Lecture in Algebra

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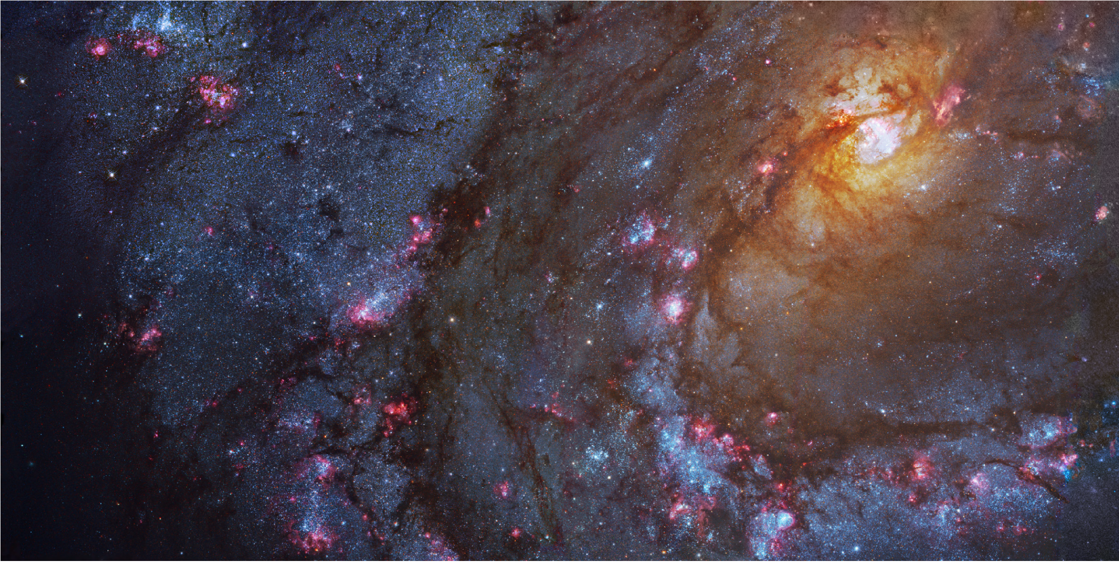
Mathematical Induction



1. Mathematical Induction

1.1 Introduction

Mathematical induction is a technique for proving results or establishing statements for natural numbers.

In 370 BC, **Plato's Parmenides** may have contained an early example of an implicit inductive proof. The earliest clear use of mathematical induction may be found in **Euclid's** proof that the number of primes is infinite.

In India, early implicit proofs by mathematical induction appear in **Bhaskara's** "cyclic method", and in the **al-Fakhri** written by **al-Karaji** around 1000 AD, who applied it to arithmetic sequences to prove the binomial theorem and properties of Pascal's triangle.

1.2 Mathematical Induction

In this section, we can use mathematical induction to prove that a propositional function (a statement) $P(n)$ is true for all integers $n \geq 1$.

Definition 1.2.1 Principal of Mathematical Induction Let $P(n)$ be a propositional function (a statement) defined for integers n, and a fixed integer a. Then, if these two conditions are true

1. $P(a)$ is true.
2. if $P(k)$ is true for some integer $k \geq a$, then $P(k+1)$ is also true.

then the $P(n)$ is true for all integers $n \geq a$.

Now, we can refine an induction proof into a 3-step procedure:

1. Verify that $P(1)$ is true.
2. Assume that $P(k)$ is true for some integer $k \geq 1$.
3. Show that $P(k+1)$ is also true.

R

1. The first step, is called the **basis step** or the **anchor step** or the **initial step**.
2. The second step, the assumption that $P(k)$ is true, is sometimes referred to as **the inductive hypothesis** or **induction hypothesis**.

■ **Example 1.1** Use mathematical induction to prove that

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

Solution: Let the statement $P(n)$ be

$$1+2+3+\ldots.+n = \frac{n(n+1)}{2}.$$

1. We show that $P(1)$ is true:

$$L.H.S = 1, \qquad R.H.S = \frac{1(1+1)}{2} = 1$$

Both sides of the statement are equal hence $P(1)$ is true.

2. We assume that $P(k)$ is true:

$$1+2+3+\ldots.+k = \frac{k(k+1)}{n}.$$

3. We show that $P(k+1)$ is true:

$$\begin{aligned} L\cdot H\cdot S &= 1+2+3+\ldots+k+(k+1) \\ &= \frac{k(k+1)}{2}+(k+1) \\ &= \frac{(k+1)}{2}[k+2] \\ &= R.H.S. \end{aligned}$$

Thus, the statement $P(n)$ is true for all positive integers n.

■

■ **Example 1.2** Use mathematical induction to prove that

$$1^2+2^2+3^2+\ldots.+n^2 = \frac{n(n+1)(2n+1)}{6}$$

Solution Let the statement $P(n)$ be

$$1^2+2^2+3^2+\ldots.+n^2 = \frac{n(n+1)(2n+1)}{6}.$$

1. At $n = 1$

$$L.H.S. = 1^2 = 1, \qquad R.H.S. = \frac{1(1+1)(2+1)}{6} = 1,$$

therefore $P(1)$ is true.

2. Let $n = k$, then $P(k)$ is true i.e.,

$$1^2 + 2^2 + 3^2 + \ldots. + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

3. At $n = k+1$

$$\begin{aligned} L.H.S. &= 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{(k+1)}{6}[k(2k+1) + 6(k+1)] \\ &= \frac{(k+1)}{6}\left[2k^2 + k + 6k + 6\right] \\ &= \frac{(k+1)}{6}\left[2k^2 + 7k + 6\right] \\ &= \frac{(k+1)}{6}[(2k+3)(k+2)] \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \\ &= R.H.S. \end{aligned}$$

So $P(k+1)$ is true and therefore, the statement $P(n)$ is true for all positive integers n.

■

■ **Example 1.3** Prove that $\left(n^3 + 2n\right)$ is divisible by 3 for all positive integers n.

Solution Suppose that $P(n)$ be

" $\left(n^3 + 2n\right)$ is divisible by 3" .

1. At $n = 1$

$$1^3 + 2(1) = 3 \text{ is divisible by } 3,$$

therefore $P(1)$ is true.

2. Let $n = k$, then $P(k)$ is true i.e.,

$$\left(k^3 + 2k\right) \quad \textit{is divisible by} \quad 3.$$

3. At $n = k + 1$

$$\begin{aligned}(k+1)^3 + 2(k+1) &= \left(k^3 + 3k^2 + 3k + 1\right) + (2k+2) \\ &= k^3 + 3k^2 + 5k + 3 \\ &= \left(k^3 + 2k\right) + \left(3k^2 + 3k + 3\right) \\ &= \left(k^3 + 2k\right) + 3\left(k^2 + k + 1\right)\end{aligned}$$

$\left(k^3 + 2k\right)$ is divisible by 3 from (2), and $3\left(k^2 + k + 1\right)$ is also divisible by 3, therefore $P(k+1)$ is true.
Thus, $P(n)$ is true for all positive integers n.

■

■ **Example 1.4** Prove that $2^{n-1} \leq n!$ for all positive integers n.

Solution: Let $P(n)$ be $2^{n-1} \leq n!$.

1. At $n = 1$, we get

$$2^{1-1} = 2^0 = 1 \leq 1! = 1,$$

then $P(1)$ is true.

2. Let $n = k$, then $P(k)$ is true i.e.,

$$2^{k-1} \leq k!$$

3. At $n = k + 1$, then we get

$$2^k = (2)\left(2^{k-1}\right) \leq (2)(k!)$$

$$\Rightarrow (2)\left(2^{k-1}\right) \leq (k+1)(k!)$$

$$\Rightarrow 2^k \leq (k+1)! \forall k \in Z^+,$$

i.e., $P(k+1)$ is true.
Thus, $P(n)$ is true for all positive integers n.

■

■ **Example 1.5** If $\mathrm{P}(n)$: "$49^n + 16^n + k$ is divisible by 64 for $n \in \mathbf{N}$" is true, then the least negative integral value of k is......

Solution For $n = 1, \mathrm{P}(1) : 65 + k$ is divisible by 64.
Thus k, should be -1 since, $65 - 1 = 64$ is divisible by 64 ■

■ **Example 1.6** State whether the following proof (by mathematical induction) is true or false for the statement.

$$\mathrm{P}(n) : 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(n+1)}{6}$$

Solution False.
Since in the inductive step both the inductive hypothesis and what is to be proved are wrong. ■

1.3 Exercises

(1) Prove each of the statements by the Principle of Mathematical Induction:

(i) 4^{n-1} is divisible by 3, for each natural number n.

(ii) 2^{3n-1} is divisible by 7, for all natural numbers n.

(iii) $n^3 - 7n + 3$ is divisible by 3, for all natural numbers n.

(iv) $3^{2n} - 1$ is divisible by 8, for all natural numbers n.

(v) $1 + 5 + 9 + ... + (4n - 3) = n(2n - 1)$ for all natural numbers n.

(vi) $2 + 4 + 6 + ... + 2n = n^2 + n$ for all natural numbers n.

Fill in the blanks in the following :

(2) If $10^n + 3.4^{n+2} + k$ is divisible by 9 for all $n \in N$, then the least positive integral value of k is......

(3) If $P(n) : 2n < n!, n \in N$, then $P(n)$ is true for all $n \geq$.......

Chapter two

Partial Fractions

2. Partial Fractions

An algebraic fraction is a fraction in which the numerator and denominator are both polynomial expressions.
In this chapter, we study a fraction and convert it into a partial fraction.
It useful give some definitions which help for understanding this subject.

2.1 Introduction

Definition 2.1.1 A polynomial of degree n in one variable x is an expression of the form

$$\begin{aligned} f(x) &= a_0 + a_1 x + \cdots + a_n x^n \\ &= \sum_{i=0}^{n} a_i x^i, \end{aligned}$$

where a_i, $i = 0, 1, \ldots, n \in \mathbb{R}$ *or* $\mathbb{C}$, are coefficients of polynomial and the degree of polynomial $deg\ f = n$.

Definition 2.1.2 Let $f(x)$ and $g(x)$ be two polynomials, then the quotient of two polynomials

$$h(x) = \frac{f(x)}{g(x)},$$

where $g(x) \neq 0$ with no common factors, is called **Rational fraction**, $f(x)$ the **numerator**, and $g(x)$ the **denominator**.

2.2 Partial fractions

To express a single rational fraction into the sum of two or more single rational fractions are called partial fraction resolution.
For example,

$$\frac{2x + x^2 - 1}{x(x^2 - 1)} = \frac{1}{x} + \frac{1}{x-1} - \frac{1}{x+1}.$$

A rational fraction is of two types:

2.2.1 Proper Fraction:

Definition 2.2.1 A rational fraction $h(x) = \dfrac{f(x)}{g(x)}$, is called a proper fraction if the degree of numerator $f(x)$ is less than the degree of denominator $g(x)$.

■ **Example 2.1** The following are proper fraction

(1) $\frac{9x^2 - 9x + 6}{(x-1)(2x-1)(x+2)}$.

(2) $\frac{6x+27}{3x^3-9x}$. ■

2.2.2 Improper Fraction:

Definition 2.2.2 A rational fraction $h(x) = \dfrac{f(x)}{g(x)}$, is called an improper fraction if the degree of numerator $f(x)$ is greater than or equal to the degree of denominator $g(x)$.

■ **Example 2.2** The following are improper fraction

(1) $\frac{9x^3-9x+6}{(x-1)(2x-1)(x+2)}$.

(2) $\frac{6x^4+27}{3x^3-9x}$. ■

R An improper fraction can be expressed, by division, as the sum of a polynomial and a proper fraction i.e.,

$$\frac{f(x)}{g(x)} = p(x) + \frac{q(x)}{g(x)},$$

where $degq(x)$ less than $deg\ g(x)$.

■ **Example 2.3** The improper fraction

$$\frac{6x^3+5x^2-7}{3x^2-2x-1},$$

can be expressed, by division, as

$$\frac{6x^3+5x^2-7}{3x^2-2x-1} = (2x+3) + \frac{8x-4}{x^2-2x-1}$$

■

2.3 Process of Finding Partial Fraction

A proper fraction $\dfrac{f(x)}{g(x)}$ can be resolved into partial fractions as:

(1) ***The denominator factor as distinct or repeated linear factors***

The rational fraction	The partial fractions
$\frac{f(x)}{(a_1x+b_1)(a_2x+b_2)\ldots}$	$\frac{A}{a_1x+b_1}+\frac{B}{a_2x+b_2}+\ldots$
$\frac{f(x)}{(ax+b)^k}$	$\frac{A_1}{(ax+b)}+\frac{A_2}{(ax+b)^2}+\ldots+\frac{A_k}{(ax+b)^k}$

(2) ***The denominator factor as distinct or repeated quadratic factors cannot be factored further***

The rational fraction	The partial fractions
$\frac{f(x)}{\left(a_1x^2+b_1x+c_1\right)\left(a_2x^2+b_2x+c_2\right)\ldots}$	$\frac{Ax+B}{a_1x^2+b_1x+c_1}+\frac{Cx+D}{a_2x^2+b_2x+c_2}+\ldots$
$\frac{f(x)}{\left(ax^2+bx+c\right)^k}$	$\frac{A_1x+B_1}{\left(ax^2+bx+c\right)}+\frac{A_2x+B_2}{\left(ax^2+bx+c\right)^2}+\ldots+\frac{A_kx+B_k}{\left(ax^2+bx+c\right)^k}$

where $A,B,C,A_1,A_2,\ldots.,A_k$ are constants whose values are to be determined.
The evaluation of the coefficients of the partial fractions is based on the following theorem:

Theorem 2.3.1 If two polynomials are equal for all values of the variables, then the coefficients having same degree on both sides are equal.

■ **Example 2.4** Resolve

$$\frac{7x-25}{x^2-7x+12}$$

into partial fractions.
Solution:

$$\frac{7x-25}{(x-3)(x-4)} = \frac{A}{(x-3)} + \frac{B}{(x-4)}.$$

Multiplying both sides by $(x-3)(x-4)$, we get

$$7x-25 = A(x-4)+B(x-3).$$

Comparing the coefficients of like powers of x on both sides, we have

$$7 = A+B,$$

$$-25 = -4A-3B.$$

Solving these equation we get $A=3$ and $B=4$. Hence the required partial fractions are:

$$\frac{7x-25}{x^2-7x+12} = \frac{3}{(x-3)} + \frac{4}{(x-4)}.$$

■

■ **Example 2.5** Resolve into partial fraction

$$\frac{8x-8}{x^3-2x^2-8x}.$$

Solution:

$$\frac{8x-8}{x^3-2x^2-8x} = \frac{8x-8}{x(x-4)(x+2)} = \frac{A}{x} + \frac{B}{(x-4)} + \frac{C}{(x+2)}.$$

Multiplying both sides by $x(x-4)(x+2)$, we get

$$8x-8 = A(x-4)(x+2)+Bx(x+2)+Cx(x-4)$$

Put $x = 0$ in the above equation, we have $A = 1$.
Put $x = 4$ in the above equation, we have $B = 1$.
Put $x = -2$ in the above equation, we have $C = -2$.
Hence the required partial fractions

$$\frac{8x-8}{x^3-2x^2-8x} = \frac{1}{x} - \frac{1}{x-4} - \frac{2}{x+2}$$

■

■ **Example 2.6** Resolve into partial fractions:

$$\frac{x^2-3x+1}{(x-1)^2(x-2)}.$$

Solution:

$$\frac{x^2-3x+1}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$$

Multiplying both sides by $(x-1)^2(x-2)$, we get

$$x^2-3x+1 = A(x-1)(x-2) + B(x-2) + C(x-1)^2,$$

Put $x = 1$ in the above equation, we have $B = 1$.
Put $x = 2$ in the above equation, we have $C = -1$.
Comparing the coefficient of like powers of x on both sides in the above equation, we get

$$A + C = 1 \Rightarrow A = 2.$$

.
Hence the required partial fractions

$$\frac{x^2-3x+1}{(x-1)^2(x-2)} = \frac{2}{x-1} + \frac{1}{(x-1)^2} + \frac{(-1)}{x-2}.$$

■

■ **Example 2.7** Express the following in partial fractions:

$$\frac{x+1}{x^3+x^2-6x}.$$

Solution:

$$\frac{x+1}{x^3+x^2-6x} = \frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{x+3}$$

Multiplying both sides by x^3+x^2-6x, we get

$$x+1 = A(x-2)(x+3) + Bx(x+3) + Cx(x-2),$$

Put $x=0$ in the above equation, we have $A = \frac{-1}{6}$.
Put $x=2$ in the above equation, we have $B = \frac{3}{10}$.
Put $x=-3$ in the above equation, we have $B = \frac{-2}{15}$.
Hence the required partial fractions

$$\frac{x+1}{x^3+x^2-6x} = \frac{\frac{-1}{6}}{x} + \frac{\frac{3}{10}}{(x-2)} + \frac{\frac{-2}{15}}{x+3}.$$

■

Now, we give some example when the denominator contains ir-reducible (repeated) quadratic factors.

■ **Example 2.8** Resolve into partial fractions:

$$\frac{9x-7}{(x+3)(x^2+1)}.$$

Solution:

$$\frac{9x-7}{(x+3)(x^2+1)} = \frac{A}{(x+3)} + \frac{Bx+C}{(x^2+1)}.$$

Multiplying both sides by $(x+3)(x^2+1)$, we get

$$9x-7 = A(x^2+1) + B(x^2+3x) + C(x+3),$$

Put $x = -3$ in the above equation, we have $A = \frac{-17}{5}$.
Comparing the coefficient of like powers of x on both sides in the above equation, we get

$$A + B = 0 \Rightarrow B = \frac{17}{5}.$$

$$3B + C = 9 \Rightarrow C = \frac{-6}{5}.$$

Hence the required partial fractions

$$\frac{9x-7}{(x+3)(x^2+1)} = \frac{\frac{(-17)}{5}}{(x+3)} + \frac{\frac{17}{5}x - \frac{6}{5}}{(x^2+1)}.$$

■

■ **Example 2.9** Resolve into partial fractions:

$$\frac{x^2+x+2}{x^2(x^2+3)^2}.$$

Solution:

$$\frac{x^2+x+2}{x^2(x^2+3)^2} = \frac{A}{x} + \frac{B}{X^2} + \frac{Cx+D}{(x^2+3)} + \frac{Ex+F}{(x^2+3)^2}.$$

Multiplying both sides by $x^2(x^2+3)^2$, we get

$$x^2+x+2 = Ax(x^2+3)^2 + B(x^2+3)^2 + (Cx+D)x^2(x^2+3) + (Ex+F)x^2.$$

Putting $x = 0$ in the above equation, we have $B = \frac{2}{9}$.
Comparing the coefficient of like powers of x on both sides in the above equation, we get Coefficient of x^5

$$A + C = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Coefficient of x^4

$$B - D = 0 \dots\dots\dots\dots\dots\dots (2)$$

Coefficient of x^3

$$6A + 3C + E = 0 \dots\dots\dots\dots (3)$$

Coefficient of x^2

$$6B + 3D + F = 1 \dots\dots\dots\dots (4)$$

Coefficient of x

$$9A = 1 \dots\dots\dots\dots\dots\dots\dots (5)$$

Hence the required partial fractions

$$\frac{x^2+x+2}{x^2(x^2+3)^2} = \frac{\frac{1}{9}}{x} + \frac{\frac{2}{9}}{X^2} - \frac{x+2}{(x^2+3)} - \frac{x-1}{(x^2+3)^2}.$$

■

■ **Example 2.10** Resolve into partial fractions:

$$\frac{x^2+1}{x^4-x^2+1}.$$

Solution:

$$\frac{x^2+1}{x^4-x^2+1} = \frac{Ax+B}{(x^2-x+1)} + \frac{Cx+D}{(x^2+x+1)}.$$

Multiplying both sides by x^4+x^2+1, we get

$$x^2+1 = (Ax+B)(x^2+x+1) + (Cx+D)(x^2-x+1),$$

Comparing the coefficient of like powers of x, we have
Coefficient of x^3:

$$A + C = 0 \dots\dots\dots\dots\dots\dots\dots\dots (1)$$

Coefficient of x^2:

$$A+B-C+D=1 \dots\dots\dots\dots\dots (2)$$

Coefficient of x:

$$A+B+C-D=0 \dots\dots\dots\dots\dots (3)$$

Constant:

$$B+D=1 \dots\dots\dots\dots\dots\dots\dots\dots (4)$$

Subtract (4) from (2) we have

$$A=C \dots\dots\dots\dots\dots\dots\dots\dots\dots (5)$$

Adding (1) and (5), we have $A=0$.
Putting the value of A and C in (3), we have

$$B=D \dots\dots\dots\dots\dots\dots\dots\dots\dots (6)$$

Adding (4) and (6)

$$B=\frac{1}{2}$$

Hence the required partial fractions

$$\frac{x^2+1}{x^4-x^2+1}=\frac{\frac{1}{2}}{(x^2-x+1)}+\frac{\frac{1}{2}}{(x^2+x+1)}.$$

■

Now, we give some example for improper rational fraction.

■ **Example 2.11** Express the following in partial fractions:

$$\frac{x^2+x+1}{x^2+2x+1}$$

Solution:
The given fraction is improper rational fraction, then we divide the numerator by the denominator

$$\begin{array}{r|l} & 1 \\ \hline x^2+2x+1 & x^2+x+1 \\ & x^2+2x+1 \\ \hline & -x \end{array}$$

$\therefore \dfrac{x^2+x+1}{x^2+2x+1} = 1 - \dfrac{x}{x^2+2x+1}$,

We decompose the proper fraction $\dfrac{x}{x^2+2x+1}$ as follow:

$$\frac{x}{x^2+2x+1} = \frac{x}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} = \frac{A(x+1)+B}{(x+1)^2}$$

$\Rightarrow x = A(x+1) + B$.

Equate the coefficients of x and x^0 (constant terms) to get:

$1 = A$ and $0 = A + B \Rightarrow A = 1$, $B = -1$

$\therefore \dfrac{x}{x^2+2x+1} = \dfrac{1}{x+1} - \dfrac{1}{(x+1)^2}$,

$\therefore \dfrac{x^2+x+1}{x^2+2x+1} = 1 - \dfrac{1}{x+1} + \dfrac{1}{(x+1)^2}$.

∎

2.4 Exercises

Express each of the following in partial fractions:

1. $\dfrac{3x+4}{x^2+x-6}$.
2. $\dfrac{2x+4}{x^3+x^2+x+1}$.
3. $\dfrac{x+4}{x^3+x^2-2x}$.
4. $\dfrac{4}{x^4+x^2-2}$.
5. $\dfrac{x^3-2x+2}{x^3-2x+1}$.

Chapter three

Matrices

3. Matrices

In 1848, G.G. Sylvester introduces the concept of matrices as the name of a group of numbers arranged in a rectangular in the form of rows and columns. In 1855, Arthur Cayley studied matrices from an algebraic perspective. In this study, he defined the process of multiplying matrices using linear transformations.

Definition 3.0.1 A matrix is a rectangular arrangement of numbers (real or complex) which may be represented as,

$$A = (a_{ij})_{m\times n} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix},$$

the general form of a matrix with m rows and n columns.

R Capital letters $A, B, \ldots$ denote matrices, whereas lower case letters $a, b, \ldots$ denote elements.

■ **Example 3.1** Build a matrix $A = (a_{ij})_{2\times 3}$, where

$$a_{ij} = \begin{cases} i+j & if\ i<j \\ i & if\ i=j \\ i-j & if\ i>j \end{cases}$$

Solution:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix},$$

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 1 & 2 & 5 \end{pmatrix},$$

■

■ **Example 3.2** Build a matrix $B = (b_{ij})_{3\times 3}$;

$$b_{ij} = \begin{cases} i+j & if\ i<j \\ 0 & if\ i=j \\ i^2-j^2 & if\ i>j \end{cases}$$

Solution:

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

$$b_{11} = 0\ ,\ b_{12} = 1+2 = 3\ ,\ b_{13} = 1+3 = 4\ ,$$
$$b_{21} = 2^2 - 1^2 = 3\ ,\ b_{22} = 0\ ,\ b_{23} = 2+3 = 5,$$
$$b_{31} = 3^2 - 1^2 = 8\ ,\ b_{32} = 3^2 - 2^2 = 5\ ,\ b_{33} = 0,$$

$$\therefore B = \begin{pmatrix} 0 & 3 & 4 \\ 3 & 0 & 5 \\ 8 & 5 & 0 \end{pmatrix}.$$

■

Definition 3.0.2 Two matrices $A_{m\times n} = (a_{ij})$ and $B_{p\times q} = (b_{kl})$ are equal, if
1- $m = p$ and $n = q$.
2- $a_{ij} = b_{kl}\ \forall i, j, k, l$.

■ **Example 3.3** Given

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \end{pmatrix},$$

and

$$C = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix},$$

disuss the possibility that

1. $A = B$.
2. $B = C$.
3. $A = C$.

Solution

1. $A = B$ is impossible because A and B are of different size.
2. Similarly, $B = C$ is impossible.
3. $A = C$ is possible.

■

Definition 3.0.3 A matrix whose elements are all zero is called a zero matrix and denoted by 0 or O.

Definition 3.0.4 A matrix with the same number of rows as columns is called a square matrix.
A square matrix with n rows and n columns is called a $n-$square matrix.

Example 3.4 The matrix

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 0 & -4 & -1 \\ 5 & 3 & 2 \end{pmatrix},$$

is a 3 square matrix. ■

Definition 3.0.5 The main diagonal or simply diagonal of a square matrix $A = (a_{ij})$ is the numbers $a_{11}, a_{22}, \ldots, a_{nn}$.

Example 3.5 In the above Example 3.4, the numbers along the main diagonal are $1, -4, 2$. ■

Definition 3.0.6 The square matrix with 1s along the main diagonal and 0s elsewhere is called the unit matrix or the identity matrix and will be denoted by I.
For any square matrix A, $AI = IA = A$.

Example 3.6 The matrix

$$I_{3\times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

is a unit matrix of type 3×3. ■

3.1 Matrix Addition

Definition 3.1.1 The sum of the two matrices A and B, written $A+B$, is the matrix obtained by adding the corresponding element from A and B i.e.,

$$A + B = (a_{ij} + b_{ij}).$$

R $A+B$ have the same type as A and B.

R The sum of two matrices with different types is not defined.

■ **Example 3.7** Let A and B;

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ -1 & 4 & 1 \end{pmatrix},$$

be two matrices, then

$$A+B = \begin{pmatrix} 2 & 4 & 1 \\ 6 & 2 & 0 \\ -1 & 8 & 2 \end{pmatrix}.$$

■

■ **Example 3.8** Let

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 4 & 5 \end{pmatrix},$$

$$B = \begin{pmatrix} 3 & 0 & -6 \\ 2 & -3 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix},$$

and

$$D = \begin{pmatrix} 0 & 5 & -2 \\ 1 & -3 & -1 \end{pmatrix}.$$

Find $A+B$ and $C+D$.

Solution

$$A+B = \begin{pmatrix} 4 & -2 & -3 \\ 2 & 1 & 6 \end{pmatrix},$$

and the sum of $C+D$ is not defined. ■

Theorem 3.1.1 Let A, B and C be matrices with the same type, then

1. $(A+B)+C = A+(B+C)$.
2. $A+B = B+A$.
3. $A+O = O+A = A$.

Where O is a zero matrix with the same type of A.

Proof. Let $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{m\times n}$ *and* $C = (c_{ij})_{m\times n}$, then

$$\begin{aligned}(i)\ (A+B)+C &= [(a_{ij})+(b_{ij})]+(c_{ij})\\ &= (a_{ij}+b_{ij})+c_{ij}\\ &= (a_{ij}+b_{ij}+c_{ij})\\ &= (a_{ij})+(b_{ij}+c_{ij})\\ &= A+(B+C)\end{aligned}$$

$$\begin{aligned}(ii)\ A+B &= (a_{ij})+(b_{ij})\\ &= (a_{ij}+b_{ij})\\ &= (b_{ij}+a_{ij})\\ &= (b_{ij})+(a_{ij}).\end{aligned}$$

(*iii*) Trivial. ∎

■ **Example 3.9** Solve

$$\begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix} + X = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix},$$

where X is a matrix.

Solution:

To solve

$$\begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix} + X = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix},$$

simply subtract the matrix

$$\begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix},$$

from both sides to get

$$X = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} - \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 0 & 1 \end{pmatrix}.$$

■

3.2 Scalar Multiplication

Definition 3.2.1 The product of a scalar k and a matrix A, written kA is the matrix obtained by multiplying each element of A by k, i.e.,

$$kA = (ka_{ij})_{m\times n}.$$

■ **Example 3.10** $3\begin{pmatrix} 1 & -2 & 0 \\ 4 & 3 & -3 \end{pmatrix} = \begin{pmatrix} 3 & -6 & 0 \\ 12 & 9 & -15 \end{pmatrix}.$

■

■ **Example 3.11** If $kA = 0$, show that either $k = 0$ or $A = 0$.

Solution:

Write $A = (a_{ij})$, so that $kA = 0$, means $ka_{ij} = 0$, for all i and j. If $k = 0$, there is nothing to do. If $k \neq 0$, then $ka_{ij} = 0$ implies that $a_{ij} = 0$, for all i and j; that is, $A = 0$. ■

3.3 Matrix Multiplication

Definition 3.3.1 Le $A_{m\times n} = (a_{ij})$ and $B_{n\times q} = \left(b_{jk}\right)$, then

$$C_{m\times p} = AB = (\sum_{j=0}^{n} a_{ij}b_{jk}).$$

■ **Example 3.12** Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & -1 \\ 5 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 4 & 5 \end{pmatrix},$$

and

$$D = \begin{pmatrix} 1 & 4 \\ 2 & -5 \\ 3 & 6 \end{pmatrix},$$

then find
(1) AB.
(2) AC.
(3) AD.

Solution

(1) $AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 12 & -1 \\ 23 & -3 \end{pmatrix}.$

(2) $AC = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & -6 & 13 \\ 3 & 10 & 29 \end{pmatrix}.$

(3) $AD = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & -5 \\ 3 & 6 \end{pmatrix}$ is not defined. ■

Theorem 3.3.1 Let A, B and C be matrices with the same type, then

(i) $(AB)C = A(BC)$
(ii) $A(B+C) = AB + AC$
(iii) $(B+C)A = BA + CA$
(iv) $k(AB) = (kA)B = A(kB)$ *where k is a scalar.*

Proof. Let $A = (a_{ij})_{m\times n}$, $B = (b_{jk})_{n\times p}$ *and* $C = (c_{kl})_{p\times q}$, then

(i) $L.H.S = (AB)C$
$= (\sum_{j=0}^{n} a_{ij} b_{jk}).(c_{kl})$
$= (\sum_{k=0}^{p}[(\sum_{j=0}^{n} a_{ij} b_{jk}).c_{kl}])$
$= (\sum_{k=0}^{p}\sum_{j=0}^{n} a_{ij} b_{jk}.c_{kl}).$

$R.H.S = A(BC)$
$= (a_{ij})(\sum_{k=0}^{p} b_{jk} c_{kl})$

$= (\sum_{j=0}^{n} a_{ij}[(\sum_{k=0}^{p} b_{jk}c_{kl})]$
$= (\sum_{j=0}^{n}\sum_{k=0}^{p} a_{ij}b_{jk}c_{kl}).$

Assuming I have written these correctly, we can make two observations: first, the summands are equivalent, as multiplication is associative. Second, the order of the summations doesn't matter when we're summing a finite number of entries. Thus, $(AB)C = A(BC)$.

(ii) Let $A = (a_{ij})_{m\times n},\ B = (b_{jk})_{n\times n}$ and $C = (c_{jk})_{n\times n},$ then
$L.H.S = A(B+C)$
$= (\sum_{j=1}^{n} a_{ij}\left(b_{jk}+c_{jk}\right))$
$= (\sum_{j=1}^{n}(a_{ij}b_{jk}+a_{ij}c_{jk}))$
$= (\sum_{j=1}^{n} a_{ij}b_{jk}) + (\sum_{j=1}^{n} a_{ij}c_{jk})$
$= AB + AC.$

(iii) In the same way.
(iv) Trivial.

■

R The matrix product is not commutative in general i.e.,

$$AB \neq BA.$$

■ **Example 3.13** Simplify the expression

$$A(BC-CD)+A(C-B)D-AB(C-D).$$

Solution
$A(BC-CD)+A(C-B)D-AB(C-D) = A(BC)-A(CD)+(AC-AB)D-(AB)C+(AB)D = ABC-ACD+ACD-ABD-ABC+ABC = 0.$ ■

■ **Example 3.14** Show that $AB = BA$ if and only if

$$(A-B)(A+B) = A^2-B^2.$$

Solution
In general the following hold

$$(A-B)(A+B)=A(A+B)-B(A+B)=A^2+AB-BA-B^2.$$

Hence if $AB = BA$, then $(A-B)(A+B) = A^2 - B^2$. Conversely, if this last equation holds, then equation becomes

$$(A-B)(A+B)=A(A+B)-B(A+B)=A^2+AB-BA-B^2.$$

This gives $0 = AB - BA$, and then $AB = Bc$. ■

3.4 Transpose

Definition 3.4.1 The *transpose of a matrix*

$$A = (a_{ij})_{m\times n},$$

written by A^T is the matrix obtained by writing the rows of A, in order, as columns, i.e.,

$$A^T = (a_{ji})_{n\times m}.$$

Example 3.15 Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \end{pmatrix},$$

then

$$A^T = \begin{pmatrix} 1 & 4 \\ 2 & -5 \\ 3 & 6 \end{pmatrix}.$$

■

The transpose operation on a matrix satisfies the following properties:

Theorem 3.4.1 Let A and B be matrices with the same type, then
1. $(A+B)^T = A^T + B^T$.
2. $(A^T)^T = A$.
3. $(kA)^T = kA^T$, *for k a scalar.*
4. $(AB)^T = B^T A^T$.

Proof. Let $A = (a_{ij})_{m\times n}, B = (b_{jk})_{m\times n}$, *then*

1. $$\begin{aligned} L.H.S &= (A+B)^T \\ &= (a_{ij}+b_{ij})^T \\ &= (a_{ji}+b_{ji}) \\ &= (a_{ji})+(b_{ji}) \\ &= A^T + B^T \\ &= R.H.S. \end{aligned}$$

2. $$\begin{aligned} L.H.S &= \left(A^T\right)^T \\ &= \left((a_{ij})^T\right)^T \\ &= (a_{ji})^T \\ &= (a_{ij}) \\ &= A = R.H.S. \end{aligned}$$
3. Exercise.
4. Exercise.

■

Definition 3.4.2 A matrix A is called *symmetric* if

$$A = A^T.$$

Definition 3.4.3 A matrix A is called *skew-symmetric* if

$$A = -A^T.$$

R A symmetric matrix A is necessarily square.

■ **Example 3.16** If A and B are symmetric $n \times n$ matrices, show that $A+B$ is symmetric.
Solution:
Since $A = A^T$ and B^T, so, we have

$$(A+B)^T = A^T + B^T = A + B.$$

Hence $A+B$ is symmetric. ■

■ **Example 3.17** Let A be a square matrix satisfies,

$$A = 2A^T.$$

show that necessarily $A = 0$.
Solution:
If we iterate the given equation, gives

$$A = 2A^T.$$

$$= 2(2A^T)^T.$$
$$= 2((2A^T)^T).$$
$$= 4A.$$

This lead to $3A = O$ and hence $A = 0$. ■

■ **Example 3.18** If A and B are two skew symmetric matrices of same order, then AB issymmetric matrix if ....
Solution
$AB = BA$. ■

3.5 The inverse of a matrix

The inverse of a square $n \times n$ matrix A is another $n \times n$ matrix denoted by A^{-1} such that

$$AA^{-1} = A^{-1}A = I.$$

where I is the $n \times n$ identity matrix. That is, multiplying a matrix by its inverse produces an identity matrix. Not all square matrices have an inverse matrix. If the determinant of the matrix is zero, then it will not have an inverse, and the matrix is said to be singular. Only non-singular matrices have inverses.

Definition 3.5.1 If A is a square matrix, a matrix B is called an inverse of A if and only if

$$AB = I \text{ and } BA = I.$$

■ **Example 3.19** Show that

$$B = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$$

is an inverse of

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Solution:

Compute AB and BA.

$$AB = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$BA = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence $AB = I = BA$, so B is indeed an inverse of A. ■

■ **Example 3.20** If

$$A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},$$

show that $A^3 = I$ and so find A^{-1}. **Solution:**
We have

$$A^2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix},$$

and so

$$A^3 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence $A^3 = I$, as asserted. This can be written as

$$A^2A = AA^2 = I,$$

so it shows that A^2 is the inverse of A. That is,

$$A^{-1} = A^2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

■

3.5.1 Adjoint of a square matrix

Let $A = (a_{ij})_{n\times n}$ be a square matrix of order n and let c_{ij} be the cofactor of a_{ij} in the determinant |A|, then the adjoint of A, denoted by adj (A), is defined as the transpose of the matrix, formed by the cofactors of the matrix.

Theorem 3.5.1 Given any non-singular matrix A, its inverse can be found from the formula

$$A^{-1} = \frac{adj\,A}{|A|}.$$

where $adj\,A$ is the adjoint matrix and $|A|$ is the determinant of A.

■ **Example 3.21** Find A^{-1} where

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}.$$

Solution:
We calculate the value of the determinant of the matrix

$$|A| = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 2 \\ 0 & -3 & -6 \\ 0 & -6 & -3 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -3 & -6 \\ -6 & -3 \end{vmatrix}$$

$$= -27 \neq 0 \quad .$$

The cofactors of the matrix

$$\Delta_{11} = (-1)^{1+1} \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} = -3,$$

$$\Delta_{12} = (-1)^{1+2} \begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} = -6,$$

$$\Delta_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 1 \\ 2 & -2 \end{vmatrix} = -6,$$

$$\Delta_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 2 \\ -2 & 1 \end{vmatrix} = -6,$$

$$\Delta_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3,$$

$$\Delta_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} = 6,$$

$$\Delta_{31} = (-1)^{3+1} \begin{vmatrix} 2 & 2 \\ 1 & -2 \end{vmatrix} = -6,$$

$$\Delta_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} = 6,$$

$$\Delta_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3.$$

So,

$$\tilde{A} = (\Delta_{ij}) = \begin{pmatrix} -3 & -6 & -6 \\ -6 & -3 & 6 \\ -6 & 6 & -3 \end{pmatrix} = -3 \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix},$$

and

$$ad\,jA = (\tilde{A})^t = -3 \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix},$$

thus

$$\begin{aligned} A^{-1} &= \frac{ad\,jA}{|A|} \\ &= \frac{-3}{-27} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \\ &= \frac{1}{9} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}. \end{aligned}$$

■

Theorem 3.5.2 All the following matrices are square matrices of the same size.

1. I is invertible and $I^{-1} = I$.
2. If A is invertible, so is A^{-1}, and $\left(A^{-1}\right)^{-1} = A$.
3. If A and B are invertible, so is AB, and

$$(AB)^{-1} = B^{-1}A^{-1}$$

.

4. If A is invertible, then $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.

3.6 Exercises

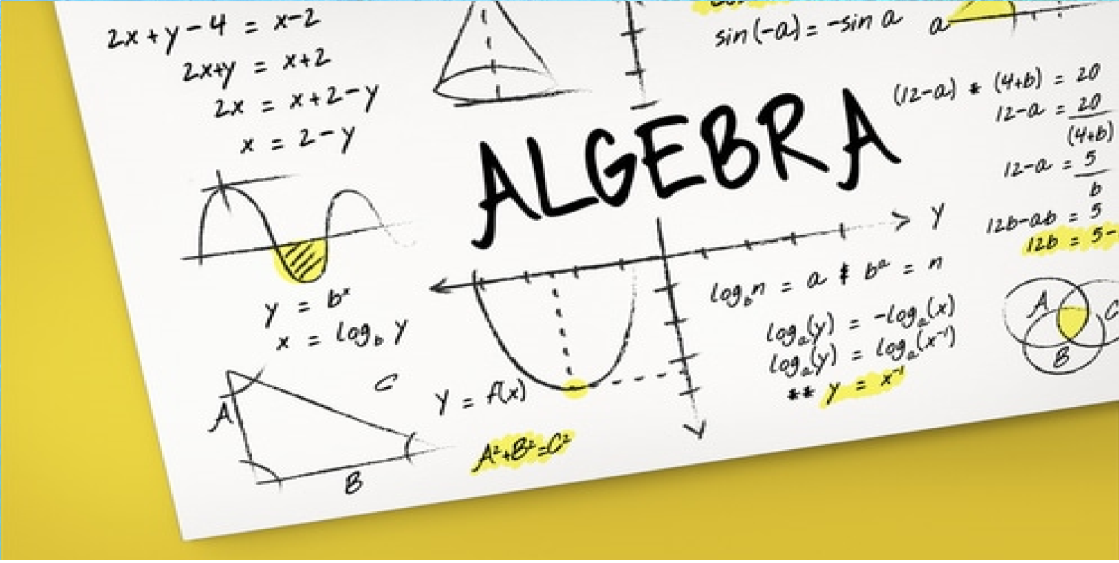
1- Prove that
(i) $(kA)^T = kA^T$, *for k a scalar.*
(ii) $(AB)^T = B^T A^T$.
(iii) I is invertible and $I^{-1} = I$.
(iv) If A is invertible, so is A^{-1}, and $\left(A^{-1}\right)^{-1} = A$.
(v) If A and B are invertible, so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.
(vi) If A is invertible, then $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.
2- Build matrices $A = (a_{ij})_{3\times 2}$, $B = (b_{ij})_{2\times 3}$;

$$a_{ij} = \begin{cases} i+j & if\ i<j \\ i & if\ i=j \\ i-j & if\ i>j \end{cases} \quad , \quad b_{ij} = \begin{cases} 2i-1 & if\ i=j \\ i+j-2 & if\ i\neq j \end{cases}$$

3- If $A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & -3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -4 & 0 & 1 \\ 2 & -1 & 3 & -1 \\ 4 & 0 & -2 & 0 \end{pmatrix}$.
Compute AB.
4- If $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}$, $B = \begin{pmatrix} -11 & -4 & 6 \\ 2 & 0 & -1 \\ 2 & 1 & -1 \end{pmatrix}$.
Compute AB^t.
6- If $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -1 & 1 \\ -4 & 3 & -2 \\ 3 & -2 & 1 \end{pmatrix}$.
Compute AB^t.
7- Find the inverse of the matrices
(i) $\begin{pmatrix} -2 & 3 \\ -5 & -6 \end{pmatrix}$.
(ii) $\begin{pmatrix} 3 & 5 \\ 7 & 9 \end{pmatrix}$.
(iii) $\begin{pmatrix} 1 & 1 & 3 \\ 2 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix}$.

Chapter Four

Complex numbers



4. Complex Numbers

Complex numbers evolved through the work of Caspar Wessel (1745–1818), Jean-Robert Argand (1768–1822), and Carl Friedrich Gauss (1777–1855).

Definition 4.0.1 A complex number is a number consisting of a real and imaginary part. Its standard form is

$$z = x + iy$$

where $x, y \in \mathbf{R}$, $Re(z) = x$, $Im(z) = y$ and $i = \sqrt{-1}$.

(R) The set of all complex number denoted by $\mathbf{C}$

Definition 4.0.2 Two complex numbers are equal if their real parts are equal and their imaginary parts are equal, i.e., if $z_1 = x_1 + iy_1$ equal $z_2 = x_2 + iy_2$, then $x_1 = x_2$ and $y_1 = y_2$.

Definition 4.0.3 The complex conjugate of a complex number $z = x + iy$, denoted by $\bar{z}$ is given by

$$\bar{z} = x - iy.$$

Definition 4.0.4 Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex number, then the addition of z_1, z_2 is given by

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2).$$

Definition 4.0.5 The absolute value or modulus of a complex number $z = x + iy$ is

$$|z| = \sqrt{x^2 + y^2}.$$

■ **Example 4.1** Find $Re(z)$, $Im(z)$, $\bar{z}$, $-z$ and z^{-1} for each comlex number z of the following:

$$1 - 2i,\ 2 + i,\ i,\ \frac{1}{1+i},\ -2$$

<u>**Solution**</u> ■

comlex number	$1-2i$	$2+i$	i	$\frac{1}{1+i}$	-2
$Re(z)=x$	1	2	0	$\frac{1}{2}$	-2
$Im(z)$	-2	1	1	$\frac{-1}{2}$	0
$\bar{z}$	$1+2i$	$2-i$	$-i$	$\frac{1+i}{2}$	-2
$-z$	$-1+2i$	$-2-i$	$-i$	$\frac{-1+i}{2}$	2
z^{-1}	$\frac{1+2i}{5}$	$-\ldots$	$-i$	$\ldots$	$\frac{-1}{2}$

4.1 The polar form of a complex number

Definition 4.1.1 Let $z = x+iy$ be a complex number, then the *polar form* of a complex number defined as follow

$$z = r(\cos\theta + i\sin\theta)$$

where

$$x = r\cos\theta\ ,\ \ y = r\sin\theta,$$

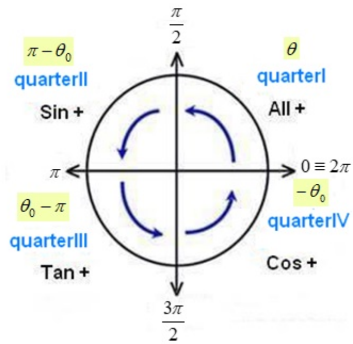
and

$$r = |z| = \sqrt{x^2+y^2}\ ,\ \ \theta = \tan^{-1}\frac{y}{x}.$$

The number r is the absolute value or modulus of z, and θ is an *argument* of z; denoted by $\arg(z)$.

R The *principal argument* of z is $-\pi \le \theta \le \pi$.

Now, we show how to determined, the principal argument according to in which quarterlies.



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■ **Example 4.2** Write each of the following comlex number z in polar

1. $1+i$.
2. $-\sqrt{3}+i$.
3. $-1-i\sqrt{3}$.
4. $1-i$.

Solution:

1. $z=1+i$.
   Since $x=1$, and $y=1$, then

$$r=\sqrt{x^2+y^2}=\sqrt{1+1}=\sqrt{2},$$

$$\begin{aligned}\sin\,\theta &= \tfrac{y}{r} = \tfrac{1}{\sqrt{2}}\,,\\ \cos\theta &= \tfrac{x}{r} = \tfrac{1}{\sqrt{2}}\,,\\ \tan\theta &= \tfrac{y}{x} = \tfrac{1}{1} = 1\,.\end{aligned}$$

$$\therefore \theta = \frac{\pi}{4},$$

$$\therefore 1+i=\sqrt{2}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}).$$

2. $z=-\sqrt{3}+i$.

Since $x = 1$, and $y = 1$, then

$$\begin{aligned} r &= \sqrt{x^2+y^2} = \sqrt{3+1} = 2\,, \\ \sin\theta &= \tfrac{y}{r} = \tfrac{1}{2}\,, \\ \cos\theta &= \tfrac{x}{r} = \tfrac{-\sqrt{3}}{2}\,, \\ \tan\theta &= \tfrac{y}{x} = \tfrac{1}{-\sqrt{3}}\,. \end{aligned}$$

$$\therefore \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6},$$

$$\therefore -\sqrt{3} + i = 2\,[\cos(\frac{5\pi}{6}) + i\sin(\frac{5\pi}{6})].$$

3. $z = -1 - i\sqrt{3}$.
   Since $x = 1$, and $y = 1$, then

$$r = \sqrt{x^2+y^2} = \sqrt{1+3} = 2\,,$$

$$\begin{aligned} \sin\theta &= \tfrac{y}{r} = \tfrac{-\sqrt{3}}{2}\,, \\ \cos\theta &= \tfrac{x}{r} = \tfrac{-1}{2}\,, \\ \tan\theta &= \tfrac{y}{x} = \tfrac{-\sqrt{3}}{-1} = \sqrt{3}\,. \end{aligned}$$

$$\therefore \theta = \frac{\pi}{3} - \pi = -\frac{2\pi}{3},$$

$$\therefore -1 - i\sqrt{3} = 2\,[\cos(-\frac{2\pi}{3}) + i\sin(-\frac{2\pi}{3})].$$

4. $z = 1 - i$.
   Since $x = 1$, and $y = 1$, then

$$r = \sqrt{x^2+y^2} = \sqrt{1+1} = \sqrt{2}\,,$$

$$\begin{aligned} \sin\theta &= \tfrac{y}{r} = \tfrac{-1}{\sqrt{2}}\,, \\ \cos\theta &= \tfrac{x}{r} = \tfrac{1}{\sqrt{2}}\,, \\ \tan\theta &= \tfrac{y}{x} = \tfrac{-1}{1} = -1\,. \end{aligned}$$

$$\therefore \theta = -\frac{\pi}{4},$$

$$\therefore 1 - i = \sqrt{2}\,[\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})].$$

■

4.1.1 Multiplication and Division of Complex Numbers

The polar form of (trigonometric form for) a complex numbers is particularly convenient for multiplying and dividing complex numbers.

Theorem 4.1.1 Let $z_1 = r_1(cos\theta_1 + i\ sin\theta_1)$ and $z_2 = r_2(cos\theta_2 + i\ sin\theta_2)$, then

1. $z_1.z_2 = r_1 r_2[cos(\theta_1 + \theta_2) + i\ sin(\theta_1 + \theta_2)]$.
2. $\frac{z_1}{z_2} = \frac{r_1}{r_2}[cos(\theta_1 - \theta_2) + i\ sin(\theta_1 - \theta_2)]$.

Proof. Suppose that $z_1 = r_1(cos\theta_1 + i\ sin\theta_1)$ and $z_2 = r_2(cos\theta_2 + i\ sin\theta_2)$, then

1. $z_1.z_2 = r_1(cos\theta_1 + i\ sin\theta_1).r_2(cos\theta_2 + i\ sin\theta_2)$
$= r_1 r_2[(cos\theta_1 cos\theta_2 - sin\theta_1 sin\theta_2) + i(cos\theta_1 sin\theta_2 + cos\theta_2 sin\theta_1)]$
$= r_1 r_2[cos(\theta_1 + \theta_2) + i\ sin(\theta_1 + \theta_2)]$.

2. Exercise. ■

■ **Example 4.3** Use an algebraic method to express the product of and in standard form.Approximate exact values with a calculator when appropriate.

$$z_1 = 25\sqrt{2}(cos\frac{-\pi}{4} + i\ sin\frac{-\pi}{4})$$

and

$$z_2 = 14(cos\frac{\pi}{3} + i\ sin\frac{\pi}{3}).$$

Solution:

$$z_1.z_2 = 14(25\sqrt{2})(cos(\frac{-\pi}{4} + \frac{\pi}{3}) + i\ sin(\frac{-\pi}{4} + \frac{\pi}{3}))$$
$$= 350\sqrt{2}(cos\frac{\pi}{12} + i\ sin\frac{\pi}{12})$$
$$\approx 478.11 + 128.11i.$$ ■

■ **Example 4.4** Use an algebraic method to express the product $\frac{z_1}{z_2}$ in standard form. Approximate exact values with a calculator when appropriate.

$$z_1 = 2\sqrt{2}(cos 135^\circ + i\ sin\ 135^\circ)$$

and

$$z_2 = 6(cos 300^\circ + i\ sin 300^\circ).$$

Solution

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{2\sqrt{2}(cos 135^\circ + i\ sin\ 135^\circ)}{6(cos 300^\circ + i\ sin 300^\circ)}\\ &= \frac{2\sqrt{2}}{6}(cos(135^\circ - 300^\circ) + i\ sin\ (135^\circ - 300^\circ))\\ &= \frac{\sqrt{2}}{3}(cos - 165^\circ + i\ sin\ \ -165^\circ)\\ &\approx -0.46 - 0.12i.\end{aligned}$$

■

4.2 De Moivre's Theorem

Theorem 4.2.1 Let $z = r(\cos\theta + i\sin\theta)$ be a complex number and n be any real number. Then

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

.

Proof. We can prove this theorem by using Mathematical Induction. ■

■ **Example 4.5** Using De Moivre's Theorem, find the value of

$$(1+i)^8.$$

Solution:
We put the complex number $z = 1+i$ in the polar form as follows:

$$r = \sqrt{1+1} = \sqrt{2},$$

$$\theta = \tan^{-1}\frac{1}{1} = \tan^{-1}1 = \frac{\pi}{4}$$

$$\therefore z = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}),$$

so

$$\therefore z^8 = (\sqrt{2})^8(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})^8 = 16(\cos 2\pi + i\sin 2\pi) = 16.$$

■

■ **Example 4.6** Using De Moivres' Theorem, reduce the complex number:

$$z = \frac{(\cos 2\theta - i\sin 2\theta)^5(\cos 3\theta + i\sin 3\theta)^7}{(\cos 4\theta + i\sin 4\theta)^{11}(\cos 5\theta - i\sin 5\theta)^9},$$

and find its value at

$$\theta = \frac{\pi}{6}.$$

Solution: We rewrite z as follow:

$$z = \frac{[\cos(-2\theta)+i\sin(-2\theta)]^5[\cos 3\theta + i\sin 3\theta]^7}{[\cos 4\theta + i\sin 4\theta]^{11}[\cos(-5\theta)+i\sin(-5\theta)]^9}$$

$$= \frac{[\cos\theta + i\sin\theta]^{-10}[\cos\theta + i\sin\theta]^{21}}{[\cos\theta + i\sin\theta]^{44}[\cos\theta + i\sin\theta)]^{-45}}$$

$$= (\cos\theta + i\sin\theta)^{12}$$

$$= \cos 12\theta + i\sin 12\theta.$$

and when $\theta = \frac{\pi}{6}$ we find

$$z = \cos(12\frac{\pi}{6}) + i\sin(12\frac{\pi}{6}) = \cos 2\pi + i\sin 2\pi = 1.$$

■

■ **Example 4.7** Using De Moivre's Theorem, reduce the complex number:

$$\frac{(1+i\tan\theta)^5}{(1-i\tan\theta)^7},$$

and find its value at

$$\theta = \frac{\pi}{6}.$$

Solution:
We rewrite z as follow:

$$z = \frac{(1+i\tan\theta)^5}{(1-i\tan\theta)^7},$$

$$= \frac{(1+i\frac{\sin\theta}{\cos\theta})^5}{(1-i\frac{\sin\theta}{\cos\theta})^7}$$

$$= \frac{(\cos\theta)^2(\cos\theta + i\sin\theta)^5}{(\cos\theta - i\sin\theta)^7}$$

$$= \frac{(\cos\theta)^2(\cos\theta + i\sin\theta)^5}{(\cos\theta + i\sin\theta)^{-7}}$$

$$= (\cos\theta)^2(\cos\theta + i\sin\theta)^{12}$$
$$= (\cos\theta)^2[\cos(12\theta) + i\sin(12\theta)].$$

and when $\theta = \frac{\pi}{6}$, we find

$$z = (\cos(\frac{\pi}{6}))^2[\cos(12\frac{\pi}{6}) + i\sin(12\frac{\pi}{6})]$$
$$= (\frac{\sqrt{3}}{2})^2[\cos 2\pi + i\sin 2\pi] = \frac{3}{4}.$$

■

4.2.1 Roots of Complex Numbers

Definition 4.2.1 A complex number $u = x + iy$ is an nth root of z if

$$u^n = 1.$$

If $z = 1$, then u is an nth root of unity.

Now, we find nth roots of a complex number as follow

Definition 4.2.2 If$z = r(cos\theta + isin\theta)$, then the n distinct complex numbers

$$r^{\frac{1}{n}}(cos\frac{\theta + 2\pi k}{n} + isin\frac{\theta + 2\pi k}{n}),$$

where $k = 0, 1, 2, \ldots, n-1$, are the nth roots of the complex number z.

■ **Example 4.8** Find the fourth roots of

$$z = 5(cos\frac{\pi}{3} + isin\frac{\pi}{3}).$$

Solution:
The fourth roots of z are the complex numbers

$$5^{\frac{1}{4}}(cos\frac{\frac{\pi}{3} + 2\pi k}{n} + isin\frac{\frac{\pi}{3} + 2\pi k}{n}),$$

where $k = 0,1,2,3$.
When $k = 0$,

$$z_1 = 5^{\frac{1}{4}}(cos\frac{\frac{\pi}{3}+2\pi(0)}{n}+isin\frac{\frac{\pi}{3}+2\pi(0)}{n}),$$

$$= 5^{\frac{1}{4}}(cos\frac{\pi}{12}+isin\frac{\pi}{12}).$$

When $k = 1$,

$$z_2 = 5^{\frac{1}{4}}(cos\frac{\frac{\pi}{3}+2\pi(1)}{n}+isin\frac{\frac{\pi}{3}+2\pi(1)}{n}),$$

$$= 5^{\frac{1}{4}}(cos\frac{7\pi}{12}+isin\frac{7\pi}{12}).$$

When $k = 2$,

$$z_3 = 5^{\frac{1}{4}}(cos\frac{\frac{\pi}{3}+2\pi(2)}{n}+isin\frac{\frac{\pi}{3}+2\pi(2)}{n}),$$

$$= 5^{\frac{1}{4}}(cos\frac{13\pi}{12}+isin\frac{13\pi}{12}).$$

When $k = 3$,

$$z_4 = 5^{\frac{1}{4}}(cos\frac{\frac{\pi}{3}+2\pi(3)}{n}+isin\frac{\frac{\pi}{3}+2\pi(3)}{n}),$$

$$= 5^{\frac{1}{4}}(cos\frac{19\pi}{12}+isin\frac{19\pi}{12}).$$

■

4.3 Exercises

A- Write the complex number $z = \frac{2}{1+i}$ in the form $z = x + iy$, and find

1. $Re(z)$.
2. $Im(z)$.
3. $\bar{z}$.
4. $|z|$.
5. $\arg(z)$.

B- Write the complex number $z = \frac{4}{-\sqrt{3}+i}$ in the form $z = x + iy$, and find

1. $Re(z)$.
2. $Im(z)$.
3. $\bar{z}$.
4. $|z|$.
5. $\arg(z)$.

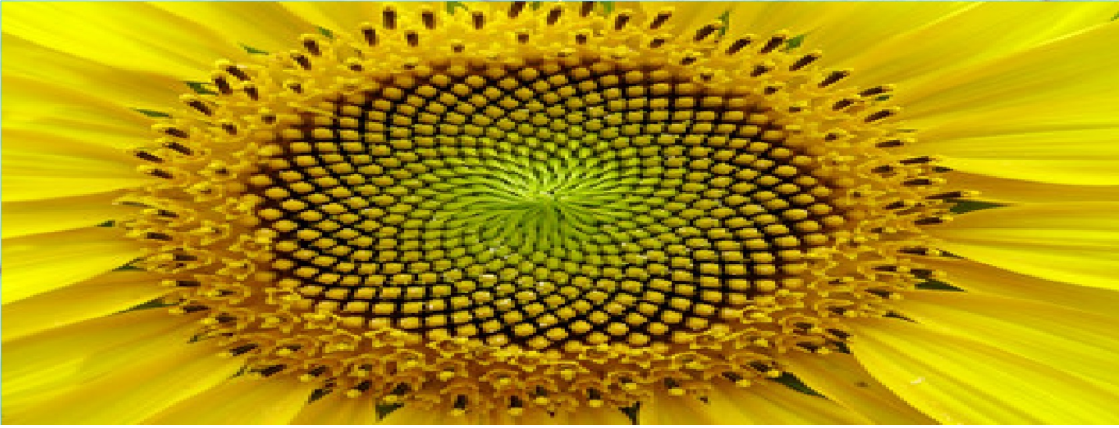
C Prove Theorem 4.2.1.

D Using De Moivre‘s Theorem, find the value of

1. $(1 + i\sqrt{3})^6$.
2. $(\sqrt{3} + i)^{12}$.

Chapter Five

Series



5. Series

In mathematics a series is defined as the sum of a sequence of numbers. It can be express by using the notation $\sum a_n$, where $\sum$ represents "sum" and a_n is the nth term of the sum, which is a generalized way of expressing the terms of the sum.

5.1 Sequence

Definition 5.1.1 A sequence is a mapping from $\mathbb{N}$ to a non-empty set X i.e.,

$$f:\mathbb{N} \rightarrow X,$$

and we can write sequence terms in the form $a_1, a_2, \ldots, a_n, \ldots$ where

$$a_1 = f(1), a_2 = f(2), \ldots, a_n = f(n), \ldots$$

We call a_1 the first term and a_n the general term.

5.2 Serie

Definition 5.2.1 A serie is a summation of sequence terms.
Infinite serie:

$$\sum_{r=1}^{\infty} a_r = a_1 + a_2 + \cdots + a_n + \ldots$$

Finite serie:

$$\sum_{r=1}^{n} a_r = a_1 + a_2 + \cdots + a_n.$$

5.3 Geometric series

Definition 5.3.1 For $r \neq 0$, the sum of the first n terms of a geometric series is:

$$\sum_{r=1}^{n} a_r = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(d^n - 1)}{d - 1}.$$

Where ad^{n-1} is a basis.

5.4 Arithmetic Series

Definition 5.4.1 Let

$$a + (a+d) + (a+2d) + \cdots + (a + (n-1)d),$$

the the summation is

$$\sum_{r=1}^{n} a_r = \frac{n}{2}[2a + (n-1)d],$$

where d is a basis $(a + (n-1)d)$ or $d = (a_{r+1} - a_r)$.

Now, we introduce a method of differences which it used to sum of finite series

5.5 The method of differences

Theorem 5.5.1 In a series

$$\sum_{r=1}^{n} a_r = a_1 + a_2 + \cdots + a_n.$$

If we can write the general term in the series in the form

$$a_r = f(r+1) - f(r).$$

Then the summation is

$$\sum_{r=1}^{n} a_r = f(n+1) - f(1).$$

Proof. Let $r = 1, 2, 3, \ldots, n$ in $a_r = f(r+1) - f(r)$, then we obtain

$$a_1 = f(2) - f(1).$$
$$a_2 = f(3) - f(2).$$
$$\vdots$$
$$a_n = f(n+1) - f(n).$$

So the summation is

$$\sum_{r=1}^{n} a_r = f(n+1) - f(1).$$

■

■ **Example 5.1** Find the summation of series

$$1+2+3+\cdots+n.$$

Solution:
The general term is $a_r = r$ and we apply the Theorem 5.5.1

$$r(r+1)-(r-1)r = r[r+1-r+1] = 2a_r.$$

Thus,

$$a_r = \frac{1}{2}[r(r+1)-(r-1)r].$$

So the summation:

$$\sum_{r=1}^{n} a_r = f(n+1)-f(1) = \frac{1}{2}n(n+1).$$

■

■ **Example 5.2** Find the summation of series

$$1\times 2+2\times 3+3\times 4+\ldots \textit{to n term}.$$

Solution:
The general term is

$$a_r = r(r+1),$$

and we apply the Theorem 5.5.1

$$r(r+1)(r+2)-(r-1)r(r+1) = r(r+1)[r+2-r+1] = 3a_r.$$

Thus,

$$a_r = \frac{1}{3}[r(r+1)(r+2)-(r-1)r(r+1)].$$

So the summation:

$$\sum_{r=1}^{n} a_r = f(n+1)-f(1) = \frac{1}{3}n(n+1)(n+2).$$

■

■ **Example 5.3** Find the summation of series

$$1\times 2\times 3+2\times 3\times 4+3\times 4\times 5+\ldots \textit{to n term}$$

Solution:
The general term is

$$a_r = r\,(r+1)\,(r+2)\,,$$

and we apply the Theorem 5.5.1

$$r\,(r+1)\,(r+2)\,(r+3)-(r-1)\,r\,(r+1)\,(r+2)=4a_r.$$

Thus,

$$a_r=\frac{1}{4}[r\,(r+1)\,(r+2)\,(r+3)-(r-1)\,r\,(r+1)\,(r+2)].$$

So the summation:

$$\sum_{r=1}^{n} a_r = f\,(n+1)-f\,(1)=\frac{1}{4}n\,(n+1)\,(n+2)(n+3).$$

■

Theorem 5.5.2 In a series

$$\sum_{r=1}^{n} a_r = a_1+a_2+\cdots+a_n.$$

If we can write the general term in this series in the form

$$a_r = f\,(r)-f\,(r+1)\,.$$

Then the summation is

$$\sum_{r=1}^{n} a_r = f\,(1)-f\,(r+1)$$

■ **Example 5.4** Find the summation of series

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\dots to\ n\ term.$$

Solution:
The general term is

$$a_r=\frac{1}{r(r+1)},$$

and by using Partial Fraction we find

$$a_r=\frac{1}{r}-\frac{1}{r+1},$$

we apply the Theorem 5.5.2

$$\sum_{r=1}^{n} a_r = f(1)-f(n+1)=\frac{n}{n+1}.$$

■

■ **Example 5.5** Find the summation of series

$$\frac{1}{2\times 3\times 4}+\frac{1}{3\times 4\times 5}+\dots to\ n\ term.$$

Solution:
The general term is

$$a_r=\frac{1}{(r+1)(r+2)(r+3)},$$

and by using Partial Fraction we find

$$a_r=\frac{\frac{1}{2}}{(r+1)(r+2)}-\frac{\frac{1}{2}}{(r+2)(r+3)},$$

we apply the Theorem 5.5.2

$$\sum_{r=1}^{n} a_r = f(1)-f(n+1)=\frac{n(n+5)}{12(n+2)(n+3)}.$$

■

■ **Example 5.6** Find the summation of series

$$\frac{1}{2!}+\frac{2}{3!}+\ldots to\ n\ term.$$

Solution:
The general term is

$$\begin{aligned} a_r &= \frac{r}{(r+1)!}, \\ &= \frac{r+1-1}{(r+1)!} \\ &= \frac{r+1}{(r+1)!}-\frac{1}{(r+1)!} \\ &= \frac{1}{r!}-\frac{1}{(r+1)!} \end{aligned}$$

we apply the Theorem 5.5.2

$$\sum_{r=1}^{n} a_r = f(1)-f(n+1) = 1-\frac{1}{(n+1)!}.$$

R We can use some laws to find the summation some series:

1. $\sum_{r=1}^{n} \mathrm{r} = \frac{1}{2}n(n+1)$.
2. $\sum_{r=1}^{n} \mathrm{r}^2 = \frac{1}{6}n(n+1)(2n+1)$.
3. $\sum_{r=1}^{n} \mathrm{r}^3 = \frac{1}{4}n^2(n+1)^2$.

■

■ **Example 5.7** Find the summation of series

$$\sum_{r=11}^{20} \mathrm{r}^2$$

Solution:

$$\sum_{r=11}^{20} \mathrm{r}^2 = \sum_{r=1}^{20} \mathrm{r}^2 - \sum_{r=1}^{10} \mathrm{r}^2 = 2485.$$

■

5.6 Exercise

A.Find the summation of series

1. $1\times 5 + 3\times 8 + 5\times 11 + \ldots\ to\ n\ terms.$
2. $1\times 3 + 2\times 5 + 3\times 7 + \ldots\ to\ n\ terms.$
3. $1\times 4 + 4\times 7 + 7\times 10 + \ldots\ to\ n\ terms.$
4. $1\times 4\times 7 + 4\times 7\times 10 + 7\times 10\times 13 + \ldots\ to\ n\ terms.$
5. $3\times 4\times 5 + 4\times 5\times 6 + 5\times 6\times 7 + \ldots\ to\ n\ terms.$
6. $1\times 1! + 2\times 2! + 3\times 3! + \ldots to\, n\, terms.$

B. Prove that

$$\frac{1}{(1)(3)} + \frac{1}{(3)(5)} + \frac{1}{(5)(7} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}.$$

5.7 Convergence or Divergence of Serie

Definition 5.7.1 Let S_n be nth partial sum of the Serie, then the Serie is Convergence if

$$\lim_{n\to\infty} S_n = L ,$$

where L is constant.
Let S_n be nth partial sum of the serie, then the serie is divergence if

$$\lim_{n\to\infty} S_n = \pm\infty .$$

Necessary condition for convergence

Definition 5.7.2 Let $\sum_{n=1}^{\infty} a_n$ be convergence serie, then

$$\lim_{n\to\infty} a_n = 0 .$$

Example 5.8 Study convergence or divergence series

$$\sum_{n=1}^{\infty} \frac{n}{3n+1}.$$

Solution:

$$\lim_{n\to\infty} \frac{n}{3n+1} = \frac{1}{3} \neq 0.$$

So the series is divergence. ∎

5.8 Tests of convergence and divergence

5.8.1 Comparison test

Definition 5.8.1 Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series, then

1. If $a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ is convergence, then $\sum_{n=1}^{\infty} a_n$ is convergence.
2. If $a_n \geq b_n$ and $\sum_{n=1}^{\infty} b_n$ is divergence, then $\sum_{n=1}^{\infty} a_n is$ divergence.

■ **Example 5.9** Study convergence or divergence series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

Solution:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots$$

$$= 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + \dots$$

We comparison by

$$\sum_{n=1}^{\infty} b_n = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$

We find $a_n \geq b_n$ and $\sum_{n=1}^{\infty} b_n$ divergence, so $\sum_{n=1}^{\infty} a_n$ is divergence.

■

5.8.2 Limit Comparison Test

Definition 5.8.2 Consider two series $\sum_{n=1}^{\infty} a_n$, and $\sum_{n=1}^{\infty} b_n$, where

$$a_n \geq 0\,,\, b_n \geq 0\; \forall n$$

If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = k,$$

where k nonzero, positive or negative number, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent or divergent together.

■ **Example 5.10** Study Convergence or divergence series

$$\sum_{n=1}^{\infty} \frac{3n^2+2n+7}{n^5+4}.$$

Solution:

We compare the series

$$\sum_{n=1}^{\infty} \frac{3n^2+2n+7}{n^5+4}$$

with a series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^3}\,,$$

then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 3$$

So $\sum_{n=1}^{\infty} a_n$ is Convergence. ■

■ **Example 5.11** Discuss the convergence or divergence of a series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Solution

We compare the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

with a series

$$\sum_{n=1}^{\infty} \log\left(1+\frac{1}{n}\right),$$

$$a_n = \frac{1}{n}\ ,\ b_n = \log(1+\frac{1}{n}),$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\log\left(1+\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{1}{\log\left(1+\frac{1}{n}\right)^n} = \frac{1}{\log e} = 1,$$

and the series $\sum_{n=1}^{\infty} \log\left(1+\frac{1}{n}\right)$ is divergent, then also $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. ∎

5.8.3 Root test

Definition 5.8.3 Let $\sum_{n=1}^{\infty} a_n$, the root test be

$$\lim_{n\to\infty} \sqrt[n]{a_n} = L = \begin{cases} L < 1 \text{ convergence} \\ L > 1 \text{ divergence} \\ L = 1 \ faild \end{cases}$$

■ **Example 5.12** Study convergence or divergence series

$$\sum_{n=1}^{\infty} \frac{x^n}{n^n}.$$

Solution:

$$\lim_{n\to\infty} \sqrt[n]{\frac{x^n}{n^n}} = 0 < 1.$$

So the Serie is convergence ■

■ **Example 5.13** Study Convergence or divergence series

$$sum_{n=1}^{\infty} \frac{1}{(logn)^n}.$$

Solution:

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{(logn)^n}} = 0 < 1.$$

So the Serie is convergence ■

5.8.4 Ratio test

Definition 5.8.4 Let $\sum_{n=1}^{\infty} a_n, \ \ a_n \geq 0$, the ratio test be

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L = \begin{cases} L < 1 \text{ convergence} \\ L > 1 \text{ divergence} \\ L = 1 \quad faild \end{cases}$$

■ **Example 5.14** Study convergence or divergence series

$$\sum_{n=1}^{\infty} \frac{3^n}{n^2}.$$

Solution:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{3^{n+1}}{(n+1)^2}}{\frac{3^n}{n^2}}$$

$$= 3 > 1.$$

So the serie is divergence. ■

5.9 Exercise

A. Study Convergence or divergence series

1. $\sum_{n=1}^{\infty} \frac{2n+7}{n+4}$.
2. $\sum_{n=1}^{\infty} \frac{3n^2+2n+7}{5n^3+8}$.
3. $\sum_{n=1}^{\infty} \frac{1}{9n+4}$.
4. $\sum_{n=1}^{\infty} \frac{33^n}{5n^3+8}$.
5. $\sum_{n=1}^{\infty} \frac{n}{4^3}$.
6. $\sum_{n=1}^{\infty} \frac{3\sqrt{n}^n}{n!}$.

Wish you all the best,

Dr.A.Elrawy

بخصوص مقرر بحتة (2) البرنامج الانجليزي – الفرقة الاولي بكلية التربية استخدم الكتاب الاتي

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مع خالص التحية والتقدير

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