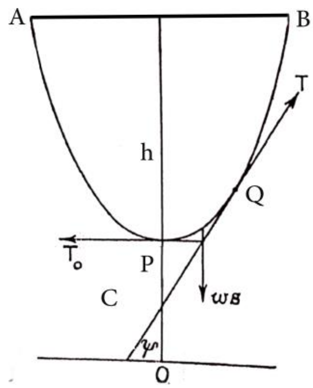
Chapter (1)

The Catenary

Definition(The common Catenary):

The catenary is the curve in which a uniform chain or string hangs when freely suspended from two points$A \& B$



Denote the tension at the lowest point y T_0 ,this will be horizontal. Let$s$ be the length of chain measured from P to any point Q .Let the tension at Qp be T and let its inclination to the horizontal be ψ.

Let the weight per unit length of the chain be ω.

The part of the chain PQ will be in equilibrium under the action of three forces, its weight ωs ,T_0,and T , the tensions at P and Q .

The intrinsic Equation of the catenary:

Resolving vertically and horizontally weget,

$$T \ \sin\psi = \omega s \qquad , \qquad T\cos\psi = T_0 \qquad 1.1$$

For convenience we introduce another constant c ,which is such that $T_{0=}\omega c$. Then

$$T\ \sin\psi=\omega s \qquad , \qquad T\cos\psi=\omega c$$

Dividing

$$s=c\ \tan\psi \qquad\qquad (i)$$

This is the intrinsic equation of the curve, (c is called the parameter of the catenary).

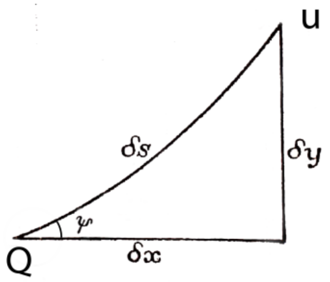
<u>The cartesian Equation of the catenary</u>:

To find the Cartesian equation of the curve we flow:

Since $\tan\psi=\frac{dy}{dx}$, then from $(i)\ \frac{dy}{dx}=\frac{s}{c}$

Consider a small element δs of a curve joining two points Q and U on the curve. Let the coordinates of Q and U be $(x,y)\&(x+\delta x,y+\delta y)$respectively. Then

$$(\delta s)^2\cong(\delta x)^2+(\delta y)^2$$



Dividing by $(\delta x)^2$ then $(\delta y)^2$ respectively we get:

$$\left(\frac{\delta s}{\delta x}\right)^2\cong 1+\left(\frac{\delta y}{\delta x}\right)^2$$

and

$$\left(\frac{\delta s}{\delta y}\right)^2 \cong \left(\frac{\delta x}{\delta y}\right)^2 + 1$$

When $\delta s, \delta x, \delta y \to 0$, the above equations becomes

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 \quad (ii)$$

and

$$\left(\frac{ds}{dy}\right)^2 = \left(\frac{dx}{dy}\right)^2 + 1 \qquad (iii)$$

(ii) gives

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{s}{c}\right)^2$$

$$\therefore \quad \frac{ds}{dx} = \frac{\sqrt{(c^2 + s^2)}}{c}$$

$$\therefore \quad dx = \frac{c\ ds}{\sqrt{(c^2 + s^2)}}$$

$$\therefore \quad x = c\ sinh^{-1}\ \frac{s}{c} \qquad (iv)$$

or $$s = c \sinh \frac{x}{c} \qquad (v)$$

provided $x = 0$ when $s = 0$.

(iii) gives

$$\left(\frac{ds}{dy}\right)^2 = 1 + \left(\frac{c}{s}\right)^2$$

$$\therefore \quad \frac{ds}{dy} = \frac{\sqrt{(c^2 + s^2)}}{s}$$

$$\therefore \; dy = \frac{s\,ds}{\sqrt{(c^2 + s^2)}}$$

$$\therefore \; y = \sqrt{(c^2 + s^2)}$$

i.e. $$y^2 = s^2 + c^2 \qquad (vi)$$

provided $y = c$ when $s = 0 \& x = 0$

Substituting from (v) in (vi)

. $$y^2 = c^2\left(1 + sinh^2\frac{x}{c}\right)$$

$$= c^2 cosh^2(\frac{x}{c})$$

$$\therefore \quad y = c\cosh\left(\frac{x}{c}\right) \qquad (viii)$$

This is the Cartesian equation of the catenary.

<u>The tension at any point</u>:

Since

$$T\ \sin\psi = \omega s \qquad , \qquad T\cos\psi = \omega c$$

then $T^2 = \omega^2(s^2 + c^2)$

which from (vi) gives

$$T^2 = \omega^2 y^2$$

$$\therefore \quad T = \omega y$$

Thus,the tension at any point of the catenary is proportional to the height of the point above the x −axis which is usually called the (directric).

The Lightning and telephone wires:

When c is large, from equation (vii),

$$\therefore \quad y = c\cosh\left(\frac{x}{c}\right) = \frac{C}{2}\left(e^{x/c} + e^{-x/c}\right)$$

$$= s = \frac{c}{2}\left\{1 + \frac{x}{c} + \frac{x^2}{2c^2} + \ + \cdots + (1 - \frac{x}{c} + \frac{x^2}{2c^2} - \cdots)\right\}$$

$$= c + \frac{x^2}{2c} + \cdots$$

i.e. $y - c \cong \frac{x^2}{2c}$ $\quad (X)$ provided c is large.

In this case the curve is approximately a parabola of latus rectum $2c$

Definition (the span): The span is distance AB ,ie. the distancebetween the two hangs points $A\&B$.

If $k$is half the span, half the length of the chain is given by:

$$s = \frac{c}{2}\left\{1 + \frac{k}{c} + \frac{k^2}{2c^2} + \frac{k^3}{6c^3} + \cdots - (1 - \frac{k}{c} + \frac{k^2}{2c^2} - \frac{k^3}{6c^3} \ldots)\right\}$$

$$= \frac{c}{2}\left\{\frac{2k}{c} + \frac{k^3}{3c^3} + \cdots\right\}$$

$= k + \frac{k^3}{6c^2}$ provided c is large.

$$\therefore \quad s - k = \frac{k^3}{6c^2} \qquad (Xi)$$

<u>Definition (the sag)</u>: The sag is the difference between the coordinates of y at values of$x$for the two point$P \& B$.Or the normal distance from the lowest point P to the span line AB.

<u>The Relation between the span and sag</u>:

If h is the sag, then for $x = 0$, $y = c$ and $x = k$, $y \cong c + \frac{k^2}{2c}$ those come from (X) .Her we can get:

$$h = \frac{k^2}{2c} \qquad (*)$$

this leads to $1/c^2 = 4h^2/k^4$

then from (Xi) we have:

$$s - k = k^3/6c^2 = (k^3/6).(1/c^2) = (k^3/6).(4h^2/k^4) = 4h^2/6k$$

$$\therefore \quad 2(s - k) = (8/3)\,.(h^2/2k)$$

this means that the difference between the length of the chain $2s$ and

the span $2k$ is equal to $2(s-k)=(8/3)\,.((sag)^2/span).$ $(**)$

The equations (*)&(**) clarify two relations between the span and sag for the catenary.

Note: when c is large as mentioned above the chain or wire represents

the Lightning and telephone wires. In this case the length of the wire

$2s$ is little bigger than the span AB .So also the sag$h$ will be small.

Examples

Many problems involving catenary cables can be solved using the following formulas:

$s=c\sinh\left(\frac{x}{c}\right)$ (i) $x=c\ sinh^{-1}\left(\frac{s}{c}\right)$ (ii)

$y^2-s^2=c^2$ (iii) $y=c\ cosh\left(\frac{x}{c}\right)$ (iv)

$T_0=\omega c$ $(v\ )$ $T=\omega y$ (vi)

$W=\omega s$ (vii)

All the parameters in the above equations have been defined before.

Example (1):

an electric power of line length $140\ m$and mass per unit length of $3\ kg/m$ is to be suspended between two towers $120\ m$ apart and of the same height. Determine the sag and maximum tension in the power line.

The solution

The sag, h, can be found from Eq(iii),provided that wecan determinethe distance, c

$y_B^2 - s_B^2 = \mathrm{c}^2$ (Eq(iii)evaluated at point B)

or

$$(h + c)^2 - (70\ m)^2 = c^2 \qquad (1)$$

The distance c ,can be determined from Eq.(i) :

$s_B = c\ \sinh(\frac{x_B}{c})$ (Eq(i)evaluated at point B)

or $\quad 70\ m = c\ \sinh\left(\frac{60\ m}{c}\right) \qquad (2)$

This equation must be solved numerically for c. An initialestimate for

c, whenthesolver on acalculator isto beused, could be

$$c = s_B = 70m$$

The solution to Eq.(2) is

$$c = 61.45\ m$$

Another possible solution is $c = -61.45\ m$,but this has no physical meaning. You can get the same result directly by using a modern calculator like($casio\ \ f_x - 991ES\ PLUS$).

$$(h + 61.45\ m)^2 - (70\ m)^2 = (61.45\ m)^2$$

Solving gives the sag:

$$h = 31.70\ m$$

The other negative root has no physical meaning.

The maximum tension, T_{max}, occurs where the cable has its steepest slope, point B (or point A). This can be calculated from Eq.(vi) :

$T_{max} = \omega y_B$(Eq(vi)evaluated at point B)

ω is given, then:

$$T_{max} = [(3\text{kg/m})(9.81\ \ \text{m/s}^2)][31.70\text{m} + 61.45\text{m}]$$

$$= 2740\ N = 2.74\ KN$$

Example (2):

A cable is supported at two points 400 ft apart and at the same elevation. If the sag is 40 ft and the weight per unit length of the cable is 4 lb/ft, determine the length of the cable and the tension at the low point, C.

The solution

The length of cable, s_B, from the low point to point B can be found from Eq. (i) provided that we can determine the distance c:

s_B = c sinh (x_B/c) (Eq. (i) evaluated at point B)

$= \text{c} \sinh (200/\text{c})$ (1)

The distance c can be determined from Eq.(iv)

$y_B = c \cosh(xB/c)$(Eq. (iv) evaluated at point B)

or,

c + 40 ft = c cosh (200 ft/c) (2)

This equation must be solved numerically for c. An initial estimate for c, when the solver on a calculator is to be used, could be

$$c = sag = 40\ ft$$

The solution to Eq.(2) is

$$c = 506.53\ ft$$

Using this value of c in Eq. (1) gives

$s_B = \text{c} \sinh(\frac{x_B}{c})$ (Eq. (1) repeated)

= (506.53 ft) sinh (200 ft/506.53 ft)

= 205.237 ft

Because the tension at the low point of the cable is horizontal, it can be found from Eq.(v):

$$T_0 = \omega c$$

$$= (4\ lb/ft)(506.53)$$

$$= 2.025\ lb.$$

Example (3) :

A 20-m chain is suspended between two points at the same elevation and with a sag of 6 m as shown. If the total mass of the chain 45 kg, determine the distance between the supports. Also determine the maximum tension.

The solution

The distance between the supports is $2x_B$, and x_B can befound from

Eq.(i), provided that we can determine the distance c;

$s_B = c\ \sinh(\frac{x_B}{c})$ (Eq(i)evaluated at point B)

since $s_B = 10\text{m}$, then:

$$10\text{m} = c\ \sinh(\frac{x_B}{c})$$

This equation can be solved explicitly for x_B by rearranging it as

$$\sinh\left(\frac{x_B}{c}\right) = \frac{10\text{m}}{c}$$

Which implies:

$$\frac{x_B}{c} = sinh^{-1}\left(\frac{10\ m}{c}\right)$$

So $x_B = c\ sinh^{-1}\left(\frac{10\ m}{c}\right)$ (1)

The distance c can be determined from Eq.(ii):

$y_B^2 - s_B^2 = c^2$ (Eq(iii)evaluated at point B)

$$(6\ m + c)^2 - (10\ m)^2 = (c)^2$$

or $36 + 12c + c^2 - 100 = c^2$

The c^2 terms cancel and resulting linear equation has the solution:

$$c = 5.333\ m$$

Substituting this value of c into Eq.(1)gives:

$$x_B = 5.333\ m\ sinh^{-1}(10\ m/5.333\ m) = 7{,}393\ m$$

Thus, the distance between supports $2x_B$ can be found:

$2x_B = 2(70393\ m) = 14.786\ m.$

The maximum tension, T_{max}, occurs where the slope of the cable is a maximum, at point B (or point A). This can be calculated from Eq.(vi):

$T_{max} = \omega y_{B}$(Eq(vi)evaluated at point B)

$$= \left(\frac{Tatol\ weight\ of\ the\ cable}{Tatol\ length\ of\ the\ cable}\right) y_B$$

$$= \left(\frac{(45.Kg)(9{,}81\ m/s^2)}{20\ m}\right)(6\ m5.333\ m) = 250\ N\ .$$

Example (4):

A certain cable will break if the maximum tension exceeds 500 N. If the cable is 50-mlong and has a mass of 50 kg, determine the greatest span possible. Also determine the sag.

The solution

The maximum tension has been specified (500 N) ,so a good place to start our solution is to see how we can use the fact that $T_{max} = 500$ N .Eq.(vii)relates the tension,T ,to they , coordinate of a point on the curve:

$$T = \omega y \text{ (Eq. (vi) repeated)}$$

The maximum tension, T_{max}, occurs where the cable has its steepest slope, point B (or point A). This can be calculated from Eq.(vi) :

$T_{max} = \omega y_{B}$(Eq(vi)evaluated at point B)

Thus, because we know the maximum tension, we can compute y_B :

$$y_B = \frac{T_{max}}{\omega} = \frac{T_{max}}{\left(\frac{Tatol\ weight\ of\ the\ \text{cable}}{Tatol\ length\ of\ the\ cable}\right)}$$

$$= \frac{500\ N}{\left(\frac{(50\ Kg)(9{,}81\ m/s^2)}{50\ m}\right)} = 50.97\ m$$

The distance between supports is $2x_B$, so we need to use the value of

y_B to determine x_B.this can be done by using Eq.(vi).provided that we can determine c :

$$y_B = c\cosh(x_B/c)\text{(Eq. (iv) evaluated at point B)}$$

We can solve this equation explicitly for x_B by rewriting it as:

$$\cosh(x_B/c) = y_B/c$$

So

$$x_B = c\ cosh^{-1}(y_B/c) \qquad (2)$$

The distance c , can be calculated from Eq.(iii) :

$y_B^2 - s_B^2 = \text{c}^2$ (Eq(iii)evaluated at point B)

$$(50.97\ m)^2 - (25\ m)^2 = (c)^2$$

The solution is

$$c = \pm 44.42\ m$$

The negative root has no physical meaning.

Substituting the value of $c = 44.42\,m$ and $y_B = 50.97$ into Eq.(2)gives:

$$x_B = 44{,}42\,m\,cochh^{-1}(50.97\,m/44.42\,m) = 23.836\,m$$

So, the distance between supports $2x_B$isknown:

$2x_B = 2(23.836\,m) = 47.7\,m.$

Since c and y_B are known, the sag can be computed:

$$h = y_B - c$$

$= (50.97m) - (44.42\,m) = 6.55\,m$.

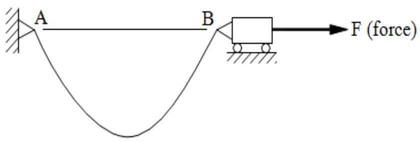
Example (5):

The cable is attached to a fixed support at A and a moveable support

at B. If the cable is80-ft long, weighs 0.3 lb/ft, and spans 50 ft, determine the force F holding the moveable support in place. Also determine the sag.

The solution

The force F acting on the moveable



support at$B$ equals the horizontal component,T_0, of tension in the cable,

$F = T_0$. Eq.(v) can be used to calculate T_0 , provided that we can

determine the distance c :

$T_0 = \omega c$ (Eq. (v) repeated)

$$= (0.3\ lb/ft)\ c = F \qquad (1)$$

The distance c ,can be calculated from Eq.(i) :

$s_B = c\ \sinh(\frac{x_B}{c})$ (Eq(i)evaluated at point B)

since $s_B = 40$ ft ,

then:

$$40\text{ft} = c\ \sinh(25\ \text{ft}/c) \qquad (2)$$

This equation must be solved numerically for c. An initial estimate for c, when the solver on a calculator is to be used, could be

$$c = x_B = 25\ ft$$

The solution to Eq.(2) is

$$c = \pm 14.229\ ft$$

The negative root has no physical meaning.

Using $c = 14.229\ ft$ in Eq.(1)gives:

$T_0 = \omega c$ (Eq. (v) repeated)

$$= (0.3\ lb/ft)(14.229\ ft)$$

$$= 4.27\ ft$$

The sag,h , can be calculated from Eq.$(iv)$and the known value of c :

$$h = y_B - c \quad = c\ \cosh(x_B/c) - c$$

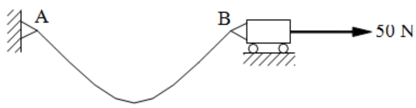
$$= (14.229\ ft)\cosh(25.ft/14.229ft) - 14.229ft$$

$= 28.2\ ft$.

Example (6):

. The cable is attached to a fixed support at A and a moveable support at B. If the cable is 40 m long, and has mass of 0.4Kg/m. If the force F holding the moveable support at the B is equal to $50\ N$ in the horizontal direction, determine the span and the sag.

The solution



The span is$2x_B$, andx_B can befound from Eq.(i), provided that we can determine The distance c;

$s_B = c\ \sinh(\frac{x_B}{c})$ (Eq(i)evaluated at point B)

This equation can be solved explicitly for x_B by rearranging it as

$$\sinh\left(\frac{x_B}{c}\right) = \frac{s_B}{c}$$

Which implies:$\frac{x_B}{c} = sinh^{-1}\left(\frac{s_B}{c}\right)$

So $\quad x_B = c\ sinh^{-1}\left(\frac{s_B}{c}\right)$

Then $\quad x_B = c\ sinh^{-1}\left(\frac{20\ \text{m}}{c}\right) \qquad (1)$

Because the 50 N force acting on the moveable support equals the horizontal component, T_0 , of the tension in the cable, Eq.(v) with $T_0 = 50\ N$ can be used to solve for c :

$T_0 = \omega c$ (Eq. (v) repeated)

or $\qquad 50\ N = [(0.4\ Kg/m)(9.81 m/s^2)]c$

solving gives:

$$c = 12.742\ m$$

Using this value of c in Eq. (1)gives:

$x_B = c\ sinh^{-1}(20\ \text{m}/c)$(Eq. (1) repeated)

$$= (12.742\ m) sinh^{-1}(20\ \text{m}/12.742\ m) = 15.708\ m$$

so, the span is

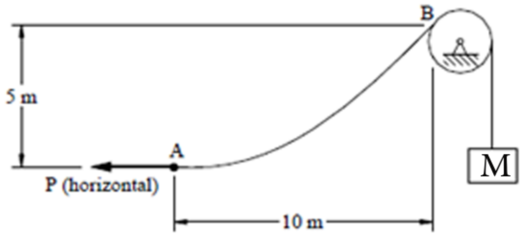
$$span = 2x_B = 2(15.708\ m) = 31.4\ m$$

The sag,h , can be calculated from Eq.(iv) and the known value of c :$h = y_B - c = c\ \cosh(x_B/c) - c$

$$= (12.742\ m)\cosh(15.708.ft/12.742\ m) - 12.742\ m = 28.2m$$

Example (7):

A cable goes over a frictionless pulley at $B$and supports a block of mass M. The other end of the cable is pulled by a horizontal force P.



If the cable has a mass per length of 0.3 kg/m, determine values of $P$and Mthat will maintain the cable in the position shown.

The solution

The force P equals T_0 ,the horizontal component of the cable tension

given$\mathrm{T_0} = \omega\mathrm{c}$(Eq. (v) repeated)

so, with $\mathrm{T_0} = \mathrm{P}$then:

$$\mathrm{P} = \omega\mathrm{c} \qquad (1)$$

Here:

$$\omega = (0.3\ Kg/m)(9.81\ m/s^2)$$

$$= 2.943\ N/m \qquad (2)$$

The value of c in Eq.(1) can be found from Eq.(iv):

$$y_B = c\cosh(x_B/c)\text{(Eq. (iv) evaluated at point}$$

or

$$5\ m + c = c\cosh(10\ m/c)$$

Solving numerically gives:

$$c = 10.743\ m$$

Using this value of c in Eq.(1)gives:

$$\text{P} = \omega\text{c}$$

$$= (2.943\ N/m)(10.743\ m)$$

$$= 31.617\ N$$

The cable tension at B must equals the weight,mg :

$$T_B = Mg$$

thus, the mass is

$$M = T_B/g$$

By Eq.(vi)

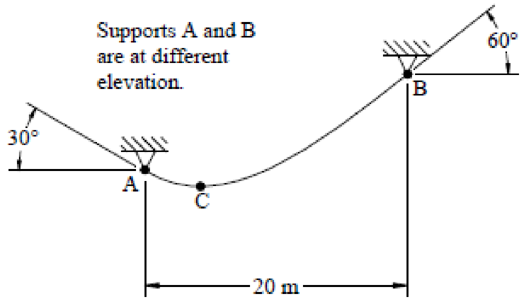
$$M = \omega y_B/g$$

By Eq. (2)

$$M = [(2.943\ \ N/m)(5\ m + 10.743\ m)]/(9.81\ N/m^2)$$

$$= 4.72\ Kg$$

Example (8):



A chain makes angles of 30° and 60° at its supports as shown.

Determine the location of the low point C of the chain relative to A. Also determine the tension at support A, if the cable has a mass per length of 0.6 kg/m.

The solution

The geometric data are shown in the figure. To determine the location of the low point C relative to A, we need to determine the coordinates xA and yA. We can get an equation for xA by using the fact that the slope is known at A:

$$-\tan 30^{\circ} = \left[\frac{dy}{dx}\right]_{atA} = \left[\frac{d(c\cosh(x/c))}{dx}\right]_{atA} \qquad \text{by Eq.}(iv)$$

$$= \sinh(x_A/c)$$

Solving for x_A gives:

$$x_A = c\sinh^{-1}(-\tan 30^{\circ}) \qquad (1)$$

Similarly at point B , we have

$$x_B = c\sinh^{-1}(\tan 60^{\circ}) \qquad (2)$$

The coordinates x_A and x_B are related to the 20-m span through the equation:

$$x_A - x_B = 20 \text{ m}$$

By substituting from Eqs.(1)&(2) we get:

$$c \sinh^{-1}(-\tan 30^{\circ}) - c \sinh^{-1}(\tan 60^{\circ}) = 20$$

Since this equation is linear in c, it is easily solved to give c = 10.717 m. Eq. (1) then gives

$x_A = c \sinh^{-1}(-\tan 30^{\circ})$(Eq (1) repeated)

$$= (10.717\, m) \sinh^{-1}(-\tan 30^{\circ})$$

$$= -5.887\, m$$

The y coordinate of point A can now be calculated fromEq. (iv):

$y_A = c\ \cosh(x_A/c)$ (Eq.(iv) evaluated at point A)

$$= (10.717\, m) \cosh(-5.887\, m/10.717 m)$$

$$= 12.375\, m \qquad (3)$$

The vertical distance between support A and the low point C is given by

$$d = y_A - c$$

$$= 12.375\, m - 10.717\, m$$

$= 1.658\, m$ (by Eq. (3))

The tension at A is given by Eq. (vi):

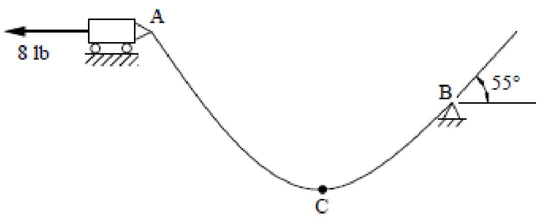
$T_A = \omega y_A$ (Eq.(vi) evaluated at point A)

$$= [(0.6\ Kg/m)(9.81\ m/s^2)](12.375\ m) = 72.8\ N.$$

Example (9):

A wire weighting 0.2 lb/ft is attached to a moveable support at A and makes an angle of 55° at a fixed support at B. Supports A and B are at different elevations. Determine the location of the low point C of the wire relative to support B. Also, determine the tension in the wire at C.

The solution



To determine the location of the low point, C , relative to the support at B, we need to determine the coordinates x_B and y_B . We can get an equation for x_B by using the fact that the slop is known at B.

$$\tan 55^{\circ} = \left[\frac{dy}{dx}\right]_{atB}$$

$= \left[\frac{d(c\cosh(x/c))}{dx}\right]_{at\boldsymbol{B}}$ by Eq.(iv)

$$= \sinh(x_B/c)$$

Thus

$$x_B = c\sinh^{-1}(\tan 55^{\circ}) \qquad (1)$$

The value of c occurring in Eq. (1) can be found by observing that the 8-lb force acting at support A equals T_0 the horizontal component of tension at A, so Eq. (v) gives

$\mathrm{T_0 = \omega c}$ (Eq. (v) repeated)

$$\therefore 8\ Ib = (0.2\ \omega\ Ib/ft)c$$

Solving gives:

$$\mathrm{C = 40\ ft} \qquad (2)$$

Using this result,$C = 40\ ft$ in Eq.(1)gives:

$x_B = c\sinh^{-1}(\tan 55^{\circ})$(Eq. (1) repeated)

$$= (40\ ft)\sinh^{-1}(\tan 55^{\circ})$$

$$= 46.169\ ft$$

The vertical distance between B and C is:

$$d = y_B - c$$

$$= c\cosh(x_B/c) - c$$

$$= (40\ ft)\cosh(46{,}169\ ft/40\ ft) - 40\ ft$$

$$= 29.7\ ft$$

Since point C is the low point of the cable, the tension there is horizontal and so must equal the horizontal component of tension at A

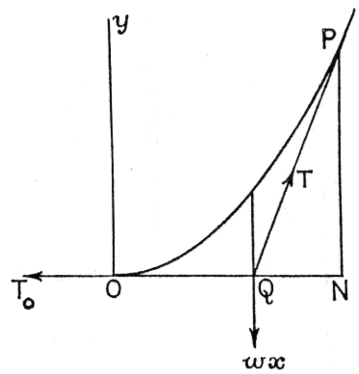
which is known to be 8 Ib that is:

$T_c = 8\ Ib$.

Worked examples

Example (1): (The suspension bridge)

If a chain supports a continuous load, uniformly distributed, the chain hangs in the form of a parabola. O is the lowest point of the chain and P any point of the chain whose coordinates referred to horizontal and



vertical through O are (x, y) The weight carried by the portion OP will be proportional to ON and acts through Q the midpoint of ON. We may call it ωx .

The other forces acting on the portion OP are T_0 the horizontal tension At O and the tension T at P, three of them must therefore meet at Q and PNQ is a triangle of forces.

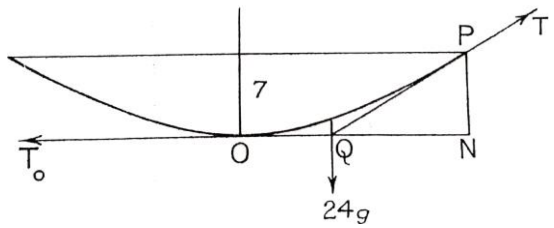
$$\therefore \quad \frac{\omega x}{PN} = \frac{T_0}{NQ} \qquad \therefore \quad T_0 y = \frac{1}{2}\omega x^2$$

Hence, if we denote T_0 by ωc , then we can get $y = \frac{x^2}{2\,c}$

this means that the curve of the chain is a parabola.

Now if the span of a suspension bridge is $96\ m$ and the sag in the chain is $7\ m$ .TheTwo branches of the chain support a load of $1000\ kg$per horizontal meter. Find the tension at the lowest and highest points. The load carried by OP is $24gkN$ .The triangle QPN is a triangle of forces.

<u>The solution</u>



$$QN = 24\,m \quad , \quad PN = 7\,m \qquad \therefore Qp = 25\,m$$

$$\frac{T_0}{24} = \frac{T}{25} = \frac{24g}{7}$$

$T = 840\ kN$, $T_0 \cong 810\ kN$.

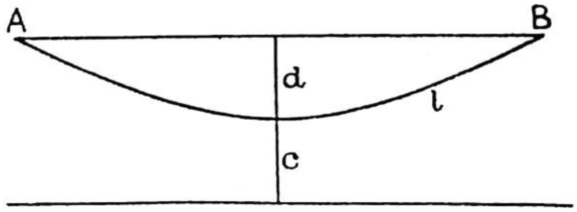
==

Example (2):

A uniform chain of length $2l$ and weight ω per unit length is suspended between two points at the same level and has a maximum depth$d$.Prove the tension at the lowest is $\omega\,(l^2 - d^2)/2d$.If $l = 50\ m$ and $d = 20\ m$ find the distance between the points of suspension.

The solution

For the catenary $y^2 = c^2 + s^2$,



At $By = c + d$, $s = l$

$$\therefore\ (c + d)^2 = c^2 + l^2$$

$$2cd = l^2 - d^2$$

$$\therefore\ c = l^2 - d^2/2d$$

$\therefore$ the tension at the lowest is $= \omega c = \omega\,(l^2 - d^2)/2d$.

If $l = 50\ m$ and $d = 20\ m$, then $c = 2500 - 400/40 = 105/2$

Now $s = c \sinh x/c$

Hence if $AB = 2x$,

$$\therefore \quad x = (105/2) \sinh^{-1}(20/21)$$

$$= (105/2) lin\left[(20/21) + \sqrt{\{1 + (20/21)^2\}}\right]$$

$$= (105/2) lin(49/21)$$

$\therefore \quad AB = 105 \times 2.303\ log_{10}(49/21) \cong 89\ m$.

==

EXERCISES:

(1)A rope has an effective length of $20\ m$and mass5 kg per miter. One end of the rope is $4\ m$ higher than the other. Find the maximum tension in the rope when the tangent at the lower end is horizontal.

(2)A uniform chain of length $2l$ has its ends fixed at two points at the same level. The sag at the middle is h .prove that the span is $[(l^2 - h)/h] lin[(l + h)/(l - h)]$.

(3)A uniform wire hangs freely from tow points at the same level $200\ m$apart. The sag is $15\ m$. Show the greatest tension is approximately $348\ \omega$ and the length of wire is approximately $203\ m$.

(4)Find approximately the greatest tension in a wire which has mass $100\ g$ per miter when it hangs with a sag of $25\ cm$ when stretched between two points at the same level $40\ m$apart.

(5)A uniform heavy chain of length $31\ m$ is suspended from tow points at the same level and$30\ m$apart. Show that the tension at the lowest point is about 1.08times the weight of the chain.

Chapter (2)

Direct Stress and Strain

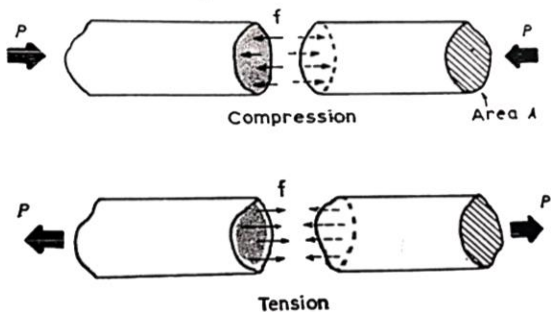
----

(1) Stress:

The ability of a structural member to withstand load or transmit force, as in a machine, depends upon its dimensions. In particular, the cross-sectional area over which the load is distributed determines the intensity or average stress in the member. If the intensity of loading is uniform the direct stress, , f is defined as the ratio of load, P ,to cross-sectional area,A , normal to the load as shown in the Fig. Thus:

$$\text{stress} = \frac{\text{load}}{\text{area}}$$

or $$f = \frac{P}{A}$$



2. 1

If the load is in pounds and the area in square inches the units of stress are pounds per square inch $(Ib/in.^2)$.There are another unit:

If,P , is expressed in Newton (N) , and A, original area, in square meters(m^2) ,the stress,f , will be expresses in N/m^2 , this unit is called Pascal (Pa) .

As Pascal is a small quantity in practice, multiples of this unit is used.

$$1\,KPa = 10^3 Pa = 10^3\,N/m^2\,(KPa = Kilo\ Pascal)$$

$$1\,MPa = 10^6 Pa = 10^6\,N/m^2$$

$$= 1\ N/mm^2\,(MPa = Mega\ Pascal)$$

$$1\,GPa = 10^9 Pa = 10^9\,N/m^2\,(GPa = Giga\ Pascal)$$

The direct stress may be tensile or compressive according as the load is a pull (tension), or push (compression). It is often convenient to consider tensile stresses and loads as positive and compressive stresses and loads as negative.

(2) Strain:

A member under any loading experiences a change in shape or size inthe case of a bar loading in tension the extension of the bar depends upon its total length. The bar is said to be strained and the strain is defined as the extension per unit of original length of the bar. Strain may produced in two ways:

1- By application of a load.

2- By a change in temperature, unaccompanied by load or stress.

If l is the original length of the bar, x the extension or contraction in length under load or temperature change, and $e$the strain, then:

$$\text{strain} = \frac{\text{change in length}}{\text{original lenth}}$$

or $$e = \frac{x}{l}$$

Strain is a ratio and has therefore no units.

Strain due to an extension is considered positive, that associated with a contraction is negative.

(3) Relation between Stress and Strain:

If the extension or compression in a member due to a load disappears on removal of the load, then the material is said to be elastic. Most metals are elastic over a limited range of stress known as the elastic range. Elastic materials, with some exception, obey Hooke's, which states that: the strain is directly proportional to the applied stress Thus

$$\frac{stress}{strain} = constant\ (E)$$

i.e.$\frac{f}{e} = E$or $e = \frac{f}{E}$

where E is the constant of proportionality, known as the modulus of elasticity or Young's modulus?

Since strain is a ratio, the units of $E$are those of stress, i.e. pounds per square inch.

Examples

Example (1):

A rubber pad for a machine mounting is to carry a load of $1000\ Ib$ and to compress$0.2in$. If the stress in the rubber is not exceed $40\ Ib/in.^2$, determine the diameter and thickness of a pad of circular cross-section.

Take E for rubber as $150\ Ib/in.^2$.

The solution

$$\text{stress} = \frac{\text{load}}{\text{area}}$$

i.e $$f = \frac{P}{A}$$

$$40 = \frac{1000}{\pi d^2/4}$$

hence $d^2 = 31.83\ in.^2$ and $d = 5.64\ in$

i.e. diameter of pad=5.64 in.

The increase in area due to compression has been neglected.

Also $stress = \frac{reduction\ in\ length}{original\ length}$

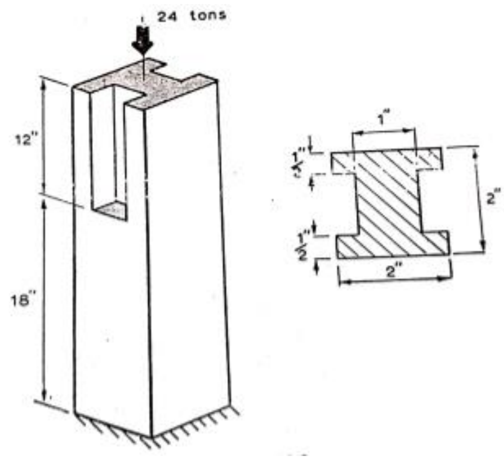
then $\frac{f}{E} = \frac{x}{l}$ this leads to $\frac{40}{150} = \frac{0.2}{l}$

therefore, thickness of pad is given by

$$l = 0.75\ in.$$

<u>Example (2):</u>

The Fig shows a steel strut with tow grooves cut out along part of its length. Calculate the total compression of the strut due to a load of $24\ tons. E = 12500\ \ ton/in.^2$



<u>The solution</u>

Suffices 1 *and* 2 denote solid and grooved portions, respectively. the load at every section is the same, $24\ ton$.

For the solid length of $18\ in.$

compression $x_1 = e_1 l = e_1 \times 18$

$$\text{stress}, f_1 = \frac{P}{A_1} = \frac{24}{2 \times 2} = 6 \ tons/in.^2$$

$$\text{strain}, e_1 = \frac{f_1}{E} = \frac{6}{E}$$

For the grooved length 0f $12\ in.$

compression $x_2 = e_2 l = e_2 \times 12$

$$\text{stress}, f_2 = \frac{P}{A_2} = \frac{24}{(4-1)(1)} = 8 \ tons/in.^2$$

$$\text{strain}, e_2 = \frac{f_2}{E} = \frac{8}{E}$$

The total compression of the strut is equal to the sum of the compressions of the solid and grooved portions. Therefore

$$x = x_1 + x_2$$

$$= (e_1 \times 18) + (e_2 \times 12)$$

$$= \left(\frac{6}{E} \times 18\right) + \left(\frac{8}{E} \times 12\right)$$

$$= \frac{204}{E}$$

$$= \frac{204}{12500}$$

$$= 0.0163\ in.$$

Note: It has been assumed here that the stress distribution is uniform over all sections, but at the change in cross-section the stress distribution is actually very complex. The assumption produces little error in the calculated compression.

Example (3):

A rod $10\ mm \times 10\ mm$ cross-section is carrying an axial tensile load $10\ KN$. In this rod the tensile stress developed is given by:

$$f = \frac{P}{A} = \frac{10\ KN}{(10\ mm \times 10\ mm)} = \frac{10 \times 10^3\ N}{100\ mm^2} = 100\ MPa$$

Example (4):

A rod $100\ mm$ in original length. When we apply an axial tensile loa $10\ KN$ .The final length of the rod after application the tensile is $100.1\ mm$ .So in this rod tensile strain is developed and is given by;

$$e = \frac{x}{l} = \frac{100.1mm - 100\ mm}{100\ mm} = \frac{0.1mm}{100\ mm} = 0.001(\text{Dimenionless})\text{Tensile.}$$

Example (5):

A rod $100\ mm$ in original length. When we apply an axial compressive load $10\ KN.$ .The final length of the rod after application compressive is $99\ mm$ .So, in this rod compressive strain is developed and is given by;

$$e = \frac{x}{l} = \frac{99\ mm - 100\ mm}{100\ mm} = \frac{-0.1mm}{100\ mm} = -0.001(\text{Dimenionless})\text{Tensile.}$$

==

Exercises

(1) A bar of 1 $in.$ diameter is subjected to a tensile load of 10000 Ib.Calculate the extensionon a 1 $ft.$ length. $E = 30 \times 10^4\ Ib/in.^2$.

(2)A light alloy bar is observed to increase in length by 0.35 per centwhen subjected to a tensile stress of 18 $ton/in.^2$ .Calculate Young' modulus for the material.

(3)A duralumnin tie, 2 ft long 1.5 $in.$ diameter, has a hole drilled out along its length .The hole is of 1 $in.$diameter and 4 $in.$ long.Calculate the total extension of the tiedue to a load of 18 $tons$. $E = 12 \times 10^4\ Ib/in.^2$.

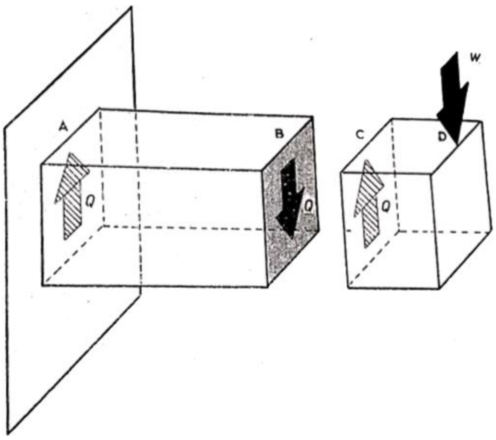
(4) A steel strut of rectangular section is made up of two lengths. The first 6 $in.$ long, has breadth 2 $in.$and depth 1.5 $in.$; the second,4 $in.$long is 1 $in.$ square. If $E = 14000\ tons/in.^2$, calculate the compression of the strut under a load of 10 $tons$.

Chapter (3)

Shear force and Bending Moment

(1) Shear force (SF):

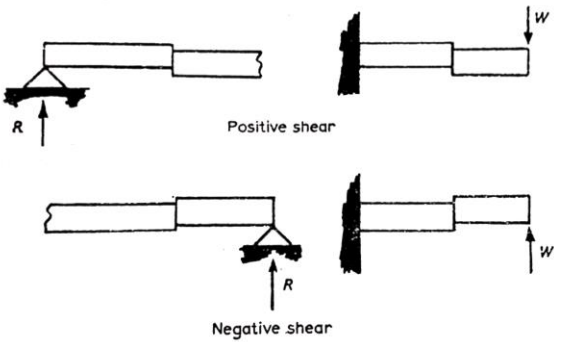
The shear force in a beam at any section is the force transverse to the beam tending to cause it to shear across the section. Fig.(3.1) shows abeam under a transverse load W at the end D ; the other end A is built in to the wall. Such a beam is called a cantilever and the load W , which is assumed to act at a point, is called a concentrated or point load.



Consider the equilibrium of any portion of beam CD . At section C for balance of forces there must be an upward force Q equal and opposite to the load W at D . This force Q is provided by the resistance of the beam to shear at the plane B ; this plane being coincident with the plane section at C . Q is the shear force at B and in this cace has the same magnitude for any section in AD . Consider now the equilibrium of the portion of beam AB . There is a downward force $Q = W$, exerted on plane B , so for balance there must be an upward force Q at A . This latter force being exerted on the beam by the wall.

Sign Convention

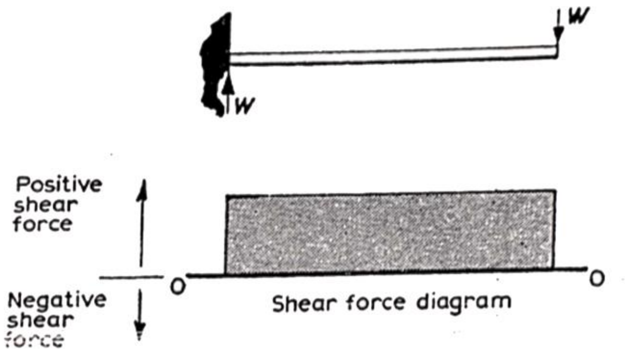
The shear force at any section is taken positive if the right-hand side tends to slide downwards relative to the lift-hand portion, fig.(3.2).A negative shear force tends to cause the right-hand portion to slide upward relative to the lift. (In some books flowed totally opposite sign convention).



If several loads act on the beam to the right-hand side of section C the shear force a C is the resultant of these loads. Thus, the shear force at any section of a loaded beam is the algebraic sum of the loads to one side of the section. It does not matter which side of the section is considered provided all loads on that side are taken into account-including the forces exerted by fixings and props.

(2) Shear Force Diagram (SFD):

The graph showing the variation of shear force along a beam is known as the shear force diagram. for the beam of Fig 3.1 the shear force was $+W$, uniform along the beam. Fig (3.3) shows the shear force diagram for this beam, $0 - 0$ being the axisof zero shear force.

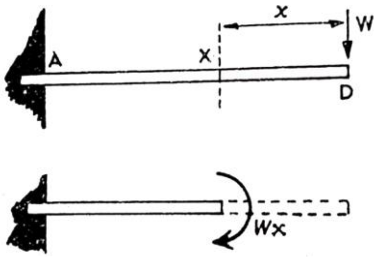


(3) Bending Moment (BM):

The bending effect at any section X of a concentrated load W at D ,

Fig.(3.4) , is measured by the applied moment Wx , where x is the perpendicular distance of the line of section of$W$ from section X .

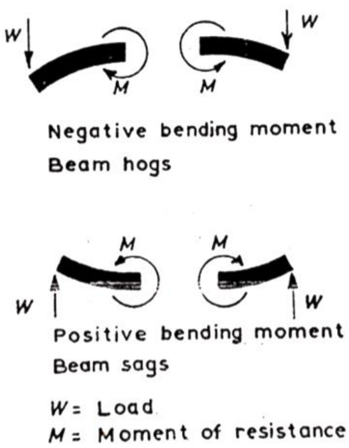
This moment is called the bending moment and is balanced by an equal and opposite moment M exered by the material of the beam at$X$, called the moment of resistance.



Sign Convention

A bending moment is taken as positive if its effect is to tend to make the beam sag at the section considered, Fig.(3.5).If the moment tends make the beam bend upward or hog at the section it is negative.

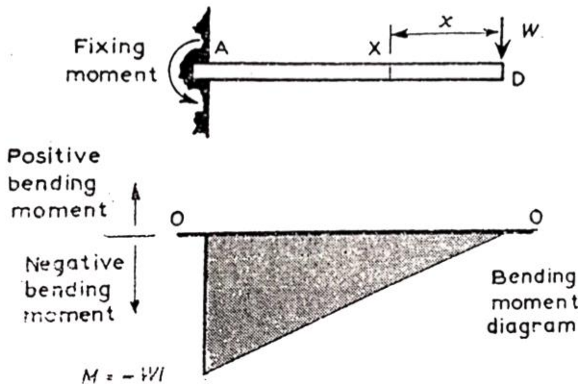
When more than one load act on a beam the bending moment at any section is the algebraic sum of the moments due to all the loads on one side of the beam. It does not matter which side of the section is considered but all loads on that side must be taken into account, including any moments exerted by fixings.



(4) Bending Moment Diagram (BMD):

The variation of bending moment along the beam is shown in a bending moment diagram. for the cantilever beam of Fig 3.1 the bending moment at any section X is given by:

bending moment$= -Wx$ (negative, since the beam hogs at X)



Since there is no other load on the beam this expression for the bending moment applies for the whole length of beam from $x = 0$ to $x = l$ .The moment is proportional to$x$ and hence the bending moment diagram is a straight line. Hence the diagram can be drawn by calculating the moment at two points and joining two corresponding points on the graph by a straight line.

At D, $x = 0$ and bending moment $= 0$

At A, $x = l$ and bending moment $= -Wl$

Since the bending moment is everywhere negative the graph plotted is below the line $0 - 0$ of zero bending moment, Fig.(3.6). At the fixed end A the wall exerts a moment Wl anticlockwise onthe beam; this is calleda fixing moment.

(5) Calculation of Beam Reactions:

When a beam is fixed at some point, or supported by props the fixings and props exert reaction forces on beam. To calculate these reactions the procedure is:

(a) equate the net vertical force to zero;

(b) equate the total moment about any convenient point to zero.

Note (1): Distinguish carefully between "taking moments" and calculating a "bending moment":

(1) The Principle of Moments states that the algebraic sum of the moments of all the forces about any point is zero, i.e. when forces on both sides of a beam section are considered.

(2) The bending Moment is the algebraic sum of the moments of forces on one side of the section about that section.

Note (2): What are the benefits of drawing shear force (SF) and bending

Moment (BM)diagram?

The benefits of drawing a variation of (SF) and (BM)in a beam as a function of $'x'$ measuredfrom one end of the beam is that it becomes easier to determine the maximum absolute value of (SF) and (BM).The (SF) and (BM) diagram gives a clear picture in our mind about the variation of (SF) and (BM) throughout the entire section of the beam.

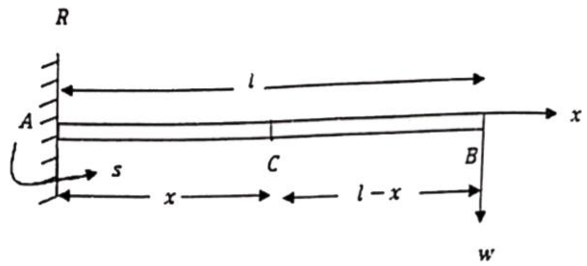
Further, the determination of value of deflection of beam subjected to a given loading where we will use the formula $EL\ \frac{d^2y}{d\,x^2} = M_x$.

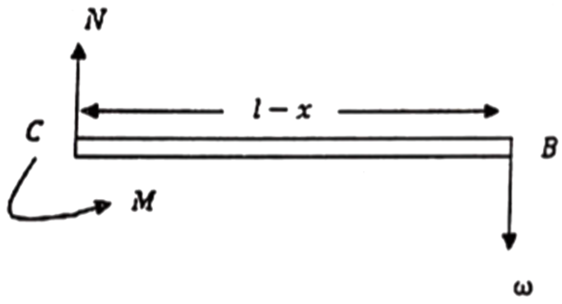
Examples

Example (1):

Draw the (SF) & (BM) diagrams at any section for a light horizontal beam AB ,its length is L .The end A of the beam is fixed at a vertical wall, while the free end $B$is loaded by a weight W .

The Solution





We take a section for the beam at C , where:

$$AC = x \qquad \& \qquad CB = L - x$$

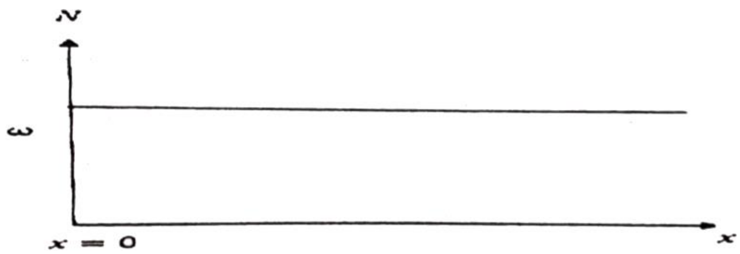
We study the equilibrium of the part CB or the part AC .

The study of the right part CB is easier than the lift part AC , because the existence of the reaction R and the couple S .

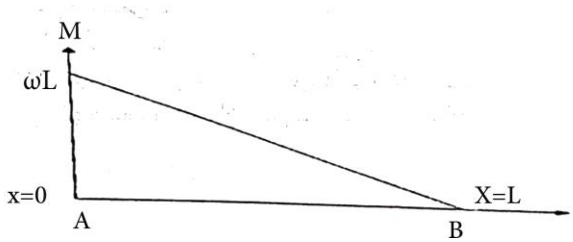
The shear force (SF) is $N = W$ (1)

and the bending moment (BM) is $M = W(L - x)(2)$

From EQ.(1) the (SF) N is constant at any section and so it is a straight line parallel to x −axis As shown in the next Fig..



But from EQ.(2) the (BN) M is depending on x , and its diagram isshown in the next Fig.



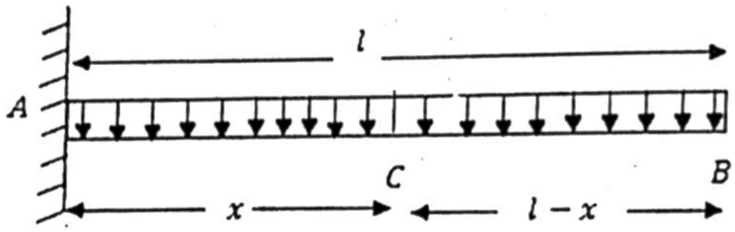
For the equilibrium of the beam AB we find that:

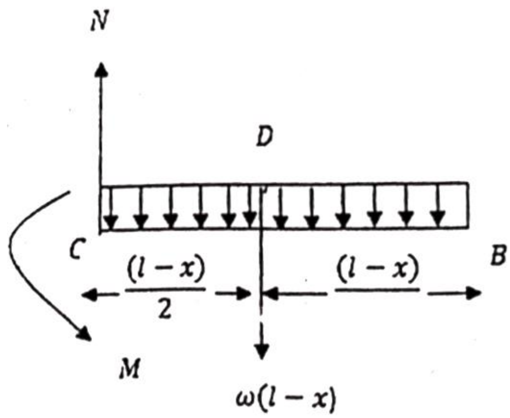
$R = W$, $S = WL.$

Example (2):

Draw the (SF) & (BM) diagrams for a light horizontal beam AB ,its length is L .The end A of the beam is fixed at a vertical wall, while the free end B is free. The beam is loaded uniformly by a weightω per unit length.

The Solution



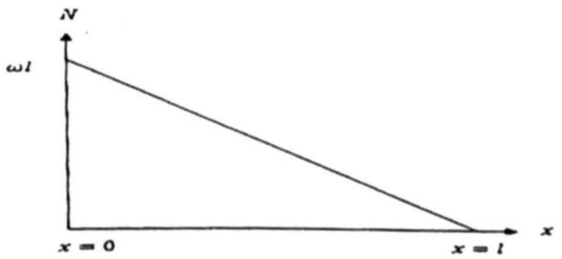


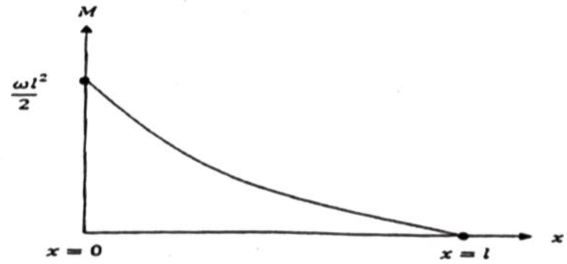
We note that the weight of the part CB is $\omega(L-x)$ and acts at its middle point. From the equilibrium of this part we find that:

The (SF) is $N=\omega(L-x)(1)$,

and the (BM) is $M=\omega(L-x)\frac{(L-x)}{2}=\frac{\omega}{2}(L-x)^2(2)$.

Then the (SFD) &(BMD) will be shown in the following Figs.

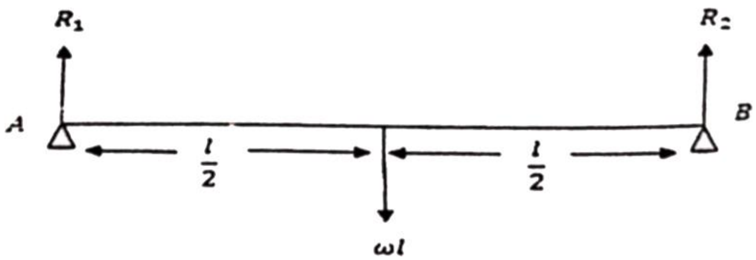




Example (3):

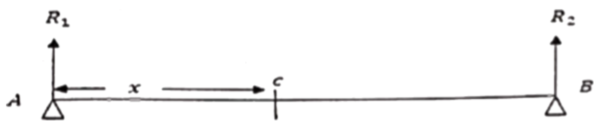
Draw the (SF) & (BM) diagrams for a heavy horizontal beam AB ,its length is L , and ω is its weigh per length. The beam is standing on two weidges in the same horizontal plane atits ends.

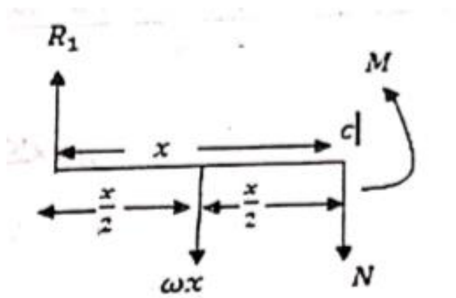
The Solution



From the symmetry we find that:

$R_1 = R_2 = \omega L/2$ Where $R_1 + R_2 = \omega L$.





By considering the equilibrium of the part AC , we find that:

$$R_1 = \omega x + N$$

$$\therefore \ \frac{1}{2}\,\omega L = \omega x + N$$

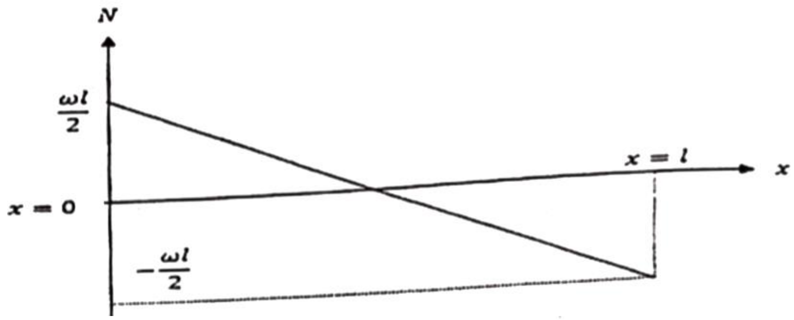
$$\therefore \ N = \ \omega\left(\frac{L}{2} - x\right)$$

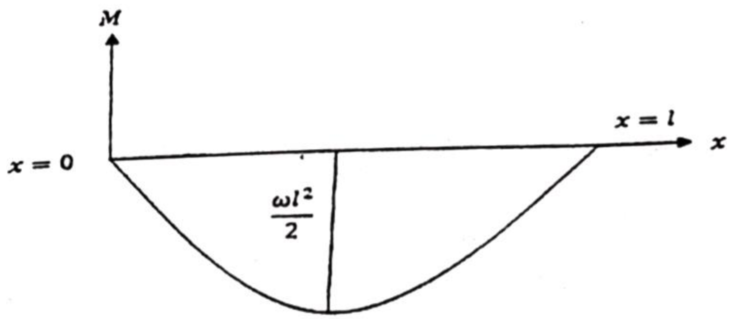
And by taking the moment about the point C , we get:

$$M + \omega x\left(\frac{x}{2}\right) = R_1 x$$

$\therefore \ M = \frac{\omega}{2}\,Lx - \frac{\omega}{2}x^2 = -\frac{\omega}{2}(x^2 - Lx).$

The (SFD) &(BMD) will be shown in the following Figs.

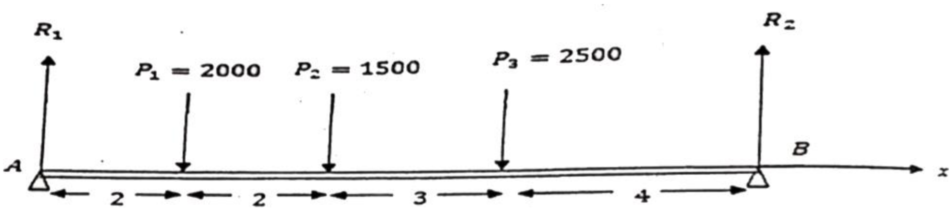




Example (4):

Draw the (SF) & (BM) diagrams for a light horizontal beam AB ,its length is $11\ ft$, and stands on two weidges in the same plane at its ends. The beam carries $p_1 = 2000, p_2 = 1500, p_3 = 2500\ lb$ at the three pointes a, b, c such that $Aa = ab = 2\ ft$ &$bc = 3\ ft$.

The Solution



From the equilibrium of the beam we get:

$$R_{1+}R_2 = 2000 + 1500 + 2500 = 6000\ lb \qquad (1)$$

By taking the moment about the point B we get:

$$11\ R_1 = 2000\ \times 9 + 1500 \times 7 + 2500 \times 4$$

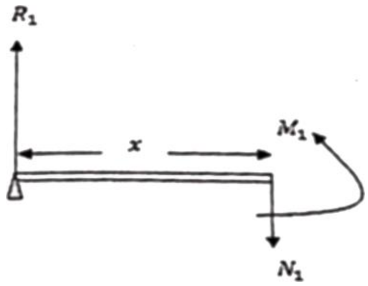
$$\therefore 11\ R_1 = 38500 \qquad \rightarrow \qquad R_1 = 3500\ lb \qquad (2)$$

By substituting from (1) in (2) we get:

$$R_2 = 2500\ lb \qquad (3)(\quad)$$

For determination the (SF) &(BM) at any point we consider the sections where:

$(i) 0 < x < 2$



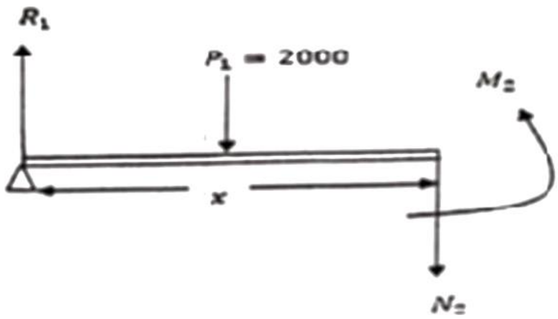
$N_1 = R_1 = 3500\ lb$

$M_1 = R_1\,x = 3500x\ lb$

$(ii) 2 < x < 4$

$N_2 = R_1 - p_1 = 3500 - 2000 = 1500\ lb$

$$M_2 = R_1\,x - 2000(x-2) = 3500\,x - 2000(x-2)\ lb$$



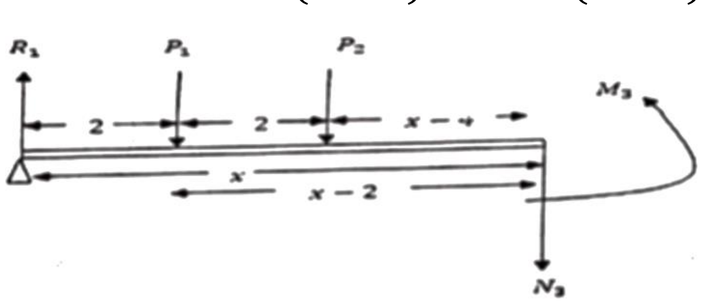
$(iii) 4 < x < 7$

$N_3 = R_1 - p_1 - p_2$

$= 3500 - 2000 - 1500 = 0\ lb$

$M_3 = R_1\,x - 2000(x-2) - 1500(x-4)$

$$= 3500\,x - 2000(x-2) - 1500(x-4)\ lb$$



$(v) 7 < x < 11$

$N_4 = R_1 - p_1 - p_2 - p_3$

$= 3500 - 2000 - 1500 - 2500 = -2500\ lb$

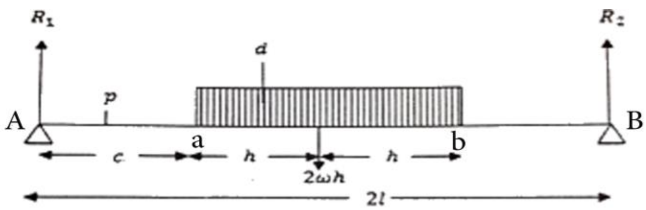
$M_4 = R_1\, x - 2000(x-2) - 1500(x-4) - 2500(x-7) -$

$= 3500\, x - 2000(x-2) - 1500(x-4) -\ lb2500(x-7)$

Example (5):

Find the (SF) and defined the maximum (BM) at a point d for a light horizontal beam AB ,its length is $2L$ and stands on two weidges in the same plane at its ends. The beam carries a movable weight$ab = 2h\omega$ where $2h(h < L)$is its length. Then draw (SFD) &(BMD), and prove that $\frac{ad}{ab} = \frac{Ad}{dB}$.

The Solution



By taking a position for the beam ABas shown in the figure such that $Aa = c$, and by finding the value of c , which makes the (BM) at d is maximum.

In case, the equilibrium of AB , we get:

$$R_1 + R_2 = 2\omega h$$

By taking the moment about the point B we get:

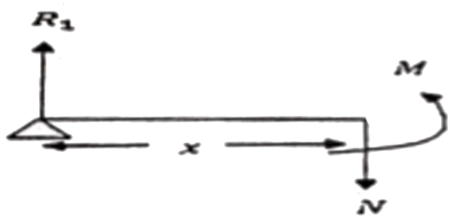
$$R_1 \times 2L = 2\omega h(2L - c - h)$$

$$\therefore\ R_1 = \frac{\omega h}{L}(2L - c - h)$$

By taking a section at $p$where,$AP = x$ & $x < c$, we get:

$$N = R_1 = \frac{\omega h}{L}(2L - c - h)$$

$M = R_1 x = \frac{\omega h}{L}(2L - c - h)x$

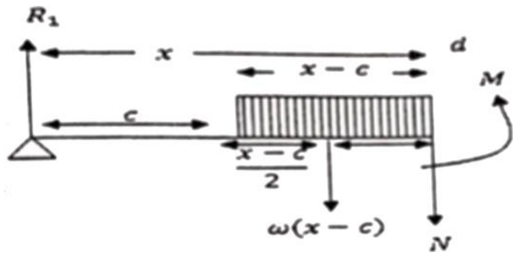


And, by taking a section at d where ,$Ad = x$ & $x > c$, we get:

$$N = R_1 - \omega(x - c)$$

$$= \frac{\omega h}{L}(2L - c - h) - \omega(x - c)$$

$$M = \frac{\omega h}{L}(2L - c - h)x - \frac{1}{2}\omega(x - c)$$



The maximum value of M will be when

$\frac{dM}{dc} = 0$, i.e.$-\frac{x\omega h}{L} + \omega(x - c) = 0$

$$\therefore \quad c = \left(1 - \frac{b}{L}\right)x$$

By substituting in M , we have:

$$M_{max} = \frac{\omega h}{L}\left(2L - h - x\left(1 - \frac{h}{L}\right)\right)x - \frac{1}{2}\omega\left(x - x\left(1 - \frac{h}{L}\right)\right)x$$

In this case, we find that :

$$\frac{ad}{db} = \frac{x-c}{2h-(x-c)} = \frac{\frac{h}{L}x}{2h-\frac{h}{L}x} = \frac{hL(x)}{\frac{h}{L}(2L-x)} = \frac{x}{2L-x} = \frac{Ad}{dB} .$$

Example (6):

AB is a beam, its length is L , and the end $B$is fixed at a vertical wall. The beam is loaded by a weight Wdistributedlinearly, by uniformly increasing, starting from zero at the free end A.Find the (SF) & (BM) then draw its diagrams.

<u>The Solution</u>

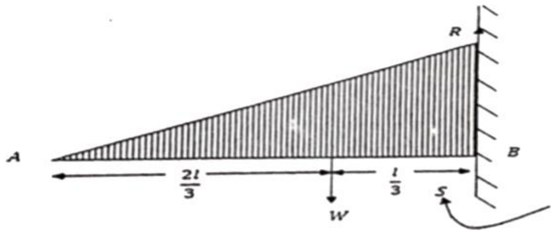
The density of loading is $\omega = \omega(x)$ at the section C ,where $AC = x$,

Then $\omega = \gamma x$ (linearly distribution)

$W = \int_0^L \omega \, dx = \int_0^L \gamma x \, dx$

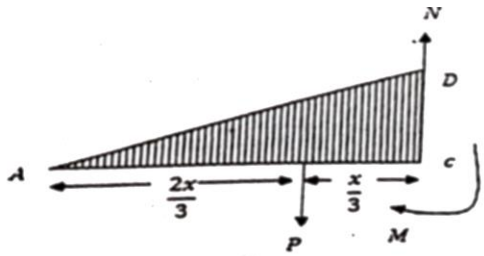
$\therefore W = \gamma L^2/2 \therefore \gamma = 2W/L^2$

$\therefore \omega = \gamma x = (2W/L^2)x$



For the section AC , we get

$$N = P = \int_0^x \omega \, dx = \int_0^x (2Wx/L^2) \, dx$$



$\therefore N = (Wx^2/L^2)$

We note that the weight P dived AC by the ratio

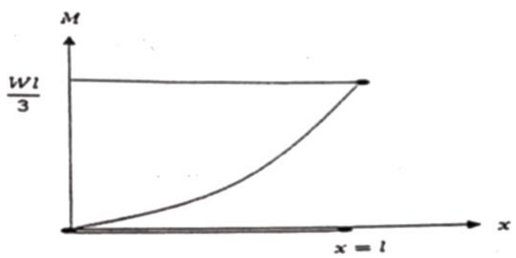
$AE = 2EC = 2x/3$

By taking the moment about C we find that

$$M = P(x/3) = (Wx^2/L^2)(x/3) = (Wx^3/3L^3)$$

We note that $R = W$, $S = WL/3$ and

$AF = 2FB = (2/3)L$.



<u>Exercises</u>

(1)Draw the (SF) & (BM) diagrams at any section for a light horizontal beam AB ,its length is L .The end A of the beam is fixed at a vertical, while the free end B is loaded by a weight ωL .

(2)Draw the (SF) & (BM) diagrams for a light horizontal beam AB ,its length is L , and stands on two weidges in the same plane at its ends. The beam carries two equal weights$p_1 = p_2 = \omega$ at the two points $C\&D$ such that $AC = DB = a,\ (a < L/2)$.

(3)Find and draw the (SF)&(BM) for a light horizontal beam AB ,its length is$10\ ft$and stands on two weidges in the same plane at its ends. The beam is loaded by a uniformally distributed weight, where $\omega = 10\ lb$ per unit length.

Chapter 62

Partial Derivatives

FUNCTIONS OF SEVERAL VARIABLES. If a real number z is assigned to each point (x, y) of a part (region) of the xy plane, then z is said to be given as a function, $z = f(x, y)$, of the independent variables x and y. The locus of all points (x, y, z) satisfying $z = f(x, y)$ is a surface in ordinary space. In a similar manner, functions $w = f(x, y, z, \ldots)$ of several independent variables may be defined, but no geometric picture is available.

There are a number of differences between the calculus of one and of two variables. Fortunately, the calculus of functions of three or more variables differs only slightly from that of functions of two variables. The study here will be limited largely to functions of two variables.

LIMITS AND CONTINUITY. We say that a function $f(x, y)$ has a limit A as $x \to x_0$ and $y \to y_0$, and we write $\lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) = A$, if, for any $\epsilon > 0$, however small, there exists a $\delta > 0$ such that, for all (x, y) satisfying

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \tag{62.1}$$

we have $|f(x, y) - A| < \epsilon$. Here, (*62.1*) defines a deleted neighborhood of (x_0, y_0), namely, all points except (x_0, y_0) lying within a circle of radius δ and center (x_0, y_0).

A function $f(x, y)$ is said to be continuous at (x_0, y_0) provided $f(x_0, y_0)$ is defined and $\lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) = f(x_0, y_0)$. (See Problems 1 and 2.)

PARTIAL DERIVATIVES. Let $z = f(x, y)$ be a function of the independent variables x and y. Since x and y are independent, we may (1) allow x to vary while y is held fixed, (2) allow y to vary while x is held fixed, or (3) permit x and y to vary simultaneously. In the first two cases, z is in effect a function of a single variable and can be differentiated in accordance with the usual rules.

If x varies while y is held fixed, then z is a function of x; its derivative with respect to x,

$$f_x(x, y) = \frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

is called *the (first) partial derivative of* $z = f(x, y)$ *with respect to* x.

If y varies while x is held fixed, z is a function of y; its derivative with respect to y,

$$f_y(x, y) = \frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

is called *the (first) partial derivative of* $z = f(x, y)$ *with respect to* y. (See Problems 3 to 8.)

If z is defined implicitly as a function of x and y by the relation $F(x, y, z) = 0$, the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ may be found using the implicit differentiation rule of Chapter 11. (See Problems 9 to 12.)

The partial derivatives defined above have simple geometric interpretations. Consider the surface $z = f(x, y)$ in Fig. 62-1. Let APB and CPD be sections of the surface cut by planes through P, parallel to xOz and yOz, respectively. As x varies while y is held fixed, P moves along the curve APB and the value of $\partial z/\partial x$ at P is the slope of the curve APB at P.

Similarly, as y varies while x is held fixed, P moves along the curve CPD and the value of $\partial z/\partial y$ at P is the slope of the curve CPD at P. (See Problem 13.)

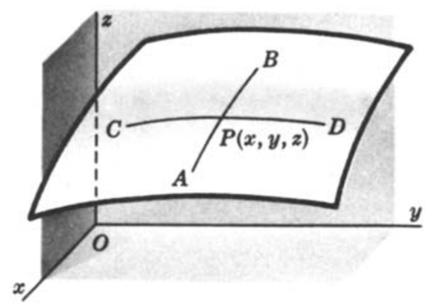


Fig. 62-1

PARTIAL DERIVATIVES OF HIGHER ORDERS. The partial derivative $\partial z/\partial x$ of $z = f(x, y)$ may in turn be differentiated partially with respect to x and y, yielding the second partial derivatives

$$\frac{\partial^2 z}{\partial x^2} = f_{xx}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \quad \text{and} \quad \frac{\partial^2 z}{\partial y\,\partial x} = f_{yx}(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

Similarly, from $\partial z/\partial y$ we may obtain

$$\frac{\partial^2 z}{\partial x\,\partial y} = f_{xy}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2} = f_{yy}(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

If $z = f(x, y)$ and its partial derivatives are continuous, the order of differentiation turns out to be immaterial; that is, $\dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{\partial^2 z}{\partial y\,\partial x}$. (See Problems 14 and 15.)

Solved Problems

1. Investigate $z = x^2 + y^2$ for continuity.

For any set of finite values $(x, y) = (a, b)$, we have $z = a^2 + b^2$. As $x \to a$ and $y \to b$, $x^2 + y^2 \to a^2 + b^2$. Hence, the function is continuous everywhere.

2. The following functions are continuous everywhere except at the origin $(0, 0)$, where they are not defined. Can they be made continuous there?

(*a*) $z = \dfrac{\sin(x+y)}{x+y}$

Let $(x, y) \to (0, 0)$ over the line $y = mx$; then $z = \dfrac{\sin(x+y)}{x+y} = \dfrac{\sin(1+m)x}{(1+m)x} \to 1$. The function may be made continuous everywhere by redefining it as $z = \dfrac{\sin(x+y)}{x+y}$ for $(x, y) \neq (0, 0)$; $z = 1$ for $(x, y) = (0, 0)$.

(*b*) $z = \dfrac{xy}{x^2 + y^2}$

Let $(x, y) \to (0, 0)$ over the line $y = mx$; the limiting value of $z = \dfrac{xy}{x^2 + y^2} = \dfrac{m}{1 + m^2}$ depends on the particular line chosen. Thus, the function cannot be made continuous at $(0, 0)$.

In Problems 3 to 7, find the first partial derivatives.

3. $z = 2x^2 - 3xy + 4y^2$

Treating y as a constant and differentiating with respect to x yield $\frac{\partial z}{\partial x} = 4x - 3y$.

Treating x as a constant and differentiating with respect to y yield $\frac{\partial z}{\partial y} = -3x + 8y$.

4. $z = \frac{x^2}{y} + \frac{y^2}{x}$

Treating y as a constant and differentiating with respect to x yield $\frac{\partial z}{\partial x} = \frac{2x}{y} - \frac{y^2}{x^2}$.

Treating x as a constant and differentiating with respect to y yield $\frac{\partial z}{\partial y} = -\frac{x^2}{y^2} + \frac{2y}{x}$.

5. $z = \sin(2x + 3y)$

$$\frac{\partial z}{\partial x} = 2\cos(2x + 3y) \quad \text{and} \quad \frac{\partial z}{\partial y} = 3\cos(2x + 3y)$$

6. $z = \arctan x^2y + \arctan xy^2$

$$\frac{\partial z}{\partial x} = \frac{2xy}{1 + x^4y^2} + \frac{y^2}{1 + x^2y^4} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{x^2}{1 + x^4y^2} + \frac{2xy}{1 + x^2y^4}$$

7. $z = e^{x^2 + xy}$

$$\frac{\partial z}{\partial x} = e^{x^2 + xy}(2x + y) = z(2x + y) \quad \text{and} \quad \frac{\partial z}{\partial y} = e^{x^2 + xy}(x) = xz$$

8. The area of a triangle is given by $K = \frac{1}{2}ab \sin C$. If $a = 20$, $b = 30$, and $C = 30°$, find:
(*a*) The rate of change of K with respect to a, when b and C are constant.
(*b*) The rate of change of K with respect to C, when a and b are constant.
(*c*) The rate of change of b with respect to a, when K and C are constant.

(*a*) $\frac{\partial K}{\partial a} = \frac{1}{2} b \sin C = \frac{1}{2}(30)(\sin 30°) = \frac{15}{2}$

(*b*) $\frac{\partial K}{\partial C} = \frac{1}{2} ab \cos C = \frac{1}{2}(20)(30)(\cos 30°) = 150\sqrt{3}$

(*c*) $b = \frac{2K}{a \sin C}$ and $\frac{\partial b}{\partial a} = -\frac{2K}{a^2 \sin C} = -\frac{2(\frac{1}{2}ab \sin C)}{a^2 \sin C} = -\frac{b}{a} = -\frac{3}{2}$

In Problems 9 to 11, find the first partial derivatives of z with respect to the independent variables x and y.

9. $x^2 + y^2 + z^2 = 25$

Solution 1: Solve for z to obtain $z = \pm\sqrt{25 - x^2 - y^2}$. Then

$$\frac{\partial z}{\partial x} = \frac{-x}{\pm\sqrt{25 - x^2 - y^2}} = -\frac{x}{z} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{-y}{\pm\sqrt{25 - x^2 - y^2}} = -\frac{y}{z}$$

Solution 2: Differentiate implicitly with respect to x, treating y as a constant, to obtain

$$2x + 2z\frac{\partial z}{\partial x} = 0 \quad \text{or} \quad \frac{\partial z}{\partial x} = -\frac{x}{z}$$

Then differentiate implicitly with respect to y, treating x as a constant:

$$2y + 2z\frac{\partial z}{\partial y} = 0 \quad \text{or} \quad \frac{\partial z}{\partial y} = -\frac{y}{z}$$

10. $x^2(2y+3z)+y^2(3x-4z)+z^2(x-2y)=xyz$

The procedure of Solution 1 of Problem 9 would be inconvenient here. Instead, differentiating implicitly with respect to x yields

$$2x(2y+3z)+3x^2\frac{\partial z}{\partial x}+3y^2-4y^2\frac{\partial z}{\partial x}+2z(x-2y)\frac{\partial z}{\partial x}+z^2=yz+xy\frac{\partial z}{\partial x}$$

so that
$$\frac{\partial z}{\partial x}=-\frac{4xy+6xz+3y^2+z^2-yz}{3x^2-4y^2+2xz-4yz-xy}$$

Differentiating implicitly with respect to y yields

$$2x^2+3x^2\frac{\partial z}{\partial y}+2y(3x-4z)-4y^2\frac{\partial z}{\partial y}+2z(x-2y)\frac{\partial z}{\partial y}-2z^2=xz+xy\frac{\partial z}{\partial y}$$

so that
$$\frac{\partial z}{\partial y}=-\frac{2x^2+6xy-8yz-2z^2-xz}{3x^2-4y^2+2xz-4yz-xy}$$

11. $xy+yz+zx=1$

Differentiating with respect to x yields $y+y\dfrac{\partial z}{\partial x}+x\dfrac{\partial z}{\partial x}+z=0$ and $\dfrac{\partial z}{\partial x}=-\dfrac{y+z}{x+y}$.

Differentiating with respect to y yields $x+y\dfrac{\partial z}{\partial y}+z+x\dfrac{\partial z}{\partial y}=0$ and $\dfrac{\partial z}{\partial y}=-\dfrac{x+z}{x+y}$.

12. Considering x and y as independent variables, find $\dfrac{\partial r}{\partial x},\dfrac{\partial r}{\partial y},\dfrac{\partial \theta}{\partial x},\dfrac{\partial \theta}{\partial y}$ when $x=e^{2r}\cos\theta$, $y=e^{3r}\sin\theta$.

First differentiate the given relations with respect to x:

$$1=2e^{2r}\cos\theta\frac{\partial r}{\partial x}-e^{2r}\sin\theta\frac{\partial \theta}{\partial x}\qquad\text{and}\qquad 0=3e^{3r}\sin\theta\frac{\partial r}{\partial x}+e^{3r}\cos\theta\frac{\partial \theta}{\partial x}$$

Then solve simultaneously to obtain $\dfrac{\partial r}{\partial x}=\dfrac{\cos\theta}{e^{2r}(2+\sin^2\theta)}$ and $\dfrac{\partial \theta}{\partial x}=-\dfrac{3\sin\theta}{e^{2r}(2+\sin^2\theta)}$.

Now differentiate the given relations with respect to y:

$$0=2e^{2r}\cos\theta\frac{\partial r}{\partial y}-e^{2r}\sin\theta\frac{\partial \theta}{\partial y}\qquad\text{and}\qquad 1=3e^{3r}\sin\theta\frac{\partial r}{\partial y}+e^{3r}\cos\theta\frac{\partial \theta}{\partial y}$$

Then solve simultaneously to obtain $\dfrac{\partial r}{\partial y}=\dfrac{\sin\theta}{e^{3r}(2+\sin^2\theta)}$ and $\dfrac{\partial \theta}{\partial y}=\dfrac{2\cos\theta}{e^{3r}(2+\sin^2\theta)}$.

13. Find the slopes of the curves cut from the surface $z=3x^2+4y^2-6$ by planes through the point $(1,1,1)$ and parallel to the coordinate planes xOz and yOz.

The plane $x=1$, parallel to the plane yOz, intersects the surface in the curve $z=4y^2-3$, $x=1$. Then $\partial z/\partial y=8y=8\times 1=8$ is the required slope.

The plane $y=1$, parallel to the plane xOz, intersects the surface in the curve $z=3x^2-2$, $y=1$. Then $\partial z/\partial x=6x=6$ is the required slope.

In Problems 14 and 15, find all second partial derivatives of z.

14. $z=x^2+3xy+y^2$

$$\frac{\partial z}{\partial x}=2x+3y\qquad \frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=2\qquad \frac{\partial^2 z}{\partial y\,\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=3$$

$$\frac{\partial z}{\partial y}=3x+2y\qquad \frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=2\qquad \frac{\partial^2 z}{\partial x\,\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=3$$

15. $z = x\cos y - y\cos x$

$$\frac{\partial z}{\partial x} = \cos y + y\sin x \qquad \frac{\partial z}{\partial y} = -x\sin y - \cos x \qquad \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = y\cos x$$

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = -\sin y + \sin x = \frac{\partial^2 z}{\partial x\,\partial y} \qquad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = -x\cos y$$

Supplementary Problems

16. Investigate each of the following to determine whether or not it can be made continuous at $(0, 0)$: $(a)\ \frac{y^2}{x^2+y^2}$, $(b)\ \frac{x-y}{x+y}$, $(c)\ \frac{x^3+y^3}{x^2+y^2}$, $(d)\ \frac{x+y}{x^2+y^2}$. *Ans.* (*a*) no; (*b*) no; (*c*) yes; (*d*) no

17. For each of the following functions, find $\partial z/\partial x$ and $\partial z/\partial y$.

(*a*) $z = x^2 + 3xy + y^2$ *Ans.* $\frac{\partial z}{\partial x} = 2x + 3y;\ \frac{\partial z}{\partial y} = 3x + 2y$

(*b*) $z = \frac{x}{y^2} - \frac{y}{x^2}$ *Ans.* $\frac{\partial z}{\partial x} = \frac{1}{y^2} + \frac{2y}{x^3};\ \frac{\partial z}{\partial y} = -\frac{2x}{y^3} - \frac{1}{x^2}$

(*c*) $z = \sin 3x \cos 4y$ *Ans.* $\frac{\partial z}{\partial x} = 3\cos 3x\cos 4y;\ \frac{\partial z}{\partial y} = -4\sin 3x\sin 4y$

(*d*) $z = \arctan\frac{y}{x}$ *Ans.* $\frac{\partial z}{\partial x} = \frac{-y}{x^2+y^2};\ \frac{\partial z}{\partial y} = \frac{x}{x^2+y^2}$

(*e*) $x^2 - 4y^2 + 9z^2 = 36$ *Ans.* $\frac{\partial z}{\partial x} = -\frac{x}{9z};\ \frac{\partial z}{\partial y} = \frac{4y}{9z}$

(*f*) $z^3 - 3x^2y + 6xyz = 0$ *Ans.* $\frac{\partial z}{\partial x} = \frac{2y(x-z)}{z^2+2xy};\ \frac{\partial z}{\partial y} = \frac{x(x-2z)}{z^2+2xy}$

(*g*) $yz + xz + xy = 0$ *Ans.* $\frac{\partial z}{\partial x} = -\frac{y+z}{x+y};\ \frac{\partial z}{\partial y} = -\frac{x+z}{x+y}$

18. (*a*) If $z = \sqrt{x^2+y^2}$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$.

(*b*) If $z = \ln\sqrt{x^2+y^2}$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1$.

(*c*) If $z = e^{x/y}\sin\frac{x}{y} + e^{y/x}\cos\frac{y}{x}$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$.

(*d*) If $z = (ax+by)^2 + e^{ax+by} + \sin(ax+by)$, show that $b\frac{\partial z}{\partial x} = a\frac{\partial z}{\partial y}$.

19. Find the equation of the line tangent to
(*a*) The parabola $z = 2x^2 - 3y^2$, $y = 1$ at the point $(-2, 1, 5)$ *Ans.* $8x + z + 11 = 0,\ y = 1$
(*b*) The parabola $z = 2x^2 - 3y^2$, $x = -2$ at the point $(-2, 1, 5)$ *Ans.* $6y + z - 11 = 0,\ x = -2$
(*c*) The hyperbola $z = 2x^2 - 3y^2$, $z = 5$ at the point $(-2, 1, 5)$ *Ans.* $4x + 3y + 5 = 0,\ z = 5$
Show that these three lines lie in the plane $8x + 6y + z + 5 = 0$.

20. For each of the following functions, find $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x\,\partial y}$, $\frac{\partial^2 z}{\partial y\,\partial x}$, and $\frac{\partial^2 z}{\partial y^2}$.

(a) $z = 2x^2 - 5xy + y^2$ *Ans.* $\dfrac{\partial^2 z}{\partial x^2} = 4;\ \dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{\partial^2 z}{\partial y\,\partial x} = -5;\ \dfrac{\partial^2 z}{\partial y^2} = 2$

(b) $z = \dfrac{x}{y^2} - \dfrac{y}{x^2}$ *Ans.* $\dfrac{\partial^2 z}{\partial x^2} = -\dfrac{6y}{x^4};\ \dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{\partial^2 z}{\partial y\,\partial x} = 2\left(\dfrac{1}{x^3} - \dfrac{1}{y^3}\right);\ \dfrac{\partial^2 z}{\partial y^2} = \dfrac{6x}{y^4}$

(c) $z = \sin 3x \cos 4y$ *Ans.* $\dfrac{\partial^2 z}{\partial x^2} = -9z;\ \dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{\partial^2 z}{\partial y\,\partial x} = -12\cos 3x \sin 4y;\ \dfrac{\partial^2 z}{\partial y^2} = -16z$

(d) $z = \arctan \dfrac{y}{x}$ *Ans.* $\dfrac{\partial^2 z}{\partial x^2} = -\dfrac{\partial^2 z}{\partial y^2} = \dfrac{2xy}{(x^2+y^2)^2};\ \dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{\partial^2 z}{\partial y\,\partial x} = \dfrac{y^2 - x^2}{(x^2+y^2)^2}$

21. (a) If $z = \dfrac{xy}{x-y}$, show that $x^2 \dfrac{\partial^2 z}{\partial x^2} + 2xy \dfrac{\partial^2 z}{\partial x\,\partial y} + y^2 \dfrac{\partial^2 z}{\partial y^2} = 0$.

(b) If $z = e^{\alpha x} \cos \beta y$ and $\beta = \pm\alpha$, show that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$.

(c) If $z = e^{-t}(\sin x + \cos y)$, show that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial z}{\partial t}$.

(d) If $z = \sin ax \sin by \sin kt\sqrt{a^2+b^2}$, show that $\dfrac{\partial^2 z}{\partial t^2} = k^2\left(\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2}\right)$.

22. For the gas formula $\left(p + \dfrac{a}{v^2}\right)(v - b) = ct$, where a, b, and c are constants, show that

$$\frac{\partial p}{\partial v} = \frac{2a(v-b) - (p + a/v^2)v^3}{v^3(v-b)} \qquad \frac{\partial v}{\partial t} = \frac{cv^3}{(p + a/v^2)v^3 - 2a(v-b)}$$

$$\frac{\partial t}{\partial p} = \frac{v-b}{c} \qquad \frac{\partial p}{\partial v}\,\frac{\partial v}{\partial t}\,\frac{\partial t}{\partial p} = -1$$

[For the last result, see Problem 11 of Chapter 64.]

Chapter 63

Total Differentials and Total Derivatives

TOTAL DIFFERENTIALS. The differentials dx and dy for the function $y = f(x)$ of a single independent variable x were defined in Chapter 28 as

$$dx = \Delta x \qquad \text{and} \qquad dy = f'(x)\,dx = \frac{dy}{dx}\,dx$$

Consider the function $z = f(x, y)$ of the two independent variables x and y, and define $dx = \Delta x$ and $dy = \Delta y$. When x varies while y is held fixed, z is a function of x only and *the partial differential of z with respect to x* is defined as $d_x z = f_x(x, y)\,dx = \frac{\partial z}{\partial x}\,dx$. Similarly, *the partial differential of z with respect to y* is defined as $d_y z = f_y(x, y)\,dy = \frac{\partial z}{\partial y}\,dy$. The *total differential dz* is defined as the sum of the partial differentials,

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy \tag{63.1}$$

For a function $w = F(x, y, z, \ldots, t)$, the total differential dw is defined as

$$dw = \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz + \cdots + \frac{\partial w}{\partial t}\,dt \tag{63.2}$$

(See Problems 1 and 2.)

As in the case of a function of a single variable, the total differential of a function of several variables gives a good approximation of the total increment of the function when the increments of the several independent variables are small.

EXAMPLE 1: When $z = xy$, $dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = y\,dx + x\,dy$; and when x and y are given increments $\Delta x = dx$ and $\Delta y = dy$, the increment Δz taken on by z is

$$\begin{aligned}\Delta z &= (x + \Delta x)(y + \Delta y) - xy = x\,\Delta y + y\,\Delta x + \Delta x\,\Delta y \\ &= x\,dy + y\,dx + dx\,dy\end{aligned}$$

A geometric interpretation is given in Fig. 63-1: dz and Δz differ by the rectangle of area $\Delta x\,\Delta y = dx\,dy$.

(See Problems 3 to 9.)

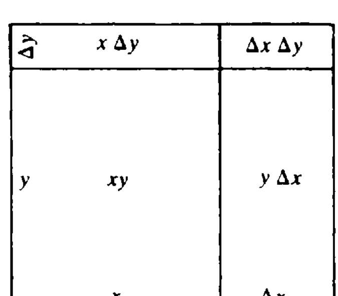


Fig. 63-1

THE CHAIN RULE FOR COMPOSITE FUNCTIONS. If $z = f(x, y)$ is a continuous function of the variables x and y with continuous partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$, and if x and y are differentiable functions $x = g(t)$ and $y = h(t)$ of a variable t, then z is a function of t and dz/dt, called the *total derivative* of z with respect to t, is given by

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \tag{63.3}$$

Similarly, if $w = f(x, y, z, \ldots)$ is a continuous function of the variables $x, y, z, \ldots$ with continuous partial derivatives, and if $x, y, z, \ldots$ are differentiable functions of a variable t, the total derivative of w with respect to t is given by

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} + \cdots \tag{63.4}$$

(See Problems 10 to 16.)

If $z = f(x, y)$ is a continuous function of the variables x and y with continuous partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$, and if x and y are continuous functions $x = g(r, s)$ and $y = h(r, s)$ of the independent variables r and s, then z is a function of r and s with

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \tag{63.5}$$

Similarly, if $w = f(x, y, z, \ldots)$ is a continuous function of the variables $x, y, z, \ldots$ with continuous partial derivatives $\partial w/\partial x$, $\partial w/\partial y$, $\partial w/\partial z, \ldots$, and if $x, y, z, \ldots$ are continuous functions of the independent variables $r, s, t, \ldots$, then

$$\begin{aligned} \frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} + \cdots \\ \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} + \cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{63.6}$$

(See Problems 17 to 19.)

Solved Problems

In Problems 1 and 2, find the total differential.

1. $z = x^3y + x^2y^2 + xy^3$

We have $\quad \dfrac{\partial z}{\partial x} = 3x^2y + 2xy^2 + y^3 \quad$ and $\quad \dfrac{\partial z}{\partial y} = x^3 + 2x^2y + 3xy^2$

Then $\quad dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy = (3x^2y + 2xy^2 + y^3)\,dx + (x^3 + 2x^2y + 3xy^2)\,dy$

2. $z = x\sin y - y\sin x$

We have $\quad \dfrac{\partial z}{\partial x} = \sin y - y\cos x \quad$ and $\quad \dfrac{\partial z}{\partial y} = x\cos y - \sin x$

Then $\quad dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy = (\sin y - y\cos x)\,dx + (x\cos y - \sin x)\,dy$

3. Compare dz and Δz, given $z = x^2 + 2xy - 3y^2$.

$$\frac{\partial z}{\partial x} = 2x + 2y \quad \text{and} \quad \frac{\partial z}{\partial y} = 2x - 6y . \quad \text{So} \quad dz = 2(x+y)\,dx + 2(x-3y)\,dy$$

Also,

$$\begin{aligned}\Delta z &= [(x+dx)^2 + 2(x+dx)(y+dy) - 3(y+dy)^2] - (x^2 + 2xy - 3y^2)\\ &= 2(x+y)\,dx + 2(x-3y)\,dy + (dx)^2 + 2\,dx\,dy - 3(dy)^2\end{aligned}$$

Thus dz and Δz differ by $(dx)^2 + 2\,dx\,dy - 3(dy)^2$.

4. Approximate the area of a rectangle of dimensions 35.02 by 24.97 units.

For dimensions x by y, the area is $A = xy$ so that $dA = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy = y\,dx + x\,dy$. With $x = 35$, $dx = 0.02$, $y = 25$, and $dy = -0.03$, we have $A = 35(25) = 875$ and $dA = 25(0.02) + 35(-0.03) = -0.55$. The area is approximately $A + dA = 874.45$ square units.

5. Approximate the change in the hypotenuse of a right triangle of legs 6 and 8 inches when the shorter leg is lengthened by $\frac{1}{4}$ inch and the longer leg is shortened by $\frac{1}{8}$ inch.

Let x, y, and z be the shorter leg, the longer leg, and the hypotenuse of the triangle. Then

$$z = \sqrt{x^2 + y^2} \qquad \frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \qquad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} \qquad dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}$$

When $x = 6$, $y = 8$, $dx = \frac{1}{4}$, and $dy = -\frac{1}{8}$, then $dz = \frac{6(\frac{1}{4}) + 8(-\frac{1}{8})}{\sqrt{6^2 + 8^2}} = \frac{1}{20}$ inch. Thus the hypotenuse is lengthened by approximately $\frac{1}{20}$ inch.

6. The power consumed in an electrical resistor is given by $P = E^2/R$ (in watts). If $E = 200$ volts and $R = 8$ ohms, by how much does the power change if E is decreased by 5 volts and R is decreased by 0.2 ohm?

We have

$$\frac{\partial P}{\partial E} = \frac{2E}{R} \qquad \frac{\partial P}{\partial R} = -\frac{E^2}{R^2} \qquad dP = \frac{2E}{R}dE - \frac{E^2}{R^2}dR$$

When $E = 200$, $R = 8$, $dE = -5$, and $dR = -0.2$, then

$$dP = \frac{2(200)}{8}(-5) - \left(\frac{200}{8}\right)^2(-0.2) = -250 + 125 = -125$$

The power is reduced by approximately 125 watts.

7. The dimensions of a rectangular block of wood were found to be 10, 12, and 20 inches, with a possible error of 0.05 in in each of the measurements. Find (approximately) the greatest error in the surface area of the block and the percentage error in the area caused by the errors in the individual measurements.

The surface area is $S = 2(xy + yz + zx)$; then

$$dS = \frac{\partial S}{\partial x}dx + \frac{\partial S}{\partial y}dy + \frac{\partial S}{dz}dz = 2(y+z)\,dx + 2(x+z)\,dy + 2(y+x)\,dz$$

The greatest error in S occurs when the errors in the lengths are of the same sign, say positive. Then

$$dS = 2(12+20)(0.05) + 2(10+20)(0.05) + 2(12+10)(0.05) = 8.4 \text{ in}^2$$

The percentage error is (error/area)(100) = (8.4/1120)(100) = 0.75%.

8. For the formula $R = E/C$, find the maximum error and the percentage error if $C = 20$ with a possible error of 0.1 and $E = 120$ with a possible error of 0.05.

Here $$dR = \frac{\partial R}{\partial E}\,dE + \frac{\partial R}{\partial C}\,dC = \frac{1}{C}\,dE - \frac{E}{C^2}\,dC$$

The maximum error will occur when $dE = 0.05$ and $dC = -0.1$; then $dR = \frac{0.05}{20} - \frac{120}{400}(-0.1) = 0.0325$ is the approximate maximum error. The percentage error is $\frac{dR}{R}(100) = \frac{0.0325}{8}(100) = 0.40625 = 0.41\%$.

9. Two sides of a triangle were measured as 150 and 200 ft, and the included angle as 60°. If the possible errors are 0.2 ft in measuring the sides and 1° in the angle, what is the greatest possible error in the computed area?

Here $$A = \frac{1}{2}xy\sin\theta \qquad \frac{\partial A}{\partial x} = \frac{1}{2}y\sin\theta \qquad \frac{\partial A}{\partial y} = \frac{1}{2}x\sin\theta \qquad \frac{\partial A}{\partial \theta} = \frac{1}{2}xy\cos\theta$$

and $$dA = \frac{1}{2}y\sin\theta\,dx + \frac{1}{2}x\sin\theta\,dy + \frac{1}{2}xy\cos\theta\,d\theta$$

When $x = 150$, $y = 200$, $\theta = 60°$, $dx = 0.2$, $dy = 0.2$, and $d\theta = 1° = \pi/180$, then

$$dA = \tfrac{1}{2}(200)(\sin 60°)(0.2) + \tfrac{1}{2}(150)(\sin 60°)(0.2) + \tfrac{1}{2}(150)(200)(\cos 60°)(\pi/180) = 161.21\text{ ft}^2$$

10. Find dz/dt, given $z = x^2 + 3xy + 5y^2$; $x = \sin t$, $y = \cos t$.

Since $$\frac{\partial z}{\partial x} = 2x + 3y \qquad \frac{\partial z}{\partial y} = 3x + 10y \qquad \frac{dx}{dt} = \cos t \qquad \frac{dy}{dt} = -\sin t$$

we have $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (2x + 3y)\cos t - (3x + 10y)\sin t$$

11. Find dz/dt, given $z = \ln(x^2 + y^2)$; $x = e^{-t}$, $y = e^t$.

Since $$\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2} \qquad \frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2} \qquad \frac{dx}{dt} = -e^{-t} \qquad \frac{dy}{dt} = e^t$$

we have $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{2x}{x^2 + y^2}(-e^{-t}) + \frac{2y}{x^2 + y^2}e^t = 2\,\frac{ye^t - xe^{-t}}{x^2 + y^2}$$

12. Let $z = f(x, y)$ be a continuous function of x and y with continuous partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$, and let y be a differentiable function of x. Then z is a differentiable function of x. Find a formula for dz/dx.

By (63.3), $$\frac{dz}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

The shift in notation from z to f is made here to avoid possible confusion arising from the use of dz/dx and $\partial z/\partial x$ in the same expression.

13. Find dz/dx, given $z = f(x, y) = x^2 + 2xy + 4y^2$, $y = e^{ax}$.

$$\frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = (2x + 2y) + (2x + 8y)ae^{ax} = 2(x + y) + 2a(x + 4y)e^{ax}$$

14. Find (*a*) dz/dx and (*b*) dz/dy, given $z = f(x, y) = xy^2 + x^2y$, $y = \ln x$.

(*a*) Here x is the independent variable:

$$\frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = (y^2 + 2xy) + (2xy + x^2)\frac{1}{x} = y^2 + 2xy + 2y + x$$

(b) Here y is the independent variable:

$$\frac{dz}{dy} = \frac{\partial f}{\partial x}\frac{dx}{dy} + \frac{\partial f}{\partial y} = (y^2 + 2xy)x + (2xy + x^2) = xy^2 + 2x^2y + 2xy + x^2$$

15. The altitude of a right circular cone is 15 inches and is increasing at 0.2 in/min. The radius of the base is 10 inches and is decreasing at 0.3 in/min. How fast is the volume changing?

Let x be the radius, and y the altitude of the cone (Fig. 63-2). From $V = \frac{1}{3}\pi x^2 y$, considering x and y as functions of time t, we have

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} = \frac{\pi}{3}\left(2xy\frac{dx}{dt} + x^2\frac{dy}{dt}\right) = \frac{\pi}{3}[2(10)(15)(-0.3) + 10^2(0.2)] = -70\pi/3 \text{ in}^3/\text{min}$$

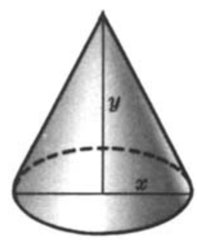


Fig. 63-2

16. A point P is moving along the curve of intersection of the paraboloid $\frac{x^2}{16} - \frac{y^2}{9} = z$ and the cylinder $x^2 + y^2 = 5$, with x, y, and z expressed in inches. If x is increasing at 0.2 in/min, how fast is z changing when $x = 2$?

From $z = \frac{x^2}{16} - \frac{y^2}{9}$, we obtain $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{x}{8}\frac{dx}{dt} - \frac{2y}{9}\frac{dy}{dt}$. Since $x^2 + y^2 = 5$, $y = \pm 1$ when $x = 2$; also, differentiation yields $x\frac{dx}{dt} + y\frac{dy}{dt} = 0$.

When $y = 1$, $\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{2}{1}(0.2) = -0.4$ and $\frac{dz}{dt} = \frac{2}{8}(0.2) - \frac{2}{9}(-0.4) = \frac{5}{36}$ in/min.

When $y = -1$, $\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = 0.4$ and $\frac{dz}{dt} = \frac{2}{8}(0.2) - \frac{2}{9}(-1)(0.4) = \frac{5}{36}$ in/min.

17. Find $\partial z/\partial r$ and $\partial z/\partial s$, given $z = x^2 + xy + y^2$; $x = 2r + s$, $y = r - 2s$.

Here $\quad \frac{\partial z}{\partial x} = 2x + y \quad \frac{\partial z}{\partial y} = x + 2y \quad \frac{\partial x}{\partial r} = 2 \quad \frac{\partial x}{\partial s} = 1 \quad \frac{\partial y}{\partial r} = 1 \quad \frac{\partial y}{\partial s} = -2$

Then

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = (2x + y)(2) + (x + 2y)(1) = 5x + 4y$$

and

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (2x + y)(1) + (x + 2y)(-2) = -3y$$

18. Find $\frac{\partial u}{\partial \rho}$, $\frac{\partial u}{\partial \beta}$, and $\frac{\partial u}{\partial \theta}$, given $u = x^2 + 2y^2 + 2z^2$; $x = \rho \sin\beta \cos\theta$, $y = \rho \sin\beta \sin\theta$, $z = \rho\cos\beta$.

$$\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \rho} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \rho} = 2x\sin\beta\cos\theta + 4y\sin\beta\sin\theta + 4z\cos\beta$$

$$\frac{\partial u}{\partial \beta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \beta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \beta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \beta} = 2x\,\rho\cos\beta\cos\theta + 4y\,\rho\cos\beta\sin\theta - 4z\,\rho\sin\beta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta} = -2x\,\rho\sin\beta\sin\theta + 4y\,\rho\sin\beta\cos\theta$$

19. Find du/dx, given $u = f(x, y, z) = xy + yz + zx$; $y = 1/x$, $z = x^2$.

From (*63.6*),

$$\frac{du}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial z}\frac{dz}{dx} = (y+z) + (x+z)\left(-\frac{1}{x^2}\right) + (y+x)2x = y + z + 2x(x+y) - \frac{x+z}{x^2}$$

20. If $z = f(x, y)$ is a continuous function of x and y possessing continuous first partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$, derive the basic formula

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y \qquad (1)$$

where ϵ_1 and $\epsilon_2 \to 0$ as Δx and $\Delta y \to 0$.

When x and y are given increments Δx and Δy respectively, the increment given to z is

$$\begin{aligned}\Delta z &= f(x+\Delta x, y+\Delta y) - f(x, y)\\ &= [f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)] + [f(x, y+\Delta y) - f(x, y)] \qquad (2)\end{aligned}$$

In the first bracketed expression, only x changes; in the second, only y changes. Thus, the law of the mean (*26.5*) may be applied to each:

$$f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) = \Delta x\, f_x(x+\theta_1\Delta x, y+\Delta y) \qquad (3)$$

$$f(x, y+\Delta y) - f(x, y) = \Delta y\, f_y(x, y+\theta_2\Delta y) \qquad (4)$$

where $0 < \theta_1 < 1$ and $0 < \theta_2 < 1$. Note that here the derivatives involved are partial derivatives.

Since $\partial z/\partial x = f_x(x, y)$ and $\partial z/\partial y = f_y(x, y)$ are, by hypothesis, continuous functions of x and y,

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} f_x(x+\theta_1\Delta x, y+\Delta y) = f_x(x, y) \qquad \text{and} \qquad \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} f_y(x, y+\theta_2\Delta y) = f_y(x, y)$$

Then $\quad f_x(x+\theta_1\Delta x, y+\Delta y) = f_x(x, y) + \epsilon_1 \quad$ and $\quad f_y(x, y+\theta_2\Delta y) = f_y(x, y) + \epsilon_2$

where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as Δx and $\Delta y \to 0$.

After making these replacements in (*3*) and (*4*) and then substituting in (*1*), we have, as required,

$$\Delta z = [f_x(x, y) + \epsilon_1]\Delta x + [f_y(x, y) + \epsilon_2]\Delta y = f_x(x, y)\Delta x + f_y(x, y)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$$

Note that the total derivative dz is a fairly good approximation of the total increment Δz when $|\Delta x|$ and $|\Delta y|$ are small.

Supplementary Problems

21. Find the total differential, given:

(*a*) $z = x^3y + 2xy^3$ *Ans.* $dz = (3x^2 + 2y^2)y\,dx + (x^2 + 6y^2)x\,dy$

(*b*) $\theta = \arctan(y/x)$ *Ans.* $d\theta = \dfrac{x\,dy - y\,dx}{x^2 + y^2}$

(*c*) $z = e^{x^2 - y^2}$ *Ans.* $dz = 2z(x\,dx - y\,dy)$

(*d*) $z = x(x^2 + y^2)^{-1/2}$ *Ans.* $dz = \dfrac{y(y\,dx - x\,dy)}{(x^2 + y^2)^{3/2}}$

22. The fundamental frequency of vibration of a string or wire of circular section under tension T is $n = \frac{1}{2rl}\sqrt{\frac{T}{\pi d}}$, where l is the length, r the radius, and d the density of the string. Find (*a*) the approximate effect of changing l by a small amount dl, (*b*) the effect of changing T by a small amount dT, and (*c*) the effect of changing l and T simultaneously.

Ans. (*a*) $-(n/l)\,dl$; (*b*) $(n/2T)\,dT$; (*c*) $n(-dl/l + dT/2T)$

23. Use differentials to compute (*a*) the volume of a box with square base of side 8.005 and height 9.996 ft; (*b*) the diagonal of a rectangular box of dimensions 3.03 by 5.98 by 6.01 ft.

Ans. (*a*) 640.544 ft^3; (*b*) 9.003 ft

24. Approximate the maximum possible error and the percentage of error when z is computed by the given formula:

(*a*) $z = \pi r^2 h$; $r = 5 \pm 0.05$, $h = 12 \pm 0.1$ *Ans.* 8.5π; 2.8%

(*b*) $1/z = 1/f + 1/g$; $f = 4 \pm 0.01$, $g = 8 \pm 0.02$ *Ans.* 0.0067; 0.25%

(*c*) $z = y/x$; $x = 1.8 \pm 0.1$, $y = 2.4 \pm 0.1$ *Ans.* 0.13; 10%

25. Find the approximate maximum percentage of error in:

(*a*) $\omega = \sqrt[3]{g/b}$ if there is a possible 1% error in measuring g and a possible $\frac{1}{2}$% error in measuring b.
$\left(\textit{Hint}: \ln \omega = \frac{1}{3}(\ln g - \ln b);\ \frac{d\omega}{\omega} = \frac{1}{3}\left(\frac{dg}{g} - \frac{db}{b}\right);\ \left|\frac{dg}{g}\right| = 0.01;\ \left|\frac{db}{b}\right| = 0.005\right)$ *Ans.* 0.005

(*b*) $g = 2s/t^2$ if there is a possible 1% error in measuring s and $\frac{1}{4}$% error in measuring t.
Ans. 0.015

26. Find du/dt, given:

(*a*) $u = x^2y^3$; $x = 2t^3$, $y = 3t^2$ *Ans.* $6xy^2t(2yt + 3x)$

(*b*) $u = x\cos y + y\sin x$; $x = \sin 2t$, $y = \cos 2t$
Ans. $2(\cos y + y\cos x)\cos 2t - 2(-x\sin y + \sin x)\sin 2t$

(*c*) $u = xy + yz + zx$; $x = e^t$, $y = e^{-t}$, $z = e^t + e^{-t}$ *Ans.* $(x + 2y + z)e^t - (2x + y + z)e^{-t}$

27. At a certain instant the radius of a right circular cylinder is 6 inches and is increasing at the rate 0.2 in/sec, while the altitude is 8 inches and is decreasing at the rate 0.4 in/s. Find the time rate of change (*a*) of the volume and (*b*) of the surface at that instant.

Ans. (*a*) 4.8π in^3/sec; (*b*) 3.2π in^2/sec

28. A particle moves in a plane so that at any time t its abscissa and ordinate are given by $x = 2 + 3t$, $y = t^2 + 4$ with x and y in feet and t in minutes. How is the distance of the particle from the origin changing when $t = 1$? *Ans.* $5/\sqrt{2}$ ft/min

29. A point is moving along the curve of intersection of $x^2 + 3xy + 3y^2 = z^2$ and the plane $x - 2y + 4 = 0$. When $x = 2$ and is increasing at 3 units/sec, find (*a*) how y is changing, (*b*) how z is changing, and (*c*) the speed of the point.

Ans. (*a*) increasing 3/2 units/sec; (*b*) increasing 75/14 units/sec at (2, 3, 7) and decreasing 75/14 units/sec at (2, 3, −7); (*c*) 6.3 units/sec

30. Find $\partial z/\partial s$ and $\partial z/\partial t$, given:

(*a*) $z = x^2 - 2y^2$; $x = 3s + 2t$, $y = 3s - 2t$ *Ans.* $6(x - 2y)$; $4(x + 2y)$

(*b*) $z = x^2 + 3xy + y^2$; $x = \sin s + \cos t$, $y = \sin s - \cos t$ *Ans.* $5(x + y)\cos s$; $(x - y)\sin t$

(*c*) $z = x^2 + 2y^2$; $x = e^s - e^t$, $y = e^s + e^t$ *Ans.* $2(x + 2y)e^s$; $2(2y - x)e^t$

(*d*) $z = \sin(4x + 5y)$; $x = s + t$, $y = s - t$ *Ans.* $9\cos(4x + 5y)$; $-\cos(4x + 5y)$

(*e*) $z = e^{xy}$; $x = s^2 + 2st$, $y = 2st + t^2$ *Ans.* $2e^{xy}[tx + (s + t)y]$; $2e^{xy}[(s + t)x + sy]$

31. (*a*) If $u = f(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$, show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

(*b*) If $u = f(x, y)$ and $x = r\cosh s$, $y = r\sinh s$, show that

$$\left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 - \frac{1}{s^2}\left(\frac{\partial u}{\partial s}\right)^2$$

32. (*a*) If $z = f(x + \alpha y) + g(x - \alpha y)$, show that $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{1}{\alpha^2}\dfrac{\partial^2 z}{\partial y^2}$. (*Hint*: Write $z = f(u) + g(v)$, $u = x + \alpha y$, $v = x - \alpha y$.)

(*b*) If $z = x^n f(y/x)$, show that $x\,\partial z/\partial x + y\,\partial z/\partial y = nz$.

(*c*) If $z = f(x, y)$ and $x = g(t)$, $y = h(t)$, show that, subject to continuity conditions

$$\frac{d^2 z}{dt^2} = f_{xx}(g')^2 + 2f_{xy}g'h' + f_{yy}(h')^2 + f_x g'' + f_y h''$$

(*d*) If $z = f(x, y)$; $x = g(r, s)$, $y = h(r, s)$, show that, subject to continuity conditions

$$\frac{\partial^2 z}{\partial r^2} = f_{xx}(g_r)^2 + 2f_{xy}g_r h_r + f_{yy}(h_r)^2 + f_x g_{rr} + f_y h_{rr}$$

$$\frac{\partial^2 z}{\partial r\,\partial s} = f_{xx}g_r g_s + f_{xy}(g_r h_s + g_s h_r) + f_{yy}h_r h_s + f_x g_{rs} + f_y h_{rs}$$

$$\frac{\partial^2 z}{\partial s^2} = f_{xx}(g_s)^2 + 2f_{xy}g_s h_s + f_{yy}(h_s)^2 + f_x g_{ss} + f_y h_{ss}$$

33. A function $f(x, y)$ is called *homogeneous of order n* if $f(tx, ty) = t^n f(x, y)$. (For example, $f(x, y) = x^2 + 2xy + 3y^2$ is homogeneous of order 2; $f(x, y) = x\sin(y/x) + y\cos(y/x)$ is homogeneous of order 1.) Differentiate $f(tx, ty) = t^n f(x, y)$ with respect to t and replace t by 1 to show that $xf_x + yf_y = nf$. Verify this formula using the two given examples. See also Problem 32(*b*).

34. If $z = \phi(u, v)$, where $u = f(x, y)$ and $v = g(x, y)$, and if $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$, show that

(*a*) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$ (*b*) $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = \left\{\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial v}{\partial x}\right)^2\right\}\left(\dfrac{\partial^2 \phi}{\partial u^2} + \dfrac{\partial^2 \phi}{\partial v^2}\right)$

35. Use (*1*) of Problem 20 to derive the chain rules (*63.3*) and (*63.5*). (*Hint*: For (*63.3*), divide by Δt.)

Chapter 64

Implicit Functions

THE DIFFERENTIATION of a function of one variable, defined implicitly by a relation $f(x, y) = 0$, was treated intuitively in Chapter 11. For this case, we state without proof:

Theorem 64.1: If $f(x, y)$ is continuous in a region including a point (x_0, y_0) for which $f(x_0, y_0) = 0$, if $\partial f/\partial x$ and $\partial f/\partial y$ are continuous throughout the region, and if $\partial f/\partial y \neq 0$ at (x_0, y_0), then there is a neighborhood of (x_0, y_0) in which $f(x, y) = 0$ can be solved for y as a continuous differentiable function of x, $y = \phi(x)$, with $y_0 = \phi(x_0)$ and $\dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y}$.

(See Problems 1 to 3.)

Extending this theorem, we have the following:

Theorem 64.2: If $F(x, y, z)$ is continuous in a region including a point (x_0, y_0, z_0) for which $F(x_0, y_0, z_0) = 0$, if $\dfrac{\partial F}{\partial x}$, $\dfrac{\partial F}{\partial y}$, and $\dfrac{\partial F}{\partial z}$ are continuous throughout the region, and if $\partial F/\partial z \neq 0$ at (x_0, y_0, z_0), then there is a neighborhood of (x_0, y_0, z_0) in which $F(x, y, z) = 0$ can be solved for z as a continuous differentiable function of x and y, $z = \phi(x, y)$, with $z_0 = \phi(x_0, y_0)$ and $\dfrac{\partial z}{\partial x} = -\dfrac{\partial F/\partial x}{\partial F/\partial z}$, $\dfrac{\partial z}{\partial y} = -\dfrac{\partial F/\partial y}{\partial F/\partial z}$.

(See Problems 4 and 5.)

Theorem 64.3: If $f(x, y, u, v)$ and $g(x, y, u, v)$ are continuous in a region including the point (x_0, y_0, u_0, v_0) for which $f(x_0, y_0, u_0, v_0) = 0$ and $g(x_0, y_0, u_0, v_0) = 0$, if the first partial derivatives of f and of g are continuous throughout the region, and if at (x_0, y_0, u_0, v_0) the determinant $J\left(\dfrac{f, g}{u, v}\right) \equiv \begin{vmatrix} \partial f/\partial u & \partial f/\partial v \\ \partial g/\partial u & \partial g/\partial v \end{vmatrix} \neq 0$, then there is a neighborhood of (x_0, y_0, u_0, v_0) in which $f(x, y, u, v) = 0$ and $g(x, y, u, v) = 0$ can be solved simultaneously for u and v as continuous differentiable functions of x and y, $u = \phi(x, y)$ and $v = \psi(x, y)$. If at (x_0, y_0, u_0, v_0) the determinant $J\left(\dfrac{f, g}{x, y}\right) \neq 0$, then there is a neighborhood of (x_0, y_0, u_0, v_0) in which $f(x, y, u, v) = 0$ and $g(x, y, u, v) = 0$ can be solved for x and y as continuous differentiable functions of u and v, $x = h(u, v)$ and $y = k(u, v)$.

(See Problems 6 and 7.)

Solved Problems

1. Use Theorem 64.1 to show that $x^2 + y^2 - 13 = 0$ defines y as a continuous differentiable function of x in any neighborhood of the point $(2, 3)$ that does not include a point of the x axis. Find the derivative at the point.

Set $f(x, y) = x^2 + y^2 - 13$. Then $f(2, 3) = 0$, while in any neighborhood of $(2, 3)$ in which the function is defined, its partial derivatives $\partial f/\partial x = 2x$ and $\partial f/\partial y = 2y$ are continuous, and $\partial f/\partial y \neq 0$. Then

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \quad \text{and} \quad \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{x}{y} = -\frac{2}{3} \text{ at } (2, 3)$$

2. Find dy/dx, given $f(x, y) = y^3 + xy - 12 = 0$.

We have $\dfrac{\partial f}{\partial x} = y$ and $\dfrac{\partial f}{\partial y} = 3y^2 + x$. So $\dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y} = -\dfrac{y}{3y^2 + x}$

3. Find dy/dx, given $e^x \sin y + e^y \sin x = 1$.

Put $f(x, y) = e^x \sin y + e^y \sin x - 1$. Then $\dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y} = -\dfrac{e^x \sin y + e^y \cos x}{e^x \cos y + e^y \sin x}$.

4. Find $\partial z/\partial x$ and $\partial z/\partial y$, given $F(x, y, z) = x^2 + 3xy - 2y^2 + 3xz + z^2 = 0$.

Treating z as a function of x and y defined by the relation and differentiating partially with respect to x and again with respect to y, we have

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = (2x + 3y + 3z) + (3x + 2z)\frac{\partial z}{\partial x} = 0 \qquad (1)$$

and

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = (3x - 4y) + (3x + 2z)\frac{\partial z}{\partial y} = 0 \qquad (2)$$

From (1), $\dfrac{\partial z}{\partial x} = -\dfrac{\partial F/\partial x}{\partial F/\partial z} = -\dfrac{2x + 3y + 3z}{3x + 2z}$. From (2), $\dfrac{\partial z}{\partial y} = -\dfrac{\partial F/\partial y}{\partial F/\partial z} = -\dfrac{3x - 4y}{3x + 2z}$.

5. Find $\partial z/\partial x$ and $\partial z/\partial y$, given $\sin xy + \sin yz + \sin zx = 1$.

Set $F(x, y, z) = \sin xy + \sin yz + \sin zx - 1$; then

$$\frac{\partial F}{\partial x} = y \cos xy + z \cos zx \qquad \frac{\partial F}{\partial y} = x \cos xy + z \cos yz \qquad \frac{\partial F}{\partial z} = y \cos yz + x \cos zx$$

and

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} = -\frac{y \cos xy + z \cos zx}{y \cos yz + x \cos zx} \qquad \frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z} = -\frac{x \cos xy + z \cos yz}{y \cos yz + x \cos zx}$$

6. If u and v are defined as functions of x and y by the equations

$$f(x, y, u, v) = x + y^2 + 2uv = 0 \qquad g(x, y, u, v) = x^2 - xy + y^2 + u^2 + v^2 = 0$$

find (a) $\partial u/\partial x$, $\partial v/\partial x$ and (b) $\partial u/\partial y$, $\partial v/\partial y$.

(a) Differentiating f and g partially with respect to x, we obtain

$$1 + 2v\frac{\partial u}{\partial x} + 2u\frac{\partial v}{\partial x} = 0 \qquad \text{and} \qquad 2x - y + 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

Solving these relations simultaneously for $\partial u/\partial x$ and $\partial v/\partial x$, we find

$$\frac{\partial u}{\partial x} = \frac{v + u(y - 2x)}{2(u^2 - v^2)} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = \frac{v(2x - y) - u}{2(u^2 - v^2)}$$

(b) Differentiating f and g partially with respect to y, we obtain

$$2y + 2v\frac{\partial u}{\partial y} + 2u\frac{\partial v}{\partial y} = 0 \qquad \text{and} \qquad -x + 2y + 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

Then

$$\frac{\partial u}{\partial y} = \frac{u(x - 2y) + 2vy}{2(u^2 - v^2)} \qquad \text{and} \qquad \frac{\partial v}{\partial y} = \frac{v(2y - x) - 2uy}{2(u^2 - v^2)}$$

7. Given $u^2 - v^2 + 2x + 3y = 0$ and $uv + x - y = 0$, find (a) $\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial y}$ and (b) $\dfrac{\partial x}{\partial u}, \dfrac{\partial y}{\partial u}, \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial v}$.

(a) Here x and y are to be considered as independent variables. Differentiate the given equations partially with respect to x, obtaining

$$2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x} + 2 = 0 \qquad \text{and} \qquad v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x} + 1 = 0$$

Solve these relations simultaneously to obtain $\frac{\partial u}{\partial x} = -\frac{u+v}{u^2+v^2}$ and $\frac{\partial v}{\partial x} = \frac{v-u}{u^2+v^2}$.

Differentiate the given equations partially with respect to y, obtaining

$$2u\frac{\partial u}{\partial y} - 2v\frac{\partial v}{\partial y} + 3 = 0 \qquad \text{and} \qquad v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y} - 1 = 0$$

Solve simultaneously to obtain $\frac{\partial u}{\partial y} = \frac{2v-3u}{2(u^2+v^2)}$ and $\frac{\partial v}{\partial y} = \frac{2u+3v}{2(u^2+v^2)}$.

(*b*) Here u and v are to be considered as independent variables. Differentiate the given equations partially with respect to u, obtaining $2u + 2\frac{\partial x}{\partial u} + 3\frac{\partial y}{\partial u} = 0$ and $v + \frac{\partial x}{\partial u} - \frac{\partial y}{\partial u} = 0$. Then $\frac{\partial x}{\partial u} = -\frac{2u+3v}{5}$ and $\frac{\partial y}{\partial u} = \frac{2(v-u)}{5}$.

Differentiate the given equations partially with respect to v, obtaining $-2v + 2\frac{\partial x}{\partial v} + 3\frac{\partial y}{\partial v} = 0$ and $u + \frac{\partial x}{\partial v} - \frac{\partial y}{\partial v} = 0$. Then $\frac{\partial x}{\partial v} = \frac{2v-3u}{5}$ and $\frac{\partial y}{\partial v} = \frac{2u(u+v)}{5}$.

Supplementary Problems

8. Find dy/dx, given
(*a*) $x^3 - x^2y + xy^2 - y^3 = 1$ (*b*) $xy - e^x \sin y = 0$ (*c*) $\ln(x^2+y^2) - \arctan y/x = 0$

Ans. (*a*) $\frac{3x^2-2xy+y^2}{x^2-2xy+3y^2}$; (*b*) $\frac{e^x \sin y - y}{x - e^x \cos y}$; (*c*) $\frac{2x+y}{x-2y}$

9. Find $\partial z/\partial x$ and $\partial z/\partial y$, given

(*a*) $3x^2 + 4y^2 - 5z^2 = 60$ *Ans.* $\partial z/\partial x = 3x/5z$; $\partial z/\partial y = 4y/5z$

(*b*) $x^2 + y^2 + z^2 + 2xy + 4yz + 8zx = 20$ *Ans.* $\frac{\partial z}{\partial x} = -\frac{x+y+4z}{4x+2y+z}$; $\frac{\partial z}{\partial y} = -\frac{x+y+2z}{4x+2y+z}$

(*c*) $x + 3y + 2z = \ln z$ *Ans.* $\frac{\partial z}{\partial x} = \frac{z}{1-2z}$; $\frac{\partial z}{\partial y} = \frac{3z}{1-2z}$

(*d*) $z = e^x \cos(y+z)$ *Ans.* $\frac{\partial z}{\partial x} = \frac{z}{1+e^x\sin(y+z)}$; $\frac{\partial z}{\partial y} = \frac{-e^x\sin(y+z)}{1+e^x\sin(y+z)}$

(*e*) $\sin(x+y) + \sin(y+z) + \sin(z+x) = 1$

Ans. $\frac{\partial z}{\partial x} = -\frac{\cos(x+y)+\cos(z+x)}{\cos(y+z)+\cos(z+x)}$; $\frac{\partial z}{\partial y} = -\frac{\cos(x+y)+\cos(y+z)}{\cos(y+z)+\cos(z+x)}$

10. Find all the first and second partial derivatives of z, given $x^2 + 2yz + 2zx = 1$.

Ans. $\frac{\partial z}{\partial x} = -\frac{x+z}{x+y}$; $\frac{\partial z}{\partial y} = -\frac{z}{x+y}$; $\frac{\partial^2 z}{\partial x^2} = \frac{x-y+2z}{(x+y)^2}$; $\frac{\partial^2 z}{\partial x\,\partial y} = \frac{x+2z}{(x+y)^2}$; $\frac{\partial^2 z}{\partial y^2} = \frac{2z}{(x+y)^2}$

11. If $F(x, y, z) = 0$ show that $\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = -1$.

12. If $z = f(x, y)$ and $g(x, y) = 0$, show that $\frac{dz}{dx} = \frac{\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}} = \frac{1}{\frac{\partial g}{\partial y}} J\left(\frac{f, g}{x, y}\right)$.

Concerning 3D-geomery course, teaching source has taken from this link:

https://www.coursehero.com/file/60350377/Analytical-Solid-Geometrypdf/

which is clearing in the next pages as:

Chapter 3

Analytical Solid Geometry

3.1 INTRODUCTION

In 1637, Rene Descartes* represented geometrical figures (configurations) by equations and vice versa. Analytical Geometry involves algebraic or analytic methods in geometry. Analytical geometry in three dimensions also known as Analytical solid** geometry or solid analytical geometry, studies geometrical objects in space involving three dimensions, which is an extension of coordinate geometry in plane (two dimensions).

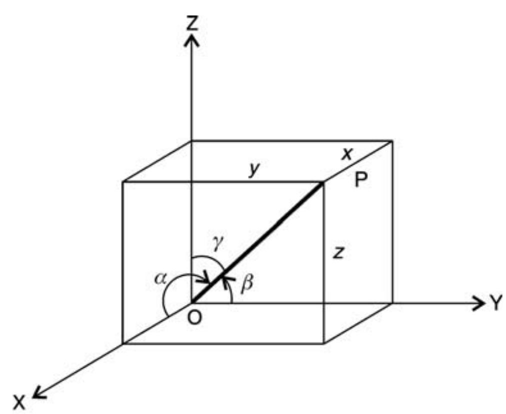


Fig. 3.1

* Rene Descartes (1596–1650) French philosopher and mathematician, latinized name for Renatus Cartesius.

** *Not* used in the sense of "non-hollowness". By a sphere or cylinder we mean a hollow sphere or cylinder.

Rectangular Cartesian Coordinates

The position (location) of a point in space can be determined in terms of its perpendicular distances (known as rectangular cartesian coordinates or simply *rectangular coordinates*) from three mutually perpendicular planes (known as **coordinate planes**). The lines of intersection of these three coordinate planes are known as *coordinate axes* and their point of intersection the *origin*.

The three axes called x-axis, y-axis and z-axis are marked positive on one side of the origin. The positive sides of axes *OX*, *OY*, *OZ* form a right handed system. The coordinate planes divide entire space into eight parts called *octants*. Thus a point P with coordinates x, y, z is denoted as $P(x, y, z)$. Here x, y, z are respectively the perpendicular distances of P from the *YZ*, *ZX* and *XY* planes. Note that a line perpendicular to a plane is perpendicular to every line in the plane.

Distance between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$. *Distance* from origin $O(0, 0, 0)$ is $\sqrt{x_2^2 + y_2^2 + z_2^2}$. Divisions of the line joining two points P_1, P_2: The coordinates of $Q(x, y, z)$, the point on P_1P_2 dividing the line segment P_1P_2 in the ratio $m : n$ are $\left(\frac{nx_1+mx_2}{m+n}, \frac{ny_1+my_2}{m+n}, \frac{nz_1+mz_2}{m+n}\right)$ or putting k for $\frac{m}{n}$, $\left(\frac{x_1+kx_2}{1+k}, \frac{y_1+ky_2}{1+k}, \frac{z_1+kz_2}{1+k}\right)$; $k \neq -1$. Coordinates of mid point are $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$.

Direction of a line: A line in space is said to be directed if it is taken in a definite sense from one extreme (end) to the other (end).

Angle between Two Lines

Two straight lines in space may or may not intersect. If they intersect, they form a plane and are said to be coplanar. If they do not intersect, they are called *skew lines*.

Angle between two intersecting (coplanar) lines is the angle between their positive directions.

Angle between two non-intersecting (non-coplanar or skew) lines is the angle between two intersecting lines whose directions are same as those of given two lines.

3.2 DIRECTION COSINES AND DIRECTION RATIOS

Direction Cosines of a Line

Let L be a directed line OP from the origin $O(0, 0, 0)$ to a point $P(x, y, z)$ and of length r (Fig. 1.2). Suppose OP makes angles α, β, γ with the positive directions of the coordinate axes. Then α, β, γ are known as the *direction angles* of L. The cosines of these angles $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are known as the *direction cosines* of the line $L(OP)$ and are in general denoted by l, m, n respectively.

Thus

$$l = \cos\alpha = \frac{x}{r}, \quad m = \cos\beta = \frac{y}{r}, \quad n = \cos\gamma = \frac{z}{r}.$$

where $$r = \sqrt{x^2 + y^2 + z^2}.$$

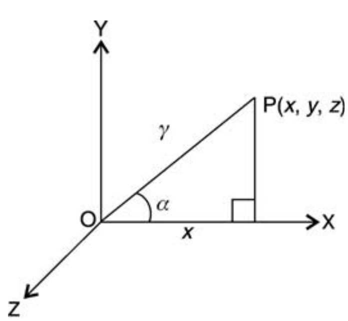


Fig. 3.2

Corollary 1: Lagrange's identity: $l^2+m^2+n^2 = 1$ i.e., sum of the squares of the direction cosines of any line is one, since $l^2 + m^2 + n^2 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = 1$.

Corollary 2: Direction cosines of the coordinate axes OX, OY, OZ are (1, 0, 0), (0, 1, 0), (0, 0, 1) respectively.

Corollary 3: The coordinates of P are (lr, mr, nr) where l, m, n are the direction cosines of OP and r is the length of OP.

Note: Direction cosines is abbreviated as DC's.

Direction Ratios

(abbreviated as DR's:) of a line L are any set of three numbers a, b, c which are proportional to l, m, n the DC's of the line L. DR's are also known as direction numbers of L. Thus $\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = k$ (proportionality constant) or $l = ak, m = bk$, $n = ck$. Since $l^2 + m^2 + n^2 = 1$ or $(ak)^2 + (bk)^2 + (ck)^2 = 1$ or $k = \frac{\pm 1}{\sqrt{a^2+b^2+c^2}}$. Then the actual direction cosines are $\cos\alpha = l = ak = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}$, $\cos\beta = m = bk = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}$, $\cos\gamma = m = ck = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$ with $a^2 + b^2 + c^2 \neq 0$. Here +ve sign corresponds to positive direction and −ve sign to negative direction.

Note 1: Sum of the squares of DR's need not be one.

Note 2: Direction of line is $[a, b, c]$ where a, b, c are DR's.

Direction cosines of the line joining two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$:

$$l = \cos\alpha = \frac{PQ}{r} = \frac{LM}{r} = \frac{OM - OL}{r} = \frac{x_2 - x_1}{r}.$$

Similarly, $m = \cos\beta = \frac{y_2-y_1}{r}$ and $n = \cos\gamma = \frac{z^2-z_1}{r}$. Then the DR's of P_1P_2 are $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$

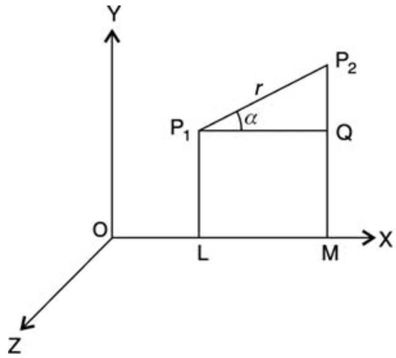


Fig. 3.3

Projections

Projection of a point P on line L is Q, the foot of the perpendicular from P to L.

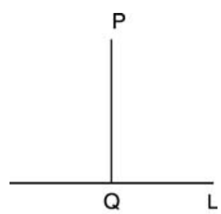


Fig. 3.4

Projection of line segment

P_1P_2 on a line L is the line segment MN where M and N are the feet of the perpendiculars from P and Q on to L. If θ is the angle between P_1P_2 and line L, then projection of P_1P_2 on $L = MN = PR = P_1P_2\cos\theta$. Projection of line segment P_1P_2 on line L with (whose) DC's l, m, n is

$$l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$$

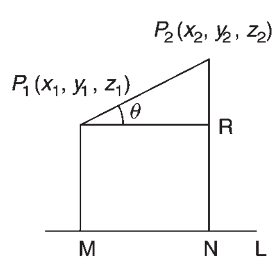


Fig. 3.5

Angle between Two Lines

Let θ be the angle between the two lines OP_1 and OP_2. Let $OP_1 = r_1, OP_2 = r_2$. Let l_1, m_1, n_1 be DC's of OP_1 and l_2, m_2, n_2 are DC's of OP_2. Then the coordinates of P_1 are l_1r_1, m_1r_1, n_1r_1 and of P_2 and l_2r_2, m_2r_2, n_2r_2.

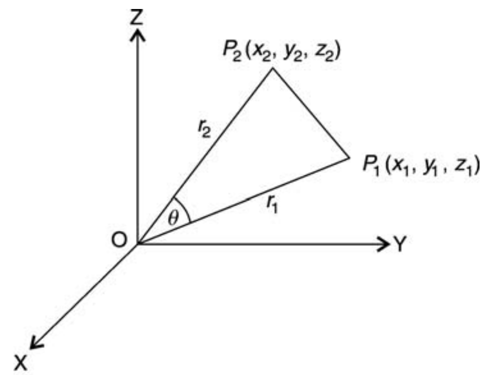


Fig. 3.6

From ΔOP_1P_2, we have

$$P_1P_2^2 = OP_1^2 + OP_2^2 - 2OP_1 \cdot OP_2 \cdot \cos\theta$$

$$(l_2r_2 - l_1r_1)^2 + (m_2r_2 - m_1r_1)^2 + (n_2r_2 - n_1r_1)^2$$
$$= \left[(l_1r_1)^2 + (m_1r_1)^2 + (n_1r_1)^2\right]$$
$$+ \left[(l_2r_2)^2 + (m_2r_2)^2 + (n_2r_2)^2\right] - 2 \cdot r_1r_2\cos\theta.$$

Using $l_1^2 + m_1^2 + n_1^2 = 1$ and $l_2^2 + m_2^2 + n_2^2 = 1$,

$$r_1^2 + r_2^2 - 2r_1r_2(l_1l_2 + m_1m_2 + n_1n_2)$$
$$= r_1^2 + r_2^2 - 2r_1r_2\cos\theta.$$

Then $\boxed{\cos\theta = l_1l_2 + m_1m_2 + n_1n_2}$

Corollary 1:

$$\sin^2\theta = 1 - \cos^2\theta = 1 - (l_1l_2 + m_1m_2 + n_1n_2)^2$$
$$= (l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2)$$
$$-(l_1l_2 + m_1m_2 + n_1n_2)^2$$
$$= (l_1m_2 - m_1l_2)^2 + (m_1n_2 - n_1m_2)^2$$
$$+(n_1l_2 - n_2l_1)^2$$

using the Lagrange's identity. Then

$$(l_1^2+m_1^2+n_1^2)(l_2^2+m_2^2+n_2^2)-(l_1l_2+m_1m_2+n_1n_2)^2)$$
$$= (l_1m_2-l_2m_1)^2+(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2.$$

Thus $\sin\theta = \sqrt{\sum(l_1m_2 - m_1l_2)^2}$

Corollary 2: $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{\sum(l_1m_2-m_1l_2)^2}}{l_1l_2+m_1m_2+n_1n_2}$.

Corollary 3: If a_1, b_1, c_1 and a_2, b_2, c_2 are DR's of OP_1 and OP_2

Then $l_1 = \frac{a_1}{\sqrt{a_1^2+b_1^2+c_1^2}}$, $m_1 = \frac{b_1}{\sqrt{a_1^2+b_1^2+c_1^2}}$, $n_1 = \frac{c_1}{\sqrt{a_1^2+b_1^2+c_1^2}}$ etc.

Then $\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$,

$$\sin\theta = \frac{\sqrt{(a_1b_2-a_2b_1)^2+(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2}}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}.$$

Corollary: Condition for perpendicularity:

The two lines are perpendicular if $\theta = 90°$. Then

$$\cos\theta = \cos 90 = 0$$

Thus $\boxed{l_1l_2 + m_1m_2 + n_1n_2 = 0}$

or $\boxed{a_1a_2 + b_1b_2 + c_1c_2 = 0}$

Corollary: Condition for parallelism:

If the two lines are parallel then $\theta = 0$. So $\sin\theta = 0$.

$$(l_1m_2 - m_1l_2)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 = 0$$

or $$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} = \frac{\sqrt{l_1^2+m_1^2+n_1^2}}{\sqrt{l_2^2+m_2^2+n_2^2}} = \frac{1}{1}.$$

Thus $\boxed{l_1 = l_2, \quad m_1 = m_2, \quad n_1 = n_2}$

or $$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

Worked Out Examples

Example 1: Find the angle between the lines $A(-3, 2, 4)$, $B(2, 5, -2)$ and $C(1, -2, 2)$, $D(4, 2, 3)$.

Solution: DR's of AB: $2-(-3)$, $5-2$, $-2-4$ $= 5, 3, -6$

DR's of CD: $3, 4, 1$. Then DC's of AB are $l_1 = \cos\alpha_1 = \frac{5}{\sqrt{5^2+3^2+6^2}} = \frac{5}{\sqrt{25+9+36}} = \frac{5}{\sqrt{70}}$ and $m_1 = \cos\beta_1 = \frac{3}{70}$, $n_1 = \cos\gamma_1 = \frac{-6}{\sqrt{70}}$. Similarly, $l_2 = \cos\alpha_2 = \frac{3}{\sqrt{3^2+4^2+1^2}} = \frac{3}{\sqrt{9+16+1}} = \frac{3}{\sqrt{26}}$, and $m_2 = \cos\beta_2 = \frac{4}{\sqrt{26}}$, $n_2 = \cos\gamma_2 = \frac{1}{\sqrt{26}}$. Now

$$\cos\theta = \cos\alpha_1\cdot\cos\alpha_2 + \cos\beta_1\cdot\cos\beta_2 + \cos\gamma_1\cdot\cos\gamma_2$$
$$= l_1l_2 + m_1m_2 + n_1n_2$$
$$\cos\theta = \frac{5}{\sqrt{70}}\cdot\frac{3}{\sqrt{26}} + \frac{3}{\sqrt{70}}\cdot\frac{4}{\sqrt{26}} - \frac{6}{\sqrt{70}}\cdot\frac{1}{\sqrt{26}}$$
$$= 0.49225$$
$$\therefore \quad \theta = \cos^{-1}(0.49225) = 60°30.7'$$

Example 2: Find the DC's of the line that is $\perp^r$ to each of the two lines whose directions are $[2, -1, 2]$ and $[3, 0, 1]$.

Solution: Let $[a, b, c]$ be the direction of the line. Since this line is $\perp^r$ to the line with direction $[2, -1, 2]$, by orthogonality

$$2a - b + 2c = 0$$

Similarly, direction $[a, b, c]$ is $\perp^r$ to direction $[3, 0, 1]$. So

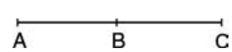
$$3a + 0 + c = 0.$$

Solving $c = -3a$, $b = -4a$ or direction $[a, b, c] = [a, -4a, -3a] = [1, -4, -3]$.

$\therefore$ DC's of the line: $\frac{1}{\sqrt{1^2+4^2+3^2}} = \frac{1}{\sqrt{26}}, \frac{-4}{\sqrt{26}}, \frac{-3}{\sqrt{26}}$.

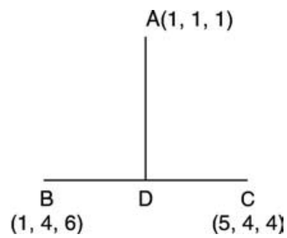
Example 3: Show that the points $A(1, 0, -2)$, $B(3, -1, 1)$ and $C(7, -3, 7)$ are collinear.

Solution: DR's of AB: $[2, -1, 3]$, DR's of AC: $[6, -3, 9]$, DR's of BC: $[4, -2, 6]$. Thus DR's of AB, AC, BC are same. Hence A, B, C are collinear.



Example 4: Find the coordinates of the foot of the perpendicular from $A(1, 1, 1)$ on the line joining $B(1, 4, 6)$ and $C(5, 4, 4)$.

Solution: Suppose D divides BC in the ratio $k : 1$. Then the coordinates of D are $\left(\frac{5k+1}{k+1}, \frac{4k+4}{k+1}, \frac{4k+6}{k+1}\right)$. DR's of AD: $\frac{4k}{k+1}, 3, \frac{3K+5}{k+1}$, DR's of BC: $4, 0, -2$ AD is $\perp^r$ BC: $16k - 6k - 10 = 0$, or $k = 1$.



Coordinates of the foot of perpendicular are (3, 4, 5).

Example 5: Show that the points $A(1, 0, 2)$, $B(3, -1, 3)$, $C(2, 2, 2)$, $D(0, 3, 1)$ are the vertices of a parallelogram.

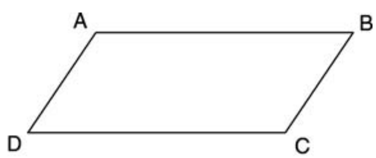


Fig. 3.7

Solution: DR's of AB are $[3-1, -1-0, 3-2] = [2, -1, 1]$. Similarly, DR's of BC are $[-1, 3, -1]$, of $CD[-2, 1, -1]$ of $DA[-1, 3, -1]$. Since DR's of AB and CD are same, they are parallel. Similarly BC and DA are parallel since DR's are same. Further AB is not $\perp^r$ to AD because

$$2(+1) + (-1)(-3) + 1(+1) = 6 \neq 0$$

Similarly, AD is not $\perp^r$ to BC because

$$2(-1) + (-1)3 + 1(-1) = -6 \neq 0.$$

Hence $ABCD$ is a parallelogram.

Exercise

1. Show that the points $A(7, 0, 10)$, $B(6, -1, 6)$, $C(9, -4, 6)$ form an isoscales right angled triangle.

   Hint: $AB^2 = BC^2 = 18$, $CA^2 = 36$, $AB^2 + BC^2 = CA^2$

2. Prove that the points $A(3, -1, 1)$, $B(5, -4, 2)$, $C(11, -13, 5)$ are collinear.

   Hint 1: $AB^2 = 14$, $BC^2 = 126$, $CA^2 = 224$, $AB + BC = 4\sqrt{14} = CA$

   Hint 2: DR's of $AB = 2, -3, 1$; BC: $6, -9, 3$; $AB\|^l$ to BC

3. Determine the internal angles of the triangle ABC where $A(2, 3, 5)$, $B(-1, 3, 2)$, $C(3, 5, -2)$.

   Hint: $AB^2 = 18$, $BC^2 = 36$, $AC^2 = 54$. DC's AB: $-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}$; BC: $\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}$; AC: $\frac{1}{3\sqrt{6}}, \frac{2}{3\sqrt{6}}, \frac{-7}{3\sqrt{6}}$.

   Ans. $\cos A = \frac{1}{\sqrt{3}}$, $\cos B = 0$ i.e., $B = 90^\circ$, $\cos C = \frac{\sqrt{6}}{3}$.

4. Show that the foot of the perpendicular from $A(0, 9, 6)$ on the line joining $B(1, 2, 3)$ and $C(7, -2, 5)$ is $D(-2, 4, 2)$.

   Hint: D divides BC in $k : 1$, $D\left(\frac{7k+1}{k+1}, \frac{-2k+2}{k+1}, \frac{5k+3}{k+1}\right)$. DR's AD: $(7k+1, -11k-7, -k-3)$, DR's BC: $6, -4, 2$. $AD \perp^r BC$: $k = -\frac{1}{3}$.

5. Find the condition that three lines with DC's l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 are concurrent.

   Hint: Line with DC's l, m, n through point of concurrency will be $\perp^r$ to all three lines, $ll_i + mm_i + nn_i = 0$, $i = 1, 2, 3$.

   Ans.
   $$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$$

6. Show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$ where $\alpha, \beta, \gamma, \delta$ are the angles which a line makes with the four diagonals of a cube.

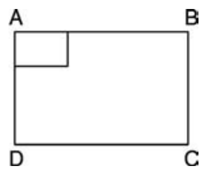
   Hint: DC's of four diagonals are (k, k, k), $(-k, k, k)$, $(k, -k, k)$, $(k, k, -k)$ where $k = \frac{1}{\sqrt{3}}$; l, m, n are DC's of line. $\cos\alpha = l.k. + mk + nk$, $\cos\beta = (-l + m + n)k$, $\cos\gamma = (l - m + n)k$, $\cos\delta = (l + m - n)k$.

7. Show that the points $A(-1, 1, 3)$, $B(1, -2, 4)$, $C(4, -1, 1)$ are vertices of a right triangle.

   Hint: DR's AB : $[2, -3, 1]$, BC : $[3, 1, -3]$, CA : $[5, -2, -2]$. AB is $\perp^r$ BC.

8. Prove that $A(3, 1, -2)$, $B(3, 0, 1)$, $C(5, 3, 2)$, $D(5, 4, -1)$ form a rectangle.

   Hint: DR's: AB: $[0, -1, 3]$; AC: $[2, 2, 4]$, $CD[0, 1, -3]$; $AD[2, 3, 1]$; $BC[2, 3, 1]$; $AB \| CD$, $AD \| BC$, $AD \perp AB$: $0 - 3 + 3 = 0$, $BC \perp DC$: $0 + 3 - 3 = 0$.



9. Find the interior angles of the triangle

   $A(3, -1, 4), B(1, 2, -4), C(-3, 2, 1).$

   Hint: DC's of AB: $(-2, 3, -8)k_1$, BC: $(-4, 0, 5)k_2$, AC: $(-6, 3, -3)k_3$ where $k_1 = \frac{1}{\sqrt{77}}$, $k_2 = \frac{1}{\sqrt{41}}$, $k_3 = \frac{1}{\sqrt{54}}$.

Ans. $\cos A = \frac{15}{\sqrt{462}}$, $\cos B = \frac{32}{\sqrt{3157}}$, $\cos C = \frac{3}{\sqrt{246}}$.

10. Determine the DC's of a line $\perp^r$ to a triangle formed by $A(2, 3, 1)$, $B(6, -3, 2)$, $C(4, 0, 3)$.

Ans. $(3, 2, 0)k$ where $k = \frac{1}{\sqrt{13}}$.

   Hint: DR: AB: $[4, -6, 1]$, BC: $[-2, 3, 1]$, CA: $[2, -3, 2]$. $[a, b, c]$ of $\perp^r$ line: $4a - 6b + c = 0$, $-2a + 3b + c = 0$, $2a - 3b + 2c = 0$.

3.3 THE PLANE

Surface is the locus of a point moving in space satisfying a single condition.

Example: Surface of a sphere is the locus of a point that moves at a constant distance from a fixed point.

Surfaces are either plane or curved. Equation of the locus of a point is the analytical expression of the given condition(s) in terms of the coordinates of the point.

Exceptional cases: Equations may have locus other than a surface. Examples: (i) $x^2 + y^2 = 0$ is z-axis (ii) $x^2 + y^2 + z^2 = 0$ is origin (iii) $y^2 + 4 = 0$ has no locus.

Plane is a surface such that the straight line PQ, joining any two points P and Q on the plane, lies completely on the plane.

General equation of first degree in x, y, z is of the form

$$Ax + By + Cz + D = 0$$

Here A, B, C, D are given real numbers and A, B, C are not all zero (i.e., $A^2 + B^2 + C^2 \neq 0$)

Book Work: Show that every equation of the first degree in x, y, z represents a plane.

Proof: Let

$$Ax + By + Cz + D = 0 \tag{1}$$

be the equation of first degree in x, y, z with the condition that not all A, B, C are zero (i.e., $A^2 + B^2 + C^2 \neq 0$). Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be any two points on the surface represented by (1). Then

$$Ax_1 + By_1 + Cz_1 + D_1 = 0 \tag{2}$$

$$Ax_2 + By_2 + Cz_2 + D_2 = 0 \tag{3}$$

Multiplying (3) by k and adding to (2), we get

$$A(x_1 + kx_2) + B(y_1 + ky_2) + C(z_1 + kz_2) + D(1 + k) = 0 \tag{4}$$

Assuming that $1 + k \neq 0$, divide (4) by $(1 + k)$.

$$A\left(\frac{x_1 + kx_2}{1 + k}\right) + B\left(\frac{y_1 + ky_2}{1 + k}\right) + C\left(\frac{z_1 + kz_2}{1 + k}\right) + D = 0$$

i.e., the point $R\left(\frac{x_1+kx_2}{1+k}, \frac{y_1+ky_2}{1+k}, \frac{z_1+kz_2}{1+k}\right)$ which is point dividing the line PQ in the ratio $k : 1$, also lies on the surface (1). Thus any point on the line joining P and Q lies on the surface i.e., line PQ completely lies on the surface. Therefore the surface by definition must be a plane.

General form of the equation of a plane is

$$Ax + By + Cz + D = 0$$

Special cases:

(i) Equation of plane passing through origin is

$$Ax + By + Cz = 0 \tag{5}$$

(ii) Equations of the coordinate planes XOY, YOZ and ZOX are respectively $z = 0$, $x = 0$ and $y=0$

(iii) $Ax + By + D = 0$ plane $\perp^r$ to xy-plane
$Ax + Cz + D = 0$ plane $\perp^r$ to xz-plane
$Ay + Cz + D = 0$ plane $\perp^r$ to yz-plane.

Similarly, $Ax + D = 0$ is $\|^l$ to yz-plane, $By + D = 0$ is $\|^l$ to zx-plane, $cz + D = 0$ is $\|^l$ to xy-plane.

One point form

Equation of a plane through a fixed point $P_1(x_1, y_1, z_1)$ and whose normal CD has DC's proportional to (A, B, C): For any point $P(x, y, z)$ on the given plane, the DR's of the line P_1P are $(x - x_1, y - y_1, z - z_1)$. Since a line perpendicular to a plane is perpendicular to every line in the plane, so ML is perpendicular to P_1, P. Thus

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0 \qquad (6)$$

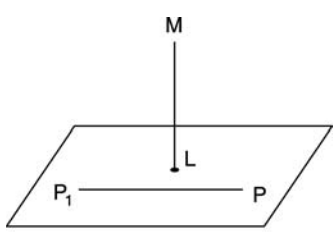


Fig. 3.8

Note 1: Rewriting (6), we get the general form of plane

$$Ax + By + Cz + D = 0 \qquad (1)$$

where $D = -ax_1 - by_1 - cz_1$

Note 2: The real numbers A, B, C which are the coefficients of x, y, z respectively in (1) are proportional to DC's of the normal ot the plane (1).

Note 3: Equation of a plane parallel to plane (1) is

$$Ax + By + Cz + D^* = 0 \qquad (7)$$

x-intercept of a plane is the point where the plane cuts the x-axis. This is obtained by putting $y = 0$, $z = 0$. Similarly, y-, z-intercepts. Traces of a plane are the lines of intersection of plane with coordinate axis.

Example: xy-trace is obtained by putting $z = 0$ in equation of plane.

Intercept form

Suppose $P(a, 0, 0)$, $Q(0, b, 0)$, $R(0, 0, c)$ are the x-, y-, z-intercepts of the plane. Then P, Q, R lies on the plane. From (1)

$$Aa + 0 + 0 + D = 0$$

or
$$A = -\frac{D}{a}.$$

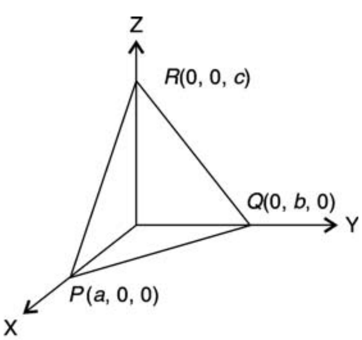


Fig. 3.9

similarly, $0 + bB + 0 + D = 0$ or $B = -\frac{D}{b}$ and $C = -\frac{D}{c}$.

Eliminating A, B, C the equation of the plane in the intercept form is

$$-\frac{D}{a}x - \frac{D}{b} - \frac{D}{c}z + D = 0$$

or
$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad (8)$$

Normal form

Let $P(x, y, z)$ be any point on the plane. Let ON be the perpendicular from origin O to the given plane. Let $ON = p$. (i.e., length of the perpendicular ON is p). Suppose l, m, n are the DC's of ON. Now ON is perpendicular to PN. Projection of OP on ON is ON itself i.e., p.

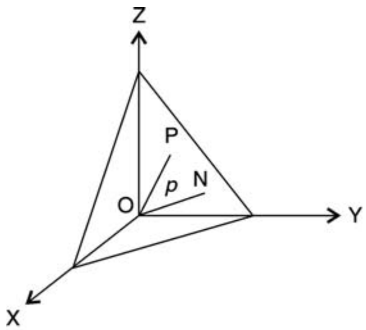


Fig. 3.10

Also the projection OP joining origin (0, 0, 0) to $P(x, y, z)$ on the line ON with DC's l, m, n is

$$l(x-0)+m(y-0)+n(z-0)$$

or
$$lx+my+nz \tag{9}$$

Equating the two projection values from (8) & (9)

$$lx+my+nz=p \tag{10}$$

Note 1: p is always positive, since p is the perpendicular distance from origin to the plane.

Note 2: Reduction from general form.

Transpose constant term to R.H.S. and make it positive (if necessary by multiplying throughout by -1). Then divide throughout by $\pm\sqrt{A^2+B^2+C^2}$. Thus the general form $Ax+By+Cz+D=0$ takes the following normal form

$$\frac{Ax}{\pm\sqrt{A^2+B^2+C^2}}+\frac{By}{\pm\sqrt{A^2+B^2+C^2}}+\frac{Cz}{\pm\sqrt{A^2+B^2+C^2}}$$
$$=\frac{-D}{\pm\sqrt{A^2+B^2+C^2}} \tag{11}$$

The sign before the radical is so chosen to make the R.H.S. in (11) positive.

Three point form

Equation of a plane passing through three given points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$:

Since the three points P_1, P_2, P_3 lie on the plane

$$Ax+By+Cz+D=0 \tag{1}$$

we have
$$Ax_1+By_1+Cz_1+D=0 \tag{12}$$
$$Ax_2+By_2+Cz_2+D=0 \tag{13}$$
$$Ax_3+By_3+Cz_3+D=0 \tag{14}$$

Eliminating A, B, C, D from (1), (12), (13), (14) (i.e., a non trivial solution A, B, C, D for the homogeneous system of 4 equations exist if the determinant coefficient is zero)

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}=0 \tag{15}$$

Equation (15) is the required equation of the plane through the 3 points P_1, P_2, P_3.

Corollary 1: Coplanarity of four given points: The four points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$, $P_4(x_4, y_4, z_4)$ are coplanar (lie in a plane) if

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}=0 \tag{16}$$

Angle between Two Given Planes

The angle between two planes

$$A_1x+B_1y+C_1z+D_1=0 \tag{17}$$
$$A_2x+B_2y+C_2z+D_2=0 \tag{18}$$

is the angle θ between their normals. Here A_1, B_1, C_1 and A_2, B_2, C_2 are the DR's of the normals respectively to the planes (17) and (18). Then

$$\cos\theta=\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}}$$

Condition for perpendicularity

If $\theta=0$ then the two planes are $\perp^r$ to each other. Then

$$A_1A_2+B_1B_2+C_1C_2=0 \tag{19}$$

Condition for parallelism

If $\theta=0$, the two planes are $\|^l$ to each other. Then

$$\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2} \tag{20}$$

Note: Thus parallel planes differ by a constant.

Although there are four constants A, B, C, D in the equation of plane, essentially three conditions are required to determine the three ratios of A, B, C, D, for example plane passing through:

a. three non-collinear points

b. two given points and $\perp^r$ to a given plane

c. a given point and $\perp^r$ to two given planes etc.

Coordinate of the Foot of the Perpendicular from a Point to a Given Plane

Let $Ax + By + Cz + D = 0$ be the given plane and $P(x_1, y_1, z_1)$ be a given point. Let PN be the perpendicular from P to the plane. Let the coordinates of the foot of the perpendicular PN be $N(\alpha, \beta, \gamma)$. Then DR's of $PN(x_1 - \alpha, y_1 - \beta, z_1 - \gamma)$ are proportional to the coefficients A, B, C i.e.,

$$x_1 - \alpha = kA, \quad y_1 - B = kB, \quad z_1 - \gamma = kC$$

or $$\alpha = x_1 - kA, \quad y_1 = \beta - kB, \quad z_1 = \gamma - kC$$

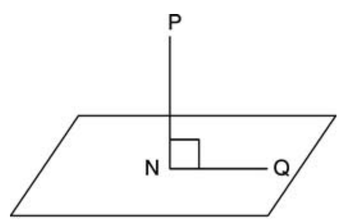


Fig. 3.11

Since N lies in the plane

$$A\alpha + B\beta + C\gamma + D = 0$$

Substituting α, β, γ,

$$A(x_1 - kA) + b(y_1 - kB) + c(z_1 - kC) + D = 0$$

Solving $$k = \frac{Ax_1 + By_1 + CZ_1 + D}{A^2 + B^2 + C^2}$$

Thus the coordinates of $N(\alpha, \beta, \gamma)$ the foot of the perpendicular from $P(x_1, y_1, z_1)$ to the plane are

$$\alpha = x_1 - \frac{A(Ax_1 + By_1 + Cz_1 + D)}{A^2 + B^2 + C^2},$$

$$\beta = y_1 - \frac{B(Ax_1 + By_1 + Cz_1 + D)}{A^2 + B^2 + C^2},$$

$$\gamma = z_1 - \frac{C(Ax_1 + By_1 + Cz_1 + D)}{A^2 + B^2 + C^2} \qquad (21)$$

Corollary 1: Length of the perpendicular from a given point to a given plane:

$$PN^2 = (x_1 - \alpha)^2 + (y_1 - \beta)^2 + (z_1 - \gamma)^2$$
$$= (kA)^2 + (kB)^2 + (kC)^2$$
$$= k^2(A^2 + B^2 + C^2)$$
$$= \left[\frac{Ax_1 + By_1 + Cz_1 + D}{A^2 + B^2 + C^2}\right]^2 (A^2 + B^2 + C^2)$$
$$= \frac{(Ax_1 + By_1 + Cz_1 + D)^2}{A^2 + B^2 + C^2}$$

or $$PN = \frac{Ax_1 + By_1 + Cz_1 + D}{\pm\sqrt{A^2 + B^2 + C^2}}.$$

The sign before the radical is chosen as positive or negative according as D is positive or negative. Thus the numerical values of the length of the perpendicular PN is

$$PN = \left|\frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}\right| \qquad (22)$$

Note: PN is obtained by substituting the coordinates (x_1, y_1, z_1) in the L.H.S. of the Equation (1) and dividing it by $\sqrt{A^2 + B^2 + C^2}$.

Equation of a plane passing through the line of intersection of two given planes $u \equiv A_1x + B_1y + C_1z + D_1 = 0$ and $v \equiv A_2x + B_2y + C_2z + D_2 = 0$ is $u + kv = 0$ where k is any constant.

Equations of the two planes bisecting the angles between two planes are

$$\frac{A_1x + B_1y + C_1z + D_1}{\sqrt{A_1^2 + B_1^2 + C_1^2}} = \pm\frac{A_2x + B_2y + C_2z + D_2}{\sqrt{A_2^2 + B_2^2 + C_2^2}}.$$

WORKED OUT EXAMPLES

Example 1: Find the equation of the plane which passes through the point (2, 1, 4) and is

a. Parallel to plane $2x + 3y + 5z + 6 = 0$

b. Perpendicular to the line joining (3, 2, 5) and (1, 6, 4)

c. Perpendicular to the two planes $7x + y + 2z = 6$ and $3x + 5y - 6z = 8$

d. Find intercept points and traces of the plane in case c.

Solution:

a. Any plane parallel to the plane

$$2x + 3y + 5z + 6 = 0$$

is given by $2x+3y+5z+k=0$ (1) (differs by a constant). Since the point (2, 1, 4) lies on the plane (1), $2(2)+3(1)+5(4)+k=0$, $k=-27$. Required equation of plane is $2x+3y+5z-27=0$.

b. Any plane through the point (2, 1, 4) is (one point form)

$$A(x-2)+B(y-1)+C(z-4)=0 \qquad (2)$$

DC's of the line joining $M(3,2,5)$ and $N(1,6,4)$ are proportional to $2,-4,1$. Since MN is perpendicular to (2), A, B, C are proportional to $2,-4,1$. Then $2(x-2)-4(y-1)+1(z-4)=0$. The required equation of plane is $2x-4y+z-4=0$.

c. The plane through (2, 1, 4) is

$$A(x-2)+B(y-1)+C(z-4)=0. \qquad (2)$$

This plane (2) is perpendicular to the two planes $7x+y+2z=6$ and $3x+5y-6z=8$. Using $A_1A_2+B_1B_2+C_1C_2=0$, we have

$$7a+b+2c=0$$

$$3a+5b-6c=0$$

Solving $\frac{a}{-6-10}=\frac{-b}{-42-8}=\frac{c}{35-3}$ or $\frac{a}{1}=\frac{b}{-3}=\frac{c}{-2}$.

Required equation of plane is

$$1(x-4)-3(y-1)-2(z-4)=0$$

or

$$x-3y-2z+7=0$$

d. x-intercept: Put $y=z=0$, $\therefore x=-7$ or $(-7, 0, 0)$ is the x-intercept. Similarly, y-intercept is $(0, \frac{7}{3}, 0)$ and z-intercept is $\left(0,0,\frac{7}{2}\right)$. xy-trace is obtained by putting $z=0$. It is $x-3y+7=0$. Similarly, yz-trace is $3y+2z-7=0$ and zx-trace is $x-2z+7=0$.

Example 2: Find the equation of the plane containing the points $P(3,-1,-4)$, $Q(-2,2,1)$, $R(0,4,-1)$.

Solution: Equation of plane through the point $P(3,-1,-4)$ is

$$A(x+3)+B(y+1)+C(z+4)=0. \qquad (1)$$

DR's of PQ: $-5, 3, 5$; DR's of PR: $-3, 5, 3$. Since line PQ and PR completely lies in the plane (1), normal to (1) is perpendicular to PQ and PR. Then

$$-5A+3B+5C=0$$

$$-3A+5B+3C=0$$

Solving $A=C=1, \qquad B=0$

$$(x-3)+0+(z+4)=0$$

Equation of the plane is

$$x+z+1=0$$

Aliter: Equation of the plane by 3-point form is

$$\begin{vmatrix} x & y & z & 1 \\ 3 & -1 & -4 & 1 \\ -2 & 2 & 1 & 1 \\ 0 & 4 & -1 & 1 \end{vmatrix}=0$$

Expanding $D_1x-D_2y+D_3z-1.D_4=0$ where

$$D_1=\begin{vmatrix} -1 & -4 & 1 \\ 2 & 1 & 1 \\ 4 & -1 & 1 \end{vmatrix}=-16, \quad D_2=\begin{vmatrix} 3 & -4 & 1 \\ -2 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix}0=0$$

$$D_3=\begin{vmatrix} 3 & -1 & 1 \\ -2 & 2 & 1 \\ 0 & 4 & 1 \end{vmatrix}=-16, \quad D_4=\begin{vmatrix} 3 & -1 & -4 \\ -2 & 2 & 1 \\ 0 & 4 & -1 \end{vmatrix}=16$$

or required equation is $x+z+1=0$.

Example 3: Find the perpendicular distance between (a) The Point $(3,2,-1)$ and the plane $7x-6y+6z+8=0$ (b) between the parallel planes $x-2y+2z-8=0$ and $x-2y+2z+19=0$ (c) find the foot of the perpendicular in case (a).

Solution:

Perpendicular distance $=\left(\frac{Ax_1+By_1+Cz_1+D}{\sqrt{A^2+B^2+C^2}}\right)$

a. Point $(3,2,-1)$, plane is $7x-6y+6z+8=0$. So perpendicular distance from $(3,2,-1)$ to plane is

$$=\frac{7(3)-6(2)+6(-1)+8}{\sqrt{7^2+6^2+6^2}}=\frac{11}{-11}=|-1|=1$$

b. x-intercept point of plane $x-2y+2z-8=0$ is $(8,0,0)$ (obtained by putting $y=0, z=0$ in the equation). Then perpendicular distance from the point (8, 0, 0) to the second plane $x - 2y + 2z + 19 = 0$ is $\frac{1.8-2.0+2.0+19}{\sqrt{1^2+2^2+2^2}} = \frac{27}{3} = 9$

c. Let $N(\alpha, \beta, \gamma)$ be the foot of the perpendicular from $P(3, 2, -1)$. DR's of PN: $3-\alpha, 2-\beta, -1-\gamma$. DR's of normal to plane are $7, -6, 6$. These are proportional. $\frac{3-\alpha}{7} = \frac{2-\beta}{-6} = \frac{-1-\gamma}{6}$ or $\alpha = 3-7k, \beta = 2+6k, \gamma = -1-6k$. Now (α, β, γ) lies on the plane. $7(3-7k) - 6(2+6k) + 6(-1-6k) + 8 = 0$ or $k = \frac{1}{11}$.
$\therefore$ the coordinates of the foot of perpendicular are $\left(\frac{26}{11}, \frac{28}{11}, \frac{-17}{11}\right)$.

Example 4: Are the points (2, 3, −5) and (3, 4, 7) on the same side of the plane $x + 2y - 2z = 9$?

Solution: Perpendicular distance of the point $(2, 3, -5)$ from the plane $x + 2y - 2z - 9 = 0$ or $-x - 2y + 2z + 9 = 0$ is $\frac{-1.2-2(3)-2(-5)+9}{\sqrt{1^2+2^2+2^2}} = -\frac{9}{3} = -3$.

$\perp^r$ distance of (3, 4, 7) is $\frac{-1.3-2.4+2.7+9}{\sqrt{1^2+2^2+2^2}} = \frac{12}{3} = 6$

$\perp^r$ distance from origin (0, 0, 0) is $\frac{0+0+0+9}{3} = 3$

So points (2, 3, −5) and (3, 4, 7) are on opposite sides of the given plane.

Example 5: Find the angle between the planes $4x - y + 8z = 9$ and $x + 3y + z = 4$.

Solution: DR's of the planes are [4, −1, 8] and [1, 3, 1]. Now

$$\cos\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}$$
$$= \frac{4.1 + 3\cdot(-1) + 1.8}{\sqrt{16+1+64}\sqrt{1+9+1}}$$
$$= \frac{9}{\sqrt{81}\sqrt{11}} = \frac{1}{\sqrt{11}} \quad \text{or} \quad \theta = \cos^{-1}\frac{1}{\sqrt{11}}.$$

Example 6: Find the equation of a plane passing through the line of intersection of the planes.

a. $3x + y - 5z + 7 = 0$ and $x - 2y + 4z - 3 = 0$ and passing through the point (−3, 2, −4)

b. $2x - 5y + z = 3$ and $x + y + 4z = 5$ and parallel to the plane $x + 3y + 6z = 1$.

Solution:

a. Equation of plane is $u + kv = 0$ i.e.,

$$(3x + y - 5z + 7) + k(x - 2y + 4z - 3) = 0.$$

Since point (−3, 2, −4) lies on the intersection plane

$$[3(-3) + 1.(2) - 5(-4) + 7] + k[1(-3) - 2(2) + 4(-4) - 3] = 0.$$

So $k = \frac{10}{13}$. Then the required plane is

$$49x - 7y - 25z + 61 = 0.$$

b. Equation of plane is $u + kv = 0$ i.e.,

$$(2x - 5y + z - 3) + k(x + y + 4z - 5) = 0$$
$$\text{or} \quad (2+k)x + (-5+k)y + (1+4k)z + (-3-5k) = 0.$$

Since this intersection plane is parallel to $x + 3y + 6z - 1 = 0$

$$\text{So} \quad \frac{2+k}{1} = \frac{-5+k}{3} = \frac{1+4k}{6} \quad \text{or} \quad k = -\frac{11}{2}.$$

Required equation of plane is $7x + 21y + 42z - 49 = 0$.

Example 7: Find the planes bisecting the angles between the planes $x + 2y + 2z = 9$ and $4x - 3y + 12z + 13 = 0$. Specify the angle θ between them.

Solution: Equations of the bisecting planes are

$$\frac{x + 2y + 2z - 9}{\sqrt{1 + 2^2 + 2^2}} = \pm\frac{4x - 3y + 12z + 13}{\sqrt{4^2 + 3^2 + 12^2}}$$
$$\frac{x + 2y + 2z - 9}{3} = \pm\frac{4x - 3y + 12z + 13}{13}$$

$$\text{or} \quad 25x + 17y + 62z - 78 = 0 \quad \text{and}$$
$$x + 35y - 10z - 156 = 0.$$

$$\cos\theta = \frac{25\cdot 1 + 17\cdot 35 - 62\times 10}{\sqrt{25^2 + 17^2 + 62^2}\sqrt{1 + 35^2 + 10^2}} = 0$$

$$\therefore \quad \theta = \frac{\pi}{2}$$

i.e, angle between the bisecting planes is $\frac{\pi}{2}$.

Example 8: Show that the planes

$$7x + 4y - 4z + 30 = 0 \quad (1)$$

$$36x - 51y + 12z + 17 = 0 \quad (2)$$

$$14x + 8y - 8z - 12 = 0 \quad (3)$$

$$12x - 17y + 4z - 3 = 0 \quad (4)$$

form four faces of a rectangular parallelopiped.

Solution: (1) and (3) are parallel since $\frac{7}{14} = \frac{4}{8} = \frac{-4}{-8} = \frac{1}{2}$. (2) and (4) are parallel since $\frac{36}{12} = \frac{-51}{-17} = \frac{12}{4} = 3$. Further (1) and (2) are $\perp^r$ since

$$7 \cdot 36 + 4(-51) - 4(12) = 252 - 204 - 48 = 0.$$

EXERCISE

1. Find the equation of the plane through $P(4, 3, 6)$ and perpendicular to the line joining $P(4, 3, 6)$ to the point $Q(2, 3, 1)$.

   Hint: DR's PQ: $[2, 0, 5]$, DR of plane through $(4, 3, 6) : x - 4, y - 3, z - 6; \perp^r$: $2(x - 4) + 0(y - 3) + 5(z - 6) = 0$

Ans. $2x + 5z - 38 = 0$

2. Find the equation of the plane through the point $P(1, 2, -1)$ and parallel to the plane $2x - 3y + 4z + 6 = 0$.

   Hint: Eq. $2x - 3y + 4z + k = 0, (1, 2, -1)$ lies, $k = 8$.

Ans. $2x - 3y + 4z + 8 = 0$

3. Find the equation of the plane that contains the three points $P(1, -2, 4)$, $Q(4, 1, 7)$, $R(-1, 5, 1)$.

   Hint: $A(x - 1) + B(y + 2) + C(z - 4) = 0$, DR: PQ: $[3, 3, 3]$, PR: $[-2, 7, -3]$. $\perp^r$ $3A + 3B + 3C = 0$, $-2A + 7B - 3C = 0$, $A = -10B, C = 9B$.

   Aliter: $$\begin{vmatrix} x & y & z & 1 \\ 1 & -2 & 4 & 1 \\ 4 & 1 & 7 & 1 \\ -1 & 5 & 1 & 1 \end{vmatrix} = 0,$$

   $$D_1 x - D_2 y + D_3 z - D_4 = 0$$

   where $D_1 = \begin{vmatrix} -2 & 4 & 1 \\ 1 & 7 & 1 \\ 5 & 1 & 1 \end{vmatrix}$ etc.

Ans. $10x - y - 9z + 24 = 0$

4. Find the equation of the plane
   a. passing through $(1, -1, 2)$ and $\perp^r$ to each of the planes $2x + 3y - 2z = 5$ and $x + 2y - 3z = 8$
   b. passing through $(-1, 3, -5)$ and parallel to the plane $6x - 3y - 2z + 9 = 0$
   c. passing through $(2, 0, 1)$ and $(-1, 2, 0)$ and $\perp^r$ to the plane $2x - 4y - z = 7$.

Ans. **a.** $5x - 4y - z = 7$
b. $6x - 3y - 2z + 5 = 0$
c. $6x + 5y - 8z = 4$

5. Find the perpendicular distance between
   a. the point $(-2, 8, -3)$ and plane $9x - y - 4z = 0$
   b. the two planes $x - 2y + 2z = 6, 3x - 6y + 6z = 2$
   c. the point $(1, -2, 3)$ and plane $2x - 3y + 2z - 14 = 0$.

Ans. (a) $\sqrt{2}$ (b) $\frac{-16}{9}$ (c) 0 i.e., lies on the plane.

6. Find the angle between the two planes
   a. $x + 4y - z = 5, y + z = 2$
   b. $x - 2y + 3z + 4 = 0, 2x + y - 3z + 7=0$

Ans. (a) $\cos\theta = \frac{1}{2}, \theta = 60^\circ$ (b) $\cos\theta = \frac{-9}{14}$.

7. Prove that the planes $5x - 3y + 4z = 1$, $8x + 3y + 5z = 4$, $18x - 3y + 13z = 6$ contain a common line.

   Hint: $u + kv = 0$ substitute in $w = 0, k = \frac{1}{2}$

8. Find the coordinates of N, the foot of the perpendicular from the point $P(-3, 0, 1)$ on the plane $4x - 3y + 2z = 19$. Find the length of this perpendicular. Find also the image of P in the plane.

   Hint: $PN = NQ$ i.e., N is the mid point.

Ans. $N(1, -3, 3)$, $\sqrt{29}$, image of P is $Q(5, -6, 5)$

9. Find the equation of the plane through the line of intersection of the two planes $x - 3y + 5z - 7 = 0$ and $2x + y - 4z + 1 = 0$ and $\perp^r$ to the plane $x + y - 2z + 4 = 0$.

Ans. $3x - 2y + z - 6 = 0$

10. A variable plane passes through the fixed point (a, b, c) and meets the coordinate axes in P, Q, R. Prove that the locus of the point common to the planes through P, Q, R parallel to the coordinate plane is $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$.

    Hint: $OP = x_1,\ OQ = y_1,\ OR = z_1,\ \frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1$, (a, b, c) lies, $\frac{a}{x_1} + \frac{b}{y_1} + \frac{c}{z_1} = 1$.

3.4 THE STRAIGHT LINE

Two surfaces will in general intersect in a curve. In particular two planes, which are not parallel, intersect in a straight line.

Example: The coordinate planes ZOX and XOY, whose equations are $y = 0$ and $z = 0$ respectively, intersect in a line the x-axis.

Straight line

The locus of two simultaneous equations of first degree in x, y, z

$$\begin{aligned} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{aligned} \quad (1)$$

is a straight line, provided $A_1 : B_1 : C_1 \neq A_2 : B_2 : C_2$ (i.e., not parallel). Equation (1) is known as the **general form** of the equation of a straight line. Thus the equation of a straight line or simply line is the pair of equations taken together i.e., equations of two planes together represent the equation of a line. However this representation is not unique, because many planes can pass through a given line. Thus a given line can be represented by different pairs of first degree equations.

Projecting planes

Of the many planes passing through a given line, those that are perpendicular to the coordinate planes are known as projecting planes and their traces give the **projections** of the line on the coordinate planes.

Symmetrical Form

The equation of line passing through a given point $P_1(x_1, y_1, z_1)$ and having direction cosines l, m, n is given by

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \quad (2)$$

since for any point $P(x, y, z)$ on the line, the DR's of PP_1: $x - x_1, y - y_1, z - z_1$ be proportional to l, m, n. Equation (2) represent two independent linear equations and are called the symmetrical (or symmetric) form of the equation of a line.

Corollary: Any point P on the line (2) is given by

$$x = x_1 + lr, \qquad y = y_1 + mr, \qquad z = z_1 + nr \quad (3)$$

for different values of r, where $r = PP_1$.

Corollary: Lines perpendicular to one of the coordinate axes:

a. $x = x_1, \frac{y-y_1}{m} = \frac{z-z_1}{n}$, ($\perp^r$ to x-axis i.e., $\|^l$ to yz-plane)

b. $y = y_1, \frac{x-x_1}{l} = \frac{z-z_1}{n}$, ($\perp^r$ to y-axis i.e., $\|^l$ to xz-plane)

c. $z = z_1, \frac{x-x_1}{l} = \frac{y-y_1}{m}$, ($\perp^r$ to z-axis i.e., $\|^l$ to xy-plane)

Corollary: Lines perpendicular to two axes

a. $x = x_1, y = y_1$ ($\perp^r$ to x- & y-axis i.e., $\|^l$ to z-axis):

b. $x = x_1, z = z_1$ ($\perp^r$ to x- & z-axis i.e., $\|^l$ to y-axis)

c. $y = y_1, z = z_1$ ($\perp^r$ to y- & z-axis i.e., $\|^l$ to x-axis)

Corollary: Projecting planes: (containing the given line)

(a) $\frac{x-x_1}{l} = \frac{y-y_1}{m}$ (b) $\frac{x-x_1}{l} = \frac{z-z_1}{n}$ (c) $\frac{y-y_1}{m} = \frac{z-z_1}{n}$.

Note: When any of the constants l, m, n are zero, the Equation (2) are equivalent to equations

$$\frac{l}{x - x_1} = \frac{m}{y - y_1} = \frac{n}{z - z_1}.$$

Example: $\frac{x}{0} = \frac{y}{2} = \frac{z}{0}$ means $\frac{0}{x} = \frac{2}{y} = \frac{0}{z}$.

Corollary: If a, b, c are the DR's of the line, then (2) takes the form $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$.

Corollary: **Two point form** of a line passing through two given points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} \tag{4}$$

since the DR's of P_1P_2 are $x_2-x_1, y_2-y_1, z_2-z_1$.

Transformation of General Form to Symmetrical Form

The general form also known as unsymmetrical form of the equation of a line can be transformed to symmetrical form by determining

(a) one point on the line, by putting say $z = 0$ and solving the simultaneous equations in x and y.

(b) the DC's of the line from the fact that this line is $\perp^r$ to both normals of the given planes.

For example,

(a) by putting $z = 0$ in the general form

$$\begin{aligned} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{aligned} \tag{2}$$

and solving the resulting equations

$$A_1x + B_1y + D_1 = 0$$

$$A_2x + B_2y + D_2 = 0,$$

we get a point on the line as

$$\left(\frac{B_1D_2 - B_2D_1}{A_1B_2 - A_2B_1}, \frac{A_2D_1 - A_1D_2}{A_1B_2 - A_2B_1}, 0\right) \tag{5}$$

(b) Using the orthogonality of the line with the two normals of the two planes, we get

$$lA_1 + mB_1 + nC_1 = 0$$

$$lA_2 + mB_2 + nC_2 = 0$$

where $(l, m, n), (A_1, B_1, C_1)$ and (A_2, B_2, C_2) are DR's of the line, normal to first plane, normal to second plane respectively. Solving, we get the DR's l, m, n of the line as

$$\frac{l}{B_1C_2 - B_2C_1} = \frac{m}{C_1A_2 - C_2A_1} = \frac{n}{A_1B_2 - A_2B_1} \tag{6}$$

Using (5) and (6), thus the given general form (2) of the line reduces to the symmetrical form

$$\frac{x - \frac{(B_2D_1 - B_1D_2)}{A_1B_2 - A_2B_1}}{B_1C_2 - B_2C_1} = \frac{y - \frac{(A_2D_1 - A_1D_2)}{A_1B_2 - A_2B_1}}{C_1A_2 - C_2A_1} = \frac{z - 0}{A_1B_2 - A_2B_1} \tag{7}$$

Note 1: In finding a point on the line, one can put $x = 0$ or $y = 0$ instead of $z = 0$ and get similar results.

Note 2: General form (2) can also be reduced to the two point form (4) (special case of symmetric form) by determining two points on the line.

Angle between a Line and a Plane

Let π be the plane whose equation is

$$Ax + By + Cz + D = 0 \tag{8}$$

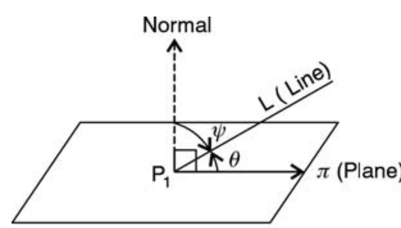


Fig. 3.12

and L be the straight line whose symmetrical form is

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \tag{2}$$

Let θ be the angle between the line L and the plane π. Let ψ be the angle between L and the normal to the plane π. Then

$$\cos\psi = \frac{lA + mB + nC}{\sqrt{l^2+m^2+n^2}\sqrt{A^2+B^2+C^2}} = \cos(90-\theta) = \sin\theta \tag{9}$$

since $\psi = 90 - \theta$. The angle between a line L and plane π is the complement of the angle between the line L and the normal to the plane). Thus θ is determined from (9).

Corollary: Line is $\|^l$ to the plane if $\theta = 0$ then $\sin\theta = 0$ i.e.,

$$\boxed{lA + mB + nC = 0} \tag{10}$$

Corollary: Line is $\perp^r$ to the plane if $\theta = \frac{\pi}{2}$, then $\sin\theta = 1$ i.e.,

$$\boxed{\frac{l}{A} = \frac{m}{B} = \frac{n}{C}} \tag{11}$$

(i.e., DR's of normal and the line are same).

Conditions for a Line L to Lie in a Plane π

If every point of line L is a point of plane π, then line L lies in plane π. Substituting any point of the line $L : (x_1 + lr, y_1 + mr, z_1 + nr)$ in the equation of the plane (8), we get

$$A(x_1 + lr) + B(y_1 + mr) + C(z_1 + nr) + D = 0$$

$$\text{or} \quad (Al + Bm + Cn)r + (Ax_1 + By_1 + Cz_1 + D) = 0 \tag{12}$$

This Equation (12) is satisfied for all values of r if the coefficient of r and constant term in (12) are both zero i.e.,

$$\boxed{\begin{array}{c} Al + Bm + Cn = 0 \quad \text{and} \\ Ax_1 + By_1 + Cz_1 + D = 0 \end{array}} \tag{13}$$

Thus the two conditions for a line L to lie in a plane π are given by (13) which geometrically mean that (i) line L is $\perp^r$ to the nomal ot the plne and (ii) a (any one) point of line L lies on the plane.

Corollary: General equation of a plane containing line L (2) is

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0 \tag{14}$$

subject to

$$Al + Bm + Cn = 0$$

Corollary: Equation of any plane through the line of intersection of the two planes

$$u \equiv A_1x + B_1y + C_1z + D_1 = 0 \quad \text{and}$$

$$v \equiv A_2x + B_2y + C_2z + D_2 = 0$$

is $u + kv = 0$ or $(A_1x + B_1y + C_1z + D_1) + k(A_2x + B_2y + C_2z + D_2) = 0$ where k is a constant.

Coplanar Lines

Consider two given straight lines L_1

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1} \tag{15}$$

and line L_2

$$\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2} \tag{16}$$

From (14), equation of any plane containing line L_1 is

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0 \tag{17}$$

subject to

$$Al_1 + Bm_1 + Cn_1 = 0 \tag{18}$$

If the plane (17) contains line L_2 also, then the point (x_2, y_2, z_2) of L_2 should also lie in the plane (17). Then

$$A(x_2 - x_1) + B(y_2 - y_1) + C(z_2 - z_1) = 0 \tag{19}$$

But the line L_2 is $\perp^r$ to the normal to the plane (17). Thus

$$Al_2 + Bm_2 + Cn_2 = 0 \tag{20}$$

Therefore the two lines L_1 and L_2 will lie in the same plane if (17), (18), (20) are simultaneously satisfied. Eliminating A, B, C from (19), (18), (20)(i.e., homogeneous system consistent if coefficient determinant is zero), we have

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \tag{21}$$

Thus (21) is the condition for coplanarity of the two lines L_1 and L_2. Now the equation of the plane containing lines L_1 and L_2 is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \tag{22}$$

which is obtained by eliminating A, B, C from (17), (18), (20).

Corollary: Condition for the two lines L_1

$$\begin{aligned} & u_1 \equiv A_1x + B_1y + C_1z + D_1 = 0, \\ & u_2 \equiv A_2x + B_2y + C_2z + D_2 = 0 \\ \text{and Line } L_2 \quad & u_3 \equiv A_3x + B_3y + C_3z + D_3 = 0, \\ & u_4 \equiv A_4x + B_4y + C_4z + D_4 = 0 \end{aligned} \tag{23}$$

to be coplanar is

$$\begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix} = 0 \tag{24}$$

If $P(\alpha, \beta, \gamma)$ is the point of intersection of the two lies, then P should satisfy the four Equations (23): $u_i|$ at $(\alpha, \beta, \gamma) = 0$ for $i = 1, 2, 3, 4$. Elimination of (α, β, γ) from these four equations leads to (24).

Corollary: The general form of equations of a line L_3 intersecting the lines L_1 and L_2 given by (23) are

$$u_1 + k_1u_2 = 0 \quad \text{and} \quad u_3 + k_2u_4 = 0 \tag{25}$$

where k_1 and k_2 are any two numbers.

Foot and length of the perpendicular from a point $P_1(\alpha, \beta, \gamma)$ to a given line L: $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$

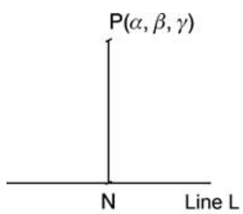


Fig. 3.13

Any point on the line L be $(x_1 + lr, y_1 + mr, z_1 + nr)$. The DR's of PN are $x_1 + lr - \alpha, y_1 + mr - \beta, z_1 + nr - \gamma$. Since PN is $\perp^r$ to line L, then

$$l(x_1 + lr - \alpha) + m(y_1 + mr - \beta) + n(z_1 + nr - \gamma) = 0.$$

Solving

$$r = \frac{l(\alpha - x_1) + m(\beta - y_1) + n(\gamma - z_1)}{l^2 + m^2 + n^2} \tag{26}$$

The coordinates of N, the foot of the perpendicular PN is $(x_1 + lr - \alpha, y_1 + mr - \beta, z_1 + nr - \gamma)$ where r is given by (26).

The length of the perpendicular PN is obtained by distance formula between P (given) and N (found).

Line of greatest slope in a plane

Let ML be the line of intersection of a horizontal plane I with slant plane II. Let P be any point on plane II. Draw $PN \perp^r$ to the line ML. Then the line of greatest slope in plane II is the line PN, because no other line in plane II through P is inclined to the horizontal plane I more steeply than PN.

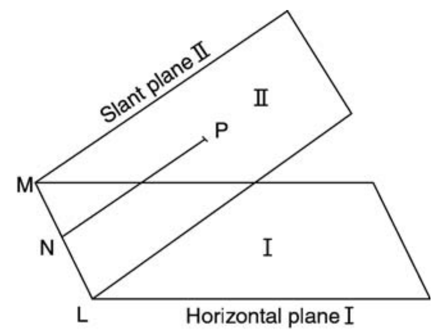


Fig. 3.14

WORKED OUT EXAMPLES

Example 1: Find the points where the line $x - y + 2z = 2$, $2x - 3y + 4z = 0$ pierces the coordinate planes.

Solution: Put $z = 0$ to find the point at which the line pierces the xy-plane: $x - y = 2$ and $2x - 3y = 0$ or $x = 6$, $y = 4$. $\therefore$ (6, 4, 0).
Put $x = 0$, $-y + 2z = 2$, $-3y + 4z = 0$ or $y = 4$, $z = 3$ $\therefore$ (0, 4, 3) is piercing point.
Put $y = 0$, $x + 2z = 2$, $2x + 4z = 0$ no unique solution.
Note that DR's of the line are [2, 0, −1]. So this line is $\perp^r$ to y-axis whose DR's are [0, 1, 0] (i.e., $2 \cdot 0 + 0 \cdot 1 + (-1) \cdot 0 = 0$). Hence the given line does not pierce the xz-plane.

Example 2: Transfer the general (unsymmetrical) form $x + 2y + 3z = 1$ and $x + y + 2z = 0$ to the symmetrical form.

Solution: Put $x = 0$, $2y + 3z = 1$, $y + 2z = 0$. Solving $z = -1$, $y = 2$. So $(0, 2, -1)$ is a point on the line. Let l, m, n be the DR's of the line. Since this line is $\perp^r$ to both normals of the given two planes, we have

$$1 \cdot l + 2 \cdot m + 3 \cdot n = 0$$
$$1 \cdot l + 1 \cdot m + 2 \cdot n = 0$$

Solving $\frac{l}{4-3} = -\frac{m}{2-3} = \frac{n}{1-2}$ or $\frac{l}{1} = \frac{m}{1} = -\frac{n}{1}$

Equation of the line passing through the point $(0, 2, -1)$ and having DR's $1, 1, -1$ is

$$\frac{x-0}{1} = \frac{y-2}{2} = \frac{z+1}{-1}$$

Aliter: Two point form.

Put $y = 0$, $x + 3z = 1$, $x + 27 = 0$. Solving $z = 1$, $x = -2$ or $(-2, 0, 1)$ is another point on the line. Now DR's of the line joining the two points $(0, 2, -1)$ and $(-2, 0, 1)$ are $-2, -2, 2$. Hence the equation of the line in the two point form is

$$\frac{x-0}{-2} = \frac{y-2}{-2} = \frac{z+1}{2} \quad \text{or} \quad \frac{x}{1} = \frac{y-2}{1} = \frac{z+1}{-1}.$$

Example 3: Find the acute angle between the lines $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ and $\frac{x}{5} = \frac{y}{4} = \frac{z}{-3}$.

Solution: DR's are $[2, 2, 1]$ and $[5, 4, -3]$. If θ is the angle between the two lines, then

$$\cos\theta = \frac{l_1l_2 + m_1m_2 + n_1n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2}\sqrt{l_2^2 + m_2^2 + n_2^2}}$$

$$= \frac{2 \cdot 5 + 2 \cdot 4 + 1 \cdot (-3)}{\sqrt{4+4+1}\sqrt{25+16+9}} = \frac{15}{3\sqrt{50}} = \frac{1}{\sqrt{2}}$$

$$\therefore \quad \theta = 45^\circ$$

Example 4: Find the equation of the plane containing the line $x = y = z$ and passing through the point $(1, 2, 3)$.

Solution: General form of the given line is

$$x - y = 0 \quad \text{and} \quad x - z = 0.$$

Equation of a plane containing this line is

$$(x - y) + k(x - z) = 0$$

Since point $(1, 2, 3)$ lies on this line, it also lies on the above plane. Then

$$(1-2) + k(1-3) = 0 \quad \text{or} \quad k = -\frac{1}{2}$$

Equation of required plane is

$$(x - y) - \frac{1}{2}(x - z) = 0$$

or

$$x - 2y + z = 0.$$

Example 5: Show that the lines $\frac{x}{1} = \frac{y+3}{2} = \frac{z+1}{3}$ and $\frac{x-3}{2} = \frac{y}{1} = \frac{z-1}{-1}$ intersect. Find the point of intersection.

Solution: Rewriting the equation in general form, we have

$$2x - y = 3, \qquad 3x - z = 1$$

and

$$x - 2y = 3, \qquad x + 2z = 5$$

If these four equations have a common solution, then the given two lines intersect. Solving, $y = -1$, then $x = 1$, $z = 2$. So the point of intersection is $(1, -1, 2)$.

Example 6: Find the acute angle between the lines $\frac{x}{3} = \frac{y}{1} = \frac{z}{0}$ and the plane $x + 2y - 7 = 0$.

Solution: DR's of the line: $[3, 1, 0]$. DR's of normal to the plane is $[1, 2, 0]$. If ψ is the angle between the line and the normal, then

$$\cos\psi = \frac{3 \cdot 1 + 1 \cdot 2 + 0 \cdot 0}{\sqrt{3^2 + 1^2 + 0^2}\sqrt{1^2 + 2^2 + 0^2}}$$

$$= \frac{5}{\sqrt{10}\sqrt{5}} = \frac{1}{\sqrt{2}} \quad \text{so} \quad \psi = 45^\circ.$$

Angle θ between the line and the plane is the complement of the angle ψ i.e., $\theta = 90 - \psi = 90 - 45 = 45^\circ$.

Example 7: Show that the lines $x + y - 3z = 0$, $2x + 3y - 8z = 1$ and $3x - y - z = 3$, $x + y - 3z = 5$ are parallel.

Solution: DR's of the first line are

l_1	m_1	n_1
1	1	−3
2	3	−8

or $\frac{l_1}{1} = \frac{m_1}{2} = \frac{n_1}{1}$

Similarly, DR's of the second line are

$$\begin{matrix} l_2 & m_2 & n_2 \\ 3 & -1 & -1 \\ 1 & 1 & -3 \end{matrix} \quad \text{or} \quad \frac{l_2}{4}=\frac{m_2}{8}=\frac{n_2}{4} \quad \text{i.e.,} \quad \frac{l_2}{1}=\frac{m_2}{2}=\frac{n_2}{1}$$

Since the DR's of the two lines are same, they are parallel.

Example 8: Find the acute angle between the lines $2x - y + 3z - 4 = 0$, $3x + 2y - z + 7 = 0$ and $x + y - 2z + 3 = 0, 4x - y + 3z + 7 = 0$.

Solution: The line represented by the two planes is perpendicular to both the normals of the two planes. If l_1, m_1, n_1 are the DR's of this line, then

$$\begin{matrix} l_1 & m_1 & n_1 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{matrix} \quad \text{or} \quad \frac{l_1}{-5} = \frac{m_1}{11} = \frac{n_1}{7}$$

Similarly, DR's of the 2nd line are

$$\begin{matrix} l_2 & m_2 & n_2 \\ 1 & +1 & -2 \\ 4 & -1 & -3 \end{matrix} \quad \text{or} \quad \frac{l_2}{-1} = \frac{m_2}{11} = \frac{n_2}{5}$$

If θ is the angle between the lines, then

$$\cos\theta = \frac{l_1 l_2 + m_1 m_2 + n_1 n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2}\sqrt{l_2^2 + m_2^2 + n_2^2}}$$

$$= \frac{5 + 121 + 35}{\sqrt{195}\sqrt{147}} = \frac{23}{3\sqrt{65}}$$

$\therefore$ So $\theta = 180^\circ 1.4'$

Example 9: Prove that the line $\frac{x-4}{2} = \frac{y-2}{3} = \frac{z-3}{6}$ lies in the plane $3x - 4y + z = 7$.

Solution: The point of the line (4, 2, 3) should also lie in the plane. So $3 \cdot 4 - 4 \cdot 2 + 1 \cdot 3 = 7$ satisfied. The line and normal to the plane are perpendicular. So $2 \cdot 3 + 3 \cdot (-4) + 6 \cdot 1 = 6 - 12 + 6 = 0$. Thus the given line completely lies in the given plane.

Example 10: Show that the lines $\frac{x-2}{2} = \frac{y-3}{-1} = \frac{z+4}{3}$ and $\frac{x-3}{1} = \frac{y+1}{3} = \frac{z-1}{-2}$ are coplanar. Find their common point and determine the equation of the plane containing the two given lines.

Solution: Here first line passes through $(2, 3, -4)$ and has DR's $l_1, m_1, n_1 : 2, -1, 3$. The second line passes through $(3, -1, 1)$ and has DR's l_2, m_2, n_2 : $1, 3, -2$. Condition for coplanarity:

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} 3-2 & -1-3 & 1+4 \\ 2 & -1 & 3 \\ 1 & 3 & 2 \end{vmatrix}$$

$$= \begin{matrix} 7+28-35 = 0 \\ \text{satisfied.} \end{matrix}$$

Point of intersection: Any point on the first line is $(2 + 2r_1, 3 - r_1 - 4 + 3r_1)$ and any point on the second line is $(3 + r_2, -1 + 3r_2, 1 - 2r_2)$. When the two lines intersect in a common point then coordinates on line (1) and line (2) must be equal, i.e., $2 + 2r_1 = 3 + r_2, 3 - r_1 = -1 + 3r_2$ and $-4 + 3r_1 = 1 - 2r_2$. Solving $r_1 = r_2 = 1$. Therefore the point of intersection is $(2 + 2 \cdot 1, 3 - 1, -4 + 3 \cdot 1) = (4, 2, -1)$.

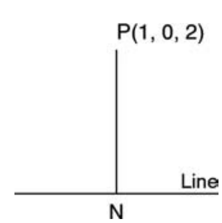
Equation of plane containing the two lines:

$$\begin{vmatrix} x-x_1 & y-y_1, & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} x-2 & y-3 & z+4 \\ 2 & -1 & 3 \\ 1 & 3 & -2 \end{vmatrix} = 0$$

Expanding $-7(x-2) - (-7)(y-3) + 7(z+4) = 0$ or $x - y - z + 3 = 0$.

Example 11: Find the coordinates of the foot of the perpendicular from $P(1, 0, 2)$ to the line $\frac{x+1}{3} = \frac{y-2}{-2} = \frac{z+1}{-1}$. Find the length of the perpendicular and its equation.

Solution: Any point N on the given line is $(3r - 1, 2 - 2r, -1 - r)$. DR's of PN are $(3r - 2, 2 - 2r, -3 - r)$. Now PN is normal to line if $3(3r - 2) + (-2)(2 - 2r) + (-1)(-3 - r) = 0$ or $r = \frac{1}{2}$. So the coordinates of N the foot of the perpendicular from P to the line are $\left(3 \cdot \frac{1}{2} - 1, 2 - 2 \cdot \frac{1}{2}, -1 - \frac{1}{2}\right)$ or $\left(\frac{1}{2}, 1, -\frac{3}{2}\right)$.



Length of the perpendicular

$$PN = \sqrt{\left(\frac{1}{2}-1\right)^2 + (1-0)^2 + \left(-\frac{3}{2}-2\right)^2}$$
$$= \sqrt{\frac{1}{4}+1+\frac{49}{4}} = \sqrt{\frac{54}{4}} = \frac{3}{2}\sqrt{6}.$$

DR's of PM with $r = \frac{1}{2}$ are $[3\cdot\frac{1}{2}-2, 2-2\cdot\frac{1}{2}, -3-\frac{1}{2}]$ i.e., DR's of PM are $\frac{1}{2}, -1, \frac{7}{2}$. And PM passes through $P(1, 0, 2)$. Therefore the equation of the perpendicular PM

$$\frac{x-1}{\frac{1}{2}} = \frac{y-0}{-1} = \frac{z-2}{\frac{7}{2}} \quad \text{or} \quad x-1 = \frac{y}{-2} = \frac{z-2}{7}.$$

Example 12: Find the equation of the line of the greatest slope through the point (2, 1, 1) in the slant plane $2x + y - 5z = 0$ to the horizontal plane $4x - 3y + 7z = 0$.

Solution: Let l_1, m_1, n_1 be the DR's of the line of intersection ML of the two given planes. Since ML is $\perp^r$ to both normals,

$$2l_1 + m_1 - 5n_1 = 0, \qquad 4l_1 - 3m_1 + 7n_1 = 0.$$

Solving $\frac{l_1}{4} = \frac{m_1}{17} = \frac{n_1}{5}$. Let PN be the line of greatest slope and let l_2, m_2, n_2 be its DR's. Since PN and ML are perpendicular

$$4l_2 + 17m_2 + 5n_2 = 0$$

Also PN is perpendicular to normal of the slant plane $2x + y - 5z = 0$. So

$$2l_2 + m_2 - 5n_2 = 0$$

Solving $\frac{l_2}{3} = \frac{m_2}{-1} = \frac{n_2}{1}$.
Therefore the equation of the line of greatest slope PN having DR's $3, -1, 1$ and passing through $P(2, 1, 1)$ is

$$\frac{x-2}{3} = \frac{y-1}{-1} = \frac{z-1}{1}.$$

EXERCISE

1. Find the points where the line $x + y + 4z = 6$, $2x - 3y - 2z = 2$ pierce the coordinate planes.

Ans. (0, −2, 2), (4, 2, 0), (2, 0, 1)

2. Transform the general form $3x + y - 2z = 7$, $6x - 5y - 4z = 7$ to symmetrical form and two point form.

   Hint: (0, 1, −3), (2, 1, 0) are two points on the line.

Ans. $\frac{x-2}{2} = \frac{y-1}{0} = \frac{z-0}{3}$

3. Show that the lines $x = y = z + 2$ and $\frac{x-1}{1} = \frac{y}{0} = \frac{z}{2}$ intersect and find the point of intersection.

   Hint: Solve $x - y = 0$, $y - z = 2$, $y = 0$, $2x - z = 2$ simultaneously.

Ans. (0, 0, −2)

4. Find the equation plane containing the line $x = y = z$ and

   a. Passing through the line $x + 1 = y + 1 = z$

   b. Parallel to the line $\frac{x+1}{3} = \frac{y}{2} = \frac{z}{-1}$.

Ans. (a) $x - y = 0$; (b) $3x - 4y + z = 0$

5. Show that the line $\frac{x+1}{1} = \frac{y}{-1} = \frac{z-2}{2}$ is in the plane $2x + 4y + z = 0$.

   Hint: $2(1) + 4(-1) + 1(2) = 0$,
   $2(-1) + 4(0) + 2 = 0$

6. Find the equation of the plane containing line $\frac{x-1}{3} = \frac{y-1}{4} = \frac{z-2}{2}$ and parallel to the line $x - 2y + 3z = 4$, $2x - 3y + 4z = 5$.

   Hint: Eq. of 2nd line $\frac{x-0}{\frac{1}{2}} = \frac{y-1}{1} = \frac{z-2}{\frac{1}{2}}$, contains 1st line: $3A + 4B + 2C = 0$. Parallel to 2nd line $A + 2B + C = 0$, $A = 0$, $B = -\frac{1}{2}C$, $D = -\frac{3}{2}C$.

Ans. $y - 2z + 3 = 0$

7. Show that the lines $x + 2y - z = 3$, $3x - y + 2z = 1$ and $2x - 2y + 3z = 2$, $x - y + z + 1 = 0$ are coplanar. Find the equation of the plane containing the two lines.

   Hint: $\frac{x-0}{3} = \frac{y-\frac{7}{3}}{-5} = \frac{z-\frac{5}{3}}{-7}$, $\frac{x-0}{1} = \frac{y-5}{+1} = \frac{z-4}{0}$.

$$\begin{vmatrix} x-0 & y-5 & z-4 \\ 3 & -5 & -7 \\ 1 & 1 & 0 \end{vmatrix} = 0, \quad \text{Expand.}$$

Ans. $7x - 7y + 8z + 3 = 0$

8. Prove that the equation of the plane through the origin containing the line $\frac{x-1}{5} = \frac{y-2}{4} = \frac{z-3}{5}$ is $x - 5y + 3z = 0$.

   Hint: $A(x-1) + B(y-2) + C(z-3) = 0$, $5A + 2B + 3C = 0$, $A + 2B + 3C = 0$,

   Expand $\begin{vmatrix} x-1 & y-2 & z-3 \\ 5 & 4 & 5 \\ 1 & 2 & 3 \end{vmatrix} = 0$

9. Find the image of the point $P(1, 3, 4)$ in the plane $2x - y + z + 3 = 0$.

   Hint: Line through P and $\perp^r$ to plane: $\frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-4}{1}$. Image Q: $(2r+1, -r+3, r+4)$. Mid point L of PQ is $(r+1, -\frac{1}{2}r+3, \frac{1}{2}r+4)$. L lies on plane, $r = -2$.

Ans. $(-3, 5, 2)$

10. Determine the point of intersection of the lines

$$\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}, \frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$$

   Hint: General points: $(r_1 + 4, -4r_1 - 3, 7r_1 - 1)$, $(2r_2 + 1, -3r_2 - 1, 8r_2 - 10)$, Equating $r_1 + 4 = 2r_2 + 1$, $-4r_1 - 3 = -3r_2 - 1$, solving $r_1 = 1, r_2 = 2$.

Ans. $(5, -7, 6)$

11. Show that the lines $\frac{x+3}{2} = \frac{y+5}{3} = \frac{z-7}{-3}$, $\frac{x+1}{4} = \frac{y+1}{5} = \frac{z+1}{-1}$ are coplanar. Find the equation of the plane containing them.

Ans. $6x - 5y - z = 0$

12. Find the equation of the line which passes through the point $(2, -1, 1)$ and intersect the lines $2x + y = 4$, $y + 2z = 0$, and $x + 3z = 4$, $2x + 5z = 8$.

Ans. $x + y + z = 2$, $x + 2z = 4$

13. Find the coordinates of the foot of the perpendicular from $P(5, 9, 3)$ to the line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$. Find the length of the perpendicular and its equations.

Ans. $(3, 5, 7)$, Length: 6, Equation $\frac{x-5}{-2} = \frac{y-9}{-4} = \frac{z-3}{4}$.

14. Find the equation of the line of greatest slope in the slant plane $2x + y - 5z = 12$ and passing through the point $(2, 3, -1)$ given that the line $\frac{x}{4} = \frac{y}{-3} = \frac{z}{7}$ is vertical.

Ans.

15. Find the angle between the line $\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6}$ and the plane $3x + y + z = 7$.

   Hint: DR's of line: 2, 3, 6; DR's of normal to plane 3, 1, 1

$$\cos(90 - \theta) = \sin\theta = \frac{2 \cdot 3 + 3 \cdot 1 + 6 \cdot 1}{\sqrt{4+9+36}\sqrt{9+1+1}}.$$

Ans. $\sin\theta = \frac{15}{7\sqrt{11}}$

16. Find the angle between the line $x + y - z = 1$, $2x - 3y + z = 2$ and the plane $3x + y - z + 5 = 0$.

   Hint: DR's of line 2, 3, 5, DR's of normal: 3, 1, −1

$$\cos(90 - \theta) = \sin\theta = \frac{2 \cdot 3 + 3 \cdot 1 + 5 \cdot (-1)}{\sqrt{4+9+25}\sqrt{9+1+1}}.$$

Ans. $\sin\theta = \frac{4}{\sqrt{38}\sqrt{11}}$

3.5 SHORTEST DISTANCE BETWEEN SKEW LINES

Skew lines: Any two straight lines which do not lie in the same plane are known as skew lines (or non-planar lines). Such lines neither intersect nor are parallel. **Shortest distance between two skew lines:**

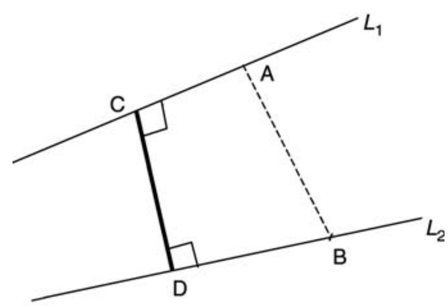


Fig. 3.15

Let L_1 and L_2 be two skew lines; L_1 passing through a given point A and L_2 through a given point

B. Shortest distance between the two skew lines L_1 and L_2 is the length of the line segment CD which is perpendicular to **both** L_1 and L_2. The equation of the shortest distance line CD can be uniquely determined since it intersects both lines L_1 and L_2 at right angles. Now CD = projection of AB on $CD = AB \cos\theta$ where θ is the angle between AB and CD. Since $\cos\theta < 1$, $CD < AB$, thus CD is the shortest distance between the lines L_1 and L_2.

Magnitude (length) and the equations of the line of shortest distance between two lines L_1 and L_2:

Suppose the equation of given line L_1 be

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \tag{1}$$

and of line L_2 be

$$\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} \tag{2}$$

Assume the equation of shortest distance line CD as

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \tag{3}$$

where (α, β, γ) and (l, m, n) are to be determined. Since CD is perpendicular to both L_1 and L_2,

$$ll_1 + mm_1 + nn_1 = 0$$

$$ll_2 + mm_2 + nn_2 = 0$$

Solving

$$\frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1}$$

$$= \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2}}$$

$$= \frac{1}{\sqrt{\sum(m_1n_2 - m_2n_1)^2}} = \frac{1}{k}$$

where $k = \sqrt{\sum(m_1n_2 - m_2n_1)^2}$

or $l = \dfrac{m_1n_2 - m_2n_1}{k}$, $m = \dfrac{n_1l_2 - n_2l_1}{k}$,

$$n = \frac{l_1m_2 - l_2m_1}{k} \tag{4}$$

Thus the DC's l, m, n of the shortest distance line CD are determined by (4).

Magnitude of shortest distance CD = projection of AB on CD where $A(x_1, y_1, z_1)$ is a point on L_1 and $B(x_2, y_2, z_2)$ is a point on L_2.

$\therefore$ shortest distance $CD =$

$$= l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1) \tag{5}$$

In the determinant form,

$$\text{Shortest distance } CD = \frac{1}{k}\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \tag{5$'$}$$

Note: If shortest distance is zero, then the two lines L_1 and L_2 are coplanar.

Equation of the line of shortest distance *CD*: Observe that CD is coplanar with both L_1 and L_2. Let P_1 be the plane containing L_1 and CD. Equation of plane P_1 containing coplanar lines L_1 and CD is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 \tag{6}$$

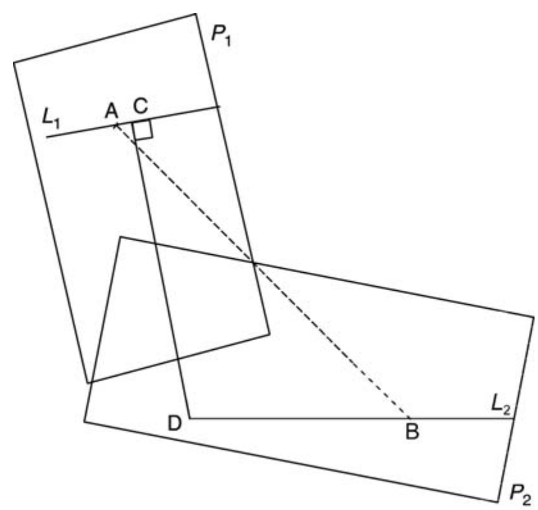


Fig. 3.16

Similarly, equation of plane P_2 containing L_2 and CD is

$$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix} = 0 \tag{7}$$

Equations (6) and (7) together give the equation of the line of shortest distance.

Points of intersection C and D with L_1 and L_2: Any general point C^* on L_1 is

$$(x_1 + l_1 r_1, \quad y_1 + m_1 r_1, \quad z_1 + n_1 r_1)$$

and any general point D^* on L_2 is

$$(x_2 + l_2 r_2, \quad y_2 + m_2 r_2, \quad z_2 + n_2 r_2)$$

DR's of C^*D^*: $(x_2 - x_1 + l_2 r_2 - l_1 r_1,\ y_2 - y_1 + m_2 r_2 - m_1 r_1,\ z_2 - z_1 + n_2 r_2 - n_1 r_1)$

If C^*D^* is $\perp^r$ to both L_1 and L_2, we get two equations for the two unknowns r_1 and r_2. Solving and knowing r_1 and r_2, the coordinates of C and D are determined. Then the magnitude of CD is obtained by length formula, and equation of CD by two point formula.

Parallel planes: Shortest distance CD = perpendicular distance from any point on L_1 to the plane parallel to L_1 and containing L_2.

Worked Out Examples

Example 1: Find the magnitude and equation of the line of shortest distance between the lines

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4},$$
$$\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}.$$

Solution: Point $A(x_1, y_1, z_1)$ on first line is $(1, 2, 3)$ and $B(x_2, y_2, z_2)$ on second line is $(2, 4, 5)$. Also (l_1, m_1, n_1) are $(2, 3, 4)$ and $(l_2, m_2, n_2) = (3, 4, 5)$. Then

$$k^2 = (m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 + (l_1 m_2 - l_2 m_1)^2$$
$$= (15 - 16)^2 + (12 - 10)^2 + (8 - 9)^2$$
$$= 1 + 4 + 1 = 6 \quad \text{or} \quad k = \sqrt{6}.$$

So DR's is of line of shortest of distance: $-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}$.

$$\text{Shortest distance} = \frac{1}{k}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}\frac{1}{\sqrt{6}}$$
$$= \frac{(15-16) - 2(10-12) + 2(8-9)}{\sqrt{6}}$$
$$= \frac{-1+4-2}{\sqrt{6}} = \frac{1}{\sqrt{6}}.$$

Equation of shortest distance line:

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{vmatrix} = 0 \quad \text{or} \quad 11x + 2y - 7z + 6 = 0$$

and

$$\begin{vmatrix} x-1 & y-4 & z-5 \\ 2 & 3 & 4 \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{vmatrix} = 0 \quad \text{or} \quad 7x + y - 5z + 7 = 0.$$

Example 2: Determine the points of intersection of the line of shortest distance with the two lines

$$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}; \ \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}.$$

Also find the magnitude and equation of shortest distance.

Solution: Any general point C^* on first line is $(3 + 3r_1, 8 - r_1, 3 + r_1)$ and any general point D^* on the second line is $(-3 - 3r_2, -7 + 2r_2, 6 - 4r_2)$. DR's of C^*D^* are $(6 + 3r_1 + 3r_2, 15 - r_1 - 2r_2, -3 + r_1 - 4r_2)$. If C^*D^* is $\perp^r$ to both the given lines, then

$$3(6+3r_1+3r_2) - 1(15-r_1-2r_2) + 1(-3+r_1-4r_2) = 0$$
$$-3(6+3r_1+3r_2) + 2(15-r_1-2r_2) + 4(-3+r_1-4r_2) = 0$$

Solving for r_1 and r_2, $11r_1 - 7r_2 = 0, +7r_1 + 29r_2 = 0$ so $r_1 = r_2 = 0$. Then the points of intersection of shortest distance line CD with the given two lines are $C(3, 8, 3)$, $D(-3, -7, 6)$.

$$\text{Length of } CD = \sqrt{(-6)^2 + (-15)^2 + (3)^2}$$
$$= \sqrt{270} = 3\sqrt{30}$$

$$\text{Equation } CD\text{: } \frac{x-3}{-3-3} = \frac{y-8}{-7-8} = \frac{z-3}{6-3}$$

$$\text{i.e., } \frac{x-3}{-6} = \frac{y-8}{-15} = \frac{z-3}{3}.$$

Example 3: Calculate the length and equation of line of shortest distance between the lines

$$5x - y - z = 0, \quad x - 2y + z + 3 = 0 \qquad (1)$$

$$7x - 4y - 2z = 0, \quad x - y + z - 3 = 0 \qquad (2)$$

Solution: Any plane containing the second line (2) is

$$(7x - 4y - 2z) + \mu(x - y + z - 3) = 0$$

or $(7 + \mu)x + (-4 - \mu)y + (-2 + \mu)z - 3\mu = 0 \qquad (3)$

DR's of first line (1) are $(l, m, n) = (-3, -6, -9)$ obtained from:

l	m	n
5	−1	−1
l	−2	1

The plane (3) will be parallel to the line (1) with $l = -3, m = -6, n = -9$ if

$$-3(7 + \mu) + 6(4 + \mu) + 9(2 - \mu) = 0 \quad \text{or} \quad \mu = \frac{7}{2}$$

Substituting μ in (3), we get the equation of a plane containing line (2) and parallel to line (1) as

$$7x - 5y + z - 7 = 0 \qquad (4)$$

To find an arbitrary point on line (1), put $x = 0$. Then $-y - z = 0$ or $y = -z$ and $-2y + z + 3 = 0$, $z = -1$, $y = 1$. $\therefore$ $(0, 1, -1)$ is a point on line (1). Now the length of the shortest distance = perpendicular distance of $(0, 1, -1)$ to plane (4)

$$= \frac{0 - 5(1) + (-1) - 7}{\sqrt{49 + 25 + 1}} = \left|\frac{-13}{\sqrt{75}}\right| = \frac{13}{\sqrt{75}} \qquad (5)$$

Equation of any plane through line (1) is

$$5x - y - z + \lambda(x - 2y + z + 3) = 0$$

or $(5 + \lambda)x + (-y - 2\lambda)y + (-1 + \lambda)z + 3\lambda = 0 \qquad (6)$

DR's of line (2) are $(l, m, n) = (2, 3, 1)$ obtained from

l	m	n
7	−4	−2
1	−1	1

plane (6) will be parallel to line (2) if

$$2(5 + \lambda) + 3(-y - 2\lambda) + 1(-1 + \lambda) = 0 \quad \text{or} \quad \lambda = 2.$$

Thus the equation of plane containing line (1) and parallel to line (2) is

$$7x - 5y + z + 6 = 0 \qquad (7)$$

Hence equation of the line of shortest distance is given by (6) and (7) together.

Aliter: A point on line (2) is $(0, -1, 2)$ obtained by putting $x = 0$ and solving (2). Then the length of shortest distance = perpendicular distance of $(0, -1, 2)$ to the plane (7) $= \frac{0+5+2+6}{\sqrt{75}} = \frac{13}{\sqrt{75}}$

Note: By reducing (1) and (2) to symmetric forms

$$\frac{x-\frac{1}{3}}{1} = \frac{y-\frac{5}{3}}{2} = \frac{z}{3}$$
$$\frac{x+4}{1} = \frac{y+7}{\frac{3}{2}} = \frac{z}{\frac{1}{2}}$$

The problem can be solved as in above worked Example 1.

Example 4: Show that the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}; \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ are coplanar.

Solution: Shortest distance between the two lines is

$$\begin{vmatrix} 2-1 & 3-2 & 4-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} \begin{matrix} =(-1)-(-2)+(-1) \\ =0 \end{matrix}$$

$\therefore$ Lines are coplanar.

Example 5: If a, b, c are the lengths of the edges of a rectangular parallelopiped, show that the shortest distance between a diagonal and an edge not meeting the diagonal is $\frac{bc}{\sqrt{b^2+c^2}} \left(\text{or} \frac{ca}{\sqrt{c^2+a^2}} \text{ or } \frac{ab}{\sqrt{a^2+b^2}}\right)$.

Solution: Choose coterminus edges OA, OB, OC along the X, Y, Z axes. Then the coordinates are $A(a, 0, 0)$, $B(0, b, 0)$, $C(0, 0, c)$, $E(a, b, 0)$, $D(0, b, c)$, $G(a, 0, c) F(a, b, c)$ etc. so that $OA = a$, $OB = b$, $OC = c$.
To find the shortest distance between a diagonal OF and an edge GC. Here GC does not interest OF

Equation of the line OF: $\frac{x-0}{a-0} = \frac{y-0}{b-0} = \frac{z-0}{c-0}$

$$\text{or} \quad \frac{x}{a} = \frac{y}{b} = \frac{z}{c} \quad (1)$$

Equation of the line GC: $\frac{x-0}{a-0} = \frac{y-0}{b-0} = \frac{z-c}{c-c}$

$$\text{or} \quad \frac{x}{1} = \frac{y}{0} = \frac{z-c}{0} \quad (2)$$

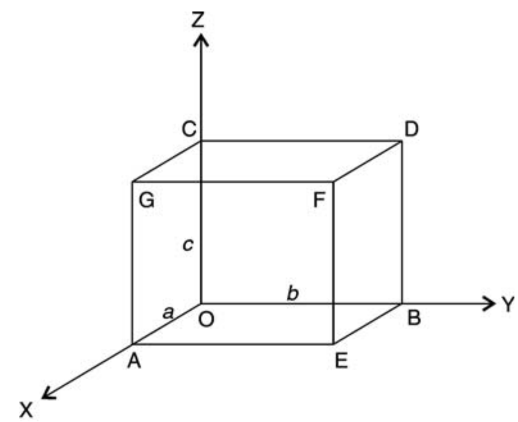


Fig. 3.17

Equation of a plane containing line (1) and parallel to (2) is

$$\begin{vmatrix} x & y & z \\ a & b & c \\ 1 & 0 & 0 \end{vmatrix} = 0 \quad \text{or} \quad cy - bz = 0 \quad (3)$$

Shortest distance = Length of perpendicular drawn from a point say $C(0, 0, c)$ to the plane (3)

$$= \frac{c \cdot 0 - b \cdot c}{\sqrt{0^2 + c^2 + b^2}} = \frac{bc}{\sqrt{c^2+b^2}}.$$

In a similar manner, it can be proved that the shortest distance between the diagonal OF and non-intersecting edges AN and AM are respectively $\frac{ca}{\sqrt{c^2+a^2}}, \frac{ab}{\sqrt{a^2+b^2}}$.

EXERCISE

1. Determine the magnitude and equation of the line of shortest distance between the lines. Find the points of intersection of the shortest distance line, with the given lines
$\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7}, \quad \frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$.

Ans. 14, $117x + 4y - 41z - 490 = 0, 9x - 4y - z = 14$, points of intersection (5, 7, 3), (9, 13, 15).

2. Calculate the length, points of intersection, the equations of the line of shortest distance between the two lines
$\frac{x+1}{2} = \frac{y+1}{3} = \frac{z+1}{4}, \quad \frac{x+1}{3} = \frac{y}{4} = \frac{z}{5}$.

Ans. $\frac{1}{\sqrt{6}}, \frac{x-\frac{5}{3}}{\frac{1}{6}} = \frac{y-3}{-\frac{1}{2}} = \frac{z-\frac{15}{2}}{\frac{1}{6}}, \left(\frac{5}{3}, 3, \frac{13}{3}\right),$ $\left(\frac{3}{2}, \frac{10}{3}, \frac{25}{6}\right)$.

3. Find the magnitude and equations of shortest distance between the two lines

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}, \quad \frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}.$$

Ans. $\frac{1}{\sqrt{6}}$, $11x + 2y - 7z + 6 = 0$, $7x + y - 5z + 7 = 0$.

4. Show that the shortest distance between the lines $\frac{x}{2} = \frac{y}{-3} = \frac{z}{1}$ and $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$ is $\frac{1}{\sqrt{3}}$ and its equations are $4x + y - 5z = 0$, $7x + y - 8z = 31$.

5. Determine the points on the lines $\frac{x-6}{3} = \frac{y-7}{-1} = \frac{z-4}{1}$, $\frac{x}{-3} = \frac{y+9}{2} = \frac{z-2}{4}$ which are nearest to each other. Hence find the shortest distance between the lines and find its equations.

Ans. $(3, 8, 3)$, $(-3, -7, 6)$, $3\sqrt{30}$, $\frac{x-3}{2} = \frac{y-8}{5} = \frac{z-3}{-1}$.

6. Prove that the shortest distance between the two lines $\frac{x-1}{3} = \frac{y-4}{2} = \frac{z-4}{-2}$, $\frac{x+1}{2} = \frac{y-1}{-4} = \frac{z+2}{1}$ is $\frac{120}{\sqrt{341}}$

   Hint: Equation of a plane passing through the first lines nad parallel to the second line is $6x + 7y + 16z = 98$. A point on second line is $(-1, 1, -2)$. Perpendicular distance $= \frac{6(-1)+7(1)+16(-2)}{\sqrt{6^2+7^2+16^2}}$.

7. Find the length and equations of shortest distance between the lines $x - y + z = 0$, $2x - 3y + 4z = 0$; and $x + y + 2z - 3 = 0$, $2x + 3y + 3z - 4 = 0$.

   Hint: Equations of two lines in symmetric form are $\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$, $\frac{x-5}{-3} = \frac{y+2}{1} = \frac{z}{1}$.

Ans. $\frac{13}{\sqrt{66}}$, $3x - y - z = 0$, $x + 2y + z - 1 = 0$.

8. Determine the magnitude and equations of the line of shortest distance between the lines $\frac{x-3}{2} = \frac{y+15}{-7} = \frac{z-9}{5}$ and $\frac{x+1}{2} = \frac{y-1}{1} = \frac{z-9}{-3}$.

Ans. $4\sqrt{3}$, $-4x + y + 3z = 0$, $4x - 5y + z = 0$ (or $x = y = z$).

9. Obtain the coordinates of the points where the line of shortest distance between the lines $\frac{x-23}{-6} = \frac{y-19}{-4} = \frac{z-25}{3}$ and $\frac{x-12}{-9} = \frac{y-1}{4} = \frac{z-5}{2}$ meets them. Hence find the shortest distance between the two lines.

Ans. $(11, 11, 31)$, $(3, 5, 7)$, 26

10. Find the shortest distance between any two opposite edges of a tetrahedron formed by the planes $x + y = 0$, $y + z = 0$, $z + x = 0$, $x + y + z = a$. Also find the point of intersection of three lines of shortest distances.

    Hint: Vertices are $(0, 0, 0)$, $(a, -a, a)$, $(-a, a, a)$, $(a, a, -a)$.

Ans. $\frac{2a}{\sqrt{6}}$, $(-a, -a, -a)$.

11. Find the shortest distance between the lines PQ and RS where $P(2, 1, 3)$, $Q(1, 2, 1)$, $R(-1, -2, -2)$, $S(-1, 4, 0)$.

Ans. $3\sqrt{2}$

3.6 THE RIGHT CIRCULAR CONE

Cone

A **cone** is a surface generated by a straight line (known as **generating line or generator**) passing through a fixed point (known as **vertex**) and satisfying a condition, for example, it may intersect a given curve (known as **guiding curve**) or touches a given surface (say a sphere). Thus cone is a set of points on its generators. Only cones with second degree equations known as quadratic cones are considered here. In particular, quadratic cones with vertex at origin are homogeneous equations of second degree.

Equation of cone with vertex at (α, β, γ) and the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, $z = 0$ as the guiding curve:

The equation of any line through vertex (α, β, γ) is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \tag{1}$$

(1) will be generator of the cone if (1) intersects the given conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \; z = 0 \tag{2}$$

Since (1) meets $z = 0$, put $z = 0$ in (1), then the point

$\left(\alpha-\frac{l\gamma}{n}, \beta-\frac{m\gamma}{n}, 0\right)$ will lie on the conic (2), if

$$a\left(\alpha-\frac{l\gamma}{n}\right)^2+2h\left(\alpha-\frac{l\gamma}{n}\right)\left(\beta-\frac{m\gamma}{n}\right)+b\left(\beta-\frac{m\gamma}{n}\right)^2+$$
$$+2g\left(\alpha-\frac{l\gamma}{n}\right)+2f\left(\beta-\frac{m\gamma}{n}\right)+c=0 \qquad (3)$$

From (1)

$$\frac{l}{n}=\frac{x-\alpha}{z-\gamma}, \qquad \frac{m}{n}=\frac{y-\beta}{z-\gamma} \qquad (4)$$

Eliminate l, m, n from (3) using (4),

$$a\left(\alpha-\frac{x-\alpha}{z-\gamma}\cdot\gamma\right)^2+$$
$$+2h\left(\alpha-\frac{x-\alpha}{z-\gamma}\cdot\gamma\right)\left(\beta-\frac{y-\beta}{z-\gamma}\cdot\gamma\right)+$$
$$+b\left(\beta-\frac{y-\beta}{z-\gamma}\cdot\gamma\right)^2+2g\left(\alpha-\frac{x-\alpha}{z-\gamma}\cdot\gamma\right)+$$
$$+2f\left(\beta-\frac{\gamma-\beta}{z-\gamma}\cdot\gamma\right)+c=0$$

or

$$a(\alpha z-x\gamma)^2+2h(\alpha z-x\gamma)(\beta z-y\gamma)+$$
$$+b(\beta z-y\gamma)^2+2g(\alpha z-x\gamma)(z-\gamma)+$$
$$+2f(\beta z-y\gamma)(z-\gamma)+c(z-\gamma)^2=0$$

or

$$a(x-\alpha)^2+b(y-\beta)^2+c(z-\gamma)^2+$$
$$+2f(z-\gamma)(y-\beta)+2g(x-\alpha)(z-\gamma)+$$
$$+2h(x-\alpha)(y-\beta)=0 \qquad (5)$$

Thus (5) is the equation of the quadratic cone with vertex at (α, β, γ) and guiding curve as the conic (2).
Special case: Vertex at origin (0, 0, 0). Put $\alpha=\beta=\gamma=0$ in (5). Then (5) reduces to

$$ax^2+by^2+cz^2+2fzy+2gxz+2hxy=0 \qquad (6)$$

Equation (6) which is a homogeneous and second degree in x, y, z is the equation of cone with vertex at origin.

Right circular cone

A right circular cone is a surface generated by a line (**generator**) through a fixed point (**vertex**) making a constant angle θ (**semi-vertical angle**) with the fixed line (**axis**) through the fixed point (**vertex**). Here the guiding curve is a circle with centre at c. Thus every section of a right circular cone by a plane perpendicular to its axis is a circle.

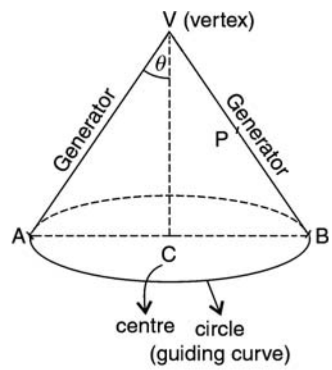


Fig. 3.18

Equation of a right circular cone: with vertex at (α, β, γ), semi vertical angle θ and equation of axis

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad (1)$$

Let $P(x, y, z)$ be any point on the generating line VB. Then the DC's of VB are proportional to $(x-\alpha, y-\beta, z-\gamma)$. Then

$$\cos\theta=\frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{\sqrt{(l^2+m^2+n^2)}\sqrt{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}}$$

Rewriting, the required equation of cone is

$$\left[l(x-\alpha)+m(y-\beta)+n(z-\gamma)\right]^2=$$
$$=(l^2+m^2+n^2)\left[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2\right]\cos^2\theta \qquad (2)$$

Case 1: If vertex is origin (0, 0, 0) then (2) reduces

$$(lx+my+nz)^2=(l^2+m^2+n^2)(x^2+y^2+z^2)\cos^2\theta \qquad (3)$$

Case 2: If vertex is origin and axis of cone is z-axis (with $l=0, m=0, n=1$) then (2) becomes

$$z^2=(x^2+y^2+z^2)\cos^2\theta \quad \text{or} \quad z^2\sec^2\theta=x^2+y^2+z^2$$
$$z^2(1+\tan^2\theta)=x^2+y^2+z^2$$

i.e.,
$$x^2+y^2=z^2\tan^2\theta \qquad (4)$$

Similarly, with y-axis as the axis of cone

$$x^2 + z^2 = y^2 \tan^2\theta$$

with x-axis as the axis of cone

$$y^2 + z^2 = x^2 \tan^2\theta.$$

If the right circular cone admits sets of three mutually perpendicular generators then the semi-vertical angle $\theta = \tan^{-1}\sqrt{2}$ (since the sum of the coefficients of x^2, y^2, z^2 in the equation of such a cone must be zero i.e., $1 + 1 - \tan^2\theta = 0$ or $\tan\theta = \sqrt{2}$).

WORKED OUT EXAMPLES

Example 1: Find the equation of cone with base curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $z = 0$ and vertex (α, β, γ). Deduce the case when base curve is $\frac{x^2}{16} + \frac{y^2}{9} = 1$, $z = 0$ and vertex at (1, 1, 1).

Solution: The equation of any generating line through the vertex (α, β, γ) is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \tag{1}$$

This generator (1) meets $z = 0$ in the point

$$\left(x = \alpha - \frac{l\gamma}{n}, \quad y = \beta - \frac{m\gamma}{n}, \quad z = 0\right) \tag{2}$$

Point (2) lies on the generating curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{3}$$

Substituting (2) in (3)

$$\frac{\left(\alpha - \frac{l\gamma}{n}\right)^2}{a^2} + \frac{\left(\beta - \frac{m\gamma}{n}\right)^2}{b^2} = 1 \tag{4}$$

Eliminating l, m, n from (4) using (1),

$$\frac{\left[\alpha - \left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right]^2}{a^2} + \frac{\left[\beta - \left(\frac{y-\beta}{z-\gamma}\right)\gamma\right]^2}{b^2} = 1$$

$$b^2\Big[\alpha(z-\gamma) - \gamma(x-\alpha)\Big]^2 + a^2\Big[\beta(z-\gamma) - \gamma(y-\beta)\Big]^2$$
$$= a^2b^2(z-\gamma)^2$$

Deduction: When $a = 4, b = 3, \alpha = 1, \beta = 1, \gamma = 1$,

$$9\Big[(z-1) - (x-1)\Big]^2 + 16\Big[(z-1) - (y-1)\Big]^2$$
$$= 144(z-1)^2$$

$$9x^2 + 16y^2 - 119z^2 - 18xz - 32yz + 288z - 144 = 0.$$

Example 2: Find the equation of the cone with vertex at (1, 0, 2) and passing through the circle $x^2 + y^2 + z^2 = 4$, $x + y - z = 1$.

Solution: Equation of generator is

$$\frac{x-1}{l} = \frac{y-0}{m} = \frac{z-2}{n} \tag{1}$$

Any general point on the line (1) is

$$(1 + lr, \quad mr, \quad 2 + nr). \tag{2}$$

Since generator (1) meets the plane

$$x + y - z = 1 \tag{3}$$

substitute (2) in (3)

$$(1 + lr) + (mr) - (2 + nr) = 1$$

or
$$r = \frac{2}{l + m - n}. \tag{4}$$

Since generator (1) meets the sphere

$$x^2 + y^2 + z^2 = 4 \tag{5}$$

substitute (2) in (5)

$$(1 + lr)^2 + (mr)^2 + (2 + nr)^2 = 4$$

or
$$r^2(l^2 + m^2 + n^2) + 2r(l + 2n) + 1 = 0 \tag{6}$$

Eliminate r from (6) using (4), then

$$\frac{4}{(l+m-n)^2}(l^2+m^2+n^2) + 2\frac{2}{(l+m-n)}(l+2n) + 1 = 0$$

$$9l^2 + 5m^2 - 3n^2 + 6lm + 2ln + 6nm = 0 \tag{7}$$

Eliminate l, m, n from (7) using (1), then

$$9\left(\frac{x-1}{r}\right)^2 + 5\left(\frac{y}{r}\right)^2 - 3\left(\frac{z-2}{r}\right)^2 + 6\left(\frac{x-1}{r}\right)\left(\frac{y}{r}\right) +$$
$$+ 2\left(\frac{x-1}{r}\right)\left(\frac{z-2}{r}\right) + 6\left(\frac{z-2}{r}\right)\left(\frac{y}{r}\right) = 0$$

or
$$9(x-1)^2 + 5y^2 - 3(z-2)^2 + 6y(x-1) +$$
$$+ 2(x-1)(x-2) + 6(z-2)y = 0$$

Vertex (0, 0, 0):

Example 3: Determine the equation of a cone with vertex at origin and base curve given by

a. $ax^2 + by^2 = 2z, \quad lx + my + nz = p$

b. $ax^2 + by^2 + cz^2 = 1, \quad lx + my + nz = p$

c. $x^2 + y^2 + z^2 = 25, \quad x + 2y + 2z = 9$

Solution: We know that the equation of a quadratic cone with vertex at origin is a homogeneous equation of second degree in x, y, z. By eliminating the non-homogeneous terms in the base curve, we get the required equation of the cone.

a. $2z$ is the term of degree one and is non homogeneous. Solving

$$\frac{lx + my + nz}{p} = 1$$

rewrite the equation

$$ax^2 + by^2 = 2 \cdot z(1) = 2z\left(\frac{lx + my + nz}{p}\right)$$

$$apx^2 + bpy^2 - 2nz^2 - 2lxz - 2myz = 0$$

which is the equation of cone.

b. Except the R.H.S. term 1, all other terms are of degree 2 (and homogeneous). Rewriting, the required equation of cone as

$$ax^2 + by^2 + cz^2 = (1)^2 = \left(\frac{lx + my + nz}{p}\right)^2$$

$$(ap^2 - l^2)x^2 + (bp^2 - m^2)y^2 + (cp^2 - n^2)z^2 -$$

$$-2lmxy - 2mnyz - 2lnxz = 0$$

c. On similar lines

$$x^2 + y^2 + z^2 = 25 = 25(1)^2 = 25\left(\frac{x + 2y + 2z}{9}\right)^2$$

$$56x^2 - 19y^2 - 19z^2 - 100xy - 200yz - 100xz = 0$$

Right circular cone:

Example 4: Find the equation of a right circular cone with vertex at (2, 0, 0), semi-vertical angle $\theta = 30^\circ$ and axis is the line $\frac{x-2}{3} = \frac{y}{4} = \frac{z}{6}$.

Solution: Here $\alpha = 2, \beta = 0, \gamma = 0, l = 3, m = 4, n = 6$

$$\frac{\sqrt{3}}{2} = \cos 30 = \cos\theta$$

$$= \frac{l(x-\alpha) + m(y-\beta) + n(z-\gamma)}{\sqrt{(l^2 + m^2 + n^2)[(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2]}}$$

$$\frac{\sqrt{3}}{2} = \frac{3(x-2) + 4y + 6z}{\sqrt{9 + 16 + 36}\sqrt{(x-2)^2 + y^2 + z^2}}$$

$$183[(x-2)^2 + y^2 + z^2] = 4[3(x-2) + 4y + 6z]^2$$

$$147x^2 + 119y^2 + 39z^2 - 192yz - 144zx - 96xy -$$

$$-588x + 192y + 288z + 588 = 0$$

Vertex (0, 0, 0):

Example 5: Find the equation of the right circular cone which passes through the line $2x = 3y = -5z$ and has $x = y = z$ as its axis.

Solution: DC's of the generator $2x = 3y = -5z$ are $\frac{1}{2}, \frac{1}{3}, -\frac{1}{5}$. DC's of axis are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$. Point of intersection of the generator and axis is (0, 0, 0). Now

$$\cos\theta = \frac{\frac{1}{2}\cdot\frac{1}{\sqrt{3}} + \frac{1}{3}\cdot\frac{1}{\sqrt{3}} - \frac{1}{5}\cdot\frac{1}{\sqrt{3}}}{\sqrt{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}\sqrt{\frac{1}{4}+\frac{1}{9}+\frac{1}{25}}} = \frac{\frac{19}{30}}{\sqrt{\frac{361}{900}}}\cdot\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Equation of cone with vertex at origin

$$\frac{1}{\sqrt{3}} = \cos\theta = \frac{\frac{1}{\sqrt{3}}(x + y + z)}{1\sqrt{x^2 + y^2 + z^2}}$$

$$x^2 + y^2 + z^2 = (x + y + z)^2$$

$$xy + yz + zx = 0.$$

Example 6: Determine the equation of a right circular cone with vertex at origin and the guiding curve circle passing through the points (1, 2, 2), (1, −2, 2)(2, −1, −2).

Solution: Let l, m, n be the DC's of OL the axis of the cone. Let θ be the semi- vertical angle. Let $A(1, 2, 2)$, $B(1, -2, 2)$, $C(2, -1, -2)$ be the three points on the guiding circle. Then the lines OA, OB, OC make the same angle θ with the axis OL. The DC's of OA, OB, OC are proportional to

(1, 2, 2)(1, −2, 2)(2, −1, −2) respectively. Then

$$\cos\theta = \frac{l(1)+m(2)+n(2)}{\sqrt{1}\cdot\sqrt{1+4+4}} = \frac{l+2m+2n}{3} \quad (1)$$

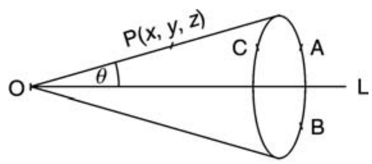


Fig. 3.19

Similarly,

$$\cos\theta = \frac{l(1)+m(-2)+n(2)}{\sqrt{1}\sqrt{1+4+4}} = \frac{l-2m+2n}{3} \quad (2)$$

$$\cos\theta = \frac{2l-m-2n}{3} \quad (3)$$

From (1) and (2), $4m = 0$ or $m = 0$.
From (2) and (3), $l + m - 4n = 0, l - 4n = 0$ or $l = 4n$.

DC's $\frac{l}{4} = \frac{m}{0} = \frac{n}{1}$ or $\frac{l}{\frac{4}{\sqrt{17}}} = \frac{m}{0} = \frac{n}{\frac{1}{\sqrt{17}}}$.

From (1) $\cos\theta = \dfrac{\frac{4}{\sqrt{17}} + 2\cdot 0 + 2\frac{1}{\sqrt{17}}}{3} = \dfrac{2}{\sqrt{17}}$.

Equation of right circular cone is

$$(l^2+m^2+n^2)(x^2+y^2+z^2)\cos^2\theta = (lx+my+nz)^2$$

$$\left(\frac{16}{17}+0+\frac{1}{17}\right)(x^2+y^2+z^2)\frac{4}{17} = \left(\frac{4}{\sqrt{17}}x+0+\frac{1}{\sqrt{17}}z\right)^2$$

$$4(x^2+y^2+z^2) = (4x+z)^2$$

$$12x^2 - 4y^2 - 3z^2 + 8xz = 0$$

is the required equation of the cone.

EXERCISE

1. Find the equation of the cone whose vertex is (3, 1, 2) and base circle is $2x^2 + 3y^2 = 1$, $z = 1$.

Ans. $2x^2 + 3y^2 + 20z^2 - 6yz - 12xz + 12x + 6y - 38z + 17 = 0$

2. Find the equation of the cone whose vertex is origin and guiding curve is $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{1} = 1, x + y + z = 1$.

Ans. $27x^2 + 32y^2 + 72(xy + yz + zx) = 0$.

3. Determine the equation of the cone with vertex at origin and guiding curve $x^2 + y^2 + z^2 - x - 1 = 0, x^2 + y^2 + z^2 + y - z = 0$.

   Hint: Guiding curve is circle in plane $x + y = 1$. Rewrite $x^2 + y^2 + z^2 - x(x + y) - (x + y)^2 = 0$.

Ans. $x^2 + 3xy - z^2 = 0$

4. Show that the equation of cone with vertex at origin and base circle $x = a, y^2 + z^2 = b^2$ is $a^2(y^2 + z^2) = b^2x^2$. Further prove that the section of the cone by a plane parallel to the XY-plane is a hyperbola.

Ans. $b^2x^2 - a^2y^2 = a^2c^2, z = c$ (put $z = c$ in equation of cone)

5. Find the equation of a cone with vertex at origin and guiding curve is the circle passing through the X, Y, Z intercepts of the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Ans. $a(b^2 + c^2)yz + b(c^2 + a^2)zx + c(a^2 + b^2)xy = 0$

6. Write the equation of the cone whose vertex is (1, 1, 0) and base is $y^2 + z^2 = 9, x = 0$.

   Hint: Substitute $\left(0, 1 - \frac{m}{l}, -\frac{n}{l}\right)$ in base curve and eliminate $\frac{m}{l} = \frac{y-1}{x-1}, \frac{n}{l} = \frac{z}{z-1}$.

Ans. $x^2 + y^2 + z^2 - 2xy = 0$

Right circular cone (R.C.C.)

7. Find the equation of R.C.C. with vertex at (2, 3, 1), axis parallel to the line $-x = \frac{y}{2} = z$ and one of its generators having DC's proportional to (1, −1, 1).

   Hint: $\cos\theta = \frac{-1-2+1}{\sqrt{6}\sqrt{3}}, l = -1, m = 2, n = 1, \alpha = 2, \beta = 3, \gamma = 1$.

Ans. $x^2 - 8y^2 + z^2 + 12xy - 12yz + 6zx - 46x + 36y + 22z - 19 = 0$

8. Determine the equation of R.C.C. with vertex at origin and passes through the point (1, 1, 2) and axis line $\frac{x}{2} = \frac{-y}{4} = \frac{z}{3}$.

Hint: $\cos\theta = \frac{2-4+6}{\sqrt{6}\sqrt{29}}$, DC's of generator: 1, 1, 2, axis: 2, −4, 3

Ans. $4x^2+40y^2+19z^2-48xy-72yz+36xz=0$

9. Find the equation of R.C.C. whose vertex is origin and whose axis is the line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ and which has semi- vertical angle of 30°

Hint: $\cos 30 = \frac{\sqrt{3}}{2} = \frac{x(1)+y(2)+z(3)}{\sqrt{(x^2+y^2+z^2)}\sqrt{1+4+9}}$

Ans. $19x^2+13y^2+3z^2-8xy-24yz-12zx=0$

10. Obtain the equation of R.C.C. generated when the straight line $2y+3z=6, x=0$ revolves about z-axis.

Hint: Vertex (0, 0, 2), generator $\frac{x}{0}=\frac{y}{3}=\frac{z-2}{-2}$, $\cos\theta=-\frac{2}{\sqrt{13}}$.

Ans. $4x^2+4y^2-9z^2+36z-36=0$

11. Lines are drawn from the origin with DC's proportional to (1, 2, 2), (2, 3, 6), (3, 4, 12). Find the equation of R.C.C.

Hint: $\cos\alpha=\frac{l+2m+2n}{3}=\frac{2l+3m+6n}{7}=\frac{3l+4m+12n}{13}$ $\frac{l}{-1}=\frac{m}{1}=\frac{n}{1}$, $\cos\alpha=\frac{1}{\sqrt{3}}$, DC's of axis: −1, 1, 1.

Ans. $xy-yz+zx=0$

12. Determine the equation of the R.C.C. generated by straight lines drawn from the origin to cut the circle through the three points (1, 2, 2), (2, 1, −2), and (2, −2, 1).

Hint: $\cos\alpha=\frac{l+2m+2n}{3}=\frac{2l+m-2n}{3}=\frac{2l-2m+n}{3}$ $\frac{l}{5}=\frac{m}{1}=\frac{n}{1}$, $\cos\alpha=\frac{5+2+2}{3\sqrt{27}}=\frac{1}{\sqrt{3}}$.

Ans. $8x^2-4y^2-4z^2+5xy+5zx+yz=0$

3.7 THE RIGHT CIRCULAR CYLINDER

A cylinder is the surface generated by a straight line (known as **generator**) which is parallel to a fixed straight line (known as **axis**) and satisfies a condition; for example, it may intersect a fixed curve (known as the **guiding curve**) or touch a given surface. A **right circular cylinder** is a cylinder whose surface is generated by revolving the generator at a fixed distance (known as the **radius**) from the axis; i.e., the guiding curve in this case is a circle. In fact, the intersection of the right circular cylinder with any plane perpendicular to axis of the cylinder is a circle.

Equation of a cylinder with generators parallel to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ and guiding curve conic $ax^2+by^2+2hxy+2gx+2fy+c=0, z=0$.

Let $P(x_1, y_1, z_1)$ be any point on the cylinder. The equation of the generator through $P(x_1, y_1, z_1)$ which is parallel to the given line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \tag{1}$$

is
$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \tag{2}$$

Since (2) meets the plane $z=0$,

$$\therefore \quad \frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{0-z_1}{n}$$

or
$$x=x_1-\frac{l}{n}z_1,\ y=y_1-\frac{m}{n}z_1 \tag{3}$$

Since this point (3) lies on the conic

$$ax^2+by^2+2hxy+2gx+2fy+c=0 \tag{4}$$

substitute (3) in (4). Then

$$a\left(x_1-\frac{l}{n}z_1\right)^2+b\left(y_1-\frac{m}{n}z_1\right)^2+$$
$$+2h\left(x_1-\frac{l}{n}z_1\right)\left(y_1-\frac{m}{n}z_1\right)+2g\left(x_1-\frac{l}{n}z_1\right)+$$
$$+2f\left(y_1-\frac{m}{n}z_1\right)+c=0.$$

The required equation of the cylinder is

$$a(nx-lz)^2+b(ny-mz)^2+2h(nx-lz)(ny-mz)+$$
$$+2ng(nx-lz)+2nf(ny-mz)+cn^2=0 \tag{5}$$

where the subscript 1 is droped because (x_1, y_1, z_1) is any general point on the cylinder.

Corollary 1: The equation of a cylinder with axis parallel to z-axis is obtained from (5) by putting $l=0, m=0, n=1$ which are the DC's of z-axis: i.e.,

$$ax^2+by^2+2hxy+2gx+2fy+c=0$$

which is **free** from z.

Thus the equation of a cylinder whose axis is paralle to x-axis (y-axis or z-axis) is obtained by eliminating the variable $x(y$ or $z)$ from the equation of the conic.

Equation of a right circular cylinder:

a. Standard form: with z-axis as axis and of radius a. Let $P(x, y, z)$ be any point on the cylinder. Then M the foot of the perpendicular PM has $(0, 0, z)$ and $PM = a$ (given). Then

$$a = PM = \sqrt{(x-0)^2 + (y-0)^2 + (z-z)^2}$$
$$x^2 + y^2 = a^2$$

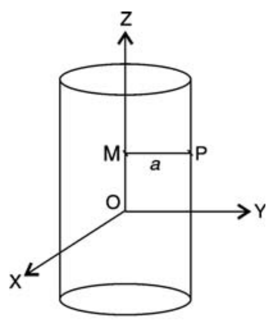


Fig. 3.20

Corollary 2: Similarly, equation of right circular cylinder with y-axis is $x^2 + z^2 = a^2$, with x-axis is $y^2 + z^2 = a^2$.

b. General form with the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ as axis and of radius a.

Axis AB passes through the point (α, β, γ) and has DR's l, m, n. Its DC's are $\frac{l}{k}, \frac{m}{k}, \frac{n}{k}$ where $k = \sqrt{l^2 + m^2 + n^2}$.

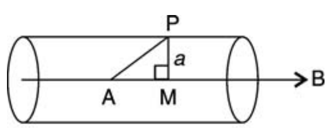


Fig. 3.21

From the right angled triangle APM

$$AP^2 = PM^2 + AM^2$$
$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2$$
$$= a^2 + \Big[l(x-\alpha) + m(y-\beta) + n(z-\gamma)\Big]^2$$

which is the required equation of the cylinder (Here AM is the projection of AP on the line AB is equal to $l(x-\alpha) + m(y-\beta) + n(z-\gamma)$).

Enveloping cylinder of a sphere is the locus of the tangent lines to the sphere which are parallel to a given line. Suppose

$$x^2 + y^2 + z^2 = a^2 \quad (1)$$

is the sphere and suppose that the generators are parallel to the given line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \quad (2)$$

Then for any point $P(x_1, y_1, z_1)$ on the cylinder, the equation of the generating line is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \quad (3)$$

Any general point on (3) is

$$(x_1 + lr, \quad y_1 + mr, \quad z_1 + nr) \quad (4)$$

By substituting (4) in (1), we get the points of intersection of the sphere (1) and the generating line (3) i.e.,

$$(x_1 + lr)^2 + (y_1 + mr)^2 + (z_1 + nr)^2 = a^2$$

Rewriting as a quadratic in r, we have

$$(l^2 + m^2 + n^2)r^2 + 2(lx_1 + my_1 + nz_1)r +$$
$$+ (x_1^2 + y_1^2 + z_1^2 - a^2) = 0 \quad (5)$$

If the roots of (5) are equal, then the generating line (3) meets (touches) the sphere in a single point i.e., when the discriminant of the quadratic in r is zero.

or $$4(lx_1 + my_1 + nz_1)^2 - 4(l^2 + m^2 + n^2) \times$$
$$\times (x_1^2 + y_1^2 + z_1^2 - a^2) = 0$$

Thus the required equation of the enveloping cylinder is

$$(lx + my + nz)^2 = (l^2 + m^2 + n^2)(x^2 + y^2 + z^2 - a^2)$$

where the subscript 1 is droped to indicate that (x, y, z) is a general point on the cylinder.

WORKED OUT EXAMPLES

Example 1: Find the equation of the quadratic cylinder whose generators intersect the curve $ax^2 +$

$by^2 + cz^2 = k, lx + my + nz = p$ and parallel to the y-axis. Deduce the case for $x^2 + y^2 + z^2 = 1$ and $x + y + z = 1$ and parallel to y-axis

Solution: Eliminate y between

$$ax^2 + by^2 + cz^2 = k \quad (1)$$

and

$$lx + my + nz = p \quad (2)$$

Solving (2) for y, we get

$$y = \frac{p - lx - nz}{m} \quad (3)$$

Substitute (3) in (1), we have

$$ax^2 + b\left(\frac{p - lx - nz}{m}\right)^2 + cz^2 = k.$$

The required equation of the cylinder is

$$(am^2 + l^2)x^2 + (bn^2 + m^2c)z^2 - 2pblx - 2npbz + 2blnxz + (bp^2 - m^2k) = 0.$$

Deduction: Put $a = 1, b = 1, c = 1, k = 1, l = m = n = p = 1$

$$2x^2 + 2z^2 - 2x - 2z + 2xz = 0$$

or

$$x^2 + z^2 + xz - x - z = 0.$$

Example 2: If l, m, n are the DC's of the generators and the circle $x^2 + y^2 = a^2$ in the XY-plane is the guiding curve, find the equation of the cylinder. Deduce the case when $a = 4, l = 1, m = 2, n = 3$.

Solution: For any point $P(x_1, y_1, z_1)$ on the cylinder, the equation of the generating line through P is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \quad (1)$$

Since the line (1) meets the guiding curve $x^2 + y^2 = a^2, z = 0$,

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{0 - z_1}{n}$$

or

$$x = x_1 - \frac{lz_1}{n}, \qquad y = y_1 - \frac{mz_1}{n} \quad (2)$$

This point (2) lies on the circle $x^2 + y^2 = a^2$ also. Substituting (2) in the equation of circle, we have

$$\left(x_1 - \frac{lz_1}{n}\right)^2 + \left(y_1 - \frac{mz_1}{n}\right)^2 = a^2$$

or

$$(nx - lz)^2 + (ny - mz)^2 = n^2a^2$$

is the equation of the cylinder.

Deduction: Equation of cylinder whose generators are parallel to the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and pass through the curve $x^2 + y^2 = 16, z = 0$. With $a = 4, l = 1, m = 2, n = 3$, the required equation of the cylinder is

$$(3x - z)^2 + (3y - 2z)^2 = 9(16) = 144$$

or

$$9x^2 + 9y^2 + 5z^2 - 6zx - 12yz - 144 = 0.$$

Example 3: Find the equation of the right circular cylinder of radius 3 and the line $\frac{x-1}{2} = \frac{y-3}{2} = \frac{z-5}{-1}$ as axis.

Solution: Let $A(1, 3, 5)$ be the point on the axis and DR's of AB are $2, 2, -1$ or DC's of AB are $\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}$. Radius $PM = 3$ given. Since AM is the projection of AP on AB, we have

$$AM = \frac{2}{3}(x - 1) + \frac{2}{3}(y - 3) - \frac{1}{3}(z - 5)$$

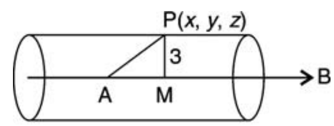


Fig. 3.22

From the right angled triangle APM

$$AP^2 = AM^2 + MP^2$$

$$(x - 1)^2 + (y - 3)^2 + (z - 5)^2 = \left[2\frac{(x-1)}{3} + 2\frac{y-3}{3} - 1\frac{(z-5)}{3}\right] + 9$$

$$9[x^2 + 1 - 2x + y^2 + 9 - 6y + z^2 + 25 - 10z] = [2x + 2y - z - 3]^2 + 81$$

$$9[x^2 + y^2 + z^2 - 2x - 6y - 10z + 35] = [4x^2 + 4y^2 + z^2 + 9 + 8xy - 4xz - 12x - 4yz - 12y + 6z] + 81$$

is the required equation of the cylinder.

Example 4: Find the equation of the enveloping cylinder of the sphere $x^2 + y^2 + z^2 - 2y - 4z - 11 = 0$ having its generators parallel to the line $x = -2y = 2z$.

Solution: Let $P(x_1, y_1, z_1)$ be any point on the cylinder. Then the equation of the generating line through P and parallel to the line $x = -2y = 2z$ or $\frac{x}{1} = \frac{y}{-\frac{1}{2}} = \frac{z}{\frac{1}{2}}$ is

$$\frac{x - x_1}{1} = \frac{y - y_1}{-\frac{1}{2}} = \frac{z - z_1}{\frac{1}{2}} \quad (1)$$

Any general point on (1) is

$$\left(x_1 + r, \quad y_1 - \frac{1}{2}r, \quad z_1 + \frac{1}{2}r\right) \quad (2)$$

The points of intersection of the line (1) and the sphere

$$x^2 + y^2 + z^2 - 2y - 4z - 11 = 0 \quad (3)$$

are obtained by substituting (2) in (3).

$$(x_1 + r)^2 + \left(y_1 - \frac{1}{2}r\right)^2 + \left(z_1 + \frac{1}{2}r^2\right)^2 - 2\left(y_1 - \frac{1}{2}r\right) - 4\left(z_1 + \frac{1}{2}r\right) - 11 = 0$$

Rewriting this as a quadratic in r

$$\frac{3}{2}r^2 + (2x_1 - y_1 + z_1 - 1)r + (x_1^2 + y_1^2 + z_1^2 - 2y_1 - 4z_1 - 11) = 0 \quad (4)$$

The generator touches the sphere (3 if (4) has equal roots i.e., discriminant is zero or

$$(2x_1 - y_1 + z_1 - 1)^2 = 4 \cdot \frac{3}{2} \cdot (x_1^2 + y_1^2 + z_1^2 - 2y_1 - 4z_1 - 11).$$

The required equation of the cylinder is

$$2x^2 + 5y^2 + 5z^2 + 4xy - 4xz + 2yz + 4x - 14y - 22z - 67 = 0.$$

EXERCISE

1. Find the equation of the quadratic cylinder whose generators intersect the curve
   a. $ax^2 + by^2 = 2z, lx + my + nz = p$ and are parallel to z-axis.
   b. $ax^2 + by^2 + cz^2 = 1, lx + my + nz = p$ and are parallel to x-axis.
   Hint: Eliminate z

Ans. **a.** $n(ax^2 + by^2) + 2lx + 2my - 2p = 0$

Hint: Eliminate x.

Ans. **b.** $(bl^2 + am^2)y^2 + (cl^2 + an^2)z^2 + 2amnyz -2ampy - 2anpz + (ap^2 - l^2) = 0$

2. If l, m, n are the DC's of the generating line and the circle $x^2 + z^2 = a^2$ in the zx-plane is the guiding curve, find the equation of the sphere.

Ans. $(mx - ly)^2 + (mz - ny)^2 = a^2m^2$

Find the equation of a right circular cylinder (4 to 9)

4. Whose axis is the line $\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z-2}{5}$ and radius is 2 units.

Ans. $26x^2 + 29y^2 + 5z^2 + 4xy + 10yz - 20zx + 150y + 30z + 75 = 0$

5. Having for its base the circle $x^2 + y^2 + z^2 = 9, x - y + z = 3$.

Ans. $x^2 + y^2 + z^2 + xy + yz - zx - 9 = 0$

6. Whose axis passes through the point (1, 2, 3) and has DC's proportional to (2, −3, 6) and of radius 2.

Ans. $45x^2 + 40y^2 + 13z^2 + 36yz - 24zx + 12xy - 42x - 280y - 126z + 294 = 0.$

7. Whose axis is the line $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{2}$ and radius 2 units.

Ans. $5x^2 + 8y^2 + 5z^2 - 4yz - 8zx - 4xy + 22x - 16y - 14z - 10 = 0$

8. The guiding curve is the circle through the three points (1, 0, 0), (0, 1, 0)(0, 0, 1).

Ans. $x^2 + y^2 + z^2 - xy - yz - zx = 1$

9. The directing curve is $x^2 + z^2 - 4x - 2z + 4 = 0$, $y = 0$ and whose axis contains the point (0, 3, 0). Also find the area of the section of the cylinder by a plane parallel to xz-plane.

   Hint: Centre of circle (2, 0, 1) radius: 1

Ans. $9x^2 + 5y^2 + 9z^2 + 12xy + 6yz - 36x - 30y - 18z + 36 = 0, \pi$

10. Find the equation of the enveloping cylinder of the sphere $x^2 + y^2 + z^2 - 2x + 4y = 1$, having its generators parallel to the line $x = y = z$.

Ans. $x^2 + y^2 + z^2 - xy - yz - zx - 2x + 7y + z - 2 = 0$.