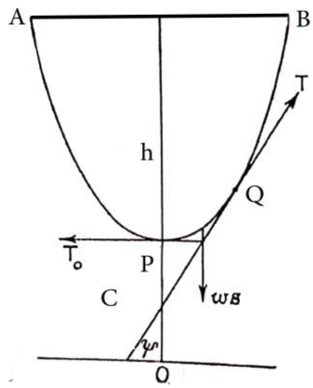
Chapter (1)

The Catenary

Definition(The common Catenary):

The catenary is the curve in which a uniform chain or string hangs when freely suspended from two points$A\&B$

Denote the tension at the lowest point y T_0 ,this will be horizontal. Let$s$ be the length of chain measured from P to any point Q .Let the tension at Qp be T and let its inclination to the horizontal be ψ.



Let the weight per unit length of the chain be ω.

The part of the chain PQ will be in equilibrium under the action of three forces, its weight ωs ,T_0,and T , the tensions at P and Q .

The intrinsic Equation of the catenary:

Resolving vertically and horizontally weget,

$$T\ \sin\psi = \omega s \qquad , \qquad T\cos\psi = T_0$$

For convenience we introduce another constant c ,which is such that $T_{0=}\omega c$. Then

$$T\ \sin\psi = \omega s \qquad , \qquad T\cos\psi = \omega c$$

Dividing

$$s = c\ \tan\psi \qquad\qquad (i)$$

This is the intrinsic equation of the curve, (c is called the parameter of the catenary).

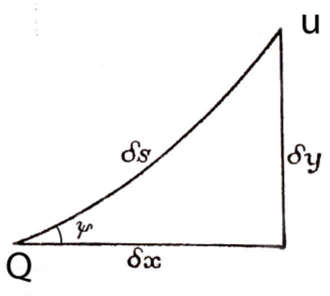
<u>The cartesian Equation of the catenary</u>:

To find the Cartesian equation of the curve we flow:

Since $\tan\psi = \frac{dy}{dx}$, then from $(i)\ \frac{dy}{dx} = \frac{s}{c}$

Consider a small element δs of a curve joining two points Q and U on the curve. Let the coordinates of Q and U be (x, y)&$(x + \delta x, y + \delta y)$respectively. Then

$$(\delta s)^2 \cong (\delta x)^2 + (\delta y)^2$$



Dividing by $(\delta x)^2$ then $(\delta y)^2$ respectively we get:

$$\left(\frac{\delta s}{\delta x}\right)^2 \cong 1 + \left(\frac{\delta y}{\delta x}\right)^2$$

and

$$\left(\frac{\delta s}{\delta y}\right)^2 \cong \left(\frac{\delta x}{\delta y}\right)^2 + 1$$

When $\delta s, \delta x, \delta y \to 0$, the above equations becomes

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 \quad (ii)$$

and

$$\left(\frac{ds}{dy}\right)^2 = \left(\frac{dx}{dy}\right)^2 + 1 \qquad (iii)$$

(ii) gives

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{s}{c}\right)^2$$

$$\therefore \quad \frac{ds}{dx} = \frac{\sqrt{(c^2 + s^2)}}{c}$$

$$\therefore \quad dx = \frac{c\ ds}{\sqrt{(c^2 + s^2)}}$$

$$\therefore \quad x = c\ sinh^{-1}\ \frac{s}{c} \qquad (iv)$$

or $\qquad s = c \sinh \frac{x}{c} \qquad (v)$

provided $x = 0$ when $s = 0$.

(iii) gives

$$\left(\frac{ds}{dy}\right)^2 = 1 + \left(\frac{c}{s}\right)^2$$

$$\therefore \quad \frac{ds}{dy} = \frac{\sqrt{(c^2 + s^2)}}{s}$$

$$\therefore \quad dy = \frac{s\,ds}{\sqrt{(c^2 + s^2)}}$$

$$\therefore \quad y = \sqrt{(c^2 + s^2)}$$

i.e. $$y^2 = s^2 + c^2 \qquad (vi)$$

provided $y = c$ when $s = 0 \& x = 0$

Substituting from (v) in (vi)

. $$y^2 = c^2\left(1 + sinh^2\frac{x}{c}\right)$$

$$= c^2 cosh^2(\frac{x}{c})$$

$$\therefore \quad y = c\cosh\left(\frac{x}{c}\right) \qquad (viii)$$

This is the Cartesian equation of the catenary.

<u>The tension at any point</u>:

Since

$$T\ \sin\psi = \omega s \qquad , \qquad T\cos\psi = \omega c$$

then $T^2 = \omega^2(s^2 + c^2)$

which from (vi) gives

$$T^2 = \omega^2 y^2$$

$$\therefore \quad T = \omega y$$

Thus,the tension at any point of the catenary is proportional to the height of the point above the x −axis which is usually called the (directric).

The Lightning and telephone wires:

When c is large, from equation (vii),

$$\therefore \quad y = c \cosh\left(\frac{x}{c}\right) = \frac{C}{2}\left(e^{x/c} + e^{-x/c}\right)$$

$$= s = \frac{c}{2}\left\{1 + \frac{x}{c} + \frac{x^2}{2c^2} + \ + \cdots + (1 - \frac{x}{c} + \frac{x^2}{2c^2} - \cdots)\right\}$$

$$= c + \frac{x^2}{2c} + \cdots$$

i.e. $\qquad y - c \cong \frac{x^2}{2c} \qquad (X) \qquad$ provided c is large.

In this case the curve is approximately a parabola of latus rectum $2c$

Definition (the span): The span is distance AB ,ie. the distancebetween the two hangs points $A\&B$.

If $k$is half the span, half the length of the chain is given by:

$$s = \frac{c}{2}\left\{1 + \frac{k}{c} + \frac{k^2}{2c^2} + \frac{k^3}{6c^3} + \cdots - (1 - \frac{k}{c} + \frac{k^2}{2c^2} - \frac{k^3}{6c^3} \ldots)\right\}$$

$$= \frac{c}{2}\left\{\frac{2k}{c} + \frac{k^3}{3c^3} + \cdots\right\}$$

$= k + \frac{k^3}{6c^2}$ provided c is large.

$$\therefore \quad s - k = \frac{k^3}{6c^2} \qquad (Xi)$$

Definition (the sag): The sag is the difference between the coordinates of y at values of x for the two point P&B. Or the normal distance from the lowest point P to the span line AB.

The Relation between the span and sag:

If h is the sag, then for $x = 0$, $y = c$ and $x = k$, $y \cong c + \frac{k^2}{2c}$ those come from (X) .Her we can get:

$$h = \frac{k^2}{2c} \qquad (*)$$

this leads to $1/c^2 = 4h^2/k^4$

then from (Xi) we have:

$$s - k = k^3/6c^2 = (k^3/6).(1/c^2) = (k^3/6).(4h^2/k^4) = 4h^2/6k$$

$$\therefore \quad 2(s - k) = (8/3)\,.\,(h^2/2k)$$

this means that the difference between the length of the chain $2s$ and

the span $2k$ is equal to $2(s-k)=(8/3)\,.((sag)^2/span)$. $(**)$

The equations (*)&(**) clarify two relations between the span and sag for the catenary.

Note: when c is large as mentioned above the chain or wire represents

the Lightning and telephone wires. In this case the length of the wire

$2s$ is little bigger than the span AB .So also the sag$h$ will be small.

Examples

Many problems involving catenary cables can be solved using the following formulas:

$$s = c\sinh\left(\frac{x}{c}\right) \qquad (i) \qquad\qquad x = c\,sinh^{-1}\left(\frac{s}{c}\right) \qquad (ii)$$

$$y^2 - s^2 = c^2 \qquad (iii) \qquad\qquad y = c\,cosh\left(\frac{x}{c}\right) \qquad (iv)$$

$$T_0 = \omega c \qquad (v) \qquad\qquad T = \omega y \qquad (vi)$$

$$W = \omega s \qquad (vii)$$

All the parameters in the above equations have been defined before.

Example (1):

an electric power of line length 140 $m$and mass per unit length of 3 kg/m is to be suspended between two towers 120 m apart and of the same height. Determine the sag and maximum tension in the power line.

The solution

The sag, h, can be found from Eq(iii),provided that wecan determinethe distance, c

$y_B^2 - s_B^2 = c^2$ (Eq(iii)evaluated at point B)

or

$$(h + c)^2 - (70\ m)^2 = c^2 \qquad (1)$$

The distance c ,can be determined from Eq.(i) :

$s_B = c\sinh(\frac{x_B}{c})$ (Eq(i)evaluated at point B)

or $$70\ m = c\sinh\left(\frac{60\ m}{c}\right) \qquad (2)$$

This equation must be solved numerically for c. An initialestimate for

c, whenthesolver on acalculator isto beused, could be

$$c = s_B = 70m$$

The solution to Eq.(2) is

$$c = 61.45\ m$$

Another possible solution is $c = -61.45\ m$,but this has no physical meaning. You can get the same result directly by using a modern calculator like($casio\ \ f_x - 991ES\ PLUS$).

$$(h + 61.45\ m)^2 - (70\ m)^2 = (61.45\ m)^2$$

Solving gives the sag:

$$h = 31.70\ m$$

The other negative root has no physical meaning.

The maximum tension, T_{max}, occurs where the cable has its steepest slope, point B (or point A). This can be calculated from Eq.(vi) :

$T_{max} = \omega y_B$(Eq(vi)evaluated at point B)

ω is given, then:

$$T_{max} = [(3\text{kg/m})(9.81\ \ \text{m/s}^2)][31.70\text{m} + 61.45\text{m}]$$

$$= 2740\ N = 2.74\ KN$$

<u>Example (2):</u>

A cable is supported at two points 400 ft apart and at the same elevation. If the sag is 40 ft and the weight per unit length of the cable is 4 lb/ft, determine the length of the cable and the tension at the low point, C.

The solution

The length of cable, s_B, from the low point to point B can be found from Eq. (i) provided that we can determine the distance c:

s_B = c sinh (x_B/c) (Eq. (i) evaluated at point B)

$= c \sinh (200/c)$ (1)

The distance c can be determined from Eq.(iv)

$y_B = c \cosh(xB/c)$(Eq. (iv) evaluated at point B)

or,

$c + 40$ ft = c cosh (200 ft/c) (2)

This equation must be solved numerically for c. An initial estimate for c, when the solver on a calculator is to be used, could be

$$c = sag = 40\ ft$$

The solution to Eq.(2) is

$$c = 506.53\ ft$$

Using this value of c in Eq. (1) gives

$s_B = c \sinh(\frac{x_B}{c})$ (Eq. (1) repeated)

= (506.53 ft) sinh (200 ft/506.53 ft)

= 205.237 ft

Because the tension at the low point of the cable is horizontal, it can be found from Eq.(v):

$$T_0 = \omega c$$

$$= (4\ lb/ft)(506.53)$$

$$= 2.025\ lb.$$

Example (3) :

A 20-m chain is suspended between two points at the same elevation and with a sag of 6 m as shown. If the total mass of the chain 45 kg, determine the distance between the supports. Also determine the maximum tension.

The solution

The distance between the supports is$2x_B$, andx_B can befound from

Eq.(i), provided that we can determine the distance c;

$s_B = c\ \sinh(\frac{x_B}{c})$ (Eq(i)evaluated at point B)

since $s_B = 10\text{m}$, then:

$$10\text{m} = c\ \sinh(\frac{x_B}{c})$$

This equation can be solved explicitly for x_B by rearranging it as

$$\sinh\left(\frac{x_B}{c}\right) = \frac{10\mathrm{m}}{c}$$

Which implies:

$$\frac{x_B}{c} = sinh^{-1}\left(\frac{10\,m}{c}\right)$$

So $\qquad x_B = c\, sinh^{-1}\left(\frac{10\,m}{c}\right) \qquad (1)$

The distance c can be determined from Eq.(ii):

$y_B^2 - s_B^2 = \mathrm{c}^2 \qquad$ (Eq(iii)evaluated at point B)

$$(6\,m + c)^2 - (10\,m)^2 = (c)^2$$

or $36 + 12c + c^2 - 100 = c^2$

The c^2 terms cancel and resulting linear equation has the solution:

$$c = 5.333\,m$$

Substituting this value of c into Eq.(1)gives:

$$x_B = 5.333\,m\, sinh^{-1}(10\,m/5.333\,m) = 7{,}393\,m$$

Thus, the distance between supports $2x_B$ can be found:

$2x_B = 2(70393\,m) = 14.786\,m.$

The maximum tension, T_{max}, occurs where the slope of the cable is a maximum, at point B (or point A). This can be calculated from Eq.(vi):

$T_{max} = \omega y_B$(Eq(vi)evaluated at point B)

$$= \left(\frac{Tatol\ weight\ of\ the\ cable}{Tatol\ length\ of\ the\ cable}\right) y_B$$

$$= \left(\frac{(45.Kg)(9{,}81\ m/s^2)}{20\ m}\right)(6\ m5.333\ m) = 250\ N\ .$$

Example (4):

A certain cable will break if the maximum tension exceeds 500 N. If the cable is 50-mlong and has a mass of 50 kg, determine the greatest span possible. Also determine the sag.

The solution

The maximum tension has been specified (500 N) ,so a good place to start our solution is to see how we can use the fact that $T_{max} = 500$ N .Eq.(vii)relates the tension,T ,to they , coordinate of a point on the curve:

$$T = \omega y \text{ (Eq. (vi) repeated)}$$

The maximum tension, T_{max}, occurs where the cable has its steepest slope, point B (or point A). This can be calculated from Eq.(vi) :

$T_{max} = \omega y_B$(Eq(vi)evaluated at point B)

Thus, because we know the maximum tension, we can compute y_B :

$$y_B = \frac{T_{max}}{\omega} = \frac{T_{max}}{\left(\frac{Tatol\ weight\ of\ the\ \text{cable}}{Tatol\ length\ of\ the\ cable}\right)}$$

$$= \frac{500\ N}{\left(\frac{(50\ Kg)(9{,}81\ m/s^2)}{50\ m}\right)} = 50.97\ m$$

The distance between supports is $2x_B$, so we need to use the value of y_B to determine x_B.this can be done by using Eq.(vi).provided that we can determine c :

$$y_B = c\cosh(x_B/c) \text{(Eq. (iv) evaluated at point B)}$$

We can solve this equation explicitly for x_B by rewriting it as:

$$\cosh(x_B/c) = y_B/c$$

So

$$x_B = c\ cosh^{-1}(y_B/c) \qquad (2)$$

The distance c , can be calculated from Eq.(iii) :

$$y_B^2 - s_B^2 = \mathrm{c}^2 \qquad \text{(Eq}(iii)\text{evaluated at point B)}$$

$$(50.97\ m)^2 - (25\ m)^2 = (c)^2$$

The solution is

$$c = \pm 44.42\ m$$

The negative root has no physical meaning.

Substituting the value of $c = 44.42\, m$ and $y_B = 50.97$ into Eq.(2)gives:

$$x_B = 44{,}42\, m\, cochh^{-1}(50.97\, m/44.42\, m) = 23.836\, m$$

So, the distance between supports $2x_B$isknown:

$2x_B = 2(23.836\, m) = 47.7\, m.$

Since c and y_B are known, the sag can be computed:

$$h = y_B - c$$

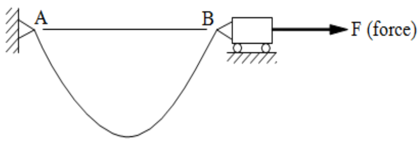
$= (50.97m) - (44.42\, m) = 6.55\, m$.

Example (5):

The cable is attached to a fixed support at A and a moveable support at B. If the cable is80-ft long, weighs 0.3 lb/ft, and spans 50 ft, determine the force F holding the moveable support in place. Also determine the sag.

The solution

The force F acting on the moveable support at$B$ equals the horizontal component,T_0, of tension in the cable,



$F = T_0$. Eq.(v) can be used to calculate T_0 , provided that we can determine the distance c :

$T_0 = \omega c$ (Eq. (v) repeated)

$$= (0.3\ lb/ft)\ c = F \qquad (1)$$

The distance c ,can be calculated from Eq.(i) :

$s_B = c\ \sinh(\frac{x_B}{c})$ (Eq(i)evaluated at point B)

since $s_B = 40$ ft ,

then:

$$40\text{ft} = c\ \sinh(25\ \text{ft}/c) \qquad (2)$$

This equation must be solved numerically for c. An initial estimate for c, when the solver on a calculator is to be used, could be

$$c = x_B = 25\ ft$$

The solution to Eq.(2) is

$$c = \pm 14.229\ ft$$

The negative root has no physical meaning.

Using $c = 14.229\ ft$ in Eq.(1)gives:

$T_0 = \omega c$ (Eq. (v) repeated)

$$= (0.3\ lb/ft)(14.229\ ft)$$

$$= 4.27\ ft$$

The sag,h , can be calculated from Eq.$(iv)$and the known value of c :

$$h = y_B - c \quad = c\ \cosh(x_B/c) - c$$

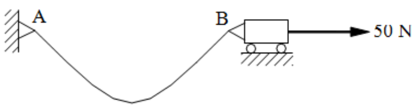
$$= (14.229\ ft)\cosh(25.ft/14.229ft) - 14.229ft$$

$= 28.2\ ft$.

Example (6):

. The cable is attached to a fixed support at A and a moveable support at B. If the cable is 40 m long, and has mass of 0.4Kg/m. If the force F holding the moveable support at the B is equal to $50\ N$ in the horizontal direction, determine the span and the sag.

The solution



The span is $2x_B$, and x_B can befound from Eq.(i), provided that we can determine The distance c;

$s_B = c\ \sinh(\frac{x_B}{c})$ (Eq(i)evaluated at point B)

This equation can be solved explicitly for x_B by rearranging it as

$$\sinh\left(\frac{x_B}{c}\right) = \frac{s_B}{c}$$

Which implies: $\frac{x_B}{c} = sinh^{-1}\left(\frac{s_B}{c}\right)$

So $x_B = c\ sinh^{-1}\left(\frac{s_B}{c}\right)$

Then $x_B = c\ sinh^{-1}\left(\frac{20\ \text{m}}{c}\right)$ (1)

Because the 50 N force acting on the moveable support equals the horizontal component, T_0 , of the tension in the cable, Eq.(v) with $T_0 = 50\ N$ can be used to solve for c :

$T_0 = \omega c$ (Eq. (v) repeated)

or $50\ N = [(0.4\ Kg/m)(9.81 m/s^2)]c$

solving gives:

$$c = 12.742\ m$$

Using this value of c in Eq. (1)gives:

$x_B = c\ sinh^{-1}(20\ \text{m}/c)$(Eq. (1) repeated)

$$= (12.742\ m) sinh^{-1}(20\ \text{m}/12.742\ m) = 15.708\ m$$

so, the span is

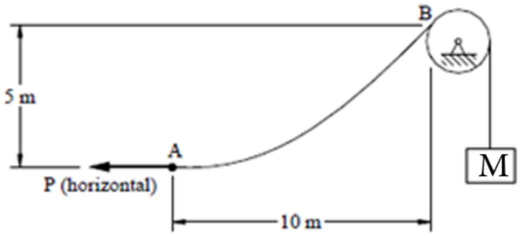
$$span = 2x_B = 2(15.708\ m) = 31.4\ m$$

The sag,h , can be calculated from Eq.(iv) and the known value of c :$h = y_B - c = c\ \cosh(x_B/c) - c$

$$= (12.742\ m)\cosh(15.708. ft/12.742\ m) - 12.742\ m = 28.2 m$$

Example (7):

A cable goes over a frictionless pulley at $B$and supports a block of mass M. The other end of the cable is pulled by a horizontal force P.



If the cable has a mass per length of 0.3 kg/m, determine values of $P$and Mthat will maintain the cable in the position shown.

The solution

The force P equals T_0 ,the horizontal component of the cable tension

given$\mathrm{T_0} = \omega c$(Eq. (v) repeated)

so, with $\mathrm{T_0} = \mathrm{P}$then:

$$\mathrm{P} = \omega c \qquad (1)$$

Here:

$$\omega = (0.3\ Kg/m)(9.81\ m/s^2)$$

$$= 2.943\ \ N/m \qquad (2)$$

The value of c in Eq.(1) can be found from Eq.(iv):

$$y_B = c\cosh(x_B/c) \text{(Eq. (iv) evaluated at point}$$

or

$$5\ m + c = c\cosh(10\ m/c)$$

Solving numerically gives:

$$c = 10.743\ m$$

Using this value of c in Eq.(1)gives:

$$\mathrm{P} = \omega\mathrm{c}$$

$$= (2.943\ N/m)(10.743\ m)$$

$$= 31.617\ N$$

The cable tension at B must equals the weight,mg :

$$T_B = Mg$$

thus, the mass is

$$M = T_B/g$$

By Eq.(vi)

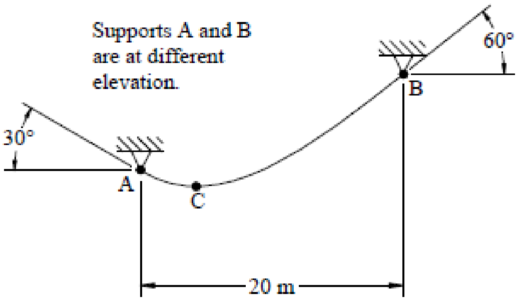
$$M = \omega y_B/g$$

By Eq. (2)

$$M = [(2.943\ \ N/m)(5\ m + 10.743\ m)]/(9.81\ N/m^2)$$

$$= 4.72\ Kg$$

<u>Example (8):</u>



A chain makes angles of 30° and 60° at its supports as shown.

Determine the location of the low point C of the chain relative to A. Also determine the tension at support A, if the cable has a mass per length of 0.6 kg/m.

<u>The solution</u>

The geometric data are shown in the figure. To determine the location of the low point C relative to A, we need to determine the coordinates xA and yA. We can get an equation

for xA by using the fact that the slope is known at A:

$$-\tan 30^{\circ} = \left[\frac{dy}{dx}\right]_{atA} = \left[\frac{d(c\cosh(x/c))}{dx}\right]_{atA} \qquad \text{by Eq.}(iv)$$

$$= \sinh(x_A/c)$$

Solving for x_A gives:

$$x_A = c\sinh^{-1}(-\tan 30^{\circ}) \qquad (1)$$

Similarly at point B , we have

$$x_B = c\sinh^{-1}(\tan 60^{\circ}) \qquad (2)$$

The coordinates x_A and x_B are related to the 20-m span through the equation:

$$x_A - x_B = 20 \text{ m}$$

By substituting from Eqs.(1)&(2) we get:

$$c \sinh^{-1}(-\tan 30^\circ) - c \sinh^{-1}(\tan 60^\circ) = 20$$

Since this equation is linear in c, it is easily solved to give c = 10.717 m. Eq. (1) then gives

$x_A = c \sinh^{-1}(-\tan 30^\circ)$(Eq (1) repeated)

$$= (10.717\ m) \sinh^{-1}(-\tan 30^\circ)$$

$$= -5.887\ m$$

The y coordinate of point A can now be calculated fromEq. (iv):

$y_A = c\ \cosh(x_A/c)$ (Eq.(iv) evaluated at point A)

$$= (10.717\ m) \cosh(-5.887\ m/10.717m)$$

$$= 12.375\ m \qquad (3)$$

The vertical distance between support A and the low point C is given by

$$d = y_A - c$$

$$= 12.375\ m - 10.717\ m$$

$= 1.658\ m$ (by Eq. (3))

The tension at A is given by Eq. (vi):

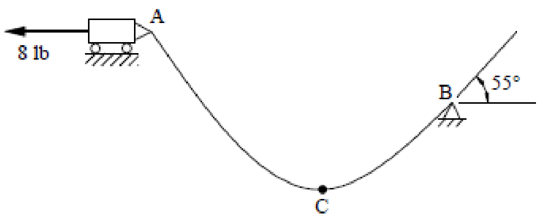
$T_A = \omega y_A$ (Eq.(vi) evaluated at point A)

$= [(0.6\ Kg/m)(9.81\ m/s^2)](12.375\ m) = 72.8\ N.$

Example (9):

A wire weighting 0.2 lb/ft is attached to a moveable support at A and makes an angle of 55° at a fixed support at B. Supports A and B are at different elevations. Determine the location of the low point C of the wire relative to support B. Also, determine the tension in the wire at C.

The solution



To determine the location of the low point, C , relative to the support at B, we need to determine the coordinates x_B and y_B . We can get an equation for x_B by using the fact that the slop is known at B.

$$\tan 55^{\circ} = \left[\frac{dy}{dx}\right]_{atB}$$

$= \left[\frac{d(c\cosh(x/c))}{dx}\right]_{at\boldsymbol{B}}$ by Eq.(iv)

$$= \sinh(x_B/c)$$

Thus

$$x_B = c\sinh^{-1}(\tan 55^{\circ}) \qquad (1)$$

The value of c occurring in Eq. (1) can be found by observing that the 8-lb force acting at support A equals T_0 the horizontal component of tension at A, so Eq. (v) gives

$\mathrm{T_0} = \omega\mathrm{c}$ (Eq. (v) repeated)

$$\therefore 8\ Ib = (0.2\ \omega\ Ib/ft)c$$

Solving gives:

$$\mathrm{C} = 40\ \mathrm{ft} \qquad (2)$$

Using this result,$C = 40\ ft$ in Eq.(1)gives:

$x_B = c\sinh^{-1}(\tan 55^{\circ})$(Eq. (1) repeated)

$$= (40\ ft)\sinh^{-1}(\tan 55^{\circ})$$

$$= 46.169\ ft$$

The vertical distance between B and C is:

$$d = y_B - c$$

$$= c\cosh(x_B/c) - c$$

$$= (40\ ft)\cosh(46{,}169\ ft/40\ ft) - 40\ ft$$

$$= 29.7\ ft$$

Since point C is the low point of the cable, the tension there is horizontal and so must equal the horizontal component of tension at A

which is known to be 8 Ib that is:

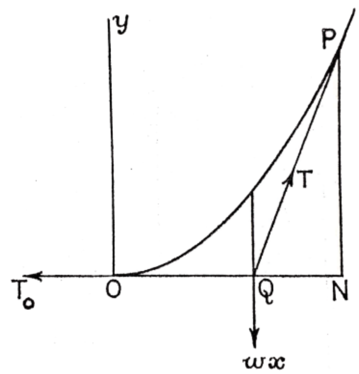
$T_c = 8\ Ib$.

Worked examples

Example (1): (The suspension bridge)

If a chain supports a continuous load, uniformly distributed, the chain hangs in the form of a parabola.

O is the lowest point of the chain and P any point of the chain whose coordinates referred to horizontal and



vertical through O are (x, y) The weight carried by the portion OP will be proportional to ON and acts through Q the midpoint of ON. We may call it ωx .

The other forces acting on the portion OP are T_0 the horizontal tension At O and the tension$T$ at P, three of them must therefore meet at Q and PNQ is a triangle of forces.

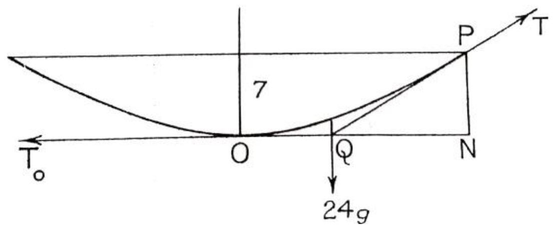
$$\therefore \quad \frac{\omega x}{PN} = \frac{T_0}{NQ} \qquad \therefore \quad T_0 y = \frac{1}{2}\omega x^2$$

Hence, if we denote T_0 by ωc , then we can get $\quad y = \frac{x^2}{2\,c}$

this means that the curve of the chain is a parabola.

Now if the span of a suspension bridge is $96\ m$ and the sag in the chain is $7\ m$ .TheTwo branches of the chain support a load of $1000\ kg$per horizontal meter. Find the tension at the lowest and highest points. The load carried by OP is $24gkN$ .The triangle QPN is a triangle of forces.

The solution



$$QN = 24\, m \quad , \quad PN = 7\, m \qquad \therefore Qp = 25\, m$$

$$\frac{T_0}{24} = \frac{T}{25} = \frac{24g}{7}$$

$T = 840\ kN$, $T_0 \cong 810\ kN$.

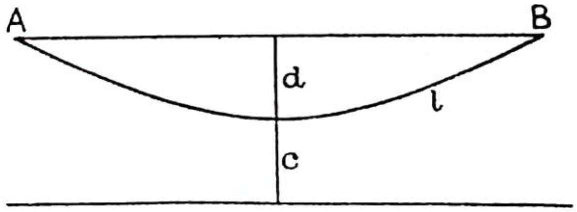
==

Example (2):

A uniform chain of length $2l$ and weight ω per unit length is suspended between two points at the same level and has a maximum depth d. Prove the tension at the lowest is $\omega\,(l^2 - d^2)/2d$. If $l = 50\ m$ and $d = 20\ m$ find the distance between the points of suspension.

The solution

For the catenary $y^2 = c^2 + s^2$,



At $B\,y = c + d$, $s = l$

$$\therefore \quad (c+d)^2 = c^2 + l^2$$

$$2cd = l^2 - d^2$$

$$\therefore \quad c = l^2 - d^2/2d$$

$\therefore$ the tension at the lowest is $= \omega c = \omega\,(l^2 - d^2)/2d$.

If $l = 50\ m$ and $d = 20\ m$, then $c = 2500 - 400/40 = 105/2$

Now $s = c \sinh x/c$

Hence if $AB = 2x$,

$$\therefore \quad x = (105/2) \sinh^{-1}(20/21)$$

$$= (105/2) lin\left[(20/21) + \sqrt{\{1 + (20/21)^2\}}\right]$$

$$= (105/2) lin(49/21)$$

$\therefore \quad AB = 105 \times 2.303\ log_{10}(49/21) \cong 89\ m$.

==

EXERCISES:

(1)A rope has an effective length of $20\ m$and mass5 kg per miter. One end of the rope is $4\ m$ higher than the other. Find the maximum tension in the rope when the tangent at the lower end is horizontal.

(2)A uniform chain of length $2l$ has its ends fixed at two points at the same level. The sag at the middle is h .prove that the span is $[(l^2 - h)/h] lin[(l + h)/(l - h)]$.

(3)A uniform wire hangs freely from tow points at the same level $200\ m$apart. The sag is $15\ m$. Show the greatest tension is approximately $348\ \omega$ and the length of wire is approximately $203\ m$.

(4)Find approximately the greatest tension in a wire which has mass $100\ g$ per miter when it hangs with a sag of $25\ cm$ when stretched between two points at the same level $40\ m$apart.

(5)A uniform heavy chain of length $31\ m$ is suspended from tow points at the same level and$30\ m$apart. Show that the tension at the lowest point is about 1.08times the weight of the chain.

Chapter (2)

Direct Stress and Strain

----

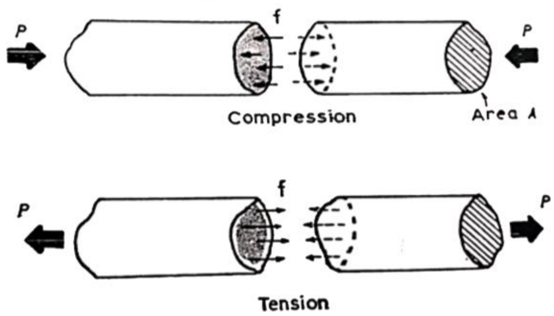
(1) Stress:

The ability of a structural member to withstand load or transmit force, as in a machine, depends upon its dimensions. In particular, the cross-sectional area over which the load is distributed determines the intensity or average stress in the member. If the intensity of loading is uniform the direct stress, , f is defined as the ratio of load, P ,to cross-sectional area,A , normal to the load as shown in the Fig. Thus:

$$\text{stress} = \frac{\text{load}}{\text{area}}$$

or

$$\mathrm{f} = \frac{\mathrm{P}}{\mathrm{A}}$$



If the load is in pounds and the area in square inches the units of stress are pounds per square inch $(Ib/in.^2)$.There are another unit:

If,P , is expressed in Newton (N) , and A, original area, in square meters(m^2) ,the stress,f , will be expresses in N/m^2 , this unit is called Pascal (Pa) .

As Pascal is a small quantity in practice, multiples of this unit is used.

$$1\,KPa = 10^3 Pa = 10^3\,N/m^2\ (KPa = Kilo\ Pascal)$$

$$1\,MPa = 10^6 Pa = 10^6\,N/m^2$$

$$= 1\ N/mm^2\ (MPa = Mega\ Pascal)$$

$$1\,GPa = 10^9 Pa = 10^9\,N/m^2\ (GPa = Giga\ Pascal)$$

The direct stress may be tensile or compressive according as the load is a pull (tension), or push (compression). It is often convenient to consider tensile stresses and loads as positive and compressive stresses and loads as negative.

(2) Strain:

A member under any loading experiences a change in shape or size inthe case of a bar loading in tension the extension of the bar depends upon its total length. The bar is said to be strained and the strain is defined as the extension per unit of original length of the bar. Strain may produced in two ways:

1- By application of a load.

2- By a change in temperature, unaccompanied by load or stress.

If l is the original length of the bar, x the extension or contraction in length under load or temperature change, and $e$the strain, then:

$$\text{strain} = \frac{\text{change in length}}{\text{original lenth}}$$

or $$e = \frac{x}{l}$$

Strain is a ratio and has therefore no units.

Strain due to an extension is considered positive, that associated with a contraction is negative.

(3) Relation between Stress and Strain:

If the extension or compression in a member due to a load disappears on removal of the load, then the material is said to be elastic. Most metals are elastic over a limited range of stress known as the elastic range. Elastic materials, with some exception, obey Hooke's, which states that: the strain is directly proportional to the applied stress Thus

$$\frac{stress}{strain} = constant\ (E)$$

i.e.$\frac{f}{e} = E$or $e = \frac{f}{E}$

where E is the constant of proportionality, known as the modulus of elasticity or Young's modulus?

Since strain is a ratio, the units of $E$are those of stress, i.e. pounds per square inch.

Examples

Example (1):

A rubber pad for a machine mounting is to carry a load of $1000\ Ib$ and to compress$0.2in$. If the stress in the rubber is not exceed $40\ Ib/in.^2$, determine the diameter and thickness of a pad of circular cross-section.

Take E for rubber as $150\ Ib/in.^2$.

The solution

$$\text{stress} = \frac{\text{load}}{\text{area}}$$

i.e $$f = \frac{P}{A}$$

$$40 = \frac{1000}{\pi d^2/4}$$

hence $d^2 = 31.83\ in.^2$ and $d = 5.64\ in$

i.e. diameter of pad=5.64 in.

The increase in area due to compression has been neglected.

Also $\quad stress = \frac{reduction\ in\ length}{original\ length}$

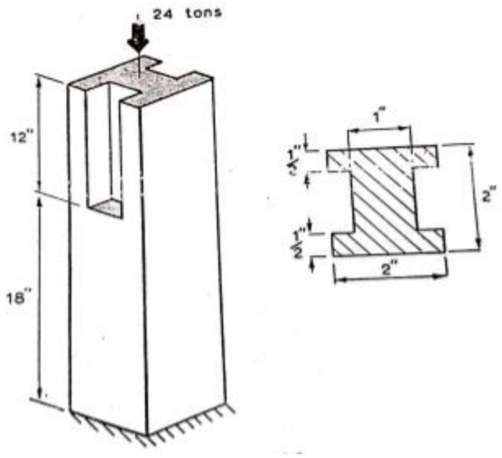
then $\quad \frac{f}{E} = \frac{x}{l} \quad$ this leads to $\frac{40}{150} = \frac{0.2}{l}$

therefore, thickness of pad is given by

$$l = 0.75\ in.$$

Example (2):

The Fig shows a steel strut with tow grooves cut out along part of its length. Calculate the total compression of the strut due to a load of $24\ tons. E = 12500\ ton/in.^2$



The solution

Suffices 1 and 2 denote solid and grooved portions, respectively. the load at every section is the same, $24\ ton$.

For the solid length of $18\ in.$

compression $x_1 = e_1 l = e_1 \times 18$

$$\text{stress}, f_1 = \frac{P}{A_1} = \frac{24}{2\times 2} = 6\ tons/in.^2$$

$$\text{strain}, e_1 = \frac{f_1}{E} = \frac{6}{E}$$

For the grooved length 0f $12\ in.$

compression $x_2 = e_2 l = e_2 \times 12$

$$\text{stress}, f_2 = \frac{P}{A_2} = \frac{24}{(4-1)(1)} = 8\ tons/in.^2$$

$$\text{strain}, e_2 = \frac{f_2}{E} = \frac{8}{E}$$

The total compression of the strut is equal to the sum of the compressions of the solid and grooved portions. Therefore

$$x = x_1 + x_2$$

$$= (e_1 \times 18) + (e_2 \times 12)$$

$$= \left(\frac{6}{E} \times 18\right) + \left(\frac{8}{E} \times 12\right)$$

$$= \frac{204}{E}$$

$$= \frac{204}{12500}$$

$$= 0.0163\ in.$$

Note: It has been assumed here that the stress distribution is uniform over all sections, but at the change in cross-section the stress distribution is actually very complex. The assumption produces little error in the calculated compression.

Example (3):

A rod $10\ mm \times 10\ mm$ cross-section is carrying an axial tensile load $10\ KN$. In this rod the tensile stress developed is given by:

$$f = \frac{P}{A} = \frac{10\ KN}{(10\ mm \times 10\ mm)} = \frac{10\ \times 10^3\ N}{100\ mm^2} = 100\ MPa$$

Example (4):

A rod $100\ mm$ in original length. When we apply an axial tensile loa $10\ KN$ .The final length of the rod after application the tensile is $100.1\ mm$ .So in this rod tensile strain is developed and is given by;

$$e = \frac{x}{l} = \frac{100.1mm - 100\ mm}{100\ mm} = \frac{0.1mm}{100\ mm} = 0.001(\text{Dimenionless})\text{Tensile.}$$

Example (5):

A rod $100\ mm$ in original length. When we apply an axial compressive load $10\ KN.$ .The final length of the rod after application compressive is $99\ mm$ .So, in this rod compressive strain is developed and is given by;

$$e = \frac{x}{l} = \frac{99\ mm - 100\ mm}{100\ mm} = \frac{-0.1mm}{100\ mm} = -0.001(\text{Dimenionless})\text{Tensile.}$$

==

Exercises

(1) A bar of 1 $in.$ diameter is subjected to a tensile load of 10000 Ib.Calculate the extensionon a 1 $ft.$ length. $E = 30 \times 10^4 \, Ib/in.^2$.

(2)A light alloy bar is observed to increase in length by 0.35 per centwhen subjected to a tensile stress of 18 $ton/in.^2$ .Calculate Young' modulus for the material.

(3)A duralumnin tie, 2 ft long 1.5 $in.$ diameter, has a hole drilled out along its length .The hole is of 1 $in.$diameter and 4 $in.$ long.Calculate the total extension of the tiedue to a load of 18 $tons$. $E = 12 \times 10^4 \, Ib/in.^2$.

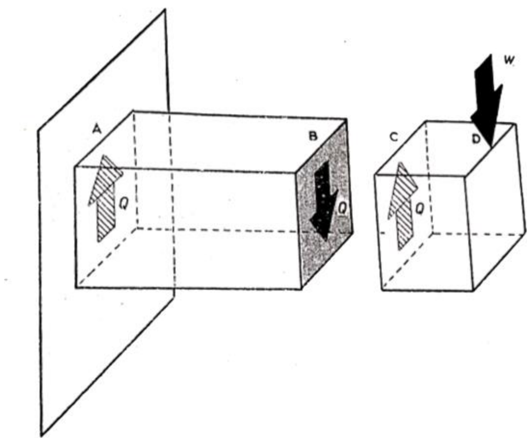
(4) A steel strut of rectangular section is made up of two lengths. The first 6 $in.$ long, has breadth 2 $in.$and depth 1.5 $in.$; the second,4 $in.$long is 1 $in.$ square. If $E = 14000 \; tons/in.^2$, calculate the compression of the strut under a load of 10 $tons$.

Chapter (3)

Shear force and Bending Moment

(1) Shear force (SF):

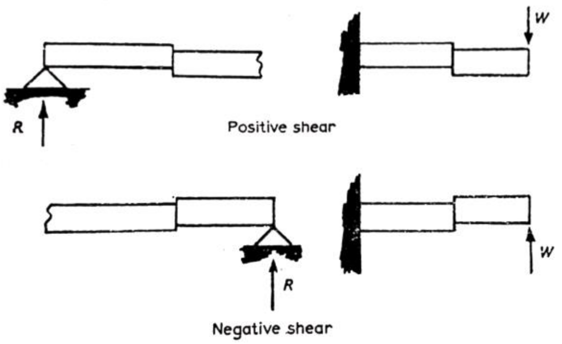
The shear force in a beam at any section is the force transverse to the beam tending to cause it to shear across the section. Fig.(3.1) shows abeam under a transverse load W at the end D ; the other end A is built in to the wall. Such a beam is called a cantilever and the load W , which is assumed to act at a point, is called a concentrated or point load.



Consider the equilibrium of any portion of beam CD . At section C for balance of forces there must be an upward force Q equal and opposite to the load W at D . This force Q is provided by the resistance of the beam to shear at the plane B ; this plane being coincident with the plane section at C . Q is the shear force at B and in this cace has the same magnitude for any section in AD . Consider now the equilibrium of the portion of beam AB . There is a downward force $Q = W$, exerted on plane B , so for balance there must be an upward force Q at A . This latter force being exerted on the beam by the wall.

Sign Convention

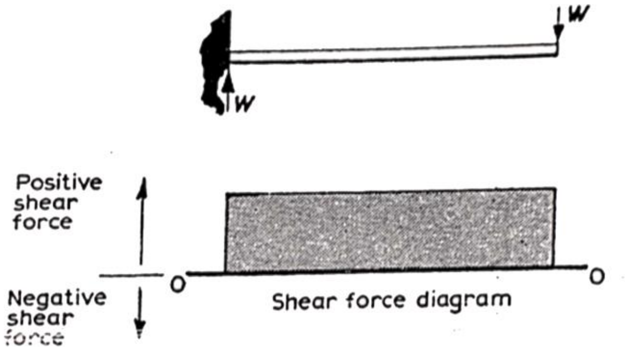
The shear force at any section is taken positive if the right-hand side tends to slide downwards relative to the lift-hand portion, fig.(3.2).A negative shear force tends to cause the right-hand portion to slide upward relative to the lift. (In some books flowed totally opposite sign convention).



If several loads act on the beam to the right-hand side of section C the shear force a C is the resultant of these loads. Thus, the shear force at any section of a loaded beam is the algebraic sum of the loads to one side of the section. It does not matter which side of the section is considered provided all loads on that side are taken into account-including the forces exerted by fixings and props.

(2) Shear Force Diagram (SFD):

The graph showing the variation of shear force along a beam is known as the shear force diagram. for the beam of Fig 3.1 the shear force was $+W$, uniform along the beam. Fig (3.3) shows the shear force diagram for this beam, $0-0$ being the axisof zero shear force.

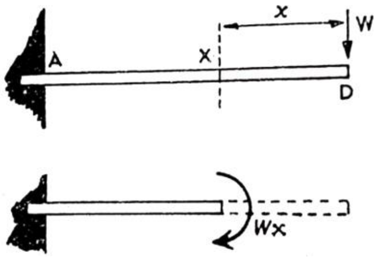


(3) Bending Moment (BM):

The bending effect at any section X of a concentrated load W at D ,

Fig.(3.4) , is measured by the applied moment Wx , where x is the perpendicular distance of the line of section of$W$ from section X .

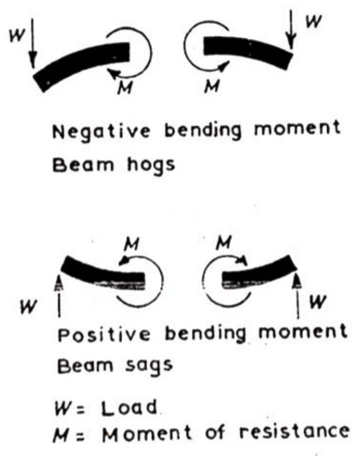
This moment is called the bending moment and is balanced by an equal and opposite moment M exered by the material of the beam at$X$, called the moment of resistance.



Sign Convention

A bending moment is taken as positive if its effect is to tend to make the beam sag at the section considered, Fig.(3.5).If the moment tends make the beam bend upward or hog at the section it is negative.

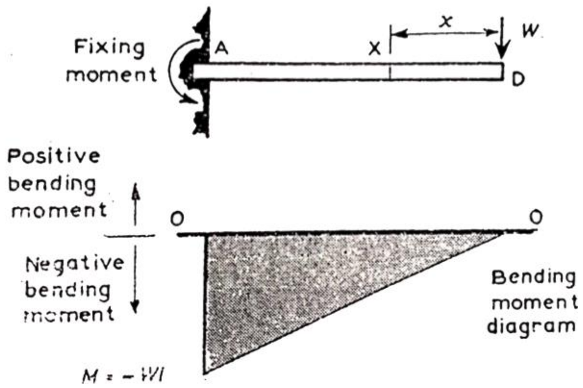
When more than one load act on a beam the bending moment at any section is the algebraic sum of the moments due to all the loads on one side of the beam. It does not matter which side of the section is considered but all loads on that side must be taken into account, including any moments exerted by fixings.



(4) Bending Moment Diagram (BMD):

The variation of bending moment along the beam is shown in a bending moment diagram. for the cantilever beam of Fig 3.1 the bending moment at any section X is given by:

bending moment$= -Wx$ (negative, since the beam hogs at X)



Since there is no other load on the beam this expression for the bending moment applies for the whole length of beam from $x = 0$ to $x = l$ .The moment is proportional to$x$ and hence the bending moment diagram is a straight line. Hence the diagram can be drawn by calculating the moment at two points and joining two corresponding points on the graph by a straight line.

At D, $x = 0$ and bending moment $= 0$

At A, $x = l$ and bending moment $= -Wl$

Since the bending moment is everywhere negative the graph plotted is below the line $0 - 0$ of zero bending moment, Fig.(3.6). At the fixed end A the wall exerts a moment Wl anticlockwise onthe beam; this is calleda fixing moment.

(5) Calculation of Beam Reactions:

When a beam is fixed at some point, or supported by props the fixings and props exert reaction forces on beam. To calculate these reactions the procedure is:

(a) equate the net vertical force to zero;

(b) equate the total moment about any convenient point to zero.

Note (1): Distinguish carefully between "taking moments" and calculating a "bending moment":

(1) The Principle of Moments states that the algebraic sum of the moments of all the forces about any point is zero, i.e. when forces on both sides of a beam section are considered.

(2) The bending Moment is the algebraic sum of the moments of forces on one side of the section about that section.

Note (2): What are the benefits of drawing shear force (SF) and bending

Moment (BM)diagram?

The benefits of drawing a variation of (SF) and (BM)in a beam as a function of $'x'$ measuredfrom one end of the beam is that it becomes easier to determine the maximum absolute value of (SF) and (BM).The (SF) and (BM) diagram gives a clear picture in our mind about the variation of (SF) and (BM) throughout the entire section of the beam.

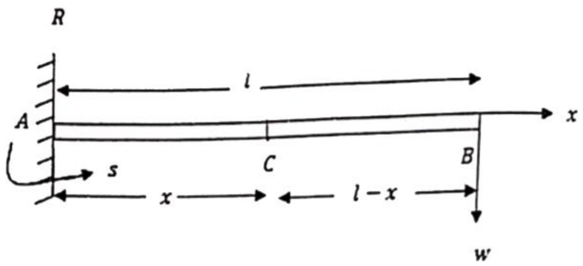
Further, the determination of value of deflection of beam subjected to a given loading where we will use the formula $EL\ \frac{d^2y}{d\,x^2} = M_x$.

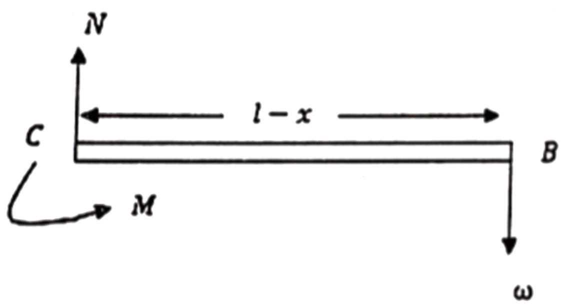
Examples

Example (1):

Draw the (SF) & (BM) diagrams at any section for a light horizontal beam AB ,its length is L .The end A of the beam is fixed at a vertical wall, while the free end $B$is loaded by a weight W .

The Solution





We take a section for the beam at C , where:

$$AC = x \qquad \& \qquad CB = L - x$$

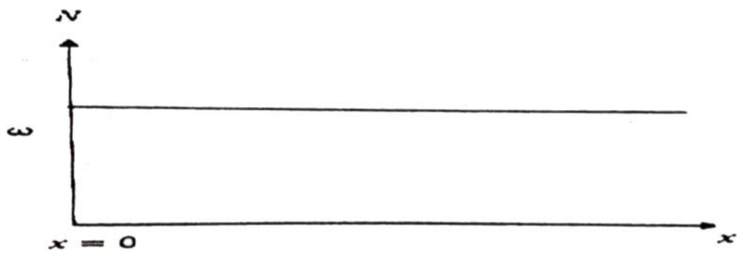
We study the equilibrium of the part CB or the part AC .

The study of the right part CB is easier than the lift part AC , because the existence of the reaction R and the couple S .

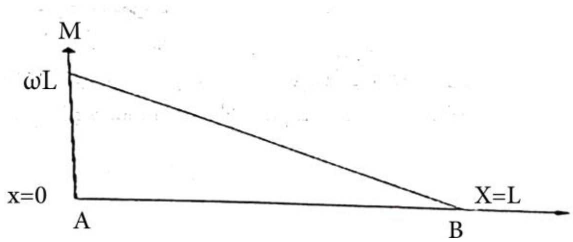
The shear force (SF) is $N = W$ (1)

and the bending moment (BM) is $M = W(L - x)(2)$

From EQ.(1) the (SF) N is constant at any section and so it is a straight line parallel to x $-$axis As shown in the next Fig..



But from EQ.(2) the (BN) M is depending on x , and its diagram isshown in the next Fig.



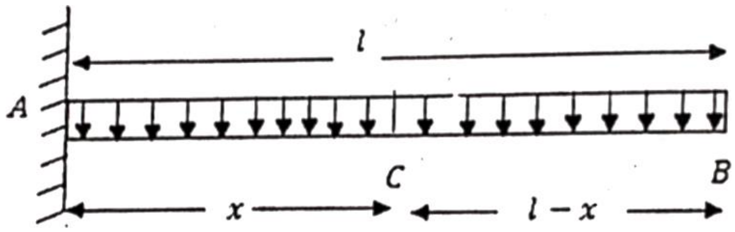
For the equilibrium of the beam AB we find that:

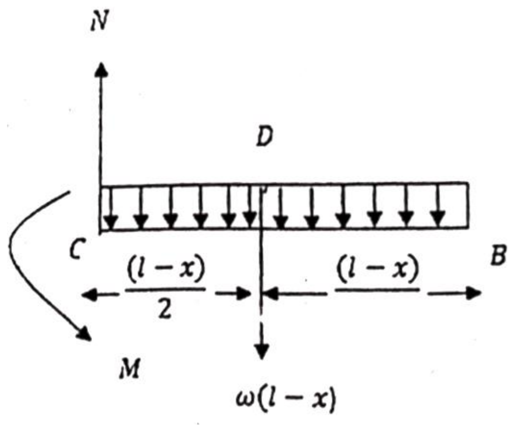
$R = W$, $S = WL.$

Example (2):

Draw the (SF) & (BM) diagrams for a light horizontal beam AB ,its length is L .The end A of the beam is fixed at a vertical wall, while the free end B is free. The beam is loaded uniformly by a weightω per unit length.

The Solution



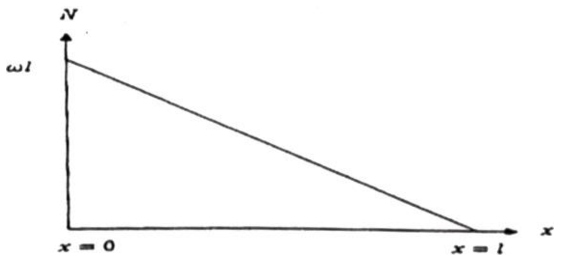


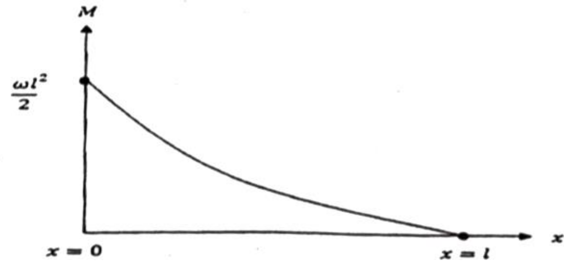
We note that the weight of the part CB is $\omega(L-x)$ and acts at its middle point. From the equilibrium of this part we find that:

The (SF) is $N = \omega(L-x) \quad (1)$,

and the (BM) is $M = \omega(L-x)\frac{(L-x)}{2} = \frac{\omega}{2}(L-x)^2 \quad (2)$.

Then the (SFD) &(BMD) will be shown in the following Figs.

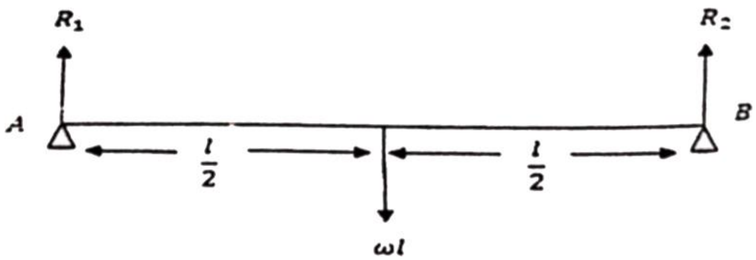




Example (3):

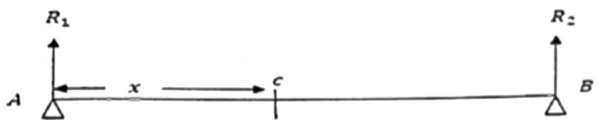
Draw the (SF) & (BM) diagrams for a heavy horizontal beam AB ,its length is L , and ω is its weigh per length. The beam is standing on two weidges in the same horizontal plane atits ends.

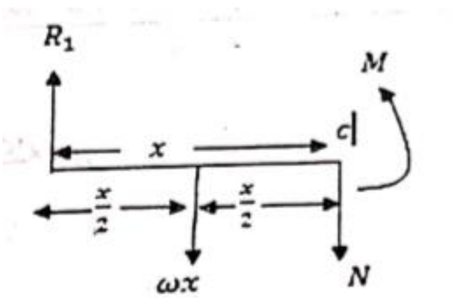
The Solution



From the symmetry we find that:

$R_1 = R_2 = \omega L/2$ Where $R_1 + R_2 = \omega L$.





By considering the equilibrium of the part AC , we find that:

$$R_1 = \omega x + N$$

$$\therefore \; \frac{1}{2}\,\omega L = \omega x + N$$

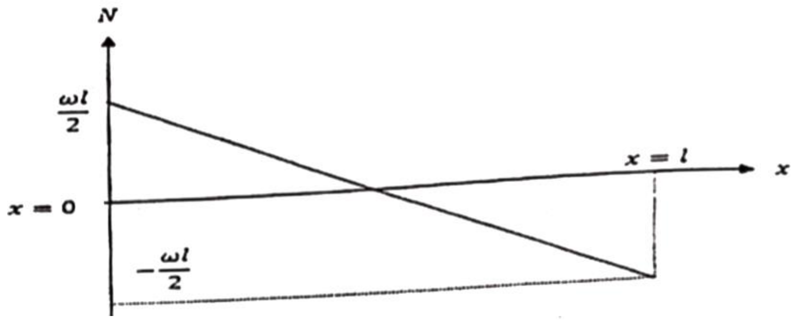
$$\therefore \; N = \; \omega\left(\frac{L}{2} - x\right)$$

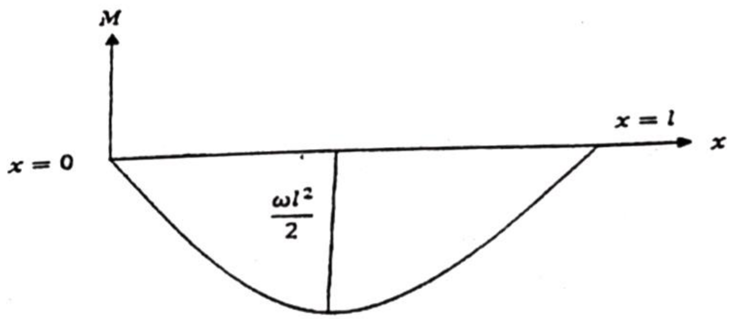
And by taking the moment about the point C , we get:

$$M + \omega x\left(\frac{x}{2}\right) = R_1 x$$

$\therefore \; M = \frac{\omega}{2}\,Lx - \frac{\omega}{2}x^2 = -\frac{\omega}{2}(x^2 - Lx).$

The (SFD) &(BMD) will be shown in the following Figs.

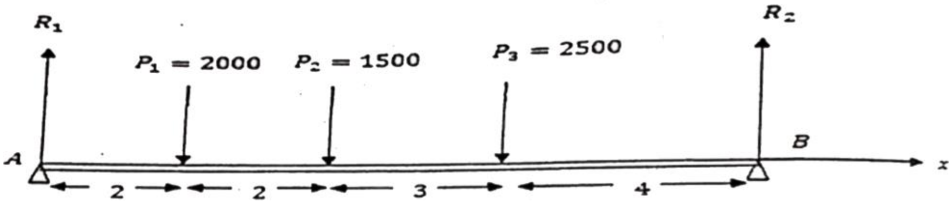




Example (4):

Draw the (SF) & (BM) diagrams for a light horizontal beam AB ,its length is $11\ ft$, and stands on two weidges in the same plane at its ends. The beam carries $p_1 = 2000, p_2 = 1500, p_3 = 2500\ lb$ at the three pointes a, b, c such that $Aa = ab = 2\ ft\ \&bc = 3\ ft$.

The Solution



From the equilibrium of the beam we get:

$$R_{1+}R_2 = 2000 + 1500 + 2500 = 6000\ lb \qquad (1)$$

By taking the moment about the point B we get:

$$11\ R_1 = 2000\ \times 9 + 1500 \times 7 + 2500 \times 4$$

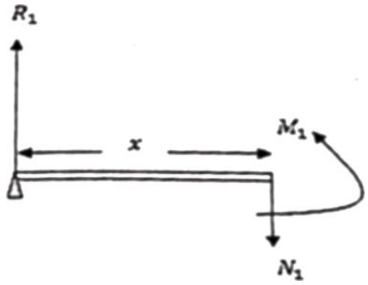
$$\therefore 11\ R_1 = 38500 \qquad \rightarrow \qquad R_1 = 3500\ lb \qquad (2)$$

By substituting from (1) in (2) we get:

$$R_2 = 2500\ lb \qquad (3)(\quad)$$

For determination the (SF) &(BM) at any point we consider the sections where:

$(i) 0 < x < 2$



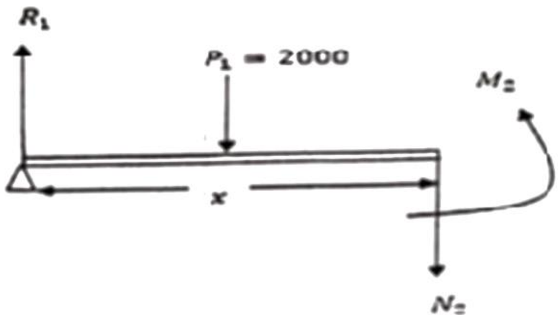
$N_{1=}R_1 = 3500\ lb$

$M_{1=}R_1\, x = 3500x\ lb$

$(ii) 2 < x < 4$

$N_2 = R_1 - p_1 = 3500 - 2000 = 1500\ lb$

$$M_{2=}R_1\, x - 2000(x-2) = 3500\, x - 2000(x-2)\ lb$$



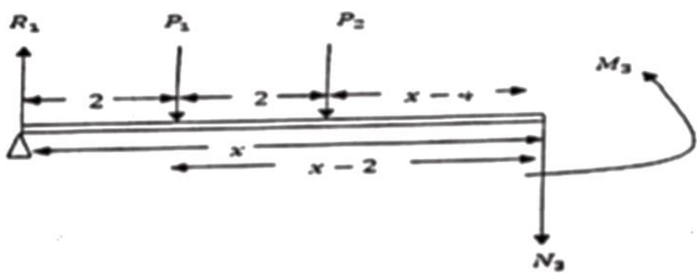
$(iii) 4 < x < 7$

$N_3 = R_1 - p_1 - p_2$

$= 3500 - 2000 - 1500 = 0\ lb$

$M_3 = R_1\, x - 2000(x-2) - 1500(x-4)$

$$= 3500\, x - 2000(x-2) - 1500(x-4)\ lb$$



$(v) 7 < x < 11$

$N_4 = R_1 - p_1 - p_2 - p_3$

$= 3500 - 2000 - 1500 - 2500 = -2500\, lb$

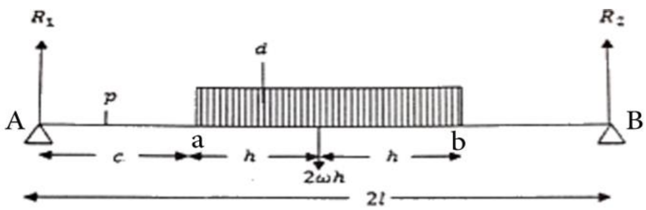
$M_4 = R_1\, x - 2000(x-2) - 1500(x-4) - 2500(x-7) -$

$= 3500\, x - 2000(x-2) - 1500(x-4) - \ lb2500(x-7)$

Example (5):

Find the (SF) and defined the maximum (BM) at a point d for a light horizontal beam AB ,its length is $2L$ and stands on two weidges in the same plane at its ends. The beam carries a movable weight$ab = 2h\omega$ where $2h(h < L)$is its length. Then draw (SFD) &(BMD), and prove that $\frac{ad}{ab} = \frac{Ad}{dB}$.

The Solution



By taking a position for the beam ABas shown in the figure such that $Aa = c$, and by finding the value of c , which makes the (BM) at d is maximum.

In case, the equilibrium of AB , we get:

$$R_1 + R_2 = 2\omega h$$

By taking the moment about the point B we get:

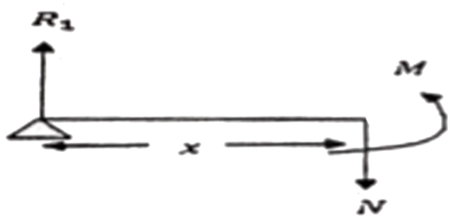
$$R_1 \times 2L = 2\omega h(2L - c - h)$$

$$\therefore\ R_1 = \frac{\omega h}{L}(2L - c - h)$$

By taking a section at $p$where,$AP = x$ & $x < c$, we get:

$$N = R_1 = \frac{\omega h}{L}(2L - c - h)$$

$M = R_1 x = \frac{\omega h}{L}(2L - c - h)x$

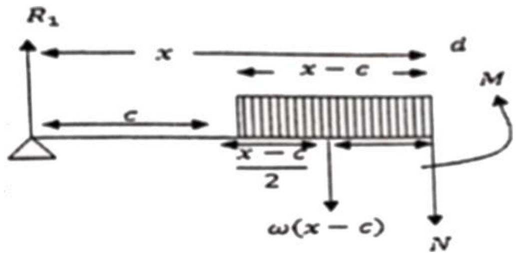


And, by taking a section at d where ,$Ad = x$ & $x > c$, we get:

$$N = R_1 - \omega(x - c)$$

$$= \frac{\omega h}{L}(2L - c - h) - \omega(x - c)$$

$$M = \frac{\omega h}{L}(2L - c - h)x - \frac{1}{2}\omega(x - c)$$



The maximum value of M will be when

$\frac{dM}{dc} = 0$, i.e.$-\frac{x\omega h}{L} + \omega(x - c) = 0$

$$\therefore \quad c = \left(1 - \frac{b}{L}\right)x$$

By substituting in M , we have:

$$M_{max} = \frac{\omega h}{L}\left(2L - h - x\left(1 - \frac{h}{L}\right)\right)x - \frac{1}{2}\omega\left(x - x\left(1 - \frac{h}{L}\right)\right)x$$

In this case, we find that :

$\frac{ad}{db} = \frac{x-c}{2h-(x-c)} = \frac{\frac{h}{L}x}{2h-\frac{h}{L}x} = \frac{hL(x)}{\frac{h}{L}(2L-x)} = \frac{x}{2L-x} = \frac{Ad}{dB}$.

<u>Example (6):</u>

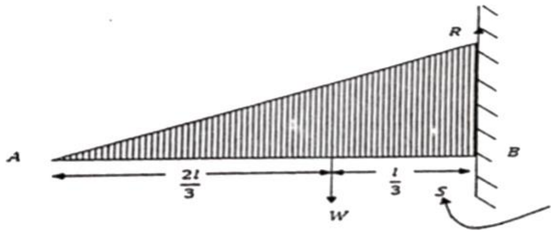
AB is a beam, its length is L , and the end $B$is fixed at a vertical wall. The beam is loaded by a weight Wdistributedlinearly, by uniformly increasing, starting from zero at the free end A.Find the (SF) & (BM) then draw its diagrams.

<u>The Solution</u>

The density of loading is $\omega = \omega(x)$ at the section C ,where $AC = x$,

Then $\omega = \gamma x$ (linearly distribution)

$W = \int_0^L \omega\, dx = \int_0^L \gamma x\, dx$

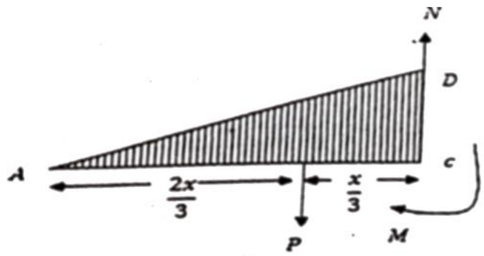


$\therefore W = \gamma L^2/2 \therefore \gamma = 2W/L^2$

$\therefore \omega = \gamma x = (2W/L^2)x$

For the section AC , we get

$$N = P = \int_0^x \omega\, dx = \int_0^x (2Wx/L^2)\, dx$$



$\therefore N = (Wx^2/L^2)$

We note that the weight P dived AC by the ratio

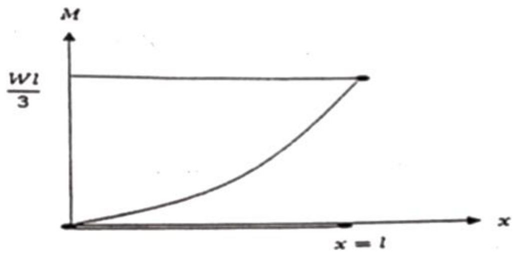
$AE = 2EC = 2x/3$

By taking the moment about C we find that

$$M = P(x/3) = (Wx^2/L^2)(x/3) = (Wx^3/3L^3)$$

We note that $R = W$, $S = WL/3$ and

$AF = 2FB = (2/3)L$.



Exercises

(1)Draw the (SF) & (BM) diagrams at any section for a light horizontal beam AB ,its length is L .The end A of the beam is fixed at a vertical, while the free end B is loaded by a weight ωL .

(2)Draw the (SF) & (BM) diagrams for a light horizontal beam AB ,its length is L , and stands on two weidges in the same plane at its ends. The beam carries two equal weights$p_1 = p_2 = \omega$ at the two points C&D such that $AC = DB = a,\ (a < L/2)$.

(3)Find and draw the (SF)&(BM) for a light horizontal beam AB ,its length is$10\ ft$and stands on two weidges in the same plane at its ends. The beam is loaded by a uniformally distributed weight, where $\omega = 10\ lb$ per unit length.

Chapter (4)

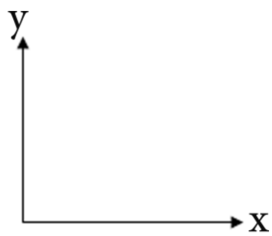
Bending of the beams

Introduction :

(∗) WE know that the axis of a beam deflects from its initial position under action of applied forces.

(∗)In this chapter we learn how to determine the elastic deflection of a beam.

Selection of co-ordinate:

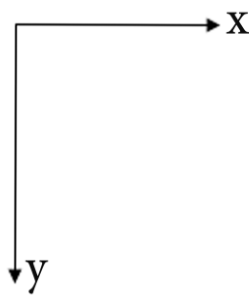


We will not introduce any other co- ordinate system. We use general co- ordinateaxes as shown in the figure. This system will be followed in deflection and in shear force and bending moment diagram. Here downward direction will be negative i.e. negative Y −axis.Therefore,downward direction of the beam will be treated as negative.

To determine the value of deflection of beam subjected to a given loading where we will use the formula $EL \frac{d^2y}{d\,x^2} = M_x$.

Some books fix a co- ordinateaxesas shown in the following figure.

Here downward direction will be positive i.e. positive Y −axis. Therefore,downward direction of the beam will be treated positive. As beam is generally deflected in downward direction and this co-ordinate system treats downward deflection is positive deflection.



To determine the value of deflection of beam subjected to a given. loading where we will use the formula $EL\ \frac{d^2y}{d\,x^2} = -M_x$.

Why to calculate the deflections?

(∗) To prevent cracking of attached brittle materials.

(∗) To make sure the structure does not deflect severely and to ‘appear' safe for its occupants.

(∗)To help analyzing statically indeterminate structures.

(∗)Information on deformation characteristics of members is essential in the study of vibrations of machines.

Now the equations that we will use for find the bending of beams are :

$$M = \pm K\ \frac{d^2y}{d\,x^2} \& N = \pm K\ \frac{d^3y}{d\,x^3}$$

Where $K = EL$ is Rigidity Flexural,M&N are (BM) and (SF)

There are boundary conditions that will be satisfied at some points at the fixed beams which are:

(i) At the free ends of the beams (BM) and (SF) are equal to zero so

$y'' = 0 \quad \& y''' = 0$.

(ii) At the fixed points there is no bending and so

$y = 0 \quad \& y' = 0$.

(iii) At the fixed points there is no bending moment and so

$y'' = 0$.

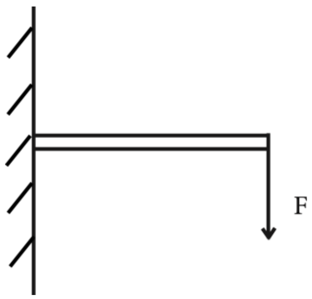
(iv) At the free end there is no shear force and so

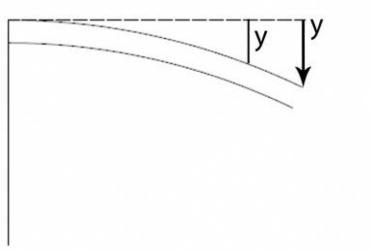
$y''' = 0$.

(v)If thebeam is simple standing at its ends the deflection and BM

Will tends to zero i.e.

$y = 0 \quad \& y'' = 0$.



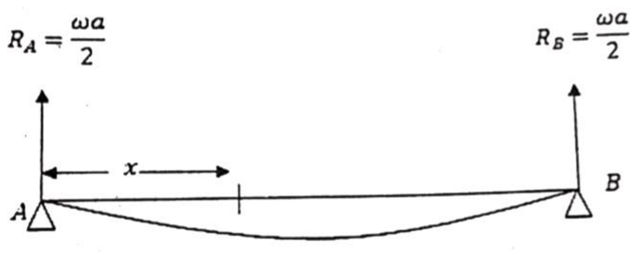


Examples

Example (1):

A uniform beam AB stands at its ends $A\&B$ on two widges in the same horizontal plane .Prove that the deflection at a distance x from one end $(A\ say)$ is equal to $\frac{\omega x}{24\ K}(x-a)(a^2+ax-x^2)$ where a is $beam's$ length and ω is weight per length.

The solution



The (BM) about the point, where $AC = x$is:

$$M = -\frac{\omega a}{2}x + \frac{\omega}{2}x^2 (1)$$

But $M = EL\ \frac{d^2y}{d\,x^2}(2)$

$\therefore\ M = EL\ \frac{d^2y}{d\,x^2} = -\frac{\omega a}{2}x + \frac{\omega}{2}x^2(3)$

By integration twice, we get:

$$EL\ \frac{dy}{d\,x} = -\frac{\omega a}{4}x^2 + \frac{\omega}{6}x^3 + c \qquad (4)$$

$$EL\, y = -\frac{\omega a}{12}x^3 + \frac{\omega}{24}x^4 + cx + c'(5)$$

From the boundary conditions:

At $A:\quad x = 0 \quad , \quad y = 0 \qquad \therefore \quad c' = 0$

At $B:\quad x = a \quad , \quad y = 0 \qquad \therefore \quad c = \frac{\omega a^3}{24}$

$\therefore$bysubstitution in (5) we get:

$EL\, y = -\frac{\omega x}{24}(a^3 + x^3 - 2ax^2)$

$$\therefore \quad y = \frac{\omega x}{24\, K}(x-a)(a^2 + ax - x^2)$$

==

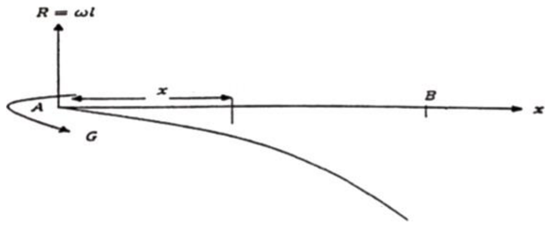
Example (2):

$AB$is a uniform beam, the end A is fixed at a vertical wall, while the end B is free. Prove that the deflection at B under the action of its weight is equal to $3/8$ the deflection if we consider the beam is light and the end B carries a weight equal to the weight of the beam.

The solution

If we consider the length of the beam is, and ω is the weight per unit length, we get:

(i)The first case (The beam is heavy):



The (BM) at a point C where $CB = l - x$ is

$$M = \frac{\omega}{2}(l-x)^2 (1)$$

$\therefore \quad EL\ \frac{d^2y}{d\,x^2} = \frac{\omega}{2}(l^2 - 2lx + x^2)(2)$

By integration we get:

$EL\ \frac{dy}{d\,x} = \frac{\omega}{2}\left(l^2x - lx^2 + \frac{x^3}{3}\right) + c$

At $\quad A: \quad x = 0 \quad , \quad y' = 0 \qquad \therefore \quad c = 0$

$$EL\ \frac{dy}{d\,x} = \frac{\omega}{2}\left(l^2x - lx^2 + \frac{x^3}{3}\right)(3)$$

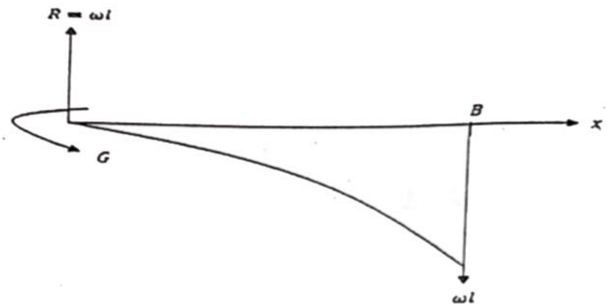
By integration again we get:

$$EL\,y = \frac{\omega}{2}\left(\frac{l^2}{2}x^2 - \frac{l}{3}x^3 + \frac{x^4}{12}\right) + c'$$

At $\quad A: \quad x = 0 \quad , \quad y = 0 \quad \therefore \quad c' = 0$

$$y_1 = (y)_{x=l} = \frac{\omega l^4}{8K} \qquad (*)$$

(ii) The second case (The beam is light):



$M = \omega l(l - x)(1)$

$$\therefore \quad EL\,\frac{d^2y}{d\,x^2} = \omega l(l - x)(2)$$

$$EL\,\frac{dy}{d\,x} = \omega l\left(lx - \frac{x^2}{2}\right) + c(3)$$

At $\quad A: \quad x = 0 \quad , \quad y' = 0 \quad \therefore \quad c = 0$

$$EL\,\frac{dy}{d\,x} = \omega l\left(lx - \frac{x^2}{2}\right)(4)$$

By integration again we get:

$$EL\,y = \omega\left(\frac{l}{2}x^2 - \frac{l}{6}x^3\right) + c'(5)$$

At $\quad A: \quad x = 0 \quad , \quad y = 0 \quad \therefore \quad c' = 0$

$$EL\,y = \omega\left(\frac{l}{2}x^2 - \frac{l}{6}x^3\right)(6)$$

$At \quad B: \quad x = l \quad , \quad y = y_2$

$$y_2 = (y)_{x=l} = \frac{\omega l^4}{3K} \qquad (**)$$

From$(*)$ and $(**)$ we get:

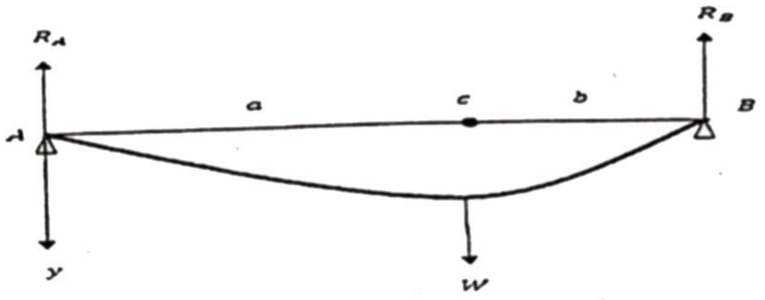
$\frac{y_1}{y_2} = \frac{3}{8} \qquad \therefore \quad y_1 = \frac{3}{8} y_2.$

==================================

Example (3):

$AB$is a light uniformbeam, its length is$(a+b)$ .The beam stands on two wedges at its ends $A\&B$ in the same horizontal plane, the beam carries a weight equal to W at the point C , where $AC = a$. Find the curve of the beam.

The solution



From the equilibrium of the beam we get:

$R_A + R_B = W$

And by taking the moment about the point B we get:

$R_A(a+b) = W(b)$,

$$\therefore R_A = \frac{Wb}{(a+b)} \& R_B = \frac{Wa}{(a+b)}$$

The curve of the beam can be determined from the two following equations:

$0 \le x \le a$ $\quad$ $,a \le x \le a+b$

$$EL\,\frac{d^2y}{d\,x^2} = -\frac{Wb}{(a+b)}x \quad ,EL\,\frac{d^2y}{d\,x^2} = -\frac{Wb}{(a+b)}x + W(x-a)$$

$$\therefore \quad EL\,\frac{dy}{d\,x} = -\frac{Wb}{(a+b)}\frac{x^2}{2} + c, EL\,\frac{dy}{d\,x} = -\frac{Wb}{(a+b)}\frac{x^2}{2} + \frac{W}{2}(x-a)^2 + c'$$

And because of the continuity at $x = a$ then $c = c'$

Again, by integrating, we have:

$$ELy = -\frac{Wb}{(a+b)}\frac{x^3}{6} + cx + D, ELy = -\frac{Wb}{(a+b)}\frac{x^3}{6} + \frac{W}{6}(x-a)^3 + cx + D'$$

And also because of the continuity at $x = a$then $D = D'$,

And soat$x = a \quad , y = 0 \quad \therefore \quad D = 0$

at$x = a \quad , y = 0 \quad \therefore \quad c = \frac{Wba+2b}{6(a+b)}$

Now we can get:

$$ELy = -\frac{Wb}{(a+b)}\frac{x^3}{6} + \frac{Wba+2b}{6(a+b)}x \quad ,0 \le x \le a$$

$$ELy = -\frac{Wb}{(a+b)}\frac{x^3}{6} + \frac{W}{6}(x-a)^3 + \frac{Wba+2b}{6(a+b)}x + D'$$

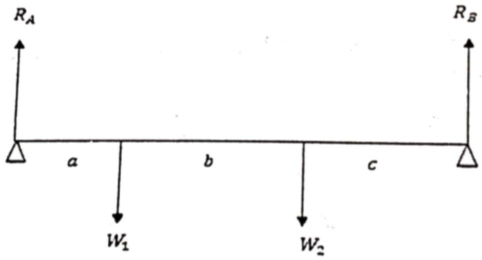
$$ELy = -\frac{Wb}{(a+b)}\frac{x^3}{6} + \frac{W}{6}(x-a)^3 + \frac{Wba+2b}{6(a+b)}x \ \ ,a \le x \le a+b$$

==

Example (4):

AB is a lightuniform beam , its length is$(a+b+c)$ .The beam stands on two wedges at its ends $A\&B$ in the same horizontal plane, The beam carries two weights equal to $W_1\&W_2$ at the points$C\&D$, where $AC = a, BC = b, CD = c$. Find the curve of the beam.

The solution



From the equilibrium of the beam we get:

$R_A + R_B = W_1 + W_2$

And by taking the moment aboutthe point A then about the point B we get

$$R_A = \frac{W_1(b+c) + W_2c}{(a+b+c)} \& R_B = \frac{W_1a + W_2(a+b)}{(a+b+c)}$$

The curve of the beam can be determined from the two following equations:

$0 \le x \le a \quad ,a \le x \le a+b \ , \qquad a+b \le x \le a+b+c$

$$EL\ \frac{d^2y}{d\,x^2} = -R_A x + W_1(x-a) + W_2(x-a-b) \quad ,$$

$$EL\ \frac{dy}{d\,x} = -R_A\frac{x^2}{2} + \frac{1}{2}W_1(x-a)^2 + \frac{1}{2}W_2(x-a-b)^2.$$

The constants had been equaled in the three sections because the continuity of the functions.

Also, we can get the deflection in the form:

$$EL\ y = -R_A\frac{x^3}{6} + \frac{1}{6}W_1(x-a)^3 + \frac{1}{6}W_2(x-a-b)^3.$$

=====================================

<u>Exercises</u>

(1)$AB$is heavy beam which the end A is fixed at a vertical wall, while the free end Bcarries a weight ωl ,where l. prove that the deflection is

$$y = \left(\frac{1}{8} + \frac{1}{3}\right)\frac{\omega l^4}{K} = \frac{11}{24}\frac{\omega l^4}{K}.$$

(2)A light alloy bar is observed to increase in length by 0.35 per centwhen subjected to a tensile stress of $18\ ton/in.^2$ .Calculate Young' modulus for the material.

(3)A duralumnin tie, $2\ ft$ long $1.5\ in.$ diameter, has a hole drilled out along its length .The hole is of $1\ in.$diameter and $4\ in.$ long.Calculate the total extension of the tiedue to a load of $18\ tons$. $E = 12 \times 10^4\ Ib/in.^2$.

(4) A steel strut of rectangular section is made up of two lengths. The

first6 in. long, has breadth 2 in.and depth 1.5 in. ;thesecond,4 in.long is1 in. square. If $E = 14000\ tons/in.^2$,calculate the compression of the strut under a load of $10\ tons$.

FACULTY OF SCIENCE QENA

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SOUTH VALLEY UNIVERSITY

جامعة جنوب الوادى

STATIC-2

2nd_Year

DR.Mohammed Abd EL-Aziz

PART II—HYDROSTATICS

CHAPTER 10

FLUID PRESSURE

10.1. Introductory

Whereas Statics deals with the equilibrium of a rigid body, Hydrostatics deals with the equilibrium of fluids. A *fluid* is defined as a body whose constituent particles act on each other with a pressure which is normal to their common surface when the fluid is at rest. When a fluid is in motion this pressure may have a component in the plane of the common surface, and this component of the pressure is called *viscosity*. A *perfect fluid* is defined as one in which there is no viscosity, and which therefore offers no resistance to the separation of its particles. Since in hydrostatics we are dealing with fluids which are at rest, we need make no distinction between viscous and non-viscous fluids, and we have that *the pressure of a fluid in equilibrium is always normal to any surface with which it is in contact.*

Fluids are divided into *liquids* and *gases*. A liquid is a fluid which may be taken as incompressible, such liquids as water or mercury can only be appreciably compressed by very great pressures. A gas, on the other hand, is easily compressible and is capable of indefinite expansion to fill the space which contains it.

10.2. Pressure at a Point

The pressure of a fluid on a given area of surface may be uniform or varying. The mean pressure on a given area is measured by dividing the total thrust on the area by the area. Thus a thrust of W lb. wt. on an area of A square inches gives a mean pressure of W/A lb. wt. per square inch.

The pressure at a point is the mean pressure over an indefinitely small area surrounding the point, and is measured in units such as lb. wt. per square inch or grammes weight per square centimetre or dynes per square centimetre.

The pressure at a point in a fluid in equilibrium is the same in all directions.

This means that if a surface could be made to pass through any point of the fluid without disturbing equilibrium, the pressure on the surface at the point would be independent of the orientation of the surface.

The theorem is proved by considering the thrusts of the remainder of the fluid on the faces of a small triangular prism of the fluid. Let the right-angled triangles ABC and DEF (Fig. 136) be end faces of the prism perpendicular to its length with BC and EF horizontal, AB

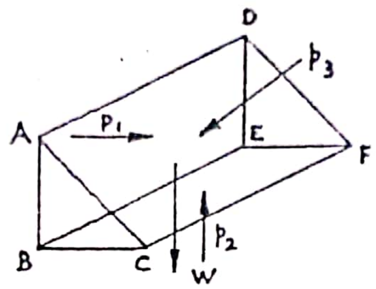

FIG. 136.

and DE vertical. Let p_1, p_2, p_3 be the mean pressures on the faces $ADEB$, $BEFC$, $ADFC$ respectively. Let w be the weight per unit volume of the fluid and l the length AD of the prism. The weight of the prism W, is given by

$$W = \tfrac{1}{2}AB \, . \, BC \, . \, l \, . \, w.$$

The thrusts on the three faces are

$$p_1 AB \, . \, l, \quad p_2 BC \, . \, l, \quad p_3 CA \, . \, l, \text{ respectively,}$$

normal to the faces, and the thrusts on the end faces ABC and DEF will be equal and opposite. Hence for equilibrium, resolving horizontally and vertically we have

$$p_1 AB \, . \, l = p_3 CA \, . \, l \cos BAC,$$
$$p_2 BC \, . \, l = p_3 CA \, . \, l \cos ACB + \tfrac{1}{2}AB \, . \, BC \, . \, l \, . \, w.$$

Hence,

$$p_1 AB \, . \, l = p_3 AB \, . \, l,$$
$$p_2 BC \, . \, l = p_3 BC \, . \, l + \tfrac{1}{2}AB \, . \, BC \, . \, l \, . \, w.$$

Therefore,

$$p_1 = p_3,$$
$$p_2 = p_3 + \tfrac{1}{2}AB \, . \, w.$$

In the limit, as AB tends to zero and the size of the prism decreases indefinitely, we have

$$p_1 = p_2 = p_3,$$

that is, the pressure at the point in the three directions considered are equal.

This result is confirmed experimentally by introducing pressure-gauges in a fluid which record the same pressures at a point for different orientations.

The pressure at a point in a fluid in equilibrium is the same for points at the same horizontal level.

This theorem is proved by considering the thrust of the remainder of the fluid on a small right circular cylinder of the fluid whose axis is horizontal (Fig. 137).

Let a be the area of each end of the cylinder, and let the mean pressures on the ends be p_1 and p_2 respectively.

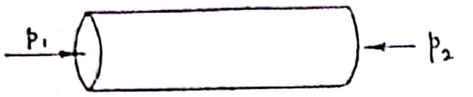

FIG. 137.

The pressure at any point on the curved surface of the cylinder will be perpendicular to the surface, and therefore perpendicular to the axis of the cylinder. Hence the only forces parallel to the axis of the cylinder will be the thrusts on the ends, and since the cylinder is in equilibrium, we have

$$p_1a = p_2a,$$

that is,

$$p_1 = p_2.$$

p_1 and p_2 are the mean pressures over the ends of the cylinder, and when the radius of the cylinder is diminished indefinitely they become the pressures at points at the same horizontal level, and the theorem is proved.

The pressure at a point in a fluid in equilibrium increases in proportion to the depth of the point.

This theorem is proved by considering the thrust of the remainder of the fluid on a small right circular cylinder of the fluid whose axis is vertical (Fig. 138). Let a be the area of each end section; let the mean pressures on the upper and lower ends be p_0 and p_1 respectively, let h be the height of the cylinder and w the density of the fluid. Then, since the pressure at any point of the curved surface is perpendicular to the axis of the cylinder, the only vertical forces are the thrusts on the ends and the weight of the cylinder and we have for equilibrium

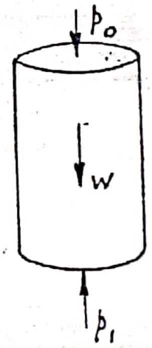

FIG. 138.

$$p_1a = p_0a + ahw,$$

that is,

$$p_1 = p_0 + hw.$$

In the limit as the radius of the cylinder diminishes indefinitely p_0 and p_1 are the pressures at points whose difference in level is h, and the theorem is proved.

Since p_0 and p_1 are pressures per unit area, it follows that the difference between p_1 and p_0 is the weight of a column of the fluid of unit cross-sectional area and height h.

Example 1. *Find the increase in pressure on descending a depth of 20 ft. in water, the density of the water being 62·5 lb./cu. ft.*

Let p_0 and p_1 be the upper and lower pressures in lb. wt./sq. ft.

The weight of a column of water of cross-sectional area 1 sq. ft. and height 20 ft. is 1250 lb.

Hence,
$$p_1 - p_0 = 1250 \text{ lb. wt./sq. ft.}$$
$$= \frac{1250}{144} \text{ lb. wt./sq. in.}$$
$$\simeq 8{\cdot}68 \text{ lb. wt./sq. in.}$$

10.3. Atmospheric Pressure

The pressure on the surface of a liquid will be that due to the atmosphere. This pressure varies with the prevailing atmospheric conditions, but is generally about $14\frac{1}{2}$ lb. wt./sq. in. The atmospheric pressure is usually given as the height of the mercury barometer, and the pressure is the weight of a column of mercury of unit cross-sectional area whose height is that of the barometer.

The average height of a mercury barometer is about 30 in. The corresponding height of a water barometer would be about 33 ft. Hence the pressure at a depth x in water when the water barometer records 33 ft. will be that due to a column of water of height $x + 33$ ft.

The additional atmospheric pressure is frequently omitted when calculating pressure in a liquid.

10.4. Transmissibility of Fluid Pressure

We have seen that pressure in a fluid at rest is the same at points at the same level, and that the difference in pressure at different levels is the product of the density and difference in height.

It follows that if an additional pressure is applied to any point of a fluid at rest in a closed vessel, this pressure is transmitted to every point of the fluid.

It follows also that the free surface of a liquid at rest under gravity must be horizontal, and that the free surfaces of liquids in inter-communicating vessels must be at the same level. This assumes, of course, that the extent of the liquid is limited and that the weights of its particles may be taken as acting in parallel directions.

The Hydraulic Press

The hydraulic press is a machine which makes use of the transmissibility of fluid pressure to obtain mechanical advantage. Essentially it consists of two inter-communicating cylinders containing liquid (Fig. 139). A force P applied by a piston in the smaller cylinder is used to overcome a resistance W to a piston in the larger cylinder. Let the cross-sectional areas of the larger and smaller cylinders be a and b respectively. The force P on the smaller piston causes an increase in pressure P/b throughout the liquid, and therefore at every point on the larger piston.

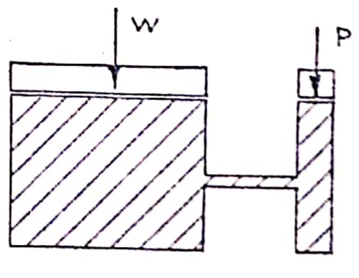

FIG. 139.

Hence, the additional thrust on this piston is $a \times \frac{P}{b} = \frac{a}{b}P$, and if this thrust equals W the mechanical advantage is a/b.

If the smaller piston descends a distance x, a volume xb of liquid moves into the larger cylinder and lifts the larger piston a distance

xb/a. Hence the velocity ratio of the machine is a/b and the principle of work is verified.

The hydraulic press is sometimes called Bramah's press, after the inventor of a U-shaped leather collar for the large piston, which is pressed against the sides of the cylinder by the liquid pressure to prevent leakage.

10.5. Density and Specific Gravity

The *density* of any substance is the mass of a unit volume of the substance. The density of water is usually taken as 62·5 lb./cu. ft. In metric units the density of water is taken as 1 gm./c.c. The density of a substance, therefore, relates to certain units of mass and of volume. In general, the density of a substance varies with its temperature, but in the case of liquids this variation is slight and is usually neglected.

The *specific gravity* of a substance is the ratio of its density to that of water and is a number independent of the units in which the densities are given. Thus, if the specific gravity of a substance is 4, its density is $4 \times 62{\cdot}5 = 250$ lb./cu. ft. or 4 gm./c.c.

Example 2. *Five litres of a liquid of specific gravity* 1·3 *are mixed with* 7 *litres of a liquid of specific gravity* 0·78. *If the bulk of the liquid shrinks* 1 *per cent. on mixing, find the specific gravity of the mixture.* (L.U.)

Let w grammes be the mass of a litre of water. Then the total weight of the mixture is

$$5 \times 1{\cdot}3w + 7 \times 0{\cdot}78w \text{ grammes.}$$

The volume of the mixture is

$$12 \times 0{\cdot}99 \text{ litres.}$$

The mass of one litre of the mixture is therefore

$$\frac{5 \times 1{\cdot}3 + 7 \times 0{\cdot}78}{12 \times 0{\cdot}99} w \text{ grammes.}$$
$$= 1{\cdot}007w \text{ grammes.}$$

Hence, the specific gravity of the mixture is 1·007.

EXERCISES 10 (*a*)

1. The pressure in a water pipe at the bottom of a building is 53 lb. wt./sq. in.; at the top it is 10 lb. wt./sq. in. Find the height of the building.
2. Find the pressure in tons/sq. in. at a depth of 1000 fathoms in sea water. (1 fathom = 6·08 ft., specific gravity of sea water 1·024.)
3. A bottle is filled with liquid of specific gravity σ and is weighed. A solid body, whose weight in air is W gm. and whose specific gravity is σ', is put into the bottle and the overflow removed. Prove that the increase in weight is $W(\sigma' - \sigma)/\sigma'$. If $W = 2{\cdot}45$, and if water is the liquid used, the increase in weight is 1·95 gm. Calculate the specific gravity of the solid. (L.U.)
4. A hollow spherical shell of external and internal radii a and b respectively has a specific gravity S. If the shell can float in water, prove that $\frac{b}{a} > \left(1 - \frac{1}{S}\right)^{\frac{1}{3}}$.

(5)

Deduce that for iron (sp. gr. 7·8), the thickness cannot exceed 0·045 of the external radius. (L.U.)

5. If 16 c.c. of sulphuric acid of specific gravity 1·85 are mixed with 7 c.c. of water, and if the specific gravity of the dilute acid is 1·64, find the contraction in volume which has taken place. (L.U.)

6. Find the atmospheric pressure which is equivalent to that at a depth of 30 in. in mercury (sp. gr. 13·6).

7. Two vertical cylinders of cross-sectional area 120 and 200 sq. in. respectively contain water and are connected at their bases by a pipe. On top of the water in the smaller cylinder is a light piston which fits the cylinder. A weight of 50 lb. is placed on the piston. Find the amount by which the piston is lowered and the free surface in the other cylinder raised.

10.6. Common Surface of Liquids

It can be shown that the common surface of two liquids of different densities which do not mix is a horizontal plane.

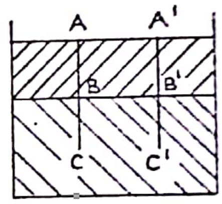

FIG. 140.

Let the densities of the liquids be ρ_1 and ρ_2, where $\rho_1 < \rho_2$. Let C and C' (Fig. 140) be two points in the same horizontal plane in the lower liquid, and let verticals through C and C' meet the common surface in B and B' and the free surface of the upper liquid in A and A' respectively. If p_0 be the pressure at the free surface and p the pressure at C and C', we have by considering the equilibrium of small columns of the liquids of heights AC and $A'C'$

$$p - p_0 = \rho_1 AB + \rho_2 BC,$$
$$p - p_0 = \rho_1 A'B' + \rho_2 B'C'.$$

Hence

$$\rho_1 AB + \rho_2(AC - AB) = \rho_1 A'B' + \rho_2(A'C' - A'B'),$$

and since $AC = A'C'$ we have

$$AB = A'B'.$$

Therefore, since the free surface is horizontal, the common surface of the liquids must be horizontal.

10.7. Measurement of Relative Density

The relative density of liquids which do not mix may be measured in a U-tube. With the heavier liquid filling the bottom of the tube, let XX' be the level of the common surface of the liquids (Fig. 141). Let ρ_1 and ρ_2 be the densities of the liquids ($\rho_1 < \rho_2$) and let their free surfaces be at heights h_1 and h_2 above XX'.

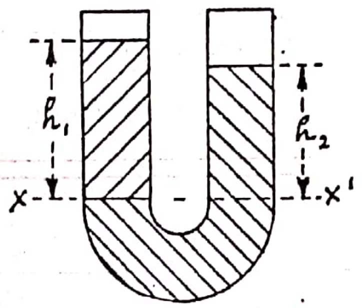

FIG. 141.

Then, since the pressure at the level XX' must be the same in the two arms of the tube, we have

$$\rho_1 h_1 = \rho_2 h_2,$$

and hence by measuring h_1 and h_2 the relative density may be found.

Similar calculations may be made when more than two columns of liquids balance in a U-tube.

Example 3. *Some mercury (sp. gr. 13·6) is poured into a uniform U-tube and into one arm alcohol (sp. gr. 0·8) is poured which occupies a length 6·8 cm. of the tube. What is the difference in level of the free surfaces of the liquids? If now chloroform (sp. gr. 1·5) is poured into the other arm until the free surfaces of the chloroform and alcohol are at the same level, what length of the tube is occupied by the chloroform?*

In the first case let h cm. be the height of the column of mercury above the level of the common surface of mercury and alcohol.

Then we have

$$6 \cdot 8 \times 0 \cdot 8w = h \times 13 \cdot 6w,$$

where w is the mass of water that would fill 1 cm. of the tube.

Hence,

$$h = 0 \cdot 4 \text{ cm.},$$

and the difference in level is $6 \cdot 8 - 0 \cdot 4 = 6 \cdot 4$ cm.

When chloroform is added let h_1 cm. be the length of tube it fills.

Then, since the free surfaces are at the same level, the pressures at depth h_1 in either arm will be the same. That is

$$h_1 \times 1 \cdot 5w = 6 \cdot 8 \times 0 \cdot 8w + (h_1 - 6 \cdot 8) \times 13 \cdot 6w,$$
$$h_1 = 7 \cdot 2 \text{ cm.}$$

10.8. Thrust on a Plane Area

If a plane area be immersed in a liquid, the pressure at any point will be perpendicular to the area and the total thrust is the sum of these parallel pressures over the area.

If the area is horizontal, the pressure at any point will be ρh, where h is the height of the free surface above the area and ρ the density of the liquid. Hence, if A be the total area, the thrust on it will be

$$\rho A h.$$

If the area is not horizontal, let $\bar{y}$ be the depth of its centroid G below the free surface (Fig. 142). Let a small element of area δA be at a depth y below the free surface. Then the thrust on the element δA will be

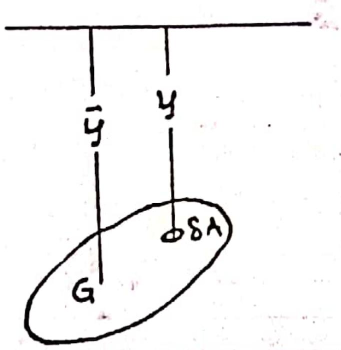

FIG. 142.

$$\rho y \delta A.$$

Summing such quantities over the whole area we have the total thrust $= \Sigma \rho y \delta A$.

But the depth of the centroid of the area is given by

$$A\bar{y} = \Sigma y \delta A.$$

Therefore, if the density of the liquid is uniform the total thrust is

$$\rho A \bar{y}.$$

Hence, the total thrust on a plane area immersed in a given liquid depends only on the area and the depth of its centroid. It follows that if the area, while remaining immersed, be rotated about any axis through its centroid, the total thrust on the area is unchanged.

We have found that the *magnitude* of the total thrust on a plane area depends on the depth of the centroid; the point where this force may be taken to act is called the *centre of pressure*, which is not at the centroid unless the area is horizontal.

10.9. Thrusts in Liquids of Differing Densities

Suppose we have a plane area immersed in two layers of liquids of densities ρ_1 and ρ_2 $(\rho_1 < \rho_2)$, so that the common surface of the liquids divides the area into two parts.

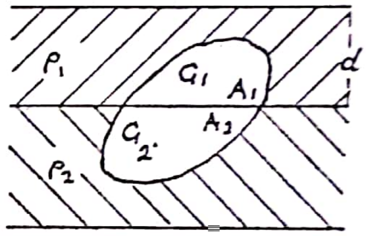

FIG. 143.

Let the areas of the two parts be A_1 and A_2, and the depths of their centroids below the free surface $\bar{x}_1$ and $\bar{x}_2$. Let d be the depth of the upper layer (Fig. 143).

Then the thrust on the area A_1

$$= A_1\bar{x}_1\rho_1,$$

which is the product of the area A_1 and the pressure at its centroid. At each point of the area A_2 there is a pressure $\rho_1 d$ due to the upper layer of liquid, equivalent to a thrust $A_2\rho_1 d$ at its centroid. In addition, due to the liquid of density ρ_2, there is a thrust $A_2\rho_2(\bar{x}_2 - d)$ on the area A_2.

Hence, the total thrust on the area A_2

$$= A_2\{\rho_1 d + \rho_2(\bar{x}_2 - d)\},$$

and this is the product of the area A_2 and the pressure at its centroid.

It should be noted that the thrust on a plane area immersed in a liquid may be greater than the entire weight of the liquid. For example, if a vessel in the form of a hollow pyramid of base area A and height h be filled with liquid of density ρ through a hole at the vertex, the base, being horizontal, is at a depth h below the free surface of the liquid, the pressure at any point of the base is ρh and the total thrust on the base is ρAh.

The weight of the liquid is $\frac{1}{3}\rho Ah$.

The difference between the thrust on the base and the weight of the liquid arises from the downward thrust of the sides of the pyramid on the liquid. Hence, the *upward* thrust of the liquid on the sides of the pyramid will be $\frac{2}{3}\rho Ah$.

Example 4. *A tank whose base is 3 ft. by 4 ft. and height 2 ft. is half filled with water. It is turned about one of the shorter edges of its base until the water just begins to overflow. Find the thrust on one of its vertical sides and on its base in this position.*

When the tank has been tilted, a diagonal of its vertical faces will be horizontal (Fig. 144).

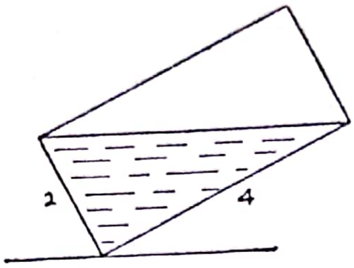

FIG. 144.

Area immersed of a vertical face $= 4$ sq. ft.,

length of the diagonal $= 2\sqrt{5}$ ft.,

maximum depth of water $= \dfrac{4}{\sqrt{5}} = 0{\cdot}8\sqrt{5}$ ft.,

depth of centroid of immersed area $= \dfrac{1}{3} \times 0{\cdot}8\sqrt{5}$ ft.,

thrust on vertical side $= 62{\cdot}5 \times 4 \times \dfrac{1}{3} \times 0{\cdot}8\sqrt{5},$

$= 149{\cdot}1$ lb. wt.

Depth of centroid of base $= 0{\cdot}4\sqrt{5}$ ft.
Area of base $= 12$ sq. ft.
Thrust on base $= 62{\cdot}5 \times 12 \times 0{\cdot}4\sqrt{5},$
$= 670{\cdot}8$ lb. wt.

Example 5. *A vertical rectangular area $ABCD$, with AB horizontal and DC below AB, is subject to the pressure of a homogeneous liquid at rest under gravity. E is the middle point of AB. Show that the ratio of the thrusts on the triangles CED and DEA lies between $2:1$ and $4:1$, depending on the depth of AB below the free surface of the liquid. Find the depth of AB when this ratio is $5:2$ and AD is 4 ft.* (L.U.)

Let h be the depth of AB below the free surface, a the length of AD and ρ the density of the liquid (Fig. 145).

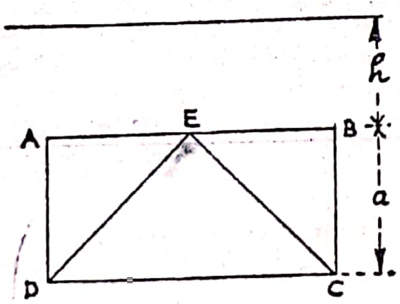

FIG. 145.

Area CED $= \dfrac{1}{2} a \, . \, AB$

area DEA $= \dfrac{1}{4} a \, . \, AB,$

depth of centroid of $CED = h + \dfrac{2}{3}a,$

depth of centroid of DEA $= h + \frac{1}{3}a,$

thrust on CED $= \frac{1}{2}\rho aAB\left(h + \frac{2}{3}a\right).$

thrust on DEA $= \frac{1}{4}\rho aAB\left(h + \frac{1}{3}a\right).$

ratio of thrusts $= \frac{2(3h + 2a)}{3h + a}.$

If $h = 0$ this ratio is $4:1$, if h is very large compared with a it is approximately $2:1$.

If the ratio is $5:2$ and $a = 4$ ft. we have

$$\frac{5}{2} = \frac{2(3h + 8)}{3h + 4},$$

and $h = 4$ ft.

Example 6. *The figure (Fig. 146) represents a mould for casting a metal hemisphere of diameter 4 ft., molten metal being poured into the mould through a small aperture at O. The metal weighs 480 lb./cu. ft.*

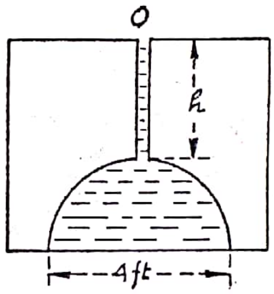

FIG. 146.

(i) *Find the upward thrust of the molten metal on the mould when h = 1.*
(ii) *If the mould weighs 8 tons, find the value of h when the mould is on the point of being lifted.* (L.U.)

When $h = 1,$
area of circular base $= 4\pi$ sq. ft.
depth below free surface $= 3$ ft.
Thrust on base $= 12\pi \times 480$ lb. wt.,
$= 5760\pi$ lb. wt.
Weight of metal $= \frac{2}{3}\pi \times 2^3 \times 480,$
$= 2560\pi$ lb. wt.
Difference $= 3200\pi$ lb. wt.
$= 4{\cdot}49$ tons wt.

This difference is the upward thrust of the metal on the mould.

If the upward thrust is 8 tons wt. we have

$$4\pi(h + 2) \times 480 - 2560\pi = 8 \times 2240,$$
$$h = 2{\cdot}3 \text{ ft.}$$

EXERCISES 10 (*b*)

1. Two liquids of specific gravities 1·2 and 0·84 are poured into the limbs of a U-tube until the differences in level of their upper surfaces is 9 in. Find the heights of these surfaces above the common surface and the pressure at the common surface.
2. A U-tube with open ends of 1 sq. in. cross-section whose vertical

branches rise to a height of 33 in., contains mercury in both branches to a height of 6·8 in. Find the greatest amount of water than can be poured into one of the branches, assuming the specific gravity of mercury to be 13·6. (L.U.)

3. A square lamina $ABCD$, of side 3 ft., is placed vertically in a homogeneous liquid with the edge AB in the surface. A straight line LM parallel to AB divides the square into two parts, so that the thrusts on the two parts are equal. Find the depth of LM. If E is a point in BC such that the thrust on the triangle ECD is one half of the thrust on the square, find the length EC. (L.U.)

4. A circular hole of 18 in. diameter is cut in the vertical side of a tank filled with liquid of specific gravity 1·1. The highest point of the hole is 27 in. below the free surface of the liquid. What force is needed to hold a plate on the side to cover the hole ?

5. A closed barrel with circular ends 30 in. in diameter and 3 ft. deep is filled with sea-water (sp. gr. 1·025) and then slowly tilted through an angle of 30°. Find the thrust of the water on the bottom of the barrel.

6. A closed cubical box of side 2 ft. is half full of oil (sp. gr. 1·3) and half full of water. It is tilted about one edge which is horizontal until the faces about this edge are inclined at 45° to the horizontal. Find the liquid thrust on one of the vertical faces of the box (i) if the oil and water are not mixed, (ii) if they are thoroughly mixed. (L.U.)

7. A rectangular area of height h is immersed vertically in water with one edge in the free surface. A horizontal line divides the area at a depth $2h/3$. Find the ratio of the thrusts on the two parts. (L.U.)

8. A cubical tank of 1 metre edge with its base horizontal has its lower half filled with water and its upper half filled with oil of specific gravity 0·8. Determine the total fluid thrusts on the upper half and on the lower half of one of its vertical sides. (L.U.)

9. A circular lamina of radius 6 in. is held in a vertical position with its centre in the plane of separation of two liquids, the upper layer being water of depth 12 in., and the lower layer oil (sp. gr. 1·2). Find the total liquid thrust on the lamina.

10. A rectangular lock gate 12 ft. wide can just bear a resultant force of 100 tons wt. If the depth of water on the lower side is 10 ft., to what depth can the water on the other side be allowed to rise ? (L.U.)

11. A vertical triangular area ABC with $AB = AC = 5$ ft., $BC = 6$ ft., has its vertex A in the free surface of a liquid of specific gravity 1·3 and its base BC horizontal. Find the thrust of the liquid. If it is lowered until A is at a depth of 4 ft. in the liquid, find the additional thrust.

12. A closed rectangular tank is 10 ft. high and stands on a base 4 ft. square. It is filled with water and tilted through an angle of 40° about one of the edges of the base. Find the thrust on the top and on the base in this position.

13. A pair of equal lock-gates are kept shut by the thrust of water 14 ft. deep. Taking the gates as rectangular and plane, each measuring 18 ft. vertical by 12 ft. horizontal, and supported by vertical hinges at their outer edges 20 ft. apart, show that the reaction between the gates across the line where they abut is equal to the resultant reaction on the hinges, and find its value. (L.U.)

14. A rectangular dam of a reservoir is 40 ft. wide and 10 ft. deep; calculate the thrust on the dam, taking 36 cu. ft. of water to weigh a ton, the surface of the water being 1 ft. below the top of the dam. (L.U.)

15. A reservoir is closed by a sluice-gate 7 ft. broad and 10 ft. high. Calculate in tons weight the pressure of the water upon it, assuming that a cubic foot of water weighs 1000 oz. and that the top of the sluice is in the surface of the water. (L.U.)

16. A square plate is immersed with an edge of length a in the surface of water; find the position of the two horizontal lines which divide the square into three rectangles, on each of which the total water pressure is the same. (L.U.)

17. A rectangular vessel contains three liquids which do not mix, of specific gravities 1·0, 1·2, 1·6, the thicknesses of which are 4, 3 and 2 in. respectively. Compare the total normal thrusts of the liquids on a side of the vessel. (L.U.)

10.10. Centre of Pressure

The centre of pressure of a plane area immersed in a liquid is the point at which the total thrust of the liquid on the area may be assumed to act. We are usually concerned with the distance of the centre of pressure from the line in which the plane of the area meets the free surface of the liquid, and we shall prove that for a uniform liquid this distance is the second moment of the area about the line divided by the first moment of the area about the same line.

Centre of Pressure of a Vertical Area

Consider a vertical plane area A completely immersed in a uniform liquid of density ρ, and let the plane in which the area lies meet the free surface of the liquid in the line XX' (Fig. 147). Let a small element of area δA be at a depth y below the free surface.

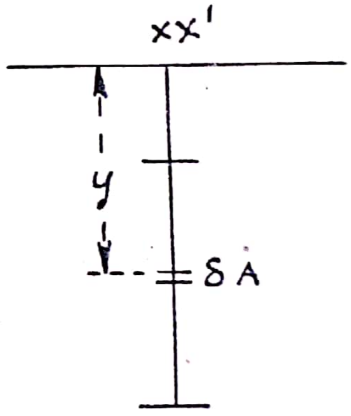

FIG. 147.

Then the thrust on the element δA is $\rho y \delta A$ and the total thrust on the area is $\Sigma \rho y \delta A$, where the summation is over the whole area. The moment of the thrust on the element δA about the line XX' is $\rho y^2 \delta A$, and the sum of the moments of the thrusts on all elements of the body is $\Sigma \rho y^2 \delta A$.

Hence, since the moment of the total thrust about XX' must be equal to this quantity, we have that the distance p of the centre of pressure from the line XX' is given by

$$p = \frac{\Sigma \rho y^2 \delta A}{\Sigma \rho y \delta A} = \frac{\Sigma y^2 \delta A}{\Sigma y \delta A},$$

$$= \frac{\text{the second moment of area about } XX'}{\text{the first moment of area about } XX'}.$$

If h be the distance of the centroid of the area from XX' and k its radius of gyration about XX' we have

$$p = \frac{Ak^2}{Ah},$$
$$= \frac{k^2}{h}.$$

Centre of Pressure of Inclined Area

Suppose the area considered is inclined at an angle θ to the horizontal (Fig. 148). As before, let the plane of the area meet the free surface of the liquid in the line XX' and let a small element of area δA be at a distance y from the line XX'. Then the thrust on an element δA is $\rho y \sin\theta\ \delta A$ and the total thrust on the area

$$= \Sigma \rho y \sin\theta\ A = \rho \sin\theta\ \Sigma y \delta A.$$

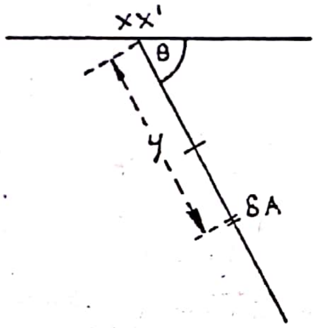

FIG. 148.

The moment of the thrust on the element δA about XX' is $\rho y^2 \sin\theta\ \delta A$ and the total moment $= \Sigma \rho y^2 \sin\theta\ \delta A$

$$= \rho \sin\theta\ \Sigma y^2 \delta A.$$

Hence, as before, the distance p of the centre of pressure of the area from XX' is given by

$$p = \frac{\rho \sin\theta\ \Sigma y^2 \delta A}{\rho \sin\theta\ \Sigma y \delta A} = \frac{k^2}{h}.$$

Hence, the position of the centre of pressure in the area is unaltered by turning the area about the line in which its plane meets the free surface of the liquid.

Distance from Centroid to Centre of Pressure

Let Ak_1^2 be the second moment of area about an axis parallel to XX' through the centroid. Then by the parallel axis theorem

$$Ak^2 = A(k_1^2 + h^2).$$

Therefore

$$p = \frac{k^2}{h} = \frac{k_1^2 + h^2}{h},$$
$$= \frac{k_1^2}{h} + h.$$

Hence, the distance of the centre of pressure from XX' is always greater than the distance of the centroid from XX', that is, the centre of pressure always lies below the centroid at a distance measured at right angles to XX' of k_1^2/h.

When the distance of the centre of pressure from XX' is known its position is in many cases evident. When the area has an axis of symmetry perpendicular to XX' the centre of pressure obviously lies on this axis.

If a triangular lamina has its base parallel to the line in which its plane cuts the free surface, the pressure on any thin strip of its area parallel to its base may be taken as acting at the mid-point of the strip, and hence the centre of pressure of the triangle must lie on the median from the mid-point of its base.

10.11. Standard Cases

Centre of Pressure of a Rectangle

Consider a rectangle $ABCD$ immersed in a uniform liquid with the side AB in the free surface (Fig. 149). Let $AB = a$, $BC = h$. The

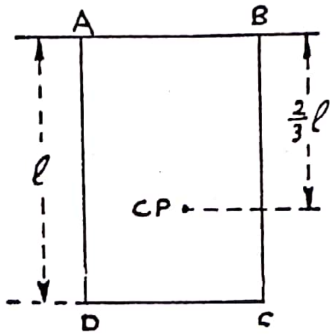

FIG. 149.

second moment of the rectangle about $AB = \dfrac{ah^3}{3}$ and its first moment about AB is $ah \times \dfrac{h}{2}$.

Hence $$p = \frac{2}{3}h.$$

Centre of Pressure of a Triangle

Consider a triangle ABC of height h with its base BC in the surface of the liquid (Fig. 150).

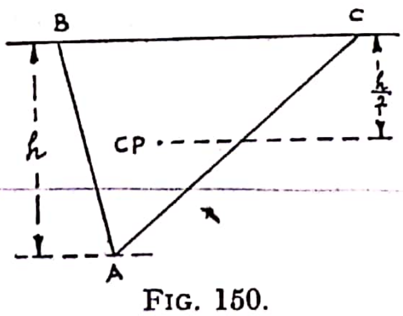

FIG. 150.

The second moment of the triangle about BC is $\dfrac{1}{2}h \,.\, BC \times \dfrac{h^2}{6}$ and the first moment about BC is $\dfrac{1}{2}h \,.\, BC \times \dfrac{h}{3}$.

Hence $$p = \frac{h}{2}.$$

(14)

If the vertex A be in the surface of the liquid and the base BC horizontal (Fig. 151), we have the second moment about a parallel to

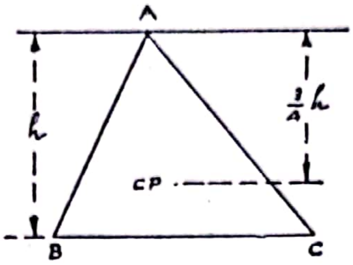

FIG. 151.

BC through A is $\frac{1}{2}h \cdot BC \times \frac{h^2}{2}$ and the first moment is $\frac{1}{2}h \cdot BC \times \frac{2h}{3}$.

Hence $$p = \frac{3h}{4}.$$

Centre of Pressure of Circular Area

Consider a circular area of radius a whose centre is distant h from the line XX' in which its plane meets the free surface (Fig. 152).

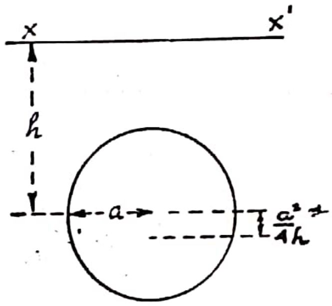

FIG. 152.

The second moment of the area about a diameter is $\pi a^2 \times \frac{a^2}{4}$, and hence about XX' $\pi a^2\left(\frac{a^2}{4} + h^2\right)$.

The first moment of the area about XX' is $\pi a^2 \times h$.

Hence $$p = \frac{h^2 + a^2/4}{h} = h + \frac{a^2}{4h}.$$

The centre of pressure is therefore at a distance $\frac{a^2}{4h}$ from the centre of the circle.

10.12. Increase of Depth

When the centre of pressure of a plane area has been found and the depth of the area is then increased, the increase in depth causes an equal additional pressure proportional to this increase at every point of the area. Hence the additional thrust due to the increase in depth may be taken as acting at the centroid of the area. By finding the position of the resultant of this additional thrust at the centroid and the original thrust at the centre of pressure, the new centre of pressure may be found.

Example 7. *A rectangle of sides a and b has its sides of length a vertical and its upper edge at a depth h below the surface of a liquid of density ρ. Find the distance of the centre of pressure from its centroid.*

Let A be the higher edge of the rectangle and B its centroid (Fig. 153).

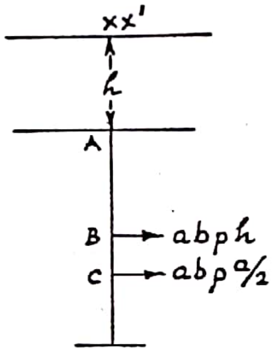

FIG. 153.

If the liquid just covered the rectangle the thrust would be $\frac{a^2b\rho}{2}$ at the centre of pressure C at $\frac{2}{3}a$ below A.

The additional head of liquid h causes a thrust $abh\rho$ at the centroid B.

The total thrust is $ab\left(h + \frac{a}{2}\right)\rho$ at a point x below B where

$$abh\rho\ x = \frac{a^2b\rho}{2}\left(\frac{a}{6} - x\right).$$

Therefore

$$x = \frac{a^2}{6(2h + a)}.$$

Example 8. *A lamina in the form of a regular hexagon is half immersed in a liquid, a diagonal being in the free surface. Prove that the centre of pressure of the immersed half is at a depth 5r/8, where r is the radius of the inscribed circle of the hexagon.* (L.U.)

The problem is not altered by taking the immersed portion of the hexagon as vertical. We have three equilateral triangles A, B, C, of area $\frac{r^2}{\sqrt{3}}$ and height r (Fig. 154).

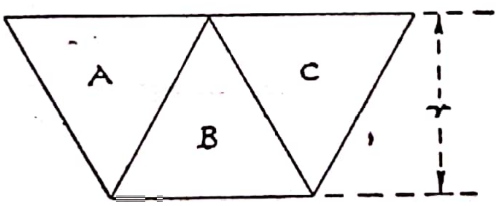

FIG. 154.

The triangles A and C have each a thrust $\frac{r^2}{\sqrt{3}} \times \frac{r}{3}\rho$ acting at a depth

$\frac{r}{2}$ below the surface. The triangle B has a thrust $\frac{r^2}{\sqrt{3}} \times \frac{2r}{3} \rho$ acting at a depth $\frac{3r}{4}$ below the surface.

Hence, the thrust on B is equal to the sum of the thrusts on A and C, and hence the total thrust is at a depth midway between the depths of the centres of pressure of A and B, that is, at a depth $\frac{1}{2}\left(\frac{r}{2} + \frac{3r}{4}\right) = \frac{5r}{8}$.

Example 9. *A circular manhole 2 ft. in diameter in the vertical side of a water tank has its centre at a depth 3·5 ft. below the surface of the water. It is covered by a circular plate held in position by bolts at the ends of its vertical diameter. Find the tensions in the bolts.*

The thrust on the plate is (Fig. 155).

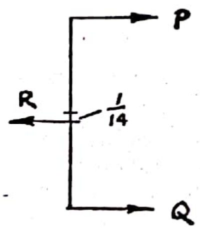

FIG. 155.

$$R = \pi \times 3{\cdot}5 \times 62{\cdot}5 \text{ lb. wt.}$$

acting at a depth

$$\frac{1}{4 \times 3{\cdot}5} = \frac{1}{14} \text{ ft.}$$

below the centre of the circle.

Forces P and Q at the end of the diameter must balance R and we have

$$P + Q = \pi \times 3{\cdot}5 \times 62{\cdot}5,$$

$$P\left(1 + \frac{1}{14}\right) = Q\left(1 - \frac{1}{14}\right),$$

$$P = 319 \text{ lb. wt.},$$

$$Q = 368 \text{ lb. wt.}$$

Example 10. *The depths of water on the two sides of a rectangular lock gate 12 ft. wide are 10 ft. and 4 ft. Find the resultant thrust on the gate and its turning moment about the bottom of the gate.*

On the deeper side the resultant thrust is

$$12 \times 10 \times 5 \times 62{\cdot}5 = 37{,}500 \text{ lb. wt.}$$

acting at a height $\frac{10}{3}$ ft. above the bottom.

On the other side the thrust is

$$12 \times 4 \times 2 \times 62{\cdot}5 = 6000 \text{ lb. wt.}$$

acting at a height $\frac{4}{3}$ ft. above the bottom.

Hence, the resultant thrust is 31,500 lb. wt. and the turning moment about the bottom is

$$37{,}500 \times \frac{10}{3} - 6000 \times \frac{4}{3} = 117{,}000 \text{ ft. lb. wt.}$$

(17)

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EXERCISES 10 (c)

1. The triangular lamina ABC is vertically immersed with the edge AB in the free surface and C at depth h. If D is the mid-point of BC, show that the depth of the centre of liquid pressure on the triangular portion ADC is $7h/12$, and find the ratio of the thrusts on ADC and ADB. (L.U.)
2. A triangle ABC, right-angled at C, is immersed vertically in water with AB in the free surface. It is then rotated about A until AC is vertical. Show that the depths of the centre of pressure on the triangle ABC in the two positions are in the ratio $2 \sin \alpha : 3$, where α is the angle BAC. (L.U.)
3. A square plate, of side a, is immersed vertically in water with one side in the surface. Calculate the ratio of the depths of the centres of liquid pressure on each of the two triangles into which the plate is divided by a diagonal. (L.U.)
4. $ABCD$ is a square lamina of side $2a$ from which the portion AEB is removed, where E is the mid-point of AD. The lamina $BCDE$ is immersed in liquid with B in the surface and BC vertical. Show that the depth of the centre of gravity of $BCDE$ is $11a/9$, and that the depth of the centre of liquid pressure on one side of $BCDE$ is $31a/22$. (L.U.)
5. ABC is a triangular lamina in which the angle A is a right-angle, $AB = AC$ and $BC = a$. It is immersed vertically in liquid with BC horizontal, and at a depth a below the surface, and with A below BC. Find the distance from A of the centre of liquid pressure on one side of the lamina. (L.U.)
6. The vertical end of a tank is a trapezium whose upper and lower horizontal sides are 12 ft. and 6 ft. respectively, and whose depth is 6 ft. Calculate the thrust on the vertical end of the tank and the depth of the centre of pressure, when the tank is filled with water. (L.U.)
7. A triangular lamina ABC, right-angled at A, is immersed in water with its plane vertical and B in the free surface. If $BC = a$ and the angle ACB is θ, find the depth of the centre of pressure of the triangle when BC is vertical. (L.U.)
8. $ABCD$ is a square lamina of side $2a$ and P is the mid-point of CD. The triangle CBP is removed, and the remaining lamina is immersed vertically in liquid with AB in the surface. Find the depth of the centre of liquid pressure. (L.U.)
9. A hollow cube of edge a with one face horizontal has its lower half filled with water, whilst the upper half is filled with liquid of specific gravity 0·9. If the liquids do not mix, show that the depth of the centre of liquid pressure on a vertical face of the cube is $149a/222$. (L.U.)
10. A square $ABCD$, of side a, is immersed vertically in liquid, with the side AB in the free surface. Show that, if P is a point in BC such that $BP = \frac{2}{3}a$, the liquid thrust on the triangle PCD is twice that on the triangle ABP. What is then the depth of the centre of pressure of the trapezium $APCD$? (L.U.)

(18)

11. A lamina $ABED$ has the shape of a square $ABCD$ of side a from which a triangular portion BCE has been removed, E being a point on CD. The lamina is immersed vertically in a uniform liquid with AB in the surface. If the depth of the centre of liquid pressure is $3a/5$, find the length of DE. (L.U.)

12. A layer of liquid of density ρ and depth a rests on top of a liquid of density 3ρ and depth greater than a. A square lamina of side $2a$ is immersed vertically with one edge in the free surface. Prove that the depth of the centre of liquid pressure is $13a/9$. (L.U.)

13. A, B, C, D are consecutive vertices of a regular hexagon of side a, and AD is a diagonal. A plane area $ABCD$ of this form is immersed vertically in liquid with BC in the surface. Show that the depth of the centre of liquid pressure is $7\sqrt{3}a/20$. (L.U.)

14. A rectangular area of height h is immersed vertically in water with one edge in the free surface. A horizontal line is drawn across the area at a depth $2h/3$. Find the thrusts on the two parts into which the area is divided, and show that the distances of the centres of pressure on the two parts from the line of division are in the ratio $4:5$. (L.U.)

15. A rectangular lock-gate 12 ft. wide can just bear a resultant force of 100 tons weight. If the depth of water on the lower side is 10 ft., to what depth can the water on the other side be allowed to rise? When this depth is as great as possible, at what height above the bottom of the gate does the resultant thrust act? (L.U.)

16. A cubical tank of 1 metre edge with its base horizontal has its lower half filled with water and its upper half filled with oil of specific gravity 0·8. Determine the total fluid thrusts on the upper and lower halves of one of the vertical sides and find the depth of the centre of pressure of the upper half. (L.U.)

17. The depths of water on the two sides of a rectangular lock-gate are h_1 and h_2. Prove that the resultant pressure on the gate acts at a point whose depth below the mean level of the surface is $(h_1^2 + h_2^2 + 4h_1h_2)/6(h_1 + h_2)$. (L.U.)

18. A trapezium is formed by cutting a square $ABCD$ of side a along the line joining A to the mid-point of BC, and is immersed in water with AD in the surface. Find the distances of the centre of pressure from AD and DC. (L.U.)

19. A cube of edge 3 ft. is full of water, and is placed with two faces vertical and four faces making 45° with the vertical. Find the thrust on one of the upper inclined faces, and the moment of this thrust about the highest edge of the cube. (L.U.)

20. A rectangular door in the vertical side of a reservoir can turn freely about its lower edge, and is fastened at its two upper corners. The door is 3 ft. wide and 6 ft. high, and its upper edge is 5 ft. below the water level. Determine the reactions at the upper corners, assuming them equal. (L.U.)

21. A dam 30 ft. high has a triangular cross-section. If the water side of the dam is vertical and the base is 20 ft. thick, determine the density of the material constituting the dam, given that the water surface is level with the top and that the resultant of the water pressure and weight of the dam passes through a point on the base 14 ft. from the water side. (Q.E.)

22. A rectangular barge 4 ft. deep by 6 ft. wide by 16 ft. long floats in

water and is loaded eccentrically so that along one side 3 ft. of the side is immersed, while along the other only 1 ft. is under water. Find (*a*) the magnitude of the total load on the barge, including its own weight, and (*b*) the magnitude and line of action of the resultant force due to water pressure on the bottom of the barge. (Q.E.)

F

CHAPTER 11

BUOYANCY

11.1. Principle of Archimedes

If a body be at rest wholly or partly immersed in a liquid, the resultant of the liquid pressures on the body is a vertical force equal to the weight of the liquid displaced by the body and acting upwards through the centre of gravity of the displaced liquid.

This resultant force is called the *force of buoyancy*, and the centre of gravity of the displaced liquid is called the *centre of buoyancy.*

The principle of Archimedes, which holds for gases as well as liquids, enables us to study floating bodies, to measure specific gravities and to find liquid thrusts on curved surfaces. The principle is established by considering the horizontal and vertical components of the thrust on a submerged surface.

Horizontal Component of Thrust on a Surface

The horizontal component of the thrust in any given direction on a submerged surface is equal to the thrust on the projection of the surface on a vertical plane perpendicular to the given direction, and acts through the centre of pressure of the projection.

Let ρ be the density of the liquid and δA a small element of the area of the surface at a depth y (Fig. 156). Let the element δA be considered as a plane area inclined at an angle θ to the plane of projection. The thrust on δA is $\rho y \delta A$ perpendicular to δA. The horizontal component in the given direction of this force is $\rho y \delta A \cos\theta$. But $\delta A \cos\theta$ is the area of the projection of δA on the perpendicular plane of projection. Hence the horizontal component of the thrust on δA is the same as the thrust on its projection; therefore the horizontal thrust on the whole surface is the same as that on its projection and acts through the centre of pressure of the projection. It follows that if a *body* is immersed in a liquid, it may be considered as two surfaces having the same horizontal projections on any vertical plane, and hence, the horizontal components of the thrusts on the two surfaces being equal and opposite, the total horizontal thrust on the body is zero.

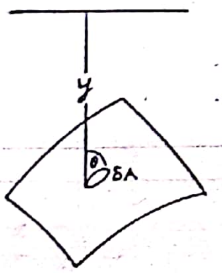

FIG. 156.

Vertical Component of Thrust on a Surface

The vertical component of the thrust on a submerged surface is equal to the weight of the liquid contained above the surface by the verticals through the boundary of the surface.

Let ρ be the density of the liquid and δA a small element of the surface at depth y inclined at an angle θ to the horizontal (Fig. 157).

δA being considered as a plane area, the thrust on the element will be $\rho y \delta A$ normal to δA, and the vertical component of this thrust will be $\rho y \delta A \cos\theta$. The projection of the area δA on a horizontal plane is $\delta A \cos\theta$, and hence $\rho y \delta A \cos\theta$ is the weight of the column of liquid which is vertically above δA. Hence, the vertical component of thrust on the whole surface is the sum of such columns, and is therefore the total weight of liquid above the surface.

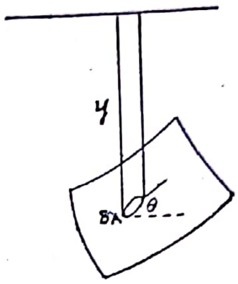

FIG. 157.

This is the downward thrust on the upper face of the surface. If the surface is of negligible thickness, the pressure at any point on its lower face will be equal and opposite to that at the same point on its upper face, and hence the vertical component of thrust on the lower face will be equal and opposite to that on the upper face.

Vertical Component of Thrust on a Body

A body may be considered as two surfaces, the dividing line being points at which vertical tangents can be drawn to the body (Fig. 158).

The downward thrust on the upper surface is equal to the weight of liquid above this surface; the upward thrust on the lower surface is equal to the weight of liquid above this surface. Hence the resultant thrust on the body is the difference between the weights of these quantities of liquid, that is, the weight of liquid that would fill the space between the two surfaces.

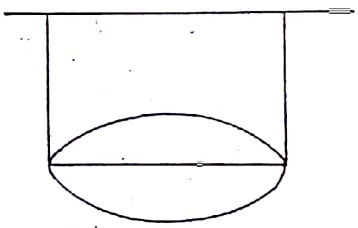
FIG. 158.

This is the force of buoyancy. This force, being the sum of the weights of small columns of liquid contained between the upper and lower surfaces, may be taken to act at the centre of gravity of this liquid, that is, at the centre of buoyancy.

If the body is not completely immersed, the upper surface must be taken as the section of the body by the free surface of the liquid, and the volume of liquid displaced is the volume of the body below this section.

11.2. Floating Bodies

If a body floats in a liquid, its weight acts downwards through its centre of gravity G, and the force of buoyancy acts upwards through the centre of buoyancy B (Fig. 159). Hence the force of buoyancy must be equal to the weight of the body, that is, the weight of the liquid displaced by the body must be equal to the weight of the whole body.

Also, the two forces must be in the same vertical line, that is, the centres of gravity of the body and of the liquid displaced must be in the same vertical line.

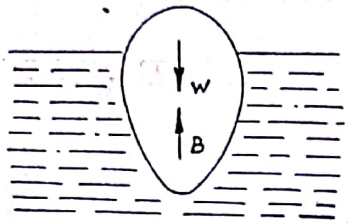

FIG. 159.

It follows that if a body of uniform material floats in water its specific gravity s must be less than unity. If V be the volume of the body and v the volume of water displaced, then $Vs = v$, and s is the fraction of the volume immersed.

Example 1. *A pontoon has vertical sides, a rectangular base* 60 *ft. long and* 12 *ft. wide and ends sloping at* 45° *to the horizontal. It floats in water with its base* 6 *in. below the surface. Find the weight of the pontoon. Find also how much it is immersed when a load of* 60 *tons is placed centrally on it.*

The weight of the pontoon (Fig. 160) is the weight of the volume of water

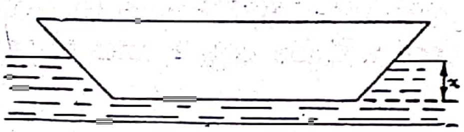

FIG. 160.

it displaces. A section of the displaced liquid parallel to a side is a trapezium of sides 60 and 61 ft. and width 6 in.

Therefore

volume displaced $= 12 \times 60{\cdot}5 \times \frac{1}{2} = 363$ cu. ft.

weight of water displaced $= 363 \times 62{\cdot}5$ lb.

$= 10{\cdot}1$ tons,

and this is the weight of the pontoon.

Let x ft. be the height of the free surface above the base when a load of 60 tons is added.

Total weight $= 70{\cdot}1$ tons,

volume displaced $= 12 \times (60 + x) \times x$,

weight of water displaced $= 12 \dfrac{(60 + x)x\ 62{\cdot}5}{2240}$ tons.

Therefore $x(x + 60) = \dfrac{70{\cdot}1 \times 2240}{12 \times 62{\cdot}5} = 209{\cdot}4$,

$x = 3{\cdot}3$ ft. approximately.

Example 2. *A beam* 4 *ft. long, of cross-section* 9 *in. by* 3 *in. and weighing* 30 *lb., floats vertically in water with a piece of iron (sp. gr.* 7·5) *attached to its lower end If the iron weighs* 10 *lb. what length of the beam is immersed ?*

Let x ft. of the beam be immersed.

Weight of water displaced by the beam $= x \times \frac{9}{12} \times \frac{3}{12} \times 62{\cdot}5$ lb. wt.

Weight of water displaced by the iron $= \frac{10}{7{\cdot}5}$ lb. wt.

Total weight of water displaced $= 40$ lb.

Therefore

$$x \times \frac{9}{12} \times \frac{3}{12} \times 62{\cdot}5 + \frac{10}{7{\cdot}5} = 40,$$

$$x = 3{\cdot}3$$

Example 3. *A 5-gm. weight is placed on the base of a conical piece of wood (sp. gr. 64/125) which floats with its base horizontal and vertex downwards partly immersed in water in a large vessel. The 5-gm. weight causes the circular base of the cone to sink to the level of the surface of the water. This weight is now removed and a brass weight (sp. gr. 8) attached to the apex under water. Find this weight if the cone sinks half as far as before.* (L.U.)

Let h cm. be the vertical height of the cone and α its semi-vertical angle.
Volume of cone $= \frac{1}{3}\pi h^3 \tan^2 \alpha$.
Let h_1 cm. be the length of the axis under water when the cone floats alone.
Volume immersed $= \frac{1}{3}\pi h_1{}^3 \tan^2 \alpha$.
This volume is $\frac{64}{125}$ of the whole,

therefore
$$h_1 = \tfrac{4}{5}h.$$

The additional volume of water displaced by the 5-gm. weight weighs 5 gm., that is,

$$\tfrac{61}{125} \times \tfrac{1}{3}\pi h^3 \tan^2 \alpha = 5 \qquad \ldots \ldots \quad (1)$$

Let w gm. be the weight of brass, the weight of water it displaces is $\frac{w}{8}$ gm. and the additional weight of water now displaced by the cone is

$$\tfrac{1}{3}\pi(\tfrac{9}{10}h)^3 \tan^2 \alpha - \tfrac{1}{3}\pi(\tfrac{4}{5}h)^3 \tan^2 \alpha.$$

Therefore
$$\tfrac{7}{8}w = \tfrac{1}{3}\pi h^3 \tan^2 \alpha\,(\tfrac{729}{1000} - \tfrac{64}{125}),$$
hence, using (1),
$$w = 2\tfrac{33}{61} \text{ gm.}$$

Flotation in Liquids of Different Densities

In establishing the principle of Archimedes it was not assumed that the liquid was of uniform density. Hence, we may use the principle to find the force of buoyancy when a body is immersed in liquid with layers of different densities, the liquid displaced in each layer being considered as having the density of the layer.

Example 4. *A cylinder of wood 12 in. long floats in water with its axis vertical and 10 in. of its length immersed. Oil of specific gravity 0·75 is poured on to the water until the top of the cylinder is in the oil surface. What is the depth of the layer of oil?*

Let x in. be the depth of the layer of oil, and A sq. ft. the cross-sectional area of the cylinder.

The specific gravity of the wood is $\frac{10}{12}$, therefore its weight is

$$\frac{10}{12}A \times 1 \times 62{\cdot}5 \text{ lb.}$$

Weight of oil displaced $\quad = 0{\cdot}75 \times A \times \frac{x}{12} \times 62{\cdot}5$ lb.

Weight of water displaced $\quad = A\left(\frac{12 - x}{12}\right) \times 62{\cdot}5$ lb.

Therefore
$$\frac{10}{12} = \frac{0{\cdot}75x}{12} + \frac{12 - x}{12},$$
$$x = 8 \text{ in.}$$

EXERCISES 11 (*a*)

1. A cubical lump of ice (sp. gr. 0·92) has embedded in it a piece of iron (sp. gr. 7·76) of mass 1 gm. The lump of ice floats in water and gradually melts, but retains its cubical form. Find the length of the edge of the cube when it sinks. (L.U.)
2. Find the weight of iron (sp. gr. 7·76) that would just be sufficient to submerge a block of wood (sp. gr. 0·72) of volume 250 cu. in. floating in water,

   (*a*) if the iron is on top of the wood and out of the water,
   (*b*) if the iron is attached to the underside of the block.
3. A hollow spherical ball is made of copper (sp. gr. 8·8), and its external diameter is 4 in. If it just floats in water completely immersed, find the thickness of the metal.
4. A closed cubical box whose edges are 3 ft. long weighs 200 lb. A piece of iron (sp. gr. 7·76) is to be attached to the mid-point of one edge so that the cube may float in sea-water (sp. gr. 1·025) with that edge 15 in. below the surface and horizontal. Find the weight of iron required.
5. A cubical block of wood weighing 50 lb. floats in water with three-quarters of its bulk immersed, and floats in oil just totally immersed when a weight of 10 lb. is placed on top of it clear of the oil. Find the specific gravity of the oil.
6. A piece of metal which weighs 10 kg. floats in mercury with 5/9ths of its volume immersed. Find the volume and density of the metal, assuming the specific gravity of the mercury to be 13·5. (L.U.)
7. A uniform solid cork cube floats in water with two faces vertical and with two-thirds of its surface area exposed. Find the specific gravity of the cork.
8. A cube of ice (sp. gr. 0·918) floats in sea-water (sp. gr. 1·026) with two faces horizontal and projects 1 cm. above the surface of the water. Find to what height it will project if transferred to float in fresh water. (L.U.)
9. A spherical piece of ice (sp. gr. 0·918) of radius 10 cm. has a lump of iron (sp. gr. 7·8) inside it and floats in water with 19/20ths of its volume immersed. Find the volume of the iron. (L.U.)
10. A square slab of wood, 2 in. thick, floats in water with three-quarters of its volume immersed. If the wood weighs 25 lb., what is the length of the side of the square? If the block is floating with its square faces horizontal and oil (sp. gr. 0·72) is poured on top of the water until the upper surface of the wood is just immersed, find the depth of the layer of oil.

11.3. Determination of Specific Gravity

The specific gravity of a solid is simply determined by weighing it in air and in water.

If W be the weight of the body and s its specific gravity, the weight of water displaced by the body is W/s. This is therefore the force of buoyancy acting on the body when it is immersed in water, and its apparent weight is then $W - W/s$.

Hence, the apparent loss of weight when the body is weighed in water is W/s. The weighing is, of course, done by suspending the body from the arm of a balance which is outside the water. If the body is weighed in a liquid of specific gravity s_1, the force of buoyancy is $\frac{W}{s} \times s_1$, and this is then the apparent loss of weight.

The specific gravity of a liquid is measured by means of a *hydrometer*. This is, essentially, a cylindrical body which floats in the liquid with its axis vertical. The length of the axis which is immersed depends on the specific gravity of the liquid, and the cylinder may be graduated so that the specific gravity of the liquid may be read off at the level of its free surface.

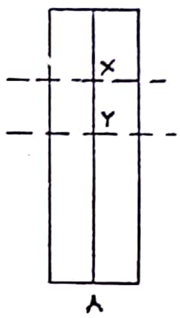

FIG. 161.

Let W be the weight of the cylinder, a its cross-sectional area and X and Y (Fig. 161) the levels of the free surface when it floats in water and in a liquid of specific gravity s respectively. Then if A be the lower end of the axis of the cylinder and ρ the density of water

$$a \, . \, AX \, . \, \rho = W,$$
$$a \, . \, AY \, . \, \rho s = W,$$
$$s = \frac{AX}{AY}.$$

In the usual type of hydrometer there is a portion of the body of different shape below the lower end of the cylinder. If V be the volume of this portion the above equations become

$$(V + aAX)\rho = W,$$
$$(V + aAY)\rho s = W,$$
$$s = \frac{V + a \, . \, AX}{V + a \, . \, AY},$$

and the cylinder is graduated accordingly.

Another hydrometer commonly used is known as Nicholson's hydrometer. This is made to float in water and in the liquid whose specific gravity is being measured with the same proportion of its volume immersed, by adding weights to a scale-pan which is above the surface.

If W be the weight of the hydrometer, V the volume immersed and w the weight to be added to keep the same volume immersed in liquid of specific gravity s, we have

$$V\rho = W,$$
$$V\rho s = W + w,$$
$$s = \frac{W + w}{W}.$$

Example 5. *A piece of wood weighs* 144 *gm. in air, and a piece of metal weighs* 36 *gm. in water. When fastened together the two weigh* 24 *gm. in water and* 8 *gm. in a solution of specific gravity* 1·1. *Find the specific gravities of the wood and the metal.* (L.U.)

Let s_1 and s_2 be the specific gravities of the wood and metal respectively, and let W gm. be the weight of the metal in air.

The weight of the two in water is 24 gm., therefore

$$144\left(1 - \frac{1}{s_1}\right) + 36 = 24,$$

$$s_1 = \frac{12}{13}.$$

The weight of the metal in water is 36 gm., therefore

$$W\left(1 - \frac{1}{s_2}\right) = 36.$$

The weight of the two in liquid of specific gravity 1·1 is 8 gm., therefore

$$144\left(1 - \frac{1{\cdot}1 \times 13}{12}\right) + W\left(1 - \frac{1{\cdot}1}{s_2}\right) = 8.$$

Hence, eliminating W we have

$$s_2 = 10.$$

EXERCISES 11 (*b*)

1. A piece of an alloy weighing 96 gm. is composed of two metals whose specific gravities are 11·4 and 7·4. If the weight of the alloy is 86 gm. in water, find the weight of each metal in the alloy. (L.U.)
2. A piece of metal weighs 11·4 gm. in air and 9·6 gm. in a liquid of specific gravity 1·2. Calculate the specific gravity of the metal. A piece of wood weighs 1·8 gm. in air. When the metal is attached to the wood the two together weigh 9 gm. in the liquid. Calculate the specific gravity of the wood. (L.U.)
3. A body weighing 12·4 gm. appears to weigh 2·4 gm. in water. It is then weighed in a mixture of a liquid *A* of specific gravity 0·8 and water, and appears to weigh 3·6 gm. Find how many c.c. of the liquid *A* are added to 10 c.c. of water to form the mixture. (L.U.)
4. A piece of a material *A* weighs 30·4 gm. in air and 26·4 gm. in water; a piece of a material *B* weighs 67·2 gm. in air and 61·2 gm. in water. A piece of an alloy formed from *A* and *B* weighs 54·6 gm. in air and 48·6 gm. in water. Find the volume of each material in the piece of alloy, assuming there to have been no diminution of volume when the alloy was made. (L.U.)
5. A cubical block of wood (sp. gr. 0·83) contains inside it a piece of lead (sp. gr. 11·35). It floats in water with one-tenth of its volume above the surface, and can just be immersed by putting a weight of 100 gm. on its upper surface. Calculate the length of an edge of the cube and the volume of the piece of lead. (L.U.)
6. A uniform rod has a weight attached to one end to make it float upright in liquid. If 3 in. of the rod is immersed when it floats in water and 3·5 in. when it floats in liquid of specific gravity 0·9, what length of it will be immersed when it floats in liquid of specific gravity 1·2? (L.U.)
7. A bottle with a long straight neck floats vertically upright in water, part of the neck alone emerging. It is then placed in a solution of specific gravity 1·2, and 3 in. more of the neck are above the surface. If the neck be circular, 1 in. in diameter, find the weight of the bottle. (L.U.)

8. A piece of glass weighs 5 gm. in air, 3·2 gm. in water, and 1·7 gm. in sulphuric acid. Find the specific gravity of the glass and of the sulphuric acid. (L.U.)
9. A mixture of two uniform substances weighs 25 gm. in water and 28 gm. in liquid of specific gravity 0·8. A mass of one of the substances of equal weight with the mixture weighs 30 gm. in the second liquid, and a mass of the other substance of equal weight with the mixture weighs 15 gm. in water. Find the specific gravity of each substance and their volumes in the mixture. (L.U.)
10. A nugget of quartz and gold weighs 13 oz. in air and 9 oz. in water; taking the specific gravities of quartz and gold to be 2·6 and 19·5 respectively, find the weight of gold in the nugget. (L.U.)

11.4. Equilibrium of a Body in a Liquid

If a body is held in any way in a liquid, the general conditions of equilibrium may be applied by resolving and taking moments of all the forces including the force of buoyancy acting at the centre of buoyancy.

Example 6. *A beam whose cross-section is a rectangle is held horizontally with a diagonal of its cross-section in the surface of water by a vertical rope attached to the mid-point of its lowest edge. Find the specific gravity of the beam.*

Consider the central cross-section of the beam. Let C be the mid-point of the horizontal diagonal, A the point of the lowest edge and B the centroid of the immersed triangle (Fig. 162).

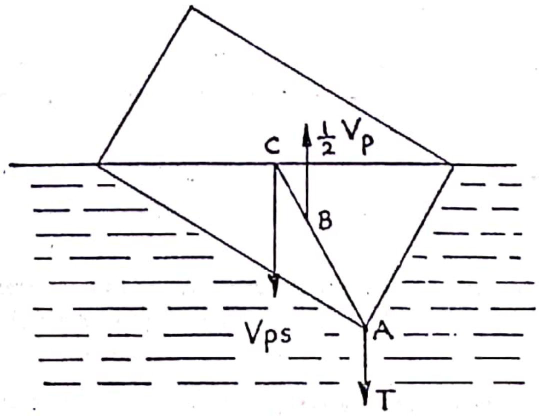

FIG. 162.

Let V be the volume of the block, s its specific gravity and ρ the density of water.

Then the weight of the block is $V\rho s$ and the force of buoyancy is $\frac{1}{2}V\rho$ acting at B.

Since $AB = \frac{2}{3}AC$, we have, by equating the moments of the weight and the buoyancy about A,

$$V\rho s = \frac{2}{3} \times \frac{1}{2} V\rho.$$

Therefore,

$$s = \frac{1}{3}.$$

Example 7. *A uniform rod of specific gravity s is free to turn about its lower end, which is fixed at a depth h in water. Prove that the rod can float in an inclined position provided $sl^2 > h^2$, where l is the length of the rod. Determine s if the rod floats with half of its length immersed.* (L.U.)

Let x be the length of the rod immersed, a its cross-sectional area and ρ the density of water (Fig. 163).

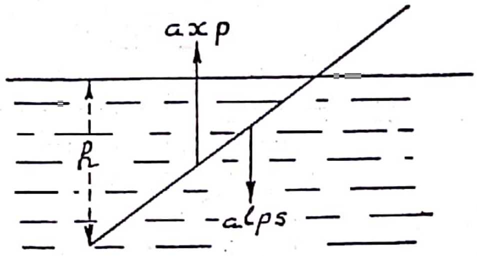

FIG. 163.

The weight of the rod $= al\rho s$,
the force of buoyancy $= ax\rho$.

These forces act at distances $l/2$ and $x/2$ respectively from the hinge, therefore, taking moments about the hinge we have

$$\frac{x}{2} \times ax\rho = \frac{l}{2} \times al\rho s,$$
$$x^2 = l^2 s,$$

x must be greater than h, and hence $l^2 s > h^2$.
Also, if $x = l/2$ we have $\qquad s = \frac{1}{4}$.

EXERCISES 11 (c)

1. ABC is a triangular lamina, the angle A being a right angle and the angle B 30°. It can turn freely in its own plane, which is vertical, about A, which is fixed in the surface of water. If it is in equilibrium with C immersed and AC making an angle of 60° with the horizontal, show that the specific gravity of the material of the lamina is $\frac{3}{8}$. (L.U.)
2. A solid lighter than water is held completely immersed in a vessel containing 100 c.c. of water by means of a string fastened to a point in the base of the vessel, and the tension in the string is 1·6 gm. wt. When 60 c.c. of a second liquid of specific gravity 1·2 are added and thoroughly mixed with the water, the tension in the string is 2·2 gm. wt. Find the specific gravity and the weight of the solid. (L.U.)
3. A bucket containing water is suspended by a cord which passes over a smooth pulley, the pulley being sufficiently small for the other end of the cord, to which is attached a ball, to hang inside the bucket. If W be the weight of the water and $s(> 2)$ the specific gravity of the ball, prove that equilibrium is possible if the weight of the ball lies between W and $Ws/(s - 2)$. (L.U.)
4. A uniform thin rod AB of length 3 ft. is freely hinged to a point A 1 ft. above the surface of a tank of deep water. The rod is in equilibrium when partially immersed and making an angle of 60° with the vertical. Find its specific gravity. (L.U.)
5. A uniform rod is free to rotate about a fixed horizontal axis through its upper end. The lower end dips into water and rests in an inclined position with one-third of its length immersed. Show that the specific gravity of the rod is $\frac{5}{9}$.

A piece of metal (sp. gr. 4·6) is now attached to the lower end of the rod, and the system rests with one-half of the length of the rod immersed. Find the ratio of the volume of the metal to the volume of the rod. (L.U.)

6. Two solid uniform spheres, each of radius 4 in., are connected by a light string and totally immersed in a tank of water. If the specific gravities of the spheres are 0·4 and 1·8, find the tension in the string and the pressure between the bottom of the tank and the heavier sphere. (L.U.)

7. A buoy is in the form of a right circular cone of height 12 ft. and base diameter 9 ft. The vertex is fastened to the bottom of the sea by a chain, and it floats at low tide with the chain slack, the axis of the cone vertical and two-thirds of it immersed. Find the weight of the buoy, assuming that the weight of the chain is negligible and that the density of the water is 64 lb./ft.^{3}. Calculate also the pull on the chain at high tide when the buoy is completely immersed. (L.U.)

8. A thin uniform rod of length $2a$ and specific gravity $\frac{3}{4}$ is hinged at one end to a point at a height $a/2$ above the surface of water, with the other end immersed. Find the inclined position of equilibrium. (L.U.)

9. The cross-section of a uniform prism is a rectangle with unequal sides, and it floats in water with two opposite edges in the surface, supported by a string attached to the edge which is out of the water. Show that the specific gravity of the prism is $\frac{2}{3}$, that the string is vertical, and the tension in it is one-fourth of the weight of the prism. (L.U.)

10. A uniform thin rod floats in water in an inclined position with one-half of its length immersed, the upper end of the rod being supported by a string. Prove that the string is vertical and that the density of the rod must be three-quarters of that of the water. (L.U.)

11.5. Thrust on a Curved Surface

Body Immersed in Liquid

When a body is immersed in a liquid, the force of buoyancy is the *resultant* of the liquid thrusts on the surfaces of the body. Hence, if the body has a plane surface and a curved surface, and the force of buoyancy and the thrust on the plane surface are known, the thrust on the curved surface is easily found. This is not a question of the equilibrium of the body as a whole, but of using the fact that the force of buoyancy is caused by the liquid thrusts on the surfaces of the body.

Example 8. *A quadrant of a solid sphere of radius a is held with one of its plane semicircular faces in the surface of water. Find the thrust on the curved surface of the quadrant.*

Assuming that there is no thrust on the horizontal face, let P be the thrust on the vertical face and B the force of buoyancy (Fig. 164).

Then if ρ be the density of water

$$P = \frac{1}{2}\pi a^2 \times \frac{4a}{3\pi} \times \rho = \frac{2}{3} a^3 \rho.$$

$$B = \frac{1}{4} \times \frac{4}{3}\pi a^3 \times \rho = \frac{\pi}{3} a^3 \rho.$$

Let X and Y be the horizontal and vertical components of the thrust on the curved surface. Then the force B is the *resultant* of the forces P, X, Y.

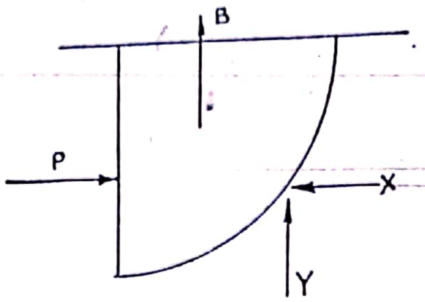

FIG. 164.

Therefore

$$X = P = \frac{2}{3} a^3 \rho,$$

$$Y = B = \frac{\pi}{3} a^3 \rho,$$

$$\sqrt{X^2 + Y^2} = \frac{1}{3}\sqrt{\pi^2 + 4} \, . \, a^3 \rho.$$

Since the pressure at any point of the curved surface is normal to the surface, the resultant thrust must pass through the centre of the sphere and its inclination to the horizontal is $\tan^{-1}\frac{Y}{X} = \tan^{-1} \pi/2$.

Body Containing Liquid

When a hollow body contains liquid, the liquid exerts thrusts on the plane and curved surfaces of the body. The liquid is in equilibrium under the action of its weight and the thrusts on the liquid exerted by the surfaces of the body, which are equal and opposite to the thrusts of the liquid on the body. This is equivalent to saying that the weight of the liquid is equal to the *resultant* of the thrust of the liquid on the plane and curved surfaces of the body. If therefore the weight of the liquid and the thrust on the plane surface of the body are known the thrust on the curved surface may be found.

Example 9. *A hollow right circular cone of height h and semi-vertical angle α is filled with liquid of density ρ and rests with a generator in contact with a horizontal plane. Find the thrust of the liquid on the curved surface of the cone.*

Let P be the liquid thrust on the base of the cone, X and Y the horizontal and vertical components of the liquid thrust on the curved surface and

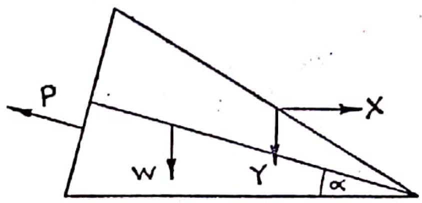

FIG. 165.

W the weight of the liquid (Fig. 165). Then W is equal to the resultant of the forces P, X and Y.

$$W = \frac{1}{3} \pi h^3 \tan^2 \alpha \, . \, \rho.$$

The depth of the centroid of the base below the free surface of the liquid, which is assumed to be at the highest point of the cone, is $h \sin \alpha$.

Hence
$$P = \pi h^2 \tan^2 \alpha \times h \sin \alpha \times \rho,$$
and this force is inclined at an angle α to the horizontal.

Therefore
$$X = P \cos \alpha = \pi\rho h^3 \tan^2 \alpha \sin \alpha \cos \alpha,$$
$$Y = W + P \sin \alpha = \pi\rho h^3 \tan^2 \alpha\left(\frac{1}{3} + \sin^2 \alpha\right),$$
$$\sqrt{(X^2 + Y^2)} = \frac{1}{3}\pi\rho h^3 \tan^2 \alpha\sqrt{(1 + 15 \sin^2 \alpha)},$$
$$= W\sqrt{(1 + 15 \sin^2 \alpha)}.$$

EXERCISES 11 (*d*)

1. A closed vessel of thin sheet metal consists of a right circular cylinder of radius a and height a closed at one end, and having the rim of the other end soldered to the rim of the base of a circular cone of radius a and height a. It is held with its axis vertical and conical part uppermost, and filled with water through a small orifice at the vertex of the cone. If the orifice is then closed and the vessel inverted, find the ratio of the thrusts on the curved conical surface in the two positions. (L.U.)
2. A thin hollow cone of base radius r and height h, just filled with liquid of density w, is fixed with its axis horizontal. Find the magnitude of the thrust on the curved surface due to the liquid. (L.U.)
3. A closed circular cylinder is completely immersed in water with its centre at a depth h and its axis inclined at an angle θ to the vertical. Calculate the resultant liquid thrust on the curved surface of the cylinder in terms of the weight W of water displaced by the cylinder. (L.U.)
4. A closed hollow hemisphere is completely filled with liquid. Compare the magnitude of the liquid thrusts on its curved surface in the three positions (i) with its plane face horizontal and uppermost, (ii) with its plane face horizontal and lowermost, (iii) with its plane face vertical. (L.U.)
5. A conical vessel contains enough fluid to fill it to a depth equal to half the depth of the vessel when the vertex is downwards. If the vessel is inverted, show that the resultant thrust of the curved surface is altered in the ratio $(23 - 12 \times 7^{\frac{1}{3}}) : 1$.
6. A closed vessel in the form of a right circular cone is placed on its flat base, and liquid is poured in through a hole at the top until the depth of liquid is one-third of the height of the cone. If W is the weight of liquid that would fill the cone, show that the pressures on the base and curved surface are respectively W and $8W/27$. (L.U.)

EXERCISES 11 (*e*)

1. A uniform cube, of edge a and weight W, floats partly immersed in a liquid of density ρ with a pair of opposite faces horizontal. A load w evenly distributed along one edge of the upper face tilts the cube until that edge is in the free surface of the liquid. Show that the

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inclination θ of the upper face of the cube to the horizontal is given by $\tan\theta = 2\left(1 - \frac{W + w}{\rho a^3}\right)$. (L.U.)

2. A uniform solid cone of volume V and density ρ_1 is held completely immersed with its axis vertical in a liquid of density $\rho_2 (> \rho_1)$ by means of a string attached to its vertex and to the base of the vessel containing the liquid. Find the tension in the string.

   The liquid slowly drains out of the vessel, and when the string becomes slack two-thirds of the axis of the cone is still immersed. Find the ratio ρ_1/ρ_2. (L.U.)

3. A uniform cube of side a and specific gravity σ floats in water with two faces horizontal. When a second liquid of specific gravity $\sigma_1 (< 1)$ is poured on to the water to a depth b the upper face of the block is in the free surface. Find the value of σ in terms of σ_1, a and b. Show also that when the depth of the liquid is $b/2$ the height of the upper face above the free surface is $b(1 - \sigma_1)/2$. (L.U.)

4. A solid hemisphere of radius 10 cm. is held with its circular base vertical and its centre in the plane of separation of two layers of liquid, the upper layer of depth 10 cm. being water and the lower layer liquid of specific gravity 1·2. Find the total thrust on the vertical plane and the magnitude and direction of the resultant thrust on the curved surface of the hemisphere. (L.U.)

5. Two equal light hemispherical shells can be fitted together to form a sphere which is water-tight on closing two small catches at A and B, the opposite ends of a diameter of the common rim. If the sphere is placed with the point A resting on a horizontal table and is filled with a weight W of water, find the least pair of equal and opposite forces that must be applied at B to prevent the hemispheres from separating when the catch B is released. (L.U.)

6. A sea-wall slopes from the bottom at 30° to the horizontal for 30 ft., and is then continued vertically upwards. Find the resultant horizontal and vertical pressures on it in tons weight per yard of its length when there is a depth of 15 ft. of water (density of sea water 64 lb./ft.^{3}). (L.U.)

7. A closed hemispherical bowl of radius 6 in. is filled with water. It is held with its plane face making an angle of 50° with the horizontal and the curved surface above this face. Find the magnitude and direction of the thrust on the curved surface.

8. A trough whose cross-section is a triangle with vertex downwards is partly filled with water of weight W. A block of wood of weight W' is then introduced and floats on the water. Show that the pressure on either of the vertical plane ends of the trough is increased in the ratio $(1 + W'/W)^{3/2}$. (L.U.)

9. A cylinder of specific gravity 0·45 floats in water with its axis vertical. If its height be 12 in. and radius 2 in., find what volume of lead (sp. gr. 11·4) must be suspended from the centre of its lower face to make it sink so that only 1 in. emerges from the water. (L.U.)

10. The cross-section of a block of wood is a trapezium $ABCD$. AB and CD are the parallel sides 10 inches apart, $AB = 16$ in. $CD = 6$ in., $AD = BC$. The specific gravity of the wood is $\frac{3}{5}$, and the block floats with the face containing AB immersed and horizontal. Show that the depth of AB is about 4·86 in. (L.U.)

11. A ship with sides practically vertical near the water-line is observed to sink 13·8 in. deeper when loaded with 400 tons of cargo in sea-water. Find the water-line area, taking the density of sea-water as 64 lb. per cubic foot. The displacement of the ship is 4500 tons; find how far it sinks when passing from salt water to fresh water. (L.U.)

12. A buoy in the form of a hollow spherical shell, of external diameter 3 ft. and uniform thickness, floats in water with half of its volume immersed. Find the thickness of the shell if the material is of specific gravity 8. (L.U.)

13. An iron cylinder of diameter 25 cm. and height 10 cm. stands in a cylindrical jar of diameter 30 cm. Mercury is now poured into the jar until the iron begins to float. Calculate the depth of the mercury. Oil is now poured on top of the mercury until the upper face of the cylinder is just level with the surface of the oil. Find the volume of oil required and the height of the base of the cylinder above the bottom of the jar. Take the specific gravities of iron, mercury and oil to be 7·9, 13·6 and 0·9 respectively. (L.U.)

14. A rectangular barge is 60 ft. long and 15 ft. broad, and when unloaded weighs 30 tons. It is floating in water of density 62·5 lb./cu. ft. Find what added load will increase the draught of the barge so that it is uniformly 3 ft. What is then the resultant force arising from water-pressure on one of the long sides of the barge, and where does it act? (Q.E.)

15. A cylindrical vessel, 6 in. in diameter and 12 in. deep, is filled with water to a depth of 9 in. A steel cylinder, of diameter 3 in. and height 4 in., is then suspended in the water with its axis vertical and its lower end 1 in. from the bottom of the vessel; the wire is attached to the top face of the cylinder. What is (*a*) the tension in the wire, (*b*) the increase in the force exerted by the water on the bottom of the vessel, and (*c*) the tension in the cylinder across a horizontal section through its centre of gravity?

Density of steel 0·283 lb./cub. in. Specific gravity of steel 7·8. (Q.E.)