looked at, $x = 0$.

The preceding discussion leads to the following definition.

Definition

A function $f(x)$ is called **differentiable** at $x = a$ if $f'(x)$ exists and $f(x)$ is called differentiable on an interval if the derivative exists for each point in that interval.

The next theorem shows us a very nice relationship between functions that are continuous and those that are differentiable.

Theorem

If $f(x)$ is differentiable at $x = a$ then $f(x)$ is continuous at $x = a$.

See the Proof of Various Derivative Formulas section of the Extras chapter to see the proof of this theorem.

Note that this theorem does not work in reverse. Consider $f(x) = |x|$ and take a look at,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} |x| = 0 = f(0)$$

So, $f(x) = |x|$ is continuous at $x = 0$ but we've just shown above in Example 4 that $f(x) = |x|$ is not differentiable at $x = 0$.

Alternate Notation

Next we need to discuss some alternate notation for the derivative. The typical derivative notation is the "prime" notation. However, there is another notation that is used on occasion so let's cover that.

Given a function $y = f(x)$ all of the following are equivalent and represent the derivative of $f(x)$ with respect to *x*.

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}\left(f(x)\right) = \frac{d}{dx}(y)$$

Because we also need to evaluate derivatives on occasion we also need a notation for evaluating derivatives when using the fractional notation. So if we want to evaluate the derivative at *x*=*a* all of the following are equivalent.

$$f'(a) = y'\big|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{dy}{dx}\right|_{x=a}$$

Note as well that on occasion we will drop the *(x)* part on the function to simplify the notation somewhat. In these cases the following are equivalent.

$$f'(x) = f'$$

As a final note in this section we'll acknowledge that computing most derivatives directly from the definition is a fairly complex (and sometimes painful) process filled with opportunities to make mistakes. In a couple of section we'll start developing formulas and/or properties that will help us to take the derivative of many of the common functions so we won't need to resort to the definition of the derivative too often.

This does not mean however that it isn't important to know the definition of the derivative! It is an important definition that we should always know and keep in the back of our minds. It is just something that we're not going to be working with all that much.

Differentiation Formulas

In the first section of this chapter we saw the definition of the derivative and we computed a couple of derivatives using the definition. As we saw in those examples there was a fair amount of work involved in computing the limits and the functions that we worked with were not terribly complicated.

For more complex functions using the definition of the derivative would be an almost impossible task. Luckily for us we won't have to use the definition terribly often. We will have to use it on occasion, however we have a large collection of formulas and properties that we can use to simplify our life considerably and will allow us to avoid using the definition whenever possible.

We will introduce most of these formulas over the course of the next several sections. We will start in this section with some of the basic properties and formulas. We will give the properties and formulas in this section in both "prime" notation and "fraction" notation.

Properties

1) $\left(f(x)\pm g(x)\right)' = f'(x)\pm g'(x)$ **OR** $\dfrac{d}{dx}\left(f(x)\pm g(x)\right) = \dfrac{df}{dx}\pm\dfrac{dg}{dx}$

In other words, to differentiate a sum or difference all we need to do is differentiate the individual terms and then put them back together with the appropriate signs. Note as well that this property is not limited to two functions.

See the Proof of Various Derivative Formulas section of the Extras chapter to see the proof of this property. It's a very simple proof using the definition of the derivative.

2) $\left(cf(x)\right)' = cf'(x)$ **OR** $\dfrac{d}{dx}\left(cf(x)\right) = c\dfrac{df}{dx}$, c is any number

In other words, we can "factor" a multiplicative constant out of a derivative if we need to. See the Proof of Various Derivative Formulas section of the Extras chapter to see the proof of this property.

Note that we have not included formulas for the derivative of products or quotients of two functions here. The derivative of a product or quotient of two functions is not the product or quotient of the derivatives of the individual pieces. We will take a look at these in the next section.

Next, let's take a quick look at a couple of basic "computation" formulas that will allow us to actually compute some derivatives.

Formulas

1) If $f(x) = c$ then $f'(x) = 0$ **OR** $\frac{d}{dx}(c) = 0$

The derivative of a constant is zero. See the [Proof of Various Derivative Formulas] section of the Extras chapter to see the proof of this formula.

2) If $f(x) = x^n$ then $f'(x) = nx^{n-1}$ **OR** $\frac{d}{dx}(x^n) = nx^{n-1}$, n is any number.

This formula is sometimes called the **power rule**. All we are doing here is bringing the original exponent down in front and multiplying and then subtracting one from the original exponent.

Note as well that in order to use this formula n must be a number, it can't be a variable. Also note that the base, the x, must be a variable, it can't be a number. It will be tempting in some later sections to misuse the Power Rule when we run in some functions where the exponent isn't a number and/or the base isn't a variable.

See the [Proof of Various Derivative Formulas] section of the Extras chapter to see the proof of this formula. There are actually three different proofs in this section. The first two restrict the formula to n being an integer because at this point that is all that we can do at this point. The third proof is for the general rule, but does suppose that you've read most of this chapter.

These are the only properties and formulas that we'll give in this section. Let's do compute some derivatives using these properties.

Example 1 Differentiate each of the following functions.

(a) $f(x) = 15x^{100} - 3x^{12} + 5x - 46$ [Solution]

(b) $g(t) = 2t^6 + 7t^{-6}$ [Solution]

(c) $y = 8z^3 - \frac{1}{3z^5} + z - 23$ [Solution]

(d) $T(x) = \sqrt{x} + 9\sqrt[3]{x^7} - \frac{2}{\sqrt[5]{x^2}}$ [Solution]

(e) $h(x) = x^{\pi} - x^{\sqrt{2}}$ [Solution]

Solution

(a) $f(x) = 15x^{100} - 3x^{12} + 5x - 46$

In this case we have the sum and difference of four terms and so we will differentiate each of the terms using the first property from above and then put them back together with the proper sign. Also, for each term with a multiplicative constant remember that all we need to do is "factor" the constant out (using the second property) and then do the derivative.

$$f'(x) = 15(100)x^{99} - 3(12)x^{11} + 5(1)x^{0} - 0$$
$$= 1500x^{99} - 36x^{11} + 5$$

Notice that in the third term the exponent was a one and so upon subtracting 1 from the original exponent we get a new exponent of zero. Now recall that $x^0 = 1$. Don't forget to do any basic arithmetic that needs to be done such as any multiplication and/or division in the coefficients.

[Return to Problems]

(b) $g(t) = 2t^6 + 7t^{-6}$

The point of this problem is to make sure that you deal with negative exponents correctly. Here is the derivative.

$$g'(t) = 2(6)t^5 + 7(-6)t^{-7}$$
$$= 12t^5 - 42t^{-7}$$

Make sure that you correctly deal with the exponents in these cases, especially the negative exponents. It is an easy mistake to "go the other way" when subtracting one off from a negative exponent and get $-6t^{-5}$ instead of the correct $-6t^{-7}$.

[Return to Problems]

(c) $y = 8z^3 - \dfrac{1}{3z^5} + z - 23$

Now in this function the second term is not correctly set up for us to use the power rule. The power rule requires that the term be a variable to a power only and the term must be in the numerator. So, prior to differentiating we first need to rewrite the second term into a form that we can deal with.

$$y = 8z^3 - \frac{1}{3}z^{-5} + z - 23$$

Note that we left the 3 in the denominator and only moved the variable up to the numerator. Remember that the only thing that gets an exponent is the term that is immediately to the left of the exponent. If we'd wanted the three to come up as well we'd have written,

$$\frac{1}{(3z)^5}$$

so be careful with this! It's a very common mistake to bring the 3 up into the numerator as well at this stage.

Now that we've gotten the function rewritten into a proper form that allows us to use the Power Rule we can differentiate the function. Here is the derivative for this part.

$$y' = 24z^2 + \frac{5}{3}z^{-6} + 1$$

[Return to Problems]

(d) $T(x) = \sqrt{x} + 9\sqrt[3]{x^7} - \dfrac{2}{\sqrt[5]{x^2}}$

All of the terms in this function have roots in them. In order to use the power rule we need to first convert all the roots to fractional exponents. Again, remember that the Power Rule requires us to have a variable to a number and that it must be in the numerator of the term. Here is the function written in "proper" form.

$$\begin{aligned} T(x) &= x^{\frac{1}{2}} + 9\left(x^7\right)^{\frac{1}{3}} - \frac{2}{\left(x^2\right)^{\frac{1}{5}}} \\ &= x^{\frac{1}{2}} + 9x^{\frac{7}{3}} - \frac{2}{x^{\frac{2}{5}}} \\ &= x^{\frac{1}{2}} + 9x^{\frac{7}{3}} - 2x^{-\frac{2}{5}} \end{aligned}$$

In the last two terms we combined the exponents. You should always do this with this kind of term. In a later section we will learn of a technique that would allow us to differentiate this term without combining exponents, however it will take significantly more work to do. Also don't forget to move the term in the denominator of the third term up to the numerator. We can now differentiate the function.

$$\begin{aligned} T'(x) &= \frac{1}{2}x^{-\frac{1}{2}} + 9\left(\frac{7}{3}\right)x^{\frac{4}{3}} - 2\left(-\frac{2}{5}\right)x^{-\frac{7}{5}} \\ &= \frac{1}{2}x^{-\frac{1}{2}} + \frac{63}{3}x^{\frac{4}{3}} + \frac{4}{5}x^{-\frac{7}{5}} \end{aligned}$$

Make sure that you can deal with fractional exponents. You will see a lot of them in this class.

[Return to Problems]

(e) $h(x) = x^{\pi} - x^{\sqrt{2}}$

In all of the previous examples the exponents have been nice integers or fractions. That is usually what we'll see in this class. However, the exponent only needs to be a number so don't get excited about problems like this one. They work exactly the same.

$$h'(x) = \pi x^{\pi-1} - \sqrt{2}x^{\sqrt{2}-1}$$

The answer is a little messy and we won't reduce the exponents down to decimals. However, this problem is not terribly difficult it just looks that way initially.

[Return to Problems]

There is a general rule about derivatives in this class that you will need to get into the habit of using. When you see radicals you should always first convert the radical to a fractional exponent and then simplify exponents as much as possible. Following this rule will save you a lot of grief in the future.

Back when we first put down the properties we noted that we hadn't included a property for products and quotients. That doesn't mean that we can't differentiate any product or quotient at this point. There are some that we can do.

Example 2 Differentiate each of the following functions.

(a) $y = \sqrt[3]{x^2}\left(2x - x^2\right)$ [Solution]

(b) $h\left(t\right) = \dfrac{2t^5 + t^2 - 5}{t^2}$ [Solution]

Solution

(a) $y = \sqrt[3]{x^2}\left(2x - x^2\right)$

In this function we can't just differentiate the first term, differentiate the second term and then multiply the two back together. That just won't work. We will discuss this in detail in the next section so if you're not sure you believe that hold on for a bit and we'll be looking at that soon as well as showing you an example of what it won't work.

It is still possible to do this derivative however. All that we need to do is convert the radical to fractional exponents (as we should anyway) and then multiply this through the parenthesis.

$$y = x^{\frac{2}{3}}\left(2x - x^2\right) = 2x^{\frac{5}{3}} - x^{\frac{8}{3}}$$

Now we can differentiate the function.

$$y' = \frac{10}{3}x^{\frac{2}{3}} - \frac{8}{3}x^{\frac{5}{3}}$$

[Return to Problems]

(b) $h\left(t\right) = \dfrac{2t^5 + t^2 - 5}{t^2}$

As with the first part we can't just differentiate the numerator and the denominator and the put it back together as a fraction. Again, if you're not sure you believe this hold on until the next section and we'll take a more detailed look at this.

We can simplify this rational expression however as follows.

$$h\left(t\right) = \frac{2t^5}{t^2} + \frac{t^2}{t^2} - \frac{5}{t^2} = 2t^3 + 1 - 5t^{-2}$$

This is a function that we can differentiate.

$$h'\left(t\right) = 6t^2 + 10t^{-3}$$

[Return to Problems]

So, as we saw in this example there are a few products and quotients that we can differentiate. If we can first do some simplification the functions will sometimes simplify into a form that can be differentiated using the properties and formulas in this section.

Before moving on to the next section let's work a couple of examples to remind us once again of some of the interpretations of the derivative.

Example 3 Is $f(x) = 2x^3 + \frac{300}{x^3} + 4$ increasing, decreasing or not changing at $x = -2$?

Solution

We know that the rate of change of a function is given by the functions derivative so all we need to do is it rewrite the function (to deal with the second term) and then take the derivative.

$$f(x) = 2x^3 + 300x^{-3} + 4 \qquad \Rightarrow \qquad f'(x) = 6x^2 - 900x^{-4} = 6x^2 - \frac{900}{x^4}$$

Note that we rewrote the last term in the derivative back as a fraction. This is not something we've done to this point and is only being done here to help with the evaluation in the next step. It's often easier to do the evaluation with positive exponents.

So, upon evaluating the derivative we get

$$f'(-2) = 6(4) - \frac{900}{32} = -\frac{129}{4} = -32.25$$

So, at $x = -2$ the derivative is negative and so the function is decreasing at $x = -2$.

Example 4 Find the equation of the tangent line to $f(x) = 4x - 8\sqrt{x}$ at $x = 16$.

Solution

We know that the equation of a tangent line is given by,

$$y = f(a) + f'(a)(x - a)$$

So, we will need the derivative of the function (don't forget to get rid of the radical).

$$f(x) = 4x - 8x^{\frac{1}{2}} \qquad \Rightarrow \qquad f'(x) = 4 - 4x^{-\frac{1}{2}} = 4 - \frac{4}{x^{\frac{1}{2}}}$$

Again, notice that we eliminated the negative exponent in the derivative solely for the sake of the evaluation. All we need to do then is evaluate the function and the derivative at the point in question, $x = 16$.

$$f(16) = 64 - 8(4) = 32 \qquad\qquad f'(x) = 4 - \frac{4}{4} = 3$$

The tangent line is then,

$$y = 32 + 3(x - 16) = 3x - 16$$

Example 5 The position of an object at any time *t* (in hours) is given by,

$$s(t) = 2t^3 - 21t^2 + 60t - 10$$

Determine when the object is moving to the right and when the object is moving to the left.

Solution

The only way that we'll know for sure which direction the object is moving is to have the velocity in hand. Recall that if the velocity is positive the object is moving off to the right and if the velocity is negative then the object is moving to the left.

So, we need the derivative since the derivative is the velocity of the object. The derivative is,

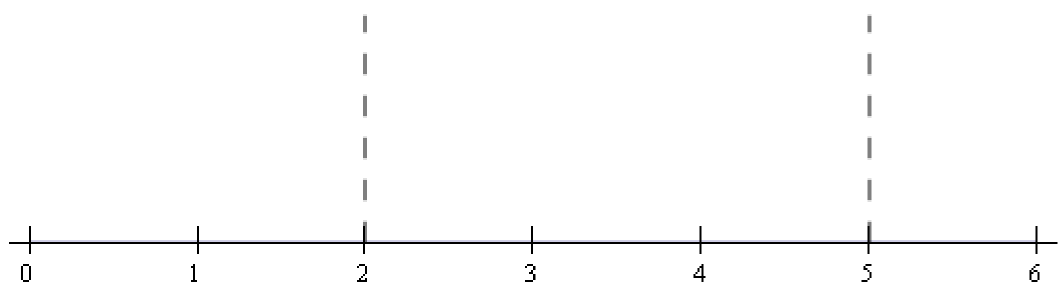
$$s'(t) = 6t^2 - 42t + 60 = 6(t^2 - 7t + 10) = 6(t-2)(t-5)$$

The reason for factoring the derivative will be apparent shortly.

Now, we need to determine where the derivative is positive and where the derivative is negative. There are several ways to do this. The method that I tend to prefer is the following.

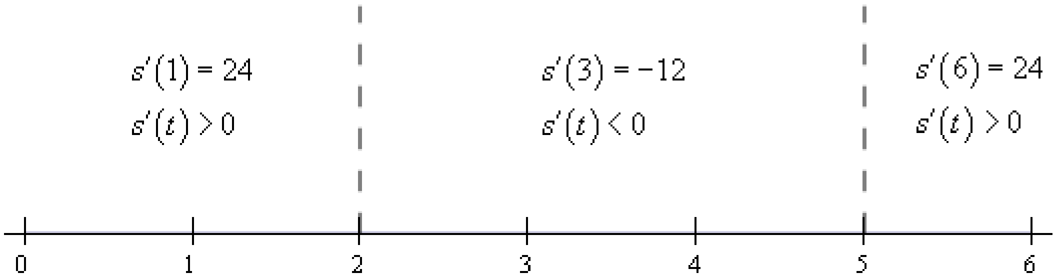
Since polynomials are continuous we know from the Intermediate Value Theorem that if the polynomial ever changes sign then it must have first gone through zero. So, if we knew where the derivative was zero we would know the only points where the derivative *might* change sign.

We can see from the factored form of the derivative that the derivative will be zero at $t = 2$ and $t = 5$. Let's graph these points on a number line.



Now, we can see that these two points divide the number line into three distinct regions. In reach of these regions we **know** that the derivative will be the same sign. Recall the derivative can only change sign at the two points that are used to divide the number line up into the regions.

Therefore, all that we need to do is to check the derivative at a test point in each region and the derivative in that region will have the same sign as the test point. Here is the number line with the test points and results shown.



Here are the intervals in which the derivative is positive and negative.

$$\begin{aligned} \text{positive} &: \quad -\infty < t < 2 \quad \& \quad 5 < t < \infty \\ \text{negative} &: \quad 2 < t < 5 \end{aligned}$$

We included negative *t*'s here because we could even though they may not make much sense for this problem. Once we know this we also can answer the question. The object is moving to the right and left in the following intervals.

$$\begin{aligned} \text{moving to the right} &: \quad -\infty < t < 2 \quad \& \quad 5 < t < \infty \\ \text{moving to the left} &: \quad 2 < t < 5 \end{aligned}$$

Make sure that you can do the kind of work that we just did in this example. You will be asked numerous times over the course of the next two chapters to determine where functions are positive and/or negative. If you need some review or want to practice these kinds of problems you should check out the Solving Inequalities section of my Algebra/Trig Review.

Product and Quotient Rule

In the previous section we noted that we had to be careful when differentiating products or quotients. It's now time to look at products and quotients and see why.

First let's take a look at why we have to be careful with products and quotients. Suppose that we have the two functions $f\left(x\right)=x^{3}$ and $g\left(x\right)=x^{6}$. Let's start by computing the derivative of the product of these two functions. This is easy enough to do directly.

$$\left(f\,g\right)'=\left(x^{3}x^{6}\right)'=\left(x^{9}\right)'=9x^{8}$$

Remember that on occasion we will drop the *(x)* part on the functions to simplify notation somewhat. We've done that in the work above.

Now, let's try the following.

$$f'\left(x\right)g'\left(x\right)=\left(3x^{2}\right)\left(6x^{5}\right)=18x^{7}$$

So, we can very quickly see that.

$$\left(f\,g\right)'\neq f'\,g'$$

In other words, the derivative of a product is not the product of the derivatives.

Using the same functions we can do the same thing for quotients.

$$\left(\frac{f}{g}\right)'=\left(\frac{x^{3}}{x^{6}}\right)'=\left(\frac{1}{x^{3}}\right)'=\left(x^{-3}\right)'=-3x^{-4}=-\frac{3}{x^{4}}$$

$$\frac{f'\left(x\right)}{g'\left(x\right)}=\frac{3x^{2}}{6x^{5}}=\frac{1}{2x^{3}}$$

So, again we can see that,

$$\left(\frac{f}{g}\right)'\neq\frac{f'}{g'}$$

To differentiate products and quotients we have the **Product Rule** and the **Quotient Rule**.

Product Rule

If the two functions *f(x)* and *g(x)* are differentiable (*i.e.* the derivative exist) then the product is differentiable and,

$$\left(f\,g\right)'=f'\,g+f\,g'$$

The proof of the Product Rule is shown in the Proof of Various Derivative Formulas section of the Extras chapter.

Quotient Rule

If the two functions *f(x)* and *g(x)* are differentiable (*i.e.* the derivative exist) then the quotient is differentiable and,

$$\left(\frac{f}{g}\right)' = \frac{f'\,g - f\,g'}{g^2}$$

Note that the numerator of the quotient rule is very similar to the product rule so be careful to not mix the two up!

The proof of the Product Rule is shown in the Proof of Various Derivative Formulas section of the Extras chapter.

Let's do a couple of examples of the product rule.

Example 1 Differentiate each of the following functions.

(a) $y = \sqrt[3]{x^2}\left(2x - x^2\right)$ [Solution]

(b) $f\left(x\right) = \left(6x^3 - x\right)\left(10 - 20x\right)$ [Solution]

Solution

At this point there really aren't a lot of reasons to use the product rule. As we noted in the previous section all we would need to do for either of these is to just multiply out the product and then differentiate.

With that said we will use the product rule on these so we can see an example or two. As we add more functions to our repertoire and as the functions become more complicated the product rule will become more useful and in many cases required.

(a) $y = \sqrt[3]{x^2}\left(2x - x^2\right)$

Note that we took the derivative of this function in the previous section and didn't use the product rule at that point. We should however get the same result here as we did then.

Now let's do the problem here. There's not really a lot to do here other than use the product rule. However, before doing that we should convert the radical to a fractional exponent as always.

$$y = x^{\frac{2}{3}}\left(2x - x^2\right)$$

Now let's take the derivative. So we take the derivative of the first function times the second then add on to that the first function times the derivative of the second function.

$$y' = \frac{2}{3}x^{-\frac{1}{3}}\left(2x - x^2\right) + x^{\frac{2}{3}}\left(2 - 2x\right)$$

This is NOT what we got in the previous section for this derivative. However, with some simplification we can arrive at the same answer.

$$y' = \frac{4}{3}x^{\frac{2}{3}} - \frac{2}{3}x^{\frac{5}{3}} + 2x^{\frac{2}{3}} - 2x^{\frac{5}{3}} = \frac{10}{3}x^{\frac{2}{3}} - \frac{8}{3}x^{\frac{5}{3}}$$

This is what we got for an answer in the previous section so that is a good check of the product rule.

[Return to Problems]

(b) $f(x) = (6x^3 - x)(10 - 20x)$

This one is actually easier than the previous one. Let's just run it through the product rule.

$$\begin{aligned} f'(x) &= (18x^2 - 1)(10 - 20x) + (6x^3 - x)(-20) \\ &= -480x^3 + 180x^2 + 40x - 10 \end{aligned}$$

Since it was easy to do we went ahead and simplified the results a little.

[Return to Problems]

Let's now work an example or two with the quotient rule. In this case, unlike the product rule examples, a couple of these functions will require the quotient rule in order to get the derivative. The last two however, we can avoid the quotient rule if we'd like to as we'll see.

Example 2 Differentiate each of the following functions.

(a) $W(z) = \dfrac{3z + 9}{2 - z}$ [Solution]

(b) $h(x) = \dfrac{4\sqrt{x}}{x^2 - 2}$ [Solution]

(c) $f(x) = \dfrac{4}{x^6}$ [Solution]

(d) $y = \dfrac{w^6}{5}$ [Solution]

Solution

(a) $W(z) = \dfrac{3z + 9}{2 - z}$

There isn't a lot to do here other than to use the quotient rule. Here is the work for this function.

$$\begin{aligned} W'(z) &= \frac{3(2 - z) - (3z + 9)(-1)}{(2 - z)^2} \\ &= \frac{15}{(2 - z)^2} \end{aligned}$$

[Return to Problems]

(b) $h(x) = \dfrac{4\sqrt{x}}{x^2 - 2}$

Again, not much to do here other than use the quotient rule. Don't forget to convert the square root into a fractional exponent.

$$\begin{aligned} h'(x) &= \frac{4\left(\frac{1}{2}\right)x^{-\frac{1}{2}}\left(x^2-2\right) - 4x^{\frac{1}{2}}\left(2x\right)}{\left(x^2-2\right)^2} \\ &= \frac{2x^{\frac{3}{2}} - 4x^{-\frac{1}{2}} - 8x^{\frac{3}{2}}}{\left(x^2-2\right)^2} \\ &= \frac{-6x^{\frac{3}{2}} - 4x^{-\frac{1}{2}}}{\left(x^2-2\right)^2} \end{aligned}$$

[Return to Problems]

(c) $f(x) = \dfrac{4}{x^6}$

It seems strange to have this one here rather than being the first part of this example given that it definitely appears to be easier than any of the previous two. In fact, it is easier. There is a point to doing it here rather than first. In this case there are two ways to do compute this derivative. There is an easy way and a hard way and in this case the hard way is the quotient rule. That's the point of this example.

Let's do the quotient rule and see what we get.

$$f'(x) = \frac{(0)\left(x^6\right) - 4\left(6x^5\right)}{\left(x^6\right)^2} = \frac{-24x^5}{x^{12}} = -\frac{24}{x^7}$$

Now, that was the "hard" way. So, what was so hard about it? Well actually it wasn't that hard, there is just an easier way to do it that's all. However, having said that, a common mistake here is to do the derivative of the numerator (a constant) incorrectly. For some reason many people will give the derivative of the numerator in these kinds of problems as a 1 instead of 0! Also, there is some simplification that needs to be done in these kinds of problems if you do the quotient rule.

The easy way is to do what we did in the previous section.

$$f'(x) = 4x^{-6} = -24x^{-7} = -\frac{24}{x^7}$$

Either way will work, but I'd rather take the easier route if I had the choice.

[Return to Problems]

(d) $y = \frac{w^6}{5}$

This problem also seems a little out of place. However, it is here again to make a point. Do not confuse this with a quotient rule problem. While you can do the quotient rule on this function there is no reason to use the quotient rule on this. Simply rewrite the function as

$$y = \frac{1}{5} w^6$$

and differentiate as always.

$$y' = \frac{6}{5} w^5$$

[Return to Problems]

Finally, let's not forget about our applications of derivatives.

Example 3 Suppose that the amount of air in a balloon at any time *t* is given by

$$V(t) = \frac{6\sqrt[3]{t}}{4t+1}$$

Determine if the balloon is being filled with air or being drained of air at $t = 8$.

Solution

If the balloon is being filled with air then the volume is increasing and if it's being drained of air then the volume will be decreasing. In other words, we need to get the derivative so that we can determine the rate of change of the volume at $t = 8$.

This will require the quotient rule.

$$V'(t) = \frac{2t^{-\frac{2}{3}}(4t+1) - 6t^{\frac{1}{3}}(4)}{(4t+1)^2}$$

$$= \frac{-16t^{\frac{1}{3}} + 2t^{-\frac{2}{3}}}{(4t+1)^2}$$

$$= \frac{-16t^{\frac{1}{3}} + \frac{2}{t^{\frac{2}{3}}}}{(4t+1)^2}$$

Note that we simplified the numerator more than usual here. This was only done to make the derivative easier to evaluate.

The rate of change of the volume at $t = 8$ is then,

$$V'(8) = \frac{-16(2) + \frac{2}{4}}{(33)^2} \qquad (8)^{\frac{1}{3}} = 2 \qquad (8)^{\frac{2}{3}} = \left((8)^{\frac{1}{3}}\right)^2 = (2)^2 = 4$$
$$= -\frac{63}{2178}$$

So, the rate of change of the volume at $t = 8$ is negative and so the volume must be decreasing. Therefore air is being drained out of the balloon at $t = 8$.

As a final topic let's note that the product rule can be extended to more than two functions, for instance.

$$(f\,g\,h)' = f'\,g\,h + f\,g'\,h + f\,g\,h'$$
$$(f\,g\,h\,w)' = f'\,g\,h\,w + f\,g'\,h\,w + f\,g\,h'\,w + f\,g\,h\,w'$$

With this section and the previous section we are now able to differentiate powers of *x* as well as sums, differences, products and quotients of these kinds of functions. However, there are many more functions out there in the world that are not in this form. The next few sections give many of these functions as well as give their derivatives.

Derivatives of Trig Functions

With this section we're going to start looking at the derivatives of functions other than polynomials or roots of polynomials. We'll start this process off by taking a look at the derivatives of the six trig functions. Two of the derivatives will be derived. The remaining four are left to the reader and will follow similar proofs for the two given here.

Before we actually get into the derivatives of the trig functions we need to give a couple of limits that will show up in the derivation of two of the derivatives.

Fact

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \qquad \lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0$$

See the Proof of Trig Limits section of the Extras chapter to see the proof of these two limits.

Before we start differentiating trig functions let's work a quick set of limit problems that this fact now allows us to do.

Example 1 Evaluate each of the following limits.

(a) $\lim\limits_{\theta \to 0} \dfrac{\sin \theta}{6\theta}$ [Solution]

(b) $\lim\limits_{x \to 0} \dfrac{\sin(6x)}{x}$ [Solution]

(c) $\lim\limits_{x \to 0} \dfrac{x}{\sin(7x)}$ [Solution]

(d) $\lim\limits_{t \to 0} \dfrac{\sin(3t)}{\sin(8t)}$ [Solution]

(e) $\lim\limits_{x \to 4} \dfrac{\sin(x-4)}{x-4}$ [Solution]

(f) $\lim\limits_{z \to 0} \dfrac{\cos(2z)-1}{z}$ [Solution]

Solution

(a) $\lim\limits_{\theta \to 0} \dfrac{\sin \theta}{6\theta}$

There really isn't a whole lot to this limit. In fact, it's only here to contrast with the next example so you can see the difference in how these work. In this case since there is only a 6 in the denominator we'll just factor this out and then use the fact.

$$\lim_{\theta \to 0} \frac{\sin \theta}{6\theta} = \frac{1}{6} \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = \frac{1}{6}(1) = 1$$

[Return to Problems]

(b) $\lim\limits_{x\to 0}\dfrac{\sin(6x)}{x}$

Now, in this case we can't factor the 6 out of the sine so we're stuck with it there and we'll need to figure out a way to deal with it. To do this problem we need to notice that in the fact the argument of the sine is the same as the denominator (*i.e.* both θ's). So we need to get both of the argument of the sine and the denominator to be the same. We can do this by multiplying the numerator and the denominator by 6 as follows.

$$\lim_{x\to 0}\frac{\sin(6x)}{x}=\lim_{x\to 0}\frac{6\sin(6x)}{6x}=6\lim_{x\to 0}\frac{\sin(6x)}{6x}$$

Note that we factored the 6 in the numerator out of the limit. At this point, while it may not look like it, we can use the fact above to finish the limit.

To see that we can use the fact on this limit let's do a **change of variables**. A change of variables is really just a renaming of portions of the problem to make something look more like something we know how to deal with. They can't always be done, but sometimes, such as this case, they can simplify the problem. The change of variables here is to let $\theta = 6x$ and then notice that as $x \to 0$ we also have $\theta \to 0$. When doing a change of variables in a limit we need to change all the *x*'s into θ's and that includes the one in the limit.

Doing the change of variables on this limit gives,

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(6x)}{x} &= 6\lim_{x\to 0}\frac{\sin(6x)}{6x} && \text{let } \theta = 6x\\ &= 6\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\\ &= 6(1)\\ &= 6\end{aligned}$$

And there we are. Note that we didn't really need to do a change of variables here. All we really need to notice is that the argument of the sine is the same as the denominator and then we can use the fact. A change of variables, in this case, is really only needed to make it clear that the fact does work.

[Return to Problems]

(c) $\lim\limits_{x\to 0}\dfrac{x}{\sin(7x)}$

In this case we appear to have a small problem in that the function we're taking the limit of here is upside down compared to that in the fact. This is not the problem it appears to be once we notice that,

$$\frac{x}{\sin(7x)} = \frac{1}{\frac{\sin(7x)}{x}}$$

and then all we need to do is recall a nice property of limits that allows us to do ,

$$\begin{aligned}\lim_{x\to 0}\frac{x}{\sin(7x)} &= \lim_{x\to 0}\frac{1}{\frac{\sin(7x)}{x}}\\ &= \frac{\lim\limits_{x\to 0}1}{\lim\limits_{x\to 0}\frac{\sin(7x)}{x}}\\ &= \frac{1}{\lim\limits_{x\to 0}\frac{\sin(7x)}{x}}\end{aligned}$$

With a little rewriting we can see that we do in fact end up needing to do a limit like the one we did in the previous part. So, let's do the limit here and this time we won't bother with a change of variable to help us out. All we need to do is multiply the numerator and denominator of the fraction in the denominator by 7 to get things set up to use the fact. Here is the work for this limit.

$$\begin{aligned}\lim_{x\to 0}\frac{x}{\sin(7x)} &= \frac{1}{\lim\limits_{x\to 0}\frac{7\sin(7x)}{7x}}\\ &= \frac{1}{7\lim\limits_{x\to 0}\frac{\sin(7x)}{7x}}\\ &= \frac{1}{(7)(1)}\\ &= \frac{1}{7}\end{aligned}$$

[Return to Problems]

(d) $\lim\limits_{t\to 0}\frac{\sin(3t)}{\sin(8t)}$

This limit looks nothing like the limit in the fact, however it can be thought of as a combination of the previous two parts by doing a little rewriting. First, we'll split the fraction up as follows,

$$\lim_{t\to 0}\frac{\sin(3t)}{\sin(8t)} = \lim_{t\to 0}\frac{\sin(3t)}{1}\frac{1}{\sin(8t)}$$

Now, the fact wants a *t* in the denominator of the first and in the numerator of the second. This is easy enough to do if we multiply the whole thing by $\frac{t}{t}$ (which is just one after all and so won't change the problem) and then do a little rearranging as follows,

$$\begin{aligned}\lim_{t\to 0}\frac{\sin(3t)}{\sin(8t)} &= \lim_{t\to 0}\frac{\sin(3t)}{1}\frac{1}{\sin(8t)}\frac{t}{t}\\ &= \lim_{t\to 0}\frac{\sin(3t)}{t}\frac{t}{\sin(8t)}\\ &= \left(\lim_{t\to 0}\frac{\sin(3t)}{t}\right)\left(\lim_{t\to 0}\frac{t}{\sin(8t)}\right)\end{aligned}$$

At this point we can see that this really is two limits that we've seen before. Here is the work for each of these and notice on the second limit that we're going to work it a little differently than we did in the previous part. This time we're going to notice that it doesn't really matter whether the sine is in the numerator or the denominator as long as the argument of the sine is the same as what's in the numerator the limit is still one.

Here is the work for this limit.

$$\begin{aligned}\lim_{t\to 0}\frac{\sin(3t)}{\sin(8t)} &= \left(\lim_{t\to 0}\frac{3\sin(3t)}{3t}\right)\left(\lim_{t\to 0}\frac{8t}{8\sin(8t)}\right)\\ &= \left(3\lim_{t\to 0}\frac{\sin(3t)}{3t}\right)\left(\frac{1}{8}\lim_{t\to 0}\frac{8t}{\sin(8t)}\right)\\ &= (3)\left(\frac{1}{8}\right)\\ &= \frac{3}{8}\end{aligned}$$

[Return to Problems]

(e) $\lim_{x\to 4}\frac{\sin(x-4)}{x-4}$

This limit almost looks the same as that in the fact in the sense that the argument of the sine is the same as what is in the denominator. However, notice that, in the limit, x is going to 4 and not 0 as the fact requires. However, with a change of variables we can see that this limit is in fact set to use the fact above regardless.

So, let $\theta = x-4$ and then notice that as $x\to 4$ we have $\theta\to 0$. Therefore, after doing the change of variable the limit becomes,

$$\lim_{x\to 4}\frac{\sin(x-4)}{x-4} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

[Return to Problems]

(f) $\lim_{z \to 0} \frac{\cos(2z) - 1}{z}$

The previous parts of this example all used the sine portion of the fact. However, we could just have easily used the cosine portion so here is a quick example using the cosine portion to illustrate this. We'll not put in much explanation here as this really does work in the same manner as the sine portion.

$$\begin{aligned} \lim_{z \to 0} \frac{\cos(2z) - 1}{z} &= \lim_{z \to 0} \frac{2(\cos(2z) - 1)}{2z} \\ &= 2 \lim_{z \to 0} \frac{\cos(2z) - 1}{2z} \\ &= 2(0) \\ &0 \end{aligned}$$

All that is required to use the fact is that the argument of the cosine is the same as the denominator.

[Return to Problems]

Okay, now that we've gotten this set of limit examples out of the way let's get back to the main point of this section, differentiating trig functions.

We'll start with finding the derivative of the sine function. To do this we will need to use the definition of the derivative. It's been a while since we've had to use this, but sometimes there just isn't anything we can do about it. Here is the definition of the derivative for the sine function.

$$\frac{d}{dx}(\sin(x)) = \lim_{h \to 0} \frac{\sin(x + h) - \sin(x)}{h}$$

Since we can't just plug in $h = 0$ to evaluate the limit we will need to use the following trig formula on the first sine in the numerator.

$$\sin(x + h) = \sin(x)\cos(h) + \cos(x)\sin(h)$$

Doing this gives us,

$$\begin{aligned} \frac{d}{dx}(\sin(x)) &= \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\ &= \lim_{h \to 0} \frac{\sin(x)(\cos(h) - 1) + \cos(x)\sin(h)}{h} \\ &= \lim_{h \to 0} \sin(x) \frac{\cos(h) - 1}{h} + \lim_{h \to 0} \cos(x) \frac{\sin(h)}{h} \end{aligned}$$

As you can see upon using the trig formula we can combine the first and third term and then factor a sine out of that. We can then break up the fraction into two pieces, both of which can be dealt with separately.

Now, both of the limits here are limits as *h* approaches zero. In the first limit we have a sin*(x)* and in the second limit we have a cos*(x)*. Both of these are only functions of *x* only and as *h* moves in towards zero this has no affect on the value of *x*. Therefore, as far as the limits are concerned, these two functions are constants and can be factored out of their respective limits. Doing this gives,

$$\frac{d}{dx}\left(\sin\left(x\right)\right) = \sin\left(x\right)\lim_{h\to 0}\frac{\cos\left(h\right)-1}{h} + \cos\left(x\right)\lim_{h\to 0}\frac{\sin\left(h\right)}{h}$$

At this point all we need to do is use the limits in the fact above to finish out this problem.

$$\frac{d}{dx}\left(\sin\left(x\right)\right) = \sin\left(x\right)\left(0\right) + \cos\left(x\right)\left(1\right) = \cos\left(x\right)$$

Differentiating cosine is done in a similar fashion. It will require a different trig formula, but other than that is an almost identical proof. The details will be left to you. When done with the proof you should get,

$$\frac{d}{dx}\left(\cos\left(x\right)\right) = -\sin\left(x\right)$$

With these two out of the way the remaining four are fairly simple to get. All the remaining four trig functions can be defined in terms of sine and cosine and these definitions, along with appropriate derivative rules, can be used to get their derivatives.

Let's take a look at tangent. Tangent is defined as,

$$\tan\left(x\right) = \frac{\sin\left(x\right)}{\cos\left(x\right)}$$

Now that we have the derivatives of sine and cosine all that we need to do is use the quotient rule on this. Let's do that.

$$\begin{aligned}\frac{d}{dx}\left(\tan\left(x\right)\right) &= \frac{d}{dx}\left(\frac{\sin\left(x\right)}{\cos\left(x\right)}\right)\\ &= \frac{\cos\left(x\right)\cos\left(x\right) - \sin\left(x\right)\left(-\sin\left(x\right)\right)}{\left(\cos\left(x\right)\right)^2}\\ &= \frac{\cos^2\left(x\right) + \sin^2\left(x\right)}{\cos^2\left(x\right)}\end{aligned}$$

Now, recall that $\cos^2(x)+\sin^2(x)=1$ and if we also recall the definition of secant in terms of cosine we arrive at,

$$\begin{aligned}\frac{d}{dx}\left(\tan(x)\right) &= \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}\\ &= \frac{1}{\cos^2(x)}\\ &= \sec^2(x)\end{aligned}$$

The remaining three trig functions are also quotients involving sine and/or cosine and so can be differentiated in a similar manner. We'll leave the details to you. Here are the derivatives of all six of the trig functions.

Derivatives of the six trig functions

$$\frac{d}{dx}\left(\sin(x)\right)=\cos(x) \qquad \frac{d}{dx}\left(\cos(x)\right)=-\sin(x)$$

$$\frac{d}{dx}\left(\tan(x)\right)=\sec^2(x) \qquad \frac{d}{dx}\left(\cot(x)\right)=-\csc^2(x)$$

$$\frac{d}{dx}\left(\sec(x)\right)=\sec(x)\tan(x) \qquad \frac{d}{dx}\left(\csc(x)\right)=-\csc(x)\cot(x)$$

At this point we should work some examples.

Example 2 Differentiate each of the following functions.

(a) $g(x)=3\sec(x)-10\cot(x)$ [Solution]

(b) $h(w)=3w^{-4}-w^2\tan(w)$ [Solution]

(c) $y=5\sin(x)\cos(x)+4\csc(x)$ [Solution]

(d) $P(t)=\dfrac{\sin(t)}{3-2\cos(t)}$ [Solution]

Solution

(a) $g(x)=3\sec(x)-10\cot(x)$

There really isn't a whole lot to this problem. We'll just differentiate each term using the formulas from above.

$$\begin{aligned}g'(x) &= 3\sec(x)\tan(x)-10\left(-\csc^2(x)\right)\\ &= 3\sec(x)\tan(x)+10\csc^2(x)\end{aligned}$$

[Return to Problems]

(b) $h(w)=3w^{-4}-w^2\tan(w)$

In this part we will need to use the product rule on the second term and note that we really will need the product rule here. There is no other way to do this derivative unlike what we saw when we first looked at the product rule. When we first looked at the product rule the only functions we knew how to differentiate were polynomials and in those cases all we really needed to do was multiply them out and we could take the derivative without the product rule. We are now getting into the point where we will be forced to do the product rule at times regardless of whether or not we want to.

We will also need to be careful with the minus sign in front of the second term and make sure that it gets dealt with properly. There are two ways to deal with this. One way it to make sure that you use a set of parenthesis as follows,

$$\begin{aligned} h'(w) &= -12w^{-5} - \left(2w\tan(w) + w^2\sec^2(w)\right) \\ &= -12w^{-5} - 2w\tan(w) - w^2\sec^2(w) \end{aligned}$$

Because the second term is being subtracted off of the first term then the whole derivative of the second term must also be subtracted off of the derivative of the first term. The parenthesis make this idea clear.

A potentially easier way to do this is to think of the minus sign as part of the first function in the product. Or, in other words the two functions in the product, using this idea, are $-w^2$ and $\tan(w)$. Doing this gives,

$$h'(w) = -12w^{-5} - 2w\tan(w) - w^2\sec^2(w)$$

So, regardless of how you approach this problem you will get the same derivative.

[Return to Problems]

(c) $y = 5\sin(x)\cos(x) + 4\csc(x)$

As with the previous part we'll need to use the product rule on the first term. We will also think of the 5 as part of the first function in the product to make sure we deal with it correctly. Alternatively, you could make use of a set of parenthesis to make sure the 5 gets dealt with properly. Either way will work, but we'll stick with thinking of the 5 as part of the first term in the product. Here's the derivative of this function.

$$\begin{aligned} y' &= 5\cos(x)\cos(x) + 5\sin(x)\left(-\sin(x)\right) - 4\csc(x)\cot(x) \\ &= 5\cos^2(x) - 5\sin^2(x) - 4\csc(x)\cot(x) \end{aligned}$$

[Return to Problems]

(d) $P(t) = \dfrac{\sin(t)}{3 - 2\cos(t)}$

In this part we'll need to use the quotient rule to take the derivative.

$$P'(t) = \frac{\cos(t)(3-2\cos(t)) - \sin(t)(2\sin(t))}{(3-2\cos(t))^2}$$
$$= \frac{3\cos(t) - 2\cos^2(t) - 2\sin^2(t)}{(3-2\cos(t))^2}$$

Be careful with the signs when differentiating the denominator. The negative sign we get from differentiating the cosine will cancel against the negative sign that is already there.

This appears to be done, but there is actually a fair amount of simplification that can yet be done. To do this we need to factor out a "-2" from the last two terms in the numerator and the make use of the fact that $\cos^2(\theta) + \sin^2(\theta) = 1$.

$$P'(t) = \frac{3\cos(t) - 2\left(\cos^2(t) + \sin^2(t)\right)}{(3-2\cos(t))^2}$$
$$= \frac{3\cos(t) - 2}{(3-2\cos(t))^2}$$

[Return to Problems]

As a final problem here let's not forget that we still have our standard interpretations to derivatives.

Example 3 Suppose that the amount of money in a bank account is given by
$$P(t) = 500 + 100\cos(t) - 150\sin(t)$$
where t is in years. During the first 10 years in which the account is open when is the amount of money in the account increasing?

Solution

To determine when the amount of money is increasing we need to determine when the rate of change is positive. Since we know that the rate of change is given by the derivative that is the first thing that we need to find.

$$P'(t) = -100\sin(t) - 150\cos(t)$$

Now, we need to determine where in the first 10 years this will be positive. This is equivalent to asking where in the interval [0, 10] is the derivative positive. Recall that both sine and cosine are continuous functions and so the derivative is also a continuous function. The Intermediate Value Theorem then tells us that the derivative can only change sign if it first goes through zero.

So, we need to solve the following equation.

$$-100\sin(t)-150\cos(t)=0$$
$$100\sin(t)=-150\cos(t)$$
$$\frac{\sin(t)}{\cos(t)}=-1.5$$
$$\tan(t)=-1.5$$

The solution to this equation is,

$$t=2.1588+2\pi n, \qquad n=0,\pm 1,\pm 2,\ldots$$
$$t=5.3004+2\pi n, \qquad n=0,\pm 1,\pm 2,\ldots$$

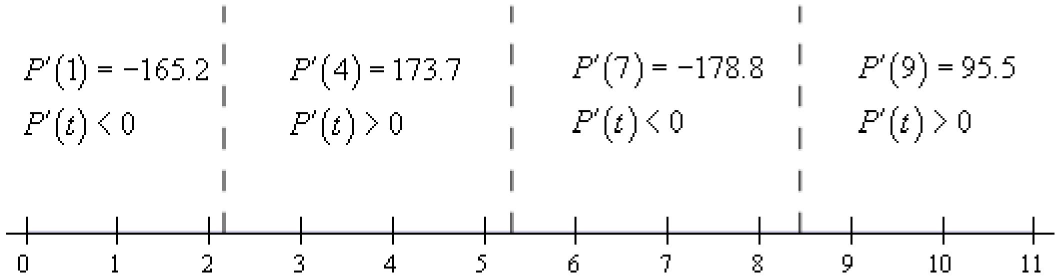
If you don't recall how to solve trig equations go back and take a look at the sections on solving trig equations in the Review chapter.

We are only interested in those solutions that fall in the range [0, 10]. Plugging in values of *n* into the solutions above we see that the values we need are,

$$t=2.1588 \qquad t=2.1588+2\pi=8.4420$$
$$t=5.3004$$

So, much like solving polynomial inequalities all that we need to do is sketch in a number line and add in these points. These points will divide the number line into regions in which the derivative must always be the same sign. All that we need to do then is choose a test point from each region to determine the sign of the derivative in that region.

Here is the number line with all the information on it.



So, it looks like the amount of money in the bank account will be increasing during the following intervals.

$$2.1588<t<5.3004 \qquad 8.4420<t<10$$

Note that we can't say anything about what is happening after $t=10$ since we haven't done any work for *t*'s after that point.

In this section we saw how to differentiate trig functions. We also saw in the last example that our interpretations of the derivative are still valid so we can't forget those.

Also, it is important that we be able to solve trig equations as this is something that will arise off and on in this course. It is also important that we can do the kinds of number lines that we used in the last example to determine where a function is positive and where a function is negative. This is something that we will be doing on occasion in both this chapter and the next.

Derivatives of Exponential and Logarithm Functions

The next set of functions that we want to take a look at are exponential and logarithm functions. The most common exponential and logarithm functions in a calculus course are the natural exponential function, $\mathbf{e}^x$, and the natural logarithm function, $\ln(x)$. We will take a more general approach however and look at the general exponential and logarithm function.

Exponential Functions
We'll start off by looking at the exponential function,

$$f(x) = a^x$$

We want to differentiate this. The power rule that we looked at a couple of sections ago won't work as that required the exponent to be a fixed number and the base to be a variable. That is exactly the opposite from what we've got with this function. So, we're going to have to start with the definition of the derivative.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x a^h - a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x\left(a^h - 1\right)}{h} \end{aligned}$$

Now, the a^x is not affected by the limit since it doesn't have any *h*'s in it and so is a constant as far as the limit is concerned. We can therefore factor this out of the limit. This gives,

$$f'(x) = a^x \lim_{h\to 0} \frac{a^h - 1}{h}$$

Now let's notice that the limit we've got above is exactly the definition of the derivative at of $f(x) = a^x$ at $x = 0$, *i.e.* $f'(0)$. Therefore, the derivative becomes,

$$f'(x) = f'(0)\, a^x$$

So, we are kind of stuck we need to know the derivative in order to get the derivative!

There is one value of *a* that we can deal with at this point. Back in the Exponential Functions section of the Review chapter we stated that $\mathbf{e} = 2.71828182845905\ldots$ What we didn't do however do actually define where **e** comes from. There are in fact a variety of ways to define **e**. Here are a three of them.

Some Definitions of e.

1. $\mathbf{e} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$
2. $\mathbf{e}$ is the unique positive number for which $\lim_{h \to 0} \frac{\mathbf{e}^h - 1}{h} = 1$
3. $\mathbf{e} = \sum_{n=0}^{\infty} \frac{1}{n!}$

The second one is the important one for us because that limit is exactly the limit that we're working with above. So, this definition leads to the following fact,

Fact 1

For the natural exponential function, $f(x) = \mathbf{e}^x$ we have $f'(0) = \lim_{h \to 0} \frac{\mathbf{e}^h - 1}{h} = 1$.

So, provided we are using the natural exponential function we get the following.

$$f(x) = \mathbf{e}^x \qquad \Rightarrow \qquad f'(x) = \mathbf{e}^x$$

At this point we're missing some knowledge that will allow us to easily get the derivative for a general function. Eventually we will be able to show that for a general exponential function we have,

$$f(x) = a^x \qquad \Rightarrow \qquad f'(x) = a^x \ln(a)$$

Logarithm Functions

Let's now briefly get the derivatives for logarithms. In this case we will need to start with the following fact about functions that are inverses of each other.

Fact 2

If *f(x)* and *g(x)* are inverses of each other then,

$$g'(x) = \frac{1}{f'(g(x))}$$

So, how is this fact useful to us? Well recall that the natural exponential function and the natural logarithm function are inverses of each other and we know what the derivative of the natural exponential function is!

So, if we have $f(x) = \mathbf{e}^x$ and $g(x) = \ln x$ then,

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\mathbf{e}^{g(x)}} = \frac{1}{\mathbf{e}^{\ln x}} = \frac{1}{x}$$

The last step just uses the fact that the two functions are inverses of each other.

Putting this all together gives,

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad\qquad x > 0$$

Note that we need to require that $x > 0$ since this is required for the logarithm and so must also be required for its derivative. In can also be shown that,

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x} \qquad\qquad x \neq 0$$

Using this all we need to avoid is $x = 0$.

In this case, unlike the exponential function case, we can actually find the derivative of the general logarithm function. All that we need is the derivative of the natural logarithm, which we just found, and the change of base formula. Using the change of base formula we can write a general logarithm as,

$$\log_a x = \frac{\ln x}{\ln a}$$

Differentiation is then fairly simple.

$$\begin{aligned} \frac{d}{dx}(\log_a x) &= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) \\ &= \frac{1}{\ln a}\frac{d}{dx}(\ln x) \\ &= \frac{1}{x \ln a} \end{aligned}$$

We took advantage of the fact that a was a constant and so $\ln a$ is also a constant and can be factored out of the derivative. Putting all this together gives,

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

Here is a summary of the derivatives in this section.

$\frac{d}{dx}(\mathbf{e}^x) = \mathbf{e}^x$	$\frac{d}{dx}(a^x) = a^x \ln a$
$\frac{d}{dx}(\ln x) = \frac{1}{x}$	$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

Okay, now that we have the derivations of the formulas out of the way let's compute a couple of derivatives.

Example 1 Differentiate each of the following functions.

(a) $R(w) = 4^w - 5\log_9 w$

(b) $f(x) = 3\mathbf{e}^x + 10x^3 \ln x$

(c) $y = \dfrac{5\mathbf{e}^x}{3\mathbf{e}^x + 1}$

Solution

(a) This will be the only example that doesn't involve the natural exponential and natural logarithm functions.

$$R'(w) = 4^w \ln 4 - \frac{5}{w \ln 9}$$

(b) Not much to this one. Just remember to use the product rule on the second term.

$$\begin{aligned} f'(x) &= 3\mathbf{e}^x + 30x^2 \ln x + 10x^3 \left(\frac{1}{x}\right) \\ &= 3\mathbf{e}^x + 30x^2 \ln x + 10x^2 \end{aligned}$$

(c) We'll need to use the quotient rule on this one.

$$\begin{aligned} y &= \frac{5\mathbf{e}^x \left(3\mathbf{e}^x + 1\right) - \left(5\mathbf{e}^x\right)\left(3\mathbf{e}^x\right)}{\left(3\mathbf{e}^x + 1\right)^2} \\ &= \frac{15\mathbf{e}^{2x} + 5\mathbf{e}^x - 15\mathbf{e}^{2x}}{\left(3\mathbf{e}^x + 1\right)^2} \\ &= \frac{5\mathbf{e}^x}{\left(3\mathbf{e}^x + 1\right)^2} \end{aligned}$$

There's really not a lot to differentiating natural logarithms and natural exponential functions at this point as long as you remember the formulas. In later sections as we get more formulas under our belt they will become more complicated.

Next, we need to do our obligatory application/interpretation problem so we don't forget about them.

Example 2 Suppose that the position of an object is given by

$$s(t) = t\mathbf{e}^t$$

Does the object ever stop moving?

Solution

First we will need the derivative. We need this to determine if the object ever stops moving since at that point (provided there is one) the velocity will be zero and recall that the derivative of the position function is the velocity of the object.

The derivative is,

$$s'(t) = \mathbf{e}^t + t\mathbf{e}^t = (1+t)\mathbf{e}^t$$

So, we need to determine if the derivative is ever zero. To do this we will need to solve,

$$(1+t)\mathbf{e}^t = 0$$

Now, we know that exponential functions are never zero and so this will only be zero at $t = -1$. So, if we are going to allow negative values of *t* then the object will stop moving once at $t = -1$. If we aren't going to allow negative values of *t* then the object will never stop moving.

Before moving on to the next section we need to go back over a couple of derivatives to make sure that we don't confuse the two. The two derivatives are,

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1} \qquad \text{Power Rule}$$

$$\frac{d}{dx}\left(a^x\right) = a^x \ln a \qquad \text{Derivative of an exponential function}$$

It is important to note that with the Power rule the exponent MUST be a constant and the base MUST be a variable while we need exactly the opposite for the derivative of an exponential function. For an exponential function the exponent MUST be a variable and the base MUST be a constant.

It is easy to get locked into one of these formulas and just use it for both of these. We also haven't even talked about what to do if both the exponent and the base involve variables. We'll see this situation in a later section.

Chapter **2**

DERIVATIVE OF FUNCTIONS

So far we have studied about the limits of a given function. We use limits foı finding the instantaneous rate of change of one quantity w.r.t. another. The process involved in it is called Differentiation. Differentiation has important applications in Engineering. For example, in finding the rate of change of surface area of a circular plate or rectangular plate while heating it, current flowing through a conductor can be determined by the rate of change flowing through it, velocity of a moving body can be obtained by finding the derivative of displacement w.r.t time etc.

If y is a function of x, and δy is the small increment in y corresponding to small increment δx in x, then $\underset{\delta x \to 0}{Lt} \frac{\delta y}{\delta x}$ (if it exists) is called the derivative or differential co-efficient of y with respect to x and is donated by $\frac{dy}{dx}$ i.e. $\frac{dy}{dx} = \underset{\delta x \to 0}{Lt} \frac{\delta y}{\delta x}$

Differentiation from first Principles

Let y = f(x) -- (1)

When x changes to x+δx, y changes to y+δy

∴ y+δy = f(x+δx) -- (2)

Subtracting (1) from (2) we get,

y+δy−y = f(x+δx) - f(x)

or δy = f(x+δx) −f(x)

Dividing both sides of δx,

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

When δx → 0

$$\underset{\delta x}{Lt} \frac{\delta y}{\delta x} = \underset{\delta \pi \to 0}{Lt} \frac{f(x+\delta x) - f(x)}{\delta x}$$

Thus if (x) is a function of x, then $\underset{\delta x \to 0}{Lt} \frac{f(x+\delta x) - f(x)}{\delta x}$ (if it exists) is called the derivative of f(x) with respect to x and is denoted by f′(x)

Hence $f'(x) = \underset{\delta x \to 0}{Lt} \frac{f(x+\delta x)-f(x)}{\delta x}$

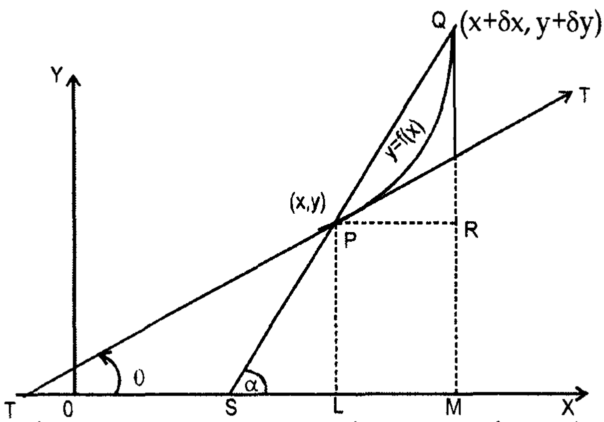
Hence from the above it is clear that following steps may be followed to find the differential co-efficient of a function of x.

(i) Let the given function y = f(x)
(ii) Given increments i.e. change x to $(x+\delta x)$ and y to $(y+\delta y)$
(iii) Subtract (i) from (ii) to find δy and simplify.
(iv) Divide both sides by δx to find incremental ratio $\frac{\delta y}{\delta x}$
(v) Proceed to find the limit as $\delta x \longrightarrow 0$

The above process of finding the derivative or differential coefficient of a function is called Differentiation from first principles or Differentiation ab-initio or Differentiation by δ-method.

Geometrical Interpretation of the Derivative

Let us consider the graph of a curve y=f(x).



Let P(x,y) be any point on a curve on a curve and $Q(x+\delta x, y+\delta y)$ be another point in the neighbourhood of P.

Draw the secant PQ making an angle a with the positive direction of x-axis, From P and Q draw PL, QM perpendicular to x-axis and PR⊥QM.

Then $PR = x + \delta x - x = \delta x$

$QR = y + \delta y - y = \delta y$

$$\tan\alpha = \frac{QR}{PR} = \frac{\delta y}{\delta x} = \frac{f(x+\delta x)-f(x)}{\delta x}$$

As $Q \to P$, $\delta x \to 0$ and secant PQ becomes the tangent TPT′ at P. Also angle $\alpha \to$ angle θ.

Slope of tangent at P = $\tan\theta$

$$= \underset{\alpha \to 0}{Lt} \tan\alpha$$

$$= \underset{\delta x \to 0}{Lt} \frac{\delta y}{\delta x}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \frac{dy}{dx} \text{ (By definition)}$$

$$= \text{The value of } \frac{dy}{dx} \text{ at P}$$

Thus, the derivative of any function at any given point represents the slope of the tangent to the curve at that point.

Example 1. To find the derivative of x^n, where n is any real number

Solution : Let $f(x) = x^n$

$\therefore$ $f(x+h) = (x+h)^n$

$$\therefore \quad f'(x) = \underset{h \to 0}{\text{Lt}} \frac{f(x+h) - f(x)}{h}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{(x+h)^n - x^n}{h}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{\left[x\left(1+\frac{h}{x}\right)\right]^n - (x)^n}{h}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{x^n\left(1+\frac{h}{x}\right)^n - (x)^n}{h}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{(x)^n\left[\left(1+\frac{h}{x}\right)^n - 1\right]}{h}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{x^n}{h}\left[\left\{1 + n\left(\frac{h}{x}\right) + \frac{n(n-1)\frac{h^2}{x^2}}{2!} + \frac{n(n-1)(n-2)\frac{h^3}{x^3}}{3!} + \ldots\right\} - 1\right]$$

[using Binomial theorem for $(1+x)^n$]

$$= \underset{h \to 0}{Lt} \frac{x^n}{h}\left[n\left(\frac{h}{x}\right)+\frac{n(n-1)}{2!}.\frac{h^2}{x^2}+\frac{n(n-1)(n-2)}{3!}.\frac{h^3}{x^3}+....\right]$$

$$= \underset{h \to 0}{Lt} x^n\left[\frac{n}{x}+\frac{n(n-1)}{2!}\frac{h}{x^2}+\frac{n(n-1)(n-2)}{3!}\frac{h^3}{x^3}+....\right]$$

$$= x^n\left(\frac{n}{x}\right)=nx^{n-1}$$

$\therefore \quad f'(x) = nx^{x-1}$

Derivative of f(x) is also denoted by $\frac{d}{dx}(f(x))$.

If y = f(x), then derivative of f(x) w.r.t x will be denoted by $\frac{dy}{dx}$

Thus for $y = x^n$ $\qquad \frac{dy}{dx} = nx^{n-1}$

Derivatives of some standard functions and some standard Rules

1. $\frac{d}{dx}$ (constant) = 0
2. $\frac{d}{dx}(x^n) = nx^{x-1}$; In particular $\frac{d}{dx}(x) = 1$
3. $\frac{d}{dx}(a^x) = a^x \log_e a$, a>0 and a≠1
4. $\frac{d}{dx}(e^x) = e^x$
5. $\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e$
6. $\frac{d}{dx}(\log x) = \frac{1}{x}$, x>0
7. $\frac{d}{dx}[u+v] = \frac{du}{dx}+\frac{dv}{dx}$
   where u and v are functions of x.
8. $\frac{d}{dx}(u-v) = \frac{du}{dx}-\frac{dv}{dx}$, where u,v are functions of x.

9. $\frac{d}{dx}(u.v) = u\frac{dv}{dx} + v\frac{du}{dx}$, where u,v are functions of x.

This rule is known as **Product Rule.**

10. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$, $v(x) \neq 0$, where u,v are functions of x.

This rule is known as **Quotient Rule.**

11. $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}(f(x))$; where c is any real number

12. $\frac{d}{dx}(\sin x) = \cos x$

13. $\frac{d}{dx}(\cos x) = -\sin x$

14. $\frac{d}{dx}(\tan x) = \sec^2 x$

15. $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

16. $\frac{d}{dx}(\sec x) = \sec x \tan x$

17. $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$

18. $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$; $|x| < 1$

19. $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$; $|x| < 1$

20. $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

21. $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$

22. $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$, $|x| > 1$

23. $\frac{d}{dx}(\text{cosec}^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}, \ |x| > 1$

24. $\frac{d}{dx}(|x|) = \frac{x}{|x|}, \ x \neq 0$

25. $\frac{d}{dx}(ax+b)^n = na(ax+b)^{n-1}$

Example 2. Find the derivative of x^3 w.r.t x.

Solution Let $y = x^3$

$$\therefore \frac{dy}{dx} = 3x^{3-1} = 3x^2 \quad , \quad \left[\because \frac{d}{dx}(x^n) = nx^{n-1}\right]$$

Example 3. Find the derivative of $x^{-\frac{3}{2}}$ w.r.t x.

Solution Let $y = x^{-\frac{3}{2}}$

$$\frac{dy}{dx} = -\frac{3}{2}(x)^{-\frac{3}{2}-1} = -\frac{3}{2}(x)^{-\frac{5}{2}}$$

Example 4. Find the derivative of $\sqrt{x}$ w.r.t x.

Solution Let $y = \sqrt{x}$

$y = (x)^{1/2}$

$$\therefore \frac{dy}{dx} = \frac{1}{2}(x)^{\frac{1}{2}-1}$$

$$= \frac{1}{2}(x)^{-\frac{1}{2}}$$

$$= \frac{1}{2(x)^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$$

Example 5 Find the derivatives of

(i) $\sqrt{x} + \frac{1}{\sqrt{x}}$ (ii) $\sqrt{x} - \frac{1}{\sqrt{x}}$ w.r.t x.

Solution (i) $y = \sqrt{x} + \frac{1}{\sqrt{x}}$

$= (x)^{1/2} + (x)^{-1/2}$

$$\frac{dy}{dx} = \frac{d}{dx}(x)^{½} + \frac{d}{dx}(x)^{-½}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}(x)^{½-1} + \left(-\frac{1}{2}\right)(x)^{-½-1}$$

$$= \frac{1}{2}(x)^{-½} - \frac{1}{2}(x)^{-\frac{3}{2}}$$

$$= \frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}$$

(ii) $$y = \sqrt{x} - \frac{1}{\sqrt{x}}$$

$$= (x)^{½} - (x)^{-½}$$

$$\frac{dy}{dx} = \frac{d}{dx}(x)^{½} - \frac{d}{dx}(x)^{-½}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}(x)^{½-1} - \left(-\frac{1}{2}\right)(x)^{-½-1}$$

$$= \frac{1}{2}(x)^{-\frac{1}{2}} + \frac{1}{2}(x)^{-\frac{3}{2}}$$

$$= \frac{1}{2\sqrt{x}} + \frac{1}{2x\sqrt{x}}$$

Example 6 Find the derivative of $3x^3+5x^2+7$ w.r.t x.

Solution Let $y = 3x^3+5x^2+7$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(3x^3) + \frac{d}{dx}(5x^2) + \frac{d}{dx}(7)$$

$$= 3\frac{d}{dx}(x^3) + 5\frac{d}{dx}(x^2) + \frac{d}{dx}(7)$$

$$= 3(3x^{3-1})+5(2x^{2-1})+0 \quad , \quad \left[\because \frac{d}{dx}(\text{constant}) = 0\right]$$

$$= 9x^2+10x$$

Example 7 Find the Derivative of

(i) $(3x+5)^7$

(ii) $(2x-3)^8$

Solution (i) Let $y = (3x+5)^7$

$$\frac{dy}{dx} = 7(3)(3x+5)^{7-1}$$

$$= 21(3x+5)^6 \qquad \left[\because \frac{d}{dx}(ax+b)^n = na(ax+b)^{n-1}\right]$$

(ii) $y = (2x-3)^8$

$$\frac{dy}{dx} = 8(2)(2x-3)^{8-1} = 16\,(2x-3)^7$$

Remark - Derivative of a function is also known as Differential co-efficient of a function.

Example 8 Find the Differential Co-efficient of $y = (x+5)^3(x-3)^7$ w.r.t x

Solution $y = (x+5)^3(x-3)^7$

$$\frac{dy}{dx} = \frac{d}{dx}\,[(x+5)^3(x-3)^7]$$

$$= (x+5)^3\frac{d}{dx}(x-3)^7 + (x-3)^7\frac{d}{dx}(x+5)^3, \qquad \left[\begin{array}{l}\text{using Product Rule}\\ \because \frac{d}{dx}(u.v) = u\frac{dv}{dx} + v.\frac{du}{dx}\end{array}\right]$$

$$= (x+5)^3(7)(x-3)^{7-1}+(x-3)^7(3)(x+5)^{3-1}$$

$$= 7(x+5)^3\,(x-3)^6+3(x+5)^2(x-3)^7$$

$$= (x+5)^2(x-3)^6[7(x+5)+3(x-3)]$$

$$= (x+5)^2(x-3)^6[7x+35+3x-9]$$

$$= (x+5)^2(x-3)^6(10x+26)$$

$$= (x+5)^2(x-3)^6(2)(5x+13)$$

$$= 2(x+5)^2(x-3)^6(5x+13)$$

Example 9 For $y = \frac{(2x^3+3)}{(x-5)}$ find $\frac{dy}{dx}$

Solution : $y = \frac{(2x^3+3)}{(x-5)}$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left[\frac{(2x^3+3)}{(x-5)}\right]$$

$$= \frac{(x-5)\frac{d}{dx}(2x^3+3) - (2x^3+3)\frac{d}{dx}(x-5)}{(x-5)^2}, \qquad \text{[using Quotient Rule]}$$

$$= \frac{(x-5)[2(3x^2)+0]-(2x^2+3)(1-0)}{(x-5)^2}$$

$$= \frac{(x-5)(6x^2)-(2x^2+3)(1)}{(x-5)^2}$$

$$= \frac{6x^3-30x^2-2x^3-3}{(x-5)^2}$$

$$= \frac{4x^3-30x^2-3}{(x-5)^2}$$

Example 10. If $f(x) = 4x^2 + 7x + x - e^x$

Find $f'(x)$ and hence $f'(2)$

Solution : $f(x) = 4x^2 + 7x - e^x$

$\therefore \quad f'(x) = 4\frac{d}{dx}(x^2) + 4\frac{d}{dx}(x) - \frac{d}{dx}(e^x)$

$= 4(2x) + 7(1) - e^x$

$= 8x + 7 - e^x$

$\therefore \quad f'(2) = 8(2) + 7 - e^2$

$= 16 + 7 - e^2$

$= 23 - e^2$

Example 11. For function $y = \sqrt{2x-3}$; find $\frac{dy}{dx}$ and hence find $\frac{dy}{dx}$ at x=6

Solution : $y = \sqrt{2x-3}$

$= (2x-3)^{1/2}$

$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(2x-3)^{1/2}$

$= \frac{1}{2}(2)(2x-3)^{½-1}$ $\qquad [\frac{d}{dx}(ax+b)^n = na(ax+b)^{n-1}]$

$= (2x-3)^{-1/2}$

$= \frac{1}{(2x-3)^{½}}$

$\frac{dy}{dx} = \frac{1}{\sqrt{2x-3}}$

Now $\left(\frac{dy}{dx}\right)_{x=6} = \frac{1}{\sqrt{2(6)-3}}$

$= \frac{1}{\sqrt{12-3}} \qquad = \frac{1}{\sqrt{9}} \qquad = \frac{1}{3}$

Example 12 If $y = 2 \sin x + 3 \cos x - \tan x$ find $\frac{dy}{dx}$.

Solution $y = 2 \sin x + 3 \cos x - \tan x$

$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}[2\sin x] + \frac{d}{dx}[3 \cos x] - \frac{d}{dx}(\tan x)$

$= 2\frac{d}{dx}(\text{Sin}x) + 3\frac{d}{dx}(\cos x) - \frac{d}{dx}(\tan x)$

$= 2 \cos x + 3(-\sin x) - \sec^2 x$

$\therefore \quad \frac{dy}{dx} = 2\cos x - 3\sin x - \sec^2 x.$

Example 13 If $y = x \sin x$, find $\frac{dy}{dx}$.

Solution $y = x \sin x$

$\therefore \quad \frac{dy}{dx} = x\frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x)$

$= x \cos x + \sin x\,(1)$

$= x \cos x + \sin x$

Example 14 If $y = \frac{2\sin x}{(\log x)}$ find $\frac{dy}{dx}$.

Solution $y = \frac{2\sin x}{\log x}$

$= \frac{\log x \frac{d}{dx}(2\sin x) - (2\sin x)\frac{d}{dx}(\log x)}{(\log x)^2}$, [using Quotient Rule]

$= \frac{(\log x)(2\cos x) - (2\sin x)\left(\frac{1}{x}\right)}{(\log x)^2}$

$$= \frac{2x(\log x)(\cos x) - 2\sin x}{x(\log x)^2}$$

$$= \frac{2x\cos x \log x - 2\sin x}{x(\log x)^2}$$

Example 15 If $y = x^5 - (3)^x + e^x + \log_5 x$ find $\frac{dy}{dx}$.

Slution $y = x^5 - (3)^x + e^x + \log_5 x$

$$\frac{dy}{dx} = 5(x)^4 - (3)^x \log 3 + e^x + \frac{1}{x}\log_5 e \quad ,$$

$$\left[\because \frac{d}{dx}(a)^x = a^x \log a, \ \frac{d}{dx}(e^x) = e^x, \ \frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e\right]$$

Example 16 Find the derivative of $x^2 \log x \sin x$ w.r.t. x.

Solution Let $y = x^2 \log x \sin x$

$$\frac{dy}{dx} = (\log x)(\sin x)\frac{d}{dx}(x^2) + x^2 \sin x \frac{d}{dx}(\log x) + x^2 \log x \frac{d}{dx}(\sin x)$$

$$\left[\because \frac{d}{dx}(uvw) = (vw)\frac{d}{dx}(u)(wu) + \frac{d}{dx}(vu) + \frac{d}{dx}(w)\right]$$

$$= \log x . \sin x \, 2x + \frac{x^2 \sin x}{x} + x^2 \log x \cos x$$

$$= 2x \log x . \sin x + x \sin x + x^2 \log x \cos x.$$

Differentiation of function of a function (Chain Rule)

If $y = f(u)$ and $u = g(x)$, then, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

similarly, If $y = f(z)$, $z = g(u)$ and $u = h(x)$.

then, $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{du} \times \frac{du}{dx}$

Example 17 If $y = \sin x^2$, then find $\frac{dy}{dx}$ by Chain Rule.

Solution Let $x^2 = u$

$\therefore$ $y = \sin u$ and $u = x^2$

Thus y is a function of u and u is a function of x. Therefore by Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{d}{du}(\sin u) \times \frac{d}{dx}(x)^2$$

$= \cos u\,(2x)$

$= (\cos x^2)(2x)$, $[\because u = x^2]$

$= 2x \cos x^2$

Example 18 If $y = \sin(2x + 7)$, then find $\frac{dy}{dx}$ by chain Rule.

Solution : Let $(2x + 7) = u$

$\therefore$ $y = \sin u$ and $u = 2x + 7$

By Chain Rule, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$= \frac{d}{du}(\sin u) \times \frac{d}{dx}(2x + 7)$$

$$= (\cos u)\left[2\frac{d}{dx}(x) + \frac{d}{dx}(7)\right]$$

$= (\cos u)(2(1) + 0)$

$= 2 \cos u$

$= 2\cos(2x + 7)$

Remark : From above example it is concluded that $\frac{d}{dx}[f(ax+b)] = af'(ax + b)$.

Example 19 If $y = \sqrt{\tan(x^2 - 5)}$, find $\frac{dy}{dx}$

Solution Let $x^2 - 5 = u$, and

$\tan u = z$

Thus, $y = \sqrt{z}$, $z = \tan u$ and $u = x^2 - 5$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{du} \times \frac{du}{dx}$$

$$= \frac{d}{dz}(z)^{\frac{1}{2}} \times \frac{d}{du}(\tan u) \times \frac{d}{dx}(x^2 - 5)$$

$$= \frac{1}{2}(z)^{\frac{1}{2}-1} \times (\sec^2 u) \times (2x - 0)$$

$$= e^{5x}\left[\frac{6x}{3x^2-5} + 5\log(3x^2-5)\right]$$

Example 23 If $y = \sqrt{x+\sqrt{x+\sqrt{x+........\infty}}}$

Show that $(2y-1)\frac{dy}{dx} = 1$

Solution $y = \sqrt{x+\sqrt{x+\sqrt{x+........\infty}}}$

$\therefore$ $y = \sqrt{x+y}$

Squaring both sides,

$y^2 = x+y$

Differentiating both sides w.r.t. x

$$2y.\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

or $2y\frac{dy}{dx} - \frac{dy}{dx} = 1$ or $(2y-1)\frac{dy}{dx} = 1$

Implicit Differentiation

So far we were dealing with explicit functions in which y was given to be the function of x i.e. y= f(x) and the corresponding differentiation is called **Explicit Differentiation**.

The function in which y is not expressed as the function of x, instead, the relation between x and y is expressed as f(x,y) = C, then ; y is called implicit function of x, and the differentiation for this type of function is called **Implicit differentiation**.

Example 24 If $2x^2 + 5xy + 5y^2 = 1$, then find $\frac{dy}{dx}$.

Solution $2x^2 + 5xy + 5y^2 = 1$

Differentiating both sides w.r.t. x, we get

$$2(2x) + 5[x.\frac{dy}{dx} + y.(1)] + 5(2y)\frac{dy}{dx} = 0$$

$$4x + 5x.\frac{dy}{dx} + 5y + 10y.\frac{dy}{dx} = 0$$

$$\therefore (5x+10y)\frac{dy}{dx} = -(4x+5y)$$

Hence, $\frac{dy}{dx} = -\frac{(4x+5y)}{(5x+10y)}$

Example 25 If $x^3 + y^3 = 3axy$; find $\frac{dy}{dx}$.

Solution $x^3+y^3 = 3axy$

Differentiating both sides w.r.t.x, we get

$$3x^2 + 3y^2 \frac{dy}{dx} = 3a\,[x.\frac{dy}{dx} + y.\ (1)]$$

$$3x^2 + 3y^2\frac{dy}{dx} = 3ax\frac{dy}{dx} + 3ay$$

$$(3y^2 - 3ax)\frac{dy}{dx} = 3ay - 3x^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{3ay - 3x^2}{3y^2 - 3ax}$$

or $$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

Example 26 If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots\ldots\infty}}}$, then find $\frac{dy}{dx}$.

Solution: $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots\ldots\infty}}}$

$y = \sqrt{\sin x + y}$

Squaring both sides, we get $y^2 = \sin x + y$

Differentiating both sides, w.r.t. x, we get

$$2y.\frac{dy}{dx} = \cos x + \frac{dy}{dx}$$

or $$(2y-1)\frac{dy}{dx} = \cos x$$

$$\frac{dy}{dx} = \frac{\cos x}{(2y-1)}$$

Differentiation of Parametric Functions -

Let y = f(x) be any function,

If two variable x and y are separately expressed in terms of a third variable t, then, the function is said to be expressed in parametric form. The third variable t is called a parameter.

The Differentiation of such functions is called Differentiation of Parametric functions, or simply Parametric Differentiation.

So, if x = f(t) and y = g(t)

Then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

Example 27 If $y = t^2 + 5t$ and $X = 5t - t^3$, find $\frac{dy}{dx}$ and hence, find $\left(\frac{dy}{dx}\right)$ at t = 1

Solution : $y = t^2+5t$

$$\therefore \frac{dy}{dt} = 2t + 5$$

and $x = 5t - t^3$

$$\therefore \frac{dx}{dt} = 5(1) - 3\tau^2$$

$$= 5 - 3t^2$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{2t+5}{5-3t^2}$$

$$\therefore \left(\frac{dy}{dx}\right)_{t=1} = \frac{2(1)+5}{5-3(1)^2} = \frac{7}{2} = 3.5$$

Examples 28 If $x = at^2$, $y = 2at$, find $\frac{dy}{dx}$.

Solution: $x = at^2$

$$\therefore \frac{dx}{dt} = a(2t) \qquad = 2at$$

and $y = 2at$

$$\therefore \frac{dy}{dt} = 2a(1) \qquad = 2a$$

Hence $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$= \frac{2a}{2at} \quad = \frac{1}{t}$

Example 29 Differentiate $(2x+1)^3$ w.r.t. $3x^2-x+8$.

Solution : Let $y = (2x+1)^3$

and $z = (3x^2 - x+8)$

To find $\frac{dy}{dz} = \frac{dy/dx}{dz/dx}$

Now, $y = (2x+1)^3$

$\therefore \quad \frac{dy}{dx} = 3(2x+1)^2 . \frac{d}{dx}(2x+1)$

$= 3(2x+1)^2 (2)$

$= 6(2x+1)^2$

and $z = (3x^2 - x + 8)$

$\frac{dz}{dx} = 3(2x) - (1) + 0 \quad = 6x - 1$

Hence, $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{6(2x+1)^2}{(6x-1)}$

Example 30 Differentiate $\tan x^2$ w.r.t. $\tan^2 x$.

Solution : Let $y = \tan x^2$ and $z = \tan^2 x$

$\therefore \quad \frac{dy}{dz} = \frac{dy/dx}{dz/dx}$

Now $y = \tan x^2$

$\therefore \quad \frac{dy}{dx} = (\sec^2 x^2)(2x)$

$= 2x \sec^2 x^2$

$z = \tan^2 x \quad , \quad \frac{dz}{dx} = 2\tan x \sec^2 x$

Hence $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{2x \sec^2 x^2}{2\tan x . \sec^2 x}$

$= \frac{x \sec^2 x^2}{\tan x . \sec^2 x}$

Logarithmic differentiation - When a given function is the product of two or more other functions, or is the product and division of two or more functions or a function raised to the power of another function, then first we take its logarithm and simplify and then differentiate. This process is called Logarithmic differentiation.

e.g. the differentiation of functions

$$x^x, \quad (\text{Sin } x)^{\cos x}, \quad \frac{x \sin x e^x}{\log x}, \quad \frac{x\sqrt{\sin x}}{(x^2+3)\tan x} \quad \text{etc.}$$

Example 31 Differentiate x^x w.r.t x.

Solution : Let $y = x^x$

$\therefore \log y = \log x^x$

$\log y = x \log x$, $[\because \log m^n = n \log m]$

Differentiating both sides w.r.t. x,

$$\frac{1}{y}\frac{dy}{dx} = x.\frac{1}{x} + (\log x) . (1)$$

$$= 1 + \log x$$

$$\therefore \frac{dy}{dx} = y\,(1 + \log x) = x^x(1 + \log x).$$

Example 32 Differentiates $x^{\sin x}$ w.r.t. x

Solution: Let $y = x^{\sin x}$

$\therefore \log y = \log x^{\sin x}$

$\log y = \sin x \log x$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \sin x . \frac{1}{x} + \log x\,(\cos x)$$

$$\frac{dy}{dx} = y\left[\frac{\sin x + x \log x(\cos x)}{x}\right]$$

$$= x^{\sin x}\left[\frac{\sin x + x(\log x)\cos x}{x}\right]$$

Example 33 Differentiate $\frac{x \sin x}{e^x \log x}$ w.r.t x

Solution Let $y = \frac{x \sin x}{e^x \log x}$

Taking log on both sides

$$\log y = \log\left[\frac{x\sin x}{e^x \log x}\right] \qquad \left[\begin{array}{l}\because \log mn = \log m + \log n \\ \& \ \log\left(\frac{m}{n}\right) = \log m - \log n\end{array}\right]$$

$$\log y = \log x + \log \sin x - \log e^x - \log (\log x)$$

Differentiating w.r.t. x, we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{\sin x}.\cos x - \frac{(1)e^x}{e^x} - \frac{1}{\log x} \times \frac{1}{x}$$

$$\frac{dy}{dx} = y\left[\frac{1}{x} + \cot x - 1 - \frac{1}{x\log x}\right]$$

or $$\frac{dy}{dx} = \frac{x\sin x}{e^x \log x}\left[\frac{1}{x} + \cot x - \frac{1}{x\log x} - 1\right]$$

Example 34 Differentiate $\sin x^{\sin x^{\sin x \cdots\infty}}$

Solution Let $y = \sin x^{\sin x^{\sin x \cdots\infty}}$

$$y = (\sin x)^y$$

taking log on both sides, we get

$$\log y = \log (\sin x)^y$$

$$\log y = y \log \sin x$$

Differentiating w.r.t. x, we have

$$\frac{1}{y}.\frac{dy}{dx} = y.\frac{1}{\sin x}.\cos x + (\log \sin x)\frac{dy}{dx}$$

$$\left(\frac{1}{y} - \log \sin x\right)\frac{dy}{dx} = y \cot x$$

$$\therefore \quad \frac{dy}{dx} = \frac{y^2 \cot x}{[1 - y\log \sin x]}$$

Example 35 Differentiae $(x)^{\sin x} + (\sin x)^x$ w.r.t x

Solution $y = (x)^{\sin x} + (\sin x)^x$

Let $u = (x)^{\sin x}$

and $v = (\sin x)^x$

$\therefore \quad y = u+v$

$$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \text{-----(1)}$$

Now $u = x^{\sin x}$

$\log u = \sin x \log x$

$$\therefore \quad \frac{1}{u}\frac{du}{dx} = \sin x . \frac{1}{x} + (\log x) \cos x$$

$$\frac{du}{dx} = u\left[\frac{\sin x}{x} + (\cos x)\log x\right]$$

$v = (\sin x)^x$

$\log v = x \log \sin x$

$$\frac{1}{v}\frac{dv}{dx} = x . \frac{1}{\sin x} . \cos x + \log(\sin x)$$

$$\frac{dv}{dx} = v[x \cot x + \log \sin x]$$

Putting the value of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in eq. (1) we get,

$$\frac{dy}{dx} = u\left[\frac{\sin x}{x} + \cos x \log x\right] + v[x \cot x + \log \sin x]$$

$$\therefore \quad \frac{dy}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \cos x \log x\right] + (\sin x)^x [x \cot x + \log \sin x]$$

Successive differentiation - The process of finding out the derivative of a given function, again and again is called successive differentiation.

If y is a given function of x, then $\frac{dy}{dx}$ is called first order derivative of y. On differentiating $\frac{dy}{dx}$, we get $\frac{d^2y}{dx^2}$ called second order derivative or second order differential co -efficient, and is read as d-two -y by d-x square.

On differentiating $\frac{d^2y}{dx^2}$ we get third order derivative or differential co efficient of third order $\frac{d^3y}{dx^3}$ (read as d - three -y by d-x-cube)

Similarly derivatives of higher order can be defined. The derivative of n th order is written as $\frac{d^ny}{dx^n}$ (read as -d-n-y by d-x-n).

We have following types of notations for derivatives:-

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3} \text{ ---- } \frac{d^ny}{dx^n}$$

$y', y'', y''', ---- y^n$

$y_1, y_2, y_3, ---- y_n$

$f'(x), f''(x), f'''(x), ---- f^n(x)$

$Dy, D^2y, D^3y, ---- D^ny$

Where, $D^r = \frac{d^r}{dx^r}$

Example 36 If $y = 6x^5 - 4x^3 + 2x^2 - 7$

Find $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \frac{d^5y}{dx^5}, \frac{d^6y}{dx^6}$

Solution $y = 6x^5 - 4x^3 + 2x^2 - 7$

$$\therefore \quad \frac{dy}{dx} = 30x^4 - 12x^2 + 4x$$

$$\frac{d^2y}{dx^2} = 120x^3 - 24x + 4$$

$$\frac{d^3y}{dx^3} = 360x^2 - 24$$

$$\frac{d^4y}{dx^4} = 720x$$

$$\frac{d^5y}{dx^5} = 720$$

$$\frac{d^6y}{dx^6} = 0$$

Remark- In the example 36, all the derivatives of y of order greater than or equal to 6 are zeros. From this we can conclude that for a polynomial of degree n, all the derivatives of order $\geq (n+1)$ are zeros.

Example 37 If $y = \log(\sin x)$, find y_3

Solution $y = \log(\sin x)$

$$y_1 = \frac{1}{\sin x} . \cos x = \cot x$$

$$y_2 = -\text{cosec}^2 x$$

$$y_3 = -2\text{cosec } x . \frac{d}{dx}(\text{cosec } x)$$

$= -2 \text{ cosec } x[-\text{cosec } x \cot x]$

$\therefore \quad y_3 = 2 \text{ cosec}^2 x \cot x.$

Example 38 If $y = \sin(\tan x)$, find $\frac{d^2y}{dx^2}$.

Solution $y = \sin(\tan x)$

$$\frac{dy}{dx} = \cos(\tan x) \cdot \frac{d}{dx}(\tan x)$$

$$= \cos(\tan x) \cdot \sec^2 x$$

$$\frac{d^2y}{dx^2} = \cos(\tan x) .2 \sec x. [\sec x. \tan x] + \sec^2 x. [-\sin(\tan x)].\sec^2 x$$

$$= 2 \sec^2 x \tan x. \cos(\tan x) - \sec^4 x. \sin(\tan x)$$

Example 39 If $x = a(\theta - \sin\theta)$

and $y = a(1 + \cos\theta)$, find $\frac{d^2y}{dx^2}$.

Solution: The given equation is in parametric form, and the parameter is θ.

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$y = a(1 + \cos\theta)$$

$$\therefore \quad \frac{dy}{d\theta} = a[-\sin\theta] = -a\sin\theta$$

and $x = a[\theta \,\square\, \sin\theta]$

$$\therefore \quad \frac{dx}{d\theta} = a[1 - \cos\theta]$$

$$\therefore \quad \frac{dy}{dx} = \frac{-a\sin\theta}{a(1-\cos\theta)} = \frac{-a\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)}{a\left[2\sin^2\frac{\theta}{2}\right]}$$

$$= -\cot\frac{\theta}{2}$$

Now, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right]$

$$= \frac{d}{dx}\left[-\cot\frac{\theta}{2}\right]$$

$$= \frac{d}{d\theta}\left[-\cot\frac{\theta}{2}\right] \times \frac{d\theta}{dx}$$

$$= -\frac{1}{2}\left[-\operatorname{cosec}^2\frac{\theta}{2}\right] \times \frac{1}{a(1-\cos\theta)}$$

$$= \frac{\operatorname{cosec}^2\frac{\theta}{2}}{2a(1-\cos\theta)}$$

$$= \frac{\operatorname{cosec}^2\frac{\theta}{2}}{2a\left(2\sin^2\frac{\theta}{2}\right)} \qquad \left[\because 1-\cos\theta = 2\sin^2\frac{\theta}{2}\right]$$

$$= \frac{1}{4a}\operatorname{cosec}^4\frac{\theta}{2}$$

Example 40 Find the second order derivative of x^x.

Solution $y = x^x$

$\therefore \log y = \log x^x$

or $\log y = x \log x$

Differentiating both sides w.r.t. x,

$$\frac{1}{y}\frac{dy}{dx} = x.\frac{1}{x} + (\log x)(1)$$

$$= 1+\log x$$

$$\therefore \frac{dy}{dx} = y(1+\log x)$$

Again Diff. w.r.t x,

$$\frac{d^2y}{dx^2} = y\left[\frac{1}{x}\right] + (1+\log x).\frac{dy}{dx}$$

$$= \frac{y}{x} + (1+\log x).y(1+\log x)$$

$$= \frac{x^x}{x} + x^x(1+\log x)^2$$

$$= x^{x-1} + x^x(1+\log x)^2$$

Example 41 If $y = a.e^{nx} + be^{-nx}$, find y_2

Solution

$y = a.e^{nx} + be^{-nx}$

$\therefore\ y_1 = ane^{nx} + be^{-nx}$

$= ane^{nx} - bne^{-nx}$

$\therefore\ y_2 = an(n)e^{nx} - bn(-n)e^{-nx}$

$= n^2a.e^{nx} + n^2be^{-nx}$

$= n^2[ae^{nx} + be^{-nx}]$

$y_2 = n^2y$

Example 42 If $y = a \cos x + b \sin x$
then show that $y_2 + m^2y = 0$

Solution

$y = a \cos mx + b \sin mx$

$y_1 = -m \sin mx + mb \cos mx$

$y_1 = -ma \sin mx + mb \cos mx$

$y_2 = -m^2a \cos mx - m^2b \sin mx$

Hence $y_2 = -m^2[a \cos x + b \sin x]$

$y_2 = -m^2y$

or $y_2 + m^2y = 0$

MULTIPLE CHOICE QUESTIONS

Q.1 If $y = x^3$; then $\frac{dy}{dx} = ?$

(a) $2x^3$ (b) $3x^2$ (c) $3x$ (d) $2x$

Q.2 If $y = \sqrt{x}$; then $\frac{dy}{dx} = ?$

(a) $\frac{1}{2}\sqrt{x}$ (b) $\frac{1}{\sqrt{x}}$ (c) $\frac{1}{2\sqrt{x}}$ (d) $2\sqrt{x}$

Q.3 If $y = \sqrt{x} - \frac{1}{\sqrt{x}}$; then $\frac{dy}{dx} = ?$

(a) $\frac{1}{\sqrt{x}} + \frac{1}{2\sqrt{x}}$ (b) $\frac{1}{2\sqrt{x}} - \frac{1}{2x}$ (c) $\frac{1}{2\sqrt{x}} + \frac{1}{2x\sqrt{x}}$ (d) None of these

Q.4 If $y =$ c (constant); then $\frac{dy}{dx} = ?$

(a) 0 (b) c (c) 1 (d) None of these

Q.5 If $f(x) = 4x^3 - 2x^2 + x + 1$; then $f'(x) = ?$

(a) $12x^2 - 4x + 2$ (b) $12x^2 - 4x + 1$ (c) $12x^3 - 4x + 1$ (d) None of these

Q.6 If $y = \sqrt{2x+3}$; then $\frac{dy}{dx}$ = ?

(a) $\frac{1}{2}\sqrt{2x+3}$ (b) $\frac{1}{\sqrt{2x+3}}$ (c) $\frac{2}{\sqrt{2x+3}}$ (d) $2\sqrt{2x+3}$

Q.7 If y = log x; x>0; then $\frac{dy}{dx}$ = ?

(a) x (b) $\frac{1}{x^2}$ (c) e^x (d) $\frac{1}{x}$

Q.8 If y = log sin x; then $\frac{dy}{dx}$ = ?

(a) cot x (b) tan x (c) $\frac{1}{\sin x}$ (d) sin x

Q.9 If $y = x^{-\frac{13}{7}}$, then $\frac{dy}{dx}$ = ?

(a) $-\frac{13}{7}x^{\frac{20}{7}}$ (b) $\frac{13}{7}x^{-\frac{20}{7}}$ (c) $\frac{13}{7}x^{-\frac{7}{20}}$ (d) $-\frac{13}{7}x^{-\frac{20}{7}}$

Q.10 If y = log sec x; then $\frac{dy}{dx}$ = ?

(a) sec x (b) $\frac{1}{\sec x}$ (c) – tan x (d) tan x

Q.11 If y = log cos x; then $\frac{dy}{dx}$ = ?

(a) $\frac{1}{\cos x}$ (b) –tan x (c) tan x (d) cot x

Q.12 If $y = e^x$; then $\frac{dy}{dx}$ = ?

(a) e^x (b) e^{-x} (c) 1 (d) 0

Q.13 If $y = e^{(3x+7)}$; then $\frac{dy}{dx}$ = ?

(a) e^{3x+7} (b) $e^{-(3x+7)}$ (c) 1 (d) $3e^{3x+7}$

Q.14 If $y = e^{-7x}$; then $\frac{dy}{dx}$ = ?

(a) e^{7x} (b) e^{-7x} (c) $-7e^{-7x}$ (d) None of these

Q.15 If $y = \log_a x$; then $\frac{dy}{dx} = ?$

(a) $\frac{1}{x}$ (b) $\frac{1}{x}\log_a e$ (c) $\frac{1}{x}\log_e a$ (d) None of these

Q.16 If $y = \log_5 \sin x$; then $\frac{dy}{dx} = ?$

(a) $\cot x \log_5 e$ (b) $\frac{1}{\sin x}$ (c) $\cot x$ (d) $\cot x \log_e 5$

Q.17 For $y = \log e^{x^3}$; then $\frac{dy}{dx} = ?$

(a) $\frac{1}{x^3}$ (b) x^3 (c) $3x^2$ (d) None of these

Q.18 For $y = a^x$; then $\frac{dy}{dx} = ?$

(a) a^x (b) $a^x \log a$ (c) $\frac{a^x}{\log a}$ (d) xa^{x-1}

Q.19 Fog $y = 3^{\sin x}$; then $\frac{dy}{dx} = ?$

(a) $3^{\sin x}$ (b) $3^{\sin x} \log 3$ (c) $3^{\sin x} \cos x \log 3$ (d) $\frac{3^{\sin x}}{\log^3}$

Q.20 For $y = x^a+a^x+e^x+a^a$; then $\frac{dy}{dx} = ?$

(a) $ax^{a-1}+a^x\log a+e^x$ (b) $ax^a+a^x\log a+e^x+a^a$
(c) $ax+a^x+e^x$ (d) $ax^{a-1}+a^x+e^x+a^a$

Q.21 For $y = (3x-5)^4$; then $\frac{dy}{dx} = ?$

(a) $4(3x-5)^3$ (b) $(3x-5)^3$ (c) $3(3x-5)^3$ (d) $12(3x-5)^3$

Q.22 If $y = \frac{x^3+2x^2-x}{\sqrt{x}}$;then $\frac{dy}{dx} = ?$

(a) $\frac{5}{2}x^{5/2}+3x^{3/2}-\frac{1}{2}\sqrt{x}$ (b) $\frac{5}{2}x^{3/2}+3x^{3/2}-1$

(c) $\frac{5}{2}x^{3/2} - 3x^{3/2} - \frac{1}{2}\sqrt{x}$ (d) $\frac{5}{2}x^{3/2} + 3\sqrt{x} - \frac{1}{2\sqrt{x}}$

Q.23 For y = x sin x; then $\frac{dy}{dx}$ = ?

(a) sin x +x (b) x cos x +1 (c) x cos x +sin x (d) x sin x +cos x

Q.24 For y = x log x; then $\frac{dy}{dx}$ = ?

(a) x +log x (b) 1 +log x (c) $x + \frac{1}{x}$ (d) $1 + \frac{1}{x}$

Q. 25 For y = x log sin x, then $\frac{dy}{dx}$ = ?

(a) x cot x +log sin x (b) $x + \frac{1}{\sin x}$ (c) x +log sin x (d) $1 + \frac{1}{\sin x}$

Q.26 If y = x $\log e^x$, then $\frac{dy}{dx}$ = ?

(a) x^2 (b) 2x (c) $x.\frac{1}{e^x} + e^x$ (d) $x+e^x$

Q.27 If $y = (2x-3)\sqrt{2x-3}$; $\frac{dy}{dx}$ = ?

(a) $3\sqrt{2x-3}$ (b) $\frac{3}{2\sqrt{2x-3}}$ (c) $\sqrt{2x-3}$ (d) $\frac{3}{2}\sqrt{2x-3}$

Q.28. Derivation of log (cos (log x)) w.r.t x is

(a) $\frac{1}{\cos(\log x)}$ (b) $\frac{1}{x\cos(\log x)}$ (c) $-\frac{1}{x}\tan(\log x)$ (d) $-\frac{1}{x}\cot(\log x)$

Q.29. $\frac{d}{dx}\log[f(x)]$ = ?

(a) $\frac{f'(x)}{f(x)}$ (b) $\frac{f(x)}{f'(x)}$ (c) $\frac{1}{f(x)}$ (d) $\frac{1}{f'(x)}$

Q.30 Derivative of x^x w.r.t x will be

(a) $x^x(x+\log x)$ (b) $\frac{x^x}{\log x}$ (c) x +log x (d) $x^x(1+\log x)$

Q.31 If y = tan(-5x+2), then $\frac{dy}{dx}$ = ?

(a) $\sec^2(-5x+2)$ (b) $-5\sec^2(-5x+2)$ (c) $-\sec^2(-5x+2)$ (d) $-5\text{cosec}^2(-5x+2)$

Q.32 If $y = a^5$; then $\frac{dy}{dx}$ = ?

(a) a^4 (b) $5a^4$ (c) a^5 (d) 0

Q.33 If $y = \sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$; then $\frac{dy}{dx}$ = ?

(a) $\tan^2 x$ (b) $\sec^2 x$ (c) 1 (d) tan x

Q.34 If $y = \log\sqrt{\frac{1-\cos}{1+\cos}}$; then $\frac{dy}{dx}$ = ?

(a) cosec x (b) sec x (c) $\frac{1+\cos}{1-\cos}$ (d) $\cot\frac{x}{2}$

Q.35 If $y = \frac{1+\tan}{1-\tan}$; then $\frac{dy}{dx}$ = ?

(a) $\tan\left(\frac{\pi}{4}+x\right)$ (b) $\cot\left(\frac{\pi}{4}+x\right)$ (c) $\sec^2\left(\frac{\pi}{4}-x\right)$ (d) $\sec^2\left(\frac{\pi}{4}+x\right)$

Q.36 If $y = \sqrt{1+\sin 2x}$; $\frac{dy}{dx}$ = ?

(a) sin x + cos x (b) sin x - cos x (c) cos x - sin x (d) 2cos 2x

Q.37 For $y = \sqrt{1-\sin x}$; $\frac{dy}{dx}$ = ?

(a) $\frac{1}{2}\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)$ (b) $-\frac{1}{2}\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)$

(c) $\pm\frac{1}{2}\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)$ (d) None of these

Q.38 If $y = \sqrt{1-\cos 2x}$; find $\frac{dy}{dx}$.

(a) $\sqrt{2}\sin x$ (b) $\sqrt{2}\cot x$ (c) cos x (d) sin x

Q.39 If $y = \sqrt{\frac{1+\cos 2x}{2}}$; find $\frac{dy}{dx}$

(a) sin x (b) -sin x (c) cos x (d) -cos x

Q.40 If y = log(cosec (3x+7)), then $\frac{dy}{dx}$ = ?

(a) -3cot(3x+7) (b) 3cot(3x+7) (c) sin(3x+7) (d) 3sin(3x+7)

Q.41 $y = \log(ax^2+bx+c)^4$; then $\frac{dy}{dx}$ = ?

(a) $\frac{1}{(ax^2+bx+c)}4$ (b) $4(ax^2+bx+c)^2$ (c) $\frac{4(2ax+b)}{(ax^2+bx+c)}$ (d) $\frac{4}{(ax^2+bx+c)}$

Q.42 For $y = \log(x+\sqrt{x^2-a^2})$; $\frac{dy}{dx}$ = ?

(a) $\frac{1}{x+\sqrt{x^2-a^2}}$ (b) $1+\frac{1}{2\sqrt{x^2-a^2}}$ (c) $1+\frac{x}{\sqrt{x^2-a^2}}$ (d) $\frac{1}{\sqrt{x^2-a^2}}$

Q.43 If y = $\sqrt{\log x+\sqrt{\log x+\sqrt{\log x.......\infty}}}$; then $(2y-1)\frac{dy}{dx}$ = ?

(a) x (b) log x+y (c) $\frac{1}{x}$ (d) $\sqrt{\log x+y}$

Q.44 If y = $\sqrt{\sin+\sqrt{\sin x+\sqrt{\sin+......\infty}}}$ then (2y–1) $\frac{dy}{dx}$ = ?

(a) sin x (b) $\sqrt{\sin x+y}$ (c) $\frac{1}{\sin x}$ (d) cos x

Q.45 If y = $\frac{1}{\sec x-\tan x}$ then $\frac{dy}{dx}$.

(a) sec x(sec x–tan x) (b) sex x (sec x +tan x)
(c) sec x + tan x (d) sec x – tan x

Q.46 If y = $\frac{\log x}{x}$; x > 0, then $\frac{dy}{dx}$?

(a) $\frac{1-\log x}{x^2}$ (b) $\frac{1+\log x}{x^2}$ (c) $\frac{1}{x^2}$ (d) $\frac{1}{x\log x}$

Q.47 If $y = \tan^2 x^3$, then $\frac{dy}{dx}$ = ?

(a) $2\tan x^3$ (b) $2\tan x^3 \sec^2 x^3$ (c) $2\tan x^2$ (d) $6x^2 \tan x^3$ sex^2 x^3

Q.48 Derivation of sin x° w.r.t. x. will be

(a) $\cos x^\circ$ (b) o (c) $\frac{\pi}{180}\cos x^\circ$ (d) $\cos\frac{\pi x}{180}$

Q.49 Derivative of $\sin\sqrt{b^2+x^2}$ w.r.t. x will be

(a) $2x\cos\sqrt{b^2+x^2}$ (b) $\frac{x\cos\sqrt{b^2+x^2}}{\sqrt{b^2+x^2}}$ (c) $\frac{\cos\sqrt{b^2+x^2}}{\sqrt{b^2+x^2}}$ (d) $\cos\sqrt{b^2+x^2}$

Q.50 For $y = \sqrt{\sin x}$; $\frac{dy}{dx} = ?$

(a) $\sqrt{\cos x}$ (b) $\frac{1}{2\sqrt{\cos x}}$ (c) $\frac{\cos x}{2\sqrt{\sin x}}$ (d) None of these

Q.51 If $x = at^2$, $y = 2at$, then $\frac{dy}{dx} = ?$

(a) t (b) t^2 (c) $\frac{2}{t}$ (d) $\frac{1}{t}$

Q.52 If $x = 5t + t^3$, $y = t^2+8t$, then $\frac{dy}{dx} = ?$

(a) $\frac{2t+8}{5+3t^2}$ (b) $\frac{5+3t^2}{2t+8}$ (c) $\frac{5-3t^2}{2t+8}$ (d) None of these

Q.53 Derivative of $(2x+3)^5$ w.r.t (x^2+3x+5) is

(a) $5(2x+3)^4$ (b) $5(2x+3)^3$ (c) $10(2x+3)^3$ (d) None of these

Q.54 Derivative of $\frac{x^2}{1+x^2}$ w.r.t x^2 will be

(a) $\frac{2x}{(1+x^2)^2}$ (b) $\frac{x}{(1+x^2)^2}$ (c) $\frac{x^2}{(1+x^2)^2}$ (d) $(1+x^2)^{-2}$

Q.55 If $\sqrt{x}+\sqrt{y}=5$, then the value of $\frac{dy}{dx}$ at (4,9) will be

(a) $-\frac{2}{3}$ (b) $\frac{3}{2}$ (c) -2 (d) $-\frac{3}{2}$

Q.56 If $3x^2+2xy+3y^2+8 = 0$ then $\frac{dy}{dx} = ?$

(a) $\frac{3x+y}{x+3y}$ (b) $\frac{-(3x+y)}{(x+3y)}$ (c) x+y (d) $\frac{x+y}{x+3y}$

Q.57 For $y = x^{\sin x}$, then $\frac{dy}{dx} = ?$

(a) $x^{\sin x}\left[\frac{\sin x + \cos x \log x}{x}\right]$

(b) $\left[\frac{\sin x + \cos . \log x}{x}\right]$

(c) $x^{\sin x}\left[\frac{\sin x + x\cos x \log x}{x}\right]$

(d) None of these

Q.58 For $y = \log\left(\frac{x^2 \sin x e^x}{\log x}\right)$, then $\frac{dy}{dx} = ?$

(a) $2 \log x + \cot x + 1 - \frac{1}{x \log x}$

(b) $\log x + \text{cosec}\, x - \frac{1}{x \log x}$

(c) $2\log x + \tan x + 1 - \frac{1}{x \log x}$

(d) None of these

Q.59 For $y = \log(\log(\log(x+1)))$, then $\frac{dy}{dx} = ?$

(a) $\frac{1}{\log(\log(x+1))}$

(b) $\frac{1}{\log(\log(x+1))(\log(x+1)}$

(c) $\frac{1}{(x+1)\log[\log(x+1)]\log(x+1)}$

(d) None of these

Q.60 Derivative of $\sin^2 x$ w.r.t. $\sin x^2$ will be

(a) $\frac{\sin x \cos x}{\cos x^2}$ (b) $\frac{\sin x}{x \cos x^2}$ (c) $\frac{\sin x}{x^2 \cos x^2}$ (d) $\frac{\sin x \cos x}{x \cos x^2}$

Q.61 For $y = \sqrt[4]{x^3}$, then $\frac{dy}{dx} = ?$

(a) $\frac{1}{4}x^{-\frac{1}{4}}$ (b) $-\frac{3}{4}x^{-\frac{1}{4}}$ (c) $\frac{4}{3}x^{-\frac{1}{4}}$ (d) $\frac{3}{4}x^{-\frac{1}{4}}$

Q.62 If $f(x) = \frac{x}{x+1}$, then $f'(2) = ?$

(a) $\frac{1}{9}$ (b) $\frac{1}{3}$ (c) 9 (d) None of these

Q.63 If $y = \sqrt{1+x^2}$, then $y\frac{dy}{dx} - x = ?$

Chapter 3

TANGENTS AND NORMALS

Tangent – Let $y = f(x)$ be a curve and let P (x_1y_1) be any point on the curve at which the function is derivable. Then the tangent to the curve $y = f(x)$ at point $P(x_1y_1)$ is the line touching the curve at point $9(x_1y_1)$.In the figure 3.1, the line MN is the tangent to the curve $y= f(x)$ at point P.

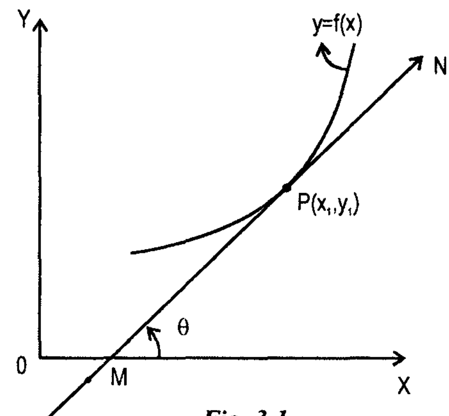


Fig. 3.1

Slope of tangent – The tangent of the angle subtended by the tangent to a curve $y = f(x)$ at a point P with the positive direction of x - axis is called slope of tangent to the curve $y = f(x)$ at point P.

In the figure 3.1, slope of tangnt MN to the curve $y = (x)$ at point P (x_1y_1) is $\tan\theta$

i.e. slope of tangent MN to curve $y = f(x)$ at $P(x_1y_1) = \tan\theta$

From our previous knowledge of differentation, we know that the derivative of a function at a point $P(x_1y_1)$ is the slope of tangent to the curve $y = f(x)$ at point $P(x_1y_1)$

$$\therefore \text{ slope of tangent to } y = f(x) \text{ at point p } (x_1y_1) = \tan\theta = \left(\frac{dy}{dx}\right)_{(x_1,y_1)}$$

EQUATION OF TANGENT

Equation of tangent to the curve $y = f(x)$ at point p (x_1y_1) is given by

$$(y-y_1) = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} (x-x_1)$$

(using slope - point form of st. line)

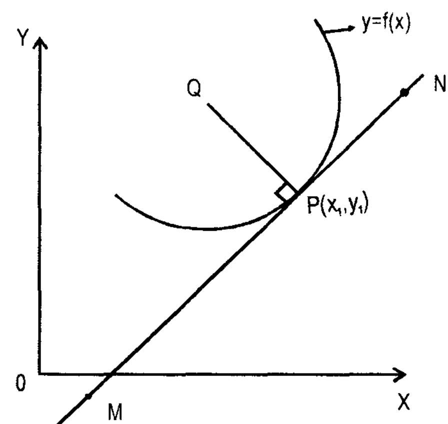


Fig. 3.2

NORMAL - Let $P(x_1y_1)$ be any point on the Curve $y = f(x)$. Then Normal to the curve at point $P(x_1y_1)$, is the line perpendicular to the tangent to curve at the pont of contact $P(x_1y_1)$.

In the figure 3.2 the line PQ represent Normal to the cure $y= f(x)$ at the point $P(x_1y_1)$ on the curve.

SLOPE OF NORMAL - From the knowledge of straight line we know that two straight lines are perpendicular to each other if the product of their slopes is –1 or their slopes are –ve reciprocal of each other.

$$\text{Thus slope of Normal at P} = \frac{-1}{\text{Slope of tangent at P}}$$

$$\therefore \quad \text{Slope of Normal at } P(x_1y_1) = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}}$$

EQUATION OF NORMAL - Equation of normal to the curve y = f(x) at the point $P(x_1y_1)$ on the curve y = f(x) is given by

$y - y_1 = (x - x_1)$ [By slope – point form of the line]

CONCLUSIONS:- $\frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}}$

(1) Slope of tangent at $P(x_1y_1) = \left(\frac{dy}{dx}\right)_{(x_1,y_1)}$

(2) Slope of Normal at $P(x_1y_1) = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}}$

(3) Equation of tangent at $P(x_1y_1)$ is $(y-y_1) = \left(\frac{dy}{dx}\right)_{(x_1,y_1)}(x-x_1)$

(4) Equation of normal at $P(x_1y_1)$ is $(y-y_1) = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}}(x-x_1)$

Remarks: 1) The point $P(x_1y_1)$ at which the tangent to the curve touches the Curve y = f(x), is called point of contact.

2) If the tangent is parallel to X - axis ($\perp$r to Y- axis), $\theta = 0°$. $\therefore$ slope of tangent to the Curve of y = f(x) will be $\frac{dy}{dx} = \tan 0° = 0$

3) If the tangent is perpendicular to the X-axis ($\perp$r to Y - axis) then $\theta = 90°$, thus slope of tangent to the cure y= f(x) will be, $\frac{dy}{dx} = \tan 90° = \infty$

Example 1 Find the slope of tangent to the curve $y = 4x^3 - 2x^2 + 3$ at the point x = 2

Solution : $y = 4x^3 - 2x^2 + 3$

$$\frac{dy}{dx} = 12x^2 - 4x$$

$\therefore$ Slope of tangent at point x = 2

$$= \left(\frac{dy}{dx}\right)_{x=2} = 12(2)^2 - 4(2)$$

$$= 48 - 8 \quad = 40$$

Example 2 Find the slpoe of tangent to the curve $x^2 - xy + y^2 - 7 = 0$ at the point (1,3).

Solution: Equation of curve is

$x^2 - xy + y^2 - 7 = 0$

Diff. w.r.t. x,

$$2x - x\frac{dy}{dx} - y.(1) + 2y\frac{dy}{dx} = 0$$

$$(2y - x)\frac{dy}{dx} = y - 2x$$

$$\therefore \quad \frac{dy}{dx} = \frac{(y-2x)}{(2y-x)}$$

$\therefore$ slope of tangent to given curve at point (1, 3) will be

$$\left(\frac{dy}{dx}\right)_{(1,3)} = \frac{3-2(1)}{2(3)-1} = \frac{1}{5}$$

$\therefore$ Slole of tangent to the given curve is $\frac{1}{5}$

Example 3 Find the slope of tangent to the curve having parametric equations $x = t^2 - t + 1$, $y = 3t + 4$ at the point t = 5.

Solution : $x = t^2 - t + 1$

$$\therefore \quad \frac{dx}{dt} = 2t - 1$$

and $y = 3t + 4$

$$\therefore \quad \frac{dy}{dt} = 3$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3}{2t-1}$$

$\therefore$ slope of tangent to the given curve at the point t = 5 will be

$$\left(\frac{dy}{dx}\right)_{t=5} = \frac{3}{2(5)-1} = \frac{3}{9} = \frac{1}{3}$$

Example 4 Find the slope of normal to the curves at the given points

(i) $y = x^3 + 4x^2 - 5$ at $x = -2$

(ii) $x^2 + y^2 = 16$ at $(3, \sqrt{7})$

(iii) $x = a(\theta + \cos\theta)$; $y = 2a \sin\theta$ at $\theta = \frac{\pi}{6}$

Solution : (i) Given curve is $y = x^3 + 4x^2 - 5$

$$\frac{dy}{dx} = 3x^2 + 8x$$

$\therefore$ slope of normal at $x = -2$

$$\frac{-1}{\left(\frac{dy}{dx}\right)_{x=-2}} = \frac{-1}{3(-2)^2 + 8(-2)}$$

$$= \frac{-1}{-4} = \frac{1}{4}$$

(ii) Given curve is $x^2 + y^2 = 16$; at $(3, \sqrt{7})$.

Diff. w.r.t. x

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

$$\therefore \left(\frac{dy}{dx}\right)_{(3,\sqrt{7})} = -\frac{3}{\sqrt{7}}$$

$\therefore$ Slope of normal to the curve at point $(3, \sqrt{7})$

$$= \frac{-1}{\left(\frac{dy}{dx}\right)_{(3,\sqrt{7})}} = \frac{-1}{(-3/\sqrt{7})} = \frac{\sqrt{7}}{3}$$ *Ans.*

(iii) Given curve is $x = a(\theta + \cos\theta)$; $y = 2a\sin\theta$ at $\theta = \frac{\pi}{6}$.

$$\frac{dx}{d\theta} = a[1 - \sin\theta]$$

and $\frac{dy}{d\theta} = 2a\cos\theta$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2a\cos\theta}{a(1-\sin\theta)}$$

$$= \frac{2\cos\theta}{(1-\sin\theta)}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{6}} = \frac{2\cos\frac{\pi}{6}}{\left(1-\sin\frac{\pi}{6}\right)}$$

$$= \frac{2\left(\frac{\sqrt{3}}{2}\right)}{1-\frac{1}{2}} = \frac{\sqrt{3}}{\frac{1}{2}} = 2\sqrt{3}$$

Example 5: Find the slope of tangent and slope of normal to the curve $y = u^2 + 1$; $u = \sin 2x$ at $x = \frac{\pi}{8}$

Solution : $y = u^2 + 1$; $u = \sin 2x$

$$\frac{dy}{du} = 2u \quad ; \quad \frac{du}{dx} = 2\cos 2x$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$= 2u \times 2\cos 2x$

$= 2\sin 2x\,(2\cos 2x)$

$= 2[2\sin 2x\cos 2x]$

$= 2\sin 4x$, $[\because 2\sin\theta\cos\theta = \sin 2\theta]$

$\therefore$ slope of tangent at $x = \frac{\pi}{8}$ will be

$$\left(\frac{dy}{dx}\right)_{x=\pi/8} = 2 \sin 4 \left(\frac{\pi}{8}\right)$$

$$= 2 \sin \frac{\pi}{2}$$

$$= 2(1) = 2$$

and alope of Normal at $x = \frac{\pi}{8}$

$$= \frac{-1}{\left(\frac{dy}{dx}\right)_{x=\pi/8}} = -\frac{1}{2}$$

Example 6 Find the point at which the curve $y = 10 + 2x - x^2$ has its slope unity.

Solution : Given curve is $y = 10 + 2x - x^2$

$$\frac{dy}{dx} = 2-2x$$

$\therefore$ Beacuse slope is unity, therefore Put $\frac{dy}{dx} = 1$

$\therefore$ or $2 - 2x = 1$

or $-2x = -1$

or $x = \frac{1}{2}$

putting $x = \frac{1}{2}$ in the given equation,

$$y = 10 + 2\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^2 = 10 + 1 - \frac{1}{4}$$

$$= 11 - \frac{1}{4} = \frac{44-1}{4} = \frac{43}{4}$$

$\therefore$ The required point is $\left(\frac{1}{2}, \frac{43}{4}\right)$.

Example 7 Find the equation of tangent to the curves at the given points -

(i) $y = x^2 - 4x + 3$ at (4,3).

(ii) $x^2 - y^2 = 9$ at (5,4)

(iii) $x = at^2$, $y = 2at$ at $t = 1$

(iv) $x = a(\theta - \sin\theta)$; $y = a(1-\cos\theta)$ at $\theta = \frac{\pi}{2}$

Solution : (i) Given curve is $y = x^2 - 4x + 3$

$$\frac{dy}{dx} = 2x - 4$$

$$\left(\frac{dy}{dx}\right)_{(4,3)} = 2(4) - 4 = 4$$

$\therefore$ Eq. of tangent at (4,3) will be

$$(y - y_1) = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} (x - x_1)$$

$(y - 3) = 4(x - 4)$ [Here $(x_1 y_1) = (4,3)$]

$y - 3 = 4x - 16$

$-4x + y - 3 + 16 = 0$

$-4x + y + 13 = 0$

or $4x - y - 13 = 0$

(ii) Given curve is $x^2 - y^2 = 9$ at (5,4)

or $2x - 2y\dfrac{dy}{dx} = 0$

$\therefore \dfrac{dy}{dx} = \dfrac{x}{y}$

Now, $\left(\dfrac{dy}{dx}\right)_{(5,4)} = \dfrac{5}{4}$

$\therefore$ e.q. of tangent at (5,4) will be

$$(y - 4) = \frac{5}{4}(x - 5)$$

or $4y - 16 = 5x - 25$

or $5x - 4y - 9 = 0$

(iii) Given curve is $x = at^2$ and $y = 2at$

$$\frac{dx}{dt} = a(2t); \quad \frac{dy}{dt} = 2a(1)$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$$

$$\therefore \left(\frac{dy}{dx}\right)_{t=1} = \frac{1}{1} = 1$$

Now, $(at^2, 2at)_{t=1}$

$= (a, 2a)$

$\therefore$ eq. of tangent at (a, 2a) will be

$(y - 2a) = 1(x - a)$

$y - 2a = x - a$

or $x - y + a = 0$

(iv) Given curve is $x = a(\theta \square \sin\theta)$ and $y = a(1 - \cos\theta)$

$$\frac{dx}{d\theta} = a[1 - \cos\theta] \text{ and } \frac{dy}{d\theta} = a[-(-\sin\theta)] = a\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1-\cos\theta)}$$

$$= \frac{\sin\theta}{(1-\cos\theta)}$$

$$\therefore \left(\frac{dy}{dx}\right)_{\theta=\pi/2} = \frac{\sin\frac{\pi}{2}}{\left(1-\cos\frac{\pi}{2}\right)} = \frac{1}{(1-0)} = 1$$

Given point at $\theta = \frac{\pi}{2}$ is $a\left(\frac{\pi}{2} - \sin\frac{\pi}{2}\right), a\left(1 - \cos\frac{\pi}{2}\right)$ or $a\left(\frac{\pi}{2} - 1\right), a$

$\therefore$ Equation of tangent to the curve at given point is

$$(y-a) = \left[x - a\left(\frac{\pi}{2} - 1\right)\right]$$

$$y - a = x - \frac{a\pi}{2} + a$$

or $2x - 2y - a\pi + 4a = 0$

Example 8 Find the equation of Normal to the curve at the indicated points:-

(i) $y = x^4 - 5x^2 + 7$ at $x = -1$

(ii) $y^2 = 12x$ at (3,6)

Solution

(i) Given curve is $y = x^4 - 5x^2 + 7$

$$\frac{dy}{dx} = 4x^3 - 10x$$

$$\therefore \left(\frac{dy}{dx}\right)_{x=-1} = 4(-1)^3 - 10(-1) = -4 + 10 = 6$$

Now , $x_1 = -1$

$\therefore \quad y_1 = (-1)^4 - 5(-1)^2 + 7$

$= 1 - 5 + 7 \quad = 3$

$\therefore$ Equation of Normal to the curve at (–1,3) will be

$$(y - 3) = \frac{-1}{\left(\frac{dy}{dx}\right)_{(-1,3)}}(x-(-1))$$

$$(y - 3) = -\frac{1}{6}(x-(-1))$$

$$6(y - 3) = -x-1$$

$$x+6y - 17 = 0$$

(ii) Given curve is $y^2 = 12x$

$$2y\frac{dy}{dx} = 12$$

$$\frac{dy}{dx} = \frac{6}{y}$$

$$\therefore \left(\frac{dy}{dx}\right)_{(3,6)} = \frac{6}{6} = 1$$

Eq. of Normal to the curve at (3,6) will be

$$(y - 6) = \frac{-1}{\left(\frac{dy}{dx}\right)_{(3,6)}}(x-3)$$

$$(y - 6) = \frac{-1}{1}(x-3)$$

$$y - 6 = -x + 3$$

or $x + y - 6 - 3 = 0$

$$x + y - 9 = 0$$

MULTIPLE CHOICE QUESTIONS

Q1. For a function y = f(x), the slope of tangent at a point P (x_1y_1) is given by

(a) $-\left(\frac{dy}{dx}\right)_{x=x_1}$ (b) $\left(\frac{dx}{dy}\right)_{x=x_1}$ (c) $\left(\frac{dy}{dx}\right)_{x=x_1}$ (d) $\frac{-1}{\left(\frac{dy}{dx}\right)_{x=x_1}}$

Q2 For a function y = f(x), the slope of Normal at apoint $P(x_1y_1)$ is given by

(a) $-\left(\frac{dy}{dx}\right)_{x=x_1}$ (b) $\left(\frac{dx}{dy}\right)_{x=x_1}$ (c) $\left(\frac{dy}{dx}\right)_{x=x_1}$ (d) $\frac{-1}{\left(\frac{dy}{dx}\right)_{x=x_1}}$

Q3. If m_1 and m_2 are respectively the slopes of tangent and Normal to the curve y = f(x), then

(a) $m_1 = m_2$ (b) $m_1 . m_2 = 1$ (c) $m_1.m_2 = -1$ (d) None of these

Q4. If the tangent at a point on a curve y = f(x) is parallel to X-axis, then its slope will be

(a) 1 (b) 0 (c) -1 (d) $\sqrt{3}$

Q5. If the tanget at apoint on a curve y = f(x) is perpendicular to Y -axis, then, its slope will be

(a) 1 (b) 0 (c) -1 (d) $\sqrt{3}$

Q6. If the tangent to be curve of y = f(x) at $P(x_1, y_1)$ is | | to Y - axis then,

(a) $\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \infty$ (b) $\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = -1$ (c) $\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 0$ (d) none of these

(e) Both (a) and (c)

Q7. If the tangent to the curve of y = f(x) at $P(x_1, y_1)$ is perpendicular r to X - axis , then

(a) $\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 0$ (b) $\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \infty$ (c) $\left(\frac{dx}{dy}\right)_{(x_1,y_1)} = \alpha$ (d) none of these

Q8. It the normal to the curve y = f(x) at P (x_1, y_1) is | | to X - axis (or $\perp$r to Y- axis)

then $\left(\frac{dy}{dx}\right)_{(x_1,y_1)}$ is equal to

(a) $\propto$ (b) 0 (c) 1 (d) -1

Q9. If the normal to the curve of y = f(x) at P (x_1y_1) is $\perp$r to x- axis (or | | to Y - axis)

then $\left(\frac{dy}{dx}\right)_{(x_1,y_1)}$ is equal to

(a) ∞ (b) 1 (c) 0 (d) -1

Q10. If the slope of tangent at a point P on a curve is $\sqrt{3}$, then the slope of normal to the curve at point P will be

(a) $\sqrt{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) $-\frac{1}{\sqrt{3}}$ (d) $-\sqrt{3}$

Q11. If the slope of normal at a point P on the curve y = f(x) is 2, then the slope of tangent to the curve at point P will be

(a) $-\frac{1}{2}$ (b) 2 (c) $\frac{1}{2}$ (d) -2

Q12. If the tangent to a curve y= f(x) at a point P makes an angle of 45° with the + ve direction of X - axis, then the slope of tangent will be

(a) 1 (b) -1 . (c) $\sqrt{3}$ (d) $\frac{1}{\sqrt{3}}$

Q13. If the normal at a point P on the curve of y= f(x) makes an angle of 135° with the +ve direction of X–axis, then the slope of normal will be

(a) -1 (b) 1 (c) $\sqrt{3}$ (d) $-\frac{1}{\sqrt{3}}$

Q14. The angle between the tangent and normal at a point P on the curve of any function f (x) is
(a) 60° (b) 0° (c) 90° (d) 180°

Q15. If the slope of a tangent to a curve is $\sqrt{3}$ then the angle that the tangent subtends with the direction of x–axis is
(a) 90° (b) 30° (c) 45° (d) 60°

Q16. If the slope of tangent to a curve y = f(x) at a point $P(x_1,y_1)$ is $\frac{1}{\sqrt{3}}$, yhen $\left(\frac{dy}{dx}\right)$ at (x_1,y_1) is

(a) $\sqrt{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) $-\sqrt{3}$ (d) $-\frac{1}{\sqrt{3}}$

Q17. If $\left(\frac{dy}{dx}\right)_P$ at a point P on the curve of a function y = f(x) is -1, then, the angle subtended by tangent at point P on the curve with +ve direction of x–axis is
(a) 45° (b) 60° (c) 135° (d) 30°

Q18. The slope of tangent to the Curve $y = x^2 + x-7$ at x= 2 is
(a) -1 (b) 5 (c) 3 (d) -3

Q19. The slope of Normal to the curve $y = 3x^2 - 2x + 8$ at $x = 1$ is

(a) -4 (b) $-\frac{1}{4}$ (c) 2 (d) 1

Q20. The equation of tangent to the curve $y = x^2 + x - 7$ at $(2, -1)$ is

(a) $x + y - 11 = 0$ (b) $5x + y - 11 = 0$
(c) $5x - y - 11 = 0$ (d) $x + y + 7 = 0$

Q21. The equation of normal to the curve $y = 3x^2 - 2x + 8$ at the point (1,9) is

(a) $x + 4y - 37 = 0$ (b) $x + y + 3 = 0$
(c) $x - y + 3 = 0$ (d) $2x + y + 7 = 0$

Q22. The equation of tangent to the curve $\sqrt{x} + \sqrt{y} = 7$ at the point (9, 16) is

(a) $4x + 3y - 14 = 0$ (b) $3x + 4y - 14 = 0$
(c) $4x - 3y + 84 = 0$ (d) $4x + 3y - 84 = 0$

Q23. The equation of normal to the curve $x^2 + y^2 = 13$ at point (2,3) is

(a) $2x - 3y = 0$ (b) $2x + 3y = 0$ (c) $3x - 2y = 0$ (d) $x - y = 0$

Q24. The equation of tangent to the curve $x = t^2 + 2t + 3$ and $y = t^2 + t - 1$ at the point $t = 1$ is

(a) $3x + 4y - 14 = 0$ (b) $3x - 4y - 14 = 0$
(c) $x - y + 7 = 0$ (d) $3x + y - 14 = 0$

Q25. The equation of Normal to the curve $x = t^2 + 1$ and $y = t + 4$ at point $t = 2$ is

(a) $4x + y - 26 = 0$ (b) $x + y - 26 = 0$
(c) $x + 4y - 6 = 0$ (d) $x + y + 9 = 0$

Q26. The equation of tangent to the curve $y = 2u$ and $u = e^x$ at point $x = 1$ is

(a) $2ex + y = 0$ (b) $2ex - y = 0$ (c) $ex - 2y = 0$ (d) $x - ey = 0$

Q27. The equation of normal to the curve $y = \sin z$; $z = x^2 + \frac{\pi}{2}$ at $x = 0$ is

(a) $x = 0$ (y -axis) (b) $y = 0$ (x - axis) (c) $x + y = \pi$ (d) $x - y = 1$

Q28. The point at which the tangent to the curve $y = x^3 - 2x^2 + 4$ is || to x-axis are

(a) (0,4); $\left(\frac{4}{3}, 4\right)$ (b) (0,2); $\left(\frac{4}{3}, 4\right)$ (c) (0, 4); $\left(\frac{4}{3}, \frac{76}{27}\right)$ (d) None of these

Q29. The point at which the tangent to the curve $y = x^3 - 2x^2 + 4$ makes an angle of 135° with the +ve direction of x -axis is

(a) (1,3) (b) (2,3) (c) (0,3) (d) None of these

Q30. The points at which the tangents to the curve $x^2 + y^2 = 4$ are $\perp$r to x - axis (|| to y-axis) are

(a) $(0, \pm 2)$ (b) $(\pm 2, 0)$ (c) $(\pm 3, 0)$ (d) $(\pm 3, 4)$

Q 31. The points at which the tangents to the curve $x^2 + y^2 = y$ are || to X - axis are

(a) $(0, \pm 2)$ (b) $(\pm 2, 0)$ (c) $(\pm 3, 0)$ (d) $(\pm 3, 4)$

Chapter **4**

RATE OF CHANGE OF QUANTITIES

To find the rate of change of one variable with respect to another variable, we take help of differentiation.

Suppose y(x) is a function of variable x, then we know by the definition of derivative of y w.r.t. x,

$$\frac{dy}{dx} = \underset{\delta x \to 0}{Lt} \frac{f(x+\delta x)-f(x)}{\delta x}$$

$$= \underset{\delta x \to 0}{Lt} \frac{(y+\delta y)-y}{\delta x}$$

$$= \underset{\delta x \to 0}{Lt} \frac{\delta y}{\delta x} = \text{Rate of change of y w.r.t. x, when } \delta x \to 0$$

Thus $\frac{dy}{dx}$ or $f'(x)$ defines the rate of increment in y w.r.t increment in x.

Remark : (1) If $\frac{dy}{dx}$ is +ve, then y increases with increase in x.

(2) If $\frac{dy}{dx}$ is -ve, then y decreases with increase in x.

(3) If x and y are functions of t(time) i.e. x=f(t) and y=g(t),

then $$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

or $$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

(Time rate of change of y = $\frac{dy}{dx}$ (Time rate of change of x))

Rate of Change of Some Important Physical Quantities

(1) Velocity - Velocity of a body is defined as the time rate of change of displacement.

$\therefore$ Velocity = $\frac{dx}{dt}$; where x denotes displacement and t denotes time.

$$\text{Velocity} = \underset{\delta t \to 0}{Lt} \frac{\delta x}{\delta t} = \frac{dx}{dt}$$

(2) Acceleration - Acceleration of a moving body is defined as the time rate of change of velocity.

$$\text{Acceleration} = \frac{\text{Change in velocity}}{\text{(Time interval during the change)}}$$

$$= \underset{\delta t \to 0}{Lt} \frac{\delta v}{\delta t} = \frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt}$$

$$\text{Also Acceleration} = \frac{dv}{dt} = \frac{dv}{dx} \times v = v\frac{dv}{dx}$$

$$= \frac{d}{dt}\left(\frac{dx}{dt}\right)$$

$$= \frac{d^2x}{dt^2}$$

$$\text{Hence Acceleration} = \frac{dv}{dt} = \frac{d^2x}{dt^2} = v\frac{dv}{dx}$$

(3) Current - Current through a conductor is defined as the rate of charge flowing per unit time through the conductor. It is denoted by I, the time must be very small i.e. $\delta t \to 0$

$$\therefore \quad I = \underset{\delta t \to 0}{Lt} \frac{\delta q}{\delta t} = \frac{dq}{dt}$$

(4) Rate of change of Area - During expansion or contraction, the surface area of a body changes with time, so, Rate of change of area can be evaluated by Differentiating Area w.r.t time.

Now, $\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$

Thus we can find $\frac{dA}{dt}$ if we know, the relation between A and x i.e. A = f(x) and

$\frac{dx}{dt}$ Rate of change of x w.r.t time t.

[Generally x denotes sides for rectangle and square and radius for circle]

(5) Rate of change of Volume - When ever, there is an increase in the volume of a body due to one or another reason, such as increase in the volume of a balloon when air is pumped in it, or increasing volume of a heap of sand when falling from a cylindrical pipe etc. we can evaluate it by differentiating V(volume) w.r.t time t.

i.e. $\frac{dV}{dt}$

If V = f(x); where V depends on x

Then $\frac{dV}{dt} = \frac{dV}{dx}.\frac{dx}{dt}$

(6) Calculation of Errors and Percentage errors - Let y = f(x) be a function of x.

Now $\frac{dy}{dx} = \underset{\delta x \to 0}{Lt} \frac{\delta y}{\delta x}$

Hence $\delta x \to 0$ means δx is very small, so for small values of δx, $\frac{\delta y}{\delta x} = \frac{dy}{dx}$

$\therefore \quad \delta y = \frac{dy}{dx}.\delta x$

So, if δx denotes the small error in x then, the corresponding error in y will be

$$\delta y = \frac{dy}{dx}.\delta x$$

$\frac{\delta x}{x}$ is called Relative error in x and $\frac{\delta x}{x}.100$ is called percentage error in x.

Example 1 The displacement of a particle at any time t is given by $S = t^3-3t^2+5t+7$, Find its velocity and acceleration at the end of 2 hrs, where S is in kilometers and t is in hrs.

Solution : $S = t^3-3t^2+5t+7$

Velocity $= v = \frac{ds}{dt} = 3t^2-6t+5$

Acceleration $= a = \frac{d^2s}{dt^2} = 6t - 6$

$\therefore$ Velocity at the end of 2 hrs $= \left(\frac{ds}{dt}\right)_{t=2} = 3(2)^2-6(2)+5$

$= 12-12+5 \quad = 5$ km/hr

Acceleration at the end of 2hrs $= \left(\frac{d^2s}{dt^2}\right)_{t=2} = 6(2)-6$

$= 12-6$

$= 6km/hr^2$

Hence, velocity $= 5km/hr$

and Acceleration $= 6km/hr^2$

Example 2 The charge flowing in a conductor is given by expression $Q = t^3+3t^2+9t+2$ Then find the current flowing in the conductor after 2 seconds, when Q is measured in coulombs and t in seconds.

Solution : Current (I) $= \frac{dQ}{dt}$

$= 3t^2+6t+9$

$\therefore \quad (I)_{t=2} = 3(2)^2+6(2)+9$

$= 12+12+9$

$= 33$ coulombs / Second

$= 33$ Ampere

Example 3 A circular plate of metal is being expanded by heating. The radius of plate increases at the rate of 0.3 cm per second. Find the rate of increase of Area when its radius become 15cm.

Solution : We know that Area of a circular plate is given by $A = \pi r^2$, where r is the radius of plate and we are given $\frac{dr}{dt} = 0.3$ cm/sec. to find $\frac{dA}{dt}$ at r = 15cm.

Let $A = \pi r^2$

$\therefore \quad \frac{dA}{dt} = \pi(2r)\frac{dr}{dt}$

$= 2\pi r\frac{dr}{dt}$

$= 2\pi r(0.3)$

$\therefore \quad \left(\frac{dA}{dt}\right)_{r=15cm} = 2\pi(15)(0.3)$

$= 30\times0.3\pi \ cm^2/sec.$

$= 9\pi \ cm^2/sec.$

$= 9\times3.14 \ cm^2/sec.$

$= 28.26 \ cm^2/sec.$

Example 4 The sides of an equilateral triangle is increasing at the rate of 2cm/sec. Find the rate at which the area increases, when the side is 5cm.

Solution : Area of an equilateral triangle having side 'a' cm is given by $A = \frac{\sqrt{3}}{4}a^2$

We are given, $\frac{da}{dt} = 2\text{cm/sec.}$ and to find $\left(\frac{dA}{dt}\right)_{a=5cm}$

$$A = \frac{\sqrt{3}}{4}a^2$$

$$\therefore \quad \frac{dA}{dt} = \frac{\sqrt{3}}{4}(2a)\frac{da}{dt}$$

$$= \frac{\sqrt{3}}{2}a(2)$$

$$\therefore \quad \left(\frac{dA}{dt}\right)_{a=5cm} = \sqrt{3}(5)$$

$$= 5\sqrt{3} \ \text{cm}^2/\text{sec.}$$

Example 5 The volume of a sphere is increasing at the rate of 0.2 cm^3 per second. Find the rate of increase of radius when the radius is 10 cm.

Solution : Volume of sphere is given by

$$V = \frac{4}{3}\pi r^3$$

We are given $\frac{dv}{dt} = 0.2\ \text{cm}^3/\text{sec.}$ and we have to find $\left(\frac{dr}{dt}\right)_{r=10}$

$$\text{Now, } V = \frac{4}{3}\pi r^3$$

$$\therefore \quad \frac{dV}{dt} = \frac{4}{3}\pi\left(3r^2\frac{dr}{dt}\right)$$

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

$$\therefore \quad 0.2 = 4\pi r^2\left(\frac{dr}{dt}\right)$$

$$\therefore \quad \frac{dr}{dt} = \frac{0.2}{4\pi r^2}$$

$$\therefore \quad \left(\frac{dr}{dt}\right)_{r=10} = \frac{0.2}{4\pi(10)^2}$$

$$= \frac{0.2}{400\pi} = \frac{1}{2000\pi} \text{ cm/sec.}$$

Example 6 Find the error and %age error in calculating the Area of a circular plate when an error of 0.4% has been committed while measuring the radius of that circle.

Solution : %age error in radius = 0.4%

$$\therefore \quad \frac{\delta r}{r} \times 100 = 0.4$$

$$\therefore \quad \delta r = \frac{0.4r}{100} \qquad \text{------ (1)}$$

Now area of circular plate i.e. $A = \pi r^2$

$$\therefore \quad \frac{dA}{dr} = 2\pi r$$

$$\text{Now } \delta A = \frac{dA}{dr}.\delta r$$

$$\therefore \quad \delta A = (2\pi r)\left[\frac{0.4r}{100}\right] \quad , \text{ using (1)}$$

$$= \frac{0.8\pi r^2}{100} = \frac{8\pi r^2}{1000}$$

$$\delta A = \frac{\pi r^2}{125}$$

and %age error in Area

$$= \frac{\delta A}{A} \times 100$$

$$= \frac{\frac{\pi r^2}{125}}{\pi r^2} \times 100 = \frac{1}{125} \times 100 = \frac{4}{5} = 0.8\%$$

Example 7 If $y = x^3 - 5$ and x changes from 3 to 2.99, what is the approximate change in y.

Solution: $y = x^3-5$

$$\frac{dy}{dx} = 3x^2$$

Now, approximate change in y can be calculated as

$$\delta y = \frac{dy}{dx}.\delta x$$

$$= 3x^2[2.99-3]\ldots\ldots \qquad [\because \delta x = 2.99-3]$$

$$= 3x^2(-0.01)$$

$$= 3(3)^2(-0.01) = -0.27 \text{ (approx.)}$$

Example 8 Use differential to find the approximate value of cube root of 28.

Solution : Let $y = (x)^{1/3}$

$\therefore$ $y+\delta y = (x+\delta x)^{1/3}$

$\therefore$ $\delta y = (x+\delta x)^{1/3} - (x)^{1/3}$

put $x+\delta x = 28$, where $x=27$ and $\delta x=1$

$\therefore$ $\delta y = (28)^{1/3} - (27)^{1/3}$

Now $\delta y = \frac{dy}{dx}.\delta x$

$$(28)^{1/3} - (27)^{1/3} = \frac{1}{3}(x)^{1/3-1}.[(x+\delta x)-x]$$

$$= \frac{1}{3}(x)^{-2/3}.[28-27]$$

$$= \frac{1}{3}(x)^{-2/3}(1)$$

$$= \frac{1}{3}(x)^{-2/3}$$

$$= \frac{1}{3}\times\frac{1}{(27)^{2/3}}$$

$$= \frac{1}{3}\frac{1}{\left[(27)^{1/3}\right]^2}$$

$$= \frac{1}{3}\frac{1}{(3)^2}$$

$$(28)^{1/3} - (27)^{1/3} = \frac{1}{27}$$

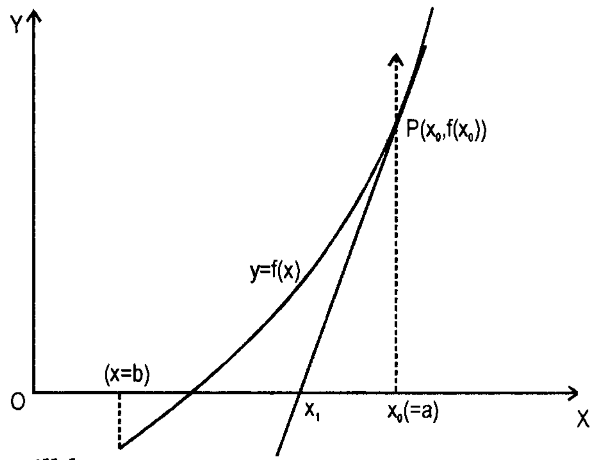
$\therefore \quad (28)^{1/3} = \frac{1}{27} + (27)^{1/3}$

$= \frac{1}{27} + 3 = \frac{1+81}{27} = \frac{82}{27} = 3.037$

Newton Raphson's Method -

Newton Raphson method is used for finding the solution of a given equation f(x)=0. In this method first of all we find two values of x (Say 'a' and 'b'), for which f(x) has opposite sign. [Say f(a) > 0 and f(b) < 0]

Now from 'a' and 'b' we select that value which makes f(x) most nearest to 0. Let it be (a). Then we denote this value by x_0, thus x_0 = a.



The equation of tangent at $(x_0, f(x_0))$, will be

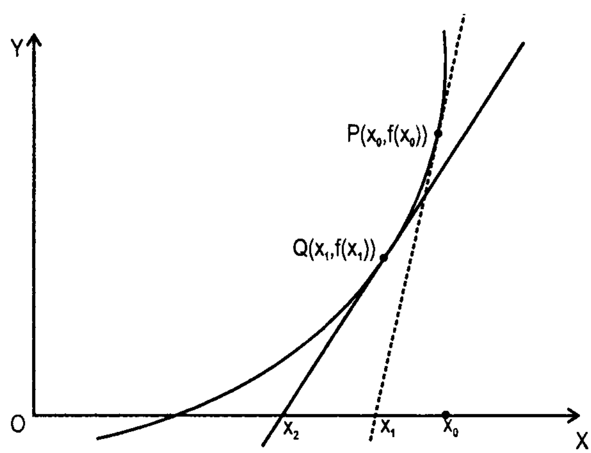
$y - f(x_0) = f'(x_0)(x - x_0)$ ---- (1)

Let this tangent intersects x-axis at $x = x_1$

$0 - f(x_0) = f'(x_0)(x_1 - x_0)$, $[\because f(x_1) = 0]$

$x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)}$

or $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$



This is the first approximation.

If $f(x_1)=0$ then $x=x_1$ will be the root of equation f(x) = 0, otherwise,

Let a tangent to the curve be drawn at point $Q(x_1, f(x_1))$ on the curve.

Then, the equation of this tangent will be $f(x)-(fx_1) = f'(x_1)(x-x_1)$.

Let this tangent cuts x-axis at point $x=x_2$

$\therefore \quad 0 - f(x_1) = f'(x_1)(x_2 - x_1)$ $[\because f(x_2)=0]$

$x_2 - x_1 = -\frac{f(x_1)}{f'(x_1)}$ or $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

This is our second approximation. If $f(x_2)=0$, then $x=x_2$ will be the root of equation $f(x) = 0$, otherwise we take third approximation and so on till we get the exact root of eq. $f(x) = 0$ or the value of x to the desired level of accuracy i.e. the value of x for which $f(x) \approx 0$.

The iterative formula for (n+1)th approximation is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad f'(x_n) \neq 0$$

Example 1 Find a root of equation $x^2-4x+1=0$ by Newton Raphson's method.

Solution : The given equation is $x^2-4x+1=0$

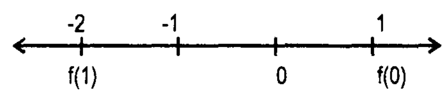
Hence $f(x) = x^2-4x+1$

For $x = 0$, $f(0) = (0)^2-4(0)+1 = 1>0$

For $x = 1$, $f(1) = (1)^2-4(1)+1 = -2<0$

$\therefore$ One of the two roots of equation, lies between 0 and 1.

Since, $f(0) = 1$, is most nearest to 0 as compared to $f(1) = -2$, therefore, let us take $x_0 = 0$



Now, by Newton Raphson formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad ------ (1)$$

Now, $f'(x) = 2x-4$

$\therefore$ for $n = 0$,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= x_0 - \frac{[x_0^2 - 4x_0 + 1]}{2x_0 - 4}$$

$$= 0 - \frac{[(0)^2 - 4(0) + 1]}{[2(0) - 4]}$$

$$= \frac{-1}{-4} \qquad = 0.25$$

Now $f(0.25) = (0.25)^2-4(0.25)+1$

$= 0.0625-1+1$

$= 0.0625 \neq 0$

Let us take second approximation,

Put $n = 1$ in (1)

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.25 - \frac{[(0.25)^2 - 4(0.25) + 1]}{[2(0.25) - 4]}$$

$$= 0.25 - \frac{[0.0625 - 1 + 1]}{[0.50 - 4]}$$

$$= 0.25 - \frac{[0.0625]}{[-3.50]}$$

$$= 0.25 + \frac{0.0625}{3.50} \qquad = 0.25 + 0.01786$$

$= 0.26786$

$= 0.26786 \simeq 0.267$, [Taking three places of decimals]

Now, $f(0.267) = (0.267)2-4(0.267)+1$

$= 0.07129 - 1.068 + 1$

$= 1.07129 - 1.068$

$= 0.00329$, which is very nearest to 0.

Thus $x = 0.267$ is the approximate value of root of equation $x^2-4x+1 = 0$

Example 2 By using Newton Raphson's method find a root of equation $x^4-11x+5=0$ near to 2 upto second approximation.

Solution : Hence, $f(x) = x^4-11x+5$

Let $x_0 = 2$

$\therefore$ By Newton Raphson's method

$$x_{x+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad ----(1)$$

and $f'(x) = 4x^3-11$

For $n = 0$,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{[x_0^4 - 11x_0 + 5]}{[4x_0^3 - 11]}$$

$\therefore \quad x_1 = 2 - \frac{[(2)^4 - 11(2) + 5]}{[4(2)^3 - 11]}$

$= 2 - \frac{[-1]}{21} \quad = 2 + \frac{1}{21} \quad = \frac{43}{21} \quad = 2.04$

f(2.04) $= (2.04)^4 - 11(2.04) + 5$

$= 17.31891 - 22.44 + 5$

$= 22.31891 - 22.44 = -0.12109$

for second approximation, Put n = 1 in ---- (1)

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= x_1 - \frac{[(x_1)^4 - 11(x_1) + 5]}{[4(x_1)^3 - 11]}$

$= 2.04 - \frac{[-0.12109]}{[2(2.04)^3 - 11]}$

$= 2.04 + \frac{0.12109}{(33.95864 - 11)}$

$= 2.04 + 0.00527 = 2.04527$

$= 2.045$ (Taking three places of decimals)

f(2.045) $= (2.045)^4 - 11(2.045) + 5$

$= 17.48938 - 22.495 + 5$

$= -0.00562$

Which is nearest to zero

$\therefore$ x = 2.045 is the approximate root of given equation.

Example 3 If x = 2.94 is the first approximated value of the root of equation $x^3 - 9x + 1 = 0$, by Newton's Raphson's method. Then the value of root of the equation $x^3 - 9x + 1 = 0$, by second approximation will be

(a) 2.9435 (b) 2.9427
(c) 2.9428 (d) 2.9423

Solution : $f(x) = x^3 - 9x + 1$

According to Question, $x_1 = 2.94$

We are to find $x_2 = ?$

By Newton Raphson's method,

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \qquad \left[\because x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\right]$

$$= 2.94 - \frac{[(2.94)^3 - 9(2.94) + 1]}{[3(2.94)^2 - 9]}$$

$$= \frac{3(2.94)^3 - 9(2.94) - (2.94)^3 + 9(2.94) - 1}{[3(2.94)^2 - 9]}$$

$$= \frac{2(2.94)^3 - 1}{3(2.94)^2 - 9} \quad = \frac{49.82436}{16.9308} \quad = 2.94282$$

$\therefore$ x_2 = 2.9428 is the second approximate value of root.

$\therefore$ (c) is the correct option.

MULTIPLE CHOICE QUESTIONS

Q.1 The displacement travelled by a train is given by the equation $S=t^3+4t^2+2t-6$. Find the velocity after 2 hr. (S in km.)

(a) 32 km/hr. (b) 30 km/hr. (c) 35 km/hr. (d) None of these

Q.2 The displacement travelled by a train is given by $S = 3t^3-4t^2+3t+10$, find its acceleration after 3hrs (S in kms.)

(a) 46 km/hr^2 (b) 40 km/hr. (c) 54 km/hr (d) 64 km/hr

Q.3 The angle x which changes half times as fast as its tangent is

(a) 60° (b) 75° (c) 30° (d) 45°

Q.4 The angle x which changes twice as fast as its sine is

(a) 60° (b) 75° (c) 30° (d) 45°

Q.5 A body is moving with displacement S given by $S=t^3+3t^2-21t+9$, (S in km, and t in hrs). After how much time, the velocity will be equal to acceleration in magnitude.

(a) 2hrs. (b) 1hr. (c) 3hrs. (d) 4hrs.

Q.6 A particle is moving with uniform retardation and with displacement given by $S = 2t^3-9t^2-60t+5$, After how much time, it will come to rest? (time is in seconds)

(a) 3 sec. (b) 5 sec. (c) 2 sec. (d) 1 sec.

Q.7 A square plate is being expanded by heating. If its side is increasing at the rate of 0.2 cm per second, then at what rate, the area is increasing when the side is 20cm?

(a) 10cm^2/sec. (b) 8 cm^2/sec. (c) 2 cm^2/sec. (d) 5 cm^2/sec.

Q.8 A circular plate of metal expands by heating so that its radius is increasing at the rate of 0.02 cm/sec. Find the rate at which the area is increasing when the radius is 7 cm.

(a) 8.8 cm^2/sec. (b) 4.8 cm^2/sec. (c) 4.9 cm^2/sec. (d) 1.4 cm^2/sec.

Q.9 A metal plate in the form of an equilateral triangle is being heated and its Area is increasing at the rate of 0.9 cm/sec. At what rate the side is increasing when it is $\sqrt{3}$ cm in length?
(a) 0.3 cm/sec. (b) 0.5 cm/sec. (c) 0.6 cm/sec. (d) None of these

Q.10 A man 2 meters high walks at a uniform speed of 8m/minute from a lamp post 6 meters high. The rate at which the length of his shadow increases is
(a) 3 m/minute (b) 4 m/minute (c) 2 m/minute (d) 5 m/minute

Q.11 A spider is rising towards ceiling at the rate of 2 m/minute, then with what rate the distance between the spider and the ceiling is decreasing?
(a) 1 m/minute (b) 4 m/minute (c) 4.2 m/minute (d) 2 m/minute

Q.12 A varying force is applied on a body of unit mass and if the force increases at the rate of 2 Newton per second, then the acceleration in the body is changing at the rate of
(a) 2 cm/sec^2 (b) 2 m/sec^2 (c) 1 m/sec^2 (d) None of these.

Q.13 The velocity of a particle moving in a st. line is given by the relation v = $\sqrt{2(\cos x + x \sin x)}$, where 'x' is the distance of the particle from a fixed point, then its acceleration will be.
(a) x cos x (b) x sin x (c) cos x (d) sin x

Q.14 The displacement of a particle at the time t is given by $S=e^{2t}\cos 2t$, then its velocity at t=0 will be
(a) 1 unit/sec. (b) 0 (c) 2 units/sec. (d) None of these

Q.15 A man 2 meters high walks at a uniform speed 'u' meters/minutes away from a lamp post of 6m high. His shadow is increasing at the rate of 2m/minutes then, the value of u will be
(a) 3 m/minutes (b) 2 m/minutes (c) 4 m/minutes (d) None of these

Q.16 The time of oscillation of a simple pendulum of length 'ℓ' is given by T = $2\pi\sqrt{\frac{\ell}{g}}$, if the relative change in length of pendulum is 4 then what will be the relative change in T?
(a) 2 (b) 4 (c) 8 (d) 1

Q.17 The time of oscillation of a simple pendulum of length 'ℓ' is given by T = $2\pi\sqrt{\frac{\ell}{g}}$, if the %age error in '$\ell$' is 5%, then the %age error in T will be
(a) 10% (b) 2.5% (c) 5% (d) 1%

Chapter **5**

MAXIMA AND MINIMA

The differentiation of various functions render a great services in solving the problems concerned with finding out the maximum and minimum values of quantities.

In the various fields of engineering and technology, we have to find the maximum or minimum values of one quantity w.r.t another quantity. For example in finding the radius and height of a cylinder that is to be manufactured with the metal sheet of given surface area, so that the capacity of cylinder is maximum. Sometimes it is necessary to find the least cost for the transmission of given horse power. We can find the dimensions of a plot of given perimeter, so that its area is maximum and so on. In this chapter we are going to solve problems concerned with maximum and minimum values of various quantities.

Increasing and Decreasing Functions

Increasing Function - Let $y = f(x)$ be a function of variable x. Then $y=f(x)$ is said to be increasing function of x, if y increases or remains same as x increases or in other words, if $x_2>x_1$ then $f(x_2) \geq f(x_1)$ where $x_1, x_2 \in D_f$.

The figure 5.1 shows the graph of an increasing function.

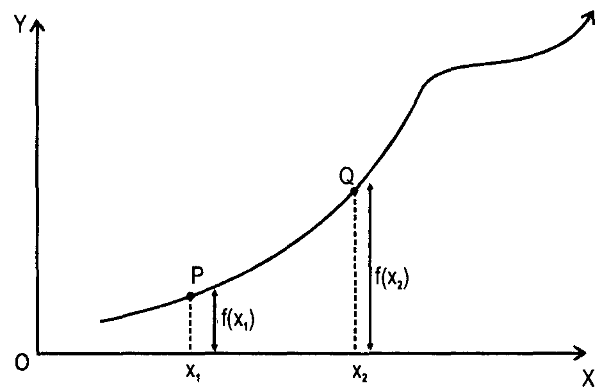


Fig. 5.1

Clearly, $x_2>x_1 \Rightarrow f(x_2) \geq f(x_1); x_1, x_2 \in D_f$.

Strictly increasing function - A function $y=f(x)$ is said to be strictly increasing if $x_2>x_1 \Rightarrow f(x_2) > f(x_1)$.

The figure 5.2 shows the graph of a strictly increasing function:

Clearly,

$x_2>x_1 \Rightarrow f(x_2) > f(x_1); x_1, x_2 \in D_f$.

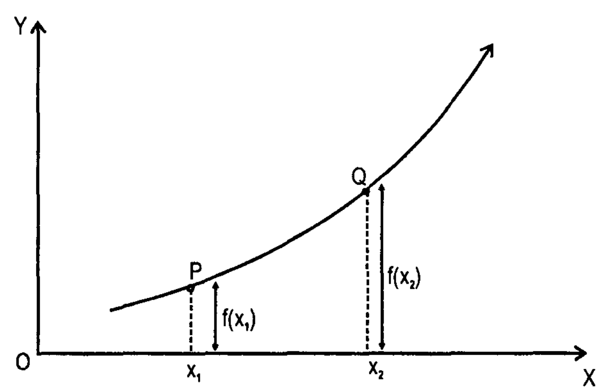


Fig. 5.2

Decreasing Function - A function y = f(x) is said to be a decreasing function if y decreases or remains same as x increases.

In other words,

$x_2 > x_1 \Rightarrow f(x_2) \le f(x_1);\ x_1, x_2 \in D_f$.

The figure 5.3 shows the graph of a decreasing function.

Clearly $x_2 > x_1 \Rightarrow f(x_2) \le f(x_1)$

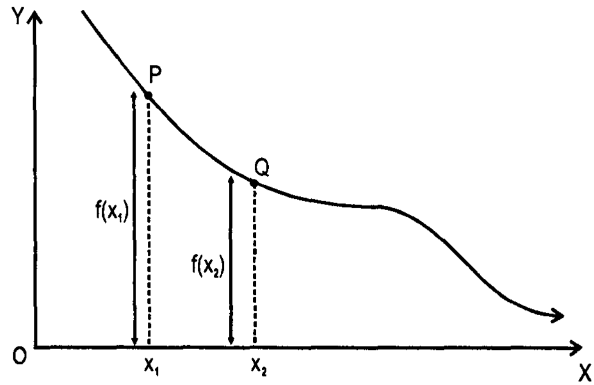


Fig. 5.3

Strictly Decreasing function - A function y = f(x) is said to be a strictly decreasing function, if $x_2 > x_1 \Rightarrow f(x_2) < f(x_1);\ x_1, x_2 \in D_f$.

The figure 5.4 shows the graph of a strictly decreasing function

Clearly,

$x_2 > x_1 \Rightarrow f(x_2) < f(x_1);\ x_1, x_2 \in D_f$.

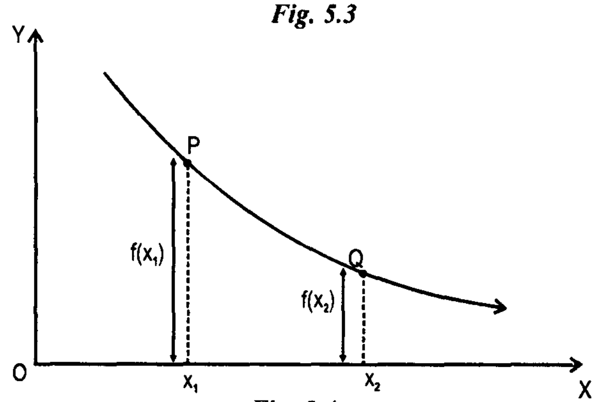


Fig. 5.4

Conditions for a function y = f(x) to be increasing or decreasing.

Let y = f(x) be an increasing function in the open interval (a,b).

Let x ∈ (a,b). From the knowledge of differentiation we know that the derivative of a function at any point P on the curve y=f(x) gives the slope of tangent to the curve at that point P.

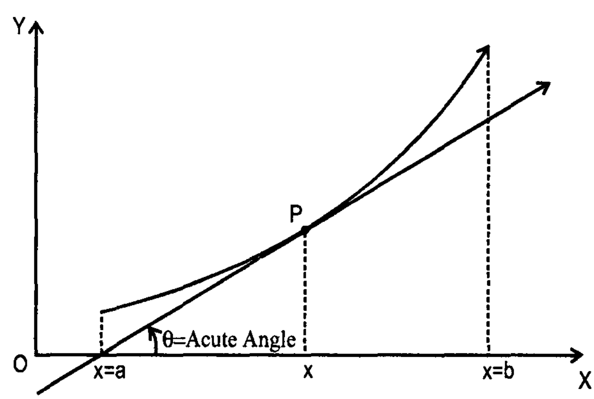
$$\therefore\ f'(x) = \frac{dy}{dx} = \tan\theta$$

Since the graph is increasing, θ will be acute i.e. in 1st quadrant, for which tanθ is non-negative.

$$\therefore\ f'(x) = \frac{dy}{dx} \ge 0$$

and for strictly increasing function

$$\frac{dy}{dx} > 0 \text{ i.e. +ve}$$

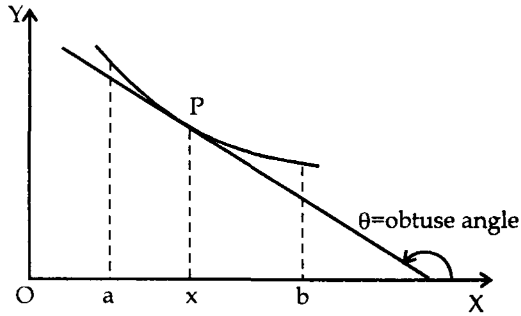


Similarly, for a decreasing function $y = f(x)$, θ is obtuse, and hence, $\tan\theta \le 0$.

$\therefore\ f'(x) = \dfrac{dy}{dx} \le 0.$

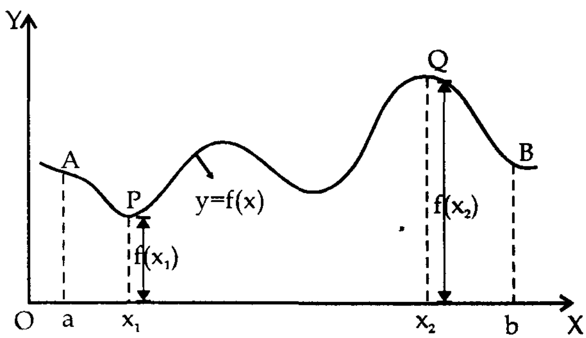
and for strictly decreasing function

$f'(x) = \dfrac{dy}{dx} < 0$ i.e -ve



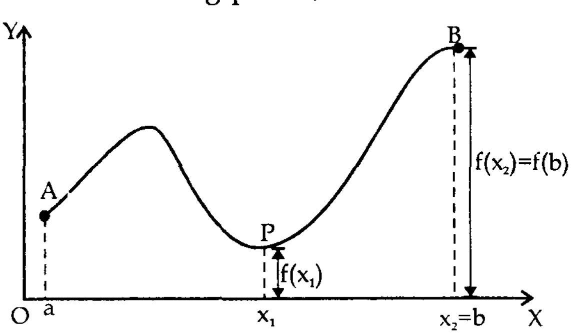
Maximum and Minimum Values of a Function -

Consider a function $y = f(x)$, having the graph as shown below-

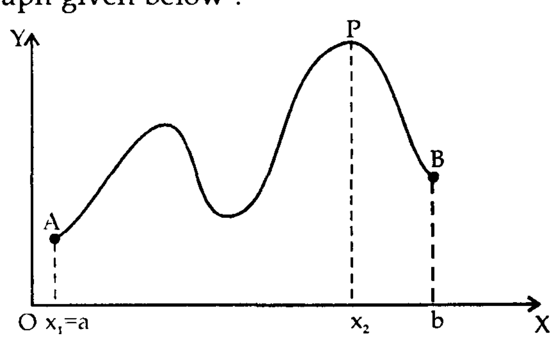


In the interval [a,b], f(x) has maximum value at the turning point $Q(x=x_2)$, where as f(x) has minimum value at the turning point $P(x=x_1)$. The points $x=x_1$ and $x=x_2$ are called point of **absolute minima** and point of **absolute maxima** respectively and the corresponding values of functions i.e. $f(x_1)$ and $f(x_2)$ are called **absolute maximum** and **absolute minimum** values of the function y=f(x) on the interval [a,b]

It is not necessary that the absolute maximum and minimum values of function occur at turning points, as is clear from the graph given below :-



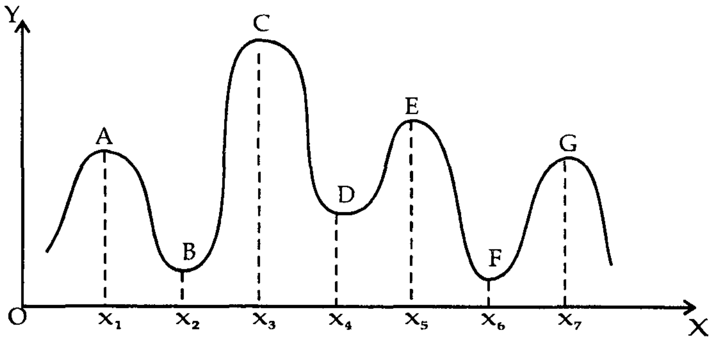
Absolute minimum value of $f(x) = f(x_1)$
Absolute maximum value of $f(x) = f(x_2)=f(b)$



Absolute minimum value of $f(x) = f(a)$
Absolute maximum value of $f(x) = f(x_2)$

From above discussion, it is clear, that a function may attain its maximum or minimum value at turning points or at the end points of the closed interval [a,b].

Local Maximum and Local Minimum Values of a Function - Let y = f(x) be a function having its graph as shown below in the figure



The turning points A,C,E and G are the highest points among all the points on the curve in their small neighbourhood. The values of x corresponding to these points i.e. $x=x_1$, x_3, x_5 and x_7 are called points of **Local Maxima** and the corresponding values of function i.e. $f(x_1)$, $f(x_3)$, $f(x_5)$ and $f(x_7)$ are called **Local Maximum** values of function.

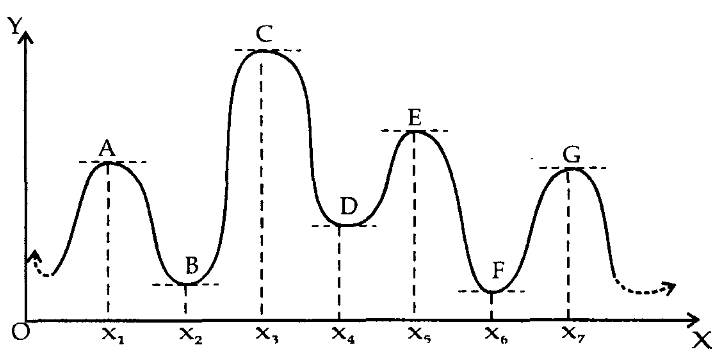
The turning points B,D and F are the lowest points among the points on the graph in their small neighborhoods. The values of x corresponding to these points i.e. $x=x_2$, x_4 and x_6 are called points of **Local Minima** and the corresponding values of function i.e. $f(x_2)$,$f(x_4)$ and $f(x_6)$ are called **Local Minimum** values of function.

Remarks : (1) If a function y=f(x) is defined on R, the set of real number, and we are to find its maximum and minimum values, then we find its local maximum and local minimum values, if they exist.

(2) If we are to find the maximum and minimum values of a function y=f(x) on [a,b], then, we find both Absolute as well as Local maximum and Local minimum values.

To find maximum/minimum values of a function y = f(x) on R [Local Maximum and Local Minimum Values]

Let y=f(x) be a function having its graph as shown in figure below-



The point A,B,C,D,E,F,G, ... are the turning points of the graph. These points are also called stationary points. On these points, the tangents to the graph are parallel to x-axis and hence, their slope will be zero i.e. at these points dy/dx = 0.

So to find these stationary points (also called critical points) we put $\frac{dy}{dx} = 0$ and find the roots of this equation, (say) x_1, x_2, x_3, x_4, ... Now we are to separate the points of local maxima and local minima from these stationery points, x_1, x_2, x_3, x_4, ...

For this we have two methods (Tests)

(i) First derivative test

(ii) Second derivative test

Here we will study only second derivative test.

Second derivative Test (Working Rule) - After finding the critical points of function f(x) by putting $\frac{dy}{dx} = 0$, say x_1, x_2, x_{-3}, — — — we separate points of maxima/ minima, by using, second derivatives as follows :

Let x_k be the critical point to be tested for maxima/minima. We find $f''(x)$ and find the value of $f''(x_{-k})$.

(i) If $f''(x_{-k})$ is –ve, then x_k is the point of maxima and $f(x_{-k})$ will be the maximum value of f(x) at that point.

(ii) If $f''(x_{-k})$ is +ve, then x_k is the point of minima and $f(x_{-k})$ will be the minimum value of f(x) at that point.

(iii) If $f''(x_{-k})=0$ then we find $f'''(x_{-k})$. If $f'''(x_{-k}) \neq 0$, then, x_k is the point of inflexion and if $f'''(x_{-k}) = 0$, then second derivative test fails.

Example 1 Find the maximum and minimum values of the function $f(x)=x^3-3x^2-24x+5$

Solution : $f(x) = x^3-3x^2-24x+5$

$\therefore \quad f'(x) = 3x^2-6x-24$

For critical points, put $f'(x) = 0$

$3x^2-6x-24 = 0$

$3(x^2-2x-8) = 0$

$x^2-2x-8 = 0$

$(x-4)(x+2) = 0$

$\therefore \quad x = 4$ and $x = -2$ are two critical points.

$f''(x) = 6x-6$

Now $f''(x)_{x=4} = 6(4)-6 = 18 > 0$

$\therefore \quad x = 4$ is the point of minima. And minimum value of the function is

$f(u) = (4)^3-3(4)^2-24(4)+5$

$= 64-48-96+5$

$= -75$

at $x = -2$, $(f''(x))_{x=-2} = 6(-2)66 = -18<0$

$\therefore$ $x = -2$ is the point of maxima. and maximum value of the function is

$f(-2) = (-2)^3-3(-2)^2-24(-2)+5$

$= 18-12+48+5 = 33$

Example 2 Find the points of maxima or minima (if it exists) and also the maximum and minimum values of the function:

$$f(x) = \sin^4x + \cos^4x, \ 0<x<\frac{\pi}{2}$$

Solution :

$$f(x) = \sin^4x + \cos^4x, \ 0<x<\frac{\pi}{2}$$

$$f'(x) = 4\sin^3x \cos x + 4\cos^3x (-\sin x)$$

$$= 4 \sin x \cos x[\sin^2x - \cos^2x]$$

$$= 4 \sin x \cos x (\sin + \cos x)(\sin x - \cos x)$$

for critical points, put $f'(x) = 0$

$\Rightarrow$ $4 \sin x \cos x (\sin x + \cos x)(\sin x - \cos x) = 0$

$\Rightarrow$ $\sin x = 0, \cos x = 0, \tan x = -1, \tan x =1$

$$\therefore \quad x = 0, \frac{\pi}{2}, \frac{\pi}{4}, \frac{-3\pi}{4},$$

But $0<x<\frac{\pi}{2}$

$\therefore$ $x = \frac{\pi}{4}$ is the only critical point in $0<x<\frac{\pi}{2}$

$$f'(x) = 4(\sin x \cos x) (\sin x + \cos x)(\sin x - \cos x)$$

$$= 2(2\sin x \cos x)(\sin^2x - \cos^2x)$$

$$= 2(\sin2x)(-\cos2x) \quad , \quad [\because \cos^2x - \sin^2x = \cos2x]$$

$$= -2\sin2x \cos2x$$

$$f'(x) = -\sin4x$$

$$\therefore \quad f''(x) = -4\cos4x$$

Put $x = \frac{\pi}{4}$

$$\therefore \quad f''\left(\frac{\pi}{4}\right) = -4 \cos 4\left(\frac{\pi}{4}\right)$$

$$= -4 \cos\pi$$

$$= (-4)(-1) = 4 = +ve$$

$\therefore$ $x = \frac{\pi}{4}$ is the point of minima and the minimum value of f(x) is

$$f\left(\frac{\pi}{4}\right) = \sin^4\left(\frac{\pi}{4}\right) + \cos^4\left(\frac{\pi}{4}\right)$$

$$= \left(\frac{1}{\sqrt{2}}\right)^4 + \left(\frac{1}{\sqrt{2}}\right)^4$$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$\therefore$ $x = \frac{\pi}{4}$ is the point of minima and the minimum value of function is $\frac{1}{2}$.

Example 3 $\frac{\log x}{x}$ has a maximum value at

(a) e
(b) e^2
(c) e^{-2}
(d) e^3

Solution $f(x) = \frac{\log x}{x}$

$$f'(x) = \frac{x.\frac{1}{x} - \log x.(1)}{x^2}$$

$$= \frac{1 - \log x}{x^2}$$

Put $f'(x) = 0$

$\Rightarrow$ $\frac{1 - \log x}{x^2} = 0$

$1 - \log x = 0$

$\log x = 1$ $[\because \log e = 1]$

$\Rightarrow$ $\log x = \log e$

$\Rightarrow$ $x = e$

$$\text{Now, } f''(x) = \frac{(x^2)\left(0 - \frac{1}{x}\right) - (1 - \log x)(2x)}{(x^2)^2}$$

$$= \frac{-x - 2x + 2x \log x}{x^4}$$

$$= \frac{x[2\log x - 3]}{x^4}$$

$\therefore \quad f''(e) = \frac{2\log e - 3}{e^3} = \frac{2-3}{e^3}$

$$= -\frac{1}{e^3} < 0$$

$\therefore$ function has maximum value at x = e.

$\therefore$ x = e is the point of maxima.

Example 4 Find the points of maxima and minima for the function $4x^3-9x^2-12x+4$ and also find the maximum and minimum values of the function.

Solution : $f(x) = 4x^3-9x^2-12x+4$

$f'(x) = 12x^2-18x-12$

$= 6[2x^2-3x-2]$

$= 6[2x^2-4x+x-2]$

$= 6[2x(x-2)+1(x-2)]$

$= 6(2x+1)(x-2)$

For critical points, $f'(x) = 0$

$\Rightarrow \quad 6(2x+1)(x-2) = 0$

either $x = -\frac{1}{2}$ or $x = 2$

$\therefore \quad x = -\frac{1}{2}, 2$ are the critical points,

Now, $\frac{d^2y}{dx^2} = 24x - 18$

$$\left(\frac{d^2y}{dx^2}\right)_{x=-\frac{1}{2}} = 24\left(-\frac{1}{2}\right) - 18 \quad = -12-18 = -30<0$$

and $$\left(\frac{d^2y}{dx^2}\right)_{x=2} = 24(2)-18 \quad = 48-18 = 30>0$$

$\therefore \quad x = -\frac{1}{2}$ is the point of maxima and the maximum vale of function at

$x = -\frac{1}{2}$ will be $f\left(-\frac{1}{2}\right) = 4\left(-\frac{1}{2}\right)^3 - 9\left(-\frac{1}{2}\right)^2 - 12\left(-\frac{1}{2}\right) + 4$

$$= 4\left(-\frac{1}{8}\right)-\frac{9}{4}+6+4$$

$$= -\frac{1}{2}-\frac{9}{4}+\frac{10}{1}$$

$$= \frac{-2-9+10}{4}$$

and the minimum value of function will be at x=2 and will be =

f(2) $= 4(2)^3-9(2)^2-12(2)+4$

$= 32-36-24+4 = -24$

Example 5 The minimum value of function x^2-4x+7 will be

(a) 2 (b) 3

(c) 4 (d) 7

Solution **Let** $y = x^2-4x+7$

$$\frac{dy}{dx} = 2x-4$$

Put $\frac{dy}{dx} = 0$

i.e. $2x - 4 = 0$

or $x = 2$

Now $\frac{d^2y}{dx^2} = 2$

$\therefore \left(\frac{d^2y}{dx^2}\right)_{x=2} = 2>0$

$\therefore$ y has minimum value at x=2 and the minimum value of f(x) will be $f(2) = (2)^2-4(2)+7 = 4-8+7 = 3$

$\therefore$ (b) will be the correct option.

Example 6 Show that none of the following functions has a maximum or minimum value.

(1) $\log x^2$ (2) e^x (3) $2x^3+3x^2+12x+5$

Solution :(1) $y = \log x^2$

$= 2 \log x$

$\therefore \frac{dy}{dx} = 2\left(\frac{1}{x}\right)=\frac{2}{x}$

$\therefore \frac{dy}{dx} = \frac{2}{x} \neq 0$ for any real value of x

$\therefore$ $y = \log x^2$ has neither maximum nor minimum values.

(2) $y = e^x$

$\frac{dy}{dx} = e^x$ $\left[\because \frac{d}{dx}(e^x) = e^x\right]$

$\neq 0$ for any real value of x

$\therefore$ $y = e^x$ has neither maximum nor minimum values.

(3) $y = 2x^3+3x^2+12x+5$

$\frac{dy}{dx} = 6x^2+6x+12$

Put $\frac{dy}{dx} = 0$

$\Rightarrow$ $6x^2+6x+12 = 0$

or $x^2 + x + 2 = 0$

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(2)}}{2(1)}$$

$x = \frac{-1 \pm \sqrt{-7}}{2}$ which is not real

$\therefore$ f(x) has neither maximum nor minimum values.

Example 7 Find two positive numbers whose sum is 18 and their product is maximum.

Solution : Let the positive number be x and y

$\therefore$ By given condition, $x+y = 18$ --- (1)

Let P denotes the product of x and y

$\therefore$ $P = xy$

$P = x(18-x)$ --- [from(1)]

$P = 18x - x^2$

$\therefore$ $\frac{dP}{dx} = 18-2x$

For maximum or minimum values of P,

put $\frac{dP}{dx} = 0$

$\Rightarrow$ $18-2x = 0$

or $x = 9$

Now, $\frac{d^2P}{dx^2} = 0-2(1) = -2$

$\therefore \quad \left(\frac{d^2P}{dx^2}\right)_{x=9} = -2<0$

$\therefore$ P will be maximum for $x = 9$

and $y = 18-x$

$= 18-9 \quad = 9$

$\therefore \quad x = 9, y = 9$

Hence the two required numbers are 9 and 9.

Example 8 Show that among all the rectangles of given area, square has the least perimeter.

Solution : Let x and y be the length and breadth of any of the rectangles of given area.

Let A (constant) be the area of all given rectangles.

$\therefore \quad xy = A \qquad --- (1)$

Now, Perimeter of rectangle is given by,

$P = 2(x+y)$

$P = 2x+2y$

$P = 2x+2\left[\frac{A}{x}\right] \quad , \qquad \left[\text{from(1) } y=\frac{A}{x}\right]$

$P = 2x+2A(x)^{-1}$

$\frac{dP}{dx} = 2+2A(-1)x^{-2}$

$= 2-\frac{2A}{x^2}$

Put $\frac{dP}{dx} = 0$

$\Rightarrow \quad \frac{2A}{x^2} = 2$

$A = x^2$

$x^2 = A$

$x = \pm\sqrt{A}$

But x, being the side of rectangle can never be negative.

$\therefore \quad x = \sqrt{A}$

Now, $\frac{d^2P}{dx^2} = 0-2A\left[\frac{-2}{x^3}\right] \qquad = \frac{4A}{x^3}$

$$\therefore \quad \left(\frac{d^2P}{dx^2}\right)_{x=\sqrt{A}} = \frac{4A}{(\sqrt{A})^3} = \frac{4A}{A^{3/2}}$$

$$= \frac{4}{\sqrt{A}} > 0$$

Hence , P will be minimum for $x = \sqrt{A}$ and $y = \frac{A}{x} = \frac{A}{\sqrt{A}} = \sqrt{A}$

i.e. the sides of rectangle are equal $(= \sqrt{A})$

i.e. the rectangle will be a square.

Example 9 The profit function of a company is given by $P(x) = 2x^3+3x^2-120x+100$ Find the maximum profit.

Solution : $P(x) = 2x^3+3x^2-120x+100$

$$\frac{dP}{dx} = 6x^2+6x-120$$

For maximum profit, Put $\frac{dP}{dx} = 0$

$\Rightarrow \quad 6x^2+6x-120 = 0$

$6[x^2+x-20] = 0$

$x^2+x-20 = 0$

$(x+5)(x-4) = 0$

$\Rightarrow \quad x = -5$ or $x = 4$

Now, $\frac{d^2P}{dx^2} = 12x + 6$

$$\left(\frac{d^2P}{dx^2}\right)_{x=-5} = 12(-5)+6 = -60+6 = -54 < 0$$

$x = 4$ is the point of minima.

$\therefore$ P will be maximum for $x = -5$

and the maximum profit will be

$= P(-5) = 2(-5)^3+3(-5)^2-120(-5)+100$

$= 2(-125)+75+600+100$

$= -250 + 75 + 600 + 100$

$= 525$

MULTIPLE CHOICE QUESTIONS

Q.1 If f(x) is an increasing function, in the neighborhood of x, then $f'(x_0)$ is equal to
(a) 0 (b) >0 (c) <0 (d) ≥ 0

Q.2 If f(x) is a decreasing function, in the neighborhood of x, then $f(x_0)$ is equal to
(a) 0 (b) >0 (c) ≤0 (d) ≥0

Q.3 If f(x) has maximum or minimum value at a point x, then, x_0 is called a
(a) point of inflextion (b) absolute point
(c) constant point (d) stationary point

Q.4 If x_0 is the point of maxima or minima of function f(x), then $f'(x_0)$ = ?
(a) +ve (b) 0 (c) –ve (d) None of these

Q.5 A point x, for which $f'(x_0) = 0$, but x_0 is neither a point of maxima nor minima, then x_0 is called
(a) constant point (b) point of inflexion (c) stationery point (d) absolute point

Q.6 If $f(x) = x^3-7x^2+8x-5$, then its stationary (critical) point will be
(a) (2,4) (b) (3,4) (c) 2/3,4) (d) (4,3)

Q.7 If f(x) = 5–x–2cos , then the only critical (stationary) point of f(x) will be
(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{4}$

Q.8 If f(x) = cos x + sin x, then the only stationary point of f(x) will be
(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{4}$

Q.9. If f(x) = log sec x + 2cos x, then the critical points of f(x) in the interval $\left[0, \frac{\pi}{2}\right]$ are
(a) $0, \frac{\pi}{2}$ (b) $0, \frac{\pi}{6}$ (c) $0, \frac{\pi}{3}$ (d) $\frac{\pi}{3}, \frac{\pi}{6}$

Q.10 If $f(x) = x^3-3x^2+6x+3$, then the stationary points of f(x) are
(a) (2,3) (b) (0,3) (c) (1,2) (d) Does not exist

Q.11 The only critical point of the function $f(x) = x^x$ is
(a) $\frac{1}{e}$ (b) e (c) e^2 (d) 0

Q.12 The only stationary point of the function $f(x) = \frac{\log x}{x}$ for x > 0, is
(a) $\frac{1}{e}$ (b) e (c) e^2 (d) 0

Q.13 The point on the curve $y = -2x^3+6x^2+8x-1$, at which the slope is maximum is
(a) x = –1 (b) x = 2 (c) x = 1 (d) x = 0