MATHEMATICAL METHODS

III-YEAR

FACULTY OF EDUCATION

MATHEMATICS DEPT.

LAPLACE TRANSFORM

There are many transform techniques which are used in the analysis and design of engineering systems. Some of the transform techniques were introduced in the beginning by great individuals which were vigorously defined and developed by mathematicians in later days. The technique of Laplace transform is one such. In this chapter and subsequent chapters to follow, we introduce the idea of Laplace transform and develop some useful results. Subsequently, we display how the Laplace transform technique is used in solving a class of problems in differential equations .

The main use of Laplace transform is the following:

Consider any variable y depending upon a time parameter t for $t \geq 0$. Let y be governed by the differential equation

$$y^{(n)}(t)+a_1 y^{(n-1)}(t)+a_2 y^{(n-2)}+\ldots+a_n y=g(t) \quad \ldots(1)$$

with $y(0)=y_0, y'(0)=y_1, y''(0)=y_2, \ldots\ldots\ldots, y^{(n-1)}(0)=y_{n-1}$ ... (2)

Suppose the solution $y(t)$ is to be obtained subject to the initial conditions (2). Earlier we were trying to write the solution $y(t)$ in terms of a Complementary Function involving 'n' arbitrary constants added to a Particular Integral in the most general form. Later, we were determining the arbitrary constants using conditions in (2) to get the required Particular Solution.

Using Laplace Transform technique we can determine directly the particular solution without finding the most general solution of (1) and then evaluating from it the arbitrary constants. Further, using Laplace Transform technique, we simultaneously use (1) and (2) and convert the problem into an algebraic problem.

We get Laplace Transform of $y(t)$ as $L(y(t))=\overline{y}(s)$ and later to get y(t), we find Inverse Laplace Transform of $\overline{y}(s)$ *i.e.*, $L^{-1}\{\overline{y}(s)\}$ and get $y(t)=L^{-1}\{\overline{y}(s)\}$.

This technique is applicable in many cases. In practical problems, in many engineering applications, where we are concerned with solution of initial value problems, this is a useful technique.

The reader will be able to appreciate the uses when he/she encounters problems in their respective branch subjects.

INTEGRAL TRANSFORMS

An improper integral of the form $\int_{-\infty}^{\infty} k(s,t) f(t)\, dt$ is called integral transform of $f(t)$ if it i convergent. It is denoted by $\bar{f}(s)$.

Thus $\bar{f}(s) = \int_{-\infty}^{\infty} k(s,t) f(t)\, dt$ ....(1)

where $k(s,t)$ is a known function of s and t, called the **kernal** of the transform.

The function $f(t)$ is called the **Inverse transform** of $\bar{f}(s)$.

Some of the well known transforms are given below.

(*i*) **Laplace Transform :**

If we take $k(s,t) = \begin{cases} e^{-st}, t \geq 0 \\ 0, \quad t < 0 \end{cases}$ then (1) becomes

$$\bar{f}(s) = \int_{0}^{\infty} f(t)\, e^{-st} dt$$

This transform is known as Laplace Transform.

(*ii*) **Fourier Transform :**

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-ist} dt$$

(*iii*) **Fourier Sine and Cosine Transforms :**

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin sx\, dx$$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos sx\, dx$$

(*iv*) **Mellin Transform :**

$$\bar{f}(s) = \int_{0}^{\infty} f(t)\, t^{s-1} dt$$

1.3 DEFINITION

Let $f(t)$ be a given function and defined for all positive values of t. Then the Laplace Transform of $f(t)$, denoted by $L\{f(t)\}$ or $\bar{f}(s)$ is defined by

$$\boxed{L\{f(t)\} = \bar{f}(s) = \int_{0}^{\infty} e^{-st} f(t)\, dt} \quad ...(1)$$

provided the integral exists. Here the parameter s is a real or complex number.

The operation of multiplying $f(t)$ by e^{-st} and integrating from 0 to ∞ is called Laplace transformation.

The relation (1) can also be written as $f(t) = L^{-1}\{\bar{f}(s)\}$.

In such a case, the function $f(t)$ is said to be *Inverse Laplace Transform* of $\bar{f}(s)$. The symbol L which transform $f(t)$ into $\bar{f}(s)$ can be called the Laplace Transform operator. The symbol L^{-1} which transforms $\bar{f}(s)$ to $f(t)$ can be called the Inverse Laplace Transform operator.

Note : $\int_0^{\infty} e^{-st} f(t)\, dt$ exists if $\int_0^{K} e^{-st} f(t)\, dt$ can actually be evaluated and its limit as $K \to \infty$ exists. Otherwise, if $f(t)$ is continuous and $\underset{t \to \infty}{\text{Lt}} [e^{-at} f(t)]$ is finite, then $L\{f(t)\}$ exists for $s > a$.

Important Observation : The Laplace Transform of a function $f(t)$ is the function $\bar{f}(s)$, which is a **Unilateral Transform**. When one says "the Laplace Transform" without qualification, the unilateral or one-sided transform is normally intended. The Laplace Transform can be alternatively defined as the **Bilateral Laplace Transform** or **Two-sided Transform** by extending the limits of integration to be the entire real axis.

The Bilateral Laplace Transform is defined as follows :

$$B\{f(s)\} = \int_{-\infty}^{\infty} e^{-st} f(t)\, dt$$

1.3.1 Sufficient Conditions For the Existence of the Laplace Transform of a Function

While finding the Laplace transforms of elementary functions, it can be noticed that the integral exists under certain conditions, such as $s > 0$ or $s > a$ etc. In general, the function $f(t)$ must satisfy the following conditions for the existence of the Laplace transform.

(*i*) The function $f(t)$ must be piece-wise continuous or sectionally continuous in any limited interval $0 < a \le t \le b$.

(*ii*) The function $f(t)$ is of exponential order.

1.4 DEFINITIONS

1. **Piece-wise Continuous Function :**

A function $f(t)$ is said to be piece-wise (or sectionally) continuous over the closed interval $[a, b]$ if it is defined on that interval and is such that the interval can be divided into a finite number of subintervals, in each of which $f(t)$ is continuous and has both right and left hand limits at every end point of the subinterval.

e.g. : (*i*) The function $f(t) = \begin{cases} t^2, 0 < t < 5 \\ 2t + 3, t > 5 \end{cases}$

is sectionally continuous for $t > 0$.

(*ii*) The function $f(t) = \dfrac{1}{t}$ is not sectionally continuous in any interval containing $t = 0$.

2. **Function of Exponential Order :**

A function $f(t)$ is said to be of exponential order a if

$$\lim_{t \to \infty} e^{-at} f(t) = \text{a finite quantity}$$

i.e. for a given positive number T, there exists a real number $M > 0$ such that

$$|e^{-at} f(t)| < M, \forall\ t \geq T$$

or $$|f(t)| < Me^{at}, \forall\ t \geq T$$

For example, $f(t) = t^2$, $\sin at$, e^{at} etc. are all of exponential order and also continuous. But $f(t) = e^{t^2}$ is not of exponential order and as such its Laplace transform does not exist.

General Properties of Laplace Transform :

A very important property is that the Laplace transformation is a linear operator, just as differentiation and integration.

1.5 LINEARITY PROPERTY

Theorem : If $L[f(t)] = \bar{f}(s)$ and $L[g(t)] = \bar{g}(s)$ then

$L[c_1 f(t) + c_2 g(t)] = c_1 L[f(t)] + c_2 L[g(t)] = c_1 \bar{f}(s) + c_2 \bar{g}(s)$, where c_1 and c_2 are constants.

Proof : By definition,

$$L[c_1 f(t) + c_2 g(t)\} = \int_0^\infty e^{-st}[c_1 f(t) + c_2 g(t)]dt = \int_0^\infty e^{-st} c_1 f(t)dt + \int_0^\infty e^{-st} c_2 g(t)dt$$

$$= c_1 \int_0^\infty e^{-st} f(t)\ dt + c_2 \int_0^\infty e^{-st} g(t)dt$$

$$= c_1 L[f(t)] + c_2 L[g(t)] = c_1 \bar{f}(s) + c_2 \bar{g}(s)$$

The above result can easily be generalized to more than two functions.

Hence the Laplace transform of the sum of two or more functions of t is the sum of the Laplace transforms of the separate functions.

1.6 LAPLACE TRANSFORM OF SOME ELEMENTARY FUNCTIONS

Elementary functions include Algebraic and transcendental functions. From the definition and by ordinary integration, we obtain the following results.

1. $L\{k\} = \dfrac{k}{s} (s > 0)$, where k is a constant.

Proof: By definition,

$$L\{k\} = \int_0^\infty e^{-st} \cdot k\ dt = k \int_0^\infty e^{-st} \cdot dt = k \left(\frac{e^{-st}}{-s} \right)_0^\infty = -\frac{k}{s}(e^{-\infty} - 1)$$

$$= -\frac{k}{s}(0-1)\ [\because e^{-\infty} = 0]$$

$$= \frac{k}{s}, \text{ if } s > 0$$

Note : The above Laplace transform does not exist for $s \leq 0$. It follows :

(*i*) For $k = 0$, $L\{0\} = 0$ (*ii*) For $k = 1$, $L\{1\} = \dfrac{1}{s}$, $s > 0$.

2. $L\{t\} = \dfrac{1}{s^2}$

Proof : $L\{t\} = \int_0^\infty e^{-st}.t\,dt = \left[t\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{(-s)^2}\right)\right]_0^\infty = \frac{1}{s^2}$, if $s > 0$

3. $L\{t^n\} = \frac{n!}{s^{n+1}}$ where n is a positive integer.

Proof : $L\{t^n\} = \int_0^\infty e^{-st}.t^n dt = \int_0^\infty t^n d\left(\frac{e^{-st}}{-s}\right)$

$$= \left[t^n\left(\frac{e^{-st}}{-s}\right)\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} \cdot nt^{n-1}dt, \text{ by parts}$$

$$= \frac{n}{s}\int_0^\infty e^{-st}t^{n-1}dt \quad [\because \text{ first portion} = 0 \text{ at both limits}]$$

$$= \frac{n}{s} L\{t^{n-1}\}$$

Similarly, $L\{t^{n-1}\} = \frac{n-1}{s} L\{t^{n-2}\}$

$$L\{t^{n-2}\} = \frac{n-2}{s} L\{t^{n-3}\}$$

By repeatedly applying this, we get

$$L\{t^n\} = \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \ldots \frac{2}{s} \cdot \frac{1}{s} L\{t^{n-n}\}$$

$$= \frac{n!}{s^n} L\{1\} = \frac{n!}{s^n} \cdot \frac{1}{s} = \frac{n!}{s^{n+1}}$$

Definition: *Gamma Function.* If $n > 0$ then the Gamma function denoted by $\Gamma(n)$ and is defined by $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx$, with $n > 0$

The following are some important properties of the Gamma function :

(i) $\Gamma(n+1) = n.\Gamma(n)$, if $n > 0$

(ii) $\Gamma(n+1) = n!$, if n is a positive integer

(iii) $\Gamma(1) = 1$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$

(iv) $\Gamma(0), \Gamma(-1), \Gamma(-2), \Gamma(-3)$, ... are all not defined.

Note: $L\{t^n\}$, where 'n' is non-negative real number *i.e*, $n > 0$ then it can be expressed in terms of Gamma function.

4. $L\{t^a\} = \frac{\Gamma(a+1)}{s^{a+1}}$, where $s > 0$ and 'a' is real number > -1.

Proof : We have $L\{t^a\} = \int_0^\infty e^{-st} t^a \, dt$

Putting $x = st$, we get

$$L\{t^n\} = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^a \frac{dx}{s} = \frac{1}{s^{a+1}} \int_0^\infty e^{-x} x^a \, ds$$

$$= \frac{1}{s^{a+1}} \int_0^\infty e^{-x} \, x^{(a+1)-1} dx = \frac{\Gamma(a+1)}{s^{a+1}}, \text{ when } a > -1$$

Cor 1 : If $a = n$ is a positive integer, $\Gamma(n+1) = n!$ in particular.

Hence $L\{t^n\} = \dfrac{\Gamma(n+1)}{s^{n+1}} = \dfrac{n!}{s^{n+1}}$ where n is a positive integer.

Cor 2 : When $n = \dfrac{1}{2}$, we have

$$L\{\sqrt{t}\} = L\{t^{1/2}\} = \frac{1}{s^{\frac{1}{2}+1}} \Gamma\left(\frac{1}{2}+1\right)$$

$$= \frac{1}{s^{3/2}} \Gamma\left(\frac{3}{2}\right) = \frac{1}{s^{3/2}} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \quad [\because \Gamma(n+1) = n \, . \, \Gamma(n), \text{ if } n > 0]$$

$$= \frac{1}{s^{3/2}} \Gamma\left(\frac{3}{2}\right) = \frac{1}{s^{3/2}} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \quad [\because \Gamma(n+1) = n \, . \, \Gamma(n), \text{ if } n > 0]$$

$$= \frac{1}{2s^{3/2}} \cdot \sqrt{\pi} \quad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$\therefore L\{\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}}$$

Cor 3 : Similarly, $L\left\{\dfrac{1}{\sqrt{t}}\right\} = \sqrt{\dfrac{\pi}{s}}$

5. $L\{e^{at}\} = \dfrac{1}{s-a}$, if $s > a$

6. $L\{\sinh at\} = \dfrac{a}{s^2 - a^2}$, if $s > |a|$

Proof: Using the linearity property of the Laplace transform, we have

$$L(\sinh at\} = L\left\{\frac{e^{at} - e^{-at}}{2}\right\}$$

$$= \frac{1}{2}\left[L\{e^{at}\} - L\{e^{-at}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{a}{s^2 - a^2}$$

7. Similarly, $L\{\cosh at\} = \dfrac{s}{s^2 - a^2}$, if $s > |a|$

Here for the existence of Laplace Transform we require $s > a$ or more precisely $Re(s) > a$.

8. $L\{\sin at\} = \dfrac{a}{s^2 + a^2}$, if $s > 0$

Proof: $L\{\sin at\} = \int_0^\infty e^{-st} \sin at \, dt$. Apply $\int e^{ax} \sin bx \, dx = \dfrac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx)$

$$= \left[\frac{e^{-st}}{s^2 + a^2}(-s \sin at - a \cos at)\right]_0^\infty$$

$$= \frac{a}{s^2 + a^2}$$

9. Similarly, $L\{\cos at\} = \dfrac{s}{s^2 + a^2}$, if $s > 0$

Alternate Method to find L{sin *at*} and L{cos *at*} :

We know that $L\{e^{at}\} = \dfrac{1}{s-a}$

Replacing a by ia, we get

$$L\{e^{iat}\} = \frac{1}{s - ia} = \frac{s + ia}{(s - ia)(s + ia)}$$

i.e., $L\{\cos at + i \sin at\} = \dfrac{s + ia}{s^2 + a^2}$

$$= \frac{s}{s^2 + a^2} + i \frac{a}{s^2 + a^2}$$

Some Elementary Functions $f(t)$ and their Laplace Transforms $L\{f(t)\}$

S.No.	$f(t)$	$\bar{f}(s)$
1.	1	$\frac{1}{s}$
2.	t	$\frac{1}{s^2}$
3.	t^2	$\frac{2!}{s^3}$
4.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$
5.	$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$
6.	$t^n \, (n = 0, 1, 2, 3, \ldots)$	$\frac{n!}{s^{n+1}}$
7.	$t^a \, (a > 0)$	$\frac{\Gamma(a+1)}{s^{a+1}}$
8.	e^{at}	$\frac{1}{s-a}$
9.	e^{-at}	$\frac{1}{s+a}$
10.	$\sin at$	$\frac{a}{s^2+a^2}$
11.	$\cos at$	$\frac{s}{s^2+a^2}$
12.	$\sinh at$	$\frac{a}{s^2-a^2}$
13.	$\cosh at$	$\frac{s}{s^2-a^2}$

SOLVED EXAMPLES

Example 1 : Find $L(5\sin t + 2\sin 3t)$

Solution : By Linearity property of Laplace Transform,

$L(5\sin t + 2\sin 3t) = 5.\ L(\sin t) + 2.\ L(\sin 3t)$

$$= 5.\frac{1}{s^2+1} + 2.\frac{3}{s^2+3^2} \quad \left[\because L(\sin at) = \frac{a}{s^2+a^2}\right]$$

$$= \frac{5}{s^2+1} + \frac{6}{s^2+9}$$

Example 2 : Find the Laplace transform of $e^{3t} - 2e^{-2t} + \sin 2t + \cos 3t + \sinh 3t - 2\cosh 4t + 9$

Solution : Let $f(t) = e^{3t} - 2e^{-2t} + \sin 2t + \cos 3t + \sinh 3t - 2\cosh 4t + 9$

Then $L\{f(t)\} = L\{e^{3t} - 2e^{-2t} + \sin 2t + \cos 3t + \sinh 3t - 2\cosh 4t + 9\}$

By Linearity property,

$$L\{f(t)\} = L\{e^{3t}\} - 2.L\{e^{-2t}\} + L\{\sin 2t\} + L\{\cos 3t\} + L\{\sinh 3t\} - 2L\{\cosh 4t\} + 9.L\{1\}$$

$$= \frac{1}{s-3} - 2\left(\frac{1}{s+2}\right) + \frac{2}{s^2+2^2} + \frac{s}{s^2+3^2} + \frac{3}{s^2-3^2} - 2\left(\frac{s}{s^2-4^2}\right) + 9\left(\frac{1}{s}\right)$$

$$= \frac{1}{s-3} - \frac{2}{s+2} + \frac{2}{s^2+4} + \frac{s}{s^2+9} + \frac{3}{s^2-9} - \frac{2s}{s^2-16} + \frac{9}{s}$$

$$= \bar{f}(s), \text{ say}$$

Example 3 : Find the Laplace transforms of:

(*i*) $\dfrac{e^{-at}-1}{a}$ (*ii*) $e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t$

(*iii*) $(t^2+1)^2$ (*iv*) $(\sin t + \cos t)^2$ (*v*) $\cos^3 2t$

(*vi*) $\cosh^2 2t$ (*vii*) $\sinh^3 2t$ (*viii*) $\cosh^3 2t$

(*ix*) $\left(\sqrt{t} + \dfrac{1}{\sqrt{t}}\right)^3$

Solution :

(*i*) By Linearity property,

$$L\left\{\frac{e^{-at}-1}{a}\right\} = \frac{1}{a}L\{e^{-at}-1\} = \frac{1}{a}\left[L\{e^{-at}\} - L\{1\}\right]$$

$$= \frac{1}{a}\left[\frac{1}{s+a} - \frac{1}{s}\right] = \frac{-1}{s(s+a)}$$

(*ii*) By Linearity property,

$$L\{e^{2t}+4t^3-2\sin 3t+3\cos 3t\} = L(e^{2t})+4L(t^3)-2L(\sin 3t)+3L(\cos 3t)$$

$$= \frac{1}{s-2}+4.\frac{3!}{s^4}-2.\frac{3}{s^2+9}+3.\frac{s}{s^2+9}$$

$$= \frac{1}{s-2}+\frac{24}{s^4}-\frac{6}{s^2+9}+\frac{3s}{s^2+9}$$

$$= \frac{1}{s-2}+\frac{24}{s^4}+\frac{3(s-2)}{s^2+9}$$

(*iii*) Let $f(t)=(t^2+1)^2=t^4+2t^2+1$

$\therefore L\{(t^2+1)^2\}=L\{t^4+2t^2+1\}=L\{t^4\}+2L\{t^2\}+L\{1\}$, **using Linearity property**

$$=\frac{4!}{s^5}+2.\frac{2!}{s^3}+\frac{1}{s} \quad \left[L\{t^n\}=\frac{n!}{s^{n+1}}\right]$$

$$=\frac{24}{s^5}+\frac{4}{s^3}+\frac{1}{s}=\frac{1}{s^5}(s^4+4s^2+24)$$

(*iv*) Since $(\sin t+\cos t)^2=\sin^2 t+\cos^2 t+2\sin t\cos t = 1+\sin 2t$

$$\therefore L\{(\sin t+\cos t)^2\}=L\{1+\sin 2t\}=\frac{1}{s}+\frac{2}{s^2+4}=\frac{s^2+2s+4}{s(s^2+4)}$$

(*v*) Since $\cos 6t=\cos 3(2t)=4\cos^3 2t-3\cos 2t$ $\quad [\because \cos 3A=4\cos^3 A-3\cos A]$

$$\therefore \cos^3 2t=\frac{1}{4}(3\cos 2t+\cos 6t)$$

Hence $L\{\cos^3 2t\}=L\left\{\frac{1}{4}(3\cos 2t+\cos 6t)\right\}=\frac{3}{4}L\{\cos 2t\}+\frac{1}{4}L\{\cos 6t\}$

$$=\frac{3}{4}.\frac{s}{s^2+4}+\frac{1}{4}.\frac{s}{s^2+36}=\frac{s}{4}\left(\frac{3}{s^2+4}+\frac{1}{s^2+36}\right)$$

$$=\frac{s}{4}\left[\frac{4s^2+112}{(s^2+4)(s^2+36)}\right]=\frac{s(s^2+28)}{(s^2+4)(s^2+36)}$$

(*vi*) Since $\cosh^2 2t=\frac{1}{2}(1+\cosh 4t)$ $\quad [\because \cosh 2A=2\cosh^2 A-1]$

$$\therefore L\{\cosh^2 2t\}=\frac{1}{2}\left[L\{1\}+L\{\cosh 4t\}\right]$$

$$=\frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2-16}\right]=\frac{s^2-8}{s(s^2-16)}$$

(*vii*) We know that $\sinh x=\frac{e^x-e^{-x}}{2}$

$$\therefore \sinh 2t=\frac{e^{2t}-e^{-2t}}{2}$$

Hence $L\{\sinh^3 2t\} = L\left\{\left(\frac{e^{2t}-e^{-2t}}{2}\right)^3\right\}$

$$= L\left\{\frac{1}{8}(e^{6t}-3e^{2t}+3e^{-2t}-e^{-6t})\right\}, \textbf{ using } (a-b)^3 \textbf{ formula}$$

$$= \frac{1}{8}\cdot\frac{1}{s-6} - \frac{3}{8}\cdot\frac{1}{s-2} + \frac{3}{8}\cdot\frac{1}{s+2} - \frac{1}{8}\cdot\frac{1}{s+6}$$

$$= \frac{1}{8}\left(\frac{1}{s-6}-\frac{1}{s+6}\right) - \frac{3}{8}\left(\frac{1}{s-2}-\frac{1}{s+2}\right)$$

$$= \frac{1}{8}\left(\frac{12}{s^2-36}-\frac{12}{s^2-4}\right) = \frac{48}{(s^2-4)(s^2-36)}$$

(*ix*) $\mathrm{L}\left\{\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)^3\right\} = \mathrm{L}\left\{(\sqrt{t})^3 + \left(\frac{1}{\sqrt{t}}\right)^3 + 3(\sqrt{t})^2\cdot\frac{1}{\sqrt{t}} + 3\sqrt{t}\left(\frac{1}{\sqrt{t}}\right)^2\right\}$, **using $(a+b)^3$ formula**

$$= \mathrm{L}\left\{t^{3/2}+t^{-3/2}+3t^{1/2}+3t^{-1/2}\right\}$$

$$= \mathrm{L}\left\{t^{3/2}\right\} + \mathrm{L}\left\{t^{-3/2}\right\} + 3\cdot\mathrm{L}\left\{\sqrt{t}\right\} + 3\cdot\mathrm{L}\left\{\frac{1}{\sqrt{t}}\right\}$$

$$= \frac{1}{s^{5/2}}\Gamma\left(\frac{5}{2}\right) + \frac{1}{s^{-1/2}}\Gamma\left(-\frac{1}{2}\right) + 3\cdot\frac{\sqrt{\pi}}{2s^{3/2}} + 3\cdot\sqrt{\frac{\pi}{s}}$$

$$= \frac{1}{s^{5/2}}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right) + \sqrt{s}(-2\sqrt{\pi}) + \frac{3\sqrt{\pi}}{2s^{3/2}} + 3\cdot\sqrt{\frac{\pi}{s}} \quad \left[\because \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}\right]$$

$$= \frac{3\sqrt{\pi}}{4s^{5/2}} - 2\sqrt{\pi s} + \frac{3\sqrt{\pi}}{2s^{3/2}} + 3\cdot\sqrt{\frac{\pi}{s}}$$

$$= \frac{\sqrt{\pi}}{4}\left[\frac{3}{s^{5/2}} + \frac{6}{s^{3/2}} + \frac{12}{\sqrt{s}} - 8\sqrt{s}\right].$$

EXERCISE 1.1

Find the Laplace Transform of the following functions:

(1) t^2+at+b

(2) $t^3+5\cos t$

(3) (i) $2e^{3t}-e^{-3t}$ (ii) 2^t

(4) $\sin t\cos t$

(5) (i) $\sin^2 t$ (ii) $\sin^2 at$

(6) $\cos^2 t$

(7) $\left\{\frac{1}{\sqrt{\pi t}}\right\}$

(8) $3\cosh 5t-4\sinh 5t$

(9) (i) $\cos h^2(2t)$ (ii) $\sinh^2(2t)$

(10) $\sinh at-\sin at$

(11) $\sin(at+b)$

(12) $\cos(wt+\theta)$

(13) (i) $\cos^3 t$ (ii) $\cos^3 3t$

(14) $\sinh^3 2t$

(15) $\sin^3 2t$

(16) (i) $\sin 2t\sin 3t$ (ii) $\cos 4t\sin 2t$

(17) (i) $\cos 5t\cos 2t$ (ii) $3\cos 3t\cos 4t$

(18) $\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)^3$

(19) $f(t)=\begin{cases}2t, 0<t<5\\ 1, t>5\end{cases}$

(20) $f(t)=\begin{cases}0, 0<t<1\\ 1, 1<t<2\\ 2, t>2\end{cases}$

(21) $f(t)=\begin{cases}e^t, 0<t<1\\ 0, t>1\end{cases}$

(22) $f(t)=\begin{cases}\cos t \text{ when } 0<1<2\pi\\ 0, \text{ when } t>2\pi\end{cases}$

ANSWERS

(1) $\frac{2}{s^3}+\frac{a}{s^2}+\frac{b}{s}$

(2) $\frac{6}{s^4}+\frac{5s}{s^2+1}$

(3) (i) $\frac{s+9}{s^2-9}$ (ii) $\frac{1}{s-\log 2}$

(4) $\frac{1}{s^2+4}$

(5) (i) $\frac{2}{s(s^2+4)}$ (ii) $\frac{2a^2}{s(s^2+4a^2)}$

(6) $\frac{s^2+2}{s(s^2+4)}$

(7) $\frac{1}{\sqrt{s}}$

(8) $\frac{3s-20}{s^2-25}$

(9) (i) $\frac{s^2-8}{s(s^2-16)}$ (ii) $\frac{8}{s(s^2-16)}$

(10) $\frac{2a^3}{s^4-a^4}$

(11) $\frac{a\cos b+s\sin b}{s^2+a^2}$

(12) $\frac{s\cos\theta-w\sin\theta}{s^2+w^2}$

(13) (i) $\frac{s^3+7s}{(s^2+1)(s^2+9)}$ (ii) $\frac{s(s^2+63)}{(s^2+9)(s^2+81)}$

(14) $\frac{48}{(s^2-4)(s^2-36)}$

(15) $\frac{48}{(s^2+4)(s^2+36)}$

(16) (i) $\frac{12s}{(s^2+1)(s^2+25)}$ (ii) $\frac{2(s^2-12)}{(s^2+36)(s^2+4)}$

(17) (i) $\frac{s(s^2+29)}{(s^2+9)(s^2+49)}$ (ii) $\frac{3s(s^2+25)}{(s^2+1)(s^2+49)}$

(18) $\frac{\sqrt{\pi}}{4}\left[\frac{3}{s^{5/2}}-\frac{6}{s^{3/2}}+\frac{12}{\sqrt{s}}+8\sqrt{s}\right]$

(19) $\frac{2}{s^2}(1-e^{-5s})-\frac{9}{s}e^{-5s}$

(20) $\frac{1}{s}(e^{-s}+e^{-2s})$

(21) $\frac{1}{1-s}(e^{1-s}-1)$ (or) $\frac{1}{s-1}[1-e^{-(s-1)}]$

(22) $\frac{s(1-e^{-2\pi s})}{1+s^2}$

1.7 FIRST TRANSLATION (OR) FIRST SHIFTING THEOREM

Theorem : If $L\{f(t)\} = \bar{f}(s)$, then $L\{e^{at} f(t)\} = \bar{f}(s-a), s-a>0$

Proof: By definition,

$$L\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\,dt = \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$= \int_0^\infty e^{-ut} f(t)\,dt, \text{ where } u = s-a$$

$$= \bar{f}(u) = \bar{f}(s-a)$$

Hence $L\{e^{at} f(t)\} = [L\{f(t)\}]_{\text{change } s \text{ to } s-a}$

Corollary 1: Using the above theorem, we have $L\{e^{-at} f(t)\} = \bar{f}(s+a), (s+a)>0$

and $L\{e^{-at} f(t)\} = [L\{f(t)\}]_{\text{change } s \text{ to } s+a}$

Corollary 2: As an application of this theorem, we obtain the following results :

(*i*) $L\{e^{at} \cdot t^n\} = [L(t^n)]_{s\to s-a} = \left[\frac{n!}{s^{n+1}}\right]_{\text{change } s \text{ to } s-a} = \frac{n!}{(s-a)^{n+1}}$

(*ii*) $L\{e^{at} \sin bt\} = [L(\sin bt)]_{s\to s-a} = \left(\frac{b}{s^2+b^2}\right)_{s\to s-a}$

(*iii*) $L\{e^{at} \cos bt\} = \frac{s-a}{(s-a)^2+b^2}$

(*iv*) $L\{e^{at} \sinh bt\} = [L(\sinh bt)]_{s\to(s-a)} = \left(\frac{b}{s^2-b^2}\right)_{s\to(s-a)}$

$$= \frac{b}{(s-a)^2-b^2}$$

(*v*) $L\{e^{at} \cosh bt\} = \frac{s-a}{(s-a)^2-b^2}$

(*vi*) $L\left\{e^{-at} t^n\right\} = \frac{n!}{(s+a)^{n+1}}$

(*vii*) $L\left\{e^{-at} \sin bt\right\} = \frac{b}{(s+a)^2+b^2}$

SOLVED EXAMPLES

Example 1 : Find the Laplace transform of

(*i*) $e^{-t}\cos 2t$

(*ii*) $e^{-3t}(2\cos 5t - 3\sin 5t)$

[JNTU 2004, 2007S, (A) June 2010, (H) Dec. 2011S, (K) June 2012 (Set No. 1)]

(*iii*) $e^{-t}(3\sin 2t - 5\cosh 2t)$

(*iv*) $e^{-at}\sinh bt$

(*v*) $e^{-at}\cosh bt$ (*vi*) $t\,e^{2t}\sin 3t$

Solution : **(*i*)** We have

$$L\{f(t)\} = L\{\cos 2t\} = \frac{s}{s^2+4} = \bar{f}(s), \text{ say}$$

Now applying First Shifting theorem, we get

$$L\{e^{-t}\cos 2t\} = L\{e^{-t}f(t)\} = \bar{f}(s+1) = \left[L\{\cos 2t\}\right]_{\text{change } s \text{ to } s+1}$$

$$= \left[\frac{s}{s^2+4}\right]_{\text{change } s \text{ to } s+1} = \frac{s+1}{(s+1)^2+4} = \frac{s+1}{s^2+2s+5}$$

(*ii*) We have

$$L\{f(t)\} = L(2\cos 5t - 3\sin 5t) = \frac{2s}{s^2+25} - \frac{3(5)}{s^2+25} = \frac{2s-15}{s^2+25} = \bar{f}(s), \text{ say}$$

Now applying First Shifting theorem, we have

$$L\{e^{-3t}(2\cos 5t - 3\sin 5t)\} = L\{e^{-3t}f(t)\} = \bar{f}(s+3) = [\bar{f}(s)]_{s\to(s+3)}$$

$$= \left(\frac{2s-15}{s^2+25}\right)_{\text{change } s \text{ to } s+3} = \frac{2(s+3)-15}{(s+3)^2+25} = \frac{2s-9}{s^2+6s+34}$$

(*iii*) We have

$$L(3\sin 2t - 5\cosh 2t) = \frac{3(2)}{s^2+4} - \frac{5s}{s^2-4} = \frac{6}{s^2+4} - \frac{5s}{s^2-4} = \bar{f}(s)$$

Using First Shifting theorem,

$$L[e^{-t}(3\sin 2t - 5\cosh 2t)] = \bar{f}(s+1) = \left[\bar{f}(s)\right]_{s\to s+1} = \left(\frac{6}{s^2+4} - \frac{5s}{s^2-4}\right)_{\text{change } s \text{ to } s+1}$$

$$= \frac{6}{(s+1)^2+4} - \frac{5(s+1)}{(s+1)^2-4} = \frac{6}{s^2+2s+5} - \frac{5(s+1)}{s^2+2s-3}$$

(*iv*) We know that $L(\sinh bt) = \frac{b}{s^2-b^2}, s > |b|$

Now applying First Shifting theorem, we have

$$L\left\{e^{-at}\sinh bt\right\} = \left(\frac{b}{s^2-b^2}\right)_{\text{change } s \text{ to } s+a} = \frac{b}{(s+a)^2-b^2}$$

(*v*) We know that $L(\cosh bt) = \frac{s}{s^2-b^2}, s > |b|$

Now applying First Shifting theorem, we have

$$L\left\{e^{-at}\cosh bt\right\} = \left(\frac{s}{s^2-b^2}\right)_{\text{change } s \text{ to } s+a} = \frac{s+a}{(s+a)^2-b^2}$$

(*vi*) Since $L\{t\} = \frac{1}{s^2}$, we have

$$L\{te^{i3t}\} = \frac{1}{(s-3i)^2}, \text{ using First Shifting Theorem}$$

$$= \frac{(s+3i)^2}{(s-3i)^2(s+3i)^2}$$

or $$L\{t\cos 3t + i\, t\sin 3t\} = \frac{(s^2-9)+i6s}{(s^2+9)^2} \quad \left[\because e^{i\theta} = \cos\theta + i\sin\theta\right]$$

Equating imaginary parts on both sides, we have $L\{t\sin 3t\} = \frac{6s}{(s^2+9)^2}$

Now applying First Shifting theorem, we have

$$L\{e^{2t} t\sin 3t\} = [L\{t\sin 3t\}]_{\text{change } s \text{ to } s-2}$$

$$= \left[\frac{6s}{(s^2+9)^2}\right]_{\text{change } s \text{ to } s-2} = \frac{6(s-2)}{[(s-2)^2+9]^2} = \frac{6(s-2)}{(s^2-4s+13)^2}$$

Example 2 : Find (*i*) $L\{(t+3)^2 e^t\}$ (*ii*) $L\{e^{-t}\cos^2 t\}$

(*iii*) $L\{e^{3t}\sin^2 t\}$

Solution : (*i*) We have $L\{f(t)\} = L\{(t+3)^2\} = L\{t^2+6t+9\} = \frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s} = \bar{f}(s)$

By First Shifting theorem,

$$L\{e^t (t+3)^2\} = L\{e^t f(t)\} = \bar{f}(s-1) = [\bar{f}(s)]_{s\to s-1} = \left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]_{\text{change } s \text{ to } s-1}$$

$$= \frac{2}{(s-1)^3} + \frac{6}{(s-1)^2} + \frac{9}{s-1} = \frac{9s^2-12s+5}{(s-1)^3}$$

(*ii*) We have

$$L\{\cos^2 t\} = L\left\{\frac{1+\cos 2t}{2}\right\} = \frac{1}{2}[L\{1\} + L\{\cos 2t\}]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right] = \frac{s^2+2}{s(s^2+4)} = \bar{f}(s)$$

Using First Shifting Theorem,

$$L\{e^{-t}\cos^2 t\} = \bar{f}(s+1) = \left[\frac{s^2+2}{s(s^2+4)}\right]_{s\to s+1}$$

$$= \frac{(s+1)^2+2}{(s+1)[(s+1)^2+4]} = \frac{s^2+2s+3}{(s+1)(s^2+2s+5)}$$

(*iii*) We have

$$L\{\sin^2 t\} = L\left\{\frac{1-\cos 2t}{2}\right\} = \frac{1}{2}[L\{1\} - L\{\cos 2t\}]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] = \frac{2}{s(s^2+4)} = \bar{f}(s), \text{ say}$$

By First Shifting Theorem,

$$L\{e^{3t}\sin^2 t\} = \bar{f}(s-3) = \left[\frac{2}{s(s^2+4)}\right]_{s\to s-3} = \frac{2}{(s-3)[(s-3)^2+4]}$$

$$= \frac{2}{(s-3)(s^2-6s+13)}$$

Example 4 : Find $L(\sqrt{t}e^{-3t})$

Solution : Let $f(t) = \sqrt{t}$. Then

$$L\{f(t)\} = L\{t^{1/2}\} = \frac{\Gamma\left(\frac{3}{2}\right)}{s^{3/2}} \quad \left[\because L\{t^a\} = \frac{\Gamma(a+1)}{s^{a+1}}\right]$$

$$= \frac{\left(\frac{3}{2}-1\right)\Gamma\left(\frac{3}{2}-1\right)}{s^{3/2}} \left[\because \Gamma(n) = (n-1)\Gamma(n-1) \text{ or } \Gamma(n+1) = n.\Gamma(n)\right]$$

$$= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}} \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$= \bar{f}(s), \text{ say}$$

By First Shifting Theorem,

$$L\{\sqrt{t}e^{-3t}\} = L\{e^{-3t}f(t)\} = \bar{f}(s+3)$$

$$= \left[\bar{f}(s)\right]_{s\to s+3} = \left(\frac{\sqrt{\pi}}{2s^{3/2}}\right)_{s\to s+3}$$

$$= \frac{\sqrt{\pi}}{2(s+3)^{3/2}}$$

Example 5 : Find (*i*) $L\{e^{4t}\sin 2t\cos t\}$ (*ii*) $L\{\cosh at \sin bt\}$.

Solution : (*i*) We have

$$L\{\sin 2t\cos t\} = L\left\{\frac{1}{2}(2\sin 2t\cos t)\right\}$$

$$= L\left\{\frac{1}{2}(\sin 3t+\sin t)\right\}, \text{ using } 2\sin A\cos B \text{ formula}$$

$$= \frac{1}{2}[L\{\sin 3t\}+L\{\sin t\}] = \frac{1}{2}\left[\frac{3}{s^2+3^2}+\frac{1}{s^2+1^2}\right]$$

By First Shifting Theorem,

$$L\{e^{4t}\sin 2t\cos t\} = \frac{1}{2}\left[\frac{3}{s^2+9}+\frac{1}{s^2+1}\right]_{s\to s-4}$$

$$= \frac{1}{2}\left[\frac{3}{(s-4)^2+9}+\frac{1}{(s-4)^2+1}\right] = \frac{1}{2}\left[\frac{3}{s^2-8s+25}+\frac{1}{s^2-8s+17}\right]$$

(*ii*) $$L\{\cosh at\sin bt\} = L\left\{\left(\frac{e^{at}+e^{-at}}{2}\right)\sin bt\right\} = \frac{1}{2}\left[L\{e^{at}\sin bt\}+L\{e^{-at}\sin bt\}\right]$$

By First Shifting Theorem,

$$L\{\cosh at\sin bt\} = \frac{1}{2}\left[\{L(\sin bt)\}_{s\to s-a}+\{L(\sin bt)\}_{s\to s+a}\right]$$

$$= \frac{1}{2}\left[\left(\frac{b}{s^2+b^2}\right)_{s\to s-a}+\left(\frac{b}{s^2+b^2}\right)_{s\to s+a}\right]$$

$$= \frac{1}{2}\left[\frac{b}{(s-a)^2+b^2}+\frac{b}{(s+a)^2+b^2}\right]$$

$$= \frac{b}{2}\left[\frac{1}{s^2-2as+a^2+b^2}+\frac{1}{s^2+2as+a^2+b^2}\right]$$

$$= \frac{b}{2}\left[\frac{2(s^2+a^2+b^2)}{[(s-a)^2+b^2][(s+a)^2+b^2]}\right]$$

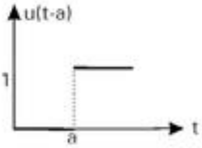
$$= \frac{b(s^2+a^2+b^2)}{[(s-a)^2+b^2]\,[(s+a)^2+b^2]}$$

1.8 UNIT STEP FUNCTION (HEAVISIDE* UNIT FUNCTION)

The unit step function is defined as H(*t*–*a*) or *u*(*t*–*a*) = $\begin{cases} 0, & \text{if } t < a \\ 1, & \text{if } t > a \end{cases}$

This is the unit step function at $t = a$.

Laplace Transform of $u(t-a)$



$$L\{u(t-a)\} = \int_0^\infty e^{-st}\, u(t-a)\, dt$$

$$= \int_0^a e^{-st} \cdot 0\, dt + \int_a^\infty e^{-st} \cdot 1\, dt = \int_a^\infty e^{-st} dt = \left(\frac{e^{-st}}{-s}\right)_a^\infty$$

$$= -\frac{1}{s}(e^{-\infty} - e^{-as}) = -\frac{1}{s}(0 - e^{-as}) = \frac{e^{-as}}{s}$$

Thus the graph of a unit step function $u(t-a)$ is a straight line parallel to t - axis from a to ∞.

Note 1 : If $a = 0$, then the unit step function $u(t-a)$ is $u(t) = \begin{cases} 0, \text{if } t < 0 \\ 1, \text{if } t > 0 \end{cases}$

Multiplying a given function $f(t)$ with the unit step function $u\,(t-a)$, several effects can be produced.

Note 2 : If $f(t) = \begin{cases} f_1(t), 0 < t < a \\ f_2(t), t > a \end{cases}$ then $f(t)$ can be written as

$$f(t) = f_1(t) + [f_2(t) - f_1(t)]\, u\,(t-a)$$

Proof : When $t < a$, $u(t-a) = 0$ so that $f(t) = f_1(t)$ for $t < a$

When $t > a$, $u(t-a) = 1$ so that $f(t) = f_1(t) + [f_2(t) - f_1(t)] = f_2(t)$ for $t > a$

1. 9 SECOND TRANSLATION (OR) SECOND SHIFTING THEOREM

Theorem : If $L\{f(t)\} = \bar{f}(s)$ and $g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$ then $L\{g(t)\} = e^{-as}\,\bar{f}(s)$

Proof : By definition,

$$L\{g(t)\} = \int_0^\infty e^{-st}\, g(t)\, dt = \int_0^a e^{-st} g(t)\, dt + \int_a^\infty e^{-st} g(t)\, dt$$

$$= \int_0^a e^{-st}.0 dt + \int_a^\infty e^{-st} f(t-a)\, dt = \int_a^\infty e^{-st} f(t-a)\, dt$$

Put $t - a = u$ so that $dt = du$.

Also $u = 0$ when $t = a$ and $u \to \infty$ when $t \to \infty$

$$\therefore \quad L\{g(t)\} = \int_0^\infty e^{-s(u+a)} f(u)\, du = e^{-as}\int_0^\infty e^{-su} f(u)\, du = e^{-as}\int_0^\infty e^{-st} f(t)\, dt$$

$$= e^{-as}\, L\{f(t)\} = e^{-as}\,\bar{f}(s)$$

Another Form of Second Shifting Theorem:

Theorem : If $L\{F(t)\} = \bar{f}(s)$ and $a > 0$, then $L\{F(t-a)\,H(t-a)\} = e^{-as}\bar{f}(s)$, where

$H(t) = \begin{cases} 1, & \text{if } t > 0 \\ 0, & \text{if } t < 0 \end{cases}$ and $H(t)$ is called Heaviside unit step function.

Proof : By definition,

$$L\{F(t-a)\,H(t-a)\} = \int_0^\infty e^{-st}F(t-a)\,H(t-a)\,dt \qquad ...(1)$$

Put $t - a = u$ so that $dt = du$. Also when $t = 0$, $u = -a$ when $t \to \infty$, $u \to \infty$

$$\text{Then } L\{F(t-a)\,H(t-a)\} = \int_{-a}^\infty e^{-s(u+a)}F(u)\,H(u)\,du, \text{ by (1)}$$

$$= \int_{-a}^0 e^{-s(u+a)}F(u)\,H(u)\,du + \int_0^\infty e^{-s(u+a)}\,F(u)\,H(u)\,du$$

$$= \int_{-a}^0 e^{-s(u+a)}F(u).0\,du + \int_0^\infty e^{-s(u+a)}F(u).1\,du, \text{ by Definition of } H(t)$$

$$= \int_0^\infty e^{-s(u+a)}F(u)\,du = e^{-sa}\int_0^\infty e^{-su}F(u)\,du$$

$$= e^{-sa}\int_0^\infty e^{-st}\,F(t)\,dt, \text{ by property of Definite Integrals}$$

$$= e^{-sa}L\{F(t)\} = e^{-as}\bar{f}(s)$$

1.10 CHANGE OF SCALE PROPERTY

Theorem : If $L\{f(t)\} = \bar{f}(s)$, then $L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$

Proof : By definition, we have $L\{f(at)\} = \int_0^\infty e^{-st}f(at)\,dt$

Put $at = x$ so that $dt = \frac{1}{a}dx$

When $t \to \infty$, $x \to \infty$ and when $t = 0$, $x = 0$.

$$\therefore\ L\{f(at)\} = \frac{1}{a}\int_0^\infty e^{-s(x/a)}f(x)\,dx = \frac{1}{a}\int_0^\infty e^{-(s/a)t}f(t)\,dt = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$$

Note : If $L\{f(t)\} = \bar{f}(s)$, then $L\left\{f\left(\frac{t}{a}\right)\right\} = a\,\bar{f}(as)$

SOLVED EXAMPLES

Example 1 : Find the Laplace transform of $g(t)$, where (*i*) $g(t) = \begin{cases} \cos\left(t - \frac{\pi}{3}\right), & \text{if } t > \frac{\pi}{3} \\ 0, & \text{if } t < \frac{\pi}{3} \end{cases}$

(*ii*) $g(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right), \text{if } t > \frac{2\pi}{3} \\ 0, \text{if } t < \frac{2\pi}{3} \end{cases}$

Solution : Let $f(t) = \cos t$

$$\therefore \; L\{f(t)\} = L\{\cos t\} = \frac{s}{s^2+1} = \bar{f}(s)$$

(*i*) Now $g(t) = \begin{cases} f\left(t - \frac{\pi}{3}\right) = \cos\left(t - \frac{\pi}{3}\right), & t > \frac{\pi}{3} \\ 0, & t < \frac{\pi}{3} \end{cases}$

Applying Second Shifting theorem, we get

$$L\{g(t)\} = e^{-\pi s/3}\left(\frac{s}{s^2+1}\right) = \frac{s\,e^{-\pi s/3}}{s^2+1}$$

(*ii*) Proceeding as above, we get

$$L\{g(t)\} = e^{-2\pi s/3}\left(\frac{s}{s^2+1}\right) = \frac{s\,e^{-2\pi s/3}}{s^2+1}$$

Example 2 : Find the Laplace transforms of (*i*) $(t-2)^3 u(t-2)$ (*ii*) $e^{-3t} u(t-2)$

Solution : (*i*) Comparing the given function with $f(t-a)\,u(t-a)$, we have $a = 2$ and $f(t) = t^3$

$$\therefore \; L\{f(t)\} = L\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4} = \bar{f}(s)$$

Now applying Second Shifting theorem, we get $L\{(t-2)^3 u(t-2)\} = e^{-2s}.\frac{6}{s^4} = \frac{6e^{-2s}}{s^4}$

(*ii*) $L\{e^{-3t} u(t-2)\} = L\{e^{-3(t-2)}\, e^{-6} u(t-2)\} = e^{-6} L\{e^{-3(t-2)}\, u(t-2)\}$

Taking $f(t) = e^{-3t}$, $\bar{f}(s) = \frac{1}{s+3}$ and using Second Shifting theorem, we have

$$L\{e^{-3t}\, u(t-2)\} = e^{-6}\, e^{-2s}.\frac{1}{s+3} = \frac{e^{-2(s+3)}}{s+3}$$

Example 3 : Find the Laplace transform of $3\cos 4\,(t-2)\, u\,(t-2)$

Solution : Let $f(t) = 3\cos 4t$. Then

$$\text{L}\{f(t)\} = 3\,.\,\text{L}\{\cos 4t\} = 3.\frac{s}{s^2+4^2} = \frac{3s}{s^2+16} = \bar{f}(s)$$

By Second Shifting theorem,

$$\text{L}\,\{3\cos 4(t-2)\, u\,(t-2)\} = \text{L}\,\{f(t-2)\, u\,(t-2)\} = e^{-2s}\,\bar{f}(s)$$

$$= e^{-2s}.\frac{3s}{s^2+16} = \frac{3se^{-2s}}{s^2+16}$$

Example 1 : Find $L\{\sin^2(at)\}$ using change of scale property .

Solution : Since $\sin^2(at) = \dfrac{1-\cos 2at}{2}$, we have

$$L\{\sin^2(at)\} = \frac{1}{2}L\{(1-\cos 2at)\} = \frac{1}{2}\left[L(1) - L\{\cos 2at\}\right] = \frac{1}{2}.\frac{1}{s} - \frac{1}{2}L\{\cos 2at\}$$

Taking $f(t) = \cos t,\ \bar{f}(s) = \dfrac{s}{s^2+1}$, and using Change of Scale property, we have

$$L\{\cos(2at)\} = \frac{1}{2a}\frac{s/2a}{(s/2a)^2+1} = \frac{s}{s^2+4a^2}$$

Hence $L\{\sin^2 at\} = \dfrac{1}{2s} - \dfrac{1}{2}\dfrac{s}{s^2+4a^2} = \dfrac{1}{2}\left[\dfrac{1}{s} - \dfrac{s}{s^2+4a^2}\right] = \dfrac{2a^2}{s(s^2+4a^2)}$

Example 2 : If $L\{f(t)\} = \dfrac{1}{s}e^{-1/s}$, prove that $L\{e^{-t}f(3t)\} = \dfrac{e^{-3/(s+1)}}{s+1}$

Solution : $L\{f(t)\} = \dfrac{1}{s}e^{-1/s} = \bar{f}(s)$

$\therefore$ By the Change of Scale property, we have $f(3t) = \dfrac{1}{3}\bar{f}\left(\dfrac{s}{3}\right) = \dfrac{1}{3}.\dfrac{3}{s}e^{-3/s} = \dfrac{1}{s}e^{-3/s}$

Now applying the First Shifting theorem, we get

$$L\{e^{-t}f(3t)\} = \frac{e^{-3/(s+1)}}{s+1}$$

Example 3 : If $L[f(t)] = \dfrac{9s^2-12s+15}{(s-1)^3}$, find $L[f(3t)]$ using change of scale property.

Solution : Given $L[f(t)] = \dfrac{9s^2-12s+15}{(s-1)^3} = \bar{f}(s)$

By Change of Scale Property,

$$L[f(3t)] = \frac{1}{3}\bar{f}\left(\frac{s}{3}\right) = \frac{1}{3}\cdot\frac{9(s/3)^2-12(s/3)+15}{\left(\frac{s}{3}-1\right)^3} = \frac{9(s^2-4s+15)}{(s-3)^3}$$

EXERCISE 1.2

Find the Laplace Transform of the following:

(1) $t^3 e^{-3t}$ [JNTU 2004 (Set No. 4)]

(2) $e^{-t}(3\cos 5t - 4\sin 5t)$

(3) $e^{2t}(3\sinh 2t - 5\cosh 2t)$

(4) $\dfrac{e^{-at}\, t^{n-1}}{(n-1)!}$

(5) (*i*) $e^{-at}\sinh bt$ [JNTU 2003] (*ii*) $e^{-t}\sin^2 t$

(6) $\cosh kt\, \sin kt$

(7) $\cosh at\, \cos at$

(8) $\sinh 3t \cdot \cos^2 t$

(9) $(1 + t e^{-t})^3$

(10) (*i*) $e^{-t}\sin t\cos t$ (*ii*) $e^{t}\cos 4t\sin t$

(11) $(t-1)^3 u(t-1)$

(12) $e^{-2t}\{1 - u(t-1)\}$

(13) $f(t) = \begin{cases} \cos\left(t - \dfrac{3\pi}{4}\right) \text{ when } t > \dfrac{3\pi}{4} \\ 0 \text{ when } t < \dfrac{3\pi}{4} \end{cases}$

(14) $f(t) = \begin{cases} \sin\left(t - \dfrac{\pi}{3}\right), t > \dfrac{\pi}{3} \\ 0, t < \dfrac{\pi}{3} \end{cases}$

(15) If $L\{f(t)\} = \bar{f}(s)$, show that $L\left\{f\left(\dfrac{t}{a}\right)\right\} = a\bar{f}(as)$

(16) If $L\{f(t)\} = \dfrac{20-4s}{s^2-4s+20}$, find (*i*) $L\{f(3t)\}$ (*ii*) $L\{e^{-t}f(2t)\}$

(17) If $L\left\{\dfrac{\sin t}{t}\right\} = \tan^{-1}\left(\dfrac{1}{s}\right)$, find (*i*) $L\left\{\dfrac{\sin at}{t}\right\}$ (*ii*) $L\left\{e^{t}\dfrac{\sin 3t}{t}\right\}$

ANSWERS

(1) $\dfrac{6}{(s+3)^4}$

(2) $\dfrac{3x-17}{s^2+2s+26}$

(3) $\dfrac{16-5s}{s^2-4s}$

(4) $\dfrac{1}{(s+a)^n}$

(5) (*i*) $\dfrac{b}{(s+a)^2-b^2}$ (*ii*) $\dfrac{s^2+2s+3}{(s+1)(s^2+2s+5)}$

(6) $\dfrac{k(s^2+2k^2)}{s^4+4k^4}$

(7) $\dfrac{s^3}{s^4+4a^4}$

(8) $\dfrac{3}{2}\left[\dfrac{1}{s^2-9} + \dfrac{s^2-13}{s^4-10s^2+169}\right]$

(9) $\dfrac{1}{s} + \dfrac{3}{(s+1)^2} + \dfrac{6}{(s+2)^3} + \dfrac{6}{(s+3)^4}$

(10) (*i*) $\dfrac{1}{s^2+2s+5}$ (*ii*) $\dfrac{3}{s^2-2s+37} - \dfrac{1}{s^2-2s+5}$

(11) $\dfrac{6e^{-s}}{s^4}$

(12) $\dfrac{1-e^{-(s+2)}}{s+2}$

(13) $\dfrac{se^{-3\pi s/4}}{s^2+1}$

(14) $\dfrac{e^{-\pi s/3}}{s^2+1}$

(16) (*i*) $\dfrac{60-4s}{s^2-12s+80}$ (*ii*) $\dfrac{4(9-s)}{s^2-6s+73}$

(17) (*i*) $\tan^{-1}\left(\dfrac{a}{s}\right)$ (*ii*) $\tan^{-1}\left(\dfrac{3}{s-1}\right)$

1.11 LAPLACE TRANSFORM OF DERIVATIVES

Theorem: If $f(t)$ is continuous and of exponential order, and $f'(t)$ is sectionally continuous then the Laplace transform of $f'(t)$ is given by $L\{f'(t)\} = s\,\bar{f}(s) - f(0)$ where $\bar{f}(s) = L\{f(t)\}$.

Proof: By definition,

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt = \int_0^\infty e^{-st} d\{f(t)\} = \operatorname*{Lim}_{P\to\infty} \int_0^P e^{-st} d\{f(t)\}$$

$$= \operatorname*{Lim}_{P\to\infty} \left\{ \left[e^{-st} f(t) \right]_0^P + \int_0^P s e^{-st} f(t)\,dt \right\}$$

$$= \operatorname*{Lim}_{P\to\infty} \left\{ e^{-sP} f(P) - f(0) + s\int_0^P e^{-st} f(t)\,dt \right\}$$

Since $f(t)$ is of exponential order, $\operatorname*{Lim}_{t\to\infty} e^{-st} f(t) = 0$

$$\therefore L\{f'(t)\} = 0 - f(0) + s\int_0^\infty e^{-st} f(t)\,dt = -f(0) + s\,L\{f(t)\} = s\,\bar{f}(s) - f(0)$$

Note : The Laplace transform of the second derivative $f''(t)$ is similarly obtained.

We have $L\{g'(t)\} = s\,L\{g(t)\} - g(0)$

Let $g(t) = f'(t)$. Then

$$L\{f''(t)\} = s\,L\{f'(t)\} - f'(0) = s[s\,L\{f(t)\} - f(0)] - f'(0)$$

$$= s^2\,L\{f(t)\} - s\,f(0) - f'(0) = s^2 \bar{f}(s) - s\,f(0) - f'(0)$$

Assuming that $f(t)$ and $f'(t)$ are continuous and are of exponential order, the Laplace transform of higher derivatives can be obtained by the same method.

1. **To prove** $L\{f^n(t)\} = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{n-1}(0)$

We know that

$$L\{f'(t)\} = s\,\bar{f}(s) - f(0) = sL\{f(t)\} - f(0) \text{ [prove it in exam.]} \qquad \dots(1)$$

$$\therefore L\{f''(t)\} = L\{[f'(t)]'\} = sL\{f'(t)\} - f'(0), \text{ using (1)}$$

$$= s\left[sL\{f(t)\} - f(0)\right] - f'(0) = s^2 \bar{f}(s) - s\,f(0) - f'(0) \qquad \dots(2)$$

Now $L\{f'''(t)\} = L\left\{\left[f''(t)\right]'\right\} = s\,L\{f''(t)\} - f''(0)$, using (1)

$$= s\left[s^2 \bar{f}(s) - sf(0) - f'(0)\right] - f''(0), \text{ using (2)}$$

$$= s^3 \bar{f}(s) - s^2 f(0) - sf'(0) - f''(0)$$

Similarly, we can prove that

$$\boxed{L\left\{f^n(t)\right\} = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{n-1}(0)}$$

Note : $L\{f^n(t)\} = s^n \bar{f}(s)$, if $f(0) = 0$ and $f'(0) = 0$, $f''(0) = 0$, ..., $f^{n-1}(0) = 0$

SOLVED EXAMPLES

Example 1 : Using the Theorem on transforms of derivatives, find the Laplace Transform of the following functions : (*i*) e^{at} (*ii*) $\cos at$ (*iii*) $t \sin at$

Solution : (*i*) Let $f(t) = e^{at}$. Then $f'(t) = a\,e^{at}$ and $f(0) = 1$.

Now $L\{f'(t)\} = s\,L\{f(t)\} - f(0)$

i.e., $L\{a\,e^{at}\} = s\,L\{e^{at}\} - 1$

i.e., $a\,L\{e^{at}\} - s\,L\{e^{at}\} = -1$ *i.e.* $(a-s)\,L\{e^{at}\} = -1$

or $L\{e^{at}\} = \dfrac{-1}{a-s} = \dfrac{1}{s-a}$

(*ii*) Let $f(t) = \cos at$. Then $f'(t) = -a\sin at, f''(t) = -a^2 \cos at$

$\therefore L\{f''(t)\} = s^2\,L\{f(t)\} - s\,f(0) - f'(0)$ ...(1)

Now $f(0) = \cos 0 = 1$ and $f'(0) = -a\sin 0 = 0$, substituting in (1), we get

$L\{-a^2 \cos at\} = s^2\,L\{\cos at\} - s(1) - 0$

i.e., $(s^2 + a^2)\,L\,\{\cos at\} = s$ or $L\{\cos at\} = \dfrac{s}{s^2 + a^2}$

(*iii*) Let $f(t) = t\sin at$. Then $f'(t) = \sin at + at\cos at$

and $f''(t) = a\cos at + a[\cos at - at\sin at] = 2a\cos at - a^2 t\sin at$

Also $f(0) = 0$ and $f'(0) = 0$

Now $L\{f''(t)\} = s^2\,L\{f(t)\} - s\,f(0) - f'(0)$

i.e., $L\{2a\cos at - a^2 t\sin at\} = s^2 L\{t\sin at\} - 0 - 0$

i.e., $2a\,L\,\{\cos at\} - a^2\,L\,\{t\sin at\} - s^2 L\,\{t\sin at\} = 0$

i.e., $-(a^2 + s^2)\,L\{t\sin at\} = \dfrac{-2a\,s}{s^2 + a^2}$ or $L\{t\sin at\} = \dfrac{2as}{(s^2 + a^2)^2}$

Example 2 : Evaluate $L\left\{\dfrac{\cos\sqrt{t}}{\sqrt{t}}\right\}$ (or) If $L\{\sin\sqrt{t}\} = \dfrac{\sqrt{\pi}}{2s^{3/2}} e^{-\frac{1}{4s}}$, find $L\left\{\dfrac{\cos\sqrt{t}}{\sqrt{t}}\right\}$.

Solution : Let $f(t) = \sin\sqrt{t}$.

Then $f'(t) = \dfrac{1}{2\sqrt{t}}\cos\sqrt{t}$. Also $f(0) = 0$.

Now $L\{f'(t)\} = s\,L\{f(t)\} - f(0)$

i.e., $L\left\{\dfrac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = s\,L\{\sin\sqrt{t}\} - 0$

or $\dfrac{1}{2} L\left\{\dfrac{\cos\sqrt{t}}{\sqrt{t}}\right\} = s.\dfrac{\sqrt{\pi}\,e^{-1/(4s)}}{2s^{3/2}}$ $\left[\because \text{L}\{\sin\sqrt{t}\} = \dfrac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}\right]$

$= \dfrac{\sqrt{\pi}\;e^{-1/(4s)}}{2s^{1/2}}$

or $L\left\{\dfrac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\dfrac{\pi}{s}}\;e^{-1/(4s)}$

EXERCISE 1.3

Using the theorem on transforms of derivatives, evaluate:

(1) If $L\{t\sin at\} = \frac{2as}{(s^2+a^2)^2}$, then prove that $L\{\sin at + at\cos at\} = \frac{2as^2}{\left(s^2+a^2\right)^2}$.

(2) Given $L\left\{\frac{1-\cos\omega t}{\omega^2}\right\} = \frac{1}{s(s^2+\omega^2)}$, show that $L\left\{\frac{t(1-\cos\omega t)}{\omega^2}\right\} = \frac{3s^2+\omega^2}{s^2(s^2+\omega^2)^2}$.

$\left[\textbf{Hint: } L\{t\sin\omega t\} = \frac{2\omega s}{(s^2+\omega^2)^2}\right]$

1.12 LAPLACE TRANSFORM OF INTEGRALS

Theorem : If $L\{f(t)\} = \bar{f}(s)$ then $L\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\bar{f}(s)$.

Proof: Let $g(t) = \int_0^t f(u)\,du$. Then

$$g'(t) = \frac{d}{dt}\left(\int_0^t f(u)\,du\right) = f(t) \text{ and } g(0) = \int_0^0 f(u)\,du = 0$$

Taking Laplace transform on both sides, we have $L\{g'(t)\} = L\{f(t)\}$.

But $L\{g'(t)\}$ is also $= s\,L\{g(t)\} - g(0) = s\,L\{g(t)\} - 0$

$\therefore L\{g'(t)\} = L\{f(t)\} = s\,L\{g(t)\}$

$$\Rightarrow L\{g(t)\} = \frac{1}{s}L\{f(t)\}$$

$$\Rightarrow L\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}L\{f(t)\} = \frac{1}{s}\bar{f}(s)$$

Note : Similarly, if $L\{f(t)\} = \bar{f}(s)$, then $L\left\{\int_0^t\int_0^t f(u)\,du\,du\right\} = \frac{1}{s^2}\bar{f}(s)$

In general, $L\left\{\int_0^t\int_0^t \underset{(n\ times)}{\cdots\cdots} \int_0^t f(u)\,du\,du \underset{(n\ times)}{\ldots du}\right\} = \frac{1}{s^n}\bar{f}(s)$

SOLVED EXAMPLES

Example 1 : Find $L\left\{\int_0^t e^{-t}\cos t\,dt\right\}$

(or) Find the Laplace transform of $f(t)=\int_0^t e^{-u}\cos u\,du$

Solution : Let $f(t)=e^{-t}\cos t$. Then

$$L\{f(t)\}=L\{e^{-t}\cos t\}=\{L(\cos t)\}_{s\to s+1}$$

$$=\left(\frac{s}{s^2+1}\right)_{s\to s+1}=\frac{s+1}{(s+1)^2+1}$$

$$=\frac{s+1}{s^2+2s+2}=\bar{f}(s)\text{, say}$$

Using the theorem of Laplace transform of the integral, we have

$$L\left\{\int_0^t f(t)\,dt\right\}=\frac{1}{s}\bar{f}(s)$$

$$\Rightarrow\quad L\left\{\int_0^t e^{-t}\cos t\,dt\right\}=\frac{1}{s}\cdot\frac{s+1}{s^2+2s+2}=\frac{s+1}{s(s^2+2s+2)}$$

Example 2 : Find (*i*) $L\left\{\int_0^t \cosh t\,dt\right\}$

(*ii*) $L\left\{\int_0^t\int_0^t \cosh at\,dt\,dt\right\}$

Solution : (*i*) Let $f(t)=\cosh t$. Then

$$L\{f(t)\}=L\{\cosh t\}=\frac{s}{s^2-1}=\bar{f}(s)$$

Using the theorem of L.T. of integral, we have

$$L\left\{\int_0^t f(t)\,dt\right\}=\frac{1}{s}\bar{f}(s)$$

$$\therefore\quad L\left\{\int_0^t \cosh t\,dt\right\}=\frac{1}{s}\bar{f}(s)=\frac{1}{s}\cdot\frac{s}{s^2-1}=\frac{1}{s^2-1}$$

(*ii*) Let $f(t) = \cosh at$. Then

$$L\{f(t)\} = L\{\cosh at\} = \frac{s}{s^2 - a^2} = \bar{f}(s)$$

Using the theorem on Laplace transform of integral,

$$L\left\{\int_0^t \cosh at\, dt\right\} = \frac{1}{s}\bar{f}(s) = \frac{1}{s}\cdot\frac{s}{s^2-a^2} = \frac{1}{s^2-a^2}$$

Applying again,

$$L\left\{\int_0^t\int_0^t \cosh at\, dt\, dt\right\} = \frac{1}{s}\cdot\frac{1}{s^2-a^2} = \frac{1}{s(s^2-a^2)}$$

1. 13 LAPLACE TRANSFORM OF $t^n \cdot f(t)$

1. **Multiplication by t**

Theorem: If $f(t)$ is sectionally continuous and of exponential order and if $L\{f(t)\} = \bar{f}(s)$ then $L\{t\, f(t)\} = -\bar{f}'(s)$

Proof: We have $\bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt$

Then by Leibnitz's rule for differentiating under the integral sign,

$$\frac{d\bar{f}}{ds} = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)\, dt = \int_0^\infty -t\, e^{-st} f(t)\, dt$$

$$= -\int_0^\infty e^{-st}\{t\, f(t)\}\, dt = -L\{t\, f(t)\}$$

Thus $L\{\, t\, f(t)\} = -\frac{d\bar{f}}{ds} = -\bar{f}'(s)$ or $L\{\, t\, f(t)\} = (-1)\frac{d}{ds}\left[\bar{f}(s)\right]$

2. Multiplication by t^n

Theorem : If $f(t)$ is sectionally continuous and of exponential order and if $L\{f(t)\} = \bar{f}(s)$, then

$$L\left\{t^n f(t)\right\} = (-1)^n \frac{d^n}{ds^n}\left[\bar{f}(s)\right], \text{ where } n = 1, 2, 3, \ldots$$

[JNTU 2003, 2003S, 2004S, 2007S, 2008S, (K) June 2012 (Set No. 3)]

Proof : By definition, we have

$$\bar{f}(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

Differentiating both sides w.r.t. 's',

$$\frac{d}{ds}\left[\bar{f}(s)\right] = \frac{d}{ds}\left[\int_0^\infty e^{-st} f(t)\, dt\right]$$

$$= \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st}\right) f(t)\, dt,$$ by Leibnitz's rule for differentiation under the integral sign

$$\therefore \quad \frac{d}{ds}\left[\bar{f}(s)\right] = \int_0^\infty -te^{-st} f(t)dt = -\int_0^\infty e^{-st}\left\{t\cdot f(t)\right\}dt$$

$$= -L\{t\, f(t)\}$$

$$\Rightarrow \quad L\left\{t\cdot f(t)\right\} = (-1)\frac{d}{ds}\bar{f}(s)$$

which proves the result for $n = 1$.

Now assume the result is true for $n = m$ (say), so that

$$\int_0^\infty e^{-st}\left\{t^m f(t)\right\}dt = (-1)^m \frac{d^m}{ds^m}\left[\bar{f}(s)\right]$$

Again differentiating w.r.t. 's',

$$\frac{d}{ds}\left[\int_0^\infty e^{-st} t^m f(t)dt\right] = (-1)^m \frac{d^{m+1}}{ds^{m+1}}\left[\bar{f}(s)\right]$$

$$\Rightarrow \quad \int_0^\infty \frac{\partial}{\partial s}(e^{-st})t^m f(t)dt = (-1)^m \frac{d^{m+1}}{ds^{m+1}}\left[\bar{f}(s)\right], \text{ by Leibnitz's rule}$$

$$\Rightarrow \quad \int_0^\infty \left(-te^{-st}\right)t^m f(t)dt = (-1)^m \frac{d^{m+1}}{ds^{m+1}}\left[\bar{f}(s)\right]$$

$$\Rightarrow \quad \int_0^\infty e^{-st}\left[t^{m+1} f(t)\right]dt = (-1)^{m+1} \frac{d^{m+1}}{ds^{m+1}}\left[\bar{f}(s)\right]$$

This shows that, if the result is true for $n = m$, it is also true for $n = m + 1$.

But it is true for $n = 1$.

Hence, it is true for $n = 1 + 1 = 2$, and $n = 2 + 1 = 3$ and so on. Thus, by the method of mathematical induction the result is true for all values of n, where n is a positive integer.

$$\therefore L\left\{t^n f(t)\right\} = (-1)^n \frac{d^n}{ds^n}\left[\bar{f}(s)\right]$$

1. 14 DIVISION BY t

Theorem: If $L\{f(t)\} = \bar{f}(s)$, then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\,ds$, provided the integral exists.

[JNTU 2003, 2003S, 2004S, 2006,2007, (H) 2011 (Set No.3)]

Proof : Given $L\{f(t)\} = \bar{f}(s)$. Then $\bar{f}(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

Integrating both sides w.r.t. 's' from $s = s$ to $s = \infty$, we get $\int_s^\infty \bar{f}(s)\,ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)\,dt\right] ds$

Interchanging the order of integration in the repeated integrals as s and t are independent variables, we have

$$\int_s^\infty \bar{f}(s)\,ds = \int_0^\infty dt \int_s^\infty e^{-st} f(t)\,ds = \int_0^\infty f(t)\left(\int_s^\infty e^{-st}\,ds\right) dt$$

$$= \int_0^\infty f(t)\left(\frac{e^{-st}}{-t}\right)_s^\infty dt = \int_0^\infty f(t)\left[\frac{0-e^{-st}}{-t}\right] dt = \int_0^\infty f(t)\frac{e^{-st}}{t}\,dt$$

$$= \int_0^\infty e^{-st}\left\{\frac{f(t)}{t}\right\} dt = L\left\{\frac{f(t)}{t}\right\}$$

Hence $\boxed{L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\,ds}$

Similarly, the result can be extended to repeated integration of the transformation.

SOLVED EXAMPLES

Example 1 : Find the Laplace Transform of

(*i*) $\dfrac{\sin t}{t}$ [JNTU 2000, (H) R15 Sept. 2017] (*ii*) $\dfrac{e^{-at}-e^{-bt}}{t}$

(*iii*) $\dfrac{\sin 3t\ \cos t}{t}$

Solution : (*i*) We know that $L\left\{\dfrac{f(t)}{t}\right\} = \displaystyle\int_s^{\infty} \bar{f}(s)\, ds$

Since $L\{f(t)\} = L\{\sin t\} = \dfrac{1}{s^2+1} = \bar{f}(s)$

$$\therefore\ L\left\{\frac{\sin t}{t}\right\} = \int_s^{\infty} \bar{f}(s)\, ds = \int_s^{\infty} \frac{1}{s^2+1}\, ds = \left(\tan^{-1} s\right)_s^{\infty} = \tan^{-1}\infty - \tan^{-1} s$$

$$= \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s \left[\because \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}\right]$$

(*ii*) Let $f(t) = e^{-at} - e^{-bt}$. Then $L\{f(t)\} = L\{e^{-at}\} - L\{e^{-bt}\} = \dfrac{1}{s+a} - \dfrac{1}{s+b} = \bar{f}(s)$

$$\therefore L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} \bar{f}(s)\, ds = \int_s^{\infty}\left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds$$

$$= \left[\log(s+a) - \log(s+b)\right]_s^{\infty} = \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^{\infty}$$

$$= \lim_{s\to\infty}\left\{\log\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right\} - \log\left(\frac{s+a}{s+b}\right) = \log 1 - \log(s+a) + \log(s+b)$$

$$= \log\left(\frac{s+b}{s+a}\right)$$

(*iii*) Let $f(t) = \sin 3t \cos t = \dfrac{1}{2}(2\sin 3t\cos t) = \dfrac{1}{2}(\sin 4t + \sin 2t)$, using $2\sin A\cos B$ formula

Since $L\{f(t)\} = L(\sin 3t\cos t) = \dfrac{1}{2}\left(\dfrac{4}{s^2+16} + \dfrac{2}{s^2+4}\right) = \bar{f}(s)$

$$\therefore\quad L\left\{\frac{\sin 3t\cos t}{t}\right\} = L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty}\bar{f}(s) ds = \frac{1}{2}\int_s^{\infty}\left(\frac{4}{s^2+16} + \frac{2}{s^2+4}\right) ds$$

$$= \frac{1}{2}\left(\tan^{-1}\frac{s}{4} + \tan^{-1}\frac{s}{2}\right)_s^{\infty} \left[\because \int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} + \frac{\pi}{2}\right) - \left(\tan^{-1}\frac{s}{4} + \tan^{-1}\frac{s}{2}\right)\right] = \frac{\pi}{2} - \frac{1}{2}\left(\tan^{-1}\frac{s}{4} + \tan^{-1}\frac{s}{2}\right)$$

Example 2 : Find (*i*) $L\left\{\frac{\cos 2t-\cos 3t}{t}\right\}$

(*ii*) $L\left\{\frac{1-e^t}{t}\right\}$

Solution : (*i*) We have

$$L\{\cos 2t-\cos 3t\} = L\{\cos 2t\}-L\{\cos 3t\} = \frac{s}{s^2+4}-\frac{s}{s^2+9} = \bar{f}(s), say$$

$$\therefore L\left\{\frac{\cos 2t-\cos 3t}{t}\right\} = \int_s^\infty \bar{f}(s)\,ds$$

$$=\int_s^\infty\left(\frac{s}{s^2+4}-\frac{s}{s^2+9}\right)ds = \frac{1}{2}\int_s^\infty\left(\frac{2s}{s^2+4}-\frac{2s}{s^2+9}\right)ds$$

$$=\frac{1}{2}[\log\ (s^2+4)-\log\ (s^2+9)]\left[\because \int\frac{f'(x)}{f(x)}dx=\log f(x)\right]$$

$$=\frac{1}{2}\left[\log\left(\frac{s^2+4}{s^2+9}\right)\right]_s^\infty = \frac{1}{2}\left[\log\left(\frac{1+4/s^2}{1+9/s^2}\right)\right]_s^\infty$$

$$=\frac{1}{2}\left[\log\left(\frac{1+0}{1+0}\right)-\log\left(\frac{1+4/s^2}{1+9/s^2}\right)\right]$$

$$=\frac{1}{2}\left[0-\log\left(\frac{s^2+4}{s^2+9}\right)\right]=\frac{1}{2}\log\left(\frac{s^2+9}{s^2+4}\right)=\log\left(\frac{s^2+9}{s^2+4}\right)^{1/2}$$

(*ii*) Since $L\{1-e^t\} = L\{1\}-L\{e^t\} = \frac{1}{s}-\frac{1}{s-1}=\bar{f}(s)$

$$\therefore L\left\{\frac{1-e^t}{t}\right\} = \int_s^\infty \bar{f}(s)ds = \int_s^\infty\left(\frac{1}{s}-\frac{1}{s-1}\right)ds$$

$$=\left[\log\left(\frac{s}{s-1}\right)\right]_s^\infty = \left[\log\left(\frac{1}{1-\frac{1}{s}}\right)\right]_s^\infty = \log 1-\log\left(\frac{s}{s-1}\right)=\log\left(\frac{s-1}{s}\right)$$

Example 3 : Find (*i*) $L\left\{\int_0^t \frac{1-e^{-t}}{t}\,dt\right\}$ (*ii*) $L\left\{\int_0^t \frac{e^t \sin t}{t}\,dt\right\}$

Solution : (*i*) Let $f(t) = 1-e^{-t}$. Then

$$\bar{f}(s) = L\{f(t)\} = L\{1-e^{-t}\} = \frac{1}{s}-\frac{1}{s+1}$$

$$\therefore\ L\left\{\frac{1-e^{-t}}{t}\right\} = L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\,ds$$

$$= \int_s^\infty \left(\frac{1}{s}-\frac{1}{s+1}\right)ds = [\log s - \log(s+1)]_s^\infty$$

$$= \left[\log\left(\frac{s}{s+1}\right)\right]_s^\infty = \left[\log\left(\frac{1}{1+1/s}\right)\right]_s^\infty = \log\left(\frac{1}{1+0}\right) - \log\left(\frac{1}{1+1/s}\right)$$

$$= 0-\log\left(\frac{s}{s+1}\right) = \log\left(\frac{s+1}{s}\right) = \bar{F}(s)$$

Using the theorem of L.T. of integral,

$$L\left\{\int_0^t \frac{1-e^{-t}}{t}\,dt\right\} = \frac{1}{s}\cdot\bar{F}(s) = \frac{1}{s}\log\left(\frac{s+1}{s}\right) = \frac{1}{s}\log\left(1+\frac{1}{s}\right)$$

(*ii*) We know that $L\{\sin t\} = \dfrac{1}{s^2+1}$

$$\therefore L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}\,ds = (\tan^{-1} s)_s^\infty = \tan^{-1}\infty - \tan^{-1} s$$

$$= \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

By First Shifting Theorem, $L\left\{e^t\cdot\frac{\sin t}{t}\right\} = (\cot^{-1} s)_{s\to s-1} = \cot^{-1}(s-1) = \bar{f}(s)$

Using the theorem of L.T. of integral, $L\left\{\int_0^t \frac{e^t \sin t}{t}\,dt\right\} = \frac{1}{s}\bar{f}(s) = \frac{1}{s}\cot^{-1}(s-1)$

Example 6 : Find $L\left[\frac{e^{-3t}\sin 2t}{t}\right]$.

Solution : We know that $L\{\sin 2t\} = \frac{2}{s^2+2^2} = \bar{f}(s)$, say

By First Shifting theorem,

$$L\{e^{-3t}\sin 2t\} = \bar{f}(s+3) = \left(\frac{2}{s^2+2^2}\right)_{s\to s+3} = \frac{2}{(s+3)^2+2^2} = \bar{g}(s), \text{ say}$$

$$\therefore\ L\left[\frac{e^{-3t}\sin 2t}{t}\right] = \int_s^\infty \bar{g}(s)ds = \int_s^\infty \frac{2}{(s+3)^2+2^2}ds. \text{ Apply } \int\frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= 2\cdot\frac{1}{2}\left[\tan^{-1}\left(\frac{s+3}{2}\right)\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}\left(\frac{s+3}{2}\right)$$

$$= \cot^{-1}\left(\frac{s+3}{2}\right) \quad \left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right]$$

Example 7 : Find $L\left[\frac{e^{-t}\sin t}{t}\right]$

Solution : We know that $L\{\sin t\} = \frac{1}{s^2+1}$

By First Shifting Theorem,

$$L\{e^{-t}\sin t\} = \left(\frac{1}{s^2+1}\right)_{s\to s+1} = \frac{1}{(s+1)^2+1} = \bar{f}(s), \text{ say}$$

$$\therefore\ L\left\{\frac{e^{-t}\sin t}{t}\right\} = \int_s^\infty \bar{f}(s)ds$$

$$= \int_s^\infty \frac{1}{(s+1)^2+1}ds = [\tan^{-1}(s+1)]_s^\infty = \tan^{-1}\infty - \tan^{-1}(s+1)$$

$$= \frac{\pi}{2} - \tan^{-1}(s+1) = \cot^{-1}(s+1) \quad \left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right]$$

Example 9 : Find (*i*) $L\left[e^{-3t}\int_0^t \frac{\sin t}{t}\,dt\right]$.

(*ii*) $L\left\{e^{-t}\left[\int_0^t \frac{\sin u}{u}\,du\right]\right\}$

Solution : (*i*) Since $L\{\sin t\} = \frac{1}{s^2+1} = \bar{f}(s)$

$$\therefore L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \bar{f}(s)\,ds = \int_s^\infty \frac{1}{s^2+1}\,ds$$

$$= (\tan^{-1}s)_s^\infty = \tan^{-1}\infty - \tan^{-1}s = \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s$$

Hence $L\left\{\int_0^t \frac{\sin t}{t}\,dt\right\} = \frac{1}{s}\cot^{-1}s$, ... (1) using the theorem of L. T. of integral

By First Shifting theorem,

$$L\left\{e^{-3t}\int_0^t \frac{\sin t}{t}\,dt\right\} = \bar{f}(s+3) = \left(\frac{1}{s}\cot^{-1}s\right)_{s\to s+3} = \frac{1}{s+3}\cot^{-1}(s+3)$$

(*ii*) From (1) above, we have $L\left[\int_0^t \frac{\sin u}{u}\,du\right] = \frac{1}{s}\cot^{-1}s = \bar{f}(s)$

By First Shifting Theorem,

$$L\left\{e^{-t}\left[\int_0^t \frac{\sin u}{u}\,du\right]\right\} = \left[\bar{f}(s)\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\cot^{-1}s\right] = \frac{1}{s+1}\cot^{-1}(s+1)$$

Example 5 : Using Laplace transform, evaluate $\int_0^\infty te^{-t}\sin t\,dt$.

<u>**Solution :**</u> We note that the given integral is same as $\int_0^\infty te^{-st}\sin t\,dt$ where $s = 1$.

$$\text{But} \quad \int_0^\infty te^{-st}\sin t\,dt = \int_0^\infty e^{-st}(t\sin t)\,dt = L\{t\sin t\} = (-1)\frac{d}{ds}\left[L(\sin t)\right]$$

$$= (-1)\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = (-1)\cdot\frac{-1}{(s^2+1)^2}\cdot 2s = \frac{2s}{(s^2+1)^2}$$

Putting $s = 1$, we get $\int_0^\infty te^{-st}\sin t\,dt = \frac{2}{(1+1)^2} = \frac{1}{2}$

1.17 LAPLACE TRANSFORM OF PERIODIC FUNCTIONS

Definition. Periodic Function

A function $f(t)$ is said to be periodic, if and only if $f(t + T) = f(t)$ for some value of T and for every value of t. The smallest positive value of T for which this equation is true for every value of t is called the peroid of the function.

Alternatively, a function $f(t)$ is said to be periodic function of peroid $T > 0$, if

$$f(t) = f(t+T) = f(t+2T) = \cdots = f(t+nT)$$

For example, (*i*) sin t and cos t are periodic functions with period 2π

(*ii*) tan t and cot t are periodic functions with period π.

Note : If T is the period of $f(x)$ then the peroid of $f(ax+b)$ is $\dfrac{T}{a}$.

Theorem: If $f(t)$ is a periodic function with period T, then $L\{f(t)\} = \dfrac{\int_0^T e^{-st} f(t)dt}{1-e^{-sT}}$.

[JNTU (A) June 2011, May 2012 (Set No. 3)]

Proof: By definition, we have

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt$$

$$= \int_0^{T} e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} e^{-st} f(t)\,dt + \ldots$$

Put $t = u + T$ in the second integral

Put $t = u + 2T$ in the third integral

Put $t = u + 3T$ in the fourth integral and so on. Then

$$L\{f(t)\} = \int_0^{T} e^{-st} f(t)\,dt + \int_0^{T} e^{-s(u+T)} f(u+T)\,du + \int_0^{T} e^{-s(u+2T)} f(u+2T)du + \ldots$$

$$= \int_0^{T} e^{-st} f(t)dt + e^{-sT}\int_0^{T} e^{-su} f(u)\,du + e^{-2sT}\int_0^{T} e^{-su} f(u)du + \ldots$$

$$[\because f(u) = f(u+T) = f(u+2T) = \ldots]$$

$$= \int_0^{T} e^{-st} f(t)\,dt + e^{-sT}\int_0^{T} e^{-st} f(t)\,dt + e^{-2sT}\int_0^{T} e^{-st} f(t)\,dt + \ldots$$

$$= (1 + e^{-sT} + e^{-2sT} + \ldots\infty)\int_0^{T} e^{-st} f(t)\,dt$$

$$\therefore\ L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^{T} e^{-st} f(t)dt$$

Hence the result follows.

Table 1.2. Table of Laplace Transform Theorems

To facilitate the reader, we listed all the theorems discussed so far in the following table :

S. No.	Theorem	$f(t)$	$L\{f(t)\} = \bar{f}(s)$
1	Linearity Property	$c_1 f(t) + c_2 g(t)$ (c_1 & c_2 are constants)	$c_1 L\{f(t)\} + c_2 L\{g(t)\}$
2	First Shifting Theorem	(*i*) $e^{at} f(t)$ (*ii*) $e^{-at} f(t)$	$\bar{f}(s-a)$ $\bar{f}(s+a)$
3	Second Shifting Th.	$g(t) = \begin{cases} f(t-a), t > a \\ 0, t < a \end{cases}$	$e^{-as} \bar{f}(s)$
4	Change of Scale property	$f(at)$	$\frac{1}{a} \bar{f}\left(\frac{s}{a}\right)$
5	Initial Value Theorem	$\underset{t \to 0}{Lt} f(t) = \underset{s \to \infty}{Lt} sL\{f(t)\}$	
6	Final Value Th.	$\underset{t \to \infty}{Lt} f(t) = \underset{s \to 0}{Lt} sL\{f(t)\}$	
7	Differentiation Theorem	(*i*) $f'(t)$ (*ii*) $f''(t)$	$s\bar{f}(s) - f(0)$ $s^2 \bar{f}(s) - sf(0) - f'(0)$
8	Multiplication Theorem	(*i*) $t.f(t)$ (*ii*) $t^n f(t)$	$-\bar{f}'(s)$ $(-1)^n \frac{d^n}{ds^n}[\bar{f}(s)]$
9	Division Theorem	$\frac{1}{t} f(t)$	$\int_s^\infty \bar{f}(s)\, ds$ or $\int_s^\infty \bar{f}(x)\, dx$
10	Integral Theorem	$\int_0^t f(x)\, dx$	$\frac{1}{s} \bar{f}(s)$
11	Fundamental Theorem for periodic functions	$f(t)$ is periodic function of period T *i.e.* $f(t+T) = f(t)$	$\frac{\int_0^T e^{-st} f(t)\, dt}{1 - e^{-sT}}$

2. 1 THE INVERSE LAPLACE TRANSFORM

In the earlier chapter we have considered Laplace Transforms of some functions $f(t)$. Let us now consider the converse namely, given $\bar{f}(s)$ how to determine $f(t)$ such that $L\{f(t)\} = \bar{f}(s)$.

The Inverse Laplace Transform is useful in solving differential equations without finding the general solution and arbitrary constants.

Definition: If $\bar{f}(s)$ is the Laplace transform of a function $f(t)$, then $f(t)$ is called the Inverse Laplace Transform of $\bar{f}(s)$ and is denoted by $L^{-1}\{\bar{f}(s)\}$ *i.e.* $f(t) = L^{-1}\{\bar{f}(s)\}$. L^{-1} is called the Inverse Laplace Transform operator.

Thus, if $L\{e^{at}\} = \dfrac{1}{s-a}$ then $L^{-1}\left\{\dfrac{1}{s-a}\right\} = e^{at}$

Since $L\{\cos at\} = \dfrac{s}{s^2+a^2}$, $\therefore\ L^{-1}\left\{\dfrac{s}{s^2+a^2}\right\} = \cos at$ etc.

We list some standard Inverse Laplace transforms as follows :

TABLE OF INVERSE LAPLACE TRANSFORMS

S.No	$\bar{f}(s)$	$L^{-1}\{\bar{f}(s)\} = f(t)$
1.	$\dfrac{1}{s}$	1
2.	$\dfrac{1}{s^{n+1}}$, n is positive integer	$\dfrac{t^n}{n!}$
3.	$\dfrac{1}{s^{n+1}}$, $n > -1$	$\dfrac{t^n}{\Gamma(n+1)}$
4.	$\dfrac{1}{s-a}$	e^{at}
5.	$\dfrac{1}{s+a}$	e^{-at}
6.	$\dfrac{1}{s^2+a^2}$	$\dfrac{1}{a}\sin at$
7.	$\dfrac{s}{s^2+a^2}$	$\cos at$

8.	$\dfrac{1}{s^2-a^2}$	$\dfrac{1}{a}\sinh at$
9.	$\dfrac{s}{s^2-a^2}$	$\cosh at$
10.	$\dfrac{1}{(s-a)^2+b^2}\ \left(\text{or } \dfrac{1}{(s+a)^2+b^2}\right)$	$\dfrac{1}{b}e^{at}\sin bt\ \left(\text{or } \dfrac{1}{b}e^{-at}\sin bt\right)$
11.	$\dfrac{s-a}{(s-a)^2+b^2}\ \left(\text{or } \dfrac{s+a}{(s+a)^2+b^2}\right)$	$e^{at}\cos bt\ \left(\text{or } e^{-at}\cos bt\right)$
12.	$\dfrac{1}{(s-a)^2-b^2}\ \left(\text{or } \dfrac{1}{(s+a)^2-b^2}\right)$	$\dfrac{1}{b}e^{at}\sinh bt\ \left(\text{or } \dfrac{1}{b}e^{-at}\sinh bt\right)$
13.	$\dfrac{s-a}{(s-a)^2-b^2}\ \left(\text{or } \dfrac{s+a}{(s+a)^2-b^2}\right)$	$e^{at}\cosh bt\ \left(\text{or } \dfrac{1}{b}e^{-at}\cosh bt\right)$
14	$\dfrac{2as}{(s^2+a^2)^2}$	$t\sin at$
15.	$\dfrac{s^2-a^2}{(s^2+a^2)^2}$	$t\cos at$

Note: Inverse Laplace Transform of a given function $\bar{f}(s)$ can be obtained either by use of the above standard results or by splitting the given function into its partial fractions and then applying the above results.

2.11 CONVOLUTION

Convolution is useful for obtaining Inverse Laplace Transform of a product of two transforms and solving ordinary differential equations.

Definition : Let $f(t)$ and $g(t)$ be two functions defined for $t > 0$. We define

$$f(t)*g(t) = \int_0^t f(u)g(t-u)du$$

assuming that the integral on the right hand side exists.

$f(t)*g(t)$ is called the convolution product of $f(t)$ and $g(t)$.

It can be proved that

(*i*) Convolution product is commutative. *i.e.* $f(t)*g(t) = g(t)*f(t)$

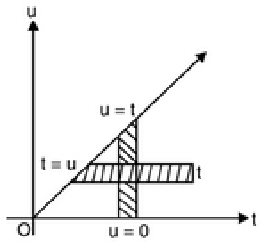
(*ii*) Convolution product is associative . *i.e.* $f(t)*(g(t)*h(t)) = (f(t)*g(t))*h(t)$

(*iii*) $f(t)*0 = 0*f(t) = 0$

Note : In general, $1*f(t) \neq f(t)$

Convolution Theorem : If $L\{f(t)\} = \bar{f}(s)$ and $L\{g(t)\} = \bar{g}(s)$ then $L\{f(t)*g(t)\} = \bar{f}(s)\cdot\bar{g}(s)$

or $L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = f(t)*g(t)$ **[JNTU 2008S, (K) June 2011 (Set No. 1), (H) June 2015]**



Proof: Let $\phi(t) = f(t)*g(t) = \int_0^t f(u)g(t-u)\,du$. Then

$$L\{\phi(t)\} = \int_0^\infty e^{-st}\left\{\int_0^t f(u)g(t-u)du\right\}dt = \int_0^\infty\int_0^t e^{-st} f(u)g(t-u)du\,dt$$

The double integral is considered within the region enclosed by the lines $u = 0$ and $u = t$.

On changing the order of integration, we get

$$L\{\phi(t)\} = \int_0^\infty\int_u^\infty e^{-st} f(u)g(t-u)dt\,du$$

$$= \int_0^\infty e^{-su} f(u)\left\{\int_u^\infty e^{-s(t-u)} g(t-u)dt\right\}du$$

$$= \int_0^\infty e^{-su} f(u)\left\{\int_0^\infty e^{-sv} g(v)dv\right\}du, \text{ on putting } t-u=v$$

$$= \int_0^\infty e^{-su} f(u)\{\bar{g}(s)\}du = \bar{g}(s)\int_0^\infty e^{-su} f(u)du = \bar{g}(s).\,\bar{f}(s)$$

$\therefore L\{\phi(t)\} = \bar{f}(s).\bar{g}(s)$ or $\phi(t) = L^{-1}\{\bar{f}(s)\,.\,\bar{g}(s)\}$ or $f(t)*g(t) = L^{-1}\{\bar{f}(s)\,.\,\bar{g}(s)\}$

Hence the theorem follows.

SOLVED EXAMPLES

Example 1 : Using Convolution theorem, find (*i*) $L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\}$ (*ii*) $L^{-1}\left\{\frac{1}{s(s^2+4)}\right\}$

Solution : (*i*) Let $\bar{f}(s) = \frac{1}{s+a}$ and $\bar{g}(s) = \frac{1}{s+b}$. Then

$$f(t) = L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$$

and $$g(t) = L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{1}{s+b}\right\} = e^{-bt}$$

$\therefore$ By Convolution theorem,

$$L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\} = L^{-1}\left\{\frac{1}{s+a}\cdot\frac{1}{s+b}\right\} = L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\}$$

$$= f(t)*g(t) = \int_0^t f(u)g(t-u)du$$

$$= \int_0^t e^{-au}\cdot e^{-b(t-u)}du = e^{-bt}\int_0^t e^{-(a-b)u}du$$

$$= e^{-bt}\left[\frac{e^{-(a-b)u}}{-(a-b)}\right]_0^t = -\frac{1}{a-b}e^{-bt}\left[e^{-(a-b)t}-1\right]$$

$$= \frac{1}{b-a}(e^{-at}-e^{-bt})$$

(*ii*) Let $\bar{f}(s)=\frac{1}{s}$ and $\bar{g}(s)=\frac{1}{s^2+4}$. Then

$$f(t)=L^{-1}\left\{\frac{1}{s}\right\}=1 \text{ and } g(t)=L^{-1}\left\{\frac{1}{s^2+2^2}\right\}=\frac{1}{2}\sin 2t$$

Applying Convolution theorem,

$$L^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+4}\right\} = L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\}$$

$$= f(t)*g(t) = \int_0^t f(u)g(t-u)du = \int_0^t 1\cdot\frac{1}{2}\sin 2(t-u)du$$

$$= \frac{1}{2}\int_0^t \sin 2(t-u)du = \frac{1}{2}\left[\frac{-\cos 2(t-u)}{-2}\right]_0^t$$

$$= \frac{1}{4}(\cos 0-\cos 2t) = \frac{1}{4}(1-\cos 2t)$$

Example 2 : Using Convolution theorem, evaluate $L^{-1}\left\{\frac{1}{s(s^2+2s+2)}\right\}$.

Solution : Since $f(t)=L^{-1}\left\{\frac{1}{s}\right\}=1$ and

$$g(t)= L^{-1}\left\{\frac{1}{s^2+2s+2}\right\} = L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} = e^{-t}L^{-1}\left\{\frac{1}{s^2+1}\right\} = e^{-t}\sin t$$

∴ By Convolution theorem, we get

$$L^{-1}\left\{\frac{1}{s(s^2+2s+2)}\right\} = L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+2s+2}\right\} = f(t)*g(t) = g(t)*f(t)$$

$$= \int_0^t g(u)f(t-u)du = \int_0^t e^{-u}\sin u\cdot 1\, du = \int_0^t e^{-u}\sin u\, du$$

$$= \left[\frac{e^{-u}}{1+1}(-\sin u-\cos u)\right]_0^t = -\frac{1}{2}\left[e^{-u}(\sin u+\cos u)\right]_0^t$$

$$= -\frac{1}{2}\left[e^{-t}(\sin t+\cos t)-1\cdot(0+1)\right] = \frac{1}{2}\left[1-e^{t}(\sin t+\cos t)\right]$$

Example 3 : Using Convolution theorem, find $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$.

Solution : Let $\bar{f}(s)=\frac{1}{s^2+a^2}$ and $\bar{g}(s)=\frac{1}{s^2+a^2}$. Then

$$f(t)=L^{-1}\left\{\frac{1}{s^2+a^2}\right\}=\frac{1}{a}\sin at \text{ and } g(t)=L^{-1}\left\{\frac{1}{s^2+a^2}\right\}=\frac{1}{a}\sin at$$

$$\therefore\quad L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}=L^{-1}\left\{\frac{1}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right\}=L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\}=f(t)*g(t)$$

$$=\int_0^t f(u)g(t-u)du=\int_0^t\frac{1}{a}\sin au\cdot\frac{1}{a}\sin a(t-u)du$$

$$=\frac{1}{2a^2}\int_0^t 2\sin\, au\, \sin\,(at-au)\, du$$

$$=\frac{1}{2a^2}\int_0^t[\cos\,(2au-at)-\cos\, at]\, du$$

$$[\because 2\sin A\sin B=\cos(A-B)-\cos(A+B)]$$

$$=\frac{1}{2a^2}\left[\frac{\sin(2au-at)}{2a}-\cos at\cdot u\right]_0^t$$

$$=\frac{1}{2a^2}\left[\frac{1}{2a}\sin at-t\cos at+\frac{1}{2a}\sin at\right]$$

Example 6 : Using Convolution theorem, find the Inverse Laplace Transform of $\frac{1}{(s^2+4)\,(s+1)^2}$.

Solution : $L^{-1}\left\{\frac{1}{(s^2+4)(s+1)^2}\right\}=L^{-1}\left\{\frac{1}{(s^2+4)(s+1)}\cdot\frac{1}{s+1}\right\}$

Consider $L^{-1}\left\{\frac{1}{(s^2+4)(s+1)}\right\}$

Since $L^{-1}\{\bar{f}(s)\}=L^{-1}\left\{\frac{1}{s^2+4}\right\}=\frac{1}{2}\sin 2t=f(t)$

and $L^{-1}\{\bar{g}(s)\}=L^{-1}\left\{\frac{1}{s+1}\right\}=e^{-t}=g(t)$,

$\therefore$ By Convolution theorem, we get

$$L^{-1}\left\{\frac{1}{(s^2+4)(s+1)}\right\}=f(t)*g(t)=\int_0^t f(u)g(t-u)\,du=\int_0^t\frac{1}{2}\sin 2ue^{-(t-u)}du$$

$$= \frac{1}{2} e^{-t} \int_0^t e^u \sin 2u \, du$$

$$= \frac{1}{2} e^{-t} \left[\frac{e^u}{1^2 + 2^2} (\sin 2u - 2\cos 2u) \right]_0^t$$

$$= \frac{1}{2} e^{-t} \left[\frac{e^t}{5} (\sin 2t - 2\cos 2t) - \frac{1}{5}(0 - 2) \right]$$

$$= \frac{1}{10} [\sin 2t - 2\cos 2t + 2e^{-t}]$$

Applying Convolution theorem again, we get

$$\mathrm{L}^{-1} \left\{ \frac{1}{(s^2+4)(s+1)^2} \right\} = \mathrm{L}^{-1} \left\{ \frac{1}{(s^2+4)\,(s+1)} \cdot \frac{1}{s+1} \right\}$$

$$= \int_0^t \frac{1}{10} (\sin 2u - 2\cos 2u + 2e^{-u})\, e^{-(t-u)} du$$

$$= \frac{e^{-t}}{10} \left[\int_0^t e^u \sin 2u \, du - 2\int_0^t e^u \cos 2u \, du + 2\int_0^t du \right]$$

$$= \frac{e^{-t}}{10} \left[\left\{ \frac{e^u}{1^2+2^2} (\sin 2u - 2\cos 2u) \right\}_0^t - 2 \left\{ \frac{e^u}{1^2+2^2} (\cos 2u + 2\sin 2u) \right\}_0^t + 2(u)_0^t \right]$$

$$= \frac{e^{-t}}{10} \left[\left\{ \frac{e^t}{5} (\sin 2t - 2\cos 2t) - \frac{1}{5}(0-2) \right\} - 2 \left\{ \frac{e^t}{5} (\cos 2t + 2\sin 2t) - \frac{1}{5}(1+0) \right\} + 2(t-0) \right]$$

$$= \frac{e^{-t}}{50} [e^t (\sin 2t - 2\cos 2t) + 2 - 2e^t (\cos 2t + 2\sin 2t) + 2 + 10t]$$

$$= \frac{e^{-t}}{50} [e^t (\sin 2t - 2\cos 2t - 2\cos 2t - 4\sin 2t) + 4 + 10t]$$

$$= \frac{e^{-t}}{50} [e^t (-3\sin 2t - 4\cos 2t) + 4 + 10t]$$

$$= \frac{e^{-t}}{50} [4 + 10t - e^t (3\sin 2t + 4\cos 2t)]$$

Example 7 : Using Laplace transform, solve $y(t) = 1 - e^{-t} + \int_0^t y(t-u)\ \sin u\, du.$

Solution : Given integral equation can be written as

$y(t) = 1 - e^{-t} + y(t) * \sin t$, using definition of convolution

Taking the Laplace Transform of both the sides, we have

$$L\{y(t)\} = L\{1\} - L\{e^{-t}\} + L\{y(t) * \sin t\}$$

$$= \frac{1}{s} - \frac{1}{s+1} + L\{y(t)\} \cdot L\{\sin t\}, \text{ using Convolution Theorem.}$$

$$= \frac{1}{s(s+1)} + L\{y(t)\} \cdot \frac{1}{s^2+1}$$

$$\Rightarrow \left(1 - \frac{1}{s^2+1}\right) L\{y(t)\} = \frac{1}{s(s+1)} \Rightarrow \frac{s^2}{s^2+1} L\{y(t)\} = \frac{1}{s(s+1)}$$

$$\Rightarrow L\{y(t)\} = \frac{s^2+1}{s^3(s+1)}$$

$$\therefore \quad y(t) = L^{-1}\left\{\frac{s^2+1}{s^3(s+1)}\right\} = L^{-1}\left\{\frac{1}{s(s+1)} + \frac{1}{s^3(s+1)}\right\}$$

$$= L^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\} + L^{-1}\left\{\frac{1}{s^3(s+1)}\right\} = 1 - e^t + L^{-1}\left\{\frac{1}{s^3(s+1)}\right\}$$

Now apply the formula $L^{-1}\left\{\frac{1}{s}\bar{f}(s)\right\}$ thrice to evaluate $L^{-1}\left\{\frac{1}{s^3(s+1)}\right\}$

[(or) Resolve $\frac{1}{s^3(s+1)}$ into partial fractions and then evaluate Inverse L.T. This part is left as an exercise to the student.]

Example 8 : Solve $\frac{dy}{dt} + 3y + 2\int_0^t y(t)dt = t$ by Laplace transform method.

[JNTU (K) June 2011 (Set No. 2)]

Solution : Taking Laplace Transform of the given equation, we get

$$L\{y'\} + 3L\{y\} + 2L\left\{\int_0^t y(t)dt\right\} = L\{t\}$$

i.e., $\left[sL\{y\} - y(0)\right] + 3L\{y\} + 2 \cdot \frac{1}{s} L\{y\} = \frac{1}{s^2}$

i.e., $\left(s + \frac{2}{s} + 3\right) L\{y\} = \frac{1}{s^2} + y(0) = \frac{1}{s^2} + a$, where $a = y(0)$, say

i.e., $\frac{s^2+3s+2}{s} L\{y\} = \frac{1}{s^2} + a \quad$ or $\quad L\{y\} = \frac{1}{s(s^2+3s+2)} + \frac{as}{s^2+2s+2}$

$$\therefore \quad y = L^{-1}\left\{\frac{1}{s(s+1)(s+2)}\right\} + a \cdot L^{-1}\left\{\frac{s}{(s+1)(s+2)}\right\}$$

$$= L^{-1}\left\{\frac{1}{2}\cdot\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\cdot\frac{1}{s+2}\right\} + aL^{-1}\left\{\frac{-1}{s+1} + \frac{2}{s+2}\right\}$$

(Resolving into partial fractions)

Example 10 : Solve the integral equation $y(t) = a\sin t - 2\int_0^t y(u)\cos(t-u)\,du$ by Laplace transform method.

Solution : By definition of convolution, the given integral equation can be written as

$y(t) = a\sin t - 2y(t) * \cos t$

Taking Laplace Transformation on both sides, we have

$L\{y(t)\} = a \,.\, L\{\sin t\} - 2 \,.\, L\{y(t) * \cos t\}$

$= a.\dfrac{1}{s^2+1} - 2 \,.\, L\{y(t)\}.\, L\{\cos t\}$, using convolution theorem

$$= \frac{a}{s^2+1} - 2L\{y(t)\}.\frac{s}{s^2+1} = \frac{a}{s^2+1} - \frac{2s}{s^2+1}L\{y(t)\}$$

$$\Rightarrow L\{y(t)\} + \frac{2s}{s^2+1}L\{y(t)\} = \frac{a}{s^2+1}$$

$$\Rightarrow \left(1 + \frac{2s}{s^2+1}\right)L\{y(t)\} = \frac{a}{s^2+1}$$

$$\Rightarrow \frac{(s+1)^2}{s^2+1}L\{y(t)\} = \frac{a}{s^2+1} \Rightarrow L\{y(t)\} = \frac{a}{(s+1)^2}$$

$\therefore \; y(t) = L^{-1}\left\{\dfrac{a}{(s+1)^2}\right\} = a \,.\, e^{-t}L^{-1}\left\{\dfrac{1}{s^2}\right\}$, using First shifting Theorem.

$= ate^{-t}$

which is the required solution.

2. 12 APPLICATIONS TO ORDINARY DIFFERENTIAL EQUATIONS

1. Solution of Ordinary Differential Equations with Constant Coefficients: Ordinary linear differential equations with constant coefficients can be easily solved by the Laplace transform method, without the necessity of first finding the general solution and then evaluating the arbitrary constants. This method is, in general, shorter than our earlier methods and is especially suitable to obtain the solution of linear non-homogeneous ordinary differential equations with constant coefficients.

Let us consider a linear differential equation with constant coefficients

$$\frac{d^n y}{dt^n}+a_1\frac{d^{n-1}y}{dt^{n-1}}+a_2\frac{d^{n-2}y}{dt^{n-2}}+....+a_{n-1}\frac{dy}{dt}+a_n y=F(t) \qquad ...(1)$$

where $F(t)$ is a function of independent variable t.

Let $y(0)=c_0, y'(0)=c_1, ..., y^{n-1}(0)=c_{n-1}$ ...(2)

be the given initial or boundary conditions, where $c_0, c_1, c_2, ..., c_{n-1}$ are constants.

If $a_1, a_2, ..., a_n$ are constants, then we use

$$L\{f^{(n)}(t)\}=s^n L\{f(t)\}-s^{n-1}f(0)-s^{n-2}f'(0)-....... -f^{n-1}(0) \qquad ...(3)$$

Taking Laplace transform of both sides of (1) and applying (3) and using conditions (2) we obtain an algebraic equation known as "subsidiary equation", from which $\bar{y}(s)=L\{y(t)\}$ is obtained. The required solution $y(t)$ is obtained by taking the inverse Laplace transform of $\bar{y}(s)$.

The special advantage of this method in solving differential equations is that the initial conditions are satisfied automatically. It is unnecessary to find the general solution and then determine the constants using the initial conditions.

Working Rule to Solve Differential Equation by Laplace Transform Method.

Step 1. Take the Laplace transform of both sides of the given differential equation.

Step 2. Use the formula

(*i*) $L\{y'(t)\}=s\bar{y}(s)-y(0)$

(*ii*) $L\{y''(t)\}=s^2\bar{y}(s)-s\cdot y(0)-y'(0)$

(*iii*) $L\{y'''(t)\}=s^3\bar{y}(s)-s^2\cdot y(0)-s\cdot y'(0)-y''(0)$

Step 3. Replace $y(0)$, $y'(0)$, $y''(0)$ with the given initial conditions.

Step 4. Transpose the terms with minus signs to the right.

Step 5. Divide by the coefficient of $\bar{y}$, getting $\bar{y}$ as a known function of s.

Step 6. Resolve this function of s (obtained in step 5) into partial fractions.

Step 7. Take the Inverse *L.T.* of $\bar{y}$ obtained in step 5. This gives y as a function of t which is the required solution of the given equation satisfying the given initial conditions.

SOLVED EXAMPLES

Example 1 : Solve $y'' + 4y = 0, y(0) = 1, y'(0) = 6$ using Laplace transform.

Solution : Given differential equation is $y'' + 4y = 0$ ... (1)

Taking Laplace Transform on both sides of (1), we get

$$L\{y'' + 4y\} = L\{0\} \Rightarrow L\{y''\} + 4L\{y\} = 0$$

$$\Rightarrow [s^2 L\{y\} - s.y(0) - y'(0)] + 4L\{y\} = 0$$

Using the given conditions, it reduces to

$$[s^2 L\{y\} - s - 6] + 4L\{y\} = 0$$

$$\Rightarrow (s^2 + 4)L\{y\} = s + 6$$

$$\Rightarrow L\{y\} = \frac{s+6}{s^2+4}$$

$$\therefore \; y = L^{-1}\left\{\frac{s+6}{s^2+4}\right\} = L^{-1}\left\{\frac{s}{s^2+4} + \frac{6}{s^2+4}\right\}$$

$$= L^{-1}\left\{\frac{s}{s^2+2^2}\right\} + 6 \,.\, L^{-1}\left\{\frac{1}{s^2+2^2}\right\}$$

$$= \cos 2t + 6.\frac{1}{2}\sin 2t = \cos 2t + 3\sin 2t$$

Example 3 : Solve the following initial value problem by using Laplace transform $4y'' + \pi^2 y = 0$, $y(0) = 2, y'(0) = 0$.

Solution : Given equation is $4y'' + \pi^2 y = 0$

Taking Laplace transform on both sides, we have

$$4L\{y''\} + \pi^2 L\{y\} = 0$$

i.e., $4[s^2 L\{y\} - sy(0) - y'(0)] + \pi^2 L[y] = 0 \Rightarrow \quad 4[s^2 L\{y\} - 2s] + \pi^2 L\{y\} = 0$

i.e., $L\{y\}[4s^2 + \pi^2] = 8s$ or $L\{y\} = \dfrac{8s}{4s^2 + \pi^2}$

$$\therefore \; y = 2L^{-1}\left\{\frac{s}{s^2 + \frac{\pi^2}{4}}\right\} = 2\cos\left(\frac{\pi t}{2}\right)$$

which is the required solution.

Example 5 : Using Laplace transform, solve

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}-3y = \sin t,\ y=\frac{dy}{dt}=0, \text{ when } t=0.$$

(or) $(D^2+2D-3)\,y=\sin x,\ y(0)=\ y'(0)=0$.

Solution : Given differential equation is $\frac{d^2y}{dt^2}+2\frac{dt}{dt}-3y=\sin t$

Taking Laplace Transform on both sides, $L\{y''\}+2L\{y\}-3L\{y\}=\frac{1}{s^2+1}$

$$\Rightarrow\quad [s^2L\{y\}-s\,y(0)-y'(0)]+2[sL\{y\}-y(0)]-3L\{y\}=\frac{1}{s^2+1}$$

Using the given conditions, it reduces to

$$s^2L\{y\}+2sL\{y\}-3L\{y\}=\frac{1}{s^2+1}$$

$$\Rightarrow\quad L\{y\}[s^2+2s-3]=\frac{1}{s^2+1}\quad\Rightarrow\quad L\{y\}=\frac{1}{(s^2+1)(s+3)(s-1)}$$

$$\Rightarrow\quad L\{y\}=\frac{\frac{-s}{10}-\frac{1}{5}}{s^2+1}-\frac{1}{40(s+3)}+\frac{1}{8(s-1)}$$, resolving into partial fractions

Taking Inverse Laplace transform, $y=\frac{-1}{10}\cos t-\frac{1}{5}\sin t-\frac{1}{40}e^{-3t}+\frac{1}{8}e^{t}$.

Example 7 : Solve the following differential equation using the Laplace transform

$$\frac{d^2y}{dt^2}+\frac{2dy}{dt}+2y=5\sin t,\ y(0)=y'(0)=0$$

Solution : Given equation is

$$y''+2y'+2y=5\sin t \qquad \dots(1)$$

Taking the Laplace Transform of both sides of (1), we get

$$L\{y''\}+2\cdot L\{y'\}+2\cdot L\{y\}=5L\{\sin t\}$$

i.e., $[s^2L\{y\}-sy(0)-y'(0)]+2[sL\{y\}-y(0)]+2L\{y\}=5\cdot\frac{1}{s^2+1}$

Using the given conditions, it reduces to

$$(s^2+2s+2)L\{y\}=\frac{5}{s^2+1}$$

$$\Rightarrow\quad L\{y\}=\frac{5}{(s^2+1)(s^2+2s+2)}$$

$$\therefore\quad y\ =L^{-1}\left\{\frac{5}{(s^2+1)(s^2+2s+2)}\right\}$$

$$=L^{-1}\left\{\frac{-2s+1}{s^2+1}+\frac{2s+3}{s^2+2s+2}\right\}$$ (Resolving into partial fractions)

$$= (-2)L^{-1}\left\{\frac{s}{s^2+1}\right\} + L^{-1}\left\{\frac{1}{s^2+1}\right\} + L^{-1}\left\{\frac{2(s+1)+1}{(s+1)^2+1}\right\}$$

$$= -2\cos t + \sin t + e^{-t}L^{-1}\left\{\frac{2s+1}{s^2+1}\right\}, \text{ using First Shifting Theorem}$$

$$= -2\cos t + \sin t + e^{-t}\left[2\cdot L^{-1}\left\{\frac{s}{s^2+1}\right\} + L^{-1}\left\{\frac{1}{s^2+1}\right\}\right]$$

$$= -2\cos t + \sin t + e^{-t}(2\cos t + \sin t)$$

Example 9 : Solve the initial value problem

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = e^{-t};\ x(0) = 0,\ x'(0) = 1.$$

(or) $y'' + 3y' + 2y = e^{-t}, y(0) = 0, y'(0) = 1$ using Laplace transform

Solution : Given differential equation is $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = e^{-t}$

Taking Laplace transform on both sides, we have

$L\{x''\} + 3L\{x'\} + 2L\{x\} = L\{e^{-t}\}$

$\Rightarrow\ [s^2L\{x\} - s.x(0) - x'(0)] + 3[sL\{x\} - x(0)] + 2L\{x\} = \frac{1}{s+1}$ ... (1)

$\Rightarrow\ [s^2L\{x\} - 1] + 3[sL\{x\}] + 2L\{x\} = \frac{1}{s+1}$, using given conditions

$\Rightarrow\ L\{x\}[s^2 + 3s + 2] = \frac{1}{s+1} + 1$

$\Rightarrow\ L\{x\} = \frac{s+2}{(s+1)^2(s+2)} = \frac{1}{(s+1)^2}$

$\Rightarrow\ x = L^{-1}\left\{\frac{1}{(s+1)^2}\right\} = t.e^{-t}$ [Taking inverse Laplace transform]

Example 15 : Solve the differential equation $\frac{d^2x}{dt^2} + 9x = \sin t$ using Laplace transform given that (*i*) $x(0) = 1,\ x\left(\frac{\pi}{2}\right) = 1$.

(*ii*) $x(0) = 1, x'(0) = 0$

Solution : (*i*) Note that the condition $x'(0)$ is not given. So let it be 'a'.

Given equation can be written as

$x'' + 9x = \sin t$

Taking Laplace Transform on both sides, we get

$L\{x''\} + 9L\{x\} = L\{\sin t\}$

i.e. $[s^2L\{x\} - s \cdot x(0) - x'(0)] + 9L\{x\} = \dfrac{1}{s^2+1}$

Using the given conditions, it reduces to

$$s^2L\{x\} - s - a + 9L\{x\} = \frac{1}{s^2+1} \quad [\text{Take } x'(0) = a]$$

i.e. $(s^2+9)\,L\{x\} = \dfrac{1}{s^2+1} + s + a$

or $L\{x\} = \dfrac{1}{(s^2+1)(s^2+9)} + \dfrac{s+a}{s^2+9} = \dfrac{1}{8}\left[\dfrac{1}{s^2+1} - \dfrac{1}{s^2+9}\right] + \dfrac{s}{s^2+9} + \dfrac{a}{s^2+9}$

$$\therefore x = \frac{1}{8}\,L^{-1}\left\{\frac{1}{s^2+1} - \frac{1}{s^2+9}\right\} + L^{-1}\left\{\frac{s}{s^2+3^2}\right\} + a \cdot L^{-1}\left\{\frac{1}{s^2+3^2}\right\}$$

$$= \frac{1}{8}\,(\sin t - \frac{1}{3}\sin 3t) + \cos 3t + \frac{a}{3}\sin 3t \qquad \text{...(1)}$$

$$= \frac{1}{8}\,\sin t + \cos 3t + \frac{1}{3}\left(a - \frac{1}{8}\right)\sin 3t$$

Given $x\left(\dfrac{\pi}{2}\right) = 1$

$$\Rightarrow \quad 1 = \frac{1}{8}\sin\frac{\pi}{2} + \cos\frac{3\pi}{2} + \frac{1}{3}\left(a - \frac{1}{8}\right)\sin\frac{3\pi}{2} = \frac{1}{8} + 0 + \frac{1}{3}\left(a - \frac{1}{8}\right)(-1)$$

$$\Rightarrow \quad 1 = \frac{1}{6} - \frac{a}{3} \Rightarrow \frac{a}{3} = \frac{1}{6} - 1 = \frac{-5}{6} \qquad \text{...(2)}$$

Hence the solution is

$$x = \frac{1}{8}(\sin t - \frac{1}{3}\sin 3t) + \cos 3t - \frac{5}{6}\sin 3t \quad [\text{ From (1) and (2) }]$$

(ii) This is left as an exercise to the reader.

Example 22 : Using Laplace transform, solve $(D^2 + 5D - 6)\ y = x^2 e^{-x}$, $y(0) = a, y'(0) = b$.

Solution : Given equation can be written as $y'' + 5y' - 6y = x^2 e^{-x}$ ... (1)

Taking the Laplace Transform of both sides, we get

$$L\{y''\} + 5L\{y'\} - 6L\{y\} = L\{x^2 e^{-x}\}$$

i.e, $[s^2 L\{y\} - s.y(0) - y'(0)] + 5\{s.L\{y\} - y(0)] - 6.L\{y\} = \left(\frac{2}{s^3}\right)_{s \to s+1}$

Using the given conditions, it reduces to

$$[s^2 L\{y\} - as - b] + 5\,[s.L\{y\} - a] - 6.\ L\ \{y\} = \frac{2}{(s+1)^3}$$

$$i.e.\ (s^2 + 5s - 6)\ L\{y\} - as - 5a - b = \frac{2}{(s+1)^3}$$

$$i.e,\ (s^2 + 5s - 6)\ L\{y\} = \frac{2}{(s+1)^3} + as + 5a + b$$

or $\quad L\{y\} = \dfrac{2}{(s+6)(s-1)(s+1)^3} + \dfrac{as}{(s+6)(s-1)} + \dfrac{5a+b}{(s+6)(s-1)}$

$$\therefore \quad y = L^{-1}\left\{\frac{2}{(s+6)(s-1)(s+1)^3}\right\} + a \cdot L^{-1}\left\{\frac{s}{(s+6)(s-1)}\right\}$$

$$+ (5a+b)\ L^{-1}\left\{\frac{1}{(s+6)(s-1)}\right\}$$

$$= L^{-1}\left\{\frac{2}{(s+6)(s-1)(s+1)^3}\right\} + \frac{a}{7} \cdot L^{-1}\left\{\frac{s}{s-1} - \frac{s}{s+6}\right\}$$

$$+(5a+b).\frac{1}{7} L^{-1}\left\{\frac{1}{s-1} - \frac{1}{s+6}\right\}$$

$$= L^{-1}\left\{\frac{2}{(s+1)(s-1)(s+1)^3}\right\} + \frac{a}{7} \cdot L^{-1}\left\{\frac{s}{s-1} - \frac{s}{s+6}\right\} + (5a+b) \cdot \frac{1}{7}(e^t - e^{-6t})$$

To evaluate the first term, use partial fractions and to evaluate the second term, use $L^{-1}\{s\bar{f}(s)\} = f'(t)$ if $f(0) = 0$

This is left as an exercise to the student.

Theorem I: *If F(t) is sectionally continuous and of exponential order a and such that* $\lim_{t \to 0} \frac{F(t)}{t}$ *exists then for* $p > a$, $L^{-1}\left[\frac{f(p)}{p}\right] = \int_0^t F(x)\,dx.$

Proof: Let $G(t) = \int_0^t F(x)\,dx$.

Then $G'(t) = F(t)$ and $G(0) = 0$.

$\therefore$ $L\{G'(t)\} = p\,L\{G(t)\} - G(0) = p\,L\{G(t)\}$

or $L\{F(t)\} = f(p) = p\,L\{G(t)\}$.

$\therefore$ $L\{G(t)\} = \frac{f(p)}{p}$. Hence $L^{-1}\left[\frac{f(p)}{p}\right] = G(t) = \int_0^t F(x)\,dx.$

Theorem II: $L^{-1}\left[\frac{f(p)}{p^2}\right] = \int_0^t \int_0^y F(x)\,dx\,dy.$

Proof: Let $G(t) = \int_0^t \int_0^y F(x)\,dx\,dy$.

Then $G'(t) = \int_0^t F(x)\,dx$, $G''(t) = F(t)$. Also $G(0) = G'(0) = 0$.

Now $L\{G''(t)\} = p^2\,L\{G(t)\} - pG(0) - G'(0) = p^2\,L\{G(t)\}$

or $L\{F(t)\} = f(p) = p^2\,L\{G(t)\}$. $\therefore$ $L\{G(t)\} = \frac{f(p)}{p^2}$

or $L^{-1}\left\{\frac{f(p)}{p^2}\right\} = G(t) = \int_0^t \int_0^y F(x)\,dx\,dy$.

This may also be written as $L^{-1}\left\{\frac{f(p)}{p^2}\right\} = \int_0^t \int_0^t F(t)\,dt^2$.

In general $L^{-1}\left\{\frac{f(p)}{p^n}\right\} = \int_0^t \int_0^t \cdots \int_0^t F(t)\,dt^n$.

Solution of Partial Differential Equations

Laplace transform is also useful in solving partial differential equations when the boundary conditions are given.

Laplace transforms of some partial derivatives:

Theorem: *If* $y(x,t)$ *is a function of* x *and* t, *then*

$$(a)\ L\left\{\frac{\partial y}{\partial t}\right\} = p\,\bar{y}(x,p) - y(x,0)$$

$$(b)\ L\left\{\frac{\partial^2 y}{\partial t^2}\right\} = p^2\,\bar{y}(x,p) - p\,y(x,0) - y_t(x,0)$$

$$(c)\ L\left\{\frac{\partial y}{\partial x}\right\} = \frac{d\,\bar{y}}{dx} \quad and \quad (d)\ L\left\{\frac{\partial^2 y}{\partial x^2}\right\} = \frac{d^2\bar{y}}{dx^2} \quad where\ L\{y(x,t)\} = \bar{y}(x,p).$$

Proof: (a)
$$L\left\{\frac{\partial y}{\partial t}\right\} = \int_0^\infty e^{-pt}\frac{\partial y}{\partial t}\,dt = \lim_{s\to\infty}\int_0^s e^{-pt}\frac{\partial y}{\partial t}\,dt$$

$$= \lim_{s\to\infty}\left[\left\{e^{-pt}\,y(x,t)\right\}_0^s + p\int_0^s e^{-pt}\,y(x,t)\,dt\right]$$

$$= p\int_0^\infty e^{-pt}\,y(x,t)\,dt - y(x,0) = p\bar{y}(x,p) - y(x,0)$$

(b) Let $V = \dfrac{\partial y}{\partial t}$.

$$\therefore \quad L\left\{\frac{\partial^2 y}{\partial t^2}\right\} = L\left\{\frac{\partial V}{\partial t}\right\} = p\,.\,L\{V\} - V(x,0)$$

$$= p\,[\,pL\{y\} - y\ (x,0)\,] - y_t(x,0) \qquad \left[\because\ V = \frac{\partial y}{\partial t} = y_t\right]$$

$$= p^2\,\bar{y}(x,p) - py(x,0) - y_t(x,0).$$

(c)
$$L\left\{\frac{\partial y}{\partial x}\right\} = \int_0^\infty e^{-pt}\frac{\partial y}{\partial x}\,dt = \frac{d}{dx}\int_0^\infty e^{-pt}\,y\,dt = \frac{d\,\bar{y}}{dx}.$$

(d)
$$L\left\{\frac{\partial^2 y}{\partial x^2}\right\} = L\left\{\frac{\partial U}{\partial x}\right\}, \text{where } U = \frac{\partial y}{\partial x}$$

$$= \frac{d}{dx}L\{U\} = \frac{d}{dx}L\left\{\frac{\partial y}{\partial x}\right\} = \frac{d}{dx}\left(\frac{d\,\bar{y}}{dx}\right) = \frac{d^2\bar{y}}{dx^2}.$$

Example 16: *Solve* $\frac{\partial y}{\partial t} = 2\frac{\partial^2 y}{\partial x^2}$ *where* $y(0,t) = 0 = y(5,t)$ *and* $y(x,0) = 10 \sin 4\pi x$.

Solution: Taking the Laplace transform of both sides of given equation, we have

$$L\left\{\frac{\partial y}{\partial t}\right\} = 2L\left\{\frac{\partial^2 y}{\partial x^2}\right\} \quad \text{or} \quad p\,\bar{y} - y(x,0) = 2\frac{d^2\bar{y}}{dx^2}$$

or $$\frac{d^2\bar{y}}{dx^2} - \frac{p}{2}\bar{y} = -5\sin 4\pi x \text{ whose general solution is}$$

$$\bar{y} = C_1 e^{\sqrt{(p/2)}\,x} + C_2\, e^{-\sqrt{(p/2)}\,x} - \frac{5\sin 4\pi x}{-(4\pi)^2 - p/2}$$

or $$\bar{y} = C_1 e^{\sqrt{(p/2)}\,x} + C_2\, e^{-\sqrt{(p/2)}\,x} + \frac{10}{32\pi^2 + p}\sin 4\pi x \qquad \ldots(1)$$

But $y(0,t) = 0 = y(5,t)$. Therefore, $\bar{y}(0,p) = 0$, $\bar{y}(5,p) = 0$.

$\therefore$ from (1), we have $0 = C_1 + C_2$

and $$0 = C_1 e^{5\sqrt{(p/2)}} + C_2 e^{-5\sqrt{(p/2)}} + \frac{10}{32\pi^2 + p}\sin 20\pi$$

$$= C_1 e^{5\sqrt{(p/2)}} + C_2 e^{-5\sqrt{(p/2)}} + 0.$$

Solving $C_1 = 0 = C_2$.

$\therefore$ from (1), we have $\bar{y} = \frac{10}{32\pi^2 + p}\sin 4\pi x$.

$\therefore$ $y = L^{-1}\left\{\frac{10}{32\pi^2 + p}\sin 4\pi x\right\}$ or $y = 10e^{-32\pi^2 t}\sin 4\pi x$,

which is the required solution.

Example 18: *Solve* $\frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} = xt$ *where* $y = 0 = \frac{\partial y}{\partial t}$ *at* $t = 0$ *and* $y(0, t) = 0$.

Solution: Taking the Laplace transform of both the sides of the given equation, we have

$$L\left\{\frac{\partial^2 y}{\partial x^2}\right\} - L\left\{\frac{\partial^2 y}{\partial t^2}\right\} = L\{xt\}$$

or $$\frac{d^2 \bar{y}}{dx^2} - [p^2 \bar{y}(x, p) - p\, y(x, 0) - y_t(x, 0)] = x\, L\{t\}$$

or $$\frac{d^2 \bar{y}}{dx^2} - p^2 \bar{y} = x/p^2$$ whose general solution is

$$\bar{y} = c_1 e^{px} + c_2 e^{-px} - x/p^4.$$

Since $\bar{y} = 0$ for all values of x, therefore, $c_1 = 0$, otherwise

$\bar{y} = \infty$, as $x \to \infty$.

$\therefore$ $$\bar{y} = c_2 e^{-px} - x/p^4.$$

Again $\bar{y} = 0$ when $x = 0$; $\therefore$ $c_2 = 0$.

$$\bar{y} = -x/p^4 \quad \text{or} \quad y = -L^{-1}\{x/p^4\} = -\frac{1}{6} xt^3,$$

which is the required solution.

FOURIER TRANSFORM AND SERIES

FOURIER TRANSFORM

Dirichlet's Conditions

A function $f(x)$ is said to satisfy Dirichlet conditions in the interval (a, b), if

(i) $f(x)$ is defined and is single-valued except possibly at a finite number of points in the interval (a, b), and

(ii) $f(x)$ and $f'(x)$ are piecewise continuous in the interval (a, b).

These conditions play an important role in the study of Fourier series and Fourier Transforms.

Fourier Transform or Complex Fourier Transform

Let $f(x)$ be a function defined on $(-\infty, \infty)$ and be piecewise continuous in each finite partial interval and absolutely integrable in $(-\infty, \infty)$, then

$$F\{f(x)\} = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{ipx} f(x)\, dx$$

is called the *Fourier Transform* of $f(x)$ and is denoted by $F\{f(x)\}$ or $\tilde{f}(p)$.

The function $f(x)$ is called the *inverse Fourier transform* of $\tilde{f}(p)$ *i.e.*,

$$f(x) = F^{-1}\{\tilde{f}(p)\}.$$

Inversion Theorem for Complex Fourier Transform

If $\tilde{f}(p)$ is the Fourier transform of $f(x)$ and if $f(x)$ satisfies the Dirichlet conditions in every finite interval $(-l, l)$ and further if $\int_{-\infty}^{\infty} |f(x)|\,dx$ is convergent, then at every point of continuity of $f(x)$,

$$f(x) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \tilde{f}(p)\, e^{-ipx}\, dp.$$

Proof: From Fourier integral formula, we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(v) \left[\int_{-\infty}^{\infty} e^{iw(x-v)} dw \right] dv$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iwx}\, dw \int_{-\infty}^{\infty} f(v)\, e^{-iwv}\, dv$$

$$= \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{-ipx}\, dp \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(x)\, e^{ipx}\, dx,$$

putting $w = -p$ so that $dw = -dp$

$$= \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{-ipx}\, \tilde{f}(p)\, dp.$$

Note: Some authors also define the Fourier transform in the following forms :

(1) $\tilde{f}(p) = \int_{-\infty}^{\infty} e^{-ipx} f(x)\, dx$

and $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(p) \cdot e^{ipx}\, dp.$

(2) $\tilde{f}(p) = \int_{-\infty}^{\infty} e^{ipx} f(x)\, dx$

and $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ipx}\, \tilde{f}(p)\, dp.$

(3) $\tilde{f}(p) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{-ipx} f(x)\, dx$

and $f(x) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \tilde{f}(p) \cdot e^{ipx}\, dp.$

Linearity Property of Fourier Transform

If $\tilde{f}(p)$ and $\tilde{g}(p)$ are Fourier transforms of $f(x)$ and $g(x)$ respectively, then

$$F\{a f(x) + bg(x)\} = a\tilde{f}(p) + b\tilde{g}(p),$$

where a and b are constants.

Proof: We have

$$F\{f(x)\} = \tilde{f}(p) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{ipx} f(x)\,dx$$

and
$$F\{g(x)\} = \tilde{g}(p) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{ipx} g(x)\,dx.$$

$\therefore$
$$F\{a f(x) + b\, g(x)\}$$
$$= \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{ipx} \{af(x) + bg(x)\}\,dx$$
$$= \frac{a}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{ipx} f(x)\,dx + \frac{b}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{ipx} g(x)\,dx$$
$$= a\tilde{f}(p) + b\tilde{g}(p).$$

Change of Scale Property

Theorem 1: (For Complex Fourier Transform). *If $\tilde{f}(p)$ is the complex Fourier transform of $f(x)$, the complex Fourier transform of $f(ax)$ is $\frac{1}{a}\tilde{f}\left(\frac{p}{a}\right)$.*

Proof. We have

$$\tilde{f}(p) = F\{f(x)\} = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{ipx} f(x)\,dx. \qquad \ldots(1)$$

Now
$$F\{f(ax)\} = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{ipx} f(ax)\,dx$$
$$= \frac{1}{a}\frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{ip(t/a)} f(t)\,dt,$$

putting $ax = t$ so that $dx = \frac{1}{a}dt$

$$= \frac{1}{a}\cdot\frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{i(p/a)t} f(t)\,dt$$
$$= \frac{1}{a}\tilde{f}\left(\frac{p}{a}\right), \text{ from (1).}$$

Shifting Property

If $\tilde{f}(p)$ is the complex Fourier transform of $f(x)$, then the complex Fourier transform of $f(x-a)$ is $e^{ipa}\tilde{f}(p)$.

Proof: We have

$$\tilde{f}(p) = F\{f(x)\} = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} e^{ipx} f(x)\,dx \qquad \ldots(1)$$

Now $$F\{f(x-a)\} = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} e^{ipx} f(x-a)\,dx = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} e^{ip(a+t)} f(t)\,dt,$$

putting $x-a=t$ so that $dx=dt$

$$= e^{ipa}\cdot\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} e^{ipt} f(t)\,dt$$

$$= e^{ipa}\tilde{f}(p), \text{ from (1).}$$

Modulation Theorem

If $\tilde{f}(p)$ is the Complex Fourier transform of $f(x)$, then the Fourier transform of $f(x)\cos ax$ is

$$\frac{1}{2}[\tilde{f}(p-a)+\tilde{f}(p+a)].$$

Proof. We have

$$\tilde{f}(p) = F\{f(x)\} = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} e^{ipx} f(x)\,dx \qquad \ldots(1)$$

Now $F\{f(x)\cos ax\}$

$$= \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} e^{ipx}.f(x)\cdot\frac{e^{iax}+e^{-iax}}{2}\,dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} e^{i(p+a)x} f(x)\,dx + \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} e^{i(p-a)x} f(x)\,dx\right]$$

$$= \frac{1}{2}[\tilde{f}(p+a)+\tilde{f}(p-a)].$$

Convolution

The function

$$H(x) = F * G = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} F(u).G(x-u)\,du$$

is called the **convolution** or **Falting** of two integrable functions F and G over the interval $(-\infty, \infty)$.

Note: Some authors also define

$$F * G = \int_{-\infty}^{\infty} F(u).G(x-u)\,du.$$

The Convolution or Falting Theorem for Fourier Transforms

If $F\{f(x)\}$ and $F\{g(x)\}$ are the Fourier transforms of the functions $f(x)$ and $g(x)$ respectively, then the Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms

i.e. $$F\{f(x) * g(x)\} = F\{f(x)\}.F\{g(x)\}.$$

Proof: We have $F\{f(x) * g(x)\}$

$$= F\left\{\frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(u)\cdot g(x-u)\,du\right\}$$

$$= \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(u)\,g(x-u)\,du\right] e^{ipx}\,dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} g(x-u)\,e^{ipx}\,dx\right] du$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} g(y)\,e^{ip(u+y)}dy\right] du,$$

putting $x - u = y$ so that $dx = dy$,

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} e^{ipu}\,g(y)\,e^{ipy}\,dy\right] du$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) \left[e^{ipu} \int_{-\infty}^{\infty} g(y)\,e^{ipy}\,dy\right] du$$

$$= \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(u) \left[e^{ipu} \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} g(x)\,e^{ipx}\,dx\right] du$$

$$= \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(u)\,[e^{ipu}\,F\{g(x)\}]\,du$$

$$= \left[\frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(u)\, e^{ipu}\, du \right] F\{g(x)\}$$

$$= \left[\frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(x)\, e^{ipx}\, dx \right] F\{g(x)\}$$

$$= F\{f(x)\} \cdot F\{g(x)\}\ .$$

Fourier Transform of The Derivatives of A Function

(a) The Fourier transform of $f'(x)$, the derivative of $f(x)$ is $-ip\,\tilde{f}(p)$, where $\tilde{f}(p)$ is the Fourier transform of $f(x)$.

Proof: By definition

$$F\{f'(x)\} = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f'(x) \cdot e^{ipx}\, dx$$

$$= \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \cdot e^{ipx}\, dx$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \frac{f(x+h)}{h} \cdot e^{ipx}\, dx - \lim_{h \to 0} \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \frac{f(x)}{h} \cdot e^{ipx}\, dx$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \frac{f(x+h)}{h} \cdot e^{ip(x+h)} e^{-iph} d(x+h) - \lim_{h \to 0} \frac{\tilde{f}(p)}{h}$$

$$= \lim_{h \to 0} \frac{e^{-iph}}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \frac{1}{h} \cdot f(y)\, e^{ipy}\, dy - \lim_{h \to 0} \frac{\tilde{f}(p)}{h}$$

$$= \lim_{h \to 0} \frac{e^{-iph}\, \tilde{f}(p)}{h} - \lim_{h \to 0} \frac{\tilde{f}(p)}{h}$$

$$= \tilde{f}(p) \cdot \lim_{h \to 0} \frac{e^{-iph} - 1}{h} = (-ip)\, \tilde{f}(p).$$

(b) The Fourier transform of $f^n(x)$, the nth derivative of $f(x)$ is $(-ip)^n$ times the Fourier transform of $f(x)$ provided that the first $(n-1)$ derivatives of $f(x)$ vanish as $x \to \pm\infty$.

Proof: By definition

$$F\{f^n(x)\} = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f^n(x) \,.\, e^{ipx}\, dx.$$

Integrating by parts, we have

$$F\{f^n(x)\} = \frac{1}{\sqrt{(2\pi)}}[f^{n-1}(x)\cdot e^{ipx}]_{-\infty}^{\infty} - \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f^{n-1}(x)\, ipe^{ipx}\, dx$$

$$= \frac{(-ip)}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f^{n-1}(x)\, e^{ipx}\, dx, \text{ since } \lim_{x\to\pm\infty} f^{n-1}(x) = 0.$$

Repeating the same process of integration by parts $(n-1)$ times more, we have

$$F\{f^n(x)\} = (-ip)^n \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(x)\, e^{ipx}\, dx$$

or $$\tilde{f}^n(p) = (-ip)^n \tilde{f}(p).$$

Illustrative Examples

Example 1: *Find the Fourier complex transform of $f(x)$, if*

$$f(x) = \begin{cases} e^{i\omega x} & a < x < b \\ 0 & x < a,\ x > b. \end{cases}$$

Solution: We have

$$F\{f(x)\} = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} e^{ipx} f(x)\, dx$$

$$= \frac{1}{\sqrt{(2\pi)}}\left[\int_{-\infty}^{a} 0\cdot e^{ipx}\, dx + \int_{a}^{b} e^{ipx}\cdot e^{i\omega x} dx + \int_{b}^{\infty} 0\cdot e^{ipx} dx\right]$$

$$= \frac{1}{\sqrt{(2\pi)}}\int_{a}^{b} e^{i(p+\omega)x} dx = \frac{1}{\sqrt{(2\pi)}}\cdot\left[\frac{e^{i(p+\omega)x}}{i(p+\omega)}\right]_a^b$$

$$= \frac{1}{\sqrt{(2\pi)}}\left|\frac{e^{i(p+\omega)a} - e^{i(p+\omega)b}}{p+\omega}\right|.$$

Example 2: *Find the Fourier transform of $F(x)$ defined by*

$$F(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a. \end{cases}$$

and hence evaluate

(a) $\int_{-\infty}^{\infty} \frac{\sin pa \cos px}{p} dp,$ *and* (b) $\int_{0}^{\infty} \frac{\sin p}{p} dp.$

Solution: We have

$$\tilde{F}(p) = \frac{1}{\sqrt{2\pi)}} \int_{-\infty}^{\infty} e^{ipx} F(x)\, dx$$

$$= \frac{1}{\sqrt{(2\pi)}} \int_{-a}^{a} e^{ipx}\, dx = \frac{1}{\sqrt{(2\pi)}} \left[\frac{e^{ipx}}{ip} \right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{(2\pi)}} \left(\frac{e^{ipa}}{ip} - \frac{e^{-ipa}}{ip} \right)$$

$$= \frac{2i \sin pa}{ip\sqrt{(2\pi)}} = \frac{2 \sin pa}{p\sqrt{(2\pi)}}, \; p \neq 0.$$

For $p = 0$, $\tilde{F}(p) = 2a / \sqrt{(2\pi)}$.

Comprehensive Exercise 1

1. (i) Find the Fourier transform of $f(x)$, if

$$f(x) = \begin{cases} \dfrac{\sqrt{(2\pi)}}{2\varepsilon}, & |x| \le \varepsilon \\ 0, & |x| > \varepsilon. \end{cases}$$

(ii) Find the Fourier transform of

$$f(x) = \begin{cases} x, & |x| \le a \\ 0, & |x| > a. \end{cases}$$

2. (i) Show that the Fourier transform of $f(x) = e^{-x^2/2}$ is $e^{-p^2/2}$.

(ii) Find the Fourier transform of the function

$$f(x) = \begin{cases} 1 + \dfrac{x}{a}, & \text{for } -a < x < 0 \\ 1 - \dfrac{x}{a}, & \text{for } 0 < x < a \\ 0, & \text{other wise.} \end{cases}$$

3. (i) Find the Fourier transform of $F(x) = \begin{cases} 1 - x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$

Example 4: *Use the method of Fourier transform to determine the displacement* $y(x,t)$ *of an infinite string, given that the string is initially at rest and that the initial displacement is* $f(x)$*,* $-\infty < x < \infty$*. Show that the solution can also be put in the form*

$$y(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)].$$

Solution: Displacement of a string is governed by one dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \ldots(1)$$

where $y(x,t)$ is the displacement at any time t.

$$-\infty < x < \infty,\ t > 0$$

and $$c^2 = \frac{T}{\rho}.$$

Taking the Fourier transform of both the sides of equation (1), we have

$$\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} \frac{\partial^2 y}{\partial t^2} e^{ipx}\,dx = c^2 \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} \frac{\partial^2 y}{\partial x^2} e^{ipx}\,dx$$

or $$\frac{d^2}{dt^2}\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} y e^{ipx}\,dx = c^2 (-ip)^2\, \tilde{y}(p,t)$$

or $$\frac{d^2 \tilde{y}(p,t)}{dt^2} = -c^2 p^2 \tilde{y}(p,t)$$

or $$\frac{d^2 \tilde{y}(p,t)}{dt^2} + c^2 p^2 \tilde{y}(p,t) = 0$$

whose solution is

$$\tilde{y}(p,t) = A\cos cpt + B\sin cpt. \qquad \ldots(2)$$

Initially the string is at rest.

$\therefore$ $$\frac{\partial y}{\partial t} = 0, \text{ at } t = 0.$$

$\therefore$ $$\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} \frac{\partial y}{\partial t}\cdot e^{ipx}\,dx = \frac{d}{dt}\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} y e^{ipx}\,dx$$

$$= \frac{d\,\tilde{y}(p,t)}{dt} = 0, \text{at } t = 0.$$

$\therefore$ from (2), we have $0 = Bcp$ or $B = 0$.

Also at $t = 0$, $y = f(x)$.

$\therefore$ at $t = 0$, $$\tilde{y}(p,0) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(u)\,e^{ipu}\,du = \tilde{f}(p).$$

$\therefore$ from (2), we have $\tilde{f}(p) = A$.

Hence $\tilde{y}(p,t) = \tilde{f}(p)\cos cpt$.

Taking the inverse Fourier transform, we have

$$y(x,t) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} \tilde{f}(p)\cos cpt\, e^{-ipx}\, dp$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)\, e^{ipu}\, du\right]\cos cpt\, e^{-ipx} dp$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)\, e^{ipu}\, du\right](e^{icpt} + e^{-icpt})\, e^{-ipx} dp$$

$$= \frac{1}{2}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(u)\, e^{-i\alpha u} du\right\}(e^{-ic\alpha t} + e^{ic\alpha t})\, e^{i\alpha x} d\alpha\right],$$

putting $p = -\alpha$ so that $dp = -d\alpha$

$$= \frac{1}{2}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\, e^{-i\alpha u}\left\{\int_{-\infty}^{\infty} e^{i\alpha(x+ct)} d\alpha\right\} du + \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\, e^{-i\alpha u}\left\{\int_{-\infty}^{\infty} e^{i\alpha(x-ct)}\, d\alpha\right\} du\right]$$

$$= \frac{1}{2}[f(x+ct) + f(x-ct)].$$

Example 5: *If the flow of heat is linear so that the variation of θ (temperature) with z and y may be neglected and if it is assumed that no heat is generated in the medium, then solve the differential equation* $\frac{\partial\theta}{\partial t} = k\frac{\partial^2\theta}{\partial x^2}$ *(one dimensional heat equation) where* $-\infty < x < \infty$ *and* $\theta = f(x)$ *where* $t = 0$, $f(x)$ *being a given function of x.*

Solution: Taking the Fourier transform of both the sides of the given equation, we have

$$\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\frac{\partial\theta}{\partial t}\, e^{ipx} dx = k\,\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\frac{\partial^2\theta}{\partial x^2}\, e^{ipx}\, dx$$

or $$\frac{d}{dt}\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\theta\cdot e^{ipx}\, dx = k(-ip)^2\,\tilde{\theta}$$ [From article 4.20 (b)]

where $\tilde{\theta} = \tilde{\theta}(p,t)$ is the Fourier transform of $\theta(x,t)$

or $$\frac{d}{dt}\tilde{\theta} = -kp^2\,\tilde{\theta}$$

whose solution is

$$\tilde{\theta} = A\,e^{-kp^2 t} \qquad \ldots(1)$$

Now at $t = 0$, $\theta = f(x)$.

$\therefore$ at $t = 0, \tilde{\theta} = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(x)\,e^{ipx}\,dx = \tilde{f}(p)$.

$\therefore$ from (1), at $t = 0, \tilde{\theta} = \tilde{f}(p) = A$;

hence from (1), $\tilde{\theta} = \tilde{f}(p)\,e^{-kp^2 t}$.

Taking the inverse Fourier transform, we have

$$\theta(x,t) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} \tilde{f}(p)\,e^{-kp^2 t}e^{-ipx}\,dp$$

Example 6: *Solve* $\frac{\partial^4 V}{\partial x^4} + \frac{\partial^2 V}{\partial y^2} = 0$, $-\infty < x < \infty$, $y \geq 0$

satisfying the conditions

(i) V *and its partial derivatives tend to zero as* $x \to \pm\infty$ *and*

(ii) $V = f(x), \frac{\partial V}{\partial y} = 0$ *on* $y = 0$.

Solution: Taking the Fourier transform of both the sides, we have

$$\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\frac{\partial^4 V}{\partial x^4}e^{ipx}dx + \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\frac{\partial^2 V}{\partial y^2}e^{ipx}dx = 0$$

or $$\frac{1}{\sqrt{(2\pi)}}\left(\frac{\partial^3 V}{\partial x^3}\cdot e^{ipx}\right)_{-\infty}^{\infty} - \frac{ip}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\frac{\partial^3 V}{\partial x^3}e^{ipx}\,dx + \frac{d^2}{dy^2}\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} V\cdot e^{ipx}\,dx = 0\,.$$

or $$-\frac{ip}{\sqrt{(2\pi)}}\left[\left(\frac{\partial^2 V}{\partial x^2}e^{ipx}\right)_{-\infty}^{\infty} - ip\int_{-\infty}^{\infty}\frac{\partial^2 V}{\partial x^2}e^{ipx}\,dx\right] + \frac{d^2\tilde{V}}{dy^2} = 0\,,$$

since $\frac{\partial^3 V}{\partial x^3} \to 0$ as $x \to \pm\infty$

or $$\frac{(ip)^2}{\sqrt{(2\pi)}}\left[\left(\frac{\partial V}{\partial x}e^{ipx}\right)_{-\infty}^{\infty} - ip\int_{-\infty}^{\infty}\frac{\partial V}{\partial x}e^{ipx}dx\right] + \frac{d^2\tilde{V}}{dy^2} = 0,$$

since $\frac{\partial^2 V}{\partial x^2} \to 0$ as $x \to \pm\infty$

or $$-\frac{(ip)^3}{\sqrt{(2\pi)}}\left[\left(Ve^{ipx}\right)_{-\infty}^{\infty} - ip\int_{-\infty}^{\infty}V\,e^{ipx}dx\right] + \frac{d^2\tilde{V}}{dy^2} = 0$$

or $$\frac{(ip)^4}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}V\,e^{ipx}\,dx + \frac{d^2\tilde{V}}{dy^2} = 0$$

or $$p^4\tilde{V} + \frac{d^2\tilde{V}}{dy^2} = 0$$

whose solution is

$$\tilde{V} = A\cos p^2 y + B\sin p^2 y. \qquad ...(1)$$

Since on $y = 0, V = f(x)$ and $\frac{\partial V}{\partial y} = 0,$

∴ taking Fourier transform, we have

on $$y = 0, \tilde{V} = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(x)\,e^{ipx}\,dx = \tilde{f}(p)$$

and $$\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\frac{\partial V}{\partial y}e^{ipx}\,dx = 0 \text{ i.e., } \frac{d\tilde{V}}{dy} = 0.$$

∴ from (1), we have

$$\tilde{f}(p) = A$$

and $$\frac{d\tilde{V}}{dy} = 0 = Bp^2 \quad \text{or} \quad B = 0.$$

∴ the solution (1) reduces to

$$\tilde{V} = \tilde{f}(p)\cos p^2 y.$$

∴ applying inversion theorem for Fourier transform, we have

$$V = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\tilde{f}(p)\cos(p^2 y)\cdot e^{-ipx}dp.$$

Fourier Series

Some Important Results

The following results are useful in Fourier Series :

(i) $\sin n\pi = 0, \cos n\pi = (-1)^n,$

$\sin\left(n+\frac{1}{2}\right)\pi = (-1)^n, \cos\left(n+\frac{1}{2}\right)\pi = 0,$ where $n \in I$.

(ii) $\int uv = uv_1 - u' v_3 + u'' v_2 - u''' v_4 + \dots,$

where $u' = \frac{du}{dx},\ u'' = \frac{d^2 u}{dx^2}, \dots$ and $v_1 = \int v\, dx,\ v_2 = \int v_1\, dx, \dots$.

(iii) $\int_0^{2\pi} \sin nx\, dx = 0.$ (iv) $\int_0^{2\pi} \cos nx\, dx = 0.$

(v) $\int_0^{2\pi} \sin^2 nx\, dx = \pi.$ (vi) $\int_0^{2\pi} \cos^2 nx\, dx = \pi.$

(vii) $\int_0^{2\pi} \sin nx \,.\, \sin mx\, dx = 0.$ (viii) $\int_0^{2\pi} \cos nx \,.\, \cos mx\, dx = 0.$

(ix) $\int_0^{2\pi} \sin nx \,.\, \cos mx\, dx = 0.$ (x) $\int_0^{2\pi} \sin nx \,.\, \cos nx\, dx = 0.$

Determination of Fourier Coefficients (Euler's Formulae)

The Fourier series is

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \ldots + a_n \cos nx + \ldots$$
$$+ b_1 \sin x + b_2 \sin 2x + \ldots + b_n \sin nx + \ldots \qquad \ldots(1)$$

To find a_0. Integrating both sides of (1) from $x = 0$ to $x = 2\pi$, we have

$$\int_0^{2\pi} f(x)\,dx = \frac{a_0}{2}\int_0^{2\pi} dx + a_1 \int_0^{2\pi} \cos x\,dx + a_2 \int_0^{2\pi} \cos 2x\,dx + \ldots$$
$$+ a_n \int_0^{2\pi} \cos nx\,dx + \ldots + b_1 \int_0^{2\pi} \sin x\,dx + b_2 \int_0^{2\pi} \sin 2x\,dx$$
$$+ \ldots + b_n \int_0^{2\pi} \sin nx\,dx + \ldots$$
$$= \frac{a_0}{2}\int_0^{2\pi} dx, \qquad \text{[Other integrals vanish]}$$

or $$\int_0^{2\pi} f(x)\,dx = \frac{a_0}{2} 2\pi.$$

$\therefore$ $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx. \qquad \ldots(2)$$

To find a_n. Multiplying each side of (1) by $\cos nx$ and integrating from $x = 0$ to $x = 2\pi$, we have

$$\int_0^{2\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_0^{2\pi} \cos nx\,dx + a_1\int_0^{2\pi} \cos x \cos nx\,dx + \ldots$$
$$+ a_n \int_0^{2\pi} \cos^2 nx\,dx \ldots + b_1 \int_0^{2\pi} \sin x nx\,dx + b_2 \int_0^{2\pi} \sin 2x \cos nx\,dx + \ldots$$
$$= a_n \int_0^{2\pi} \cos^2 nx\,dx \qquad \text{[Other integrals vanish]}$$
$$= a_n \pi.$$

$\therefore$ $$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx. \qquad \ldots(3)$$

By taking $n = 1, 2, \ldots$ we get the values of $a_1, a_2, \ldots$.

To find b_n. Multiplying each side of (1) by $\sin nx$ and integrating from $x = 0$ to $x = 2\pi$, we have

$$\int_0^{2\pi} f(x)\sin nx\,dx = \frac{a_0}{2}\int_0^{2\pi} \sin nx\,dx + a_1\int_0^{2\pi} \cos x \sin nx\,dx + \ldots$$
$$+ a_n \int_0^{2\pi} \cos nx \sin nx\,dx + \ldots$$
$$+ b_1 \int_0^{2\pi} \sin x \sin nx\,dx + \ldots + b_n \int_0^{2\pi} \sin^2 nx\,dx + \ldots$$
$$= b_n \int_0^{2\pi} \sin^2 nx\,dx \qquad \text{[Other integrals vanish]}$$
$$= b_n \pi.$$

$\therefore \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx.$ ...(4)

By taking $n = 1, 2, \ldots$ we get the values of $b_1, b_2, \ldots\ldots$.

Note. To get similar formula of a_0, $\frac{1}{2}$ has been written with a_0 in Fourier series.

If we write the Fourier series as $f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$,

then the values of constants is given as :

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx \,;\; a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \text{ and } b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx.$$

Fourier Series Expansion In The Interval $\alpha < x < \alpha + 2\pi$

The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx. \qquad \ldots(1)$$

To find a_0. Integrating both sides of (1) from $x = \alpha$ to $x = \alpha + 2\pi$, we have

$$\int_\alpha^{2\pi+\alpha} f(x)\,dx = \frac{a_0}{2}\int_\alpha^{2\pi+\alpha} dx + \int_\alpha^{2\pi+\alpha}\left\{\sum_{n=1}^{\infty} a_n \cos nx\right\}dx + \int_\alpha^{2\pi+\alpha}\left\{\sum_{n=1}^{\infty} b_n \sin nx\right\}dx$$

$$= \frac{a_0}{2}\cdot 2\pi + 0 + 0.$$

$$\therefore \qquad a_0 = \frac{1}{\pi}\int_\alpha^{2\pi+\alpha} f(x)\,dx.$$

To find a_n. Multiplying both sides of (1) by $\cos nx$ and then integrating from $x = \alpha$ to $x = \alpha + 2\pi$, we have

$$\int_\alpha^{2\pi+\alpha} f(x)\cos nx\,dx = \frac{\alpha_0}{2}\int_\alpha^{2\pi+\alpha}\cos nx\,dx + \int_\alpha^{2\pi+\alpha}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)\cos nx\,dx + \int_\alpha^{2\pi+\alpha}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)\cos nx\,dx$$

$$= 0 + \pi a_n + 0.$$

$$\therefore \qquad a_n = \frac{1}{\pi}\int_\alpha^{2\pi+\alpha} f(x)\cos nx\,dx.$$

Similarly $\qquad b_n = \frac{1}{\pi}\int_\alpha^{2\pi+\alpha} f(x)\sin nx\,dx.$

Note 1. Putting $\alpha = 0$, the interval becomes $0 < x < 2\pi$ and

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx,\quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \text{ and } b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx.$$

Note 2. Putting $\alpha = -\pi$, the interval becomes $-\pi < x < \pi$ and

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx,\quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \text{ and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

Fourier Series For Discontinuous Functions

Let the function $f(x)$ be defined by $f(x) = \begin{cases} f_1(x), & \alpha < x < x_0 \\ f_2(x), & x_0 < x < \alpha + 2\pi, \end{cases}$

in the interval $(\alpha, \alpha + 2\pi)$. Here x_0 is the point of discontinuity.

The Fourier series for $f(x)$ in such cases is obtained in the usual way. The values of a_0, a_n, b_n are given by

$$a_0 = \frac{1}{\pi}\left[\int_{\alpha}^{x_0} f_1(x)\,dx + \int_{x_0}^{\alpha+2\pi} f_2(x)\,dx\right];$$

$$a_n = \frac{1}{\pi}\left[\int_{\alpha}^{x_0} f_1(x)\cos nx\,dx + \int_{x_0}^{\alpha+2\pi} f_2(x)\cos nx\,dx\right];$$

$$b_n = \frac{1}{\pi}\left[\int_{\alpha}^{x_0} f_1(x)\sin nx\,dx + \int_{x_0}^{\alpha+2\pi} f_2(x)\sin n\,x\,dx\right].$$

At the point of discontinuity, $x = \alpha$, the Fourier series gives the value of $f(x)$ as the arithmetic mean of left and right limits.

$\therefore$ At $x = \alpha$, $f(x) = \frac{1}{2}[f(\alpha - 0) + f(\alpha + 0)]$.

Illustrative Examples

Example 1: *Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to $x = \pi$.*

Deduce that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$.

Solution: The Fourier series for $f(x)$ in $(-\pi, \pi)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

Here
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} (x - x^2)\, dx = \frac{1}{2\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi} = -\frac{\pi^2}{3};$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\cos nx\, dx$$

$$= \frac{1}{\pi}\left[(x - x^2)\frac{\sin nx}{n} - (1 - 2x)\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{-4(-1)^n}{n^2}; \qquad [\because \cos n\pi = (-1)^n]$$

and
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\sin nx\, dx$$

$$= \frac{1}{\pi}\left[(x - x^2)\left(-\frac{\cos nx}{n}\right) - (1 - 2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= -2(-1)^n / n$$

$\therefore$ The required Fourier series is

$$x - x^2 = -\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \ldots\right]$$
$$+ 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots\right]. \quad \ldots(1)$$

Deduction: Putting $x = 0$ in (1), we get

$$0 = -\frac{\pi^2}{3} + 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\right) \quad \text{or} \quad \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}.$$

Example 2: *Find the Fourier series expansion for $f(x)$, if*

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi. \end{cases} \quad \textit{Deduce that } \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}.$$

Solution: The Fourier series of $f(x)$ in $(-\pi, \pi)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

Then $$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} (-\pi)\,dx + \int_0^{\pi} x\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\,(x)_{-\pi}^{0} + \left(\frac{x^2}{2}\right)_0^{\pi}\right] = \frac{1}{\pi}\left[-\left(\pi^2 + \frac{\pi^2}{2}\right)\right] = -\frac{\pi}{2};$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\cos nx\,dx + \int_0^{\pi} x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left(\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{n^2}\cos n\pi - \frac{1}{n^2}\right] = \frac{1}{\pi n^2}(\cos n\pi - 1);$$

and $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\sin nx\,dx + \int_0^{\pi} x\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi\cos nx}{n}\right)_{-\pi}^{0} + \left(-\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(0 - \cos n\pi) - \frac{\pi}{n}\cos n\pi\right] = \frac{1}{n}(1 - 2\cos n\pi).$$

∴ The required Fourier series is

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right)$$

$$+\left(3\sin x - \frac{2\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots\right) \qquad \ldots(1)$$

Deduction: Putting $x = 0$ in (1), we get

$$f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\infty\right). \qquad \ldots(2)$$

But $f(x)$ is discontinuous at $x = 0$, and we have $f(0-0) = -\pi$ and $f(0+0) = 0$.

∴ $$f(0) = \frac{1}{2}[f(0-0) + f(0+0)] = -(\pi/2) \qquad \ldots(3)$$

Hence from (2) and (3), we have

$$-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right] \quad \text{or} \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}.$$

Example 3: *Obtain the Fourier series of* $f(x) = \left(\frac{\pi - x}{2}\right)$ *in the interval* $(0, 2\pi)$ *and hence deduce*

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Solution: The Fourier series for $f(x)$ in $(0, 2\pi)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Here $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)dx$$

$$= \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{2\pi} = \frac{1}{2\pi}[2\pi^2 - 2\pi^2] = 0;$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{2\pi}\int_0^{2\pi}(\pi - x)\cos nx\,dx$$

$$= \frac{1}{2\pi}\left[(\pi - x)\frac{\sin nx}{n} - (-1)\left\{\frac{-\cos nx}{n^2}\right\}\right]_0^{2\pi} = 0;$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{2\pi}\int_0^{2\pi}(\pi - x)\sin nx\,dx$$

$$= \frac{1}{2\pi}\left[-(\pi - x)\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right]_0^{2\pi} = \frac{1}{n}.$$

$\therefore$ $$f(x) = \frac{\pi - x}{2} = 0 + 0 + \sum_{n=1}^{\infty}\frac{1}{n}\sin nx = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots \qquad \ldots(1)$$

Deduction: Putting $x = \frac{\pi}{2}$ in (1), we get $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

Comprehensive Exercise 1

1. Find a series of sines and cosines of multiples of x which will represent $\frac{\pi}{2\sinh\pi}e^x$ in the interval $-\pi < x < \pi$.
2. Find the Fourier series of the function defined as
$$f(x) = \begin{cases} x + \pi\ ; & 0 \le x \le \pi \\ -x - \pi\ ; & -\pi \le x < 0 \end{cases} \text{ and } f(x + 2\pi) = f(x).$$
3. Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$.
4. Find the Fourier series of the function
$$f(x) = \begin{cases} -1 \text{ for } -\pi < x < -\pi/2 \\ 0 \text{ for } -\pi/2 < x < \pi/2 \\ +1 \text{ for } \pi/2 < x < \pi. \end{cases}$$

Change of Period

Generally in engineering problems, the period of the given function is not always 2π but is T or $2c$. To use the formula of a_n and b_n this period must be converted to the length 2π. The independent variable x is also to be changed proportionally.

Let the function $f(x)$ be defined in the interval $(0, 2c)$.

$\because$ $2c$ is the interval for the variable x.

$\therefore$ 2π is the interval for the variable $= \dfrac{2\pi x}{2c} = \dfrac{\pi x}{c}$.

Putting $z = \dfrac{\pi x}{c}$ or $x = \dfrac{zc}{\pi}$, the function $f(x)$ of period $2c$ is transformed to the function $f\left(\dfrac{cz}{\pi}\right)$ or $F(z)$ of period 2π.

Now $F(z)$ can be expanded in the Fourier series as

$$F(z) = f\left(\frac{cz}{\pi}\right) = \frac{a_0}{2} + a_1 \cos z + a_2 \cos 2z + \ldots + b_1 \sin z + b_2 \sin 2z + \ldots$$

where $a_0 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} F(z)\, dz = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f\left(\dfrac{cz}{\pi}\right) dz$

$$= \frac{1}{\pi}\int_0^{2c} f(x) \frac{\pi}{c}\, dx = \frac{1}{c}\int_0^{2c} f(x)\, dx; \qquad \left[\text{Putting } z = \frac{\pi x}{c}\right]$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} F(z) \cos nz\, dz = \frac{1}{\pi}\int_0^{2\pi} f\left(\frac{cz}{\pi}\right) \cos nx\, dz$$

$$= \frac{1}{\pi}\int_0^{2c} f(x) \cos \frac{n\pi x}{c} \frac{\pi}{c}\, dx = \frac{1}{c}\int_0^{2\pi} f(x) \cos \frac{n\pi x}{c}\, dx. \qquad \left[\text{Putting } z = \frac{\pi x}{c}\right]$$

Similarly, $b_n = \dfrac{1}{c}\displaystyle\int_0^{2c} f(x) \sin \dfrac{n\pi x}{c}\, dx.$

Illustrative Examples

Example 4: *Obtain Fourier Series for the function* $f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2-x), & 1 \le x \le 2. \end{cases}$

Deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots.$

Solution: Here the function is defined in the interval $(0, 2)$. Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\pi x + b_n \sin n\pi x). \quad \ldots(1)$$

Then, we have $a_0 = \dfrac{1}{c}\displaystyle\int_0^{2c} f(x)\,dx$, where $c = 1$

$$= \frac{1}{1}\int_0^2 f(x)\,dx = \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx$$

$$= \pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[2x - \frac{x^2}{2}\right]_1^2 = \frac{\pi}{2} + \frac{\pi}{2} = \pi;$$

$$a_n = \frac{1}{c}\int_0^{2c} f(x)\cos\frac{n\pi x}{c}\,dx = \frac{1}{1}\int_0^2 f(x)\cos n\pi x\,dx$$

$$= \int_0^1 \pi x\cos n\pi x\,dx + \int_1^2 \pi(2-x)\cos n\pi x\,dx$$

$$= \pi\left[x\frac{\sin n\pi x}{n\pi} + \frac{\cos n\pi x}{(n\pi)^2}\right]_0^1 + \pi\left[-(2-x)\frac{\sin n\pi x}{n\pi} - \frac{\cos n\pi x}{(n\pi)^2}\right]_1^2$$

$$= \pi\left[\frac{\cos n\pi}{(n\pi)^2} - \frac{1}{(n\pi)^2}\right] + \pi\left[-\frac{1}{(n\pi)^2} + \frac{\cos n\pi}{(n\pi)^2}\right] = \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right]$$

$$= \begin{cases} -\dfrac{4}{\pi n^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even}; \end{cases}$$

and

$$b_n = \frac{1}{c}\int_0^{2c} f(x)\sin\frac{n\pi x}{c}\,dx = \frac{1}{1}\int_0^2 f(x)\sin n\pi x\,dx$$

$$= \pi\int_0^1 x\sin n\pi x\,dx + \pi\int_0^2 (2-x)\sin n\pi x\,dx$$

$$= \pi\left[x\left(\frac{-\cos n\pi x}{n\pi}\right) + \frac{\sin n\pi x}{(n\pi)^2}\right]_0^1 + \pi\left[(2-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - \frac{\sin n\pi x}{(n\pi)^2}\right]_1^2$$

$$= -\frac{\cos n\pi}{n} + \frac{\cos n\pi}{n} = 0.$$

Putting these values of a_0, a_n, b_n in (1), we get the required Fourier series as

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos(5\pi x)}{5^2} + \ldots\right] \quad \ldots(2)$$

Deduction: Putting $x = 0$ in (2), we get

$$f(0) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right] \quad \text{or} \quad 0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right]$$

or
$$\frac{\pi}{2} = \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right] \quad \text{or} \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Example 5: *Find the Fourier series corresponding to the function $f(x)$ defined in $(-2, 2)$ as follows*

$$f(x) = \begin{cases} 2, & \text{if } -2 \le x \le 0 \\ x, & \text{if } \ 0 < x < 2. \end{cases}$$

Solution: Here the interval is $(-2, 2)$ and $l = 2$.

Let
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right].$$

Then
$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{1}{2}\left[\int_{-2}^{0} 2\,dx + \int_0^2 x\,dx\right]$$

$$= \frac{1}{2}\left[(2x)_{-2}^{0} + \left(\frac{x^2}{2}\right)_0^2\right] = \frac{1}{2}[4 + 2] = 3;$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{2}\left[\int_{-2l}^{0} 2\cos\frac{n\pi x}{2}\,dx + \int_0^2 x\cos\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[\frac{4}{n\pi}\left(\sin\frac{n\pi x}{2}\right)_{-2}^{0} + \left(x\frac{2}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)_0^2\right]$$

$$= \frac{1}{2}\left[\frac{4}{n^2\pi^2}\cos n\pi - \frac{4}{n^2\pi^2}\right] = \frac{2}{n^2\pi^2}[(-1)^n - 1]$$

$$= \begin{cases} -\dfrac{4}{n^2\pi^2}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even;} \end{cases}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{1}{2}\int_{-2}^{0} 2\sin\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_0^2 x\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[2\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right)\right]_{-2}^{0} + \frac{1}{2}\left[x\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right) + (1)\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_0^2$$

$$=\frac{1}{2}\left[2\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right)\right]_{-2}^{0}+\frac{1}{2}\left[x\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right)+(1)\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_{0}^{2}$$

$$=\frac{1}{2}\left[-\frac{4}{n\pi}+\frac{4}{n\pi}\cos n\pi\right]+\frac{1}{2}\left[-\frac{4}{n\pi}\cos n\pi+\frac{4}{n^2\pi^2}\sin n\pi\right]=\frac{1}{2}\left[-\frac{4}{n\pi}\right]=-\frac{2}{n\pi}.$$

The required Fourier series is

$$f(x)=\frac{a_0}{2}+a_1\cos\frac{\pi x}{l}+a_2\cos\frac{2\pi x}{l}+a_3\cos\frac{3\pi x}{l}+\ldots$$
$$+b_1\sin\frac{\pi x}{l}+b_2\sin\frac{2\pi x}{l}+b_3\sin\frac{3\pi x}{l}+\ldots$$

$$=\frac{3}{2}-\frac{4}{\pi^2}\left\{\frac{1}{1^2}\cos\frac{\pi x}{2}+\frac{1}{3^2}\cos\frac{3\pi x}{2}+\ldots\right\}-\frac{2}{\pi}\left\{\frac{1}{1}\sin\frac{\pi x}{2}+\frac{1}{2}\sin\frac{2\pi x}{2}+\frac{1}{3}\sin\frac{3\pi x}{2}+\ldots\right\}.$$

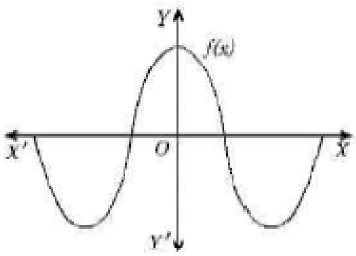
Fourier Series Expansion of An Even or an Odd Function in $(-\pi, \pi)$

A function $f(x)$ is called an even (symmetric) function if $f(-x) = f(x)$.

The area under such a curve from $-\pi$ to π is double the area from 0 to π.

$$\therefore \qquad \int_{-\pi}^{\pi} f(x)\,dx = 2\int_{0}^{\pi} f(x)\,dx.$$

Even Function



A function $f(x)$ is called an odd (or skew symmetric) function if $f(-x) = -f(x)$.

Here the area under the curve from $-\pi$ to π is zero.

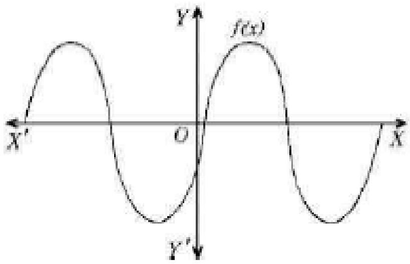
$$\therefore \qquad \int_{-\pi}^{\pi} f(x)\,dx = 0.$$

Expansion Of An Even Function In $(-\pi, \pi)$:

Here

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx.$$

Odd Function



As $f(x)$ and $\cos nx$ are both even functions, therefore the product of $f(x).\cos nx$ is also an even function.

$$\therefore \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx.$$

As sin nx is an odd function so $f(x) \cdot \sin nx$ is
also an odd function. We need not to calculate b_n. It saves our labour a lot.

$\therefore$ $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0.$$

Thus the series of the even function will contain only cos terms.

Expansion Of An Odd Function In $(-\pi, \pi)$:

Here $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = 0\ ;$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx = 0 \qquad [\because f(x) \cdot \cos nx \text{ is an odd function}]$$

and $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x) \sin nx \, dx$$

$[\because f(x) \cdot \sin nx$ is an even function$]$

Thus the series of the odd function will contain only sin terms.

Illustrative Examples

Example 6: *Obtain Fourier's series for the expansion of* $f(x) = x \sin x$ *in the interval* $-\pi < x < \pi$. *Hence deduce that*

$$\frac{\pi - 2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots$$

Solution: Here $f(x) = x \sin x$ being an even function of x. The Fourier series is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$f(x) = \frac{1}{\pi}\int_0^{\pi} f(u)\,du + \frac{2}{\pi}\sum_{n=1}^{\infty} \cos n\,x \int_0^{\pi} f(u)\cos nu\,du,$$

where $f(u) = u \sin u$.

Now $\int_0^{\pi} u \sin u\,du = [-u\cos u + \sin u]_0^{\pi} = \pi$

and $\int_0^{\pi} u \sin u \cos nu\,du = \frac{1}{2}\int_0^{\pi} u\,[\sin (n+1)u - \sin (n-1)u]\,du$

$$= \frac{1}{2}\left[\frac{-u\cos (n+1)u}{n+1} + \frac{\sin (n+1)u}{(n+1)^2}\right]_0^{\pi} - \frac{1}{2}\left[-\frac{u\cos (n-1)u}{n-1} + \frac{\sin (n-1)u}{(n-1)^2}\right]_0^{\pi}$$

$$= \frac{\pi}{2}\left[\frac{\cos (n-1)\pi}{n-1} - \frac{\cos (n+1)\pi}{n+1}\right] = \frac{\pi}{2}\left[-\frac{\cos n\pi}{n-1} + \frac{\cos n\pi}{n+1}\right]$$

$$= \frac{\pi \cos n\pi}{1-n^2},\ \text{when } n > 1.$$

When $n = 1$, we have

$$\int_0^{\pi} u \sin u \cos u\,du = \frac{1}{2}\int_0^{\pi} u \sin 2u\,du = \frac{1}{2}\left[-\frac{u\cos 2u}{2} + \frac{\sin 2u}{4}\right]_0^{\pi} = -\frac{\pi}{4}.$$

$\therefore$
$$x \sin x = 1 + \frac{2}{\pi}\left(-\frac{\pi}{4}\cos x + \sum_{n=2}^{\infty} \frac{\pi \cos n\pi}{1-n^2}\cos nx\right)$$

$$= 1 + 2\left[-\frac{1}{4}\cos x - \frac{1}{1.3}\cos 2x + \frac{1}{2.4}\cos 3x - \frac{1}{3.5}\cos 4x + \dots\right]$$

Deduction: Putting $x = \frac{1}{2}\pi$, we get

$$\frac{\pi}{2} = 1 + 2\left\{\frac{1}{1.3} - \frac{1}{3\cdot 5} + \frac{1}{5.7} - \dots\right\} \text{ or } \frac{\pi}{4} = \frac{1}{2} + \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots$$

or
$$\frac{\pi - 2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots.$$

Example 7: Obtain Fourier Series of the function

$$f(x) = \begin{cases} x, & -\pi < x < 0 \\ -x, & 0 < x < \pi. \end{cases}$$

Solution: We have $f(x) = \begin{cases} x, & -\pi < x < 0 \\ -x, & 0 < x < \pi \end{cases}$ and $f(-x) = \begin{cases} -x, & \pi > x > 0 \\ x, & 0 > x > -\pi \end{cases}$

Thus $f(x)$ is an even fucntion in $(-\pi, \pi)$ and so $b_n = 0$.

The Fourier series in this case is $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$.

$$\text{Here } a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} -x\,dx = -\frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = -\frac{1}{\pi}\left[\frac{\pi^2}{2}\right] = -\frac{\pi}{2};$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} -x\cos nx\,dx = -\frac{2}{\pi}\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= -\frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}[1 - (-1)^n] = \begin{cases} 0, & n \text{ is even} \\ 4/\pi n^2, & n \text{ is odd} \end{cases}$$

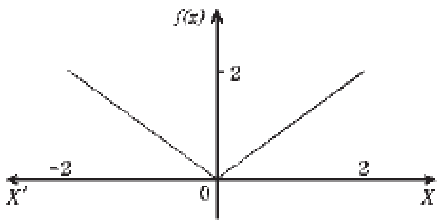
$\therefore$ The required series is $f(x) = -\frac{\pi}{2} + \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right]$

Example 8: A periodic function of period 4 is defined as

$f(x) = |x|, -2 < x < 2$. Obtain its Fourier series expansion.

Solution: We have

$$f(x) = \begin{cases} |x|, & -2 < x < 2 \\ x, & 0 < x < 2 \\ -x, & 2 < x < 0. \end{cases}$$



Here $f(x)$ is an even function in the interval $(-2, 2)$ and so $b_n = 0$.

The Fourier series in this case is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$.

$$\text{Here } a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\,dx = \frac{1}{2}\int_0^2 x\,dx + \frac{1}{2}\int_{-2}^{0} (-x)\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2 + \frac{1}{2}\left[\frac{-x^2}{2}\right]_{-2}^{0} = \frac{1}{4}(4-0) + \frac{1}{4}(0+4) = 2;$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx = \frac{1}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{-2}^{0} (-x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[x\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right]_0^2$$

$$+\frac{1}{2}\left[(-x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right)-(-1)\left(-\frac{4}{n^2\pi^2}\right)\cos\frac{n\pi x}{2}\right]_{-2}^{0}$$

$$=\frac{1}{2}\left[0+\frac{4}{n^2\pi^2}(-1)^n-\frac{4}{n^2\pi^2}\right]+\frac{1}{2}\left[0-\frac{4}{n^2\pi^2}+\frac{4}{n^2\pi^2}(-1)^n\right]$$

$$=\frac{1}{2}\frac{4}{n^2\pi^2}[(-1)^n-1-1+(-1)^n]=\frac{4}{n^2\pi^2}[(-1)^n-1]$$

$$=\begin{cases}-\dfrac{8}{n^2\pi^2}, & \text{if } n \text{ is odd}\\ 0, & \text{if } n \text{ is even.}\end{cases}$$

∴ The required Fourier series is $f(x)=\dfrac{a_0}{2}+a_1\cos\dfrac{\pi x}{c}+a_2\cos\dfrac{2\pi x}{c}+\ldots$

$$=1-\frac{8}{\pi^2}\left[\frac{\cos\frac{\pi x}{2}}{1^2}+\frac{\cos\frac{3\pi x}{2}}{3^2}+\frac{\cos\frac{5\pi x}{2}}{5^2}+\ldots\right].$$

Half Range Fourier Cosine and Sine Series [Interval $(0, T)$]

For the interval $(0, T)$ the **Half Range Fourier Cosine Series** is

$$f(x) = \frac{a_0}{2} + a_1 \cos \frac{\pi x}{T} + a_2 \cos \frac{2\pi x}{T} + \ldots + a_n \cos \frac{n \pi x}{T} + \ldots$$

where $$a_0 = \frac{2}{T} \int_0^T f(x)\, dx, a_n = \frac{2}{T} \int_0^T f(x) \cos \frac{n\pi x}{T}\, dx.$$

Thus when $f(x)$ is an even function with period T, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{T},$$

where $$a_0 = \frac{2}{T} \int_0^T f(x)\, dx,\ a_n = \frac{2}{T} \int_0^T f(x) \cos \frac{n\pi x}{T}\, dx .$$

In this case $b_n = 0$.

Again for the interval $(0, T)$ the **Half Range Fourier sine series** is

$$f(x) = b_1 \sin \frac{\pi x}{T} + b_2 \sin \frac{2\pi x}{T} + \ldots + b_n \sin \frac{n\pi x}{T} + \ldots,$$

where $$b_n = \frac{2}{T} \int_0^T f(x) \sin \frac{n\pi x}{T}\, dx.$$

Thus when $f(x)$ is an odd function with period T, we have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{T}, \text{ where } b_n = \frac{2}{T} \int_0^T f(x) \sin \frac{n\pi x}{T}\, dx.$$

In this case $a_0 = 0 = a_n$.

Remark: While expanding a function in $(0, T)$ as a series of sines or cosines, we only see if it is an odd or even function of period $2T$. It makes no matter if the function is odd or even or neither.

Illustrative Examples

Example 9: *Find the Fourier half-range cosine series of the function*

$$f(t) = \begin{cases} 2t, & 0 < t < 1 \\ 2(2-t), & 1 < t < 2. \end{cases}$$

Solution: The Fourier half range cosine series in interval $(0, T)$ is

$$f(t) = \frac{a_0}{2} + a_1 \cos \frac{\pi t}{T} + a_2 \cos \frac{2\pi t}{T} + a_3 \cos \frac{3\pi t}{T} + \ldots \qquad \ldots(1)$$

Here $T = 2$. We have

$$a_0 = \frac{2}{T}\int_0^T f(t)\, dt = \frac{2}{2}\int_0^1 2t\, dt + \frac{2}{2}\int_1^2 2(2-t)\, dt$$

$$= \left[t^2\right]_0^1 + \left[2\left(2t - \frac{t^2}{2}\right)\right]_1^2 = 1 + \left[(4t - t^2)\right]_1^2 = 1 + (8 - 4 - 4 + 1) = 2,$$

$$a_n = \frac{2}{T}\int_0^T f(t) \cos \frac{n\pi t}{T}\, dt = \frac{2}{2}\int_0^1 2t \cos \frac{n\pi t}{2}\, dt + \frac{2}{2}\int_1^2 2(2-t) \cos \frac{n\pi t}{2}\, dt$$

$$= \left[2t\left(\frac{2}{n\pi}\sin \frac{n\pi t}{2}\right) - (2)\left(-\frac{4}{n^2\pi^2}\cos \frac{n\pi t}{2}\right)\right]_0^1$$

$$+ \left[(4-2t)\left(\frac{2}{n\pi}\sin \frac{n\pi t}{2}\right) - (-2)\left(-\frac{4}{n^2\pi^2}\cos \frac{n\pi t}{2}\right)\right]_1^2$$

$$= \left[\frac{4}{n\pi}\sin \frac{n\pi}{2} + \frac{8}{n^2\pi^2}\cos \frac{n\pi}{2} - \frac{8}{n^2\pi^2}\right]$$

$$+ \left[0 - \frac{8}{n^2\pi^2}\cos n\pi - \frac{4}{n\pi}\sin \frac{n\pi}{2} + \frac{8}{n^2\pi^2}\cos \frac{n\pi}{2}\right]$$

$$= \frac{8}{n^2\pi^2}\cos \frac{n\pi}{2} - \frac{8}{n^2\pi^2} - \frac{8}{n^2\pi^2}\cos n\pi + \frac{8}{n^2\pi^2}\frac{\cos n\pi}{2}$$

$$= \frac{8}{n^2\pi^2}\left[2\cos \frac{n\pi}{2} - 1 - \cos n\pi\right]$$

$\therefore$
$$f(x) = 1 + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(2\cos \frac{n\pi}{2} - 1 - \cos n\pi\right)\cos \frac{n\pi t}{2}$$

Example 10: *(i) Express* $f(x) = x$ *as a half-range sine series in* $0 < x < 2$.

(ii) Express $f(x) = x$ *as a half-range cosine series in* $0 < x < 2$.

Solution: (i) The Fourier sine series for $f(x)$ in $(0, 2)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2} \text{ where } b_n = \frac{2}{2}\int_0^2 f(x)\sin \frac{n\pi x}{2}\, dx = \int_0^2 x \sin \frac{n\pi x}{2}\, dx$$

$$= \left[-\frac{2x}{n\pi}\cos \frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin \frac{n\pi x}{2}\right]_0^2 = -\frac{4(-1)^n}{n\pi}.$$

$\therefore$
$$b_1 = 4/\pi,\ b_2 = -4/2\pi,\ b_3 = 4/3\pi,\ b_4 = -4/4\pi, \text{ etc.}$$

Hence, the required half-range Fourier sine series for $f(x)$ in $(0, 2)$ is

$$f(x) = \frac{4}{\pi}\left[\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \frac{1}{4}\sin\frac{4\pi x}{2} + \ldots\right].$$

(ii) The Fourier cosine series for $f(x)$ in $(0, 2)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$$

where $a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^2 x\,dx = 2$ and $a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$

$$= \int_0^2 x\cos\frac{n\pi x}{2}\,dx = \left[\frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right]_0^2 = \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right].$$

$\therefore \quad a_1 = -8/\pi^2, a_2 = 0, a_3 = -8/3^2\pi^2, a_4 = 0, a_5 = -8/5^2\pi^2$, etc.

Hence, the required half-range Fourier series for $f(x)$ in $(0, 2)$ is

$$f(x) = 1 - \frac{8}{\pi^2}\left[\frac{\cos \pi x/2}{1^2} + \frac{\cos 3\pi x/2}{3^2} + \frac{\cos 5\pi x/2}{5^2} + \ldots\right]$$

Parseval's Formula

If the Fourier series for $f(x)$ converges uniformly in $(-l, l)$, then

$$\int_{-l}^{l} [f(x)]^2\,dx = l\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\right].$$

Proof: The Fourier series for $f(x)$ in $(-l, l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right), \quad \ldots(1)$$

where $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$; ...(2) $\quad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$ ...(3)

and $\quad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx.$ ...(4)

Multiplying (1) by $f(x)$, we get

$$[f(x)]^2 = \frac{a_0}{2} f(x) + \sum_{n=1}^{\infty} a_n f(x)\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n f(x)\sin\frac{n\pi x}{l} \quad \ldots(5)$$

Integrating (5) term by term from $-l$ to l, we get

$$\int_{-l}^{l} [f(x)]^2\,dx = \frac{a_0}{2}\int_{-l}^{l} f(x)\,dx + \sum_{n=1}^{\infty} a_n \int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$+ \sum_{n=1}^{\infty} b_n \int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx \quad \ldots(6)$$

Putting the values of a_0, a_n, b_n from (2), (3) and (4) in (6), we get

$$\int_{-l}^{l} [f(x)]^2\,dx = \frac{l a_0^2}{2} + \sum_{n=1}^{\infty} l\,a_n^2 + \sum_{n=1}^{\infty} l\,b_n^2$$

or

$$\int_{-l}^{l} [f(x)]^2\,dx = l\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\right].$$

This is the **Parseval's formula.**

Corollary 1: If the Fourier series for $f(x)$ in the interval $0 < x < 2l$ converges uniformly, then $\int_{0}^{2l} [f(x)]^2\,dx = l\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\right].$

Corollary 2: If the half range Fourier cosine series for $f(x)$ in the interval $0 < x < l$ converges uniformly, then $\int_{-l}^{l} [f(x)]^2\,dx = \frac{l}{2}\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right].$

Corollary 3: If the half range Fourier sine series for $f(x)$ in the interval $0 < x < l$ converges uniformly, then $\int_{0}^{l} [f(x)]^2\,dx = \frac{l}{2}\sum_{n=1}^{\infty} b_n^2$.

Corollary 4: Root Mean Square Value. The root mean square value or the effective value of the function $f(x)$ denoted by $[f(x)]_{rms}$ over an interval (a, b) is defined as

$$[f(x)]_{rms} = \left\{ \frac{\int_a^b [f(x)]^2 \, dx}{b - a} \right\}^{1/2} .$$

The root mean square value of a periodic function is frequently used in electric circuit theory and in the theory of mechanical vibrations.

Chapter

Beta and Gamma Functions

1 Beta Function

Euler's Integrals : Beta and Gamma Functions:

Definition: *The definite integral*

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \quad \textit{for} \quad m>0, n>0$$

is called the **Beta function** *and is denoted by* **B** (m, n) [*read as "Beta m, n"*] .

Thus $\mathbf{B}(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx,$

where m, n are any positive numbers, integral or fractional. Beta function is also called the **Eulerian integral of the first kind.**

2 Some Simple Properties of Beta Function

(i) Symmetry of Beta function *i.e.*, B $(m, n) = B(n, m)$.

We have $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx,$ by the def. of the Beta function

$$= \int_0^1 (1-x)^{m-1} \{1-(1-x)\}^{n-1} dx \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^1 (1-x)^{m-1} x^{n-1} dx = \int_0^1 x^{n-1} (1-x)^{m-1} dx$$

$= \mathbf{B}(n, m)$, by the def. of Beta function.

Hence $\mathbf{B}(m, n) = \mathbf{B}(n, m)$.

(ii) If *m* or *n* is a positive integer, *B* (*m*, *n*) can be evaluated in an explicit form.

Case I. When *n* is a positive integer. If $n = 1$, the result is obvious because

$$\mathbf{B}(m, 1) = \int_0^1 x^{m-1} (1-x)^{1-1} dx = \int_0^1 x^{m-1} dx = \left[\frac{x^m}{m}\right]_0^1 = \frac{1}{m}.$$

So let us take $n > 1$. We have

$$\mathbf{B}(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

$$= \int_0^1 (1-x)^{n-1} x^{m-1} dx$$

$$= \left[(1-x)^{n-1} \cdot \frac{x^m}{m}\right]_0^1 - \int_0^1 (n-1)(1-x)^{n-2}(-1) \cdot \frac{x^m}{m} dx,$$

integrating by parts taking x^{m-1} as the second function

$$= 0 + \frac{n-1}{m} \cdot \int_0^1 x^m (1-x)^{n-2} dx \qquad [\because n > 1]$$

$$= \frac{n-1}{m} \cdot \int_0^1 x^{(m+1)-1} (1-x)^{(n-1)-1} dx$$

$$= \frac{n-1}{m} \mathbf{B}(m+1, n-1).$$

By the repeated application of this process, we get

$$\mathbf{B}(m, n) = \frac{n-1}{m} \cdot \frac{n-2}{m+1} \cdot \frac{n-3}{m+2} \cdots\cdots \frac{1}{m+n-2} \mathbf{B}(m+n-1, 1)$$

$$= \frac{n-1}{m} \cdot \frac{n-2}{m+1} \cdot \frac{n-3}{m+2} \cdots\cdots \frac{1}{m+n-2} \int_0^1 x^{m+n-2} (1-x)^0 dx$$

$$= \frac{n-1}{m} \cdot \frac{n-2}{m+1} \cdot \frac{n-3}{m+2} \cdots\cdots \frac{1}{m+n-2} \int_0^1 x^{m+n-2} dx$$

$$= \frac{(n-1)!}{m(m+1)(m+2)\cdots\cdots(m+n-2)} \cdot \left[\frac{x^{m+n-1}}{m+n-1}\right]_0^1.$$

$$\therefore \qquad \mathbf{B}(m, n) = \frac{1}{m(m+1)(m+2)\cdots\cdots(m+n-2)(m+n-1)}.$$

Case II. When *m* is a positive integer. Since the Beta function is symmetrical in *m* and *n* *i.e.*, $\mathbf{B}(m, n) = \mathbf{B}(n, m)$, therefore by case I, we conclude that

$$\mathbf{B}(m, n) = \frac{(m-1)!}{n(n+1)(n+2)\ldots\ldots(n+m-2)(n+m-1)}.$$

(iii) If both m and n are positive integers, then

$$\mathbf{B}(m, n) = \frac{(m-1)!\,(n-1)!}{(m+n-1)!}$$

From (ii), we have

$$\mathbf{B}(m, n) = \frac{(n-1)!}{m(m+1)(m+2)\ldots\ldots(m+n-2)(m+n-1)}$$

$$= \frac{(n-1)!\,(m-1)!}{(m+n-1)(m+n-2)\ldots\ldots(m+1)\,m\,(m-1)!},$$

writing the denominator in the reversed order and multiplying the Nr and Dr by $(m-1)!$

$$= \frac{(m-1)!\,(n-1)!}{(m+n-1)!}.$$

Illustrative Examples

Example 1: *Express the following integrals in terms of Beta function :*

(i) $\int_0^1 x^m (1-x^2)^n \, dx, m > -1, n > -1$; **(Lucknow 2010)** *(ii)* $\int_0^1 \frac{x^2}{\sqrt{(1-x^5)}} \, dx$

(iii) $\int_0^1 x^{m-1} (1-x^2)^{n-1} \, dx$.

Solution: (i) We have

$$\int_0^1 x^m (1-x^2)^n \, dx = \int_0^1 x^{m-1} (1-x^2)^n . x \, dx \qquad \textbf{[Note]}$$

$$= \int_0^1 y^{(m-1)/2} (1-y)^n \cdot \frac{dy}{2},$$

putting $x^2 = y$ so that $2x\,dx = dy$

$$= \frac{1}{2} \int_0^1 y^{(m-1)/2} (1-y)^n \, dy$$

$$= \frac{1}{2} \int_0^1 y^{[(m+1)/2]-1} (1-y)^{(n+1)-1} \, dy$$

$$= \frac{1}{2} \mathbf{B}\left(\frac{1}{2}(m+1), n+1\right).$$

(ii) We have

$$\int_0^1 \frac{x^2}{\sqrt{(1-x^5)}} \, dx = \int_0^1 x^2 (1-x^5)^{-1/2} \, dx$$

$$= \int_0^1 x^2 \cdot \frac{1}{x^4} (1 - x^5)^{-1/2} . x^4 \, dx = \int_0^1 x^{-2} (1 - x^5)^{-1/2} x^4 \, dx$$

$$= \int_0^1 y^{-2/5} (1 - y)^{-1/2} \cdot \frac{1}{5} dy, \quad \text{putting } x^5 = y \text{ so that } 5x^4 \, dx = dy$$

$$= \frac{1}{5} \int_0^1 y^{-2/5} (1 - y)^{-1/2} \, dy = \frac{1}{5} \int_0^1 y^{(3/5) - 1} (1 - y)^{(1/2) - 1} \, dy$$

$$= \frac{1}{5} \mathbf{B} \left(\frac{3}{5}, \frac{1}{2} \right) .$$

(iii) Proceed as in part (i).

Example 2: *Prove that*

$$\int_0^a (a - x)^{m - 1} . x^{n - 1} \, dx = a^{m + n - 1} \mathbf{B} (m, n) = \frac{a^{m+n-1} \Gamma m \Gamma n}{\Gamma (m + n)} .$$

Solution: We have

$$\int_0^a (a - x)^{m - 1} x^{n - 1} \, dx$$

$$= \int_0^1 (a - ay)^{m - 1} (ay)^{n - 1} a \, dy, \quad \text{putting } x = ay$$

$$= \int_0^1 a^{(m - 1) + (n - 1) + 1} (1 - y)^{m - 1} y^{n - 1} \, dy$$

$$= a^{m + n - 1} \int_0^1 y^{n - 1} (1 - y)^{m - 1} \, dy$$

$$= a^{m + n - 1} \mathbf{B} (n, m) = a^{m + n - 1} \mathbf{B} (m, n) \qquad [\because \mathbf{B} (m, n) = \mathbf{B} (n, m)]$$

$$= \frac{a^{m + n - 1} \Gamma m \Gamma n}{\Gamma (m + n)} . \qquad \left[\because \mathbf{B} (m, n) = \frac{\Gamma m \Gamma n}{\Gamma (m + n)} \right]$$

Example 3: *Show that if m, n are positive, then*

$$\int_a^b (x - a)^{m - 1} (b - x)^{n - 1} \, dx = (b - a)^{m + n - 1} . \mathbf{B} (m, n)$$

$$= (b - a)^{m + n - 1} \cdot \frac{\Gamma m \Gamma n}{\Gamma (m + n)} .$$

Solution: The given integral is

$$\int_a^b (x - a)^{m - 1} (b - x)^{n - 1} \, dx .$$

Put $x = a + (b - a) y$ so that $dx = (b - a) dy$.

Also when $x = a$, $y = 0$ and when $x = b$, $y = 1$.

$$\therefore \quad \int_a^b (x - a)^{m - 1} (b - x)^{n - 1} \, dx$$

$$= \int_0^1 [(b - a) y]^{m - 1} [b - a - (b - a) y]^{n - 1} . (b - a) \, dy$$

$$= \int_0^1 (b - a)^{m - 1} . y^{m - 1} . (b - a)^{n - 1} . (1 - y)^{n - 1} . (b - a) \, dy$$

$$= (b-a)^{m+n-1} \int_0^1 y^{m-1} (1-y)^{n-1}\, dy = (b-a)^{m+n-1}\, \mathbf{B}(m, n)$$

$$= (b-a)^{m+n-1} \cdot \frac{\Gamma m \,\Gamma n}{\Gamma (m+n)} \cdot \qquad \left[\because \mathbf{B}(m,n) = \frac{\Gamma m\, \Gamma n}{\Gamma (m+n)}\right]$$

Example 4: *Show that* $\displaystyle\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\, dx = \frac{1}{(a+b)^m \cdot a^n}\, \mathbf{B}(m, n)$.

Solution: The given integral

$$I = \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\, dx$$

$$= \int_0^1 \left(\frac{x}{a+bx}\right)^{m-1} \cdot \left(\frac{1-x}{a+bx}\right)^{n-1} \cdot \frac{1}{(a+bx)^2}\, dx.$$ **[Note]**

Put $\dfrac{x}{a+bx} = \dfrac{y}{a+b}$ so that $\dfrac{(a+bx).1 - x.b}{(a+bx)^2}\, dx = \dfrac{dy}{a+b}$

i.e., $\dfrac{1}{(a+bx)^2}\, dx = \dfrac{dy}{a(a+b)}$.

Further $\dfrac{1-x}{a+bx} = \dfrac{1}{a}\,\dfrac{a-ax}{a+bx} = \dfrac{1}{a}\left[\dfrac{a+bx-ax-bx}{a+bx}\right] = \dfrac{1}{a}\left[1 - \dfrac{x(a+b)}{a+bx}\right] = \dfrac{1-y}{a}$.

Also when $x = 0$, $y = 0$ and when $x = 1$, $y = 1$.

$$\therefore \quad I = \int_0^1 \left(\frac{y}{a+b}\right)^{m-1} \left(\frac{1-y}{a}\right)^{n-1} \cdot \frac{dy}{a(a+b)}$$

$$= \frac{1}{(a+b)^m \cdot a^n} \int_0^1 y^{m-1}(1-y)^{n-1}\, dy = \frac{\mathbf{B}(m,n)}{(a+b)^m \cdot a^n}.$$

Example 5: *Prove that* $\dfrac{\mathbf{B}(m+1, n)}{\mathbf{B}(m, n)} = \dfrac{m}{m+n}$.

Solution: We have $\mathbf{B}(m+1, n) = \mathbf{B}(n, m+1)$

[By the symmetry of Beta function]

$$= \int_0^1 x^{n-1}(1-x)^{(m+1)-1}\, dx = \int_0^1 (1-x)^m\, x^{n-1}\, dx$$ **[Note]**

$$= \left[(1-x)^m \cdot \frac{x^n}{n}\right]_0^1 - \int_0^1 m(1-x)^{m-1}(-1) \cdot \frac{x^n}{n}\, dx,$$

(integrating by parts)

$$= 0 + \frac{m}{n}\int_0^1 x^{n-1}. x(1-x)^{m-1}\, dx$$

$$= \frac{m}{n}\int_0^1 x^{n-1}[1-(1-x)](1-x)^{m-1}\, dx$$

$$= \frac{m}{n}\left[\int_0^1 x^{n-1}(1-x)^{m-1}\,dx - \int_0^1 x^{n-1}(1-x)^m\,dx\right]$$

$$= \frac{m}{n}[\mathbf{B}(n,m) - \mathbf{B}(n,m+1)] = \frac{m}{n}\mathbf{B}(m,n) - \frac{m}{n}\mathbf{B}(m+1,n)$$

or $\left(1+\frac{m}{n}\right)\mathbf{B}(m+1,n) = \frac{m}{n}\mathbf{B}(m,n)$ [By transposition]

or $(n+m)\,\mathbf{B}(m+1,n) = m\,\mathbf{B}(m,n)$ or $\dfrac{\mathbf{B}(m+1,n)}{\mathbf{B}(m,n)} = \dfrac{m}{m+n}$.

3 Another Form of Beta Function

$$\mathbf{B}(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx,\ m>0,\ n>0.$$

Proof: By the definition of Beta function, we have

$$\mathbf{B}(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx.$$

Put $x = \frac{1}{1+y}$ so that $dx = -\frac{1}{(1+y)^2}\,dy$.

Also when $x \to 0$, $y \to \infty$ and when $x = 1$, $y = 0$.

$$\therefore \quad \mathbf{B}(m,n) = \int_\infty^0 \frac{1}{(1+y)^{m-1}}\cdot\left[1-\frac{1}{1+y}\right]^{n-1}\cdot\left[-\frac{1}{(1+y)^2}\right]dy$$

$$= \int_0^\infty \frac{1}{(1+y)^{m-1+2}}\cdot\frac{y^{n-1}}{(1+y)^{n-1}}\,dy = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx. \qquad \ldots(1)$$

[By a property of definite integrals]

Again since Beta function is symmetrical in m and n, we have

$$\mathbf{B}(m,n) = \mathbf{B}(n,m) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx, \text{ by (1).}$$

Thus $\mathbf{B}(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx,\ m>0,\ n>0.$

Illustrative Examples

Example 6: *Prove that* $\int_0^\infty \frac{x^{m-1} - x^{n-1}}{(1+x)^{m+n}}\,dx = 0,\ m>0,\ n>0.$

Solution: The given integral is $= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx - \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$

$$= \mathbf{B}(m, n) - \mathbf{B}(n, m)$$

$$= \mathbf{B}(m, n) - \mathbf{B}(m, n) = 0 .$$

Example 7: *Express* $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx$ *in terms of Beta function where* $m > 0$, $n > 0$, $a > 0$, $b > 0$.

Solution: In the given integral put $bx = ay$ *i.e.*, $x = (a/b)\,y$ so that $dx = (a/b)\,dy$. When $x = 0$, $y = 0$ and when $x \to \infty$, $y \to \infty$.

$$\therefore \quad \int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx = \int_0^\infty \left(\frac{a}{b}\,y\right)^{m-1} \cdot \frac{1}{(a+ay)^{m+n}} \cdot \frac{a}{b}\,dy$$

$$= \int_0^\infty \frac{a^{m-1}\,y^{m-1}\,a}{b^{m-1}.\,a^{m+n}\,(1+y)^{m+n}\,b}\,dy = \frac{1}{a^n b^m}\int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$$

$$= \frac{1}{a^n b^m}\,\mathbf{B}(m, n).$$ [By article 3]

4 Gamma Function

Definition: *The definite integral*

$$\int_0^\infty e^{-x}\,x^{n-1}\,dx, \text{ for } n > 0$$

is called the ***Gamma Function*** *and is denoted by* $\Gamma(n)$ [*read as "Gamma n"*].

Thus $\quad \boldsymbol{\Gamma(n) = \int_0^\infty e^{-\infty} x^{n-1} dx, \text{ for } n > 0.}$

Gamma function is also called **Eulerian integral of the second kind**.

5 Elementary Properties of Gamma Function

(i) $\Gamma(n+1) = n\,\Gamma(n)$, *where* $n > 0$

and (ii) $\Gamma(n) = (n-1)\,!$, *where n is a positive integer.*

Proof: By the definition of gamma function, we have

$$\Gamma(n+1) = \int_0^\infty e^{-x}\,x^{(n+1)-1}\,dx = \int_0^\infty x^n\,e^{-x}\,dx$$

$$= [-e^{-x}\,x^n]_0^\infty + \int_0^\infty e^{-x}.\,nx^{n-1}\,dx, \qquad \ldots(1)$$

integrating by parts taking e^{-x} as the second function.

Now $$\lim_{x \to \infty} \frac{x^n}{e^x} = \lim_{x \to \infty} \frac{x^n}{1 + x + (x^2/2!) + \ldots + (x^n/n!) + \ldots}$$

$$= \lim_{x \to \infty} \frac{1}{\frac{1}{x^n} + \frac{1}{x^{n-1}} + \ldots + \frac{1}{n!} + \frac{x}{(n+1)!} + \ldots\ldots} = \frac{1}{\infty} = 0.$$

$\therefore$ from (1), we get $\Gamma(n+1) = 0 + n \int_0^\infty e^{-x} x^{n-1}\, dx$, $[\because n > 0]$

$= n\Gamma(n)$, which proves the result (i).

(ii) We have $\Gamma(n) = \Gamma[(n-1)+1] = (n-1)\Gamma(n-1)$. $[\because \Gamma(n+1) = n\Gamma(n)]$

Similarly $\Gamma(n-1) = (n-2)\Gamma(n-2), \ldots$ etc.

Hence if n is a +ive integer, then proceeding as above, we get

$$\Gamma(n) = (n-1)(n-2)\ldots 2.1\,\Gamma(1).$$

But $$\Gamma(1) = \int_0^\infty e^{-x} x^{1-1}\, dx = \int_0^\infty e^{-x} x^0\, dx = \int_0^\infty e^{-x}.1\, dx$$

$$= \left[\frac{e^{-x}}{-1}\right]_0^\infty = -\left[\lim_{x \to \infty} \frac{1}{e^x} - e^0\right] = -[0-1] = 1.$$

Hence $\Gamma(n) = (n-1)(n-2)\ldots 2.1.1 = (n-1)!$ if n is a + ive integer.

Remember: $\Gamma(n) = (n-1)\Gamma(n-1)$, where $n > 1$ and $\Gamma(1) = 1$.

Also it may be remarked that $\Gamma(0) = \infty$ and $\Gamma(-n) = \infty$ where n is a positive integer.

6 Some Transformations of Gamma Function

We have $$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx. \quad \ldots(1)$$

(i) Put $x = ay$ so that $dx = a\, dy$; when $x = 0$, $y = 0$ and when $x \to \infty$, $y \to \infty$.

$\therefore$ $$\Gamma(n) = \int_0^\infty e^{-ay} a^n y^{n-1}\, dy.$$

Hence $$\int_0^\infty e^{-ay} y^{n-1}\, dy = \frac{\Gamma(n)}{a^n}.$$ **(Remember)**

(ii) In (1) if we put $x = \log(1/y)$ or $y = e^{-x}$ so that $dy = -e^{-x}\, dx$,

then $$\Gamma(n) = -\int_1^0 \left(\log \frac{1}{y}\right)^{n-1} dy = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy.$$

(iii) In (1) if we put $x^n = y$ so that $nx^{n-1}\, dx = dy$, we get

$$\Gamma(n) = \frac{1}{n} \int_0^\infty e^{-(y)^{1/n}}\, dy$$

or $$\int_0^\infty e^{-(y)^{1/n}}\, dy = n\Gamma(n) = \Gamma(n+1).$$

7 Relation between Beta and Gamma Functions

$$\mathbf{B}\,(\boldsymbol{m},\boldsymbol{n})=\frac{\Gamma\,(\boldsymbol{m})\,\Gamma\,(\boldsymbol{n})}{\Gamma\,(\boldsymbol{m}+\boldsymbol{n})},\ \textit{where } \boldsymbol{m}>\mathbf{0},\ \boldsymbol{n}>\mathbf{0}.$$

(Agra 2001; Lucknow 07; Kanpur 09; Garhwal 10; Kumaun 15)

Proof: We have

$$\frac{\Gamma\,(m)}{z^m}=\int_0^\infty e^{-zx}\,x^{m-1}\,dx. \qquad \text{[See article 6, part (ii)]}$$

$$\therefore \qquad \Gamma\,(m)=z^m\int_0^\infty e^{-zx}.\,x^{m-1}\,dx=\int_0^\infty z^m\,e^{-zx}\,x^{m-1}\,dx.$$

Multiplying both sides by $e^{-z}\,z^{n-1}$, we get

$$\Gamma\,(m)\,e^{-z}\,z^{n-1}=\int_0^\infty e^{-z(1+x)}\,z^{m+n-1}\,x^{m-1}\,dx. \qquad ...(1)$$

Now integrating both sides of (1) with respect to z from 0 to ∞, we get

$$\Gamma\,(m)\int_0^\infty e^{-z}\,z^{n-1}\,dz=\int_0^\infty\left[\int_0^\infty e^{-z(1+x)}\,z^{m+n-1}\,x^{m-1}\,dx\right]dz$$

or

$$\Gamma\,(m)\,\Gamma\,(n)=\int_0^\infty\left[\int_0^\infty e^{-z(1+x)}\,z^{m+n-1}\,dz\right]x^{m-1}\,dx$$

$$=\int_0^\infty\frac{\Gamma\,(m+n)}{(1+x)^{m+n}}\,x^{m-1}\,dx \qquad \text{[By article 6, part (ii)]}$$

$$=\Gamma\,(m+n)\int_0^\infty\frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$=\Gamma\,(m+n)\,.\,B\,(m,n),\ \text{by article 3}.$$

$$\therefore \qquad \mathbf{B}\,(m,n)=\frac{\Gamma\,(m)\,\Gamma\,(n)}{\Gamma\,(m+n)}.$$

Thus remember that $\displaystyle\int_0^1 x^{m-1}\,(1-x)^{n-1}\,dx=\frac{\Gamma\,(m)\,\Gamma\,(n)}{\Gamma\,(m+n)}$.

Corollary: $\boldsymbol{\Gamma\,(n)\,\Gamma\,(1-n)=\dfrac{\pi}{\sin n\pi}}$, **where** $\boldsymbol{0<n<1}$.

Proof: We know that $\displaystyle\mathbf{B}\,(m,n)=\int_0^\infty\frac{x^{n-1}}{(1+x)^{m+n}}\,dx$, [See article 3]

and $\qquad \mathbf{B}\,(m,n)=\dfrac{\Gamma\,(m)\,\Gamma\,(n)}{\Gamma\,(m+n)}$, where $m>0$ and $n>0$.

$$\therefore \qquad \frac{\Gamma\,(m)\,\Gamma\,(n)}{\Gamma\,(m+n)}=\int_0^\infty\frac{x^{n-1}}{(1+x)^{m+n}}\,dx.$$

Putting $m+n=1$ or $m=1-n$ in the above relation, we get

$$\frac{\Gamma\,(1-n)\,\Gamma\,(n)}{\Gamma\,(1)}=\int_0^\infty\frac{x^{n-1}}{1+x}\,dx,\ \text{where } 0<n<1.$$

[Note that $m>0\Rightarrow 1-n>0\Rightarrow n<1$.]

But $\Gamma(1) = 1$. Also

$$\int_0^\infty \frac{x^{n-1}}{1+x}\, dx = \frac{\pi}{\sin n\pi}.$$ **(Remember)**

$$\therefore \qquad \Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}, \text{ where } 0 < n < 1.$$

8 The Value of $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof: We know that $B(m, n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$ ...(1)

If we take $m = \frac{1}{2}, n = \frac{1}{2}$, then from (1), we have

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{\Gamma(1)} = \left[\Gamma\left(\frac{1}{2}\right)\right]^2 \qquad [\because \Gamma(1) = 1]$$

Thus $$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = B\left(\frac{1}{2}, \frac{1}{2}\right) = \int_0^1 x^{1/2-1}(1-x)^{1/2-1}\, dx,$$

by the definition of Beta function

$$= \int_0^1 x^{-1/2}(1-x)^{-1/2}\, dx.$$

Now put $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\, d\theta$.

Also when $x = 0, \theta = 0$ and when $x = 1, \theta = \frac{1}{2}\pi$.

$$\therefore \qquad \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \int_0^{\pi/2} \frac{1}{\sin\theta}\cdot\frac{1}{\cos\theta}\cdot 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} d\theta = 2\,[\theta]_0^{\pi/2}$$

$$= 2\left[\frac{1}{2}\pi - 0\right] = \pi.$$

Taking square root of both the sides, we get

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$ **(Remember)**

Important Deduction: *To prove that* $\int_0^\infty e^{-x^2}\, dx = \dfrac{\sqrt{\pi}}{2}$.

Proof: Let $I = \int_0^\infty e^{-x^2}\, dx$.

Put $x^2 = z$ so that $2x\, dx = dz$

or $$dx = \frac{1}{2x}\,dz = \frac{1}{2\sqrt{z}}\,dz = \frac{1}{2}\,z^{-1/2}\,dz\,.$$

Also when $x = 0, z = 0$ and when $x \to \infty, z \to \infty$.

$$\therefore \qquad I = \int_0^\infty e^{-z}\,\frac{1}{2}\,z^{-1/2}\,dz$$

$$= \frac{1}{2}\int_0^\infty e^{-z}\,z^{1/2-1}\,dz = \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

Hence $$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{\mathbf{2}}.$$ **(Remember)**

9 $$\int_0^{\pi/2} \cos^m\theta \sin^n\theta\,d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)},\quad m > -1, n > -1$$

Proof: Put $\sin^2\theta = x$ so that $2\sin\theta\cos\theta\,d\theta = dx$

or $$2\sin\theta.\sqrt{(1-\sin^2\theta)}\,d\theta = dx \quad\text{or}\quad 2\,x^{1/2}.\sqrt{(1-x)}\,d\theta = dx\,.$$

$$\therefore \qquad d\theta = \frac{dx}{2\,x^{1/2}\,(1-x)^{1/2}}.$$

Also when $\theta = 0$, $x = 0$ and when $\theta = \frac{1}{2}\pi$, $x = 1$.

$$\therefore \qquad \int_0^{\pi/2} \cos^m\theta\sin^n\theta\,d\theta = \int_0^{\pi/2} (1-\sin^2\theta)^{m/2}.\sin^n\theta\,d\theta$$

$$= \int_0^1 (1-x)^{m/2}.\,x^{n/2}.\frac{dx}{2\,x^{1/2}\,(1-x)^{1/2}}$$

$$= \frac{1}{2}\int_0^1 x^{(n-1)/2}\,(1-x)^{(m-1)/2}\,dx$$

$$= \frac{1}{2}\int_0^1 x^{\{(n+1)/2\}-1}\,(1-x)^{\{(m+1)/2\}-1}\,dx$$

$$= \frac{1}{2}\,B\left(\frac{m+1}{2},\frac{n+1}{2}\right),\quad \text{provided } m > -1 \text{ and } n > -1$$

$$= \frac{1}{2}\,\frac{\Gamma\frac{1}{2}(m+1)\,\Gamma\frac{1}{2}(n+1)}{\Gamma\frac{1}{2}(m+1+n+1)} \qquad \left[\because\ \mathbf{B}(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}\right]$$

$$= \frac{\Gamma\frac{1}{2}(m+1)\,\Gamma\frac{1}{2}(n+1)}{2\,\Gamma\frac{1}{2}(m+n+2)}.$$

10 Some Important Transformations of Beta Function

Beta function can be transformed into many other forms. A few of them are given below.

(i) We know that $\int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}}\, dy = \mathbf{B}(m, n)$.

Now $$\int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}}\, dy = \int_0^{1} \frac{y^{m-1}}{(1+y)^{m+n}}\, dy + \int_1^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}}\, dy .$$

Making the substitution $y = 1/x$ in the last integral, we get

$$\int_1^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}}\, dy = \int_0^{1} \frac{x^{n-1}\, dx}{(1+x)^{m+n}} .$$

$$\therefore \quad \mathbf{B}(m, n) = \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}}\, dy$$

$$= \int_0^{1} \frac{y^{m-1}}{(1+y)^{m+n}}\, dy + \int_0^{1} \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

$$= \int_0^{1} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx + \int_0^{1} \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

$$= \int_0^{1} \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx .$$

Hence $$\boldsymbol{\int_0^{1} \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx = \mathbf{B}(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}} .$$

(ii) We know that $\int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx = \mathbf{B}(m, n)$.

If we put $x = \frac{ay}{b}$, so that $dx = \frac{a}{b}\, dy$, we get

$$\int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx = a^m\, b^n \int_0^{\infty} \frac{y^{m-1}}{(ay+b)^{m+n}}\, dy .$$

$$\therefore \quad \int_0^{\infty} \frac{y^{m-1}}{(ay+b)^{m+n}}\, dy = \frac{1}{a^m\, b^n} \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx = \frac{1}{a^m\, b^n}\, \mathbf{B}(m, n) .$$

Hence $$\boldsymbol{\int_0^{\infty} \frac{y^{m-1}}{(ay+b)^{m+n}}\, dy = \frac{\Gamma(m)\,\Gamma(n)}{a^m b^n\, \Gamma(m+n)}} .$$

Again putting $y = \tan^2 \theta$ *i.e.*, $dy = 2 \tan\theta \sec^2\theta\, d\theta$ in the integral just obtained, we get

$$\boldsymbol{\int_0^{\pi/2} \frac{\sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta}{(a \sin^2\theta + b \cos^2\theta)^{m+n}} = \frac{\Gamma(m)\,.\Gamma(n)}{2\, a^m b^n\, \Gamma(m+n)}} .$$

(iii) We know that $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \mathbf{B}(m,n)$.

Putting $x = \sin^2\theta$, so that $dx = 2\sin\theta\cos\theta\,d\theta$, we have

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta.$$

$$\therefore \qquad \int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta = \frac{\mathbf{B}(m,n)}{2} = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}.$$

This result may also be written in the form

$$\int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\cdot\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)},$$

by putting $2m-1 = p$ and $2n-1 = q$.

(iv) We know that $\int_0^1 y^{m-1}(1-y)^{n-1}\,dy = \mathbf{B}(m,n)$.

Putting $y = \frac{x-b}{a-b}$, so that $dy = \frac{dx}{a-b}$, we have

$$\int_0^1 y^{m-1}.(1-y)^{n-1}\,dy = \int_b^a \left(\frac{x-b}{a-b}\right)^{m-1}\left(\frac{a-x}{a-b}\right)^{n-1}\cdot\frac{dx}{a-b}$$

$$= \frac{1}{(a-b)^{m+n-1}}\int_b^a (x-b)^{m-1}(a-x)^{n-1}\,dx.$$

$$\therefore \qquad \int_b^a (x-b)^{m-1}(a-x)^{n-1}\,dx = (a-b)^{m+n-1}\int_0^1 y^{m-1}(1-y)^{n-1}\,dy$$

or

$$\int_b^a (x-b)^{m-1}(a-x)^{n-1}\,dx = (a-b)^{m+n-1}\,\mathbf{B}(m,n) = (a-b)^{m+n-1}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

11 Duplication Formula

$$\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m), \text{ where } m > 0.$$

Proof: We know that

$$\mathbf{B}(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \text{ where } m > 0,\ n > 0.$$

If we take $n = m$, then

$$\mathbf{B}(m, m) = \frac{[\Gamma(m)]^2}{\Gamma(2m)}. \quad \text{...(1)}$$

Again by the definition of Beta function, we have

$$\mathbf{B}(m, m) = \int_0^1 x^{m-1}(1-x)^{m-1}\, dx.$$

Let us put $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\, d\theta$.

Also when $x = 0, \theta = 0$ and when $x = 1, \theta = \frac{1}{2}\pi$.

Then $$\mathbf{B}(m, m) = \int_0^{\pi/2} \sin^{2(m-1)}\theta . \cos^{2(m-1)}\theta . 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta . \cos^{2m-1}\theta\, d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta\cos\theta)^{2m-1}\, d\theta$$

$$= 2\int_0^{\pi/2} \left(\frac{\sin 2\theta}{2}\right)^{2m-1} d\theta = \frac{1}{2^{2m-2}}\int_0^{\pi/2} \sin^{2m-1} 2\theta\, d\theta$$

$$= \frac{1}{2^{2m-2}}\int_0^{\pi} \sin^{2m-1}\phi \cdot \frac{d\phi}{2},$$

putting $2\theta = \phi$ so that $d\theta = \frac{1}{2}d\phi$

$$= \frac{1}{2^{2m-1}}\int_0^{\pi} \sin^{2m-1}\phi\, d\phi$$

$$= \frac{1}{2^{2m-1}} \cdot 2\int_0^{\pi/2} \sin^{2m-1}\phi\, d\phi \quad \textbf{(Note)}$$

$$= \frac{1}{2^{2m-2}}\int_0^{\pi/2} \sin^{2m-1}\phi . \cos^0\phi\, d\phi \quad \textbf{(Note)}$$

$$= \frac{1}{2^{2m-2}} \frac{\Gamma\frac{1}{2}(2m-1+1)\,\Gamma\frac{1}{2}(0+1)}{2\,\Gamma\frac{1}{2}(2m-1+0+2)} = \frac{1}{2^{2m-1}} \cdot \frac{\Gamma(m)\,\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})}$$

$$= \frac{1}{2^{2m-1}} \cdot \frac{\Gamma(m)\sqrt{\pi}}{\Gamma(m+\frac{1}{2})}. \quad \text{...(2)}$$

$[\because \Gamma(\frac{1}{2}) = \sqrt{\pi}]$

Now equating the two values of $B(m, m)$ obtained in (1) and (2), we get

$$\frac{[\Gamma(m)]^2}{\Gamma(2m)} = \frac{1}{2^{2m-1}} \cdot \frac{\Gamma(m).\sqrt{\pi}}{\Gamma(m+\frac{1}{2})}$$

or $$\Gamma(m)\,\Gamma(m+\frac{1}{2}) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m). \quad \textbf{(Remember)}$$

12 $\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right)=\frac{(2\pi)^{(n-1)/2}}{n^{1/2}}$,

where n is a positive integer.

Proof: Let $A=\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right)$. ...(1)

Writing the above expression in the reverse order, we have

$$A=\Gamma\left(1-\frac{1}{n}\right)\Gamma\left(1-\frac{2}{n}\right)\ldots\Gamma\left(1-\frac{n-2}{n}\right)\Gamma\left(1-\frac{n-1}{n}\right). \quad ...(2)$$

Multiplying (1) and (2), we get

$$A^2=\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(1-\frac{2}{n}\right).....\Gamma\left(\frac{n-1}{n}\right)\Gamma\left(1-\frac{n-1}{n}\right)$$

$$=\frac{\pi^{n-1}}{\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{n-1}{n}\pi}. \quad \text{[See corollary of article 7]} \quad ...(3)$$

To calculate this expression, we factorize $1-x^{2n}$.

Now the roots of the equation $x^{2n}-1=0$ are given by

$$x=(1)^{1/2n}=(\cos 2r\pi+i\sin 2r\pi)^{1/2n}$$

$$=\cos\frac{r\pi}{n}+i\sin\frac{r\pi}{n}, \quad \text{where } r=0,1,2,\ldots,2n-1.$$

Hence, we have

$$1-x^{2n}=(1-x)(1+x)\left(x-\cos\frac{\pi}{n}-i\sin\frac{\pi}{n}\right)\left(x-\cos\frac{\pi}{n}+i\sin\frac{\pi}{n}\right)\ldots$$

$$\ldots\left(x-\cos\frac{n-1}{n}\pi-i\sin\frac{n-1}{n}\pi\right)\left(x-\cos\frac{n-1}{n}\pi+i\sin\frac{n-1}{n}\pi\right)$$

$$=(1-x^2)\left(1-2x\cos\frac{\pi}{n}+x^2\right)\left(1-2x\cos\frac{2\pi}{n}+x^2\right)\ldots$$

$$\ldots\left(1-2x\cos\frac{n-1}{n}\pi+x^2\right).$$

$$\therefore \quad \frac{1-x^{2n}}{1-x^2}=\left(1-2x\cos\frac{\pi}{n}+x^2\right)\left(1-2x\cos\frac{2\pi}{n}+x^2\right)\ldots\left(1-2x\cos\frac{n-1}{n}\pi+x^2\right).$$

Putting $x=1$ and $x=-1$ respectively, we have in the limit,

$$n=\left(2-2\cos\frac{\pi}{n}\right)\left(2-2\cos\frac{2\pi}{n}\right)\ldots\left(2-2\cos\frac{n-1}{n}\pi\right)$$

and $$n=\left(2+2\cos\frac{\pi}{n}\right)\left(2+2\cos\frac{2\pi}{n}\right)\ldots\left(2+2\cos\frac{n-1}{n}\pi\right).$$

Multiplying these, we get

$$n^2 = 2^{2n-2} \sin^2 \frac{\pi}{n} \cdot \sin^2 \frac{2\pi}{n} \ldots \sin^2 \frac{n-1}{n} \pi$$

or $$n = 2^{n-1} . \sin \frac{\pi}{n} \cdot \sin \frac{2\pi}{n} \ldots \sin \frac{n-1}{n} \pi .$$

Hence, from (3), we get

$$A^2 = \frac{\pi^{n-1}}{n / 2^{n-1}} = \frac{(2\pi)^{n-1}}{n} \quad \text{or} \quad A = \frac{(2\pi)^{(n-1)/2}}{n^{1/2}} .$$

Remark: The value of $\sin \frac{\pi}{n} \sin \frac{2\pi}{n} \ldots \sin \frac{n-1}{n} \pi$ can also be found by using the trigonometrical identity

$$\frac{\sin n\theta}{\sin \theta} = 2^{n-1} \sin \left(\theta + \frac{\pi}{n} \right) \sin \left(\theta + \frac{2\pi}{n} \right) \sin \left(\theta + \frac{3\theta}{n} \right) \ldots \sin \left(\theta + \frac{n-1}{n} \pi \right) .$$

From the above identity, we have

$$\frac{\sin n\theta}{n\theta} \cdot \frac{\theta}{\sin \theta} \cdot n = 2^{n-1} \sin \left(\theta + \frac{\pi}{n} \right) \sin \left(\theta + \frac{2\pi}{n} \right) \ldots \sin \left(\theta + \frac{n-1}{n} \pi \right) .$$

Taking limit as $\theta \to 0$, we get

$$n = 2^{n-1} \sin \frac{\pi}{n} \sin \frac{2\pi}{n} \sin \frac{3\pi}{n} \ldots \sin \frac{n-1}{n} \pi .$$

13 **(i)** $\int_0^\infty e^{-ax} \cos bx . x^{m-1}\, dx = \frac{\Gamma m}{k^m} \cos m\alpha$

(ii) $\int_0^\infty e^{-ax} \sin bx . x^{m-1} dx = \frac{\Gamma m}{k^m} \sin m\alpha,$

where $k = \sqrt{(a^2 + b^2)}$ **and** $\alpha = \tan^{-1} \left(\frac{a}{b} \right)$

Proof: We have

$$\int_0^\infty e^{-ax} e^{ibx} x^{m-1} dx = \int_0^\infty e^{-(a-ib)x} x^{m-1} dx = \frac{\Gamma(m)}{(a-ib)^m}$$

[See article 6, part (i)]

$$= (a - ib)^{-m} \Gamma(m) . \qquad \ldots(1)$$

Let us first separate $(a - ib)^{-m}$ into real and imaginary parts.

Put $a = k \cos \alpha$ and $b = k \sin \alpha$ so that

$$\alpha = \tan^{-1} (b / a) \text{ and } k = \sqrt{(a^2 + b^2)} .$$

Then $$(a - ib)^{-m} = [k (\cos \alpha - i \sin \alpha)]^{-m}$$

$$= k^{-m} (\cos \alpha - i \sin \alpha)^{-m}$$

$$= k^{-m} (\cos m\alpha + i \sin m\alpha), \text{ by De-Moivre's theorem.}$$

Now from (1), we have

$$\int_0^\infty e^{-ax} e^{ibx} x^{m-1} dx = k^{-m} (\cos m\alpha + i \sin m\alpha) \Gamma(m)$$

or $$\int_0^\infty e^{-ax} (\cos bx + i \sin bx) x^{m-1} dx = \frac{\Gamma(m)}{k^m} (\cos m\alpha + i \sin m\alpha),$$

$$[\because e^{i\theta} = \cos\theta + i\sin\theta, \text{ by Euler's theorem}]$$

or $$\int_0^\infty e^{-ax} \cos bx \,.\, x^{m-1} dx + i \int_0^\infty e^{-ax} \sin bx \,.\, x^{m-1} dx$$

$$= \frac{\Gamma(m)}{k^m} \cos m\alpha + i \frac{\Gamma(m)}{k^m} \sin m\alpha. \quad \text{...(2)}$$

Equating real and imaginary parts in (2), we get

$$\int_0^\infty e^{-ax} \cos bx \,.\, x^{m-1} dx = \frac{\Gamma(m)}{k^m} \cos m\alpha,$$

and $$\int_0^\infty e^{-ax} \sin bx \,.\, x^{m-1} dx = \frac{\Gamma(m)}{k^m} \sin m\alpha,$$

where $k = \sqrt{(a^2 + b^2)}$ and $\alpha = \tan^{-1}(b/a)$.

Deductions: **(i)** If we put $a = 0$, then $\alpha = \pi/2$ and $k = b$.

Hence $$\int_0^\infty x^{m-1} \cos bx \, dx = \frac{\Gamma(m)}{b^m} \cos \frac{m\pi}{2}$$

and $$\int_0^\infty x^{m-1} \sin bx \, dx = \frac{\Gamma(m)}{b^m} \sin \frac{m\pi}{2}.$$

(ii) If we put $m = 1$, then

$$\int_0^\infty e^{-ax} \cos bx \, dx = \frac{\Gamma(1)}{k} \cos\alpha = \frac{k\cos\alpha}{k^2} = \frac{a}{a^2 + b^2}$$

and $$\int_0^\infty e^{-ax} \sin bx \, dx = \frac{\Gamma(1)}{k} \sin\alpha = \frac{k\sin\alpha}{k^2} = \frac{b}{a^2 + b^2}.$$

Illustrative Examples

Example 8: *Evaluate the following integrals:*

(i) $\displaystyle\int_0^\infty \frac{x^8 (1 - x^6)}{(1 + x)^{24}} dx$,

(ii) $\displaystyle\int_0^\infty \frac{x^4 (1 + x^5)}{(1 + x)^{15}} dx$,

(iii) $\displaystyle\int_0^\infty \frac{x \, dx}{1 + x^6}$.

Solution: **(i)** We have

$$\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx = \int_0^\infty \frac{x^8\,dx}{(1+x)^{24}} - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}\,dx$$

$$= \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}}\,dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}\,dx$$

$$= \mathbf{B}(9,15) - \mathbf{B}(15,9), \qquad \text{[By article 3]}$$

$$= \mathbf{B}(9,15) - \mathbf{B}(9,15),$$

by symmetry of Beta function

$$= 0.$$

(ii) We have $\displaystyle\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\,dx = \int_0^\infty \frac{x^4\,dx}{(1+x)^{15}} + \int_0^\infty \frac{x^9\,dx}{(1+x)^{15}}$

$$= \int_0^\infty \frac{x^{5-1}}{(1+x)^{5+10}}\,dx + \int_0^\infty \frac{x^{10-1}}{(1+x)^{10+5}}\,dx$$

$$= \mathbf{B}(5,10) + \mathbf{B}(10,5) = \mathbf{B}(5,10) + \mathbf{B}(5,10)$$

$$= 2\,\mathbf{B}(5,10) = 2\,\frac{\Gamma 5\,\Gamma 10}{\Gamma 15}$$

$$= 2\cdot\frac{4\cdot 3\cdot 2\cdot 1}{14\cdot 13\cdot 12\cdot 11\cdot 10} = \frac{1}{5005}.$$

(iii) Let $\displaystyle I = \int_0^\infty \frac{x\,dx}{1+x^6}.$

Put $x^6 = y$ or $x = y^{1/6}$, so that $dx = \frac{1}{6}\,y^{-5/6}\,dy$.

$$\therefore \quad I = \frac{1}{6}\int_0^\infty \frac{y^{1/6}.\,y^{-5/6}}{1+y}\,dy = \frac{1}{6}\int_0^\infty \frac{y^{-2/3}}{1+y}\,dy$$

$$= \frac{1}{6}\int_0^\infty \frac{y^{(1/3)-1}}{(1+y)^{(1/3)+(2/3)}}\,dy = \frac{1}{6}\,B\left(\frac{1}{3},\frac{2}{3}\right), \qquad \text{[By article 3]}$$

$$= \frac{1}{6}\,\frac{\Gamma\frac{1}{3}\,\Gamma\frac{2}{3}}{\Gamma\left(\frac{1}{3}+\frac{2}{3}\right)} = \frac{1}{6}\,\frac{\Gamma\frac{1}{3}\,\Gamma\left(1-\frac{1}{3}\right)}{\Gamma 1}$$

$$= \frac{1}{6}\cdot\frac{\pi}{\sin\frac{1}{3}\pi} \qquad \left[\because\ \Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{1}{6}\cdot\frac{\pi}{(\sqrt{3}/2)} = \frac{1}{6}\cdot\frac{2\pi}{\sqrt{3}} = \frac{\pi}{3\sqrt{3}}.$$

Example 9: *Show that* $\displaystyle\int_0^1 \frac{dx}{(1-x^n)^{1/2}} = \frac{\sqrt{\pi}\,\Gamma(1/n)}{n\,\Gamma(1/n+1/2)}.$

Solution: Let $x^n = \sin^2 \theta$ *i.e.*, $x = \sin^{2/n} \theta$ so that $dx = \frac{2}{n} \sin^{(2/n)-1} \theta \cos \theta \, d\theta$.

Then
$$\int_0^1 \frac{dx}{\sqrt{(1-x^n)}} = \frac{2}{n} \int_0^{\pi/2} \frac{\sin^{(2/n)-1} \theta \cos \theta \, d\theta}{\cos \theta}$$
$$= \frac{2}{n} \int_0^{\pi/2} \sin^{(2/n)-1} \theta \cos^0 \theta \, d\theta$$
$$= \frac{2}{n} \cdot \frac{\Gamma(1/n)\,\Gamma(\frac{1}{2})}{2\,\Gamma(1/n+1/2)} = \frac{\sqrt{\pi}}{n} \cdot \frac{\Gamma(1/n)}{\Gamma(1/n+1/2)}.$$

Example 10: *Evaluate* $\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} \, dx$.

Solution: We have
$$\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} \, dx = \int_0^1 \frac{x^{m-1} \, dx}{(1+x)^{m+n}} + \int_0^1 \frac{x^{n-1} \, dx}{(1+x)^{m+n}}. \qquad \text{...(1)}$$

Now in the second integral on the R.H.S. of (1), we put $x = 1/y$ so that $dx = -(1/y^2)\,dy$; also when $x \to 0$, $y \to \infty$ and when $x = 1$, $y = 1$.

$$\therefore \quad \int_0^1 \frac{x^{n-1} \, dx}{(1+x)^{m+n}} = \int_\infty^1 \frac{(1/y)^{n-1}}{(1+1/y)^{m+n}} \left(-\frac{1}{y^2} \, dy \right)$$
$$= -\int_\infty^1 \frac{y^{m+n} \, dy}{(1+y)^{m+n} . \, y^{n-1} . \, y^2} = \int_1^\infty \frac{y^{m-1}}{(1+y)^{m+n}} \, dy$$

(Note)

$$= \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} \, dx. \qquad \left[\because \int_a^b f(x) \, dx = \int_a^b f(y) \, dy \right]$$

Now from (1), we have
$$\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} \, dx = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} \, dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} \, dx$$
$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} \, dx,$$

by a property of definite integrals

$$= \mathbf{B}(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}. \qquad \text{[Refer articles 3 and 7]}$$

Example 11: *Show that* $\int_0^{\pi/2} (\tan x)^n \, dx = \frac{\pi}{2} \sec \frac{n\pi}{2}$, *where* $-1 < n < 1$.

Solution: We have

$$\int_0^{\pi/2} \tan^n x\,dx = \int_0^{\pi/2} \frac{\sin^n x}{\cos^n x}\,dx = \int_0^{\pi/2} \sin^n x \cos^{-n} x\,dx$$

$$= \frac{\Gamma\frac{1}{2}(n+1)\,.\,\Gamma\frac{1}{2}(-n+1)}{2\,\Gamma\frac{1}{2}(n-n+2)},$$

where $-n+1>0$ *i.e.*, $n<1$ and $n+1>0$ *i.e.*, $n>-1$

$$= \frac{1}{2}\Gamma\frac{1}{2}(n+1)\,\Gamma\frac{1}{2}(1-n)$$

$$= \frac{1}{2}\Gamma\frac{1}{2}(n+1)\,\Gamma\left[1-\frac{1}{2}(n+1)\right] = \frac{1}{2}\frac{\pi}{\sin\frac{1}{2}(n+1)\pi},$$

$$\left[\because\ \Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}, \text{ by cor. to article 7}\right]$$

$$= \frac{\pi}{2}\cdot\frac{1}{\sin\left(\frac{1}{2}\pi+\frac{1}{2}n\pi\right)} = \frac{\pi}{2}\cdot\frac{1}{\cos\left(\frac{1}{2}n\pi\right)}$$

$$= \frac{\pi}{2}\sec\frac{n\pi}{2}, \quad \text{where } -1<n<1.$$

Example 12: *Prove that*

(i) $\displaystyle\int_0^\infty x^{2n-1} e^{-ax^2}\,dx = \frac{\Gamma(n)}{2a^n}$;

(ii) $\displaystyle\int_0^\infty x^m e^{-ax^n}\,dx = \frac{\Gamma[(m+1)/n]}{na^{(m+1)/n}}$;

(iii) $\displaystyle\int_0^1 \frac{dx}{\sqrt{(-\log x)}} = \sqrt{\pi}$.

Solution: **(i)** Let $I = \int_0^\infty x^{2n-1} e^{-ax^2}\,dx = \int_0^\infty x^{2n-2} e^{-ax^2}\,x\,dx$.

Put $ax^2 = z$ so that $2ax\,dx = dz$. When $x=0$, $z=0$ and when $x\to\infty$, $z\to\infty$.

$$\therefore \quad I = \int_0^\infty \left(\frac{z}{a}\right)^{n-1} e^{-z}\frac{1}{2a}\,dz = \frac{1}{2a^n}\int_0^\infty e^{-z} z^{n-1}\,dz$$

$$= \frac{1}{2a^n}\Gamma(n), \text{ by definition of Gamma function.}$$

(ii) Let $\displaystyle I = \int_0^\infty x^m e^{-ax^n}\,dx = \int_0^\infty \frac{x^m}{x^{n-1}} e^{-ax^n} x^{n-1}\,dx$ **[Note]**

$$= \int_0^\infty x^{m-n+1} e^{-ax^n} x^{n-1}\,dx.$$

Put $ax^n = t$ so that $na\,x^{n-1}\,dx = dt$. Also when $x=0$, $t=0$ and when $x\to\infty$, $t\to\infty$.

$$\therefore \quad I = \int_0^\infty \left(\frac{t}{a}\right)^{(m-n+1)/n} e^{-t}\cdot\frac{1}{na}\,dt, \qquad \left[\because\ a x^n = t \Rightarrow x = \left(\frac{t}{a}\right)^{1/n}\right]$$

$$= \frac{1}{na \,.\, a^{(m-n+1)/n}} \int_0^{\infty} t^{\{(m+1)/n\}-1} e^{-t}\, dt$$

$$= \frac{1}{na^{(m+1)/n}} \Gamma\{(m+1)/n\}, \text{ by the definition of Gamma function.}$$

(iii) Let $\quad I = \int_0^1 \frac{dx}{\sqrt{(-\log x)}} = \int_0^1 \frac{dx}{\sqrt{\{\log (1/x)\}}} = \int_0^1 \left(\log \frac{1}{x}\right)^{-1/2} dx\,.$

Put $\log (1/x) = y$ *i.e.*, $1/x = e^{y}$ *i.e.*, $x = e^{-y}$ so that $dx = -e^{-y}\, dy$.

Also when $x \to \infty$, $y \to \infty$ and when $x = 1$, $y = 0$.

$\therefore \quad I = -\int_{\infty}^{0} y^{-1/2} e^{-y}\, dy = \int_0^{\infty} e^{-y}\, y^{1/2-1}\, dy$

$= \Gamma\left(\frac{1}{2}\right)$, by the def. of Gamma function

$= \sqrt{\pi}\,.$

Example 13: *Evaluate the integral*

$$\int_a^b (x-a)^p (b-x)^q\, dx, \text{ where } p \text{ and } q \text{ are positive integers.}$$

Solution: Let $\quad I = \int_a^b (x-a)^p (b-x)^q\, dx\,.$

Put $\quad x = a\cos^2\theta + b\sin^2\theta$ so that

$dx = -2a\cos\theta\sin\theta\, d\theta + 2b\sin\theta\cos\theta\, d\theta$

i.e., $\quad dx = 2(b-a)\cos\theta\sin\theta\, d\theta\,.$

Also $\quad x - a = a\cos^2\theta + b\sin^2\theta - a = b\sin^2\theta - a(1-\cos^2\theta)$

$= b\sin^2\theta - a\sin^2\theta = (b-a)\sin^2\theta$

and $\quad b - x = b - a\cos^2\theta - b\sin^2\theta = b(1-\sin^2\theta) - a\cos^2\theta$

$= (b-a)\cos^2\theta\,.$

To find the limits for θ, when $x = a$, we have

$a = a\cos^2\theta + b\sin^2\theta$

i.e., $\quad (b-a)\sin^2\theta = 0$ *i.e.*, $\sin^2\theta = 0$ as $a \neq b$ *i.e.*, $\theta = 0$

and when $x = b$, we have

$b = a\cos^2\theta + b\sin^2\theta$

i.e., $\quad (a-b)\cos^2\theta = 0$ *i.e.*, $\cos^2\theta = 0$ as $a \neq b$ *i.e.*, $\theta = \pi/2\,.$

Thus the new limits for θ are 0 to $\pi/2$. Hence the given integral

$$I = \int_0^{\pi/2} (b-a)^p \sin^{2p}\theta \,.\, (b-a)^q \cos^{2q}\theta \,.\, 2(b-a)\cos\theta\sin\theta\, d\theta$$

$$= 2(b-a)^{p+q+1} \int_0^{\pi/2} \sin^{2p+1}\theta \cos^{2q+1}\theta\, d\theta$$

$$= 2\,(b-a)^{p+q+1}\,\frac{\Gamma\{\frac{1}{2}(2p+1+1)\}\,\Gamma\{\frac{1}{2}(2q+1+1)\}}{2\,\Gamma\,(2p+1+2q+1+2)},$$

provided $2p+1>-1$ and $2q+1>-1$ *i.e.*, $p>-1$ and $q>-1$ which is so because p and q are given to be + ive integers

$$= (b-a)^{p+q+1}\,\frac{\Gamma\,(p+1)\,\Gamma\,(q+1)}{\Gamma\,(p+q+1+1)}$$

$$= (b-a)^{p+q+1}\,\frac{p\,!\,q\,!}{(p+q+1)\,!},$$

because $\Gamma\,(n+1) = n\,!$ if n is a positive integer.

Example 14: *Find the value of* $\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{3}{9}\right)\ldots\Gamma\left(\frac{8}{9}\right)$.

Solution: We know that

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{(n-1)/2}}{n^{1/2}},$$

where n is a positive integer.

Putting $n = 9$ in the above relation, we get

$$\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\ldots\Gamma\left(\frac{8}{9}\right) = \frac{(2\,\pi)^{(9-1)/2}}{9^{1/2}} = \frac{(2\,\pi)^4}{3} = \frac{16}{3}\,\pi^4.$$

Example 15: *Show that*

(i) $2^n\,\Gamma\,(n+\frac{1}{2}) = 1.3.5\ldots.(2n-1)\,\sqrt{\pi}$, *where n is a + ive integer,*

(ii) $\Gamma\left(\frac{3}{2}-x\right)\Gamma\left(\frac{3}{2}+x\right) = \left(\frac{1}{4}-x^2\right)\pi\ \sec\ \pi x$, *provided* $-1<2x<1$.

Solution: **(i)** We have

$$\Gamma\,(n+\frac{1}{2}) = (n-\frac{1}{2})\,\Gamma\,(n-\frac{1}{2}) = \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\Gamma\left(n-\frac{3}{2}\right)$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\ldots\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdot\frac{2n-5}{2}\ldots\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}$$

$$= \frac{1}{2^n}\,(2n-1)\,(2n-3)\,(2n-5)\ldots3.1.\sqrt{\pi}.$$

$\therefore$ $$2^n\,\Gamma\,(n+\frac{1}{2}) = 1.3.5\ldots.(2n-1)\,\sqrt{\pi}.$$

(ii) We have

$$\Gamma\left(\frac{3}{2}-x\right)\Gamma\left(\frac{3}{2}+x\right) = (\frac{1}{2}-x)\,\Gamma\,(\frac{1}{2}-x)\,.\,(\frac{1}{2}+x)\,\Gamma\,(\frac{1}{2}+x)$$

$$=\left(\frac{1}{4}-x^2\right)\Gamma\left(\frac{1-2\,x}{2}\right)\Gamma\left(\frac{1+2\,x}{2}\right)$$

$$=\left(\frac{1}{4}-x^2\right)\Gamma\left(\frac{1-2\,x}{2}\right)\Gamma\left(1-\frac{1-2\,x}{2}\right)$$

$$=\left(\frac{1}{4}-x^2\right)\frac{\pi}{\sin\left(\frac{1-2\,x}{2}\,\pi\right)}$$

$$=\left(\frac{1}{4}-x^2\right)\cdot\frac{\pi}{\sin\left(\frac{1}{2}\,\pi - x\pi\right)}$$

$$=\left(\frac{1}{4}-x^2\right)\cdot\frac{\pi}{\cos\,x\,\pi}=\left(\frac{1}{4}-x^2\right)\cdot\pi\sec\pi\,x\,.$$

Example 16: *With certain restrictions on the values of a, b, m and n, prove that*

$$\int_0^\infty\int_0^\infty e^{-(a\,x^2+by^2)}\,x^{2\,m-1}\,y^{2\,n-1}\,dx\,dy=\frac{\Gamma\,(m)\,\Gamma\,(n)}{4\,a^m\,b^n}\,.$$

Solution: Let us denote the given integral by I . Then

$$I=\int_0^\infty e^{-ax^2}\,x^{2\,m-1}\,dx\times\int_0^\infty e^{-by^2}\,y^{2\,n-1}\,dy=I_1\times I_2\,.$$

To evaluate I_1 , put $ax^2=t$ so that $2ax\,dx=dt$.

$$\therefore \qquad I_1=\int_0^\infty e^{-t}\,(t\,/\,a)^{(2\,m-1)/2}\cdot\frac{dt}{2\,\sqrt{(at)}}$$

$$=\frac{1}{2a^m}\int_0^\infty e^{-t}\,t^{\,m-1}\,dt$$

$$=\frac{\Gamma\,(m)}{2a^m},\text{ provided } a \text{ and } m \text{ are +ive.}$$

Similarly, $I_2=\dfrac{\Gamma\,(n)}{2b^n}$, provided b and n are +ive.

Hence $\qquad I=\dfrac{\Gamma\,m\,\Gamma\,n}{4a^m\,b^n}\cdot$

Example 17: *Show that the sum of the series*

$$\frac{1}{n+1}+m\,\frac{1}{n+2}+\frac{m\,(m+1)}{2\,!}\cdot\frac{1}{n+3}+\frac{m\,(m+1)\,(m+2)}{3\,!}\cdot\frac{1}{n+4}+\ldots$$

is $\dfrac{\Gamma\,(n+1)\,\Gamma\,(1-m)}{\Gamma\,(n-m+2)}$ *, where* $-1<n<1$.

Solution: We have

$$\frac{\Gamma(n+1)\,\Gamma(1-m)}{\Gamma(n-m+2)} = \mathbf{B}(n+1, 1-m)$$

$$= \int_0^1 x^n (1-x)^{-m}\, dx$$

$$= \int_0^1 x^n \left[1 + mx + \frac{m(m+1)}{2!} x^2 + \frac{m(m+1)(m+2)}{3!} x^3 + \ldots\right] dx$$

$$= \int_0^1 \left[x^n + mx^{n+1} + \frac{m(m+1)}{2!} x^{n+2} + \frac{m(m+1)(m+2)}{3!} x^{n+3} + \ldots\right] dx$$

$$= \left[\frac{x^{n+1}}{n+1} + m\frac{x^{n+2}}{n+2} + \frac{m(m+1)}{2!}\frac{x^{n+3}}{n+3} + \frac{m(m+1)(m+2)}{3!}\frac{x^{n+4}}{n+4} + \ldots\right]_0^1$$

$$= \frac{1}{n+1} + m.\frac{1}{n+2} + \frac{m(m+1)}{2!}\cdot\frac{1}{n+3} + \frac{m(m+1)(m+2)}{3!}\frac{1}{n+4} + \ldots$$

Example 18: *Prove that* $\int_0^\infty \frac{\sin bz}{z}\, dz = \frac{\pi}{2}\cdot$

Solution: We have

$$I = \int_0^\infty \int_0^\infty e^{-xz} \sin bz\, dx\, dz$$

$$= \int_0^\infty \left[\frac{e^{-xz}}{-z}\right]_0^\infty \sin bz\, dz \text{, on first integrating w.r.t. } x$$

$$= \int_0^\infty \frac{\sin bz}{z}\, dz\,. \qquad \ldots(1)$$

Again on first integrating w.r.t. z, we have

$$I = \int_0^\infty \int_0^\infty e^{-xz} \sin bz\, dx\, dz = \int_0^\infty \left[\int_0^\infty e^{-xz} \sin bz\, dz\right] dx$$

$$= \int_0^\infty \frac{b}{b^2 + x^2}\, dx\,,$$ [See article 13, Deduction (ii)]

$$= \left[\tan^{-1}\frac{x}{b}\right]_0^\infty = \frac{\pi}{2}\cdot \qquad \ldots(2)$$

Hence equating the two values (1) and (2) of I, we have

$$\int_0^\infty \frac{\sin bz}{z}\, dz = \frac{\pi}{2}\cdot$$

Example 19: *Show that*

$$\int_0^\infty \cos(bz^{1/n})\,dz = \frac{1}{b^n}\Gamma(n+1)\,.\cos\frac{n\pi}{2}.$$

Solution: Put $z^{1/n} = x$ *i.e.*, $z = x^n$, so that $dz = nx^{n-1}\,dx$.

$$\therefore \qquad \int_0^\infty \cos(bz^{1/n})\,dz = \int_0^\infty \cos(bx)\,.\,nx^{n-1}\,dx$$

$$= n\int_0^\infty x^{n-1}\cos(bx)\,dx$$

$$= \text{real part of } n\int_0^\infty e^{-ibx}\,x^{n-1}\,dx$$

$$= \text{real part of } n\,\frac{\Gamma(n)}{(ib)^n}$$

$$= \text{real part of } \frac{n\,\Gamma(n)}{b^n}\cdot\left(\cos\frac{1}{2}\pi + i\sin\frac{1}{2}\pi\right)^{-n}$$

$$= \text{real part of } \frac{\Gamma(n+1)}{b^n}\left(\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right)$$

$$= \frac{1}{b^n}\cdot\Gamma(n+1)\,.\cos\left(\frac{n\pi}{2}\right).$$

Example 20: *Show that* $\displaystyle\int_0^\infty \frac{x^c}{c^x}\,dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}}$, $c > 1$.

Solution: We have

$$I = \int_0^\infty \frac{x^c}{c^x}\,dx = \int_0^\infty \frac{x^c}{e^{\log c^x}}\,dx$$

$$= \int_0^\infty \frac{x^c}{e^{x\log c}}\,dx = \int_0^\infty e^{-x\log c}\,x^c\,dx.$$

Put $x\log c = y$ so that $(\log c)\,dx = dy$.

When $x = 0$, we have $y = 0$ and when $x \to \infty$, $y \to \infty$.

Also $c > 1 \Rightarrow \log c > 0$.

$$\therefore \qquad I = \int_0^\infty e^{-y}\left(\frac{y}{\log c}\right)^c \frac{dy}{\log c}$$

$$= \frac{1}{(\log c)^{c+1}}\int_0^\infty e^{-y}\,y^{(c+1)-1}\,dy$$

$$= \frac{1}{(\log c)^{c+1}}\,\Gamma(c+1),$$

provided $c + 1 > 0$ which is so because $c > 1$.

Comprehensive Exercise 1

1. Prove that

(i) $\int_0^a \dfrac{dx}{(a^n - x^n)^{1/n}} = \dfrac{1}{n} \cdot \dfrac{\pi}{\sin (\pi / n)}$.

(ii) $\int_0^2 (8 - x^3)^{-1/3}\, dx = \dfrac{2\pi}{3\sqrt{3}}$.

2. Show that $\int_0^1 \dfrac{x^{m-1} (1-x)^{n-1}}{(a+x)^{m+n}}\, dx = \dfrac{\mathbf{B}(m, n)}{a^n (1+a)^m} = \dfrac{\Gamma(m)\Gamma(n)}{a^n (1+a)^m \Gamma(m+n)}$.

[Hint. Put $\dfrac{x(1+a)}{a+x} = y$].

3. Show that $\int_0^{\pi/2} \sin^p \theta \cos^q \theta\, d\theta = \dfrac{1}{2} B\left(\dfrac{p+1}{2}, \dfrac{q+1}{2}\right), p > -1, q > -1.$

Deduce that $\int_0^2 x^4 (8 - x^3)^{-1/3}\, dx = \dfrac{16}{3} B\left(\dfrac{5}{3}, \dfrac{2}{3}\right)$.

4. Prove that $B(m, n) = B(m+1, n) + B(m, n+1)$ for $m > 0, n > 0$.

5. Prove that

(i) $\int_0^\infty e^{-ax} x^{n-1}\, dx = \dfrac{\Gamma(n)}{a^n}$. (ii) $\int_0^1 \left(\log \dfrac{1}{x}\right)^{n-1} dx = \Gamma(n)$.

6. Show that, if $m > -1$, then

$$\int_0^\infty x^m e^{-n^2 x^2}\, dx = \frac{1}{2\, n^{m+1}} \Gamma\left(\frac{m+1}{2}\right).$$

7. Prove that $\int_0^1 x^m (1 - x^n)^p\, dx = \dfrac{1}{n} B\left(\dfrac{m+n}{n}, p+1\right)$

8. Prove that $\int_0^1 (1 - x^n)^{1/n}\, dx = \dfrac{1}{n} \dfrac{[\Gamma(1/n)]^2}{2\Gamma(2/n)}$.

9. Show that $\Gamma(0 \cdot 1)\Gamma(0 \cdot 2)\Gamma(0 \cdot 3) \ldots \Gamma(0 \cdot 9) = \dfrac{(2\pi)^{9/2}}{\sqrt{(10)}}$.

10. Show that $\int_0^{\pi/2} \sqrt{(\sin \theta)}\, d\theta \times \int_0^{\pi/2} \dfrac{d\theta}{\sqrt{(\sin \theta)}} = \pi.$

11. Show that $\int_0^1 \dfrac{x^2\, dx}{(1 - x^4)^{1/2}} \times \int_0^1 \dfrac{dx}{(1 + x^4)^{1/2}} = \dfrac{\pi}{4\sqrt{2}}$.

12. Show that $\Gamma\left(\dfrac{1}{4}\right)\Gamma\left(\dfrac{3}{4}\right) = 2\int_0^{\pi/2} \sqrt{(\tan \theta)}\, d\theta = 4\int_0^\infty \dfrac{x^2\, dx}{1 + x^4} = \pi\sqrt{2}$.

13. Show that the perimeter of a loop of the curve $r^n = a^n \cos n\theta$ is

$$\frac{a}{n} \cdot 2^{(1/n)-1} \cdot \frac{[\Gamma(1/2n)]^2}{\Gamma(1/n)}.$$

14. Prove that $\int_0^1 \frac{dx}{\sqrt{(1-x^4)}} = \frac{\sqrt{2}}{8\sqrt{\pi}} \left[\Gamma\left(\frac{1}{4}\right)\right]^2.$

15. Show that $\int_0^{\pi/2} \sin^p \theta \, d\theta = \frac{1}{2} B\left(\frac{1}{2}, \frac{p+1}{2}\right).$

16. Show that

(i) $\int_0^\infty xe^{-\alpha x} \cos \beta x \, dx = \frac{(\alpha^2 - \beta^2)}{(\alpha^2 + \beta^2)^2}$

(ii) $\int_0^\infty xe^{-\alpha x} \sin \beta x \, dx = \frac{2\alpha\beta}{(\alpha^2 + \beta^2)^2}.$

17. Prove that $\int_{-\infty}^\infty \cos\left(\frac{1}{2}\pi x^2\right) dx = 1.$

18. Show that $\mathbf{B}(m, m) \, . \, \mathbf{B}\left(m + \frac{1}{2}, m + \frac{1}{2}\right) = \frac{\pi m^{-1}}{2^{4m-1}}.$

19. Prove that $\int_0^\pi \frac{\sin^{n-1} x \, dx}{(a + b\cos x)^n} = \frac{2^{n-1}}{(a^2 - b^2)^{n/2}} \cdot \mathbf{B}\left(\frac{n}{2}, \frac{n}{2}\right)$, $a > b$.

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. For $m > 0, n > 0$,

(a) $B(m, n) = \frac{\Gamma(m)}{\Gamma(n)}$

(b) $B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

(c) $B(m, n) = \frac{\Gamma(n)}{\Gamma(m)}$

(d) $B(m, n) = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)}$

2. The value of the integral $\int_0^\infty e^{-x} x^{-1/2} \, dx$ is

(a) $\frac{\sqrt{\pi}}{2}$

(b) $\frac{\pi}{2}$

(c) $\sqrt{\pi}$

(d) π

3. For $m > 0, n > 0,$

(a) $B(m, n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$

(b) $B(m, n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx$

(c) $B(m, n) = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$

(d) $B(m, n) = \int_0^\infty \frac{x^{m}}{(1+x)^{m+n}} dx$

4. If $a > 0$ and $n > 0,$ then the value of the integral

$$\int_0^\infty e^{-ax} x^{n-1} dx \text{ is}$$

(a) $a^n \Gamma(n)$

(b) $a^{-n} \Gamma(n)$

(c) $\frac{\Gamma(n)}{2a^n}$

(d) $\frac{\Gamma(n)}{n^2}$

5. The value of $\int_0^1 x^4 (1-x)^3 dx$ is

(a) $\frac{1}{280}$

(b) $\frac{1}{180}$

(c) $\frac{1}{380}$

(d) $\frac{1}{80}$

6. The value of the integral $\int_0^{\pi/2} \sin^4 x \cos^2 x \, dx$ is

(a) $\frac{\pi}{4}$

(b) $\frac{\pi}{8}$

(c) $\frac{\pi}{16}$

(d) $\frac{\pi}{32}$

7. The value of $\int_0^\infty \frac{x^{m-1} dx}{(1+x)^{m+n}}$ is

(a) $\Gamma(m) + \Gamma(n)$

(b) $\frac{\Gamma(m)}{\Gamma(n)}$

(c) $\Gamma(m) . \Gamma(n)$

(d) $\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

8. If $m, n > 0,$ then the value of $\int_0^1 x^{n-1} \left(\log \frac{1}{x}\right)^{m-1} dx$ is equal to

(a) $\frac{\Gamma(m)}{n^m}$

(b) $\frac{\Gamma(n)}{n^m}$

(c) $\frac{\Gamma(n)}{m^n}$

(d) $\frac{\Gamma(m)}{m^m}$

9. The value of $\Gamma\left(\frac{1}{4}\right) \cdot \Gamma\left(\frac{3}{4}\right)$ is ....

(a) π (b) $\frac{\pi}{\sqrt{2}}$

(c) $\pi\sqrt{2}$ (d) $\sqrt{\frac{\pi}{2}}$

10. The value of integral $\int_0^1 \frac{dx}{\sqrt{-\log x}}$ is

(a) $\sqrt{\pi}$ (b) π

(c) $\sqrt{\frac{\pi}{2}}$ (d) none of these

11. If $m > 0, n > 0$, then $B(m, n)$ is defined as

(a) $\int_0^1 x^m (1-x)^n \, dx$ (b) $\int_0^1 x^{m-1} (1-x)^n \, dx$

(c) $\int_0^1 x^{m-1} (1-x)^{n-1} \, dx$ (d) $\int_0^1 x^{m-1} (1-x)^{n+1} \, dx$

12. $\int_0^{\pi/2} \cos^m \theta \sin^n \theta \, d\theta = \dfrac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$ when

(a) $m > 0, \ n > 0$ (b) $m > -1, \ n > -1$

(c) $m > -\frac{1}{2}, \ n > -\frac{1}{2}$ (d) $0 < m < 1, 0 < n < 1$

13. The value of $\int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx$ is

(a) 1 (b) zero

(c) $(n-1)!$ (d) Γn

14. The value of $\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{3}{9}\right)\ldots\Gamma\left(\frac{8}{9}\right)$ is

(a) $\frac{1}{6}\pi^3$ (b) $16\pi^4$

(c) $\frac{1}{3}\pi^4$ (d) $\frac{16}{3}\pi^4$

15. Integral $\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$ is equal to

(a) $B(m+1, n+1)$ (b) $B(m, n+1)$

(c) $B(m, n)$ (d) none of these

16. Value of integral $\int_0^{\pi/2} \cos^m \theta \sin^n \theta \, d\theta$ is

(a) $\dfrac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$

(b) $\dfrac{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$

(c) $\dfrac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+1}{2}\right)}$

(d) $\dfrac{\Gamma\left(\frac{m-1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{2\Gamma\left(\frac{m-n-2}{2}\right)}$

17. The integral $\int_0^\infty x^{n-1} e^{-x} \, dx$ is known as

(a) Beta function
(b) Gamma function
(c) Beta and Gamma function
(d) none of these

18. The value of $\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)$ shall be

(a) $\dfrac{\sqrt{\pi}}{2}$

(b) $\dfrac{\pi}{\sqrt{2}}$

(c) $\dfrac{\sqrt{3}\,\pi}{2}$

(d) $\dfrac{2\pi}{\sqrt{3}}$

Fill in the Blank(s)

Fill in the blanks "......" so that the following statements are complete and correct.

1. The definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, for $m > 0, n > 0$ is called the ......

2. The definite integral $\int_0^\infty e^{-x} x^{n-1}\,dx$, for $n > 0$ is called the ....

3. $B(m, n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\ldots\ldots}$.

4. $\dfrac{B(m+1, n)}{B(m, n)} = \dfrac{m}{\ldots\ldots}$.

5. For $m > 0, n > 0, \int_0^\infty \dfrac{x^{m-1} - x^{n-1}}{(1+x)^{m+n}}\,dx = \ldots\ldots.$

6. For $n > 0, \Gamma(n+1) = \ldots\ldots\,\Gamma(n)$.

7. If n is a positive integer, then $\Gamma(n) = \ldots\ldots.$

8. If $0 < n < 1$, then $\Gamma(n)\,\Gamma(1-n) = \ldots\ldots$

9. $\Gamma\left(\frac{1}{2}\right) = \ldots\ldots$

10. If $m > -1, n > -1$, then $\int_0^{\pi/2} \cos^m \theta \sin^n \theta \, d\theta = \dfrac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\ldots\ldots}$

11. $\int_0^{\infty} e^{-x^2}\, dx = \ldots\ldots$

12. For $m > 0, \Gamma(m)\,\Gamma\left(m + \frac{1}{2}\right) = \dfrac{\sqrt{\pi}}{\ldots\ldots}\Gamma(2m)$.

13. For $a > 0, n > 0, \int_0^{\infty} e^{-ax} x^{n-1}\, dx = \dfrac{\Gamma(n)}{\ldots\ldots}$.

14. The value of $\Gamma\left(\frac{1}{3}\right)\cdot\Gamma\left(\frac{2}{3}\right)$ is ....

True or False

Write 'T' for true and 'F' for false statement.

1. $\int_0^{\infty} e^{-x} x^{1/2}\, dx = \Gamma\left(\frac{1}{2}\right)$.

2. $\int_0^{\infty} \dfrac{x\, dx}{1+x^6} = \dfrac{\pi}{3\sqrt{3}}$.

3. $\int_0^{\infty} \dfrac{x^8(1-x^6)}{(1+x)^{24}}\, dx = 1$.

4. For $m > 0, n > 0, B(m, n) = \int_0^{\infty} \dfrac{x^{n-1}}{(1+x)^{m+n}}\, dx$.

5. For $m > 0, n > 0, \int_0^{\infty} \dfrac{x^{m-1}}{(1+x)^{m+n}}\, dx = \int_0^{\infty} \dfrac{x^{n-1}}{(1+x)^{m+n}}\, dx$.

6. $\Gamma(6) = 120$.

7. For $m > 0, n > 0, B(m, n) = \dfrac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m+n)}$.

8. $\int_0^{\infty} e^{-x^2}\, dx = \dfrac{\sqrt{\pi}}{4}$.

9. $B(m+1, n) + B(m, n+1) = B(m+1, n+1)$.

3

Green Function

Structure

3.1. Introduction. This chapter contains methods to obtain Green function for a given non-homogeneous linear second order boundary value problem and reduction of boundary value problem to Fredholm integral equation with Green function as kernel.

31.1. Objective. The objective of these contents is to provide some important results to the reader like:

(i) Construction of Green function.

(ii) Reduction of boundary value problem to Fredholm integral equation with Green function as kernel.

3.1.2. Keywords. Green function, Integral Equations, Boundary Conditions.

3.2. Construction of Green function. Consider a differential equation of order n

$$L(u) = p_0(x)\, u^n + p_1(x)\, u^{n-1} + p_2(x)\, u^{n-2} + ... + p_n(x)\, u = 0 \qquad (1)$$

where the functions $p_0(x), p_1(x)$, $p_2(x), \dots, p_n(x)$ are continuous on [a, b], $p_0(x) \neq 0$ on [a, b], and the boundary conditions

$$V_k(u) = \alpha_k u(a) + \alpha_k^1 u'(a) + \alpha_k^2 u''(a) + \dots + \alpha_k^{n-1} u^{n-1}(a)$$

$$+\beta_k u(b) + \beta_k^1 u'(b) + \beta_k^2 u''(b) + \dots + \beta_k^{n-1} u^{n-1}(b) \qquad (2)$$

for k = 1, 2,…,n, where the linear forms $V_1, V_2, \dots, V_n$ in u(a), $u'(a), \dots, u^{n-1}(a)$, u(b), $u'(b), \dots, u^{n-1}(b)$ are linearly independent.

The homogeneous boundary value problem (1), (2) contains only a trivial solution $u(x) \equiv 0$.

Green's function of the boundary value problem (1), (2) is the function G (x, ξ) constructed for any point ξ, a < ξ < b satisfying the following properties :

1. G(x, ξ) is continuous in x for fixed ξ and has continuous derivatives with regard to x upto order $(n-2)$ inclusive for $a \le x \le b$.
2. Its $(n-1)$th derivative with regard to x at the point x = ξ has a discontinuity of first kind, the jump being equal to $-\frac{1}{[p_0(x)]_{x=\xi}}$, that is,

$$\left\{\frac{\partial^{n-1}}{\partial x^{n-1}} G(x,\xi)\right\}_{x=\xi+0} - \left\{\frac{\partial^{n-1}}{\partial x^{n-1}} G(x,\xi)\right\}_{x=\xi-0} = -\frac{1}{p_0(\xi)} \qquad (3)$$

   where $G|_{x=\xi+0}$ defines the limit of G(x, ξ) as $x \to \xi$ from the right and $G|_{x=\xi-0}$ defines the limit of G(x, ξ) as $x \to \xi$ from the left.

3. In each of the intervals [a, ξ) and (ξ, b] the function G(x, ξ), considered as a function of x, is a solution of the equation (1)

$$L(G) = 0 \qquad (4)$$

4. The function G(x, ξ) satisfies the boundary conditions (2)

$$V_k(G) = 0, \; k = 1, 2, 3, \dots, n, \qquad (5)$$

If the boundary value problem (1), (2) contains only the trivial solution $u(x) \equiv 0$ then the operator L contains one and only one Green's function G(x, ξ).

Consider $u_1(x), u_2(x), \dots, u_n(x)$ be linearly independent solutions of the equation L(u) = 0. From the condition 1, the unknown Green's function G(x, ξ) must have the representation on the intervals [a, ξ) and (ξ, b]

$$G(x, \xi) = a_1u_1(x) + a_2u_2(x) + \dots + a_nu_n(x), \; a \le x < \xi$$

and

$$G(x, \xi) = b_1u_1(x) + b_2u_2(x) + \dots + b_nu_n(x), \; \xi \le x < b,$$

where $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ are some functions of ξ.

From the condition 1, the continuity of the function $G(x, \xi)$ and of its first $(n-2)$ derivatives with regard to x at the point $x = \xi$ yields

$$[b_1u_1(\xi) + b_2u_2(\xi) + \dots + b_nu_n(\xi)] - [a_1u_1(\xi) + a_2u_2(\xi) + \dots + a_nu_n(\xi)] = 0$$

$$[b_1u_1'(\xi) + b_2u_2'(\xi) + \dots + b_nu_n'(\xi)] - [a_1u_1'(\xi) + a_2u_2'(\xi) + \dots + a_nu_n'(\xi)] = 0$$

$$[b_1u_1''(\xi) + b_2u_2''(\xi) + \dots + b_nu_n''(\xi)] - [a_1u_1''(\xi) + a_2u_2''(\xi) + \dots + a_nu_n''(\xi)] = 0$$

$$\dots \qquad \dots \qquad \dots \qquad \dots \qquad \dots$$

$$[b_1u_1^{n-2}(\xi) + b_2u_2^{n-2}(\xi) + \dots + b_nu_n^{n-2}(\xi)] - [a_1u_1^{n-2}(\xi) + a_2u_2^{n-2}(\xi) + \dots + a_nu_n^{n-2}(\xi)] = 0$$

Also, $[b_1u_1^{n-1}(\xi) + b_2u_2^{n-1}(\xi) + \dots + b_nu_n^{n-1}(\xi)] - [a_1u_1^{n-1}(\xi) + a_2u_2^{n-1}(\xi) + \dots + a_nu_n^{n-1}(\xi)] = -\dfrac{1}{p_0(\xi)}$

Assume $C_k(\xi) = b_k(\xi) - a_k(\xi)$, k = 1, 2, … , n ; then the system of linear equations in $C_k(\xi)$ are obtained

$$C_1u_1(\xi) + C_2u_2(\xi) + \dots + C_nu_n(\xi) = 0$$

$$C_1u_1'(\xi) + C_2u_2'(\xi) + \dots + C_nu_n'(\xi) = 0$$

$$\dots \qquad \dots \qquad \dots \qquad \dots \qquad \dots$$

$$C_1u_1^{n-2}(\xi) + C_2u_2^{n-2}(\xi) + \dots + C_nu_n^{n-2}(\xi) = 0$$

$$C_1u_1^{n-1}(\xi) + C_2u_2^{n-1}(\xi) + \dots + C_nu_n^{n-1}(\xi) = -\frac{1}{p_0(\xi)} \qquad (6)$$

The determinant of the system is equal to the value of the Wronskian $W(u_1, u_2, \dots, u_n)$ at the point $x = \xi$ and is therefore different from zero.

From the boundary conditions (2), we have

$$V_k(u) = A_k(u) + B_k(u) \qquad (7)$$

where $A_k(u) = \alpha_k u(a) + \alpha^1{}_k u'(a) + \alpha^2{}_k u''(a) + \dots\dots + \alpha_k^{n-1}u^{n-1}(a)$

$$B_k(u) = \beta_k u(b) + \beta^1{}_k u'(b) + \beta^2{}_k u''(b) + \dots\dots + \beta_k^{n-1}u^{n-1}(b)$$

Using the condition 4, we have

$V_k(G) = a_1A_k(u_1) + a_2A_k(u_2) + \dots + a_nA_k(u_n) + \dots + b_1B_k(u_1) + b_2B_k(u_2) + \dots + b_nB_k(u_n) = 0$,

where k = 1, 2, …, n.

Since $a_k = b_k - c_k$, so we have

$$(b_1 - c_1)A_k(u_1) + (b_2 - c_2)A_k(u_2) + \dots + (b_n - c_n)A_k(u_n) + b_1B_k(u_1) + b_2B_k(u_2) + \dots + b_nB_k(u_n) = 0$$

$$\Rightarrow \quad b_1V_k(u_1) + b_2V_k(u_2) + \dots + b_nV_k(u_n) = c_1A_k(u_1) + c_2A_k(u_2) + \dots + c_nA_k(u_n) \qquad (8)$$

which is a linear system in the quantities $b_1, b_2, \ldots, b_n$. The determinant of the system is different from zero, that is,

$$\begin{vmatrix} V_1(u_1) & V_1(u_2) & \cdots & V_1(u_n) \\ V_2(u_1) & V_2(u_2) & \cdots & V_2(u_n) \\ \cdots & \cdots & \cdots & \cdots \\ V_n(u_1) & V_n(u_2) & \cdots & V_n(u_n) \end{vmatrix} \neq 0$$

The system of equations (8) contain a unique solution in $b_1(\xi), b_2(\xi), \ldots, b_n(\xi)$ and since

$a_k(\xi) = b_k(\xi) - c_k(\xi)$, it follows that the quantities $a_k(\xi)$ are defined uniquely.

I. If the boundary value problem (1), (2) is self – adjoint, then Green's function is symmetric, that is, $G(x, \xi) = G(\xi, x)$. The converse is true as well.

II. If at one of the extremities of an interval [a, b], the coefficient of the derivative vanishes. For example, $p_0(a) = 0$, then the natural boundary condition for the boundedness of the solution x = a is imposed, and at the other extremity the ordinary boundary condition is specified.

3.2.1. Particular case. We shall construct the Green's Function $G(x, \xi)$ for a given number ξ, for the second differential equation

$$L(u) + \phi(x) = 0 \tag{1}$$

where

$$L \equiv \frac{d}{dx}\left(p\,\frac{d}{dx}\right) + q \tag{2}$$

Together with the homogenous boundary conditions of the form

$$\alpha u + \beta \frac{du}{dx} = 0 \tag{3}$$

The Green's function $G(x, \xi)$ constructed for any point $\xi, a < \xi < b$ contains the following properties:

1. $G_1(\xi) = G_2(\xi);$ it follows that the function $G(x, \xi)$ is continuous in x for fixed ξ, in particular, continuous at the point x = ξ.

2. The derivatives of G(which are of finite magnitude) are continuous at every point within the range of x except at x = ξ where it is continuous so that

$$G_2'(\xi) - G_1'(\xi) - \frac{1}{p(\xi)}$$

3. The functions G_1 and G_2 satisfy homogenous conditions at the end points x = a and x = b respectively.

4. The function G_1 and G_2 satisfy the homogenous equations LG = 0 in their defined intervals except at z = ξ, that is, $LG_1 = 0, x < \xi$, $L\,G_2 = 0,\ x > \xi$.

Consider the Green's function $G(x, \xi)$ exists, then the solution of the given differential equation can be transformed to the relation

$$u(x) = \int_a^b G(x, \xi)\, \phi(\xi)\, d\xi \tag{4}$$

Consider two linearly independent solutions of the homogeneous equation $L(u) = 0$. *Let* $u = v_1(x)$ and $u = u_2(x)$ be the non-trivial solution of the equation, which satisfy the homogenous conditions at x = a and x = b respectively.

Consider the Green's functions for the problem from the conditions III and IV, in the form

$$G(x, \xi) = \begin{cases} C_1 u_1(x), & x < \xi \\ C_2\, u_2(x), & x < \xi \end{cases} \tag{5}$$

where the constant C_1 and C_2 are chosen in a manner that the conditions I and II are fulfilled. Thus, we have

$$C_2 u_2(\xi) - C_1 u_1(\xi) = 0$$

$$C_2 u_2'(\xi) - C_1 u_1'(\xi) - \frac{1}{p(\xi)} \tag{6}$$

The determinant of the system (6) is the Wronskian $W\left[u_1(\xi),\ u_2(\xi)\right]$ evaluated at the point x = ξ for linearly independent solution $u_1(x)$ and $u_2(x)$, and, hence it is different from zero $W(\xi) \neq 0$

$$W\left[u_1(\xi), u_2(\xi)\right] = \begin{vmatrix} u_1(\xi) & u_2(\xi) \\ u_1'(\xi) & u_2'(\xi) \end{vmatrix} = u_1(\xi) u_2'(\xi) - u_2(\xi) u_1'(\xi) \tag{7}$$

By using Abel's formula, we notice that the expression has the value {C/p(ξ)}, where C is a constant independent of ξ, that is,

$$u_1(\xi) u_2'(\xi) - u_2(\xi) u_1'(\xi) = \frac{C}{p(\xi)} \tag{8}$$

From the system (6), we have

$$C_1 = -\frac{1}{C}\, u_2(\xi)\ ,\ C_2 = -\frac{1}{C}\, u_1(\xi)$$

Thus the relation (5) reduces to

$$G(x, \xi) = \begin{cases} -\dfrac{1}{C} u_1(x)\, u_2(\xi), & x < \xi \\ -\dfrac{1}{C} u_1(\xi) u_2(x), & x > \xi \end{cases} \tag{9}$$

This result breaks down iff C vanishes, so that u_1 and u_2 are linearly dependent, and hence are each multiples of a certain non-trivia function U(x). In this case, the function u(x) satisfies the equation L(u) = 0 together with the end conditions at x = a, x = b.

Converse. The integral equation

$$u(x) = \int_a^b G(x, \xi)\, \phi(\xi)\, d\xi \tag{10}$$

where G(x, ξ) are defined by the relation (9), satisfy the differential equation

$$L(u) + \phi(x) = 0 \tag{11}$$

together with the prescribed boundary condition.

We know that

$$u(x) = -\frac{1}{C}\left[\int_a^x u_1(\xi)\, u_2(x)\, \phi(\xi)\, d\xi + \int_x^b u_1(x)\, u_2(\xi)\, \phi(\xi)\, d\xi\right] \tag{12}$$

$$u'(x) = -\frac{1}{C}\left[\int_a^x u_2'(x)\, u_1(\xi)\, \phi(\xi)\, d\xi + \int_x^b u_1'(x)\, u_2(\xi)\, \phi(\xi)\, d\xi\right] \tag{13}$$

$$u''(x) = -\frac{1}{C}\left[\int_a^x u_2''(x)\, u_1(\xi)\, \phi(\xi)\, d\xi + \int_x^b u_1''(x)\, u_2(\xi)\, \phi(\xi)\, d\xi\right]$$

$$-\frac{1}{C}\left[u_2'(x)\, u_1(x) - u_1'(x)\, u_2(x)\right]\phi(x) \tag{14}$$

Since L(u) $\equiv$ $p(x)u''(x) + p(x)u'(x) + q(x)\, u(x)$

Thus,

$$Lu(x) = -\frac{1}{C}\left[\int_a^x \{Lu_2(x)\}\, u_1(\xi)\, \phi(\xi)\, d\xi + \int_x^b \{Lu_2(x)\}\, u_2(\xi)\, \phi(\xi)\, d\xi\right] - \frac{1}{C}\left[p(x).\frac{C}{p(x)}\phi(x)\right]$$

Again, $u_1(x)$ and $u_2(x)$ satisfy L (u) = 0, hence the first two terms vanish identically.

So, L u(x) = $-\phi$(x) $\Rightarrow$ L u(x) + ϕ(x) = 0

Therefore, a function u(x) satisfying (10) also satisfies the differential equation (11)

Again from (12) and (13), we have

$$u(a) = -\frac{u_1(a)}{C}\int_a^b u_2(\xi)\, \phi(\xi)\, d\xi$$

$$u'(a) = -\frac{u_1'(b)}{C}\int_a^b u_2(\xi)\, \phi(\xi)\, d\xi$$

which shows that the function u defined by (11) satisfies the same homogeneous condition at x = a as the function u_1.

Note. Let $\phi(x) = \lambda r(x) u(x) - f(x)$.

From the differential equation (1), we have

$$Lu(x) + \lambda r(x) u(x) = f(x) \qquad (15)$$

The corresponding Fredholm integral equation becomes

$$u(x) = \lambda \int_a^b G(x, \xi) r(\xi) u(\xi) d\xi - \int_a^b G(x, \xi) f(\xi) d\xi \qquad (16)$$

where $G(x, \xi)$ is the Green's function.

From (9), it follows that $G(x, \xi)$ is symmetric but the kernel $K(x, \xi)$ $\{= G(x, \xi) r(\xi)\}$ is not symmetric unless r(x) is a constant.

Consider $\sqrt{\{r(x)\}}u(x) = V(x)$ with the assumption that r(x) is non – negative over (a, b). This equation (16) may be expressed as

$$\frac{V(x)}{\sqrt{r(x)}} = \lambda \int_a^b G(x, \xi) \sqrt{r(\xi)}\, V(\xi) d\xi - \int_a^b G(x, \xi) f(\xi) d\xi$$

or
$$V(x) = \lambda \int_a^b K(x, \xi) V(\xi) d\xi - \int_a^b K(x, \xi) \frac{f(\xi)}{\sqrt{r(\xi)}} d\xi, \qquad (17)$$

where $K(x, \xi) = \sqrt{\{r(x) r(\xi)\}}\, G(x, \xi)$ and hence possesses the same symmetry as $G(x, \xi)$.

3.2.2. Example. Construct an integral equation corresponding to the boundary value problem.

$$x^2 \frac{d^2u}{dx^2} + x\frac{du}{dx} + (\lambda x^2 - 1)\, u = 0, \qquad (1)$$

$$u(0) = 0, u(1) = 0 \qquad (2)$$

Solution. The differential equation (1) may be written as

$$\frac{d}{dx}\left(x\frac{du}{dx}\right) + \left(-\frac{1}{x} + \lambda x\right) u = 0.$$

or
$$\left[\frac{d}{dx}\left(x\frac{du}{dx}\right) - \frac{u}{x}\right] + \lambda x u = 0.$$

Comparing with the equation (15), we have

$$p = x,\ q = -\frac{1}{x},\ r = x \qquad (3)$$

The general solution of the homogeneous equation

$$L(u) = 0 \quad \Rightarrow \quad \left\{\frac{d}{dx}\left(x\frac{du}{dx}\right) - \frac{u}{x}\right\} = 0$$ is given by

$$u(x) = C_1 x + C_2\left(\frac{1}{x}\right)$$

Consider $u = u_1(x)$ and $u = u_2(x)$ be the non – trivial solutions of the equation, which satisfy the conditions at x = 0 and x = 1 respectively then

$$u_1(x) = x \qquad \text{and } u_2(x) = \frac{1}{x} - x\,.$$

The Wronskian of $u_1(x)$ and $u_2(x)$ is given by

$$W[u_1(x), u_2(x)] = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} = x\left(-\frac{1}{x^2} - 1\right) - \left(\frac{1}{x} - x\right) = -\frac{2}{x}$$

So, $$u_1(x)u_2'(x) - u_2(x)u_1'(x) = -\frac{2}{x} \Rightarrow C = -2$$

Thus from the relation (19), we have

$$G(x, \xi) = \begin{cases} \dfrac{1}{2}\dfrac{x}{\xi}(1-\xi^2)\,, & x < \xi\,, \\ \dfrac{1}{2}\dfrac{\xi}{x}(1-x^2)\,, & x > \xi\,, \end{cases} \tag{4}$$

Therefore, from (16), the corresponding Fredholm integral equation becomes

$u(x) = \lambda\int_0^1 G(x, \xi)\, \xi\, u(\xi)\, d\xi$, where the Green's function $G(x, \xi)$ is defined by the relation (4).

3.2.3. Example. Construct Green's function for the homogeneous boundary value problem

$$\frac{d^4u}{dx^4} = 0 \text{ with the conditions } u(0) = u'(0) = 0,\ u(1) = u'(1) = 0.$$

Solution. The differential equation is given by

$$\frac{d^4u}{dx^4} = 0 \tag{1}$$

We notice that the boundary value problem contains only a trivial solution. The fundamental system of solutions for the differential equation (1) is

$$u_1(x) = 1,\ u_2(x) = x,\ u_3(x) = x^2,\ u_4(x) = x^3 \tag{2}$$

Its general solution is of the form

$$u(x) = A + Bx + Cx^2 + Dx^3,$$

where A, B, C, D are arbitrary constants. The boundary conditions give the relations for determining the constants A, B, C, D :

$$u(0) = 0 \quad \Rightarrow \quad A = 0,\quad u'(0) = 0 \quad \Rightarrow \quad B = 0$$

$u(1) = 0 \quad \Rightarrow \quad A + B + C + D = 0, \; u'(1) = 0 \quad \Rightarrow \quad B + 2C + 3D = 0$

$\Rightarrow \quad A = B = C = D = 0.$

Thus the boundary value problem has only a zero solution $u(x) \equiv 0$ and hence we can construct a unique Green's function for it.

Construction of Green's Function: **C**onsider the unknown Green's function $G(x, \xi)$ must have the representation on the interval $[0, \xi)$ and $(\xi, 1]$.

$$G(x, \xi) = \begin{cases} a_1.1 + a_2.x + a_3.x^2 + a_4.x^3, & 0 \le x \le \xi \\ b_1.1 + b_2.x + b_3.x^2 + b_4.x^3, & \xi \le x \le 1 \end{cases} \tag{3}$$

where $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ are the unknown functions of ξ.

Consider $\quad C_k = b_k(\xi) - a_k(\xi), \; k = 1, 2, 3, 4, \ldots \quad (4)$

The system of linear equations for determining the functions $C_k(\xi)$ become

$$C_1 + C_2\xi + C_3\xi^2 + C_4\xi^3 = 0$$

$$C_2 + 2C_3\xi + 3C_4\xi^2 = 0$$

$$2C_3 + 6C_4\xi = 0$$

$$6C_4 = 1$$

$$\Rightarrow \quad C_4(\xi) = \frac{1}{6}, \; C_3(\xi) = -\frac{1}{2}\xi, \; C_2(\xi) = \frac{1}{2}\xi^2, \; C_1(\xi) = -\frac{1}{6}\xi^3 \tag{5}$$

From the property 4 of Green's function, it must satisfy the boundary conditions :

$$G(0, \xi) = 0, \; G'_x(0, \xi) = 0$$

$$G(1, \xi) = 0, \; G'_x(1, \xi) = 0$$

The relations reduce to

$$a_1 = 0, \; a_2 = 0$$

$$b_1 + b_2 + b_3 + b_4 = 0$$

$$b_2 + 2b_3 + 3b_4 = 0 \tag{6}$$

From the relation (4), (5) and (6), we have

$$C_1 = b_1(\xi) - a_1(\xi) \Rightarrow b_1(\xi) = -\frac{1}{6}\xi^3$$

or $\quad C_2 = b_2(\xi) - a_2(\xi) \Rightarrow b_2(\xi) = \frac{1}{2}\xi^2$

or $b_3 + b_4 = \frac{1}{6}\xi^3, \frac{1}{2}\xi^2, 2b_3 + 3b_4 = -\frac{1}{2}\xi^2$

$\Rightarrow$ $b_4(\xi) = \frac{1}{2}\xi^2 - \frac{1}{3}\xi^3$ and $b_3(\xi) = \frac{1}{2}\xi^3 - \xi^2$

or $C_3(\xi) = b_3(\xi) - a_3(\xi)$

$\Rightarrow$ $a_3(\xi) = b_3(\xi) - C_3(\xi) = \frac{1}{2}\xi^3 - \xi^2 + \frac{1}{2}\xi$

and $C_4(\xi) = b_4(\xi) - a_4(\xi)$

$\Rightarrow$ $a_4(\xi) = b_4(\xi) - C_4(\xi) = \frac{1}{2}\xi^2 - \frac{1}{3}\xi^3 - \frac{1}{6}$

Substituting the value of the constants a_1, a_2, a_3, a_4, b_1, b_2, C_3, C_4 in the relation (3), the Green's function $G(x, \xi)$ is obtained as

$$G(x,\xi) = \begin{cases} \left(\frac{1}{2}\xi - \xi^2 + \frac{1}{2}\xi^3\right)x^2 - \left(\frac{1}{6} - \frac{1}{2}\xi^2 + \frac{1}{3}\xi^3\right)x^3, & 0 \le x \le \xi \\ -\frac{1}{6}\xi^3 + \frac{1}{2}\xi^2 x + \left(\frac{1}{2}\xi^3 - \xi^2\right)x^2 + \left(\frac{1}{2}\xi^2 - \frac{1}{3}\xi^3\right)x^3, & \xi \le x \le 1 \end{cases}$$

The expression $G(x, \xi)$ may be transformed to

$$G(x, \xi) = \left(\frac{1}{2}x - x^2 + \frac{1}{2}x^3\right)\xi^2 - \left(\frac{1}{6} - \frac{1}{2}x^2 + \frac{1}{3}x^3\right)\xi^3, \quad \xi \le x \le 1$$

$\Rightarrow$ $G(x, \xi) = G(\xi, x)$, that is, Green's function is symmetric.

3.2.4. Example. Construct Green's function for the equation $x\frac{d^2u}{dx^2} + \frac{du}{dx} = 0$ with the conditions u(x) is bounded as $x \to 0$, $u(1) = \mu u'(1)$, $\mu \ne 0$.

Solution. The differential equation is given by $x\frac{d^2u}{dx^2} + \frac{du}{dx} = 0$ (1)

or $\left(\frac{d^2u/dx^2}{du/dx}\right)dx = -\frac{1}{x}dx$

or $\log\frac{du}{dx} = -\log x + \log A$

or $\frac{du}{dx} = \frac{A}{x}$

or $u(x) = A\log x + B$ (2)

The conditions u(x) is bounded as $x \to 0$ and $u(1) = \mu u'(1)$, $\mu \neq 0$ has only a trivial solution $u(x) \equiv 0$, hence we can construct a (unique) Green's function $G(x, \xi)$

Consider the function $G(x, \xi)$ as:

$$G(x, \xi) = \begin{cases} a_1 + a_2 \log x, & 0 < x \leq \xi \\ b_1 + b_2 \log x, & \xi \leq x \leq 1 \end{cases} \quad (3)$$

where a_1, a_2, b_1, b_2 are unknown functions of ξ.

Consider $C_k = b_k(\xi) - a_k(\xi)$, k = 1, 2,…

From the continuity of $G(x, \xi)$ for $x = \xi$, we obtain

$$b_1 + b_2 \log \xi - a_1 - a_2 \log \xi = 0$$

and the jump $G'_x(x, \xi)$ at the point $x = \xi$ is equal to $\frac{1}{\xi}$ so that

$$b_2 . \frac{1}{\xi} - a_2 . \frac{1}{\xi} = -\frac{1}{\xi}$$

Putting $C_1 = b_1 - a_1$, $C_2 = b_2 - a_2$ (4)

$\Rightarrow$ $C_1 + C_2 \log \xi = 0$, $C_2 = -1$.

Hence $C_1 = \log \xi$ and $C_2 = -1$ (5)

The boundedness of the function $G(x, \xi)$ as $x \to 0$ gives $a_2 = 0$

Also, $G(x, \xi) = \mu G'_x(x, \xi)$, $b_1 = \mu b_2$

$\Rightarrow$ $a_1 = -(\mu + \log \xi)$, $a_2 = 0$, $b_1 = -1$, $b_2 = -\mu$

Substituting the value of the constants a_1, a_2, b_1, b_2 in the relation (3), the Green's function is obtained as

$$G(x, \xi) = \begin{cases} -(\mu + \log \xi), & 0 < x \leq \xi \\ -(1 + \mu \log x), & \xi \leq x \leq 1 \end{cases}.$$

3.2.5. Exercise.

1. Construct the Green's function for the boundary value problem $u''(x) + \mu^2 u = 0$ with the conditions u(0) = u(1) = 0.

Answer. $$G(x, \xi) = \begin{cases} \dfrac{\sin \mu(\xi - 1) \sin \mu x}{\mu \sin \mu}, & 0 \leq x \leq \xi \\ \dfrac{\sin \mu\xi \sin \mu(x-1)}{\mu \sin \mu}, & \xi < x \leq 1 \end{cases}$$

2. Find the Green's function for the boundary value problem $\frac{d^2u}{dx^2} - u(x) = 0$ with the conditions u(0) = u(1) = 0.

Answer. $G(x, \xi) = \begin{cases} \frac{\sinh x \sinh(\xi - 1)}{\sinh 1} &, \quad 0 \le x \le \xi \\ \frac{\sinh \xi \sinh(x-1)}{\sinh 1} &, \quad \xi \le x \le 1 \end{cases}$.

3.2.6. Article. If u(x) has continuous first and second derivatives, and satisfies the boundary value problem $\frac{d^2u}{dx^2} + \lambda u = 0$ with u(0) = u(l) = 0 then u(x) is continuous and satisfies the homogeneous linear integral equation $u(x) = \lambda \int_0^1 G(x, \xi)\, u(\xi)\, d\xi$.

Solution : The differential equation may be written as

$$\frac{d^2u}{dx^2} + \lambda u = 0 \;\Rightarrow\; \frac{d^2u}{dx^2} = -\lambda u \tag{1}$$

By integrating with regard to x over the interval (0, x) two times, we obtain

$$\frac{du}{dx} = -\lambda \int_0^x u(\xi)\, d\xi + C$$

or
$$u(x) = -\lambda \int_0^x (x-\xi)\, u(\xi)\, d\xi + C_x + D \tag{2}$$

where C and D are the integration constants, to be determined by the boundary conditions.

$$u(0) = 0 \quad \Rightarrow \quad D = 0$$

$$u(l) = 0 \quad \Rightarrow \quad -\lambda \int_0^l (l-\xi)\, u(\xi)\, d\xi + Cl = 0$$

$$\Rightarrow \quad C = \frac{\lambda}{l} \int_0^l (l-\xi)\, u(\xi)\, d\xi$$

Substituting the value of the constants C and D in (2), we have

$$u(x) = -\lambda \int_0^x (x-\xi)\, u(\xi)\, d\xi + \frac{\lambda}{l} \int_0^l x(l-\xi)\, u(\xi)\, d\xi$$

or
$$u(x) = -\lambda \int_0^x (x-\xi)\, u(\xi)\, d\xi + \frac{\lambda}{l} \int_0^x x(l-\xi)\, u(\xi)\, d\xi + \frac{\lambda}{l} \int_x^l x(l-\xi)\, u(\xi)\, d\xi$$

or
$$u(x) = \lambda \int_0^x \frac{\xi}{l}(l-x)\, u(\xi)\, d\xi + \lambda \int_x^l \frac{x}{l}(l-\xi)\, u(\xi)\, d\xi$$

or
$$u(x) = \lambda \int_0^l G(x, \xi)\, u(\xi)\, d\xi$$

where $$G(x, \xi) = \begin{cases} \dfrac{\xi}{l}(l-x) & , \ x > \xi \\ \dfrac{x}{l}(l-\xi) & , \ x < \xi \end{cases}.$$

3.2.7. Exercise.

1. Construct the Green's function for the boundary value problem $\dfrac{d^2u}{dx^2}+\mu^2 u = 0$ with the conditions u(0) = u(l) = 0.

   Answer. $$G(x, \xi) = \begin{cases} a_1\cos\mu x + a_2\sin\mu x = -\dfrac{\sin\mu(\xi-l)\sin\mu x}{\mu\sin\mu l} & , \quad 0 \le x < \xi \\ b_1\cos\mu x + b_2\sin\mu x = -\dfrac{\sin\mu\xi\sin\mu(x-l)}{\mu\sin\mu l} & , \quad \xi < x \le l \end{cases}$$

2. Construct the Green's function for the boundary value problem $\dfrac{d^2u}{dx^2}$ = 0 with the conditions u(0) = $u'(1)$ and $u'(0)$ = u(1).

   Answer. $$G(x, \xi) = \begin{cases} (-\xi+2)x+(-\xi+1) & , \quad 0 \le x < \xi \\ (-\xi+1)x+1 & , \quad \xi < x \le 1 \end{cases}$$

3. Construct the Green's function for the boundary value problem $\dfrac{d^3u}{dx^3}$ = 0 with the boundary conditions u(0) = $u'(1)$ = 0 and $u'(0)$ = u(1).

   Answer. $$G(x, \xi) = \begin{cases} \dfrac{1}{2}x(\xi-1)[x-x\xi+2\xi] & 0 \le x < \xi \\ \dfrac{1}{2}\xi[x(2-x)(\xi-2)+\xi] & \xi < x \le 1 \end{cases}$$

4. Construct the Green's function for the boundary value problem $x^2\dfrac{d^2u}{dx^2}+x\dfrac{du}{dx}-u$ = 0 with u(x) is bounded as $x \to 0$ and u(1) = 0.

   Answer. $$G(x, \xi) = \begin{cases} \dfrac{1}{2}x\left(\dfrac{1}{\xi^2}-1\right) & , \quad 0 \le x < \xi \\ \dfrac{1}{2}\left(\dfrac{1}{x}-x\right) & , \quad \xi < x \le 1 \end{cases}$$

5. Construct the Green's function for the boundary value problem $\dfrac{d^2u}{dx^2}-u$ = 0 with the conditions u(0) = $u'(0)$ and u(l) + λ $u'(l)$ = 0.

Answer. G(x, ξ) = $\begin{cases} -\frac{1}{2}\left(\frac{1-\lambda}{1+\lambda}\right)e^{x+\xi-2l}+\frac{1}{2}e^{x-\xi} & , \ 0 \le x < \xi \\ -\frac{1}{2}\left(\frac{1-\lambda}{1+\lambda}\right)e^{x+\xi-2l}+\frac{1}{2}e^{\xi-x} & , \ \xi < x \le 1 \end{cases}$, where $|\lambda| \neq 1$.

6. Using Green's function, solve the boundary value problem $u''(x) - u(x) = x$ with boundary conditions u(0) = u(1) = 0.

Answer. Here, G(x, ξ) = $\begin{cases} -\frac{\sinh x \sinh(\xi-1)}{\sinh 1} & , \ 0 \le x < \xi \\ -\frac{\sinh \xi \sinh(x-1)}{\sinh 1} & , \ \xi < x \le 1 \end{cases}$ and the solution of the given boundary value problem is given by u(x) = $\int_0^1 G(x, \xi)\xi \, d\xi$, so u(x) = $\frac{\sinh x}{\sinh 1} - x$.

7. Using Green's function, solve the boundary value problem $\frac{d^2u}{dx^2} + u = x$ with the boundary conditions u(0) = 0 and u($\pi/2$) = 0.

Answer. Here, G(x, ξ) = $\begin{cases} \cos\xi \sin x & , \ 0 \le x < \xi \\ \sin\xi \cos x & , \ \xi < x \le \pi/2 \end{cases}$ and u(x) = $\int_0^{\pi/2} G(x, \xi)\, \xi \, d\xi$, implies

u(x) = x − $\frac{\pi}{2}$sin x.

8. Solve the boundary value problem using Green's function

$$\frac{d^2u}{dx^2} + u = x^2 \text{ ; } u(0) = u(\pi/2) = 0.$$

Answer. u(x) = $-\left[2\cos x + \sin x\left(2 - \frac{\pi^2}{4}\right) + x^2 - 2\right]$.

3.3. Construction of Green's function when the boundary value problem contains a parameter.

Consider a differential equation of order n

$$L(u) - \lambda h = h(x) \quad (1)$$

with $$V_k(u) = 0, \ k = 1, 2, 3, \dots, n \quad (2)$$

where $$L(u) \equiv p_0(x)u^n(x) + p_1(x)u^{n-1}(x) + \dots + p_n(x)u(x) \quad (3)$$

and $$V_k(u) \equiv \alpha_k u(a) + \alpha_k^1 u'(a) + \dots + \alpha_k^{n-1} u^{n-1}(a) + \dots + \beta_k u(b) + \beta_k^1 u'(b) + \dots + \beta_k^{n-1} u^{n-1}(b) + \dots \quad (4)$$

where the linear forms $V_1, V_2, \dots, V_n$ in u(a), u'(a), …, u^{n-1}(a), u(b), u'(b), …, u^{n-1}(b) are linearly independent, h(x) is a given continuous function of x, λ is some non–zero numerical parameter.

For $h(x) \equiv 0$, the equation (1) reduces to homogeneous boundary value problem

$$L(u) = \lambda u,$$
$$V_k(u) = 0, k = 1, 2, 3, \ldots, n \quad (5)$$

Those values of λ for which the boundary value problem (5) has non trivial solutions u(x) are called the eigenvalues. The non–trivial solutions are called the associated eigen functions.

If the boundary value problem

$$L(u) = 0,$$
$$V_k(u) = 0, k = 1, 2, \ldots, n \quad (6)$$

contains the Green's function $G(x, \xi)$, then the boundary value problem (1) and (2) is equivalent to the Fredholm integral equation

$$u(x) = \lambda\int_a^b G(x, \xi)\, u(\xi)\, d\xi + f(x) \quad (7)$$

where $$f(x) = \int_a^b G(x, \xi)\; h(\xi)\; d\xi \quad (8)$$

In particular, the homogeneous boundary value problem (5) is equivalent to the homogeneous integral equation

$$u(x) = \lambda\int_a^b G(x, \xi)\; u(\xi)\; d\xi \quad (9)$$

Since $G(x, \xi)$ is a continuous kernel, therefore the Fredholm homogeneous integral equation of second kind (9) can have at most a countable number of eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$ which do not have a finite limit point. For all values of λ different from the eigen values, the non-homogeneous equation (7) has a solution for any continuous function f(x). Thus the solution is given by

$$u(x) = \lambda\int_a^b R(x, \xi; \lambda)\; f(\xi)\, d\xi + f(\xi) \quad (10)$$

where $R(x, \xi ; \lambda)$ is the resolvent kernel of the kernel $G(x, \xi)$. The function $R(x, \xi ; \lambda)$ is a meromorphic function of λ for any fixed values of x and ξ in [a, b]. The eigen values of the homogeneous integral equation (9) may by the pole of this function.

3.3.1. Example. Reduce the boundary value problem $\frac{d^2u}{dx^2} + \lambda u = x$, $u(0) = u(\pi/2) = 0$, to an integral equation using Green's function.

Solution. Consider the associated boundary value problem

$$\frac{d^2u}{dx^2} = 0 \quad (1)$$

whose general solution is given by u(x) = Ax + B

The boundary conditions u(0) = 0, u($\pi/2$) = 0 yields only the trivial solution u(x) $\equiv$ 0. Therefore, the Green's function G(x, ξ) exists for the associated boundary value problem

$$G(x,\ \xi) = \begin{cases} a_1x + a_2 \ , & 0 \le x < \xi \\ b_1x + b_2 \ , & \xi < x \le \pi/2 \end{cases} \qquad (2)$$

The Green's function G(x, ξ) must satisfy the following properties :

(I) The function G(x, ξ) is continuous at x = ξ, that is,

$$b_1\xi + b_2 = a_1\xi + a_2$$

$$\Rightarrow \quad (b_1 - a_1)\ \xi + (b_2 - a_2) = 0 \qquad (3)$$

(II) The derivative G(x, ξ) has a discontinuity of magnitude $-\left\{\dfrac{1}{p_0(\xi)}\right\}$ at the point x = ξ,

that is, $$\left(\frac{\partial G}{\partial x}\right)_{x=\xi+0} - \left(\frac{\partial G}{\partial x}\right)_{x=\xi-0} = -1 \Rightarrow b_1 - a_1 = -1 \qquad (4)$$

(III) The function G(x, ξ) must satisfy the boundary conditions

$$G(0,\ \xi) = 0 \qquad \Rightarrow \qquad a_2 = 0 \qquad (5)$$

$$G(\pi/2,\ \xi) = 0 \qquad \Rightarrow \qquad b_1\left(\frac{\pi}{2}\right) + b_2 = 0 \qquad (6)$$

Solving the equations (3), (4), (5) and (6), we have

$$a_1 = 1 - \frac{2\xi}{\pi},\ a_2 = 0,\ b_2 = \xi,\ b_1 = -\frac{2\xi}{\pi}.$$

Substituting the value of the constants in (2), the required Green's function G(x, ξ) is obtained

$$G(x,\ \xi) = \begin{cases} \left(1 - \dfrac{2\xi}{\pi}\right)x \ , & 0 \le x < \xi \\ \left(1 - \dfrac{2x}{\pi}\right)\xi \ , & \xi < x \le \pi/2 \end{cases} \qquad (7)$$

Consider the Green's function G(x, ξ) given by the relation (7) as the kernel of the integral equation, we obtain the integral equation for u(x) :

$$u(x) = f(x) - \lambda\int_0^{\pi/2} G(x,\xi)\ u(\xi)\ d\xi, \text{ where } f(x) = \int_0^{\pi/2} G(x,\xi)\ \xi\ d\xi$$

or $$f(x) = \int_0^x \left(1 - \frac{2x}{\pi}\right)\xi^2\ d\xi + \int_x^{\pi/2}\left(1 - \frac{2\xi}{\pi}\right)x\xi d\xi$$

or $$f(x) = \frac{1}{3}\left(1-\frac{2x}{\pi}\right)x^3 + x\left(\frac{1}{2}\xi^2 - \frac{2}{3\pi}\xi^3\right)_x^{\pi/2}$$

or $$f(x) = \frac{1}{3}x^3 - \frac{2}{3\pi}x^4 + \frac{\pi^2 x}{24} - \frac{1}{2}x^3 + \frac{2}{3\pi}x^4$$

or $$f(x) = \frac{\pi^2}{24}x - \frac{x^3}{6}$$

Thus, the given boundary value problem has been reduced to an integral equation

$$u(x) + \lambda\int_0^{\pi/2} G(x\,,\,\xi)\ u(\xi)\ d\xi\ = \frac{\pi^2}{24}x - \frac{1}{6}x^3.$$

3.3.2. Exercise.

1. Reduce the boundary value problem $\frac{d^2u}{dx^2} + xu\ = 1$, $u(0) = u(1) = 0$ to an integral equation.

Answer. $G(x,\ \xi) = \int_0^x \xi(1-x)\ d\xi + \int_x^1 x(1-\xi)\ d\xi\ = \frac{1}{2}x(1-x)$, and the required integral equation is $u(x) = \int_0^1 G(x\,,\,\xi)\ \xi u(\xi)\ d\xi - \frac{1}{2}x(1-x)$

2. Reduce the boundary value problem to an integral equation

$$\frac{d^2u}{dx^2} = \lambda u + 1,\ u(0) = u'(0) = 0,\ u''(1) = u'''(1) = 0$$

Answer. $u(x) = \lambda\int_0^1 G(x\,,\,\xi)\ u(\xi)\ d\xi + f(x)$, where $f(x) = \frac{1}{24}x^2(x^2 - 4x + 6)$

3. Reduce the boundary value problem $\frac{d^2u}{dx^2} + \frac{\pi^2}{4}u = \lambda u + \cos\frac{\pi x}{2}$, with $u(-1) = u(1)$ and $u'(-1) = u'(1)$ to an integral equation.

Answer. Here, $$G(x,\ \xi) = \begin{cases} \frac{1}{\pi}\sin\frac{\pi}{2}(x-\xi), & -1 \le x < \xi \\ \frac{1}{\pi}\sin\frac{\pi}{2}(\xi - x), & \xi < x \le 1 \end{cases}$$

and $$u(x) = \lambda\int_{-1}^1 G(x\,,\,\xi)\ u(\xi)\ d\xi - \left(\frac{x}{\pi}\sin\frac{\pi x}{2} + \frac{2}{\pi^2}\cos\frac{\pi x}{2}\right).$$

4. Reduce the following boundary value problems to integral equations.

(a) $u'' + \lambda u\ = 2x + 1$, $u(0) = u'(1)$, $u'(0) = u(1)$

(b) $u'' + \lambda u = e^x$, $u(0) = u''(0)$, $u(1) = u'(1)$.

Answer. (a) Here, $G(x,\ \xi) = \begin{cases} -\{(\xi-2)x+(\xi-1)\} &, \ 0 \le x < \xi \\ -\{(\xi-1)x-1\} &, \ \xi < x \le 1 \end{cases}$ and the boundary value problem reduces to the integral equation

$$u(x) = -\lambda\int_0^1 G(x\,,\,\xi)\,u(\xi)\,d\xi - \frac{1}{6}(2x^3+3x^2-17x-5)\,.$$

(b) Here, $G(x,\ \xi) = \begin{cases} -(1+x)\xi &, \ 0 \le x < \xi \\ -(1+\xi)x &, \ \xi < x \le 1 \end{cases}$ and the boundary value problem reduces to

$$u(x) = -\lambda\int_0^1 G(x\,,\,\xi)\,u(\xi)\,d\xi - e^x\,.$$

5. Reduce the Bessel's differential equation $x^2\dfrac{d^2u}{dx^2} + x\dfrac{du}{dx} + (\lambda x^2 - 1)u = 0$ with the conditions $u(0) = 0$, $u(1) = 0$ into an integral equation.

Answer.: The standard equation of Bessel's equation is given by

Here, $G(x,\ \xi) = \begin{cases} \dfrac{x}{2\xi}(1-\xi^2)\,, & 0 \le x < \xi \\ \dfrac{\xi}{2x}(1-x^2)\,, & \xi < x \le 1 \end{cases}$ and the integral equation can be obtained as

$$u(x) = \lambda\int_0^1 G(x\,,\,\xi)\,r(\xi)\,u(\xi)\,d\xi\,.$$

6. Determine the Green's function $G(x,\ \xi)$ for the differential equation $\left[\dfrac{d}{dx}\left(x\dfrac{d}{dx}\right) - \dfrac{n^2}{x}\right]u = 0$ with the conditions $u(0) = 0$ and $u(1) = 0$.

Answer. $G(x,\ \xi) = \begin{cases} \dfrac{x^n}{2n\xi^n}(1-\xi^{2n})\,, & x < \xi \\ \dfrac{\xi^n}{2nx^n}(1-x^{2n})\,, & x > \xi. \end{cases}$

3.4. Non–homogeneous ordinary Equation. The boundary value problem associated with a non – homogenous ordinary differential equation of second order is

$$Ly \equiv A_0(x)\frac{d^2y}{dx^2} + A_1(x)\frac{dy}{dx} + A_2(x)\,y = f(x),\ a < x < b \qquad (1)$$

with boundary conditions
$$\left.\begin{aligned} \alpha_1 y(a) + \alpha_2 y'(a) &= 0 \\ \beta_1 y(b) + \beta_2 y'(b) &= 0 \end{aligned}\right\} \qquad (2)$$

where α_1, α_2, β_1 and β_2 are constants.

3.4.1. Self–Adjoint Operator. The operator L is said to be self – adjoint if for any two functions say u(x) and v(x) operated on L, the expression (vLu − uLv) dx is an exact differential that is, there exist a function g such that dg = (vLu − uLv) dx.

3.4.2. Green's Function Method. Green's function method is an important method to solve B.V.P. associated with non–homogeneous ordinary or partial differential equation . Here we shall show that a B.V.P. will be reduced to a Fredholm integral equation whose kernel is Green's function. We shall be using a special type of B.V.P. namely Sturm – Liouville's problem.

3.4.3. Theorem. Show that the differential operator L of the Sturm – Liouville's Boundary value problem (S.L.B.V.P.)

$$Ly = \frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] + \left[q(x)+\lambda p(x)\right] y(x) = 0 \qquad (1)$$

with
$$\left.\begin{aligned} \alpha_1 y(a)+\alpha_2 y'(a) &= 0 \\ \beta_1 y(b)+\beta_2 y'(b) &= 0 \end{aligned}\right\} \qquad (2)$$

where α, β, α_2 and β_2 are constants is self adjoint.

Proof. Let u and v be two solutions of the given S.L.B.V.P. then

$$Lu = \frac{d}{dx}\left[r(x)\frac{du}{dx}\right] + \left[g(x)+\lambda p(x)\right] u(x) = 0$$

and
$$Lv = \frac{d}{dx}\left[r(x)\frac{dv}{dx}\right] + \left[q(x)+\lambda p(x)\right] v(x) = 0$$

So,

$$vLu - uLv = v\frac{d}{dx}\left[r(x)\frac{du}{dx}\right] + \left[q(x)+\lambda p(x)\right] u(x)v - \left[u\frac{d}{dx}\left[r(x)\frac{dv}{dx}\right] + \left[q(x)+\lambda p(x)\right]v(x)u\right]$$

$$= v\frac{d}{dx}\left[r(x)\frac{du}{dx}\right] - u\frac{d}{dx}\left[r(x)\frac{dv}{dx}\right]$$

$$= \left[v\frac{d}{dx}\left(r(x)\frac{du}{dx}\right) + \left(r(x)\frac{du}{dx}\right)\frac{dv}{dx}\right] - \left[u\frac{d}{dx}\left(r(x)\frac{dv}{dx}\right) + \left(r(x)\frac{dv}{dx}\right)\frac{du}{dx}\right]$$

$$= \frac{d}{dx}\left[r(x)\,v(x)\frac{du}{dx}\right] - \frac{d}{dx}\left[r(x)\,u(x)\frac{dv}{dx}\right]$$

$$= \frac{d}{dx}\left[r(x)\,v(x)\frac{du}{dx} - r(x)\,u(x)\frac{dv}{dx}\right] = \frac{d}{dx}\left[r(x)\left(v(x)\frac{du}{dx} - u(x)\frac{dv}{dx}\right)\right] = \frac{dg}{dx}$$

where $g = r(x)\left(v(x)\frac{du}{dx} - u(x)\frac{dv}{dx}\right)$. Then, (vLu − uLv) dx = dg

Hence operator in equation (1) is self – adjoint.

3.4.4. Construction of Green's function by variation of parameter method.

Consider the non – homogeneous differential equation

$$Lu = \frac{d}{dx}\left[r(x)\ \frac{du}{dx}\right] + \left[q(x)+\lambda p(x)\right]\ u(x)\ = f(x) \qquad (1)$$

subject to boundary condition:

$$\left.\begin{aligned}\alpha_1 u(a)+\alpha_2 u'(a)\ &=\ 0\\ \beta_1 u(b)+\beta_2 u'(b)\ &=\ 0\end{aligned}\right\} \qquad (*)$$

Construct the Green's function and show that

$$u(x) = -\int_a^b G(x\ ,\ \xi)\ \ f(\xi)\ d\xi \qquad (**)$$

where $G(x,\ \xi)$ is the Green's function defined above.

Solution. Let $v_1(x)$ and $v_2(x)$ be two linearly independent solution of the homogeneous differential equation.

$$Lu = \frac{d}{dx}\left[r(x)\frac{du}{dx}\right] +\ \left[q(x)+\lambda p(x)\right]\ u(x) = 0 \qquad (2)$$

Then the general solution of (2) by the method of variation of parameters is

$$u(x) = a_1(x)\ v_1(x) + a_2(x)\ v_2(x) \qquad (3)$$

where the unknown variables $a_1(x)$ and $a_2(x)$ are to be determined. We assume that neither the solution $v_1(x)$ nor $v_2(x)$ satisfy both the boundary conditions at x = a and x = b but the general solution u(x) satisfies these conditions.

Now, we differentiate (3) w.r.t. x and obtain

$$u'(x)\ =\ a_1'v_1 + a_1v_1' + a_2'v_2 + a_2v_2' \qquad (4)$$

Let us equate to zero the terms involving derivatives of parameter, that is,

$$a_1'(x)v_1(x) + a_2'(x)v_2(x)\ = 0 \qquad (5)$$

which yields

$$u'(x)\ =\ a_1(x)\ v_1'(x) + a_2(x)\ v_2'(x) \qquad (6)$$

Putting the values of u(x) and $u'(x)$ from (3) and (6) respectively in equation (1), we obtain

$$Lu = \frac{d}{dx}\left[r(x)\ \left(a_1v_1' + a_2v_2'\right)\right] + \left[q(x)+\lambda p(x)\right](a_1v_1 + a_2v_2) = f(x)$$

or $a_1\left[\frac{d}{dx}(r\,v_1') + v_1(q+\lambda p)\right] + a_2\frac{d}{dx}\left[(r\,v_2') + v_2(q+\lambda p)\right] + \left(a_1'v_1' + a_2'v_2'\right)\, r(x) = f(x)$ (7)

Since v_1 and v_2 are solutions of homogeneous equation (2), so by (7), we get

$$\left(a_1'v_1' + a_2'v_2'\right) r(x) = f(x)$$

$$\Rightarrow \quad a_1'(x)\, v_1'(x) + a_2'(x)\, v_2'(x) = \frac{f(x)}{r(x)} \qquad (8)$$

Equations (5) and equation (8) can be solved to get

$$a_1'(x) = \frac{f(x)\, v_2(x)}{r(x)\left[v_2 v_1' - v_1 v_2'\right]} \quad \text{and} \quad a_2'(x) = \frac{-f(x)\, v_1(x)}{r(x)\left[v_2 v_1' - v_1 v_2'\right]} \qquad (9)$$

Now the operator L is exact and we have proved that

$$v_2\, L\, v_1 - v_1\, L\, v_2 = \frac{d}{dx}\left[r(x)\, \left(v_2 v_1' - v_1 v_2'\right)\right] \qquad (10)$$

Since v_1 and v_2 are solutions of Sturm – Liouville homogeneous differential equation so that $Lv_1 = 0$ and $Lv_2 = 0$ and thus equation (10) gives

$$\frac{d}{dx}\left[r(x)\, \left(v_2 v_1' - v_1 v_2'\right)\right] = 0$$

$$\Rightarrow \quad r(x)\, \left(v_2 v_1' - v_1 v_2'\right) = \text{constant} = -\beta \text{ (say)} \qquad (11)$$

Thus, equation (9) becomes

$$\left.a_1'(x) = \frac{-f(x)\, v_2(x)}{\beta} \quad \text{and} \quad a_2'(x) = \frac{f(x)\, v_1(x)}{\beta}\right] \qquad (12)$$

Integrating (12), we get

$$a_1(x) = -\frac{1}{\beta}\int_{c_1}^{x} f(\xi)\, v_2(\xi)\; d\xi \qquad (13)$$

and $$a_2(x) = \frac{1}{\beta}\int_{c_2}^{x} f(\xi)\, v_1(\xi)\; d\xi \qquad (14)$$

where c_1 and c_2 are arbitrary constants to be determined from the boundary condition on $a_1(x)$ and $a_2(x)$. These conditions are to be imposed in accordance with our earlier assumption that $v_1(x)$ and $v_2(x)$ does not satisfy boundary conditions but the final solution u(x) satisfies boundary conditions in equation (*). So, that

$$\alpha_1 u(a) + \alpha_2 u'(a) = 0 \qquad (15)$$

$$\beta_1 u(b) + \beta_2 u'(b) = 0 \qquad (16)$$

Using (3) and (6) in equation (15), we obtain

$$0 = \alpha_1 u(a) + \alpha_2 u'(a)$$

$$= \alpha_1 \left[a_1(a)v_1(a) + a_2(a)v_2(a)\right] + \alpha_2 \left[a_1(a)v_1'(a) + a_2(a)v_2'(a)\right]$$

$$= a_1(a)\left[\alpha_1 v_1(a) + \alpha_2 v_1'(a)\right] + a_2(a)\left[\alpha_1 v_2(a) + \alpha_2 v_2'(a)\right]$$

Let us now assume that $v_2(x)$ satisfies first boundary condition of (*) but $v_1(x)$ does not satisfy it, then

$$\alpha_1 v_2(a) + \alpha_2 v_2'(a) = 0$$
$$\alpha_1 v_1(a) + \alpha_2 v_2'(a) \neq 0$$

so that $a_1(a)\ \left[\alpha_1 v_1(a) + \alpha_2 v_1'(a)\right] = 0 \Rightarrow a_1(a) = 0$

Using this condition in (13), we get

$$0 = a_1(a) = -\frac{1}{\beta}\int_{c_1}^{a} f(\xi)\, v_2(\xi)\ d\xi \text{ which is satisfied when } c_1 = a$$

Thus, the solution in (13) is :

$$a_1(x) = -\frac{1}{\beta}\int_{a}^{x} f(\xi)\, v_2(\xi)\ d\xi \tag{17}$$

Similarly, using (3) and (6) in (16), we obtain $c_2 = b$ and the solution in (14) is :

$$a_2(x) = \frac{1}{\beta}\int_{b}^{x} f(\xi)\, v_1(\xi)\ d\xi = -\frac{1}{\beta}\int_{x}^{b} f(\xi)\, v_1(\xi)\ d\xi \tag{18}$$

The final solution of the non – homogeneous B.V.P. is

$$u(x) = a_1(x)\, v_1(x) + a_2(x)\, v_2(x)$$

$$= -\frac{1}{\beta}\ v_1(x)\int_{a}^{x} f(\xi)\, v_2(\xi)\, d\xi - \frac{1}{\beta}\ v_2(x)\int_{x}^{b} f(\xi)\, v_1(\xi)\, d\xi$$

$$= -\int_{a}^{x} \frac{v_1(x)\, v_2(\xi)}{\beta}\ f(\xi)\, d\xi - \int_{x}^{b} \frac{v_2(x)\, v_1(\xi)}{\beta}\ f(\xi)\, d\xi = -\int_{a}^{b} G(x, \xi)\, f(\xi)\ d\xi$$

where $G(x, \xi) = \begin{cases} \dfrac{1}{\beta} v_1(x)\, v_2(\xi) & \xi \le x \le b \\ \dfrac{1}{\beta} v_2(x)\, v_1(\xi) & a \le x \le \xi \end{cases}$

3.5. Basic Properties of Green's Function.

3.5.1. Theorem. The Green function $G(x, \xi)$ is symmetric in x and ξ, that is, $G(x, \xi) = G(\xi, x)$.

Proof. Interchanging x and ξ in G(x, ξ) defined above :

$$G(x, \xi) = \begin{cases} \frac{1}{\beta} v_1(\xi)\, v_2(x) & x \le \xi \\ \frac{1}{\beta} v_1(x)\, v_2(\xi) & \xi \le x \end{cases} = G(\xi, x).$$

3.5.2. Theorem. The function G(x, ξ) satisfies the boundary conditions given in equation (*).

Proof. Consider

$$\alpha_1 G(a, \xi) + \alpha_2 G'(a, \xi) = \alpha_1 \left[\frac{v_2(a)v_1(\xi)}{\beta}\right] + \alpha_2 \left[\frac{v_2'(a)v_1(\xi)}{\beta}\right]$$

$$= \frac{1}{\beta}\left[\alpha_1 v_2(a) + \alpha_2 v_2'(a)\right] v_1(\xi)$$

$$= \frac{1}{\beta}[0] v_1(\xi) = 0 \quad x \le \xi \le b$$

Again, $\beta_1 G(b, \xi) + \beta_2 G'(b, \xi) = \beta_1 \left[\frac{v_1(b)v_2(\xi)}{\beta}\right] + \beta_2 \left[\frac{v_1'(b)v_2(\xi)}{\beta}\right]$

$$= \frac{1}{\beta}\left[\beta_1 v_1(b) + \beta_2 v_1'(b)\right] v_2(\xi)$$

$$= \frac{1}{\beta}[0] v_2(\xi) = 0 \quad , a \le \xi \le x.$$

3.5.3. Theorem. The function G(x, ξ) is continuous in [a, b]

Proof. Clearly, G(x, ξ) is continuous at every point of [a, b] except possibly at x = ξ. By definition of G(x, ξ), it can be observed that both branches have same value at x = ξ given by $\frac{1}{\beta}\left[v_1(\xi)\, v_2(\xi)\right]$.

Hence G(x, ξ) is continuous in [a, b].

3.5.4. Theorem. $\frac{\partial G}{\partial x}$ has a jump discontinuity at x = ξ, given by

$$\left.\frac{\partial G}{\partial x}\right|_{x = \xi^+} - \left.\frac{\partial G}{\partial x}\right|_{x = \xi^-} = -\frac{1}{r(\xi)}$$

where r(x) is the co – efficient of $u''(x)$ in equation (1).

Proof. We have $\left.\frac{\partial G}{\partial x}\right|_{\substack{x = \xi^+ \\ (x > \xi)}} - \left.\frac{\partial G}{\partial x}\right|_{\substack{x = \xi^- \\ (x < \xi)}} = \frac{1}{\beta}\left[v_1'(x)\, v_2(\xi)\right]_{x = \xi} - \frac{1}{\beta}\left[v_2'(x)\, v_1(\xi)\right]_{x = \xi}$

$$= \frac{1}{\beta}\left[v_1'(\xi)\, v_2(\xi) - v_2'(\xi)\, v_1(\xi)\right]$$

$$= \frac{1}{\beta}\left[\frac{-\beta}{r(\xi)}\right] = -\frac{1}{r(\xi)} \qquad \text{[By equation (11)]}$$

3.6. Fredholm Integral Equation and Green's Function. Consider the general boundary value problem

$$A_0(x)\, \frac{d^2y}{dx^2} + A_1(x)\frac{dy}{dx} + A_2(x)\, y + \lambda\, p(x)\, y = h(x) \tag{1}$$

with boundary conditions: y(a) = 0, y(b) = 0. (2)

We shall show that it reduces to Fredholm integral equation with the Green's function as its kernel.

To make the above operator in (1) as a self – adjoint operator, we shift the term λ p(x)y to the right side and then divide it by $\frac{r(x)}{A_0(x)}$.

The solution of (1) in terms of Green's function is

$$y(x) = -\int_a^b G(x\,,\,\xi)\;\; f(\xi)\, d\xi \;\text{ where } f(x) = h(x) - \lambda\, p(x)\, y(x)$$

or $$y(x) = -\int_a^b G(x\,,\,\xi)\;\left[h(\xi) - \lambda\, p(\xi)\, y\right] d\xi$$

$$= -\int_a^b G(x\,,\,\xi)\;\; h(\xi)\, d\xi + \lambda\int_a^b G(x\,,\,\xi)\;\; p(\xi)\, y(\xi)\, d\xi$$

$$= K(x) + \lambda \int_a^b G(x\,,\,\xi)\;\; p(\xi)\, y(\xi)\, d\xi \tag{3}$$

where $$K(x) = -\int_a^b G(x\,,\,\xi)\;\; h(\xi)\, d\xi \tag{4}$$

This is a Fredholm integral equation of the second kind with kernel K(x, ξ) = G(x, ξ) p(ξ) and a non – homogeneous term K(x).

Now, multiplying equation (3) by $\sqrt{p(x)}$, we get

$$\sqrt{p(x)}\;\; y(x) = \sqrt{p(x)}\;\; K(x) + \lambda\int_a^b \sqrt{p(x)\, p(\xi)}\;\; G(x,\, \xi)\sqrt{p(\xi)}\;\; y(\xi)\, d\,\xi$$

Let us use, u(x) = $\sqrt{p(x)}$ y(x) and g(x) = $\sqrt{p(x)}$ K(x)

Then, $u(x) = g(x) + \lambda\int_a^b \sqrt{p(x)\,p(\xi)}\; G(x, \xi)\, u(\xi)\, d\xi$ (5)

Here the kernel of Fredholm integral equation of second kind is symmetric that is,

$$K(x, \xi) = \sqrt{p(x)\,p(\xi)}\; G(x, \xi) \quad (6)$$

is symmetric, since $G(x, \xi)$ is symmetric.

Remark : We had obtained the required result in equation (3). We had proceed to obtain equation (5) just to get the kernel in more symmetric form.

3.7. Check Your Progress.

1. Solve the boundary value problem using Green's function $\frac{d^2u}{dx^2} - u = -2e^x$ with boundary conditions $u(0) = u'(0)$, $u(l) + u'(l) = 0$.

Answer. $G(x, \xi) = \begin{cases} \frac{1}{2}e^{x-\xi} & , \; 0 \le x < \xi \\ \frac{1}{2}e^{-(x-\xi)} & , \; \xi < x \le l \end{cases}$ and $u(x) = -[(l - x)e^x + \sinh x]$

2. Solve the boundary value problem using Green's function $\frac{d^4u}{dx^4} = 1$, with boundary conditions $u(0) = u'(0) = u''(1) = u'''(1) = 0$.

Answer. $G(x, \xi) = \begin{cases} \frac{1}{6}x^2(3\xi - x) & , \; 0 \le x < \xi \\ \frac{1}{6}\xi^2(3\xi - x) & , \; \xi < x \le 1 \end{cases}$ and $u(x) = \frac{1}{24}x^2(x^2 - 4x + 6)$.

3.8. Summary. In this chapter, we discussed various methods to construct Green function for a given non-homogeneous linear second order boundary value problem and then boundary value problem can be reduced to Fredholm integral equation with Green function as kernel and hence can be solbed using the methods studied in the previous chapter.

Books Suggested:

1. Jerri, A.J., Introduction to Integral Equations with Applications, A Wiley-Interscience Publication, 1999.
2. Kanwal, R.P., Linear Integral Equations, Theory and Techniques, Academic Press, New York.
3. Lovitt, W.V., Linear Integral Equations, McGraw Hill, New York.
4. Hilderbrand, F.B., Methods of Applied Mathematics, Dover Publications.

Sturm-Liouville System

Definition 6.1 (Sturm-Liouville Boundary Value Problem (SL-BVP)) *With the notation*

$$\mathcal{L}[y] \equiv \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)\,y, \tag{6.1}$$

consider the Sturm-Liouville equation

$$\mathcal{L}[y] + \lambda r(x)\,y = 0, \tag{6.2}$$

where $p > 0$, $r \geq 0$, *and* p, q, r *are continuous functions on interval* $[a, b]$; *along with the boundary conditions*

$$a_1 y(a) + a_2 p(a) y'(a) = 0, \quad b_1 y(b) + b_2 p(b) y'(b) = 0, \tag{6.3}$$

where $a_1^2 + a_2^2 \neq 0$ *and* $b_1^2 + b_2^2 \neq 0$.

The problem of finding a complex number μ *if any, such that the BVP* (6.2)-(6.3) *with* $\lambda = \mu$, *has a non-trivial solution is called a Sturm-Liouville Eigen Value Problem (SL-EVP). Such a value* μ *is called an eigenvalue and the corresponding non-trivial solutions* $y(.;\mu)$ *are called eigenfunctions. Further,*

(i) *An SL-EVP is called a* regular *SL-EVP if* $p > 0$ *and* $r > 0$ *on* $[a, b]$.

(ii) *An SL-EVP is called a* singular *SL-EVP if (i)* $p > 0$ *on* (a, b) *and* $p(a) = 0 = p(b)$, *and (ii)* $r \geq 0$ *on* $[a, b]$.

(iii) *If* $p(a) = p(b)$, $p > 0$ *and* $r > 0$ *on* $[a, b]$, p, q, r *are continuous functions on* $[a, b]$, *then solving Sturm-Liouville equation* (6.2) *coupled with boundary conditions*

$$y(a) = y(b), \quad y'(a) = y'(b), \tag{6.4}$$

is called a periodic *SL-EVP.*

We are not going to discuss singular SL-BVPs. Before we discuss further, let us completely study two examples that are representatives of their class of problems.

6.1 Two examples

Example 6.2 *For* $\lambda \in \mathbb{R}$, *solve*

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(\pi) = 0. \tag{6.5}$$

For reasons that will be clear later on, it is enough to consider $\lambda \in \mathbb{R}$.

Case 1. *Let* $\lambda < 0$. *Then* $\lambda = -\mu^2$, *where* μ *is real and non-zero. The general solution of ODE in* (6.5) *is given by*

$$y(x) = Ae^{\mu x} + Be^{-\mu x} \tag{6.6}$$

This y *satisfies boundary conditions in* (6.5) *if and only if* $A = B = 0$. *That is,* $y \equiv 0$. *Therefore, there are no negative eigenvalues.*

Case 2. *Let* $\lambda = 0$. *In this case, it easily follows that trivial solution is the only solution of*

$$y'' = 0, \quad y(0) = 0, \quad y'(\pi) = 0. \tag{6.7}$$

Thus, 0 *is not an eigenvalue.*

Case 3. *Let* $\lambda > 0$. *Then* $\lambda = \mu^2$, *where* μ *is real and non-zero. The general solution of ODE in* (6.5) *is given by*

$$y(x) = A\cos(\mu x) + B\sin(\mu x) \tag{6.8}$$

This y *satisfies boundary conditions in* (6.5) *if and only if* $A = 0$ *and* $B\cos(\mu\pi) = 0$. *But* $B\cos(\mu\pi) = 0$ *if and only if, either* $B = 0$ *or* $\cos(\mu\pi) = 0$.

The condition $A = 0$ *and* $B = 0$ *means* $y \equiv 0$. *This does not yield any eigenvalue. If* $y \not\equiv 0$, *then* $b \neq 0$. *Thus* $\cos(\mu\pi) = 0$ *should hold. This last equation has solutions given by* $\mu = \frac{2n-1}{2}$, *for* $n = 0, \pm 1, \pm 2, \ldots$. *Thus eigenvalues are given by*

$$\lambda_n = \frac{2n-1}{2}, \quad n = 0, \pm 1, \pm 2, \ldots \tag{6.9}$$

and the corresponding eigenfunctions are given by

$$\phi_n(x) = B\sin\left(\frac{2n-1}{2}x\right), \quad n = 0, \pm 1, \pm 2, \ldots \tag{6.10}$$

Note: All the eigenvalues are positive. The eigenfunctions corresponding to each eigenvalue form a one dimensional vector space and so the eigenfunctions are unique upto a constant multiple.

Example 6.3 *For* $\lambda \in \mathbb{R}$, *solve*

$$y'' + \lambda y = 0, \quad y(0) - y(\pi) = 0, \quad y'(0) - y'(\pi) = 0. \tag{6.11}$$

This is not a SL-BVP. It is a mixed boundary condition unlike the separated BC above. These boundary conditions are called periodic boundary conditions.

Case 1. *Let* $\lambda < 0$. *Then* $\lambda = -\mu^2$, *where* μ *is real and non-zero. In this case, it can be easily verified that trivial solution is the only solution of the BVP* (6.11).

Case 2. *Let* $\lambda = 0$. *In this case, general solution of ODE in* (6.11) *is given by*

$$y(x) = A + Bx \tag{6.12}$$

This y *satisfies the BCs in* (6.11) *if and only if* $B = 0$. *Thus* A *remains arbitrary.*

Thus 0 *is an eigenvalue with eigenfunction being any non-zero constant. Note that eigenvalue is simple. An eigenvalue is called simple eigenvalue if the corresponding eigenspace is of dimension one, otherwise eigenvalue is called multiple eigenvalue.*

Case 3. *Let* $\lambda > 0$. *Then* $\lambda = \mu^2$, *where* μ *is real and non-zero. The general solution of ODE in* (6.11) *is given by*

$$y(x) = A\cos(\mu x) + B\sin(\mu x) \tag{6.13}$$

This y *satisfies boundary conditions in* (6.11) *if and only if*

$$\begin{aligned} A\sin(\mu\pi) + B(1 - \cos(\mu\pi)) &= 0, \\ A(1 - \cos(\mu\pi)) - B\sin(\mu\pi) &= 0. \end{aligned}$$

This has non-trivial solution for the pair (A, B) *if and only if*

$$\begin{vmatrix} \sin(\mu\pi) & 1-\cos(\mu\pi) \\ 1-\cos(\mu\pi) & -\sin(\mu\pi) \end{vmatrix} = 0. \tag{6.14}$$

That is, $\cos(\mu\pi) = 1$*. This further implies that* $\mu = \pm 2n$ *with* $n \in \mathbb{N}$*, and hence* $\lambda = 4n^2$ *with* $n \in \mathbb{N}$.
Thus positive eigenvalues are given by

$$\lambda_n = 4n^2, \quad n \in \mathbb{N}. \tag{6.15}$$

and the eigenfunctions corresponding to λ_n *are given by*

$$\phi_n(x) = \cos(2nx), \quad \psi_n(x) = \sin(2nx), \quad n \in \mathbb{N}. \tag{6.16}$$

Note: All the eigenvalues are non-negative. There are two linearly independent eigenfunctions, namely $\cos(2nx)$ *and* $\sin(2nx)$ *corresponding to each positive eigenvalue* $\lambda_n = 4n^2$*. Compare these properties with that of previous example.*