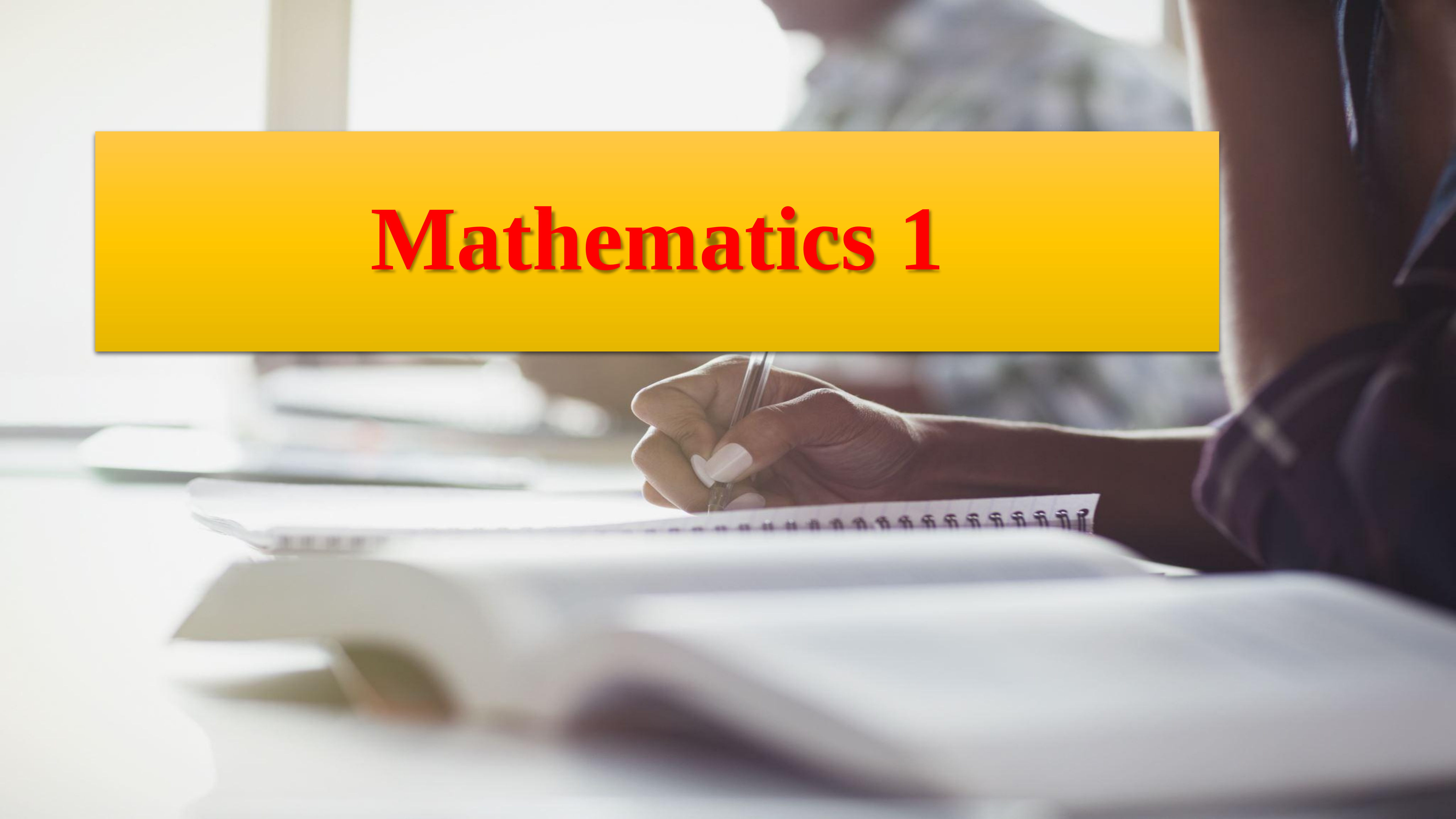
Mathematics 1



Functions

Functions are a tool for describing the real world in mathematical terms.

Definition:

If a variable y depends on a variable x in such away that each variable of x determines exactly one value of y, then we say that y is a function of x.

$$y = f(x).$$

Functions

A function can be represented by:

- **Numerically (by tables).**
- **Geometrically (by graph).**
- **Algebraically (by equation).**
- **Verbally.**

Functions

A function can be represented by:

- **Numerically (by tables).**

Table 0.1.1

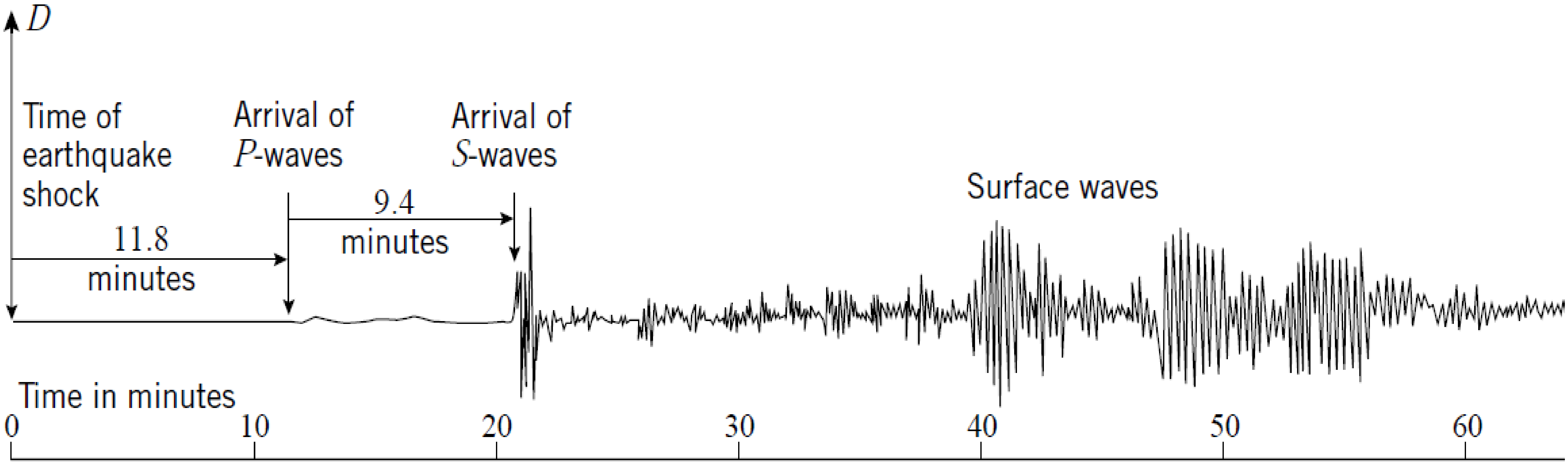
INDIANAPOLIS 500
QUALIFYING SPEEDS

YEAR t	SPEED S (mi/h)
1994	228.011
1995	231.604
1996	233.100
1997	218.263
1998	223.503
1999	225.179
2000	223.471
2001	226.037
2002	231.342
2003	231.725

Functions

A function can be represented by:

- **Geometrically (by graph)**



Functions

A function can be represented by:

- Algebraically (by equation)

$$F = G\frac{m_1 m_2}{r^2}$$

- Verbally.

The gravitational force of attraction between two bodies in the universe is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

Functions

Definition:

A function f is a rule that associates a unique output with each input. If the input is denoted by x, then the out put is denoted by f(x).

Functions

► **Example** $y = 3x^2 - 4x + 2$

$$f(0) = 3(0)^2 - 4(0) + 2 = 2$$

f associates $y = 2$ with $x = 0$.

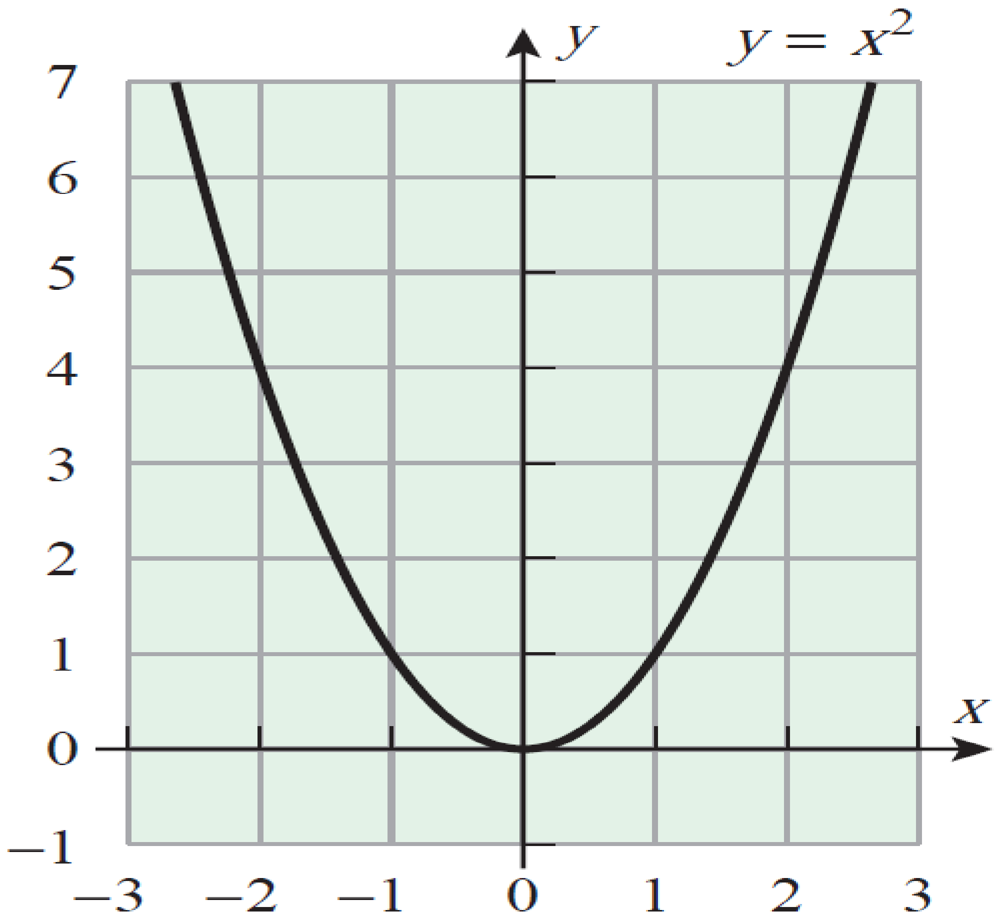
$$f(-1.7) = 3(-1.7)^2 - 4(-1.7) + 2 = 17.47$$

f associates $y = 17.47$ with $x = -1.7$.

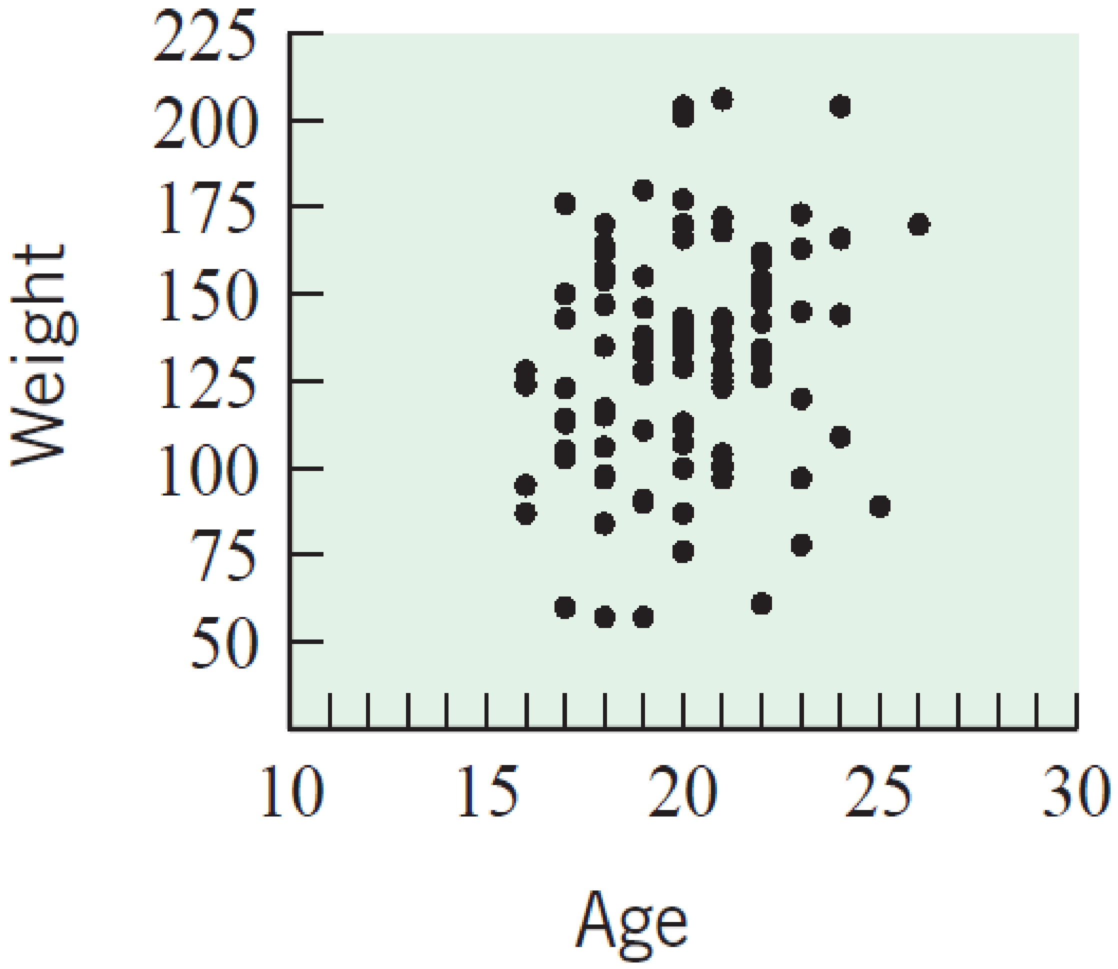
$$f(\sqrt{2}) = 3(\sqrt{2})^2 - 4\sqrt{2} + 2 = 8 - 4\sqrt{2}$$

f associates $y = 8 - 4\sqrt{2}$ with $x = \sqrt{2}$.

Functions



function

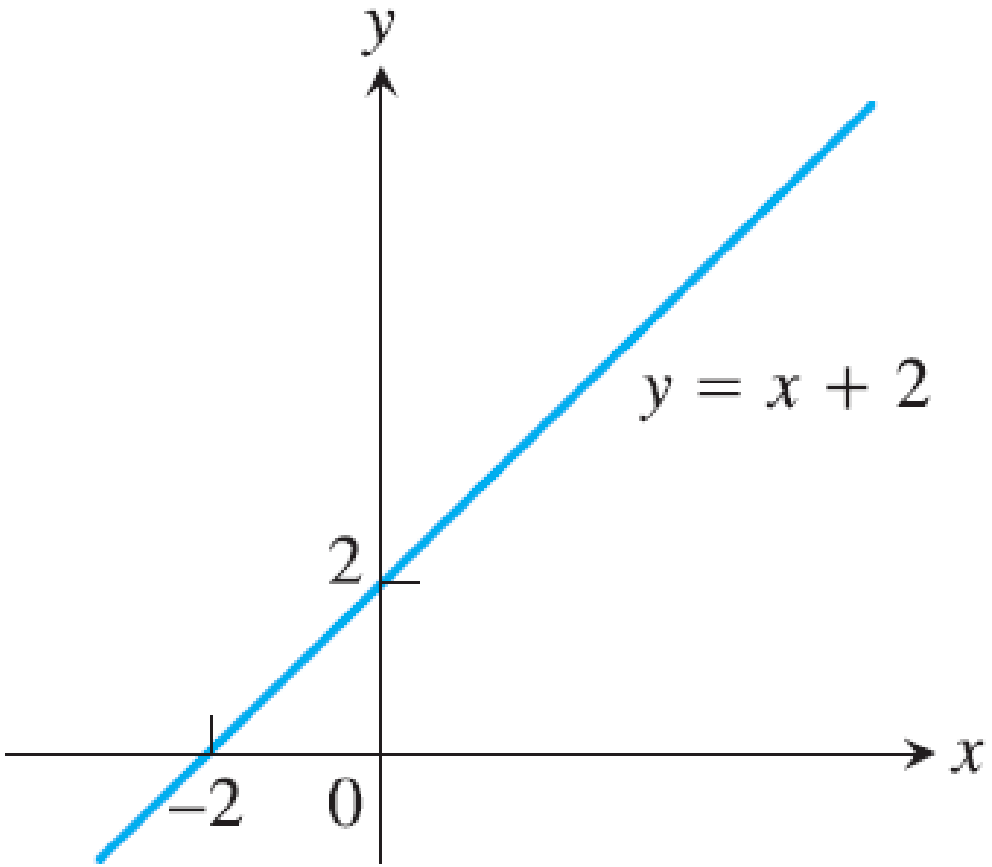


Not function

Graphs of Functions

If f(x) is a function, then the graph of f(x) in the xy-plane is defined to be the graph of the equation y=f(x).

Example: $f(x) = x + 2$



Graphs of Functions

Example: Graph the function $y = x^2$ over the interval [-2,2].

- Make a table of xy-pairs that satisfy the equation.
- Plot the points (x,y) whose coordinates appear in the table.
- Draw a smooth curve through the plotted points.

Graphs of Functions

Example: Graph the function y = x^2 over the interval [-2,2].

- Make a table of xy-pairs that satisfy the equation.

x	$y = x^2$
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

Graphs of Functions

Example: Graph the function y = x^2 over the interval [-2,2].

- Plot the points (x,y) whose coordinates appear in the table.
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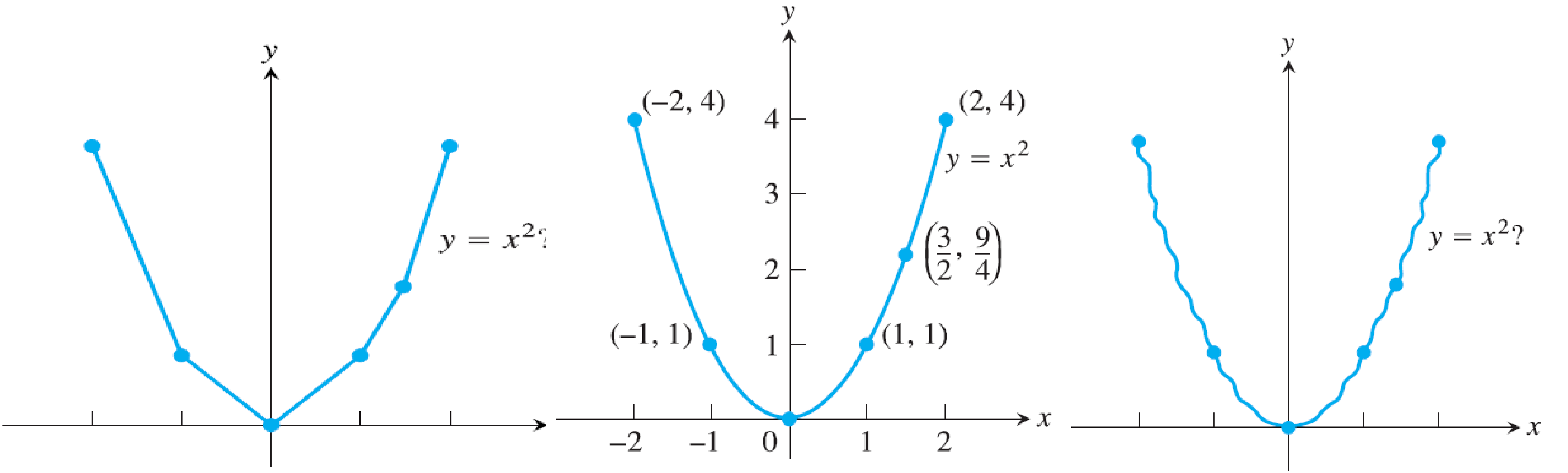
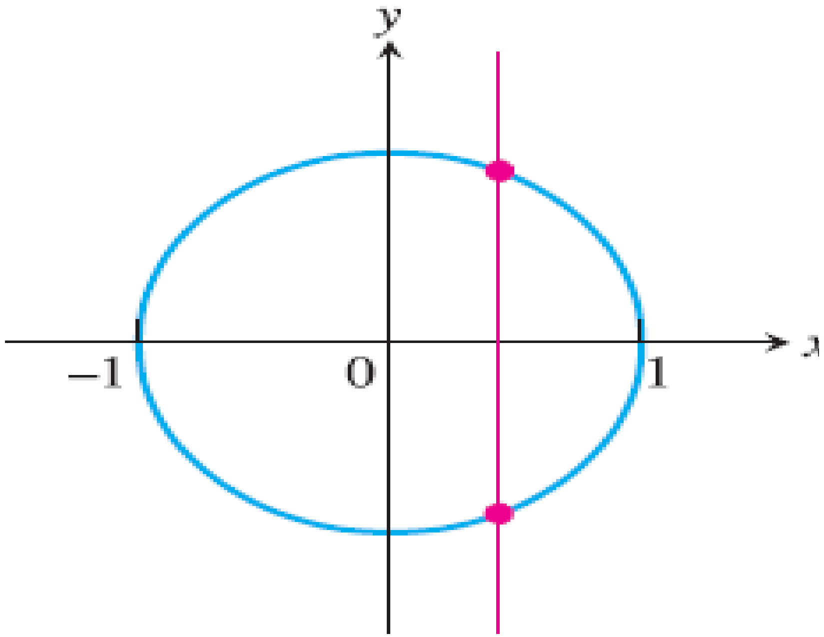


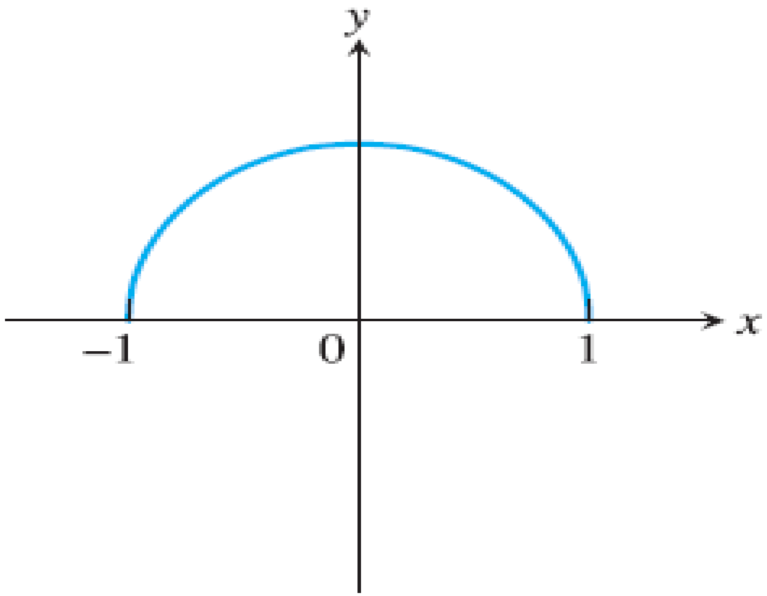
The Vertical Line Test for a Function

Not every curve in the xy-plane is the graph of a function.

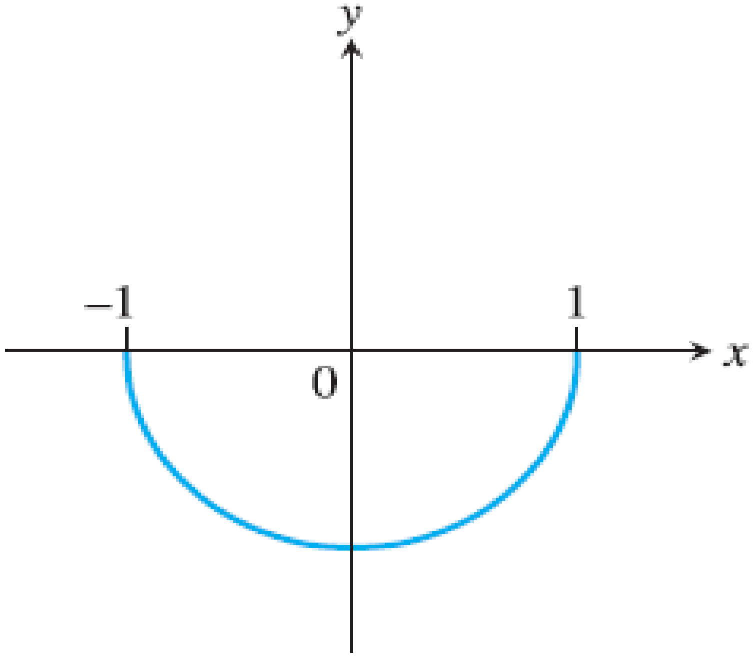
Example: A circle cannot be the graph of a function since some vertical lines intersect the circle twice.



$x^2 + y^2 = 1$



$y = \sqrt{1 - x^2}$



$y = -\sqrt{1 - x^2}$

Domain and Range

If x and y are related by the equation y=f(x), then the set of all allowable inputs (x-values) is called the domain of the function f(x).

$$y = f(x).$$

The set of outputs (y-values) that results when x various over the domain is called the range of the function f(x).

Domain and Range

Example:

Find the domain and the range of the function f(x) defined by

x	0	1	2	3
y	3	4	-1	6

Answer:

The domain is the set : {0,1,2,3}

The range is the set : {-1,3,4,6}

Domain and Range

Example:

Find the domain of the functions:

(a) $f(x) = x^3$

(b) $f(x) = 1/[(x-1)(x-3)]$

(c) $f(x) = \sqrt{x^2 - 5x + 6}$

Answer:

(a)The domain is the set : $(-\infty, \infty)$

Domain and Range

Example:

Find the domain of

(b) $f(x) = \dfrac{1}{[(x-1)(x-3)]}$

Answer:

Let x= 3

$$\mathbf{f(x) = \frac{1}{[(3-1)(3-3)]} = \frac{1}{[2(0)]}}$$

$$\mathbf{= \frac{1}{0} = \infty}$$

Domain and Range

Example:

Find the domain of

(b) $f(x) = \dfrac{1}{[(x-1)(x-3)]}$

Answer:

The domain is $\{x: x \neq 1 \text{ and } x \neq 3\}$

$= (-\infty, 1) \cup (1,3) \cup (3, \infty)$

Let $x = 1$

$$f(x) = \frac{1}{[(1-1)(1-3)]} = \frac{1}{[0(-2)]}$$

$$= \frac{1}{0} = \infty$$

Domain and Range

Example:

Find the domain and the range of

(c) f(x) = $\sqrt{1-x^2}$

Answer:

$$1-x^2 \geq 0$$

$$(1-x)(1+x) \geq 0$$

$x = 1$ and $x = -1$

$let\ x = 2,$

Then f(x)= $\sqrt{1-4}$

$=\sqrt{-3} = \sqrt{-1}\sqrt{3} = \boldsymbol{i}\sqrt{3}$

$\therefore f(x)\ not\ real\ at\ x = 2$

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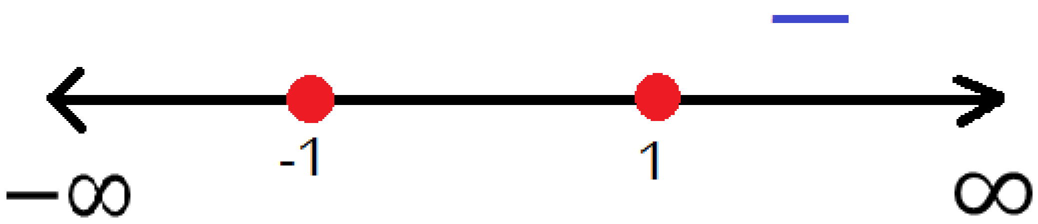

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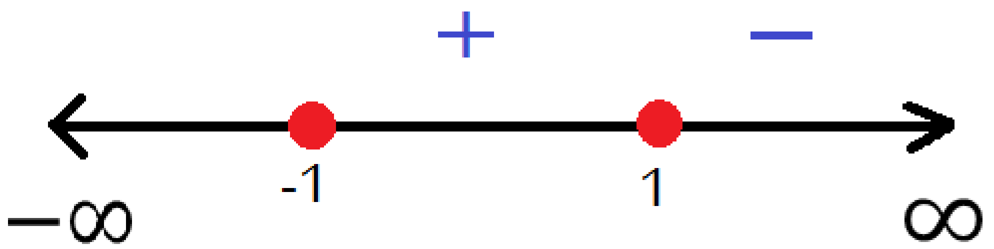


Domain and Range

Example:

Find the domain and the range of

(c) f(x) = $\sqrt{1-x^2}$

Answer:

$$1-x^2 \geq 0$$

$$(1-x)(1+x) \geq 0$$

$x = 1$ and $x = -1$

$$The\ domain\ is\ [-1,1].$$

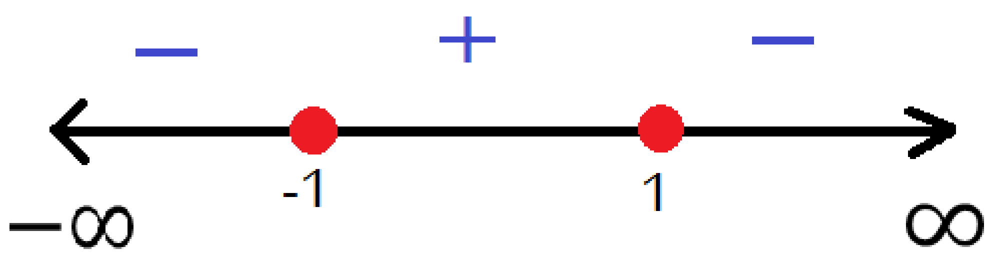
The range is [0,1].

$$let\ x = 2,$$

Then f(x)= $\sqrt{1-4}$

$=\sqrt{-3} = \sqrt{-1}\sqrt{3} = \boldsymbol{i}\sqrt{3}$

$$\therefore f(x)\ not\ real\ at\ x = 2$$



Domain and Range

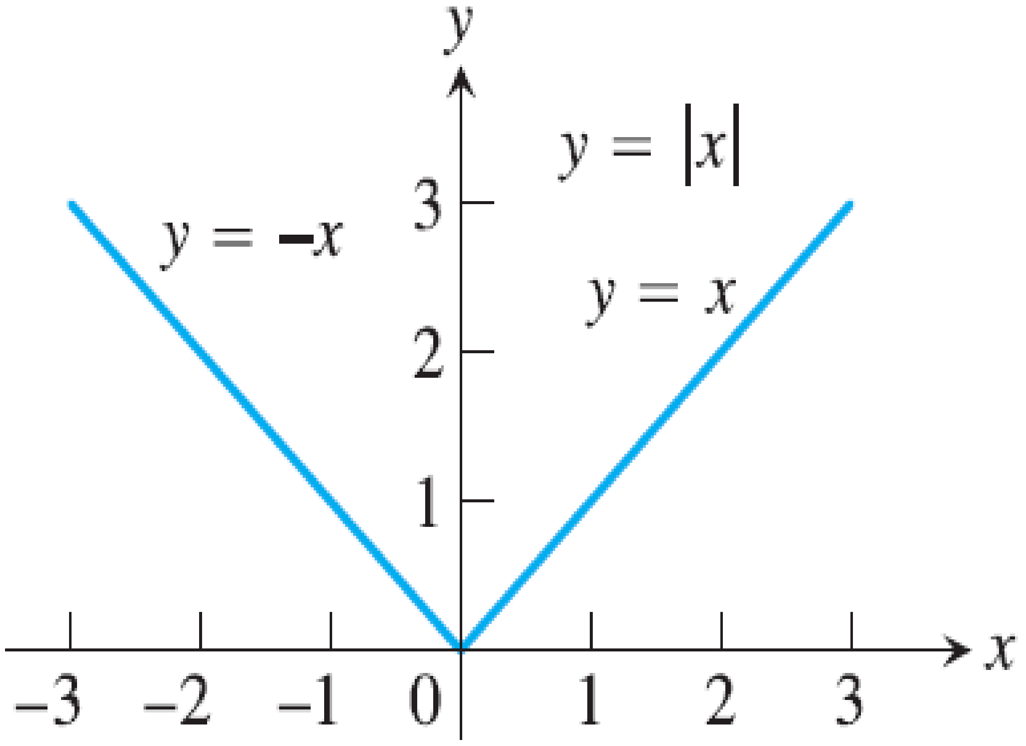
Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$

Piecewise-Defined Functions

Sometimes a function is described by using different formulas on different parts of its domain.

Example 1: Absolute value function

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0, \end{cases}$$



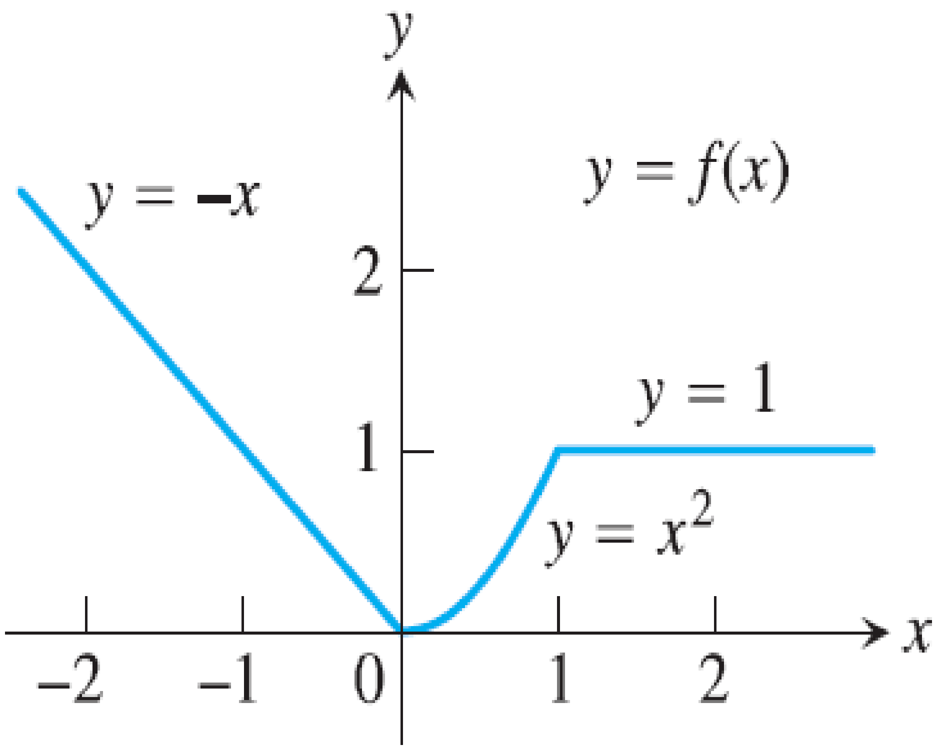
Domain: $(-\infty, \infty)$

Range: $[0, \infty)$

Piecewise-Defined Functions

Example 2:

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$



Domain: $(-\infty, \infty)$

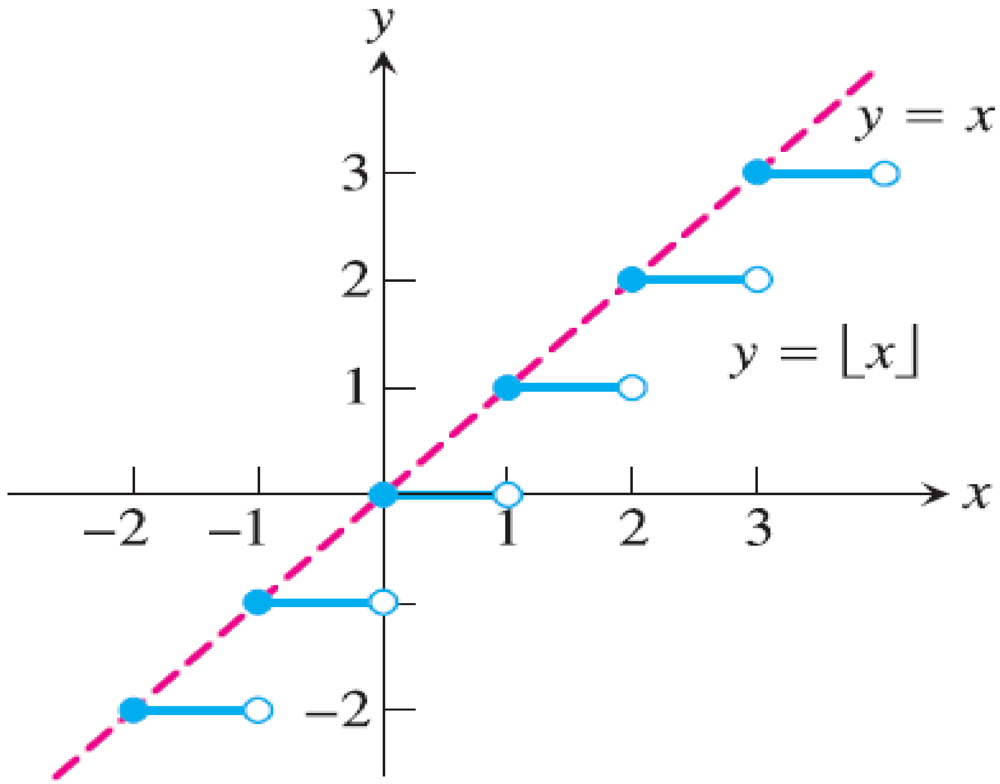
Range: $[0, \infty)$

Piecewise-Defined Functions

Example 3: Greatest integer function or the integer floor function.

The function whose value at any number *x* is the *greatest integer less than or equal to x* is called the greatest integer function or the integer floor function.

$$\lfloor 2.4 \rfloor = 2, \qquad \lfloor -0.3 \rfloor = -1$$
$$\lfloor 2 \rfloor = 2, \qquad \lfloor -1.2 \rfloor = -2,$$
$$\lfloor 1.9 \rfloor = 1, \qquad \lfloor -2 \rfloor = -2.$$
$$\lfloor 0.2 \rfloor = 0,$$
$$\lfloor 0 \rfloor = 0,$$



Piecewise-Defined Functions

Example 4: Least integer function or the integer ceiling function.

The function whose value at any number x is the *smallest integer greater than or equal to* x is called the least integer function or the integer ceiling function.

$\lceil 2.4 \rceil = 3$

$\lceil 2 \rceil = 2$

$\lceil 1.9 \rceil = 2$

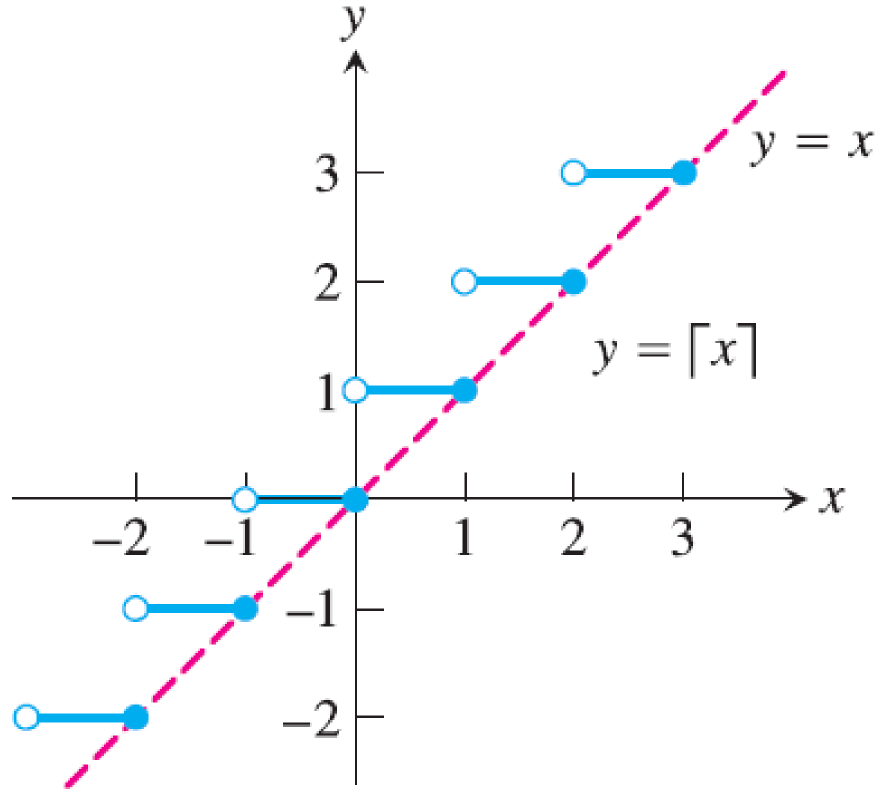
$\lceil 0.2 \rceil = 1$

$\lceil 0 \rceil = 0$

$\lceil -0.3 \rceil = 0$

$\lceil -1.2 \rceil = -1$

$\lceil -2 \rceil = -2.$



Properties of functions

Increasing and Decreasing Functions

DEFINITIONS Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

If the graph of a function *climbs or rises* as you move from left to right, we say that the function is *increasing.*

If the graph descends or falls as you move from left to right, the function is decreasing.

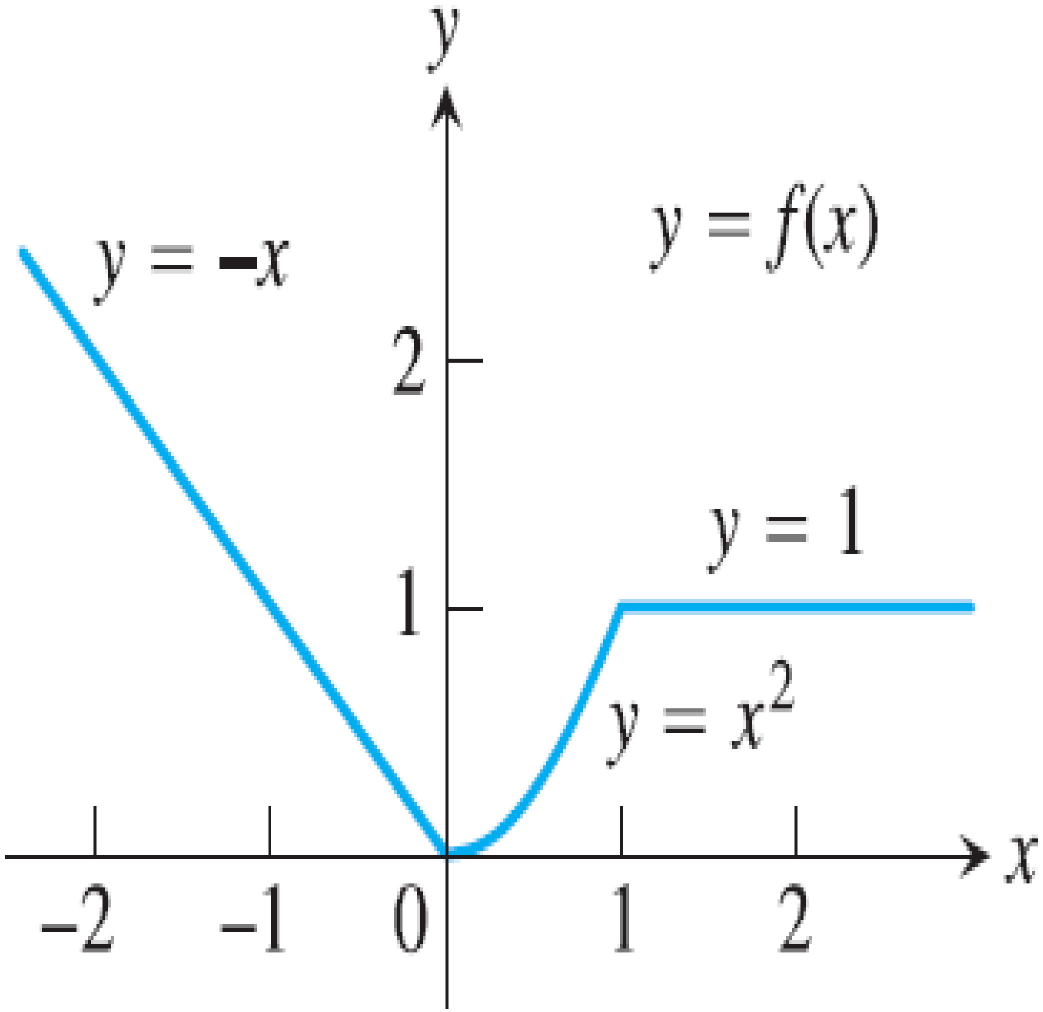
Increasing and Decreasing Functions

Example:

The function is decreasing on (-∞,0].

The function is increasing on [0, 1].

The function is neither increasing nor decreasing on the interval [1, ∞).



Problems

Functions

In Exercises 1–6, find the domain and range of each function.

1. $f(x) = 1 + x^2$

2. $f(x) = 1 - \sqrt{x}$

3. $F(x) = \sqrt{5x + 10}$

4. $g(x) = \sqrt{x^2 - 3x}$

5. $f(t) = \dfrac{4}{3 - t}$

6. $G(t) = \dfrac{2}{t^2 - 16}$

Piecewise-Defined Functions

Graph the functions in Exercises 25–28.

25. $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$

26. $g(x) = \begin{cases} 1 - x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$

27. $F(x) = \begin{cases} 4 - x^2, & x \le 1 \\ x^2 + 2x, & x > 1 \end{cases}$

28. $G(x) = \begin{cases} 1/x, & x < 0 \\ x, & 0 \le x \end{cases}$

Functions and Graphs

Find the domain and graph the functions in Exercises 15–20.

15. $f(x) = 5 - 2x$

16. $f(x) = 1 - 2x - x^2$

17. $g(x) = \sqrt{|x|}$

18. $g(x) = \sqrt{-x}$

19. $F(t) = t/|t|$

20. $G(t) = 1/|t|$

21. Find the domain of $y = \dfrac{x + 3}{4 - \sqrt{x^2 - 9}}$.

22. Find the range of $y = 2 + \dfrac{x^2}{x^2 + 4}$.

23. Graph the following equations and explain why they are not graphs of functions of x.

a. $|y| = x$

b. $y^2 = x^2$

24. Graph the following equations and explain why they are not graphs of functions of x.

a. $|x| + |y| = 1$

b. $|x + y| = 1$

Problems

Functions

In Exercises 1–6, find the domain and range of each function.

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Piecewise-Defined Functions

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a. $|y| = x$ **b.** $y^2 = x^2$

24. Graph the following equations and explain why they are not graphs of functions of x.

a. $|x| + |y| = 1$ **b.** $|x + y| = 1$

Problems

The Greatest and Least Integer Functions

33. For what values of x is

a. $\lfloor x \rfloor = 0$?　　**b.** $\lceil x \rceil = 0$?

34. What real numbers x satisfy the equation $\lfloor x \rfloor = \lceil x \rceil$?

35. Does $\lceil -x \rceil = -\lfloor x \rfloor$ for all real x? Give reasons for your answer.

36. Graph the function

$$f(x) = \begin{cases} \lfloor x \rfloor, & x \geq 0 \\ \lceil x \rceil, & x < 0. \end{cases}$$

Why is $f(x)$ called the *integer part* of x?

Increasing and Decreasing Functions

Graph the functions in Exercises 37–46. What symmetries, if any, do the graphs have? Specify the intervals over which the function is increasing and the intervals where it is decreasing.

37. $y = -x^3$

38. $y = -\dfrac{1}{x^2}$

39. $y = -\dfrac{1}{x}$

40. $y = \dfrac{1}{|x|}$

41. $y = \sqrt{|x|}$

42. $y = \sqrt{-x}$

43. $y = x^3/8$

44. $y = -4\sqrt{x}$

45. $y = -x^{3/2}$

46. $y = (-x)^{2/3}$

Problems

Even and Odd Functions

In Exercises 47–58, say whether the function is even, odd, or neither. Give reasons for your answer.

47. $f(x) = 3$

48. $f(x) = x^{-5}$

49. $f(x) = x^2 + 1$

50. $f(x) = x^2 + x$

51. $g(x) = x^3 + x$

52. $g(x) = x^4 + 3x^2 - 1$

53. $g(x) = \dfrac{1}{x^2 - 1}$

54. $g(x) = \dfrac{x}{x^2 - 1}$

55. $h(t) = \dfrac{1}{t - 1}$

56. $h(t) = |t^3|$

57. $h(t) = 2t + 1$

58. $h(t) = 2|t| + 1$

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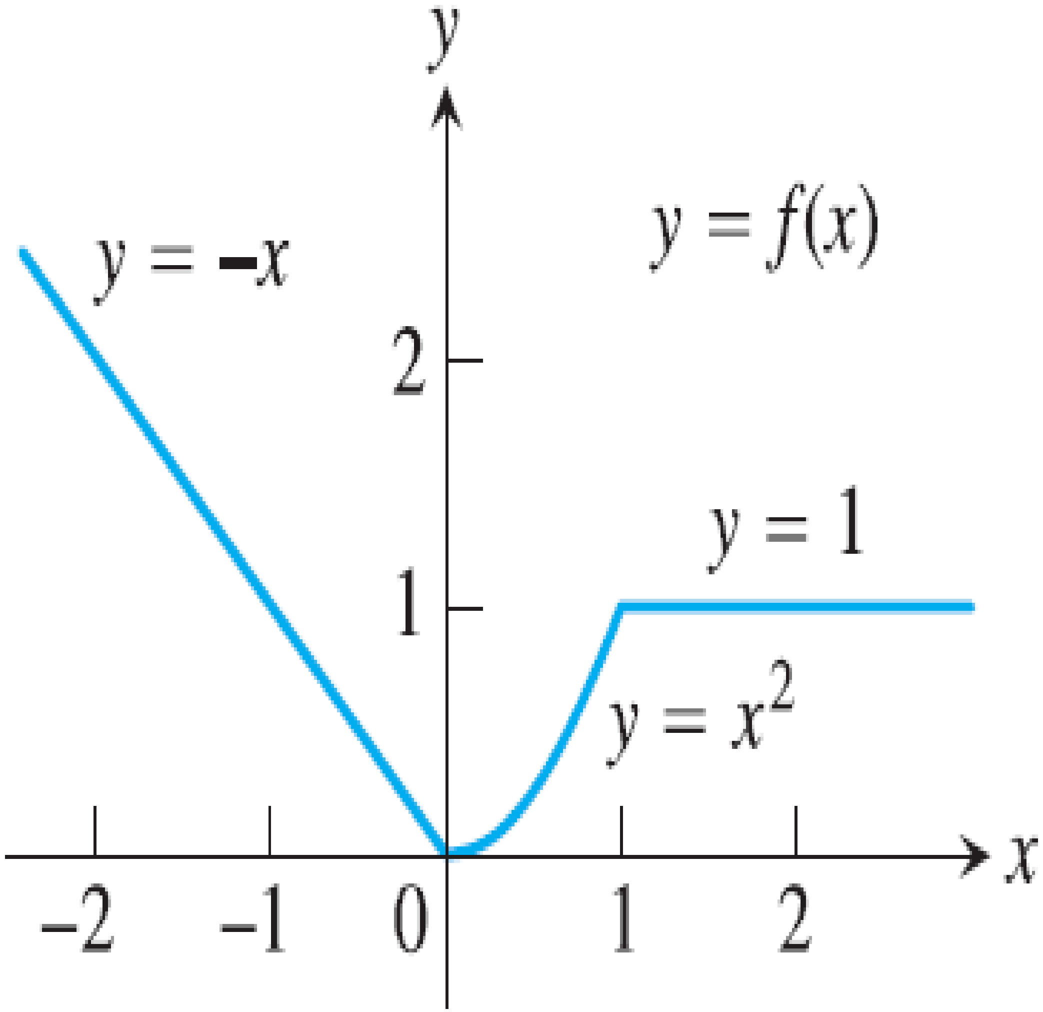


Even Functions and Odd Functions: Symmetry

DEFINITIONS A function $y = f(x)$ is an

even function of x if $f(-x) = f(x)$,

odd function of x if $f(-x) = -f(x)$,

for every x in the function's domain.

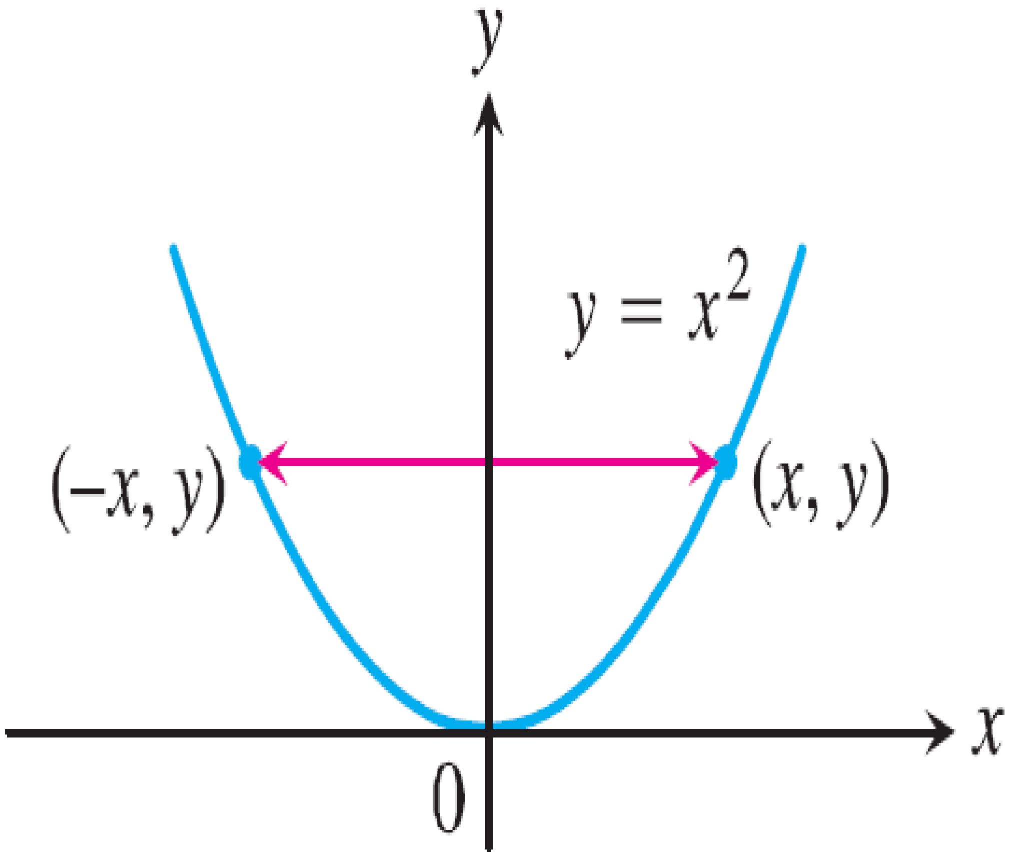
Example:

y= x^2	y= x
y= x^4	y= x^3
Even	**Odd**

Even Functions and Odd Functions: Symmetry

The graphs of *even* and *odd* functions have characteristic symmetry properties.

The graph of an even function is symmetric about the y-axis. Since f(-x) = f(x) , a point (x,y) lies on the graph if and only if the point (-x, y) lies on the graph.

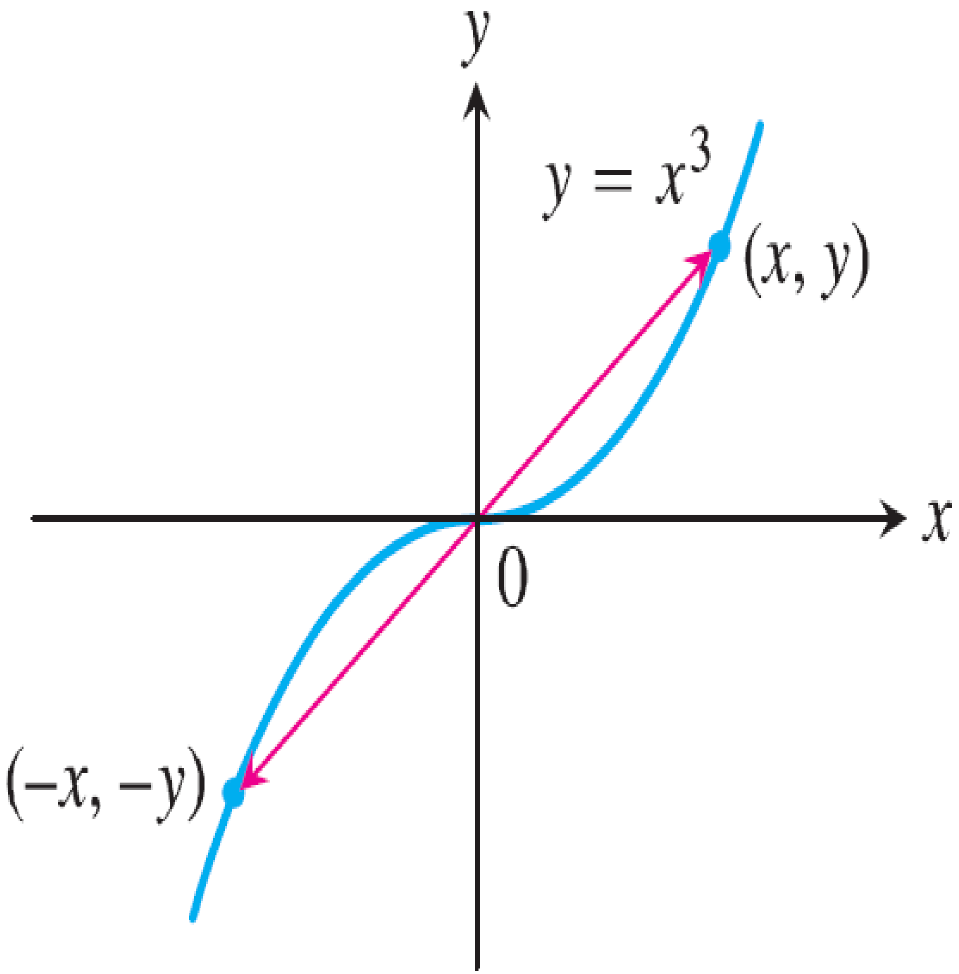


A reflection across the y-axis leaves the graph unchanged.

Even Functions and Odd Functions: Symmetry

The graphs of *even* and *odd* functions have characteristic symmetry properties.

The graph of an odd function is symmetric about the origin. Since *f(-x) = - f(x)* , a point *(x,y)* lies on the graph if and on1y if the point *(-x, -y)* lies on the graph.



Equivalently, a graph is symmetric about the origi if a rotation of 180 about the origin leaves the graph unchanged.

Notice that the definitions imply that both x and -x must be in the domain of the function f.

Even Functions and Odd Functions: Symmetry

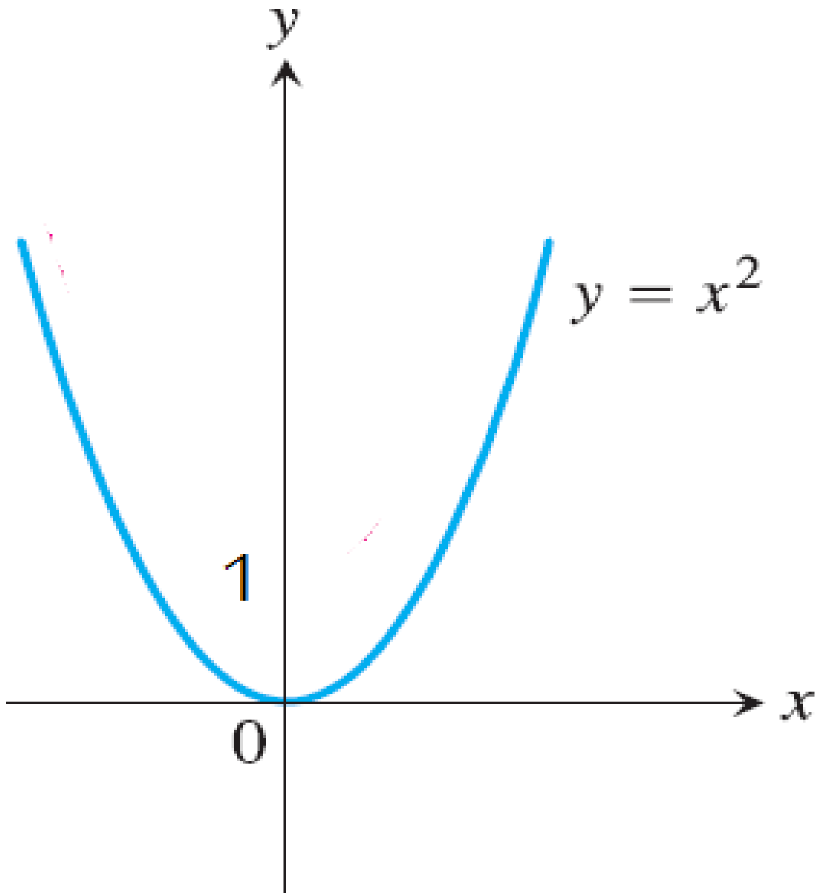
Example: $f(x) = x^2 + 1$

Answer:

$$f(-x) = (-x)^2 + 1 = x^2 + 1$$

$\therefore f(-x) = f(x)$ ⇨ Even function

symmetry about y-axis



Even Functions and Odd Functions: Symmetry

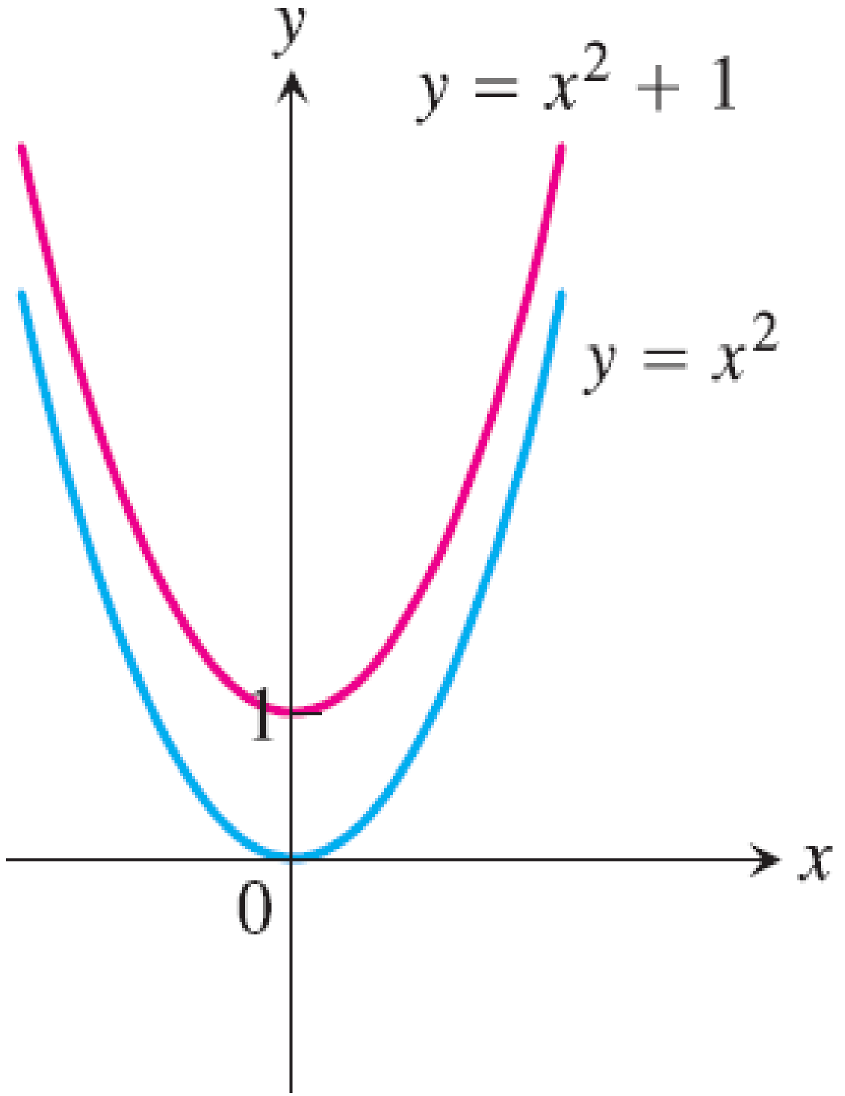
Example: $f(x) = x^2 + 1$

Answer:

$$f(-x) = (-x)^2 + 1 = x^2 + 1$$

$$\therefore f(-x) = f(x) \Rightarrow \text{Even function}$$

symmetry about y-axis



Even Functions and Odd Functions: Symmetry

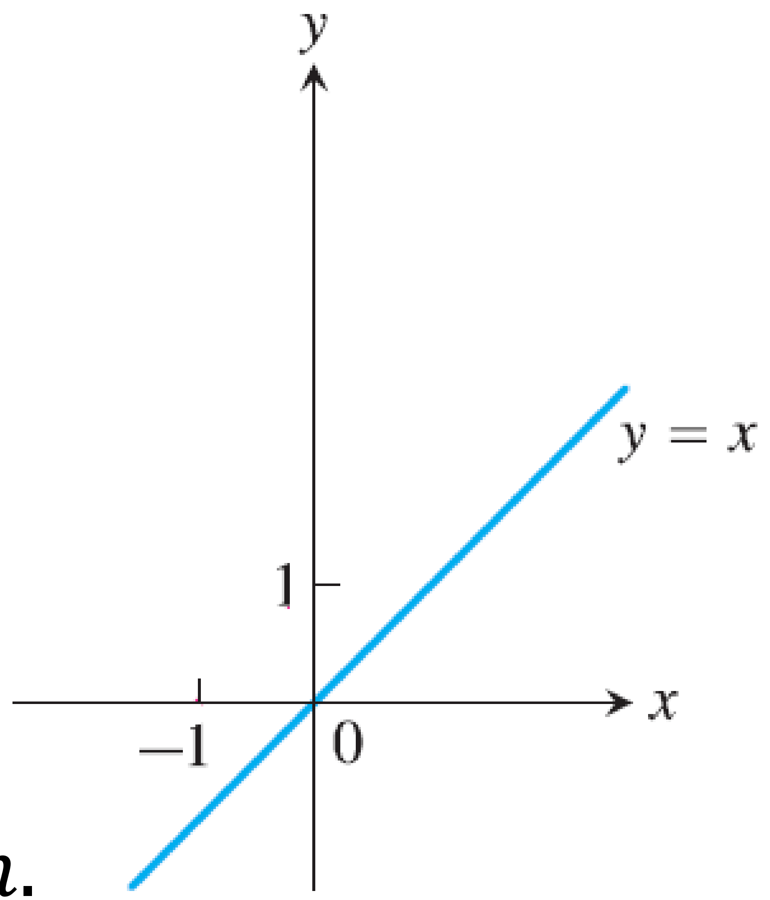
Example: $f(x) = x + 1$

$$f(-x) = -x + 1$$

$$-f(x) = -x - 1$$

$f(-x) \neq f(x)$, and $f(-x) \neq -f(x)$,

$\therefore$ *The function* $f(x)$ *is not odd and not even.*

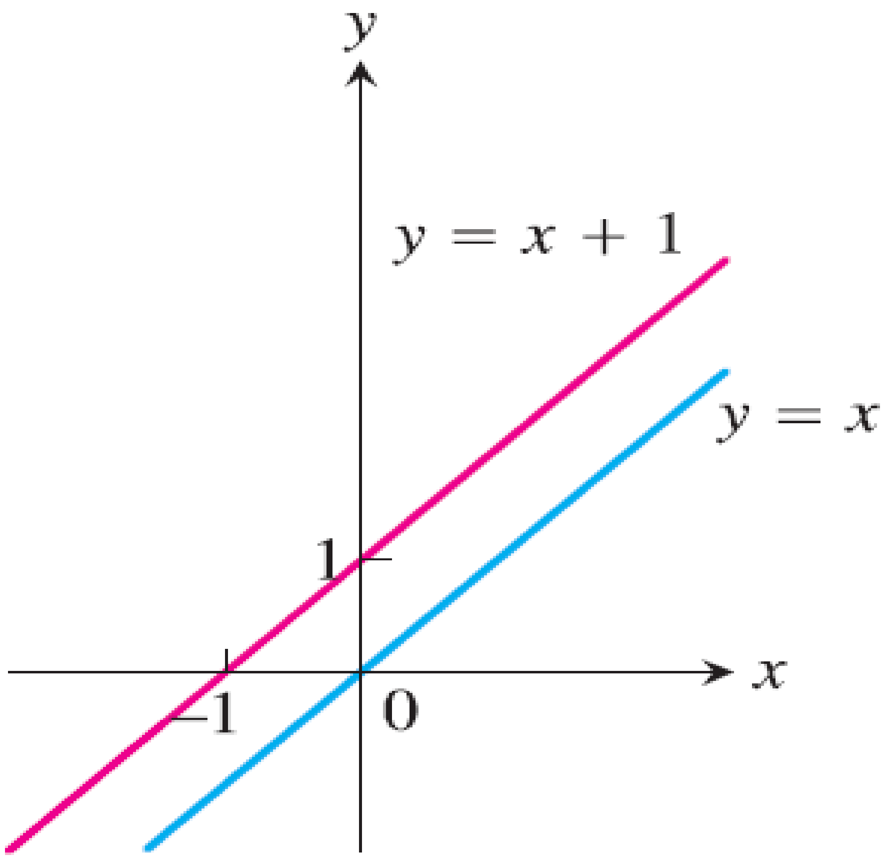


Even Functions and Odd Functions: Symmetry

Example: $f(x) = x + 1$

$f(-x) = -x + 1$

$-f(x) = -x - 1$



$f(-x) \neq f(x)$, and $f(-x) \neq -f(x)$,

$\therefore$ *The function $f(x)$ is not odd and not even.*

The symmetry about the orgini is lost

Combining Functions

Here, we study the main ways functions are combined or transformed to form new functions.

Like numbers, functions can be added, subtracted, multiplied, and divided (except where the denominator is zero) to produce new functions.

Combining Functions

If f and g are functions, then for every $\boldsymbol{x}$ that belongs to the domains of both $\boldsymbol{f}$ and **g** (that is, for $\boldsymbol{x \in D(f) \cap D(g)}$), we define functions $\boldsymbol{f}$ **+ g**, $\boldsymbol{f}$ **- g**, and $\boldsymbol{fg}$ by the formulas

$$(f + g)(x) = f(x) + g(x).$$

$$(f - g)(x) = f(x) - g(x).$$

$$(fg)(x) = f(x)g(x).$$

Combining Functions

At any point in $D(f) \cap D(g))$ *at* which $g(x)$ # 0, we can also define the function $\frac{f}{g}$ by the formula

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad (\text{where } g(x) \neq 0).$$

Functions can also be multiplied by constants: If **c** is a real number, then the function **cf** is defined for all x in the domain of f by

$$(cf)(x) = cf(x).$$

Combining Functions

Example: The functions defined by the formulas

$$f(x) = \sqrt{x} \qquad \text{and} \qquad g(x) = \sqrt{1 - x}$$

$$D(f) = [0, \infty)$$

$$1 - x \geq 0$$

$$D(g) = (-\infty, 1]$$

So, the points common to these domains are the points

$$[0, \infty) \cap (-\infty, 1] = [0, 1].$$

Combining Functions

Example: The functions defined by the formulas

$$f(x) = \sqrt{x} \qquad \text{and} \qquad g(x) = \sqrt{1 - x}$$

Function	Formula	Domain
$f + g$	$(f + g)(x) = \sqrt{x} + \sqrt{1 - x}$	$D(f) \cap D(g) = [0, 1]$
$f - g$	$(f - g)(x) = \sqrt{x} - \sqrt{1 - x}$	$[0, 1]$
$g - f$	$(g - f)(x) = \sqrt{1 - x} - \sqrt{x}$	$[0, 1]$

Combining Functions

Example: The functions defined by the formulas

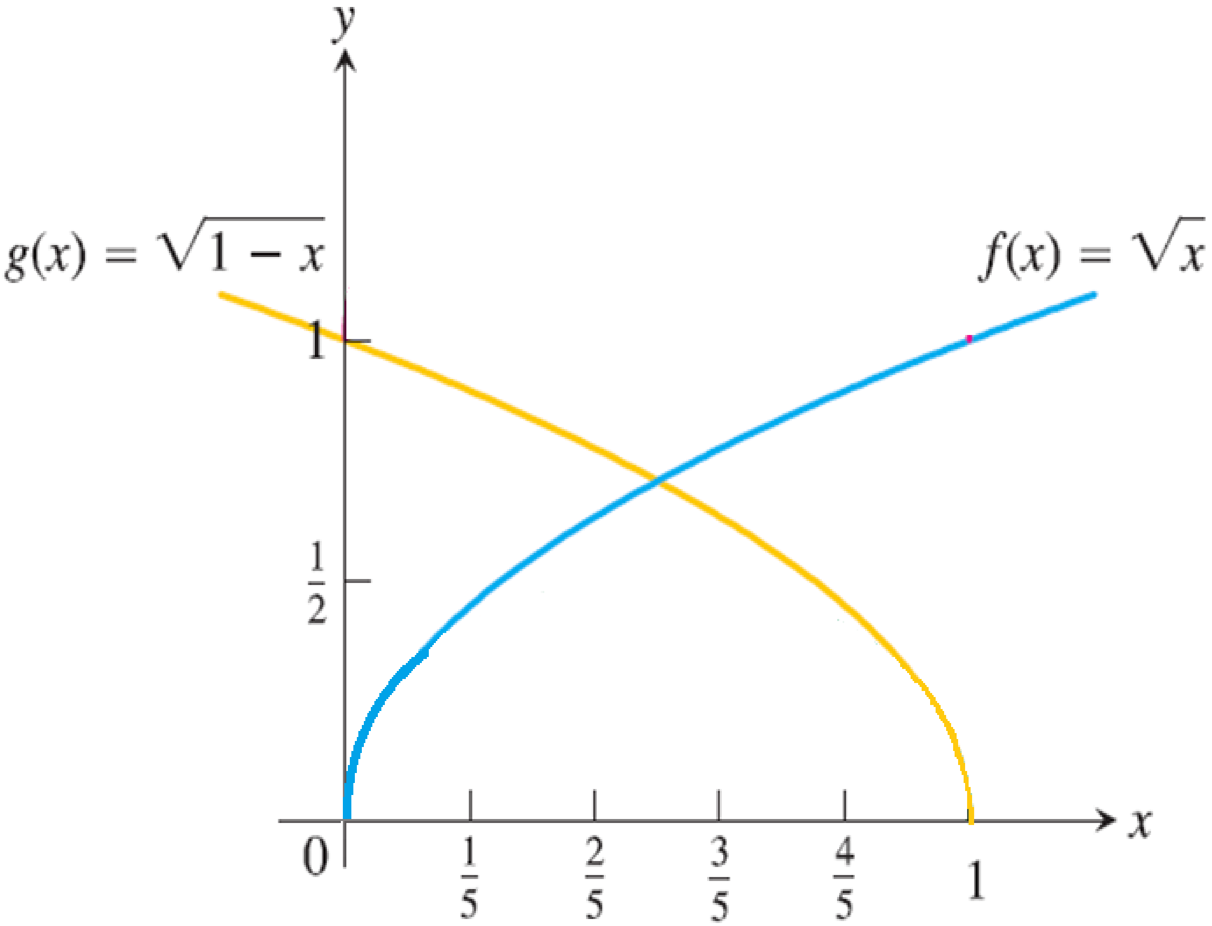
$$f(x) = \sqrt{x} \qquad \text{and} \qquad g(x) = \sqrt{1 - x}$$

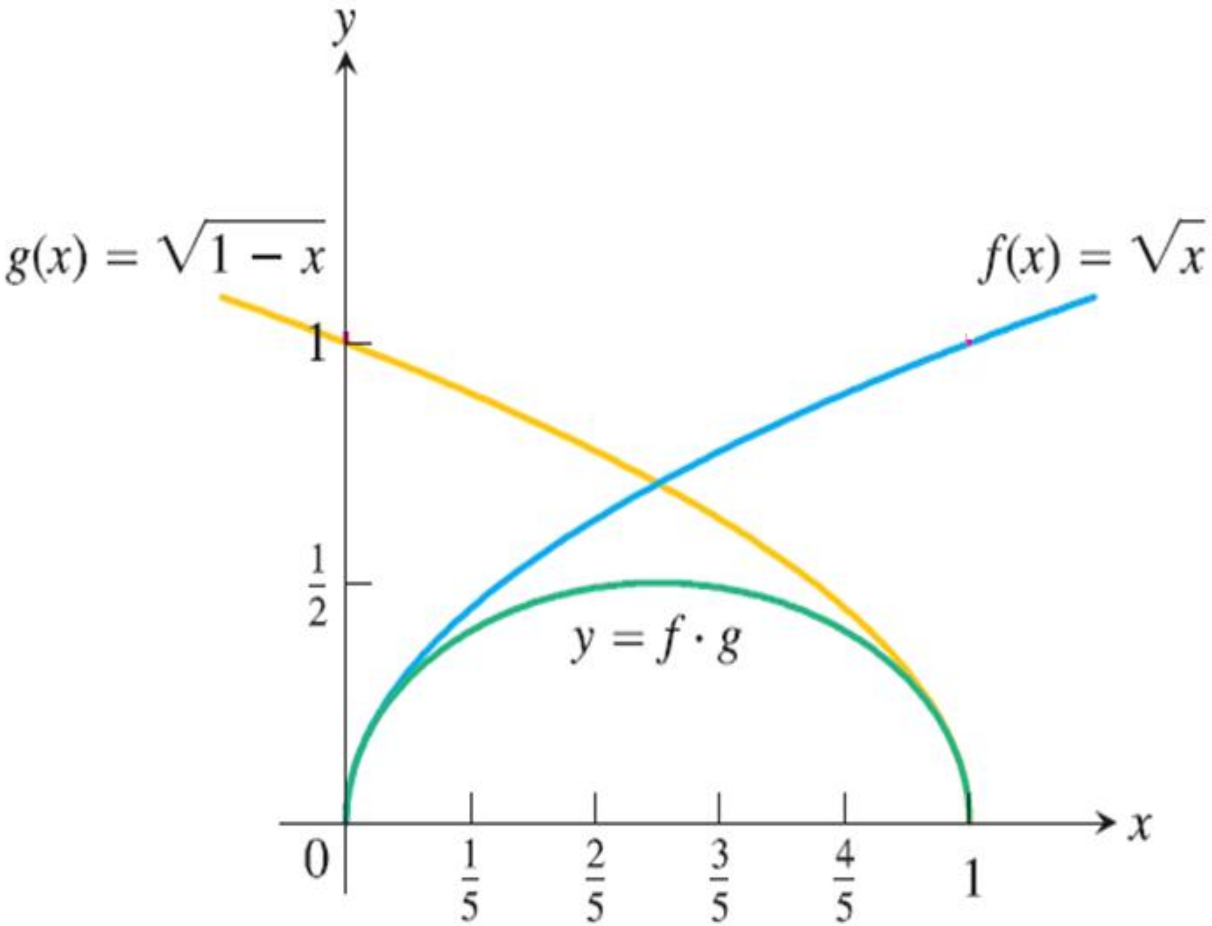
Function	Formula	Domain
$f \cdot g$	$(f \cdot g)(x) = f(x)g(x) = \sqrt{x(1-x)}$	$[0, 1]$
f/g	$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1-x}}$	$[0, 1)$ ($x = 1$ excluded)
g/f	$\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1-x}{x}}$	$(0, 1]$ ($x = 0$ excluded)

Combining Functions

Example 1: The functions defined by the formulas

$$f(x) = \sqrt{x} \quad \text{and} \quad g(x) = \sqrt{1 - x}$$

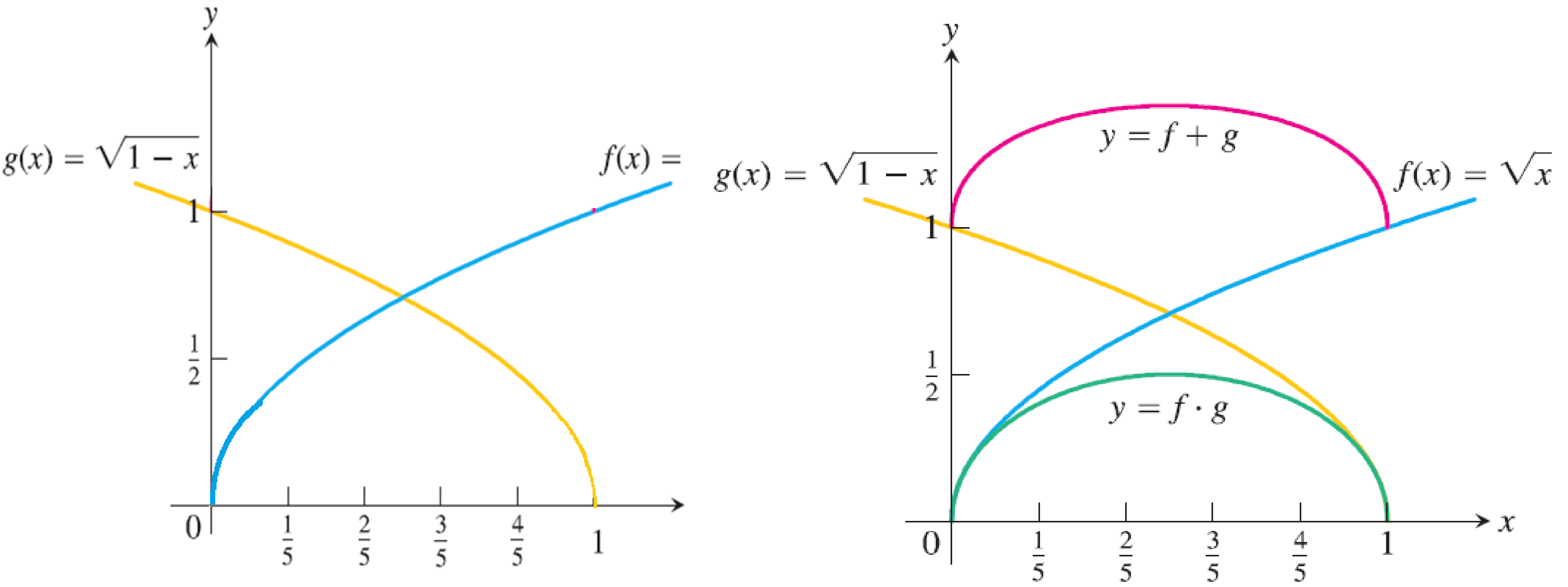




Combining Functions

Example 1: The functions defined by the formulas

$$f(x) = \sqrt{x} \qquad \text{and} \qquad g(x) = \sqrt{1 - x}$$



Combining Functions

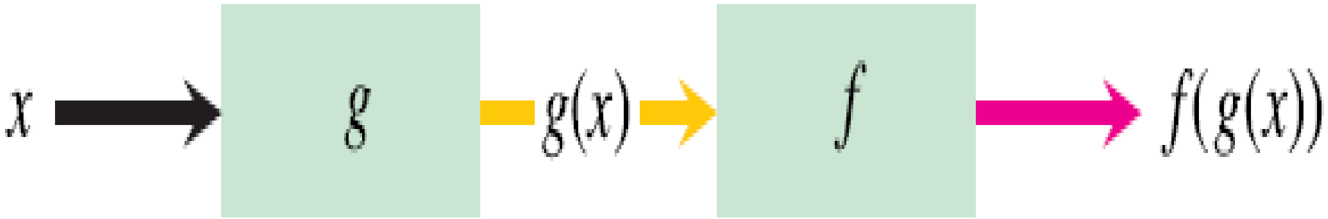
Composition is another method for combining functions.

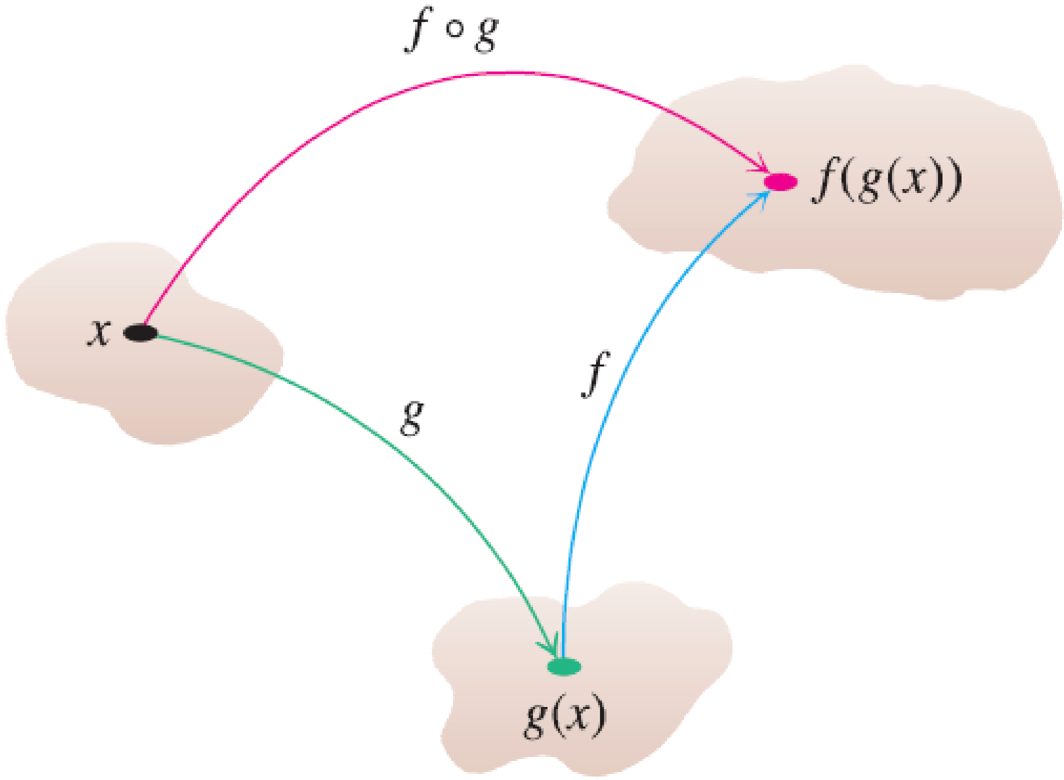
DEFINITION If f and g are functions, the **composite** function $f \circ g$ ("f composed with g") is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.

Combining Functions





The definition implies that ***f o g*** can be formed when the range of **g** lies in the domain of ***f***. To find **(f *o* g)*(x)*,** *first* find **g(x)** and *second* find ***f(g(x))***.

Combining Functions

The functions ***f o g*** and ***g o f*** are usually quite different.

To evaluate the composite function ***g o f*** (when defined), we find ***f(x)*** first and then ***g(f(x))***.

The domain of **g o f** is the set of numbers **x** in the domain of **f** such that **f(x)** lies in the domain of **g**.

Combining Functions

EXAMPLE 2 If $f(x) = \sqrt{x}$ and $g(x) = x + 1$, find

(a) $(f \circ g)(x)$ **(b)** $(g \circ f)(x)$ **(c)** $(f \circ f)(x)$ **(d)** $(g \circ g)(x)$.

Solution

Composite	Domain
(a) $(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x+1}$	$[-1, \infty)$
(b) $(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
(c) $(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
(d) $(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x + 1) + 1 = x + 2$	$(-\infty, \infty)$

Shifting a Graph of a Function

A common way to obtain a new function from an existing one is by adding a constant to each output of the existing function, or to its input variable.

The graph of the new function is the graph of the original function shifted vertically or horizontally, as follows.

Shift Formulas

Vertical Shifts

$y = f(x) + k$	Shifts the graph of f *up* k units if $k > 0$
	Shifts it *down* $\|k\|$ units if $k < 0$

Horizontal Shifts

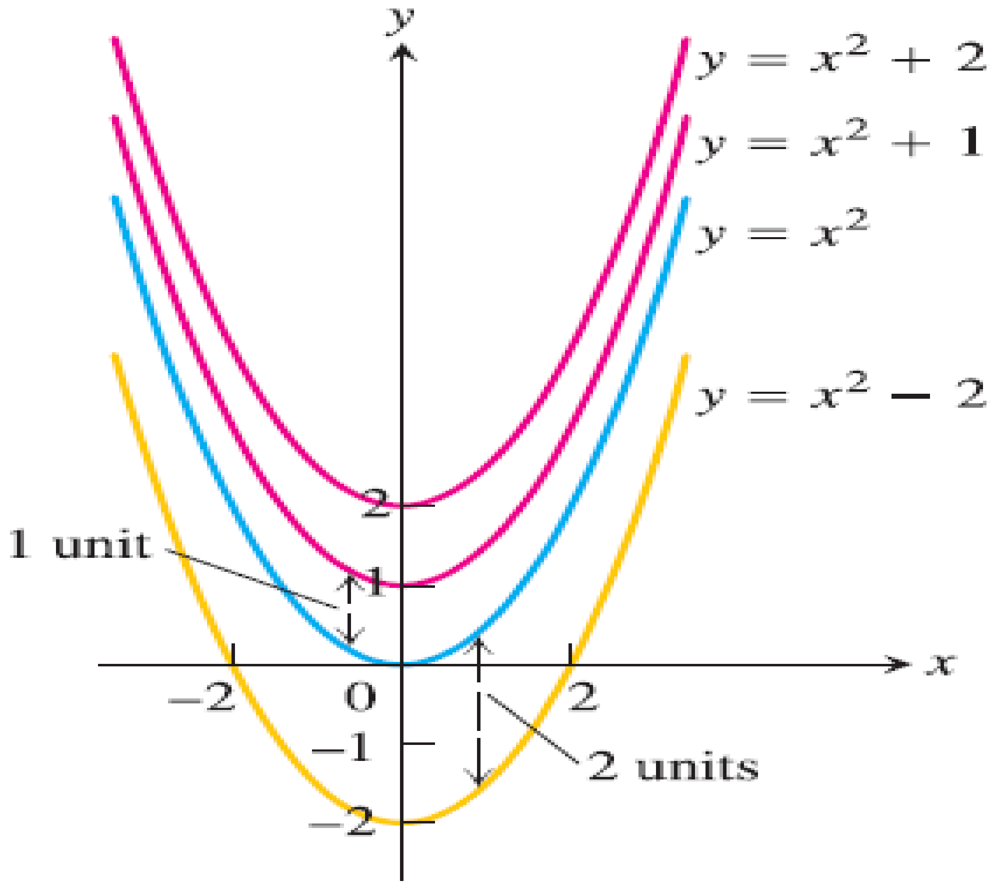
$y = f(x + h)$	Shifts the graph of f *left* h units if $h > 0$
	Shifts it *right* $\|h\|$ units if $h < 0$

Shifting a Graph of a Function

Example :

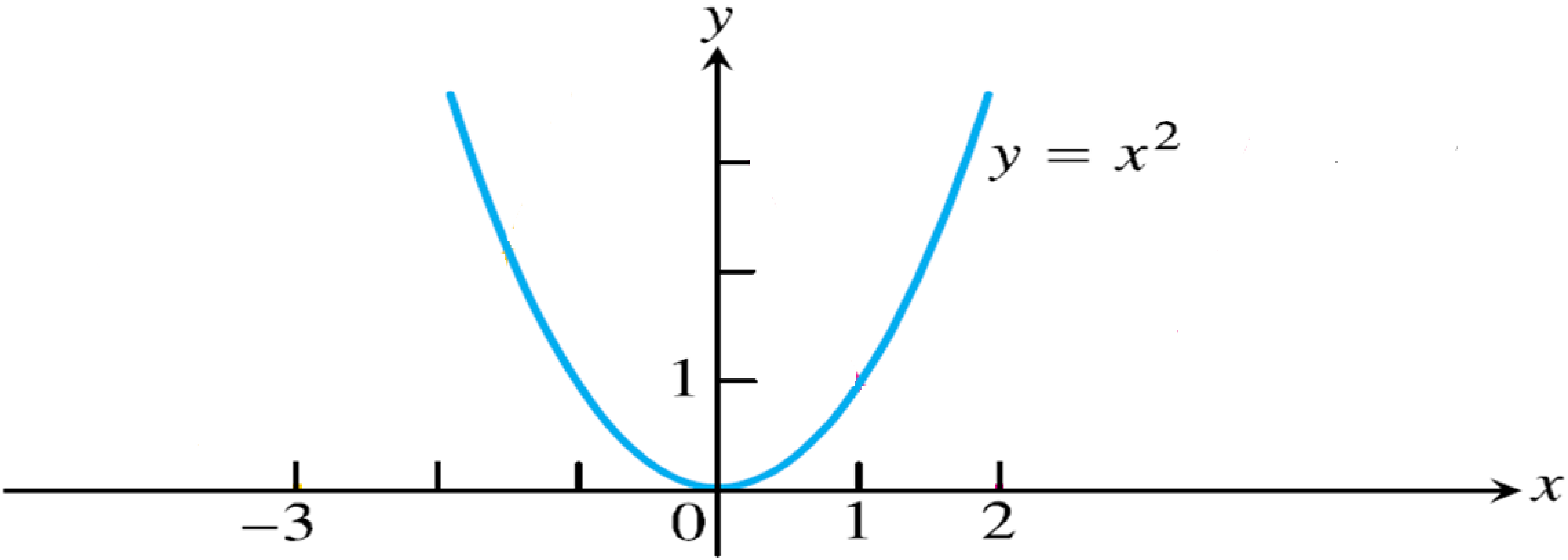
(a) Adding **1** to the right-hand side of the formula $y = x^2$ to get $y = x^2 + 1$ shifts the graph up **1** unit.

(b) Adding **-2** to the right-hand side of the formula $y = x^2$ to get $y = x^2 - 2$ shifts the graph down **2** units.



Shifting a Graph of a Function

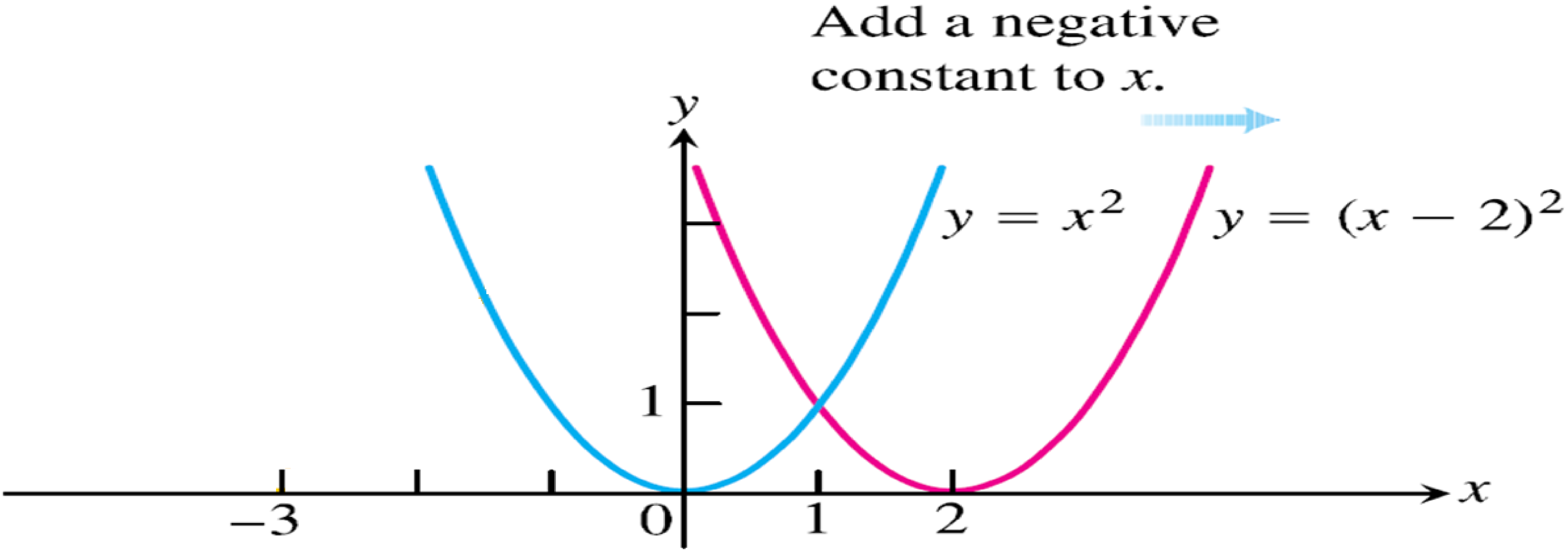
Example: $y = x^2$



Shifting a Graph of a Function

Example :

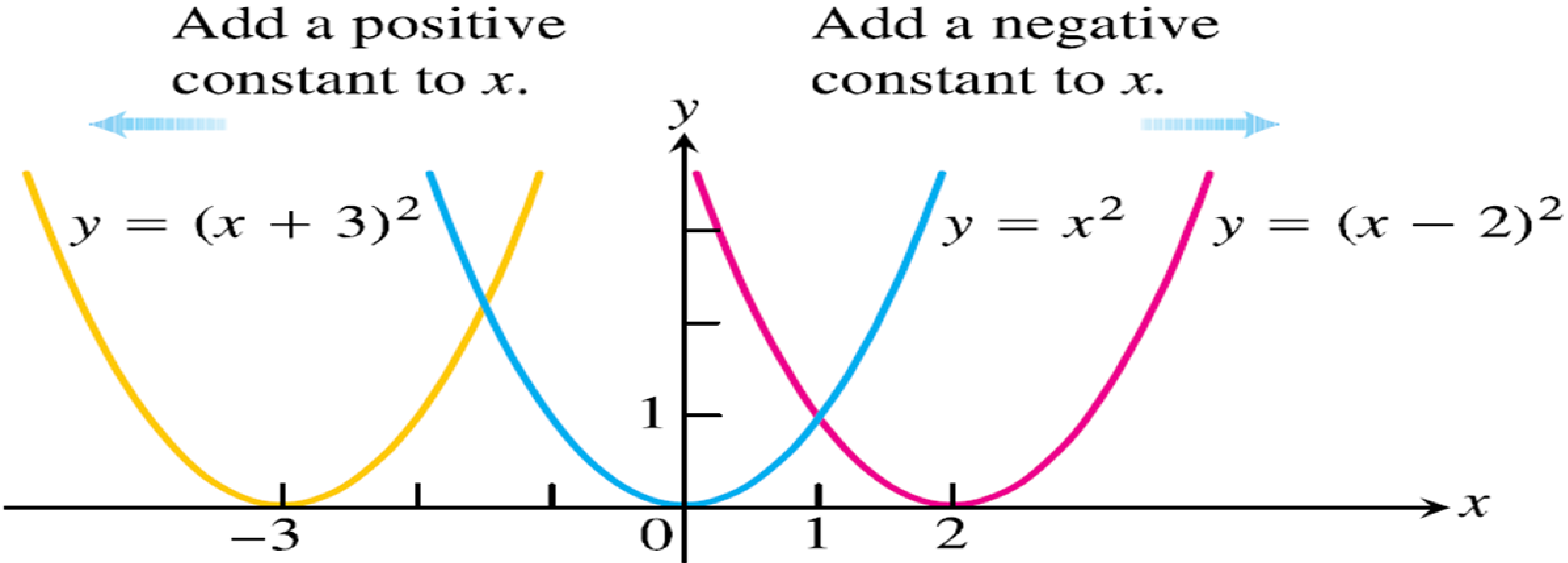
Adding **-2** to x in $y = x^2$ to get $y=(x\ -2)^2$ shifts the graph **2** units to the right.



Shifting a Graph of a Function

Example: $y = x^2$

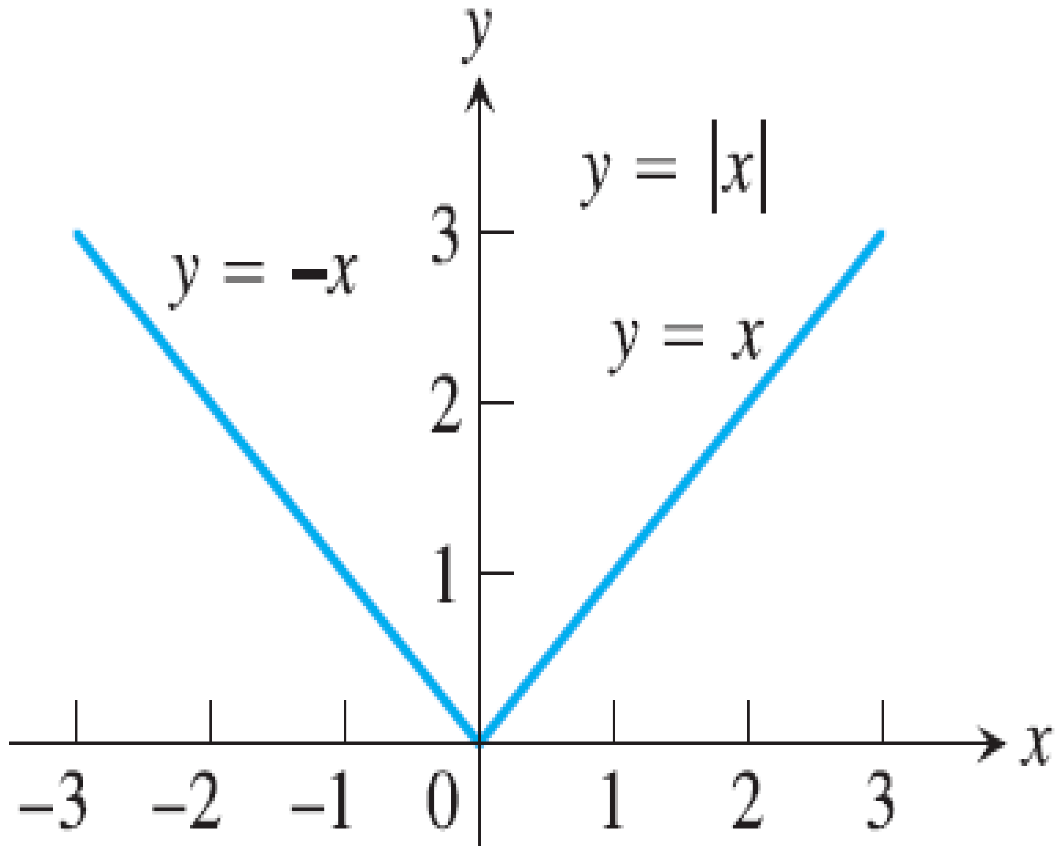
Adding **3** to x in $y = x^2$ to get $y=(x + 3)^2$ shifts the graph **3** units to the left.

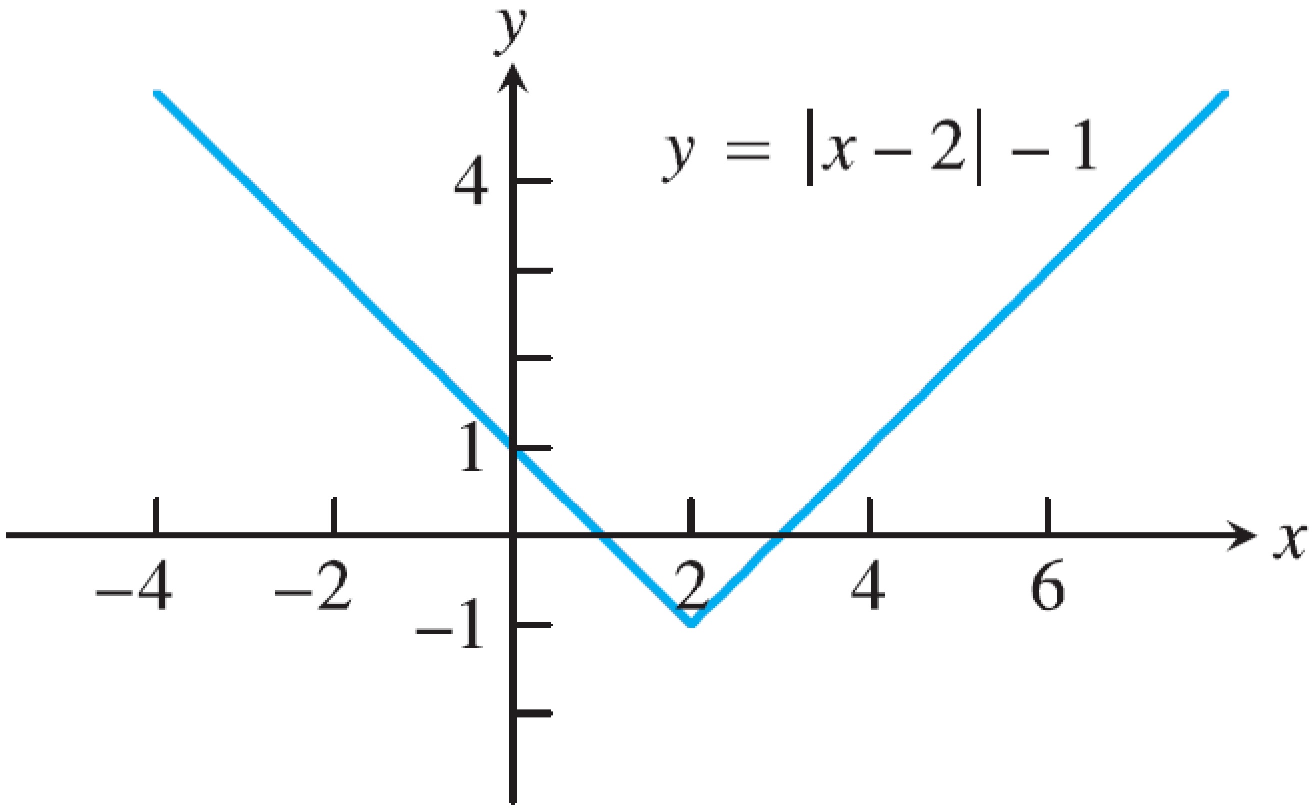


Shifting a Graph of a Function

Example: $y = |x|$

Adding -2 to *x* in ***y* = |*x*|**, and then adding **-1** to the result, gives $y = |x - 2| - 1$. This shifts the graph **2** units to the right and **1** unit down





Scaling and Reflecting a Graph of a Function

To scale the graph of a function $y = f(x)$ is to stretch or compress it, vertically or horizontally. This is accomplished by multiplying the function f, or the independent variable x, by an appropriate constant c. Reflections across the coordinate axes are special cases where $c = -1$.

Scaling and Reflecting a Graph of a Function

Vertical and Horizontal Scaling and Reflecting Formulas

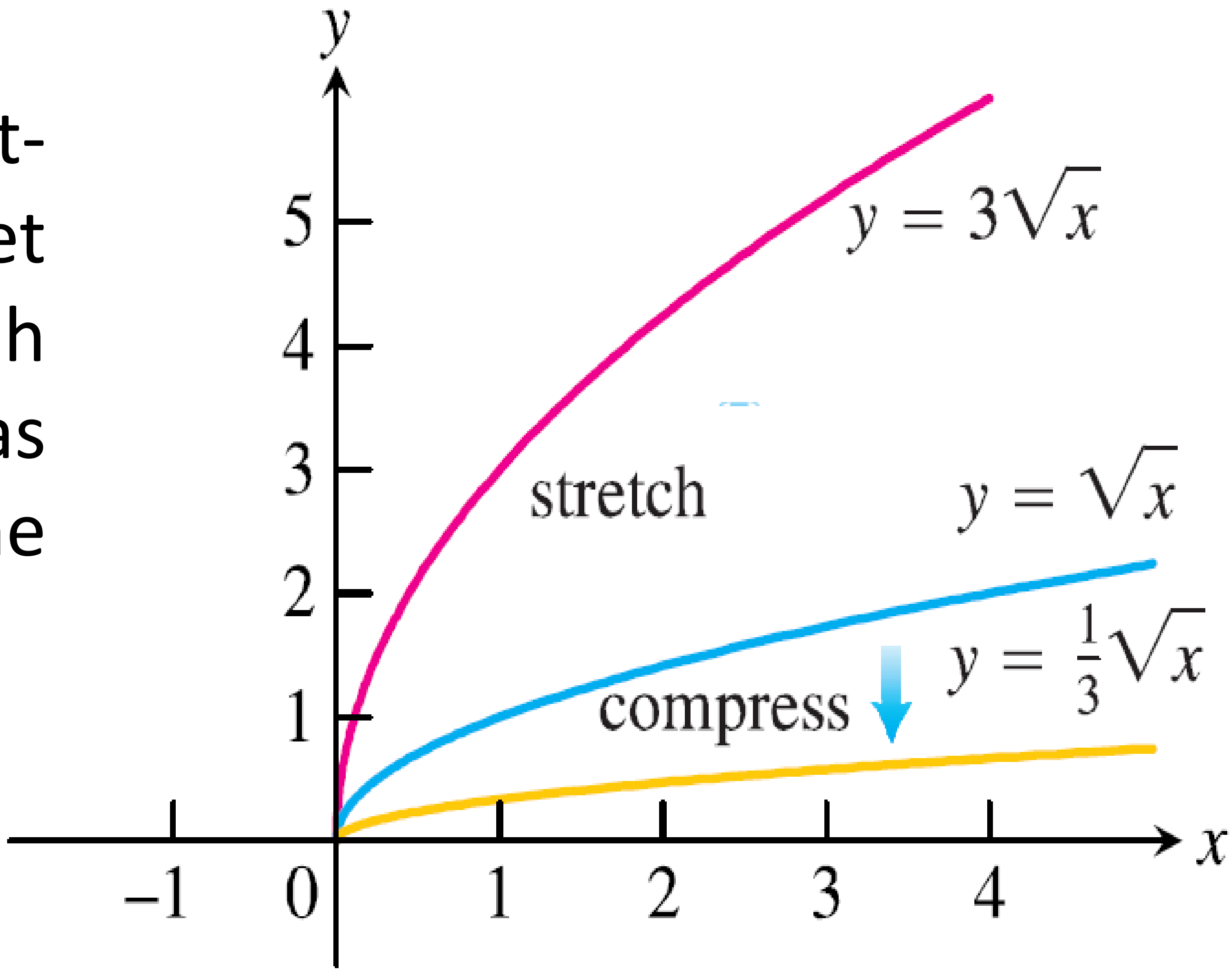
For $c > 1$, the graph is scaled:

$y = cf(x)$	Stretches the graph of f vertically by a factor of c.
$y = \frac{1}{c} f(x)$	Compresses the graph of f vertically by a factor of c.
$y = f(cx)$	Compresses the graph of f horizontally by a factor of c.
$y = f(x/c)$	Stretches the graph of f horizontally by a factor of c.

Scaling a Graph of a Function

Example: $y = \sqrt{x}$

Vertical: Multiplying the right-hand side of $y = \sqrt{x}$ by 3 to get $y = 3\sqrt{x}$ stretches the graph vertically by a factor of 3, whereas multiplying by 1/3 compresses the graph by a factor of 3

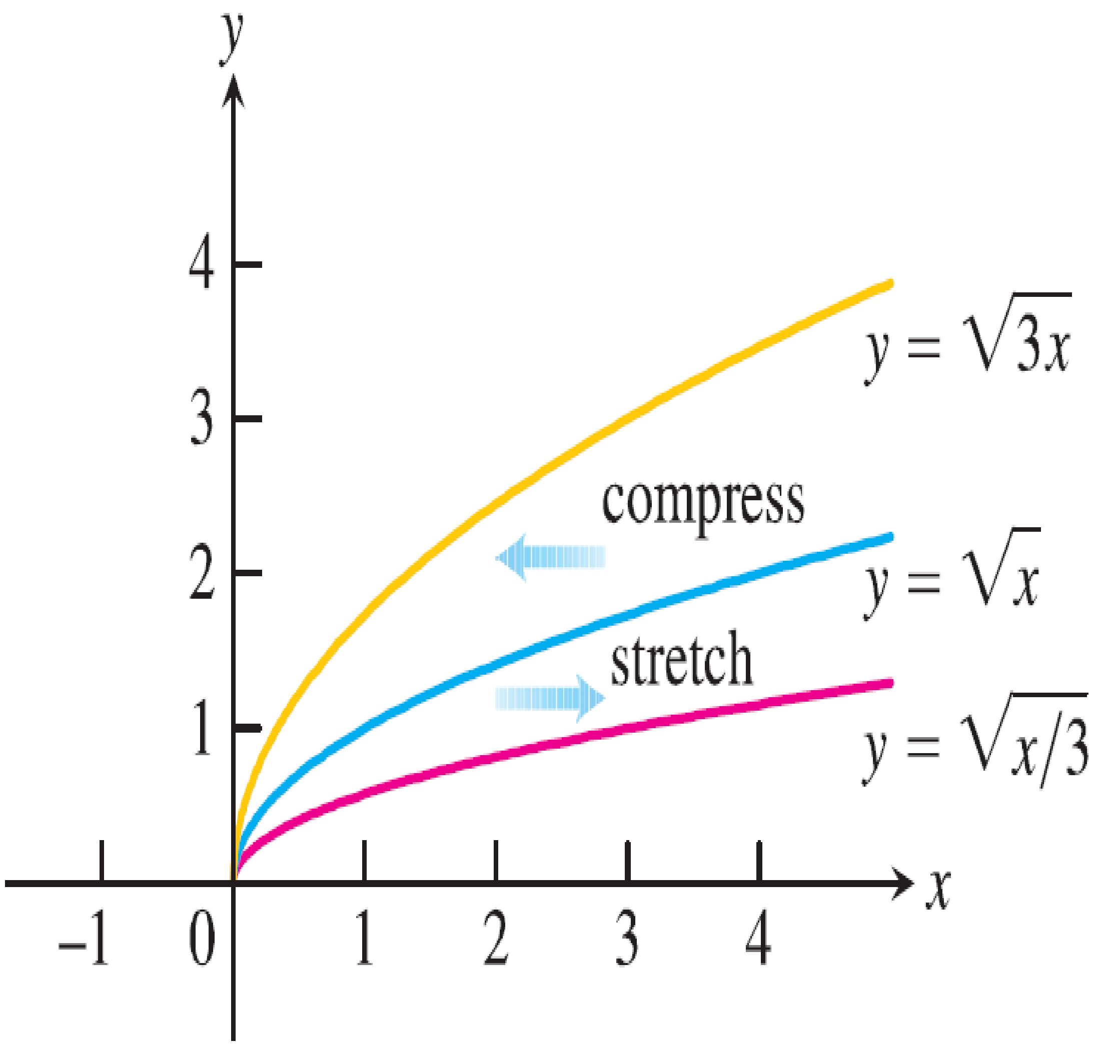


Scalling a Graph of a Function

Example: $y = \sqrt{x}$

Horizontal: The graph of $y = \sqrt{3x}$ is a horizontal compression of the graph of $y = \sqrt{x}$ by a factor of 3, and $y = \sqrt{x/3}$ is a horizontal stretching by a factor of 3.

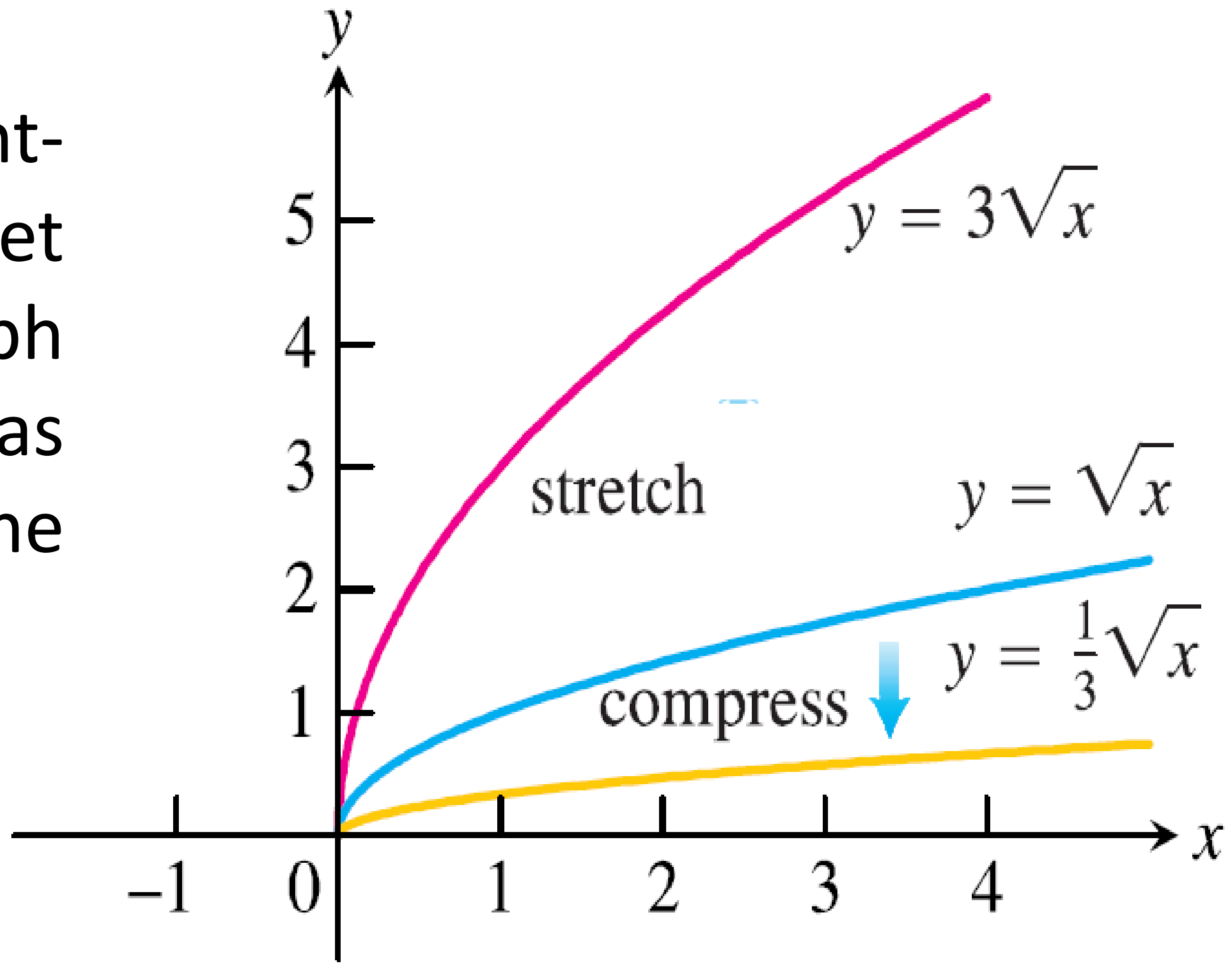
Note that $y = \sqrt{3x} = \sqrt{3}\ \sqrt{x}$ so a horizontal compression may correspond to a vertical stretching by a different scaling factor. Likewise, a horizontal stretching may correspond to a vertical compression by a different scaling factor.



Shifting a Graph of a Function

Example: $y = \sqrt{x}$

Vertical: Multiplying the right-hand side of $y = \sqrt{x}$ by 3 to get $y = 3\sqrt{x}$ stretches the graph vertically by a factor of 3, whereas multiplying by 1/3 compresses the graph by a factor of 3.



Reflecting a Graph of a Function

Vertical and Horizontal Scaling and Reflecting Formulas

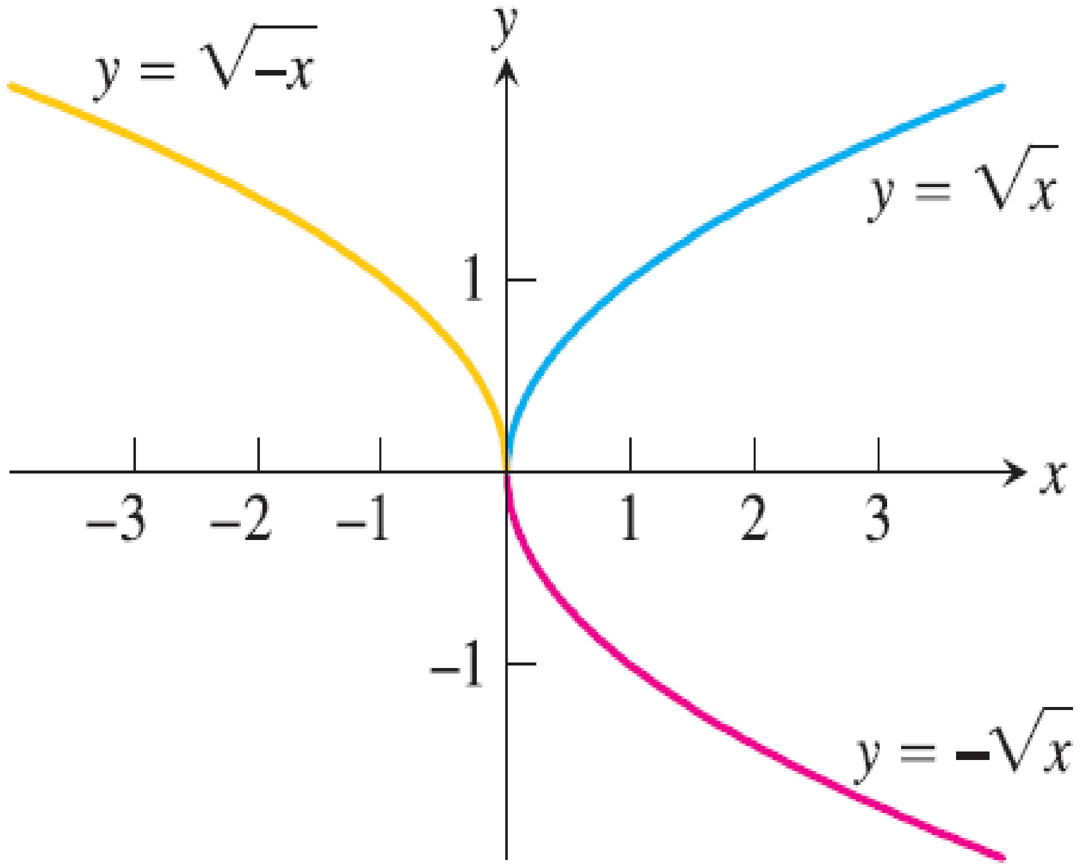
For $c = -1$, the graph is reflected:

$y = -f(x)$	Reflects the graph of f across the x-axis.
$y = f(-x)$	Reflects the graph of f across the y-axis.

Reflecting a Graph of a Function

Example: $y = \sqrt{x}$

Reflection: The graph of y = - $\sqrt{x}$ is a reflection of $y = \sqrt{x}$ across the x-axis, and $y = h$ is a reflection across the y-axis



Shifting a Graph of a Function

Example: Given the function f(x) $=x^4 - 4x^3$+10, find formulas to

(a) compress the graph horizontally by a factor of 2 followed by a reflection across the y-axis.

(b) compress the graph vertically by a factor of 2 followed by a reflection across the x-axis

Solution

(a) We multiply *x* by 2 to get the horizontal compression, and by -1 to give reflection across the y-axis. The formula is obtained by substituting -2x for *x* in the right-hand side of the equation for *f:*

$$\begin{aligned} y = f(-2x) &= (-2x)^4 - 4(-2x)^3 + 10 \\ &= 16x^4 + 32x^3 + 10. \end{aligned}$$

Shifting a Graph of a Function

Example: Given the function f(x) $=x^4 - 4x^3$+10, find formulas to

(a) compress the graph horizontally by a factor of 2 followed by a reflection across the y-axis.

(b) compress the graph vertically by a factor of 2 followed by a reflection across the x-axis

Solution

(b) The formula is

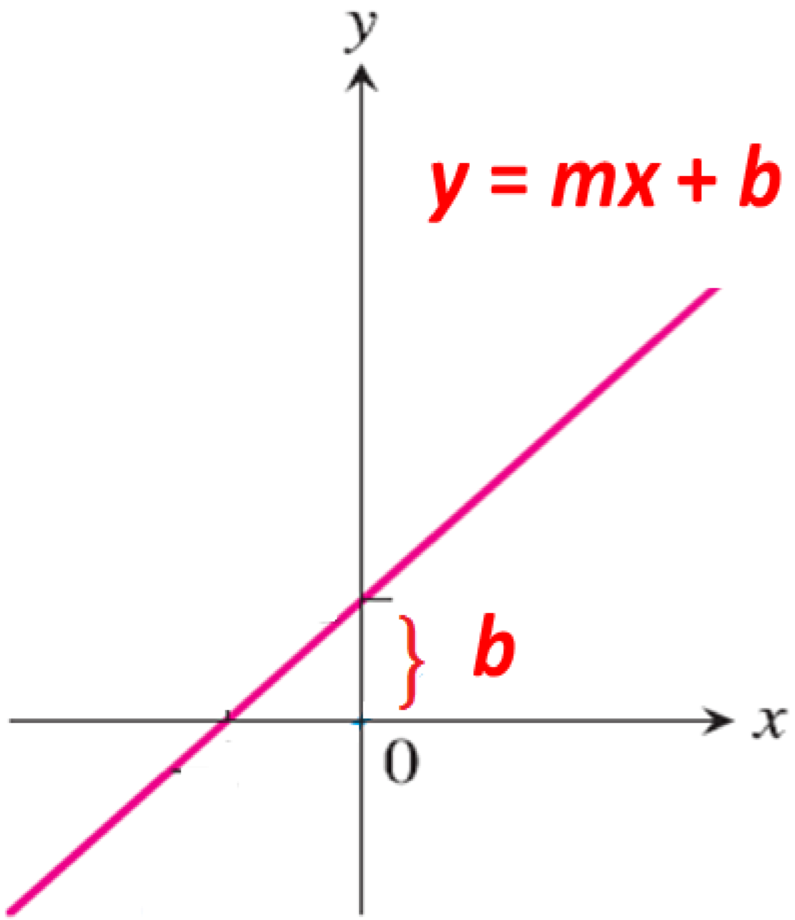
$$y = -\frac{1}{2}f(x) = -\frac{1}{2}x^4 + 2x^3 - 5.$$

Families Of Functions

Linear Functions

A function of the form ***$f(x) = mx + b$***

for constants **m** and **b**, is called a linear function.



Families Of Functions

Linear Functions

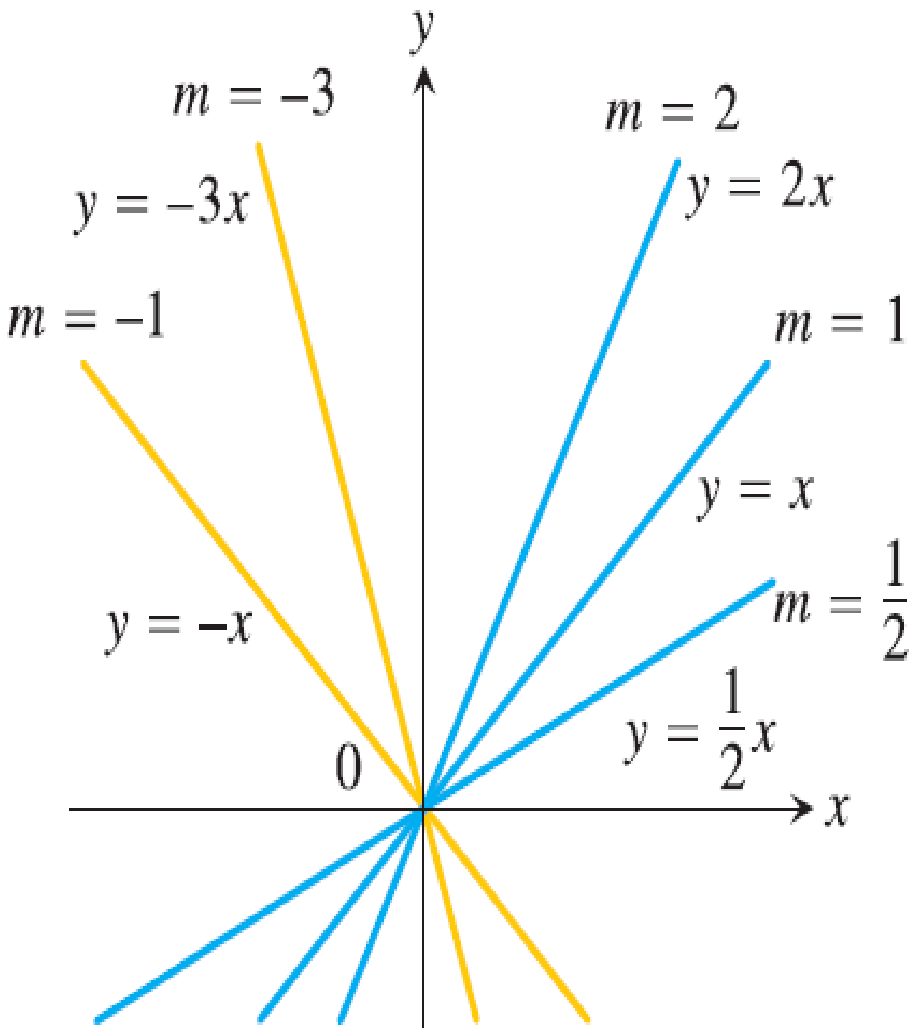
A function of the form $y = mx + b$

for constants **m** and **b**, is called a linear function.

***Let b* = 0** $\therefore f(x) = mx$

***Let b* = 0, and m=1**

$\therefore f(x) = x$ → **Identity function**



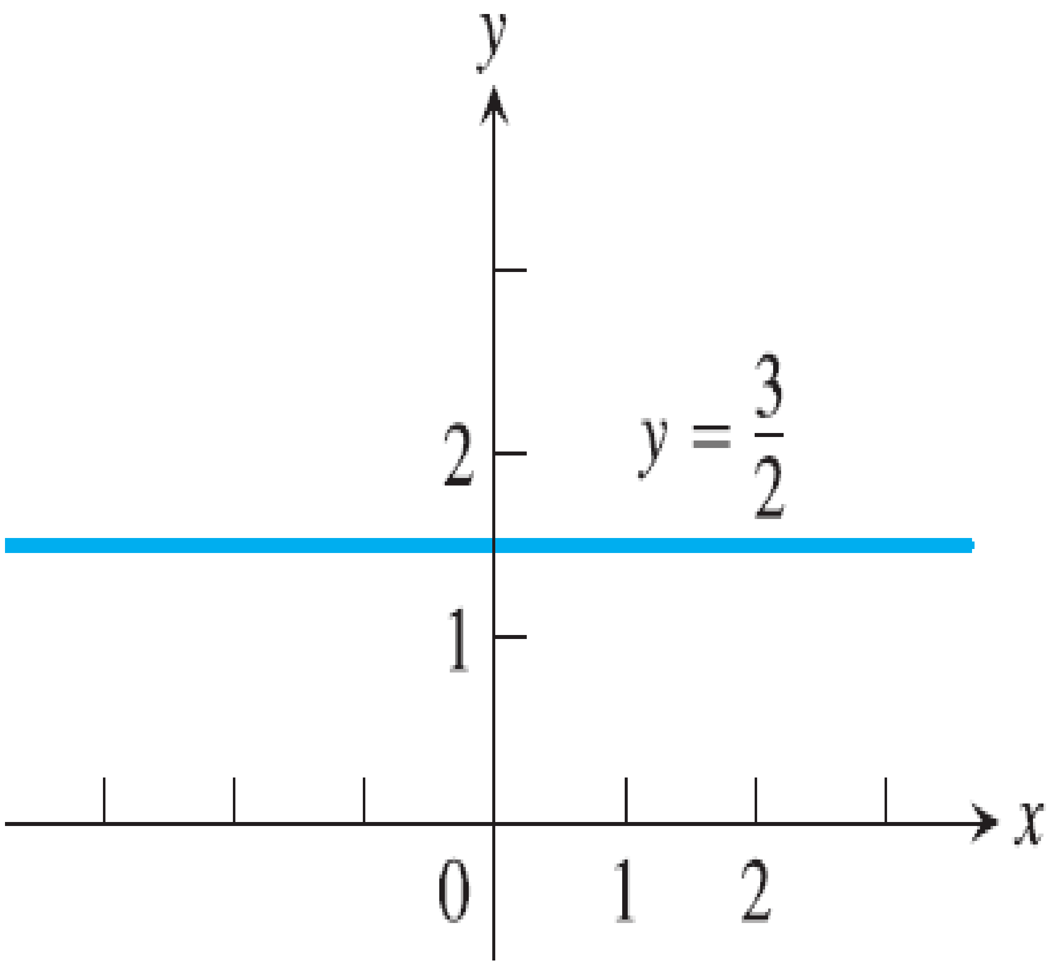
Common Functions

Linear Functions

A function of the form $f(x) = mx + b$ for constants m and b, is called a linear function.

***Let m* = 0, and b $\neq$ 0**

$\therefore f(x) = b$ ➡ **Constant function**



Common Functions

Linear Functions A function of the form $f(x) = mx + b$

Let $b = 0$ $\therefore y = mx$ → *proportionality relationship*

DEFINITION Two variables y and x are **proportional** (to one another) if one is always a constant multiple of the other; that is, if $y = kx$ for some nonzero constant k.

If the variable y is proportional to the reciprocal $\frac{1}{x}$, then sometimes it is said that y is inversely proportional to x.

Common Functions

Polynomials A function p is a polynomial if

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where **n** is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are real constants (called the **coefficients** of the polynomial).

If $a_n \neq 0$ and $n > 0$, then n is called **the degree of the polynomial**.

Example: Polynomials of degree 2

$$p(x) = ax^2 + bx + c$$

quadratic functions

$$p(x) = x^2 - 5x + 6$$

Common Functions

Polynomials $$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Example: Polynomials of degree 3

$$p(x) = ax^3 + bx^2 + cx + d$$ **cubic functions**

Example: Polynomials of degree 1

$$p(x) = mx + b$$

Linear functions with m ≠ 0 are polynomials of degree 1.

All polynomial have domain (-∞, ∞).

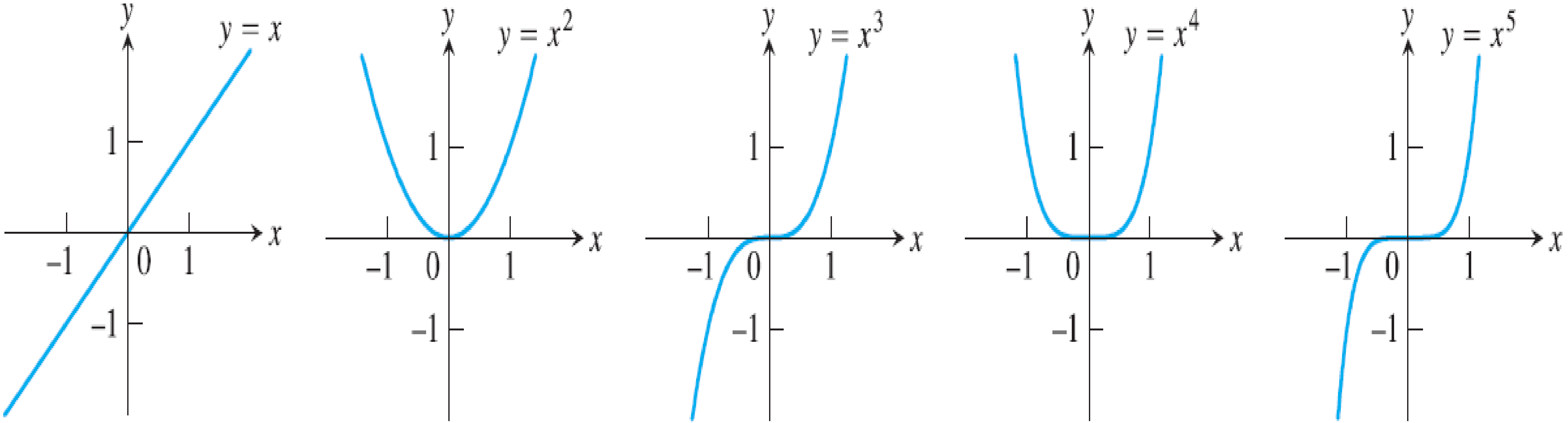
Common Functions

Power Functions

A function $f(x) = x^a$, where *a* is a constant, is called a **power function.**

There are several important cases to consider:

(a) *a* = *n*, a positive integer. $f(x) = x^n$ Domain (-∞, ∞).



Common Functions

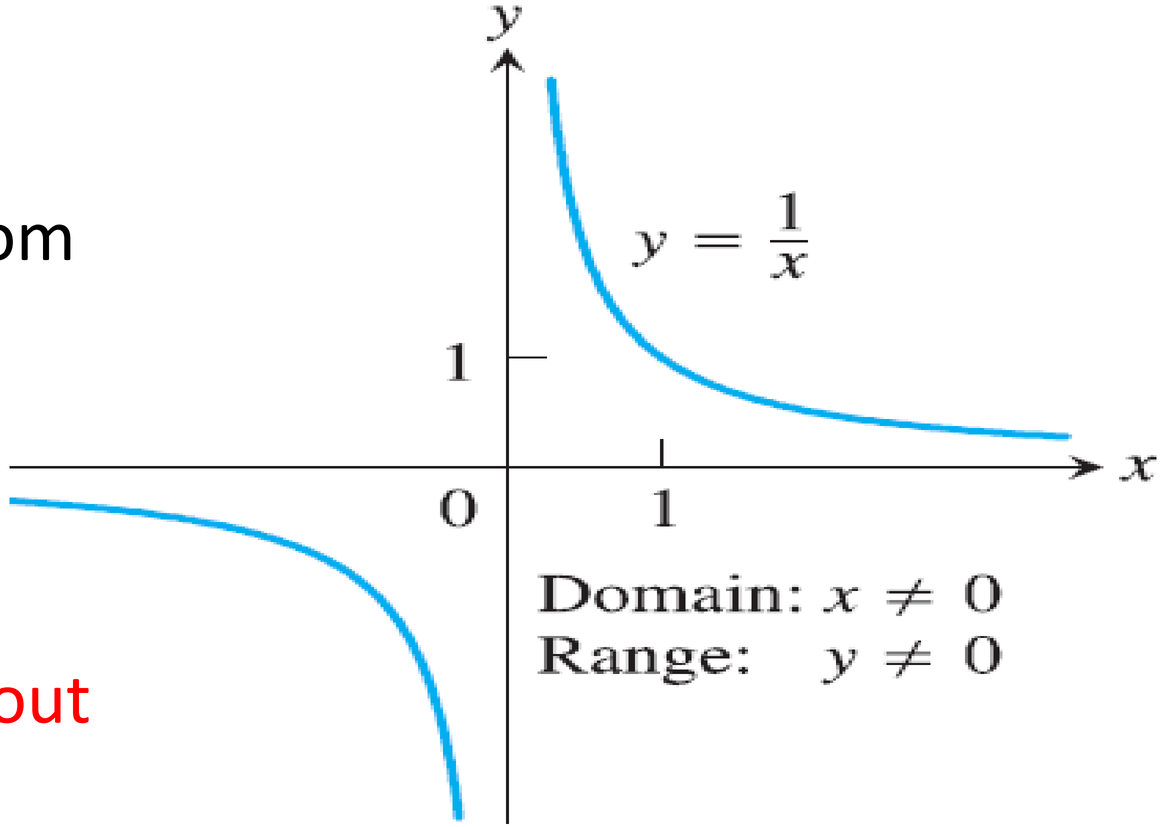
Power Functions

A function $f(x) = x^a$, where *a* is a constant, is called a **power function.**

There are several important cases to consider:

(i) a = -1 $\Rightarrow$ $f(x) = x^{-1} = \frac{1}{x}$

The graph of f(x) = $\frac{1}{x}$ is the hyperbola xy = 1, which approaches the coordinate axes far from the origin.



This function is decreasing on the intervals (-∞,0) and (0, ∞)

The graph of the function f is symmetric about the origin and it is odd function.

Common Functions

Power Functions

A function $f(x) = x^{a}$,where *a* is a constant, is called a **power function.**

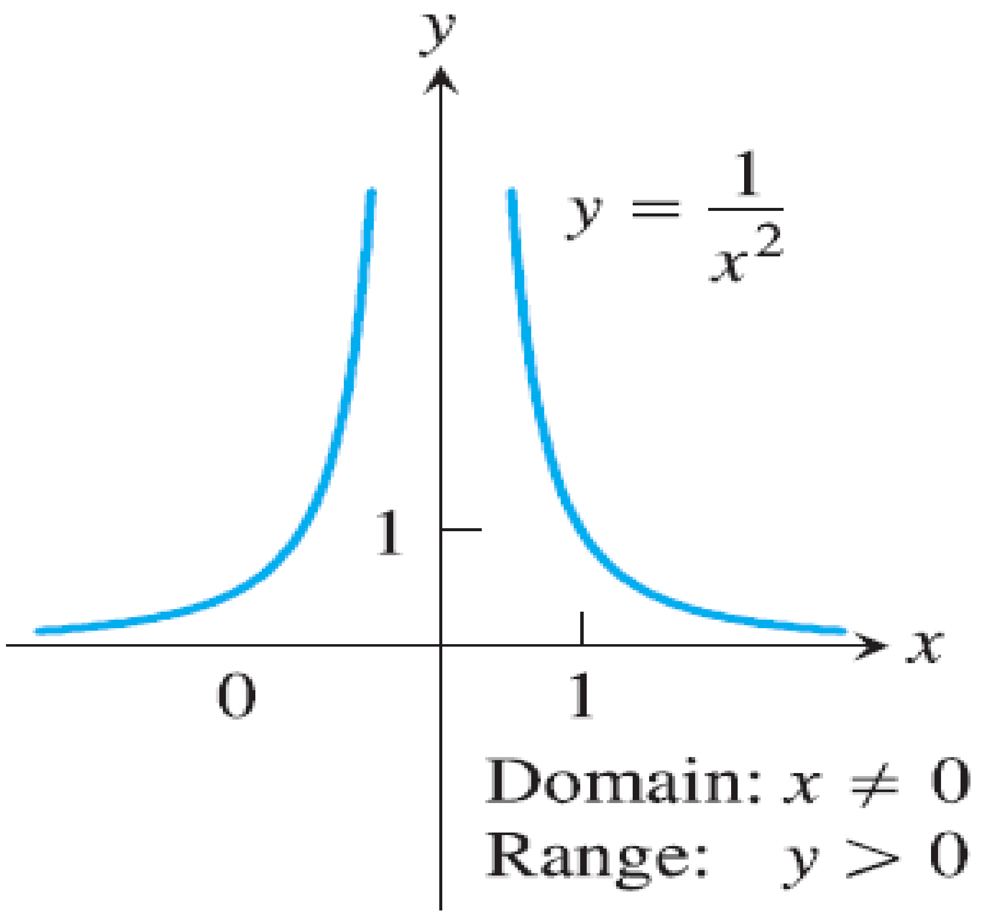
There are several important cases to consider:

(ii) *a* = -2 $\quad f(x) = x^{-2} = \frac{1}{x^2}$

The graph of f(x) = $\frac{1}{x^2}$ approaches the coordinate axes.

This function is increasing on (-∞,0) and decreasing on (0, ∞)

The graph of the function f is symmetric about the y-axis and it is even function.

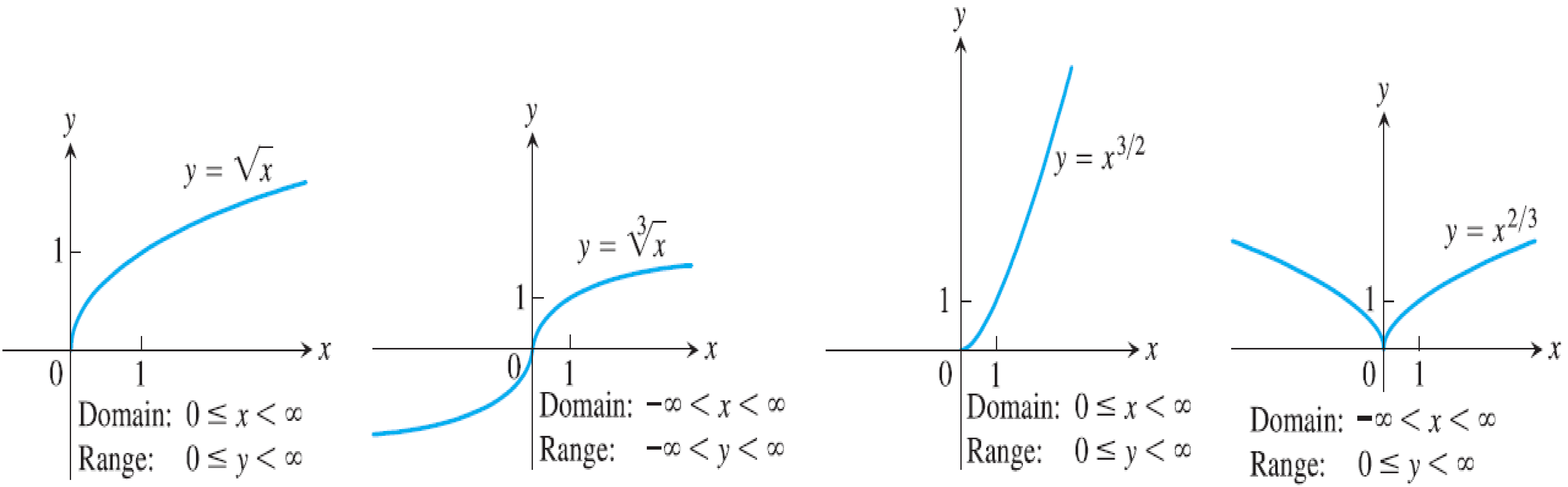


Power Functions

A function $f(x) = x^a$, where a is a constant, is called a **power function.**

There are several important cases to consider:

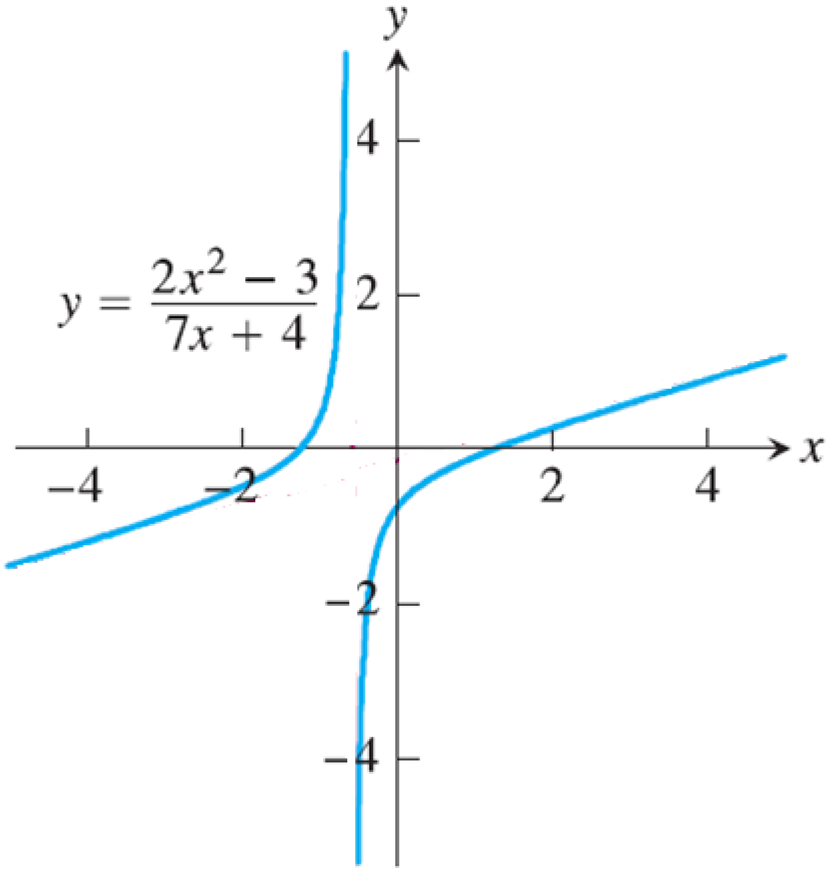
(iii) $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$ and $\frac{2}{3}$ (Recall that $x^{3/2} = (x^{1/2})^3$ and $x^{2/3} = (x^{1/3})^2$.)

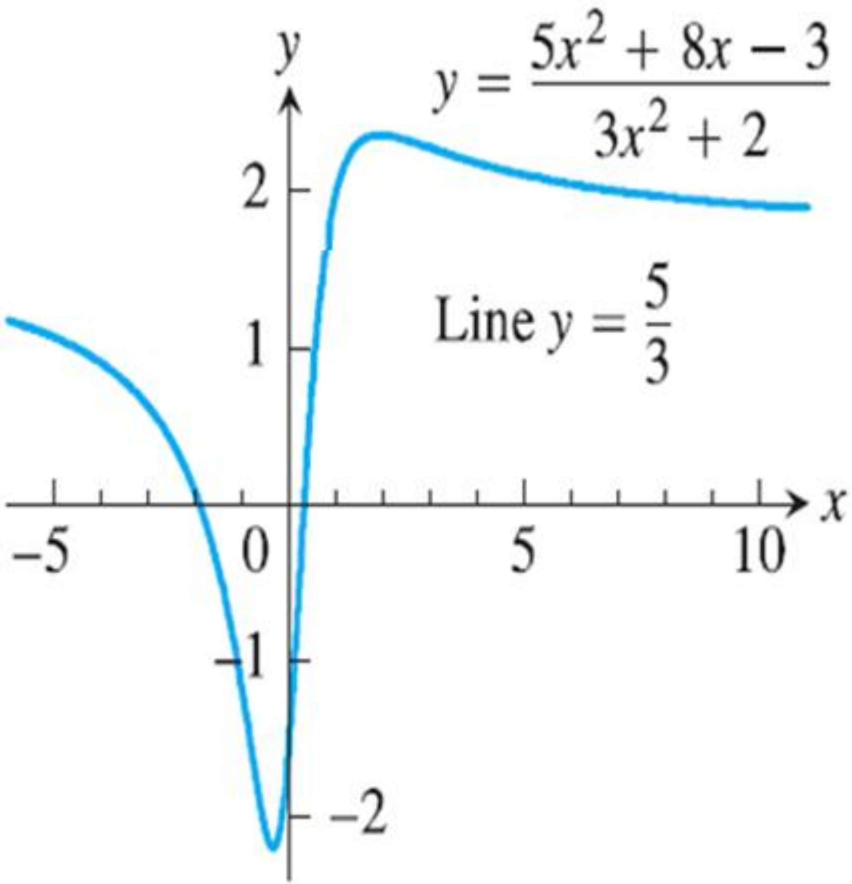


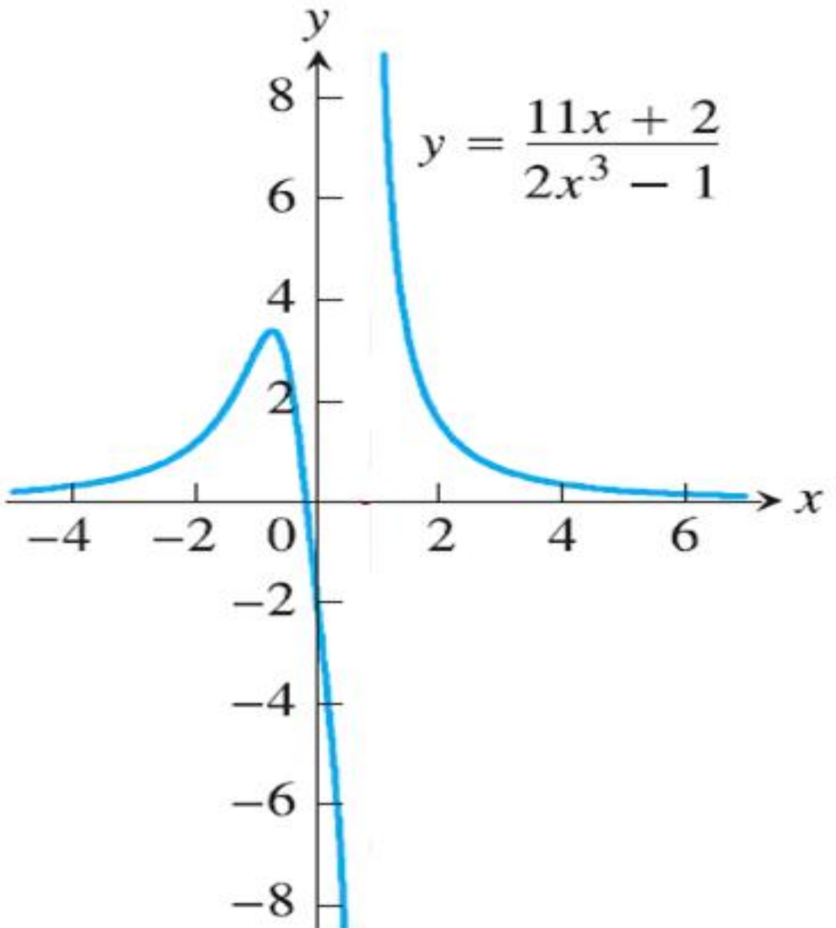
Common Functions

Rational Functions

A rational function is a quotient or ratio $f(x) = \frac{p(x)}{q(x)}$, where ***p(x)*** and ***q(x)*** are **polynomials**. The domain of a rational function is the set of all real *x* for which ***q(x)***$\neq 0$.



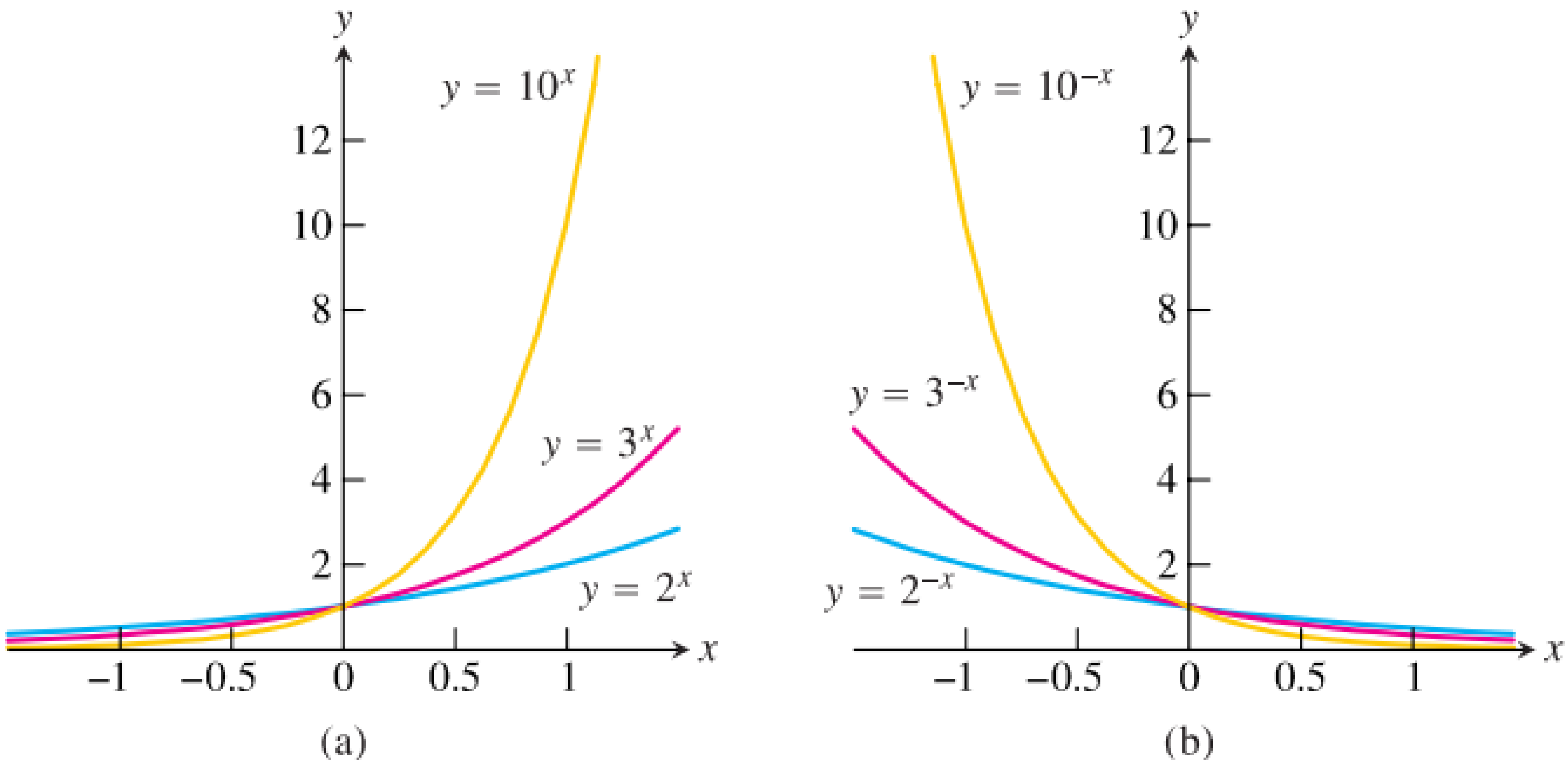




Common Functions

Trigonometric Functions: The six basic trigonometric functions: **sin x, cos x, tan x, cosec x, sec x, cot x.**

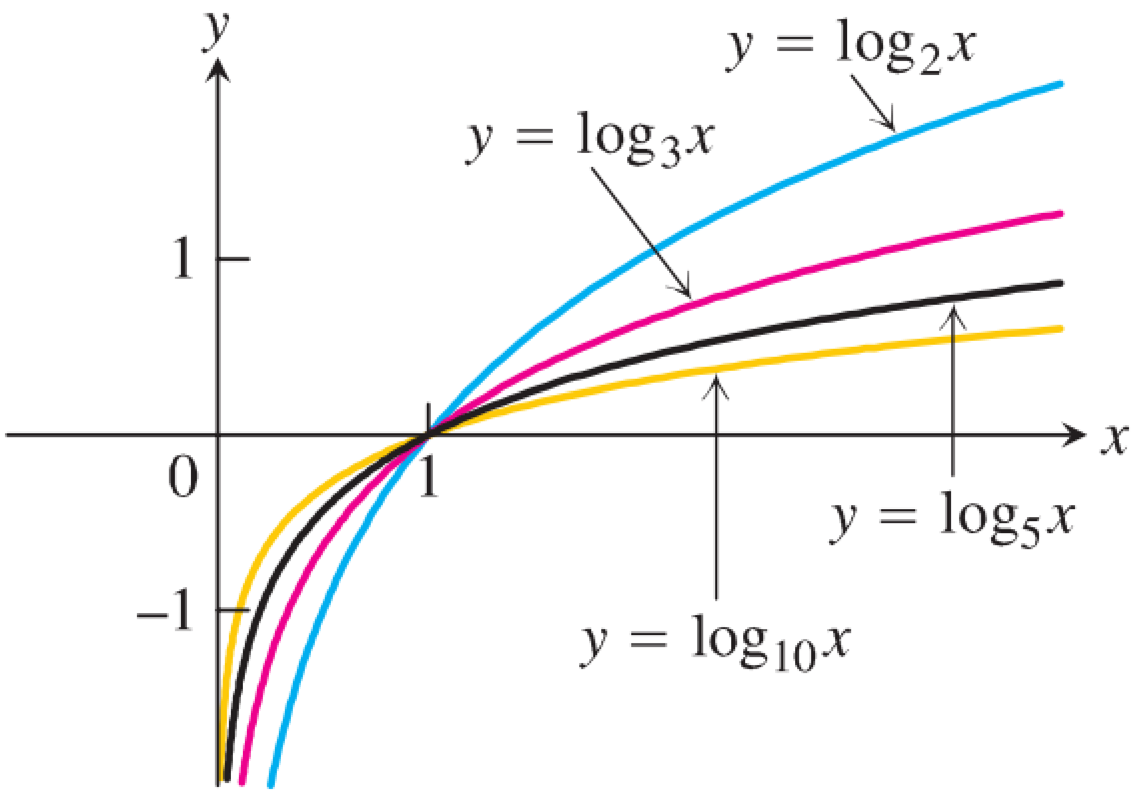
Exponential Functions: $f(x) = a^x$, where the base $a > 0$ is a positive constant. All exponential functions have domain (-∞,∞) and range (0,∞).



Common Functions

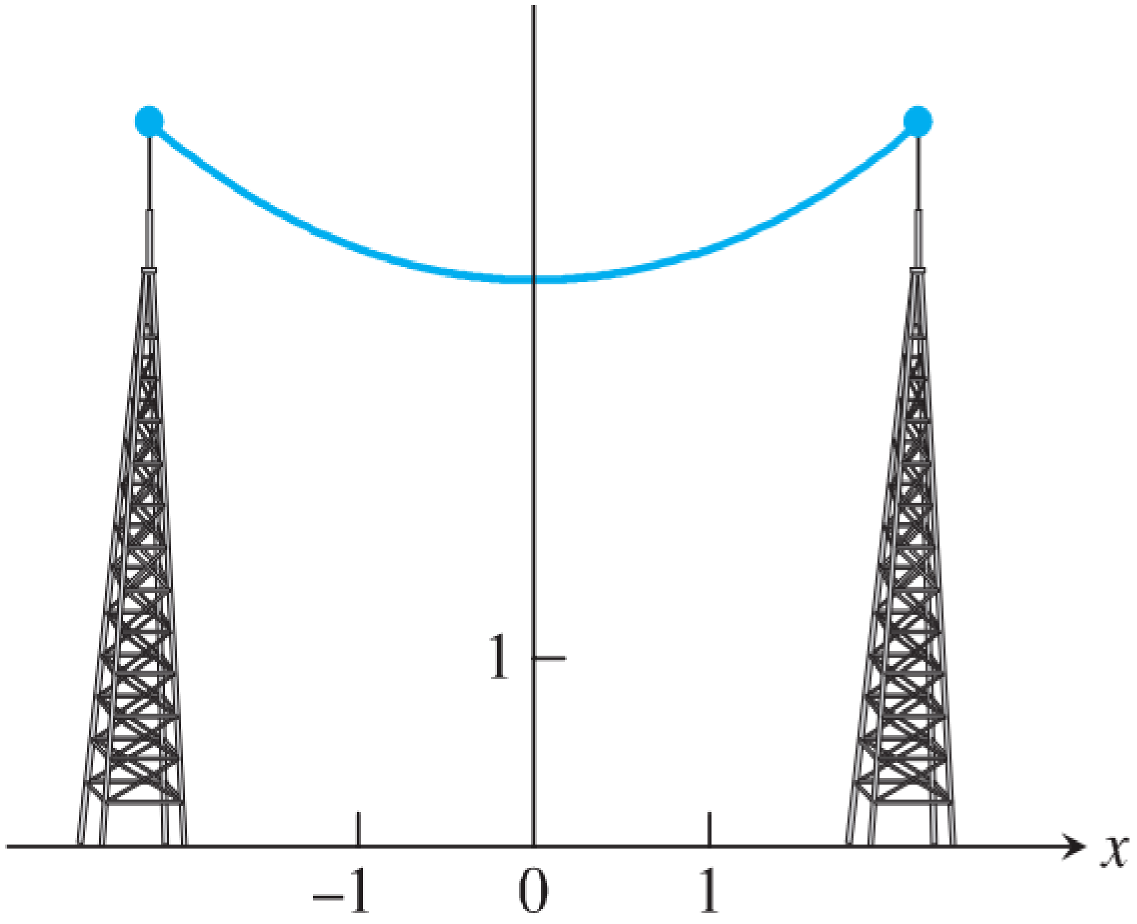
Logarithmic Functions: $f(x) = log_a x$, where the base $a \neq 1$ is a positive constant. They are the inverse functions of the exponential.

The domain is (0, ∞) and the range is (- ∞, ∞).



Common Functions

Transcendental Functions: These are functions that are not algebraic. They include the trigonometric, inverse trigonometric, exponential, and logarithmic functions, and many other functions as well.

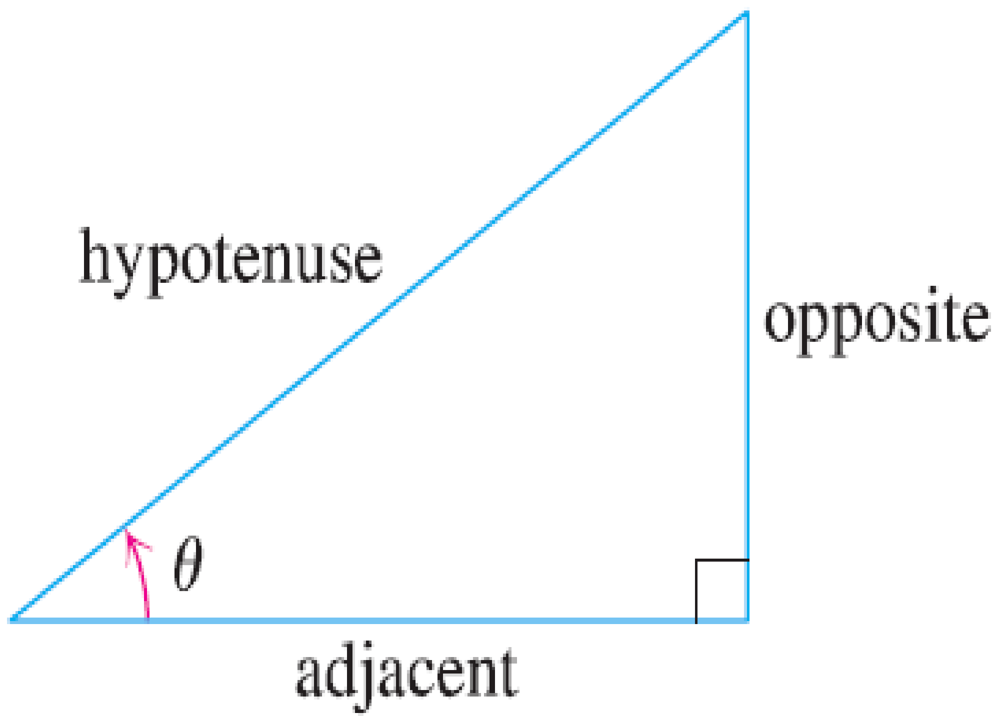


The Six Basic Trigonometric Functions

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$



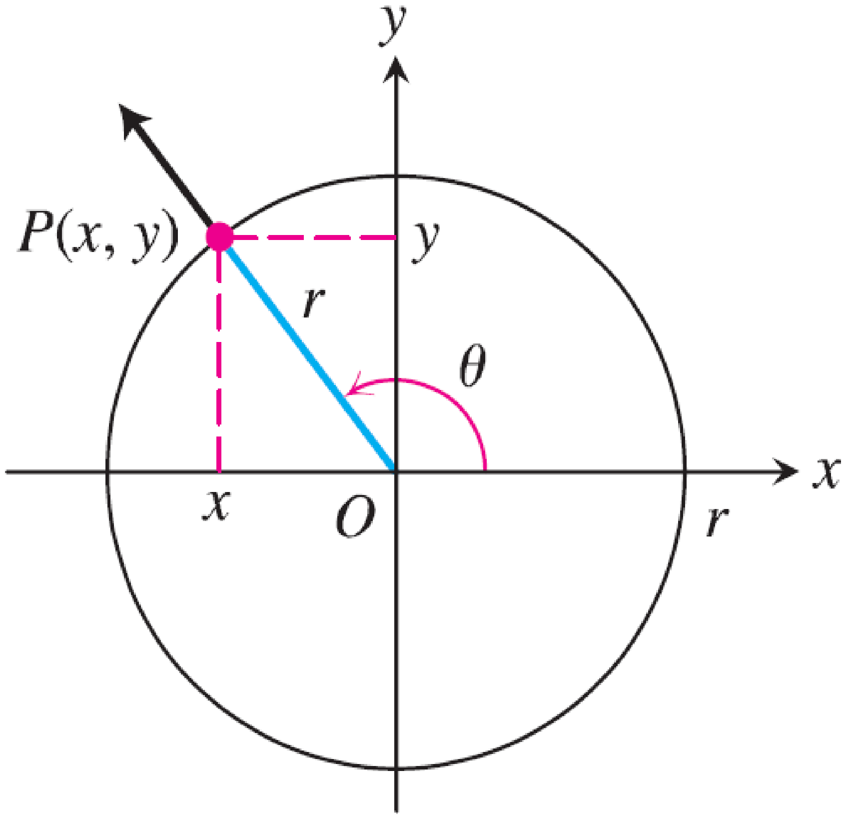
The Six Basic Trigonometric Functions

We extend this definition to obtuse and negative angles by first placing the angle in standard position in a circle of radius *r*. We then define the trigonometric functions in terms of the coordinates of the point *P(x,y)* where the angle's terminal ray intersects the circle.

sine: $\sin\theta = \dfrac{y}{r}$ **cosecant:** $\csc\theta = \dfrac{r}{y}$

cosine: $\cos\theta = \dfrac{x}{r}$ **secant:** $\sec\theta = \dfrac{r}{x}$

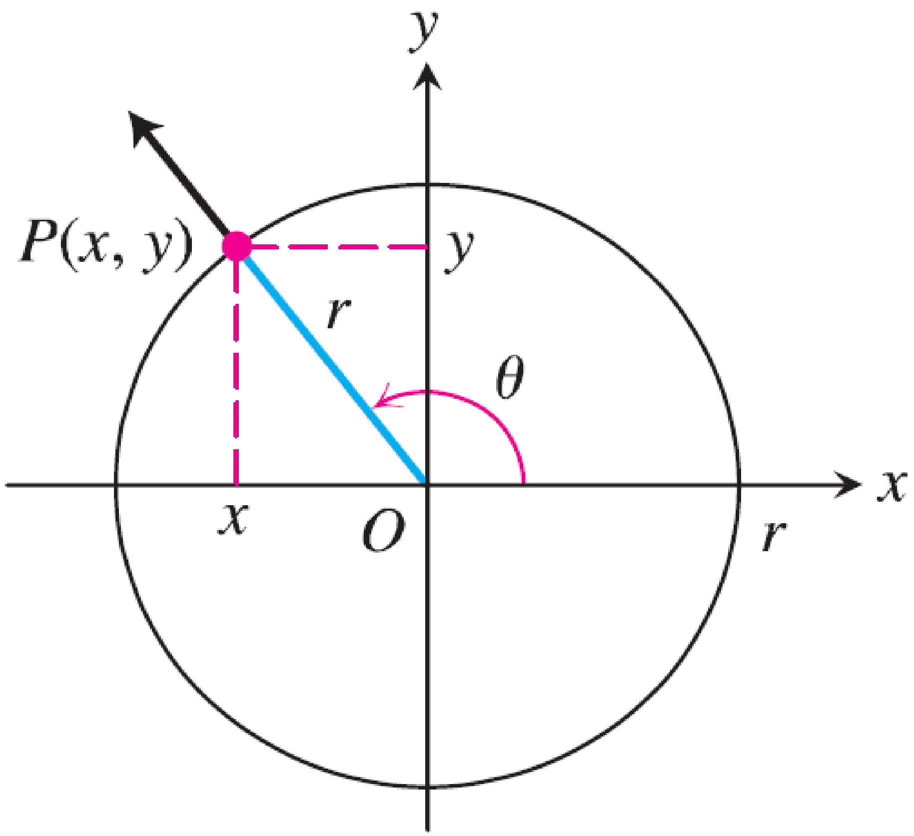
tangent: $\tan\theta = \dfrac{y}{x}$ **cotangent:** $\cot\theta = \dfrac{x}{y}$



The Six Basic Trigonometric Functions

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$



tan θ and sec θ are not defined if x = cos θ = 0. This means they are not defined if θ is $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$. Similarly, cot θ and csc θ are not defined for values of θ for which y =sin θ = 0,namely $\theta = 0, \pm\pi, \pm 2\pi, \ldots$.

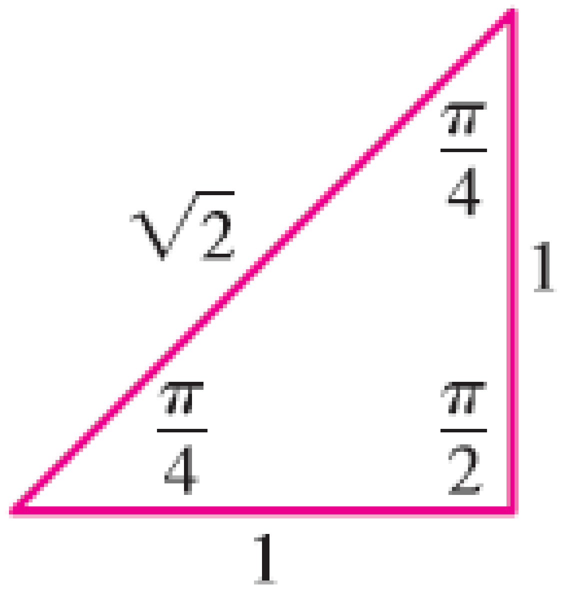
The Six Basic Trigonometric Functions

Example:

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\tan\frac{\pi}{4} = 1$$



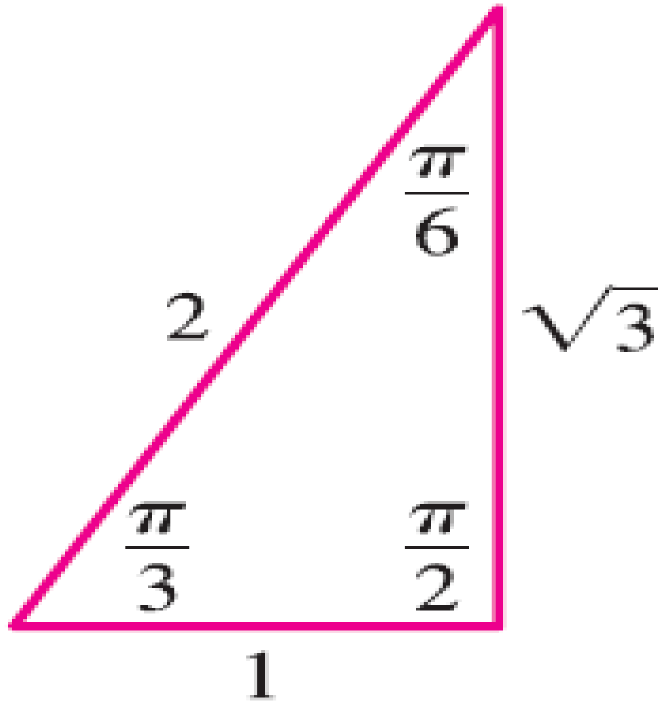
The Six Basic Trigonometric Functions

Example:

$$\sin\frac{\pi}{6} = \frac{1}{2}$$

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$



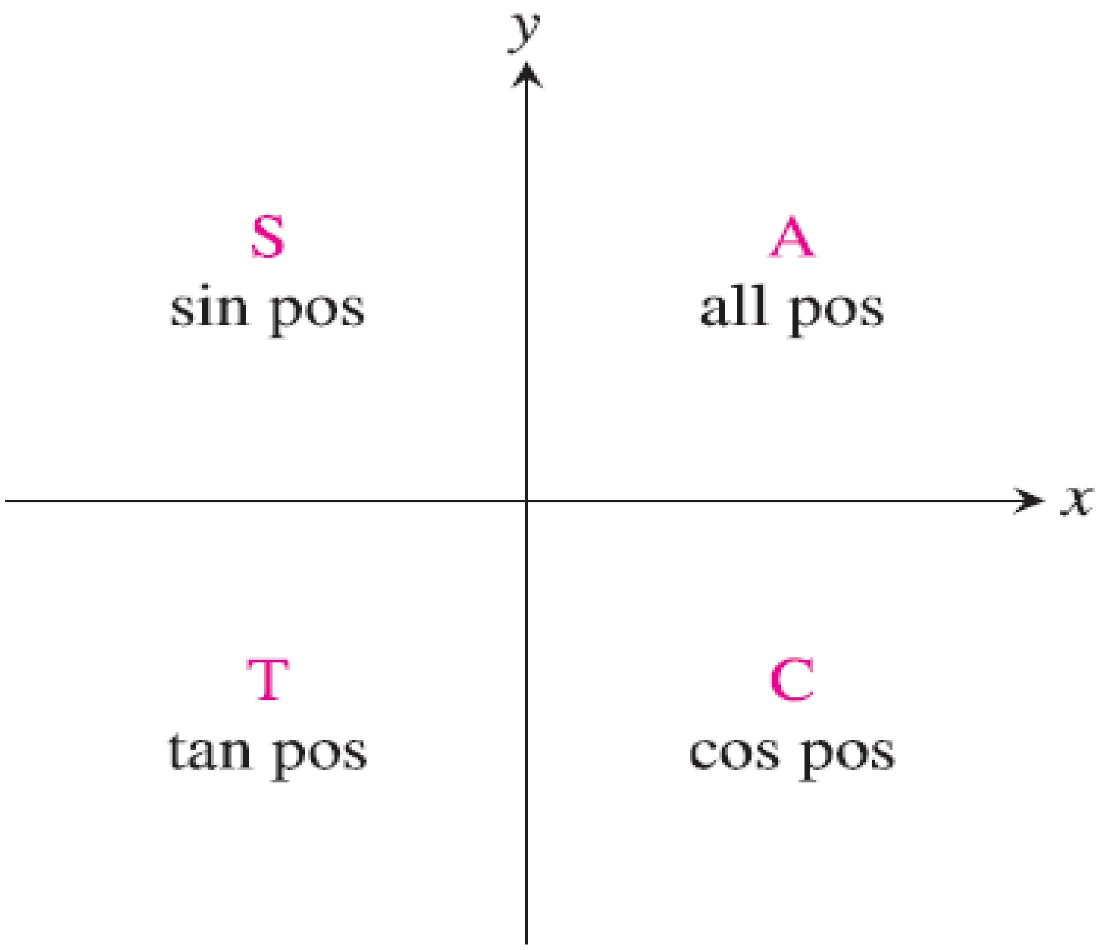
The CAST Rule

The CAST rule is useful for remembering when the basic trigonometric functions are positive or negative.

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{2\pi}{3} = -\frac{1}{2}$$

$$\tan\frac{2\pi}{3} = -\sqrt{3}.$$



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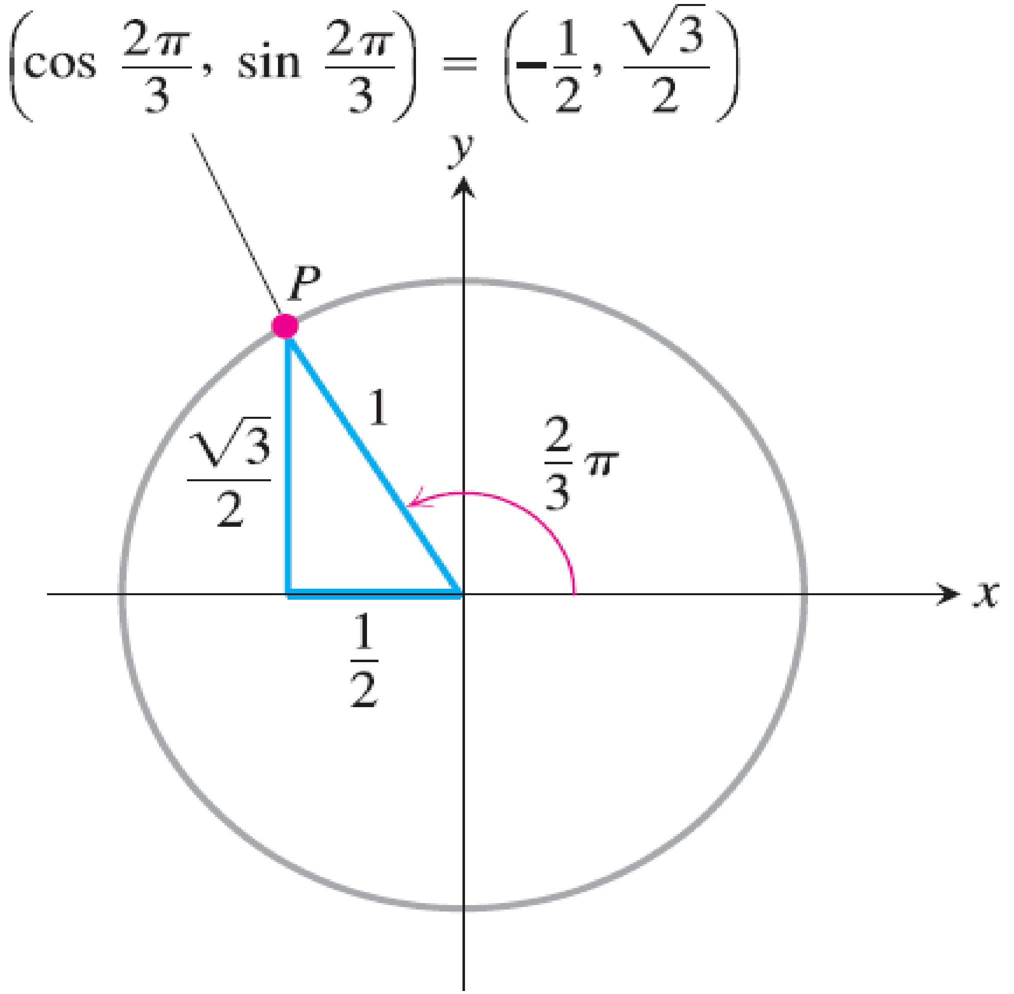


Periodicity and Graphs of the Trigonometric Functions

When an angle of measure θ and an angle of measure $\theta + 2\pi$ are in standard position, their terminal rays coincide.

The two angles therefore have the same trigonometric function values:

$\sin(\theta + 2\pi) = \sin\theta$, $\tan(\theta + \pi) = \tan\theta$, and so on.

Similarly, $\cos(\theta - 2\pi) = \cos\theta$, $\sin(\theta - 2\pi) = \sin\theta$, and so on. We describe this repeating behavior by saying that the six basic trigonometric functions are periodic.

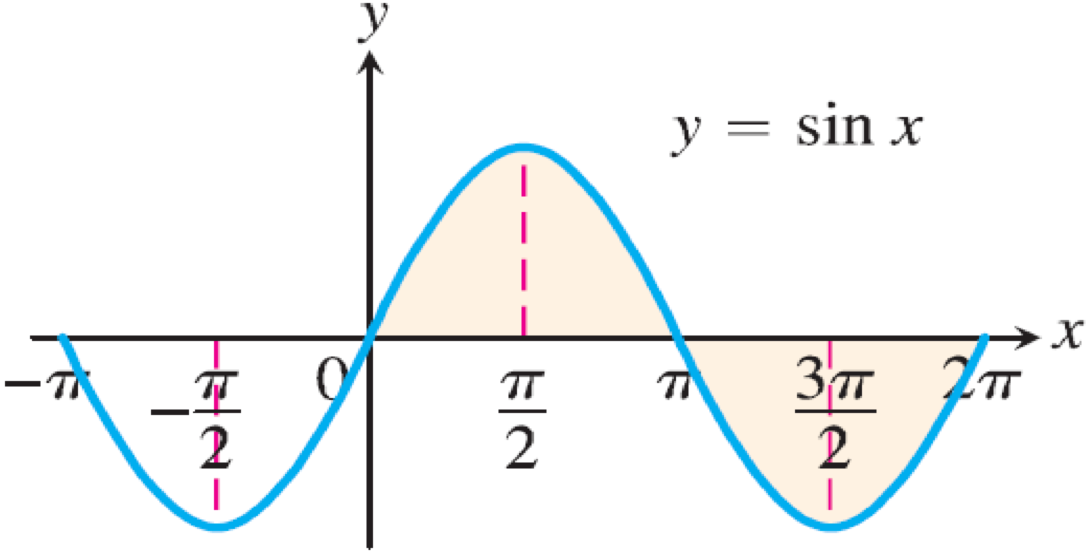
Periodicity and Graphs of the Trigonometric Functions

DEFINITION A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

Period 2π :

$$\sin(x + 2\pi) = \sin x$$

$$\sin(-x) = -\sin x$$ → Odd



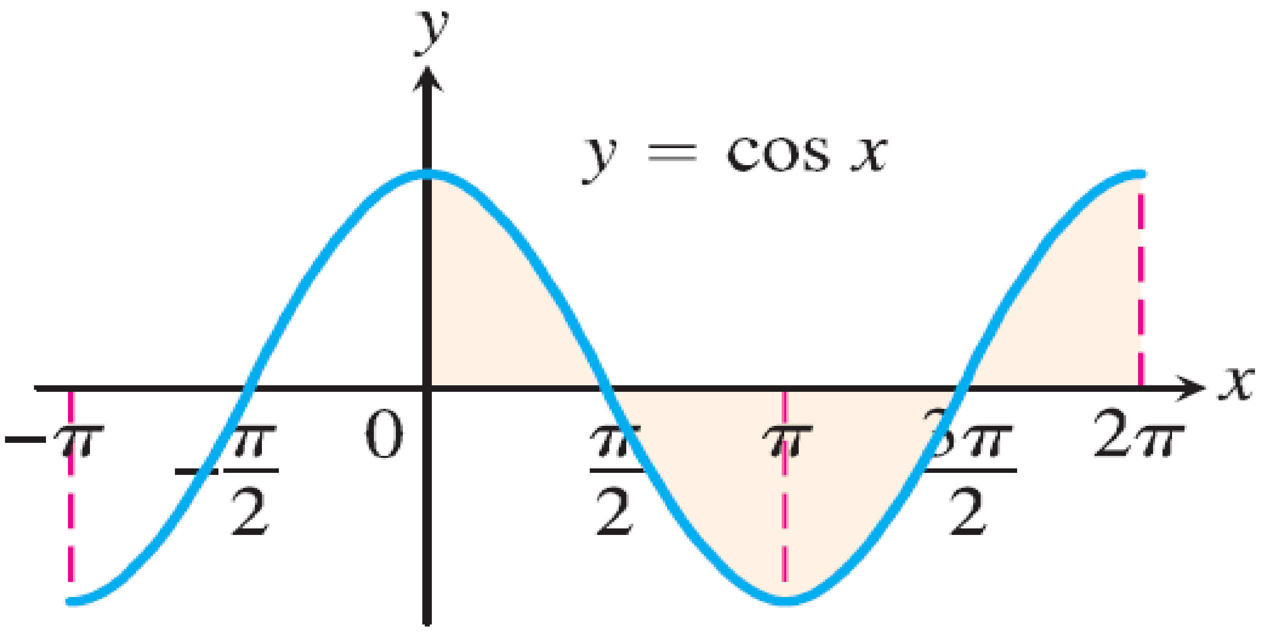
Domain: $-\infty < x < \infty$
Range: $-1 \leq y \leq 1$
Period: 2π

Periodicity and Graphs of the Trigonometric Functions

Period 2π :

$$\cos(x + 2\pi) = \cos x$$

$$\cos(-x) = \cos x$$ → Even



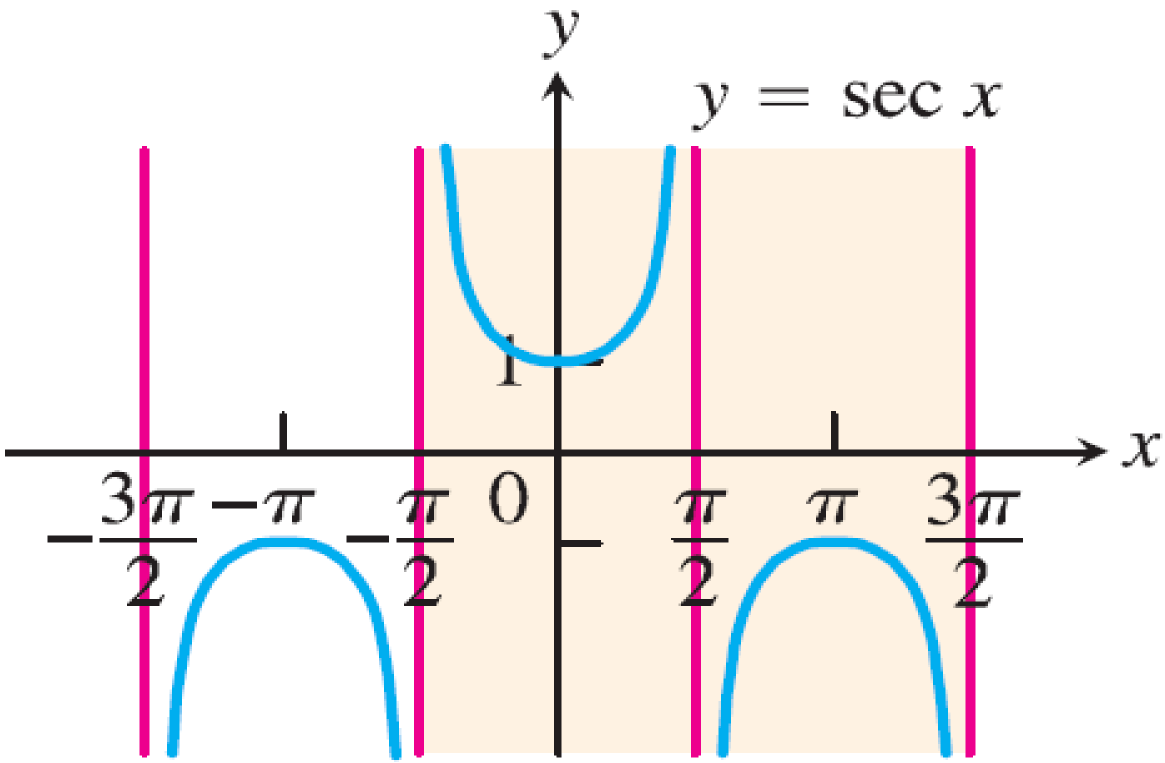
Domain: $-\infty < x < \infty$
Range: $-1 \leq y \leq 1$
Period: 2π

Periodicity and Graphs of the Trigonometric Functions

Period 2π :

$$\sec(x + 2\pi) = \sec x$$

$$\sec(-x) = \sec x$$ ⇒ Even



Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

Range: $y \leq -1$ or $y \geq 1$

Period: 2π

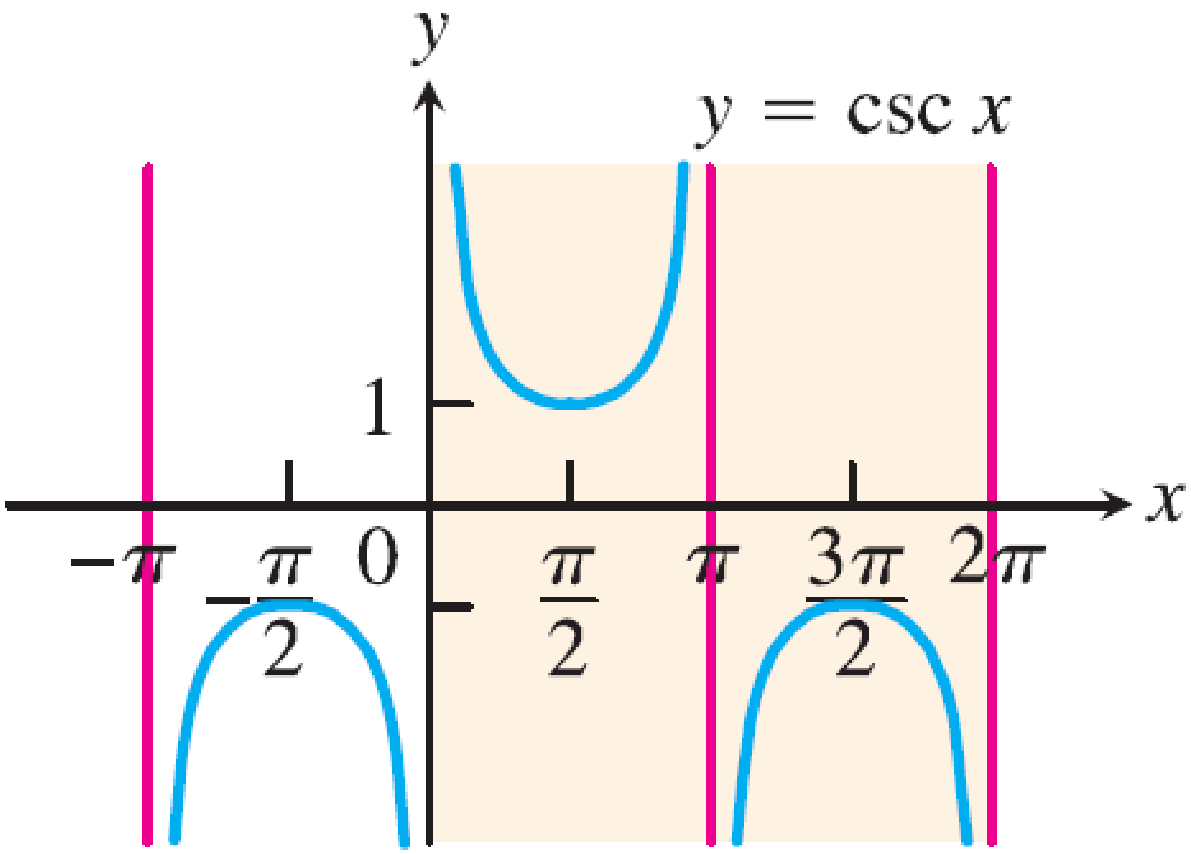
Periodicity and Graphs of the Trigonometric Functions

Period 2π :

$$\csc(x + 2\pi) = \csc x$$

$$\csc(-x) = -\csc x$$

Odd



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$

Range: $y \leq -1$ or $y \geq 1$

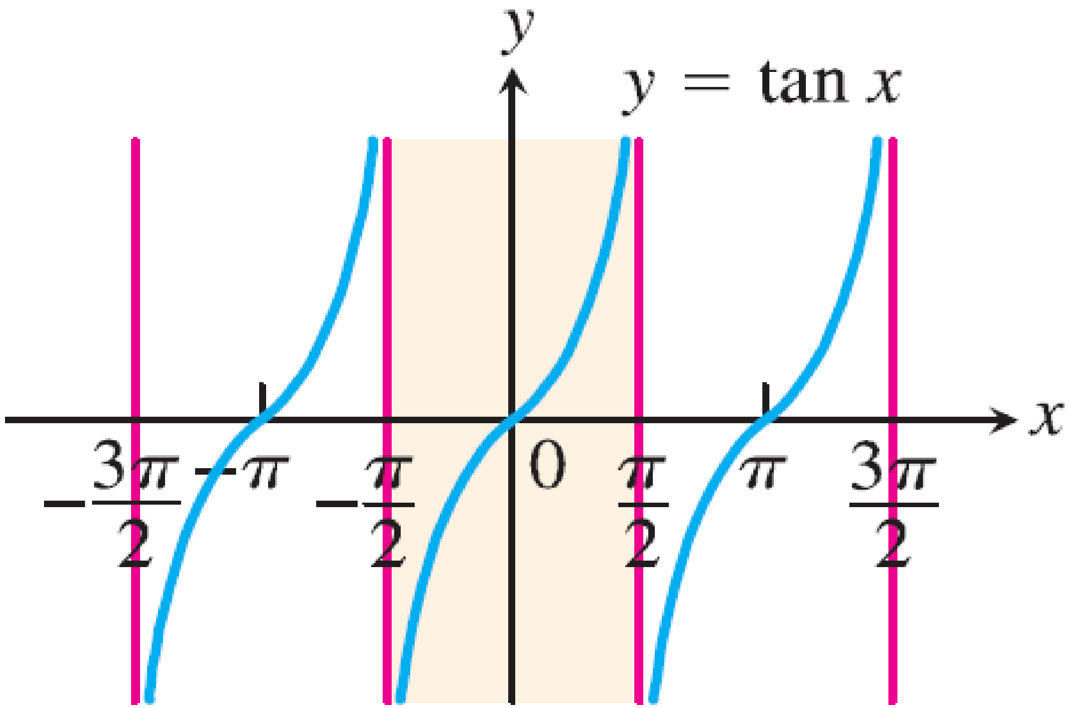
Period: 2π

Periodicity and Graphs of the Trigonometric Functions

Period π :

$$\tan(x + \pi) = \tan x$$

$$\tan(-x) = -\tan x$$ ➡ Odd



Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

Range: $-\infty < y < \infty$

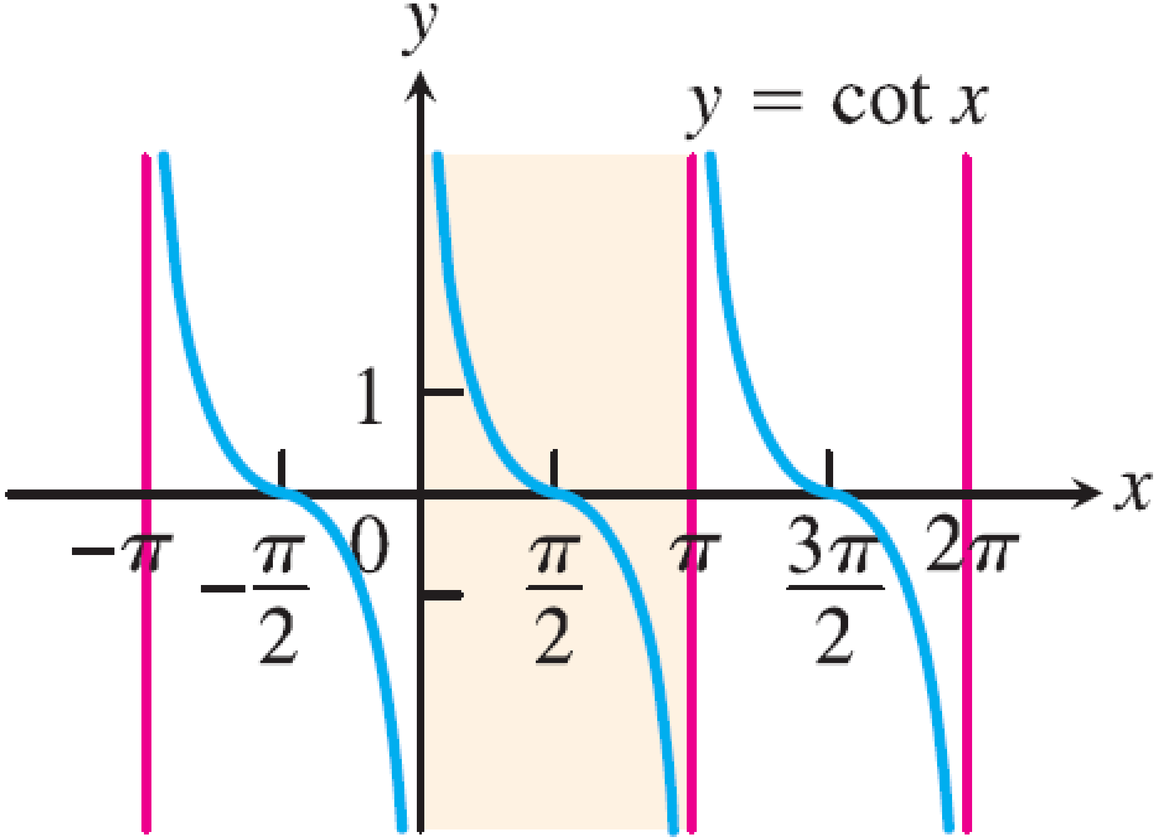
Period: π

Periodicity and Graphs of the Trigonometric Functions

Period π :

$$\cot(x+\pi) = \cot x$$

$$\cot(-x) = -\cot x$$ Odd



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$

Range: $-\infty < y < \infty$

Period: π

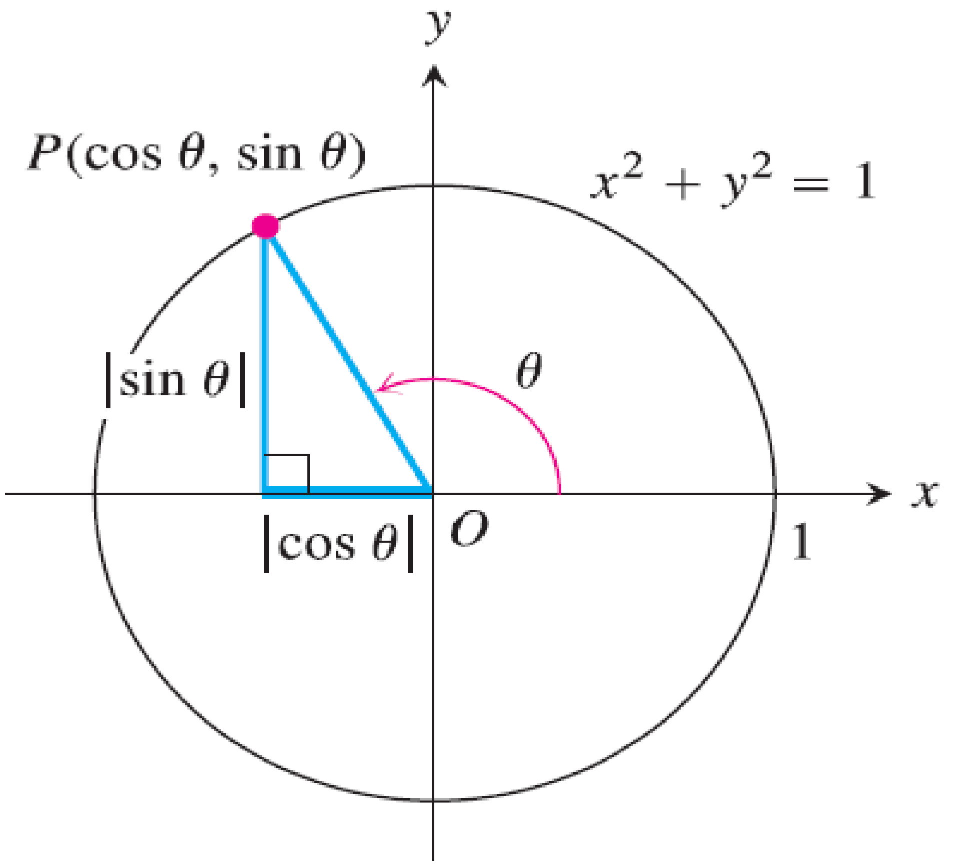
Trigonometric Identities

The coordinates of any point *P(x, y)* in the plane can be expressed in terms of the point's distance ***r*** from the origin and the angle θ that ray *OP* makes with the positive x-axis

$$x = r\cos\theta$$

$$y = r\sin\theta$$

When ***r* = 1** we can apply the Pythagorean theorem



$$\cos^2\theta + \sin^2\theta = 1$$

Trigonometric Identities

$$cos^2\theta + sin^2\theta = 1 \qquad divid\ over\ cos^2\theta$$

$$1 + tan^2\theta = sec^2\theta$$

$$cos^2\theta + sin^2\theta = 1 \qquad divid\ over\ sin^2\theta$$

$$cot^2\theta + 1 = csc^2\theta$$

The following formulas hold for all angles A and B

$$\sin(A \pm B) = \sin A\ \cos B \pm \cos A\ \sin B$$

$$cos\ (A \pm B) = cos\, A\ \cos B \mp \sin A\ \sin B$$

Double-Angle Formulas

$$\cos 2\theta = cos^2\theta - sin^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

Half-Angle Formulas

$$cos^2\theta + sin^2\theta = 1 \quad (1)$$

$$cos^2\theta - sin^2\theta = \cos 2\theta \quad (2)$$

By summing Eq. (1) and Eq.(2)

$$cos^2\theta + sin^2\theta + cos^2\theta - sin^2\theta = 1 + \cos 2\theta$$

$$2cos^2\theta = 1 + \cos 2\theta$$

$$cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

Double-Angle Formulas

Half-Angle Formulas

$$cos^2\theta + sin^2\theta = 1 \quad (1)$$

$$cos^2\theta - sin^2\theta = \cos 2\theta \quad (2)$$

By subtracting Eq. (2) from Eq.(1)

$$sin^2\theta = \frac{1-\cos 2\theta}{2}$$

INVERSE FUNCTIONS

$$y = f(x)$$

This equation is better for computing y if x is known.

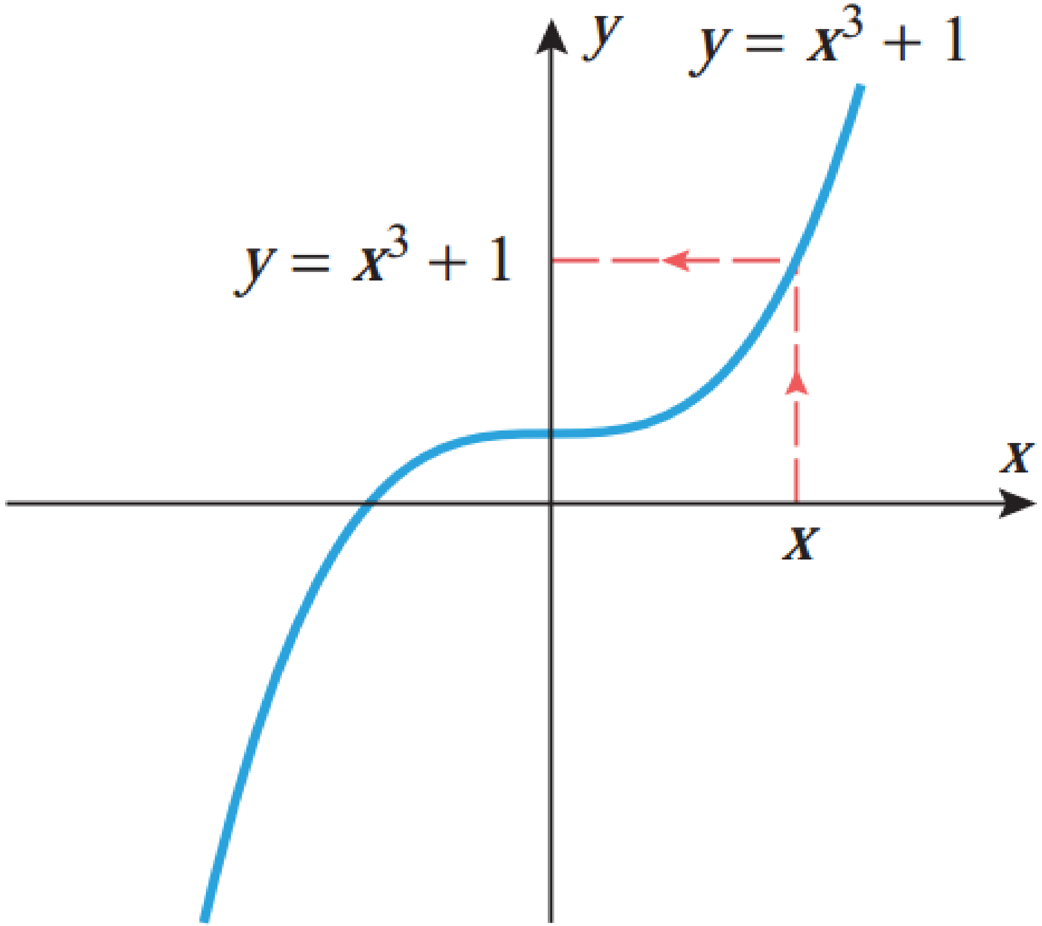
$$x = g(y)$$

This equation is better for computing x if y is known.

INVERSE FUNCTIONS

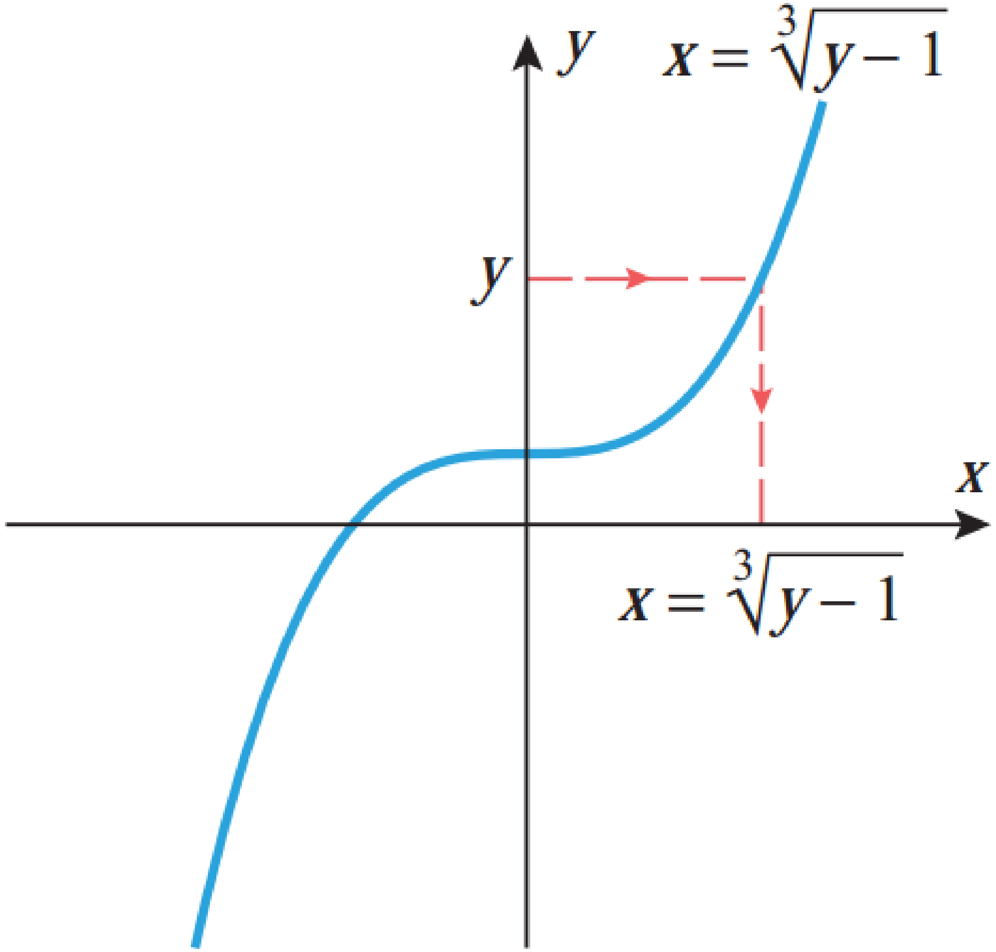
Example : $y = x^3 + 1$

$$\boxed{y = f(x)}$$



$x = \sqrt[3]{y-1}$

$$\boxed{x = g(y)}$$



INVERSE FUNCTIONS

Example : $y = x^3 + 1$ $\qquad$ $x = \sqrt[3]{y - 1}$

$$\boxed{y = f(x)} \qquad \boxed{x = g(y)}$$

When these functions are composed in either order, they <u>cancel out the effect</u> of one another:

$$g(f(x)) = \sqrt[3]{f(x) - 1} = \sqrt[3]{(x^3 + 1) - 1} = x$$

$$f(g(y)) = [g(y)]^3 + 1 = (\sqrt[3]{y - 1})^3 + 1 = y$$

INVERSE FUNCTIONS

0.4.1 DEFINITION If the functions f and g satisfy the two conditions

$$g(f(x)) = x \text{ for every } x \text{ in the domain of } f$$
$$f(g(y)) = y \text{ for every } y \text{ in the domain of } g$$

then we say that f ***is an inverse of*** g and g ***is an inverse of*** f or that f ***and*** g ***are inverse functions***.

If a function f has an inverse, then that inverse is unique.

Thus, if a function f has an inverse, then we denote it by the symbol f^{-1}.

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in the domain of } f$$
$$f(f^{-1}(y)) = y \quad \text{for every } y \text{ in the domain of } f^{-1}$$

INVERSE FUNCTIONS

Changing The independent Variable :

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in the domain of } f$$
$$f(f^{-1}(y)) = y \quad \text{for every } y \text{ in the domain of } f^{-1}$$

If we want to graph the functions f and f^{-1} together in the same xy-coordinate system, then we use x as the independent variable and y as the dependent variable for both functions.

Ex: To graph the functions: $f(x) = x^3 + 1$ and $f^{-1}(y) = \sqrt[3]{y-1}$

we would change the independent variable y to x, use y as the dependent variable for both functions, and graph the equations

$$y = x^3 + 1 \quad \text{and} \quad y = \sqrt[3]{x-1}$$

INVERSE FUNCTIONS

Changing The independent Variable :

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in the domain of } f$$
$$f(f^{-1}(y)) = y \quad \text{for every } y \text{ in the domain of } f^{-1}$$

using x as the independent variable for both f and f^{-1}:

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in the domain of } f$$
$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in the domain of } f^{-1}$$

INVERSE FUNCTIONS

Example: Confirm each of the following:

(a) The inverse of $f(x) = 2x$ is $f^{-1}(x) = \frac{1}{2}x$.

(b) The inverse of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$.

Solution:

(a)

$$f^{-1}(f(x)) = f^{-1}(2x) = \tfrac{1}{2}(2x) = x$$

$$f(f^{-1}(x)) = f\left(\tfrac{1}{2}x\right) = 2\left(\tfrac{1}{2}x\right) = x$$

(b)

$$f^{-1}(f(x)) = f^{-1}(x^3) = \left(x^3\right)^{1/3} = x$$

$$f(f^{-1}(x)) = f(x^{1/3}) = \left(x^{1/3}\right)^3 = x$$

INVERSE FUNCTIONS

Domain and Range of Inverse functions:

From:

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in the domain of } f$$
$$f(f^{-1}(y)) = y \quad \text{for every } y \text{ in the domain of } f^{-1}$$

$$\text{domain of } f^{-1} = \text{range of } f$$
$$\text{range of } f^{-1} = \text{domain of } f$$

INVERSE FUNCTIONS

A Method For Finding Inverse Functions

0.4.2 THEOREM *If an equation $y = f(x)$ can be solved for x as a function of y, say $x = g(y)$, then f has an inverse and that inverse is $g(y) = f^{-1}(y)$.*

This theorem provides us with the following procedure for finding the inverse of a function.

A Procedure for Finding the Inverse of a Function f

Step 1. Write down the equation $y = f(x)$.

Step 2. If possible, solve this equation for x as a function of y.

Step 3. The resulting equation will be $x = f^{-1}(y)$, which provides a formula for f^{-1} with y as the independent variable.

Step 4. If y is acceptable as the independent variable for the inverse function, then you are done, but if you want to have x as the independent variable, then you need to interchange x and y in the equation $x = f^{-1}(y)$ to obtain $y = f^{-1}(x)$.

INVERSE FUNCTIONS

A Method For Finding Inverse Functions

Example Find a formula for the inverse of $f(x) = \sqrt{3x - 2}$ with x as the independent variable, and state the domain of f^{-1}.

Solution Following the procedure, we first write $y = \sqrt{3x - 2}$

Then we solve this equation for x as a function of y:

$$y^2 = 3x - 2$$

$$x = \tfrac{1}{3}(y^2 + 2)$$

which tells us that $f^{-1}(y) = \tfrac{1}{3}(y^2 + 2)$

Since we want x to be the independent variable $f^{-1}(x) = \tfrac{1}{3}(x^2 + 2)$

The domain of f^{-1} is the range of f, whereas the range of f(x) is $[0, \infty)$.

INVERSE FUNCTIONS

Existence Of Inverse Functions:

This procedure can fail for two reasons. The function *f* may not have an inverse, or it may have an inverse but the equation *y = f(x)* cannot be solved explicitly for *x* as a function of *y*.

Thus, it is important to establish conditions that ensure the existence of an inverse, even if it cannot be found explicitly.

0.4.3 THEOREM *A function has an inverse if and only if it is one-to-one.*

A function that assigns distinct outputs to distinct inputs is said to be one-to-one or invertible.

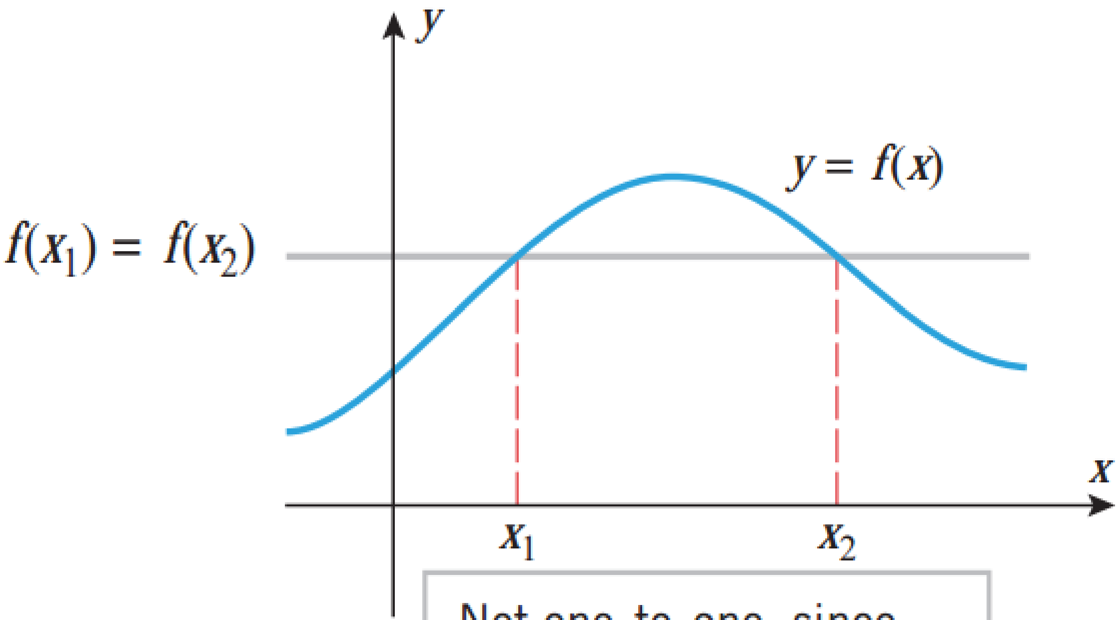
Algebraically, a function *f(x)* is one-to-one if and only if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

INVERSE FUNCTIONS

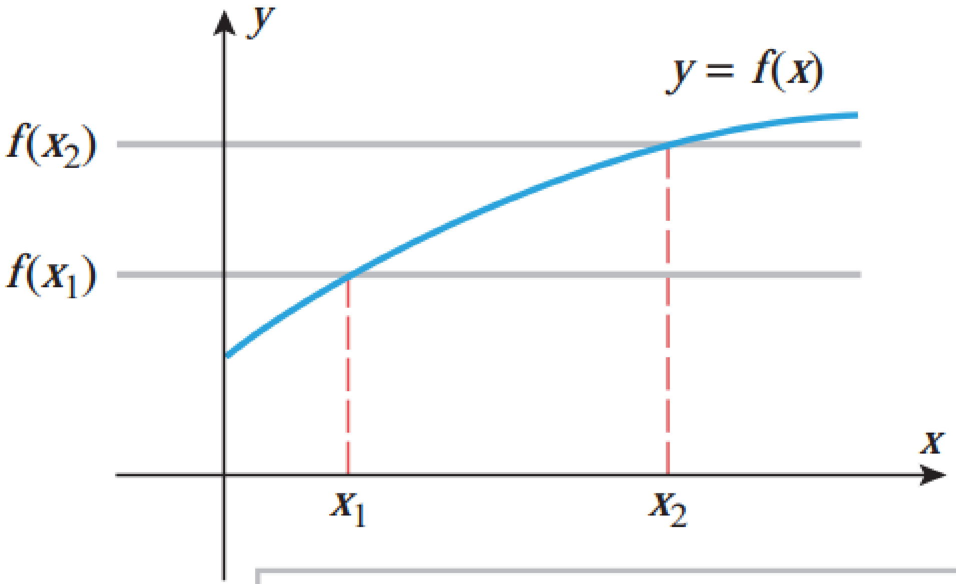
Existence Of Inverse Functions:

The Horizontal Line Test:

0.4.4 THEOREM (***The Horizontal Line Test***) *A function has an inverse function if and only if its graph is cut at most once by any horizontal line.*



Not one-to-one, since $f(x_1) = f(x_2)$ and $x_1 \neq x_2$

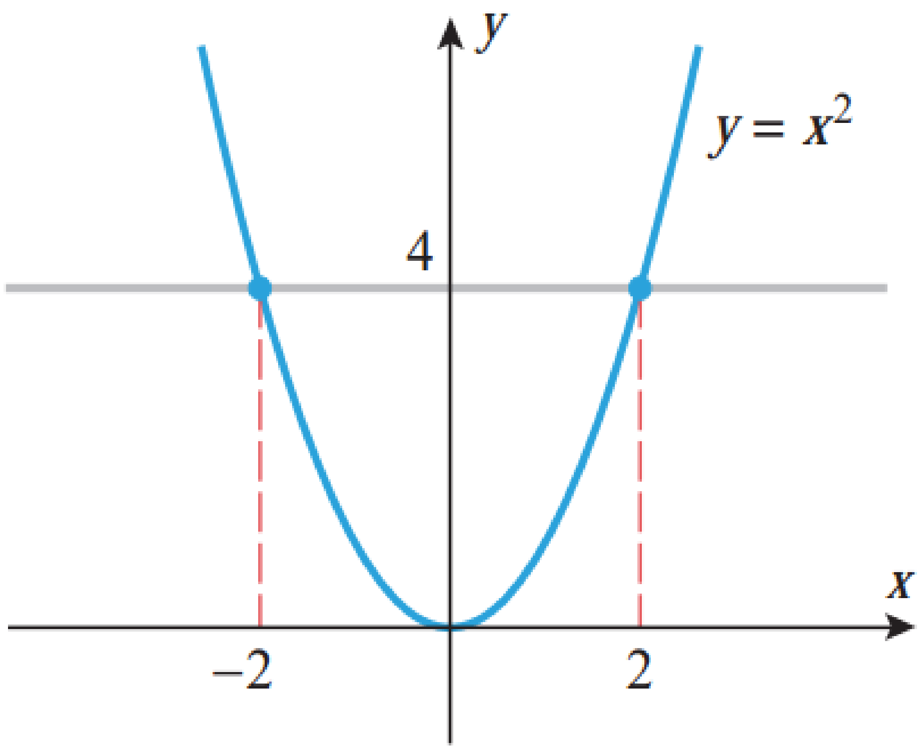


One-to-one, since $f(x_1) \neq f(x_2)$ if $x_1 \neq x_2$

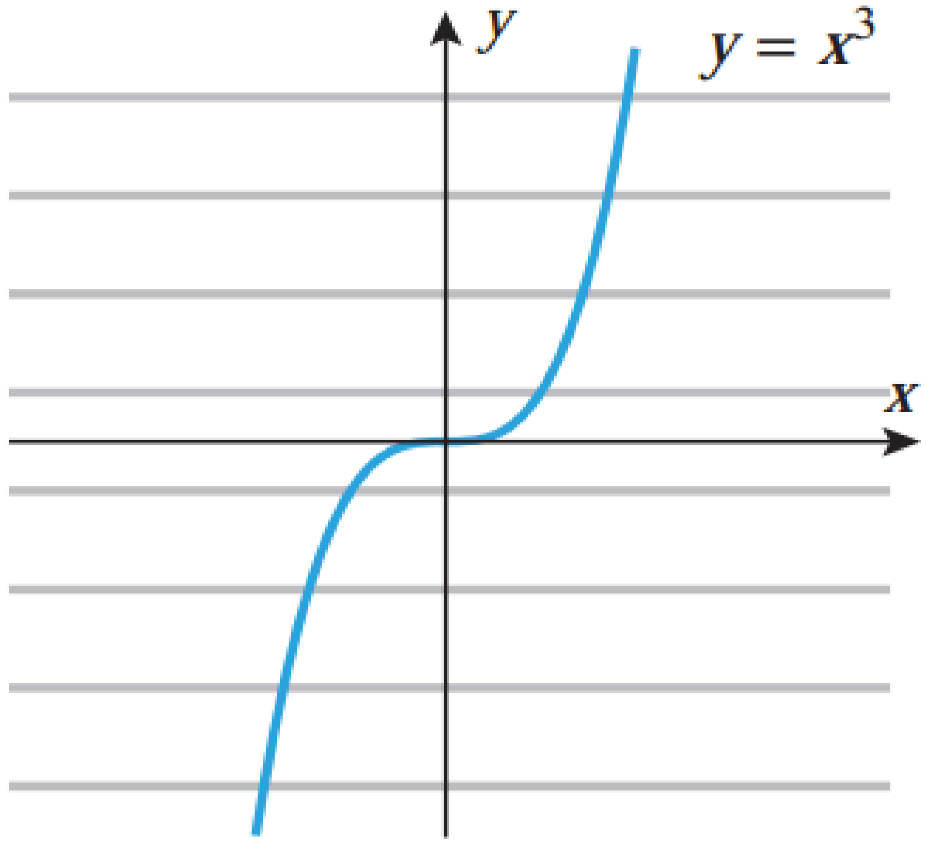
INVERSE FUNCTIONS

Existence Of Inverse Functions:

Example: $y = x^2, and\ y = x^3$



$y = x^2$ **is not invertible.**

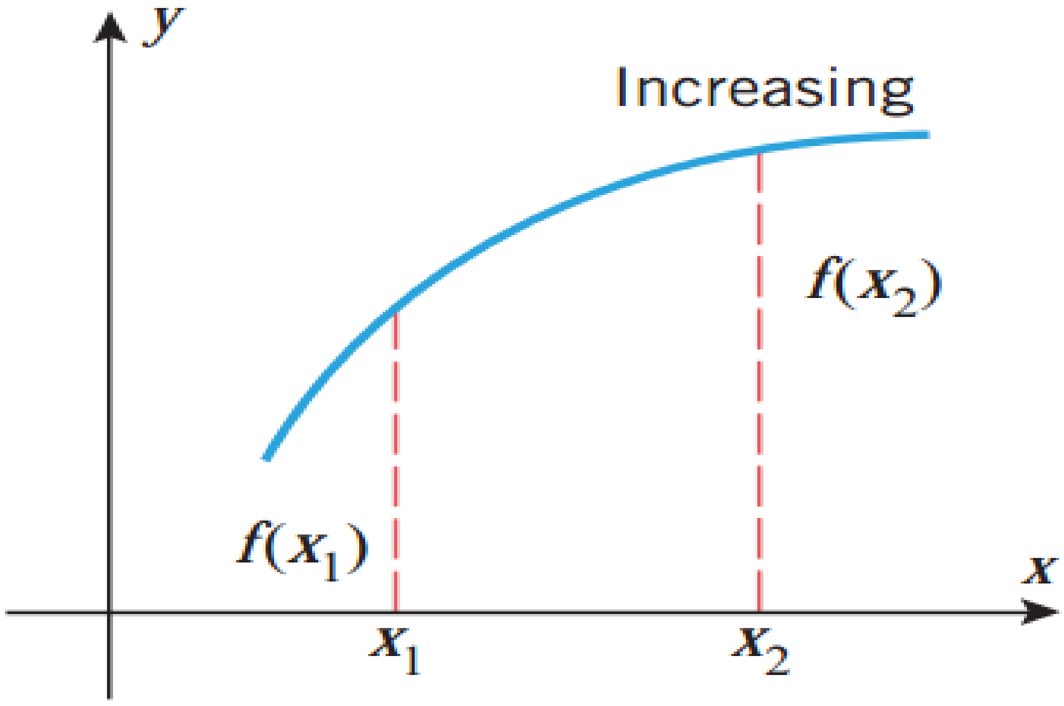


$y = x^3$ **is invertible.**

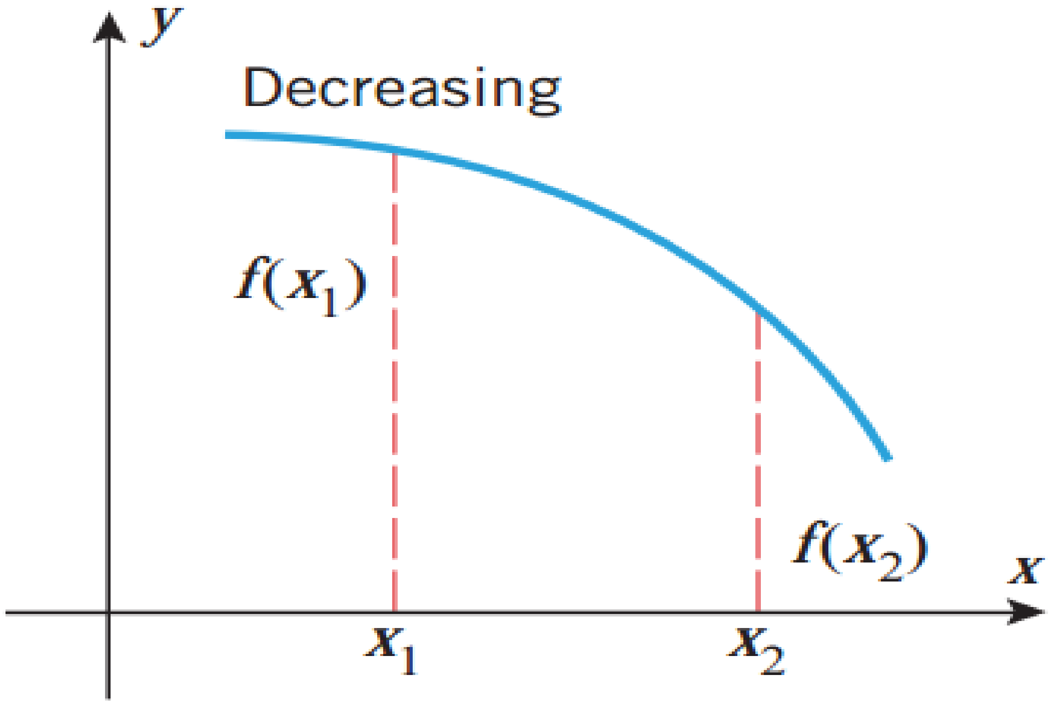
INVERSE FUNCTIONS

Existence Of Inverse Functions:

Increasing Or Decreasing functions are invertible.



$f(x_1) < f(x_2)$ if $x_1 < x_2$

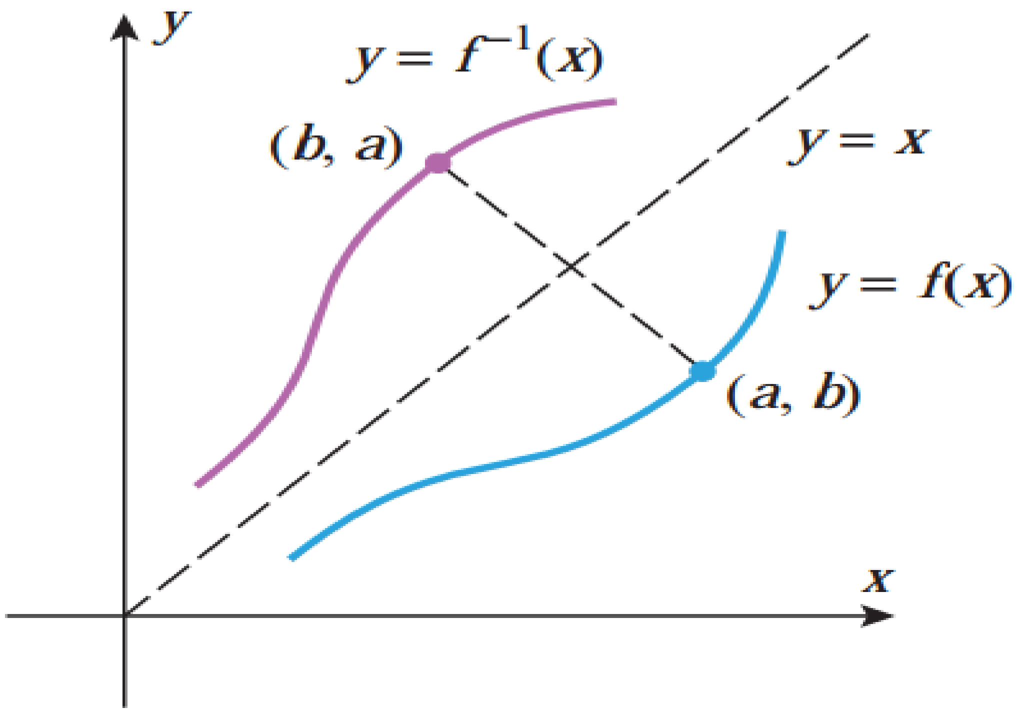


$f(x_1) > f(x_2)$ if $x_1 < x_2$

INVERSE FUNCTIONS

Graphs Of Inverse Functions

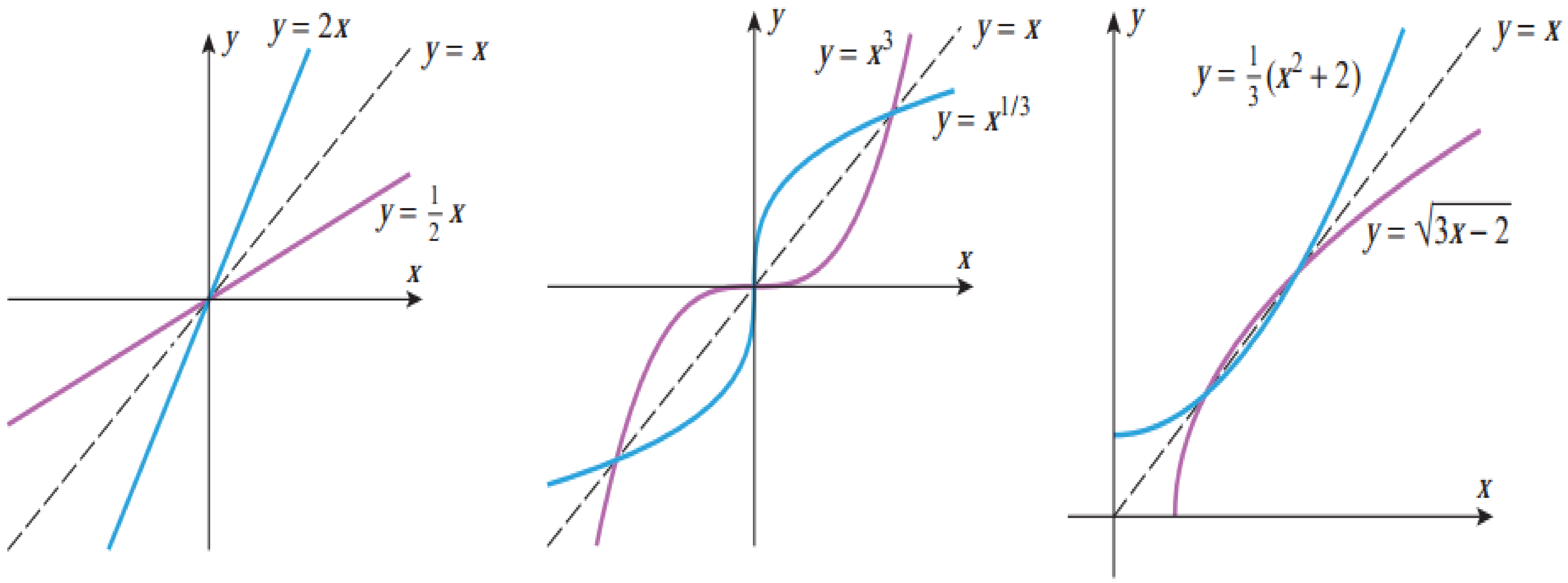
0.4.5 THEOREM *If f has an inverse, then the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of one another about the line $y = x$; that is, each graph is the mirror image of the other with respect to that line.*



The graphs of f and f^{-1} are reflections about $y = x$.

INVERSE FUNCTIONS

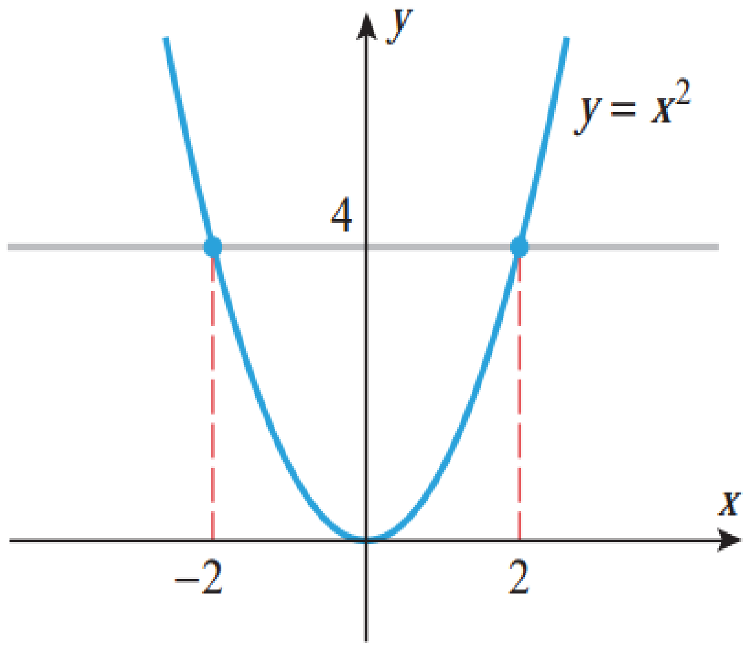
Graphs Of Inverse Functions



Restricting Domains For Invertibility

Sometimes it is possible to create an invertible function from a function that is not invertible by restricting the domain appropriately.

Example: $y = x^2$,

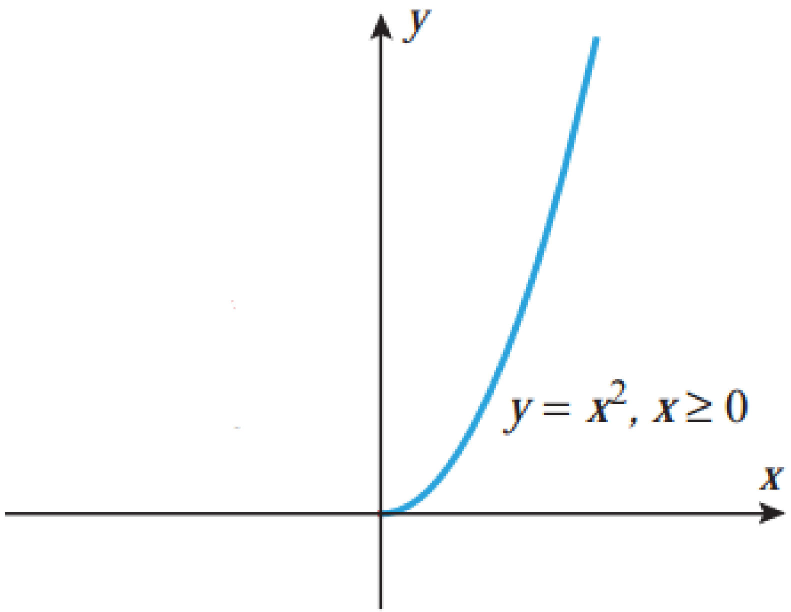


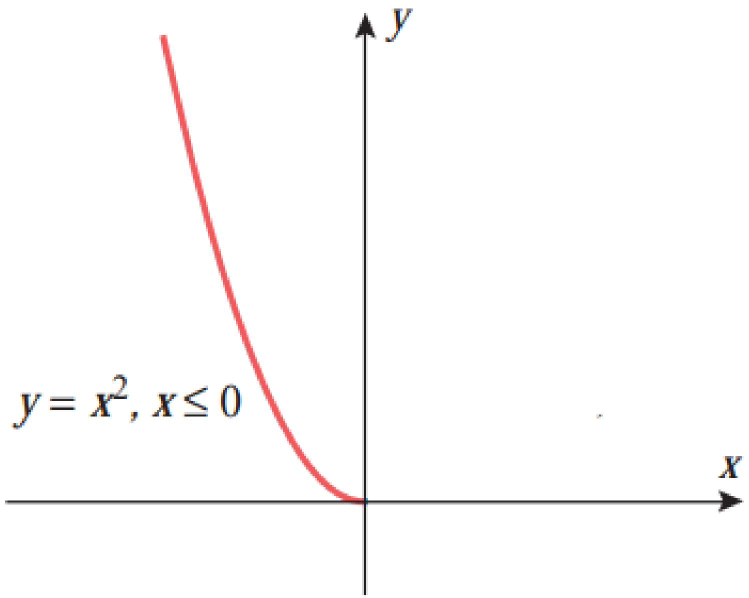
$y = x^2$ is not invertible.

However, consider the restricted functions

$$f_1(x) = x^2, \quad x \geq 0 \quad \text{and} \quad f_2(x) = x^2, \quad x \leq 0$$

These restricted functions are each one-to-one (invertible).





INVERSE FUNCTIONS

Restricting Domains For Invertibility

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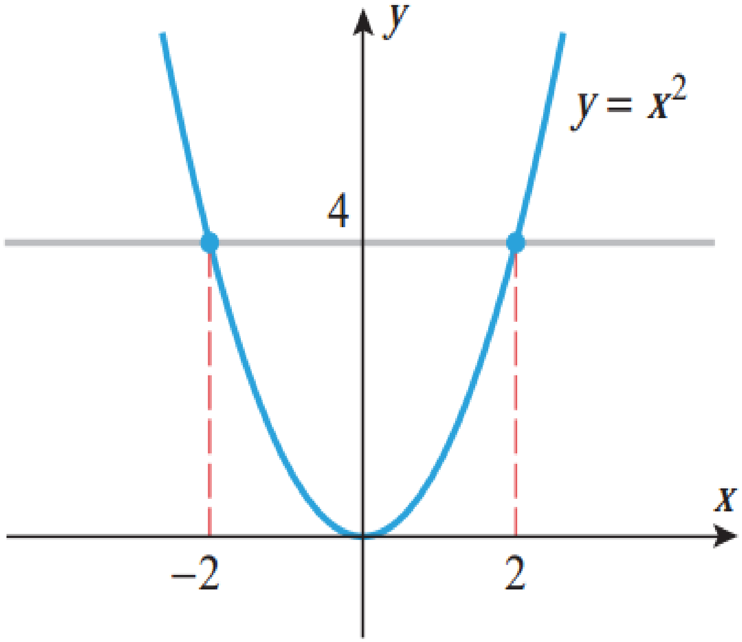
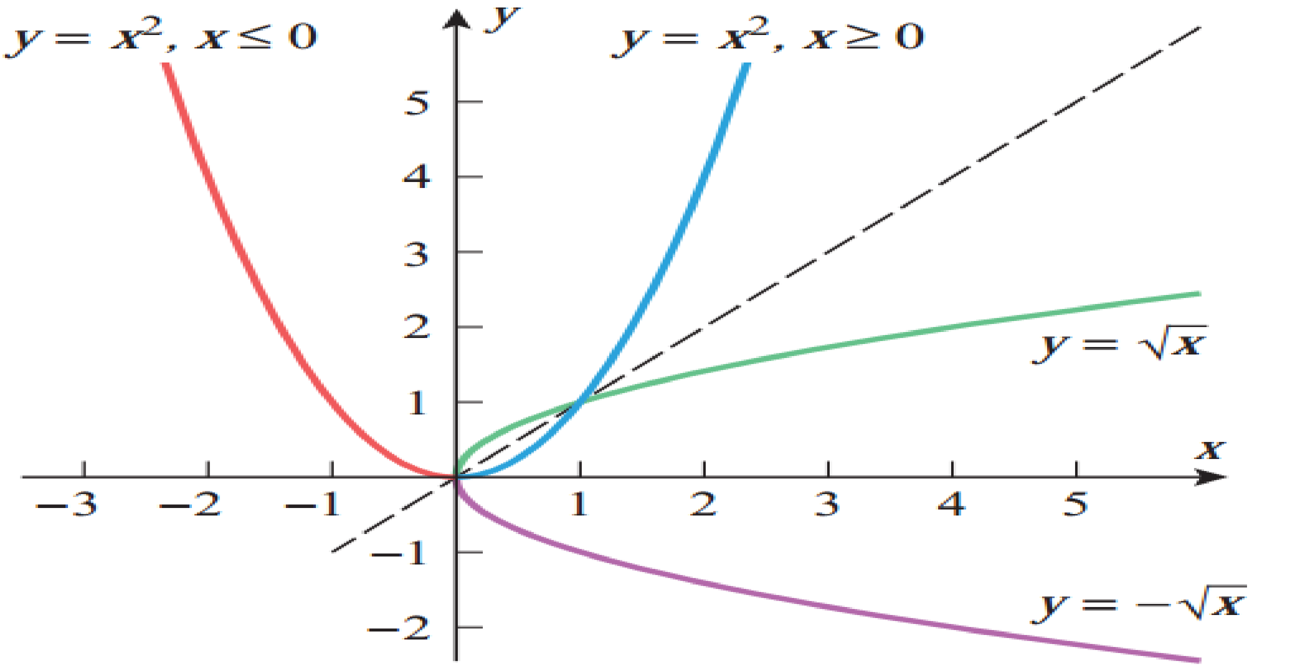


$y = x^2$ is not invertible.

However, consider the restricted functions

$$f_1(x) = x^2, \quad x \geq 0 \quad \text{and} \quad f_2(x) = x^2, \quad x \leq 0$$

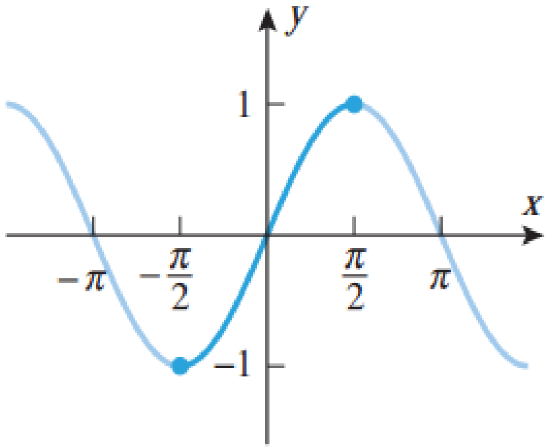
Their inverses are $f_1^{-1}(x) = \sqrt{x}$ and $f_2^{-1}(x) = -\sqrt{x}$



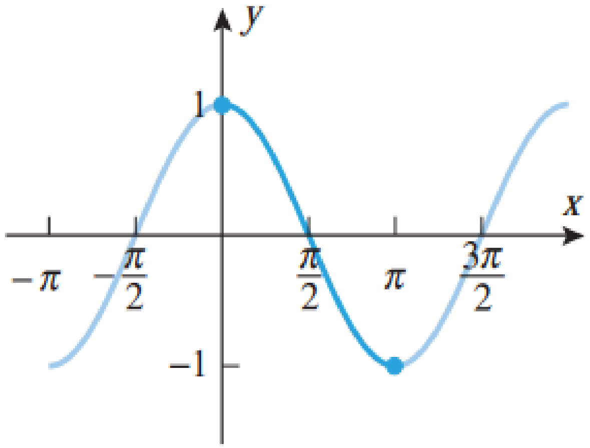
INVERSE FUNCTIONS

Inverse Trigonometric Functions

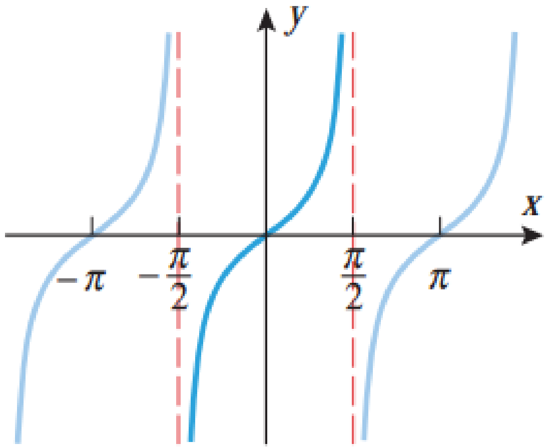
The six basic trigonometric functions do not have inverses because their graphs repeat periodically and hence do not pass the horizontal line test.



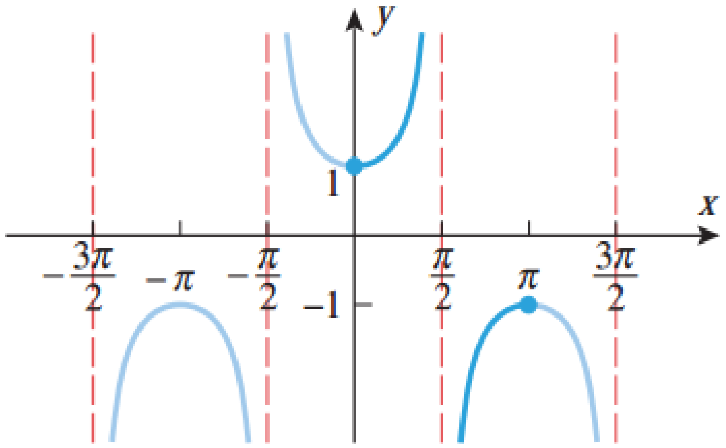
$y = \sin x$
$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$



$y = \cos x$
$0 \leq x \leq \pi$



$y = \tan x$
$-\frac{\pi}{2} < x < \frac{\pi}{2}$



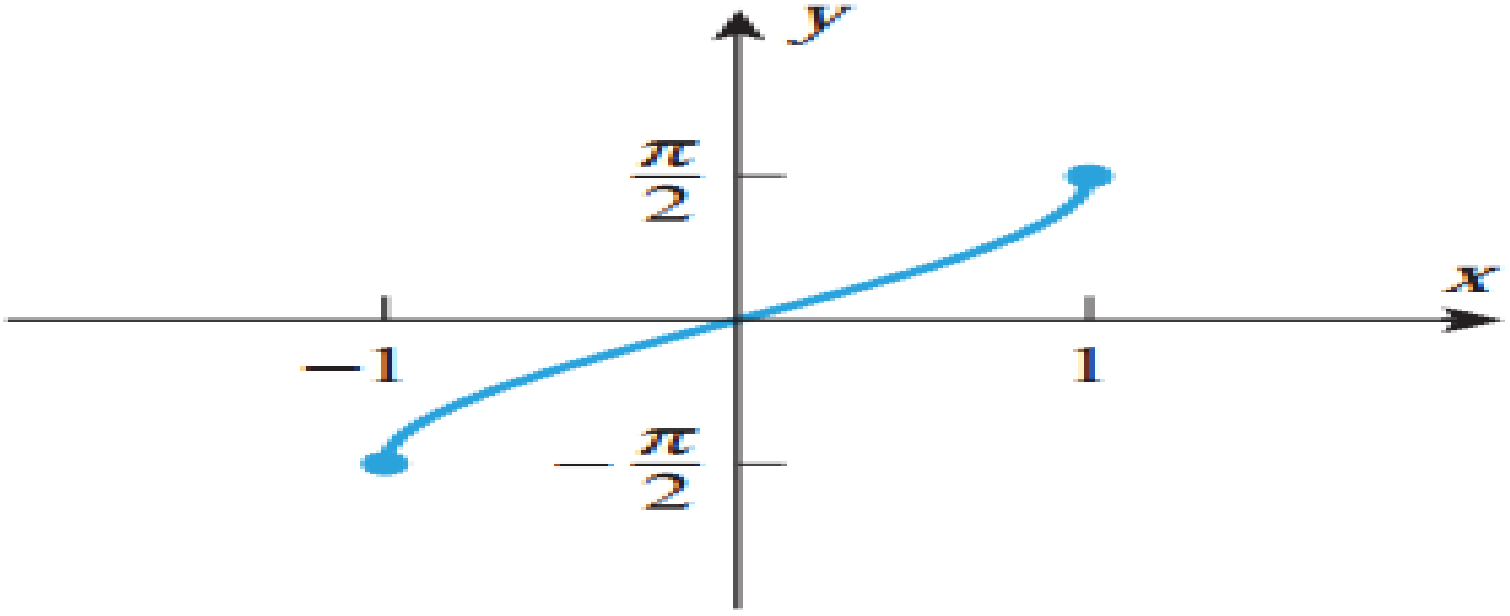
$y = \sec x$
$0 \leq x \leq \pi,\ x \neq \frac{\pi}{2}$

To circumvent this problem we will restrict the domains of the trigonometric functions to produce one-to-one functions and then define the "inverse trigonometric functions" to be the inverses of these restricted functions.

Inverse Trigonometric Functions

0.4.6 DEFINITION The ***inverse sine function***, denoted by $\sin^{-1}$, is defined to be the inverse of the restricted sine function

$$\sin x, \quad -\pi/2 \le x \le \pi/2$$



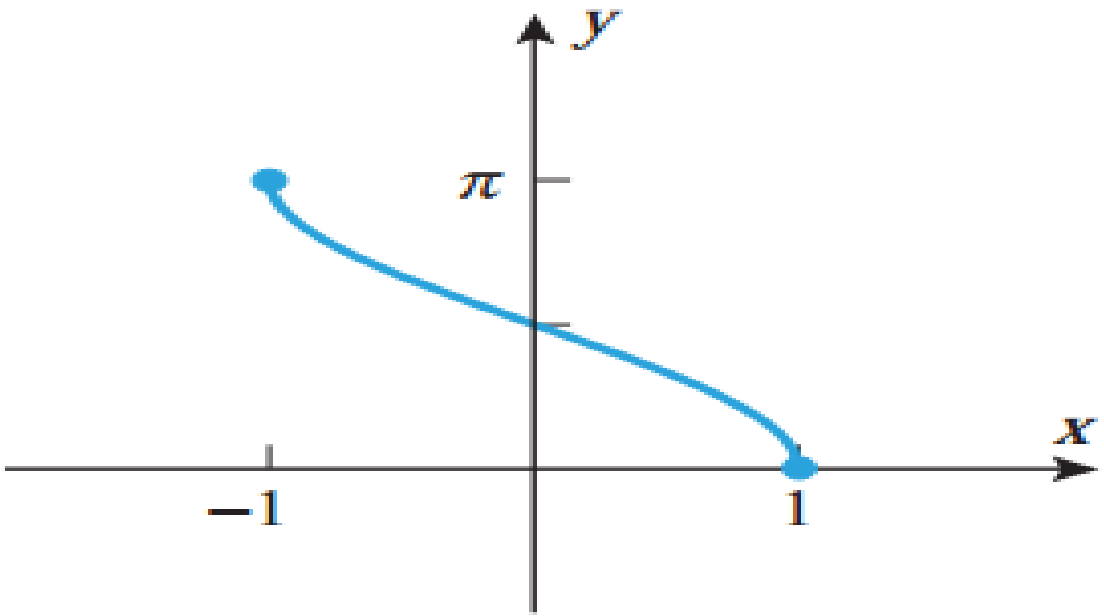
$y = \sin^{-1} x$

INVERSE FUNCTIONS

Inverse Trigonometric Functions

0.4.7 DEFINITION The ***inverse cosine function***, denoted by $\cos^{-1}$, is defined to be the inverse of the restricted cosine function

$$\cos x, \quad 0 \le x \le \pi$$



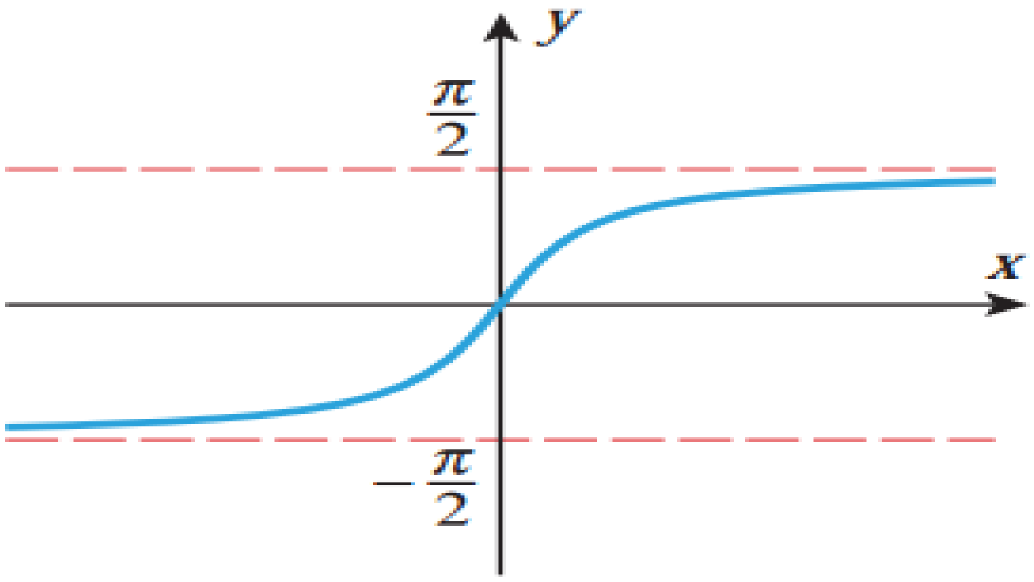
$y = \cos^{-1} x$

INVERSE FUNCTIONS

Inverse Trigonometric Functions

0.4.8 DEFINITION The ***inverse tangent function***, denoted by $\tan^{-1}$, is defined to be the inverse of the restricted tangent function

$$\tan x, \quad -\pi/2 < x < \pi/2$$



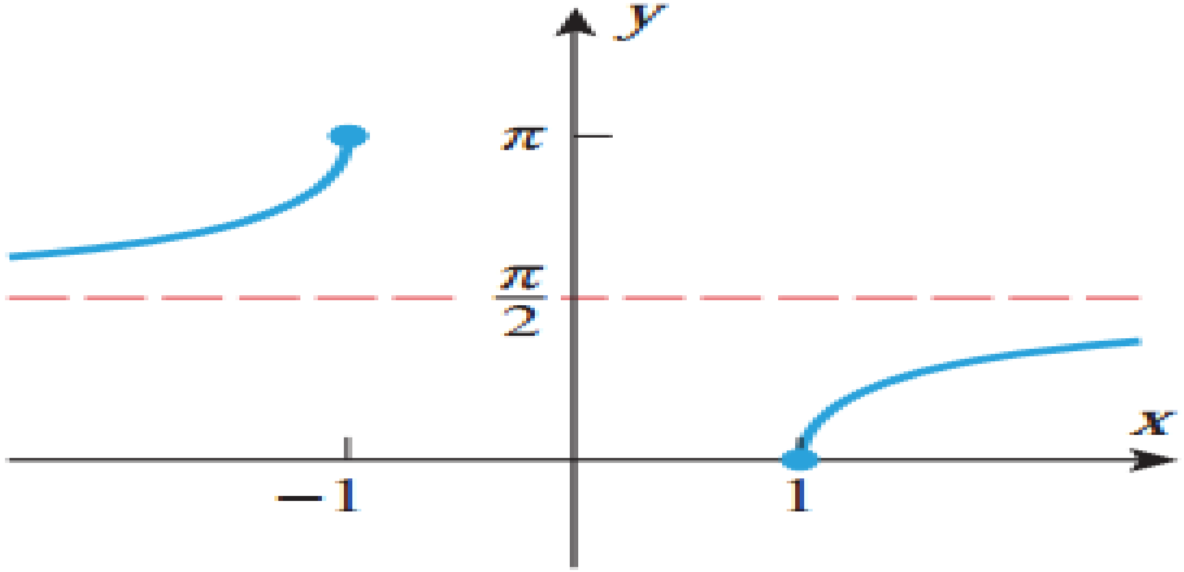
$y = \tan^{-1} x$

INVERSE FUNCTIONS

Inverse Trigonometric Functions

0.4.9 DEFINITION* The ***inverse secant function***, denoted by $\sec^{-1}$, is defined to be the inverse of the restricted secant function

$$\sec x, \quad 0 \le x \le \pi \text{ with } x \neq \pi/2$$



$y = \sec^{-1} x$

INVERSE FUNCTIONS

Inverse Trigonometric Functions

Properties of inverse trigonometric functions

FUNCTION	DOMAIN	RANGE	BASIC RELATIONSHIPS
$\sin^{-1}$	$[-1, 1]$	$[-\pi/2, \pi/2]$	$\sin^{-1}(\sin x) = x$ if $-\pi/2 \le x \le \pi/2$ $\sin(\sin^{-1} x) = x$ if $-1 \le x \le 1$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$	$\cos^{-1}(\cos x) = x$ if $0 \le x \le \pi$ $\cos(\cos^{-1} x) = x$ if $-1 \le x \le 1$
$\tan^{-1}$	$(-\infty, +\infty)$	$(-\pi/2, \pi/2)$	$\tan^{-1}(\tan x) = x$ if $-\pi/2 < x < \pi/2$ $\tan(\tan^{-1} x) = x$ if $-\infty < x < +\infty$
$\sec^{-1}$	$(-\infty, -1] \cup [1, +\infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$	$\sec^{-1}(\sec x) = x$ if $0 \le x \le \pi, x \ne \pi/2$ $\sec(\sec^{-1} x) = x$ if $\lvert x \rvert \ge 1$

Problems

EXERCISE SET

1. In (a)–(d), determine whether f and g are inverse functions.

(a) $f(x) = 4x,\ g(x) = \frac{1}{4}x$

(b) $f(x) = 3x + 1,\ g(x) = 3x - 1$

(c) $f(x) = \sqrt[3]{x - 2},\ g(x) = x^3 + 2$

(d) $f(x) = x^4,\ g(x) = \sqrt[4]{x}$

3. In each part, use the horizontal line test to determine whether the function f is one-to-one.

(a) $f(x) = 3x + 2$

(b) $f(x) = \sqrt{x - 1}$

(c) $f(x) = |x|$

(d) $f(x) = x^3$

(e) $f(x) = x^2 - 2x + 2$

(f) $f(x) = \sin x$

9–16 Find a formula for $f^{-1}(x)$. ■

9. $f(x) = 7x - 6$

10. $f(x) = \dfrac{x + 1}{x - 1}$

11. $f(x) = 3x^3 - 5$

12. $f(x) = \sqrt[5]{4x + 2}$

13. $f(x) = 3/x^2, \quad x < 0$

14. $f(x) = 5/(x^2 + 1), \quad x \geq 0$

15. $f(x) = \begin{cases} 5/2 - x, & x < 2 \\ 1/x, & x \geq 2 \end{cases}$

16. $f(x) = \begin{cases} 2x, & x \leq 0 \\ x^2, & x > 0 \end{cases}$

17–20 Find a formula for $f^{-1}(x)$, and state the domain of the function f^{-1}. ■

17. $f(x) = (x + 2)^4, \quad x \geq 0$

18. $f(x) = \sqrt{x + 3}$

19. $f(x) = -\sqrt{3 - 2x}$

20. $f(x) = x - 5x^2, \quad x \geq 1$

Exponential And Logarithmic Functions

If b is a nonzero real number, then nonzero *integer* powers of b are defined by

$$b^n = \underbrace{b \times b \times \cdots \times b}_{n \text{ factors}} \quad \text{and} \quad b^{-n} = \frac{1}{b^n}$$

If $\mathbf{n = 0}$, then $\boldsymbol{b^0 = 1}$.

Also, if $\frac{p}{q}$ is a positive rational number, then

$$b^{p/q} = \sqrt[q]{b^p} = \left(\sqrt[q]{b}\right)^p \quad \text{and} \quad b^{-p/q} = \frac{1}{b^{p/q}}$$

$$b^p b^q = b^{p+q}, \quad \frac{b^p}{b^q} = b^{p-q}, \quad \left(b^p\right)^q = b^{pq} \qquad \boldsymbol{y = b^{-x} = \frac{1}{b^x}}$$

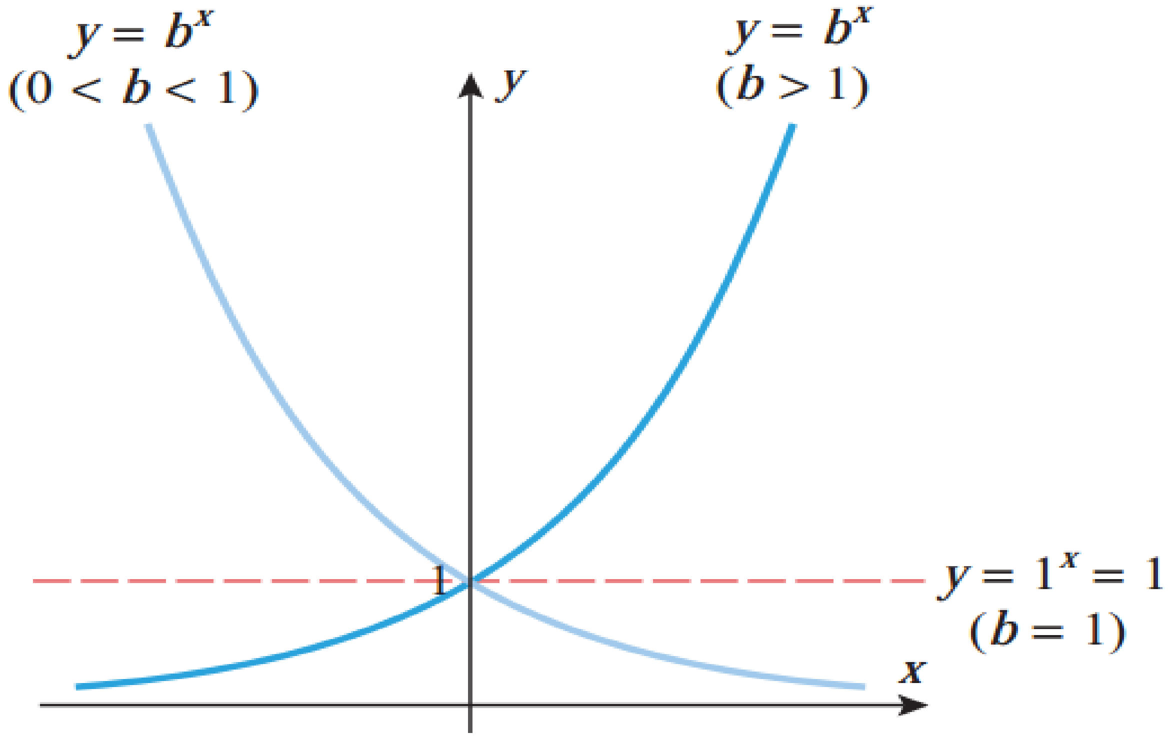
Exponential And Logarithmic Functions

The Family Of Exponential Functions

A function of the form $y = b^x$, where $b > 0$, is called an *exponential function with base b*.

Example: $f(x) = 2^x, \quad f(x) = \left(\frac{1}{2}\right)^x, \quad f(x) = \pi^x$

Note that an exponential function has a constant base and variable exponent.



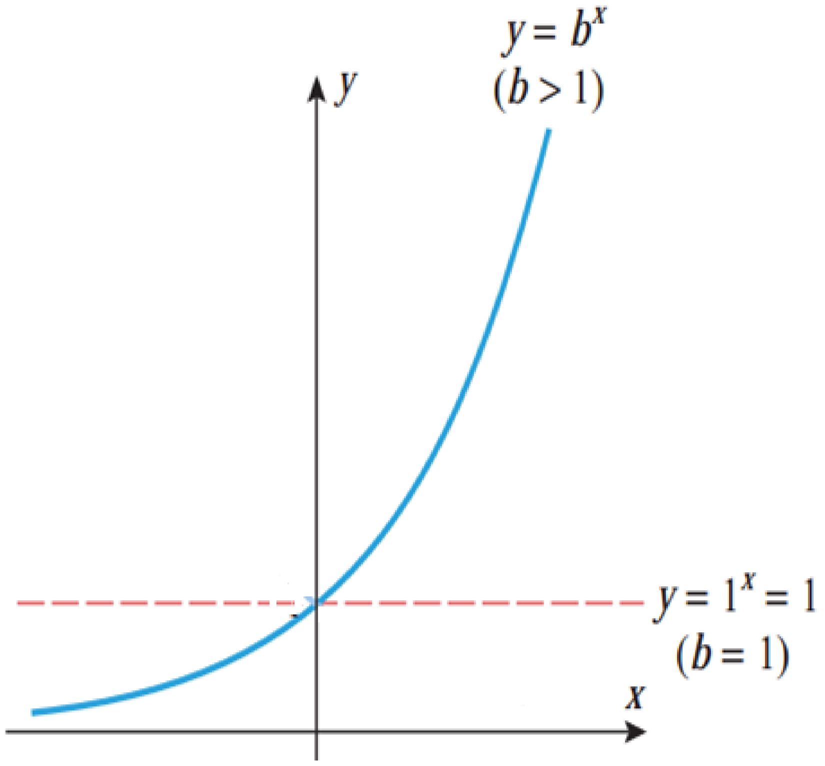
Exponential And Logarithmic Functions

The graph of $y = b^x$ has the following properties:

- The graph passes through (0, 1) because $b^0 = 1$.
- If $\mathbf{b > 1}$, the value of b^x increases as x increases.

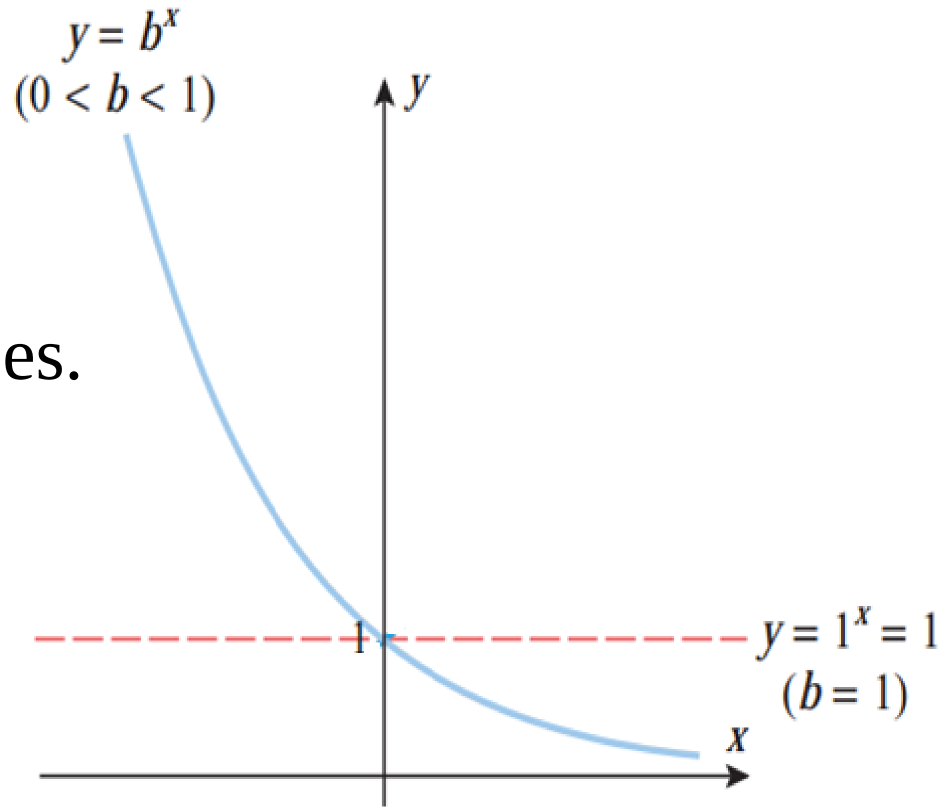
As you traverse the graph of $y = b^x$ from left to right, the values of b^x increase indefinitely.

If you traverse the graph from right to left, the values of b^x decrease toward zero but never reach zero.



Exponential And Logarithmic Functions

The graph of $y = b^x$ has the following properties:



- The graph passes through (0, 1) because $b^0 = 1$.
- If 0 < b < 1, the value of b^x decreases as x increases.

As you traverse the graph of $y = b^x$ from **left to right**, the values of b^x decrease toward zero but never reach zero.

If you traverse the graph from **right to left**, the values of b^x increase indefinitely.

- If b = 1, then the value of b^x is constant.

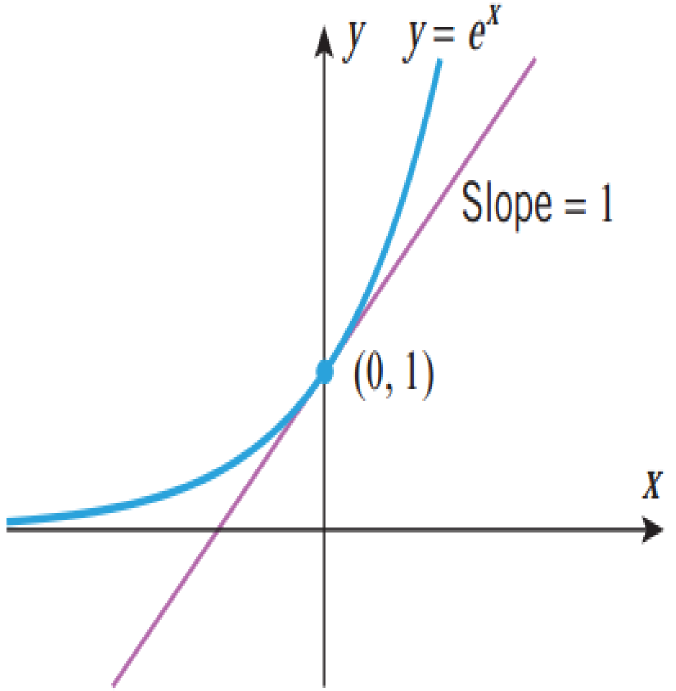
The Natural Exponential Function

Among all possible bases for exponential functions there is one particular base that plays a special role in calculus. That base, denoted by the letter ***e***, is a certain irrational number whose value to six decimal places is

$$e \approx 2.718282$$

- The function $y = e^x$ is called **the natural exponential function.**

- The tangent line to the graph of $y = e^x$ ***at*** $(0, 1)$ *has slope* 1.



The tangent line to the graph of $y = e^x$ at $(0, 1)$ has slope 1.

Logarithmic Functions

- A logarithm is an exponent.
- If $b > 0$ *and* $b \neq 1$, *then for a positive value of* x *the expression*

$$y = log_b x$$

(read "the logarithm to the base b of x") denotes that exponent to which b must be raised to produce x.

$$y = log_b x \quad \Rightarrow \quad b^y = x$$

Example :

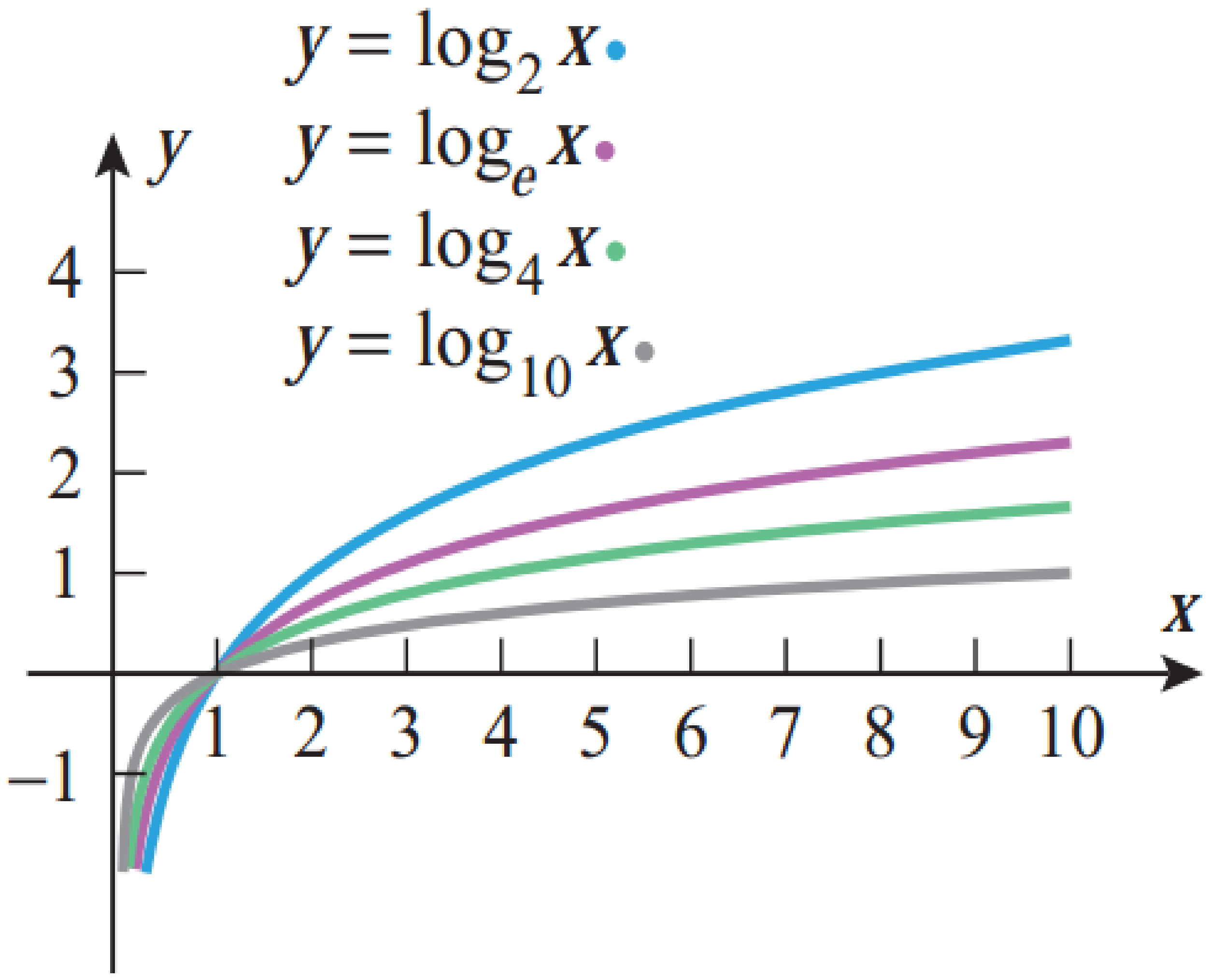
$\log_{10} 100 = 2$ $\log_{10}(1/1000) = -3$ $\log_2 16 = 4$ $\log_b 1 = 0$

$\boxed{10^2 = 100}$ $\boxed{10^{-3} = 1/1000}$ $\boxed{2^4 = 16}$ $\boxed{b^0 = 1}$

Logarithmic Functions

The graphs of $f(x) = log_b x$ for various values of b.

They all pass through the point (1,0).



Logarithmic Functions

Properties of logarithms :

$$(a) \quad \log_b(ac) = \log_b a + \log_b c$$ Product property

$$(b) \quad \log_b(a/c) = \log_b a - \log_b c$$ Quotient property

$$(c) \quad \log_b(a^r) = r \log_b a$$ Power property

$$(d) \quad \log_b(1/c) = -\log_b c$$ Reciprocal property

Example : Simplify the expression $\log \dfrac{xy^5}{\sqrt{z}}$

$$\log \frac{xy^5}{\sqrt{z}} = \log xy^5 - \log \sqrt{z} = \log x + \log y^5 - \log z^{1/2}$$
$$= \log x + 5 \log y - \tfrac{1}{2} \log z$$

Logarithmic Functions

$$f(x) = log_b x$$

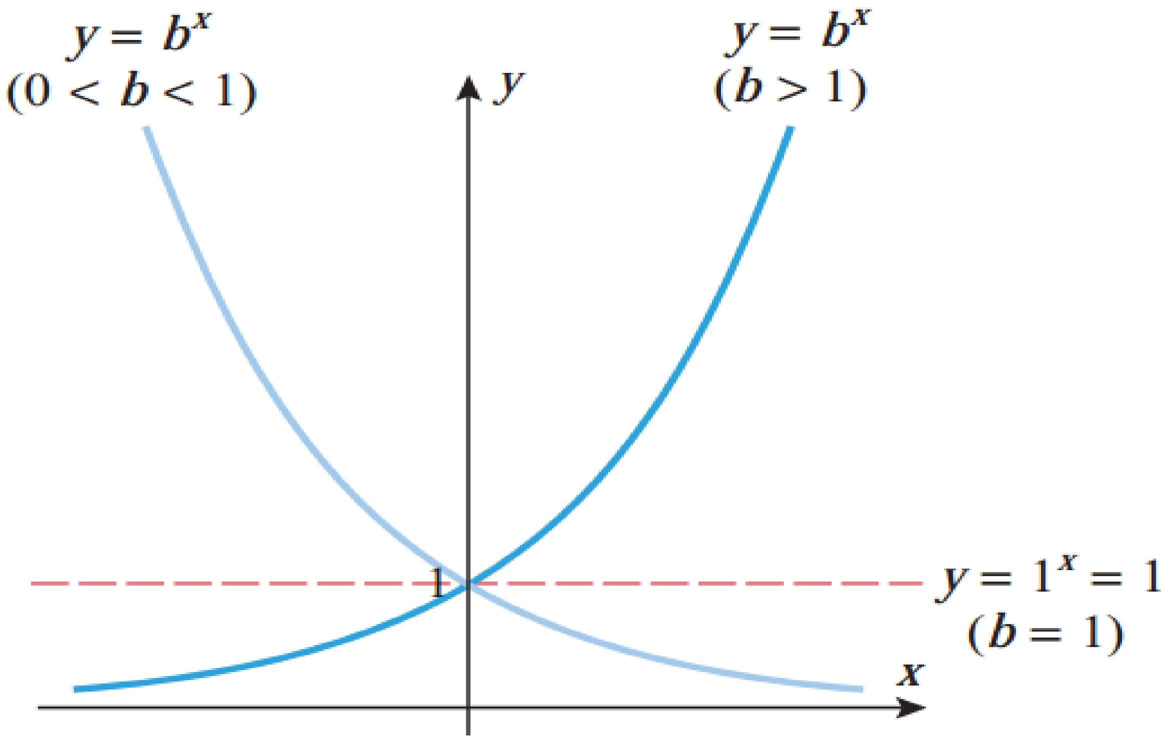
is called the *logarithmic function with base b.*

- Logarithmic functions can also be viewed as inverses of exponential functions.

$$Why\,?$$

If $b > 0$ *and* $b \neq 1$,

then the graph of $y = b^x$ passes the horizontal line test, so b^x has an inverse.



Logarithmic Functions

0.5.1 THEOREM *If $b > 0$ and $b \neq 1$, then b^x and $\log_b x$ are inverse functions.*

Proof:

This inverse with x as the independent variable by solving the equation.

$$x = b^y$$

Apply log_b on both sides

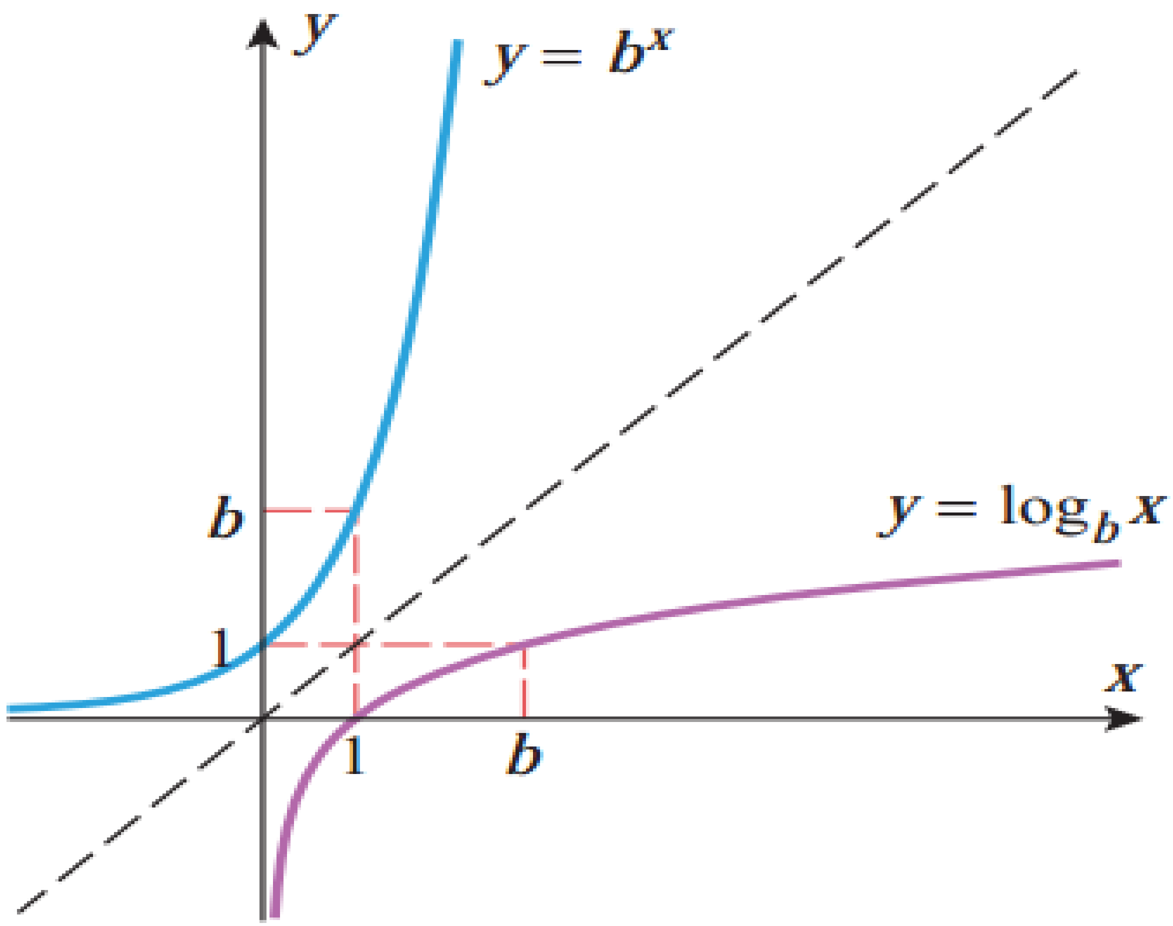
$$log_b\, x = log_b\, b^y. \quad \Rightarrow \quad log_b\, x = y\, log_b\, \text{b},$$

where $log_b\, \mathbf{b} = 1.$

$$\therefore y = log_b x$$

Logarithmic Functions

The graphs of $y = b^x$ and $y = log_b x$ are reflections of one another about the line $y = x$.



The Natural Logarithm

The function $\boldsymbol{log_e x}$ is the inverse of the natural exponential function $\boldsymbol{e^x}$.

$$\boldsymbol{\ln x = log_e x}$$

Example :

$\ln 1 = 0,$ $\boxed{\text{Since } e^0 = 1}$

$\ln e = 1,$ $\boxed{\text{Since } e^1 = e}$

$\ln 1/e = -1,$ $\boxed{\text{Since } e^{-1} = 1/e}$

$\ln(e^2) = 2$ $\boxed{\text{Since } e^2 = e^2}$

Logarithmic Functions

It also follows from the cancellation properties of inverse functions.

Example :

$$\log_b(b^x) = x \quad \text{for all real values of } x$$
$$b^{\log_b x} = x \quad \text{for } x > 0$$

In the special case where **b = e**, these equations become

$$\ln(e^x) = x \quad \text{for all real values of } x$$
$$e^{\ln x} = x \quad \text{for } x > 0$$

In words, the functions $\boldsymbol{b^x}$ and $\boldsymbol{\log_b x}$ cancel out the effect of one another when composed in either order.

Example :

$$\log 10^x = x,\ \ 10^{\log x} = x,\ \ \ln e^x = x,\ \ e^{\ln x} = x,\ \ \ln e^5 = 5,$$

Logarithmic Functions

The Natural Logarithm

Example : Prove that $y = \ln x,$ if and only if $x = e^y.$

Solution:

$x = e^y.$ **Apply *ln* on both sides**

$\ln x = \ln e^y.$

$\ln x = y \ln e \Rightarrow \ln e = 1$

$\therefore \ln x = y$

$y = \ln x,$ **take exponent of both sides**

$e^y = e^{\ln x} \Rightarrow e^{\ln x} = x$

$\therefore e^y = x$

Logarithmic Functions

Example : **Solve** $\dfrac{e^x - e^{-x}}{2} = 1$ for x.

Solution: Multiplying both sides of the given equation by 2 yields

$$e^x - e^{-x} = 2 \Rightarrow e^x - \frac{1}{e^x} = 2$$

Multiplying through by e^x yields $\Rightarrow e^x e^x - \dfrac{e^x}{e^x} = 2e^x \Rightarrow e^{2x} - 1 = 2e^x$

$$e^{2x} - 2e^x - 1 = 0 \Rightarrow (e^x)^2 - 2e^x - 1 = 0$$

let $u = e^x \Rightarrow u^2 - 2u - 1 = 0 \Rightarrow$ quadratic formula yields

$$u = \frac{2 \pm \sqrt{4+4}}{2} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2} \Rightarrow \text{since } u = e^x \Rightarrow e^x = 1 \pm \sqrt{2}$$

But e^x cannot be negative, so we discard the negative value $1 - \sqrt{2}$; thus, $\Rightarrow e^x = 1 + \sqrt{2}$

$$\ln e^x = \ln(1 + \sqrt{2}) \Rightarrow x = \ln(1 + \sqrt{2}) \approx 0.881$$

Limits and Continuity

The concept of a "limit" is the fundamental building block on which all calculus concepts are based.

1.1.1 LIMITS (AN INFORMAL VIEW) If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but not equal to a), then we write

$$\lim_{x \to a} f(x) = L \tag{6}$$

which is read "the limit of $f(x)$ as x approaches a is L" or "$f(x)$ approaches L as x approaches a." The expression in (6) can also be written as

$$f(x) \to L \quad \text{as} \quad x \to a \tag{7}$$

The limit describes the behavior of f close to a but not at a.

Limits and Continuity

1.1.2 ONE-SIDED LIMITS (AN INFORMAL VIEW) If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but greater than a), then we write

$$\lim_{x \to a^+} f(x) = L \tag{14}$$

and if the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but less than a), then we write

$$\lim_{x \to a^-} f(x) = L \tag{15}$$

Expression (14) is read "the limit of $f(x)$ as x approaches a from the right is L" or "$f(x)$ approaches L as x approaches a from the right." Similarly, expression (15) is read "the limit of $f(x)$ as x approaches a from the left is L" or "$f(x)$ approaches L as x approaches a from the left."

The one-sided limits in (14) and (15) can also be written as

$f(x) \to L$ as $x \to a^+$ and $f(x) \to L$ as $x \to a^-$ respectively.

Limits and Continuity

1.1.3 THE RELATIONSHIP BETWEEN ONE-SIDED AND TWO-SIDED LIMITS The two-sided limit of a function $f(x)$ exists at a if and only if both of the one-sided limits exist at a and have the same value; that is,

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

Example: Explain why $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

Solution

$$f(x) = \frac{|x|}{x} = \begin{cases} \frac{x}{x}, & x > 0 \\ \frac{-(-x)}{-x}, & x < 0 \end{cases} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

$$\lim_{x \to 0^+} \frac{|x|}{x} = 1 \quad \text{and} \quad \lim_{x \to 0^-} \frac{|x|}{x} = -1$$

Thus, the one-sided limits at 0 are not the same.

Infinite Limits

1.1.4 INFINITE LIMITS (AN INFORMAL VIEW) The expressions

$$\lim_{x\to a^-} f(x) = +\infty \quad \text{and} \quad \lim_{x\to a^+} f(x) = +\infty$$

denote that $f(x)$ increases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\lim_{x\to a} f(x) = +\infty$$

Similarly, the expressions

$$\lim_{x\to a^-} f(x) = -\infty \quad \text{and} \quad \lim_{x\to a^+} f(x) = -\infty$$

denote that $f(x)$ decreases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\lim_{x\to a} f(x) = -\infty$$

Infinite Limits

1.1.4 INFINITE LIMITS (AN INFORMAL VIEW) The expressions

$$\lim_{x \to a^-} f(x) = +\infty \quad \text{and} \quad \lim_{x \to a^+} f(x) = +\infty$$

denote that $f(x)$ increases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\lim_{x \to a} f(x) = +\infty$$

Similarly, the expressions

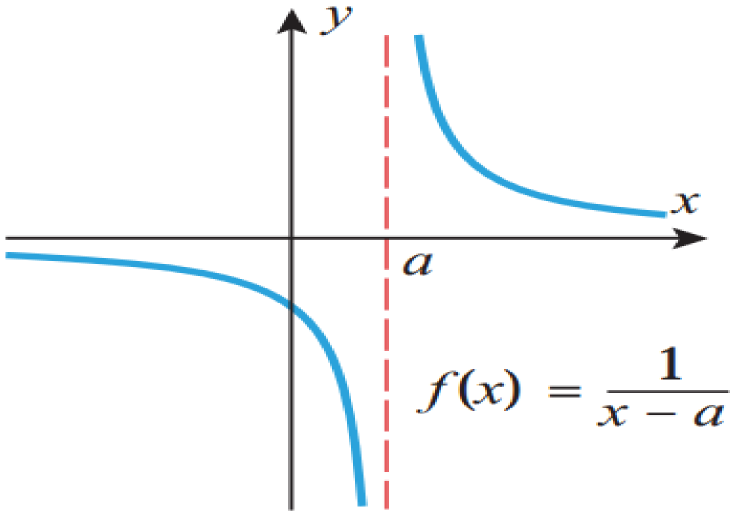
$$\lim_{x \to a^-} f(x) = -\infty \quad \text{and} \quad \lim_{x \to a^+} f(x) = -\infty$$

denote that $f(x)$ decreases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\lim_{x \to a} f(x) = -\infty$$

Infinite Limits

Example: Describe the limits at $x = a$ in appropriate limit notation.



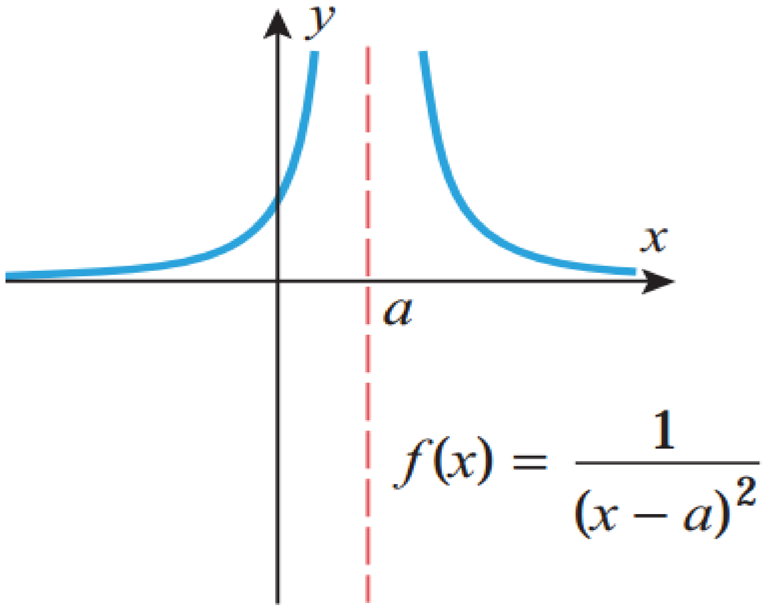
Solution:

$$\lim_{x \to a^+} \frac{1}{x-a} = +\infty \quad \text{and} \quad \lim_{x \to a^-} \frac{1}{x-a} = -\infty$$

The function increases without bound as x approaches a from the right and decreases without bound as x approaches a from the left.

Infinite Limits

Example: Describe the limits at $x = a$ in appropriate limit notation.



Solution

$$\lim_{x \to a^+} \frac{1}{(x-a)^2} = +\infty \quad \text{and} \quad \lim_{x \to a^-} \frac{1}{(x-a)^2} = +\infty$$

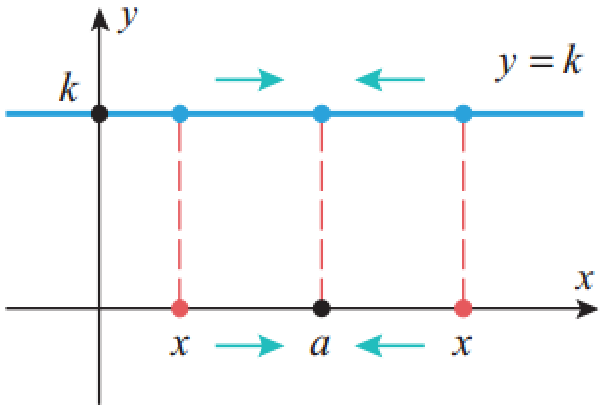
$$\lim_{x \to a} \frac{1}{(x-a)^2} = \lim_{x \to a^+} \frac{1}{(x-a)^2} = \lim_{x \to a^-} \frac{1}{(x-a)^2} = +\infty$$

The function increases without bound as x approaches a from both the right and the left.

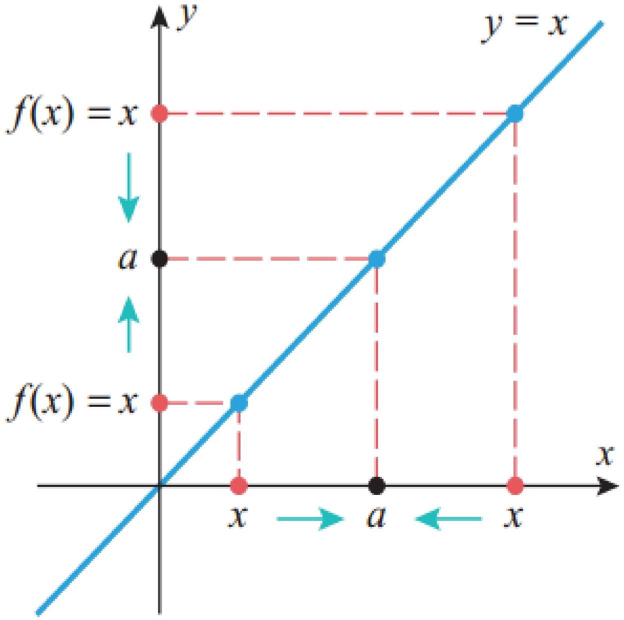
Computing Limits

1.2.1 THEOREM *Let a and k be real numbers.*

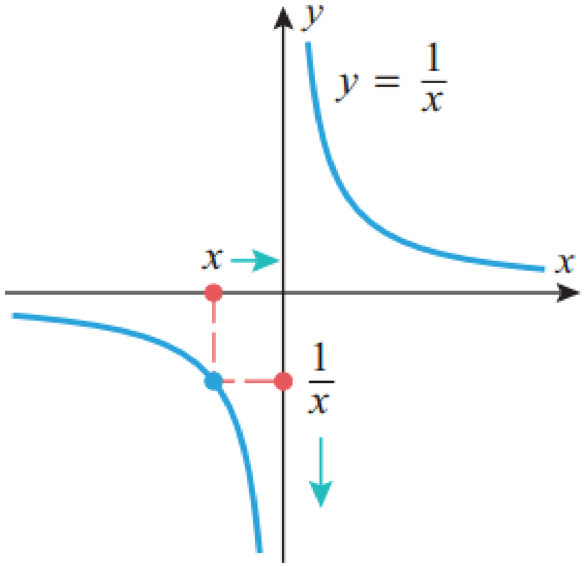
(a) $\lim_{x \to a} k = k$ (b) $\lim_{x \to a} x = a$ (c) $\lim_{x \to 0^-} \dfrac{1}{x} = -\infty$ (d) $\lim_{x \to 0^+} \dfrac{1}{x} = +\infty$



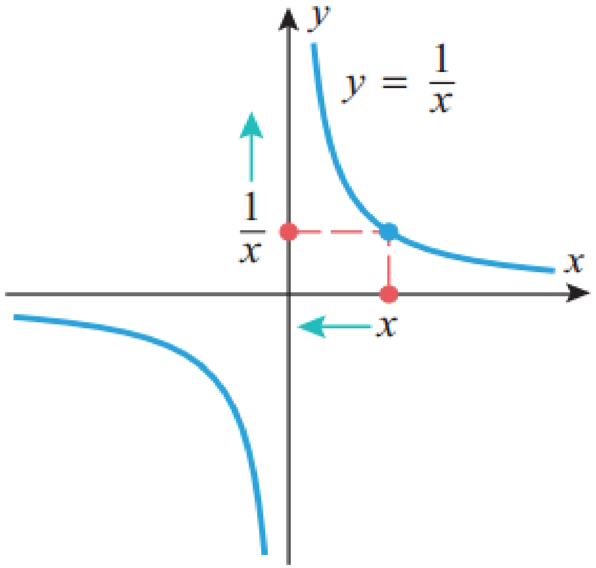
$\lim_{x \to a} k = k$



$\lim_{x \to a} x = a$



$\lim_{x \to 0^-} \dfrac{1}{x} = -\infty$



$\lim_{x \to 0^+} \dfrac{1}{x} = +\infty$

Computing Limits

THEOREM *Let a be a real number, and suppose that*

$$\lim_{x \to a} f(x) = L_1 \quad \textit{and} \quad \lim_{x \to a} g(x) = L_2$$

$$(a) \quad \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L_1 + L_2$$

$$(b) \quad \lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L_1 - L_2$$

$$(c) \quad \lim_{x \to a} [f(x)g(x)] = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right) = L_1 L_2$$

$$(d) \quad \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L_1}{L_2}, \quad \textit{provided } L_2 \neq 0$$

$$(e) \quad \lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L_1}, \quad \textit{provided } L_1 > 0 \textit{ if } n \textit{ is even.}$$

Computing Limits

Example: Find $\lim_{x \to 5} (x^2 - 4x + 3)$.

Solution:

$$\lim_{x \to 5} (x^2 - 4x + 3) = \lim_{x \to 5} x^2 - \lim_{x \to 5} 4x + \lim_{x \to 5} 3$$

$$= \lim_{x \to 5} x^2 - 4 \lim_{x \to 5} x + \lim_{x \to 5} 3 \;= 5^2 - 4(5) + 3 = 8$$

Example: Find $\lim_{x \to 2} \dfrac{5x^3 + 4}{x - 3}$.

Solution:

$$\lim_{x \to 2} \frac{5x^3 + 4}{x - 3} = \frac{\lim_{x \to 2} (5x^3 + 4)}{\lim_{x \to 2} (x - 3)} = \frac{5 \cdot 2^3 + 4}{2 - 3} = -44$$

Computing Limits

Example: Find $\lim_{x \to 3} \dfrac{x^2 - 6x + 9}{x - 3}$

Solution:

$$\lim_{x \to 3} \frac{x^2 - 6x + 9}{x - 3} = \lim_{x \to 3} \frac{(x-3)^2}{x - 3} = \lim_{x \to 3} \frac{(x-3)(x-3)}{x - 3} = \lim_{x \to 3} (x - 3) = 0$$

Example: Find $\lim_{x \to -4} \dfrac{2x + 8}{x^2 + x - 12}$

Solution:

$$\lim_{x \to -4} \frac{2x + 8}{x^2 + x - 12} = \lim_{x \to -4} \frac{2(x+4)}{(x+4)(x-3)} = \lim_{x \to -4} \frac{2}{x - 3} = -\frac{2}{7}$$

Limits and Continuity

Example: Find $\lim\limits_{x \to 1} \dfrac{x-1}{\sqrt{x}-1}$

Solution: $\lim\limits_{x \to 1} \dfrac{x-1}{\sqrt{x}-1} = \lim\limits_{x \to 1} \dfrac{(x-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)} = \lim\limits_{x \to 1} \dfrac{(x-1)(\sqrt{x}+1)}{x-1}$

$$= \lim_{x \to 1} \sqrt{x}+1 = 2$$

Example: Find $\lim\limits_{x \to +\infty} \dfrac{3x+5}{6x-8}$.

Solution: $\lim\limits_{x \to +\infty} \dfrac{3x+5}{6x-8}$ Divide each term by x. $= \lim\limits_{x \to +\infty} \dfrac{3+\frac{5}{x}}{6-\frac{8}{x}}$

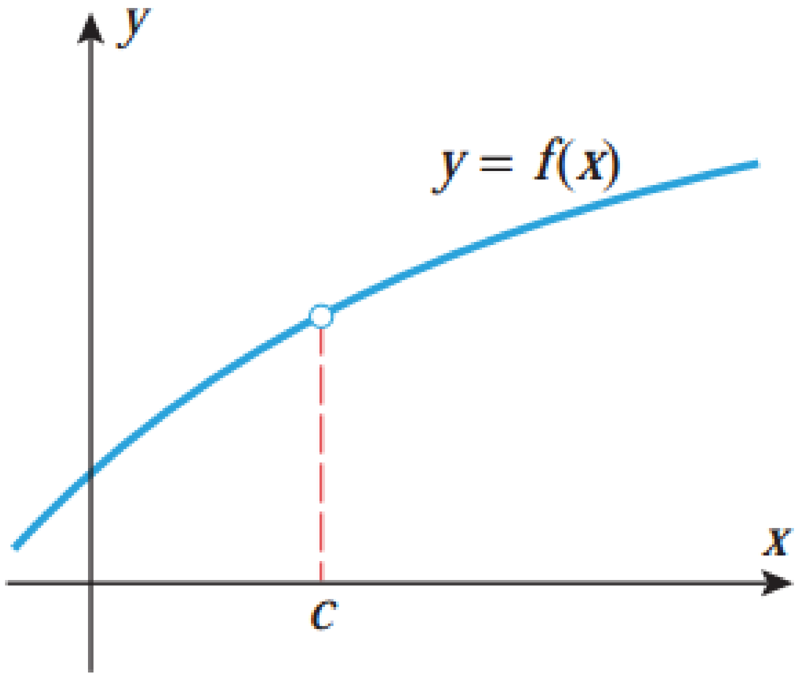
$$= \frac{\lim\limits_{x \to +\infty} \left(3+\frac{5}{x}\right)}{\lim\limits_{x \to +\infty} \left(6-\frac{8}{x}\right)} = \frac{\lim\limits_{x \to +\infty} 3 + \lim\limits_{x \to +\infty} \frac{5}{x}}{\lim\limits_{x \to +\infty} 6 - \lim\limits_{x \to +\infty} \frac{8}{x}} = \frac{3+5\lim\limits_{x \to +\infty} \frac{1}{x}}{6-8\lim\limits_{x \to +\infty} \frac{1}{x}} = \frac{3+0}{6-0} = \frac{1}{2}$$

Continuity:

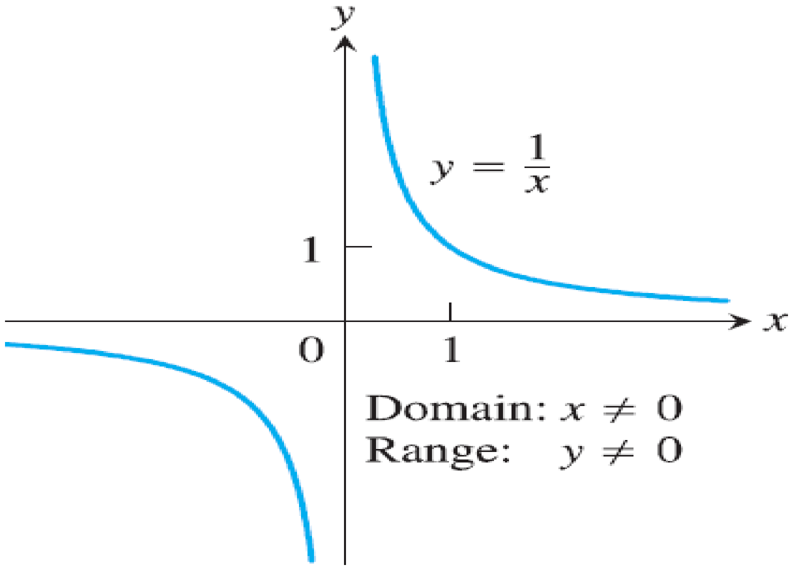
The graph of a function can be described as a "continuous curve" if it has no breaks or holes.

The graph of a function has a break or hole if any of the following conditions occur:

1- The function f is undefined at c.



Example: $f(x) = \dfrac{1}{x}$

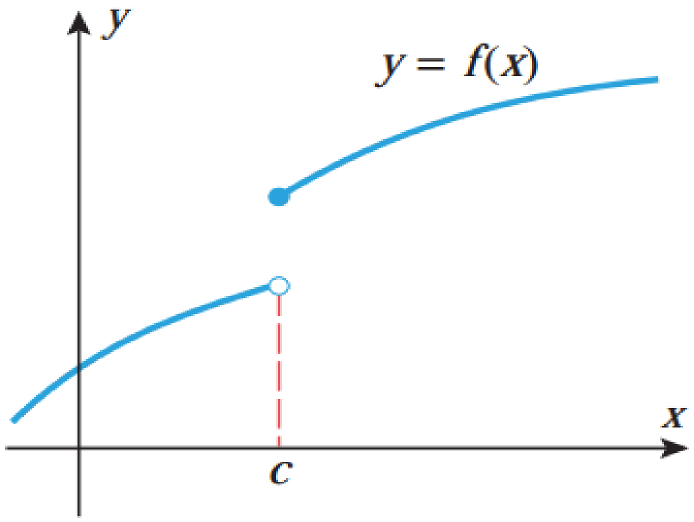


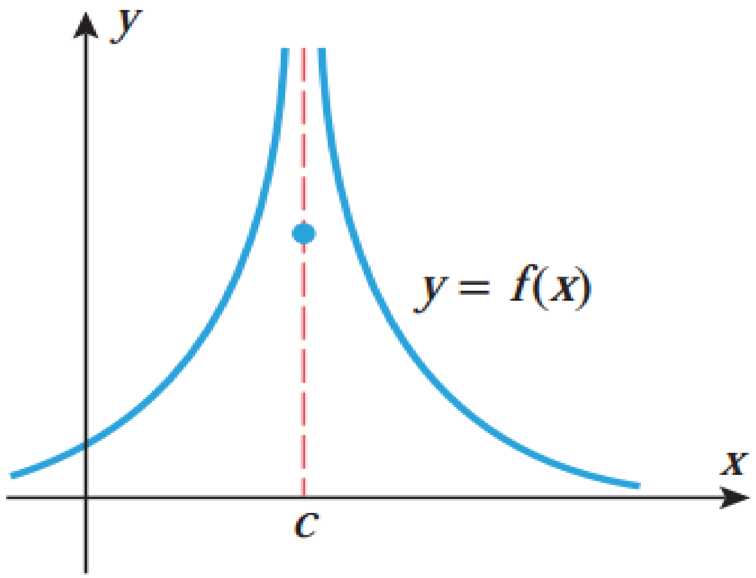
Continuity:

The graph of a function can be described as a "continuous curve" if it has no breaks or holes.

The graph of a function has a break or hole if any of the following conditions occur:

2-The limit of f(x) does not exist as x approaches c.





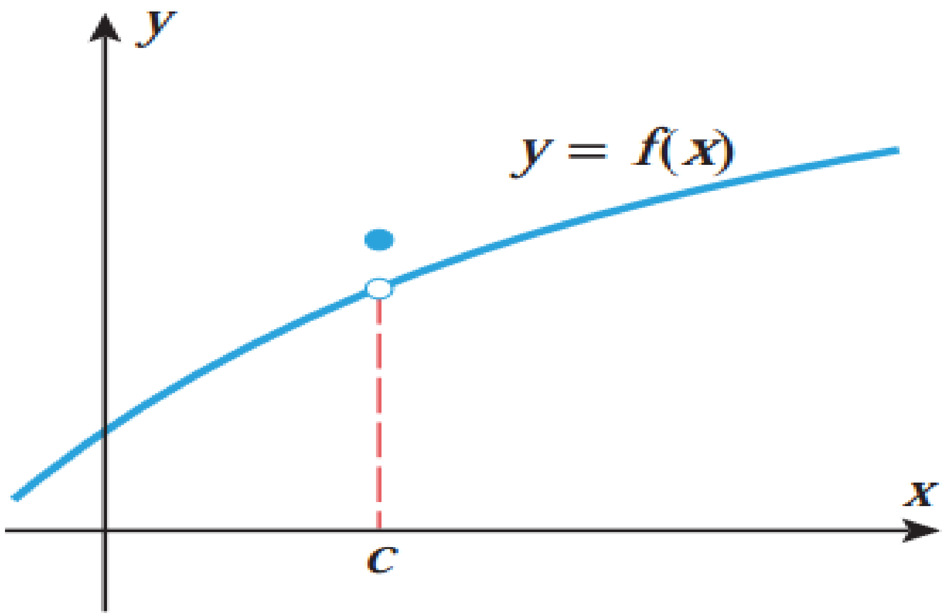
Limits and Continuity

Continuity:

The graph of a function can be described as a “continuous curve” if it has no breaks or holes.

The graph of a function has a break or hole if any of the following conditions occur:

3- The value of the function and the value of the limit at c are different.



Continuity:

1.5.1 DEFINITION A function f is said to be ***continuous at*** $x = c$ provided the following conditions are satisfied:

1. $f(c)$ is defined.

2. $\lim_{x \to c} f(x)$ exists.

3. $\lim_{x \to c} f(x) = f(c)$.

If one or more of the conditions of this definition fails to hold, then we will say that f has a ***discontinuity at*** $x = c$.

Continuity:

Example: Determine whether the following function is continuous at $x = 2$.

$$f(x) = \frac{x^2 - 4}{x - 2}$$

Solution:

we must check whether the limit of the function as $x \to 2$ is the same as the value of the function at $x = 2$.

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{x - 2} = \lim_{x \to 2} (x + 2) = 4$$

At $x =$ 2. $\Rightarrow$ $f(x) = \dfrac{x^2 - 4}{x - 2}$ $\Rightarrow$ $f(x) = \infty$

$\therefore$ The function f is undefined at $x = 2$, and hence is not continuous at $x = 2$.

Continuity on an interval

1.5.2 DEFINITION A function f is said to be ***continuous on a closed interval*** $[a, b]$ if the following conditions are satisfied:

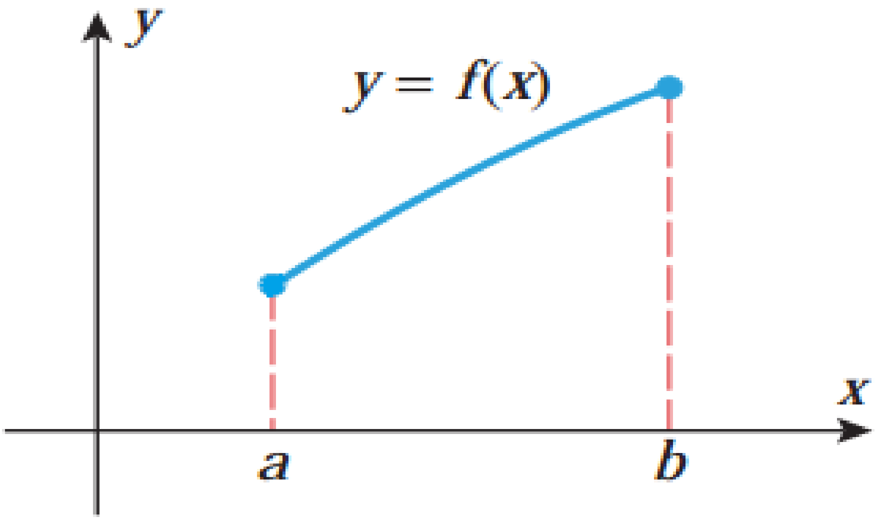
1. f is continuous on (a, b).
2. f is continuous from the right at a.
3. f is continuous from the left at b.

The function is continuous at the left endpoint because

$$\lim_{x \to a^+} f(x) = f(a)$$

The function is continuous at the right endpoint because

$$\lim_{x \to b^-} f(x) = f(b)$$



Continuity on an interval

Example: What can you say about the continuity of the function $f(x) = \sqrt{9 - x^2}$?

Solution: The domain of this function is $[-3, 3]$.

The continuity of *f will be investigated* on the open interval $(-3, 3)$ and at the two endpoints. If c is any point in the interval $(-3, 3)$, then

$$\lim_{x \to c} f(x) = \lim_{x \to c} \sqrt{9 - x^2} = \sqrt{\lim_{x \to c} (9 - x^2)} = \sqrt{9 - c^2} = f(c)$$

$\therefore$ f is continuous at each point in the interval $(-3, 3)$.

At the endpoints: $$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \sqrt{9 - x^2} = \sqrt{\lim_{x \to 3^-} (9 - x^2)} = 0 = f(3)$$

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} \sqrt{9 - x^2} = \sqrt{\lim_{x \to -3^+} (9 - x^2)} = 0 = f(-3)$$

$\therefore$ f is continuous on the closed interval $[-3,3]$.

Continuity of polynomials and rational functions

1.5.4 THEOREM

(*a*) *A polynomial is continuous everywhere.*

(*b*) *A rational function is continuous at every point where the denominator is nonzero, and has discontinuities at the points where the denominator is zero.*

The function is continuous everywhere when it is continuous at an *arbitrary* point.

If p(x) is a polynomial and **a** is any real number, then

$$\lim_{x \to a} p(x) = p(a)$$

This shows that polynomials are continuous everywhere.

Continuity of polynomials and rational functions

Example: For what values of x is there a discontinuity in the graph of

$$y = \frac{x^2 - 9}{x^2 - 5x + 6}?$$

Solution:

The function is continuous at every number where the denominator is nonzero.

$$x^2 - 5x + 6 = 0$$

$$(x - 2)(x - 3) = 0$$

yields discontinuities at $x = 2$ and at $x = 3$.

Some properties of continuous functions

1.5.3 THEOREM *If the functions f and g are continuous at c, then*

(a) $f + g$ *is continuous at* c.

(b) $f - g$ *is continuous at* c.

(c) fg *is continuous at* c.

(d) f/g *is continuous at* c *if* $g(c) \neq 0$ *and has a discontinuity at* c *if* $g(c) = 0$.

The absolute value of a continuous function is continuous.

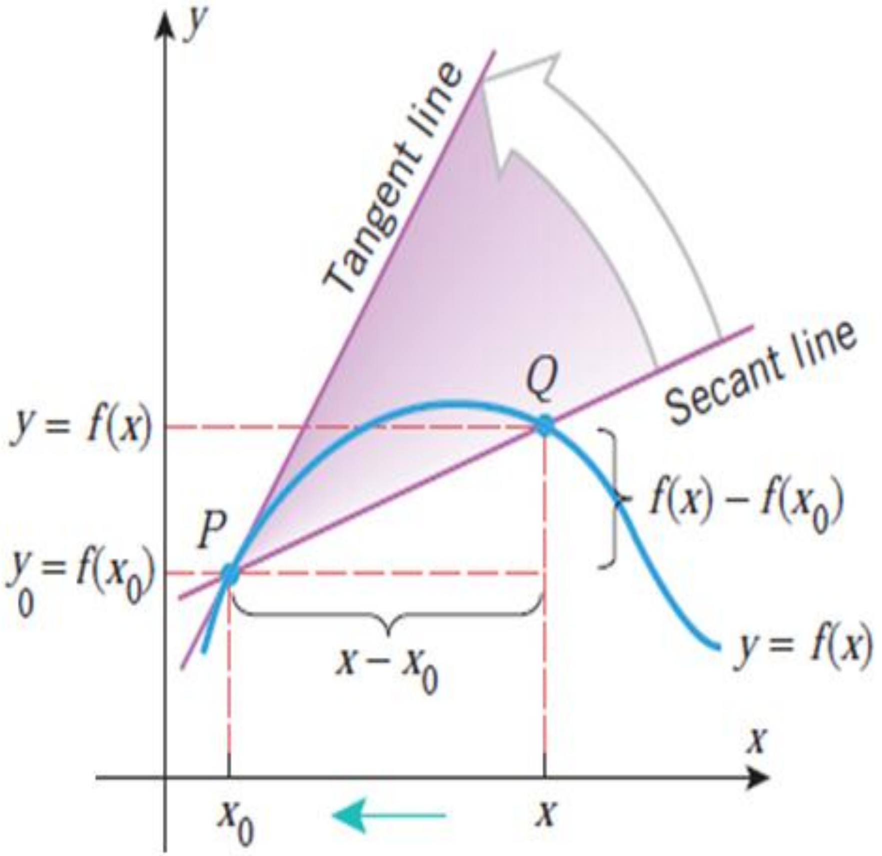
Rates of change

Tangent Lines:

What is the mathematical definition of the tangent line to a curve $y = f(x)$ at a point $P(x0, f(x0))$ on the curve?

Consider a point $Q(x, f(x))$ on the curve that is distinct from P, and compute the slope m_{PQ} of the secant line through P and Q:

$$m_{PQ} = \frac{f(x) - f(x_0)}{x - x_0}$$



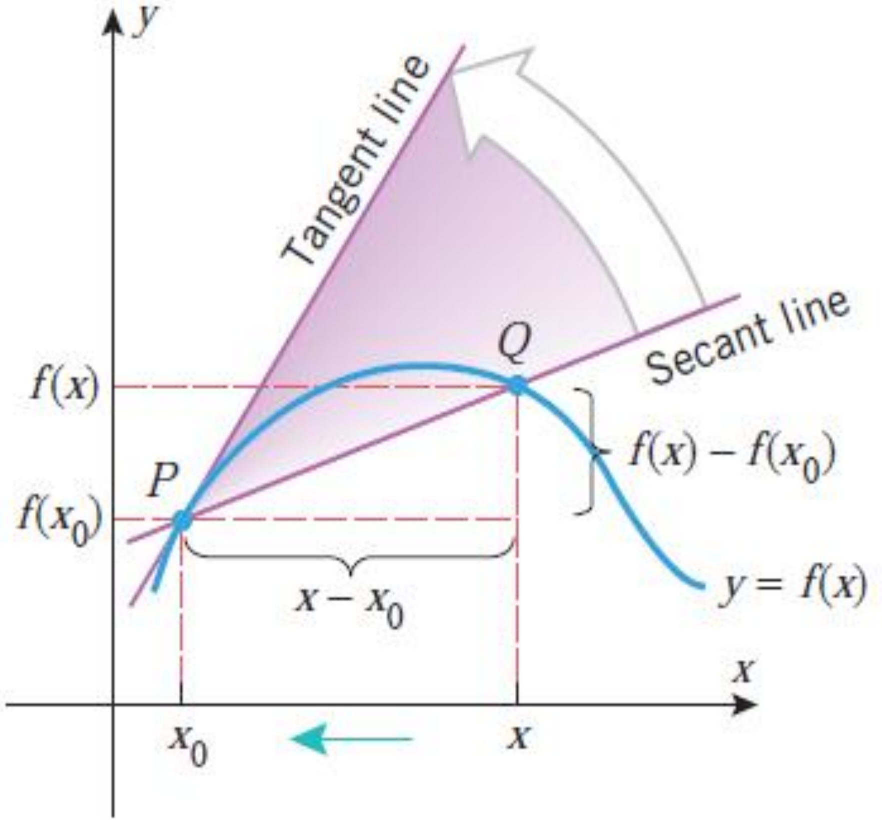
If we let x approach x_0, then the point Q will move along the curve and approach the point P.

Rates of change

Tangent Lines:

If the secant line through ***P*** and ***Q*** approaches a limiting position as $x \to x_0$, then we will regard that position to be the position of the tangent line at ***P***.

So, if the slope m_{PQ} of the secant line through P and Q approaches a limit as $x \to x_0$, then we regard that limit to be the slope m_{tan} of the tangent line at P. Thus, we make the following definition.



Thus, we make the following definition.

Rates of change

Tangent Lines:

2.1.1 DEFINITION Suppose that x_0 is in the domain of the function f. The ***tangent line*** to the curve $y = f(x)$ at the point $P(x_0, f(x_0))$ is the line with equation

$$y - f(x_0) = m_{\tan}(x - x_0)$$

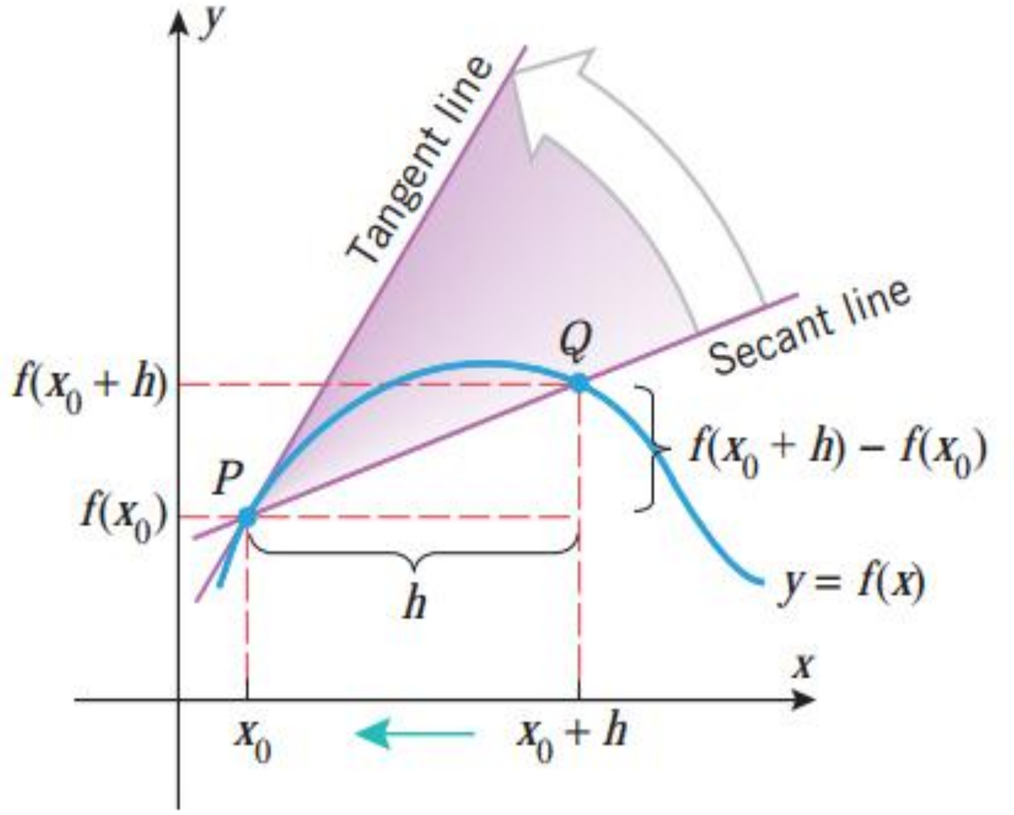
where

$$m_{\tan} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{1}$$

provided the limit exists. For simplicity, we will also call this the tangent line to $y = f(x)$ at x_0.

Tangent Lines

$$m_{\tan} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$



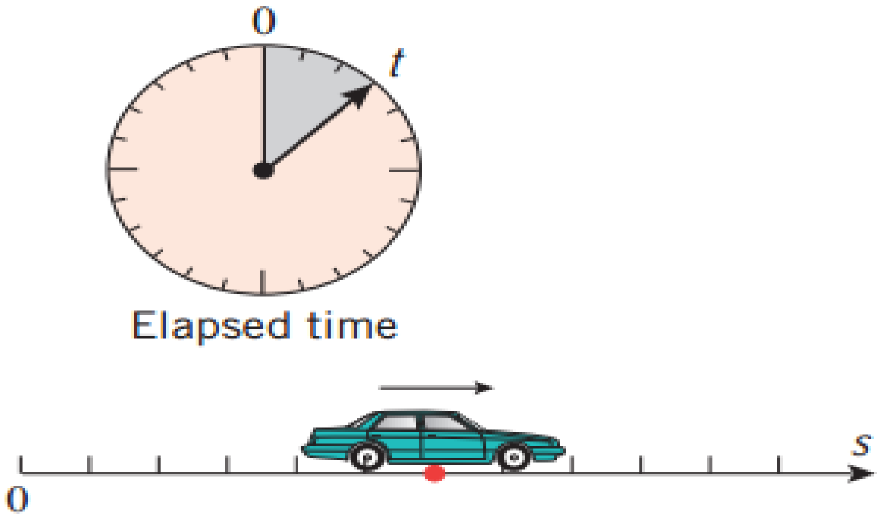
Let h denote the difference

$$h = x - x_0 \;\Rightarrow\; x = x_0 + h \;\Rightarrow\; x \to x_0 \equiv h \to 0$$

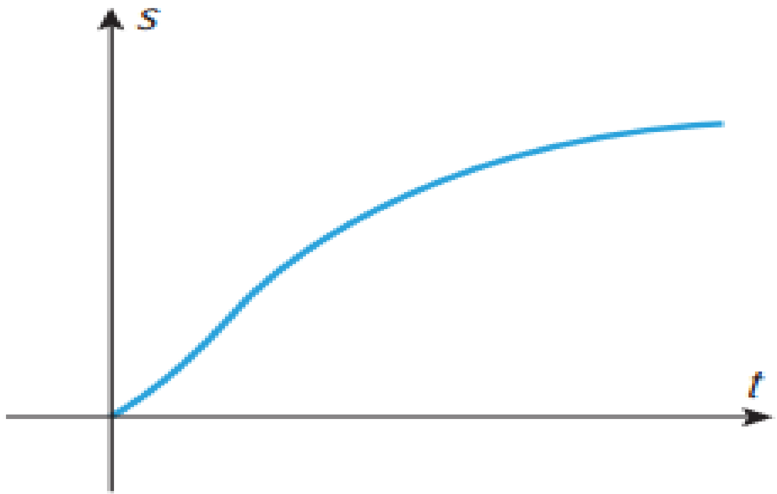
$$m_{\tan} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

This equation expresses the slope of the tangent line as a limit of slopes of secant lines.

Rates of change



Car moves only in the positive direction.



Position versus time curve

The *average velocity* over a time interval $[t_0, t_0 + h]$, $h > 0$, is defined to be:

$$v_{\text{ave}} = \frac{\text{change in position}}{\text{time elapsed}} = \frac{f(t_0 + h) - f(t_0)}{h}$$

Velocity can be viewed as ***rate of change***—the rate of change of position with respect to time.

Rates of change

Slopes and Rates of Change

Rates of change occur in other applications.

For example:

- A microbiologist might be interested in the rate at which the number of bacteria in a colony changes with time.
- A medical researcher might be interested in the rate at which the radius of an artery changes with the concentration of alcohol in the bloodstream.
- An economist might be interested in the rate at which production cost changes with the quantity of a product that is manufactured.
- An engineer might be interested in the rate at which the length of a metal rod changes with temperature.

Rates of change

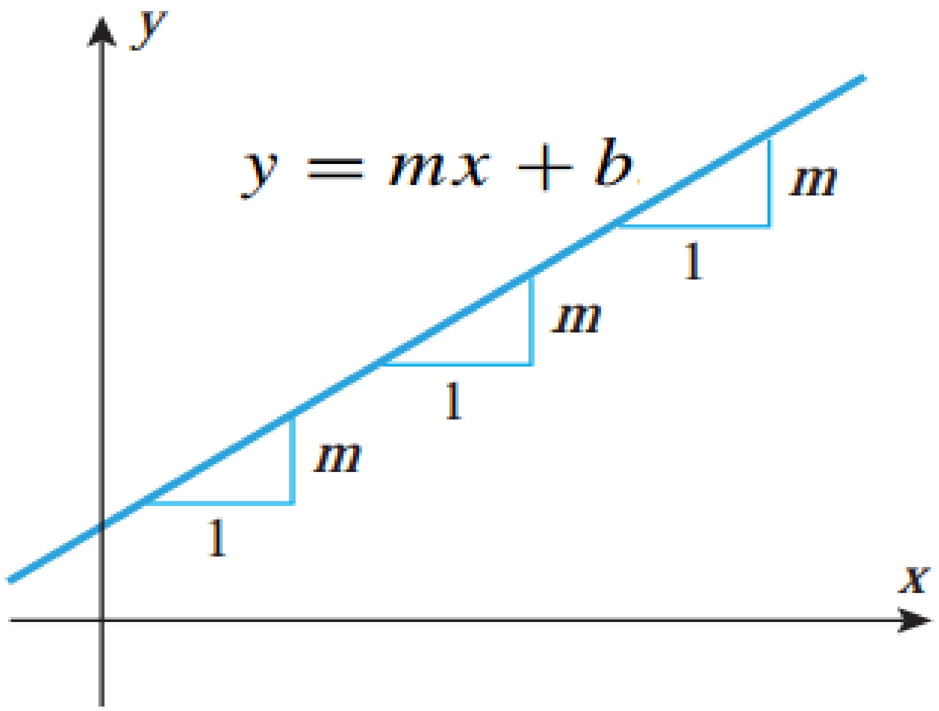
Slopes and Rates of Change

If y is a linear function $y = mx + b$, the slope m is the natural measure of the rate of change of y with respect to x.

In this Figure:

Each 1-unit increase in x anywhere along the line produces an m-unit change in y.

y changes at a constant rate with respect to x along the line and that m measures this rate of change.



A 1-unit increase in x always produces an m-unit change in y.

Slopes and Rates of Change

Example: Find the rate of change of y with respect to x if

(a) $y = 2x - 1$ (b) $y = -5x + 1$

Solution:

(a) The rate of change of $\boldsymbol{y}$ with respect to $\boldsymbol{x}$ is $\boldsymbol{m} = \mathbf{2}$.

So, each 1-unit increase in x produces a 2-unit increase in y.

(b) The rate of change of $\boldsymbol{y}$ with respect to $\boldsymbol{x}$ is $\boldsymbol{m} = \mathbf{-5}$.

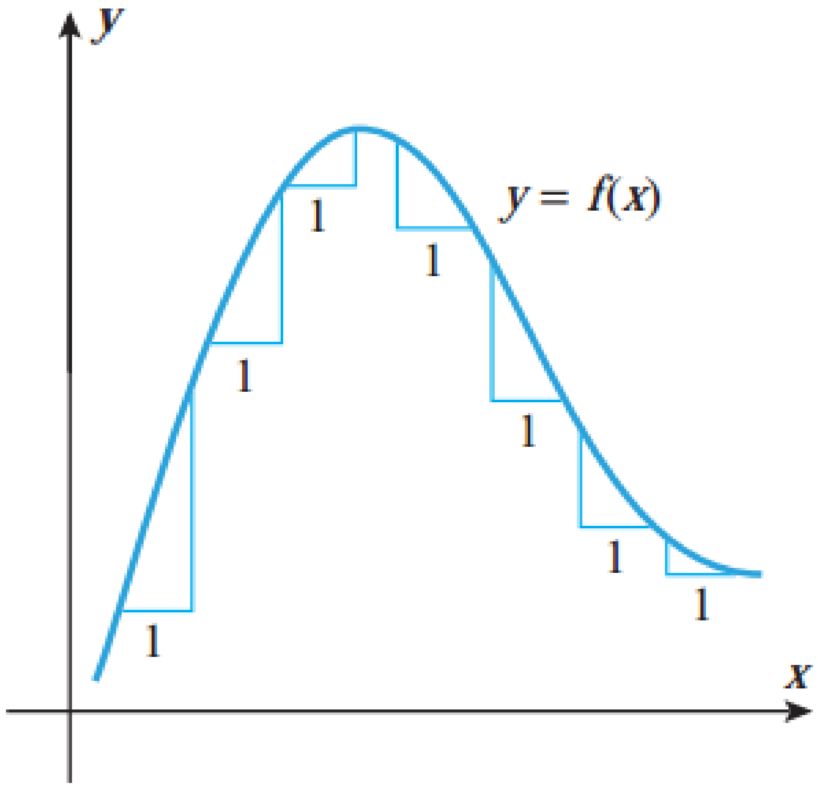
So, each 1-unit increase in x produces a 5-unit decrease in y.

Rates of Change

Although the rate of change of y with respect to x is **constant** along a nonvertical line $y = mx + b$, this is not true for a general (non-linear) curve $y = f(x)$.

For example:

The change in y that results from a 1-unit increase in x tends to have greater magnitude in regions where the curve rises or falls rapidly than in regions where it rises or falls slowly.



Rates of change

Geometrically:

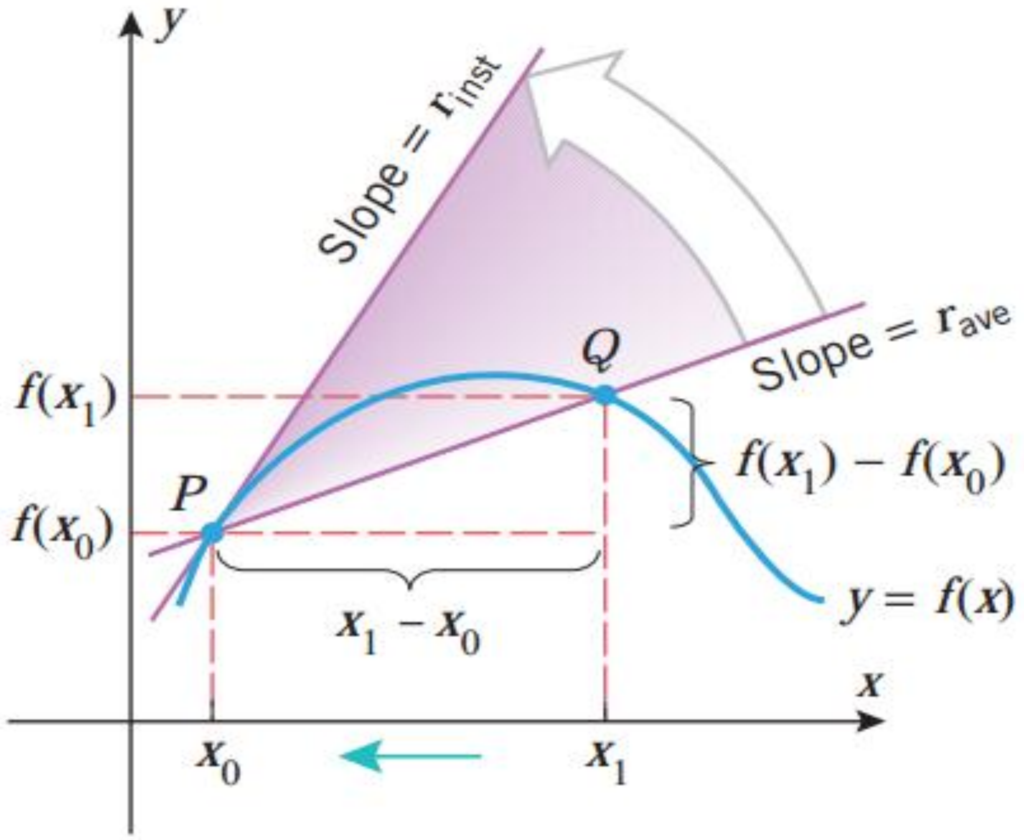
The **average rate** of change of y with respect to x over the interval $[x_0, x_1]$ is the slope of the secant line through the points $P(x0, f(x0))$ and $Q(x1, f(x1))$.

If $y = f(x)$, *then the* ***average rate*** *of change of* y *with respect to* x *over the interval* $[x_0, x_1]$ to be

$$r_{\text{ave}} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Let: $h = x_1 - x_0 \quad \Rightarrow \quad x_1 = x_0 + h$

$$r_{\text{ave}} = \frac{f(x_0 + h) - f(x_0)}{h}$$



Rates of change

Geometrically:

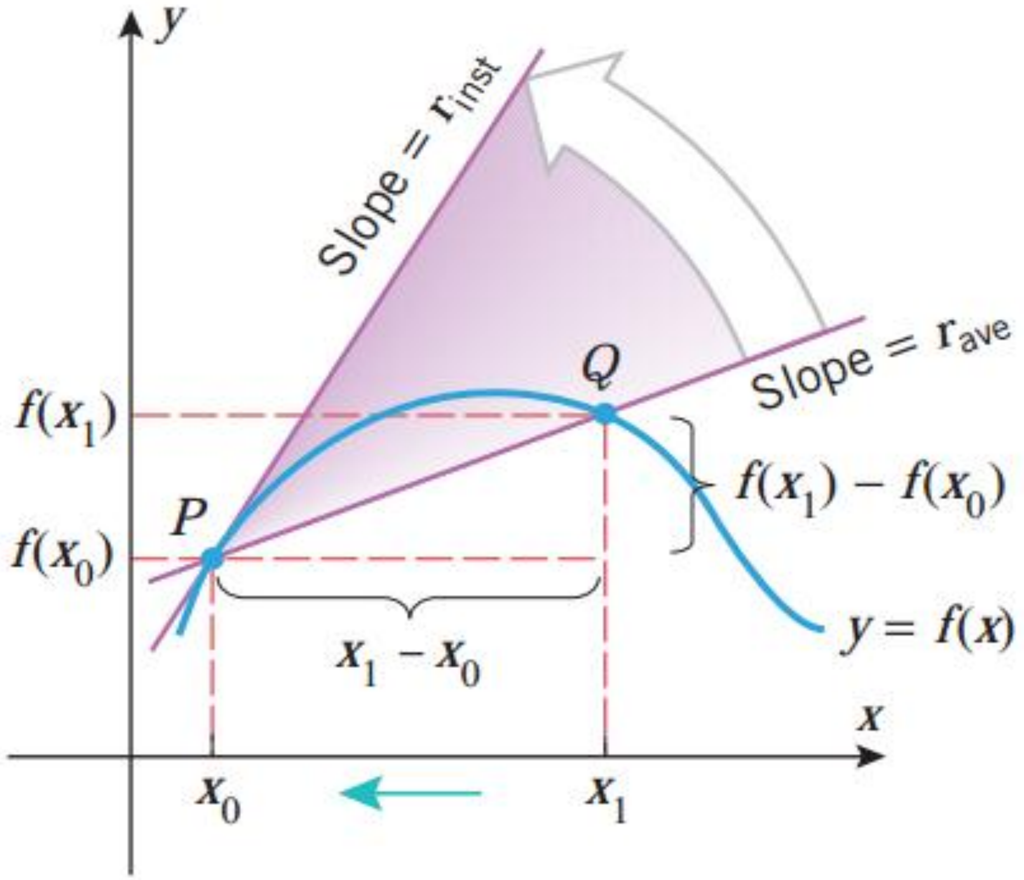
The instantaneous rate of change of y with respect to x at x_0 is the slope of the tangent line at the point *P(x0, f(x0))* (since it is the limit of the slopes of the secant lines through *P*).

The instantaneous rate of change of y *with respect to* x *at* x_0 is

$$r_{\text{inst}} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Let: $h = x_1 - x_0 \Rightarrow x_1 = x_0 + h$

$$r_{\text{inst}} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$



Rates of change

Example: Let $y = x^2 + 1$.

(a) Find the average rate of change of y with respect to x over the interval $[3, 5]$.

(b) Find the instantaneous rate of change of y with respect to x when $x = -4$.

Solution:

(a) $f(x) = x^2 + 1$, $x_0 = 3$, and $x_1 = 5$

$$r_{\text{ave}} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(5) - f(3)}{5 - 3} = \frac{26 - 10}{2} = 8$$

(b) $f(x) = x^2 + 1$ and $x_0 = -4$.

$$r_{\text{inst}} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1 \to -4} \frac{f(x_1) - f(-4)}{x_1 - (-4)} = \lim_{x_1 \to -4} \frac{(x_1^2 + 1) - 17}{x_1 + 4}$$

$$= \lim_{x_1 \to -4} \frac{x_1^2 - 16}{x_1 + 4} = \lim_{x_1 \to -4} \frac{(x_1 + 4)(x_1 - 4)}{x_1 + 4} = \lim_{x_1 \to -4} (x_1 - 4) = -8$$

The Derivative Function

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

It can be interpreted either as the slope of the tangent line to the curve $y = f(x)$ at $\boldsymbol{x = x_0}$

or

the instantaneous rate of change of y with respect to $\boldsymbol{x}$ at $\boldsymbol{x = x_0}$.

2.2.1 DEFINITION The function f' defined by the formula

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{2}$$

is called the ***derivative of f with respect to x***. The domain of f' consists of all x in the domain of f for which the limit exists.

The Derivative Function

Finding an Equation for the Tangent Line to $y = f(x)$ at $x = x_0$.

Step 1. Evaluate $f(x_0)$; the point of tangency is $(x_0, f(x_0))$.

Step 2. Find $f'(x)$ and evaluate $f'(x_0)$, which is the slope m of the line.

Step 3. Substitute the value of the slope m and the point $(x_0, f(x_0))$ into the point-slope form of the line

$$y - f(x_0) = f'(x_0)(x - x_0)$$

or, equivalently,

$$y = f(x_0) + f'(x_0)(x - x_0) \tag{3}$$

$$y = \quad b \quad + \quad m \quad x$$

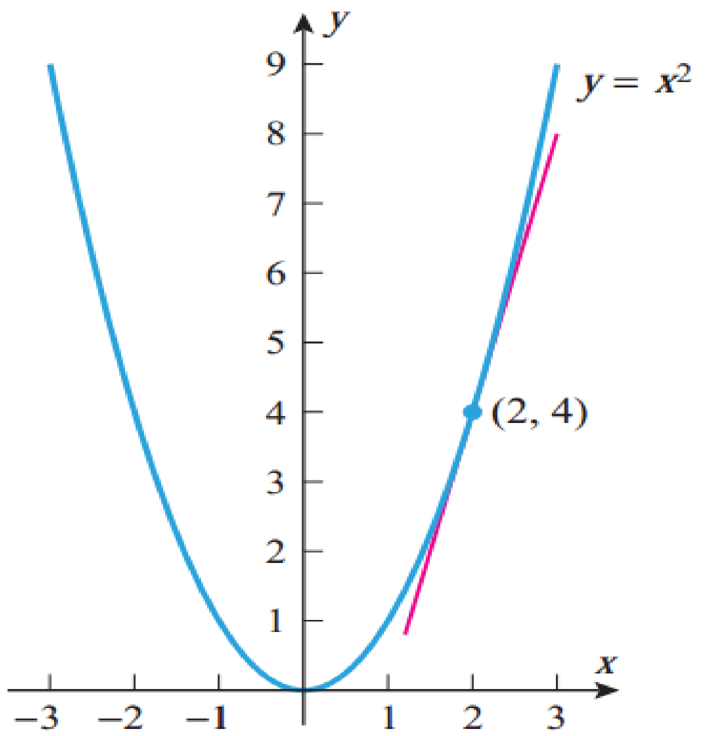
The Derivative Function

Example: Find the derivative with respect to x of $f(x) = x^2$, and use it to find the equation of the tangent line to $y = x^2$ at $x = 2$.

Solution:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h} = \lim_{h\to 0} (2x + h) = 2x$$



Thus, the slope of the tangent line to $y = x^2$ at $x = 2$ is

$$\because m = f'(x_0) = 2x_0 = f'(2) = 2(2) = 4$$

$\because f(x_0) = 4$ if $x_0 = 2$, the equation of the tangent line is

$$\because y - f(x_0) = m(x - x_0) \Rightarrow y - 4 = 4(x - 2) \Rightarrow y - 4 = 4x - 8$$

$$\therefore y = 4x - 4$$

Differentiability

It is possible that the limit that defines the derivative of a function f may not exist at certain points in the domain of f. At such points the derivative is undefined.

2.2.2 DEFINITION A function f is said to be ***differentiable at*** $\boldsymbol{x_0}$ if the limit

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{5}$$

exists. If f is differentiable at each point of the open interval (a, b), then we say that it is ***differentiable on*** $\boldsymbol{(a, b)}$, and similarly for open intervals of the form $(a, +\infty)$, $(-\infty, b)$, and $(-\infty, +\infty)$. In the last case we say that f is ***differentiable everywhere***.

THEOREM *If a function f is differentiable at x_0, then f is continuous at x_0.*

Differentiability

Example: Prove that $f(x) = |x|$ is not differentiable at $x = 0$

Solution: According to the differentiability definition, that *f(x) is differentiable if the following limit exists.*

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{|h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

$$\frac{|h|}{h} = \begin{cases} 1, & h > 0 \\ -1, & h < 0 \end{cases} \Rightarrow \lim_{h \to 0^+} \frac{|h|}{h} = 1 \quad \text{and} \quad \lim_{h \to 0^-} \frac{|h|}{h} = -1$$

Since these one-sided limits are not equal, the two-sided limit does not exist, and hence f is not differentiable at $x = 0$.

Derivatives at the endpoints of an interval

If a function f is defined on a closed interval [*a, b*] but not outside that interval, then f' is not defined at the endpoints of the interval because derivatives are two-sided limits.

To deal with this we define left-hand derivatives and right-hand derivatives by

$$f'_{-}(x) = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} \quad \text{and} \quad f'_{+}(x) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

respectively. These are called ***one-sided derivatives***.

In general, we will say that f is *differentiable* on an interval of the form [*a, b*], [*a*, + ∞*)*, *(*-∞*, b*], [*a, b), or (a, b*] if it is differentiable at all points inside the interval and the appropriate one-sided derivative exists at each included endpoint.

Techniques Of Differentiation

Derivative of a function.

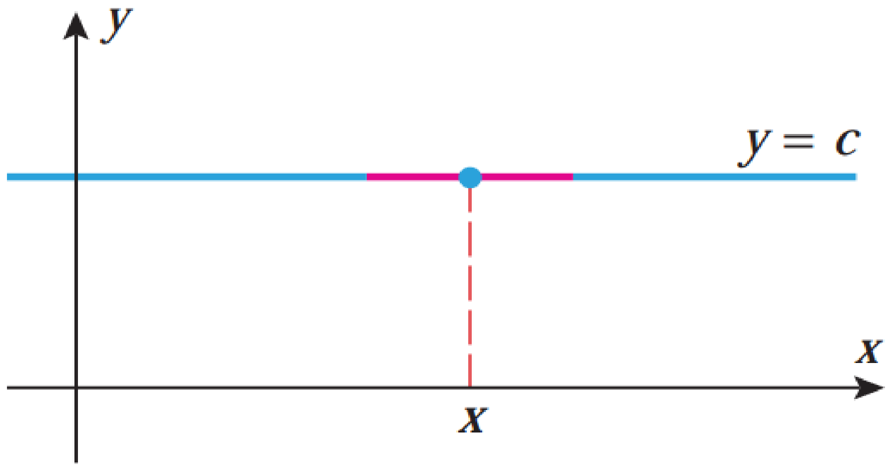
2.3.1 THEOREM *The derivative of a constant function is 0; that is, if c is any real number, then*

$$\frac{d}{dx}[c] = 0 \tag{1}$$

Proof: *The constant function is* $f(x) = c$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0$$



Example:

$$\frac{d}{dx}[1] = 0, \quad \frac{d}{dx}[-3] = 0, \quad \frac{d}{dx}[\pi] = 0, \quad \frac{d}{dx}\left[-\sqrt{2}\right] = 0$$

The tangent line to the graph of $f(x) = c$ has slope 0 for all x.

Techniques Of Differentiation

Derivative of a function.

1- $$\frac{d}{dx}[x] = 1$$

2- $$\frac{d}{dx}[x^n] = nx^{n-1}$$

3- $$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

4- $$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

5- $$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$$

Example: $$\frac{d}{dx}[x^2] = 2x \qquad \frac{d}{dx}[x^3] = 3x^2 \qquad \frac{d}{dx}[x^5] = 5x^4 \qquad \frac{d}{dt}[t^{12}] = 12t^{11}$$

Example: Find $\frac{d}{dx}[\sqrt{x}]$

Solution: $$\frac{d}{dx}[\sqrt{x}] = \frac{d}{dx}[x^{1/2}] = \frac{1}{2}x^{(1/2)-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

Techniques Of Differentiation

Derivative of a function.

Example: Find $\frac{d}{dx}[\sqrt[3]{x}]$

Solution:

$$\frac{d}{dx}[\sqrt[3]{x}] = \frac{d}{dx}[x^{1/3}] = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$$

Example: Find $\frac{d}{dx}[2x^6 + x^{-9}]$

Solution:

$$\frac{d}{dx}[2x^6 + x^{-9}] = \frac{d}{dx}[2x^6] + \frac{d}{dx}[x^{-9}]$$

$$= 12x^5 + (-9)x^{-10} = 12x^5 - 9x^{-10}$$

Techniques Of Differentiation

Example: Find dy/dx if $y = 3x^8 - 2x^5 + 6x + 1$.

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}[3x^8 - 2x^5 + 6x + 1]$$

$$= \frac{d}{dx}[3x^8] - \frac{d}{dx}[2x^5] + \frac{d}{dx}[6x] + \frac{d}{dx}[1]$$

$$= 24x^7 - 10x^4 + 6$$

Derivative of a product

2.4.1 THEOREM (*The Product Rule*) *If f and g are differentiable at x, then so is the product $f \cdot g$, and*

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)] \tag{1}$$

Techniques Of Differentiation

Example: Find dy/dx if $y = (4x^2 - 1)(7x^3 + x)$.

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}[(4x^2 - 1)(7x^3 + x)] = (4x^2 - 1)\frac{d}{dx}[7x^3 + x] + (7x^3 + x)\frac{d}{dx}[4x^2 - 1]$$

$$= (4x^2 - 1)(21x^2 + 1) + (7x^3 + x)(8x)$$

Derivative of a quotient

2.4.2 THEOREM (*The Quotient Rule*) *If f and g are both differentiable at x and if $g(x) \neq 0$, then f/g is differentiable at x and*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2} \qquad (2)$$

Techniques Of Differentiation

Example: Find $y'(x)$ for $y = \dfrac{x^3 + 2x^2 - 1}{x + 5}$.

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x^3 + 2x^2 - 1}{x + 5}\right] = \frac{(x+5)\dfrac{d}{dx}[x^3 + 2x^2 - 1] - (x^3 + 2x^2 - 1)\dfrac{d}{dx}[x+5]}{(x+5)^2}$$

$$= \frac{(x+5)(3x^2 + 4x) - (x^3 + 2x^2 - 1)(1)}{(x+5)^2}$$

Example: Find dy/dx $f(x) = \dfrac{x^2 - 1}{x^4 + 1}$.

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x^2 - 1}{x^4 + 1}\right] = \frac{(x^4+1)\dfrac{d}{dx}[x^2 - 1] - (x^2 - 1)\dfrac{d}{dx}[x^4 + 1]}{(x^4+1)^2}$$

$$= \frac{(x^4+1)(2x) - (x^2 - 1)(4x^3)}{(x^4+1)^2}$$

Techniques Of Differentiation

Derivatives of trigonometric functions

$$\frac{d}{dx}[\sin x] = \cos x$$

$$\frac{d}{dx}[\cos x] = -\sin x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\csc x] = -\csc x \cot x$$

Example: Find dy/dx if $y = x \sin x$.

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}[x \sin x] = x\frac{d}{dx}[\sin x] + \sin x\frac{d}{dx}[x] = x\cos x + \sin x$$

Techniques Of Differentiation

Example: Find dy/dx if $y = \dfrac{\sin x}{1+\cos x}$.

Solution:

$$\frac{dy}{dx} = \frac{(1+\cos x)\cdot \frac{d}{dx}[\sin x] - \sin x \cdot \frac{d}{dx}[1+\cos x]}{(1+\cos x)^2} = \frac{(1+\cos x)(\cos x) - (\sin x)(-\sin x)}{(1+\cos x)^2}$$

The chain rule:

2.6.1 THEOREM (*The Chain Rule*) *If g is differentiable at x and f is differentiable at $g(x)$, then the composition $f \circ g$ is differentiable at x. Moreover, if*

$$y = f(g(x)) \quad \text{and} \quad u = g(x)$$

then $y = f(u)$ and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \tag{1}$$

Techniques Of Differentiation

Example: Find dy/dx if $y = \cos(x^3)$.

Solution:

Let $u = x^3 \Rightarrow y = \cos u$ **Apply chain rule**

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d}{du}[\cos u] \cdot \frac{d}{dx}[x^3] = (-\sin u) \cdot (3x^2) = -3x^2 \sin(x^3)$$

Example: Find dw/dt if $w = \tan x$ and $x = 4t^3 + t$.

Solution:

$$\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt} = \frac{d}{dx}[\tan x] \cdot \frac{d}{dt}[4t^3 + t] = (\sec^2 x) \cdot (12t^2 + 1)$$

$$= [\sec^2(4t^3 + t)] \cdot (12t^2 + 1)$$

Techniques Of Differentiation

Example: Find (a) $\frac{d}{dx}[\sin(2x)]$ (b) $\frac{d}{dx}[\tan(x^2+1)]$

(c) $\frac{d}{dx}\left[\sqrt{x^3+\csc x}\right]$ (d) $\frac{d}{dx}[x^2-x+2]^{3/4}$

Solution: **(a)** Let $u = 2x \Rightarrow \frac{du}{dx} = 2$

$y = \sin u \Rightarrow \frac{dy}{du} = \cos u$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2\cos 2x$$

(b) Let $u = x^2+1 \Rightarrow \frac{du}{dx} = 2x$

$y = \tan u \Rightarrow \frac{dy}{du} = \sec^2 u$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \sec^2(x^2+1) \cdot 2x$$

Techniques Of Differentiation

Example: Find (a) $\frac{d}{dx}[\sin(2x)]$ (b) $\frac{d}{dx}[\tan(x^2+1)]$

(c) $\frac{d}{dx}\left[\sqrt{x^3+\csc x}\right]$ (d) $\frac{d}{dx}[x^2-x+2]^{3/4}$

Solution: **(c)** Let $u = x^3 + \csc x$

$$\frac{d}{dx}[\sqrt{u}] = \frac{1}{2\sqrt{u}}\frac{du}{dx} = \frac{1}{2\sqrt{x^3+\csc x}} \cdot \frac{d}{dx}[x^3+\csc x]$$

$$= \frac{1}{2\sqrt{x^3+\csc x}} \cdot (3x^2 - \csc x \cot x)$$

(d) Let $u = x^2 - x + 2$

$$\frac{d}{dx}[u^{3/4}] = \frac{3}{4}u^{-1/4}\frac{du}{dx} = \frac{3}{4}u^{-1/4}\frac{du}{dx} = \frac{3}{4}(x^2-x+2)^{-1/4} \cdot \frac{d}{dx}[x^2-x+2]$$

$$= \frac{3}{4}(x^2-x+2)^{-1/4}(2x-1)$$

Techniques Of Differentiation

Derivatives of logarithmic functions

$$\frac{d}{dx}[\ln x] = \frac{1}{x}, \quad x > 0$$

$$\frac{d}{dx}[\log_b x] = \frac{1}{x \ln b}, \quad x > 0$$

$$\frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx}$$

$$\frac{d}{dx}[\log_b u] = \frac{1}{u \ln b} \cdot \frac{du}{dx}$$

Example: Find dy/dx if $y = \ln(x^2 + 1)$

Solution: $u = x^2 + 1$

$$\frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx} = \frac{1}{x^2+1} \cdot \frac{d}{dx}[x^2+1] = \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}$$

Techniques Of Differentiation

Derivatives of logarithmic functions

Example: Find dy/dx if $y = 2\ln x + \ln(\sin x) - \frac{1}{2}\ln(1+x)$

Solution:

$$\frac{d}{dx}\left[2\ln x + \ln(\sin x) - \frac{1}{2}\ln(1+x)\right] = \frac{2}{x} + \frac{\cos x}{\sin x} - \frac{1}{2(1+x)}$$

Derivatives of exponential functions

$$\frac{d}{dx}[b^x] = b^x \ln b$$

$$\frac{d}{dx}[b^u] = b^u \ln b \cdot \frac{du}{dx}$$

$$\frac{d}{dx}[e^x] = e^x$$

$$\frac{d}{dx}[e^u] = e^u \cdot \frac{du}{dx}$$

Techniques Of Differentiation

Derivatives of exponential functions

$$\frac{d}{dx}[b^x] = b^x \ln b$$

$$\frac{d}{dx}[e^x] = e^x$$

$$\frac{d}{dx}[b^u] = b^u \ln b \cdot \frac{du}{dx}$$

$$\frac{d}{dx}[e^u] = e^u \cdot \frac{du}{dx}$$

Example: Find dy/dx for

(a) 2^x (b) e^{-2x} (3) e^{x^3} (4) $e^{\cos x}$

Solution:

(a) $\frac{d}{dx}[2^x] = 2^x \ln 2$

(b) $\frac{d}{dx}[e^{-2x}] = e^{-2x} \cdot \frac{d}{dx}[-2x] = -2e^{-2x}$

(c) $\frac{d}{dx}[e^{x^3}] = e^{x^3} \cdot \frac{d}{dx}[x^3] = 3x^2 e^{x^3}$

(d) $\frac{d}{dx}[e^{\cos x}] = e^{\cos x} \cdot \frac{d}{dx}[\cos x] = -(\sin x)e^{\cos x}$

Higher Derivatives

If f' is differentiable, then its derivative is denoted by f'' and is called the *second derivative* of *f*.

$$f'' = (f')'$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{d}{dx}[f(x)]\right] = \frac{d^2}{dx^2}[f(x)]$$

As long as we have differentiability, we can continue the process of differentiating to obtain third, fourth, fifth, and even higher derivatives of *f*.

$$f''' = (f'')', \quad f^{(4)} = (f''')', \quad f^{(5)} = (f^{(4)})', \ldots$$

$$\frac{d^n y}{dx^n} = f^{(n)}(x) = \frac{d^n}{dx^n}[f(x)]$$

Higher Derivatives

Example: Find $f^{(5)}(x)$ If $f(x) = 3x^4 - 2x^3 + x^2 - 4x + 2$.

Solution:

$$f'(x) = 12x^3 - 6x^2 + 2x - 4$$

$$f''(x) = 36x^2 - 12x + 2$$

$$f'''(x) = 72x - 12$$

$$f^{(4)}(x) = 72$$

$$f^{(5)}(x) = 0$$

Applications of differentiation:

- L'Hopital's Rule
- Related rates
- Mean Value Theorem
- Maximum and minimum

L'Hôpital's Rule

3.6.1 THEOREM (*L'Hôpital's Rule for Form 0/0*) *Suppose that f and g are differentiable functions on an open interval containing $x = a$, except possibly at $x = a$, and that*

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

If $\lim_{x \to a} [f'(x)/g'(x)]$ exists, or if this limit is $+\infty$ or $-\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as $x \to a^-$, $x \to a^+$, $x \to -\infty$, or as $x \to +\infty$.

L'Hôpital's Rule

3.6.2 THEOREM (*L'Hôpital's Rule for Form* ∞/∞) *Suppose that f and g are differentiable functions on an open interval containing $x = a$, except possibly at $x = a$, and that*

$$\lim_{x \to a} f(x) = \infty \quad \textit{and} \quad \lim_{x \to a} g(x) = \infty$$

If $\lim_{x \to a}[f'(x)/g'(x)]$ exists, or if this limit is $+\infty$ or $-\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as $x \to a^-$, $x \to a^+$, $x \to -\infty$, or as $x \to +\infty$.

L'Hôpital's Rule

Example: Find the limit $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2}$ L'Hôpital's rule.

Solution:

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \frac{0}{0}$$

Applying L'Hôpital's rule

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{\frac{d}{dx}[x^2 - 4]}{\frac{d}{dx}[x - 2]} = \lim_{x \to 2} \frac{2x}{1} = 4$$

L'Hôpital's Rule

Example: Find the limits

(a) $\lim_{x \to 0} \frac{\sin 2x}{x}$ (b) $\lim_{x \to \pi/2} \frac{1 - \sin x}{\cos x}$

(c) $\lim_{x \to 0} \frac{e^x - 1}{x^3}$ (d) $\lim_{x \to 0^-} \frac{\tan x}{x^2}$

Solution:

(a) $$\lim_{x \to 0} \frac{\sin 2x}{x} = \lim_{x \to 0} \frac{\frac{d}{dx}[\sin 2x]}{\frac{d}{dx}[x]} = \lim_{x \to 0} \frac{2\cos 2x}{1} = 2$$

$y = \sin(2x)$. Let $u = 2x$

$\frac{du}{dx} = 2$

$y = \sin(u)$

$\frac{dy}{du} = \cos(u)$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 2\cos(u) = 2\cos(2x)$

(b) $$\lim_{x \to \pi/2} \frac{1 - \sin x}{\cos x} = \lim_{x \to \pi/2} \frac{\frac{d}{dx}[1 - \sin x]}{\frac{d}{dx}[\cos x]} = \lim_{x \to \pi/2} \frac{-\cos x}{-\sin x} = \frac{0}{-1} = 0$$

L'Hôpital's Rule

Example: Find the limits

$$\text{(a)}\quad \lim_{x\to 0}\frac{\sin 2x}{x} \qquad \text{(b)}\quad \lim_{x\to \pi/2}\frac{1-\sin x}{\cos x}$$

$$\text{(c)}\quad \lim_{x\to 0}\frac{e^x-1}{x^3} \qquad \text{(d)}\quad \lim_{x\to 0^-}\frac{\tan x}{x^2}$$

Solution:

$$\textbf{(c)}\quad \lim_{x\to 0}\frac{e^x-1}{x^3}=\lim_{x\to 0}\frac{\frac{d}{dx}[e^x-1]}{\frac{d}{dx}[x^3]}=\lim_{x\to 0}\frac{e^x}{3x^2}=+\infty$$

$$\textbf{(d)}\quad \lim_{x\to 0^-}\frac{\tan x}{x^2}=\lim_{x\to 0^-}\frac{\frac{d}{dx}[\tan x]}{\frac{d}{dx}[x^2]}=\lim_{x\to 0^-}\frac{\sec^2 x}{2x}=-\infty$$

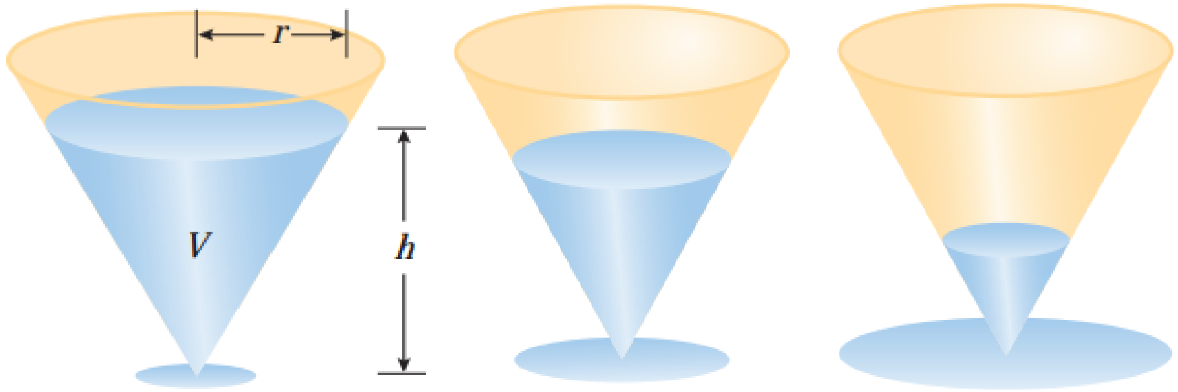
Related rates

- **In related rates problems, it is required to find the rate at which some quantity is changing by relating the quantity to other quantities whose rates of change are known.**

Example: The following figure shows a liquid draining through a conical filter

As the liquid drains:

Its volume V, height h, and radius r are functions of the elapsed time t. These variables are related by



$$V = \frac{\pi}{3} r^2 h$$

To find the rate of change of the volume V with respect to the time t

$$\frac{dV}{dt} = \frac{\pi}{3}\left[r^2 \frac{dh}{dt} + h\left(2r\frac{dr}{dt}\right)\right] = \frac{\pi}{3}\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right)$$

Related rates

Example: Suppose that x and y are differentiable functions of t and are related by the equation $y = x^3$. Find $\frac{dy}{dt}$ at time $t = 1$ if $x = 2$ and $\frac{dx}{dt} = 4$ at time $t = 1$.

Solution:

$$y = x^3$$

$$\frac{dy}{dt} = \frac{d}{dt}[x^3] = 3x^2 \frac{dx}{dt}$$

at time $t = 1$

$$\left.\frac{dy}{dt}\right|_{t=1} = 3(2)^2 \left.\frac{dx}{dt}\right|_{t=1} = 12 \cdot 4 = 48$$

The Mean Value Theorem

THEOREM 4—The Mean Value Theorem Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c). \tag{1}$$

Example: $y = x^2$ on $[0,2]$.

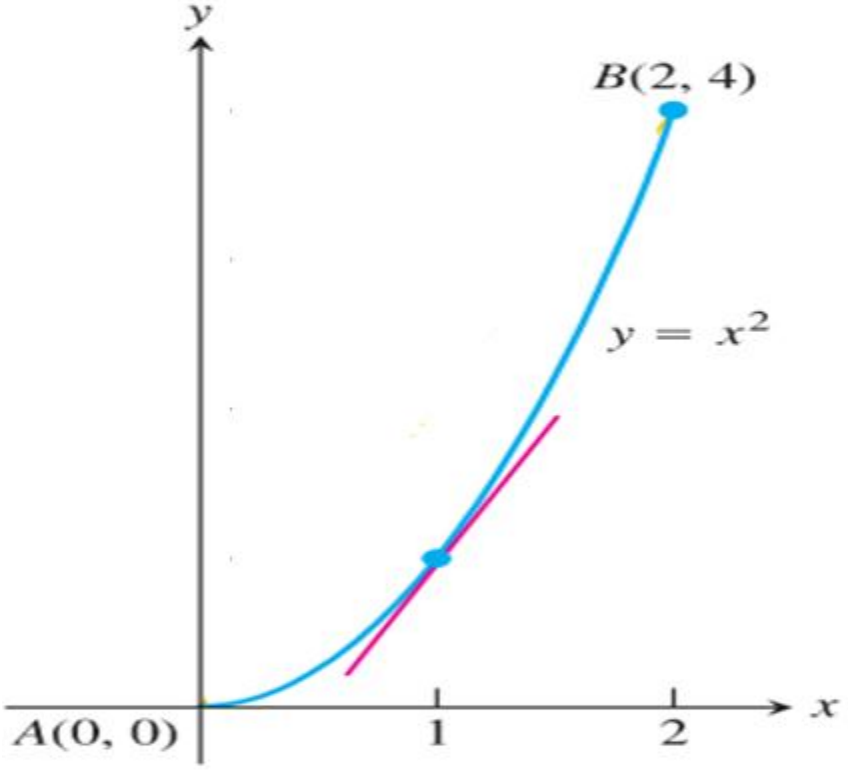
$$\frac{f(b)-f(a)}{b-a} = \frac{f(2)-f(0)}{2-0} = \frac{4-0}{2-0} = 2 \qquad (1)$$

$$f' = 2x \qquad (2)$$

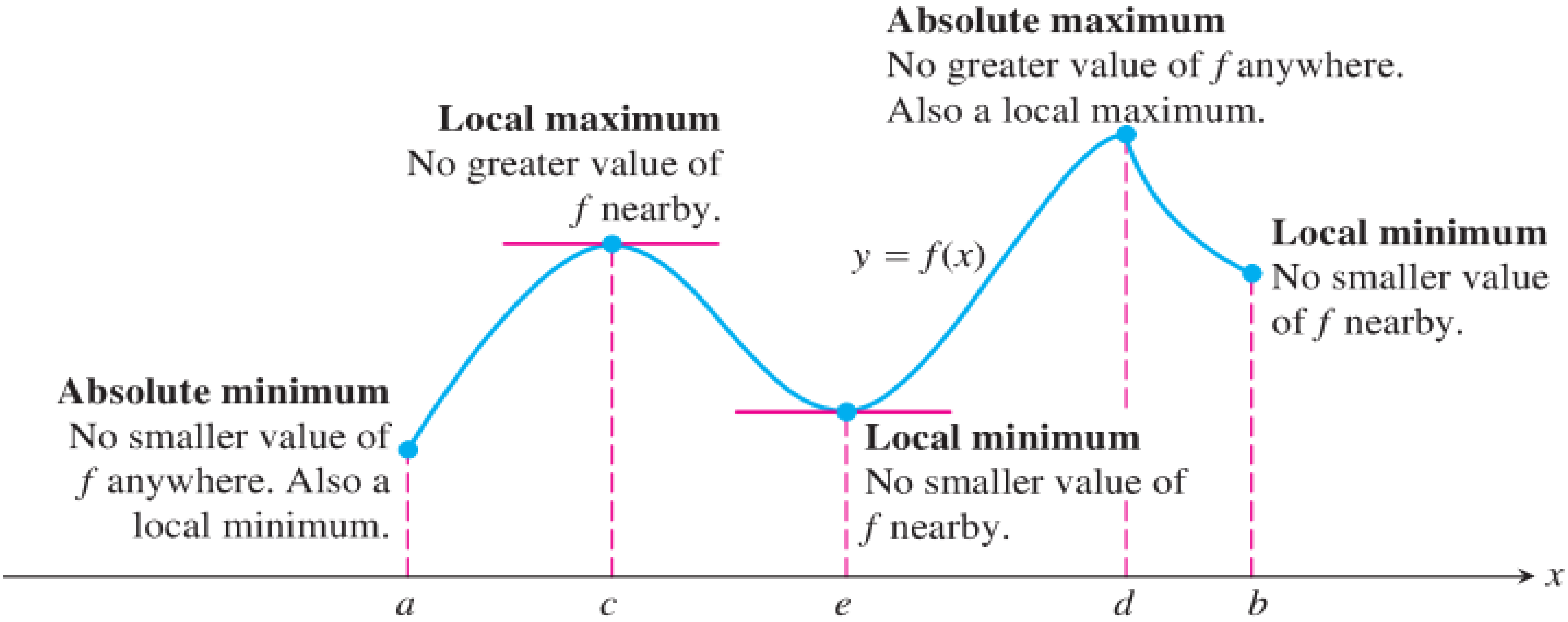
$$from\ (1),(2) and\ the\ mean\ value\ theorem$$

$$f'(c) = 2c = 2$$

$$\therefore c = 1$$



Maximum and minimum



Maximum and minimum

Finding maximum and minimum values is one of the most important applications of the derivative.

DEFINITIONS Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

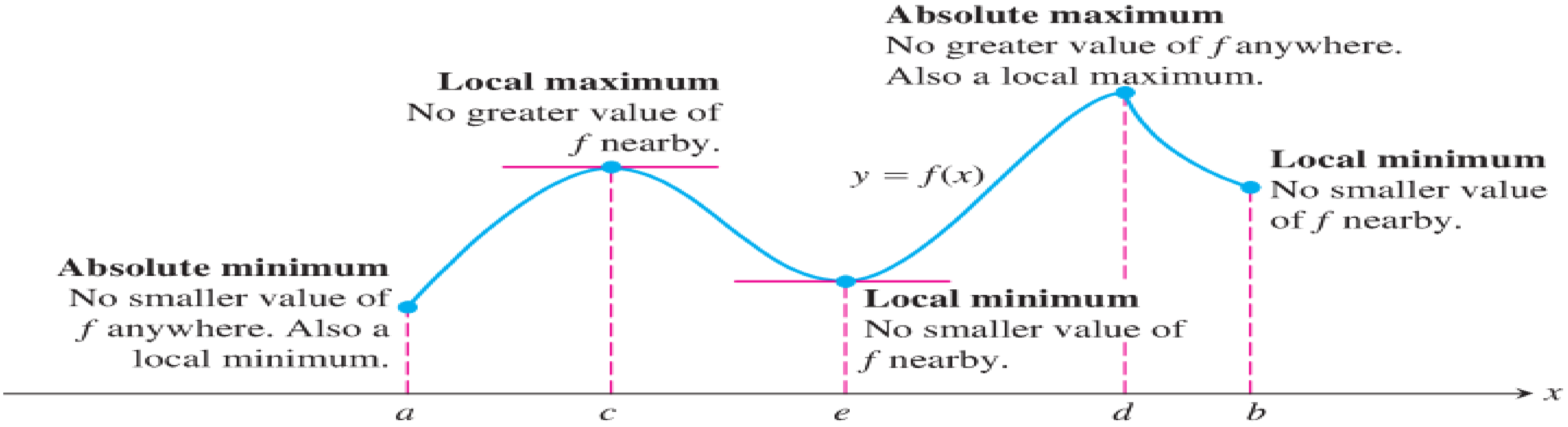
and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

Maximum and minimum values are called **extreme values** of the function *f(x)*.

Absolute maxima or minima are also referred to as **global** maxima or minima

Maximum and minimum



The next theorem explains why we usually need to investigate only a few values to find a function's extrema.

THEOREM 2—The First Derivative Theorem for Local Extreme Values If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$

Maximum and minimum

THEOREM 2—The First Derivative Theorem for Local Extreme Values If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$

Theorem 2 says that a function's first derivative is always zero at an interior point where the function has a local extreme value and the derivative is defined.

Hence the only places where a function *f(x)* can possibly have an extreme value (local or global) are

1. interior points where $f' = 0$,
2. interior points where f' is undefined,
3. endpoints of the domain of f.

Maximum and minimum

DEFINITION An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

How to Find the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1. Evaluate f at all critical points and endpoints.
2. Take the largest and smallest of these values.

Maximum and minimum

Example: Find the absolute maximum and minimum values of $y = x^2$ on [-2,1].

Solution:

$$f'(x) = \frac{dy}{dx} = 2x$$

The critical point is $f'(x) = 0$

$f'(x) = 2x = 0$ → ∴ x = 0 is the critical point.

Critical point value: $f(0) = (0)^2 = 0$

Endpoint values : $f(-2) = (-2)^2 = 4$

Endpoint values : $f(1) = (1)^2 = 1$

The function has an absolute maximum value of 4 at $x = -2$ and an absolute minimum value of 0 at x = 0.

Algbera

Determinants

The determinant of a square matrix **A** is a single number, denoted by det(**A**) or |**A**|

For $A_{2\times 2}$,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$|A| = a_{11}a_{22} - a_{12}a_{21}$$

Determinants

Example: Find |A|, where

$$A = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$$

Solution

$$|A| = 3(3) - 4(2) = 1$$

Determinants

- Determinate of 3x3 matrix

- Signs in the (i,j) cofactors

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Determinants

- For $A_{3\times3}$,

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$|A|= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33}-a_{23}a_{32})-a_{12}(a_{21}a_{33}-a_{23}a_{31})+a_{13}(a_{21}a_{32}-a_{22}a_{31})$$

Determinants

Ex. Find |A|,

$$A=\begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$$

$$|A| = 3\begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} + 0\begin{vmatrix} 1 & 0 \\ -2 & 3 \end{vmatrix}$$

$$=3(0-3)+2(0+2) = -5$$

Determinants

Or Expanding using 3rd column

$$A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$$

$$|A| = 0 \begin{vmatrix} 1 & 0 \\ -2 & 3 \end{vmatrix} - (1) \begin{vmatrix} 3 & -2 \\ -2 & 3 \end{vmatrix} + 0 \begin{vmatrix} 3 & -2 \\ 1 & 0 \end{vmatrix}$$

$$= -(9-4) = -5$$

Properties of Determinants

- $|A^T| = |A|$

$$\begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 3 \end{vmatrix} = 1$$

- For square matrices A and B, $|AB| = |A||B|$

$$A = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 4 & 2 \end{bmatrix} \qquad AB = \begin{bmatrix} 19 & 8 \\ 14 & 6 \end{bmatrix}$$

$$|AB| = 114 - 112 = 2 \qquad |A| = 9 - 8 = 1,\ |B| = 2$$

- For square matrix A and positive integer n, $|A^n| = |A|^n$

Properties of Determinants

- For diagonal or triangular matrix A,

  $|A| =$ product of diagonal entries

$$\begin{vmatrix} 3 & 0 & 0 \\ 2 & 2 & 0 \\ 5 & 2 & 4 \end{vmatrix} = 3 \times 2 \times 4 = 24$$

$$\begin{vmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{vmatrix} = -1 \times 5 \times 4 = -20$$

Properties of Determinants

- Interchange of two rows (columns) changes the sign of the determinant

$$\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = 8 - 9 = -1 \qquad \begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix} = 9 - 8 = 1$$

- Addition of a multiple of any row to another row will leave the value of the determinant unchanged

$$\begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix} = 1 \qquad \begin{vmatrix} 3 & 4 \\ 2 + 5 \times 3 & 3 + 5 \times 4 \end{vmatrix} = 1$$

Properties of Determinants

- The determinant vanishes if
  - It has a zero row (column)

$$\begin{vmatrix} 0 & 0 \\ 2 & 3 \end{vmatrix} = 0$$

  - A row (column) is multiple of another one

$$\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0$$

  - If a row (column) is a linear combination of other rows (columns)

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 7 & 5 \end{vmatrix} = 0$$

The Inverse of a Matrix

2.2 Theorem

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

The Inverse of a Matrix

2.2 Inverse of 2×2 matrices

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

The Inverse of a Matrix

Example: Find A^{-1} , if $A = \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}$

Solution:

$$A^{-1} = \frac{1}{8-6}\begin{bmatrix} 4 & -6 \\ -1 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 & -6 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

Note that: $|A| = 8 - 6 = 2$ $\quad |A^{-1}| = 2 - \frac{3}{2} = \frac{1}{2}$

i.e. $|A^{-1}| = 1/|A|$

The Inverse of a Matrix

Adjugate (classical adjoint) Method

$$(i,j) - \text{entry of } A^{-1} = \frac{C_{ji}}{|A|}$$

$$\mathrm{A}^{-1} = \frac{\mathrm{adj(A)}}{|\mathrm{A}|} = \frac{\mathrm{C}^{\mathrm{T}}}{|\mathrm{A}|} = \frac{1}{|\mathrm{A}|}\begin{bmatrix} \mathrm{C}_{11} & \mathrm{C}_{12} & \mathrm{C}_{13} & \cdots \\ \mathrm{C}_{21} & \mathrm{C}_{22} & \mathrm{C}_{23} & \cdots \\ \mathrm{C}_{31} & \mathrm{C}_{32} & \mathrm{C}_{33} & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{bmatrix}$$

If $|A| = 0$,

$\Rightarrow$ A has no inverse (non invertible or Singular)

Inverse of a Matrix

Ex. Find A^{-1} if A=

$$\begin{pmatrix} 2 & 1 & 1 \\ 6 & 2 & 1 \\ -2 & 2 & 1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} & - \begin{vmatrix} 6 & 1 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 6 & 2 \\ -2 & 2 \end{vmatrix} \\ - \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix} & - \begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & - \begin{vmatrix} 2 & 1 \\ 6 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 6 & 2 \end{vmatrix} \end{pmatrix}^{T} = \frac{1}{8} \begin{bmatrix} 0 & -8 & 16 \\ 1 & 4 & -6 \\ -1 & 4 & -2 \end{bmatrix}^{T}$$

$$= \begin{bmatrix} 0 & 1/8 & -1/8 \\ -1 & 1/2 & 1/2 \\ 2 & -3/4 & -1/4 \end{bmatrix}$$

Inverse of a Matrix

Note that

1. If k is a scalar, $(kA)^{-1} =$ $(1/k)A^{-1}$

2. If n is positive integer, $(A^n)^{-1} =$ $(A^{-1})^n$

3. If $|A| = 0$ A is singular (non invertible or has no inverse)

4. $|A^{-1}| =$ $1/|A|$

Properties of Inverse of a Matrix

Example: Evaluate $|A|$, $|A^{-1}|$, where

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution :

$|A| = \quad 6$

$|A^{-1}| = \quad 1/6$

Solve a System of linear Equation

Systems of Linear Equations

Definition:

A System of Linear Equations is a collection of one or more linear equations that sharing the same variables

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

<u>Ex.</u> Find the solution of the following linear system,

$$2x + y + z \quad = 1$$

$$6x + 2y + z = -1$$

$$-2x + 2y + z = 7$$

Carl Friedrich Gauss

German

Gauss Elimination with Back Substitution

Gauss Elimination with Back Substitution

Example: Find the solution of the following linear system

Solution

$$2x + y + z \ \ = 1$$

$$6x + 2y + z = -1$$

$$-2x + 2y + z = 7$$

$$\Rightarrow \quad \begin{bmatrix} 2 & 1 & 1 & 1 \\ 6 & 2 & 1 & -1 \\ -2 & 2 & 1 & 7 \end{bmatrix}$$

Pivot -3 × + +

$$\sim \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & -1 & -2 & -4 \\ 0 & 3 & 2 & 8 \end{bmatrix} \underset{R_3+R_1 \to R_3}{\overset{R_2-3R_1 \to R_2}{\sim}} \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & -1 & -2 & -4 \\ 0 & 0 & -4 & -4 \end{bmatrix}$$

3× +

$R_3+3R_2 \to R_3$

$$\begin{aligned} 2x + y + z &= 1 \\ -y - 2z &= -4 \\ -4z &= -4 \end{aligned}$$

$$\Rightarrow \ x = -1, y = 2, z = 1$$

https://www.youtube.com/watch?v=t96K1trPXy4

https://www.youtube.com/watch?v=VRCG_zi3lDw&t=12s

https://www.youtube.com/watch?v=lFnX9q2ET5g&t=139s