# Integral Transforms and Their Applications

Faculty of Education
Math. department
Course Code: Pure14
4th Year -1st term

# 2

# *Fourier Transforms and Their Applications*

> "The profound study of nature is the most fertile source of mathematical discoveries."
>
> Joseph Fourier

> "The theory of Fourier series and integrals has always had major difficulties and necessitated a large mathematical apparatus in dealing with questions of convergence. It engendered the development of methods of summation, although these did not lead to a completely satisfactory solution of the problem. .... For the Fourier transform, the introduction of distributions (hence, the space **S** ) is inevitable either in an explicit or hidden form. .... As a result one may obtain all that is desired from the point of view of the continuity and inversion of the Fourier transform."
>
> Laurent Schwartz

## 2.1 Introduction

Many linear boundary value and initial value problems in applied mathematics, mathematical physics, and engineering science can be effectively solved by the use of the Fourier transform, the Fourier cosine transform, or the Fourier sine transform. These transforms are very useful for solving differential or integral equations for the following reasons. First, these equations are replaced by simple algebraic equations, which enable us to find the solution of the transform function. The solution of the given equation is then obtained in the original variables by inverting the transform solution. Second, the Fourier transform of the elementary source term is used for determination of the fundamental solution that illustrates the basic ideas behind the construction and implementation of Green's functions. Third, the transform solution combined with the convolution theorem provides an elegant representation of the solution for the boundary value and initial value problems.

We begin this chapter with a formal derivation of the Fourier integral formulas. These results are then used to define the Fourier, Fourier cosine, and Fourier sine transforms. This is followed by a detailed discussion of the basic operational properties of these transforms with examples. Special attention is given to convolution and its main properties. Sections 2.10 and 2.11 deal with applications of the Fourier transform to the solution of ordinary differential equations and integral equations. In Section 2.12, a wide variety of partial differential equations are solved by the use of the Fourier transform method. The technique that is developed in this and other sections can be applied with little or no modification to different kinds of initial and boundary value problems that are encountered in applications. The Fourier cosine and sine transforms are introduced in Section 2.13. The properties and applications of these transforms are discussed in Sections 2.14 and 2.15. This is followed by evaluation of definite integrals with the aid of Fourier transforms. Section 2.17 is devoted to applications of Fourier transforms in mathematical statistics. The multiple Fourier transforms and their applications are discussed in Section 2.18.

## 2.2 The Fourier Integral Formulas

A function $f(x)$ is said to satisfy *Dirichlet's conditions* in the interval $-a < x < a$, if

(i) $f(x)$ has only a finite number of finite discontinuities in $-a < x < a$ and has no infinite discontinuities.

(ii) $f(x)$ has only a finite number of maxima and minima in $-a < x < a$.

From the theory of Fourier series we know that if $f(x)$ satisfies the Dirichlet conditions in $-a < x < a$, it can be represented as the complex Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} a_n \exp(in\pi x/a), \tag{2.2.1}$$

where the coefficients are

$$a_n = \frac{1}{2a} \int_{-a}^{a} f(\xi) \exp(-in\pi\xi/a) d\xi. \tag{2.2.2}$$

This representation is evidently periodic of period $2a$ in the interval. However, the right hand side of (2.2.1) cannot represent $f(x)$ *outside* the interval $-a < x < a$ unless $f(x)$ is periodic of period $2a$. Thus, problems on finite intervals lead to Fourier series, and problems on the whole line $-\infty < x < \infty$ lead to the

Fourier integrals. We now attempt to find an integral representation of a non-periodic function $f(x)$ in $(-\infty, \infty)$ by letting $a \to \infty$. As the interval grows $(a \to \infty)$ the values $k_n = \frac{n\pi}{a}$ become closer together and form a dense set. If we write $\delta k = (k_{n+1} - k_n) = \frac{\pi}{a}$ and substitute coefficients $a_n$ into (2.2.1), we obtain

$$f(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (\delta k) \left[ \int_{-a}^{a} f(\xi) \exp(-i\xi k_n) d\xi \right] \exp(ixk_n). \tag{2.2.3}$$

In the limit as $a \to \infty, k_n$ becomes a continuous variable $k$ and $\delta k$ becomes $dk$. Consequently, the sum can be replaced by the integral in the limit and (2.2.3) reduces to the result

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(\xi) e^{-ik\xi} d\xi \right] e^{ikx} dk. \tag{2.2.4}$$

This is known as the celebrated *Fourier integral formula*. Although the above arguments do not constitute a rigorous proof of (2.2.4), the formula is correct and valid for functions that are piecewise continuously differentiable in every finite interval and is absolutely integrable on the whole real line.

A function $f(x)$ is said to be *absolutely integrable* on $(-\infty, \infty)$ if

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty \tag{2.2.5}$$

exists.

It can be shown that the formula (2.2.4) is valid under more general conditions. The result is contained in the following theorem:

**_THEOREM 2.2.1_**

If $f(x)$ satisfies Dirichlet's conditions in $(-\infty, \infty)$, and is absolutely integrable on $(-\infty, \infty)$, then the Fourier integral (2.2.4) converges to the function $\frac{1}{2}[f(x+0) + f(x-0)]$ at a finite discontinuity at $x$. In other words,

$$\frac{1}{2}[f(x+0) + f(x-0)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \left[ \int_{-\infty}^{\infty} f(\xi) e^{-ik\xi} d\xi \right] dk. \tag{2.2.6}$$

This is usually called the *Fourier integral theorem.*

If the function $f(x)$ is continuous at point $x$, then $f(x+0) = f(x-0) = f(x)$, then (2.2.6) reduces to (2.2.4).

The Fourier integral theorem was originally stated in Fourier's famous treatise entitled *La Théorie Analytique da la Chaleur* (1822), and its deep significance was recognized by mathematicians and mathematical physicists. Indeed, this theorem is one of the most monumental results of modern mathematical analysis and has widespread physical and engineering applications.

We express the exponential factor $\exp[ik(x-\xi)]$ in (2.2.4) in terms of trigonometric functions and use the even and odd nature of the cosine and the sine functions respectively as functions of $k$ so that (2.2.4) can be written as

$$f(x) = \frac{1}{\pi}\int_0^\infty dk \int_{-\infty}^\infty f(\xi)\cos k(x-\xi)d\xi. \tag{2.2.7}$$

This is another version of the *Fourier integral formula.* In many physical problems, the function $f(x)$ vanishes very rapidly as $|x|\to\infty$, which ensures the existence of the repeated integrals as expressed.

We now assume that $f(x)$ is an even function and expand the cosine function in (2.2.7) to obtain

$$f(x) = f(-x) = \frac{2}{\pi}\int_0^\infty \cos kx\, dk \int_0^\infty f(\xi)\cos k\xi\, d\xi. \tag{2.2.8}$$

This is called the *Fourier cosine integral formula.*

Similarly, for an odd function $f(x)$, we obtain the *Fourier sine integral formula*

$$f(x) = -f(-x) = \frac{2}{\pi}\int_0^\infty \sin\, kx\, dk \int_0^\infty f(\xi)\sin k\xi\, d\xi. \tag{2.2.9}$$

These integral formulas were discovered independently by Cauchy in his work on the propagation of waves on the surface of water.

## 2.3 Definition of the Fourier Transform and Examples

We use the Fourier integral formula (2.2.4) to give a formal definition of the Fourier transform.

***DEFINITION 2.3.1*** *The Fourier transform of $f(x)$ is denoted by $\mathscr{F}\{f(x)\} = F(k)$, $k\in\mathbb{R}$, and defined by the integral*

$$\mathscr{F}\{f(x)\} = F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-ikx} f(x)dx, \tag{2.3.1}$$

*where $\mathscr{F}$ is called the Fourier transform operator or the Fourier transformation and the factor $\frac{1}{\sqrt{2\pi}}$ is obtained by splitting the factor $\frac{1}{2\pi}$ involved in*

*(2.2.4). This is often called the complex Fourier transform. A sufficient condition for $f(x)$ to have a Fourier transform is that $f(x)$ is absolutely integrable on $(-\infty, \infty)$. The convergence of the integral (2.3.1) follows at once from the fact that $f(x)$ is absolutely integrable. In fact, the integral converges uniformly with respect to $k$.*

Thus, the definition of the Fourier transform is restricted to absolutely integrable functions. This restriction is too strong for many physical applications. Many simple and common functions, such as constant function, trigonometric functions $\sin ax$, $\cos ax$, exponential functions, and $x^n H(x)$ do not have Fourier transforms, even though they occur frequently in applications. The integral in (2.3.1) fails to converge when $f(x)$ is one of the above elementary functions. This is a very unsatisfactory feature of the theory of Fourier transforms. However, this unsatisfactory feature can be resolved by means of a natural extension of the definition of the Fourier transform of a generalized function, $f(x)$ in (2.3.1). We follow Lighthill (1958) and Jones (1982) to discuss briefly the theory of the Fourier transforms of good functions.

*The inverse Fourier transform, denoted by $\mathscr{F}^{-1}\{F(k)\} = f(x)$, is defined by*

$$\mathscr{F}^{-1}\{F(k)\} = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx}\, F(k)\, dk, \tag{2.3.2}$$

*where $\mathscr{F}^{-1}$ is called the inverse Fourier transform operator.*

Clearly, both $\mathscr{F}$ and $\mathscr{F}^{-1}$ are linear integral operators. In applied mathematics, $x$ usually represents a space variable and $k(=\frac{2\pi}{\lambda})$ is a wavenumber variable where $\lambda$ is the wavelength. However, in electrical engineering, $x$ is replaced by the time variable $t$ and $k$ is replaced by the frequency variable $\omega(=2\pi\nu)$ where $\nu$ is the frequency in cycles per second. The function $F(\omega) = \mathscr{F}\{f(t)\}$ is called the *spectrum* of the *time signal function* $f(t)$. In electrical engineering literature, the Fourier transform pairs are defined slightly differently by

$$\mathscr{F}\{f(t)\} = F(\nu) = \int_{-\infty}^{\infty} f(t) e^{-2\pi\nu i t} dt, \tag{2.3.3}$$

and

$$\mathscr{F}^{-1}\{F(\nu)\} = f(t) = \int_{-\infty}^{\infty} F(\nu) e^{2\pi i\nu t} d\nu = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega, \tag{2.3.4}$$

where $\omega = 2\pi\nu$ is called the *angular frequency*. The Fourier integral formula implies that any function of time $f(t)$ that has a Fourier transform can be equally specified by its spectrum. Physically, the signal $f(t)$ is represented as an integral superposition of an infinite number of sinusoidal oscillations with different frequencies $\omega$ and complex amplitudes $\frac{1}{2\pi}F(\omega)$. Equation (2.3.4) is called the *spectral resolution* of the signal $f(t)$, and $\frac{F(\omega)}{2\pi}$ is called the *spectral density.* In summary, the Fourier transform maps a function ( or signal) of time $t$ to a function of frequency $\omega$. In the same way as the Fourier series expansion of a periodic function decomposes the function into harmonic components, the Fourier transform generates a function (or signal) of a continuous variable whose value represents the frequency content of the original signal. This led to the successful use of the Fourier transform to analyze the form of time-varying signals in electrical engineering and seismology.

Next we give examples of Fourier transforms.

***Example 2.3.1***

Find the Fourier transform of $\exp(-ax^2)$. In fact, we prove

$$F(k) = \mathscr{F}\{\exp(-ax^2)\} = \frac{1}{\sqrt{2a}} \exp\left(-\frac{k^2}{4a}\right), \quad a > 0. \tag{2.3.5}$$

Here we have, by definition,

$$\begin{aligned} F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx-ax^2} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-a\left(x+\frac{ik}{2a}\right)^2 - \frac{k^2}{4a}\right] dx \\ &= \frac{1}{\sqrt{2\pi}} \exp(-k^2/4a) \int_{-\infty}^{\infty} e^{-ay^2} dy = \frac{1}{\sqrt{2a}} \exp\left(-\frac{k^2}{4a}\right), \end{aligned}$$

in which the change of variable $y = x + \frac{ik}{2a}$ is used. The above result is correct, but the change of variable can be justified by the method of complex analysis because $(ik/2a)$ is complex. If $a = \frac{1}{2}$

$$\mathscr{F}\{e^{-x^2/2}\} = e^{-k^2/2}. \tag{2.3.6}$$

This shows $\mathscr{F}\{f(x)\} = f(k)$. Such a function is said to be *self-reciprocal* under the Fourier transformation. Graphs of $f(x) = \exp(-ax^2)$ and its Fourier transform is shown in Figure 2.1 for $a = 1$. ▯

***Example 2.3.2***

Find the Fourier transform of $\exp(-a|x|)$, i.e.,

$$\mathscr{F}\{\exp(-a|x|)\} = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{(a^2+k^2)}, \quad a > 0. \tag{2.3.7}$$

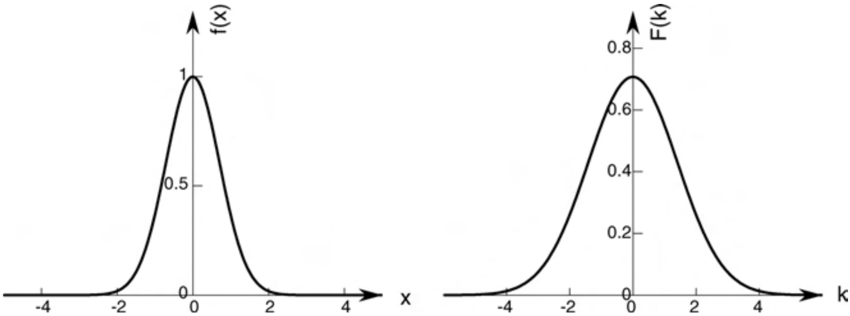


**Figure 2.1** Graphs of $f(x) = \exp(-ax^2)$ and $F(k)$ with $a = 1$.

Here we can write

$$\begin{aligned}\mathscr{F}\left\{e^{-a|x|}\right\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|-ikx}dx \\ &= \frac{1}{\sqrt{2\pi}}\left[\int_0^{\infty} e^{-(a+ik)x}dx + \int_{-\infty}^{0} e^{(a-ik)x}dx\right] \\ &= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{a+ik} + \frac{1}{a-ik}\right] = \sqrt{\frac{2}{\pi}}\frac{a}{(a^2+k^2)}.\end{aligned}$$

We note that $f(x) = \exp(-a|x|)$ decreases rapidly at infinity, it is not differentiable at $x = 0$. Graphs of $f(x) = \exp(-a|x|)$ and its Fourier transform is displayed in Figure 2.2 for $a = 1$. $\square$

***Example 2.3.3***

Find the Fourier transform of

$$f(x) = \left(1 - \frac{|x|}{a}\right) H\left(1 - \frac{|x|}{a}\right),$$

where $H(x)$ is the *Heaviside unit step function* defined by

$$H(x) = \begin{Bmatrix} 1, & x > 0 \\ 0, & x < 0 \end{Bmatrix}. \tag{2.3.8}$$

Or, more generally,

$$H(x-a) = \begin{Bmatrix} 1, & x > a \\ 0, & x < a \end{Bmatrix}, \tag{2.3.9}$$

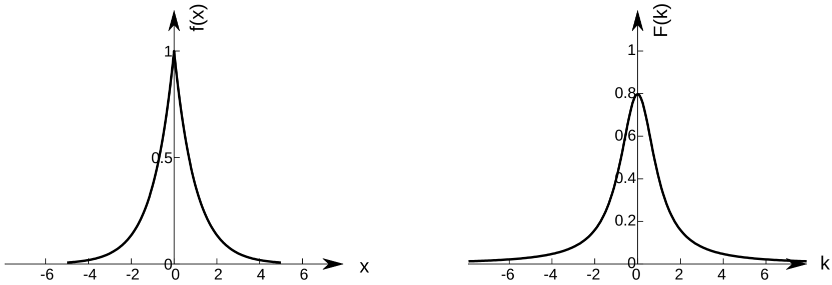


**Figure 2.2** Graphs of $f(x)=\exp(-a|x|)$ and $F(k)$ with $a=1$.

where $a$ is a fixed real number. So the Heaviside function $H(x-a)$ has a finite discontinuity at $x=a$.

$$\begin{aligned}\mathscr{F}\{f(x)\} &= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{-ikx}\left(1-\frac{|x|}{a}\right)dx = \frac{2}{\sqrt{2\pi}}\int_0^a\left(1-\frac{x}{a}\right)\cos kx\,dx \\ &= \frac{2a}{\sqrt{2\pi}}\int_0^1 (1-x)\cos(akx)dx = \frac{2a}{\sqrt{2\pi}}\int_0^1 (1-x)\frac{d}{dx}\left(\frac{\sin akx}{ak}\right)dx \\ &= \frac{2a}{\sqrt{2\pi}}\int_0^1 \frac{\sin(akx)}{ak}dx = \frac{a}{\sqrt{2\pi}}\int_0^1 \frac{d}{dx}\left[\frac{\sin^2\left(\frac{akx}{2}\right)}{\left(\frac{ak}{2}\right)^2}\right]dx \\ &= \frac{a}{\sqrt{2\pi}}\,\frac{\sin^2\left(\frac{ak}{2}\right)}{\left(\frac{ak}{2}\right)^2}. \end{aligned} \tag{2.3.10}$$

▯

***Example 2.3.4***

Find the Fourier transform of the characteristic function $\chi_{[-a,a]}(x)$, where

$$\chi_{[-a,a]}(x) = H(a-|x|) = \left\{\begin{array}{ll} 1, & |x|<a \\ 0, & |x|>a \end{array}\right\}. \tag{2.3.11}$$

We have

$$F_a(k) = \mathscr{F}\{\chi_{[-a,a]}(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \chi_{[-a,a]}(x)\, dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-ikx}\, dx = \sqrt{\frac{2}{\pi}} \left(\frac{\sin ak}{k}\right). \quad (2.3.12)$$

Graphs of $f(x) = \chi_{[-a,a]}(x)$ and its Fourier transform are shown in Figure 2.3 for $a = 1$.

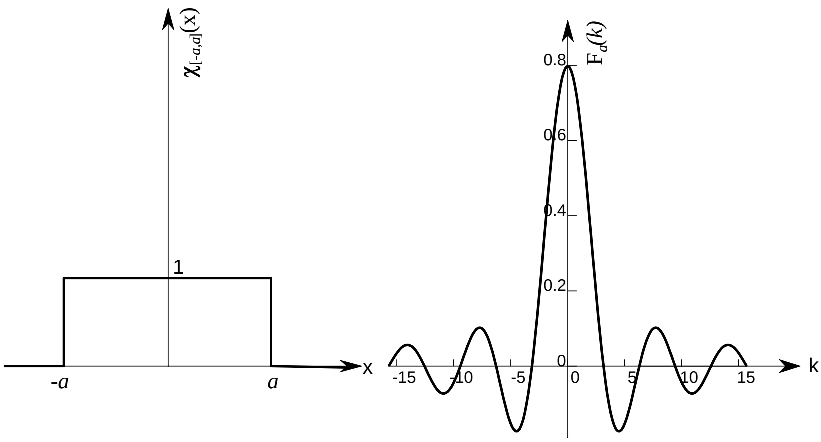


**Figure 2.3** Graphs of $\chi_{[-a,a]}(x)$ and $F_a(k)$ with $a = 1$.

□

## 2.4 Fourier Transforms of Generalized Functions

The natural way to define the Fourier transform of a generalized function, is to treat $f(x)$ in (2.3.1) as a generalized function. The advantage of this is that every generalized function has a Fourier transform and an inverse Fourier transform, and that the ordinary functions whose Fourier transforms are of interest form a subset of the generalized functions. We would not go into great detail, but refer to the famous books of Lighthill (1958) and Jones (1982) for the introduction to the subject of generalized functions.

A *good function*, $g(x)$ is a function in $C^\infty(\mathbb{R})$ that decays sufficiently rapidly that $g(x)$ and all of its derivatives decay to zero faster than $|x|^{-N}$ as $|x| \to \infty$ for all $N > 0$.

**DEFINITION 2.4.1** *Suppose a real or complex valued function $g(x)$ is defined for all $x \in \mathbb{R}$ and is infinitely differentiable everywhere, and suppose that each derivative tends to zero as $|x| \to \infty$ faster that any positive power of $\left(x^{-1}\right)$, or in other words, suppose that for each positive integer $N$ and $n$,*

$$\lim_{|x| \to \infty} x^N g^{(n)}(x) = 0,$$

*then $g(x)$ is called a good function.*

Usually, the class of good functions is represented by $\mathbb{S}$. The good functions play an important role in Fourier analysis because the inversion, convolution, and differentiation theorems as well as many others take simple forms with no problem of convergence. The rapid decay and infinite differentiability properties of good functions lead to the fact that the Fourier transform of a good function is also a good function.

Good functions also play an important role in the theory of generalized functions. A good function of bounded support is a special type of good function that also plays an important part in the theory of generalized functions. Good functions also have the following important properties. The sum (or difference) of two good functions is also a good function. The product and convolution of two good functions are good functions. The derivative of a good function is a good function; $x^n g(x)$ is a good function for all non-negative integers $n$ whenever $g(x)$ is a good function. A good function belongs to $L^p$ (a class of $p^{\text{th}}$ power Lebesgue integrable functions) for every $p$ in $1 \le p \le \infty$. The integral of a good function is not necessarily good. However, if $\phi(x)$ is a good function, then the function $g$ defined for all $x$ by

$$g(x) = \int_{-\infty}^{x} \phi(t)\, dt$$

is a good function if and only if $\int_{-\infty}^{\infty} \phi(t)\, dt$ exists.

Good functions are not only continuous, but are also uniformly continuous in $\mathbb{R}$ and absolutely continuous in $\mathbb{R}$. However, a good function cannot be necessarily represented by a Taylor series expansion in every interval. As an example, consider a good function of bounded support

$$g(x) = \left\{ \begin{array}{lr} \exp[-(1-x^2)^{-1}], & \text{if } |x| < 1 \\ 0, & \text{if } |x| \ge 1 \end{array} \right\}.$$

The function $g$ is infinitely differentiable at $x = \pm 1$, as it must be in order to be good. It does not have a Taylor series expansion in every interval, because a Taylor expansion based on the various derivatives of $g$ for any point having $|x| > 1$ would lead to zero value for all $x$.

For example, $\exp(-x^2)$, $x\exp(-x^2)$, $\left(1+x^2\right)^{-1}\exp(-x^2)$, and $\operatorname{sech}^2 x$ are good functions, while $\exp(-|x|)$ is not differentiable at $x = 0$, and the function $\left(1+x^2\right)^{-1}$ is not a good function as it decays too slowly as $|x| \to \infty$.

A sequence of good functions, $\{f_n(x)\}$ is called *regular* if, for any good function $g(x)$,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, g(x)\, dx \tag{2.4.1}$$

exists. For example, $f_n(x) = \frac{1}{n}\phi(x)$ is a regular sequence for any good function $\phi(x)$, if

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, g(x)\, dx = \lim_{n\to\infty} \frac{1}{n} \int_{-\infty}^{\infty} \phi(x)\, g(x)\, dx = 0\,.$$

Two regular sequences of good functions are equivalent if, for any good function $g(x)$, the limit (2.4.1) exists and is the same for each sequence.

A *generalized function*, $f(x)$, is a regular sequence of good functions, and two generalized functions are equal if their defining sequences are equivalent. Generalized functions are, therefore, only defined in terms of their action on integrals of good functions if

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, g(x)\, dx = \lim_{n\to\infty} \langle f_n, g\rangle \tag{2.4.2}$$

for any good function, $g(x)$, where the symbol $\langle f, g\rangle$ is used to denote the action of the generalized function $f(x)$ on the good function $g(x)$, or $\langle f, g\rangle$ represents the number that $f$ associates with $g$. If $f(x)$ is an ordinary function such that $\left(1+x^2\right)^{-N} f(x)$ is integrable in $(-\infty, \infty)$ for some $N$, then the generalized function $f(x)$ equivalent to the ordinary function is defined as any sequence of good functions $\{f_n(x)\}$ such that, for any good function $g(x)$,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, g(x)\, dx = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx \tag{2.4.3}$$

For example, the generalized function equivalent to zero can be represented by either of the sequences $\left\{\frac{\phi(x)}{n}\right\}$ and $\left\{\frac{\phi(x)}{n^2}\right\}$.

The unit function, $I(x)$, is defined by

$$\int_{-\infty}^{\infty} I(x)\, g(x)\, dx = \int_{-\infty}^{\infty} g(x)\, dx \tag{2.4.4}$$

for any good function $g(x)$. A very important and useful good function that defines the unit function is $\left\{\exp\left(-\frac{x^2}{4n}\right)\right\}$. Thus, the unit function is the generalized function that is equivalent to the ordinary function $f(x)=1$.

The *Heaviside function*, $H(x)$, is defined by

$$\int_{-\infty}^{\infty} H(x)\, g(x)\, dx = \int_0^{\infty} g(x)\, dx. \tag{2.4.5}$$

The generalized function $H(x)$ is equivalent to the ordinary unit function

$$H(x) = \begin{cases} 0, & x<0 \\ 1, & x>0 \end{cases} \tag{2.4.6}$$

since generalized functions are defined through the action on integrals of good functions, the value of $H(x)$ at $x=0$ does not have significance here.

The *sign function*, sgn(x), is defined by

$$\int_{-\infty}^{\infty} \mathrm{sgn(x)\, g(x)\, dx} = \int_0^{\infty} g(x)\, dx - \int_{-\infty}^{0} g(x)\, dx \tag{2.4.7}$$

for any good function $g(x)$. Thus, sgn(x) can be identified with the ordinary function

$$\mathrm{sgn(x)} = \begin{cases} -1, & x<0, \\ +1, & x>0. \end{cases} \tag{2.4.8}$$

In fact, $\mathrm{sgn(x)} = 2\,\mathrm{H(x)} - \mathrm{I(x)}$ can be seen as follows:

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{sgn(x)\, g(x)\, dx} &= \int_{-\infty}^{\infty} \left[2\, H(x) - I(x)\right]\, g(x)\, dx \\
&= 2\int_{-\infty}^{\infty} H(x)\, g(x)\, dx - \int_{-\infty}^{\infty} I(x)\, g(x)\, dx \\
&= 2\int_0^{\infty} g(x)\, dx - \int_{-\infty}^{\infty} g(x)\, dx \\
&= \int_0^{\infty} g(x)\, dx - \int_{-\infty}^{0} g(x)\, dx
\end{aligned}$$

In 1926, Dirac introduced the delta function, $\delta(x)$, having the following properties

$$\begin{gathered}
\delta(x) = 0, \qquad x \neq 0, \\
\int_{-\infty}^{\infty} \delta(x) dx = 1.
\end{gathered} \tag{2.4.9}$$

The Dirac delta function, $\delta(x)$ is defined so that for any good function $\phi(x)$,

$$\int_{-\infty}^{\infty} \delta(x)\,\phi(x)\,dx = \phi(0).$$

There is no ordinary function equivalent to the delta function.

The properties (2.4.9) cannot be satisfied by any ordinary functions in classical mathematics. Hence, the delta function is not a function in the classical sense. However, it can be treated as a function in the generalized sense, and in fact, $\delta(x)$ is called a *generalized function* or *distribution*. The concept of the delta function is clear and simple in modern mathematics. It is very useful in physics and engineering. Physically, the delta function represents a point mass, that is a particle of unit mass located at the origin. In this context, it may be called a *mass-density* function. This leads to the result for a point particle that can be considered as the limit of a sequence of continuous distributions which become more and more concentrated. Even though $\delta(x)$ is not a function in the classical sense, it can be approximated by a sequence of ordinary functions. As an example, we consider the sequence

$$\delta_n(x) = \sqrt{\frac{n}{\pi}}\,\exp(-nx^2), \quad n = 1, 2, 3, \ldots. \tag{2.4.10}$$

Clearly, $\delta_n(x) \to 0$ as $n \to \infty$ for any $x \neq 0$ and $\delta_n(0) \to \infty$ as $n \to \infty$ as shown in Figure 2.4. Also, for all $n = 1, 2, 3, \ldots$,

$$\int_{-\infty}^{\infty} \delta_n(x)dx = 1$$

and

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x)dx = \int_{-\infty}^{\infty} \delta(x)dx = 1$$

as expected. So the delta function can be considered as the limit of a sequence of ordinary functions, and we write

$$\delta(x) = \lim_{n\to\infty} \sqrt{\frac{n}{\pi}}\exp(-nx^2). \tag{2.4.11}$$

Sometimes, the delta function $\delta(x)$ is defined by its fundamental property

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a), \tag{2.4.12}$$

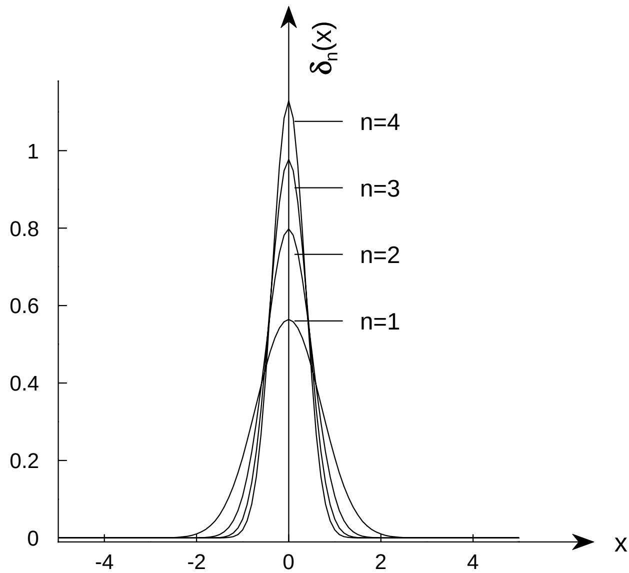


**Figure 2.4** The sequence of delta functions, $\delta_n(x)$.

where $f(x)$ is continuous in any interval containing the point $x = a$. Clearly,

$$\int_{-\infty}^{\infty} f(a)\delta(x-a)\,dx = f(a)\int_{-\infty}^{\infty}\delta(x-a)\,dx = f(a). \tag{2.4.13}$$

Thus, (2.4.12) and (2.4.13) lead to the result

$$f(x)\delta(x-a) = f(a)\delta(x-a). \tag{2.4.14}$$

The following results are also true

$$x\,\delta(x) = 0 \tag{2.4.15}$$

$$\delta(x-a) = \delta(a-x). \tag{2.4.16}$$

Result (2.4.16) shows that $\delta(x)$ is an even function.

Clearly, the result

$$\int_{-\infty}^{x}\delta(y)\,dy = \begin{Bmatrix} 1, & x>0 \\ 0, & x<0 \end{Bmatrix} = H(x)$$

shows that

$$\frac{d}{dx}H(x) = \delta(x). \tag{2.4.17}$$

The Fourier transform of the Dirac delta function is

$$\mathscr{F}\{\delta(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\delta(x)\,dx = \frac{1}{\sqrt{2\pi}}. \tag{2.4.18}$$

Hence,

$$\delta(x) = \mathscr{F}^{-1}\left\{\frac{1}{\sqrt{2\pi}}\right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}\,dk. \tag{2.4.19}$$

This is an integral representation of the *delta function* extensively used in quantum mechanics. Also, (2.4.19) can be rewritten as

$$\delta(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}\,dx. \tag{2.4.20}$$

The Dirac delta function, $\delta(x)$, is defined so that for any good function $g(x)$,

$$\langle \delta,\, g\rangle = \int_{-\infty}^{\infty} \delta(x)\,g(x)\,dx = g(0). \tag{2.4.21}$$

Derivatives of generalized functions are defined by the derivatives of any equivalent sequences of good functions. We can integrate by parts using any member of the sequences and assuming $g(x)$ vanishes at infinity. We can obtain this definition as follows:

$$\begin{aligned}\langle f',\, g\rangle &= \int_{-\infty}^{\infty} f'(x)\,g(x)\,dx \\ &= [f(x)\,g(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\,g'(x)\,dx = -\langle f,\, g'\rangle\,.\end{aligned}$$

The derivative of a generalized function $f$ is the generalized function $f'$ defined by

$$\langle f',\, g\rangle = -\langle f,\, g'\rangle \tag{2.4.22}$$

for any good function $g$.

The differential calculus of generalized functions can easily be developed with locally integrable functions. To every locally integrable function $f$, there corresponds a *generalized function* (or *distribution*) defined by

$$\langle f,\, \phi\rangle = \int_{-\infty}^{\infty} f(x)\,\phi(x)\,dx \tag{2.4.23}$$

where $\phi$ is a test function in $\mathbb{R} \to \mathbb{C}$ with bounded support ($\phi$ is infinitely differentiable with its derivatives of all orders exist and are continuous).

The derivative of a generalized function $f$ is the generalized function $f'$ defined by

$$\langle f',\, \phi\rangle = -\langle f,\, \phi'\rangle \tag{2.4.24}$$

for all test functions $\phi$. This definition follows from the fact that

$$\begin{aligned}\langle f',\, \phi\rangle &= \int_{-\infty}^{\infty} f'(x)\,\phi(x)\,dx \\ &= [f(x)\,\phi(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\,\phi'(x)\,dx = -\langle f,\, \phi'\rangle\end{aligned}$$

which was obtained from integration by parts and using the fact that $\phi$ vanishes at infinity.

It is easy to check that $H'(x) = \delta(x)$, for

$$\begin{aligned}\langle H',\, \phi\rangle &= \int_{-\infty}^{\infty} H'(x)\,\phi(x)\,dx = -\int_{-\infty}^{\infty} H(x)\,\phi'(x)\,dx \\ &= -\int_{0}^{\infty} \phi'(x)\,dx = -[\phi(x)]_0^{\infty} = \phi(0) = \langle\delta,\, \phi\rangle\,.\end{aligned}$$

Another result is

$$\langle\delta',\, \phi\rangle = -\int_{-\infty}^{\infty} \delta(x)\,\phi'(x)\,dx = -\phi'(0)\,.$$

It is easy to verify

$$f(x)\,\delta(x) = f(0)\,\delta(x)\,.$$

We next define $|x| = x\,\mathrm{sgn(x)}$ and calculate its derivative as follows. We have

$$\begin{aligned}\frac{d}{dx}|x| = \frac{d}{dx}\{x\,\mathrm{sgn(x)}\} &= x\,\frac{d}{dx}\{\mathrm{sgn(x)}\} + \mathrm{sgn(x)}\,\frac{\mathrm{dx}}{\mathrm{dx}} \\ &= x\,\frac{d}{dx}\{2\,H(x) - I(x)\} + \mathrm{sgn(x)} \\ &= 2x\,\delta(x) + \mathrm{sgn(x)} = \mathrm{sgn(x)}\end{aligned} \tag{2.4.25}$$

which is, by $\mathrm{sgn(x)} = 2\,\mathrm{H(x)} - \mathrm{I(x)}$ and $x\,\delta(x) = 0$.

Similarly, we can show that

$$\frac{d}{dx}\{\mathrm{sgn(x)}\} = 2H'(x) = 2\delta(x). \tag{2.4.26}$$

If we can show that (2.3.1) holds for good functions, it follows that it holds for generalized functions.

***THEOREM 2.4.1***
The Fourier transform of a good function is a good function.

**PROOF** The Fourier transform of a good function $f(x)$ exists and is given by

$$\mathscr{F}\{f(x)\} = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx. \tag{2.4.27}$$

Differentiating $F(k)$ $n$ times and integrating $N$ times by parts, we get

$$\begin{aligned} \left|F^{(n)}(k)\right| &\le \left| \frac{(-1)^N}{(-ik)^N} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \frac{d^N}{dx^N} \{(-ix)^n f(x)\}\, dx \right| \\ &\le \frac{1}{|k|^N} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left| \frac{d^N}{dx^N} \{x^n f(x)\} \right| dx. \end{aligned}$$

Evidently, all derivatives tend to zero as fast as $|k|^{-N}$ as $|k| \to \infty$ for any $N > 0$ and hence, $F(k)$ is a good function. ∎

***THEOREM 2.4.2***
If $f(x)$ is a good function with the Fourier transform (2.4.27), then the inverse Fourier transform is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} F(k)\, dk. \tag{2.4.28}$$

**PROOF** For any $\epsilon > 0$, we have

$$\mathscr{F}\left\{e^{-\epsilon x^2} F(-x)\right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx - \epsilon x^2} \left\{\int_{-\infty}^{\infty} e^{ixt} f(t)\, dt\right\} dx.$$

Since $f$ is a good function, the order of integration can be interchanged to obtain

$$\mathscr{F}\left\{e^{-\epsilon x^2} F(-x)\right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)\, dt \int_{-\infty}^{\infty} e^{-i(k-t)x - \epsilon x^2}\, dx$$

which is, by similar calculation used in Example 2.3.1,

$$= \frac{1}{\sqrt{4\pi\epsilon}} \int_{-\infty}^{\infty} \exp\left[-\frac{(k-t)^2}{4\epsilon}\right] f(t)\, dt\,.$$

Using the fact that

$$\frac{1}{\sqrt{4\pi\epsilon}} \int_{-\infty}^{\infty} \exp\left[-\frac{(k-t)^2}{4\epsilon}\right] dt = 1,$$

we can write

$$\mathscr{F}\left\{e^{-\epsilon x^2}F(-x)\right\}-f(k)\,.\,1$$
$$=\frac{1}{\sqrt{4\pi\epsilon}}\int_{-\infty}^{\infty}[f(t)-f(k)]\exp\left[-\frac{(k-t)^2}{4\epsilon}\right]dt. \quad (2.4.29)$$

Since $f$ is a good function, we have

$$\left|\frac{f(t)-f(k)}{t-k}\right|\le\max_{x\in\mathbb{R}}|f'(x)|\,.$$

It follows from (2.4.29) that

$$\left|\mathscr{F}\left\{e^{-\epsilon x^2}F(-x)\right\}-f(k)\right|$$
$$\le\frac{1}{\sqrt{4\pi\epsilon}}\max_{x\in\mathbb{R}}|f'(x)|\int_{-\infty}^{\infty}|t-k|\;\exp\left[-\frac{(t-k)^2}{4\epsilon}\right]dt$$
$$=\frac{1}{\sqrt{4\pi\epsilon}}\max_{x\in\mathbb{R}}|f'(x)|\;4\epsilon\int_{-\infty}^{\infty}|\alpha|\;\;e^{-\alpha^2}d\alpha\to 0$$

as $\epsilon\to 0$, where $\alpha=\frac{t-k}{2\sqrt{\epsilon}}$.
Consequently,

$$f(k)=\mathscr{F}\{F(-x)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}\,F(-x)\,dx$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}\,F(x)\,dx$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx}\,dx\int_{-\infty}^{\infty}e^{-i\xi x}\,f(\xi)\,d\xi.$$

Interchanging $k$ with $x$, this reduces to the Fourier integral formula (2.2.4) and hence, the theorem is proved. ∎

***Example 2.4.1***

The Fourier transform of a constant function $c$ is

$$\mathscr{F}\{c\}=\sqrt{2\pi}.c.\delta(k). \quad (2.4.30)$$

In the ordinary sense

$$\mathscr{F}\{c\}=\frac{c}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}\,dx$$

is not a well defined (divergent) integral. However, treated as a generalized function, $c=c\,I(x)$ and we consider $\left\{\exp\left(-\frac{x^2}{4n}\right)\right\}$ as an equivalent sequence to the unit function, $I(x)$. Thus,

$$\mathscr{F}\left\{c\,\exp\left(-\frac{x^2}{4n}\right)\right\} = \frac{c}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-ikx-\frac{x^2}{4n}\right)dx$$

which is, by Example 2.3.1,

$$\begin{aligned} &= c\sqrt{2n}\,\exp(-nk^2) = \sqrt{2\pi}.c.\sqrt{\frac{n}{\pi}}\,\exp(-nk^2) \\ &= \sqrt{2\pi}.c.\delta_n(k) = \sqrt{2\pi}.c.\delta(k) \quad \text{as} \quad n\to\infty, \end{aligned}$$

since $\{\delta_n(k)\} = \left\{\sqrt{\frac{n}{\pi}}\exp\left(-nk^2\right)\right\}$ is a sequence equivalent to the delta function defined by (2.4.10).

□

***Example 2.4.2***
Show that

$$\mathscr{F}\{e^{-ax}H(x)\} = \frac{1}{\sqrt{2\pi}(ik+a)}, \quad a>0. \tag{2.4.31}$$

We have, by definition,

$$\mathscr{F}\{e^{-ax}H(x)\} = \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\exp\{-x(ik+a)\}dx = \frac{1}{\sqrt{2\pi}(ik+a)}.$$

□

***Example 2.4.3***
By considering the function (see Figure 2.5)

$$f_a(x) = e^{-ax}H(x) - e^{ax}H(-x), \quad a>0, \tag{2.4.32}$$

find the Fourier transform of sgn(x). In Figure 2.5, the vertical axis (y-axis) represents $f_a(x)$ and the horizontal axis represents the x-axis.

We have, by definition,

$$\begin{aligned} \mathscr{F}\{f_a(x)\} &= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\exp\{(a-ik)x\}dx \\ &\quad +\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\exp\{-(a+ik)x\}dx \\ &= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{a+ik}-\frac{1}{a-ik}\right] = \sqrt{\frac{2}{\pi}}\cdot\frac{(-ik)}{a^2+k^2}. \end{aligned}$$

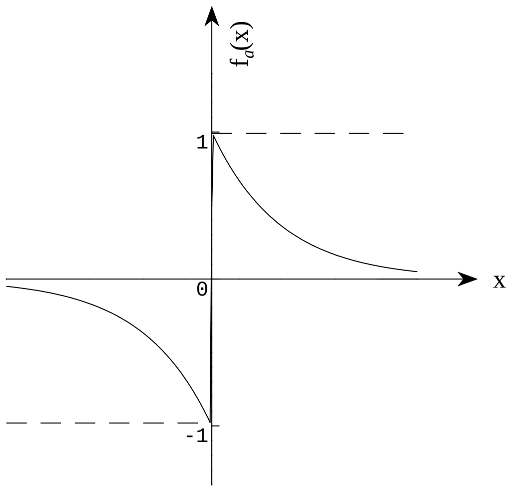


**Figure 2.5** Graph of the function $f_a(x)$.

In the limit as $a \to 0$, $f_a(x) \to \text{sgn(x)}$ and then

$$\mathscr{F}\{\text{sgn(x)}\} = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\text{ik}}.$$

Or,

$$\mathscr{F}\left\{\sqrt{\frac{\pi}{2}} i\, \text{sgn(x)}\right\} = \frac{1}{k}.$$

▯

## 2.5 Basic Properties of Fourier Transforms

***THEOREM 2.5.1***

If $\mathscr{F}\{f(x)\} = F(k)$, then

| | | | |
|---|---|---|---|
| (a) | (Shifting) | $\mathscr{F}\{f(x-a)\}=e^{-ika}\mathscr{F}\{f(x)\},$ | (2.5.1) |
| (b) | (Scaling) | $\mathscr{F}\{f(ax)\}=\dfrac{1}{\lvert a\rvert}\,F(\dfrac{k}{a}),$ | (2.5.2) |
| (c) | (Conjugate) | $\mathscr{F}\{\overline{f(-x)}\}=\overline{\mathscr{F}\{f(x)\}},$ | (2.5.3) |
| (d) | (Translation) | $\mathscr{F}\{e^{iax}f(x)\}=F(k-a),$ | (2.5.4) |
| (e) | (Duality) | $\mathscr{F}\{F(x)\}=f(-k),$ | (2.5.5) |
| (f) | (Composition) | $\int_{-\infty}^{\infty} F(k)g(k)e^{ikx}dk=\int_{-\infty}^{\infty} f(\xi)G(\xi-x)d\xi,$ where $G(k)=\mathscr{F}\{g(x)\}$. | (2.5.6) |

**PROOF** (a) We obtain, from the definition,

$$\begin{aligned}\mathscr{F}\{f(x-a)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}f(x-a)dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ik(\xi+a)}f(\xi)d\xi, \qquad (x-a=\xi) \\ &= e^{-ika}\mathscr{F}\{f(x)\}.\end{aligned}$$

The proofs of results (b)–(d) follow easily from the definition of the Fourier transform. We give a proof of the duality (e) and composition (f).

We have, by definition,

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}F(k)dk=\mathscr{F}^{-1}\{F(k)\}.$$

Interchanging $x$ and $k$, and then replacing $k$ by $-k$, we obtain

$$f(-k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}F(x)dx=\mathscr{F}\{F(x)\}.$$

To prove (f), we have

$$\int_{-\infty}^{\infty} F(k)g(k)\, e^{ikx} dk = \int_{-\infty}^{\infty} g(k)\, e^{ikx} dk \;\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik\xi} f(\xi) d\xi$$
$$= \int_{-\infty}^{\infty} f(\xi) d\xi \;\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik(\xi - x)} g(k) dk$$
$$= \int_{-\infty}^{\infty} f(\xi) G(\xi - x) d\xi.$$

In particular, when $x = 0$

$$\int_{-\infty}^{\infty} F(k) g(k) dk = \int_{-\infty}^{\infty} f(\xi) G(\xi) d\xi.$$

▮

***THEOREM 2.5.2***
If $f(x)$ is piecewise continuously differentiable and absolutely integrable, then

(i) $F(k)$ is bounded for $-\infty < k < \infty$,

(ii) $F(k)$ is continuous for $-\infty < k < \infty$.

**PROOF** It follows from the definition that

$$|F(k)| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |e^{-ikx}| |f(x)| dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)| dx = \frac{c}{\sqrt{2\pi}},$$

where $c = \int_{-\infty}^{\infty} |f(x)| dx =$ constant. This proves result (i).

To prove (ii), we have

$$|F(k+h) - F(k)| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |e^{-ihx} - 1| |f(x)| dx$$
$$\leq \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} |f(x)| dx.$$

Since $\lim_{h\to 0} |e^{-ihx} - 1| = 0$ for all $x \in \mathbb{R}$, we obtain

$$\lim_{h\to 0} |F(k+h) - F(k)| \le \lim_{h\to 0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |e^{-ihx} - 1||f(x)|dx = 0.$$

This shows that $F(k)$ is continuous. ▮

***THEOREM 2.5.3***

(*Riemann-Lebesgue Lemma*). If $F(k) = \mathscr{F}\{f(x)\}$, then

$$\lim_{|k|\to\infty} |F(k)| = 0. \tag{2.5.7}$$

**PROOF** Since $e^{-ikx} = -e^{-ikx-i\pi}$, we have

$$\begin{aligned} F(k) &= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik(x+\frac{\pi}{k})} f(x)dx \\ &= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f\left(x - \frac{\pi}{k}\right) dx. \end{aligned}$$

Hence,

$$\begin{aligned} F(k) &= \frac{1}{2} \left\{ \frac{1}{\sqrt{2\pi}} \left[ \int_{-\infty}^{\infty} e^{-ikx} f(x)dx - \int_{-\infty}^{\infty} e^{-ikx} f\left(x - \frac{\pi}{k}\right) dx \right] \right\} \\ &= \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \left[ f(x) - f\left(x - \frac{\pi}{k}\right) \right] dx. \end{aligned}$$

Therefore,

$$|F(k)| \le \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \left| f(x) - f\left(x - \frac{\pi}{k}\right) \right| dx.$$

Thus, we obtain

$$\lim_{|k|\to\infty} |F(k)| \le \frac{1}{2\sqrt{2\pi}} \lim_{|k|\to\infty} \int_{-\infty}^{\infty} \left| f(x) - f\left(x - \frac{\pi}{k}\right) \right| dx = 0.$$

▮

***THEOREM 2.5.4***
If $f(x)$ is continuously differentiable and $f(x) \to 0$ as $|x| \to \infty$, then

$$\mathscr{F}\{f'(x)\} = (ik)\mathscr{F}\{f(x)\} = ik\,F(k). \tag{2.5.8}$$

**PROOF** We have, by definition,

$$\mathscr{F}\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f'(x)dx$$

which is, integrating by parts,

$$= \frac{1}{\sqrt{2\pi}} \left[f(x)e^{-ikx}\right]_{-\infty}^{\infty} + \frac{ik}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)dx$$
$$= (ik)F(k).$$

If $f(x)$ is continuously $n$-times differentiable and $f^{(k)}(x) \to 0$ *as* $|x| \to \infty$ for $k = 1, 2, \ldots, (n-1)$, then the Fourier transform of the $n$th derivative is

$$\mathscr{F}\{f^{(n)}(x)\} = (ik)^n \mathscr{F}\{f(x)\} = (ik)^n F(k). \tag{2.5.9}$$

A repeated application of Theorem 2.5.4 to higher derivatives gives the result.

The operational results similar to those of (2.5.8) and (2.5.9) hold for partial derivatives of a function of two or more independent variables. For example, if $u(x,t)$ is a function of space variable $x$ and time variable $t$, then

$$\mathscr{F}\left\{\frac{\partial u}{\partial x}\right\} = ik\,U(k,t), \quad \mathscr{F}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -k^2\,U(k,t)\,,$$
$$\mathscr{F}\left\{\frac{\partial u}{\partial t}\right\} = \frac{dU}{dt}, \qquad \mathscr{F}\left\{\frac{\partial^2 u}{\partial t^2}\right\} = \frac{d^2U}{dt^2}\,,$$

where $U(k,t) = \mathscr{F}\,\{u(x,t)\}$. ∎

***DEFINITION 2.5.1*** *The convolution of two integrable functions $f(x)$ and $g(x)$, denoted by $(f * g)(x)$, is defined by*

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi, \tag{2.5.10}$$

*provided the integral in (2.5.10) exists, where the factor $\frac{1}{\sqrt{2\pi}}$ is a matter of choice. In the study of convolution, this factor is often omitted as this factor* *does not affect the properties of convolution. We will include or exclude the factor $\frac{1}{\sqrt{2\pi}}$ freely in this book.*

We give some examples of convolution.

***Example 2.5.1***

Find the convolution of

(a) $f(x) = \cos x$ and $g(x) = \exp(-a|x|)$, $a > 0$,

(b) $f(x) = \chi_{[a,b]}(x)$ and $g(x) = x^2$,

where $\chi_{[a,b]}(x)$ is the characteristic function of the interval $[a,b] \subseteq \mathbb{R}$ defined by

$$\chi_{[a,b]}(x) = \left\{ \begin{matrix} 1, & a \le x \le b \\ 0, & \text{otherwise} \end{matrix} \right\}.$$

(a) We have, by definition,

$$\begin{aligned}(f * g)(x) &= \int_{-\infty}^{\infty} f(x-\xi)\ g(\xi)d\xi = \int_{-\infty}^{\infty} \cos(x-\xi)\ e^{-a|\xi|}d\xi \\ &= \int_{-\infty}^{0} \cos(x-\xi)\ e^{a\xi}d\xi + \int_{0}^{\infty} \cos(x-\xi)\ e^{-a\xi}d\xi \\ &= \int_{0}^{\infty} \cos(x+\xi)\ e^{-a\xi}d\xi + \int_{0}^{\infty} \cos(x-\xi)\ e^{-a\xi}d\xi \\ &= 2\cos x \int_{0}^{\infty} \cos\xi\ e^{-a\xi}d\xi \quad = \quad \frac{2a\cos x}{(1+a^2)}.\end{aligned}$$

(b) We have

$$\begin{aligned}(f * g)(x) &= \int_{-\infty}^{\infty} f(x-\xi)\ g(\xi)d\xi = \int_{-\infty}^{\infty} \chi_{[a,b]}(x-\xi)\ g(\xi)d\xi \\ &= \int_a^b \xi^2\ d\xi = \frac{1}{3}\left(b^3 - a^3\right).\end{aligned}$$

▯

***THEOREM 2.5.5***

(*Convolution Theorem*). If $\mathscr{F}\{f(x)\} = F(k)$ and $\mathscr{F}\{g(x)\} = G(k)$, then

$$\mathscr{F}\{f(x) * g(x)\} = F(k)G(k), \tag{2.5.11}$$

or,

$$f(x) * g(x) = \mathscr{F}^{-1}\{F(k)G(k)\}, \tag{2.5.12}$$

or, equivalently,

$$\int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi = \int_{-\infty}^{\infty} e^{ikx}F(k)G(k)dk. \tag{2.5.13}$$

**PROOF** We have, by the definition of the Fourier transform,

$$\begin{aligned}
\mathscr{F}\{f(x) * g(x)\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}dx \int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ik\xi}g(\xi)d\xi \int_{-\infty}^{\infty} e^{-ik(x-\xi)}f(x-\xi)dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ik\xi}g(\xi)d\xi \int_{-\infty}^{\infty} e^{-ik\eta}f(\eta)d\eta = G(k)F(k),
\end{aligned}$$

where, in this proof, the factor $\frac{1}{\sqrt{2\pi}}$ is included in the definition of the convolution. This completes the proof. ∎

The convolution has the following algebraic properties:

$$f * g = g * f \quad \text{(Commutative)}, \tag{2.5.14}$$

$$f * (g * h) = (f * g) * h \quad \text{(Associative)}, \tag{2.5.15}$$

$$(\alpha f + \beta g) * h = \alpha\,(f * h) + \beta\,(g * h) \quad \text{(Distributive)}, \tag{2.5.16}$$

$$f * \sqrt{2\pi}\delta = f = \sqrt{2\pi}\delta * f \quad \text{(Identity)}, \tag{2.5.17}$$

where $\alpha$ and $\beta$ are constants.

We give proofs of (2.5.15) and (2.5.16). If $f * (g * h)$ exists, then

$$\begin{aligned}
[f * (g * h)]\,(x) &= \int_{-\infty}^{\infty} f(x-\xi)(g * h)(\xi)d\xi \\
&= \int_{-\infty}^{\infty} f(x-\xi) \int_{-\infty}^{\infty} g(\xi - t)h(t)\,dt \;\; d\xi
\end{aligned}$$

$$= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(x-\xi)\, g(\xi - t) d\xi \right] h(t) dt$$

$$= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(x-t-\eta) \;\; g(\eta) d\eta \right] h(t) dt \quad (\text{put } \xi - t = \eta)$$

$$= \int_{-\infty}^{\infty} [(f * g)(x-t)]\, h(t) dt$$

$$= [(f * g) * h](x),$$

where, in the above proof, under suitable assumptions, the interchange of the order of integration can be justified.

Similarly, we prove (2.5.16) using the right-hand side of (2.5.16), that is,

$$\begin{aligned} \alpha(f * h) + \beta(g * h) &= \alpha \int_{-\infty}^{\infty} f(x-\xi) h(\xi) d\xi + \beta \int_{-\infty}^{\infty} g(x-\xi) h(\xi) d\xi \\ &= \int_{-\infty}^{\infty} [\alpha f(x-\xi) + \beta g(x-\xi)]\, h(\xi) d\xi \\ &= [(\alpha f + \beta g) * h](x). \end{aligned}$$

In view of the commutative property of the convolution, (2.5.13) can be written as

$$\int_{-\infty}^{\infty} f(\xi) g(x-\xi) d\xi = \int_{-\infty}^{\infty} e^{ikx} F(k) G(k) dk. \tag{2.5.18}$$

This is valid for all real $x$, and hence, putting $x = 0$ gives

$$\int_{-\infty}^{\infty} f(\xi) g(-\xi) d\xi = \int_{-\infty}^{\infty} f(x) g(-x) dx = \int_{-\infty}^{\infty} F(k) G(k) dk. \tag{2.5.19}$$

We substitute $g(x) = \overline{f(-x)}$ to obtain

$$G(k) = \mathscr{F}\{g(x)\} = \mathscr{F}\left\{\overline{f(-x)}\right\} = \overline{\mathscr{F}\{f(x)\}} = \overline{F(k)}.$$

Evidently, (2.5.19) becomes

$$\int_{-\infty}^{\infty} f(x)\, \overline{f(x)} dx = \int_{-\infty}^{\infty} F(k)\, \overline{F(k)} dk \tag{2.5.20}$$

or,

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(k)|^2 dk. \tag{2.5.21}$$

This is well known as *Parseval's relation.*

For square integrable functions $f(x)$ and $g(x)$, the *inner product* $\langle f, g\rangle$ is defined by

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\,\overline{g(x)}dx \tag{2.5.22}$$

so the *norm* $\|f\|_2$ is defined by

$$\|f\|_2^2 = \langle f, f\rangle = \int_{-\infty}^{\infty} f(x)\,\overline{f(x)}dx = \int_{-\infty}^{\infty} |f(x)|^2 dx. \tag{2.5.23}$$

The function space $L^2(\mathbb{R})$ of all complex-valued Lebesgue square integrable functions with the inner product defined by (2.5.22) is a complete normed space with the norm (2.5.23). In terms of the norm, the Parseval relation takes the form

$$\|f\|_2 = \|F\|_2 = \|\mathscr{F} f\|_2. \tag{2.5.24}$$

This means that the Fourier transform action is *unitary*. Physically, the quantity $\|f\|_2$ is a measure of energy and $\|F\|_2$ represents the *power spectrum* of $f$.

***THEOREM 2.5.6***

(*General Parseval's Relation*). If $\mathscr{F}\{f(x)\} = F(k)$ and $\mathscr{F}\{g(x)\} = G(k)$ then

$$\int_{-\infty}^{\infty} f(x)\,\overline{g(x)}dx = \int_{-\infty}^{\infty} F(k)\,\overline{G(k)}dk. \tag{2.5.25}$$

**PROOF** We proceed formally to obtain

$$\int_{-\infty}^{\infty} F(k)\,\overline{G(k)}dk = \int_{-\infty}^{\infty} dk \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iky} f(y)\ \ dy\,\overline{\int_{-\infty}^{\infty} e^{-ikx} g(x)\ \ dx}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\ \ dy \int_{-\infty}^{\infty} \overline{g(x)}dx \int_{-\infty}^{\infty} e^{ik(x-y)}dk$$

$$= \int_{-\infty}^{\infty} \overline{g(x)}\ \ dx \int_{-\infty}^{\infty} \delta(x-y) f(y)\ \ dy = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\ \ dx.$$

In particular, when $g(x) = f(x)$, the above result agrees with (2.5.20).

We now use an indirect method to obtain the Fourier transform of sgn(x), that is,

$$\mathscr{F}\{\text{sgn(x)}\} = \sqrt{\frac{2}{\pi}}\,\frac{1}{\text{ik}}. \tag{2.5.26}$$

From (2.4.26), we find

$$\mathscr{F}\left\{\frac{d}{dx}\text{sgn(x)}\right\} = \mathscr{F}\{2H'(x)\} = 2\mathscr{F}\{\delta(x)\} = \sqrt{\frac{2}{\pi}},$$

which is, by (2.5.8),

$$ik\ \mathscr{F}\{\text{sgn(x)}\} = \sqrt{\frac{2}{\pi}},$$

or

$$\mathscr{F}\{\text{sgn(x)}\} = \sqrt{\frac{2}{\pi}}\cdot\frac{1}{\text{ik}}.$$

The Fourier transform of $H(x)$ follows from (2.4.30) and (2.5.26):

$$\begin{aligned}\mathscr{F}\{H(x)\} &= \frac{1}{2}\mathscr{F}\{1+\text{sgn(x)}\} = \frac{1}{2}[\mathscr{F}\{1\} + \mathscr{F}\{\text{sgn(x)}\}]\\ &= \sqrt{\frac{\pi}{2}}\left[\delta(k) + \frac{1}{i\pi k}\right].\end{aligned} \tag{2.5.27}$$

∎

## 2.6 Poisson's Summation Formula

A class of functions designated as $L^p(\mathbb{R})$ is of great importance in the theory of Fourier transformations, where $p(\geq 1)$ is any real number. We denote the vector space of all complex-valued functions $f(x)$ of the real variable $x$. If $f$ is a locally integrable function such that $|f|^p \in L(\mathbb{R})$, then we say $f$ is $p$-th power Lebesgue integrable. The set of all such functions is written $L^p(\mathbb{R})$. The number $||f||_p$ is called the $L^p$-norm of $f$ and is defined by

$$||f||_p = \left[\int_{-\infty}^{\infty} |f(x)|^p\ dx\right]^{\frac{1}{p}} < \infty. \tag{2.6.1}$$

Suppose $f$ is a Lebesgue integrable function on $\mathbb{R}$. Since $\exp(-ikx)$ is continuous and bounded, the product $\exp(-ikx)\,f(x)$ is locally integrable for any $k \in \mathbb{R}$. Also, $|\exp(-ikx)| \leq 1$ for all $k$ and $x$ on $\mathbb{R}$. Consider the inner product

$$\langle f,\, e^{ikx}\rangle = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx, \qquad k \in \mathbb{R}. \tag{2.6.2}$$

Clearly,

$$\left|\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx\right| \leq \int_{-\infty}^{\infty} |f(x)|\, dx = ||f||_1 < \infty. \tag{2.6.3}$$

This means that integral (2.6.2) exists for all $k \in \mathbb{R}$, and was used to define the Fourier transform, $F(k) = \mathscr{F}\{f(x)\}$ without the factor $\frac{1}{\sqrt{2\pi}}$.

Although the theory of Fourier series is a very important subject, a detailed study is beyond the scope of this book. Without rigorous analysis, we can establish a simple relation between the Fourier transform of functions in $L^1(\mathbb{R})$ and the Fourier series of related periodic functions in $L^1(-a,\, a)$ of period $2a$. If $f(x) \in L^1(-a,\, a)$ and is defined by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \qquad (-a \leq x \leq a), \tag{2.6.4}$$

where the Fourier coefficients $c_n$ is given by

$$c_n = \frac{1}{2a}\int_{-a}^{a} f(x)\, e^{-ikx}\, dx. \tag{2.6.5}$$

***THEOREM 2.6.1***

If $f(x) \in L^1(\mathbb{R})$, then the series

$$\sum_{n=-\infty}^{\infty} f(x + 2na) \tag{2.6.6}$$

converges absolutely for almost all $x$ in $(-a,\, a)$ and its sum $g(x) \in L^1(-a,\, a)$ with $g(x + 2a) = g(x)$ for $x \in \mathbb{R}$.

If $a_n$ denotes the Fourier coefficient of a function $g$, then

$$a_n = \frac{1}{2a}\int_{-a}^{a} g(x)\, e^{-inx}\, dx = \frac{1}{2a}\int_{-\infty}^{\infty} f(x)\, e^{-inx}\, dx = \frac{1}{2a} F(n).$$

**PROOF** We have

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \int_{-a}^{a} |f(x + 2na)|\, dx &= \lim_{N\to\infty} \sum_{n=-N}^{N} \int_{-a}^{a} |f(x + 2na)|\, dx \\
&= \lim_{N\to\infty} \sum_{n=-N}^{N} \int_{(2n-1)a}^{(2n+1)a} |f(t)|\, dt \\
&= \lim_{N\to\infty} \int_{-(2N+1)a}^{(2N+1)a} |f(t)|\, dt \\
&= \int_{-\infty}^{\infty} |f(t)|\, dt < \infty.
\end{aligned}$$

It follows from Lebesgue's theorem on monotone convergence that

$$\int_{-a}^{a}\left[\sum_{n=-\infty}^{\infty}|f(x+2na)|\right]\,dx=\sum_{n=-\infty}^{\infty}\int_{-a}^{a}|f(x+2na)|\,dx<\quad\infty.$$

Hence, the series $\sum_{n=-\infty}^{\infty} f(x+2na)$ converges absolutely for almost all $x$ in $(-a,\,a)$. If $g_N(x)=\sum_{n=-N}^{N} f(x+2na)$, $\lim_{N\to\infty} g_N(x)=g(x)$, where $g\in\mathbb{L}^1(-a,\,a)$, and $g(x+2a)=g(x)$.

Moreover,

$$\begin{aligned}||g||_1&=\int_{-a}^{a}|g(x)|\,dx=\int_{-a}^{a}\left|\sum_{n=-\infty}^{\infty} f(x+2na)\right|\,dx\\&\le\int_{-a}^{a}\sum_{n=-\infty}^{\infty}|f(x+2na)|\,dx\\&=\sum_{n=-\infty}^{\infty}\int_{-a}^{a}|f(x+2na)|\,dx\\&=\int_{-\infty}^{\infty}|f(x)|\,dx=||f||_1.\end{aligned}$$

▮

We consider the Fourier series of $g(x)$ given by

$$g(x)=\sum_{m=-\infty}^{\infty} c_m\,\exp(im\pi x/a), \tag{2.6.7}$$

where the coefficients $c_m$ for $m=0,\pm1,\pm2,...$ are given by

$$c_m=\frac{1}{2a}\int_{-a}^{a} g(x)\exp(-im\pi x/a)dx. \tag{2.6.8}$$

We replace $g(x)$ by the limit of the sum

$$g(x)=\lim_{N\to\infty}\sum_{n=-N}^{N} f(x+2na), \tag{2.6.9}$$

so that (2.6.8) reduces to

$$\begin{aligned}
c_m &= \frac{1}{2a} \lim_{N\to\infty} \sum_{n=-N}^{N} \int_{-a}^{a} f(x+2na)\exp(-im\pi x/a)\,dx \\
&= \frac{1}{2a} \lim_{N\to\infty} \sum_{n=-N}^{N} \int_{(2n-1)a}^{(2n+1)a} f(y)\exp(-im\pi y/a)\;\;dy \\
&= \frac{1}{2a} \lim_{N\to\infty} \int_{-(2N+1)a}^{(2N+1)a} f(x)\exp(-im\pi x/a)\;\;dx \\
&= \frac{\sqrt{2\pi}}{2a} F\left(\frac{m\pi}{a}\right), \qquad (2.6.10)
\end{aligned}$$

where $F\left(\frac{m\pi}{a}\right)$ is the discrete Fourier transform of $f(x)$.

Evidently,

$$\sum_{n=-\infty}^{\infty} f(x+2na) = g(x) = \sum_{n=-\infty}^{\infty} \frac{\sqrt{2\pi}}{2a} F\left(\frac{n\pi}{a}\right)\;\exp(in\pi x/a). \qquad (2.6.11)$$

We let $x=0$ in (2.6.11) to obtain the *Poisson summation formula*

$$\sum_{n=-\infty}^{\infty} f(2na) = \sum_{n=-\infty}^{\infty} \frac{\sqrt{2\pi}}{2a}\, F\left(\frac{n\pi}{a}\right). \qquad (2.6.12)$$

When $a=\pi$, this formula becomes

$$\sum_{n=-\infty}^{\infty} f(2\pi n) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} F(n). \qquad (2.6.13)$$

When $2a=1$, formula (2.6.12) becomes

$$\sum_{n=-\infty}^{\infty} f(n) = \sqrt{2\pi} \sum_{n=-\infty}^{\infty} F(2n\pi). \qquad (2.6.14)$$

To obtain a more general formula, we assume that $a$ is a given positive constant, and write $g(x)=f(ax)$ for all $x$. Then

$$f\left(a.\frac{2\pi n}{a}\right) = g\left(\frac{2\pi n}{a}\right),$$

and we define the Fourier transform of $f(x)$ without the factor $\frac{1}{\sqrt{2\pi}}$ so that

$$\begin{aligned}
F(n) &= \int_{-\infty}^{\infty} e^{-inx} f(x)\, dx = \int_{-\infty}^{\infty} e^{-inx} f\left(a.\frac{x}{a}\right)\, dx \\
&= \int_{-\infty}^{\infty} e^{-inx} g\left(\frac{x}{a}\right)\, dx \\
&= a \int_{-\infty}^{\infty} e^{-i(an)y}\, g(y)\, dy \\
&= a\, G(an).
\end{aligned}$$

Consequently, equality (2.6.13) reduces to

$$\sum_{n=-\infty}^{\infty} g\left(\frac{2\pi n}{a}\right) = \frac{a}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} G(an). \tag{2.6.15}$$

Putting $b = \frac{2\pi}{a}$ in (2.6.15) gives

$$\sum_{n=-\infty}^{\infty} g(bn) = \sqrt{2\pi}\, b^{-1} \sum_{n=-\infty}^{\infty} G(2\pi b^{-1} n). \tag{2.6.16}$$

When $b = 2\pi$, result (2.6.16) becomes (2.6.13). We apply these formulas to prove the following series

(a) $$\sum_{n=-\infty}^{\infty} \frac{1}{(n^2 + b^2)} = \frac{\pi}{b} \coth(\pi b), \tag{2.6.17}$$

(b) $$\sum_{n=-\infty}^{\infty} \exp(-\pi n^2 t) = \frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{\pi n^2}{t}\right), \tag{2.6.18}$$

(c) $$\sum_{n=-\infty}^{\infty} \frac{1}{(x + n\pi)^2} = \operatorname{cosec}^2(x). \tag{2.6.19}$$

To prove (a), we write $f(x) = (x^2 + b^2)^{-1}$ so that $F(k) = \sqrt{\frac{\pi}{2}}\, \frac{1}{b} \exp(-b|k|)$. We now use (2.6.14) to derive

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{1}{(n^2 + b^2)} &= \frac{\pi}{b} \sum_{n=-\infty}^{\infty} \exp(-2|n|\pi b) \\
&= \frac{\pi}{b} \left[ \sum_{n=0}^{\infty} \exp(-2n\pi b) + \sum_{n=1}^{\infty} \exp(2n\pi b) \right]
\end{aligned}$$

which is, by writing $r = \exp(-2\pi b)$,

$$\begin{aligned}
&= \frac{\pi}{b} \left[ \sum_{n=0}^{\infty} r^n + \sum_{n=1}^{\infty} \left(\frac{1}{r}\right)^n \right] = \frac{\pi}{b} \left( \frac{r}{1-r} + \frac{1}{1-r} \right) \\
&= \frac{\pi}{b} \left( \frac{1+r}{1-r} \right) = \frac{\pi}{b} \coth(\pi b).
\end{aligned}$$

It follows from (2.6.14) that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n^2+b^2)} = \frac{\pi}{b}\frac{\left(1+e^{-2\pi b}\right)}{\left(1-e^{-2\pi b}\right)}.$$

Or,

$$2\sum_{n=1}^{\infty} \frac{1}{(n^2+b^2)} + \frac{1}{b^2} = \frac{\pi}{b}\frac{\left(1+e^{-2\pi b}\right)}{\left(1-e^{-2\pi b}\right)}.$$

It turns out that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{(n^2+b^2)} &= \frac{\pi}{2b}\left[\frac{\left(1+e^{-2\pi b}\right)}{\left(1-e^{-2\pi b}\right)} - \frac{1}{\pi b}\right]\\ &= \frac{\pi^2}{x}\left[\frac{(1+e^{-x})}{(1-e^{-x})} - \frac{2}{x}\right], \qquad (2\pi b = x)\\ &= \frac{\pi^2}{x^2}\left[\frac{x\,(1+e^{-x}) - 2\,(1-e^{-x})}{(1-e^{-x})}\right]\\ &= \left(\frac{\pi}{x}\right)^2\left[\frac{x^3\left(\frac{1}{2}-\frac{1}{3}\right) - \frac{x^4}{12} + \ldots.}{x - \frac{x^2}{2!} + \frac{x^3}{3!} - \ldots.}\right].\end{aligned}$$

In the limit as $b \to 0\,(x \to 0)$, we obtain the well-known result

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{2.6.20}$$

To prove (b), we assume $f(x) = \exp(-\pi t x^2)$ so that $F(k) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{k^2}{4\pi t}\right)$. Thus, the Poisson formula (2.6.14) gives

$$\sum_{n=-\infty}^{\infty} \exp(-\pi t n^2) = \frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} \exp(-\pi n^2/t).$$

This identity plays an important role in number theory and in the theory of elliptic functions. The *Jacobi theta function* $\Theta(s)$ is defined by

$$\Theta(s) = \sum_{n=-\infty}^{\infty} \exp(-\pi s n^2), \qquad s > 0, \tag{2.6.21}$$

so that (2.6.16) gives the *functional equation* for the theta function

$$\sqrt{s}\,\Theta(s) = \Theta\left(\frac{1}{s}\right). \tag{2.6.22}$$

The theta function $\Theta(s)$ also extends to complex values of $s$ when $Re(s) > 0$ and the functional equation is still valid for complex $s$. The theta function is closely related to the *Riemann zeta function* $\zeta(s)$ defined for $Re(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{2.6.23}$$

An integral representation of $\zeta(s)$ can be found from the result

$$\int_0^{\infty} x^{s-1}\, e^{-nx} dx = \frac{\Gamma(s)}{n^s}, \qquad Re(s) > 0,$$

where the *gamma function* $\Gamma(s)$ is defined by

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt, \qquad Re(s) > 0.$$

Summing both sides of this result and interchanging the order of summation and integration, which is permissible for $Re(s) > 1$, gives

$$\Gamma(s)\,\zeta(s) = \int_0^{\infty} x^{s-1} \frac{dx}{e^x - 1}, \qquad Re(s) > 1. \tag{2.6.24}$$

It turns out that $\zeta(s)$, $\Theta(s)$, and $\Gamma(s)$ are related by the following identity:

$$\zeta(s)\Gamma(s/2) = \frac{1}{2}\pi^{s/2} \int_0^{\infty} x^{s/2-1} \left[\Theta(x) - 1\right] dx, \qquad Re(s) > 1. \tag{2.6.25}$$

Considering the complex integral in a suitable closed contour $C$

$$I = \frac{1}{2\pi i} \int_C \frac{z^{s-1}}{e^{-z} - 1} dz,$$

and using the Cauchy residue theorem with all zeros of $(e^{-z} - 1)$ at $z = 2\pi i n$, $n = \pm 1, \pm 2, ..., \pm N$ gives

$$I = -2 \sin\left(\frac{\pi s}{2}\right) \sum_{n=1}^{\infty} (2\pi n)^{s-1}.$$

To prove (c), we use the Fourier transform of the function $f(x) = (1 - |x|)$ $H\,(1 - |x|)$ to obtain the result. In the limit as $N \to \infty$, the sum of the residues is convergent so that the integral gives the relation

$$2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \zeta(1 - s) = \frac{\zeta(s)}{\Gamma(1 - s)}. \tag{2.6.26}$$

In view of another relation for the gamma function, $\Gamma(1 + z)\Gamma(-z) = -\frac{\pi}{\sin \pi z}$, the relation (2.6.26) leads to a famous functional relation for $\zeta(s)$ in the form

$$\pi^s \zeta(1 - s) = 2^{1-s} \Gamma(s) \cos\left(\frac{\pi s}{2}\right) \zeta(s). \tag{2.6.27}$$

## 2.7 The Shannon Sampling Theorem

An analog signal $f(t)$ is a continuous function of time $t$ defined in $-\infty < t < \infty$, with the exception of perhaps a countable number of jump discontinuities. Almost all analog signals $f(t)$ of interest in engineering have finite energy. By this we mean that $f \in L^2(-\infty, \infty)$. The norm of $f$ defined by

$$||f|| = \left[\int_{-\infty}^{\infty} |f(x)|^2 \, dx\right]^{\frac{1}{2}} \tag{2.7.1}$$

represents the square root of the total energy content of the signal $f(t)$. The *spectrum* of a signal $f(t)$ is represented by its Fourier transform $F(\omega)$, where $\omega$ is called the *frequency*. The frequency is measured by $\nu = \frac{\omega}{2\pi}$ in terms of Hertz.

A continuous signal $f(t)$ is called *band limited* if its Fourier transform $F(\omega)$ is zero except in a finite interval, that is, if

$$F_a(\omega) = 0 \qquad \text{for} \quad |\omega| > a. \tag{2.7.2}$$

Then $a(> 0)$ is called the *cutoff frequency.*

In particular, if

$$F(\omega) = \begin{Bmatrix} 1, & |\omega| \le a \\ 0, & |\omega| > a \end{Bmatrix} \tag{2.7.3}$$

then $F(\omega)$ is called a *gate function* and is denoted by $F_a(\omega)$, and the band limited signal is denoted by $f_a(t)$. If $a$ is the smallest value for which (2.7.2) holds, it is called the *bandwidth* of the signal. Even if an analog signal $f(t)$ is not band-limited, we can reduce it to a band-limited signal by what is called an *ideal low-pass filtering.* To reduce $f(t)$ to a band-limited signal $f_a(t)$ with bandwidth less than or equal to $a$, we consider

$$F_a(\omega) = \begin{Bmatrix} F(\omega), & |\omega| \le a \\ 0, & |\omega| > a \end{Bmatrix} \tag{2.7.4}$$

and find the low-pass filter function $f_a(t)$ by the inverse Fourier transform

$$f_a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} F_a(\omega) d\omega = \frac{1}{2\pi}\int_{-a}^{a} e^{i\omega t} F_a(\omega) d\omega. \tag{2.7.5}$$

This function $f_a(t)$ is called the *Shannon sampling function.* When $a = \pi$, $f_\pi(t)$ is called the *Shannon scaling function.* The band-limited signal $f_a(t)$ is given by

$$f_a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega = \frac{1}{2\pi}\int_{-a}^{a} e^{i\omega t} d\omega = \frac{\sin at}{\pi t}. \tag{2.7.6}$$

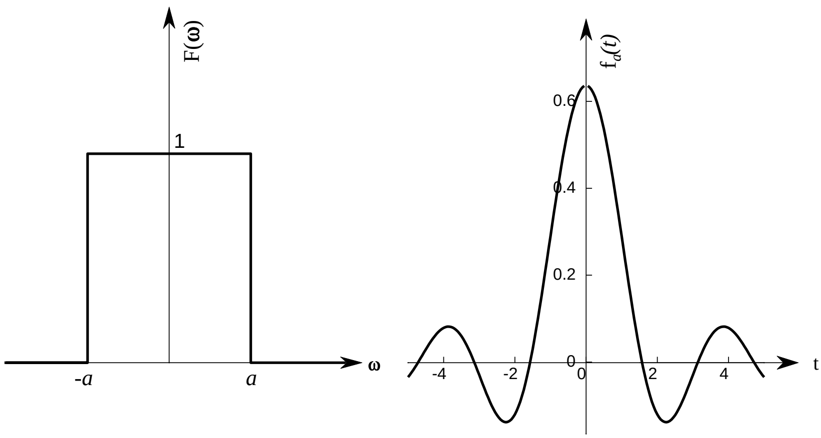


**Figure 2.6** The gate function and its Fourier transform.

Both $F(\omega)$ and $f_a(t)$ are shown in Figure 2.6 for $a = 2$.

Consider the limit as $a \to \infty$ of the Fourier integral for $-\infty < \omega < \infty$

$$1 = \lim_{a\to\infty} \int_{-\infty}^{\infty} e^{-i\omega t} f_a(t) dt = \lim_{a\to\infty} \int_{-\infty}^{\infty} e^{-i\omega t} \frac{\sin at}{\pi t} dt$$
$$= \int_{-\infty}^{\infty} e^{-i\omega t} \left[ \lim_{a\to\infty} \frac{\sin at}{\pi t} \right] dt = \int_{-\infty}^{\infty} e^{-i\omega t} \delta(t) dt.$$

Clearly, the delta function $\delta(t)$ can be thought of as the limit of the sequence of functions $f_a(t)$. More precisely,

$$\delta(t) = \lim_{a\to\infty} \left( \frac{\sin at}{\pi t} \right). \tag{2.7.7}$$

We next consider the band-limited signal

$$f_a(t) = \frac{1}{2\pi} \int_{-a}^{a} F(\omega) e^{i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, F_a(\omega)\, e^{i\omega t} d\omega,$$

which is, by the Convolution Theorem,

$$f_a(t) = \int_{-\infty}^{\infty} f(\tau) f_a(t-\tau) d\tau = \int_{-\infty}^{\infty} \frac{\sin a(t-\tau)}{\pi(t-\tau)} f(\tau) d\tau. \tag{2.7.8}$$

This integral represents the *sampling integral representation* of the band-limited signal $f_a(t)$.

**Example 2.7.1**

(*Synthesis and Resolution of a Signal; Physical Interpretation of Convolution*). In electrical engineering problems, a time-dependent electric, optical or electromagnetic *pulse* is usually called a *signal*. Such a signal can be considered as a superposition of plane waves of all real frequencies so that it can be represented by the inverse Fourier transform

$$f(t) = \mathscr{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega, \tag{2.7.9}$$

where $F(\omega) = \mathscr{F}\{f(t)\}$, the factor $(1/2\pi)$ is introduced because the angular frequency $\omega$ is related to linear frequency $\nu$ by $\omega = 2\pi\nu$, and negative frequencies are introduced for mathematical convenience so that we can avoid dealing with the cosine and sine functions separately. Clearly, $F(\omega)$ can be represented by the Fourier transform of the signal $f(t)$ as

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt. \tag{2.7.10}$$

This represents the *resolution* of the signal into its angular frequency components, and (2.7.9) gives a *synthesis* of the signal from its individual components.

Consider a simple electrical device such as an amplifier with an input signal $f(t)$, and an output signal $g(t)$. For an input of a single frequency $\omega$, $f(t) = e^{i\omega t}$. The amplifer will change the amplitude and may also change the phase so that the output can be expressed in terms of the input, the amplitude and the phase modifying function $\Phi(\omega)$ as

$$g(t) = \Phi(\omega)f(t), \tag{2.7.11}$$

where $\Phi(\omega)$ is usually known as the *transfer function* and is, in general, a complex function of the real variable $\omega$. This function is generally independent of the presence or absence of any other frequency components. Thus, the total output may be found by integrating over the entire input as modified by the amplifier

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(\omega)F(\omega)\, e^{i\omega t}d\omega. \tag{2.7.12}$$

Thus, the total output signal can readily be calculated from any given input signal $f(t)$. On the other hand, the transfer function $\Phi(\omega)$ is obviously characteristic of the amplifier device and can, in general, be obtained as the Fourier transform of some function $\phi(t)$ so that

$$\Phi(\omega) = \int_{-\infty}^{\infty} \phi(t)e^{-i\omega t}dt. \tag{2.7.13}$$

The Convolution Theorem 2.5.5 allows us to rewrite (2.7.12) as

$$g(t) = \mathscr{F}^{-1}\{\Phi(\omega)F(\omega)\} = f(t) * \phi(t) = \int_{-\infty}^{\infty} f(\tau)\phi(t-\tau)d\tau. \tag{2.7.14}$$

Physically, this result represents an output signal $g(t)$ as the integral superposition of an input signal $f(t)$ modified by $\phi(t-\tau)$. Linear translation invariant systems, such as *sensors* and *filters*, are modeled by the convolution equations $g(t) = f(t) * \phi(t)$, where $\phi(t)$ is the system impulse response function. In fact (2.7.14) is the most general mathematical representation of an output (effect) function in terms of an input (cause) function modified by the amplifier where $t$ is the time variable. Assuming the principle of causality, that is, every effect has a cause, we must require $\tau < t$. The principle of causality is imposed by requiring

$$\phi(t-\tau) = 0 \qquad \text{when } \tau > t. \tag{2.7.15}$$

Consequently, (2.7.14) gives

$$g(t) = \int_{-\infty}^{t} f(\tau)\phi(t-\tau)d\tau. \tag{2.7.16}$$

In order to determine the significance of $\phi(t)$, we use an impulse function $f(\tau) = \delta(\tau)$ so that (2.7.16) becomes

$$g(t) = \int_{-\infty}^{t} \delta(\tau)\phi(t-\tau)d\tau = \phi(t)H(t). \tag{2.7.17}$$

This recognizes $\phi(t)$ as the output corresponding to a unit impulse at $t = 0$, and the Fourier transform of $\phi(t)$ is

$$\Phi(\omega) = \mathscr{F}\{\phi(t)\} = \int_{0}^{\infty} \phi(t)e^{-i\omega t}dt, \tag{2.7.18}$$

with $\phi(t) = 0$ for $t < 0$. ▯

***Example 2.7.2***
(*The Series Sampling Expansion of a Bandlimited Signal*). Consider a band-limited signal $f_a(t)$ with Fourier transform $F(\omega) = 0$ for $|\omega| > a$. We write the Fourier series expansion of $F(\omega)$ on the interval $-a < \omega < a$ in terms of the orthogonal set of functions $\left\{\exp\left(-\frac{in\pi\omega}{a}\right)\right\}$ in the form

$$F(\omega) = \sum_{n=-\infty}^{\infty} a_n \exp\left(-\frac{in\pi}{a}\omega\right), \tag{2.7.19}$$

where the Fourier coefficients $a_n$ are given by

$$a_n = \frac{1}{2a} \int_{-a}^{a} F(\omega) \exp\left(\frac{in\pi}{a}\omega\right) d\omega = \frac{1}{2a} \; f_a\left(\frac{n\pi}{a}\right). \tag{2.7.20}$$

Thus, the Fourier series expansion (2.7.19) becomes

$$F(\omega) = \frac{1}{2a} \sum_{n=-\infty}^{\infty} f_a\left(\frac{n\pi}{a}\right) \exp\left(-\frac{in\pi}{a}\omega\right). \tag{2.7.21}$$

The signal function $f_a(t)$ is obtained by multiplying (2.7.21) by $e^{i\omega t}$ and integrating over $(-a, a)$ so that

$$\begin{aligned}
f_a(t) &= \int_{-a}^{a} F(\omega) e^{i\omega t} d\omega \\
&= \frac{1}{2a} \int_{-a}^{a} e^{i\omega t} d\omega \left[\sum_{n=-\infty}^{\infty} f_a\left(\frac{n\pi}{a}\right) \exp\left(-\frac{in\pi}{a}\omega\right)\right] \\
&= \frac{1}{2a} \sum_{n=-\infty}^{\infty} f_a\left(\frac{n\pi}{a}\right) \int_{-a}^{a} \exp\left[i\omega\left(t - \frac{n\pi}{a}\right)\right] d\omega \\
&= \sum_{n=-\infty}^{\infty} f_a\left(\frac{n\pi}{a}\right) \frac{\sin a\left(t - \frac{n\pi}{a}\right)}{a\left(t - \frac{n\pi}{a}\right)} \\
&= \sum_{n=-\infty}^{\infty} f_a\left(\frac{n\pi}{a}\right) \frac{\sin(at - n\pi)}{(at - n\pi)}.
\end{aligned} \tag{2.7.22}$$

This result is the main content of the sampling theorem. It simply states that a band-limited signal $f_a(t)$ can be reconstructed from the infinite set of discrete samples of $f_a(t)$ at $t = 0, \pm\frac{\pi}{a}, \ldots$ . In practice, a discrete set of samples is useful in the sense that most systems receive discrete samples $\{f(t_n)\}$ as an input. The sampling theorem can be realized physically. Modern telephone equipment employs sampling to send messages over wires. In fact, it seems that sampling is audible on some transoceanic cable calls.

Result (2.7.22) can be obtained from the convolution theorem by using discrete input samples

$$\sum_{n=-\infty}^{\infty} \frac{\pi}{a} f_a\left(\frac{n\pi}{a}\right) \delta\left(t - \frac{n\pi}{a}\right) = f(t). \tag{2.7.23}$$

Hence, the sampling expansion (2.7.8) gives the band-limited signal

$$f_a(t) = \int_{-\infty}^{\infty} \frac{\sin a(t-\tau)}{\pi(t-\tau)} \left[ \sum_{n=-\infty}^{\infty} \frac{\pi}{a} f_a\left(\frac{n\pi}{a}\right) \delta\left(\tau - \frac{n\pi}{a}\right) \right] d\tau$$

$$= \sum_{n=-\infty}^{\infty} f_a\left(\frac{n\pi}{a}\right) \int_{-\infty}^{\infty} \frac{\sin a(t-\tau)}{a(t-\tau)} \delta\left(\tau - \frac{n\pi}{a}\right) d\tau$$

$$= \sum_{n=-\infty}^{\infty} f_a\left(\frac{n\pi}{a}\right) \frac{\sin a\left(t - \frac{n\pi}{a}\right)}{a\left(t - \frac{n\pi}{a}\right)}. \tag{2.7.24}$$

$\square$

In general, the output can be best described by taking the Fourier transform of (2.7.14) so that

$$G(\omega) = F(\omega)\Phi(\omega), \tag{2.7.25}$$

where $\Phi(\omega)$ is called the *transfer function* of the system. Thus, the output can be calculated from (2.7.25) by the Fourier inversion formula

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, \Phi(\omega)\, e^{i\omega t}\, d\omega, \tag{2.7.26}$$

Obviously, the transfer function $\Phi(\omega)$ is a characteristic of a linear system. A linear system is a *filter* if it possesses signals of certain frequencies and attenuates others. If the transfer function

$$\Phi(\omega) = 0 \qquad |\omega| \ge \omega_0, \tag{2.7.27}$$

then $\phi(t)$, the Fourier inverse of $\Phi(\omega)$, is called a *low-pass filter.*

On the other hand, if the transfer function

$$\Phi(\omega) = 0 \qquad |\omega| \le \omega_1, \tag{2.7.28}$$

then $\phi(t)$ is a *high-pass filter.* A *bandpass filter* possesses a band $\omega_0 \le |\omega| \le \omega_1$. It is often convenient to express the system transfer function $\Phi(\omega)$ in the complex form

$$\Phi(\omega) = A(\omega) \exp[-i\theta(\omega)], \tag{2.7.29}$$

where $A(\omega)$ is called the *amplitude* and $\theta(\omega)$ is called the *phase* of the transfer function. Obviously, the system impulse response $\phi(t)$ is given by the inverse Fourier transform

$$\phi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\omega) \exp[i\{\omega t - \theta(\omega)\}]\, d\omega. \tag{2.7.30}$$

For a unit step function as the input $f(t) = H(t)$, we have

$$F(\omega) = \hat{H}(\omega) = \left(\pi\delta(\omega) + \frac{1}{i\omega}\right),$$

where $\hat{H}(\omega) = \mathscr{F}\{H(t)\}$ and the associated output $g(t)$ is then given by

$$\begin{aligned} g(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(\omega)\hat{H}(\omega)e^{i\omega t}\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\pi\delta(\omega) + \frac{1}{i\omega}\right) A(\omega)\exp[i\{\omega t - \theta(\omega)\}]\,d\omega \\ &= \frac{1}{2}A(0) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{A(\omega)}{\omega}\exp\left[i\left\{\omega t - \theta(\omega) - \frac{\pi}{2}\right\}\right]\,d\omega\,. \end{aligned} \quad (2.7.31)$$

We next give another characterization of a filter in terms of the amplitude of the transfer function.

A filter is called *distortionless* if its output $g(t)$ to an arbitrary input $f(t)$ has the same form as the input, that is,

$$g(t) = A_0 f(t - t_0). \quad (2.7.32)$$

Evidently,

$$G(\omega) = A_0 e^{-i\omega t_0}\,F(\omega) = \Phi(\omega)\,F(\omega)$$

where

$$\Phi(\omega) = A_0 e^{-i\omega t_0}$$

represents the transfer function of the distortionless filter. It has a constant amplitude $A_0$ and a linear phase shift $\theta(\omega) = \omega t_0$.

However, in general, the amplitude $A(\omega)$ of a transfer function is not constant, and the phase $\theta(\omega)$ is not a linear function.

A filter with constant amplitude, $|\theta(\omega)| = A_0$ is called an *all-pass filter*. It follows from Parseval's formula that the energy of the output of such a filter is proportional to the energy of its input.

A filter whose amplitude is constant for $|\omega| < \omega_0$ and zero for $|\omega| > \omega_0$ is called an *ideal low-pass filter*. More explicitly, the amplitude is given by

$$A(\omega) = A_0\hat{H}(\omega_0 - |\omega|) = A_0\hat{\chi}_{\omega_0}(\omega)\,, \quad (2.7.33)$$

where $\hat{\chi}_{\omega_0}(\omega)$ is a rectangular pulse. So, the transfer function of the low-pass filter is

$$\Phi(\omega) = A_0\hat{\chi}_{\omega_0}(\omega)\exp(-i\omega t_0)\,. \quad (2.7.34)$$

Finally, the *ideal high-pass filter* is characterized by its amplitude given by

$$A(\omega) = A_0\hat{H}(|\omega| - \omega_0) = A_0\hat{\chi}_{\omega_0}(\omega)\,, \quad (2.7.35)$$

where $A_0$ is a constant. Its transfer function is given by

$$\Phi(\omega) = A_0 \left[1 - \hat{\chi}_{\omega_0}(\omega)\right] \exp(-i\omega t_0). \tag{2.7.36}$$

**Example 2.7.3**

(*Bandwidth and Bandwidth Equation*). The Fourier spectrum of a signal (or waveform) gives an indication of the frequencies that exist during the total duration of the signal (or *waveform*). From the knowledge of the frequencies that are present, we can calculate the average frequency and the spread about that average. In particular, if the signal is represented by $f(t)$, we can define its Fourier spectrum by

$$F(\nu) = \int_{-\infty}^{\infty} e^{-2\pi i\nu t} f(t)\, dt. \tag{2.7.37}$$

Using $|F(\nu)|^2$ for the density in frequency, the average frequency is denoted by $<\nu>$ and defined by

$$<\nu> = \int_{-\infty}^{\infty} \nu\, |F(\nu)|^2\, d\nu. \tag{2.7.38}$$

The bandwidth is then the *root mean square* (RMS) deviation at about the average, that is,

$$B^2 = \int_{-\infty}^{\infty} (\nu - <\nu>)^2\, d\nu. \tag{2.7.39}$$

Expressing the signal in terms of its amplitude and phase

$$f(t) = a(t)\, \exp\{i\theta t\}, \tag{2.7.40}$$

the instantaneous frequency, $\nu(t)$ is the frequency at a particular time defined by

$$\nu(t) = \frac{1}{2\pi}\, \theta'(t). \tag{2.7.41}$$

Substituting (2.7.37) and (2.7.40) into (2.7.38) gives

$$<\nu> = \frac{1}{2\pi} \int_{-\infty}^{\infty} \theta'(t)\, a^2(t)\, dt = \int_{-\infty}^{\infty} \nu(t)\, a^2(t)\, dt. \tag{2.7.42}$$

This formula states that the average frequency is the average value of the instantaneous frequency weighted by the square of the amplitude of the signal.

We next derive the bandwidth equation in terms of the amplitude and phase of the signal in the form

$$B^2 = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \left[\frac{a'(t)}{a(t)}\right]^2 a^2(t)\, dt + \int_{-\infty}^{\infty} \left[\frac{1}{2\pi}\, \theta'(t) - <\nu>\right]^2 a^2(t)\, dt. \tag{2.7.43}$$

A straightforward but lengthy way to derive it is to substitute (2.7.40) into (2.7.39) and simplify. However, we give an elegant derivation of (2.7.43) by representing the frequency by the operator

$$\nu = \frac{1}{2\pi i}\,\frac{d}{dt}. \tag{2.7.44}$$

We calculate the average by sandwiching the operator between the complex conjugate of the signal and the signal. Thus,

$$\begin{aligned}
<\nu> &= \int_{-\infty}^{\infty} \nu\,|F(\nu)|^2\,d\nu = \int_{-\infty}^{\infty} \bar{f}(t)\left[\frac{1}{2\pi i}\,\frac{d}{dt}\right] f(t)\,dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} a(t)\left\{-ia'(t) + a(t)\theta'(t)\right\}\,dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} -\frac{1}{2}i\left[\frac{d}{dt}a^2(t)\right]\,dt + \frac{1}{2\pi}\int_{-\infty}^{\infty} a^2(t)\theta'(t)\,dt \qquad (2.7.45) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \theta'(t)a^2(t)\,dt \qquad (2.7.46)
\end{aligned}$$

provided the first integral in (2.7.44) vanishes if $a(t) \to 0$ as $|t| \to \infty$.

It follows from the definition (2.7.39) of the bandwidth that

$$\begin{aligned}
B^2 &= \int_{-\infty}^{\infty} (\nu - <\nu>)^2\,|F(\nu)|^2\,d\nu \\
&= \int_{-\infty}^{\infty} \bar{f}(t)\left[\frac{1}{2\pi i}\,\frac{d}{dt} - <\nu>\right]^2 f(t)\,dt \\
&= \int_{-\infty}^{\infty} \left|\left[\frac{1}{2\pi i}\,\frac{d}{dt} - <\nu>\right] f(t)\right|^2\,dt \\
&= \int_{-\infty}^{\infty} \left|\frac{1}{2\pi i}\,\frac{a'(t)}{a(t)} + \frac{1}{2\pi}\theta'(t) - <\nu>\right|^2\,a^2(t)\,dt \\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \left[\frac{a'(t)}{a(t)}\right]^2 a^2(t)\,dt + \int_{-\infty}^{\infty} \left[\frac{1}{2\pi}\,\theta'(t) - <\nu>\right]^2 a^2(t)\,dt.
\end{aligned}$$

This completes the derivation. ▯

Physically, the second term in equation (2.7.43) gives averages of all of the deviations of the instantaneous frequency from the average frequency. In electrical engineering literature, the spread of frequency about the *instantaneous frequency*, which is defined as an average of the frequencies that exist at a particular time, is called *instantaneous bandwidth*, given by

$$\sigma_{\nu/t}^2 = \frac{1}{(2\pi)^2}\left[\frac{a'(t)}{a(t)}\right]^2. \tag{2.7.47}$$

In the case of a chirp with a Gaussian envelope

$$f(t) = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} \exp\left[-\frac{1}{2}\alpha t^2 + \frac{1}{2}i\beta\alpha t^2 + 2\pi i\nu_0 t\right], \tag{2.7.48}$$

where its *Fourier spectrum* is given by

$$F(\nu) = (\alpha\pi)^{\frac{1}{4}} \left(\frac{1}{\alpha - i\beta}\right)^{\frac{1}{2}} \exp\left[-2\pi^2(\nu - \nu_0)^2/(\alpha - i\beta)\right]. \tag{2.7.49}$$

The *energy density spectrum* of the signal is

$$|F(\nu)|^2 = 2\left(\frac{\alpha\pi}{\alpha^2 + \beta^2}\right)^{\frac{1}{2}} \exp\left[-\frac{4\alpha\pi^2(\nu - \nu_0)^2}{\alpha^2 + \beta^2}\right]. \tag{2.7.50}$$

Finally, the average frequency $<\nu>$ and the bandwidth square are respectively given by

$$<\nu> = \nu_0 \quad \text{and} \quad B^2 = \frac{1}{8\pi^2}\left(\alpha + \frac{\beta^2}{\alpha}\right). \tag{2.7.51}$$

A large bandwidth can be achieved in two very qualitatively different ways. The amplitude modulation can be made large by taking $\alpha$ large, and the frequency modulation can be small by letting $\beta \to 0$. It is possible to make the frequency modulation large by making $\beta$ large and $\alpha$ very small. These two extreme situations are physically very different even though they produce the same bandwidth.

***Example 2.7.4***

Find the transfer function and the corresponding *impulse response function* of the RLC circuit governed by the differential equation

$$L\,\frac{d^2q}{dt^2} + R\,\frac{dq}{dt} + \frac{1}{C}\,q = e\,(t) \tag{2.7.52}$$

where $q\,(t)$ is the charge, $R$, $L$, $C$ are constants, and $e\,(t)$ is the given voltage (input).

Equation (2.7.25) provides the definition of the *transfer function* in the frequency domain

$$\Phi\,(\omega) = \frac{G\,(\omega)}{F\,(\omega)} = \frac{\mathscr{F}\,\{g\,(t)\}}{\mathscr{F}\,\{f\,(t)\}}, \tag{2.7.53}$$

where $\phi\,(t) = \mathscr{F}^{-1}\,\{\Phi\,(\omega)\}$ is called the *impulse response function*.

Taking the Fourier transfrom of (2.7.52) gives

$$\left(-L\omega^2 + Ri\omega + \frac{1}{C}\right) Q\,(\omega) = E\,(\omega)\,. \tag{2.7.54}$$

Thus, the transfer function is

$$\Phi(\omega) = \frac{Q(\omega)}{E(\omega)} = \frac{-C}{LC\omega^2 - iRC\omega - 1}$$
$$= \frac{i}{2L\beta}\left[\frac{1}{\omega - i(\alpha+\beta)} - \frac{1}{\omega - i(\alpha-\beta)}\right], \tag{2.7.55}$$

where

$$\alpha = \frac{R}{2L} \quad \text{and} \quad \beta = \left[\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}\right]^{\frac{1}{2}}. \tag{2.7.56}$$

The inverse Fourier transform of (2.7.55) yields the impulse response function

$$\phi(t) = \frac{1}{2\beta L}\left(e^{\beta t} - e^{-\beta t}\right) e^{-\alpha t} H(t). \tag{2.7.57}$$

□

## 2.8 Gibbs' Phenomenon

We now examine the so-called the *Gibbs jump phenomenon* which deals with the limiting behavior of a band-limited signal $f_{\omega_0}(t)$ represented by the sampling integral representation (2.7.8) at a point of discontinuity of $f(t)$. This phenomenon reveals the intrinsic overshoot near a jump discontinuity of a function associated with the Fourier series. More precisely, the partial sums of the Fourier series overshoot the function near the discontinuity, and the overshoot continues no matter how many terms are taken in the partial sum. However, the Gibbs phenomenon does not occur if the partial sums are replaced by the Cesaro means, the average of the partial sums.

In order to demonstrate the Gibbs phenomenon, we rewrite (2.7.8) in the convolution form

$$f_{\omega_0}(t) = \int_{-\infty}^{\infty} f(\tau)\,\frac{\sin\omega_0(t-\tau)}{\pi(t-\tau)}\,d\tau = (f * \delta_{\omega_0})(t), \tag{2.8.1}$$

where

$$\delta_{\omega_0}(t) = \frac{\sin\omega_0 t}{\pi t}. \tag{2.8.2}$$

Clearly, at every point of continuity of $f(t)$, we have

$$\begin{aligned}\lim_{\omega_0\to\infty} f_{\omega_0}(t) &= \lim_{\omega_0\to\infty}\left(f*\delta_{\omega_0}\right)(t) = \lim_{\omega_0\to\infty}\int_{-\infty}^{\infty} f(\tau)\,\frac{\sin\omega_0(t-\tau)}{\pi(t-\tau)}\,d\tau\\ &= \int_{-\infty}^{\infty} f(\tau)\left[\lim_{\omega_0\to\infty}\frac{\sin\omega_0(t-\tau)}{\pi(t-\tau)}\right]d\tau\\ &= \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\,d\tau = f(t)\,. \end{aligned}\tag{2.8.3}$$

We now consider the limiting behavior of $f_{\omega_0}(t)$ at the point of discontinuity $t=t_0$. To simplify the calculation, we set $t_0=0$ so that we can write $f(t)$ as a sum of a continuous function, $f_c(t)$ and a suitable step function

$$f(t) = f_c(t) + \left[f(0+) - f(0-)\right]\,H(t).\tag{2.8.4}$$

Replacing $f(t)$ by the right hand side of (2.8.4) in Equation (2.8.1) yields

$$\begin{aligned}f_{\omega_0}(t) &= \int_{-\infty}^{\infty} f_c(\tau)\,\frac{\sin\omega_0(t-\tau)}{\pi(t-\tau)}\,d\tau\\ &\qquad + \left[f(0+)-f(0-)\right]\int_{-\infty}^{\infty} H(\tau)\,\frac{\sin\omega_0(t-\tau)}{\pi(t-\tau)}\,d\tau\\ &= f_c(t) + \left[f(0+)-f(0-)\right]\,H_{\omega_0}(t)\,,\end{aligned}\tag{2.8.5}$$

where

$$\begin{aligned}H_{\omega_0}(t) &= \int_{-\infty}^{\infty} H(\tau)\,\frac{\sin\omega_0(t-\tau)}{\pi(t-\tau)}\,d\tau = \int_{0}^{\infty}\frac{\sin\omega_0(t-\tau)}{\pi(t-\tau)}\,d\tau\\ &= \int_{-\infty}^{\omega_0 t}\frac{\sin x}{\pi x}\,dx \qquad (\text{putting } \omega_0(t-\tau)=x)\\ &= \left(\int_{-\infty}^{0} + \int_{0}^{\omega_0 t}\right)\left(\frac{\sin x}{\pi x}\right)dx = \left(\int_{0}^{\infty} + \int_{0}^{\omega_0 t}\right)\left(\frac{\sin x}{\pi x}\right)dx\\ &= \frac{1}{2} + \frac{1}{\pi}\,si(\omega_0 t)\,,\end{aligned}\tag{2.8.6}$$

and the function $si(t)$ is defined by

$$si(t) = \int_0^t \frac{\sin x}{x}\,dx\,.\tag{2.8.7}$$

Note that

$$H_{\omega_0}\left(\frac{\pi}{\omega_0}\right) = \frac{1}{2} + \int_0^{\pi}\frac{\sin x}{\pi x}\,dx > 1\,,\quad H_{\omega_0}\left(-\frac{\pi}{\omega_0}\right) = \frac{1}{2} - \int_0^{\pi}\frac{\sin x}{\pi x}\,dx < 0\,.$$

Clearly, for a fixed $\omega_0$, $\frac{1}{\pi}si(\omega_0 t)$ attains its maximum at $t=\frac{\pi}{\omega_0}$ in $(0,\infty)$ and minimum at $t=-\frac{\pi}{\omega_0}$, since for a larger $t$ the integrand oscillates with decreasing amplitudes. The function $H_{\omega_0}(t)$ is shown in Figure 2.7 since $H_{\omega_0}(0)=\frac{1}{2}$

and $f_c(0) = f(0-)$ and

$$f_{\omega_0}(0) = f_c(0) + \frac{1}{2}\left[f(0+) - f(0-)\right] = \frac{1}{2}\left[f(0+) + f(0-)\right] .$$

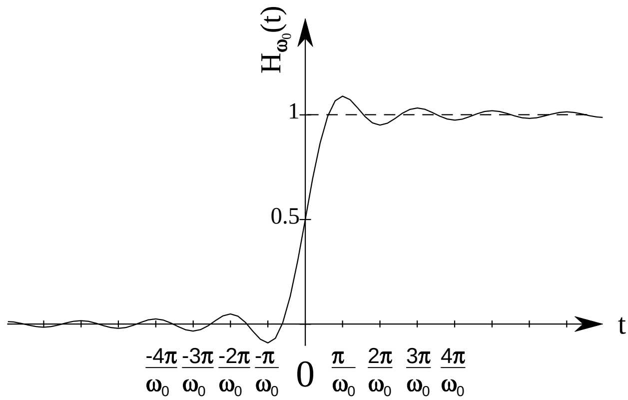


**Figure 2.7** Graph of $H_{\omega_0}(t)$.

Thus, the graph of $H_{\omega_0}(t)$ shows that as $\omega_0$ increases, the time scale changes, and the ripples remain the same. In the limit $\omega_0 \to \infty$, the convergence of $H_{\omega_0}(t) = (H * \delta_{\omega_0})(t)$ to $H(t)$ exhibits the intrinsic overshoot leading to the classical Gibbs phenomenon.

***Example 2.8.1***

(*The Square Wave Function and the Gibbs Phenomenon*). Consider the single-pulse square function defined by

$$f(x) = \left\{ \begin{array}{ll} 1, & -a < x < a \\ \frac{1}{2}, & x = \pm a \\ 0, & |x| > a \end{array} \right\} .$$

The graph of $f(x)$ is given in Figure 2.8.

Thus,

$$F(k) = \mathscr{F}\{f(x)\} = \sqrt{\frac{2}{\pi}} \left(\frac{\sin ak}{k}\right) .$$

▯

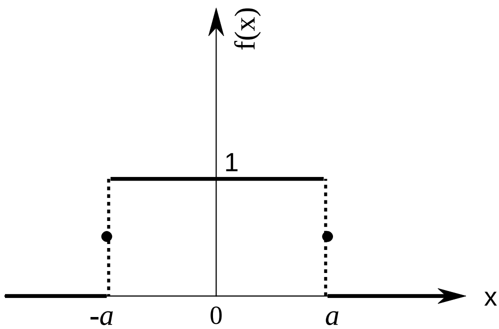


**Figure 2.8** The square wave function.

We next define a function $f_\lambda(x)$ by the integral

$$f_\lambda(x) = \int_{-\lambda}^{\lambda} F(k)\, e^{ikx}\, dk$$

.

As $|\lambda| \to \infty$, $f_\lambda(x)$ will tend pointwise to $f(x)$ for all $x$. Convergence occurs even at $x = \pm\, a$ because the function $f(x)$ is defined to have a value "half way up the step" at these points. Let us examine the behavior of $f_\lambda(x)$ as $|\lambda| \to \infty$ in a region just one side of one of the discontinuities, that is, for $x \in (0, a)$. For a fixed $\lambda$, the difference, $f_\lambda(x) - f(x)$, oscillates above and below the value 0 as $x \to a$, attaining a maximum positive value at some point, say $x = x_\lambda$. Then the quantity $f_\lambda(x_\lambda) - f(x_\lambda)$ is called the *overshoot*.

As $|\lambda| \to \infty$, so the period of the oscillations tends to zero and so also $x_\lambda \to a$; however, the value of the overshoot $f_\lambda(x_\lambda) - f(x_\lambda)$ does not tend to zero but instead tends to a finite limit. The existence of this non-zero, finite, limiting value for the overshoot is known as the *Gibbs phenomenon.* This phenomenon also occurs in an almost identical manner in the Fourier synthesis of periodic functions using Fourier series.

## 2.9 Heisenberg's Uncertainty Principle

If $f \in L^2(\mathbb{R})$, then $f$ and $F(k) = \mathscr{F}\{f(x)\}$ cannot both be essentially localized. In other words, it is not possible that the widths of the graphs of $|f(x)|^2$ and $|F(k)|^2$ can both be made arbitrarily small. This fact underlines the Heisenberg uncertainty principle in quantum mechanics and the bandwidth theorem in signal analysis. If $|f(x)|^2$ and $|F(k)|^2$ are interpreted as weighting functions, then the weighted means (averages) $<x>$ and $<k>$ of $x$ and $k$ are given by

$$<x> = \frac{1}{||f||_2^2} \int_{-\infty}^{\infty} x\,|f(x)|^2\,dx, \tag{2.9.1}$$

$$<k> = \frac{1}{||F||_2^2} \int_{-\infty}^{\infty} k\,|F(k)|^2\,dk. \tag{2.9.2}$$

Corresponding measures of the widths of these weight functions are given by the second moments about the respective means. Usually, it is convenient to define widths $\triangle x$ and $\triangle k$ by

$$(\triangle x)^2 = \frac{1}{||f||_2^2} \int_{-\infty}^{\infty} (x-<x>)^2\;|f(x)|^2\,dx, \tag{2.9.3}$$

$$(\triangle k)^2 = \frac{1}{||F||_2^2} \int_{-\infty}^{\infty} (k-<k>)^2\;|F(k)|^2\,dk. \tag{2.9.4}$$

The essence of the *Heisenberg principle* and the bandwidth theorems lies in the fact that the product $(\triangle x)(\triangle k)$ will never less than $\frac{1}{2}$. Indeed,

$$(\triangle x)(\triangle k) \geq \frac{1}{2}, \tag{2.9.5}$$

where equality in (2.9.5) holds only if $f(x)$ is a Gaussian function given by $f(x) = C\exp(-ax^2)$, $a > 0$.

We next state the Heisenberg inequality theorem as follows:

***THEOREM 2.9.1***

(*Heisenberg Inequality*). If $f(x)$, $x\,f(x)$ and $k\,F(k)$ belong to $L^2(\mathbb{R})$ and $\sqrt{x}|f(x)|\to 0$ as $|x|\to\infty$, then

$$(\triangle x)^2(\triangle k)^2 \geq \frac{1}{4}, \tag{2.9.6}$$

where $(\triangle x)^2$ and $(\triangle k)^2$ are defined by (2.9.3) and (2.9.4) respectively. Equality in (2.9.6) holds only if $f(x)$ is a *Gaussian function* given by $f(x)=C\,e^{-ax^2}$, $a > 0$.

**PROOF** If the averages are $<x>$ and $<k>$, then the average location of $\exp(-i<k>x)f(x+<x>)$ is zero. Hence, it is sufficient to prove the theorem around the zero mean values, that is, $<x>=<k>=0$. Since $||f||_2 = ||F||_2$, we have

$$||f||_2^4(\triangle x)^2(\triangle k)^2 = \int_{-\infty}^{\infty} |xf(x)|^2 dx \int_{-\infty}^{\infty} |kF(k)|^2 dk.$$

Using $ikF(k) = \mathscr{F}\{f'(x)\}$ and the Parseval formula $||f'(x)||_2 = ||ikF(k)||_2$, we obtain

$$\begin{aligned}
||f||_2^4 (\triangle x)^2 (\triangle k)^2 &= \int_{-\infty}^{\infty} |xf(x)|^2 dx \int_{-\infty}^{\infty} |f'(x)|^2 dx \\
&\geq \left| \int_{-\infty}^{\infty} \left\{ xf(x)\,\overline{f'(x)} \right\} dx \right|^2, \quad \text{(see Debnath (2002))} \\
&\geq \left| \int_{-\infty}^{\infty} x.\frac{1}{2} \left\{ f'(x)\,\overline{f(x)} + \overline{f'(x)}\,f(x) \right\} \right|^2 \\
&= \frac{1}{4} \left[ \int_{-\infty}^{\infty} x \left( \frac{d}{dx} |f|^2 \right) dx \right]^2 \\
&= \frac{1}{4} \left\{ \left[ x|f(x)|^2 \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} |f|^2 dx \right\}^2 = \frac{1}{4} ||f||_2^4 .
\end{aligned}$$

in which $\sqrt{x} f(x) \to 0$ as $|x| \to \infty$ was used to eliminate the integrated term. This completes the proof.

If we assume $f'(x)$ is proportional to $x\,f(x)$, that is, $f'(x) = b\,x\,f(x)$, where $b$ is a constant of proportionality, this leads to the *Gaussian signals*

$$f(x) = C \exp(-ax^2),$$

where $C$ is a constant of integration and $a = -\frac{b}{2} > 0$. ∎

In 1924, Heisenberg first formulated the uncertainty principle between the position and momentum in quantum mechanics. This principle has an important interpretation as an uncertainty of both the position and momentum of a particle described by a wave function $\psi \in L^2(\mathbb{R})$. In other words, it is not possible to determine the position and momentum of a particle exactly and simultaneously.

In signal processing, time and frequency concentrations of energy of a signal $f$ are also governed by the Heisenberg uncertainty principle. The average or expectation values of time $t$ and frequency $\omega$, are respectively defined by

$$< t > = \frac{1}{||f||_2^2} \int_{-\infty}^{\infty} t|f(t)|^2 dt, \qquad < \omega > = \frac{1}{||F||_2^2} \int_{-\infty}^{\infty} \omega |F(\omega)|^2 d\omega, \quad (2.9.7)$$

where the energy of a signal $f(t)$ is well localized in time, and its Fourier transform $F(\omega)$ has an energy concentrated in a small frequency domain.
The variances around these average values are given respectively by

$$\begin{aligned}
\sigma_t^2 &= \frac{1}{||f||_2^2} \int_{-\infty}^{\infty} (t- < t >)^2 |f(t)|^2 dt, \\
\sigma_\omega^2 &= \frac{1}{2\pi ||F||_2^2} \int_{-\infty}^{\infty} (\omega- < \omega >)^2 |F(\omega)|^2 d\omega.
\end{aligned} \quad (2.9.8)$$

Remarks:

1. In a time-frequency analysis of signals, the measure of the resolution of a signal $f$ in the time or frequency domain is given by $\sigma_t$ and $\sigma_\omega$. Then, the joint resolution is given by the product $(\sigma_t)$ $(\sigma_\omega)$ which is governed by the Heisenberg uncertainty principle. In other words, the product $(\sigma_t)$ $(\sigma_\omega)$ cannot be arbitrarily small and is always greater than the minimum value $\frac{1}{2}$ which is attained for the Gaussian signal.

2. In many applications in science and engineering, signals with a high concentration of energy in the time and frequency domains are of special interest. The uncertainty principle can also be interpreted as a measure of this concentration of the second moment of $f^2(t)$ and its energy spectrum $F^2(\omega)$.

## 2.10 Applications of Fourier Transforms to Ordinary Differential Equations

We consider the $n$th order linear ordinary differential equation with constant coefficients

$$Ly(x) = f(x), \tag{2.10.1}$$

where $L$ is the $n$th order differential operator given by

$$L \equiv a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0, \tag{2.10.2}$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants, $D \equiv \frac{d}{dx}$ and $f(x)$ is a given function.

Application of the Fourier transform to both sides of (2.10.1) gives

$$[a_n(ik)^n + a_{n-1}(ik)^{n-1} + \cdots + a_1(ik) + a_0]Y(k) = F(k),$$

where $\mathscr{F}\{y(x)\} = Y(k)$ and $\mathscr{F}\{f(x)\} = F(k)$.

Or, equivalently

$$P(ik)Y(k) = F(k),$$

where

$$P(z) = \sum_{r=0}^{n} a_r z^r.$$

Thus,

$$Y(k) = \frac{F(k)}{P(ik)} = F(k)Q(k), \tag{2.10.3}$$

where $Q(k) = \frac{1}{P(ik)}$.

Applying the Convolution Theorem 2.5.5 to (2.10.3) gives the formal solution

$$y(x) = \mathscr{F}^{-1}\{F(k)\,Q(k)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) q(x-\xi) d\xi, \tag{2.10.4}$$

provided $q(x) = \mathscr{F}^{-1}\{Q(k)\}$ is known explicitly.

In order to give a physical interpretation of the solution (2.10.4), we consider the differential equation with a suddenly applied impulse function $f(x) = \delta(x)$ so that

$$L\{G(x)\} = \delta(x). \tag{2.10.5}$$

The solution of this equation can be written from the inversion of (2.10.3) in the form

$$G(x) = \mathscr{F}^{-1}\left\{\frac{1}{\sqrt{2\pi}}\,Q(k)\right\} = \frac{1}{\sqrt{2\pi}}\,q(x). \tag{2.10.6}$$

Thus, the solution (2.10.4) takes the form

$$y(x) = \int_{-\infty}^{\infty} f(\xi) G(x-\xi) d\xi. \tag{2.10.7}$$

Clearly, $G(x)$ behaves like a *Green's function*, that is, it is the response to a *unit impulse*. In any physical system, $f(x)$ usually represents the *input function*, while $y(x)$ is referred to as the *output* obtained by the superposition principle. The Fourier transform of $\{\sqrt{2\pi}G(x)\} = q(x)$ is called the *admittance*. In order to find the reponse to a given input, we determine the Fourier transform of the input function, multiply the result by the admittance, and then apply the inverse Fourier transform to the product so obtained.

We illustrate these ideas by solving a simple problem in the electrical circuit theory.

**Example 2.10.1**

(*Electric Current in a Simple Circuit*). The current $I(t)$ in a simple circuit containing the resistance $R$ and inductance $L$ satisfies the equation

$$L\,\frac{dI}{dt} + RI = E(t), \tag{2.10.8}$$

where $E(t)$ is the applied electromagnetic force and $R$ and $L$ are constants.

With $E(t) = E_0 \exp(-a|t|)$, we use the Fourier transform with respect to time $t$ to obtain

$$(ikL + R)\hat{I}(k) = E_0 \sqrt{\frac{2}{\pi}}\,\frac{a}{(a^2+k^2)}.$$

Or,

$$\hat{I}(k) = \frac{aE_0}{iL}\sqrt{\frac{2}{\pi}}\frac{1}{\left(k - \frac{Ri}{L}\right)\,(k^2 + a^2)},$$

where $\mathscr{F}\{I(t)\} = \hat{I}(k)$. The inverse Fourier transform gives

$$I(t) = \frac{aE_0}{i\pi L}\int_{-\infty}^{\infty}\frac{\exp(ikt)dk}{\left(k - \frac{Ri}{L}\right)(k^2 + a^2)}. \tag{2.10.9}$$

This integral can be evaluated by the Cauchy Residue Theorem. For $t > 0$

$$\begin{aligned} I(t) &= \frac{aE_0}{i\pi L}\cdot 2\pi i\left[\text{Residue at}\, k = \frac{Ri}{L} + \text{Residue at}\, k = ia\right] \\ &= \frac{2aE_0}{L}\left[\frac{e^{-\frac{R}{L}t}}{\left(a^2 - \frac{R^2}{L^2}\right)} - \frac{e^{-at}}{2a\left(a - \frac{R}{L}\right)}\right] \\ &= E_0\left[\frac{e^{-at}}{R - aL} - \frac{2aLe^{-\frac{R}{L}t}}{R^2 - a^2L^2}\right]. \end{aligned} \tag{2.10.10}$$

Similarly, for $t < 0$, the Residue Theorem gives

$$\begin{aligned} I(t) &= -\frac{aE_0}{i\pi L}\cdot 2\pi i[\text{Residue at}\, k = -ia] \\ &= -\frac{2aE_0}{L}\left[\frac{-Le^{at}}{(aL + R)2a}\right] = \frac{E_0 e^{at}}{(aL + R)}. \end{aligned} \tag{2.10.11}$$

At $t = 0$, the current is continuous and therefore,

$$I(0) = \lim_{t\to 0} I(t) = \frac{E_0}{R + aL}.$$

If $E(t) = \delta(t)$, then $\hat{E}(k) = \frac{1}{\sqrt{2\pi}}$ and the solution is obtained by using the inverse Fourier transform

$$I(t) = \frac{1}{2\pi i L}\int_{-\infty}^{\infty}\frac{e^{ikt}}{k - \frac{iR}{L}}dk,$$

which is, by the Theorem of Residues,

$$\begin{aligned} &= \frac{1}{L}[\text{Residue at}\, k = iR/L] \\ &= \frac{1}{L}\exp\left(-\frac{Rt}{L}\right). \end{aligned} \tag{2.10.12}$$

Thus, the current tends to zero as $t \to \infty$ as expected. ▯

***Example 2.10.2***
Find the solution of the ordinary differential equation

$$-\frac{d^2u}{dx^2}+a^2u=f(x), \quad -\infty<x<\infty \tag{2.10.13}$$

by the Fourier transform method.

Application of the Fourier transform to (2.10.13) gives

$$U(k)=\frac{F(k)}{k^2+a^2}.$$

This can readily be inverted by the Convolution Theorem 2.5.5 to obtain

$$u(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)g(x-\xi)d\xi, \tag{2.10.14}$$

where $g(x)=\mathscr{F}^{-1}\left\{\frac{1}{k^2+a^2}\right\}=\frac{1}{a}\sqrt{\frac{\pi}{2}}\exp(-a|x|)$ by Example 2.3.2. Thus, the final solution is

$$u(x)=\frac{1}{2a}\int_{-\infty}^{\infty} f(\xi)e^{-a|x-\xi|}\,d\xi. \tag{2.10.15}$$

$\square$

***Example 2.10.3***
(*The Bernoulli-Euler Beam Equation*). We consider the vertical deflection $u(x)$ of an infinite beam on an elastic foundation under the action of a prescribed vertical load $W(x)$. The deflection $u(x)$ satisfies the ordinary differential equation

$$EI\frac{d^4u}{dx^4}+\kappa u=W(x), \quad -\infty<x<\infty. \tag{2.10.16}$$

where $EI$ is the flexural rigidity and $\kappa$ is the foundation modulus of the beam. We find the solution assuming that $W(x)$ has a compact support and $u, u', u'', u'''$ all tend to zero as $|x|\to\infty$.

We first rewrite (2.10.16) as

$$\frac{d^4u}{dx^4}+a^4u=w(x) \tag{2.10.17}$$

where $a^4=\kappa/EI$ and $w(x)=W(x)/EI$. Use of the Fourier transform to (2.10.17) gives

$$U(k)=\frac{W(k)}{k^4+a^4}.$$

The inverse Fourier transform gives the solution

$$\begin{aligned} u(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{W(k)}{k^4 + a^4} e^{ikx}\, dk \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{k^4 + a^4} dk \int_{-\infty}^{\infty} w(\xi) e^{-ik\xi}\, d\xi \\ &= \int_{-\infty}^{\infty} w(\xi) G(\xi, x)\, d\xi, \end{aligned} \tag{2.10.18}$$

where

$$G(\xi, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ik(x-\xi)}}{k^4 + a^4} dk = \frac{1}{\pi} \int_{0}^{\infty} \frac{\cos k(x-\xi)\, dk}{k^4 + a^4}. \tag{2.10.19}$$

The integral can be evaluated by the Theorem of Residues or by using the table of Fourier integrals. We simply state the result

$$G(\xi, x) = \frac{1}{2a^3} \exp\left(-\frac{a}{\sqrt{2}}|x-\xi|\right) \sin\left[\frac{a(x-\xi)}{\sqrt{2}} + \frac{\pi}{4}\right]. \tag{2.10.20}$$

In particular, we find the explicit solution due to a concentrated load of unit strength acting at some point $x_0$, that is, $w(x) = \delta(x - x_0)$. Then the solution for this case becomes

$$u(x) = \int_{-\infty}^{\infty} \delta(\xi - x_0) G(x, \xi)\, d\xi = G(x, x_0). \tag{2.10.21}$$

Thus, the kernel $G(x, \xi)$ involved in the solution (2.10.18) has the physical significance of being the deflection, as a function of $x$, due to a unit point load acting at $\xi$. Thus, the deflection due to a point load of strength $w(\xi)\, d\xi$ at $\xi$ is $w(\xi)\, d\xi \cdot G(x, \xi)$, and hence, (2.10.18) represents the superposition of all such incremental deflections.

The reader is referred to a more general dynamic problem of an infinite Bernoulli-Euler beam with damping and elastic foundation that has been solved by Stadler and Shreeves (1970), and also by Sheehan and Debnath (1972). These authors used the Fourier-Laplace transform method to determine the steady state and the transient solutions of the beam problem. ▯

## 2.11 Solutions of Integral Equations

The method of Fourier transforms can be used to solve simple integral equations of the convolution type. We illustrate the method by examples.

We first solve the *Fredholm integral equation* with convolution kernel in the form

$$\int_{-\infty}^{\infty} f(t)g(x-t)\,dt + \lambda f(x) = u(x), \tag{2.11.1}$$

where $g(x)$ and $u(x)$ are given functions and $\lambda$ is a known parameter.

Application of the Fourier transform to (2.11.1) gives

$$\sqrt{2\pi}F(k)G(k) + \lambda F(k) = U(k).$$

Or,

$$F(k) = \frac{U(k)}{\sqrt{2\pi}G(k) + \lambda}. \tag{2.11.2}$$

The inverse Fourier transform leads to a formal solution

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{U(k)e^{ikx}\,dk}{\sqrt{2\pi}G(k) + \lambda}. \tag{2.11.3}$$

In particular, if $g(x) = \frac{1}{x}$ so that

$$G(k) = -i\sqrt{\frac{\pi}{2}}\,\mathrm{sgn}\,\mathrm{k},$$

then the solution becomes

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{U(k)e^{ikx}\,dk}{\lambda - i\pi\,\mathrm{sgn}\,\mathrm{k}}. \tag{2.11.4}$$

If $\lambda = 1$ and $g(x) = \frac{1}{2}\left(\frac{x}{|x|}\right)$ so that $G(k) = \frac{1}{\sqrt{2\pi}}\frac{1}{(ik)}$, solution (2.11.3) reduces to the form

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (ik)\frac{U(k)e^{ikx}\,dk}{(1+ik)} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathscr{F}\{u'(x)\}\mathscr{F}\{\sqrt{2\pi}\,e^{-x}\}e^{ikx}\,dk \\ &= u'(x) * \sqrt{2\pi}\,e^{-x} = \int_{-\infty}^{\infty} u'(\xi)\exp(\xi - x)\,d\xi. \end{aligned} \tag{2.11.5}$$

***Example 2.11.1***
Find the solution of the integral equation

$$\int_{-\infty}^{\infty} f(x-\xi) f(\xi)\, d\xi = \frac{1}{x^2+a^2}. \tag{2.11.6}$$

Application of the Fourier transform gives

$$\sqrt{2\pi} F(k)\, F(k) = \sqrt{\frac{\pi}{2}}\,\frac{e^{-a|k|}}{a}.$$

Or,

$$F(k) = \frac{1}{\sqrt{2a}} \exp\left\{-\frac{1}{2}a|k|\right\}. \tag{2.11.7}$$

The inverse Fourier transform gives the solution

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}}\,\frac{1}{\sqrt{2a}} \int_{-\infty}^{\infty} \exp\left(ikx - \frac{1}{2}a|k|\right) dk \\ &= \frac{1}{2\sqrt{\pi a}}\left[\int_0^{\infty} \exp\left\{-k\left(\frac{a}{2}+ix\right)\right\} dk + \int_0^{\infty} \exp\left\{-k\left(\frac{a}{2}-ix\right)\right\} dk\right] \\ &= \frac{1}{2\sqrt{\pi a}}\left[\frac{4a}{(4x^2+a^2)}\right] = \sqrt{\frac{a}{\pi}}\cdot\frac{2}{(4x^2+a^2)}. \end{aligned}$$

□

***Example 2.11.2***
Solve the integral equation

$$\int_{-\infty}^{\infty} \frac{f(t)\, dt}{(x-t)^2+a^2} = \frac{1}{(x^2+b^2)},\ b>a>0. \tag{2.11.8}$$

Taking the Fourier transform, we obtain

$$\sqrt{2\pi}\, F(k)\, \mathscr{F}\left\{\frac{1}{x^2+a^2}\right\} = \sqrt{\frac{\pi}{2}}\,\frac{e^{-b|k|}}{b},$$

or,

$$\sqrt{2\pi}\, F(k) \sqrt{\frac{\pi}{2}}\cdot\frac{e^{-a|k|}}{a} = \sqrt{\frac{\pi}{2}}\,\frac{e^{-b|k|}}{b}.$$

Thus,

$$F(k) = \frac{1}{\sqrt{2\pi}}\left(\frac{a}{b}\right) \exp\{-|k|(b-a)\}. \tag{2.11.9}$$

The inverse Fourier transform leads to the solution

$$\begin{aligned}
f(x) &= \frac{a}{2\pi b}\int_{-\infty}^{\infty} \exp[ikx - |k|(b-a)]dk \\
&= \frac{a}{2\pi b}\left[\int_0^{\infty} \exp[-k\{(b-a)+ix\}]dk + \int_0^{\infty} \exp[-k\{(b-a)-ix\}]\right] dk \\
&= \frac{a}{2\pi b}\left[\frac{1}{(b-a)+ix} + \frac{1}{(b-a)-ix}\right] \\
&= \left(\frac{a}{\pi b}\right)\frac{(b-a)}{(b-a)^2+x^2}. \qquad (2.11.10)
\end{aligned}$$

□

***Example 2.11.3***
Solve the integral equation

$$f(t) + 4\int_{-\infty}^{\infty} e^{-a|x-t|} f(t)dt = g(x). \qquad (2.11.11)$$

Application of the Fourier transform gives

$$\begin{aligned}
F(k) + 4\sqrt{2\pi}F(k)\cdot\frac{2a}{\sqrt{2\pi}(a^2+k^2)} &= G(k) \\
F(k) &= \frac{(a^2+k^2)}{a^2+k^2+8a}\,G(k). \qquad (2.11.12)
\end{aligned}$$

The inverse Fourier transform gives

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{(a^2+k^2)G(k)}{a^2+k^2+8a}e^{ikx}dk. \qquad (2.11.13)$$

In particular, if $a = 1$ and $g(x) = e^{-|x|}$ so that $G(k) = \sqrt{\frac{2}{\pi}}\frac{1}{1+k^2}$, then solution (2.11.13) becomes

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{ikx}}{k^2+3^2}\,dk. \qquad (2.11.14)$$

For $x > 0$, we use a semicircular closed contour in the lower half of the complex plane to evaluate (2.11.14). It turns out that

$$f(x) = \frac{1}{3}\,e^{-3x}. \qquad (2.11.15)$$

Similarly, for $x<0$, a semicircular closed contour in the upper half of the complex plane is used to evaluate (2.11.14) so that

$$f(x)=\frac{1}{3}\,e^{3x}, \quad x<0. \tag{2.11.16}$$

Thus, the final solution is

$$f(x)=\frac{1}{3}\exp(-3|x|). \tag{2.11.17}$$

□

## 2.12 Solutions of Partial Differential Equations

In this section we illustrate how the Fourier transform method can be used to obtain the solution of boundary value and initial value problems for linear partial differential equations of different kinds.

***Example 2.12.1***
(*Dirichlet's Problem in the Half-Plane*). We consider the solution of the Laplace equation in the half-plane

$$u_{xx}+u_{yy}=0, \quad -\infty<x<\infty, \quad y\geq 0, \tag{2.12.1}$$

with the boundary conditions

$$u(x,0)=f(x), \qquad -\infty<x<\infty, \tag{2.12.2}$$
$$u(x,y)\to 0 \text{ as } |x|\to\infty, \quad y\to\infty. \tag{2.12.3}$$

We introduce the Fourier transform with respect to $x$

$$U(k,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}u(x,y)dx \tag{2.12.4}$$

so that (2.12.1)–(2.12.3) becomes

$$\frac{d^2U}{dy^2}-k^2U=0, \tag{2.12.5}$$

$$U(k,0)=F(k), \quad U(k,y)\to 0 \qquad \text{as } y\to\infty. \tag{2.12.6ab}$$

Thus, the solution of this transformed system is

$$U(k,y)=F(k)e^{-|k|y}. \tag{2.12.7}$$

Application of the Convolution Theorem 2.5.5 gives the solution

$$u(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)g(x-\xi)d\xi, \tag{2.12.8}$$

where

$$g(x)=\mathscr{F}^{-1}\{e^{-|k|y}\}=\sqrt{\frac{2}{\pi}}\frac{y}{(x^2+y^2)}. \tag{2.12.9}$$

Consequently, the solution (2.12.8) becomes

$$u(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(\xi)d\xi}{(x-\xi)^2+y^2}, \quad y>0. \tag{2.12.10}$$

This is the well-known *Poisson integral formula* in the half-plane. It is noted that

$$\lim_{y\to 0+} u(x,y)=\int_{-\infty}^{\infty} f(\xi)\left[\lim_{y\to 0+}\frac{y}{\pi}\cdot\frac{1}{(x-\xi)^2+y^2}\right]d\xi=\int_{-\infty}^{\infty} f(\xi)\delta(x-\xi)d\xi, \tag{2.12.11}$$

where Cauchy's definition of the delta function is used, that is,

$$\delta(x-\xi)=\lim_{y\to 0+}\frac{y}{\pi}\cdot\frac{1}{(x-\xi)^2+y^2}. \tag{2.12.12}$$

This may be recognized as a solution of the Laplace equation for a dipole source at $(x,y)=(\xi,0)$.

In particular, when

$$f(x)=T_0 H(a-|x|) \tag{2.12.13}$$

the solution (2.12.10) reduces to

$$\begin{aligned} u(x,y) &= \frac{yT_0}{\pi}\int_{-a}^{a}\frac{d\xi}{(\xi-x)^2+y^2} \\ &= \frac{T_0}{\pi}\left[\tan^{-1}\left(\frac{x+a}{y}\right)-\tan^{-1}\left(\frac{x-a}{y}\right)\right] \\ &= \frac{T_0}{\pi}\tan^{-1}\left(\frac{2ay}{x^2+y^2-a^2}\right). \end{aligned} \tag{2.12.14}$$

The curves in the upper half-plane for which the steady state temperature is constant are known as *isothermal curves.* In this case, these curves represent a family of circular arcs

$$x^2+y^2-\alpha y=a^2 \tag{2.12.15}$$

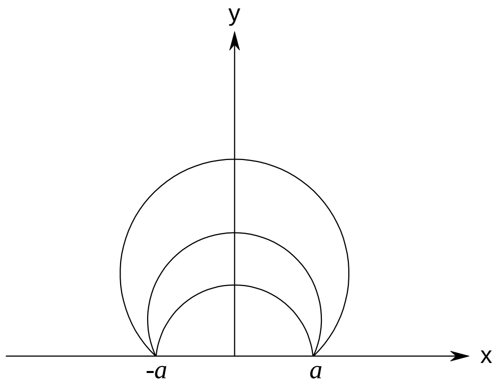


**Figure 2.9** A family of circular arcs.

with centers on the $y$-axis and the fixed end points on the $x$-axis at $x=\pm a$. The graphs of the arcs are is displayed in Figure 2.9.

Another special case deals with

$$f(x)=\delta(x). \tag{2.12.16}$$

The solution for this case follows from (2.12.10) and is

$$u(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\delta(\xi)d\xi}{(x-\xi)^2+y^2}=\frac{y}{\pi}\frac{1}{(x^2+y^2)}. \tag{2.12.17}$$

Further, we can readily deduce the solution of the *Neumann problem* in the half-plane from the solution of the Dirichlet problem. ▯

***Example 2.12.2***

(*Neumann's Problem in the Half-Plane*). Find a solution of the Laplace equation

$$u_{xx}+u_{yy}=0, \quad -\infty<x<\infty, \quad y>0, \tag{2.12.18}$$

with the boundary condition

$$u_y(x,0)=f(x), \quad -\infty<x<\infty. \tag{2.12.19}$$

This condition specifies the normal derivative on the boundary, and physically, it describes the fluid flow or, heat flux at the boundary.

We define a new function $v(x,y)=u_y(x,y)$ so that

$$u(x,y)=\int^{y}v(x,\eta)d\eta, \tag{2.12.20}$$

where an arbitrary constant can be added to the right-hand side. Clearly, the function $v$ satisfies the Laplace equation

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=\frac{\partial^2 u_y}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}=\frac{\partial}{\partial y}(u_{xx}+u_{yy})=0,$$

with the boundary condition

$$v(x,0)=u_y(x,0)=f(x) \text{ for } -\infty<x<\infty.$$

Thus, $v(x,y)$ satisfies the Laplace equation with the Dirichlet condition on the boundary. Obviously, the solution is given by (2.12.10); that is,

$$v(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(\xi)d\xi}{(x-\xi)^2+y^2}. \tag{2.12.21}$$

Then the solution $u(x,y)$ can be obtained from (2.12.20) in the form

$$\begin{aligned} u(x,y) &= \int^{y} v(x,\eta)d\eta=\frac{1}{\pi}\int^{y}\eta\, d\eta\int_{-\infty}^{\infty}\frac{f(\xi)d\xi}{(x-\xi)^2+\eta^2} \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}f(\xi)d\xi\int^{y}\frac{\eta\, d\eta}{(x-\xi)^2+\eta^2}, \quad y>0 \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}f(\xi)\log[(x-\xi)^2+y^2]d\xi, \end{aligned} \tag{2.12.22}$$

where an arbitrary constant can be added to this solution. In other words, the solution of any Neumann problem is uniquely determined up to an arbitrary constant. □

***Example 2.12.3***

(*The Cauchy Problem for the Diffusion Equation*). We consider the initial value problem for a one-dimensional diffusion equation with no sources or sinks

$$u_t=\kappa u_{xx}, \quad -\infty<x<\infty, \quad t>0, \tag{2.12.23}$$

where $\kappa$ is a diffusivity constant with the initial condition

$$u(x,0)=f(x), \quad -\infty<x<\infty. \tag{2.12.24}$$

We solve this problem using the Fourier transform in the space variable $x$ defined by (2.12.4). Application of this transform to (2.12.23)–(2.12.24) gives

$$U_t=-\kappa k^2 U, \quad t>0, \tag{2.12.25}$$

$$U(k,0)=F(k). \tag{2.12.26}$$

The solution of the transformed system is

$$U(k,t) = F(k)\, e^{-\kappa k^2 t}. \tag{2.12.27}$$

The inverse Fourier transform gives the solution

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) \exp[(ikx - \kappa k^2 t)]dk$$

which is, by the Convolution Theorem 2.5.5,

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) g(x-\xi) d\xi, \tag{2.12.28}$$

where

$$g(x) = \mathscr{F}^{-1}\{e^{-\kappa k^2 t}\} = \frac{1}{\sqrt{2\kappa t}} \exp\left(-\frac{x^2}{4\kappa t}\right), \text{ by (2.3.5).}$$

Thus, solution (2.12.28) becomes

$$u(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{\infty} f(\xi) \exp\left[-\frac{(x-\xi)^2}{4\kappa t}\right] d\xi. \tag{2.12.29}$$

The integrand involved in the solution consists of the initial value $f(x)$ and *Green's function* (or, *elementary solution*) $G(x-\xi, t)$ of the diffusion equation for the infinite interval:

$$G(x-\xi, t) = \frac{1}{\sqrt{4\pi\kappa t}} \exp\left[-\frac{(x-\xi)^2}{4\kappa t}\right]. \tag{2.12.30}$$

So, in terms of $G(x-\xi, t)$, solution (2.12.29) can be written as

$$u(x,t) = \int_{-\infty}^{\infty} f(\xi) G(x-\xi, t) d\xi \tag{2.12.31}$$

so that, in the limit as $t \to 0+$, this formally becomes

$$u(x,0) = f(x) = \int_{-\infty}^{\infty} f(\xi) \lim_{t\to 0+} G(x-\xi, t) d\xi.$$

The limit of $G(x-\xi, t)$ represents the Dirac delta function

$$\delta(x-\xi) = \lim_{t\to 0+} \frac{1}{2\sqrt{\pi\kappa t}} \exp\left[-\frac{(x-\xi)^2}{4\,\kappa t}\right]. \tag{2.12.32}$$

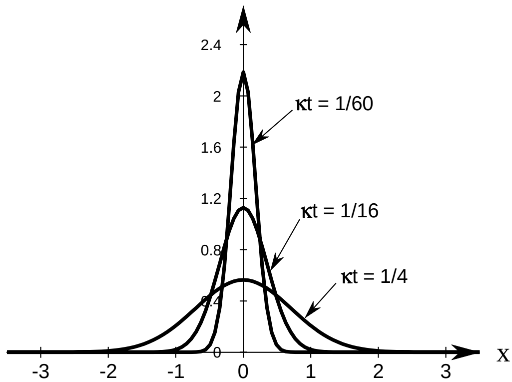


**Figure 2.10** Graphs of $G(x,t)$ against $x$.

Graphs of $G(x,t)$ are shown in Figure 2.10 for different values of $\kappa t$.

It is important to point out that the integrand in (2.12.31) consists of the initial temperature distribution $f(x)$ and Green's function $G(x-\xi,t)$ which represents the temperature response along the rod at time $t$ due to an initial unit impulse of heat at $x=\xi$. The physical meaning of the solution (2.12.31) is that the initial temperature distribution $f(x)$ is decomposed into a spectrum of impulses of magnitude $f(\xi)$ at each point $x=\xi$ to form the resulting temperature $f(\xi)G(x-\xi,t)$. Thus, the resulting temperature is integrated to find solution (2.12.31). This is called the *principle of integral superposition.*

We make the change of variable

$$\frac{\xi - x}{2\sqrt{\kappa t}} = \zeta, \quad d\zeta = \frac{d\xi}{2\sqrt{\kappa t}}$$

to express solution (2.12.29) in the form

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + 2\sqrt{\kappa t}\,\zeta) \exp(-\zeta^2) d\zeta. \tag{2.12.33}$$

The integral solution (2.12.33) or (2.12.29) is called the *Poisson integral representation* of the temperature distribution. This integral is convergent for all time $t > 0$, and the integrals obtained from (2.12.33) by differentiation under the integral sign with respect to $x$ and $t$ are uniformly convergent in the neighborhood of the point $(x,t)$. Hence, the solution $u(x,t)$ and its derivatives of all orders exist for $t > 0$.

Finally, we consider a special case involving discontinuous initial condition in the form

$$f(x) = T_0 H(x)\,, \tag{2.12.34}$$

where $T_0$ is a constant. In this case, solution (2.12.29) becomes

$$u(x,t) = \frac{T_0}{2\sqrt{\pi\kappa t}} \int_0^\infty \exp\left[-\frac{(x-\xi)^2}{4\,\kappa t}\right] d\xi. \tag{2.12.35}$$

Introducing the change of variable $\eta = \frac{\xi - x}{2\sqrt{\kappa t}}$, we can express solution (2.12.35) in the form

$$\begin{aligned} u(x,t) &= \frac{T_0}{\sqrt{\pi}} \int_{-x/2\sqrt{\kappa t}}^{\infty} e^{-\eta^2} d\eta = \frac{T_0}{2}\, erfc\left(-\frac{x}{2\sqrt{\kappa t}}\right) \\ &= \frac{T_0}{2}\left[1 + erf\left(\frac{x}{2\sqrt{\kappa t}}\right)\right]. \end{aligned} \tag{2.12.36}$$

The solution given by equation (2.12.36) with $T_0 = 1$ is shown in Figure 2.11.

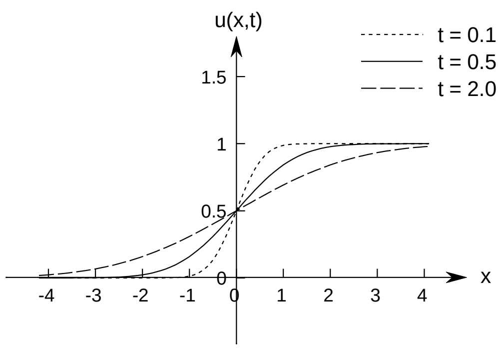


**Figure 2.11** The time development of solution (2.12.36).

□

If $f(x) = \delta(x)$, then the fundamental solution (2.7.29) is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi\,\kappa\,t}} \exp\left(-\frac{x^2}{4\kappa\,t}\right).$$

***Example 2.12.4***
(*The Cauchy Problem for the Wave Equation*). Obtain the d'Alembert solution of the initial value problem for the wave equation

$$u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0, \tag{2.12.37}$$

with the arbitrary but fixed initial data

$$u(x,0)=f(x), \quad u_t(x,0)=g(x), \quad -\infty<x<\infty. \tag{2.12.38ab}$$

Application of the Fourier transform $\mathscr{F}\{u(x,t)\}=U(k,t)$ to this system gives

$$\frac{d^2U}{dt^2}+c^2k^2U=0,$$
$$U(k,0)=F(k), \quad \left(\frac{dU}{dt}\right)_{t=0}=G(k).$$

The solution of the transformed system is

$$U(k,t)=A\,e^{ickt}+B\,e^{-ickt},$$

where $A$ and $B$ are constants to be determined from the transformed data so that $A+B=F(k)$ and $A-B=\frac{1}{ikc}G(k)$. Solving for $A$ and $B$, we obtain

$$U(k,t)=\frac{1}{2}F(k)(e^{ickt}+e^{-ickt})+\frac{G(k)}{2ick}(e^{ickt}-e^{-ickt}). \tag{2.12.39}$$

Thus, the inverse Fourier transform of (2.12.39) yields the solution

$$u(x,t)=\frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(k)\{e^{ik(x+ct)}+e^{ik(x-ct)}\}dk\right]$$
$$+\frac{1}{2c}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{G(k)}{ik}\{e^{ik(x+ct)}-e^{ik(x-ct)}\}dk\right]. \tag{2.12.40}$$

We use the following results

$$f(x)=\mathscr{F}^{-1}\{F(k)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}F(k)dk,$$

$$g(x)=\mathscr{F}^{-1}\{G(k)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}G(k)dk,$$

to obtain the solution in the final form

$$\begin{aligned}u(x,t) &= \frac{1}{2}[f(x-ct)+f(x+ct)]+\frac{1}{2c}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}G(k)dk\int_{x-ct}^{x+ct}e^{ik\xi}d\xi\\ &= \frac{1}{2}[f(x-ct)+f(x+ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}d\xi\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ik\xi}G(k)dk\right]\\ &= \frac{1}{2}[f(x-ct)+f(x+ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}g(\xi)d\xi.\end{aligned} \tag{2.12.41}$$

This is the well known *d'Alembert's solution* of the wave equation.

The method and the form of the solution reveal several important features of the wave equation. First, the method of solution essentially proves the existence of the d'Alembert solution and the solution is unique provided $f(x)$ is twice continuously differentiable and $g(x)$ is continuously differentiable. Second, the terms involving $f(x \pm ct)$ in (2.12.41) show that disturbances are propagated along the characteristics with constant velocity $c$. Both terms combined together suggest that the value of the solution at position $x$ and at time $t$ depends only on the initial values of $f(x)$ at $x - ct$ and $x + ct$ and the values of $g(x)$ between these two points. The interval $(x - ct, x + ct)$ is called the *domain of dependence* of the variable $(x, t)$. Finally, the solution depends continuously on the initial data, that is, the problem is well posed. In other words, a small change in either $f(x)$ or $g(x)$ results in a correspondingly small change in the solution $u(x, t)$.

In particular, if $f(x) = \exp(-x^2)$ and $g(x) \equiv 0$, the time development of solution (2.12.41) with $c = 1$ is shown in Figure 2.12. In this case, the solution becomes

$$u(x,t) = \frac{1}{2}[e^{-(x-t)^2} + e^{-(x+t)^2}]. \tag{2.12.42}$$

As shown in Figure 2.12, the initial form $f(x) = \exp(-x^2)$ is found to split into two similar waves propagating in opposite direction with unit velocity.
□

***Example 2.12.5***

(*The Schrödinger Equation in Quantum Mechanics*). The time-dependent Schrödinger equation of a particle of mass $m$ is

$$i\hbar\,\psi_t = \left[V(x) - \frac{\hbar^2}{2m}\nabla^2\right]\psi = H\psi, \tag{2.12.43}$$

where $h = 2\pi\hbar$ is the *Planck constant*, $\psi(\mathbf{x}, t)$ is the wave function, $V(x)$ is the potential, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the three-dimensional *Laplacian*, and $H$ is the *Hamiltonian*.

If $V(\mathbf{x}) = \text{constant} = V$, we can seek a *plane wave solution* of the form

$$\psi(\mathbf{x}, t) = A\exp[i(\boldsymbol{\kappa}\cdot\mathbf{x} - \omega t)], \tag{2.12.44}$$

where $A$ is a constant amplitude, $\boldsymbol{\kappa} = (k, l, m)$ is the wavenumber vector, and $\omega$ is the frequency.

Substituting this solution into (2.12.43), we conclude that this solution is possible provided the following relation is satisfied:

$$i\hbar(-i\omega) = V - \frac{\hbar^2}{2m}(i\kappa)^2, \quad \kappa^2 = k^2 + l^2 + m^2.$$

Or,

$$\hbar\omega = V + \frac{\hbar^2\kappa^2}{2m}. \tag{2.12.45}$$

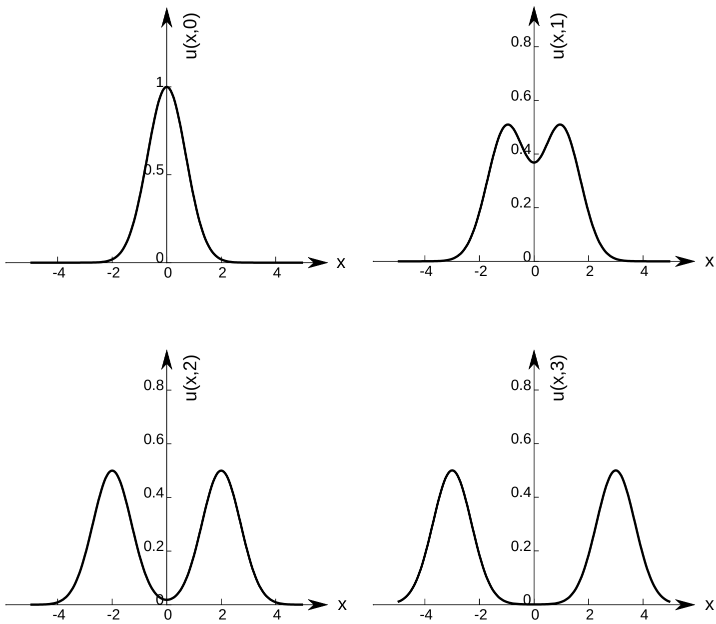


**Figure 2.12** The time development of solution (2.12.42).

This is called the *dispersion relation* and shows that the sum of the potential energy $V$ and the kinetic energy $\frac{(\hbar\kappa)^2}{2m}$ is equal to the total energy $\hbar\omega$. Further, the kinetic energy

$$K.E. = \frac{1}{2m}(\hbar\kappa)^2 = \frac{p^2}{2m}, \tag{2.12.46}$$

where $p = \hbar\kappa$ is the momentum of the particle.

The phase velocity, $C_p$ and the group velocity, $C_g$ of the wave are defined by

$$C_p = \frac{\omega}{\kappa}\,\hat{\boldsymbol{\kappa}}, \qquad C_g = \nabla_{\boldsymbol{\kappa}}\omega(\boldsymbol{\kappa}), \tag{2.12.47ab}$$

where $\boldsymbol{\kappa}$ is the wavenumber vector and $\kappa = |\boldsymbol{\kappa}|$ and $\hat{\boldsymbol{\kappa}}$ is the unit wavenumber vector.

In the one-dimensional case, the phase velocity is

$$C_p = \frac{\omega}{k} \tag{2.12.48}$$

and the group velocity is

$$C_g = \frac{\partial\omega}{\partial k} = \frac{\hbar k}{m} = \frac{p}{m} = \frac{mv}{v} = v. \tag{2.12.49}$$

This shows that the group velocity is equal to the classical particle velocity $v$.

We now use the Fourier transform method to solve the one-dimensional Schrödinger equation for a free particle ($V \equiv 0$), that is,

$$i\hbar\,\psi_t = -\frac{\hbar^2}{2m}\psi_{xx}, \quad -\infty < x < \infty, \quad t > 0, \tag{2.12.50}$$

$$\psi(x,0) = \psi_0(x), \quad -\infty < x < \infty, \tag{2.12.51}$$

$$\psi(x,t) \to 0 \quad \text{as } |x| \to \infty. \tag{2.12.52}$$

Application of the Fourier transform to (2.12.50)–(2.12.52) gives

$$\Psi_t = -\frac{i\hbar k^2}{2m}\Psi, \qquad \Psi(k,0) = \Psi_0(k). \tag{2.12.53}$$

The solution of this transformed system is

$$\Psi(k,t) = \Psi_0(k)\exp(-i\alpha k^2 t), \quad \alpha = \frac{\hbar}{2m}. \tag{2.12.54}$$

The inverse Fourier transform gives the formal solution

$$\begin{aligned}\psi(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi_0(k)\exp\{ik(x-\alpha kt)\}dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-iky}\psi(y,0)dy\int_{-\infty}^{\infty}\exp\{ik(x-\alpha kt)\}dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(y,0)dy\int_{-\infty}^{\infty}\exp\{ik(x-y-\alpha kt)\}dk. \end{aligned} \tag{2.12.55}$$

We rewrite the integrand of the second integral in (2.12.55) as follows

$$\begin{aligned}&\exp[ik(x-y-\alpha kt)] \\ &\quad = \exp\left[-i\alpha t\left\{k^2 - 2k\cdot\frac{x-y}{2\alpha t} + \left(\frac{x-y}{2\alpha t}\right)^2 - \left(\frac{x-y}{2\alpha t}\right)^2\right\}\right] \\ &\quad = \exp\left[-i\alpha t\left\{k - \frac{x-y}{2\alpha t}\right\}^2\right]\exp\left[\frac{i(x-y)^2}{4\alpha t}\right] \\ &\quad = \exp\left[\frac{i(x-y)^2}{4\alpha t}\right]\exp(-i\alpha t\xi^2), \quad \xi = k - \frac{x-y}{2\alpha t}.\end{aligned}$$

Using this result in (2.12.55), we obtain

$$\psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left[\frac{i(x-y)^2}{4\alpha t}\right]\psi(y,0)dy \int_{-\infty}^{\infty} \exp(-i\alpha t\xi^2)d\xi$$
$$= \frac{1}{2\pi}\sqrt{\frac{\pi}{2\alpha t}}(1-i)\int_{-\infty}^{\infty} \exp\left[\frac{i(x-y)^2}{4\alpha t}\right]\psi(y,0)dy$$
$$= \frac{(1-i)}{2\sqrt{2\alpha\pi t}}\int_{-\infty}^{\infty} \exp\left[\frac{i(x-y)^2}{4\alpha t}\right]\psi(y,0)dy. \quad (2.12.56)$$

This is the integral solution of the problem. ▯

***Example 2.12.6***

(*Slowing Down of Neutrons*). We consider the problem of slowing down neutrons in an infinite medium with a source of neutrons governed by

$$u_t = u_{xx} + \delta(x)\delta(t), \quad -\infty < x < \infty, \quad t > 0, \quad (2.12.57)$$
$$u(x,0) = \delta(x), \quad -\infty < x < \infty, \quad (2.12.58)$$
$$u(x,t) \to 0 \quad \text{as } |x| \to \infty \text{ for } t > 0, \quad (2.12.59)$$

where $u(x,t)$ represents the number of neutrons per unit volume per unit time, which reach the age $t$, and $\delta(x)\delta(t)$ is the source function.

Application of the Fourier transform method gives

$$\frac{dU}{dt} + k^2 U = \frac{1}{\sqrt{2\pi}}\,\delta(t),$$
$$U(k,0) = \frac{1}{\sqrt{2\pi}}.$$

The solution of this transformed system is

$$U(k,t) = \frac{1}{\sqrt{2\pi}}\,e^{-k^2 t},$$

and the inverse Fourier transform gives the solution

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2 t}dk = \frac{1}{\sqrt{2\pi}}\mathscr{F}^{-1}\left\{e^{-k^2 t}\right\}$$
$$= \frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{x^2}{4t}\right). \quad (2.12.60)$$

▯

**Example 2.12.7**
(*One-Dimensional Wave Equation*). Obtain the solution of the one-dimensional wave equation

$$u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0, \tag{2.12.61}$$

$$u(x,0) = 0, \quad u_t(x,0) = \delta(x), \quad -\infty < x < \infty. \tag{2.12.62ab}$$

Making reference to Example 2.12.4, we find $f(x) \equiv 0$ and $g(x) = \delta(x)$ so that $F(k) = 0$ and $G(k) = \frac{1}{\sqrt{2\pi}}$. The solution for $U(k,t)$ is given by

$$U(k,t) = \frac{1}{2c\sqrt{2\pi}} \left[ \frac{e^{ickt}}{ik} - \frac{e^{-ickt}}{ik} \right].$$

Thus, the inverse Fourier transform gives

$$\begin{aligned} u(x,t) &= \frac{1}{2c\sqrt{2\pi}} \mathscr{F}^{-1} \left\{ \frac{e^{ickt}}{ik} - \frac{e^{-ickt}}{ik} \right\} \\ &= \frac{1}{2c\sqrt{2\pi}} \left[ \sqrt{\frac{\pi}{2}} \{\mathrm{sgn(x+ct) - sgn(x-ct)}\} \right] \\ &= \frac{1}{4c} [\mathrm{sgn(x+ct) - sgn(x-ct)}] \\ &= \begin{cases} \dfrac{1-1}{4c} = 0, & |x| > ct > 0 \\ \dfrac{1+1}{4c} = \dfrac{1}{2c}, & |x| < ct. \end{cases} \end{aligned}$$

In other words, the solution can be written in the form

$$u(x,t) = \frac{1}{2c} H(c^2t^2 - x^2).$$

□

**Example 2.12.8**
(*Linearized Shallow Water Equations in a Rotating Ocean*). The horizontal equations of motion of a uniformly rotating inviscid homogeneous ocean of constant depth $h$ are

$$u_t - f\upsilon = -g\,\eta_x, \tag{2.12.63}$$

$$\upsilon_t + fu = 0, \tag{2.12.64}$$

$$\eta_t + hu_x = 0, \tag{2.12.65}$$

where $f = 2\Omega \sin\theta$ is the Coriolis parameter, which is constant in the present problem, $g$ is the acceleration due to gravity, $\eta(x,t)$ is the free surface elevation, $u(x,t)$ and $\upsilon(x,t)$ are the velocity fields. The wave motion is generated by the prescribed free surface elevation at $t=0$ so that the initial conditions are

$$u(x,0)=0=v(x,0), \quad \eta(x,0)=\eta_0 H(a-|x|), \tag{2.12.66abc}$$

and the velocity fields and free surface elevation function vanish at infinity.

We apply the Fourier transform with respect to $x$ defined by

$$\mathscr{F}\{f(x,t)\}=F(k,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}f(x,t)dx \tag{2.12.67}$$

to the system (2.12.63)–(2.12.65) so that the system becomes

$$\begin{aligned}\frac{dU}{dt}-fV &= -gikE\\ \frac{dV}{dt}+fU &= 0\\ \frac{dE}{dt} &= -hikU\end{aligned}$$

$$U(k,0)=0=V(k,0), \quad E(k,0)=\sqrt{\frac{2}{\pi}}\,\eta_0\left(\frac{\sin ak}{k}\right), \tag{2.12.68abc}$$

where $E(k,t)=\mathscr{F}\{\eta(x,t)\}$.

Elimination of $U$ and $V$ from the transformed system gives a single equation for $E(k,t)$ as

$$\frac{d^3E}{dt^3}+\omega^2\frac{dE}{dt}=0, \tag{2.12.69}$$

where $\omega^2=(f^2+c^2k^2)$ and $c^2=gh$. The general solution of (2.12.69) is

$$E(k,t)=A+B\cos\omega t+C\sin\omega t, \tag{2.12.70}$$

where $A$, $B$, and $C$ are arbitrary constants to be determined from (2.12.68c) and

$$\left(\frac{d^2E}{dt^2}\right)_{t=0}=-c^2k^2E(k,0)=-c^2k^2\cdot\sqrt{\frac{2}{\pi}}\,\eta_0\,\frac{\sin ak}{k},$$

which gives

$$B=\sqrt{\frac{2}{\pi}}\,\eta_0\left(\frac{\sin ak}{k}\right)\cdot\left(\frac{c^2k^2}{\omega^2}\right).$$

Also $\left(\frac{dE}{dt}\right)_{t=0}=0$ gives $C\equiv 0$ and (2.12.68c) implies $A+B=\sqrt{\frac{2}{\pi}}\eta_0\frac{\sin ak}{k}$. Consequently, the solution (2.12.70) becomes

$$E(k,t)=\sqrt{\frac{2}{\pi}}\,\eta_0\left(\frac{\sin ak}{k}\right)\frac{f^2+c^2k^2\cos\omega t}{(f^2+c^2k^2)}. \tag{2.12.71}$$

Similarly

$$U(k,t) = \sqrt{\frac{2}{\pi}}\frac{\eta_0 \sin ak}{ih} \cdot \frac{c^2 \sin \omega t}{\sqrt{c^2k^2+f^2}}, \tag{2.12.72}$$

$$V(k,t) = \frac{1}{f}\left(\frac{dU}{dt} + gik\,E\right). \tag{2.12.73}$$

The inverse Fourier transform gives the formal solution for $\eta(x,t)$

$$\eta(x,t) = \left(\frac{\eta_0}{\pi}\right)\int_{-\infty}^{\infty} \frac{\sin ak}{k} \cdot \frac{f^2 + c^2k^2 \cos \omega t}{(f^2+c^2k^2)} e^{ikx}\,dk. \tag{2.12.74}$$

Similar integral expressions for $u(x,t)$ and $v(x,t)$ can be obtained. □

***Example 2.12.9***

(*Sound Waves Induced by a Spherical Body*). We consider propagation of sound waves in an unbounded fluid medium generated by an impulsive radial acceleration of a sphere of radius $a$. Such waves are assumed to be spherically symmetric and the associated velocity potential on the pressure field $p(r,t)$ satisfies the wave equation

$$\frac{\partial^2 p}{\partial t^2} = c^2\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right)\right], \tag{2.12.75}$$

where $c$ is the speed of sound. The boundary condition required for the problem is

$$\frac{1}{\rho_0}\left(\frac{\partial p}{\partial r}\right) = -a_0\,\delta(t) \qquad \text{on } r = a, \tag{2.12.76}$$

where $\rho_0$ is the mean density of the fluid and $a_0$ is a constant.

Application of the Fourier transform of $p(r,t)$ with respect to time $t$ gives

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dP}{dr}\right) = -k^2 P(r,\omega), \tag{2.12.77}$$

$$\frac{dP}{dr} = -\frac{a_0\rho_0}{\sqrt{2\pi}}, \quad \text{on } r = a, \tag{2.12.78}$$

where $\mathscr{F}\{p(r,t)\} = P(r,\omega)$ and $k^2 = \frac{\omega^2}{c^2}$.

The general solution of (2.12.77)–(2.12.78) is

$$P(r,\omega) = \frac{A}{r}e^{ikr} + \frac{B}{r}e^{-ikr}, \tag{2.12.79}$$

where $A$ and $B$ are arbitrary constants.

The inverse Fourier transform gives the solution

$$p(r,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ \frac{A}{r} e^{i(\omega t + kr)} + \frac{B}{r} e^{i(\omega t - kr)} \right] d\omega. \tag{2.12.80}$$

The first term of the integrand represents incoming spherical waves generated at infinity and the second term corresponds to outgoing spherical waves due to the impulsive radial acceleration of the sphere. Since there is no disturbance at infinity, we impose the *Sommerfeld radiation condition* at infinity to eliminate the incoming waves so that $A = 0$, and $B$ is calculated using (2.12.78). Thus, the inverse Fourier transform gives the formal solution

$$p(r,t) = \left( \frac{a_0 \rho_0 a^2}{2\pi r} \right) \int_{-\infty}^{\infty} \frac{\exp\left[ i\omega \left\{ t - \frac{r-a}{c} \right\} \right] d\omega}{\left( 1 + \frac{i\omega a}{c} \right)}. \tag{2.12.81}$$

We next choose a closed contour with a semicircle in the upper half plane and the real $\omega$-axis. Using the Cauchy theory of residues, we calculate the residue contribution from the pole at $\omega = ic/a$. Finally, it turns out that the final solution is

$$u(r,t) = \left( \frac{\rho_0 a_0 ca}{r} \right) \exp\left[ -\frac{c}{a} \left( t - \frac{r-a}{c} \right) \right] H\left( t - \frac{r-a}{c} \right). \tag{2.12.82}$$

□

**Example 2.12.10**
(*The Linearized Korteweg-de Vries Equation*). The linearized KdV equation for the free surface elevation $\eta(x,t)$ in an inviscid water of constant depth $h$ is

$$\eta_t + c\eta_x + \frac{ch^2}{6} \eta_{xxx} = 0, \quad -\infty < x < \infty, \quad t > 0, \tag{2.12.83}$$

where $c = \sqrt{gh}$ is the shallow water speed.

Solve equation (2.12.83) with the initial condition

$$\eta(x,0) = f(x), \quad -\infty < x < \infty. \tag{2.12.84}$$

Application of the Fourier transform $\mathscr{F}\{\eta(x,t)\} = E(k,t)$ to the KdV system gives the solution for $E(k,t)$ in the form

$$E(k,t) = F(k) \exp\left[ ikct \left( \frac{k^2 h^2}{6} - 1 \right) \right].$$

The inverse transform gives

$$\eta(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) \exp\left[ ik \left\{ (x - ct) + \left( \frac{cth^2}{6} \right) k^2 \right\} \right] dk. \tag{2.12.85}$$

In particular, if $f(x)=\delta(x)$, then (2.12.85) reduces to the Airy integral

$$\eta(x,t)=\frac{1}{\pi}\int_0^\infty \cos\left[k(x-ct)+\left(\frac{cth^2}{6}\right)k^3\right]dk \tag{2.12.86}$$

which is, in terms of the Airy function,

$$=\left(\frac{cth^2}{2}\right)^{-\frac{1}{3}} Ai\left[\left(\frac{cth^2}{2}\right)^{-\frac{1}{3}}(x-ct)\right], \tag{2.12.87}$$

where the Airy function $Ai(az)$ is defined by

$$Ai(az)=\frac{1}{2\pi a}\int_{-\infty}^{\infty}\exp\left[i\left(kz+\frac{k^3}{3a^3}\right)\right]dk=\frac{1}{\pi a}\int_0^\infty \cos\left(kz+\frac{k^3}{3a^3}\right)dk. \tag{2.12.88}$$

□

***Example 2.12.11***

(*Biharmonic Equation in Fluid Mechanics*). Usually, the biharmonic equation arises in fluid mechanics and in elasticity. The equation can readily be solved by using the Fourier transform method. We first derive a biharmonic equation from the *Navier-Stokes equations* of motion in a viscous fluid which is given by

$$\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}=\mathbf{F}-\frac{1}{\rho}\nabla p+\nu\nabla^2\mathbf{u}, \tag{2.12.89}$$

where $\mathbf{u}=(u,v,w)$ is the velocity field, $\boldsymbol{F}$ is the external force per unit mass of the fluid, $p$ is the pressure, $\rho$ is the density and $\nu$ is the kinematic viscosity of the fluid.

The conservation of mass of an incompressible fluid is described by the *continuity equation*

$$\text{div }\mathbf{u}=0. \tag{2.12.90}$$

In terms of some representative length scale $L$ and velocity scale $U$, it is convenient to introduce the nondimensional flow variables

$$\mathbf{x}'=\frac{\mathbf{x}}{L},\quad t'=\frac{Ut}{L},\quad \mathbf{u}'=\frac{\mathbf{u}}{U},\quad p'=\frac{p}{\rho U^2}. \tag{2.12.91}$$

In terms of these nondimensional variables, equation (2.12.89) without the external force can be written, dropping the primes, as

$$\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}=-\nabla p+\frac{1}{R}\nabla^2\mathbf{u}, \tag{2.12.92}$$

where $R=UL/\nu$ is called the *Reynolds number*. Physically, it measures the ratio of inertial forces of the order $U^2/L$ to viscous forces of the order $\nu U/L^2$, and it has special dynamical significance. This is one of the most fundamental nondimensional parameters for the specification of the dynamical state of viscous flow fields.

In the absence of the external force, $\mathbf{F}=\mathbf{0}$, it is preferable to write the Navier-Stokes equations (2.12.89) in the form (since $\mathbf{u}\times\boldsymbol{\omega}=\frac{1}{2}\nabla u^2-\mathbf{u}\cdot\nabla\mathbf{u}$)

$$\frac{\partial\mathbf{u}}{\partial t}-\mathbf{u}\times\boldsymbol{\omega}=-\nabla\left(\frac{p}{\rho}+\frac{1}{2}u^2\right)-\nu\nabla^2\mathbf{u}, \tag{2.12.93}$$

where $\boldsymbol{\omega}=\text{curl }\mathbf{u}$ is the *vorticity vector* and $u^2=\boldsymbol{u}\cdot\boldsymbol{u}$.

We can eliminate the pressure $p$ from (2.12.93) by taking the curl of (2.12.93), giving

$$\frac{\partial\boldsymbol{\omega}}{\partial t}-\text{curl}(\mathbf{u}\times\boldsymbol{\omega})=\nu\nabla^2\boldsymbol{\omega} \tag{2.12.94}$$

which becomes, by div $\mathbf{u}=0$ and div $\boldsymbol{\omega}=0$,

$$\frac{\partial\boldsymbol{\omega}}{\partial t}=(\boldsymbol{\omega}\cdot\nabla)\mathbf{u}-(\mathbf{u}\cdot\nabla)\boldsymbol{\omega}+\nu\nabla^2\boldsymbol{\omega}. \tag{2.12.95}$$

This is universally known as the *vorticity transport equation*. The left hand-side represents the rate of change of vorticity. The first two terms on the right-hand side represent the rate of change of vorticity due to stretching and twisting of vortex lines. The last term describes the diffusion of vorticity by molecular viscosity.

In case of two-dimensional flow, $(\boldsymbol{\omega}\cdot\nabla)\mathbf{u}=0$, equation (2.12.95) becomes

$$\frac{D\boldsymbol{\omega}}{dt}=\frac{\partial\boldsymbol{\omega}}{\partial t}+(\mathbf{u}\cdot\nabla)\boldsymbol{\omega}=\nu\nabla^2\boldsymbol{\omega}, \tag{2.12.96}$$

where $\mathbf{u}=(u,v,0)$ and $\boldsymbol{\omega}=(0,0,\zeta)$, and $\zeta=v_x-u_y$. Equation (2.12.96) shows that only convection and conduction occur. In terms of the stream function $\psi(x,y)$ where

$$u=\psi_y,\quad v=-\psi_x,\quad \boldsymbol{\omega}=-\nabla^2\psi, \tag{2.12.97}$$

which satisfy (2.12.90) identically, equation (2.12.96) assumes the form

$$\frac{\partial}{\partial t}\left(\nabla^2\psi\right)+\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\nabla^2\psi=\nu\nabla^4\psi. \tag{2.12.98}$$

In case of slow motion (velocity is small) or in case of a very viscous fluid ($\nu$ very large), the Reynolds number $R$ is very small. For a steady flow in such cases of an incompressible viscous fluid, $\frac{\partial}{\partial t}\equiv 0$, while $(\mathbf{u}\cdot\nabla)\boldsymbol{\omega}$ is negligible in comparison with the viscous term. Consequently, (2.12.98) reduces to the standard *biharmonic equation*

$$\nabla^4\psi=0. \tag{2.12.99}$$

Or, more explicitly,

$$\nabla^2(\nabla^2)\psi\equiv\psi_{xxxx}+2\psi_{xxyy}+\psi_{yyyy}=0. \tag{2.12.100}$$

We solve this equation in a semi-infinite viscous fluid bounded by an infinite horizontal plate at $y=0$, and the fluid is introduced normally with a prescribed velocity through a strip $-a<x<a$ of the plate. Thus, the required boundary conditions are

$$u \equiv \frac{\partial \psi}{\partial y} = 0, \quad v \equiv \frac{\partial \psi}{\partial x} = H(a-|x|)f(x) \quad \text{on } y=0, \tag{2.12.101ab}$$

where $f(x)$ is a given function of $x$.

Furthermore, the fluid is assumed to be at rest at large distances from the plate, that is,

$$(\psi_x, \psi_y) \to (0,0) \quad \text{as } y \to \infty \qquad \text{for } -\infty < x < \infty. \tag{2.12.102}$$

To solve the biharmonic equation (2.12.100) with the boundary conditions (2.12.101ab) and (2.12.102), we introduce the Fourier transform with respect to $x$

$$\Psi(k,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \psi(x,y)dx. \tag{2.12.103}$$

Thus, the Fourier transformed problem is

$$\left(\frac{d^2}{dy^2} - k^2\right)^2 \Psi(k,y) = 0, \tag{2.12.104}$$

$$\frac{d\Psi}{dy} = 0, \quad (ik)\Psi = F(k), \quad y=0, \tag{2.12.105ab}$$

where

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-ikx} f(x)dx. \tag{2.12.106}$$

In view of the Fourier transform of (2.12.102), the bounded solution of (2.12.104) is

$$\Psi(k,y) = (A + B|k|y)\exp(-|k|y), \tag{2.12.107}$$

where $A$ and $B$ can be determined from (2.12.105ab) so that $A = B = (ik)^{-1}F(k)$. Consequently, the solution (2.12.107) becomes

$$\Psi(k,y) = (ik)^{-1}(1+|k|y)F(k)\exp(-|k|y). \tag{2.12.108}$$

The inverse Fourier transform gives the formal solution

$$\psi(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)G(k)\exp(ikx)dk, \tag{2.12.109}$$

where

$$G(k) = (ik)^{-1}(1 + |k|y)\exp(-|k|y)$$

so that

$$\begin{aligned} g(x) &= \mathscr{F}^{-1}\{G(k)\} = \mathscr{F}^{-1}\{(ik)^{-1}\exp(-|k|y)\} \\ &\qquad\qquad + y\,\mathscr{F}^{-1}\{(ik)^{-1}|k|\exp(-|k|y)\} \\ &= \mathscr{F}_s^{-1}\{k^{-1}\exp(-ky)\} + y\,\mathscr{F}_s^{-1}\{e^{-ky}\}, \end{aligned}$$

which is, by (2.13.7) and (2.13.8),

$$= \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{y}\right) + \sqrt{\frac{2}{\pi}}\frac{xy}{(x^2+y^2)}. \tag{2.12.110}$$

Using the Convolution Theorem 2.5.5 in (2.12.109) gives the final solution

$$\psi(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x-\xi)\left[\tan^{-1}\left(\frac{\xi}{y}\right) + \frac{y\xi}{\xi^2+y^2}\right]d\xi. \tag{2.12.111}$$

In particular, if $f(x) = \delta(x)$, then solution (2.12.111) becomes

$$\psi(x,y) = \frac{1}{\pi}\left[\tan^{-1}\left(\frac{x}{y}\right) + \frac{xy}{x^2+y^2}\right]. \tag{2.12.112}$$

The velocity fields $u$ and $v$ can be determined from (2.12.112). $\square$

***Example 2.12.12***

(*Biharmonic Equation in Elasticity*). We derive the biharmonic equation in elasticity from the two-dimensional equilibrium equations and the compatibility condition. In two-dimensional elastic medium, the strain components $e_{xx}, e_{xy}, e_{yy}$ in terms of the displacement functions $(u, v, 0)$ are

$$e_{xx} = \frac{\partial u}{\partial x}, \quad e_{yy} = \frac{\partial v}{\partial y}, \quad e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right). \tag{2.12.113}$$

Differentiating these results gives the *compatibility condition*

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = 2\,\frac{\partial^2 e_{xy}}{\partial x \partial y}. \tag{2.12.114}$$

In terms of the *Poisson ratio* $\nu$ and *Young's modulus* $E$ of the elastic material, the strain component in the $z$ direction is expressed in terms of stress components

$$Ee_{zz} = \sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy}). \tag{2.12.115}$$

In the case of plane strain, $e_{zz} = 0$, so that

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}). \tag{2.12.116}$$

Substituting this result in other stress-strain relations, we obtain the strain components $e_{xx}, e_{xy}, e_{yy}$ that are related to stress components $\sigma_{xx}, \sigma_{xy}, \sigma_{yy}$ by

$$Ee_{xx} = \sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz}) = (1-\nu^2)\sigma_{xx} - \nu(1+\nu)\sigma_{yy}, \quad (2.12.117)$$
$$Ee_{yy} = \sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz}) = (1-\nu^2)\sigma_{yy} - \nu(1+\nu)\sigma_{xx}, \quad (2.12.118)$$
$$Ee_{xy} = (1+\nu)\sigma_{xy}. \quad (2.12.119)$$

Putting (2.12.117)-(2.12.119) into (2.12.114) gives

$$\frac{\partial^2}{\partial y^2}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})] + \frac{\partial^2}{\partial x^2}[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})] = 2(1+\nu)\frac{\partial^2 \sigma_{xy}}{\partial x \partial y}. \quad (2.12.120)$$

The basic differential equations for the stress components $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ in the medium under the action of body forces $X$ and $Y$ are

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \rho X = \rho \frac{\partial^2 u}{\partial t^2}, \quad (2.12.121)$$
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \rho Y = \rho \frac{\partial^2 v}{\partial t^2}, \quad (2.12.122)$$

where $\rho$ is the mass density of the elastic material.

The equilibrium equations follow from (2.12.121)–(2.12.122) in the absence of the body forces ($X = Y = 0$) as

$$\frac{\partial}{\partial x}\sigma_{xx} + \frac{\partial}{\partial y}\sigma_{xy} = 0, \quad (2.12.123)$$
$$\frac{\partial}{\partial x}\sigma_{xy} + \frac{\partial}{\partial y}\sigma_{yy} = 0. \quad (2.12.124)$$

It is obvious that the expressions

$$\sigma_{xx} = \frac{\partial^2 \chi}{\partial y^2}, \quad \sigma_{xy} = -\frac{\partial^2 \chi}{\partial x \partial y}, \quad \sigma_{yy} = \frac{\partial^2 \chi}{\partial x^2} \quad (2.12.125)$$

satisfy the equilibrium equations for any arbitrary function $\chi(x, y)$. Substituting from equations (2.12.125) into the compatibility condition (2.12.120), we see that $\chi$ must satisfy the *biharmonic equation*

$$\frac{\partial^4 \chi}{\partial x^4} + 2\frac{\partial^4 \chi}{\partial x^2 \partial y^2} + \frac{\partial^4 \chi}{\partial y^4} = 0, \quad (2.12.126)$$

which may be written symbolically as

$$\nabla^4 \chi = 0. \quad (2.12.127)$$

The function $\chi$ was first introduced by Airy in 1862 and is known as the *Airy stress function.*

We determine the stress distribution in a semi-infinite elastic medium bounded by an infinite plane at $x=0$ due to an external pressure to its surface. The $x$-axis is normal to this plane and assumed positive in the direction into the medium. We assume that the external surface pressure $p$ varies along the surface so that the boundary conditions are

$$\sigma_{xx} = -p(y), \quad \sigma_{xy} = 0 \quad \text{on } x=0 \quad \text{for all } y \text{ in } (-\infty, \infty). \tag{2.12.128}$$

We derive solutions so that stress components $\sigma_{xx}, \sigma_{yy}$, and $\sigma_{xy}$ all vanish as $x \to \infty$.

In order to solve the biharmonic equation (2.12.127), we introduce the Fourier transform $\tilde{\chi}(x,k)$ of the *Airy stress function* with respect to $y$ so that (2.12.127)–(2.12.128) reduce to

$$\left(\frac{d^2}{dx^2} - k^2\right)^2 \tilde{\chi} = 0, \tag{2.12.129}$$

$$k^2 \tilde{\chi}(0,k) = \tilde{p}(k), \quad (ik)\left(\frac{d\tilde{\chi}}{dx}\right)_{x=0} = 0, \tag{2.12.130}$$

where $\tilde{p}(k) = \mathscr{F}\{p(y)\}$. The bounded solution of the transformed problem is

$$\tilde{\chi}(x,k) = (A + Bx)\exp(-|k|x), \tag{2.12.131}$$

where $A$ and $B$ are constants of integration to be determined from (2.12.130). It turns out that $A = \tilde{p}(k)/k^2$ and $B = \tilde{p}(k)/|k|$ and hence, the solution becomes

$$\tilde{\chi}(x,k) = \frac{\tilde{p}(k)}{k^2}\{1 + |k|x\}\exp(-|k|x). \tag{2.12.132}$$

The inverse Fourier transform yields the formal solution

$$\chi(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\tilde{p}(k)}{k^2}(1 + |k|x)\exp(iky - |k|x)dk. \tag{2.12.133}$$

The stress components are obtained from (2.12.125) in the form

$$\sigma_{xx}(x,y) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k^2 \tilde{\chi}(x,k)\exp(iky)dk, \tag{2.12.134}$$

$$\sigma_{xy}(x,y) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (ik)\left(\frac{d\tilde{\chi}}{dx}\right)\exp(iky)dk, \tag{2.12.135}$$

$$\sigma_{yy}(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d^2\tilde{\chi}}{dx^2}\exp(iky)dk, \tag{2.12.136}$$

where $\tilde{\chi}(x,k)$ are given by (2.12.132). In particular, if $p(y)=P\delta(y)$ so that $\tilde{p}(k)=P(2\pi)^{-\frac{1}{2}}$. Consequently, from (2.12.133)–(2.12.136) we obtain

$$\chi(x,y) = \frac{P}{2\pi}\int_{-\infty}^{\infty} k^{-2}(1+|k|x)\exp(iky-|k|x)dk$$

$$= \frac{P}{\pi}\int_{0}^{\infty} k^{-2}(1+kx)\cos ky \exp(-kx)dk. \tag{2.12.137}$$

$$\sigma_{xx} = -\frac{P}{\pi}\int_{0}^{\infty}(1+kx)e^{-kx}\cos ky\, dk = -\frac{2Px^3}{\pi(x^2+y^2)^2}. \tag{2.12.138}$$

$$\sigma_{xy} = -\frac{Px}{\pi}\int_{0}^{\infty} k\sin ky \exp(-kx)dk = -\frac{2Px^2y}{\pi(x^2+y^2)^2}. \tag{2.12.139}$$

$$\sigma_{yy} = -\frac{P}{\pi}\int_{0}^{\infty}(1-kx)\exp(-kx)\cos ky\, dk = -\frac{2Pxy^2}{\pi(x^2+y^2)^2}. \tag{2.12.140}$$

Another physically realistic pressure distribution is

$$p(y) = PH(|a|-y), \tag{2.12.141}$$

where $P$ is a constant, so that

$$\tilde{p}(k) = \sqrt{\frac{2}{\pi}}\frac{P}{k}\sin ak. \tag{2.12.142}$$

Substituting this value for $\tilde{p}(k)$ into (2.12.133)–(2.12.136), we obtain the integral expression for $\chi, \sigma_{xx}, \sigma_{xy}$, and $\sigma_{yy}$.

It is noted here that if a point force of magnitude $P_0$ acts at the origin located on the boundary, then we put $P=(P_0/2a)$ in (2.12.142) and find

$$\tilde{p}(k) = \lim_{a\to 0}\sqrt{\frac{2}{\pi}}\frac{P_0}{2}\left(\frac{\sin ak}{ak}\right) = \frac{P_0}{\sqrt{2\pi}}. \tag{2.12.143}$$

Thus, the stress components can also be written in this case. ▯

## 2.13 Fourier Cosine and Sine Transforms with Examples

The Fourier cosine integral formula (2.2.8) leads to the *Fourier cosine transform* and its inverse defined by

$$\mathscr{F}_c\{f(x)\} = F_c(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos kx\, f(x)dx, \tag{2.13.1}$$

$$\mathscr{F}_c^{-1}\{F_c(k)\} = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos kx\, F_c(k)dk, \tag{2.13.2}$$

where $\mathscr{F}_c$ is the Fourier cosine transform operator and $\mathscr{F}_c^{-1}$ is its inverse operator.

Similarly, the Fourier sine integral formula (2.2.9) leads to the *Fourier sine transform* and its inverse defined by

$$\mathscr{F}_s\{f(x)\} = F_s(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin kx f(x)dx, \tag{2.13.3}$$

$$\mathscr{F}_s^{-1}\{F_s(k)\} = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin kx\; F_s(k)dk, \tag{2.13.4}$$

where $\mathscr{F}_s$ is the Fourier sine transform operator and $\mathscr{F}_s^{-1}$ is its inverse.

***Example 2.13.1***

Show that

$$\text{(a) } \mathscr{F}_c\{e^{-ax}\} = \sqrt{\frac{2}{\pi}} \frac{a}{(a^2+k^2)}, \quad (a > 0). \tag{2.13.5}$$

$$\text{(b) } \mathscr{F}_s\{e^{-ax}\} = \sqrt{\frac{2}{\pi}} \frac{k}{(a^2+k^2)}, \quad (a > 0). \tag{2.13.6}$$

We have

$$\begin{aligned}
\mathscr{F}_c\{e^{-ax}\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos kx\, dx \\
&= \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^\infty [e^{-(a-ik)x} + e^{-(a+ik)x}]dx \\
\mathscr{F}_c\{e^{-ax}\} &= \frac{1}{2}\sqrt{\frac{2}{\pi}} \left[\frac{1}{a-ik} + \frac{1}{a+ik}\right] = \sqrt{\frac{2}{\pi}} \frac{a}{(a^2+k^2)}.
\end{aligned}$$

The proof of the other result is similar and hence left to the reader. ▯

***Example 2.13.2***
Show that

$$\mathscr{F}_s^{-1}\left\{\frac{1}{k}\exp(-sk)\right\} = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{s}\right). \tag{2.13.7}$$

We have the standard definite integral

$$\sqrt{\frac{\pi}{2}}\mathscr{F}_s^{-1}\{\exp(-sk)\} = \int_0^\infty \exp(-sk)\sin kx\, dk = \frac{x}{s^2+x^2}. \tag{2.13.8}$$

Integrating both sides with respect to $s$ from $s$ to $\infty$ gives

$$\begin{aligned}\int_0^\infty \frac{e^{-sk}}{k}\sin kx\, dk &= \int_s^\infty \frac{x ds}{x^2+s^2} = \left[\tan^{-1}\frac{s}{x}\right]_s^\infty \\ &= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{x}\right) = \tan^{-1}\left(\frac{x}{s}\right).\end{aligned} \tag{2.13.9}$$

Thus,

$$\begin{aligned}\mathscr{F}_s^{-1}\left\{\frac{1}{k}\exp(-sk)\right\} &= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{k}\exp(-sk)\sin xk\, dk \\ &= \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{s}\right).\end{aligned}$$

▯

***Example 2.13.3***
Show that

$$\mathscr{F}_s\{erfc(ax)\} = \sqrt{\frac{2}{\pi}}\frac{1}{k}\left[1-\exp\left(-\frac{k^2}{4a^2}\right)\right]. \tag{2.13.10}$$

We have

$$\begin{aligned}\mathscr{F}_s\{erfc(ax)\} &= \sqrt{\frac{2}{\pi}}\int_0^\infty erfc(ax)\sin kx\, dx \\ &= \frac{2\sqrt{2}}{\pi}\int_0^\infty \sin kx\, dx \int_{ax}^\infty e^{-t^2}\, dt.\end{aligned}$$

Interchanging the order of integration, we obtain

$$\begin{aligned}\mathscr{F}_s\{erf(ax)\} &= \frac{2\sqrt{2}}{\pi}\int_0^\infty \exp(-t^2)dt \int_0^{t/a} \sin kx\, dx \\ &= \frac{2\sqrt{2}}{\pi k}\int_0^\infty \exp(-t^2)\left\{1-\cos\left(\frac{kt}{a}\right)\right\}dt \\ &= \frac{2\sqrt{2}}{\pi k}\left[\frac{\sqrt{\pi}}{2}-\frac{\sqrt{\pi}}{2}\exp\left(-\frac{k^2}{4a^2}\right)\right].\end{aligned}$$

Thus,

$$\mathscr{F}_s\{erfc(ax)\} = \sqrt{\frac{2}{\pi}}\frac{1}{k}\left[1-\exp\left(-\frac{k^2}{4a^2}\right)\right].$$

□

## 2.14 Properties of Fourier Cosine and Sine Transforms

***THEOREM 2.14.1***

If $\mathscr{F}_c\{f(x)\} = F_c(k)$ and $\mathscr{F}_s\{f(x)\} = F_s(k)$, then

$$\mathscr{F}_c\{f(ax)\} = \frac{1}{a}F_c\left(\frac{k}{a}\right), \quad a > 0. \tag{2.14.1}$$

$$\mathscr{F}_s\{f(ax)\} = \frac{1}{a}F_s\left(\frac{k}{a}\right), \quad a > 0. \tag{2.14.2}$$

Under appropriate conditions, the following properties also hold:

$$\mathscr{F}_c\{f'(x)\} = k\ F_s(k) - \sqrt{\frac{2}{\pi}}\,f(0), \tag{2.14.3}$$

$$\mathscr{F}_c\{f''(x)\} = -k^2\ F_c(k) - \sqrt{\frac{2}{\pi}}\,f'(0), \tag{2.14.4}$$

$$\mathscr{F}_s\{f'(x)\} = -k\ F_c(k), \tag{2.14.5}$$

$$\mathscr{F}_s\{f''(x)\} = -k^2\ F_s(k) + \sqrt{\frac{2}{\pi}}\,k\,f(0). \tag{2.14.6}$$

These results can be generalized for the cosine and sine transforms of higher order derivatives of a function. They are left as exercises.

***THEOREM 2.14.2***

(*Convolution Theorem for the Fourier Cosine Transform*). If $\mathscr{F}_c\{f(x)\} = F_c(k)$ *and* $\mathscr{F}_c\{g(x)\} = G_c(k)$, then

$$\mathscr{F}_c^{-1}\{F_c(k)G_c(k)\} = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(\xi)[g(x+\xi) + g(|x-\xi|)]d\xi. \qquad (2.14.7)$$

Or, equivalently,

$$\int_0^\infty F_c(k)G_c(k)\cos kx\, dk = \frac{1}{2}\int_0^\infty f(\xi)[g(x+\xi) + g(|x-\xi|)]d\xi. \qquad (2.14.8)$$

**PROOF** Using the definition of the inverse Fourier cosine transform, we have

$$\begin{aligned}\mathscr{F}_c^{-1}\{F_c(k)G_c(k)\} &= \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(k)G_c(k)\cos kx\, dk \\ &= \left(\frac{2}{\pi}\right)\int_0^\infty G_c(k)\cos kx\, dk \int_0^\infty f(\xi)\cos k\xi\, d\xi.\end{aligned}$$

Hence,

$$\begin{aligned}\mathscr{F}_c^{-1}\{F_c(k)G_c(k)\} &= \left(\frac{2}{\pi}\right)\int_0^\infty f(\xi)d\xi \int_0^\infty \cos kx \cos k\xi\, G_c(k)dk \\ &= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty f(\xi)d\xi \sqrt{\frac{2}{\pi}}\int_0^\infty [\cos k(x+\xi) + \cos k(|x-\xi|)]G_c(k)dk \\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty f(\xi)[g(x+\xi) + g(|x-\xi|)]d\xi,\end{aligned}$$

in which the definition of the inverse Fourier cosine transform is used. This proves (2.14.7).

It also follows from the proof of Theorem 2.14.2 that

$$\int_0^\infty F_c(k)G_c(k)\cos kx\, dk = \frac{1}{2}\int_0^\infty f(\xi)[g(x+\xi) + g(|x-\xi|)]d\xi.$$

This proves result (2.14.8).

Putting $x = 0$ in (2.14.8), we obtain

$$\int_0^\infty F_c(k)G_c(k)dk = \int_0^\infty f(\xi)g(\xi)d\xi = \int_0^\infty f(x)g(x)dx.$$

Substituting $g(x)=\overline{f(x)}$ gives, since $G_c(k)=\overline{F_c(k)}$,

$$\int_0^\infty |F_c(k)|^2 dk = \int_0^\infty |f(x)|^2 dx. \tag{2.14.9}$$

This is the *Parseval relation* for the Fourier cosine transform.

Similarly, we obtain

$$\int_0^\infty F_s(k)G_s(k)\cos kx\, dk$$
$$= \sqrt{\frac{2}{\pi}}\int_0^\infty G_s(k)\cos kx\, dk \int_0^\infty f(\xi)\sin k\xi\, d\xi$$

which is, by interchanging the order of integration,

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(\xi)d\xi \int_0^\infty G_s(k)\sin k\xi \cos kx\, dk$$
$$= \frac{1}{2}\int_0^\infty f(\xi)d\xi \sqrt{\frac{2}{\pi}}\int_0^\infty G_s(k)[\sin k(\xi+x)+\sin k(\xi-x)]dk$$
$$= \frac{1}{2}\int_0^\infty f(\xi)[g(\xi+x)+g(\xi-x)]d\xi,$$

in which the inverse Fourier sine transform is used. Thus, we find

$$\int_0^\infty F_s(k)G_s(k)\cos kx\, dk = \frac{1}{2}\int_0^\infty f(\xi)[g(\xi+x)+g(\xi-x)]d\xi. \tag{2.14.10}$$

Or, equivalently,

$$\mathscr{F}_c^{-1}\{F_s(k)G_s(k)\} = \frac{1}{\sqrt{2\pi}}\int_0^\infty f(\xi)[g(\xi+x)+g(\xi-x)]d\xi. \tag{2.14.11}$$

Result (2.14.10) or (2.14.11) is also called the *Convolution Theorem* of the Fourier cosine transform.

Putting $x=0$ in (2.14.10) gives

$$\int_0^\infty F_s(k)G_s(k)dk = \int_0^\infty f(\xi)g(\xi)d\xi = \int_0^\infty f(x)g(x)dx.$$

Replacing $g(x)$ by $\overline{f(x)}$ gives the *Parseval relation* for the Fourier sine transform

$$\int_0^\infty |F_s(k)|^2 dk = \int_0^\infty |f(x)|^2 dx. \tag{2.14.12}$$

■

## 2.15 Applications of Fourier Cosine and Sine Transforms to Partial Differential Equations

***Example 2.15.1***

(*One-Dimensional Diffusion Equation on a Half Line*). Consider the initial-boundary value problem for the one-dimensional diffusion equation in $0 < x < \infty$ with no sources or sinks:

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0, \tag{2.15.1}$$

where $\kappa$ is a constant, with the initial condition

$$u(x,0) = 0, \quad 0 < x < \infty, \tag{2.15.2}$$

and the boundary conditions

$$\text{(a) } u(0,t) = f(t), \quad t \geq 0, \quad u(x,t) \to 0 \quad \text{as } x \to \infty, \tag{2.15.3}$$

or,

$$\text{(b) } u_x(0,t) = f(t), \quad t \geq 0, \quad u(x,t) \to 0 \quad \text{as } x \to \infty. \tag{2.15.4}$$

This problem with the boundary conditions (2.15.3) is solved by using the Fourier sine transform

$$U_s(k,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin kx \, u(x,t) \, dx.$$

Application of the Fourier sine transform gives

$$\frac{dU_s}{dt} = -\kappa \, k^2 \, U_s(k,t) + \sqrt{\frac{2}{\pi}} \, \kappa \, k \, f(t), \tag{2.15.5}$$

$$U_s(k,0) = 0. \tag{2.15.6}$$

The bounded solution of this differential system with $U_s(k,0)=0$ is

$$U_s(k,t)=\sqrt{\frac{2}{\pi}}\,\kappa\,k\int_0^t f(\tau)\exp[-\kappa(t-\tau)k^2]\,d\tau. \tag{2.15.7}$$

The inverse transform gives the solution

$$\begin{aligned} u(x,t) &= \sqrt{\frac{2}{\pi}}\,\kappa\int_0^t f(\tau)\mathscr{F}_s^{-1}\{k\exp[-\kappa(t-\tau)k^2]\}d\tau \\ &= \frac{x}{\sqrt{4\pi\kappa}}\int_0^t f(\tau)\exp\left[-\frac{x^2}{4\kappa(t-\tau)}\right]\frac{d\tau}{(t-\tau)^{3/2}} \end{aligned} \tag{2.15.8}$$

in which $\mathscr{F}_s^{-1}\{k\exp(-t\kappa k^2)\}=\frac{x}{2\sqrt{2}}\cdot\frac{\exp(-x^2/4\kappa t)}{(\kappa t)^{3/2}}$ is used.

In particular, $f(t)=T_0=\text{constant}$, (2.15.7) reduces to

$$U_s(k,t)=\sqrt{\frac{2}{\pi}}\frac{T_0}{k}[1-\exp(-\kappa\,t\,k^2)]. \tag{2.15.9}$$

Inversion gives the solution

$$u(x,t)=\left(\frac{2T_0}{\pi}\right)\int_0^\infty \frac{\sin kx}{k}[1-\exp(-\kappa\,t\,k^2)]dk. \tag{2.15.10}$$

Making use of the integral

$$\int_0^\infty e^{-k^2a^2}\,\frac{\sin kx}{k}dk=\frac{\pi}{2}\,erf\left(\frac{x}{2a}\right), \tag{2.15.11}$$

the solution becomes

$$\begin{aligned} u(x,t) &= \frac{2T_0}{\pi}\left[\frac{\pi}{2}-\frac{\pi}{2}\,erf\left(\frac{x}{2\sqrt{\kappa t}}\right)\right] \\ &= T_0\,erfc\left(\frac{x}{2\sqrt{\kappa t}}\right), \end{aligned} \tag{2.15.12}$$

where the *error function*, *erf*$(x)$ is defined by

$$erf(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-\alpha^2}d\alpha, \tag{2.15.13}$$

so that

$$erf(0)=0,\ erf(\infty)=\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-\alpha^2}\,d\alpha=1,\ \text{and}\ erf(-x)=-erf(x),$$

and the *complementary error function*, $erfc(x)$ is defined by

$$erfc(x)=1-erf(x)=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-\alpha^2}\,d\alpha, \tag{2.15.14}$$

so that

$$erfc(x)=1-erf(x), \quad erfc(0)=1, \quad erfc(\infty)=0,$$

and

$$erfc(-x)=1-erf(-x)=1+erf(x)=2-erfc(x).$$

Equation (2.15.1) with boundary condition (2.15.4) is solved by the Fourier cosine transform

$$U_c(k,t)=\sqrt{\frac{2}{\pi}}\int_0^\infty \cos kx\, u(x,t)dx.$$

Application of this transform to (2.15.1) gives

$$\frac{dU_c}{dt}+\kappa k^2 U_c=-\sqrt{\frac{2}{\pi}}\,\kappa f(t). \tag{2.15.15}$$

The solution of (2.15.15) with $U_c(k,0)=0$ is

$$U_c(k,t)=-\sqrt{\frac{2}{\pi}}\,\kappa\int_0^t f(\tau)\exp[-k^2\kappa(t-\tau)]d\tau. \tag{2.15.16}$$

Since

$$\mathscr{F}_c^{-1}\{\exp(-t\kappa k^2)\}=\frac{1}{\sqrt{2\kappa t}}\exp\left(-\frac{x^2}{4\kappa t}\right), \tag{2.15.17}$$

the inverse Fourier cosine transform gives the final form of the solution

$$u(x,t)=-\sqrt{\frac{\kappa}{\pi}}\int_0^t \frac{f(\tau)}{\sqrt{t-\tau}}\exp\left[-\frac{x^2}{4\kappa(t-\tau)}\right]d\tau. \tag{2.15.18}$$

□

***Example 2.15.2***
(*The Laplace Equation in the Quarter Plane*). Solve the Laplace equation

$$u_{xx}+u_{yy}=0, \quad 0<x,\ y<\infty, \tag{2.15.19}$$

with the boundary conditions

$$u(0,y)=a, \qquad u(x,0)=0, \tag{2.15.20a}$$
$$\nabla u \to 0 \qquad \text{as } r=\sqrt{x^2+y^2}\to\infty, \tag{2.15.20b}$$

where $a$ is a constant.

We apply the Fourier sine transform with respect to $x$ to find

$$\frac{d^2U_s}{dy^2} - k^2\,U_s + \sqrt{\frac{2}{\pi}}\,ka = 0.$$

The solution of this inhomogeneous equation is

$$U_s(k,y) = Ae^{-ky} + \sqrt{\frac{2}{\pi}}\cdot\frac{a}{k},$$

where $A$ is a constant to be determined from $U_s(k,0)=0$. Consequently,

$$U_s(k,y) = \frac{a}{k}\sqrt{\frac{2}{\pi}}(1-e^{-ky}). \tag{2.15.21}$$

The inverse transformation gives the formal solution

$$u(x,y) = \frac{2a}{\pi}\int_0^\infty \frac{1}{k}(1-e^{-ky})\sin kx\,dk$$

Or,

$$\begin{aligned} u(x,y) &= \frac{2a}{\pi}\left[\int_0^\infty \frac{\sin kx}{k}dk - \int_0^\infty \frac{1}{k}e^{-ky}\sin kx\,dk\right] \\ &= a - \frac{2a}{\pi}\left(\frac{\pi}{2} - \tan^{-1}\frac{y}{x}\right) = \frac{2a}{\pi}\tan^{-1}\left(\frac{y}{x}\right), \end{aligned} \tag{2.15.22}$$

in which (2.13.9) is used. ▯

***Example 2.15.3***

(*The Laplace Equation in a Semi-Infinite Strip with the Dirichlet Data*). Solve the Laplace equation

$$u_{xx}+u_{yy}=0, \quad 0<x<\infty, \quad 0<y<b, \tag{2.15.23}$$

with the boundary conditions

$$u(0,y)=0,\ u(x,y)\to 0 \quad \text{as} \quad x\to\infty \text{ for } 0<y<b \tag{2.15.24}$$
$$u(x,b)=0,\ u(x,0)=f(x) \quad \text{for} \quad 0<x<\infty. \tag{2.15.25}$$

In view of the Dirichlet data, the Fourier sine transform with respect to $x$ can be used to solve this problem. Applying the Fourier sine transform to (2.15.23)–(2.15.25) gives

$$\frac{d^2 U_s}{dy^2} - k^2 U_s = 0, \tag{2.15.26}$$

$$U_s(k,b) = 0, \quad U_s(k,0) = F_s(k). \tag{2.15.27}$$

The solution of (2.15.26) with (2.15.27) is

$$U_s(k,y) = F_s(k)\,\frac{\sinh[k(b-y)]}{\sinh kb}. \tag{2.15.28}$$

The inverse Fourier sine transform gives the formal solution

$$\begin{aligned} u(x,y) &= \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(k) \frac{\sinh[k(b-y)]}{\sinh kb} \sin kx\, dk \\ &= \frac{2}{\pi} \int_0^\infty \left[ \int_0^\infty f(l) \sin kl\, dl \right] \frac{\sinh[k(b-y)]}{\sinh kb} \sin kx\, dk. \end{aligned} \tag{2.15.29}$$

In the limit as $kb \to \infty$, $\frac{\sinh[k(b-y)]}{\sinh kb} \sim \exp(-ky)$, hence the above problem reduces to the corresponding problem in the quarter plane, $0 < x < \infty, 0 < y < \infty$. Thus, solution (2.15.29) becomes

$$\begin{aligned} u(x,y) &= \frac{2}{\pi} \int_0^\infty f(l) dl \int_0^\infty \sin kl \sin kx \exp(-ky) dk \\ &= \frac{1}{\pi} \int_0^\infty f(l) dl \int_0^\infty \{\cos k(x-l) - \cos k(x+l)\} \exp(-ky) dk \\ &= \frac{1}{\pi} \int_0^\infty f(l) \left[ \frac{y}{(x-l)^2 + y^2} - \frac{y}{(x+l)^2 + y^2} \right] dl. \end{aligned} \tag{2.15.30}$$

This is the exact integral solution of the problem. If $f(x)$ is an odd function of $x$, then solution (2.15.30) reduces to the solution (2.12.10) of the same problem in the half plane. □

## 2.16 Evaluation of Definite Integrals

The Fourier transform can be employed to evaluate certain definite integrals. Although the method of evaluation may not be very rigorous, it is quite simple and straightforward. The method can be illustrated by means of examples.

***Example 2.16.1***
Evaluate the integral

$$I(a,b) = \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}, \quad a>0, b>0. \tag{2.16.1}$$

If we write $f(x) = e^{-a|x|}$ and $g(x) = e^{-b|x|}$ then $F(k) = \sqrt{\frac{2}{\pi}} \frac{a}{(k^2+a^2)}$, $G(k) = \sqrt{\frac{2}{\pi}} \frac{b}{(k^2+b^2)}$. The Convolution Theorem 2.5.5 gives (2.5.19), that is,

$$\int_{-\infty}^{\infty} F(k)G(k)dk = \int_{-\infty}^{\infty} f(x)g(-x)dx.$$

Or, equivalently,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dk}{(k^2+a^2)(k^2+b^2)} &= \frac{\pi}{2ab} \int_{-\infty}^{\infty} e^{-|x|(a+b)} dx \\ &= \frac{\pi}{ab} \int_0^{\infty} e^{-(a+b)x} dx = \frac{\pi}{ab(a+b)}. \end{aligned} \tag{2.16.2}$$

This is the desired result.

Further

$$\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}. \tag{2.16.3}$$

$\square$

***Example 2.16.2***
Show that

$$\int_0^{\infty} \frac{x^{-p}dx}{(a^2+x^2)} = \frac{\pi}{2} a^{-(p+1)} \sec\left(\frac{\pi p}{2}\right). \tag{2.16.4}$$

We write

$$f(x) = e^{-ax} \quad \text{so that} \quad F_c(k) = \sqrt{\frac{2}{\pi}} \frac{a}{(a^2+k^2)}.$$

$$g(x) = x^{p-1} \quad \text{so that} \quad G_c(k) = \sqrt{\frac{2}{\pi}} k^{-p} \Gamma(p) \cos\left(\frac{\pi p}{2}\right).$$

Using Parseval's result for the Fourier cosine transform gives

$$\int_0^{\infty} F_c(k)G_c(k)dk = \int_0^{\infty} f(x)g(x)dx.$$

Or,

$$\frac{2a}{\pi}\cos\left(\frac{\pi p}{2}\right)\Gamma(p)\int_0^\infty \frac{k^{-p}dk}{k^2+a^2} = \int_0^\infty x^{p-1}e^{-ax}dx$$

$$= \frac{1}{a^p}\int_0^\infty e^{-t}t^{p-1}dt = \frac{\Gamma(p)}{a^p}, \quad (ax=t).$$

Thus,

$$\int_0^\infty \frac{k^{-p}dk}{a^2+k^2} = \frac{\pi}{2\,a^{p+1}}\sec\left(\frac{\pi p}{2}\right).$$

$\square$

***Example 2.16.3***

If $a>0, b>0$, show that

$$\int_0^\infty \frac{x^2\,dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2(a+b)}. \tag{2.16.5}$$

We consider

$$\mathscr{F}_s\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\frac{k}{k^2+a^2} = F_s(k)$$

$$\mathscr{F}_s\{e^{-bx}\} = \sqrt{\frac{2}{\pi}}\frac{k}{k^2+b^2} = G_s(k).$$

Then the Convolution Theorem for the Fourier cosine transform gives

$$\int_0^\infty F_s(k)G_s(k)\cos kx\,dk = \frac{1}{2}\int_0^\infty g(\xi)[f(\xi+x)+f(\xi-x)]d\xi.$$

Putting $x=0$ gives

$$\int_0^\infty F_s(k)G_s(k)dk = \int_0^\infty g(\xi)f(\xi)d\xi,$$

or,

$$\int_0^\infty \frac{k^2\,dk}{(k^2+a^2)(k^2+b^2)} = \frac{\pi}{2}\int_0^\infty e^{-(a+b)\xi}d\xi = \frac{\pi}{2(a+b)}.$$

$\square$

***Example 2.16.4***
Show that

$$\int_0^\infty \frac{x^2 dx}{(x^2+a^2)^4} = \frac{\pi}{(2a)^5}, \quad a > 0. \tag{2.16.6}$$

We write $f(x) = \frac{1}{2(x^2+a^2)}$ so that $f'(x) = -\frac{x}{(x^2+a^2)^2}$, and $\mathscr{F}\{f(x)\} = F(k) = \sqrt{\frac{\pi}{2}}\left(\frac{1}{2a}\right)\exp(-a|k|)$.

Making reference to the Parseval relation (2.4.19), we obtain

$$\int_{-\infty}^{\infty} |f'(x)|^2 dx = \int_{-\infty}^{\infty} |\mathscr{F}\{f'(x)\}|^2 dk = \int_{-\infty}^{\infty} |(ik)\mathscr{F}\{f(x)\}|^2 dk.$$

Thus,

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^4}\,dx = \frac{\pi}{2}\int_{-\infty}^{\infty} k^2 \cdot \frac{1}{(2a)^2}\exp(-2a|k|)dk$$

$$= \frac{\pi}{(2a)^2}\int_0^\infty k^2 \exp(-2ak)dk = \frac{2\pi}{(2a)^5}.$$

This gives the desired result. ∎

## 2.17 Applications of Fourier Transforms in Mathematical Statistics

In probability theory and mathematical statistics, the characteristic function of a random variable is defined by the Fourier transform or by the Fourier-Stieltjes transform of the distribution function of a random variable. Many important results in probability theory and mathematical statistics can be obtained, and their proofs can be simplified with rigor by using the methods of characteristic functions. Thus, the Fourier transforms play an important role in probability theory and mathematical statistics.

***DEFINITION 2.17.1*** *(Distribution Function). The distribution function $F(x)$ of a random variable $X$ is defined as the probability, that is, $F(x) = P(X < x)$ for every real number $x$.*

It is immediately evident from this definition that the distribution function satisfies the following properties:

(i) $F(x)$ is a non-decreasing function, that is, $F(x_1) \leq F(x_2)$ if $x_1 < x_2$.

(ii) $F(x)$ is continuous only from the left at a point $x$, that is, $F(x-0) = F(x)$, but $F(x+0) \neq F(x)$.

(iii) $F(-\infty) = 0$ and $F(+\infty) = 1$.

If $X$ is a continuous variable and if there exists a non-negative function $f(x)$ such that for every real $x$ the following relation holds:

$$F(x) = \int_{-\infty}^{x} f(x)dx, \tag{2.17.1}$$

where $F(x)$ is the distribution function of the random variable $X$, then the function $f(x)$ is called the *probability density* or simply the *density function* of the random variable $X$.

It is immediately obvious that every density function $f(x)$ satisfies the following properties:

(i)

$$F(+\infty) = \int_{-\infty}^{\infty} f(x)dx = 1. \tag{2.17.2a}$$

(ii) For every real $a$ and $b$ where $a < b$,

$$P(a \leq X \leq b) = F(b) - F(a) = \int_{a}^{b} f(x)dx. \tag{2.17.2b}$$

(iii) If $f(x)$ is continuous at some point $x$, then $F'(x) = f(x)$.

It is noted that every real function $f(x)$ which is non-negative, and integrable over the whole real line and satisfies (2.17.2ab), is the probability density function of a continuous random variable $X$. On the other hand, the function $F(x)$ defined by (2.17.1) satisfies all properties of a distribution function.

***DEFINITION 2.17.2*** *(Characteristic Function). If $X$ is a continuous random variable with the density function $f(x)$, then the characteristic function, $\phi(t)$ of the random variable $X$ or of the distribution function $F(x)$ is defined by the formula*

$$\phi(t) = E(\exp(itX)) = \int_{-\infty}^{\infty} f(x)\exp(itx)dx, \tag{2.17.3}$$

where $E[g(X)]$ is called the *expected value* of the random variable $g(X)$.

In problems of mathematical statistics, it is convenient to define the Fourier transform of $f(x)$ and its inverse in a slightly different way by

$$\mathscr{F}\{f(x)\} = \phi(t) = \int_{-\infty}^{\infty} \exp(itx) f(x) dx, \tag{2.17.4}$$

$$\mathscr{F}^{-1}\{\phi(t)\} = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-itx) \phi(t) dt. \tag{2.17.5}$$

Evidently, the characteristic function of $F(x)$ is the Fourier transform of the density function $f(x)$. The Fourier transform of the distribution function follows from the fact that

$$\mathscr{F}\{F'(x)\} = \mathscr{F}\{f(x)\} = \phi(t),$$

or,

$$\mathscr{F}\{F(x)\} = it^{-1}\phi(t). \tag{2.17.6}$$

The *composition of two distribution functions* $F_1(x)$ and $F_2(x)$ is defined by

$$F(x) = F_1(x) * F_2(x) = \int_{-\infty}^{\infty} F_1(x-y) F_2'(y) dy. \tag{2.17.7}$$

Thus, the Fourier transform of (2.17.7) gives

$$\begin{aligned} i\,t^{-1}\phi(t) &= \mathscr{F}\left\{ \int_{-\infty}^{\infty} F_1(x-y) F_2'(y) dy \right\} \\ &= \mathscr{F}\{F_1(x)\}\mathscr{F}\{f_2(x)\} = it^{-1}\phi_1(t)\phi_2(t), \end{aligned}$$

whence an important result follows:

$$\phi(t) = \phi_1(t)\phi_2(t), \tag{2.17.8}$$

where $\phi_1(t)$ and $\phi_2(t)$ are the characteristic functions of the distribution functions $F_1(x)$ and $F_2(x)$ respectively.

The *nth moment* of a random variable $X$ is defined by

$$m_n = E[X^n] = \int_{-\infty}^{\infty} x^n f(x) dx, \quad n = 1, 2, 3, \ldots. \tag{2.17.9}$$

provided this integral exists. The first moment $m_1$ (or simply $m$) is called the *expectation* of $X$ and has the form

$$m = E(X) = \int_{-\infty}^{\infty} x\, f(x) dx. \tag{2.17.10}$$

Thus, the moment of any order $n$ is calculated by evaluating the integral (2.17.9). However, the evaluation of the integral is, in general, a difficult task. This difficulty can be resolved with the help of the characteristic function defined by (2.17.4). Differentiating (2.17.4) $n$ times and putting $t=0$ gives a fairly simple formula

$$m_n = \int_{-\infty}^{\infty} x^n f(x)dx = (-i)^n \phi^{(n)}(0), \tag{2.17.11}$$

where $n=1,2,3,\ldots$.

When $n=1$, the expectation of a random variable $X$ becomes

$$m_1 = E(X) = \int_{-\infty}^{\infty} x f(x)dx = (-i)\phi'(0). \tag{2.17.12}$$

Thus, the simple formula (2.17.11) involving the derivatives of the characteristic function provides for the existence and the computation of the moment of any arbitrary order.

Similarly, the variance $\sigma^2$ of a random variable is given in terms of the characteristic function as

$$\begin{aligned}\sigma^2 &= \int_{-\infty}^{\infty} (x-m)^2 f(x)dx = m_2 - m_1^2 \\ &= \{\phi'(0)\}^2 - \phi''(0).\end{aligned} \tag{2.17.13}$$

***Example 2.17.1***

Find the moments of the normal distribution defined by the density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-m)^2}{2\sigma^2}\right\}. \tag{2.17.14}$$

The characteristic function of the normal distribution is the Fourier transform of $f(x)$, which is

$$\phi(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} \exp\left[-\frac{(x-m)^2}{2\sigma^2}\right] dx.$$

We substitute $x-m=y$ and use Example 2.3.1 to obtain

$$\phi(t) = \frac{\exp(itm)}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ity} \exp\left(-\frac{y^2}{2\sigma^2}\right) dy = \exp\left(itm - \frac{1}{2}t^2\sigma^2\right). \tag{2.17.15}$$

Thus,

$$\begin{aligned} m_1 &= (-i)\phi'(0) = m, \\ m_2 &= -\phi''(0) = (m^2 + \sigma^2), \\ m_3 &= m(m^2 + 3\sigma^2). \end{aligned}$$

Finally, the variance of the normal distribution is

$$m_2 - m_1^2 = \sigma^2. \tag{2.17.16}$$

▯

The above discussion reveals that characteristic functions are very useful for investigation of certain problems in mathematical statistics. We close this section by discussing more properties of characteristic functions.

***THEOREM 2.17.1***
(*Addition Theorem*). The characteristic function of the sum of a finite number of independent random variables is equal to the product of their characteristic functions.

**PROOF** Suppose $X_1, X_2, \ldots, X_n$ are $n$ independent random variables and $Z = X_1 + X_2 + \cdots + X_n$. Further, suppose $\phi_1(t), \phi_2(t), \ldots, \phi_n(t)$, and $\phi(t)$ are the characteristic functions of $X_1, X_2, \ldots, X_n$ and $Z$ respectively.

Then we have

$$\phi(t) = E[\exp(itZ)] = E\left[\exp\left\{it\left(X_1 + X_2 + \cdots + X_n\right)\right\}\right],$$

which is, by the independence of the random variables,

$$\begin{aligned} &= E(e^{itX_1})E(e^{itX_2}) \cdots E(e^{itX_n}) \\ &= \phi_1(t)\,\phi_2(t) \cdots \phi_n(t). \end{aligned} \tag{2.17.17}$$

This proves the *Addition Theorem.* ▮

***Example 2.17.2***
Find the expected value and the standard deviation of the sum of $n$ independent normal random variables.

Suppose $X_1, X_2, \ldots, X_n$ are $n$ independent random variables with the normal distributions $N(m_r, \sigma_r)$, where $r = 1, 2, \ldots, n$. The respective characteristic functions of these distributions are

$$\phi_r(t) = \exp\left[itm_r - \frac{1}{2}t^2\sigma_r^2\right], \quad r = 1, 2, 3, \ldots, n. \tag{2.17.18}$$

Because of the independence of $X_1, X_2, \ldots, X_n$, the random variable $Z = X_1 + X_2 + \cdots + X_n$ has the characteristic function

$$\begin{aligned}\phi(t) &= \phi_1(t)\phi_2(t)\cdots\phi_n(t)\\ &= \exp\left[it(m_1+m_2+\cdots+m_n) - \frac{1}{2}(\sigma_1^2+\sigma_2^2+\cdots+\sigma_n^2)t^2\right]. \quad (2.17.19)\end{aligned}$$

This represents the characteristic function of the normal distribution $N(m_1 + \cdots + m_n, \sqrt{\sigma_1^2+\cdots+\sigma_n^2})$. Thus, the expected value of $Z$ is $(m_1+m_2+\cdots+m_n)$ and its standard deviation is $(\sigma_1^2+\sigma_2^2+\cdots+\sigma_n^2)^{\frac{1}{2}}$. ☐

Finally, we state the fundamental *Central Limit Theorems* without proof.

***THEOREM 2.17.2***

(*The Lévy-Cramér Theorem*). Suppose $\{X_n\}$ is a sequence of random variables, $F_n(x)$ and $\phi_n(t)$ are respectively the distribution and characteristic functions of $X_n$. Then the sequence $\{F_n(x)\}$ is convergent to a distribution function $F(x)$ if and only if the sequence $\{\phi_n(t)\}$ is convergent at every point $t$ on the real line to a function $\phi(t)$ continuous in some neighborhood of the origin. The limit function $\phi(t)$ is then the characteristic function of the limit distribution function $F(x)$, and the convergence $\phi_n(t) \to \phi(t)$ is uniform in every finite interval on the $t$-axis.

***THEOREM 2.17.3***

(*The Central Limit Theorem in Probability*). Supose $f(x)$ is a nonnegative absolutely integrable function in $\mathbb{R}$ and has the following properties:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1, \quad \int_{-\infty}^{\infty} x\,f(x)\,dx = 1, \quad \int_{-\infty}^{\infty} x^2\,f(x)\,dx = 1.$$

If $f^n = f * f * \ldots * f$ is the convolution product of $f$ with itself $n$ times, then

$$\lim_{n\to\infty} \int_{a\sqrt{n}}^{b\sqrt{n}} f^n(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2}\,dx \qquad -\infty < a < b < \infty. \quad (2.17.20)$$

For a proof of the theorem, we refer the reader to Chandrasekharan (1989).

All these ideas developed in this section can be generalized for the multidimensional distribution functions by the use of multiple Fourier transforms. We refer interested readers to Lukacs (1960).

## 2.18 Multiple Fourier Transforms and Their Applications

***DEFINITION 2.18.1*** *Under the assumptions on $f(\mathbf{x})$ similar to those made for the one dimensional case, the multiple Fourier transform of $f(\mathbf{x})$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is the n-dimensional vector, is defined by*

$$\mathscr{F}\{f(\mathbf{x})\} = F(\boldsymbol{\kappa}) = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\{-i(\boldsymbol{\kappa}\cdot\mathbf{x})\} f(\mathbf{x}) d\mathbf{x}, \tag{2.18.1}$$

*where $\boldsymbol{\kappa} = (k_1, k_2, \ldots, k_n)$ is the n-dimensional transform vector and $\boldsymbol{\kappa}\cdot\mathbf{x} = (k_1x_1 + k_2x_2 + \cdots + k_nx_n)$.*

*The inverse Fourier transform is similarly defined by*

$$\mathscr{F}^{-1}\{F(\boldsymbol{\kappa})\} = f(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\{i(\boldsymbol{\kappa}\cdot\mathbf{x})\}\, F(\boldsymbol{\kappa})\, d\boldsymbol{\kappa}. \tag{2.18.2}$$

*In particular, the double Fourier transform is defined by*

$$\mathscr{F}\{f(x,y)\} = F(k,\ell) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\{-i(\boldsymbol{\kappa}\cdot\mathbf{r})\}\, f(x,y)\, dxdy, \tag{2.18.3}$$

*where $\mathbf{r} = (x, y)$ and $\boldsymbol{\kappa} = (k, \ell)$.*

*The inverse Fourier transform is given by*

$$\mathscr{F}^{-1}\{F(k,\ell)\} = f(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\{i(\boldsymbol{\kappa}\cdot\mathbf{r})\}\, F(k,\ell)\, dk\, d\ell. \tag{2.18.4}$$

*Similarly, the three-dimensional Fourier transform and its inverse are defined by the integrals*

$$\mathscr{F}\{f(x,y,z)\} = F(k,\ell,m) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\{-i(\boldsymbol{\kappa}\cdot\mathbf{r})\}\, f(x,y,z)\, dx\, dy\, dz, \tag{2.18.5}$$

$$\mathscr{F}^{-1}\{F(k,\ell,m)\} = f(x,y,z) \quad \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\{i(\boldsymbol{\kappa}\cdot\mathbf{r})\}\, F(k,\ell,m)\, dk\, d\ell\, dm. \tag{2.18.6}$$

The operational properties of these multiple Fourier transforms are similar to those of the one-dimensional case. In particular, results (2.4.7) and (2.4.8) relating the Fourier transforms of derivatives to the Fourier transforms of given functions are valid for the higher dimensional case as well. In higher dimensions, they are applied to the transforms of partial derivatives of $f(\mathbf{x})$ under the assumptions that $f$ and its partial derivatives vanish at infinity.

We illustrate the multiple Fourier transform method by the following examples of applications:

***Example 2.18.1***

(*The Dirichlet Problem for the Three-Dimensional Laplace Equation in the Half-Space*). The boundary value problem for $u(x,y,z)$ satisfies the following equation and boundary conditions:

$$\nabla^2 u \equiv u_{xx} + u_{yy} + u_{zz} = 0, \quad -\infty < x, y < \infty, \quad z > 0, \tag{2.18.7}$$

$$u(x,y,0) = f(x,y) \qquad -\infty < x, y < \infty \tag{2.18.8}$$

$$u(x,y,z) \to 0 \qquad \text{as } r = \sqrt{x^2+y^2+z^2} \to \infty. \tag{2.18.9}$$

We use the double Fourier transform defined by (2.18.3) to the system (2.18.7)–(2.18.9) which reduces to

$$\frac{d^2U}{dz^2} - \kappa^2 U = 0 \qquad \text{for} \qquad z > 0, \quad (\kappa^2 = k^2 + l^2)$$

$$U(k,\ell,0) = F(k,\ell).$$

Thus, the solution of this transformed problem is

$$U(k,\ell,z) = F(k,\ell)\exp(-|\boldsymbol{\kappa}|z) = F(k,\ell)G(k,\ell), \tag{2.18.10}$$

where $\boldsymbol{\kappa} = (k,\ell)$ and $G(k,\ell) = \exp(-|\boldsymbol{\kappa}|z)$ so that

$$g(x,y) = \mathscr{F}^{-1}\{\exp(-|\boldsymbol{\kappa}|z)\} = \frac{z}{(x^2+y^2+z^2)^{3/2}}. \tag{2.18.11}$$

Applying the Convolution Theorem to (2.18.10), we obtain the formal solution

$$u(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)g(x-\xi,y-\eta,z)\,d\xi\,d\eta$$

$$= \frac{z}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(\xi,\eta)\,d\xi\,d\eta}{[(x-\xi)^2+(y-\eta)^2+z^2]^{3/2}}. \tag{2.18.12}$$

□

***Example 2.18.2***
(*The Two-Dimensional Diffusion Equation*). We solve the two-dimensional diffusion equation

$$u_t = K\nabla^2 u, \quad -\infty < x, y < \infty, \quad t > 0, \tag{2.18.13}$$

with the initial and boundary conditions

$$u(x, y, 0) = f(x, y) \quad -\infty < x, y < \infty, \tag{2.18.14}$$
$$u(x, y, t) \to 0 \quad \text{as} \quad r = \sqrt{x^2 + y^2} \to \infty, \tag{2.18.15}$$

where $K$ is the diffusivity constant.

The double Fourier transform of $u(x, y, t)$ defined by (2.18.3) is used to reduce the system (2.18.13)–(2.18.14) into the form

$$\frac{dU}{dt} = -\kappa^2 KU, \qquad t > 0,$$
$$U(k, \ell, 0) = F(k, \ell).$$

The solution of this system is

$$U(k, \ell, t) = F(k, \ell)\exp(-tK\kappa^2) = F(k, \ell)G(k, \ell), \tag{2.18.16}$$

where

$$G(k, \ell) = \exp(-K\kappa^2 t),$$

so that

$$g(x, y) = \mathscr{F}^{-1}\{\exp(-tK\kappa^2)\} = \frac{1}{2Kt}\exp\left(-\frac{x^2 + y^2}{4Kt}\right). \tag{2.18.17}$$

Finally, the Convolution Theorem gives the formal solution

$$u(x, y, t) = \frac{1}{4\pi Kt}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi, \eta)\exp\left[-\frac{(x-\xi)^2 + (y-\eta)^2}{4Kt}\right] d\xi\, d\eta. \tag{2.18.18}$$

Or, equivalently,

$$u(x, y, t) = \frac{1}{4\pi Kt}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{r}')\exp\left\{-\frac{|\mathbf{r} - \mathbf{r}'|^2}{4Kt}\right\} d\mathbf{r}', \tag{2.18.19}$$

where $\mathbf{r}' = (\xi, \eta)$.

We make the change of variable $(\mathbf{r}' - \mathbf{r}) = \sqrt{4Kt}\,\mathbf{R}$ to reduce (2.18.19) in the form

$$u(x, y, t) = \frac{1}{\pi\sqrt{4Kt}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{r} + \sqrt{4Kt}\,\mathbf{R})\exp(-R^2)\, d\mathbf{R}. \tag{2.18.20}$$

Similarly, the formal solution of the initial value problem for the three-dimensional diffusion equation

$$u_t = K(u_{xx} + u_{yy} + u_{zz}), \quad -\infty < x, y, z < \infty, \quad t > 0 \qquad (2.18.21)$$

$$u(x, y, z, 0) = f(x, y, z), \quad -\infty < x, y, z < \infty \qquad (2.18.22)$$

is given by

$$u(x, y, z, t) = \frac{1}{(4\pi K t)^{3/2}} \int \int_{-\infty}^{\infty} \int f(\xi, \eta, \zeta) \exp\left(-\frac{r^2}{4Kt}\right) d\xi \, d\eta \, d\zeta, \qquad (2.18.23)$$

where

$$r^2 = (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2.$$

Or, equivalently,

$$u(x, y, z, t) = \frac{1}{(4\pi K t)^{3/2}} \int \int_{-\infty}^{\infty} \int f(\mathbf{r}') \exp\left\{-\frac{|\mathbf{r} - \mathbf{r}'|^2}{4Kt}\right\} d\xi \, d\eta \, d\zeta, \qquad (2.18.24)$$

where $\mathbf{r} = (x, y, z)$ and $\mathbf{r}' = (\xi, \eta, \zeta)$.

Making the change of variable $\mathbf{r}' - \mathbf{r} = \sqrt{4tK}\mathbf{R}$, solution (2.18.24) reduces to

$$u(x, y, z, t) = \frac{1}{\pi^{3/2} 4Kt} \int \int_{-\infty}^{\infty} \int f(\mathbf{r} + \sqrt{4Kt}\,\mathbf{R}) \exp(-R^2) \, d\mathbf{R}. \qquad (2.18.25)$$

This is known as the *Fourier solution.* ▯

***Example 2.18.3***

(*The Cauchy Problem for the Two-Dimensional Wave Equation*). The initial value problem for the wave equation in two dimensions is governed by

$$u_{tt} = c^2(u_{xx} + u_{yy}), \quad -\infty < x, y < \infty, \quad t > 0, \qquad (2.18.26)$$

with the initial data

$$u(x, y, 0) = 0, \quad u_t(x, y, 0) = f(x, y), \quad -\infty < x, y < \infty, \qquad (2.18.27ab)$$

where $c$ is a constant. We assume that $u$ and its first partial derivatives vanish at infinity.

We apply the two-dimensional Fourier transform defined by (2.18.3) to the system (2.18.26)–(2.18.27ab), which becomes

$$\frac{d^2U}{dt^2} + c^2\kappa^2 U = 0, \quad \kappa^2 = k^2 + \ell^2,$$

$$U(k, \ell, 0) = 0, \quad \left(\frac{dU}{dt}\right)_{t=0} = F(k, \ell).$$

The solution of this transformed system is

$$U(k,\ell,t) = F(k,\ell)\frac{\sin(c\kappa t)}{c\kappa}. \tag{2.18.28}$$

The inverse Fourier transform gives the formal solution

$$u(x,y,t) = \frac{1}{2\pi c}\int_{-\infty}^{\infty}\int \exp(i\boldsymbol{\kappa}\cdot\mathbf{r})\,\frac{\sin(c\kappa t)}{\kappa}\,F(\boldsymbol{\kappa})d\boldsymbol{\kappa} \tag{2.18.29}$$

$$= \frac{1}{4i\pi c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{F(\boldsymbol{\kappa})}{\kappa}\,\left[\exp\left\{i\kappa\left(\frac{\boldsymbol{\kappa}\cdot\mathbf{r}}{\kappa}+ct\right)\right\}\right.$$

$$\left. - \exp\left\{i\kappa\left(\frac{\boldsymbol{\kappa}\cdot\mathbf{r}}{\kappa}-ct\right)\right\}\right]d\boldsymbol{\kappa}. \tag{2.18.30}$$

The form of this solution reveals an interesting feature of the wave equation. The exponential terms $\exp\left\{i\kappa\left(ct \pm \frac{\boldsymbol{\kappa}\cdot\mathbf{r}}{\kappa}\right)\right\}$ involved in the integral solution (2.18.30) represent plane wave solutions of the wave equation (2.18.26). Thus, the solutions remain constant on the planes $\boldsymbol{\kappa}\cdot\mathbf{r} =$ constant that move parallel to themselves with velocity $c$. Evidently, solution (2.18.30) represents a superposition of the plane wave solutions traveling in all possible directions.

Similarly, the solution of the Cauchy problem for the three-dimensional wave equation

$$u_{tt} = c^2(u_{xx}+u_{yy}+u_{zz}),\ -\infty < x,y,z < \infty,\ t > 0, \tag{2.18.31}$$

$$u(x,y,z,0) = 0,\ u_t(x,y,z,0) = f(x,y,z),\ -\infty < x,y,z < \infty \tag{2.18.32ab}$$

is given by

$$u(\mathbf{r},t) = \frac{1}{2ic(2\pi)^{3/2}}\int\int_{-\infty}^{\infty}\int\frac{F(\boldsymbol{\kappa})}{\kappa}\left[\exp\left\{i\kappa\left(\frac{\boldsymbol{\kappa}\cdot\mathbf{r}}{\kappa}+ct\right)\right\}\right.$$

$$\left. - \exp\left\{i\kappa\left(\frac{\boldsymbol{\kappa}\cdot\mathbf{r}}{\kappa}-ct\right)\right\}\right]d\boldsymbol{\kappa}, \tag{2.18.33}$$

where $\mathbf{r} = (x,y,z)$ and $\boldsymbol{\kappa} = (k,\ell,m)$.

In particular, when $f(x,y,z) = \delta(x)\delta(y)\delta(z)$ so that $F(\boldsymbol{\kappa}) = (2\pi)^{-3/2}$, solution (2.18.33) becomes

$$u(\mathbf{r},t) = \frac{1}{(2\pi)^3}\int\int_{-\infty}^{\infty}\int\left(\frac{\sin c\kappa t}{c\kappa}\right)\,\exp(i(\boldsymbol{\kappa}\cdot\mathbf{r}))d\boldsymbol{\kappa}. \tag{2.18.34}$$

In terms of the spherical polar coordinates $(\kappa,\theta,\phi)$ where the polar axis (the $z$-axis) is taken along the $\mathbf{r}$ direction with $\boldsymbol{\kappa}\cdot\mathbf{r} = \kappa r\cos\theta$, we write (2.18.34) in the form

$$\begin{aligned} u(r,t) &= \frac{1}{(2\pi)^3}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{\infty} \exp(i\kappa r\ \cos\theta)\,\frac{\sin c\kappa t}{c\kappa}\cdot \kappa^2\ \sin\theta\, d\kappa \\ &= \frac{1}{2\pi^2 cr}\int_0^{\infty} \sin(c\kappa t)\sin(\kappa r)\ d\kappa \\ &= \frac{1}{8\pi^2 cr}\int_{-\infty}^{\infty} [e^{i\kappa(ct-r)} - e^{i\kappa(ct+r)}]\ d\kappa. \end{aligned}$$

Or,

$$u(r,t) = \frac{1}{4\pi cr}[\delta(ct-r) - \delta(ct+r)]. \tag{2.18.35}$$

For $t > 0,\ ct + r > 0$ so that $\delta(ct+r) = 0$ and hence,

$$u(\mathbf{r},t) = \frac{1}{4\pi cr}\,\delta(ct-r) = \frac{1}{4\pi c^2 r}\,\delta(t - \frac{r}{c}). \tag{2.18.36}$$

□

***Example 2.18.4***
(*The Three-Dimensional Poisson Equation*). The solution of the Poisson equation

$$-\nabla^2 u = f(\mathbf{r}), \tag{2.18.37}$$

where $\mathbf{r} = (x,y,z)$ is given by

$$u(\mathbf{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\mathbf{r},\boldsymbol{\xi}) f(\boldsymbol{\xi}) d\boldsymbol{\xi}, \tag{2.18.38}$$

where the Green's function $G(\mathbf{r},\boldsymbol{\xi})$ of the operator, $-\nabla^2$, is

$$G(\mathbf{r},\boldsymbol{\xi}) = \frac{1}{4\pi}\frac{1}{|\mathbf{r}-\boldsymbol{\xi}|}. \tag{2.18.39}$$

To obtain the fundamental solution, we need to solve the equation

$$-\nabla^2\ G(\mathbf{r},\boldsymbol{\xi}) = \delta(x-\xi)\delta(y-\eta)\delta(z-\zeta), \qquad \mathbf{r} \neq \boldsymbol{\xi}. \tag{2.18.40}$$

Application of the three-dimensional Fourier transform defined by (2.18.5) to (2.18.40) gives

$$\kappa^2\ \hat{G}(\boldsymbol{\kappa},\boldsymbol{\xi}) = \frac{1}{(2\pi)^{3/2}}\ \exp(-i\boldsymbol{\kappa}\cdot\boldsymbol{\xi}), \tag{2.18.41}$$

where $\hat{G}(\boldsymbol{\kappa}, \boldsymbol{\xi}) = \mathscr{F}\{G(\mathbf{r}, \boldsymbol{\xi})\}$ and $\boldsymbol{\kappa} = (k, \ell, m)$.

The inverse Fourier transform gives the formal solution

$$G(\mathbf{r}, \boldsymbol{\xi}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\{i\boldsymbol{\kappa} \cdot (\mathbf{r} - \boldsymbol{\xi})\} \frac{d\boldsymbol{\kappa}}{\kappa^2}$$

$$= \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(i\boldsymbol{\kappa} \cdot \mathbf{x}) \frac{d\boldsymbol{\kappa}}{\kappa^2}, \tag{2.18.42}$$

where $\mathbf{x} = |\mathbf{r} - \boldsymbol{\xi}|$.

We evaluate this integral using polar coordinates in the $\boldsymbol{\kappa}$-space with the axis along the $\mathbf{x}$-axis. In terms of spherical polar coordinates $(\kappa, \theta, \phi)$ so that $\boldsymbol{\kappa} \cdot \mathbf{x} = \kappa R \cos\theta$ where $R = |\mathbf{x}|$. Thus, (2.18.42) becomes

$$G(\mathbf{r}, \xi) = \frac{1}{(2\pi)^3} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{\infty} \exp(i\kappa R\ \cos\theta)\, \kappa^2 \sin\theta \cdot \frac{d\kappa}{\kappa^2}$$

$$= \frac{1}{(2\pi)^2} \int_0^{\infty} 2\, \frac{\sin\,(\kappa R)}{\kappa R}\ d\kappa = \frac{1}{4\pi R} = \frac{1}{4\pi |\mathbf{r} - \boldsymbol{\xi}|}, \tag{2.18.43}$$

provided $R > 0$.

In electrodynamics, the fundamental solution (2.18.43) has a well-known interpretation. Physically, it represents the potential at point $\mathbf{r}$ generated by the unit point charge distribution at point $\boldsymbol{\xi}$. This is what can be expected because $\delta(\mathbf{r} - \boldsymbol{\xi})$ is the charge density corresponding to a unit point charge at $\boldsymbol{\xi}$.

The solution of (2.18.37) is then given by

$$u(\mathbf{r}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\mathbf{r}, \boldsymbol{\xi}) f(\boldsymbol{\xi}) d\boldsymbol{\xi} = \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(\boldsymbol{\xi}) d\boldsymbol{\xi}}{|\mathbf{r} - \boldsymbol{\xi}|}. \tag{2.18.44}$$

The integrand in (2.18.44) consists of the given charge distribution $f(\mathbf{r})$ at $\mathbf{r} = \boldsymbol{\xi}$ and Green's function $G(\mathbf{r}, \boldsymbol{\xi})$. Physically, $G(\mathbf{r}, \boldsymbol{\xi})\ f(\boldsymbol{\xi})$ represents the resulting potentials due to elementary point charges, and the total potential due to a given charge distribution $f(\mathbf{r})$ is then obtained by the integral superposition of the resulting potentials. This is called the *principle of superposition.*
□

***Example 2.18.5***

(*The Two-Dimensional Helmholtz Equation*). To find the fundamental solution of the two-dimensional Helmholtz equation

$$-\nabla^2 G + \alpha^2 G = \delta(x - \xi)\delta(y - \eta), \quad -\infty < x, y < \infty. \tag{2.18.45}$$

It is convenient to make the change of variables $x-\xi=x^*,\ \ y-\eta=y^*$. Consequently, (2.18.45) reduces to the form, dropping the asterisks,

$$G_{xx}+G_{yy}-\alpha^2 G=-\delta(x)\delta(y). \tag{2.18.46}$$

Application of the double Fourier transform $\hat{G}(\boldsymbol{\kappa})=\mathscr{F}\{G(x,y)\}$ to (2.18.46) gives

$$\hat{G}(\boldsymbol{\kappa})=\frac{1}{2\pi}\frac{1}{(\kappa^2+\alpha^2)}, \tag{2.18.47}$$

where $\boldsymbol{\kappa}=(k,\ell)$ and $\kappa^2=k^2+\ell^2$.

The inverse Fourier transform yields the solution

$$G(x,y)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\kappa^2+\alpha^2)^{-1}\exp(i\boldsymbol{\kappa}\cdot\mathbf{x})\,dk\,d\ell. \tag{2.18.48}$$

In terms of polar coordinates $(x,y)=r(\cos\theta,\sin\theta),\ (k,\ell)=\rho(\cos\phi,\sin\phi)$, the integral solution (2.18.48) becomes

$$G(x,y)=\frac{1}{4\pi^2}\int_0^{\infty}\frac{\rho d\rho}{(\rho^2+\alpha^2)}\int_0^{2\pi}\exp\{ir\rho\ \cos(\phi-\theta)\}d\phi,$$

which is, replacing the second integral by $2\pi J_0(r\rho)$,

$$=\frac{1}{2\pi}\int_0^{\infty}\frac{\rho\ J_0(r\rho)d\rho}{(\rho^2+\alpha^2)}. \tag{2.18.49}$$

In terms of the original coordinates, the fundamental solution of (2.18.45) is given by

$$G(\mathbf{r},\boldsymbol{\xi})=\frac{1}{2\pi}\int_0^{\infty}\frac{\rho\ J_0\left[\rho\left\{(x-\xi)^2+(y-\eta)^2\right\}^{\frac{1}{2}}\right]d\rho}{(\rho^2+\alpha^2)}. \tag{2.18.50}$$

Accordingly, the solution of the inhomogeneous equation

$$(\nabla^2-\alpha^2)u=-f(x,y) \tag{2.18.51}$$

is

$$u(x,y)=\int\!\!\int_{-\infty}^{\infty}G(\mathbf{r},\boldsymbol{\xi})f(\boldsymbol{\xi})d\boldsymbol{\xi}, \tag{2.18.52}$$

where $G(\mathbf{r},\boldsymbol{\xi})$ is given by (2.18.50).

Since the integral solution (2.18.49) does not exist for $\alpha = 0$, Green's function for the two-dimensional Poisson equation (2.18.51) cannot be derived from (2.18.49). Instead, we differentiate (2.18.49) with respect to $r$ to obtain

$$\frac{\partial G}{\partial r} = \frac{1}{2\pi}\int_0^\infty \frac{\rho^2 J_0'(r\rho)d\rho}{(\rho^2+\alpha^2)}$$

which is, for $\alpha = 0$,

$$\frac{\partial G}{\partial r} = \frac{1}{2\pi}\int_0^\infty J_0'(r\rho)d\rho = -\frac{1}{2\pi r}.$$

Integrating this result gives

$$G(r,\theta) = -\frac{1}{2\pi}\log r.$$

In terms of the original coordinates, the Green's function becomes

$$G(\mathbf{r},\boldsymbol{\xi}) = -\frac{1}{4\pi}\log[(x-\xi)^2+(y-\eta)^2]. \tag{2.18.53}$$

This is Green's function for the two-dimensional Poisson equation $\nabla^2 = -f(x,y)$. Thus, the solution of the Poisson equation is

$$u(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\mathbf{r},\boldsymbol{\xi})f(\boldsymbol{\xi})d\boldsymbol{\xi}. \tag{2.18.54}$$

▯

***Example 2.18.6***

(*Diffusion of Vorticity from a Vortex Sheet*). We solve the two-dimensional vorticity equation in the $x, y$ plane given by

$$\zeta_t = \nu\nabla^2\zeta \tag{2.18.55}$$

with the initial condition

$$\zeta(x,y,0) = \zeta_0(x,y), \tag{2.18.56}$$

where $\zeta = v_x - u_y$.

Application of the double Fourier transform defined by

$$\hat{\zeta}(k,\ell,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[-i(kx+\ell y)]\zeta(x,y,t)dx\,dy$$

to (2.18.55)–(2.18.56) gives

$$\frac{d\hat{\zeta}}{dt} = -\nu(k^2+\ell^2)\hat{\zeta},$$
$$\hat{\zeta}(k,\ell,0) = \hat{\zeta}_0(k,\ell).$$

Thus, the solution of the transformed system is

$$\hat{\zeta}(k,\ell,t) = \hat{\zeta}_0(k,\ell)\exp[-\nu(k^2+\ell^2)t]. \qquad (2.18.57)$$

The inversion theorem for Fourier transform gives the formal solution

$$\zeta(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{\zeta}_0(k,\ell)\exp[i(\kappa\cdot\mathbf{r})-\nu\kappa^2 t]\,dk\,d\ell, \qquad (2.18.58)$$

where $\boldsymbol{\kappa} = (k,\ell)$ and $\kappa^2 = k^2+\ell^2$.

In particular, if $\zeta_0(x,y) = V\delta(x)$ represents a vortex sheet of constant strength $V$ per unit width in the plane $x=0$, we find $\hat{\zeta}_0(k,\ell) = V\delta(\ell)$ and hence,

$$\zeta(x,y,t) = \frac{V}{2\pi}\int_{-\infty}^{\infty}\exp\{ikx-\nu k^2 t\}dk$$
$$= \frac{V}{2\sqrt{\pi\nu t}}\exp\left(-\frac{x^2}{4\nu t}\right). \qquad (2.18.59)$$

Apart from a constant, the velocity field is given by

$$u(x,t)=0, \quad v(x,t) = \frac{V}{\sqrt{\pi}}\,erf\left(\frac{x}{2\sqrt{\nu t}}\right). \qquad (2.18.60)$$

$\square$

## 2.19 Exercises

1. Find the Fourier transforms of each of the following functions:

(a) $f(x) = \dfrac{1}{1+x^2}$,

(b) $f(x) = \dfrac{x}{1+x^2}$,

(c) $f(x) = \delta^{(n)}(x)$,

(d) $f(x) = x\,\exp(-a|x|),\; a > 0$,

(e) $f(x) = e^x\,\exp(-e^x)$,

(f) $f(x) = x\,\exp\left(-\dfrac{ax^2}{2}\right),\; a > 0$,

(g) $f(x) = x^2\,\exp\left(-\dfrac{1}{2}x^2\right)$,

(h) $f(x) = \begin{Bmatrix} 1-|x|, & |x| \le 1 \\ 0, & |x| > 1 \end{Bmatrix}$,

(i) $f(x) = \begin{Bmatrix} 1-x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{Bmatrix}$,

(j) $h_n(x) = (-1)^n\,\exp\left(\dfrac{1}{2}x^2\right) \times \left(\dfrac{d}{dx}\right)^n \exp(-x^2)$,

(k) $f(x) = \chi_{[a,b]}(x)\,e^{i\alpha x}$,

(l) $f(x) = \dfrac{cos}{sin}\,(ax^2)$.

2. Show that

(a) $\mathscr{F}\{\delta(x-ct)+\delta(x+ct)\} = \sqrt{\dfrac{2}{\pi}}\,\cos(kct)$,

(b) $\mathscr{F}\{H(ct-|x|)\} = \sqrt{\dfrac{2}{\pi}}\,\dfrac{\sin kct}{k}$,

(c) $\mathscr{F}\left\{f\left(\dfrac{x}{a}+b\right)\right\} = a\,\exp(iabk)\,F(ak)$,

(d) $\mathscr{F}\{e^{ibx}f(ax)\} = \dfrac{1}{a}\,F\left(\dfrac{k+b}{a}\right)$.

3. Show that

(a) $i\dfrac{d}{dk}F(k) = \mathscr{F}\{x\,f(x)\}$,

(b) $i^n\dfrac{d^n}{dk^n}F(k) = \mathscr{F}\{x^n\,f(x)\}$.

4. Use exercise 3(b) to find the Fourier transform of $f(x) = x^2\,\exp(-ax^2)$.

5. Prove the following:

(a) $\mathscr{F}\left\{(a^2-x^2)^{-\frac{1}{2}}H(a-|x|)\right\} = \sqrt{\dfrac{\pi}{2}}\,J_0(ak),\; a > 0.$

(b) $\mathscr{F}\{P_n(x)\ H(1-|x|)\} = (-i)^n \dfrac{1}{\sqrt{k}}\ J_{n+\frac{1}{2}}(k),$

where $P_n(x)$ is the Legendre polynomial of degree $n$.

(c) If $f(x)$ has a finite discontinuity at a point $x=a$, then

$$\mathscr{F}\{f'(x)\} = (ik)\ F(k) - \frac{1}{\sqrt{2\pi}}\exp(-ika)[f]_a,$$

where $[f]_a = f(a+0) - f(a-0)$.

Generalize this result for $\mathscr{F}\{f^{(n)}(x)\}$.

6. Find the convolution $(f*g)(x)$ if

(a) $f(x) = e^{ax}, \quad g(x) = \chi_{[0,\infty]}(x),\ a \neq 0,$

(b) $f(x) = \sin bx, \quad g(x) = \exp(-a|x|),\ a > 0,$

(c) $f(x) = \chi_{[a,b]}(x), \quad g(x) = x^2,$

(d) $f(x) = \exp(-x^2), \quad g(x) = \exp(-x^2).$

7. Prove the following results for the convolution:

(a) $\delta(x) * f(x) = f(x),$ (b) $\delta'(x) * f(x) = f'(x),$

(c) $\dfrac{d}{dx}\{f(x) * g(x)\} = f'(x) * g(x) = f(x) * g'(x),$

(d) $\displaystyle\int_{-\infty}^{\infty} (f*g)(x)dx = \int_{-\infty}^{\infty} f(u)du \int_{-\infty}^{\infty} g(v)dv\ ,$

(e) $\dfrac{d^2}{dx^2}\,(f*g)\ (x) = (f'*g')\ (x) = (f''*g)\ (x),$

(f) $(f*g)^{(n+l)}\ (x) = f^{(n)}\ (x) * g^{(l)}\ (x),$

(g) If $f$ and $g$ are both even or both odd, then $(f*g)(x)$ is even,

(h) If $f$ is even or $g$ is odd, or vice versa, then $(f*g)(x)$ is odd,

(i) If $g(x) = \dfrac{1}{2a}\,H(a-|x|)$, then $(f*g)(x)$ is the average of the function $f(x)$ in $[x-a,\, x+a]$,

(j) If $G_t(x) = \dfrac{1}{\sqrt{4\pi kt}}\displaystyle\int_{-\infty}^{\infty} f(\xi)\exp\left[-\frac{(x-\xi)^2}{4kt}\right] d\xi,$

then $G_t(x) * G_s(x) = G_{t+s}(x).$

8. Use the Fourier transform to solve the following ordinary differential equations in $-\infty < x < \infty$:

(a) $y''(x) - y(x) + 2f(x) = 0$, where $f(x) = 0$ when $x < -a$ and when $x > a$, and $y(x)$ and its derivatives vanish at $x = \pm\infty$,

(b) $2y''(x)+xy'(x)+y(x)=0$, (c) $y''(x)+x\,y'(x)+y(x)=0$,
(d) $y''(x)+x\,y'(x)+x\,y(x)=0$, (e) $\ddot{y}(t)+2\alpha\,\dot{y}(t)+\omega^2 y(t)=f(t)$.

9. Solve the following integral equations for an unknown function $f(x)$:

(a) $\displaystyle\int_{-\infty}^{\infty} \phi(x-t)f(t)dt = g(x)$.

(b) $\displaystyle\int_{-\infty}^{\infty} \exp(-at^2)f(x-t)dt = \exp(-bx^2),\ a>b>0$.

(c) $\displaystyle\int_{-\infty}^{\infty} f(x-t)f(t)dt = \frac{b}{(x^2+b^2)}$.

(d) $\displaystyle\int_{-\infty}^{\infty} \frac{f(t)dt}{(x-t^2)+a^2} = \frac{\sqrt{2\pi}}{(x^2+b^2)}$ for $b>a>0$.

(e) $\displaystyle\frac{1}{\pi}\oint_{-\infty}^{\infty} \frac{f(t)dt}{x-t} = \phi(x)$,

where the integral in (e) is treated as the Cauchy Principal value.

10. Solve the Cauchy problem for the Klein-Gordon equation

$$u_{tt}-c^2\ u_{xx}+a^2\ u=0,\ -\infty<x<\infty,\ t>0.$$
$$u(x,0)=f(x),\ \left(\frac{\partial u}{\partial t}\right)_{t=0}=g(x)\ \text{ for } -\infty<x<\infty.$$

11. Solve the telegraph equation

$$u_{tt}-c^2\ u_{xx}+u_t-au_x=0,\quad -\infty<x<\infty,\ t>0.$$
$$u(x,0)=f(x),\ \left(\frac{\partial u}{\partial t}\right)_{t=0}=g(x)\quad \text{for}\quad -\infty<x<\infty.$$

Show that the solution is unstable when $c^2<a^2$. If $c^2>a^2$, show that the bounded integral solution is

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)\exp[-k^2(c^2-a^2)t+ik(x+at)]dk$$

where $A(k)$ is given in terms of the transformed functions of the initial data. Hence, deduce the asymptotic solution as $t\to\infty$ in the form

$$u(x,t)=A(0)\sqrt{\frac{\pi}{2(c^2-a^2)t}}\ \exp\left[-\frac{(x+at)^2}{4(c^2-a^2)t}\right].$$

12. Solve the equation

$$u_{tt} + u_{xxxx} = 0, \quad -\infty < x < \infty, \ t > 0$$
$$u(x,0) = f(x), \ u_t(x,0) = 0 \quad \text{for} \quad -\infty < x < \infty.$$

13. Find the solution of the dissipative wave equation

$$u_{tt} - c^2 \ u_{xx} + \alpha u_t = 0, \ -\infty < x < \infty, \ t > 0,$$
$$u(x,0) = f(x), \left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x) \quad \text{for} \quad -\infty < x < \infty,$$

where $\alpha > 0$ is the dissipation parameter.

14. Obtain the Fourier cosine transforms of the following functions:
(a) $f(x) = x \exp(-ax), \ a > 0,$ (b) $f(x) = e^{-ax} \cos x, \ a > 0,$
(c) $f(x) = \dfrac{1}{x},$ (d) $K_0(ax),$
where $K_0(ax)$ is the *modified Bessel function.*

15. Find the Fourier sine transform of the following functions:
(a) $f(x) = x \exp(-ax), \ a > 0,$ (b) $f(x) = \dfrac{1}{x} \exp(-ax), \ a > 0,$
(c) $f(x) = \dfrac{1}{x},$ (d) $f(x) = \dfrac{x}{a^2 + x^2}.$

16. (a) If $F(k) = \mathscr{F}\{\exp(-ax^2)\}, \ a > 0,$ show that $F(k)$ satisfies the differential equation

$$2a \frac{dF}{dk} + k\,F(k) = 0 \ \text{with} \ F(0) = \frac{1}{\sqrt{2a}}.$$

(b) If $F_c(k) = \mathscr{F}_c\{\exp(-ax^2)\}$, show that $F_c(k)$ satisfies the equation

$$\frac{dF_c}{dk} + \left(\frac{k}{2a}\right) F_c = 0 \ \text{with} \ F_c(0) = 1.$$

17. Prove the following for the Fourier sine transform

(a) $$\int_0^\infty F_s(k) G_c(k) \sin kx \ dk = \frac{1}{2} \int_0^\infty g(\xi)[f(\xi + x) - f(\xi - x)] d\xi,$$

(b) $$\int_0^\infty F_c(k) G_s(k) \sin kx \ dk = \frac{1}{2} \int_0^\infty f(\xi)[g(\xi + x) - g(\xi - x)] d\xi.$$

18. Solve the integral equation

$$\int_0^\infty f(x) \sin kx \ dk = \left\{ \begin{array}{cc} 1 - k, & 0 \le k < 1 \\ 0, & k > 1 \end{array} \right\}.$$

19. Solve Example 2.15.1 with the boundary data

$$u(0,t)=0,\ u(x,t)\to 0 \quad \text{as} \quad x\to\infty, \quad \text{for} \quad t>0.$$

20. Apply the Fourier cosine transform to find the solution $u(x,y)$ of the problem

$$u_{xx}+u_{yy}=0,\ 0<x<\infty,\ 0<y<\infty,$$
$$u(x,0)=H(a-x), a>x;\ u_x(0,y)=0,\ 0<x,y<\infty.$$

21. Use the Fourier cosine (or sine) transform to solve the following integral equation:

(a) $\displaystyle\int_0^\infty f(x)\cos kx\ dx=\sqrt{\frac{\pi}{2k}}$, (b) $\displaystyle\int_0^\infty f(x)\sin kx\ dx=\frac{a}{a^2+k^2}$,

(c) $\displaystyle\int_0^\infty f(x)\sin kx dx=\frac{\pi}{2}\,J_0(ak)$, (d) $\displaystyle\int_0^\infty f(x)\cos kx\ dx=\frac{\sin ak}{k}$.

22. Solve the diffusion equation in the semi-infinite line

$$u_t=\kappa\, u_{xx},\ 0\le x<\infty,\ t>0,$$

with the boundary and initial data

$$u(0,t)=0 \quad \text{for} \quad t>0,$$
$$u(x,t)\to 0 \quad \text{as} \quad x\to\infty \quad \text{for} \quad t>0,$$
$$u(x,0)=f(x) \quad \text{for} \quad 0<x<\infty.$$

23. Use the Parseval formula to evaluate the following integrals with $a>0$ and $b>0$:

(a) $\displaystyle\int_{-\infty}^\infty \frac{dx}{(x^2+a^2)^2}$, (c) $\displaystyle\int_{-\infty}^\infty \frac{\sin^2 ax}{x^2}dx$,

(b) $\displaystyle\int_{-\infty}^\infty \frac{\sin ax}{x(x^2+b^2)}dx$ (d) $\displaystyle\int_{-\infty}^\infty \frac{\exp(-bx^2)dx}{(x^2+a^2)}$.

24. Show that

$$\int_0^\infty \frac{\sin ax\sin bx}{x^2}dx=\frac{\pi}{2}\min(a,b).$$

25. If $f(x) = \exp(-ax)$ and $g(x) = H(t-x)$, show that

$$\int_0^\infty \frac{\sin tx}{x(x^2+a^2)}dx = \frac{\pi}{2a^2}[1-\exp(-at)].$$

26. Use the Poisson summation formula to find the sum of each of the following series with non-zero $a$:

(a) $\sum_{n=-\infty}^{\infty} \frac{1}{(1+n^2a^2)}$, (b) $\sum_{n=1}^{\infty} \frac{\sin an}{n}$,

(c) $\sum_{n=1}^{\infty} \frac{\sin^2 an}{n^2}$, (d) $\sum_{n=-\infty}^{\infty} \frac{a}{n^2+a^2}$.

27. The Fokker-Planck equation (Reif, 1965) is used to describe the evolution of probability distribution functions $u(x,t)$ in nonequilibrium statistical mechanics and has the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + x\right)u.$$

The fundamental solution of this equation is defined by the equation

$$\left[\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + x\right)\right]G(x,\xi;t,\tau) = \delta(x-\xi)\delta(t-\tau).$$

Show that the fundamental solution is

$$G(x,\xi;t,\tau) = [2\pi\{1-\exp[-2(t-\tau)]\}]^{-\frac{1}{2}}\exp\left[-\frac{\{x-\xi\ \exp[-(t-\tau)]\}^2}{2[1-\exp\{-2(t-\tau)\}]}\right].$$

Hence, derive

$$\lim_{t\to\infty} G(x,\xi;t,\tau) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right).$$

With the initial condition $u(x,0) = f(x)$, show that the function $u(x,t)$ tends to the normal distribution as $t \to \infty$, that is,

$$\lim_{t\to\infty} u(x,t) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)\int_{-\infty}^{\infty} f(\xi)d\xi.$$

28. The transverse vibration of an infinite elastic beam of mass $m$ per unit length and the bending stiffness $EI$ is governed by

$$u_{tt} + a^2\ u_{xxxx} = 0, \quad \left(a^2 = \frac{EI}{m}\right),\ -\infty < x < \infty,\ t > 0.$$

Solve this equation subject to the boundary and initial data

$$u(0,t) = 0 \quad \text{for all} \quad t > 0,$$
$$u(x,0) = \phi(x), \; u_t(x,0) = \psi''(x) \quad \text{for} \quad 0 < x < \infty.$$

Show that the Fourier transform solution is

$$U(k,t) = \mathbf{\Phi}(k)\cos(atk^2) - \mathbf{\Psi}(k)\sin(atk^2).$$

Find the integral solution for $u(x,t)$.

29. Solve the Lamb (1904) problem in geophysics that satisfies the Helmholtz equation in an infinite elastic half-space

$$u_{xx} + u_{zz} + \frac{\omega^2}{{c_2}^2}u = 0, \; -\infty < x < \infty, \; z > 0,$$

where $\omega$ is the frequency and $c_2$ is the shear wave speed.

At the surface of the half-space ($z = 0$), the boundary condition relating the surface stress to the impulsive point load distribution is

$$\mu\frac{\partial u}{\partial z} = -P\delta(x) \quad \text{at} \quad z = 0,$$

where $\mu$ is one of the Lamé's constants, $P$ is a constant and

$$u(x,z) \to 0 \quad \text{as} \quad z \to \infty \quad \text{for} \quad -\infty < x < \infty.$$

Show that the solution in terms of polar coordinates is

$$\begin{aligned} u(x,z) &= \frac{P}{2i\mu}{H_0}^{(2)}\left(\frac{\omega r}{c_2}\right) \\ &\sim \frac{P}{2i\mu}\left(\frac{2c_2}{\pi\omega r}\right)^{\frac{1}{2}} \exp\left(\frac{\pi i}{4} - \frac{i\omega r}{c_2}\right) \quad \text{for} \quad \omega r >> c_2. \end{aligned}$$

30. Find the solution of the Cauchy-Poisson problem (Debnath, 1994, p. 83) in an inviscid water of infinite depth which is governed by

$$\phi_{xx} + \phi_{zz} = 0, \; -\infty < x < \infty, \; -\infty < z \le 0, \; t > 0,$$
$$\left.\begin{aligned} \phi_z - \eta_t &= 0 \\ \phi_t + g\eta &= 0 \end{aligned}\right\} \text{on } z = 0, \; t > 0,$$
$$\phi_z \to 0 \quad \text{as} \quad z \to -\infty,$$
$$\phi(x,0,0) = 0 \quad \text{and} \quad \eta(x,0) = P\delta(x),$$

where $\phi = \phi(x,z,t)$ is the velocity potential, $\eta(x,t)$ is the free surface elevation, and $P$ is a constant.

Derive the asymptotic solution for the free surface elevation in the limit as $t \to \infty$.

31. Obtain the solutions for the velocity potential $\phi(x,z,t)$ and the free surface elevation $\eta(x,t)$ involved in the two-dimensional surface waves in water of finite (or infinite) depth $h$. The governing equation, boundary, and free surface conditions and initial conditions (see Debnath 1994, p. 92) are

$$\phi_{xx}+\phi_{zz}=0,\ -h\le z\le 0,\ -\infty<x<\infty,\ t>0$$
$$\left.\begin{aligned}\phi_t+g\eta&=-\frac{P}{\rho}p(x)\exp(i\,\omega\,t)\\ \phi_z-\eta_t&=0\end{aligned}\right\}\ z=0,\ t>0$$
$$\phi(x,z,0)=0=\eta(x,0)\ \text{for all } x \text{ and } z.$$

32. Solve the steady-state surface wave problem (Debnath, 1994, p. 47) on a running stream of infinite depth due to an external steady pressure applied to the free surface. The governing equation and the free surface conditions are

$$\phi_{xx}+\phi_{zz}=0,\ -\infty<x<\infty,\ -\infty<z<0,\ t>0,$$
$$\left.\begin{aligned}\phi_x+U\phi_x+g\eta&=-\frac{P}{\rho}\,\delta(x)\exp(\epsilon t)\\ \eta_t+U\eta_x&=\phi_z\end{aligned}\right\}\ z=0,\ (\epsilon>0),$$
$$\phi_z\to 0\quad\text{as}\quad z\to-\infty.$$

where $U$ is the stream velocity, $\phi(x,z,t)$ is the velocity potential, and $\eta(x,t)$ is the free surface elevation.

33. Use the Fourier sine transform to solve the following initial and boundary value problem for the wave equation:

$$u_{tt}=c^2u_{xx},\ 0<x<\infty,\ t>0,$$
$$u(x,0)=0,\ u_t(x,0)=0\quad\text{for}\quad 0<x<\infty,$$
$$u(0,t)=f(t)\quad\text{for}\quad t>0,$$

where $f(t)$ is a given function.

34. Solve the following initial and boundary value problem for the wave equation using the Fourier cosine transform:

$$u_{tt}=c^2u_{xx},\ 0<x<\infty,\ t>0,$$
$$u(0,t)=f(t)\quad\text{for}\quad t>0,$$
$$u(x,0)=0,u_t(x,0)=0\quad\text{for}\quad 0<x<\infty,$$

where $f(t)$ is a known function.

35. Apply the Fourier transform to solve the initial value problem for the dissipative wave equation

$$u_{tt} = c^2 u_{xx} + \alpha\, u_{xxt}, \;\; -\infty < x < \infty, \;\; t > 0,$$
$$u(x,0) = f(x), \;\; u_t(x,0) = \alpha f''(x) \quad \text{for} \quad -\infty < x < \infty,$$

where $\alpha$ is a positive constant.

36. Use the Fourier sine transform to solve the initial and boundary value problem for free vibrations of a semi-infinite string:

$$u_{tt} = c^2 u_{xx}, \;\; 0 < x < \infty, \;\; t > 0,$$
$$u(0,t) = 0, \;\; t \geq 0,$$
$$u(x,0) = f(x) \quad \text{and} \quad u_t(x,0) = g(x) \quad \text{for} \quad 0 < x < \infty.$$

37. The static deflection $u(x,y)$ in a thin elastic disk in the form of a quadrant satisfies the boundary value problem

$$u_{xxxx} + 2\, u_{xxyy} + u_{yyyy} = 0, \;\; 0 < x < \infty, \;\; 0 < y < \infty,$$
$$u(0,y) = u_{xx}(0,y) = 0 \quad \text{for} \quad 0 < y < \infty,$$
$$u(x,0) = \frac{ax}{1+x^2}, \;\; u_{yy}(x,0) = 0 \quad \text{for} \quad 0 < x < \infty,$$

where $a$ is a constant, and $u(x,y)$ and its derivatives vanish as $x \to \infty$ and $y \to \infty$.

Use the Fourier sine transform to show that

$$u(x,y) = \frac{a}{2}\int_0^\infty (2+ky)\exp[-(1+y)k]\sin kx \; dx$$
$$= \frac{ax}{x^2+(1+y)^2} + \frac{axy(1+y)}{[x^2+(1+y)^2]^2}$$

38. In exercise 37, replace the conditions on $y=0$ with the conditions

$$u(x,0) = 0, \;\; u_{yy}(x,0) = \frac{ax}{(1+x^2)^2} \quad \text{for} \quad 0 < x < \infty.$$

Show that the solution is

$$u(x,y) = -\frac{ax}{4}\int_0^\infty \exp[-(1+y)k]\sin kx \; dk$$
$$= -\frac{1}{4}\frac{axy}{[x^2+(1+y)^2]}.$$

39. In exercise 37, solve the biharmonic equation in $0<x<\infty,\ 0<y<b$ with the boundary conditions

$$u(0,y) = a\sin y,\ u_{xx}(0,y)=0 \quad \text{for} \quad 0<y<b,$$
$$u(x,0) = u_{yy}(x,0)=u(x,b)=u_{yy}(x,b)=0 \quad \text{for} \quad 0<x<\infty,$$

and $u(x,y),\ u_x(x,y)$ vanish as $x\to\infty$.

40. Use the Fourier transform to solve the boundary value problem

$$u_{xx}+u_{yy}=-x\,\exp(-x^2),\ -\infty<x<\infty,\ 0<y<\infty,$$

$u(x,0)=0$, for $-\infty<x<\infty$, $u$ and its derivative vanish as $y\to\infty$. Show that

$$u(x,y)=\frac{1}{\sqrt{4\pi}}\int_0^\infty [1-\exp(-ky)]\frac{\sin kx}{k}\,\exp\left(-\frac{k^2}{4}\right)dk.$$

41. Using the definition of the characteristic function for the discrete random variable $X$

$$\phi(t)=E[\exp(itX)]=\sum_r p_r\,\exp(itx_r)$$

where $p_r=P(X=x_r)$, show that the characteristic function of the binomial distribution

$$p_r=\binom{n}{r}p^r(1-p)^{n-r}$$

is

$$\phi(t)=[1+p(e^{it}-1)]^n.$$

Find the moments.

42. Show that the characteristic function of the Poisson distribution

$$p_r=P(X=r)=\frac{\lambda^r}{r!}\,e^{-\lambda},\ r=0,1,2,\ldots$$

is

$$\phi(t)=\exp[\lambda(e^{it}-1)].$$

Find the moments.

43. Find the characteristic function of

(a) The gamma distribution whose density function is

$$f(x)=\frac{a^p}{\Gamma(p)}\,x^{p-1}e^{-ax}H(x),$$

(b) The beta distribution whose density function is

$$f(x) = \left\{ \begin{array}{ll} \dfrac{x^{p-1}(1-x)^{q-1}}{B(p,q)} & \text{for} \quad 0 < x < 1, \\ 0 & \text{for} \quad x < 0 \text{ and } \quad x > 1 \end{array} \right\},$$

(c) The Cauchy distribution whose density function is

$$f(x) = \frac{1}{\pi} \frac{\lambda}{[\lambda^2 + (x-\mu)^2]},$$

(d) The Laplace distribution whose density function is

$$f(x) = \frac{1}{2\lambda} \exp\left(-\frac{|x-u|}{\lambda}\right), \qquad \lambda > 0.$$

44. Find the density function of the random variable $X$ whose characteristic function is

$$\phi(t) = (1 - |t|) H(1 - |t|).$$

45. Find the characteristic function of uniform distribution whose density function is

$$f(x) = \left\{ \begin{array}{ll} 0, & x < 0 \\ 1, & 0 \le x \le a \\ 0, & x > a \end{array} \right\}.$$

46. Solve the *initial value problem* (Debnath, 1994, p. 115) for the two-dimensional surface waves at the free surface of a running stream of velocity $U$. The problem satisfies the equation, boundary, and initial conditions

$$\phi_{xx} + \phi_{zz} = 0, \ -\infty < x < \infty, \ -h \le z \le 0, \quad t > 0,$$

$$\left. \begin{array}{l} \phi_x + U\phi_x + g\eta = -\dfrac{P}{\rho}\, \delta(x) \exp(i\,\omega\, t) \\ \eta_t + U\eta_x - \phi_z = 0 \end{array} \right\} \text{ on } z = 0, \ t > 0,$$

$$\phi(x, z, 0) = \eta(x, 0) = 0, \quad \text{for all } x \text{ and } z.$$

47. Apply the Fourier tranform to solve the equation

$$u_{xxxx} + u_{yy} = 0, \ -\infty < x < \infty, \ y \ge 0,$$

satisfying the conditions

$$u(x, 0) = f(x), \ u_y(x, 0) = 0 \quad \text{for} \quad -\infty < x < \infty,$$

$u(x, y)$ and its partial derivatives vanish as $|x| \to \infty$.

48. The transverse vibration of a thin membrane of great extent satisfies the wave equation

$$c^2(u_{xx}+u_{yy})=u_{tt},\ -\infty<x,y<\infty,\ t>0,$$

with the initial and boundary conditions

$$\begin{aligned} &u(x,y,t)\to 0 \quad \text{as} \quad |x|\to\infty,\ |y|\to\infty \quad \text{for all} \quad t\geq 0,\\ &u(x,y,0)=f(x,y),\ u_t(x,y,0)=0 \quad \text{for all} \quad x,y. \end{aligned}$$

Apply the double Fourier transform method to solve this problem.

49. Solve the diffusion problem with a source $q(x,t)$

$$\begin{aligned} &u_t=\kappa\ u_{xx}+q(x,t),\ -\infty<x<\infty,\ t>0,\\ &u(x,0)=0 \quad \text{for} \quad -\infty<x<\infty. \end{aligned}$$

Show that the solution is

$$u(x,t)=\frac{1}{\sqrt{4\pi\,t\,\kappa}}\int_0^t (t-\tau)^{-\frac{1}{2}}\ d\tau\int_{-\infty}^{\infty} q(k,\tau)\exp\left[-\frac{(x-k)^2}{4\kappa(t-\tau)}\right]dk.$$

50. The function $u(x,t)$ satisfies the diffusion problem in a half-line

$$\begin{aligned} &u_t=\kappa\,u_{xx}+q(x,t),\ 0\leq x<\infty,\ t>0,\\ &u(x,0)=0,\ u(0,t)=0 \quad \text{for} \quad x\geq 0 \quad \text{and} \quad t>0. \end{aligned}$$

Show that

$$u(x,t)=\sqrt{\frac{2}{\pi}}\int_0^t d\tau\int_0^{\infty} Q_s(k,\tau)\exp[-\kappa k^2(t-\tau)]\sin kx\ dk,$$

where $Q_s(k,t)$ is the Fourier sine transform of $q(x,t)$.

51. Apply the triple Fourier transform to solve the initial value problem

$$\begin{aligned} &u_t=\kappa(u_{xx}+u_{yy}+u_{zz}),\ -\infty<x,\ y,\ z<\infty,\ t>0,\\ &u(\mathbf{x},0)=f(\mathbf{x}) \quad \text{for all} \quad x,y,z, \end{aligned}$$

where $\mathbf{x}=(x,y,z)$.

52. Use the Fourier transform with respect to $t$ and Laplace transform with respect to $x$ to solve the telegraph equation

$$\begin{aligned} &u_{tt}+a\ u_t+bu=c^2\ u_{xx}, \quad 0<x<\infty,\ -\infty<t<\infty,\\ &u(0,t)=f(t),\ u_x(0,t)=g(t), \quad \text{for} \quad -\infty<t<\infty, \end{aligned}$$

where $a,\ b,\ c$ are constants and $f(t)$ and $g(t)$ are arbitrary functions of time $t$.

53. Determine the steady-state temperature distribution in a disk occupying the semi-infinite strip $0 < x < \infty$, $0 < y < 1$ if the edges $x = 0$ and $y = 0$ are insulated, and the edge $y = 1$ is kept at a constant temperature $T_0\, H(a - x)$. Assuming that the disk loses heat due to its surroundings according to Newton's law with proportionality constant $h$, solve the boundary value problem

$$\begin{aligned}
&u_{xx} + u_{yy} - hu = 0, \ 0 < x < \infty, \ 0 < y < 1, \\
&u(x,1) = T_0\, H(a - x), \quad \text{for} \quad 0 < x < \infty, \\
&u_x(0,y) = 0 = u_y(x,0) \quad \text{for} \quad 0 < x < \infty, \ 0 < y < 1.
\end{aligned}$$

54. Use the double Fourier transform to solve the following equations:

    (a) $u_{xxxx} - u_{yy} + 2u = f(x,y)$,
    (b) $u_{xx} + 2u_{yy} + 3u_x - 4u = f(x,y)$,

    where $f(x,y)$ is a given function.

55. Use the Fourier transform to solve the Rossby wave problem in an inviscid $\beta$-plane ocean bounded by walls at $y = 0$ and $y = 1$ where $y$ and $x$ represent vertical and horizontal directions. The fluid is initially at rest and then, at $t = 0+$, an arbitrary disturbance localized to the vicinity of $x = 0$ is applied to generate Rossby waves. This problem satisfies the Rossby wave equation

$$\frac{\partial}{\partial t}[(\nabla^2 - \kappa^2)\psi] + \beta\psi_x = 0, \ -\infty < x < \infty, \ 0 \le y \le 1, \ t > 0,$$

with the boundary and initial conditions

$$\begin{aligned}
&\psi_x(x,y) = 0 \quad \text{for} \quad 0 < x < \infty, \ y = 0 \quad \text{and} \quad y = 1, \\
&\psi(x,y,t) = \psi_0(x,y) \ \text{at} \ t = 0 \quad \text{for all } x \text{ and } y.
\end{aligned}$$

56. Find the transfer function and the corresponding impulse response function of the input and output of the $RC$ circuit governed by the equation

$$R\,\frac{dq}{dt} + \frac{1}{C}\,q\,(t) = e\,(t)\,,$$

where $R$, $C$ are constants, $q\,(t)$ is the electric charge and $e\,(t)$ is the given voltage.

57. Prove the Poisson summation formula for the Fourier cosine transform $\mathscr{F}_c\,\{f(x)\} = F_c(k)$ in the form

$$\sqrt{a}\left[\frac{1}{2}f(0) + \sum_{n=1}^{\infty} f(na)\right] = \sqrt{b}\left[\frac{1}{2}F_c(0) + \sum_{n=1}^{\infty} F_c(nb)\right],$$

where $ab = 2\pi$ and $a > 0$.

Apply this formula to the following examples:

(a) $f(x) = e^{-x}$, $\quad F_c(k) = \sqrt{\frac{2}{\pi}}\,(1+k^2)^{-1}$,

(b) $f(x) = \exp(-\frac{1}{2}x^2)$, $\quad F_c(k) = \exp(-\frac{1}{2}k^2)$,

(c) $f(x) = \exp(-\frac{1}{2}x^2)\cos\,\alpha\,x$, $\quad F_c(k) = \exp\left[-\frac{1}{2}(\alpha^2+k^2)\right]\cosh(k\alpha)$,

(d) $f(x) = \begin{cases} \frac{2^{\frac{1}{2}-\nu}}{\Gamma(\nu+\frac{1}{2})}(1-x^2)^{\nu-\frac{1}{2}}, & 0 \le x < 1, \\ 0, & x \ge 1. \end{cases}$

$$F_c(k) = k^{-\nu} J_\nu(k), \quad k > 0; \quad F_c(0) = \frac{1}{2^\nu \Gamma(\nu+1)}.$$

# 3

# *Laplace Transforms and Their Basic Properties*

"What we know is not much. What we do not know is immense."

Pierre-Simon Laplace

"The algebraic analysis soon makes us forget the main object [of our research] by focusing our attention on abstract combinations and it is only at the end that we return to the original objective. But in abandoning oneself to the operations of analysis, one is led to the generality of this method and the inestimable advantage of transforming the reasoning by mechanical procedures to results often inaccessible by geometry....No other language has the capacity for the elegance that arises from a long sequence of expressions linked one to the other and all stemming from one fundamental idea."

Pierre-Simon Laplace

"... For Laplace, on the contrary, mathematical analysis was an instrument that he bent to his purposes for the most varied applications, but always subordinating the method itself to the content of each question. Perhaps posterity will...."

Simeon-Denis Poisson

## 3.1 Introduction

In this chapter, we present the formal definition of the Laplace transform and calculate the Laplace transforms of some elementary functions directly from the definition. The existence conditions for the Laplace transform are stated in Section 3.3. The basic operational properties of the Laplace transforms including convolution and its properties, and the differentiation and integration of Laplace transforms are discussed in some detail. The inverse Laplace transform is introduced in Section 3.7, and four methods of evaluation of the inverse transform are developed with examples. The Heaviside Expansion Theorem and the Tauberian theorems for the Laplace transform are discussed.

## 3.2 Definition of the Laplace Transform and Examples

We start with the *Fourier Integral Formula* (2.2.4), which expresses the representation of a function $f_1(x)$ defined on $-\infty < x < \infty$ in the form

$$f_1(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}dk \int_{-\infty}^{\infty} e^{-ikt} f_1(t)dt. \tag{3.2.1}$$

We next set $f_1(x) \equiv 0$ in $-\infty < x < 0$ and write

$$f_1(x) = e^{-cx} f(x)H(x) = e^{-cx} f(x), \qquad x > 0, \tag{3.2.2}$$

where $c$ is a positive fixed number, so that (3.2.1) becomes

$$f(x) = \frac{e^{cx}}{2\pi}\int_{-\infty}^{\infty} e^{ikx}dk \int_{0}^{\infty} \exp\{-t(c+ik)\} f(t)dt. \tag{3.2.3}$$

With a change of variable, $c + ik = s$, $i\,dk = ds$ we rewrite (3.2.3) as

$$f(x) = \frac{e^{cx}}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp\{(s-c)x\}ds \int_{0}^{\infty} e^{-st} f(t)dt. \tag{3.2.4}$$

Thus, the *Laplace transform* of $f(t)$ is formally defined by

$$\mathscr{L}\{f(t)\} = \bar{f}(s) = \int_{0}^{\infty} e^{-st} f(t)dt, \qquad \text{Re}\, s > 0, \tag{3.2.5}$$

where $e^{-st}$ is the *kernel* of the transform and $s$ is the *transform variable* which is a complex number. Under broad conditions on $f(t)$, its transform $\bar{f}(s)$ is analytic in $s$ in the half-plane, where $\text{Re}\, s > a$.

Result (3.2.4) then gives the formal definition of the *inverse Laplace transform*

$$\mathscr{L}^{-1}\{\bar{f}(s)\} = f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\bar{f}(s)\, ds, \qquad c > 0. \tag{3.2.6}$$

Obviously, $\mathscr{L}$ and $\mathscr{L}^{-1}$ are linear integral operators.

Using the definition (3.2.5), we can calculate the Laplace transforms of some simple and elementary functions.

**Example 3.2.1**

If $f(t) = 1$ for $t > 0$, then

$$\bar{f}(s) = \mathscr{L}\{1\} = \int_0^\infty e^{-st} dt = \frac{1}{s}. \tag{3.2.7}$$

▯

**Example 3.2.2**

If $f(t) = e^{at}$, where $a$ is a constant, then

$$\mathscr{L}\{e^{at}\} = \bar{f}(s) = \int_0^\infty e^{-(s-a)t} dt = \frac{1}{s-a}, \qquad s > a. \tag{3.2.8}$$

▯

**Example 3.2.3**

If $f(t) = \sin at$, where $a$ is a real constant, then

$$\begin{aligned} \mathscr{L}\{\sin at\} &= \int_0^\infty e^{-st} \sin at \, dt = \frac{1}{2i} \int_0^\infty [e^{-t(s-ia)} - e^{-t(s+ia)}] \, dt \\ &= \frac{1}{2i} \left[ \frac{1}{s-ia} - \frac{1}{s+ia} \right] = \frac{a}{s^2+a^2}. \end{aligned} \tag{3.2.9}$$

Similarly,

$$\mathscr{L}\{\cos at\} = \frac{s}{s^2+a^2}. \tag{3.2.10}$$

▯

**Example 3.2.4**

If $f(t) = \sinh at$ or $\cosh at$, where $a$ is a real constant, then

$$\mathscr{L}\{\sinh at\} = \int_0^\infty e^{-st} \sinh at \, dt = \frac{a}{s^2-a^2}, \tag{3.2.11}$$

$$\mathscr{L}\{\cosh at\} = \int_0^\infty e^{-st} \cosh at \, dt = \frac{s}{s^2-a^2}. \tag{3.2.12}$$

□

***Example 3.2.5***
If $f(t) = t^n$, where $n$ is a positive integer, then

$$\bar{f}(s) = \mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}. \tag{3.2.13}$$

We recall (3.2.7) and formally differentiate it with respect to $s$. This gives

$$\int_0^\infty t\, e^{-st}\, dt = \frac{1}{s^2}, \tag{3.2.14}$$

which means that

$$\mathscr{L}\{t\} = \frac{1}{s^2}. \tag{3.2.15}$$

Differentiating (3.2.14) with respect to $s$ gives

$$\mathscr{L}\{t^2\} = \int_0^\infty t^2 e^{-st}\, dt = \frac{2}{s^3}. \tag{3.2.16}$$

Similarly, differentiation of (3.2.7) $n$ times yields

$$\mathscr{L}\{t^n\} = \int_0^\infty t^n e^{-st}\, dt = \frac{n!}{s^{n+1}}. \tag{3.2.17}$$

□

***Example 3.2.6***
If $a(> -1)$ is a real number, then

$$\mathscr{L}\{t^a\} = \frac{\Gamma(a+1)}{s^{a+1}}, \qquad (s > 0). \tag{3.2.18}$$

We have

$$\mathscr{L}\{t^a\} = \int_0^\infty t^a e^{-st}\, dt,$$

which is, by putting $st = x$,

$$= \frac{1}{s^{a+1}} \int_0^\infty x^a e^{-x} dx = \frac{\Gamma(a+1)}{s^{a+1}},$$

where $\Gamma(a)$ represents the *gamma function* defined by the integral

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x}\, dx, \qquad a > 0. \tag{3.2.19}$$

It can be shown that the gamma function satisfies the relation

$$\Gamma(a+1) = a\Gamma(a). \tag{3.2.20}$$

Obviously, result (3.2.18) is an extension of (3.2.17). The latter is a special case of the former when $a$ is a positive integer.

In particular, when $a = -\frac{1}{2}$, result (3.2.18) gives

$$\mathscr{L}\left\{\frac{1}{\sqrt{t}}\right\} = \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}, \quad \text{where} \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{3.2.21}$$

Similarly,

$$\mathscr{L}\left\{\sqrt{t}\right\} = \frac{\Gamma\left(\frac{3}{2}\right)}{s^{3/2}} = \frac{\sqrt{\pi}}{2}\frac{1}{s^{3/2}}, \tag{3.2.22}$$

where

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

□

***Example 3.2.7***

If $f(t) = erf\left(\frac{a}{2\sqrt{t}}\right)$, then

$$\mathscr{L}\left\{erf\left(\frac{a}{2\sqrt{t}}\right)\right\} = \frac{1}{s}\left(1 - e^{-a\sqrt{s}}\right), \tag{3.2.23}$$

where $erf(t)$ is the *error function* defined by (2.5.13).

To prove (3.2.23), we begin with the definition (3.2.5) so that

$$\mathscr{L}\left\{erf\left(\frac{a}{2\sqrt{t}}\right)\right\} = \int_0^\infty e^{-st}\left[\frac{2}{\sqrt{\pi}}\int_0^{a/2\sqrt{t}} e^{-x^2}\, dx\right] dt,$$

which is, by putting $x = \frac{a}{2\sqrt{t}}$ or $t = \frac{a^2}{4x^2}$ and interchanging the order of integration,

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} dx \int_0^{a^2/4x^2} e^{-st} dt$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} \frac{1}{s} \left\{1 - \exp\left(-\frac{a^2 s}{4x^2}\right)\right\} dx$$

$$= \frac{1}{s} \cdot \frac{2}{\sqrt{\pi}} \left[\int_0^\infty e^{-x^2} dx - \int_0^\infty \exp\left\{-\left(x^2 + \frac{sa^2}{4x^2}\right)\right\} dx\right],$$

where the integral

$$\int_0^\infty \exp\left\{-\left(x^2 + \frac{\alpha^2}{x^2}\right)\right\} dx = \frac{1}{2}\left[\int_0^\infty \left(1 - \frac{\alpha}{x^2}\right) \exp\left[-\left(x + \frac{\alpha}{x}\right)^2 + 2\alpha\right]\right.$$

$$\left. + \int_0^\infty \left(1 + \frac{\alpha}{x^2}\right) \exp\left[-\left(x - \frac{\alpha}{x}\right)^2 - 2\alpha\right]\right] dx,$$

which is, by putting $y = \left(x \pm \frac{\alpha}{x}\right)$, $dy = \left(1 \mp \frac{\alpha}{x^2}\right) dx$, and observing that the first integral vanishes,

$$= \frac{1}{2} e^{-2\alpha} \int_{-\infty}^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2} e^{-2\alpha}, \qquad \alpha = \frac{a\sqrt{s}}{2}.$$

Consequently,

$$\mathscr{L}\left\{erf\left(\frac{a}{2\sqrt{t}}\right)\right\} = \frac{1}{s}\frac{2}{\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} e^{-a\sqrt{s}}\right] = \frac{1}{s}[1 - e^{-a\sqrt{s}}].$$

We use (3.2.23) to find the Laplace transform of the complementary error function defined by (2.10.14) and obtain

$$\mathscr{L}\left\{erfc\left(\frac{a}{2\sqrt{t}}\right)\right\} = \frac{1}{s} e^{-a\sqrt{s}}. \tag{3.2.24}$$

The proof of this result follows from $erfc(x) = 1 - erf(x)$ and $\mathscr{L}\{1\} = \frac{1}{s}$.

□

***Example 3.2.8***

If $f(t) = J_0(at)$ is a *Bessel function* of order zero, then

$$\mathscr{L}\{J_0(at)\} = \frac{1}{\sqrt{s^2 + a^2}}. \tag{3.2.25}$$

Using the series representation of $J_0(at)$, we obtain

$$\begin{aligned}\mathscr{L}\{J_0(at)\} &= \mathscr{L}\left[1 - \frac{a^2t^2}{2^2} + \frac{a^4t^4}{2^2\cdot 4^2} - \frac{a^6t^6}{2^2\cdot 4^2\cdot\cdot 6^2} + \cdots\right] \\ &= \frac{1}{s} - \frac{a^2}{2^2}\frac{2!}{s^3} + \frac{a^4}{2^2\cdot 4^2}\cdot\frac{4!}{s^5} - \frac{a^6}{2^2\cdot 4^2\cdot 6^2}\cdot\frac{6!}{s^7} + \cdots \\ &= \frac{1}{s}\left[1 - \frac{1}{2}\left(\frac{a^2}{s^2}\right) + \frac{1\cdot 3}{2\cdot 4}\left(\frac{a^4}{s^4}\right) - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{a^6}{s^6}\right) + \cdots\right] \\ &= \frac{1}{s}\left[\left(1 + \frac{a^2}{s^2}\right)^{-\frac{1}{2}}\right] = \frac{1}{\sqrt{a^2+s^2}}.\end{aligned}$$

▯

## 3.3 Existence Conditions for the Laplace Transform

A function $f(t)$ is said to be of *exponential order* $a(>0)$ on $0 \le t < \infty$ if there exists a positive constant $K$ such that for all $t > T$

$$|f(t)| \le Ke^{at}, \tag{3.3.1}$$

and we write this symbolically as

$$f(t) = O(e^{at}) \qquad \text{as} \quad t \to \infty. \tag{3.3.2}$$

Or, equivalently,

$$\lim_{t\to\infty} e^{-bt}|f(t)| \le K \lim_{t\to\infty} e^{-(b-a)t} = 0, \quad b > a. \tag{3.3.3}$$

Such a function $f(t)$ is simply called an *exponential order* as $t \to \infty$, and clearly, it does not grow faster than $Ke^{at}$ as $t \to \infty$.

***THEOREM 3.3.1***

If a function $f(t)$ is continuous or piecewise continuous in every finite interval $(0, T)$, and of exponential order $e^{at}$, then the Laplace transform of $f(t)$ exists for all $s$ provided Re $s > a$.

**PROOF** We have

$$\begin{aligned}|\bar{f}(s)| = \left|\int_0^\infty e^{-st} f(t)dt\right| &\le \int_0^\infty e^{-st}|f(t)|dt \qquad (3.3.4) \\ &\le K\int_0^\infty e^{-t(s-a)}dt = \frac{K}{s-a}, \text{ for Re } s > a.\end{aligned}$$

Thus, the proof is complete.

It is noted that the conditions as stated in Theorem 3.3.1 are sufficient rather than necessary conditions.

It also follows from (3.3.4) that $\lim_{s\to\infty} |\bar{f}(s)| = 0$, that is, $\lim_{s\to\infty} \bar{f}(s) = 0$. This result can be regarded as the limiting property of the Laplace transform. However, $\bar{f}(s) = s$ or $s^2$ is not the Laplace transform of any continuous (or piecewise continuous) function because $\bar{f}(s)$ does not tend to zero as $s \to \infty$.

Further, a function $f(t) = \exp(at^2)$, $a > 0$ cannot have a Laplace transform even though it is continuous but is *not* of the exponential order because

$$\lim_{t\to\infty} \exp(at^2 - st) = \infty.$$

∎

## 3.4 Basic Properties of Laplace Transforms

***THEOREM 3.4.1*** *(Heaviside's First Shifting Theorem).*
If $\mathscr{L}\{f(t)\} = \bar{f}(s)$, then

$$\mathscr{L}\{e^{-at} f(t)\} = \bar{f}(s+a), \tag{3.4.1}$$

where $a$ is a real constant.

**PROOF** We have, by definition,

$$\mathscr{L}\{e^{-at} f(t)\} = \int_0^\infty e^{-(s+a)t} f(t) dt = \bar{f}(s+a).$$

∎

***Example 3.4.1***
The following results readily follow from (3.4.1)

$$\mathscr{L}\{t^n e^{-at}\} = \frac{n!}{(s+a)^{n+1}}, \tag{3.4.2}$$

$$\mathscr{L}\{e^{-at} \sin bt\} = \frac{b}{(s+a)^2 + b^2}, \tag{3.4.3}$$

$$\mathscr{L}\{e^{-at} \cos bt\} = \frac{s+a}{(s+a)^2 + b^2}. \tag{3.4.4}$$

***THEOREM 3.4.2***
If $\mathscr{L}\{f(t)\} = \bar{f}(s)$, then the Second Shifting property holds:

$$\mathscr{L}\{f(t-a)H(t-a)\} = e^{-as}\bar{f}(s) = e^{-as}\mathscr{L}\{f(t)\}, \qquad a > 0. \tag{3.4.5}$$

Or, equivalently,

$$\mathscr{L}\{f(t)H(t-a)\} = e^{-as}\mathscr{L}\{f(t+a)\}. \tag{3.4.6}$$

where $H(t-a)$ is the Heaviside unit step function defined by (2.3.9).

It follows from the definition that

$$\begin{aligned}\mathscr{L}\{f(t-a)H(t-a)\} &= \int_0^\infty e^{-st} f(t-a)H(t-a)dt \\ &= \int_a^\infty e^{-st} f(t-a)dt,\end{aligned}$$

which is, by putting $t - a = \tau$,

$$= e^{-sa}\int_0^\infty e^{-s\tau} f(\tau)d\tau = e^{-sa}\bar{f}(s).$$

We leave it to the reader to prove (3.4.6).

In particular, if $f(t) = 1$, then

$$\mathscr{L}\{H(t-a)\} = \frac{1}{s}\exp(-sa). \tag{3.4.7}$$

$\square$

***Example 3.4.2***
Use the shifting property (3.4.5) or (3.4.6) to find the Laplace transform of

$$\text{(a)} \quad f(t) = \left\{\begin{array}{ll} 1, & 0 < t < 1 \\ -1, & 1 < t < 2 \\ 0, & t > 2 \end{array}\right\}, \qquad \text{(b)} \quad g(t) = \sin t\, H(t-\pi).$$

To find $\mathscr{L}\{f(t)\}$, we write $f(t)$ as

$$f(t) = 1 - 2\,H(t-1) + H(t-2).$$

Hence,

$$\begin{aligned}\bar{f}(s) = \mathscr{L}\{f(t)\} &= \mathscr{L}\{1\} - 2\,\mathscr{L}\{H(t-1)\} + \mathscr{L}\{H(t-2)\} \\ &= \frac{1}{s} - \frac{2\,e^{-s}}{s} + \frac{e^{-2s}}{s}.\end{aligned}$$

To obtain $\mathscr{L}\{g(t)\}$, we use (3.4.6) so that

$$\bar{g}(s) = \mathscr{L}\{\sin t\, H(t-\pi)\} = -e^{-\pi s}\,\mathscr{L}\{\cos t\} = -\frac{s\,e^{-\pi s}}{s^2+1}.$$

▯

**Scaling Property:**

$$\mathscr{L}\{f(at)\} = \frac{1}{|a|}\,\bar{f}\left(\frac{s}{a}\right), \quad a \neq 0. \tag{3.4.8}$$

***Example 3.4.3***

Show that the Laplace transform of the square wave function $f(t)$ defined by

$$f(t) = H(t) - 2H(t-a) + 2H(t-2a) - 2H(t-3a) + \cdots \tag{3.4.9}$$

is

$$\bar{f}(s) = \frac{1}{s}\tanh\left(\frac{as}{2}\right). \tag{3.4.10}$$

The graph of $f(t)$ is shown in Figure 3.1.

$$\begin{aligned} f(t) &= H(t) - 2H(t-a) = 1 - 2\cdot 0 = 1, \qquad 0 < t < a \\ f(t) &= H(t) - 2H(t-a) + 2H(t-2a) \\ &= 1 - 2\cdot 1 + 2\cdot 0 = -1, \qquad 0 < a < t < 2a. \end{aligned}$$

Thus,

$$\begin{aligned} \bar{f}(s) &= \frac{1}{s} - 2\cdot\frac{e^{-as}}{s} + 2\cdot\frac{e^{-2as}}{s} - 2\cdot\frac{e^{-3as}}{s} + \cdots \\ &= \frac{1}{s}[1 - 2r(1 - r + r^2 - \cdots)], \qquad \text{where} \quad r = e^{-as} \\ &= \frac{1}{s}\left[1 - \frac{2r}{1+r}\right] = \frac{1}{s}\left[1 - \frac{2e^{-as}}{1+e^{-as}}\right] \\ &= \frac{1}{s}\left(\frac{1-e^{-as}}{1+e^{-as}}\right) = \frac{1}{s}\left(\frac{e^{\frac{sa}{2}} - e^{-\frac{as}{2}}}{e^{\frac{sa}{2}} + e^{-\frac{as}{2}}}\right) = \frac{1}{s}\tanh\left(\frac{as}{2}\right). \end{aligned}$$

▯

***Example 3.4.4***

(*The Laplace Transform of a Periodic Function*). If $f(t)$ is a periodic function of period $a$, and if $\mathscr{L}\{f(t)\}$ exists, show that

$$\mathscr{L}\{f(t)\} = [1 - \exp(-as)]^{-1}\int_0^a e^{-st} f(t)\,dt. \tag{3.4.11}$$

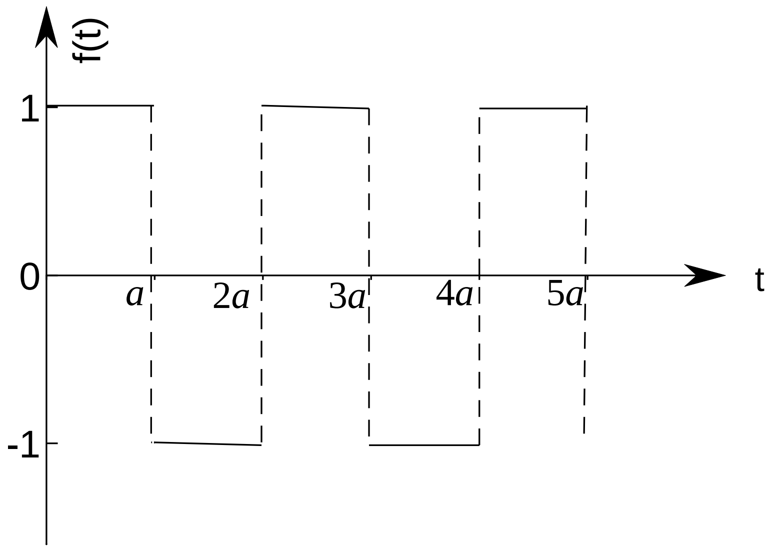


**Figure 3.1** Square wave function.

We have, by definition,

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \int_0^a e^{-st} f(t)dt + \int_a^\infty e^{-st} f(t)dt.$$

Letting $t = \tau + a$ in the second integral gives

$$\bar{f}(s) = \int_0^a e^{-st} f(t)dt + \exp(-sa) \int_0^\infty e^{-s\tau} f(\tau + a)d\tau,$$

which is, due to $f(\tau + a) = f(\tau)$ and replacing the dummy variable $\tau$ by $t$ in the second integral,

$$= \int_0^a e^{-st} f(t)dt + \exp(-sa) \int_0^\infty e^{-st} f(t)dt.$$

Finally, combining the second term with the left hand side, we obtain (3.4.11).

In particular, we calculate the Laplace transform of a rectified sine wave, that is, $f(t) = |\sin at|$. This is a periodic function with period $\dfrac{\pi}{a}$. We have

$$\int_0^{\frac{\pi}{a}} e^{-st} \sin at \, dt = \left[ \frac{e^{-st}(-a\cos at - s\sin at)}{(s^2 + a^2)} \right]_0^{\frac{\pi}{a}} = \frac{a\left\{1 + \exp\left(-\frac{s\pi}{a}\right)\right\}}{(s^2 + a^2)}.$$

Clearly, the property (3.4.11) gives

$$\begin{aligned}
\mathscr{L}\{f(t)\} &= \frac{a}{(s^2+a^2)}\cdot\frac{1+\exp\left(-\frac{s\pi}{a}\right)}{1-\exp\left(-\frac{s\pi}{a}\right)} \\
&= \frac{a}{(s^2+a^2)}\left[\frac{\exp\left(\frac{s\pi}{2a}\right)+\exp\left(-\frac{s\pi}{2a}\right)}{\exp\left(\frac{2\pi}{2a}\right)-\exp\left(-\frac{s\pi}{2a}\right)}\right] \\
&= \frac{a}{s^2+a^2}\coth\left(\frac{\pi s}{2a}\right).
\end{aligned}$$

□

***THEOREM 3.4.3*** *(Laplace Transforms of Derivatives).*
If $\mathscr{L}\{f(t)\}=\bar{f}(s)$, then

$$\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\}-f(0)=s\bar{f}(s)-f(0), \tag{3.4.12}$$
$$\mathscr{L}\{f''(t)\} = s^2\mathscr{L}\{f(t)\}-sf(0)-f'(0)=s^2\bar{f}(s)-s\,f(0)-f'(0). \tag{3.4.13}$$

More generally,

$$\mathscr{L}\{f^{(n)}(t)\}=s^n\bar{f}(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0), \tag{3.4.14}$$

where $f^{(r)}(0)$ is the value of $f^{(r)}(t)$ at $t=0,\ r=0,1,...,(n-1)$.

**PROOF** We have, by definition,

$$\mathscr{L}\{f'(t)\}=\int_0^\infty e^{-st}f'(t)dt,$$

which is, integrating by parts,

$$\begin{aligned}
&= \left[e^{-st}f(t)\right]_0^\infty + s\int_0^\infty e^{-st}f(t)dt \\
&= s\bar{f}(s)-f(0),
\end{aligned}$$

in which we assumed $f(t)\,e^{-st}\to 0$ as $t\to\infty$.

Similarly,

$$\begin{aligned}
\mathscr{L}\{f''(t)\} &= s\mathscr{L}\{f'(t)\}-f'(0), \qquad \text{by (3.4.12)} \\
&= s[s\ \bar{f}(s)-f(0)]-f'(0) \\
&= s^2\bar{f}(s)-sf(0)-f'(0),
\end{aligned}$$

where we have assumed $e^{-st}f'(t) \to 0$ as $t \to \infty$.

A similar procedure can be used to prove the general result (3.4.14).

It may be noted that similar results hold when the Laplace transform is applied to partial derivatives of a function of two or more independent variables. For example, if $u(x,t)$ is a function of two variables $x$ and $t$, then

$$\mathscr{L}\left\{\frac{\partial u}{\partial t}\right\} = s\bar{u}(x,s) - u(x,0), \tag{3.4.15}$$

$$\mathscr{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\} = s^2\bar{u}(x,s) - s\ u(x,0) - \left[\frac{\partial u}{\partial t}\right]_{t=0}, \tag{3.4.16}$$

$$\mathscr{L}\left\{\frac{\partial u}{\partial x}\right\} = \frac{d\bar{u}}{dx}, \quad \mathscr{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \frac{d^2\bar{u}}{dx^2}. \tag{3.4.17}$$

Results (3.4.12) to (3.4.14) imply that the Laplace transform reduces the operation of differentiation into algebraic operation. In view of this, the Laplace transform can be used effectively to solve ordinary or partial differential equations. $\blacksquare$

***Example 3.4.5***
Use (3.4.14) to find $\mathscr{L}\{t^n\}$.

Here $f(t) = t^n$, $f'(t) = nt^{n-1}, \cdots, f^{(n)}(t) = n!$ and $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$.

Thus,

$$\mathscr{L}\{n!\} = s^n \mathscr{L}\{t^n\}.$$

Or,

$$\mathscr{L}\{t^n\} = \frac{n!}{s^n}\mathscr{L}\{1\} = \frac{n!}{s^{n+1}}.$$

$\square$

## 3.5 The Convolution Theorem and Properties of Convolution

***THEOREM 3.5.1*** *(Convolution Theorem).*
If $\mathscr{L}\{f(t)\} = \bar{f}(s)$ and $\mathscr{L}\{g(t)\} = \bar{g}(s)$, then

$$\mathscr{L}\{f(t) * g(t)\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\} = \bar{f}(s)\bar{g}(s). \tag{3.5.1}$$

Or, equivalently,

$$\mathscr{L}^{-1}\{\bar{f}(s)\bar{g}(s)\} = f(t) * g(t), \tag{3.5.2}$$

where $f(t)*g(t)$ is called the *convolution* of $f(t)$ and $g(t)$ and is defined by the integral

$$f(t)*g(t)=\int_0^t f(t-\tau)g(\tau)d\tau. \tag{3.5.3}$$

The integral in (3.5.3) is often referred to as the *convolution integral* (or *Faltung*) and is denoted simply by $(f*g)(t)$.

**PROOF** We have, by definition,

$$\mathscr{L}\{f(t)*g(t)\}=\int_0^\infty e^{-st}dt\int_0^t f(t-\tau)g(\tau)d\tau, \tag{3.5.4}$$

where the region of integration in the $\tau-t$ plane is as shown in Figure 3.2. The integration in (3.5.4) is first performed with respect to $\tau$ from $\tau=0$ to $\tau=t$ of the vertical strip and then from $t=0$ to $\infty$ by moving the vertical strip from $t=0$ outwards to cover the whole region under the line $\tau=t$.

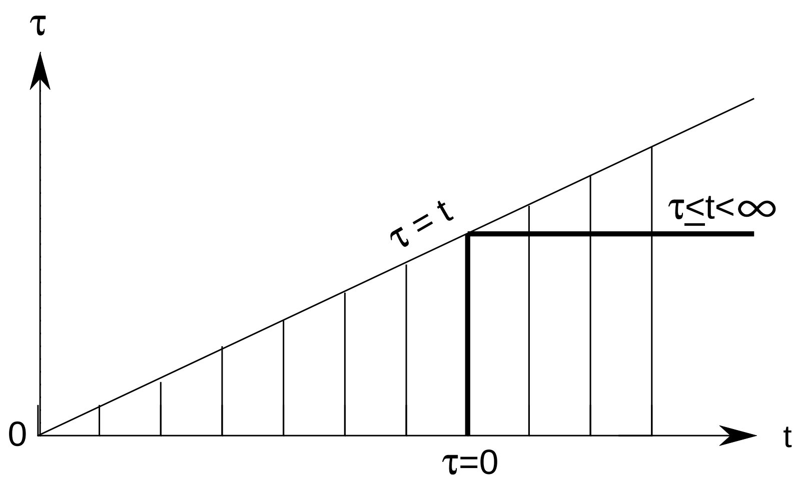


**Figure 3.2** Region of integration.

We now change the order of integration so that we integrate first along the horizontal strip from $t=\tau$ to $\infty$ and then from $\tau=0$ to $\infty$ by moving the horizontal strip vertically from $\tau = 0$ upwards. Evidently, (3.5.4) becomes

$$\mathscr{L}\{f(t)*g(t)\} = \int_0^\infty g(\tau)d\tau \int_{t=\tau}^\infty e^{-st} f(t-\tau)d\tau,$$

which is, by the change of variable $t - \tau = x$,

$$\begin{aligned}\mathscr{L}\{f(t)*g(t)\} &= \int_0^\infty g(\tau)d\tau \int_0^\infty e^{-s(x+\tau)} f(x)dx \\ &= \int_0^\infty e^{-s\tau} g(\tau)d\tau \int_0^\infty e^{-sx} f(x)dx = \bar{g}(s)\,\bar{f}(s).\end{aligned}$$

This completes the proof. ∎

**PROOF (Second Proof.)** We have, by definition,

$$\begin{aligned}\bar{f}(s)\bar{g}(s) &= \int_0^\infty e^{-s\sigma} f(\sigma)d\sigma \int_0^\infty e^{-s\mu} g(\mu)d\mu \\ &= \int_0^\infty \int_0^\infty e^{-s(\sigma+\mu)} f(\sigma)g(\mu)d\sigma\, d\mu, \end{aligned} \tag{3.5.5}$$

where the double integral is taken over the entire first quadrant $R$ of the $\sigma - \mu$ plane bounded by $\sigma = 0$ and $\mu = 0$ as shown in Figure 3.3(a).

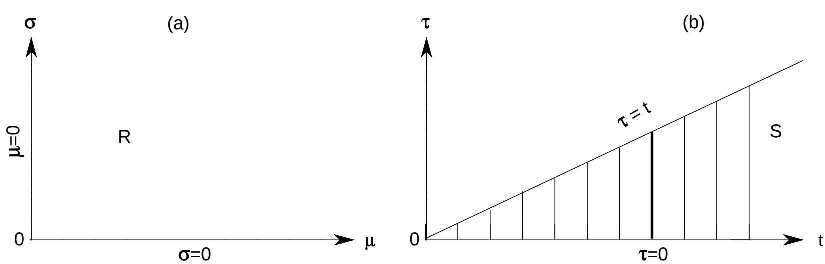


**Figure 3.3** Regions of integration.

We make the change of variables $\mu = \tau$, $\sigma = t - \mu = t - \tau$ so that the axes $\sigma = 0$ and $\mu = 0$ transform into the lines $\tau = 0$ and $\tau = t$, respectively, as shown in

Figure 3.3(b) in the $\tau - t$ plane. Consequently, (3.5.5) becomes

$$\begin{aligned}\bar{f}(s)\bar{g}(s) &= \int_0^\infty e^{-st}dt \int_{\tau=0}^{\tau=t} f(t-\tau)g(\tau)d\tau \\ &= \mathscr{L}\left\{\int_0^t f(t-\tau)g(\tau)d\tau\right\} \\ &= \mathscr{L}\{f(t)*g(t)\}.\end{aligned}$$

This proves the theorem. ∎

***Note:*** A more rigorous proof of the convolution theorem can be found in any standard treatise (see Doetsch, 1950) on Laplace transforms. The convolution operation has the following properties:

$$f(t)*\{g(t)*h(t)\} = \{f(t)*g(t)\}*h(t), \qquad \text{(Associative)}, \quad (3.5.6)$$

$$f(t)*g(t) = g(t)*f(t), \qquad \text{(Commutative)}, \quad (3.5.7)$$

$$f(t)*\{ag(t) + bh(t)\} = af(t)*g(t) + bf(t)*h(t), \qquad \text{(Distributive)}, \quad (3.5.8)$$

$$f(t)*\{ag(t)\} = \{af(t)\}*g(t) = a\{f(t)*g(t)\}, \quad (3.5.9)$$

$$\mathscr{L}\{f_1*f_2*f_3*\cdots*f_n\} = \bar{f}_1(s)\bar{f}_2(s)\cdots\bar{f}_n(s), \quad (3.5.10)$$

$$\mathscr{L}\{f^{*n}\} = \{\bar{f}(s)\}^n, \quad (3.5.11)$$

where $a$ and $b$ are constants. $f^{*n} = f*f*\cdots*f$ is sometimes called the *nth convolution.*

***Remark:*** By virtue of (3.5.6) and (3.5.7), it is clear that the set of all Laplace transformable functions forms a commutative semigroup with respect to the operation $*$. The set of all Laplace transformable functions does not form a group because $f*g^{-1}$ does not, in general, have a Laplace transform.

We now prove the associative property. We have

$$f(t)*\{g(t)*h(t)\} = \int_0^t f(\tau) \int_0^{t-\tau} g(t-\sigma-\tau)h(\sigma)d\sigma\ d\tau \quad (3.5.12)$$

$$= \int_0^t h(\sigma) \int_0^{t-\sigma} g(t-\tau-\sigma)f(\tau)d\tau\ d\sigma$$

$$= h(t)*\{f(t)*g(t)\} = \{f(t)*g(t)\}*h(t), \quad (3.5.13)$$

where (3.5.13) is obtained from (3.5.12) by interchanging the order of integration combined with the fact that $0 \le \sigma \le t-\tau$ and $0 \le \tau \le t$ imply $0 \le \tau \le t-\sigma$ and $0 \le \sigma \le t$. Properties (3.5.10) and (3.5.11) follow immediately from the associative law of the convolution.

To prove (3.5.7), we recall the definition of the convolution and make a change of variable $t-\tau=t'$. This gives

$$f(t)*g(t)=\int_0^t f(t-\tau)g(\tau)d\tau=\int_0^t g(t-t')f(t')dt'=g(t)*f(t).$$

The proofs of (3.5.8)–(3.5.9) are very simple and hence, may be omitted.

***Example 3.5.1***

Obtain the convolutions

(a) $t*e^{at}$, (b) $(\sin at*\sin at)$, (c) $\frac{1}{\sqrt{\pi t}}*e^{at}$,

(d) $1*\frac{a}{2}\frac{e^{-a^2/4t}}{\sqrt{\pi t^3}}$, (e) $\cos t*e^{2t}$, (f) $t*t*t$.

We have

(a) $t*e^{at}=\int_0^t \tau e^{a(t-\tau)}d\tau=e^{at}\int_0^t \tau e^{-a\tau}d\tau=\frac{1}{a^2}(e^{at}-at-1).$

(b) $\sin at*\sin at=\int_0^t \sin a\tau\ \sin a(t-\tau)d\tau=\frac{1}{2a}(\sin at-at\cos at).$

(c) $\frac{1}{\sqrt{\pi t}}*e^{at}=\frac{1}{\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{\tau}}e^{a(t-\tau)}d\tau,$

which is, by putting $\sqrt{a\tau}=x$,

$$\frac{1}{\sqrt{\pi t}}*e^{at}=\frac{2e^{at}}{\sqrt{\pi a}}\int_0^{\sqrt{at}} e^{-x^2}dx=\frac{e^{at}}{\sqrt{a}}\,erf\left(\sqrt{at}\right).$$

(d) We have

$$1*\frac{a}{2}\frac{e^{-a^2/4t}}{\sqrt{\pi t^3}}=\frac{a}{2\sqrt{\pi}}\int_0^t \frac{e^{-a^2/4\tau}}{\tau^{3/2}}d\tau,$$

which is, by letting $\frac{a}{2\sqrt{\tau}}=x$,

$$=\frac{2}{\sqrt{\pi}}\int_{\frac{a}{2\sqrt{t}}}^{\infty} e^{-x^2}dx=erfc\left(\frac{a}{2\sqrt{t}}\right).$$

$$\text{(e)}\ \cos t * e^{2t} = \int_0^t \cos(t-\tau)\, e^{2\tau}\, d\tau = \frac{1}{2}\int_0^t e^{2\tau}\left\{e^{i(t-\tau)} + e^{-i(t-\tau)}\right\} d\tau$$

$$= \left[\frac{e^{i(t-\tau)+2\tau}}{2(2-i)} + \frac{e^{-i(t-\tau)+2\tau}}{2(2+i)}\right] = \frac{2}{5}\, e^{2t} + \frac{1}{5}\,(\sin t - 2\cos t)\,.$$

$$\text{(f)}\ (t*t)*t = \left[\int_0^t (t-\tau)\,\tau\, d\tau\right] * t = \frac{1}{6}t^3 * t$$

$$= \frac{1}{6}\int_0^t (t-\tau)\,\tau^3\, d\tau = \frac{t^5}{5!}.$$

□

***Example 3.5.2***

Using the Convolution Theorem 3.5.1, prove that

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \tag{3.5.14}$$

where $\Gamma(m)$ is the gamma function, and $B(m,n)$ is the beta function defined by

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, \quad (m>0, n>0). \tag{3.5.15}$$

To prove (3.5.14), we consider

$$f(t) = t^{m-1}\ (m>0) \quad \text{and} \quad g(t) = t^{n-1},\ (n>0).$$

Evidently, $\bar{f}(s) = \dfrac{\Gamma(m)}{s^m}$ and $\bar{g}(s) = \dfrac{\Gamma(n)}{s^n}$.

We have

$$f*g = \int_0^t \tau^{m-1}(t-\tau)^{n-1}d\tau = \mathscr{L}^{-1}\{\bar{f}(s)\bar{g}(s)\}$$

$$= \Gamma(m)\Gamma(n)\mathscr{L}^{-1}\{s^{-(m+n)}\}$$

$$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\, t^{m+n-1}.$$

Letting $t=1$, we derive the result

$$\int_0^1 \tau^{m-1}(1-\tau)^{n-1}d\tau = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},$$

which proves the result (3.5.14). □

## 3.6 Differentiation and Integration of Laplace Transforms

***THEOREM 3.6.1***
If $f(t) = O(e^{at})$ as $t \to \infty$, then the *Laplace integral*

$$\int_0^\infty e^{-st} f(t)dt, \tag{3.6.1}$$

is uniformly convergent with respect to $s$ provided $s \geq a_1$ where $a_1 > a$.

**PROOF** Since

$$|e^{-st} f(t)| \leq K e^{-t(s-a)} \leq K e^{-t(a_1-a)} \quad \text{for all}\, s \geq a_1$$

and $\int_0^\infty e^{-t(a_1-a)}dt$ exists for $a_1 > a$, by Weierstrass' test, the Laplace integral is uniformly convergent for all $s > a_1$ where $a_1 > a$. This completes the proof.

∎

In view of the uniform convergence of (3.6.1), differentiation of (3.2.5) with respect to $s$ within the integral sign is permissible. Hence,

$$\begin{aligned}\frac{d}{ds}\bar{f}(s) &= \frac{d}{ds}\int_0^\infty e^{-st} f(t)dt = \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)dt \\ &= -\int_0^\infty t f(t) e^{-st} dt = -\mathscr{L}\{t f(t)\}.\end{aligned} \tag{3.6.2}$$

Similarly, we obtain

$$\frac{d^2}{ds^2}\bar{f}(s) = (-1)^2 \mathscr{L}\{t^2 f(t)\}, \tag{3.6.3}$$

$$\frac{d^3}{ds^3}\bar{f}(s) = (-1)^3 \mathscr{L}\{t^3 f(t)\}. \tag{3.6.4}$$

More generally,

$$\frac{d^n}{ds^n}\bar{f}(s) = (-1)^n \mathscr{L}\{t^n f(t)\}. \tag{3.6.5}$$

Results (3.6.5) can be stated in the following theorem:

***THEOREM 3.6.2*** *(Derivatives of the Laplace Transform).*
If $\mathscr{L}\{f(t)\} = \bar{f}(s)$, then

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\bar{f}(s), \tag{3.6.6}$$

where $n = 0, 1, 2, 3, \ldots$.

***Example 3.6.1***
Show that

(a) $\mathscr{L}\{t^n e^{-at}\} = \dfrac{n!}{(s+a)^{n+1}}$, (b) $\mathscr{L}\{t\cos at\} = \dfrac{s^2-a^2}{(s^2+a^2)^2}$,

(c) $\mathscr{L}\{t\sin at\} = \dfrac{2as}{(s^2+a^2)^2}$, (d) $\mathscr{L}\{t\,f'(t)\} = -\left\{s\dfrac{d}{ds}\bar{f}(s) + \bar{f}(s)\right\}$.

(a) Application of Theorem 3.6.2 gives

$$\mathscr{L}\{t^n e^{-at}\} = (-1)^n \frac{d^n}{ds^n}\cdot\frac{1}{(s+a)} = (-1)^{2n}\frac{n!}{(s+a)^{n+1}}.$$

(b)

$$\mathscr{L}\{t\cos at\} = (-1)\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) = \frac{s^2-a^2}{(s^2+a^2)^2}.$$

Results (c) and (d) can be proved similarly. ▯

***THEOREM 3.6.3*** *(Integral of the Laplace Transform).*
If $\mathscr{L}\{f(t)\} = \bar{f}(s)$, then

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)ds. \tag{3.6.7}$$

**PROOF** In view of the uniform convergence of (3.6.1), $\bar{f}(s)$ can be integrated with respect to $s$ in $(s,\infty)$ so that

$$\int_s^\infty \bar{f}(s)ds = \int_s^\infty ds \int_0^\infty e^{-st} f(t)dt$$
$$= \int_0^\infty f(t)dt \int_s^\infty e^{-st} ds$$
$$= \int_0^\infty \frac{f(t)}{t} e^{-st} dt = \mathscr{L}\left\{\frac{f(t)}{t}\right\}.$$

This proves the theorem. ∎

***Example 3.6.2***
Show that

(a) $\mathscr{L}\left\{\frac{\sin at}{t}\right\} = \tan^{-1}\left(\frac{a}{s}\right)$, (b) $\mathscr{L}\left\{\frac{e^{-a^2/4t}}{\sqrt{\pi t^3}}\right\} = \frac{2}{a}\exp(-a\sqrt{s})$.

(a) Using (3.6.7), we obtain

$$\mathscr{L}\left\{\frac{\sin at}{t}\right\} = a\int_s^\infty \frac{ds}{s^2+a^2} = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \tan^{-1}\left(\frac{a}{s}\right).$$

(b) $\mathscr{L}\left\{\frac{1}{t}\cdot\frac{e^{-a^2/4t}}{\sqrt{\pi t}}\right\} = \int_s^\infty \bar{f}(s)ds = \int_s^\infty \frac{e^{-a\sqrt{s}}}{\sqrt{s}} ds$, by Table B-4 of Laplace transforms,

which is, by putting $a\sqrt{s} = x$,

$$= \frac{2}{a}\int_{a\sqrt{s}}^\infty e^{-x}dx = \frac{2}{a}\exp(-a\sqrt{s}).$$

□

***THEOREM 3.6.4*** *(The Laplace Transform of an Integral).*
If $\mathscr{L}\{f(t)\} = \bar{f}(s)$, then

$$\mathscr{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{\bar{f}(s)}{s}. \tag{3.6.8}$$

**PROOF** We write

$$g(t) = \int_0^t f(\tau)d\tau$$

so that $g(0) = 0$ and $g'(t) = f(t)$. Then it follows from (3.4.10) that

$$\bar{f}(s) = \mathscr{L}\{f(t)\} = \mathscr{L}\{g'(t)\} = s\,\bar{g}(s) = s\,\mathscr{L}\left\{\int_0^t f(\tau)d\tau\right\}.$$

Dividing both sides by $s$, we obtain (3.6.8).

It is noted that the Laplace transform of an integral corresponds to the division of the transform of its integrand by $s$. Result (3.6.8) can be used for evaluation of the inverse Laplace transform. ∎

***Example 3.6.3***

Use result (3.6.8) to find

(a) $\mathscr{L}\left\{\int_0^t \tau^n e^{-a\tau} d\tau\right\}$, (b) $\mathscr{L}\{Si(at)\} = \mathscr{L}\left\{\int_0^t \frac{\sin a\tau}{\tau} d\tau\right\}$.

(a) We know

$$\mathscr{L}\{t^n e^{-at}\} = \frac{n!}{(s+a)^{n+1}}.$$

It follows from (3.6.8) that

$$\mathscr{L}\left\{\int_0^t \tau^n e^{-a\tau} d\tau\right\} = \frac{n!}{s(s+a)^{n+1}}.$$

(b) Using (3.6.8) and Example 3.6.2(a), we obtain

$$\mathscr{L}\left\{\int_0^t \frac{\sin a\tau}{\tau} d\tau\right\} = \frac{1}{s}\tan^{-1}\left(\frac{a}{s}\right).$$

□

## 3.7 The Inverse Laplace Transform and Examples

It has already been demonstrated that the Laplace transform $\bar{f}(s)$ of a given function $f(t)$ can be calculated by direct integration. We now look at the inverse problem. Given a Laplace transform $\bar{f}(s)$ of an unknown function $f(t)$, how can we find $f(t)$? This is essentially concerned with the solution of the integral equation

$$\int_0^\infty e^{-st} f(t)dt = \bar{f}(s). \tag{3.7.1}$$

At this stage, it is rather difficult to handle the problem as it is. However, in simple cases, we can find the inverse transform from Table B-4 of Laplace transforms. For example

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1, \quad \mathscr{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos\ at.$$

In general, the inverse Laplace transform can be determined by using four methods: (i) Partial Fraction Decomposition, (ii) the Convolution Theorem, (iii) Contour Integration of the Laplace Inversion Integral, and (iv) Heaviside's Expansion Theorem.

(i) *Partial Fraction Decomposition Method*
If

$$\bar{f}(s) = \frac{\bar{p}(s)}{\bar{q}(s)}, \tag{3.7.2}$$

where $\bar{p}(s)$ and $\bar{q}(s)$ are polynomials in $s$, and the degree of $\bar{p}(s)$ is less than that of $\bar{q}(s)$, the method of partial fractions may be used to express $\bar{f}(s)$ as the sum of terms which can be inverted by using a table of Laplace transforms. We illustrate the method by means of simple examples.

***Example 3.7.1***
To find

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s-a)}\right\},$$

where $a$ *is* a constant, we write

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{s(s-a)}\right\} &= \mathscr{L}^{-1}\left[\frac{1}{a}\left\{\frac{1}{s-a}-\frac{1}{s}\right\}\right]\\ &= \frac{1}{a}\left[\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{s}\right\}\right]\\ &= \frac{1}{a}(e^{at}-1).\end{aligned}$$

□

***Example 3.7.2***
Show that

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\} = \frac{1}{b^2-a^2}\left(\frac{\sin at}{a} - \frac{\sin bt}{b}\right).$$

We write

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\} = \frac{1}{b^2-a^2}\left[\mathscr{L}^{-1}\left\{\frac{1}{s^2+a^2}-\frac{1}{s^2+b^2}\right\}\right]$$
$$= \frac{1}{(b^2-a^2)}\left(\frac{\sin at}{a}-\frac{\sin bt}{b}\right).$$

▯

***Example 3.7.3***
Find

$$\mathscr{L}^{-1}\left\{\frac{s+7}{s^2+2s+5}\right\}.$$

We have

$$\mathscr{L}^{-1}\left\{\frac{s+7}{(s+1)^2+4}\right\} = \mathscr{L}^{-1}\left\{\frac{s+1+6}{(s+1)^2+2^2}\right\}$$
$$= \mathscr{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\} + 3\mathscr{L}^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\}$$
$$= e^{-t}\cos 2t + 3e^{-t}\sin 2t.$$

▯

***Example 3.7.4***
Evaluate the following inverse Laplace transform

$$\mathscr{L}^{-1}\left\{\frac{2s^2+5s+7}{(s-2)(s^2+4s+13)}\right\}.$$

We have

$$\mathscr{L}^{-1}\left\{\frac{2s^2+5s+7}{(s-2)(s^2+4s+13)}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s-2}+\frac{s+2}{(s+2)^2+3^2}+\frac{1}{(s+2)^2+3^2}\right\}$$
$$= \mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} + \mathscr{L}^{-1}\left\{\frac{s+2}{(s+2)^2+3^2}\right\}$$
$$+\frac{1}{3}\mathscr{L}^{-1}\left\{\frac{3}{(s+2)^2+3^2}\right\}$$
$$= e^{2t} + e^{-2t}\cos 3t + \frac{1}{3}\,e^{-2t}\sin 3t.$$

▯

(ii) *Convolution Theorem*

We shall apply the convolution theorem for calculation of inverse Laplace transforms.

***Example 3.7.5***

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s-a)}\right\} = 1 * e^{at} = \int_0^t e^{a\tau}\,d\tau = \frac{(e^{at}-1)}{a}. \qquad \square$$

***Example 3.7.6***

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\} &= t * \frac{\sin at}{a} \\ &= \frac{1}{a}\int_0^t (t-\tau)\sin a\tau\; d\tau \\ &= \frac{t}{a}\int_0^t \sin a\tau\, d\tau - \frac{1}{a}\int_0^t \tau \sin a\tau\, d\tau \\ &= \frac{1}{a^2}\left(t - \frac{1}{a}\sin at\right).\end{aligned}$$

$\square$

***Example 3.7.7***

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} &= \frac{\sin at}{a} * \frac{\sin at}{a} \\ &= \frac{1}{a^2}\int_0^t \sin a\tau \sin a(t-\tau)d\tau \\ &= \frac{1}{2a^3}(\sin at - at\cos at).\end{aligned}$$

$\square$

***Example 3.7.8***

$$
\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}(s-a)}\right\} &= \frac{1}{\sqrt{\pi t}} * e^{at}, (a>0)\\
&= \frac{1}{\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{\tau}}\, e^{a(t-\tau)} d\tau\\
&= \frac{2\,e^{at}}{\sqrt{\pi a}}\int_0^{\sqrt{at}} e^{-x^2} dx, \quad \left(\text{putting } \sqrt{a\tau}=x\right)\\
&= \frac{e^{at}}{\sqrt{a}}\, erf(\sqrt{at}). \qquad (3.7.3)
\end{aligned}
$$

□

***Example 3.7.9***
Show that

$$
\mathscr{L}^{-1}\left\{\frac{1}{s}\, e^{-a\sqrt{s}}\right\} = erfc\left(\frac{a}{2\sqrt{t}}\right). \qquad (3.7.4)
$$

In view of Example 3.6.2(b), and the Convolution Theorem 3.5.1, we obtain

$$
\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{1}{s}\, e^{-a\sqrt{s}}\right\} &= 1 * \frac{a}{2}\frac{e^{-a^2/4t}}{\sqrt{\pi t^3}}\\
&= \frac{a}{2\sqrt{\pi}}\int_0^t \frac{e^{-a^2/4\tau}}{\tau^{3/2}} d\tau,
\end{aligned}
$$

which is, by putting $\frac{a}{2\sqrt{\tau}} = x$,

$$
= \frac{2}{\sqrt{\pi}}\int_{\frac{a}{2\sqrt{t}}}^{\infty} e^{-x^2} dx = erfc\left(\frac{a}{2\sqrt{t}}\right).
$$

□

***Example 3.7.10***
Show that

$$
\mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}+a}\right\} = \frac{1}{\sqrt{\pi t}} - a\exp(ta^2)\, erfc(a\sqrt{t}). \qquad (3.7.5)
$$

We have

$$\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}+a}\right\} &= \mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}} - \frac{a}{\sqrt{s}(\sqrt{s}+a)}\right\} \\
&= \mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}}\right\} - a\mathscr{L}^{-1}\left\{\frac{\sqrt{s}-a}{\sqrt{s}(s-a^2)}\right\} \\
&= \mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}}\right\} - a\mathscr{L}^{-1}\left\{\frac{1}{s-a^2}\right\} + a^2\mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}(s-a^2)}\right\} \\
&= \frac{1}{\sqrt{\pi t}} - a\exp(a^2 t) + a\exp(a^2 t)\, erf(a\sqrt{t}), \qquad \text{by (3.7.3)} \\
&= \frac{1}{\sqrt{\pi t}} - a\exp(a^2 t)\, erfc(a\sqrt{t}).
\end{aligned}$$

□

***Example 3.7.11***

If $f(t) = \mathscr{L}^{-1}\{\bar{f}(s)\}$, then

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\,\bar{f}(s)\right\} = \int_0^t f(x)dx. \tag{3.7.6}$$

We have, by the Convolution Theorem with $g(t) = 1$ so that $\bar{g}(s) = \frac{1}{s}$,

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\,\bar{f}(s)\right\} = \int_0^t f(t-\tau)d\tau,$$

which is, by putting $t - \tau = x$,

$$= \int_0^t f(x)dx.$$

(iii) *Contour Integration of the Laplace Inversion Integral*

In Section 3.2, in inverse Laplace transform is defined by the complex integral formula

$$\mathscr{L}^{-1}\{\bar{f}(s)\} = f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\bar{f}(s)ds, \tag{3.7.7}$$

where $c$ is a suitable real constant and $\bar{f}(s)$ is an analytic function of the complex variable $s$ in the right half-plane Re $s > a$.

The details of evaluation of (3.7.7) depend on the nature of the singularities of $\bar{f}(s)$. Usually, $\bar{f}(s)$ is a single valued function with a finite or enumerably infinite number of polar singularities. Often it has branch points. The path of integration is the straight line $L$ (see Figure 3.4(a)) in the complex $s$-plane with equation $s = c + iR$, $-\infty < R < \infty$, Re $s = c$ being chosen so that all the singularities of the integrand of (3.7.7) lie to the left of the line $L$. This line is called by *Bromwich Contour*. In practice, the Bromwich Contour is closed by an arc of a circle of radius $R$ as shown in Figure 3.4(a), and then the limit as $R \to \infty$ is taken to expand the contour of integration to infinity so that all the singularities of $\bar{f}(s)$ lie inside the contour of integration.

When $\bar{f}(s)$ has a branch point at the origin, we draw the modified contour of integration by making a cut along the negative real axis and a small semicircle $\gamma$ surrounding the origin as shown in Figure 3.4(b).

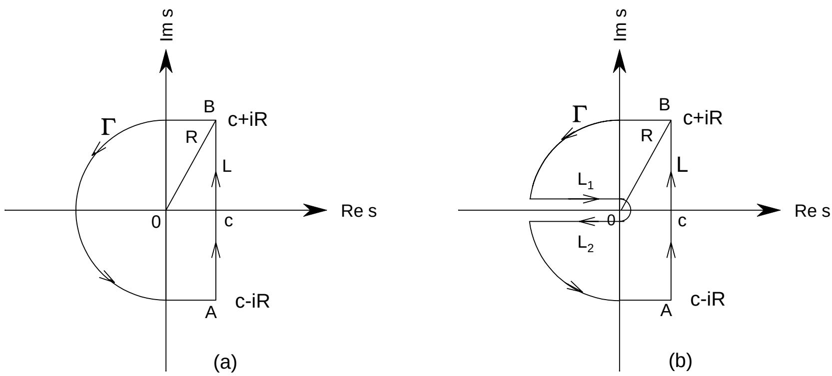


**Figure 3.4** The Bromwich contour and the contour of integration.

In either case, the Cauchy Residue Theorem is used to evaluate the integral

$$\int_L e^{st}\bar{f}(s)ds + \int_\Gamma e^{st}\bar{f}(s)ds = \int_C e^{st}\bar{f}(s)ds$$
$$= 2\pi i \times [\text{sum of the residues of } e^{st}\bar{f}(s) \text{ at the poles inside } C]. \quad (3.7.8)$$

Letting $R \to \infty$, the integral over $\Gamma$ tends to zero, and this is true in most problems of interest. Consequently, result (3.7.7) reduces to the form

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{c-iR}^{c+iR} e^{st}\bar{f}(s)ds = \text{sum of the residues of } e^{st}\bar{f}(s) \text{ at the poles of } \bar{f}(s). \quad (3.7.9)$$

We illustrate the above method of evaluation by simple examples. □

***Example 3.7.12***
If $\bar{f}(s) = \frac{s}{s^2+a^2}$, show that

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\bar{f}(s)ds = \cos at.$$

Clearly, the integrand has two simple poles at $s = \pm ia$ and the residues at these poles are

$$\begin{aligned} R_1 &= \text{Residue of } e^{st}\bar{f}(s) \text{ at } s = ia \\ &= \lim_{s\to ia}(s - ia)\frac{s\,e^{st}}{(s^2+a^2)} = \frac{1}{2}e^{iat}. \end{aligned}$$

$$\begin{aligned} R_2 &= \text{Residue of } e^{st}\bar{f}(s) \text{ at } s = -ia \\ &= \lim_{s\to -ia}(s + ia)\frac{s\,e^{st}}{(s^2+a^2)} = \frac{1}{2}e^{-iat}. \end{aligned}$$

Hence,

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\bar{f}(s)ds = R_1 + R_2 = \frac{1}{2}(e^{iat} + e^{-iat}) = \cos at,$$

as obtained earlier.

If $\bar{g}(s) = e^{st}\bar{f}(s)$ has a pole of order $n$ at $s = z$, then the residue $R_1$ of $\bar{g}(s)$ at this pole is given by the formula

$$R_1 = \lim_{s\to z}\frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}[(s-z)^n\bar{g}(s)]. \tag{3.7.10}$$

This is obviously true for a simple pole $(n = 1)$ and for a double pole $(n = 2)$.
▯

***Example 3.7.13***
Evaluate

$$\mathscr{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}.$$

Clearly

$$\bar{g}(s) = e^{st}\bar{f}(s) = \frac{s\,e^{st}}{(s^2+a^2)^2}$$

has double poles at $s = \pm ia$. The residue formula (3.7.10) for double poles gives

$$\begin{aligned} R_1 &= \lim_{s\to ia}\frac{d}{ds}\left[(s-ia)^2\frac{s\,e^{st}}{(s^2+a^2)^2}\right] \\ &= \lim_{s\to ia}\frac{d}{ds}\left[\frac{s\,e^{st}}{(s+ia)^2}\right] = \frac{t\,e^{iat}}{4ia}. \end{aligned}$$

Similarly, the residue at the double pole at $s=-ia$ is $(-t\,e^{-iat})/4ia$.

Thus,

$$f(t)=\text{Sum of the residues}=\frac{t}{4ia}(e^{iat}-e^{-iat})=\frac{t}{2a}\sin at, \tag{3.7.11}$$

as given in Table B-4 of Laplace transforms. ▯

***Example 3.7.14***

Evaluate

$$\mathscr{L}^{-1}\left\{\frac{\cosh(\alpha x)}{s\cosh(\alpha\ell)}\right\}, \qquad \alpha=\sqrt{\frac{s}{a}}.$$

We have

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\,\frac{\cosh(\alpha x)}{\cosh(\alpha\ell)}\,\frac{ds}{s}.$$

Clearly, the integrand has simple poles at $s=0$ and $s=s_n=-(2n+1)^2\dfrac{a\pi^2}{4\ell^2}$, where $n=$ 0,1,2,....

$R_1=$ Residue at the pole $s=0$ is 1, and $R_n=$ Residue at the pole $s=s_n$ is

$$\frac{\exp(-s_n t)\cosh\left\{i(2n+1)\dfrac{\pi x}{2\ell}\right\}}{\left[s\dfrac{d}{ds}\left\{\cosh l\sqrt{\dfrac{s}{a}}\right\}\right]_{s=s_n}}$$
$$=\frac{4(-1)^{n+1}}{(2n+1)\pi}\exp\left[-\left\{\frac{(2n+1)\pi}{2\ell}\right\}^2 at\right]\cos\left\{(2n+1)\frac{\pi x}{2\ell}\right\}.$$

Thus,

$$\begin{aligned} f(t) &= \text{Sum of the residues at the poles}\\ &= 1+\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)}\exp\left[-(2n+1)^2\frac{\pi^2 at}{4\ell^2}\right]\\ &\qquad\qquad \times\cos\left\{(2n+1)\frac{\pi x}{2\ell}\right\}, \end{aligned} \tag{3.7.12}$$

as given later by the Heaviside Expansion Theorem. ▯

***Example 3.7.15***

Show that

$$f(t) = \mathscr{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{s}\exp(st - a\sqrt{s})ds$$
$$= erfc\left(\frac{a}{2\sqrt{t}}\right). \tag{3.7.13}$$

The integrand has a branch point at $s = 0$. We use the contour of integration as shown in Figure 3.4(b) which excludes the branch point at $s = 0$. Thus, the Cauchy Fundamental Theorem gives

$$\frac{1}{2\pi i}\left[\int_L + \int_\Gamma + \int_{L_1} + \int_{L_2} + \int_\gamma\right]\exp(st - a\sqrt{s})\,\frac{ds}{s} = 0. \tag{3.7.14}$$

It is shown that the integral on $\Gamma$ tends to zero as $R \to \infty$, and that on $L$ gives the Bromwich integral. We now evaluate the remaining three integrals in (3.7.14). On $L_1$, we have $s = re^{i\pi} = -r$ and

$$\int_{L_1}\exp(st - a\sqrt{s})\,\frac{ds}{s} = \int_{-\infty}^{0}\exp(st - a\sqrt{s})\,\frac{ds}{s} = -\int_0^\infty \exp\{-(rt + ia\sqrt{r})\}\,\frac{dr}{r}.$$

On $L_2$, $s = re^{-i\pi} = -r$ and

$$\int_{L_2}\exp(st - a\sqrt{s})\frac{ds}{s} = \int_0^{-\infty}\exp(st - a\sqrt{s})\,\frac{ds}{s} = \int_0^\infty \exp\{-rt + ia\sqrt{r}\}\,\frac{dr}{r}.$$

Thus, the integrals along $L_1$ and $L_2$ combined yield

$$-2i\int_0^\infty e^{-rt}\sin(a\sqrt{r})\,\frac{dr}{r} = -4i\int_0^{-\infty} e^{-x^2 t}\,\frac{\sin ax}{x}dx, \qquad (\sqrt{r} = x). \tag{3.7.15}$$

Integrating the following standard integral with respect to $\beta$

$$\int_0^\infty e^{-x^2\alpha^2}\cos(2\beta x)dx = \frac{\sqrt{\pi}}{2\alpha}\exp\left(-\frac{\beta^2}{\alpha^2}\right), \tag{3.7.16}$$

we obtain

$$\frac{1}{2}\int_0^\infty e^{-x^2\alpha^2}\,\frac{\sin 2\beta x}{x}dx = \frac{\sqrt{\pi}}{2\alpha}\int_0^\beta \exp\left(-\frac{\beta^2}{\alpha^2}\right)d\beta$$

$$= \frac{\sqrt{\pi}}{2}\int_0^{\beta/\alpha} e^{-u^2}\,du, \qquad (\beta=\alpha u)$$

$$= \frac{\pi}{4}\,erf\left(\frac{\beta}{\alpha}\right). \tag{3.7.17}$$

In view of (3.7.17), result (3.7.15) becomes

$$-4i\int_0^\infty \exp(-tx^2)\,\frac{\sin ax}{x}\,dx = -2\pi i\; erf\left(\frac{a}{2\sqrt{t}}\right). \tag{3.7.18}$$

Finally, on $\gamma$, we have $s=re^{i\theta}$, $ds=ire^{i\theta}d\theta$, and

$$\int_\gamma |\exp(st-a\sqrt{s})|\frac{ds}{s} = i\int_\pi^{-\pi}\exp\left(rt\cos\theta - a\sqrt{r}\cos\frac{\theta}{2}\right)d\theta$$

$$= i\int_{-\pi}^{\pi} d\theta = 2\pi i, \tag{3.7.19}$$

in which the limit as $r\to 0$ is used and integration from $\pi$ to $-\pi$ is interchanged to make $\gamma$ in the counterclockwise direction.

Thus, the final result follows from (3.7.14), (3.7.18), and (3.7.19) in the form

$$\mathscr{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp(st-a\sqrt{s})\,\frac{ds}{s}$$

$$= \left[1 - erf\left(\frac{a}{2\sqrt{t}}\right)\right] = erfc\left(\frac{a}{2\sqrt{t}}\right).$$

(iv) *Heaviside's Expansion Theorem*

Suppose $\bar{f}(s)$ is the Laplace transform of $f(t)$, which has a Maclaurin power series expansion in the form

$$f(t)=\sum_{r=0}^\infty a_r\,\frac{t^r}{r!}. \tag{3.7.20}$$

Taking the Laplace transform, it is possible to write formally

$$\bar{f}(s)=\sum_{r=0}^\infty \frac{a_r}{s^{r+1}}. \tag{3.7.21}$$

Conversely, we can derive (3.7.20) from a given expansion (3.7.21). This kind of expansion is useful for determining the behavior of the solution for small time. Further, it provides an alternating way to prove the Tauberian theorems. ▯

***THEOREM 3.7.1***

(*Heaviside's Expansion Theorem*). If $\bar{f}(s)=\dfrac{\bar{p}(s)}{\bar{q}(s)}$, where $\bar{p}(s)$ and $\bar{q}(s)$ are polynomials in $s$ and the degree of $\bar{q}(s)$ is higher than that of $\bar{p}(s)$, then

$$\mathscr{L}^{-1}\left\{\frac{\bar{p}(s)}{\bar{q}(s)}\right\}=\sum_{k=1}^{n}\frac{\bar{p}(\alpha_k)}{\bar{q}'(\alpha_k)}\exp(t\alpha_k), \tag{3.7.22}$$

where $\alpha_k$ are the distinct roots of the equation $\bar{q}(s)=0$.

**PROOF** Without loss of generality, we can assume that the leading coefficient of $\bar{q}(s)$ is unity and write distinct factors of $\bar{q}(s)$ so that

$$\bar{q}(s)=(s-\alpha_1)(s-\alpha_2)\cdots(s-\alpha_k)\cdots(s-\alpha_n). \tag{3.7.23}$$

Using the rules of partial fraction decomposition, we can write

$$\bar{f}(s)=\frac{\bar{p}(s)}{\bar{q}(s)}=\sum_{k=1}^{n}\frac{A_k}{(s-\alpha_k)}, \tag{3.7.24}$$

where $A_k$ are arbitrary constants to be determined. In view of (3.7.23), we find

$$\bar{p}(s)=\sum_{k=1}^{n}A_k(s-\alpha_1)(s-\alpha_2)\cdots(s-\alpha_{k-1})(s-\alpha_{k+1})\cdots(s-\alpha_n).$$

Substitution of $s=\alpha_k$ gives

$$\bar{p}(\alpha_k)=A_k(\alpha_k-\alpha_1)(\alpha_k-\alpha_2)\cdots(\alpha_k-\alpha_{k+1})\cdots(\alpha_k-\alpha_n), \tag{3.7.25}$$

where $k=1,2,3,\ldots,n$.

Differentiation of (3.7.23) yields

$$\bar{q}'(s)=\sum_{k=1}^{n}(s-\alpha_1)(s-\alpha_2)\cdots(s-\alpha_{k-1})(s-\alpha_{k+1})\cdots(s-\alpha_n),$$

whence it follows that

$$\bar{q}'(\alpha_k)=(\alpha_k-\alpha_1)(\alpha_k-\alpha_2)\cdots(\alpha_k-\alpha_{k-1})(\alpha_k-\alpha_{k+1})\cdots(\alpha_k-\alpha_n). \tag{3.7.26}$$

From (3.7.25) and (3.7.26), we find

$$A_k = \frac{\bar{p}(\alpha_k)}{\bar{q}'(\alpha_k)},$$

and hence,

$$\frac{\bar{p}(s)}{\bar{q}(s)} = \sum_{k=1}^{n} \frac{\bar{p}(\alpha_k)}{\bar{q}'(\alpha_k)} \frac{1}{(s-\alpha_k)}. \tag{3.7.27}$$

Inversion gives immediately

$$\mathscr{L}^{-1}\left\{\frac{\bar{p}(s)}{\bar{q}(s)}\right\} = \sum_{k=1}^{n} \frac{\bar{p}(\alpha_k)}{\bar{q}'(\alpha_k)} \exp(t\alpha_k).$$

This proves the theorem. We give some examples of this theorem. ∎

***Example 3.7.16***

We consider

$$\mathscr{L}^{-1}\left\{\frac{s}{s^2-3s+2}\right\}.$$

Here $\bar{p}(s) = s$, and $\bar{q}(s) = s^2 - 3s + 2 = (s-1)(s-2)$. Hence,

$$\mathscr{L}^{-1}\left\{\frac{s}{s^2-3s+2}\right\} = \frac{\bar{p}(2)}{\bar{q}'(2)} e^{2t} + \frac{\bar{p}(1)}{\bar{q}'(1)} e^{t} = 2\,e^{2t} - e^{t}.$$

□

***Example 3.7.17***

Use Heaviside's power series expansion to evaluate

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\frac{\sinh x\sqrt{s}}{\sinh\sqrt{s}}\right\}, \qquad 0 < x < 1, \qquad s > 0.$$

We have

$$\begin{aligned}
\frac{1}{s}\frac{\sinh x\sqrt{s}}{\sinh\sqrt{s}} &= \frac{1}{s}\left(\frac{e^{x\sqrt{s}} - e^{-x\sqrt{s}}}{e^{\sqrt{s}} - e^{-\sqrt{s}}}\right) \\
&= \frac{1}{s}\frac{e^{-(1-x)\sqrt{s}} - e^{-(1+x)\sqrt{s}}}{1 - e^{-2\sqrt{s}}} \\
&= \frac{1}{s}\left[e^{-(1-x)\sqrt{s}} - e^{-(1+x)\sqrt{s}}\right]\left(1 - e^{-2\sqrt{s}}\right)^{-1} \\
&= \frac{1}{s}\left[e^{-(1-x)\sqrt{s}} - e^{-(1+x)\sqrt{s}}\right]\sum_{n=0}^{\infty}\exp(-2n\sqrt{s}) \\
&= \frac{1}{s}\sum_{n=0}^{\infty}\left[\exp\{-(1-x+2n)\sqrt{s}\} - \exp\{-(1+x+2n)\sqrt{s}\}\right].
\end{aligned}$$

Hence,

$$\begin{aligned}\mathscr{L}^{-1} &\left\{\frac{1}{s}\frac{\sinh x\sqrt{s}}{\sinh\sqrt{s}}\right\}\\ &= \mathscr{L}^{-1}\left\{\frac{1}{s}\sum_{n=0}^{\infty}\left[\exp\{-(1-x+2n)\sqrt{s}\} - \exp\{-(1+x+2n)\sqrt{s}\}\right]\right\}\\ &= \sum_{n=0}^{\infty}\left[\mathit{erfc}\left(\frac{1-x+2n}{2\sqrt{t}}\right) - \mathit{erfc}\left(\frac{1+x+2n}{2\sqrt{t}}\right)\right].\end{aligned}$$

□

***Example 3.7.18***

If $\alpha = \sqrt{\dfrac{s}{a}}$, show that

$$\mathscr{L}^{-1}\left[\frac{\cosh\alpha x}{s\cosh\alpha\ell}\right] = 1 - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k\cos\left\{\left(k+\frac{1}{2}\right)\frac{\pi x}{\ell}\right\}\exp\left[-(2k+1)^2\frac{a\pi^2 t}{4\ell^2}\right]}{(2k+1)}. \tag{3.7.28}$$

In this case, we write

$$\mathscr{L}^{-1}\{\bar{f}(s)\} = \mathscr{L}^{-1}\left\{\frac{\bar{p}(s)}{\bar{q}(s)}\right\} = \mathscr{L}^{-1}\left\{\frac{\cosh\alpha x}{s\cosh\alpha\ell}\right\}.$$

Clearly, the zeros of $\bar{f}(s)$ are at $s=0$ and at the roots of $\cosh\alpha\ell = 0$, that is, at $s = s_k = a\left(k+\frac{1}{2}\right)^2\left(\frac{\pi i}{\ell}\right)^2$, $k = 0, 1, 2, \ldots$. Thus,

$$\alpha_k = \sqrt{\frac{s_k}{a}} = \left(k+\frac{1}{2}\right)\frac{\pi i}{\ell}, \qquad k = 0, 1, 2, \ldots.$$

Here $\bar{p}(s) = \cosh(\alpha x)$, $\bar{q}(s) = s\cosh(\alpha\ell)$. In order to apply the Heaviside Expansion Theorem, we need

$$\bar{q}'(s) = \frac{d}{ds}(s\cosh\alpha\ell) = \cosh(\alpha\ell) + \frac{1}{2}\alpha\ell\sinh(\alpha\ell).$$

For the zero $s=0$, $\bar{q}'(0) = 1$, and for the zeros at $s = s_k$,

$$\begin{aligned}\bar{q}'(s_k) &= \frac{1}{2}\left(k+\frac{1}{2}\right)\pi i\cdot\sinh\left[\left(k+\frac{1}{2}\right)\pi i\right]\\ &= (2k+1)\frac{\pi i}{4}\cdot i\sin\left[\left(k+\frac{1}{2}\right)\pi\right]\\ &= -(2k+1)\frac{\pi}{4}\cdot\cos k\pi = (-1)^{k+1}(2k+1)\frac{\pi}{4}.\end{aligned}$$

Consequently,

$$\mathscr{L}^{-1}\left\{\frac{\cosh \alpha x}{s\cosh \alpha \ell}\right\} = 1 + \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(2k+1)}\cosh\left[(2k+1)\frac{\pi i x}{2\ell}\right]\exp(ts_k)$$

$$= 1 - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{(2k+1)}\cos\left[(2k+1)\frac{\pi x}{2\ell}\right]$$

$$\times \exp\left[-\left(k+\frac{1}{2}\right)^2\frac{\pi^2 a t}{\ell^2}\right].$$

□

## 3.8 Tauberian Theorems and Watson's Lemma

These theorems give the behavior of object functions in terms of the behavior of transform functions. Particularly, they determine the value of the object functions $f(t)$ for large and small values of time $t$. Tauberian theorems are extremely useful and have frequent applications.

***THEOREM 3.8.1*** **(***The Initial Value Theorem***).**
If $\mathscr{L}\{f(t)\} = \bar{f}(s)$ exists, then

$$\lim_{s\to\infty}\bar{f}(s) = 0. \tag{3.8.1}$$

In addition, if $f(t)$ and its derivatives exist as $t \to 0$, we obtain the *Initial Value Theorem*:

$$(i)\quad \lim_{s\to\infty}[s\bar{f}(s)] = \lim_{t\to 0} f(t) = f(0) \tag{3.8.2}$$

$$(ii)\quad \lim_{s\to\infty}[s^2\bar{f}(s) - sf(0)] = \lim_{t\to 0} f'(t) = f'(0), \quad \text{and} \tag{3.8.3}$$

$$(iii)\quad \lim_{s\to\infty}[s^{n+1}\bar{f}(s) - s^n\bar{f}(s) - \cdots - sf^{(n-1)}(0)] = f^{(n)}(0). \tag{3.8.4}$$

Results (3.8.2)–(3.8.4), which are true under fairly general conditions, determine the initial values $f(0), f'(0), \ldots, f^{(n)}(0)$ of the function $f(t)$ and its derivatives from the Laplace transform $\bar{f}(s)$.

**PROOF** To prove (3.8.1), we use the fact that the Laplace integral (3.2.5) is uniformly convergent with respect to the parameter $s$. Hence, it is permissible to take the limit $s \to \infty$ under the sign of integration so that

$$\lim_{s\to\infty} \bar{f}(s) = \int_0^\infty (\lim_{s\to\infty} e^{-st}) f(t)dt = 0.$$

Next, we use the same argument to obtain

$$\lim_{s\to\infty} \mathscr{L}\{f'(t)\} = \int_0^\infty (\lim_{s\to\infty} e^{-st}) f'(t)dt = 0.$$

Then it follows from result (3.4.10) that

$$\lim_{s\to\infty} [s\bar{f}(s) - f(0)] = 0,$$

and hence, we obtain (3.8.2), that is,

$$\lim_{s\to\infty} [s\bar{f}(s)] = f(0) = \lim_{t\to 0} f(t).$$

A similar argument combined with Theorem 3.4.2 leads to (3.8.3) and (3.8.4). ■

***Example 3.8.1***

Verify the truth of Theorem 3.8.1 for $\bar{f}(s) = (n+1)!\, s^{-(n+1)}$ where $n$ is a positive integer. Clearly, $f(t) = t^n$. Thus, we have

$$\begin{aligned}\lim_{s\to\infty} \bar{f}(s) &= \lim_{s\to\infty} \frac{(n+1)!}{s^{n+1}} = 0,\\ \lim_{s\to\infty} s\bar{f}(s) &= 0 = f(0).\end{aligned}$$

□

***Example 3.8.2***

Find $f(0)$ and $f'(0)$ when

(a) $\bar{f}(s) = \dfrac{1}{s(s^2+a^2)}$, (b) $\bar{f}(s) = \dfrac{2s}{s^2-2s+5}$.

(a) It follows from (3.8.2) and (3.8.3) that

$$\begin{aligned}f(0) &= \lim_{s\to\infty} [s\bar{f}(s)] = \lim_{s\to\infty} \frac{1}{s^2+a^2} = 0.\\ f'(0) &= \lim_{s\to\infty} [s^2\bar{f}(s) - sf(0)] = \lim_{s\to\infty} \frac{s}{s^2+a^2} = 0.\end{aligned}$$

(b)
$$f(0) = \lim_{s\to\infty} \frac{2s^2}{s^2-2s+5} = 2.$$
$$f'(0) = \lim_{s\to\infty} [s^2\bar{f}(s) - sf(0)] = \lim_{s\to\infty} \left[\frac{2s^3}{s^2-2s+5} - 2s\right] = 4.$$

□

***THEOREM 3.8.2*** **(***The Final Value Theorem***).**
If $\bar{f}(s) = \dfrac{\bar{p}(s)}{\bar{q}(s)}$, where $\bar{p}(s)$ and $\bar{q}(s)$ are polynomials in $s$, and the degree of $\bar{p}(s)$ is less than that of $\bar{q}(s)$, and if all roots of $\bar{q}(s)=0$ have negative real parts with the possible exception of one root which may be at $s=0$, then

$$\text{(i)}\quad \lim_{s\to 0} \bar{f}(s) = \int_0^\infty f(t)dt, \qquad \text{and} \tag{3.8.5}$$

$$\text{(ii)}\quad \lim_{s\to 0}[s\bar{f}(s)] = \lim_{t\to\infty} f(t), \tag{3.8.6}$$

provided the limits exist.

Result (3.8.6) is true under more general conditions, and known as the *Final Value Theorem*. This theorem determines the final value of $f(t)$ at infinity from its Laplace transform at $s=0$. However, if $\bar{f}(s)$ is more general than the rational function as stated above, a statement of a more general theorem is needed with appropriate conditions under which it is valid.

**PROOF** To prove (i), we use the same argument as employed in Theorem 3.8.1 and find

$$\lim_{s\to 0} \bar{f}(s) = \int_0^\infty \left(\lim_{s\to 0} \exp(-st)\right) f(t)dt = \int_0^\infty f(t)\,dt.$$

As before, we can use result (3.4.12) to obtain

$$\lim_{s\to 0} \mathscr{L}\{f'(t)\} = \lim_{s\to 0}[s\bar{f}(s) - f(0)] = \int_0^\infty \left(\lim_{s\to 0} \exp(-st)\right) f'(t)dt$$
$$= \int_0^\infty f'(t)dt = f(\infty) - f(0) = \lim_{t\to\infty}[f(t) - f(0)].$$

Thus, it follows immediately that

$$\lim_{s\to 0}[s\bar{f}(s)] = \lim_{t\to\infty} f(t) = f(\infty).$$

■

***Example 3.8.3***

Find $f(\infty)$, if it exists, from the following functions:

(a) $\bar{f}(s) = \dfrac{1}{s(s^2+2s+2)}$, (b) $\bar{f}(s) = \dfrac{1}{s-a}$,

(c) $\bar{f}(s) = \dfrac{s+a}{s^2+b^2}, (b \neq 0)$, (d) $\bar{f}(s) = \dfrac{s}{s-2}$.

(a) Clearly, $\bar{q}(s) = 0$ has roots at $s = 0$ and $s = -1 \pm i$, and the conditions of Theorem 3.8.2 are satisfied. Thus,

$$\lim_{s\to 0}[s\bar{f}(s)] = \lim_{s\to 0} \frac{1}{s^2+2s+2} = \frac{1}{2} = f(\infty).$$

(b) Here $\bar{q}(s) = 0$ has a real positive root at $s = a$ if $a > 0$, and a real negative root if $a < 0$. Thus, when $a < 0$

$$\lim_{s\to 0}[s\bar{f}(s)] = \lim_{s\to 0} \frac{s}{s-a} = 0 = f(\infty).$$

If $a > 0$, the Final Value Theorem does not apply. In fact,

$$f(t) = \mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \to \infty \qquad \text{as } t \to \infty.$$

(c) Here $\bar{q}(s) = 0$ has purely imaginary roots at $s = \pm ib$ which do not have negative real parts. The Final Value Theorem does not apply. In fact, $f(t) = \cos bt + \frac{a}{b}\sin bt$ and $\lim_{t\to\infty} f(t)$ does not exist. However, $f(t)$ is bounded and oscillatory for all $t > 0$.

(d) The Final Value Theorem does not apply as $\bar{q}(s) = 0$ has a positive root at $s = 2$.

□

***Watson's Lemma.*** If (i) $f(t) = O(e^{at})$ as $t \to \infty$, that is, $|f(t)| \le K\,\exp(at)$ for $t > T$ where $K$ and $T$ are constants, and (ii) $f(t)$ has the expansion

$$f(t) = t^{\alpha}\left[\sum_{r=0}^{n} a_r t^r + R_{n+1}(t)\right] \text{ for } 0 < t < T \text{ and } \alpha > -1, \tag{3.8.7}$$

where $|R_{n+1}(t)| < A\,t^{n+1}$ for $0 < t < T$ and $A$ is a constant, then the Laplace transform $\bar{f}(s)$ has the *asymptotic expansion*

$$\bar{f}(s) \sim \sum_{r=0}^{n} a_r \frac{\Gamma(\alpha+r+1)}{s^{\alpha+r+1}} + O\left(\frac{1}{s^{\alpha+n+2}}\right) \qquad \text{as } s \to \infty. \tag{3.8.8}$$

**PROOF** We have, for $s > a$,

$$\begin{aligned}\bar{f}(s) &= \int_0^T e^{-st} f(t)dt + \int_T^\infty e^{-st} f(t)dt \\ &= \int_0^T e^{-st}\, t^\alpha \left(\sum_{r=0}^n a_r t^r\right) dt + \int_0^T e^{-st}\, t^\alpha R_{n+1}(t)dt \\ &\qquad + \int_T^\infty e^{-st} f(t)dt. \qquad (3.8.9)\end{aligned}$$

The general term of the first integral in (3.8.9) can be written as

$$\begin{aligned}\int_0^T a_r\, e^{-st} t^{\alpha+r} dt &= \int_0^\infty a_r\, e^{-st} t^{\alpha+r} dt - \int_T^\infty a_r\, e^{-st} t^{\alpha+r} dt \\ &= a_r \frac{\Gamma(\alpha+r+1)}{s^{\alpha+r+1}} + O(e^{-Ts}). \qquad (3.8.10)\end{aligned}$$

As $s \to \infty$, the second integral in(3.8.9) is less in magnitude than

$$A \int_0^T e^{-st} t^{\alpha+n+1} dt = O\left(\frac{1}{s^{\alpha+n+2}}\right), \qquad (3.8.11)$$

and the magnitude of the third integral in (3.8.9) is

$$\left|\int_T^\infty e^{-st} f(t)dt\right| \le K \int_T^\infty e^{-(s-a)t} dt = K \exp[-(s-a)T], \qquad (3.8.12)$$

which is exponentially small as $s \to \infty$.

Finally, combining (3.8.10), (3.8.11), and (3.8.12), we obtain

$$\bar{f}(s) \sim \sum_{r=0}^n a_r \frac{\Gamma(\alpha+r+1)}{s^{\alpha+r+1}} + O\left(\frac{1}{s^{\alpha+n+2}}\right) \qquad \text{as } s \to \infty.$$

This completes the proof of Watson's lemma. ▮

This lemma is one of the most widely used methods for finding asymptotic expansions. In order to further expand its applicability, this lemma has subsequently been generalized and its converse has also been proved. The reader is referred to Erdélyi (1956), Copson (1965), Wyman (1964), Watson (1981), Ursell (1990), and Wong (1989).

***Example 3.8.4***
Find the asymptotic expansion of the *parabolic cylinder function* $D_\nu(s)$, which is valid for $\mathrm{Re}(\nu) < 0$, given by

$$D_\nu(s) = \frac{\exp\left(-\frac{s^2}{4}\right)}{\Gamma(-\nu)} \int_0^\infty \exp\left[-\left(st + \frac{t^2}{2}\right)\right] \frac{dt}{t^{\nu+1}}. \tag{3.8.13}$$

To find the asymptotic behavior of $D_\nu(s)$ as $s \to \infty$, we expand $\exp\left(-\frac{1}{2}t^2\right)$ as a power series in $t$ in the form

$$\exp\left(-\frac{1}{2}t^2\right) = \sum_{n=0}^\infty (-1)^n \frac{t^{2n}}{2^n\, n!}. \tag{3.8.14}$$

According to Watson's lemma, as $s \to \infty$,

$$\begin{aligned} D_\nu(s) &\sim \frac{\exp\left(-\frac{s^2}{4}\right)}{\Gamma(-\nu)} \sum_{n=0}^\infty \frac{(-1)^n}{2^n n!} \int_0^\infty t^{2n-\nu-1} e^{-st}\, dt \\ &= \frac{\exp\left(-\frac{s^2}{4}\right)}{\Gamma(-\nu)} \sum_{n=0}^\infty \frac{(-1)^n}{2^n n!} \frac{\Gamma(2n-\nu)}{s^{2n-\nu}}. \end{aligned} \tag{3.8.15}$$

This result is also valid for $\mathrm{Re}(\nu) \geq 0$. $\square$

## 3.9 Exercises

1. Find the Laplace transforms of the following functions:

   (a) $2t + a\sin at$,
   (b) $(1-2t)\exp(-2t)$,
   (c) $t\cos at$,
   (d) $t^{3/2}$,
   (e) $H(t-3)\exp(t-3)$,
   (f) $H(t-a)\sinh(t-a)$,
   (g) $(t-3)^2 H(t-3)$,
   (h) $t\,H(t-a)$,
   (i) $(1+2\,at)\,t^{-\frac{1}{2}}\exp(at)$,
   (j) $a\,\cos^2\omega t$.

2. If $n$ is a positive integer, show that $\mathscr{L}\{t^{-n}\}$ does not exist.

3. Use result (3.4.12) to find (a) $\mathscr{L}\{\cos at\}$ and (b) $\mathscr{L}\{\sin at\}$.

4. Use the Maclaurin series for $\sin at$ and $\cos at$ to find the Laplace transforms of these functions.

5. Show that $\mathscr{L}\left[\frac{1}{t}\{\exp(-at)-\exp(-bt)\}\right]=\log\left(\frac{s+b}{s+a}\right)$.

6. Show that $\mathscr{L}\left\{\int_0^t \frac{s(u)}{u}du\right\}=\frac{1}{s}\int_s^\infty \bar{f}(x)dx$.

7. Obtain the inverse Laplace transforms of the following functions:

(a) $\frac{s}{(s^2+a^2)(s^2+b^2)}$, (b) $\frac{1}{s^2(s^2+c^2)}$, (c) $\frac{1}{s^2}\exp(-as)$,

(d) $\frac{1}{(s-1)^2(s-2)}$, (e) $\frac{1}{s^2+2s+5}$, (f) $\frac{1}{s^2(s+1)(s+2)}$,

(g) $\frac{1}{s(s-a)^2}$, (h) $\frac{1}{s^2(s-a)^2}$, (i) $\frac{1}{s^2(s-a)}$.

8. Use the Convolution Theorem to find the inverse Laplace transforms of the following functions:

(a) $\frac{s^2}{(s^2+a^2)^2}$, (b) $\frac{1}{s\sqrt{s+4}}$, (c) $\frac{\bar{f}(s)}{s}$,

(d) $\frac{s}{(s^2+a^2)^2}$, (e) $\left(\frac{\omega}{s^2+\omega^2}\right)\bar{f}(s)$, (f) $\frac{1}{(s^2+a^2)^2}$,

(g) $\frac{s}{(s-a)(s^2+b^2)}$, (h) $\frac{1}{(s+1)^2}$, (i) $\frac{1}{s}\exp(-a\sqrt{s})$,

(j) $\frac{1}{s^2(s^2+a^2)}$, (k) $\frac{(s^2-a^2)}{(s^2+a^2)^2}$, (l) $\frac{1}{2}\ln(1+\frac{a^2}{s^2})$.

9. Show that

(a) $\mathscr{L}\{\exp(-t^2)\}=\frac{\sqrt{\pi}}{2}\exp\left(\frac{s^2}{4}\right)\left(1-erf\,\frac{s}{2}\right)$,

(b) $\mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}-\sqrt{a}}\right\}=\sqrt{a}\exp(at)+\frac{1}{\sqrt{\pi t}}+\sqrt{a}\exp(at)erf(\sqrt{at})$,

(c) 
$$\mathscr{L}^{-1}\left\{\frac{\sinh\left(\frac{sx}{a}\right)}{s^2\cosh\left(\frac{sb}{2a}\right)}\right\}=\frac{x}{a}+\sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{4b}{a\pi^2}\right)(2n+1)^{-2}$$
$$\times\left[\sin\left\{(2n+1)\frac{\pi x}{b}\right\}\cos\left\{(2n+1)\frac{\pi at}{b}\right\}\right],$$

(d) $\mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s^2+a^2}}\right\}=\frac{1}{\pi}\int_{-1}^{1}\frac{e^{iatx}}{\sqrt{1-x^2}}dx$.

10. Show that

(a) $\mathscr{L}\left\{\frac{1}{t}(\sin at - at\cos at)\right\} = \tan^{-1}\left(\frac{a}{s}\right) - \frac{as}{s^2+a^2},$

(b) $\mathscr{L}\left\{\int_0^t \frac{1}{\tau}(\sin a\tau - a\tau\cos a\tau)d\tau\right\} = \frac{1}{s}\left[\tan^{-1}\left(\frac{a}{s}\right) - \frac{as}{s^2+a^2}\right].$

11. Using the Heaviside power series expansion, evaluate the inverse Laplace transforms of the following functions:

(a) $\frac{1}{\sqrt{s^2+a^2}},$ (b) $\tan^{-1}\left(\frac{a}{s}\right),$ (c) $\sinh^{-1}\left(\frac{1}{s}\right),$

(d) $\frac{1}{s}\,\text{cosech}\,(x\sqrt{s}),$ (e) $\frac{1}{s}\exp\left(-\frac{1}{s}\right),$ (f) $\sin^{-1}\left(\frac{a}{s}\right).$

12. If $\mathscr{L}\{f(t)\} = \bar{f}(s)$, show that

(i) $\mathscr{L}^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(\tau)d\tau,$

(ii) $\mathscr{L}^{-1}\left\{\frac{\bar{f}(s)}{s^2}\right\} = \int_0^t\left\{\int_0^{t_1} f(\tau)d\tau\right\}dt_1 = \int_0^t (t-\tau)f(\tau)d\tau,$

(iii) $\mathscr{L}^{-1}\left\{\frac{\bar{f}(s)}{s^3}\right\} = \int_0^t\int_0^{t_1}\int_0^{t_2} f(\tau)d\tau dt_1\,dt_2 = \int_0^t \frac{1}{2}(t-\tau)^2 f(\tau)d\tau,$

and in general

(iv) $$\mathscr{L}^{-1}\left\{\frac{\bar{f}(s)}{s^n}\right\} = \int_0^t\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_{n-1}} f(\tau)d\tau\,dt_1\cdots dt_{n-1} = \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!}f(\tau)d\tau.$$

13. The staircase function $f(t) = [t]$ represents the greatest integer less than or equal to $t$. Find its Laplace transform.

14. Use the convolution theorem to prove the identity

$$\int_0^t J_0(\tau)J_0(t-\tau)d\tau = \sin t.$$

15. Show that

(a) $\mathscr{L}\{t\,H(t-a)\} = \left(\dfrac{1}{s^2}+\dfrac{a}{s}\right)\exp(-sa)$,

(b) $\mathscr{L}\{t^n\exp(at)\} = n!(s-a)^{-(n+1)}$.

16. If $\mathscr{L}\{f(t)\}=\bar{f}(s)$ and $f(t)$ has a finite discontinuity at $t=a$, show that

$$\mathscr{L}\{f'(t)\} = s\bar{f}(s) - f(0) - \exp(-sa)[f]_a,$$

where $[f]_a = f(a+0) - f(a-0)$.

17. If $f(t) = H\left(t-\dfrac{\pi}{2}\right)\sin t$, find its Laplace transforms.

18. Establish the following results:

(a) $\mathscr{L}\{\sin^2 at\} = \dfrac{2a^2}{s(s^2+4a^2)}$,

(b) $\mathscr{L}\{I_0(x)\} = \dfrac{1}{\sqrt{s^2+a^2}}$,

(c) $\mathscr{L}\{|\sin at|\} = \dfrac{a}{s^2+a^2}\coth\left(\dfrac{\pi s}{2a}\right)$, $s>0$,

(d) $\mathscr{L}\left\{\displaystyle\int_0^t \frac{\sin ax}{x}\,dx\right\} = \dfrac{1}{s}\tan^{-1}\left(\dfrac{a}{s}\right)$,

(e) $\mathscr{L}\left\{\dfrac{d}{dt}(f*g)\right\} = g(0)\bar{f}(s) + \mathscr{L}\{f*g'\} = s\bar{f}(s)\bar{g}(s)$
$= \mathscr{L}\{f'*g\} + f(0)\bar{g}(s)$.

19. Establish the following results:

(a) $\mathscr{L}\{t^2 f''(t)\} = s^2\dfrac{d^2}{ds^2}\bar{f}(s) + 4s\dfrac{d}{ds}\bar{f}(s) + 2\bar{f}(s)$,

(b) $\mathscr{L}\{t^m f^{(n)}(t)\} = (-1)^m\dfrac{d^m}{ds^m}\left[s^n\bar{f}(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)\right]$.

20. (a) Show that $f(t)=\sin(a\sqrt{t})$ satisfies the differential equation

$$4t\,f''(t) + 2f'(t) + a^2 f(t) = 0.$$

Use this differential equation to show that

(b) $\mathscr{L}\{\sin\sqrt{t}\} = \dfrac{1}{2}\Gamma\left(\dfrac{1}{2}\right)s^{-3/2}\exp\left(-\dfrac{1}{4s}\right)$, $s>0$,

(c) $\mathscr{L}\left\{\dfrac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \Gamma\left(\dfrac{1}{2}\right)\dfrac{1}{\sqrt{s}}\exp\left(-\dfrac{1}{4s}\right)$, $s>0$.

21. Establish the following results:

(a) $\mathscr{L}\left\{\int_t^\infty \frac{f(x)}{x}dx\right\} = \frac{1}{s}\int_0^s \bar{f}(x)dx,$

(b) $\mathscr{L}\left\{\int_0^\infty \frac{f(x)}{x}dx\right\} = \frac{1}{s}\int_0^\infty \bar{f}(x)dx.$

22. Use exercise 21(a) to find the Laplace transform of

(a) the *cosine integral* defined by

$$Ci(t) = \int_\infty^t \frac{\cos x}{x}dx, \quad t > 0,$$

(b) the *exponential integral* defined by

$$\mathrm{E}i(t) = \int_t^\infty \frac{e^{-x}}{x}dx, \ t > 0.$$

23. Show that

(a) $\mathscr{L}\{t\,e^{-bt}\cos at\} = \frac{(s+b)^2 - a^2}{[(s+b)^2 + a^2]^2},$

(b) $\mathscr{L}\left\{\frac{\cos at - \cos bt}{t}\right\} = \frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right),$

(c) $\mathscr{L}\{L_n(t)\} = \frac{1}{s}\left(\frac{s-1}{s}\right)^n$, where $L_n(t)$ are the Laguerre polynomials of degree $n$.

24. If $\mathscr{L}\{f(t)\} = \bar{f}(s)$ and $\mathscr{L}\{g(x,t)\} = \bar{h}(s)\exp\{-x\bar{h}(s)\}$, prove that

(a) $\mathscr{L}\left\{\int_0^\infty g(x,t)f(x)dx\right\} = \bar{h}(s)\bar{f}\{\bar{h}(s)\}.$

(b) $\mathscr{L}\left\{\int_0^\infty J_0(2\sqrt{xt})f(x)dx\right\} = \frac{1}{s}\,\bar{f}\left(\frac{1}{s}\right)$, when $g(x,t) = J_0(2\sqrt{xt})$.

25. Use Exercise 24(b) to show that

(a) $\int_0^\infty J_0(2\sqrt{xt})\sin\left(\frac{x}{a}\right)dx = a\cos at, \qquad (a \neq 0),$

(b) $\int_0^\infty J_0(2\sqrt{xt})e^{-x}x^n\,dx = n!\,e^{-t}L_n(t).$

26. Find the Laplace transform of the *triangular wave* function defined over $(0, 2a)$ by

$$f(t) = \begin{Bmatrix} t, & 0 < t < a \\ 2a - t, & a < t < 2a \end{Bmatrix}.$$

27. Use the Initial Value Theorem to find $f(0)$, and $f'(0)$ from the following functions:

(a) $\bar{f}(s) = \dfrac{s}{s^2 - 5s + 12}$, (b) $\bar{f}(s) = \dfrac{1}{s(s^2 + a^2)}$,

(c) $\bar{f}(s) = \dfrac{\exp(-sa)}{s^2 + 3s + 5}$, $a > 0$, (d) $\bar{f}(s) = \dfrac{s^2 - 1}{(s^2 + 1)}$.

28. Use the Final Value Theorem to find $f(\infty)$, if it exists, from the following functions:

(a) $\bar{f}(s) = \dfrac{1}{s(s^2 + as + b)}$, (b) $\bar{f}(s) = \dfrac{s + 2}{s^2 + 4}$,

(c) $\bar{f}(s) = \dfrac{1}{1 + as}$, (d) $\bar{f}(s) = \dfrac{3}{(s^2 + 4)^2}$.

29. If $\mathscr{L}\{f(t)\} = \bar{f}(s)$ and $\mathscr{L}\{g(t)\} = \bar{g}(s)$, establish Duhamel's integrals:

$$\mathscr{L}^{-1}\{s\bar{f}(s)\,\bar{g}(s)\} = \begin{Bmatrix} f(0)g(t) + \int_0^t f'(\tau)g(t-\tau)d\tau \\ \\ g(0)f(t) + \int_0^t g'(\tau)f(t-\tau)d\tau \end{Bmatrix}.$$

30. Using Watson's lemma, find the asymptotic expansion of

(a) $\bar{f}(s) = \int_0^\infty (1 + t^2)^{-1}\exp(-st)dt$, as $s \to \infty$,

(b) $K_0(s) = \int_0^\infty (t^2 - 1)^{-\frac{1}{2}}\exp(-st)dt$, as $s \to \infty$,

where $K_0(s)$ is the *modified Bessel* function.

31. Find the asymptotic expansion of $\bar{f}(s)$ as $s \to \infty$ when $f(t)$ is given by

(a) $(1+t)^{-1}$, (b) $\sin 2\sqrt{t}$,

(c) $\log(1+t)$, (d) $J_0(at)$.

32. Use the shifting property (3.4.5) or (3.4.6) to obtain the Laplace transform of the following functions:

(a) $f(t)=(t-a)^n\,H(t-a)$, (b) $f(t)=t^2\,H(t-a)$,

(c) $f(t)=\begin{Bmatrix} t, & 0\le t\le a \\ 0, & t\ge a\end{Bmatrix}$, (d) $f(x)=\begin{Bmatrix} w_0\left(1-\frac{2x}{l}\right), & 0<x<\frac{l}{2} \\ 0, & \frac{l}{2}<x<l\end{Bmatrix}$,

(e) $f(t)=\cos 2t\,H(t-\pi)$, (f) $f(t)=\begin{Bmatrix} 2, & 0\le t\le a \\ -2, & t\ge a\end{Bmatrix}$.

33. For the square wave function $f(t)$ given by $f(t)=a\,H(t)-a\,H(t-a)$, show that

$$\bar{f}(s)=\frac{a}{s\,(1+e^{-as})}.$$

34. If $f(t)=a\,H(t)-2a\,H(t-1)+a\,H(t-2)$, show that

$$\bar{f}(s)=\frac{a}{s}\left(1-2\,e^{-s}+e^{-2s}\right).$$

35. If $f(t)=\begin{Bmatrix} \frac{\sin t}{t}, & t\neq 0 \\ 1, & t=0\end{Bmatrix}$, show that $\bar{f}(s)=\tan^{-1}\left(\frac{1}{s}\right)$

36. If $f_p(t)=t^{p-1}\,e^{-t}\,H(t)$, show that $(f_p * f_q)(t)$ exists if and only if $p$ and $q$ are both positive.

Hence, derive the following results

(a) $(f_p * f_q)(t)=B(p,\,q)\,f_{p+q}(t)$.

(b) $f_p'(t)=(p-1)\,f_{p-1}(t)-f_p(t)$.

(c) $(f_p * f_q)'(t)=(p-1)\,B(p-1,\,q)\,f_{p+q-1}(t)-B(p,\,q)\,f_{p+q}(t)$.

(d) $(f_p * f_q)'(t)=B(p,\,q)\left[(p+q-1)f_{p+q-1}(t)-f_{p+q}(t)\right]$.

37. A family $\{h_p(t): p>0\}$ of functions on $\mathbb{R}$ is called a *convolution semi-group* if $h_p * h_q=h_{p+q}$ for all $p,\,q>0$. Show that $h_p(t)=\frac{f_p(t)}{\Gamma(p)}$ defines a convolution semi-group where $f_p(t)$ is defined in Exercise 36.

38. Using the change of variables, $s=c+i\omega$, show that the inverse Laplace transformation is a Fourier transformation, that is,

(i) $f(t)=\mathscr{L}^{-1}\left\{\bar{f}(s)\right\}=\frac{e^{ct}}{2\pi}\int\limits_{-\infty}^{\infty}\bar{f}(c+i\omega)e^{i\omega t}d\omega$.

(ii) $f(t) = \frac{1}{\pi} e^{ct} \operatorname{Re} \int_0^\infty \bar{\mathrm{f}}(\mathrm{c} + \mathrm{i}\omega)\, \mathrm{e}^{\mathrm{i}\omega \mathrm{t}} \mathrm{d}\omega.$

Hence, for real $f(t)$, show that

(iii) $\mathscr{F}_c \left\{ e^{ct} f(t) \right\} = 2 \operatorname{Re} \left[ \bar{\mathrm{f}}(\mathrm{c} + \mathrm{i}\omega) \right],$

(iv) $\mathscr{F}_s \left\{ e^{ct} f(t) \right\} = 2 \operatorname{Im} \left[ \bar{\mathrm{f}}(\mathrm{c} + \mathrm{i}\omega) \right].$

# 4

# *Applications of Laplace Transforms*

> "Mathematical sciences have attracted special attention since great antiquity, they are attracting still more attention today because of their influence on industry and the arts. The agreement of theory and practice brings most beneficial results, and it is not exclusively the practical side which gains; science is advancing under its influence as it discovers new objects of study and new aspects of the mathematical sciences...."
>
> P. L. Chebyshev

> "... partial differential equations are the basis of all physical theorems. In the theory of sound in gases, liquids and solids, in the investigations of elasticity, in optics, everywhere partial differential equations formulate basic laws of nature which can be checked against experiments."
>
> Bernhard Riemann

## 4.1 Introduction

Many problems of physical interest are described by ordinary or partial differential equations with appropriate initial or boundary conditions. These problems are usually formulated as *initial value problems, boundary value problems,* or *initial-boundary value problems* that seem to be mathematically more rigorous and physically realistic in applied and engineering sciences. The Laplace transform method is particularly useful for finding solutions of these problems. The method is very effective for the solution of the response of a linear system governed by an ordinary differential equation to the *initial data* and/or to an *external disturbance* (or *external input function*). More precisely, we seek the solution of a linear system for its state at subsequent time $t > 0$ due to the initial state at $t = 0$ and/or to the disturbance applied for $t > 0$.

This chapter deals with the solutions of ordinary and partial differential equations that arise in mathematical, physical, and engineering sciences. The applications of Laplace transforms to the solutions of certain integral equations and boundary value problems are also discussed in this chapter. It is shown by examples that the Laplace transform can also be used effectively for evaluating certain definite integrals. We also give a few examples of solutions of difference and differential equations using the Laplace transform technique. The effective use of the joint Laplace and Fourier transform is illustrated by solving several initial-boundary value problems. Application of Laplace transforms to the problem of summation of infinite series in closed form is presented with examples. Finally, it is noted that the examples given in this chapter are only representative of a wide variety of problems which can be solved by the use of the Laplace transform method.

## 4.2 Solutions of Ordinary Differential Equations

As stated in the introduction of this chapter, the Laplace transform can be used as an effective tool for analyzing the basic characteristics of a linear system governed by the differential equation in response to initial data and/or to an external disturbance. The following examples illustrate the use of the Laplace transform in solving certain initial value problems described by ordinary differential equations.

***Example 4.2.1***
(*Initial Value Problem*). We consider the first-order ordinary differential equation

$$\frac{dx}{dt} + px = f(t), \qquad t > 0, \tag{4.2.1}$$

with the initial condition

$$x(t=0) = a, \tag{4.2.2}$$

where $p$ and $a$ are constants and $f(t)$ *is* an external input function so that its Laplace transform exists.

Application of the Laplace transform $\bar{x}(s)$ of the function $x(t)$ gives

$$s\,\bar{x}(s) - x(0) + p\,\bar{x}(s) = \bar{f}(s),$$

or

$$\bar{x}(s) = \frac{a}{s+p} + \frac{\bar{f}(s)}{s+p}. \tag{4.2.3}$$

The inverse Laplace transform together with the Convolution Theorem leads to the solution

$$x(t) = ae^{-pt} + \int_0^t f(t-\tau)e^{-p\tau}\,d\tau. \tag{4.2.4}$$

Thus, the solution naturally splits into two terms—the first term corresponds to the response of the initial condition and the second term is entirely due to the external input function $f(t)$.

In particular, if $f(t) = q =$ constant, then the solution (4.2.4) becomes

$$x(t) = \frac{q}{p} + \left(a - \frac{q}{p}\right) e^{-pt}. \tag{4.2.5}$$

The first term of this solution is independent of time $t$ and is usually called the *steady-state solution.* The second term depends on time $t$ and is called the *transient solution.* In the limit as $t \to \infty$, the transient solution decays to zero if $p > 0$ and the steady-state solution is attained. On the other hand, when $p < 0$, the transient solution grows exponentially as $t \to \infty$, and the solution becomes unstable.

Equation (4.2.1) describes the law of natural growth or decay process with an external forcing function $f(t)$ according as $p > 0$ or $< 0$. In particular, if $f(t) = 0$ and $p > 0$, the resulting equation (4.2.1) occurs very frequently in chemical kinetics. Such an equation describes the rate of chemical reactions. □

**_Example 4.2.2_**

(*Second Order Ordinary Differential Equation*). The second order linear ordinary differential equation has the general form

$$\frac{d^2x}{dt^2} + 2p\frac{dx}{dt} + qx = f(t), \qquad t > 0. \tag{4.2.6}$$

The initial conditions are

$$x(t) = a, \quad \frac{dx}{dt} = \dot{x}(t) = b \qquad \text{at } t = 0, \tag{4.2.7ab}$$

where $p, q, a$ and $b$ are constants.

Application of the Laplace transform to this general initial value problem gives

$$s^2\bar{x}(s) - s\,x(0) - \dot{x}(0) + 2p\{s\,\bar{x}(s) - x(0)\} + q\bar{x}(s) = \bar{f}(s).$$

The use of (4.2.7ab) leads to the solution for $\bar{x}(s)$ as

$$\bar{x}(s) = \frac{(s+p)a + (b+pa) + \bar{f}(s)}{(s+p)^2 + n^2}, \qquad n^2 = q - p^2. \tag{4.2.8}$$

The inverse transform gives the solution in three distinct forms depending on

$q> = <p^2$, and they are

$$x(t) = ae^{-pt}\cos nt + \frac{1}{n}(b+pa)e^{-pt}\sin nt + \frac{1}{n}\int_0^t f(t-\tau)e^{-p\tau}\sin n\tau d\tau, \qquad \text{when } n^2 = q - p^2 > 0, \tag{4.2.9}$$

$$x(t) = ae^{-pt} + (b+pa)\,t\,e^{-pt} + \int_0^t f(t-\tau)\,\tau\,e^{-p\tau}d\tau, \qquad \text{when } n^2 = q-p^2 = 0, \tag{4.2.10}$$

$$x(t) = ae^{-pt}\cosh mt + \frac{1}{m}(b+pa)\,e^{-pt}\sinh mt + \frac{1}{m}\int_0^t f(t-\tau)\,e^{-p\tau}\sinh m\tau d\tau, \quad \text{when } m^2 = p^2 - q > 0. \tag{4.2.11}$$

□

***Example 4.2.3***

(*Higher Order Ordinary Differential Equations*). We solve the linear equation of order $n$ with constant coefficients as

$$f(D)\{x(t)\} \equiv D^n x + a_1 D^{n-1}x + a_2 D^{n-2}x + \cdots + a_n x = \phi(t), \quad t > 0, \tag{4.2.12}$$

with the initial conditions

$$x(t) = x_0, \quad Dx(t) = x_1, \quad D^2x(t) = x_2, \ldots, D^{n-1}x(t) = x_{n-1}, \quad \text{at } t = 0, \tag{4.2.13}$$

where $D = \dfrac{d}{dt}$ is the differential operator and $x_0, x_1, \ldots, x_{n-1}$ are constants. We take the Laplace transform of (4.2.12) to get

$$\begin{aligned}(s^n\,\bar{x} - s^{n-1}x_0 - s^{n-2}x_1 - \cdots - s\,x_{n-2} - x_{n-1}) \\ +a_1\left(s^{n-1}\,\bar{x} - s^{n-2}x_0 - s^{n-3}x_1 - \cdots - x_{n-2}\right) \\ +a_2\left(x^{n-2}\,\bar{x} - s^{n-3}x_0 - \cdots - x_{n-3}\right) \\ + \cdots + a_{n-1}(s\,\bar{x} - x_0) + a_n\,\bar{x} = \bar{\phi}(s).\end{aligned} \tag{4.2.14}$$

Or,

$$
\begin{aligned}
(s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n)\, \bar{x}(s) \\
&= \bar{\phi}(s) + (s^{n-1} + a_1 s^{n-2} + \cdots + a_{n-1}) x_0 \\
&\quad + (s^{n-2} + a_1 s^{n-3} + \cdots + a_{n-2}) x_1 + \cdots + (s + a_1) x_{n-2} + x_{n-1} \\
&= \bar{\phi}(s) + \bar{\psi}(s), \qquad (4.2.15)
\end{aligned}
$$

where $\bar{\psi}(s)$ is made up of all terms on the right hand side of (4.2.15) except $\bar{\phi}(s)$, and is a polynomial in $s$ of degree $(n-1)$.

Hence,

$$\bar{f}(s)\, \bar{x}(s) = \bar{\phi}(s) + \bar{\psi}(s),$$

where

$$\bar{f}(s) = s^n + a_1 s^{n-1} + \cdots + a_n.$$

Thus, the Laplace transform solution, $\bar{x}(s)$ is

$$\bar{x}(s) = \frac{\bar{\phi}(s) + \bar{\psi}(s)}{\bar{f}(s)}. \qquad (4.2.16)$$

Inversion of (4.2.16) yields

$$x(t) = \mathscr{L}^{-1}\left\{\frac{\bar{\phi}(s)}{\bar{f}(s)}\right\} + \mathscr{L}^{-1}\left\{\frac{\bar{\psi}(s)}{\bar{f}(s)}\right\}. \qquad (4.2.17)$$

The inverse operation on the right can be carried out by partial fraction decomposition, by the Heaviside Expansion Theorem, or by contour integration. □

***Example 4.2.4***

(*Third Order Ordinary Differential Equations*). We solve

$$(D^3 + D^2 - 6D)x(t) = 0, \qquad D \equiv \frac{d}{dt}, \qquad t > 0, \qquad (4.2.18)$$

with the initial data

$$x(0) = 1,\ \dot{x}(0) = 0,\ \text{and}\ \ddot{x}(0) = 5. \qquad (4.2.19)$$

The Laplace transform of equation (4.2.18) gives

$$[s^3\bar{x} - s^2 x(0) - s\,\dot{x}(0) - \ddot{x}(0)] + [s^2\bar{x} - s\,x(0) - \dot{x}(0)] - 6[s\,\bar{x} - x(0)] = 0.$$

In view of the initial conditions, we find

$$\bar{x}(s) = \frac{s^2 + s - 1}{s(s^2 + s - 6)} = \frac{s^2 + s - 1}{s(s+3)(s-2)}.$$

Or,

$$\bar{x}(s) = \frac{1}{6} \cdot \frac{1}{s} + \frac{1}{3} \cdot \frac{1}{s+3} + \frac{1}{2} \cdot \frac{1}{s-2}.$$

Inverting gives the solution

$$x(t) = \frac{1}{6} + \frac{1}{3}\,e^{-3t} + \frac{1}{2}\,e^{2t}. \tag{4.2.20}$$

□

***Example 4.2.5***

(*System of First Order Ordinary Differential Equations*). Consider the system

$$\left.\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + b_1(t) \\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + b_2(t) \end{aligned}\right\} \tag{4.2.21ab}$$

with the initial data

$$x_1(0) = x_{10} \quad \text{and} \quad x_2(0) = x_{20}; \tag{4.2.22ab}$$

where $a_{11}$, $a_{12}$, $a_{21}$, $a_{22}$ are constants.

Introducing the matrices

$$x \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \frac{dx}{dt} \equiv \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix}, \qquad A \equiv \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

$$b(t) \equiv \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix} \quad \text{and} \quad x_0 = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix},$$

we can write the above system in a matrix differential system as

$$\frac{dx}{dt} = A\,x + b(t), \qquad x(0) = x_0. \tag{4.2.23ab}$$

We take the Laplace transform of the system with the initial conditions to get

$$\begin{aligned} (s - a_{11})\bar{x}_1 - a_{12}\bar{x}_2 &= x_{10} + \bar{b}_1(s), \\ -a_{21}\bar{x}_1 + (s - a_{22})\bar{x}_2 &= x_{20} + \bar{b}_2(s). \end{aligned}$$

The solutions of this algebraic system are

$$\bar{x}_1(s) = \frac{\begin{vmatrix} x_{10} + \bar{b}_1(s) & -a_{12} \\ x_{20} + \bar{b}_2(s) & s - a_{22} \end{vmatrix}}{\begin{vmatrix} s - a_{11} & -a_{12} \\ -a_{21} & s - a_{22} \end{vmatrix}}, \quad \bar{x}_2(s) = \frac{\begin{vmatrix} s - a_{11} & x_{10} + \bar{b}_1(s) \\ -a_{21} & x_{20} + \bar{b}_2(s) \end{vmatrix}}{\begin{vmatrix} s - a_{11} & -a_{12} \\ -a_{21} & s - a_{22} \end{vmatrix}}. \tag{4.2.24ab}$$

Expanding these determinants, results for $\bar{x}_1(s)$ and $\bar{x}_2(s)$ can readily be inverted, and the solutions for $x_1(t)$ and $x_2(t)$ can be found in closed forms. ▯

**Example 4.2.6**
Solve the matrix differential system

$$\frac{dx}{dt} = Ax, \qquad x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{4.2.25}$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}.$$

This system is equivalent to

$$\frac{dx_1}{dt} - x_2 = 0,$$
$$\frac{dx_2}{dt} + 2x_1 - 3x_2 = 0,$$

with

$$x_1(0) = 0 \quad \text{and} \quad x_2(0) = 1.$$

Taking the Laplace transform of the coupled system with the given initial data, we find

$$s\,\bar{x}_1 - \bar{x}_2 = 0,$$
$$2\bar{x}_1 + (s-3)\bar{x}_2 = 1.$$

This system has the solutions

$$\bar{x}_1(s) = \frac{1}{s^2 - 3s + 2} = \frac{1}{s-2} - \frac{1}{s-1},$$
$$\bar{x}_2(s) = \frac{s}{s^2 - 3s + 2} = \frac{2}{s-2} - \frac{1}{s-1}.$$

Inverting these results, we obtain

$$x_1(t) = e^{2t} - e^t, \qquad x_2(t) = 2\,e^{2t} - e^t.$$

In matrix notation, the solution is

$$x(t) = \begin{pmatrix} e^{2t} - e^t \\ 2\,e^{2t} - e^t \end{pmatrix}. \tag{4.2.26}$$

▯

**Example 4.2.7**
(*Second Order Coupled Differential System*). Solve the system

$$\left.\begin{aligned} \frac{d^2x_1}{dt^2} - 3x_1 - 4x_2 = 0 \\ \frac{d^2x_2}{dt^2} + x_1 + x_2 = 0 \end{aligned}\right\} \qquad t > 0, \tag{4.2.27}$$

with the initial conditions

$$x_1(t) = x_2(t) = 0; \qquad \frac{dx_1}{dt} = 2 \quad \text{and} \quad \frac{dx_2}{dt} = 0 \text{ at } t = 0. \tag{4.2.28}$$

The use of the Laplace transform to (4.2.27) with (4.2.28) gives

$$\begin{aligned} (s^2 - 3)\bar{x}_1 - 4\bar{x}_2 = 2 \\ \bar{x}_1 + (s^2 + 1)\bar{x}_2 = 0. \end{aligned}$$

Then

$$\bar{x}_1(s) = \frac{2(s^2+1)}{(s^2-1)^2} = \frac{(s+1)^2 + (s-1)^2}{(s^2-1)^2} = \frac{1}{(s-1)^2} + \frac{1}{(s+1)^2}.$$

Hence, the inversion yields

$$x_1(t) = t(e^t + e^{-t}). \tag{4.2.29}$$

$$\bar{x}_2(s) = \frac{-2}{(s^2-1)^2} = \frac{1}{2}\left[\frac{1}{s-1} - \frac{1}{s+1} - \frac{1}{(s-1)^2} - \frac{1}{(s+1)^2}\right],$$

which can be readily inverted to find

$$x_2(t) = \frac{1}{2}(e^t - e^{-t} - t\,e^t - t\,e^{-t}). \tag{4.2.30}$$

$\square$

**Example 4.2.8**
(*The Harmonic Oscillator in a Non-Resisting Medium*). The differential equation of the oscillator in the presence of an external driving force $F\,f(t)$ is

$$\frac{d^2x}{dt^2} + \omega^2 x = F\,f(t), \tag{4.2.31}$$

where $\omega$ is the frequency and $F$ is a constant.

The initial conditions are

$$x(t) = a, \quad \dot{x}(t) = U \qquad \text{at } t = 0, \tag{4.2.32}$$

where $a$ and $U$ are constants.

Taking the Laplace transform of (4.2.31) with the initial conditions, we obtain

$$(s^2+\omega^2)\bar{x}(s)=sa+U+F\,\bar{f}(s).$$

Or,

$$\bar{x}(s)=\frac{as}{s^2+\omega^2}+\frac{U}{s^2+\omega^2}+\frac{F\,\bar{f}(s)}{s^2+\omega^2}. \tag{4.2.33}$$

Inversion together with the convolution theorem yields

$$x(t) = a\cos\omega t+\frac{U}{\omega}\sin\omega t+\frac{F}{\omega}\int_0^t f(t-\tau)\sin\,\omega\tau d\tau \tag{4.2.34}$$

$$= A\ \cos(\omega t-\phi)+\frac{F}{\omega}\int_0^t f(t-\tau)\ \sin\,\omega\tau d\tau, \tag{4.2.35}$$

where $A=\left(a^2+\dfrac{U^2}{\omega^2}\right)^{1/2}$ and $\phi=\tan^{-1}\left(\dfrac{U}{\omega a}\right)$.

The solution (4.2.35) consists of two terms. The first term represents the response to the initial data, and it describes *free oscillations* with amplitude $A$, phase $\phi$, and frequency $\omega$, which is called the *natural frequency* of the oscillator. The second term arises in response to the external force, and hence, it represents the *forced oscillations.* In order to investigate some interesting features of solution (4.2.35), we select the following cases of interest:

*(i) Zero Forcing Function.*
In this case, solution (4.2.35) reduces to

$$x(t)=A\ \cos(\omega t-\phi). \tag{4.2.36}$$

This represents simple harmonic motion with amplitude $A$, frequency $\omega$ and phase $\phi$. Evidently, the motion is oscillatory.

*(ii) Steady Forcing Function, that is,* $f(t)=1$.
In this case, solution (4.2.35) becomes

$$x-\frac{F}{\omega^2}=A\cos(\omega t-\phi)-\frac{F}{\omega^2}\cos\omega t. \tag{4.2.37}$$

In particular, when the particle is released from rest, $U=0$, (4.2.37) takes the form

$$x-\frac{F}{\omega^2}=\left(a-\frac{F}{\omega^2}\right)\cos\omega t. \tag{4.2.38}$$

This corresponds to free oscillations with the natural frequency $\omega$ and displays a shift in the equilibrium position from the origin to the point $\dfrac{F}{\omega^2}$.

*(iii) Periodic Forcing Function, that is,* $f(t) = \cos\omega_0 t$.
The transform solution can readily be found form (4.2.33) in the form

$$\begin{aligned}\bar{x}(s) &= \frac{as}{s^2+\omega^2} + \frac{U}{s^2+\omega^2} + \frac{Fs}{(s^2+\omega_0^2)(s^2+\omega^2)} \\ &= \frac{as}{s^2+\omega^2} + \frac{U}{s^2+\omega^2} + \frac{Fs}{(\omega_0^2-\omega^2)}\left(\frac{1}{s^2+\omega^2} - \frac{1}{s^2+\omega_0^2}\right). \quad (4.2.39)\end{aligned}$$

Inversion yields the solution

$$\begin{aligned}x(t) &= a\,\cos\omega t + \frac{U}{\omega}\sin\,\omega t + \frac{F}{(\omega_0^2-\omega^2)}(\cos\,\omega t - \cos\,\omega_0 t) \quad (4.2.40)\\ &= A\,\cos(\omega t - \phi) + \frac{F}{(\omega_0^2-\omega^2)}\cos\,\omega_0 t, \quad (4.2.41)\end{aligned}$$

where $A = \left\{\left(a + \frac{F}{\omega_0^2-\omega^2}\right)^2 + \frac{U^2}{\omega^2}\right\}^{1/2}$ and $\tan\,\phi = \frac{U}{\omega} \div \left(a + \frac{F}{\omega_0^2-\omega^2}\right)$.

It is noted that solution (4.2.41) consists of free oscillations of period $\left(\frac{2\pi}{\omega}\right)$ and forced oscillations of period $\left(\frac{2\pi}{\omega_0}\right)$, which is the same as that of the external periodic force. If $\omega_0 < \omega$, the phase of the forced oscillations is the same as that of the external periodic force. If $\omega_0 > \omega$, the forced term suffers from a phase change by an amount $\pi$. In other words, the forced motion is in phase or 180° out of phase with the external force according as $\omega >$ or $< \omega_0$.

When $\omega = \omega_0$, result (4.2.40) can be written as

$$\begin{aligned}x(t) &= a\,\cos\omega t + \frac{U}{\omega}\sin\ \omega t + \frac{Ft}{(\omega_0+\omega)}\left[\frac{\sin\left\{\frac{1}{2}(\omega-\omega_0)t\right\}\sin\left\{\frac{1}{2}(\omega+\omega_0)t\right\}}{\frac{1}{2}(\omega_0-\omega)t}\right]\\ &= a\cos\omega t + \frac{U}{\omega}\sin\omega t + \frac{Ft}{2\omega}\sin\omega t = A\cos(\omega t - \phi) + \frac{Ft}{2\omega}\sin\omega t, \quad (4.2.42)\end{aligned}$$

where

$$A^2 = \left(a^2 + \frac{U^2}{\omega^2}\right) \quad \text{and } \tan\,\phi = \frac{U}{a\omega}.$$

This solution clearly shows that the amplitude of the forced motion increases with $t$. Thus, if the natural frequency is equal to the forcing frequency, the oscillations become unbounded, which is physically undesirable. This phenomenon is usually called *resonance,* and the corresponding frequency $\omega = \omega_0$ is referred to as the *resonant frequency* of the system. It may be emphasized that at the resonant frequency, the solution of the problem becomes mathematically invalid for large times, and hence, it is physically unrealistic. In most dynamical systems, this kind of situation is resolved by including dissipating and/or nonlinear effects. □

**Example 4.2.9**

(*Harmonic Oscillator in a Resisting Medium*). The differential equation of the oscillator in a resisting medium where the resistance is proportional to velocity is given by

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = F\, f(t), \tag{4.2.43}$$

where $k\,(>0)$ is a constant of proportionality and the right hand side represents the external driving force. The initial state of the system is

$$x(t) = a, \quad \frac{dx}{dt} = U \qquad \text{at } t = 0. \tag{4.2.44}$$

In view of the initial conditions, the Laplace transform solution of equation (4.2.43) is obtained as

$$\begin{aligned} \bar{x}(s) &= \frac{a(s+2k) + U + F\,\bar{f}(s)}{(s^2 + 2ks + \omega^2)} \\ &= \frac{a(s+k) + (U+ak) + \bar{F}(s)}{(s+k)^2 + n^2}, \end{aligned} \tag{4.2.45}$$

where $n^2 = \omega^2 - k^2$.

Three possible cases deserve attention:

*(i)* $k < \omega$ *(small damping).*
In this case, $n^2 = \omega^2 - k^2 > 0$ and the inversion of (4.2.45) along with the Convolution Theorem yields

$$x(t) = a\,e^{-kt}\cos nt + \frac{(U+ak)}{n}e^{-kt}\sin nt + \frac{F}{n}\int_0^t f(t-\tau)e^{-k\tau}\sin n\tau\, d\tau. \tag{4.2.46}$$

This is the most general solution of the problem for an arbitrary form of the external driving force.

*(ii)* $k = \omega$ *(critical damping) so that* $n^2 = 0$.
The solution for this case can readily be obtained from (4.2.45) by inversion and has the form

$$x(t) = a\,e^{-kt} + (U+ak)\,t\,e^{-kt} + F\int_0^t f(t-\tau)\,\tau e^{-kt}\,d\tau. \tag{4.2.47}$$

*(iii)* $k > \omega$ *(large damping).*
Set $n^2 = -(k^2 - \omega^2) = -m^2$ so that $m^2 = k^2 - \omega^2 > 0$.

The transformed solution (4.2.45) assumes the form

$$\bar{x}(s) = \frac{a(s+k) + (U+ak) + F\ \bar{f}(s)}{(s+k)^2 - m^2}. \tag{4.2.48}$$

After inversion, it turns out that

$$x(t) = a\,e^{-kt}\cosh mt + \left(\frac{U+ak}{m}\right)e^{-kt}\sinh mt + \frac{F}{m}\int_0^t f(t-\tau)e^{-k\tau}\sinh m\tau d\tau. \tag{4.2.49}$$

In order to examine the characteristic features of the problem, it is necessary to specify the nature and functional form of $f(t)$ involved in the external force term. Suppose the external driving force is zero. The solution can readily be written down in all three cases.

For $0 < k < \omega$, the solution is

$$x(t) = e^{-kt}\left(a\ \cos nt + \frac{U+ak}{n}\sin nt\right) = A\,e^{-kt}\ \cos(nt-\phi), \tag{4.2.50}$$

where $A = \left\{a^2 + \frac{(U+ak)^2}{n^2}\right\}^{1/2}$ and $\phi = \tan^{-1}\left(\frac{U+ak}{an}\right)$.

Like the harmonic oscillator in a vacuum, the motion is oscillatory with the time-dependent amplitude $Ae^{-kt}$ and the modified frequency

$$n = (\omega^2 - k^2)^{1/2} = \omega\left(1 - \frac{1}{2}\frac{k^2}{\omega^2} + \cdots\right), \qquad 0 < k < \omega.$$

This means that, when the resistance is small, the modified frequency (or the undamped natural frequency) is obviously smaller than the natural frequency, $\omega$. Although the small resistance produces an insignificant effect on the frequency, the amplitude is radically modified. It should also be noted that the amplitude decays exponentially to zero as time $t \to \infty$. The phase of the motion is also changed by the small resistance. Thus, the motion is called the *damped oscillatory motion,* and depicted by Figure 4.1.

At the critical case, $\omega = k$, and hence, $n = 0$. The solution can readily be found from (4.2.47) with $F = 0$, and has the form

$$x(t) = a\,e^{-kt} + (ak+U)\,t\,e^{-kt}. \tag{4.2.51}$$

The motion ceases to be oscillatory and decays very rapidly as $t \to \infty$.

If damping is large with no external force, solution (4.2.49) reduces to

$$x(t) = a\,e^{-kt}\ \cosh mt + \left(\frac{ak+U}{m}\right)e^{-kt}\ \sinh mt. \tag{4.2.52}$$

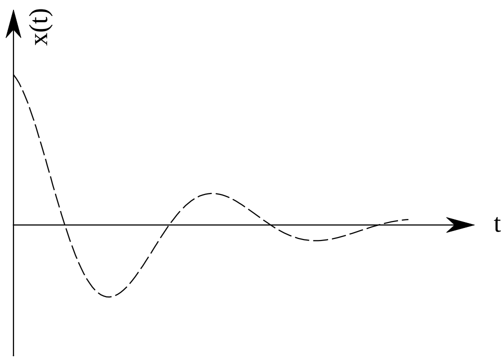


**Figure 4.1** Damped oscillatory motion.

Using $\begin{matrix}\cosh\\ \sinh\end{matrix}\, mt = \frac{1}{2}(e^{mt} \pm e^{-mt})$, we can write the solution as

$$x(t) = A\, e^{-(k-m)t} + B\, e^{-(k+m)t}, \tag{4.2.53}$$

where $A = \frac{1}{2}\left(a + \frac{ak+U}{m}\right)$ and $B = \frac{1}{2}\left(a - \frac{ak+U}{m}\right)$.

The above solution suggests that the motion is no longer oscillatory and in fact, it decays very rapidly as $t \to \infty$. ▯

***Example 4.2.10***

(*Harmonic Oscillator in a Resisting Medium with an External Periodic Force*), The motion is governed by the equation

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = F\cos\omega_0 t, \qquad k > 0 \tag{4.2.54}$$

with the initial data

$$x(0) = a \quad \text{and} \quad \dot{x}(0) = U.$$

The transformed solution for the case of small damping $(k < \omega)$ is

$$\begin{aligned}\bar{x}(s) &= \frac{a(s+k) + (U+ak)}{(s+k)^2 + n^2} + \frac{Fs}{\{(s+k)^2 + n^2\}(s^2 + \omega_0^2)}\\ &= \frac{a(s+k) + (U+ak)}{(s+k)^2 + n^2} + F\left[\frac{As - B}{(s+k)^2 + n^2} - \frac{As - C}{s^2 + \omega_0^2}\right],\end{aligned} \tag{4.2.55}$$

where

$$A = \frac{\omega_0^2 - \omega^2}{(\omega^2 - \omega_0^2)^2 + 4k^2\omega_0^2}, \quad B = \frac{2k\omega^2}{(\omega^2 - \omega_0^2)^2 + 4k^2\omega_0^2},$$

and

$$C = \frac{2\,k\,\omega_0^2}{(\omega^2-\omega_0^2)^2+4k^2\omega_0^2} \quad \text{with } \omega^2 = n^2+k^2.$$

The expression for $\bar{x}(s)$ can be inverted to obtain the solution

$$x(t) = (a+FA)\,e^{-kt}\,\cos nt + \frac{1}{n}(U+ak-FAk-FB)\,e^{-kt}\,\sin nt$$
$$-AF\,\cos\omega_0 t + \frac{CF}{\omega_0}\sin\omega_0 t. \tag{4.2.56}$$

It is convenient to write it in the form

$$x(t) = A_1\,\cos(\omega_0 t-\phi_1) + A_2\,e^{-kt}\,\cos(nt-\phi_2), \tag{4.2.57}$$

where

$$A_1^2 = F^2\left(A^2+\frac{C^2}{\omega_0^2}\right) = \frac{F^2}{(\omega^2-\omega_0^2)^2+4k^2\omega_0^2}, \tag{4.2.58}$$

$$\tan\,\phi_1 = -\frac{C}{A\omega_0} = \frac{2k\omega_0}{\omega^2-\omega_0^2}, \tag{4.2.59}$$

$$A_2^2 = (a+FA)^2 + \frac{1}{n^2}(U+ak-kFa-FB)^2, \tag{4.2.60}$$

and

$$\tan\,\phi_2 = \frac{U+ak-kFA-FB}{n(a+FB)}. \tag{4.2.61}$$

This form of solution (4.2.57) lends itself to some interesting physical interpretations. First, the displacement field $x(t)$ essentially consists of the steady state and the transient terms, which are independently modified by the damping and driving forces involved in the equation of motion. In the limit as $t\to\infty$, the latter decays exponentially to zero. Consequently, the ultimate steady state is attained in the limit, and represented by the first term of (4.2.57). In fact, the steady-state solution is denoted by $x_{st}(t)$ and given by

$$x_{st}(t) = A_1\,\cos(\omega_0 t-\phi_1), \tag{4.2.62}$$

where $A_1$ is the amplitude, $\omega_0$ is the frequency, and $\phi_1$ represents the phase lag given by

$$\begin{aligned}
\phi_1 &= \tan^{-1}\left\{\frac{2k\omega_0}{(\omega^2-\omega_0^2)}\right\} && \text{when } \omega_0<\omega,\\
&= \pi - \tan^{-1}\left\{\frac{2k\omega_0}{(\omega_0^2-\omega^2)}\right\} && \text{when } \omega_0>\omega,\\
&= \frac{\pi}{2} && \text{as } \omega_0\to\omega.
\end{aligned}$$

It should be noted that the frequency of the steady-state solution is the same as that of the external driving force, but the amplitude and the phase are modified by the parameters $\omega$, $k$ and $\omega_0$. It is of interest to examine the nature of the amplitude and the phase with respect to the forcing frequency $\omega_0$. For a low frequency $(\omega_0 \to 0)$, $A_1 = \dfrac{F}{\omega^2}$ and $\phi_1 = 0$. As $\omega_0 \to \omega$, the amplitude of the motion is still bounded and equal to $\left(\dfrac{F}{2k\omega}\right)$ if $k \neq 0$. The displacement suffers from a phase lag of $\pi/2$. Further, we note that

$$\frac{dA_1}{d\omega_0} = \frac{2\,\omega_0 F\left(\omega^2 - \omega_0^2 - 2k^2\right)}{\left\{(\omega^2 - \omega_0^2)^2 + 4k^2\omega_0^2\right\}^{3/2}}. \tag{4.2.63}$$

It follows that $A_1$ has a minimum at $\omega_0 = 0$ with minimum value $\dfrac{F}{\omega^2}$, and a maximum at $\omega_0 = (\omega^2 - 2k^2)^{1/2}$ with maximum value $\dfrac{F}{2k(\omega^2 - 2k^2)^{1/2}}$ provided $2k^2 < \omega^2$. If $2k^2 > \omega^2$, $A_1$ has no maximum and gradually decreases. The non-dimensional amplitude $A^* = \left(\dfrac{2A_1\omega^2}{F}\right)$ is plotted against the non-dimensional frequency $\omega^* = \dfrac{\omega_0}{\omega}$ for a given value of $\frac{k}{\omega}\,(<1)$ in Figure 4.2.

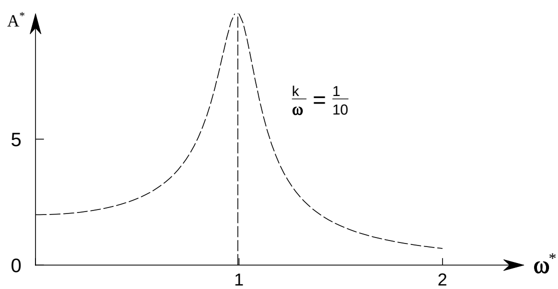


**Figure 4.2** Amplitude versus frequency with damping.

In the absence of the damping term, the amplitude $A_1$ becomes

$$A_1 = \frac{F}{\left|\omega^2 - \omega_0^2\right|},$$

which is unbounded at $\omega_0 = \omega$ and shown in Figure 4.3.

This situation has already been encountered earlier, and the frequency $\omega_0 = \omega$ was defined as the *resonant frequency*. The difficulty for the resonant case has been resolved by the inclusion of small damping effect.

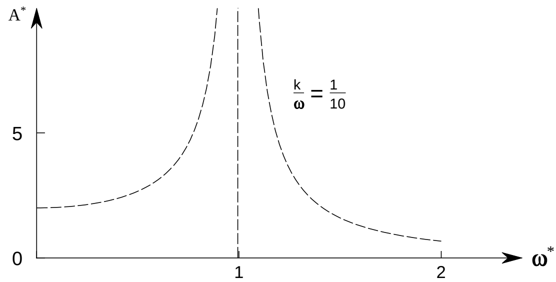


**Figure 4.3** Amplitude versus frequency without damping.

At the critical case ($k^2 = \omega^2$), the solution is found from (4.2.55) by inversion and has the form

$$x(t) = A_1 \cos(\omega_0 t - \phi) + (a + FA)\, e^{-kt} \\ + t(U + ak - FAk - FB)\, e^{-kt}. \qquad (4.2.64)$$

The transient term of this solution decays as $t \to \infty$ and the steady state is attained.

The solution for the case of high damping ($k^2 > \omega^2$) is obtained from (4.2.55) as

$$x(t) = (a + FA)\, e^{-kt} \cosh mt + \frac{1}{m}(U + ak - FAk - FB)\, e^{-kt} \sinh mt \\ - AF \cos \omega_0 t + \frac{CF}{\omega_0} \sin \omega_0 t \qquad (4.2.65)$$

where $m^2 = -n^2 = k^2 - \omega^2 > 0$. This result is somewhat similar to that of (4.2.56) or (4.2.57) with the exception that the transient term decays very rapidly as $t \to \infty$. Like previous cases, the steady state is reached in the limit. □

***Example 4.2.11***

Obtain the solution of the Bessel equation

$$t\, \frac{d^2x}{dt^2} + \frac{dx}{dt} + a^2 t\, x(t) = 0, \qquad x(0) = 1. \qquad (4.2.66)$$

Application of the Laplace transform gives

$$\mathscr{L}\left\{t \frac{d^2x}{dt^2}\right\} + \mathscr{L}\left\{\frac{dx}{dt}\right\} + a^2\, \mathscr{L}\{t\, x(t)\} = 0.$$

Or,

$$-\frac{d}{ds}\left[\mathscr{L}\left\{\frac{d^2x}{dt^2}\right\}\right] + s\, \bar{x}(s) - x(0) - a^2\, \frac{d\bar{x}}{ds} = 0.$$

Or,

$$-\frac{d}{ds}[s^2\,\bar{x} - s\,x(0) - \dot{x}(0)] + s\,\bar{x}(s) - 1 - a^2\frac{d\bar{x}}{ds} = 0.$$

Thus,

$$(s^2 + a^2)\frac{d\bar{x}}{ds} + s\,\bar{x} = 0.$$

Or,

$$\frac{d\bar{x}}{\bar{x}} = -\frac{s\,ds}{s^2 + a^2}.$$

Integration gives the solution for $\bar{x}(s)$

$$\bar{x}(s) = \frac{A}{\sqrt{s^2 + a^2}},$$

where $A$ is an integrating constant. By the inverse transformation, we obtain the solution

$$x(t) = A\,J_0(at).$$

□

***Example 4.2.12***

Find the solution of the initial value problem

$$\frac{d^2x}{dt^2} + t\,\frac{dx}{dt} - 2x = 2, \qquad x(0) = \dot{x}(0) = 0.$$

Taking the Laplace transform yields

$$\mathscr{L}\left\{\frac{d^2x}{dt^2}\right\} + \mathscr{L}\left\{t\,\frac{dx}{dt}\right\} - 2\,\bar{x}(s) = \frac{2}{s}.$$

Or,

$$s^2\bar{x} - \frac{d}{ds}\{s\,\bar{x}(s)\} - 2\bar{x} = \frac{2}{s}$$
$$\frac{d\bar{x}}{ds} + \left(\frac{3}{s} - s\right)\bar{x} = -\frac{2}{s^2}.$$

This is a first order linear equation, which can be solved by the method of the integrating factor. The integrating factor is $s^3\exp\left(-\frac{1}{2}s^2\right)$. Multiplying the equation by the integrating factor and integrating, it turns out that

$$\bar{x}(s) = \frac{2}{s^3} + \frac{A}{s^3}\,\exp\left(\frac{s^2}{2}\right),$$

where $A$ is an integrating constant. As $\bar{x}(s) \to \infty$ as $s \to \infty$, we must have $A \equiv 0$. Thus, $\bar{x}(s) = \dfrac{2}{s^3}$. Inverting, we get the solution

$$x(t) = t^2.$$

□

**Example 4.2.13**
(*Current and Charge in a Simple Electric Circuit*). The current in a circuit (see Figure 4.4) containing inductance $L$, resistance $R$, and capacitance $C$ with an applied voltage $E(t)$ is governed by the equation

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int_0^t I\,dt = E(t), \tag{4.2.67}$$

where $L$, $R$, and $C$ are constants and $I(t)$ is the current that is related to the accumulated charge $Q$ on the condenser at time $t$ by

$$Q(t) = \int_0^t I(t)\,dt \qquad \text{so} \quad \text{that} \quad \frac{dQ}{dt} = I(t). \tag{4.2.68}$$

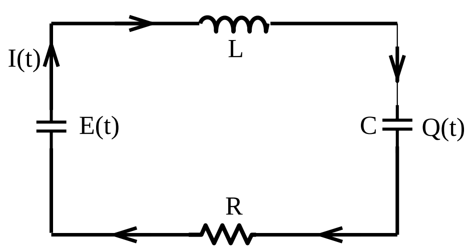


**Figure 4.4** Simple electric circuit.

If the circuit is without a condenser ($C \to \infty$), equation (4.2.67) reduces to

$$L\frac{dI}{dt} + RI = E(t), \qquad t > 0. \tag{4.2.69}$$

This can easily be solved with the initial condition $I(t=0) = I_0$. However, we solve the system (4.2.67)–(4.2.68) with the initial data

$$I(t=0) = 0, \qquad Q(t=0) = 0. \tag{4.2.70}$$

Then, in the limit $C \to \infty$, the solution of the system reduces to that of (4.2.69).

Application of the Laplace transform to (4.2.67) with (6.2.70) gives

$$\bar{I}(s) = \frac{1}{L} \frac{s\bar{E}(s)}{\left(s^2 + \frac{R}{L}s + \frac{1}{CL}\right)} = \frac{1}{L} \cdot \frac{(s+k-k)\bar{E}(s)}{(s+k)^2 + n^2}, \tag{4.2.71}$$

where $k = \dfrac{R}{2L}$, $\omega^2 = \dfrac{1}{LC}$ and $n^2 = \omega^2 - k^2$.

Inversion of (4.2.71) gives the current field for three cases:

$$I(t) = \frac{1}{L}\int_0^t E(t-\tau)\left(\cos n\tau - \frac{k}{n}\sin n\tau\right)e^{-k\tau}\,d\tau, \qquad \text{if } \omega^2 > k^2 \tag{4.2.72}$$

$$= \frac{1}{L}\int_0^t E(t-\tau)(1-k\tau)e^{-k\tau}\,d\tau, \qquad \text{if } \omega^2 = k^2 \tag{4.2.73}$$

$$= \frac{1}{L}\int_0^t E(t-\tau)\left(\cosh m\tau - \frac{k}{m}\sinh m\tau\right)e^{-k\tau}\,d\tau, \text{ if } k^2 > \omega^2 \tag{4.2.74}$$

where $m^2 = -n^2$.

In particular, if $E(t) =$ constant $= E_0$, then the solution can be obtained directly from (4.2.71) by inversion as

$$I(t) = \frac{E_0}{nL}\exp\left(-\frac{Rt}{2L}\right)\sin nt, \qquad \text{if } n^2 = \frac{1}{CL} - \left(\frac{R}{2L}\right)^2 > 0, \tag{4.2.75}$$

$$= \frac{E_0}{L}\,t\,\exp\left(-\frac{Rt}{2L}\right), \qquad \text{if } \left(\frac{R}{2L}\right)^2 = \frac{1}{CL}, \tag{4.2.76}$$

$$= \frac{E_0}{mL}\exp\left(-\frac{Rt}{2L}\right)\sinh mt, \quad \text{if } m^2 = \left(\frac{R}{2L}\right)^2 - \frac{1}{CL} > 0. \tag{4.2.77}$$

It may be observed that the solution for the case of low resistance ($R^2C < 4L$), or small damping, describes a damped sinusoidal current with slowly decaying amplitude. In fact, the rate of damping is proportional to $\dfrac{R}{L}$, and when this quantity is large, the attenuation of the current is very rapid. The frequency of the oscillating current field is

$$n = \left(\frac{1}{CL} - \frac{R^2}{4L^2}\right)^{1/2},$$

which is called the *natural frequency* of the current field. If $\dfrac{R^2}{4L^2} << \dfrac{1}{CL}$, the frequency $n$ is approximately equal to

$$n \sim \frac{1}{\sqrt{CL}}.$$

The case, $\dfrac{R^2}{4L^2} = \dfrac{1}{CL}$, corresponds to *critical damping*, and the solution for this case decays exponentially with time.

The last case, $R^2C > 4L$, corresponds to high resistance or high damping. The current related to this case has the form

$$I(t) = \frac{E_0}{2mL}\left[e^{-\left(\frac{R}{2L}-m\right)t} - e^{-\left(\frac{R}{2L}+m\right)t}\right]. \tag{4.2.78}$$

It may be recognized that the solution is no longer oscillatory and decays exponentially to zero as $t \to \infty$. This is really expected in an electrical circuit with a very high resistance. If $C \to \infty$, the circuit is free from a condenser and $m \to \dfrac{R}{2L}$. Consequently, solution (4.2.77) reduces to

$$I(t) = \frac{E_0}{R}\left[1 - \exp\left(-\frac{Rt}{L}\right)\right]. \tag{4.2.79}$$

This is identical with the solution of equation (4.2.69).

We consider another special case where the alternating voltage is applied to the circuit so that

$$E(t) = E_0 \sin\omega_0 t. \tag{4.2.80}$$

The transformed solution for $\bar{I}(s)$ follows from (4.2.71) as

$$\bar{I}(s) = \left(\frac{E_0\omega_0}{L}\right)\frac{s}{\{(s+k)^2 + n^2\}\left(s^2 + \omega_0^2\right)}. \tag{4.2.81}$$

Using the rules of partial fractions, it turns out that

$$\bar{I}(s) = \left(\frac{E_0\omega_0}{L}\right)\left[\frac{As - B}{(s+k)^2 + n^2} - \frac{As - C}{s^2 + \omega_0^2}\right], \tag{4.2.82}$$

where $(A, B, C) \equiv \dfrac{(\omega_0^2 - \omega^2,\ 2k\omega^2,\ 2k\omega_0^2)}{(\omega^2 - \omega_0^2)^2 + 4k^2\omega_0^2}$.

The inversion of (4.2.82) can be completed by Table B-4 of Laplace transforms, and the solution for $I(t)$ assumes three distinct forms according to $\omega^2 > = < k^2$.

The solution for the case of low resistance $(\omega^2 > k^2)$ is

$$I(t) = \left(\frac{E_0\omega_0}{L}\right)\left[A\,e^{-kt}\cos nt - \frac{1}{n}(Ak + B)e^{-kt}\sin nt - A\cos\omega_0 t + \frac{C}{\omega_0}\sin\omega_0 t\right], \tag{4.2.83}$$

which has the equivalent form

$$I(t) = A_1 \sin(\omega_0 t - \phi_1) + A_2\, e^{-kt}\cos(nt - \phi_2), \tag{4.2.84}$$

where

$$A_1^2 = \frac{E_0^2}{L^2}\left(A^2\omega_0^2 + C^2\right) = \frac{E_0^2\omega_0^2}{L^2\left\{\left(\omega^2-\omega_0^2\right)^2 + 4k^2\omega_0^2\right\}},\ \tan\phi_1 = \frac{A\omega_0}{C}, \tag{4.2.85}$$

$$A_2^2 = \left(\frac{E_0^2\omega_0^2}{L^2}\right)\left[A^2 + \frac{1}{n^2}(Ak+B)^2\right] \text{ and } \tan\phi_2 = -\frac{(Ak+B)}{An}. \tag{4.2.86}$$

The current field consists of the steady-state and transient components. The latter decays exponentially in a time scale of the order $\frac{L}{R}$. Consequently, the steady current field is established in the electric circuit and describes the sinusoidal current with constant amplitude and phase lagging by an angle $\phi_1$. The frequency of the steady oscillating current is the same as that of the applied voltage.

In the critical situation ($\omega^2 = k^2$), the current field is derived from (4.2.82) by inversion and has the form

$$I(t) = A_1\sin(\omega_0 t - \phi_1) + \left(\frac{E_0\omega_0}{L}\right)\left[Ae^{-kt} - (Ak+B)te^{-kt}\right]. \tag{4.2.87}$$

This result suggests that the transient component of the current dies out exponentially in the limit as $t\to\infty$. Eventually, the steady oscillating current is set up in the circuit and described by the first term of (4.2.87). Finally, the solution related to the case of high resistance ($\omega^2 < k^2$) can be found by direct inversion of (4.2.82) and is given by

$$\begin{aligned} I(t) &= A_1\,\sin(\omega_0 t - \phi_1) \\ &\quad + \left(\frac{E_0\omega_0}{L}\right)\left[A\cosh\,mt - \frac{1}{m}(Ak+B)\sinh mt\right]e^{-kt}. \end{aligned} \tag{4.2.88}$$

This solution is somewhat similar to (4.2.84) with the exception of the form of the transient term which, of course, decays very rapidly as $t\to\infty$. Consequently, the steady current field is estalished in the circuit and has the same value as in (4.2.84).

Finally, we close this example by suggesting a similarity between this electric circuit system and the mechanical system as described in Example 4.2.9. Differentiation of (4.2.67) with respect to $t$ gives a second order equation for the current field as

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = \frac{dE}{dt}. \tag{4.2.89}$$

Also, an equation for the charge field $Q(t)$ can be found from (4.2.67) and (4.2.68) as

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t). \tag{4.2.90}$$

Writing $2k = \frac{R}{L}$ and $\omega^2 = \frac{1}{LC}$, the above equation can be put into the form

$$\left(\frac{d^2}{dt^2} + 2k\,\frac{d}{dt} + \omega^2\right)\binom{I}{Q} = \frac{1}{L}\begin{pmatrix}\frac{dE}{dt}\\ E\end{pmatrix}. \tag{4.2.91ab}$$

These equations are very similar to equation (4.2.43) for a harmonic oscillator. ▯

***Example 4.2.14***

(*Current and Charge in an Electrical Network*). An electrical network is a combination of several interrelated simple electric circuits. Consider a more general network consisting of two electric circuits coupled by the mutual inductance $M$ with resistances $R_1$ and $R_2$, capacitances $C_1$ and $C_2$, and self-inductances $L_1$ and $L_2$ as shown in Figure 4.5. A time-dependent voltage $E(t)$ is applied to the first circuit at time $t = 0$, when charges and currents are zero.

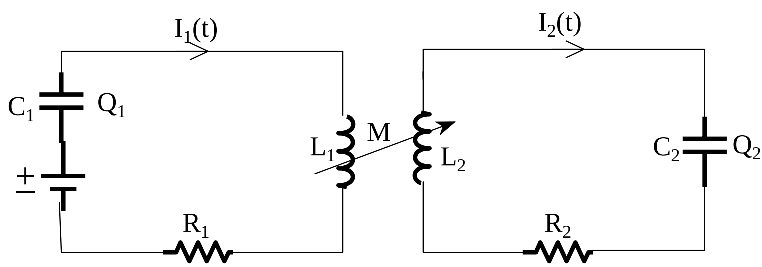


**Figure 4.5** Two coupled electric circuits.

The charge and current fields in the network are governed by the system of ordinary differential equations

$$L_1\,\frac{dI_1}{dt} + R_1 I_1 + M\frac{dI_2}{dt} + \frac{Q_1}{C_1} = E(t), \qquad t > 0 \tag{4.2.92}$$

$$M\,\frac{dI_1}{dt} + L_2\frac{dI_2}{dt} + R_2 I_2 + \frac{Q_2}{C_2} = 0, \qquad t > 0 \tag{4.2.93}$$

with

$$\frac{dQ_1}{dt} = I_1 \quad \text{and} \quad \frac{dQ_2}{dt} = I_2.$$

The initial conditions are

$$I_1 = 0,\ Q_1 = 0,\ I_2 = 0,\ Q_2 = 0 \qquad \text{at}\ t = 0. \tag{4.2.94}$$

Eliminating the currents from (4.2.92) and (4.2.93), we obtain

$$\left(L_1D^2 + R_1D + \frac{1}{C_1}\right)Q_1 + MD^2Q_2 = E(t), \tag{4.2.95}$$

$$MD^2Q_1 + \left(L_2D^2 + R_2D + \frac{1}{C_2}\right)Q_2 = 0, \tag{4.2.96}$$

where $D \equiv \dfrac{d}{dt}$.

The Laplace transform can be used to solve this system for $Q_1$ and $Q_2$. Similarly, we can find solutions for the current fields $I_1$ and $I_2$ independently or from the charge fields. We leave it as an exercise for the reader.

In the absence of the external voltage ($E = 0$) with $R_1 = R_2 = 0$, $L_1 = L_2 = L$ and $C_1 = C_2 = C$, addition and subtraction of (4.2.95) and (4.2.96) give

$$\ddot{Q}_+ + \alpha^2 Q_+ = 0, \quad \ddot{Q}_- + \beta^2 Q_- = 0, \tag{4.2.97ab}$$

where

$$Q_+ = Q_1 + Q_2, \quad Q_- = Q_1 - Q_2,$$

$$\alpha^2 = [C(L+M)]^{-1}, \quad \text{and} \quad \beta^2 = [\mathrm{C(L-M)}]^{-1}.$$

Clearly, the system executes uncoupled simple harmonic oscillations with frequencies $\alpha$ and $\beta$. Hence, the normal modes can be generated in this freely oscillatory electrical system.

Finally, in the absence of capacitances ($C_1 \to \infty$, $C_2 \to \infty$), the above network reduces to a simple one that consists of two electric circuits coupled by the mutual inductance $M$ with inductances $L_1$ and $L_2$, and resistances $R_1$ and $R_2$. As shown in Figure 4.6, an external voltage is applied to the first circuit at time $t = 0$.

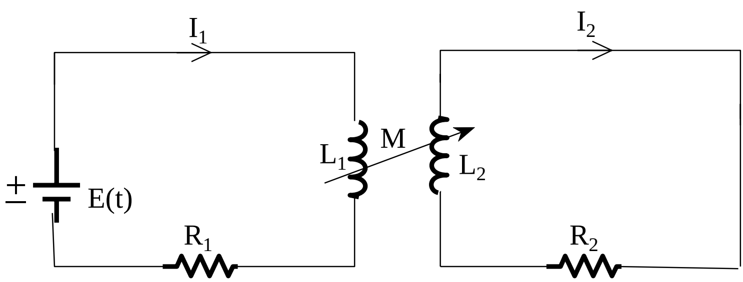


**Figure 4.6** Two coupled electric circuits without capacitances.

The current fields in the network are governed by a pair of coupled ordinary differential equations

$$L_1 \frac{dI_1}{dt} + R_1 I_1 + M \frac{dI_2}{dt} = E(t), \qquad t > 0, \tag{4.2.98}$$

$$M \frac{dI_1}{dt} + L_2 \frac{dI_2}{dt} + R_2 I_2 = 0, \qquad t > 0, \tag{4.2.99}$$

where $I_1(t)$ and $I_2(t)$ are the currents in the first and the second circuits, respectively. The initial conditions are

$$I_1(0) = I_2(0) = 0. \tag{4.2.100}$$

We shall not pursue the problem further because the transform method of solution is a simple exercise. $\square$

***Example 4.2.15***

(*Linear Dynamical Systems and Signals.* In physical and engineering sciences, a large number of linear dynamical systems with a time dependent *input signal* $f(t)$ that generates an *output signal* $x(t)$ can be described by the ordinary differential equation with constant coefficients

$$(D^n + a_{n-1}D^{n-1} + \cdots + a_0)\, x(t) = (D^m + b_{m-1}D^{m-1} + \cdots + b_0)\, f(t), \tag{4.2.101}$$

where $D \equiv \dfrac{d}{dt}$ is the differential operator, $a_r$ and $b_r$ are constants.

We apply the Laplace transform to find the output $x(t)$ so that (4.2.101) becomes

$$\bar{p}_n(s)\, \bar{x}(s) - \bar{R}_{n-1} = \bar{q}_m(s)\, \bar{f}(s) - \bar{S}_{m-1}, \tag{4.2.102}$$

where

$$\bar{p}_n(s) = s^n + a_{n-1}\, s^{n-1} + \cdots + a_0, \quad \bar{q}_m(s) = s^m + a_{m-1}\, s^{m-1} + \cdots + b_0,$$

$$\bar{R}_{n-1}(s) = \sum_{r=0}^{n-1} s^{n-r-1}\, x^{(r)}(0), \qquad \bar{S}_{m-1}(s) = \sum_{r=0}^{m-1} s^{m-r-1}\, f^{(r)}(0).$$

It is convenient to express (4.2.102) in the form

$$\bar{x}(s) = \bar{h}(s)\, \bar{f}(s) + \bar{g}(s), \tag{4.2.103}$$

where

$$\bar{h}(s) = \frac{\bar{q}_m(s)}{\bar{p}_n(s)} \quad \text{and} \quad \bar{g}(s) = \frac{\bar{R}_{n-1}(s) - \bar{S}_{m-1}(s)}{\bar{p}_n(s)}, \tag{4.2.104ab}$$

and $\bar{h}(s)$ is usually called the *transfer function.*

The inverse Laplace transform combined with the Convolution Theorem leads to the formal solution

$$x(t) = \int_0^t f(t-\tau)\, h(\tau) d\tau \; + \; g(t). \tag{4.2.105}$$

With zero initial data, $\bar{g}(s) = 0$, the transfer function takes the simple form

$$\bar{h}(s) = \frac{\bar{x}(s)}{\bar{f}(s)}. \tag{4.2.106}$$

If $f(t) = \delta(t)$ so that $\bar{f}(s) = 1$, then the output function is

$$x(t) = \int_0^t \delta(t-\tau)\, h(\tau) d\tau = h(t), \tag{4.2.107}$$

and $h(t)$ is known as the *impulse response*. ▯

***Example 4.2.16***

(*Delay Differential Equations*). In many problems, the derivatives of the unknown function $x(t)$ are related to its value at different times $t - \tau$. This leads us to consider differential equations of the form

$$\frac{dx}{dt} + a\, x(t-\tau) = f(t), \tag{4.2.108}$$

where $a$ is a constant and $f(t)$ is a given function. Equations of this type are called *delay differential equations*. In general, initial value problems for these equations involve the specification of $x(t)$ in the interval $t_0 - \tau \le t < t_0$, and this information combined with the equation itself is sufficient to determine $x(t)$ for $t > t_0$.

We show how equation (4.2.108) can be solved by the Laplace transform when $t_0 = 0$ and $x(t) = x_0$ for $t \le 0$. In view of the initial condition, we can write

$$x(t-\tau) = x(t-\tau)\, H(t-\tau)$$

so equation (4.2.108) is equivalent to

$$\frac{dx}{dt} + a\, x(t-\tau)\, H(t-\tau) = f(t). \tag{4.2.109}$$

Application of the Laplace transform to (4.2.109) gives

$$s\, \bar{x}(s) - x_0 + a \exp(-\tau s)\, \bar{x}(s) = \bar{f}(s).$$

Or,

$$\bar{x}(s) = \frac{x_0 + \bar{f}(s)}{\{s + a\exp(-\tau s)\}} \tag{4.2.110}$$

$$= \frac{1}{s}\{x_0 + \bar{f}(s)\}\left[1 + \frac{a}{s}\,\exp(-\tau s)\right]^{-1}$$

$$= \frac{1}{s}\{x_0 + \bar{f}(s)\}\sum_{n=0}^{\infty}(-1)^n\left(\frac{a}{s}\right)^n\exp(-n\tau s). \tag{4.2.111}$$

The inverse Laplace transform gives the formal solution

$$x(t) = \mathscr{L}^{-1}\left[\frac{1}{s}\{x_0 + \bar{f}(s)\}\sum_{n=0}^{\infty}(-1)^n\left(\frac{a}{s}\right)^n\exp(-n\tau s)\right]. \tag{4.2.112}$$

In order to write an explicit solution, we choose $x_0 = 0$ and $f(t) = t$, and hence, (4.2.112) becomes

$$x(t) = \mathscr{L}^{-1}\left[\frac{1}{s^3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{a}{s}\right)^n\exp(-n\tau s)\right]$$

$$= \sum_{n=0}^{\infty}(-1)^n a^n\,\frac{(t - n\tau)^{n+2}}{(n+2)!}\,H(t - n\tau), \quad t > 0. \tag{4.2.113}$$

□

***Example 4.2.17***

(*The Renewal Equation in Statistics*). The random function $X(t)$ of time $t$ represents the number of times some event has occurred between time 0 and time $t$, and is usually referred to as a *counting process*. A random variable $X_n$ that records the time it assumes for $X$ to get the value $n$ from the $n-1$ is referred to as an *inter-arrival time*. If the random variables $X_1, X_2, X_3, \ldots$ are independent and identically distributed, then the counting process $X(t)$ is called a *renewal process*. We represent their common probability distribution function by $F(t)$ and the density function by $f(t)$ so that $F'(t) = f(t)$. The *renewal function* is defined by the expected number of times the event being counted occurs by time $t$ and is denoted by $r(t)$ so that

$$r(t) = E\{X(t)\} = \int_0^{\infty} E\{X(t)|X_1 = x\}f(x)dx, \tag{4.2.114}$$

where $E\{X(t)|X_1 = x\}$ is the conditional expected value of $X(t)$ under the condition that $X_1 = x$ and has the value

$$E\{X(t)|X_1 = x\} = [1 + r(t - x)]\,H(t - x). \tag{4.2.115}$$

Thus,

$$r(t) = \int_0^t \{1 + r(t-x)\} f(x) dx.$$

Or,

$$r(t) = F(t) + \int_0^t r(t-x)\, f(x) dx. \tag{4.2.116}$$

This is called the *renewal equation* in mathematical statistics. We solve the equation by taking the Laplace transform with respect to $t$, and the Laplace transformed equation is

$$\bar{r}(s) = \overline{F}(s) + \bar{r}(s)\, \bar{f}(s).$$

Or,

$$\bar{r}(s) = \frac{\overline{F}(s)}{1 - \bar{f}(s)}. \tag{4.2.117}$$

The inverse transform gives the formal solution of the renewal function

$$r(t) = \mathscr{L}^{-t} \left\{ \frac{\overline{F}(s)}{1 - \bar{f}(s)} \right\}. \tag{4.2.118}$$

▯

## 4.3 Partial Differential Equations, Initial and Boundary Value Problems

The Laplace transform method is very useful in solving a variety of partial differential equations with assigned initial and boundary conditions. The following examples illustrate the use of the Laplace transform method.

***Example 4.3.1***
(*First-Order Initial-Boundary Value Problem*). Solve the equation

$$u_t + x u_x = x, \qquad x > 0, \quad t > 0, \tag{4.3.1}$$

with the initial and boundary conditions

$$u(x,0) = 0 \qquad \text{for} \quad x > 0, \tag{4.3.2}$$
$$u(0,t) = 0 \qquad \text{for} \quad t > 0. \tag{4.3.3}$$

We apply the Laplace transform of $u(x,t)$ with respect to $t$ to obtain

$$s\,\bar{u}(x,s)+x\,\frac{d\bar{u}}{dx}=\frac{x}{s}, \qquad \bar{u}(0,s)=0.$$

Using the integrating factor $x^s$, the solution of this transformed equation is

$$\bar{u}(x,s)=A\,x^{-s}+\frac{x}{s(s+1)},$$

where $A$ is a constant of integration. Since $\bar{u}(0,s)=0$, $A=0$ for a bounded solution. Consequently,

$$\bar{u}(x,s)=\frac{x}{s(s+1)}=x\left(\frac{1}{s}-\frac{1}{s+1}\right).$$

The inverse Laplace transform gives the solution

$$u(x,t)=x(1-e^{-t}). \tag{4.3.4}$$

□

***Example 4.3.2***

Find the solution of the equation

$$x\,u_t+u_x=x, \qquad x>0, \quad t>0 \tag{4.3.5}$$

with the same initial and boundary conditions (4.3.2) and (4.3.3).

Application of the Laplace transform with respect to $t$ to (4.3.5) with the initial conditon gives

$$\frac{d\bar{u}}{dx}+x\,s\,\bar{u}=\frac{x}{s}.$$

Using the integrating factor $\exp\left(\frac{1}{2}\,x^2 s\right)$ gives the solution

$$\bar{u}(x,s)=\frac{1}{s^2}+A\exp\left(-\frac{1}{2}\,s\,x^2\right),$$

where $A$ is an integrating constant. Since $\bar{u}(0,s)=0$, $A=-\dfrac{1}{s^2}$ and hence, the solution is

$$\bar{u}(x,s)=\frac{1}{s^2}\left[1-\exp\left(-\frac{1}{2}\,x^2\,s\right)\right]. \tag{4.3.6}$$

Finally, we obtain the solution by inversion

$$u(x,t)=t-\left(t-\frac{1}{2}\,x^2\right)H\left(t-\frac{x^2}{2}\right). \tag{4.3.7}$$

Or, equivalently,

$$u(x,t) = \left\{ \begin{array}{ll} t, & 2t < x^2 \\ \frac{1}{2}x^2, & 2t > x^2 \end{array} \right\}. \tag{4.3.8}$$

□

**Example 4.3.3**

(*The Heat Conduction Equation in a Semi-Infinite Medium*). Solve the equation

$$u_t = \kappa\, u_{xx}, \quad x > 0, \quad t > 0 \tag{4.3.9}$$

with the initial and boundary conditions

$$u(x,0) = 0, \qquad x > 0 \tag{4.3.10}$$

$$u(0,t) = f(t), \qquad t > 0 \tag{4.3.11}$$

$$u(x,t) \to 0 \qquad \text{as } x \to \infty, \quad t > 0. \tag{4.3.12}$$

Application of the Laplace transform with respect to $t$ to (4.3.9) gives

$$\frac{d^2\bar{u}}{dx^2} - \frac{s}{\kappa}\,\bar{u} = 0. \tag{4.3.13}$$

The general solution of this equation is

$$\bar{u}(x,s) = A \exp\left(-x\sqrt{\frac{s}{\kappa}}\right) + B \exp\left(x\sqrt{\frac{s}{\kappa}}\right). \tag{4.3.14}$$

where $A$ and $B$ are integrating constants. For a bounded solution, $B \equiv 0$, and using $\bar{u}(0,s) = \bar{f}(s)$, we obtain the solution

$$\bar{u}(x,s) = \bar{f}(s) \exp\left(-x\sqrt{\frac{s}{\kappa}}\right). \tag{4.3.15}$$

The inversion theorem gives the solution

$$u(x,t) = \frac{x}{2\sqrt{\pi\kappa}} \int_0^t f(t-\tau)\tau^{-3/2} \exp\left(-\frac{x^2}{4\kappa\tau}\right) d\tau, \tag{4.3.16}$$

which is, by putting $\lambda = \frac{x}{2\sqrt{\kappa\tau}}$, or, $d\lambda = -\frac{x}{4\sqrt{\kappa}}\,\tau^{-3/2} d\tau$,

$$= \frac{2}{\sqrt{\pi}} \int_{\frac{x}{2\sqrt{\kappa t}}}^{\infty} f\left(t - \frac{x^2}{4\kappa\lambda^2}\right) e^{-\lambda^2}\, d\lambda. \tag{4.3.17}$$

This is the formal solution of the problem.

In particular, if $f(t) = T_0 =$ constant, solution (4.3.17) becomes

$$u(x,t) = \frac{2T_0}{\sqrt{\pi}} \int_{\frac{x}{\kappa t}}^{\infty} e^{-\lambda^2} d\lambda = T_0 \, erfc\left(\frac{x}{2\sqrt{\kappa t}}\right). \tag{4.3.18}$$

Clearly, the temperature distribution tends asymptotically to the constant value $T_0$ as $t \to \infty$.

We consider another physical problem that is concerned with the determination of the temperature distribution in a semi-infinite solid when the rate of flow of heat is prescribed at the end $x = 0$. Thus, the problem is to solve diffusion equation (4.3.9) subject to conditions (4.3.10) and (4.3.12)

$$-k\left(\frac{\partial u}{\partial x}\right) = g(t) \quad \text{at}\, x = 0, \quad t > 0, \tag{4.3.19}$$

where $k$ is a constant that is called *thermal conductivity.*

Application of the Laplace transform gives the solution of the transformed problem

$$\bar{u}(x,s) = \frac{1}{k}\sqrt{\frac{\kappa}{s}}\; \bar{g}(s) \exp\left(-x\sqrt{\frac{s}{\kappa}}\right). \tag{4.3.20}$$

The inverse Laplace transform yields the solution

$$u(x,t) = \frac{1}{k}\sqrt{\frac{\kappa}{\pi}} \int_0^t g(t-\tau)\, \tau^{-\frac{1}{2}} \exp\left(-\frac{x^2}{4\kappa t}\right) d\tau, \tag{4.3.21}$$

which is, by the change of variable $\lambda = \dfrac{x}{2\sqrt{\kappa \tau}}$,

$$= \frac{x}{k\sqrt{\pi}} \int_{\frac{x}{\sqrt{4\kappa t}}}^{\infty} g\left(t - \frac{x^2}{4\kappa\lambda^2}\right) \lambda^{-2} e^{-\lambda^2}\, d\lambda. \tag{4.3.22}$$

In particular, if $g(t) = T_0 =$ constant, the solution becomes

$$u(x,t) = \left(\frac{T_0 x}{k\sqrt{\pi}}\right) \int_{\frac{x}{\sqrt{4\kappa t}}}^{\infty} \lambda^{-2} e^{-\lambda^2}\, d\lambda.$$

Integrating this result by parts gives the solution

$$u(x,t) = \frac{T_0}{\kappa}\left[2\sqrt{\frac{kt}{\pi}} \exp\left(-\frac{x^2}{4\kappa t}\right) - x\; erfc\left(\frac{x}{2\sqrt{\kappa t}}\right)\right]. \tag{4.3.23}$$

Alternatively, the heat conduction problem (4.3.9)–(4.3.12) can be solved by using fractional derivatives (see Chapter 5 or Debnath, 1978). We recall (4.3.15) and rewrite it

$$\frac{\partial \bar{u}}{\partial x} = -\sqrt{\frac{s}{\kappa}}\,\bar{u}. \tag{4.3.24}$$

In view of (3.9.21), this can be expressed in terms of fractional derivative of order $\frac{1}{2}$ as

$$\frac{\partial u}{\partial x} = -\frac{1}{\sqrt{\kappa}}\mathscr{L}^{-1}\left\{\sqrt{s}\,\bar{u}(x,s)\right\} = -\frac{1}{\sqrt{\kappa}}\;{}_0D_t^{\frac{1}{2}}u(x,t). \tag{4.3.25}$$

Thus, the heat flux is expressed in terms of the fractional derivative. In particular, when $u(0,t) =$ constant $= T_0$, then the heat flux at the surface is

$$-k\left(\frac{\partial u}{\partial x}\right)_{x=0} = \frac{k}{\sqrt{\kappa}}D_t^{\frac{1}{2}}T_0 = \frac{kT_0}{\sqrt{\pi\kappa\,t}}. \tag{4.3.26}$$

□

***Example 4.3.4***
(*Diffusion Equation in a Finite Medium*). Solve the diffusion equation

$$u_t = \kappa\, u_{xx}, \quad 0 < x < a, \quad t > 0, \tag{4.3.27}$$

with the initial and boundary conditions

$$u(x,0) = 0, \quad 0 < x < a, \tag{4.3.28}$$
$$u(0,t) = U, \quad t > 0, \tag{4.3.29}$$
$$u_x(a,t) = 0, \quad t > 0, \tag{4.3.30}$$

where $U$ is a constant.

We introduce the Laplace transform of $u(x,t)$ with respect to $t$ to obtain

$$\frac{d^2\bar{u}}{dx^2} - \frac{s}{\kappa}\,\bar{u} = 0, \qquad 0 < x < a, \tag{4.3.31}$$

$$\bar{u}(0,s) = \frac{U}{s}, \quad \left(\frac{d\bar{u}}{dx}\right)_{x=a} = 0. \tag{4.3.32ab}$$

The general solution of (4.3.31) is

$$\bar{u}(x,s) = A\cosh\left(x\sqrt{\frac{s}{\kappa}}\right) + B\sinh\left(x\sqrt{\frac{s}{\kappa}}\right), \tag{4.3.33}$$

where $A$ and $B$ are constants of integration. Using (4.3.32ab), we obtain the values of $A$ and $B$ so that the solution (4.3.33) becomes

$$\bar{u}(x,s) = \frac{U}{s} \cdot \frac{\cosh\left[(a-x)\sqrt{\frac{s}{\kappa}}\right]}{\cosh\left(a\sqrt{\frac{s}{\kappa}}\right)}. \tag{4.3.34}$$

The inverse Laplace transform gives the solution

$$u(x,t) = U\mathscr{L}^{-1}\left\{\frac{\cosh(a-x)\sqrt{\frac{s}{\kappa}}}{s\cosh\left(a\sqrt{\frac{s}{\kappa}}\right)}\right\}. \tag{4.3.35}$$

The inversion can be carried out by the Cauchy Residue Theorem to obtain

$$u(x,t) = U\left[1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\cos\left\{\frac{(2n-1)(a-x)\pi}{2a}\right\} \times \exp\left\{-(2n-1)^2\left(\frac{\pi}{2a}\right)^2\kappa t\right\}\right], \tag{4.3.36}$$

which is, by expanding the cosine term,

$$= U\left[1 - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin\left\{\left(\frac{2n-1}{2a}\right)\pi x\right\} \times \exp\left\{-(2n-1)^2\left(\frac{\pi}{2a}\right)^2\kappa t\right\}\right]. \tag{4.3.37}$$

This result can be obtained by the method of separation of variables. ☐

***Example 4.3.5***

(*Diffusion in a Finite Medium*). Solve the one-dimensional diffusion equation in a finite medium $0 < z < a$, where the concentration function $C(z,t)$ satisfies the equation

$$C_t = \kappa\, C_{zz}, \quad 0 < z < a, \quad t > 0, \tag{4.3.38}$$

and the initial and boundary data

$$C(z,0) = 0 \quad \text{for } 0 < z < a, \tag{4.3.39}$$

$$C(z,t) = C_0 \quad \text{for } z = a, \quad t > 0, \tag{4.3.40}$$

$$\frac{\partial C}{\partial z} = 0 \quad \text{for } z = 0, \quad t > 0, \tag{4.3.41}$$

where $C_0$ is a constant.

Application of the Laplace transform of $C(z,t)$ with respect to $t$ gives

$$\frac{d^2\bar{C}}{dz^2} - \left(\frac{s}{\kappa}\right)\bar{C} = 0, \quad 0 < z < a,$$
$$\bar{C}(a,s) = \frac{C_0}{s}, \quad \left(\frac{d\bar{C}}{dz}\right)_{z=0} = 0.$$

The solution of this system is

$$\bar{C}(z,s) = \frac{C_0 \cosh\left(z\sqrt{\frac{s}{\kappa}}\right)}{s\cosh\left(a\sqrt{\frac{s}{\kappa}}\right)}, \tag{4.3.42}$$

which is, by writing $\alpha = \sqrt{\frac{s}{\kappa}}$,

$$\begin{aligned}
&= \frac{C_0}{s}\frac{(e^{\alpha z}+e^{-\alpha z})}{(e^{\alpha a}+e^{-\alpha a})} \\
&= \frac{C_0}{s}[\exp\{-\alpha(a-z)\}+\exp\{-\alpha(a+z)\}]\sum_{n=0}^{\infty}(-1)^n\exp(-2n\alpha a) \\
&= \frac{C_0}{s}\left\{\sum_{n=0}^{\infty}(-1)^n\exp[-\alpha\{(2n+1)a-z\}]\right. \\
&\qquad\left.+\sum_{n=0}^{\infty}(-1)^n\exp[-\alpha\{(2n+1)a+z\}]\right\}.
\end{aligned} \tag{4.3.43}$$

Using the result (3.7.4), we obtain the final solution

$$\begin{aligned}
C(z,t) = C_0&\left\{\sum_{n=0}^{\infty}(-1)^n\left[erfc\left\{\frac{(2n+1)a-z}{2\sqrt{\kappa t}}\right\}\right.\right. \\
&\qquad\left.\left.+\,erfc\left\{\frac{(2n+1)a+z}{2\sqrt{\kappa t}}\right\}\right]\right\}.
\end{aligned} \tag{4.3.44}$$

This solution represents as infinite series of complementary error functions. The successive terms of this series are in fact the concentrations at depths $a-z, a+z, 3a-z, 3a+z, \ldots$ in the medium. The series converges rapidly for all except large values of $\left(\frac{\kappa t}{a^2}\right)$. □

***Example 4.3.6***

(*The Wave Equation for the Transverse Vibration of a Semi-Infinite String*). Find the displacement of a semi-infinite string which is initially at rest in its equilibrium position. At time $t=0$, the end $x=0$ is constrained to move so that the displacement is $u(0,t)=Af(t)$ for $t\geq 0$, where $A$ is a constant. The problem is to solve the one-dimensional wave equation

$$u_{tt}=c^2u_{xx}, \quad 0\leq x<\infty, \quad t>0, \tag{4.3.45}$$

with the boundary and initial conditions

$$u(x,t)=Af(t) \qquad \text{at } x=0, \qquad t\geq 0, \tag{4.3.46}$$

$$u(x,t)\to 0 \qquad \text{as } x\to\infty, \qquad t\geq 0, \tag{4.3.47}$$

$$u(x,t)=0=\frac{\partial u}{\partial t} \qquad \text{at } t=0 \qquad \text{for } 0<x<\infty. \tag{4.3.48ab}$$

Application of the Laplace transform of $u(x,t)$ with respect to $t$ gives

$$\begin{aligned} \frac{d^2\bar{u}}{dx^2}-\frac{s^2}{c^2}\,\bar{u}=0, &\qquad \text{for } 0\leq x<\infty, \\ \bar{u}(x,s)=A\bar{f}(s) &\qquad \text{at } x=0, \\ \bar{u}(x,s)\to 0 &\qquad \text{as } x\to\infty. \end{aligned}$$

The solution of this differential system is

$$\bar{u}(x,s)=A\bar{f}(s)\exp\left(-\frac{xs}{c}\right). \tag{4.3.49}$$

Inversion gives the solution

$$u(x,t)=Af\left(t-\frac{x}{c}\right)H\left(t-\frac{x}{c}\right). \tag{4.3.50}$$

In other words, the solution is

$$u(x,t)=\begin{bmatrix} Af\left(t-\frac{x}{c}\right), & t>\frac{x}{c} \\ 0, & t<\frac{x}{c} \end{bmatrix}. \tag{4.3.51}$$

This solution represents a wave propagating at a velocity $c$ with the characteristic $x=ct$. $\square$

***Example 4.3.7***

(*Potential and Current in an Electric Transmission Line*). We consider a transmission line which is a model of co-axial cable containing resistance $R$, inductance $L$, capacitance $C$, and leakage conductance $G$. The current $I(x,t)$ and potential $V(x,t)$ at a point $x$ and time $t$ in the line satisfy the coupled equations

$$L\,\frac{\partial I}{\partial t} + RI = -\frac{\partial V}{\partial x}, \tag{4.3.52}$$

$$C\,\frac{\partial V}{\partial t} + GV = -\frac{\partial I}{\partial x}. \tag{4.3.53}$$

If $I$ or $V$ is eliminated from these equations, both $I$ and $V$ satisfy the same equation in the form

$$\frac{1}{c^2}\,u_{tt} - u_{xx} + au_t + bu = 0 \tag{4.3.54}$$

where $c^2 = (LC)^{-1}, a = LG + RC$, and $b = RG$. Equation (4.3.54) is called the *telegraph equation.*

Or, equivalently, the telegraph equation can be written in the form

$$u_{tt} = c^2\,u_{xx} - (p+q)\,u_t - p\,q\,u \tag{4.3.55}$$

where $ac^2 = \frac{R}{C} + \frac{G}{C} = p + q$ and $bc^2 = pq$.

For a lossless transmission line, $R = 0$ and $G = 0$, $I$ or $V$ satisfies the classical wave equation

$$u_{tt} = c^2 u_{xx}. \tag{4.3.56}$$

The solution of this equation with the initial and boundary data is obtained from Example 4.3.6 using the boundary conditions in the potential $V(x, t)$:

$$\text{(i)}\quad V(x,t) = V_0 f(t) \qquad \text{at} \quad x = 0,\; t > 0. \tag{4.3.57}$$

This corresponds to a signal at the end $x = 0$ for $t > 0$, and $V(x, t) \to 0$ as $x \to \infty$ for $t > 0$.

A special case when $f(t) = H(t)$ is also of interest. The solution for this special case is given by

$$V(x,t) = V_0 f\left(t - \frac{x}{c}\right) H\left(t - \frac{x}{c}\right). \tag{4.3.58}$$

This represents a wave propagating at a speed $c$ with the characteristic $x = ct$.

Similarly, the solution associated with the boundary data

$$\text{(ii)}\quad \mathrm{V(x,t)} = V_0 \cos\omega t \qquad \text{at} \quad \mathrm{x} = 0 \text{ for } \mathrm{t} > 0 \tag{4.3.59}$$

$$V(x,t) \to 0 \qquad \text{as} \quad \mathrm{x} \to \infty \text{ for } \mathrm{t} > 0 \tag{4.3.60}$$

can readily be obtained from Example 4.3.6.

For ideal submarine cable (or the *Kelvin ideal cable*), $L = 0$ and $G = 0$ equation (4.3.54) reduces to the classical diffusion equation

$$u_t = \kappa\,u_{xx}, \tag{4.3.61}$$

where $\kappa = a^{-1} = (RC)^{-1}$.

The method of solution is similar to that discussed in Example 4.3.3. Using the boundary data (i), the solution for the potential $V(x,t)$ is given by

$$V(x,t) = V_0 \, erfc\left(\frac{x}{2\sqrt{\kappa t}}\right). \tag{4.3.62}$$

The current field is given by

$$I(x,t) = -\frac{1}{R}\left(\frac{\partial V}{\partial x}\right) = \frac{V_0}{R}(\pi \kappa t)^{-1/2} \exp\left(-\frac{x^2}{4\kappa t}\right). \tag{4.3.63}$$

For very large $x$, the asymptotic representation of the complementary error function is

$$erfc(x) \sim \frac{1}{x\sqrt{\pi}} \exp(-x^2), \quad x \to \infty. \tag{4.3.64}$$

In view of this asymptotic representation, solution (4.3.62) becomes

$$V(x,t) \sim \frac{2V_0}{x}\left(\frac{\kappa t}{\pi}\right)^{1/2} \exp\left(-\frac{x^2}{4\kappa t}\right). \tag{4.3.65}$$

For any $t > 0$, no matter how small, solution (4.3.62) reveals that $V(x,t) > 0$ for all $x > 0$, even though $V(x,t) \to 0$ as $x \to \infty$ Thus, the signal applied at $t = 0$ propagates with the infinite speed although its amplitude is very small for large $x$. Physically, the infinite speed is unrealistic and is essentially caused by the neglect of the first term in equation (4.3.54). In a real cable, the presence of some inductance would set a limit to the speed of propagation.

Instead of the Kelvin cable, a non-inductive leady cable ($L = 0$ and $G \neq 0$) is of interest. The equation for this case is obtained from (4.3.54) in the form

$$V_{xx} - a\,V_t - b\,V = 0, \tag{4.3.66}$$

with zero initial conditions, and with the boundary data

$$V(0,t) = H(t) \text{ and } V(x,t) \to 0 \text{ as } x \to \infty. \tag{4.3.67ab}$$

The Laplace transformed problem is

$$\frac{d^2\overline{V}}{dx^2} = (sa + b)\overline{V}, \tag{4.3.68}$$

$$\overline{V}(0,s) = \frac{1}{s}, \quad \overline{V}(x,s) \to 0 \quad \text{as } x \to \infty. \tag{4.3.69ab}$$

Thus, the solution is given by

$$\overline{V}(x,s) = \frac{1}{s} \exp[-x(sa + b)^{1/2}]. \tag{4.3.70}$$

With the aid of a standard table of the inverse Laplace transform, the solution is given by

$$V(x,t) = \frac{1}{2}\, e^{x\sqrt{b}}\, erfc\left(\frac{x}{2}\sqrt{\frac{a}{t}} + \sqrt{\frac{bt}{a}}\right) + \frac{1}{2}\, e^{-x\sqrt{b}}\, erfc\left(\frac{x}{2}\sqrt{\frac{a}{t}} - \frac{bt}{a}\right). \tag{4.3.71}$$

When $G=0\,(b=0)$, the solution becomes identical with (4.3.62).

For the Heaviside distortionless cable, $\dfrac{R}{L} = \dfrac{G}{C} = k =$ constant, the potential $V(x,t)$ and the current $I(x,t)$ satisfies the same equation

$$u_{tt} + 2ku_t + k^2 u = c^2 u_{xx}, \quad 0 \le x < \infty, \quad t > 0. \tag{4.3.72}$$

We solve this equation with the initial data (4.3.48ab) and the boundary condition (4.3.57). Application of the Laplace transform with respect to $t$ to (4.3.72) gives

$$\frac{d^2\overline{V}}{dx^2} = \left(\frac{s+k}{c}\right)^2 \overline{V}. \tag{4.3.73}$$

The solution for $\overline{V}(x,s)$ with the transformed boundary condition (4.3.56) is

$$\overline{V}(x,s) = V_0 \bar{f}(s) \exp\left[-\left(\frac{s+k}{c}\right)x\right]. \tag{4.3.74}$$

This can easily be inverted to obtain the final solution

$$V(x,t) = V_0 \exp\left(-\frac{kx}{c}\right) f\left(t - \frac{x}{c}\right) H\left(t - \frac{x}{c}\right). \tag{4.3.75}$$

This solution represents the signal that propagates with velocity $c = (LC)^{-1/2}$ with exponentially decaying amplitude, but with no distortion. Thus, the signals can propagate along the Heaviside distortionless line over long distances if appropriate boosters are placed at regular intervals in order to increase the strength of the signal so as to counteract the effects of attenuation. □

***Example 4.3.8***

Find the bounded solution of the axisymmetric heat conduction equation

$$u_t = \kappa\left(u_{rr} + \frac{1}{r}u_r\right), \quad 0 \le r < a, \quad t > 0, \tag{4.3.76}$$

with the initial and boundary data

$$u(r,0) = 0 \qquad \text{for } 0 < r < a, \tag{4.3.77}$$

$$u(r,t) = f(t) \quad \text{at } r = a \text{ for } t > 0, \tag{4.3.78}$$

where $\kappa$ and $T_0$ are constants.

Application of the Laplace transform to (4.3.76) gives

$$\frac{d^2\bar{u}}{dr^2} + \frac{1}{r}\frac{d\bar{u}}{dr} - \frac{s}{\kappa}\bar{u} = 0.$$

Or,

$$r^2\frac{d^2\bar{u}}{dr^2} + r\frac{d\bar{u}}{dr} - r^2\left(\frac{s}{\kappa}\right)\bar{u} = 0. \tag{4.3.79}$$

This is the standard Bessel equation with the solution

$$\bar{u}(r,s) = AI_0\left(r\sqrt{\frac{s}{\kappa}}\right) + BK_0\left(r\sqrt{\frac{s}{\kappa}}\right), \tag{4.3.80}$$

where $A$ and $B$ are constants of integration and $I_0(x)$ and $K_0(x)$ are the modified Bessel functions of zero order.

Since $K_0(\alpha r)$ is unbounded at $r = 0$, for the bounded solution $B \equiv 0$, and hence, the solution is

$$\bar{u}(r,s) = AI_0(kr), \quad k = \sqrt{\frac{s}{\kappa}}.$$

In view of the transformed boundary condition $\bar{u}(a,s) = \bar{f}(s)$, we obtain

$$\bar{u}(r,s) = \bar{f}(s)\,\frac{I_0(kr)}{I_0(ka)} = \bar{f}(s)\bar{g}(s), \tag{4.3.81}$$

where $\bar{g}(s) = \dfrac{I_0(kr)}{I_0(ka)}$.

By Convolution Theorem 3.5.1, the solution takes the form

$$u(r,t) = \int_0^t f(t-\tau)g(\tau)d\tau, \tag{4.3.82}$$

where

$$g(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\,\frac{I_0(kr)}{I_0(ka)}ds. \tag{4.3.83}$$

This complex integral can be evaluated by the theory of residues where the poles of the integrand are at the points $s = s_n = -\kappa\alpha_n^2$, $n = 1, 2, 3, \ldots$ and $\alpha_n$ are the roots of $J_0(a\alpha) = 0$. The residue at pole $s = s_n$ is

$$\left(\frac{2i\kappa\alpha_n}{a}\right)\frac{I_0(ir\alpha_n)}{I_0'(ia\alpha_n)}\exp(-\kappa t\alpha_n^2) = \left(\frac{2\kappa\alpha_n}{a}\right)\frac{J_0(r\alpha_n)}{J_1(a\alpha_n)}\exp(-\kappa\, t\,\alpha_n^2),$$

so that

$$g(t)=\left(\frac{2\kappa}{a}\right)\sum_{n=1}^{\infty}\frac{\alpha_n J_0(r\alpha_n)}{J_1(a\alpha_n)}\exp(-\kappa\, t\, \alpha_n^2).$$

Thus, solution (4.3.82) becomes

$$u(r,t)=\left(\frac{2\kappa}{a}\right)\sum_{n=1}^{\infty}\frac{\alpha_n J_0(r\alpha_n)}{J_1(a\alpha_n)}\int_0^t f(t-\tau)\exp(-\kappa\,\tau\,\alpha_n^2)\,d\tau, \tag{4.3.84}$$

where the summation is taken over the positive roots of $J_0(a\alpha)=0$.

In particular, if $f(t)=T_0$, then the solution (4.3.84) reduces to

$$\begin{aligned} u(r,t) &= \left(\frac{2T_0}{a}\right)\sum_{n=1}^{\infty}\frac{J_0(r\alpha_n)}{\alpha_n J_1(a\alpha_n)}(1-e^{-\kappa\, t\, \alpha_n^2}) \\ &= T_0\left[1-\frac{2}{a}\sum_{n=1}^{\infty}\frac{J_0(r\alpha_n)}{\alpha_n J_1(a\alpha_n)}e^{-\kappa\, t\, \alpha_n^2}\right]. \end{aligned} \tag{4.3.85}$$

▯

***Example 4.3.9***

(*Inhomogeneous Partial Differential Equation*). We solve the inhomogeneous problem

$$u_{xt}=-\omega\sin\omega t, \quad t>0 \tag{4.3.86}$$

$$u(x,0)=x, \quad u(0,t)=0. \tag{4.3.87ab}$$

Application of the Laplace transform with respect to $t$ gives

$$\frac{d\bar{u}}{dx}=\frac{s}{s^2+\omega^2},$$

which admits the general solution

$$\bar{u}(x,s)=\frac{sx}{s^2+\omega^2}+A,$$

where $A$ is a constant. Since $\bar{u}(0,s)=0$, $A=0$ and hence, the solution is obtained by inversion as

$$u(x,t)=x\cos\omega t. \tag{4.3.88}$$

▯

***Example 4.3.10***

(*Inhomogeneous Wave Equation*). Find the solution of

$$\frac{1}{c^2}u_{tt}-u_{xx}=k\sin\left(\frac{\pi x}{a}\right), \quad 0<x<a,\ t>0, \tag{4.3.89}$$

$$u(x,0)=0=u_t(x,0), \qquad 0<x<a, \tag{4.3.90}$$

$$u(0,t)=0=u(a,t), \qquad t>0, \tag{4.3.91}$$

where $c, k$, and $a$ are constants.

Application of the Laplace transforms gives

$$\frac{d^2\bar{u}}{dx^2} - \frac{s^2}{c^2}\bar{u} = -\frac{k}{s}\sin\left(\frac{\pi x}{a}\right), \tag{4.3.92}$$

$$\bar{u}(0, s) = 0 = \bar{u}(a, s). \tag{4.3.93}$$

The general solution of equation (4.3.92) is

$$\bar{u}(x, s) = A\exp\left(\frac{sx}{c}\right) + B\exp\left(-\frac{sx}{c}\right) + \frac{k\sin\left(\frac{\pi x}{a}\right)}{a^2 s\left(s^2 + \frac{\pi^2 c^2}{a^2}\right)}. \tag{4.3.94}$$

In view of (4.3.93), $A = B = 0$, and hence, the solution (4.3.94) becomes

$$\bar{u}(x, s) = \frac{k}{\pi^2 c^2}\sin\left(\frac{\pi x}{a}\right)\left[\frac{1}{s} - \frac{s}{s^2 + \frac{\pi^2 c^2}{a^2}}\right], \tag{4.3.95}$$

which, by inversion, gives the solution,

$$u(x, t) = \frac{k}{(\pi c)^2}\left[1 - \cos\left(\frac{\pi c t}{a}\right)\right]\sin\left(\frac{\pi x}{a}\right). \tag{4.3.96}$$

$\square$

***Example 4.3.11***

(*The Stokes Problem and the Rayleigh Problem in Fluid Dynamics*). Solve the Stokes problem, which is concerned with the unsteady boundary layer flows induced in a semi-infinite viscous fluid bounded by an infinite horizontal disk at $z = 0$ due to non-torsional oscillations of the disk in its own plane with a given frequency $\omega$.

We solve the boundary layer equation in fluid dynamics

$$u_t = \nu\, u_{zz}, \quad z > 0, \quad t > 0, \tag{4.3.97}$$

with the boundary and initial conditions

$$u(z, t) = U_0\, e^{i\omega t} \qquad \text{on } z = 0,\ t > 0, \tag{4.3.98}$$

$$u(z, t) \to 0 \qquad \text{as } z \to \infty,\ t > 0, \tag{4.3.99}$$

$$u(z, t) \to 0 \qquad \text{at } t \le 0 \text{ for all } z > 0, \tag{4.3.100}$$

where $u(z, t)$ is the velocity of fluid of kinematic viscosity $\nu$ and $U_0$ is a constant.

The Laplace transform solution of the problem with the transformed boundary conditions is

$$\overline{u}(z,s) = \frac{U_0}{(s-i\omega)} \exp\left(-z\sqrt{\frac{s}{\nu}}\right). \tag{4.3.101}$$

Using a standard table of inverse Laplace transforms, we obtain the solution

$$\begin{aligned} u(z,t) = \frac{U_0}{2}\, e^{i\omega t}[&\exp(-\lambda z)\, erfc(\zeta - \sqrt{i\omega t}) \\ &+ \exp(\lambda z)\, erfc(\zeta + \sqrt{i\omega t})], \end{aligned} \tag{4.3.102}$$

where $\zeta = z/(2\sqrt{\nu t})$ is called the *similarity variable* of the viscous boundary layer theory and $\lambda = (i\omega/\nu)^{\frac{1}{2}}$. The result (4.3.101) describes the unsteady boundary layer flow.

In view of the asymptotic formula for the complementary error function

$$erfc(\zeta \mp \sqrt{i\omega t}) \sim (2,0) \quad \text{as} \quad t \to \infty, \tag{4.3.103}$$

the above solution for $u(z,t)$ has the asymptotic representation

$$u(z,t) \sim U_0 \exp(i\omega t - \lambda z) = U_0 \exp\left[i\omega t - \left(\frac{\omega}{2\nu}\right)^{\frac{1}{2}}(1+i)z\right]. \tag{4.3.104}$$

This is called the *Stokes steady-state solution.* This represents the propagation of shear waves which spread out from the oscillating disk with velocity $(\omega/k) = \sqrt{2\nu\omega}$ and exponentially decaying amplitude. The boundary layer associated with the solution has thickness of the order $\sqrt{\nu/\omega}$ in which the shear oscillations imposed by the disk decay exponentially with distance $z$ from the disk. This boundary layer is called the *Stokes layer.* In other words, the thickness of the Stokes layer is equal to the depth of penetration of vorticity which is essentially confined to the immediate vicinity of the disk for high frequency $\omega$.

The Stokes problem with $\omega = 0$ becomes the *Rayleigh problem.* In other words, the motion is generated in the fluid from rest by moving the disk impulsively in its own plane with constant velocity $U_0$. In this case, the Laplace transformed solution is

$$\overline{u}(z,s) = \frac{U_0}{s} \exp\left(-z\sqrt{\frac{s}{\nu}}\right). \tag{4.3.105}$$

Hence, the inversion gives the Rayleigh solution

$$u(z,t) = U_0\, erfc\left(\frac{z}{2\sqrt{\nu t}}\right). \tag{4.3.106}$$

This describes the growth of a boundary layer adjacent to the disk. The associated boundary layer is called the *Rayleigh layer* of thickness of the order

$\delta \sim \sqrt{\nu t}$, which grows with increasing time. The rate of growth is of the order $d\delta/dt \sim \sqrt{\nu/t}$, which diminishes with increasing time.

The vorticity of the unsteady flow is given by

$$\frac{\partial u}{\partial z} = \frac{U_0}{\sqrt{\pi \nu t}} \exp(-\zeta^2), \tag{4.3.107}$$

which decays exponentially to zero as $z >> \delta$.

Note that the vorticity is everywhere zero at $t = 0$. This implies that it is generated at the disk and diffuses outward within the Rayleigh layer. The total viscous diffusion time is $T_d \sim \left(\delta^2/\nu\right)$.

Another physical quantity related to the Stokes and Rayleigh problems is the *skin friction* on the disk defined by

$$\tau_0 = \mu \left(\frac{\partial u}{\partial z}\right)_{z=0}, \tag{4.3.108}$$

where $\mu = \nu\rho$ is the dynamic viscosity and $\rho$ is the density of the fluid. The skin friction can readily be calculated from the flow field given by (4.3.104) or (4.3.106). □

## 4.4 Solutions of Integral Equations

***DEFINITION 4.4.1*** *An equation in which the unknown function occurs under an integral is called an integral equation.*

*An equation of the form*

$$f(t) = h(t) + \lambda \int_a^b k(t,\tau) f(\tau)\, d\tau, \tag{4.4.1}$$

*in which $f$ is the unknown function, $h(t), k(t,\tau)$; and the limits of integration $a$ and $b$ are known; and $\lambda$ is a constant, is called the linear integral equation of the second kind or the linear Volterra integral equation. The function $k(t,\tau)$ is called the kernel of the equation. Such an equation is said to be homogeneous or inhomogeneous according to $h(t) = 0$ or $h(t) \neq 0$. If the kernel of the equation has the form $k(t,\tau) = g(t-\tau)$, the equation is referred to as the convolution integral equation.*

In this section, we show how the Laplace transform method can be applied successfully to solve the convolution integral equations. This method is simple and straightforward, and can be illustrated by examples.

To solve the convolution integral equation of the form

$$f(t) = h(t) + \lambda \int_0^t g(t-\tau)f(\tau)\,d\tau, \tag{4.4.2}$$

we take the Laplace transform of this equation to obtain

$$\bar{f}(s) = \bar{h}(s) + \lambda \mathscr{L}\left\{\int_0^t g(t-\tau)f(\tau)\,d\tau\right\},$$

which is, by the Convolution Theorem,

$$\bar{f}(s) = \bar{h}(s) + \lambda \bar{f}(s)\bar{g}(s).$$

Or,

$$\bar{f}(s) = \frac{\bar{h}(s)}{1-\lambda\bar{g}(s)}. \tag{4.4.3}$$

Inversion gives the formal solution

$$f(t) = \mathscr{L}^{-1}\left\{\frac{\bar{h}(s)}{1-\lambda\bar{g}(s)}\right\}. \tag{4.4.4}$$

In many simple cases, the right-hand side can be inverted by using partial fractions or the theory of residues. Hence, the solution can readily be found.

***Example 4.4.1***
Solve the integral equation

$$f(t) = a + \lambda \int_0^t f(\tau)\,d\tau. \tag{4.4.5}$$

We take the Laplace transform of (4.4.5) to find

$$\bar{f}(s) = \frac{a}{s-\lambda},$$

whence, by inversion, it follows that

$$f(t) = a\exp(\lambda t). \tag{4.4.6}$$

□

***Example 4.4.2***
Solve the integro-differential equation

$$f(t) = a\sin t + 2\int_0^t f'(\tau)\sin(t-\tau)\,d\tau, \quad f(0)=0. \tag{4.4.7}$$

Taking the Laplace transform, we obtain

$$\bar{f}(s)=\frac{a}{s^2+1}+2\mathscr{L}\{f'(t)\}\mathscr{L}\{\sin t\}$$

Or,

$$\bar{f}(s)=\frac{a}{s^2+1}+2\frac{\{s\bar{f}(s)-f(0)\}}{s^2+1}.$$

Hence, by the initial condition,

$$\bar{f}(s)=\frac{a}{(s-1)^2}.$$

Inversion yields the solution

$$f(t)=a\,t\exp(t). \tag{4.4.8}$$

□

***Example 4.4.3***
Solve the integral equation

$$f(t)=a\,t^n-e^{-bt}-c\int_0^t f(\tau)\,e^{c(t-\tau)}\,d\tau. \tag{4.4.9}$$

Taking the Laplace transform, we obtain

$$\bar{f}(s)=\frac{a\,n!}{s^{n+1}}-\frac{1}{s+b}-\bar{f}(s)\frac{c}{s-c}$$

so that we have

$$\begin{aligned}
\bar{f}(s) &= \left(\frac{s-c}{s}\right)\left[\frac{a\,n!}{s^{n+1}}-\frac{1}{s+b}\right]\\
&= \frac{a\,n!}{s^{n+1}}-\frac{(ac)n!}{s^{n+2}}-\frac{1}{s}\left[\frac{s+b-c-b}{s+b}\right]\\
&= \frac{a\,n!}{s^{n+1}}-\frac{(ac)n!}{s^{n+2}}-\frac{1}{s}+\frac{c+b}{b}\left[\frac{1}{s}-\frac{1}{s+b}\right]\\
&= \frac{a\,n!}{s^{n+1}}-\frac{(ac)n!}{s^{n+2}}-\frac{1}{s}+\left(1+\frac{c}{b}\right)\frac{1}{s}-\left(1+\frac{c}{b}\right)\frac{1}{s+b}\\
&= \frac{a\,n!}{s^{n+1}}-\frac{(ac)n!}{s^{n+2}}+\frac{c}{bs}-\left(1+\frac{c}{b}\right)\frac{1}{s+b}
\end{aligned}$$

Inversion yields the solution

$$f(t)=at^n-\frac{n!ac}{(n+1)!}t^{n+1}+\frac{c}{b}-\left(1+\frac{c}{b}\right)e^{-bt}.$$

□

## 4.5 Solutions of Boundary Value Problems

The Laplace transform technique is also very useful in finding solutions of certain simple boundary value problems that arise in many areas of applied mathematics and engineering sciences. We illustrate the method by solving boundary value problems in the theory of deflection of elastic beams.

A horizontal beam experiences a vertical deflection due to the combined effect of its own weight and the applied load on the beam. We consider a beam of length $\ell$ and its equilibrium position is taken along the horizontal $x$-axis.

***Example 4.5.1***

(*Deflection of Beams*). The differential equation for the vertical deflection $y(x)$ of a uniform beam under the action of a transverse load $W(x)$ per unit length at a distance $x$ from the origin on the $x$-axis of the beam is

$$El\frac{d^4y}{dx^4} = W(x), \qquad \text{for} \quad 0 < x < \ell, \tag{4.5.1}$$

where $E$ is *Young's modulus*, $I$ is the *moment of inertia* of the cross section about an axis normal to the plane of bending and $EI$ is called the *flexural rigidity* of the beam.

Some physical quantities associated with the problem are $y'(x), M(x) = EIy''(x)$ and $S(x) = M'(x) = EIy'''(x)$, which respectively represent the slope, bending moment, and shear at a point.

It is of interest to find the solution of (4.5.1) subject to a given loading function and simple boundary conditions involving the deflection, slope, bending moment and shear. We consider the following cases:

(i) Concentrated load on a clamped beam of length $\ell$, that is,
$W(x) \equiv W\delta(x-a)$,
$y(0) = y'(0) = 0$ and $y(\ell) = y'(\ell) = 0$,
where $W$ is a constant and $0 < a < \ell$.

(ii) Distributed load on a uniform beam of length $\ell$ clamped at $x = 0$ and unsupported at $x = \ell$, that is,
$W(x) = WH(x-a)$,
$y(0) = y'(0) = 0$, and $M(\ell) = S(\ell) = 0$.

(iii) A uniform semi-infinite beam freely hinged at $x = 0$ resting horizontally on an elastic foundation and carrying a load $W$ per unit length.

In order to solve the problem, we use the Laplace transform $\bar{y}(s)$ of $y(x)$ defined by

$$\bar{y}(s) = \int_0^\infty e^{-sx} y(x)\, dx. \tag{4.5.2}$$

In view of this transformation, equation (4.5.1) becomes

$$EI[s^4\bar{y}(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)] = \overline{W}(s). \tag{4.5.3}$$

The solution of the transformed deflection function $\bar{y}(s)$ for case (i) is

$$\bar{y}(s) = \frac{y''(0)}{s^3} + \frac{y'''(0)}{s^4} + \frac{W}{EI}\frac{e^{-as}}{s^4}. \tag{4.5.4}$$

Inversion gives

$$y(x) = y''(0)\frac{x^2}{2} + y'''(0)\frac{x^3}{6} + \frac{W}{6EI}(x-a)^3 H(x-a). \tag{4.5.5}$$

$$y'(x) = y''(0)x + \frac{1}{2}x^2 y'''(0) + \frac{W}{2EI}(x-a)^2 H(x-a). \tag{4.5.6}$$

The conditions $y(\ell) = y'(\ell) = 0$ require that

$$y''(0)\frac{\ell^2}{2} + y'''(0)\frac{\ell^3}{6} + \frac{W}{6EI}(\ell - a)^3 = 0,$$

$$y''(0)\ell + y'''(0)\frac{\ell^2}{2} + \frac{W}{2EI}(\ell - a)^2 = 0.$$

These algebraic equations determine the value of $y''(0)$ and $y'''(0)$. Solving these equations, it turns out that

$$y''(0) = \frac{Wa(\ell - a)^2}{EI\,\ell^2} \quad \text{and} \quad y'''(0) = -\frac{W(\ell - a)^2(\ell + 2a)}{EI\,\ell^3}.$$

Thus, the final solution for case (i) is

$$y(x) = \frac{W}{2EI}\left[\frac{a(\ell - a)^2 x^2}{\ell^2} - \frac{(\ell - a)^2(\ell + 2a)x^3}{3\ell^3} + \frac{(x-a)^3 H(x-a)}{3}\right]. \tag{4.5.7}$$

It is now possible to calculate the bending moment and shear at any point of the beam, and, in particular, at the ends.

The solution for case (ii) follows directly from (4.5.3) in the form

$$\overline{y}(s) = \frac{y''(0)}{s^3} + \frac{y'''(0)}{s^4} + \frac{W}{EI}\frac{e^{-as}}{s^5}. \tag{4.5.8}$$

The inverse transformation yields

$$y(x) = \frac{1}{2}y''(0)x^2 + \frac{1}{6}y'''(0)x^3 + \frac{W}{24EI}(x-a)^4 H(x-a), \tag{4.5.9}$$

where $y''(0)$ and $y'''(0)$ are to be determined from the remaining boundary conditions $M(\ell) = S(\ell) = 0$, that is, $y''(\ell) = y'''(\ell) = 0$.

From (4.5.9) with $y''(\ell) = y'''(\ell) = 0$, it follows that

$$y''(0) + y'''(0)\ell + \frac{W}{2EI}(\ell - a)^2 = 0$$
$$y'''(0) + \frac{W}{EI}(\ell - a) = 0$$

which give

$$y''(0) = \frac{W(\ell - a)(\ell + a)}{2EI} \text{ and } y'''(0) = -\frac{W}{EI}(\ell - a).$$

Hence, the solution for $y(x)$ for case (ii) is

$$y(x) = \frac{W}{2EI}\left[\frac{(\ell^2 - a^2)x^2}{2} - (\ell - a)\frac{x^3}{3} + \frac{W}{12}(x - a)^4 H(x - a)\right]. \qquad (4.5.10)$$

The shear, $S$, and the bending moment, $M$, at the origin, can readily be calculated from the solution.

The differential equation for case (iii) takes the form

$$EI\frac{d^4y}{dx^4} + ky = W, \quad x > 0, \qquad (4.5.11)$$

where the second term on the left-hand side represents the effect of elastic foundation and $k$ is a positive constant.

Writing $\left(\frac{k}{EI}\right) = 4\omega^4$, equation (4.5.11) becomes

$$\left(\frac{d^4}{dx^4} + 4\omega^4\right) y(x) = \frac{W}{EI}, \quad x > 0. \qquad (4.5.12)$$

This has to be solved subject to the boundary conditions

$$y(0) = y''(0) = 0, \qquad (4.5.13)$$
$$y(x) \text{ is finite as } x \to \infty. \qquad (4.5.14)$$

Using the Laplace transform with respect to $x$ to (4.5.12), we obtain

$$(s^4 + 4\omega^4)\,\bar{y}(s) = \left(\frac{W}{EI}\right)\frac{1}{s} + sy'(0) + y'''(0). \qquad (4.5.15)$$

In view of the Tauberian Theorem 3.8.2 (ii), that is,

$$\lim_{s\to 0} s\,\bar{y}(s) = \lim_{x\to\infty} y(x),$$

it follows that $\bar{y}(s)$ must be of the form

$$\bar{y}(s) = \frac{W}{EI}\frac{1}{s(s^4+4\omega^4)}, \tag{4.5.16}$$

which gives

$$\lim_{x\to\infty} y(x) = \frac{W}{k}. \tag{4.5.17}$$

We now write (4.5.16) as

$$\bar{y}(s) = \frac{W}{EI}\frac{1}{4\omega^4}\left[\frac{1}{s} - \frac{s^3}{s^4+4\omega^4}\right]. \tag{4.5.18}$$

Using the standard table of inverse Laplace transforms, we obtain

$$\begin{aligned} y(x) &= \frac{W}{k}(1-\cos\omega x\,\cosh\,\omega x) \\ &= \frac{W}{k}\left[1 - \frac{1}{2}\,e^{-\omega x}\cos\omega x - \frac{1}{2}\,e^{\omega x}\cos\omega x\right]. \end{aligned} \tag{4.5.19}$$

In view of (4.5.17), the final solution is

$$y(x) = \frac{W}{k}\left(1 - \frac{1}{2}\,e^{-\omega x}\cos\omega x\right). \tag{4.5.20}$$

□

## 4.6 Evaluation of Definite Integrals

The Laplace transform can be employed to evaluate easily certain definite integrals containing a parameter. Although the method of evaluation may not be very rigorous, it is quite simple and straightforward. The method is essentially based upon the permissibility of interchange of the order of integration, that is,

$$\mathscr{L}\int_a^b f(t,x)\,dx = \int_a^b \mathscr{L}f(t,x)\,dx, \tag{4.6.1}$$

and may be well described by considering some important integrals.

***Example 4.6.1***
Evaluate the integral

$$f(t) = \int_0^\infty \frac{\sin tx}{x(a^2+x^2)}\,dx. \tag{4.6.2}$$

We take the Laplace transform of (4.6.2) with respect to $t$ and interchange the order of integration, which is permissible due to uniform convergence, to obtain

$$\begin{aligned}\bar{f}(s) &= \int_0^\infty \frac{dx}{x(a^2+x^2)} \int_0^\infty e^{-st} \sin tx \, dt \\ &= \int_0^\infty \frac{dx}{(a^2+x^2)(x^2+s^2)} \\ &= \frac{1}{s^2-a^2} \int_0^\infty \left( \frac{1}{a^2+x^2} - \frac{1}{x^2+s^2} \right) dx \\ &= \frac{1}{s^2-a^2} \left( \frac{1}{a} - \frac{1}{s} \right) \frac{\pi}{2} \\ &= \frac{\pi}{2} \frac{1}{s(s+a)} = \frac{\pi}{2} \left( \frac{1}{s} - \frac{1}{s+a} \right).\end{aligned}$$

Inversion gives the value of the given integral

$$f(t) = \frac{\pi}{2a}(1 - e^{-at}). \tag{4.6.3}$$

▯

***Example 4.6.2***
Evaluate the integral

$$f(t) = \int_0^\infty \frac{\sin^2 tx}{x^2} \, dx. \tag{4.6.4}$$

A procedure similar to the above integral with $2\sin^2 tx = 1 - \cos(2\,tx)$ gives

$$\begin{aligned}\bar{f}(s) &= \frac{1}{2} \int_0^\infty \frac{1}{x^2} \left( \frac{1}{s} - \frac{s}{4x^2+s^2} \right) dx = \frac{2}{s} \int_0^\infty \frac{dx}{4x^2+s^2} \\ &= \frac{1}{s} \int_0^\infty \frac{dy}{y^2+s^2} = \frac{1}{s^2} \left[ \tan^{-1} \frac{y}{s} \right]_0^\infty = \pm \frac{\pi}{2s^2}\end{aligned}$$

according as $s >$ or $< 0$. The inverse transform yields

$$f(t) = \frac{\pi t}{2} \, sgn \, t. \tag{4.6.5}$$

▯

***Example 4.6.3***

Show that

$$\int_0^\infty \frac{x \sin xt}{x^2+a^2}\,dx = \frac{\pi}{2}\,e^{-at}, \quad (a, t > 0). \tag{4.6.6}$$

Suppose

$$f(t) = \int_0^\infty \frac{x \sin xt}{x^2+a^2}\,dx.$$

Taking the Laplace transform with respect to $t$ gives

$$\begin{aligned}\bar{f}(s) &= \int_0^\infty \frac{x^2\,dx}{(x^2+a^2)(x^2+s^2)}\\ &= \int_0^\infty \frac{dx}{x^2+s^2} - \frac{a^2}{s^2-a^2}\int_0^\infty \left(\frac{1}{x^2+a^2} - \frac{1}{x^2+s^2}\right)dx\\ &= \frac{\pi}{2s}\left(1 - \frac{a}{s+a}\right) = \frac{\pi}{2}\,\frac{1}{(s+a)}.\end{aligned}$$

Taking the inverse transform, we obtain

$$f(t) = \frac{\pi}{2}\,e^{-at}.$$

□

## 4.7 Solutions of Difference and Differential-Difference Equations

Like differential equations, the difference and differential-difference equations describe mechanical, electrical, and electronic systems of interest. These equations also arise frequently in problems of economics and business, and particularly in problems concerning interest, annuities, amortization, loans, and mortgages. Thus, for the study of the above systems or problems, it is often necessary to solve difference or differential-difference equations with prescribed initial data. This section is essentially devoted to the solution of simple difference and differential-difference equations by the Laplace transform technique.

Suppose $\{u_r\}_{r=1}^{\infty}$ is a given sequence. We introduce the difference operators $\Delta$, $\Delta^2$, $\Delta^3$,...,$\Delta^n$ defined by

$$\Delta u_r = u_{r+1} - u_r, \tag{4.7.1}$$

$$\Delta^2 u_r = \Delta(\Delta u_r) = \Delta(u_{r+1} - u_r) = u_{r+2} - 2u_{r+1} + u_r, \tag{4.7.2}$$

$$\Delta^3 u_r = \Delta^2(u_{r+1} - u_r) = u_{r+3} - 3\,u_{r+2} + 3\,u_{r+1} - u_r. \tag{4.7.3}$$

More generally,

$$\Delta^n u_r = \Delta^{n-1}(u_{r+1} - u_r) = \sum_{k=0}^{n}(-1)^k \binom{n}{k} u_{r+n-k}. \tag{4.7.4}$$

These expressions are usually called the *first, second, third,* and *nth finite* differences respectively. Any equation expressing a relation between finite differences is called a *difference equation.* The highest order finite difference involved in the equation is referred to as its *order.* A difference equation containing the derivatives of the unknown function is called the *differential-difference equation.* Thus, the differential-difference equation has two distinct orders—one is related to the highest order finite difference and the other is associated with the highest order derivatives. Equations

$$\Delta u_r - u_r = 0, \tag{4.7.5}$$

$$\Delta^2 u_r - 2\Delta u_r = 0, \tag{4.7.6}$$

are the examples of difference equations of the first and second order, respectively. The most general linear $n$th order difference equation has the form

$$a_0 \Delta^n u_r + a_1 \Delta^{n-1} u_r + \cdots + a_{n-1}\Delta u_r + a_n u_r = f(n), \tag{4.7.7}$$

where $a_0, a_1, \ldots, a_n$ and $f(n)$ are either constants or functions of non-negative integer $n$. Like ordinary differential equations, (4.7.7) is called a *homogeneous* or *inhomogeneous* according to $f(n) = 0$ or $\neq 0$.

The following equations

$$u'(t) - u(t-1) = 0, \tag{4.7.8}$$

$$u'(t) - au(t-1) = f(t), \tag{4.7.9}$$

are the examples of the differential-difference equations, where $f(t)$ is a given function of $t$. The study of the above equation is facilitated by introducing the function

$$S_n(t) = H(t-n) - H(t-n-1), \quad n \le t < n+1, \tag{4.7.10}$$

where $n$ is a non-negative integer and $H(t)$ is the Heaviside unit step function.

The Laplace transform of $S_n(t)$ is given by

$$\overline{S}_n(s) = \mathscr{L}\{S_n(t)\} = \int_0^\infty e^{-st}\{H(t-n) - H(t-n-1)\}\,dt$$
$$= \int_n^{n+1} e^{-st}dt = \frac{1}{s}(1-e^{-s})e^{-ns} = \overline{S}_0(s)\exp(-ns), \qquad (4.7.11)$$

where $\overline{S}_0(s)$ is equal to $\frac{1}{s}(1-e^{-s})$.

We next define the function $u(t)$ by a series

$$u(t) = \sum_{n=0}^{\infty} u_n S_n(t), \qquad (4.7.12)$$

where $\{u_n\}_{n=0}^{\infty}$ is a given sequence. It follows that $u(t) = u_n$ in $n \le t < n+1$ and represents a staircase function. Further

$$u(t+1) = \sum_{n=0}^{\infty} u_n S_n(t+1) = \sum_{n=0}^{\infty} u_n[H(t+1-n) - H(t-n)]$$
$$= \sum_{n=1}^{\infty} u_n S_{n-1}(t) = \sum_{n=0}^{\infty} u_{n+1} S_n(t). \qquad (4.7.13)$$

Similarly,

$$u(t+2) = \sum_{n=0}^{\infty} u_{n+2} S_n(t). \qquad (4.7.14)$$

More generally,

$$u(t+k) = \sum_{n=0}^{\infty} u_{n+k} S_n(t). \qquad (4.7.15)$$

The Laplace transform of $u(t)$ is given by

$$\bar{u}(s) = \mathscr{L}\{u(t)\} = \int_0^\infty e^{-st} u(t)\,dt = \sum_{n=0}^{\infty} u_n \int_0^\infty e^{-st} S_n(t)\,dt$$
$$= \frac{1}{s}(1-e^{-s}) \sum_{n=0}^{\infty} u_n \exp(-ns).$$

Thus,

$$\bar{u}(s) = \frac{1}{s}(1-e^{-s})\zeta(s) = \bar{S}_0(s)\zeta(s), \qquad (4.7.16)$$

where $\zeta(s)$ represents the *Dirichlet function* defined by

$$\zeta(s) = \sum_{n=0}^{\infty} u_n \exp(-ns). \qquad (4.7.17)$$

We thus deduce

$$u(t) = \mathscr{L}^{-1}\{\bar{S}_0(s)\zeta(s)\}. \tag{4.7.18}$$

In particular, if $u_n = a^n$ is a geometric sequence, then

$$\zeta(s) = \sum_{n=0}^{\infty}(ae^{-s})^n = \frac{1}{1 - ae^{-s}} = \frac{e^s}{e^s - a}. \tag{4.7.19}$$

Thus, we obtain from (4.7.16) that

$$\mathscr{L}\{a^n\} = \bar{S}_0(s)\zeta(s) = \bar{S}_0(s)\,\frac{e^s}{e^s - a}, \tag{4.7.20}$$

so that

$$\mathscr{L}^{-1}\left\{\bar{S}_0(s)\,\frac{e^s}{e^s - a}\right\} = a^n. \tag{4.7.21}$$

From the identity,

$$\sum_{n=0}^{\infty}(n+1)(ae^{-s})^n = (1 - ae^{-s})^{-2}, \tag{4.7.22}$$

it further follows that

$$\mathscr{L}\{(n+1)a^n\} = \bar{S}_0(s)(1 - ae^{-s})^{-2} = \frac{e^{2s}\,\bar{S}_0(s)}{(e^s - a)^2}. \tag{4.7.23}$$

Thus,

$$\mathscr{L}^{-1}\left\{\frac{e^{2s}\,\bar{S}_0(s)}{(e^s - a)^2}\right\} = (n+1)a^n. \tag{4.7.24}$$

We deduce from (4.7.22) that

$$\sum_{n=0}^{\infty} na^n e^{-ns} = \frac{ae^s}{(1 - ae^{-s})^2}. \tag{4.7.25}$$

Hence,

$$\mathscr{L}\{na^n\} = \bar{S}_0(s)\,\frac{ae^s}{(e^s - a)^2}. \tag{4.7.26}$$

Therefore,

$$\mathscr{L}^{-1}\left\{\frac{a\bar{S}_0(s)e^s}{(e^s - a)^2}\right\} = na^n. \tag{4.7.27}$$

***THEOREM 4.7.1***

If $\bar{u}(s) = \mathscr{L}\{u(t)\}$, then

$$\mathscr{L}\{u(t+1)\} = e^s[\bar{u}(s) - u_0\bar{S}_0(s)], \qquad u_0 = u(0). \tag{4.7.28}$$

**PROOF** We have

$$\begin{aligned}\mathscr{L}\{u(t+1)\} &= \int_0^\infty e^{-st}u(t+1)\,dt = e^s\int_1^\infty e^{-s\tau}u(\tau)\,d\tau \\ &= e^s\left[\bar{u}(s) - \int_0^1 e^{-s\tau}u(\tau)\,d\tau\right] \\ &= e^s\left[\bar{u}(s) - u(0)\int_0^1 e^{-s\tau}\,d\tau\right] = e^s[\bar{u}(s) - u_0\bar{S}_0(s)].\end{aligned}$$

This proves the theorem.

In view of this theorem, we derive

$$\begin{aligned}\mathscr{L}\{u(t+2)\} &= e^s[\mathscr{L}\{u(t+1)\} - u(1)\bar{S}_0(s)] \\ &= e^{2s}[\bar{u}(s) - u(0)\bar{S}_0(s)] - e^s u_1\bar{S}_0(s) \\ &= e^{2s}[\bar{u}(s) - (u_0 + u_1e^{-s})\bar{S}_0(s)], \qquad u(1) = u_1. \quad (4.7.29)\end{aligned}$$

Similarly,

$$\mathscr{L}\{u(t+3)\} = e^{3s}[\bar{u}(s) - (u_0 + u_1e^{-s} + u_2e^{-2s})\bar{S}_0(s)]. \tag{4.7.30}$$

More generally, if $k$ is an integer,

$$\mathscr{L}\{u(t+k)\} = e^{ks}\left(\bar{u}(s) - \bar{S}_0(s)\sum_{r=0}^{k-1}u_r e^{-rs}\right). \tag{4.7.31}$$

∎

***Example 4.7.1***

Solve the difference equation

$$\Delta u_n - u_n = 0, \tag{4.7.32}$$

with the initial condition $u_0 = 1$.

We take the Laplace transform of the equation to obtain

$$\mathscr{L}\{u_{n+1}\} - 2\mathscr{L}\{u_n\} = 0,$$

which is, by (4.7.28),

$$e^s[\bar{u}(s) - u_0\bar{S}_0(s)] - 2\,\bar{u}(s) = 0.$$

Thus,

$$\bar{u}(s) = \frac{e^s\,\bar{S}_0(s)}{e^s - 2}.$$

Inversion with (4.7.21) gives the solution

$$u_n = 2^n. \tag{4.7.33}$$

□

**_Example 4.7.2_**
Show that the solution of the difference equation

$$\Delta^2 u_n - 2\Delta u_n = 0 \tag{4.7.34}$$

is

$$u_n = A + B\,3^n, \tag{4.7.35}$$

where $A = \frac{1}{2}(3u_0 - u_1)$ and $B = \frac{1}{2}(u_1 - u_0)$.

The given equation is

$$u_{n+2} - 4u_{n+1} + 3u_n = 0.$$

Taking the Laplace transform, we obtain

$$e^{2s}[\bar{u}(s) - (u_0 + u_1 e^{-s})\bar{S}_0(s)] - 4\,e^s[\bar{u}(s) - u_0\bar{S}_0(s)] + 3\bar{u}(s) = 0$$

or,

$$(e^{2s} - 4e^s + 3)\bar{u}(s) = [u_0(e^{2s} - 4e^s) + u_1 e^s]\bar{S}_0(s).$$

Hence,

$$\begin{aligned}\bar{u}(s) &= \bar{S}_0(s)\left[\frac{u_0(e^{2s} - 4e^s) + u_1 e^s}{(e^s - 1)(e^s - 3)}\right]\\ &= \bar{S}_0(s)\left[\frac{(3u_0 - u_1)e^s}{2(e^s - 1)} + \frac{(u_1 - u_0)e^s}{2(e^s - 3)}\right].\end{aligned}$$

The inverse Laplace transform combined with (4.7.21) gives

$$u_n = A + B\,3^n.$$

□

**_Example 4.7.3_**
Solve the difference equation

$$u_{n+2} - 2\lambda u_{n+1} + \lambda^2 u_n = 0, \tag{4.7.36}$$

with $u_0 = 0$ and $u_1 = 1$.

The Laplace transformed equation is

$$e^{2s}[\bar{u}(s) - e^{-s}\,\bar{S}_0(s)] - 2\lambda\bar{u}(s)e^s + \lambda^2\bar{u}(s) = 0$$

or,

$$\bar{u}(s) = \frac{e^s\, \bar{S}_0(s)}{(e^s - \lambda)^2}.$$

The inverse transform gives the solution

$$u_n = \frac{1}{\lambda}\, n\, \lambda^n = n\, \lambda^{n-1}. \tag{4.7.37}$$

□

**Example 4.7.4**
Solve the differential-difference equation

$$u'(t) = u(t-1), \quad u(0) = 1. \tag{4.7.38}$$

Application of the Laplace transform gives

$$s\bar{u}(s) - u(0) = e^{-s}[\bar{u}(s) - u(0)\bar{S}_0(s)],$$

or,

$$\bar{u}(s)(s - e^{-s}) = 1 + \frac{e^{-s}}{s}(e^{-s} - 1).$$

Or,

$$\begin{aligned}
\bar{u}(s) &= \left\{\frac{1}{s - e^{-s}} - \frac{e^{-s}}{s(s - e^{-s})}\right\} + \frac{e^{-2s}}{s(s - e^{-s})} \\
&= \frac{1}{s} + \frac{e^{-2s}}{s^2}\left(1 - \frac{e^{-s}}{s}\right)^{-1} \\
&= \frac{1}{s} + \frac{e^{-2s}}{s^2} + \frac{e^{-3s}}{s^3} + \frac{e^{-4s}}{s^4} + \cdots + \frac{e^{-ns}}{s^n} + \cdots .
\end{aligned}$$

In view of the result

$$\mathscr{L}^{-1}\left\{\frac{e^{-as}}{s^n}\right\} = \frac{(t-a)^{n-1}}{\Gamma(n)} H(t-a), \tag{4.7.39}$$

we obtain the solution

$$u(t) = 1 + \frac{(t-2)}{1!} + \frac{(t-3)^2}{2!} + \cdots + \frac{(t-n)^{n-1}}{(n-1)!}, \quad t > n. \tag{4.7.40}$$

□

**Example 4.7.5**
Solve the differential-difference equation

$$u'(t) - \alpha\, u(t-1) = \beta, \quad u(0) = 0. \tag{4.7.41}$$

Application of the Laplace transform yields

$$s\,\bar{u}(s) - u(0) - \alpha\, e^{-s}[\bar{u}(s) - u(0)\bar{S}_0(s)] = \frac{\beta}{s}.$$

Or,

$$\begin{aligned}\bar{u}(s) &= \frac{\beta}{s(s-\alpha e^{-s})} = \frac{\beta}{s^2}\left(1 - \frac{\alpha}{s}e^{-s}\right)^{-1}\\ &= \beta\left[\frac{1}{s^2} + \frac{\alpha\, e^{-s}}{s^3} + \frac{\alpha^2\, e^{-2s}}{s^4} + \cdots + \frac{\alpha^n\, e^{-ns}}{s^{n+2}} + \cdots\right].\end{aligned}$$

Inverting with the help of (4.7.39), we obtain the solution

$$u(t) = \beta\left[t + \frac{\alpha(t-1)^2}{\Gamma(3)} + \alpha^2\frac{(t-2)^3}{\Gamma(4)} + \cdots + \frac{\alpha^n(t-n)^{n+1}}{\Gamma(n+2)}\right],\ t > n. \quad (4.7.42)$$

□

## 4.8 Applications of the Joint Laplace and Fourier Transform

***Example 4.8.1***
(*The Inhomogeneous Cauchy Problem for the Wave Equation*). Use the joint Fourier and Laplace transform method to solve the Cauchy problem for the wave equation as stated in Example 2.12.4. with an inhomogeneous term, $q(x,t)$.

We define the joint Fourier and Laplace transform of $u(x,t)$ by

$$\bar{U}(k,s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\,dx \int_0^{\infty} e^{-st}u(x,t)\,dt. \quad (4.8.1)$$

The transformed inhomogeneous Cauchy problem has the solution in the form

$$\bar{U}(k,s) = \frac{sF(k) + G(k) + \bar{Q}(k,s)}{(s^2 + c^2k^2)}, \quad (4.8.2)$$

where $\bar{Q}(k,s)$ is the joint transform of the inhomogeneous term, $q(x,t)$ present on the right side of the wave equation.

The joint inverse transform gives the solution as

$$
\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \mathscr{L}^{-1}\left[\frac{sF(k)+G(k)+\bar{Q}(k,s)}{s^2+c^2k^2}\right] dk \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[F(k)\cos ckt + \frac{G(k)}{ck}\sin ckt\right] e^{ikx}\, dk \\
&\qquad + \frac{1}{ck}\int_0^t \sin ck(t-\tau)\, Q(k,\tau)\, d\tau \\
&= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)(e^{ickt}+e^{-ickt})e^{ikx}\, dk \\
&\quad + \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{G(k)}{ick}(e^{ickt}-e^{-ickt})e^{ikx}\, dk \\
&\quad + \frac{1}{\sqrt{2\pi}}\frac{1}{2c}\int_0^t d\tau \int_{-\infty}^{\infty} \frac{Q(k,\tau)}{ik}\left[e^{ick(t-\tau)}+e^{-ick(t-\tau)}\right] e^{ikx} dk \\
&= \frac{1}{2}[f(x-ct)+f(x+ct)] + \frac{1}{\sqrt{2\pi}}\frac{1}{2c}\int_{-\infty}^{\infty} G(k)\, dk \int_{x-ct}^{x+ct} e^{ik\xi}\, d\xi \\
&\quad + \frac{1}{2c}\int_0^t d\tau \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, Q(k,\tau)\, dk \int_{x-c(t-\tau)}^{x+c(t-\tau)} e^{ik\xi}\, d\xi \\
&= \frac{1}{2}[f(x-ct)+f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi \\
&\quad + \frac{1}{2c}\int_0^t d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} q(\xi,\tau)\, d\xi. \qquad (4.8.3)
\end{aligned}
$$

This is identical with the d'Alembert solution (2.12.41) when $q(x,t) \equiv 0$. □

***Example 4.8.2***

(*Dispersive Long Water Waves in a Rotating Ocean*). We use the joint Laplace and Fourier transform to solve the linearized horizontal equations of motion and the continuity equation in a rotating inviscid ocean. These equations in a rotating coordinate system (see Proudman, 1953; Debnath and Kulchar, 1972) are given by

$$\frac{\partial \mathbf{u}}{\partial t} + f\,\hat{\mathbf{k}} \times \mathbf{u} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho h}\boldsymbol{\tau}, \qquad (4.8.4)$$

$$\nabla \cdot \mathbf{u} = -\frac{1}{h}\frac{\partial \zeta}{\partial t}, \qquad (4.8.5)$$

where $\mathbf{u} = (u, v)$ is the horizontal velocity field, $\hat{\mathbf{k}}$ is the unit vector normal to the horizontal plane, $f = 2\Omega \sin\phi$ is the constant Coriolis parameter, $\rho$ is the constant density of water, $\zeta(x, t)$ is the vertical free surface elevation, $\tau = (\tau^x, \tau^y)$ represents the components of wind stress in the $x$ and $y$ directions, and the pressure is given by the hydrostatic equation

$$p = p_0 + g\rho(\zeta - z), \tag{4.8.6}$$

where $z$ is the depth of water below the mean free surface and $g$ is the acceleration due to gravity.

Equation (4.8.4)–(4.8.5) combined with (4.8.6) reduce to the form

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \zeta}{\partial x} + \frac{\tau^x}{\rho h}, \tag{4.8.7}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \zeta}{\partial y} + \frac{\tau^y}{\rho h}, \tag{4.8.8}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{1}{h}\frac{\partial \zeta}{\partial t}. \tag{4.8.9}$$

It follows from (4.8.7)–(4.8.8) that

$$Du = -g\left(\frac{\partial^2}{\partial x \partial t} + f\frac{\partial}{\partial y}\right)\zeta + \frac{1}{\rho h}\left(\frac{\partial \tau^x}{\partial t} + f\tau^y\right), \tag{4.8.10}$$

$$Dv = -g\left(\frac{\partial^2}{\partial y \partial t} - f\frac{\partial}{\partial x}\right)\zeta + \frac{1}{\rho h}\left(\frac{\partial \tau^y}{\partial t} - f\tau^x\right), \tag{4.8.11}$$

where the differential operator $D$ is

$$D \equiv \left(\frac{\partial^2}{\partial t^2} + f^2\right). \tag{4.8.12}$$

Elimination of $u$ and $v$ from (4.8.9)–(4.8.11) gives

$$\left(\nabla^2 - \frac{1}{c^2}D\right)\zeta_t = E(x, y, t), \tag{4.8.13}$$

where $c^2 = gh$ and $\nabla^2$ is the horizontal Laplacian, and $E(x, y, t)$ is a known forcing function given by

$$E(x, y, t) = \frac{1}{\rho c^2}\left[\frac{\partial^2 \tau^x}{\partial x \partial t} + \frac{\partial^2 \tau^y}{\partial y \partial t} + f\left(\frac{\partial \tau^y}{\partial x} - \frac{\partial \tau^x}{\partial y}\right)\right]. \tag{4.8.14}$$

Further, we assume that the conditions are uniform in the $y$ direction and the wind stress acts only in the $x$ direction so that $\tau^x$ and $E$ are given functions of $x$ and $t$ only. Consequently, equation (4.8.13) becomes

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\left(\frac{\partial^2}{\partial t^2} + f^2\right)\right]\zeta_t = \frac{1}{\rho c^2}\left(\frac{\partial^2 \tau^x}{\partial x \partial t}\right).$$

Integrating this equation with respect to $t$ gives

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\left(\frac{\partial^2}{\partial t^2} + f^2\right)\right]\zeta = \frac{1}{\rho c^2}\left(\frac{\partial \tau^x}{\partial x}\right). \tag{4.8.15}$$

Similarly, the velocity $u(x,t)$ satisfies the equation

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\left(\frac{\partial^2}{\partial t^2} + f^2\right)\right]u = -\frac{1}{\rho h c^2}\left(\frac{\partial \tau^x}{\partial t}\right). \tag{4.8.16}$$

If the right-hand side of equations (4.8.15) and (4.8.16) is zero, these equations are known as the *Klein-Gordon equations*, which have received extensive attention in quantum mechanics and in applied mathematics.

Equation (4.8.15) is to be solved subject to the following boundary and initial conditions

$$|\zeta| \quad \text{is} \quad \text{bounded} \quad \text{as } |x| \to \infty, \tag{4.8.17}$$

$$\zeta(x,t) = 0 \quad \text{at} \quad t = 0 \text{ for all real } x. \tag{4.8.18}$$

Before we solve the initial value problem, we seek a plane wave solution of the homogeneous equation (4.8.15) in the form

$$\zeta(x,t) = A\exp\{i(\omega t - kx)\}, \tag{4.8.19}$$

where $A$ is a constant amplitude, $\omega$ is the frequency, and $k$ is the wavenumber. Such a solution exists provided the dispersion relation

$$\omega^2 = c^2k^2 + f^2 \tag{4.8.20}$$

is satisfied. Thus, the phase and the group velocities of waves are given by

$$C_p = \frac{\omega}{k} = \left(c^2 + \frac{f^2}{k^2}\right)^{\frac{1}{2}}, \quad C_g = \frac{\partial\omega}{\partial k} = \frac{c^2k}{(c^2k^2+f^2)^{\frac{1}{2}}}. \tag{4.8.21ab}$$

Thus, the waves are dispersive in a rotating ocean ($f \neq 0$). However, in a non-rotating ocean ($f = 0$) all waves would propagate with constant velocity $c$, and they are non-dispersive shallow water waves. Further, $C_pC_g = c^2$ whence it follows that the phase velocity has a minimum of $c$ and the group velocity a maximum. The short waves will be observed first at a given point, even though they have the smallest phase velocity.

Application of the joint Laplace and Fourier transform to (4.8.15) together with (4.8.17)–(4.8.18) give the transformed solution

$$\tilde{\bar{\zeta}}(k,s) = -\frac{Ac^2}{(s^2+a^2)}\tilde{\bar{f}}(k,s), \qquad a^2 = (c^2k^2+f^2), \tag{4.8.22}$$

where

$$f(x,t) = \frac{1}{\rho c^2}\left(\frac{\partial \tau^x}{\partial x}\right)H(t). \tag{4.8.23}$$

The inverse transforms combined with the Convolution Theorem of the Laplace transform lead to the formal solution

$$\zeta(x,t) = -\frac{Ac}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(k^2 + \frac{f^2}{c^2}\right)^{-\frac{1}{2}} e^{ikx}\, dk \int_0^t \tilde{f}(k, t-\tau) \sin a\tau\, d\tau. \quad (4.8.24)$$

In general, this integral cannot be evaluated unless $f(x,t)$ is prescribed. Even if some particular form of $f$ is given, an exact evaluation of (4.8.24) is almost a formidable task. Hence, it is necessary to resort to asymptotic methods (see Debnath and Kulchar, 1972).

To investigate the solution, we choose a particular form of the wind stress distribution

$$\frac{\tau^x}{\rho c^2} = A\, e^{i\omega t} H(t) H(-x), \quad (4.8.25)$$

where $A$ is a constant and $\omega$ is the frequency of the applied disturbance. Thus,

$$\frac{1}{\rho c^2}\left(\frac{\partial \tau^x}{\partial x}\right) = -Ae^{i\omega t} H(t)\delta(-x). \quad (4.8.26)$$

In this case, solution (4.8.24) reduces to the form

$$\begin{aligned}\zeta(x,t) &= \frac{Ac}{\sqrt{2\pi}} \int_0^t e^{i\omega(t-\tau)} H(t-\tau) \mathscr{F}^{-1}\left[\frac{\sin a\tau}{\sqrt{k^2 + \frac{f^2}{c^2}}}\right] d\tau \\ &= \frac{Ac}{2} \int_0^t e^{i\omega(t-\tau)} H(t-\tau) J_0\left\{\frac{f}{c}(c^2\tau^2 - x^2)^{\frac{1}{2}}\right\} \\ &\qquad \times H(c\tau - |x|)\, d\tau, \end{aligned} \quad (4.8.27)$$

where $J_0(z)$ is the zero-order Bessel function of the first kind.

When $\omega \equiv 0$, this solution is identical with that of Crease (1956) who obtained the solution using the Green's function method. In this case, the solution becomes

$$\zeta = \frac{Ac}{2} \int_0^t H(t-\tau) J_0\left[f\left\{\tau^2 - \frac{x^2}{c^2}\right\}^{\frac{1}{2}}\right] H\left(\tau - \frac{|x|}{c}\right) d\tau. \quad (4.8.28)$$

In terms of non-dimensional parameters $f\tau = \alpha$, $ft = a$, and $\frac{fx}{c} = b$, solution (4.8.28) assumes the form

$$\left(\frac{2f}{Ac}\right)\zeta = \int_0^a H(a-\alpha) J_0\left[(\alpha^2 - b^2)^{\frac{1}{2}}\right] H(\alpha - |b|)\, d\alpha. \quad (4.8.29)$$

Or, equivalently,

$$\left(\frac{2f}{Ac}\right)\zeta = \int_{|b|}^{d} J_0\left[(\alpha^2 - b^2)^{\frac{1}{2}}\right] d\alpha, \tag{4.8.30}$$

where $d = \max(|b|, a)$. This is the basic solution of the problem.

In order to find the solution of (4.8.16), we first choose

$$\frac{1}{\rho c^2}\left(\frac{\partial \tau^x}{\partial t}\right) = A\delta(t)H(-x), \tag{4.8.31}$$

so that the joint Laplace and Fourier transform of this result is $A\mathscr{F}\{H(-x)\}$. Thus, the transformed solution of (4.8.16) is

$$\bar{\tilde{u}}(k,s) = \frac{Ac^2}{h}\mathscr{F}\{H(-x)\}\frac{1}{(s^2+\omega^2)}, \quad \omega^2 = (ck)^2 + f^2. \tag{4.8.32}$$

The inverse transforms combined with the Convolution Theorem lead to the solution

$$u(x,t) = \frac{Ac}{2h}\int_{-\infty}^{\infty} H(-\xi)\, J_0\left[f\left\{t^2 - \left(\frac{x-\xi}{c}\right)^2\right\}^{\frac{1}{2}}\right] \times H\left(t - \frac{(x-\xi)}{c}\right) d\xi, \tag{4.8.33}$$

which is, by the change of variable $(x-\xi)f = c\alpha$, with $a = ft$ and $b = (fx/c)$,

$$= \frac{Ac^2}{2hf}\int_{b}^{\infty} J_0\left[(a^2-\alpha^2)^{\frac{1}{2}}\right] H(a-|\alpha|)\, d\alpha. \tag{4.8.34}$$

For the case $b > 0$, solution (4.8.34) becomes

$$u(x,t) = \frac{Ac^2}{2hf}H(a-b)\int_{b}^{a} J_0\left\{(a^2-\alpha^2)^{\frac{1}{2}}\right\} d\alpha. \tag{4.8.35}$$

When $b < 0$, the velocity field is

$$\begin{aligned} u(x,t) &= \frac{Ac^2}{2hf}\left[\int_{-a}^{a} J_0\left\{(a^2-\alpha^2)^{\frac{1}{2}}\right\} d\alpha - H(a-|b|)\int_{-a}^{b} J_0\left\{(a^2-\alpha^2)^{\frac{1}{2}}\right\} d\alpha\right] \\ &= \frac{gA}{2f}\left[2\sin a - H(a-|b|)\int_{|b|}^{a} J_0\left\{(a^2-\alpha^2)^{\frac{1}{2}}\right\} d\alpha\right], \end{aligned} \tag{4.8.36}$$

which is, for $a < |b|$,

$$u(x,t) = \left(\frac{gA}{2f}\right)\sin a. \tag{4.8.37}$$

Finally, it can be shown that the velocity transverse to the direction of propagation is

$$v = \left(-\frac{gA}{2f}\right)\int_0^a d\beta \int_b^\infty J_0\left\{(\beta^2-\alpha^2)^{\frac{1}{2}}\right\} H(\beta-|\alpha|)\,d\alpha. \tag{4.8.38}$$

If $b > 0$, that is, $x$ is outside the generating region, then

$$\left(\frac{2f}{gA}\right) v = -H(a-b)\int_b^a d\beta \int_b^\beta J_0\left\{(\beta^2-\alpha^2)^{\frac{1}{2}}\right\} d\alpha,$$

which becomes, after some simplification,

$$= -\left[(1-\cos a) - \int_0^b d\alpha \int_\alpha^a J_0\left\{(\beta^2-\alpha^2)^{\frac{1}{2}}\right\}\right] H(a-b). \tag{4.8.39}$$

For $b < 0$, it is necessary to consider two cases: (i) $a < |b|$ and (ii) $a > |b|$. In the former case, (4.8.38) takes the form

$$\left(\frac{2f}{gA}\right) v = -\int_0^a d\beta \int_{-\beta}^\beta J_0\left\{(\beta^2-\alpha^2)^{\frac{1}{2}}\right\} d\alpha = -2(1-\cos b). \tag{4.8.40}$$

In the latter case, the final form of the solution is

$$\left(\frac{2f}{gA}\right) v = -(1-\cos b) + \int_0^{|b|} d\alpha \int_\alpha^a J_0\left\{(\beta^2-\alpha^2)^{\frac{1}{2}}\right\} d\beta. \tag{4.8.41}$$

Finally, the steady-state solutions are obtained in the limit as $t \to \infty (b \to \infty)$

$$\begin{aligned}
\zeta &= \frac{Ac}{2f}\exp(-|b|), \\
u &= \frac{Ag}{2f}\sin ft, \\
v &= \frac{Ag}{2f}\begin{bmatrix} \cos ft - \exp(-b), & b > 0 \\ \cos ft + \exp(-|b|) - 2, & b < 0 \end{bmatrix}.
\end{aligned} \tag{4.8.42}$$

Thus, the steady-state solutions are attained in a rotating ocean. This shows a striking contrast with the corresponding solutions in the non-rotating ocean where an ever-increasing free surface elevation is found. The terms $\sin ft$ and $\cos ft$ involved in the steady-state velocity field represent inertial oscillations with frequency $f$. ▯

***Example 4.8.3***

(*One-Dimensional Diffusion Equation on a Half Line*). Solve the equation

$$u_t = \kappa u_{xx}, \quad 0 < x < \infty, \quad t > 0, \tag{4.8.43}$$

with the boundary data

$$\left.\begin{aligned} u(x,t) &= f(t) \quad \text{for } x = 0 \\ u(x,t) &\to 0 \qquad \text{as } x \to \infty \end{aligned}\right\} \quad t > 0 \tag{4.8.44ab}$$

and the initial condition

$$u(x,t) = 0 \quad \text{at } t = 0 \quad \text{for} \quad 0 < x < \infty. \tag{4.8.45}$$

We use the joint Fourier sine and Laplace transform defined by

$$\overline{U}_s(k,s) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-st}\, dt \int_0^\infty u(x,t) \sin kx\, dx, \tag{4.8.46}$$

so that the solution of the transformed problem is

$$\overline{U}_s(k,s) = \sqrt{\frac{2}{\pi}}\, (\kappa k) \frac{\bar{f}(s)}{(s + k^2\kappa)}. \tag{4.8.47}$$

The inverse transform yields the solution

$$u(x,t) = \left(\frac{2\kappa}{\pi}\right) \int_0^\infty k \sin kx\, dk \int_0^t f(t-\tau) \exp(-\kappa\tau k^2)\, d\tau.$$

In particular, if $f(t) = T_0 = \text{constant}$, then the solution becomes

$$u(x,t) = \frac{2T_0}{\pi} \int_0^\infty \frac{\sin kx}{k} (1 - e^{-\kappa k^2 t})\, dk. \tag{4.8.48}$$

Making use of the integral (2.15.11) gives the solution

$$\begin{aligned} u(x,t) &= \frac{2T_0}{\pi} \left[\frac{\pi}{2} - \frac{\pi}{2} \, erf\left(\frac{x}{2\sqrt{\kappa t}}\right)\right] \\ &= T_0 \, erfc\left(\frac{x}{2\sqrt{\kappa t}}\right). \end{aligned} \tag{4.8.49}$$

This is identical with (2.15.12). ▯

**Example 4.8.4**
(*The Bernoulli-Euler Equation on an Elastic Foundation*). Solve the equation

$$EI\,\frac{\partial^4 u}{\partial x^4} + \kappa u + m\,\frac{\partial^2 u}{\partial t^2} = W\delta(t)\delta(x), \quad -\infty < x < \infty, \quad t > 0, \tag{4.8.50}$$

with the initial data

$$u(x,0) = 0 \quad \text{and} \quad u_t(x,0) = 0. \tag{4.8.51}$$

We use the joint Laplace and Fourier transform (4.8.1) to find the solution of the transformed problem in the form

$$\overline{U}(k,s) = \frac{W}{m\sqrt{2\pi}}\frac{1}{(s^2 + a^2k^4 + \omega^2)}, \tag{4.8.52}$$

where

$$a^2 = \frac{EI}{m} \quad \text{and} \quad \omega^2 = \frac{\kappa}{m}.$$

The inverse Laplace transform gives

$$U(k,t) = \frac{W}{m\sqrt{2\pi}}\left(\frac{\sin\alpha t}{\alpha}\right), \quad \alpha = (a^2k^4 + \omega^2)^{\frac{1}{2}}. \tag{4.8.53ab}$$

Then the inverse Fourier transform yields the formal solution

$$u(x,t) = \frac{W}{2\pi m}\int_{-\infty}^{\infty} e^{ikx}\left(\frac{\sin\alpha t}{\alpha}\right)dk. \tag{4.8.54}$$

▯

**Example 4.8.5**
(*The Cauchy-Poisson Wave Problem in Fluid Dynamics*). We consider the two-dimensional Cauchy-Poisson problem for an inviscid liquid of infinite depth with a horizontal free surface. We assume that the liquid has constant density $\rho$ and negligible surface tension. Waves are generated on the surface of water initially at rest for time $t < 0$ by the prescribed free surface displacement at $t = 0$.

In terms of the velocity potential $\phi(x,z,t)$ and the free surface elevation $\eta(x,t)$, the linearized surface wave motion in Cartesian coordinates $(x,y,z)$ is governed by the following equation and free surface and boundary conditions:

$$\nabla^2\phi = \phi_{xx} + \phi_{zz} = 0, \quad -\infty < z \le 0, \quad -\infty < x < \infty,\ t > 0, \tag{4.8.55}$$

$$\left.\begin{aligned}\phi_z - \eta_t = 0\\ \phi_t + g\eta = 0\end{aligned}\right\} \quad \text{on } z = 0, t > 0, \tag{4.8.56ab}$$

$$\phi_z \to 0 \quad \text{as } z \to -\infty. \tag{4.8.57}$$

The initial conditions are

$$\phi(x, 0, 0) = 0 \quad \text{and} \quad \eta(x, 0) = \eta_0(x), \tag{4.8.58}$$

where $\eta_0(x)$ is a given function with compact support.

We introduce the Laplace transform with respect to $t$ and the Fourier transform with respect to $x$ defined by

$$[\bar{\tilde{\phi}}(k, z, s), \bar{\tilde{\eta}}(k, s)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\, dx \int_0^{\infty} e^{-st} [\phi, \eta]\, dt. \tag{4.8.59}$$

The use of joint transform to the above system gives

$$\bar{\tilde{\phi}}_{zz} - k^2 \bar{\tilde{\phi}} = 0, \quad -\infty < z \le 0, \tag{4.8.60}$$

$$\left.\begin{aligned}\bar{\tilde{\phi}}_z = s\bar{\tilde{\eta}} - \tilde{\eta}_0(k)\\ s\bar{\tilde{\phi}} + g\bar{\tilde{\eta}} = 0\end{aligned}\right\} \quad \text{on } z = 0, \tag{4.8.61ab}$$

$$\bar{\tilde{\phi}}_z \to 0 \quad \text{as } z \to -\infty. \tag{4.8.62}$$

The bounded solution of (4.8.60) is

$$\bar{\tilde{\phi}}(k, s) = \bar{A} \exp(|k| z) \tag{4.8.63}$$

where $\overline{A} = \overline{A}(s)$ is an arbitrary function of $s$, and $\tilde{\eta}_0(k) = \mathscr{F}\{\eta_0(x)\}$.

Substituting (4.8.63) into (4.8.61ab) and eliminating $\bar{\tilde{\eta}}$ from the resulting equations gives $\bar{A}$. Hence, the solutions for $\bar{\tilde{\phi}}$ and $\bar{\tilde{\eta}}$ are

$$[\bar{\tilde{\phi}}, \bar{\tilde{\eta}}] = \left[-\frac{g\,\tilde{\eta}_0 \exp(|k|z)}{s^2 + \omega^2}, \frac{s\,\tilde{\eta}_0}{s^2 + \omega^2}\right], \tag{4.8.64ab}$$

where the dispersion relation for deep water waves is

$$\omega^2 = g|k|. \tag{4.8.65}$$

The inverse Laplace and Fourier transforms give the solutions

$$\phi(x,z,t) = -\frac{g}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sin \omega t}{\omega} \exp(ikx + |k|z)\, \tilde{\eta}_0(k)\, dk \qquad (4.8.66)$$

$$\eta(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\eta}_0(k) \cos \omega t\, e^{ikx}\, dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \tilde{\eta}_0(k) [e^{i(kx-\omega t)} + e^{i(kx+\omega t)}]\, dk, \qquad (4.8.67)$$

in which $\tilde{\eta}_0(-k) = \tilde{\eta}_0(k)$ is assumed.

Physically, the first and second integrals of (4.8.67) represent waves traveling in the positive and negative directions of $x$ respectively with phase velocity $\left(\frac{\omega}{k}\right)$. These integrals describe superposition of all such waves over the wavenumber spectrum $0 < k < \infty$.

For the classical Cauchy-Poisson wave problem, $\eta(x) = a\,\delta(x)$ where $\delta(x)$ is the Dirac delta function so that $\tilde{\eta}_0(k) = \left(a/\sqrt{2\pi}\right)$. Thus, solution (4.8.67) becomes

$$\eta(x,t) = \frac{a}{2\pi} \int_{0}^{\infty} \left[e^{i(kx-\omega t)} + e^{i(kx+\omega t)}\right] dk. \qquad (4.8.68)$$

The wave integrals (4.8.66) and (4.8.67) represent the exact solution for the velocity potential $\phi$ and the free surface elevation $\eta$ for all $x$ and $t > 0$. However, they do not lend any physical interpretations. In general, the exact evaluation of these integrals is almost a formidable task. So it is necessary to resort to asymptotic methods. It would be sufficient for the determination of the principal features of the wave motions to investigate (4.8.67) or (4.8.68) asymptotically for large time $t$ and large distance $x$ with $(x/t)$ held fixed. The asymptotic solution for this kind of problem is available in many standard books (for example, see Debnath, 1994, p 85). We state the stationary phase approximation of a typical wave integral, for $t \to \infty$,

$$\eta(x,t) = \int_{a}^{b} f(k) \exp[itW(k)]\, dk \qquad (4.8.69)$$

$$\sim f(k_1) \left[\frac{2\pi}{t|W''(k_1)|}\right]^{\frac{1}{2}} \exp\left[i\left\{tW(k_1) + \frac{\pi}{4} sgn\, W''(k_1)\right\}\right], \qquad (4.8.70)$$

where $W(k) = \frac{kx}{t} - \omega(k), x > 0$ and $k = k_1$ is a stationary point that satisfies the equation

$$W'(k_1) = \frac{x}{t} - \omega'(k_1) = 0, \quad a < k_1 < b. \qquad (4.8.71)$$

Application of (4.8.70) to (4.8.67) shows that only the first integral in (4.8.67) has a stationary point for $x>0$. Hence, the stationary phase approximation gives the asymptotic solution, as $t\to\infty, x>0$,

$$\eta(x,t)\sim\left[\frac{1}{t|\omega''(k_1)|}\right]^{\frac{1}{2}}\tilde{\eta}_0(k_1)\exp[i\{(k_1x-t\omega(k_1)\} \\ +\frac{i\pi}{4}sgn\{-\omega''(k_1)\}], \quad (4.8.72)$$

where $k_1=(gt^2/4x^2)$ is the root of the equation $\omega'(k)=\dfrac{x}{t}$.

On the other hand, when $x<0$, only the second integral of (4.8.67) has a stationary point $k_1=(gt^2/4x^2)$, and hence, the same result (4.8.70) can be used to obtain the asymptotic solution for $t\to\infty$ and $x<0$ as

$$\eta(x,t)\sim\left[\frac{1}{t|\omega''(k_1)|}\right]^{\frac{1}{2}}\tilde{\eta}_0(k_1)\exp[i\{t\omega(k_1)-k_1|x|\} \\ +\frac{i\pi}{4}sgn\,\omega''(k_1)]. \quad (4.8.73)$$

In particular, for the classical Cauchy-Poisson solution (4.8.68), the asymptotic representation for $\eta(x,t)$ follows from (4.8.73) in the form

$$\eta(x,t)\sim\frac{at}{2\sqrt{2\pi}}\frac{\sqrt{g}}{x^{3/2}}\cos\left(\frac{gt^2}{4x}\right), \quad gt^2>>4x \quad (4.8.74)$$

and a similar result for $x<0$ and $t\to\infty$. ▯

## 4.9 Summation of Infinite Series

With the aid of Laplace transforms, Wheelon (1954) first developed a direct method to the problem of summing infinite series in closed form. His method is essentially based on the operation that is contained in the summation of both sides of a Laplace transform with respect to the transform variable $s$, which is treated as the dummy index of summation $n$. This is followed by an interchange of summation and integration that leads to the desired sum as the integral of a geometric or exponential series, which can be summed in closed form. We next discuss this procedure in some detail.

If $\bar{f}(s)=\mathscr{L}\{f(x)\}$, then

$$\sum_{n=1}^{\infty}a_n\,\bar{f}(n)=\sum_{n=1}^{\infty}a_n\int_0^{\infty}f(x)e^{-nx}\,dx. \quad (4.9.1)$$

In many cases, it is possible to interchange the order of summation and integration so that (4.9.1) gives

$$\sum_{n=1}^{\infty} a_n \bar{f}(n) = \int_0^{\infty} f(t)b(t)\,dt, \tag{4.9.2}$$

where

$$b(t) = \sum_{n=1}^{\infty} a_n \exp(-nt). \tag{4.9.3}$$

We now assume $f(t) = \dfrac{1}{\Gamma(p)} t^{p-1} \exp(-xt)$ so that $\bar{f}(n) = (n+x)^{-p}$. Consequently, (4.9.2) becomes

$$\sum_{n=1}^{\infty} a_n \bar{f}(n) = \sum_{n=1}^{\infty} \frac{a_n}{(n+x)^p} = \frac{1}{\Gamma(p)} \int_0^{\infty} b(t)\, t^{p-1} \exp(-xt)\,dt. \tag{4.9.4}$$

This shows that a general series has been expressed in terms of an integral. We next illustrate the method by simple examples.

***Example 4.9.1***

Show that the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{4.9.5}$$

Putting $x = 0, p = 2$, and $a_n = 1$ for all $n$, we find, from (4.9.3) and (4.9.4),

$$b(t) = \sum_{n=1}^{\infty} \exp(-nt) = \frac{1}{e^t - 1}, \tag{4.9.6}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \int_0^{\infty} \frac{t\,dt}{e^t - 1} = \zeta(2) = \frac{\pi^2}{6}, \tag{4.9.7}$$

in which the following standard result is used

$$\int_0^{\infty} \frac{t^{p-1}}{e^{at} - 1}\,dt = \frac{\Gamma(p)}{a^p} \zeta(p), \tag{4.9.8}$$

where $\zeta(p)$ is the *Riemann zeta function* defined below by (4.9.10).

Similarly, we can show

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{\Gamma(3)} \int_0^{\infty} \frac{t^2\,dt}{e^t - 1} = \zeta(3). \tag{4.9.9}$$

More generally, we obtain, from (4.9.8),

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{\Gamma(p)} \int_0^{\infty} \frac{t^{p-1}\,dt}{e^t - 1} = \zeta(p). \tag{4.9.10}$$

□

***Example 4.9.2***
Show that

$$\sum_{n=1}^{\infty} \frac{1}{n} \exp(-an) = -\log(1 - e^{-a}). \tag{4.9.11}$$

We put $x = 0, p = 1$, and $a_n = \exp(-an)$ so that

$$b(t) = \sum_{n=1}^{\infty} \exp[-n(t+a)] = \frac{1}{e^{a+t} - 1}. \tag{4.9.12}$$

Then result (4.9.4) gives

$$\sum_{n=1}^{\infty} \frac{1}{n} \exp(-an) = \int_0^{\infty} \frac{dt}{e^{a+t} - 1}, \qquad \exp(-t) = x,$$

$$= \int_0^{1} \frac{dx}{e^a - x} = -\log(1 - e^{-a}).$$

□

***Example 4.9.3***
Show that

$$\sum_{n=1}^{\infty} \frac{1}{(n^2 + x^2)} = \frac{1}{2x^2}(\pi x \coth \pi x - 1). \tag{4.9.13}$$

We set

$$f(t) = \frac{1}{x} \sin xt,\ \bar{f}(n) = \frac{1}{n^2 + x^2}, \text{ and } a_n = 1 \quad \text{for all } n.$$

Clearly

$$b(t) = \sum_{n=1}^{\infty} \exp(-nt) = \frac{1}{e^t - 1}.$$

Thus,

$$\sum_{n=1}^{\infty} \frac{1}{(n^2 + x^2)} = \frac{1}{x} \int_0^{\infty} \frac{\sin xt}{e^t - 1}\,dt = \frac{1}{2x^2}(\pi x \ \coth \ \pi x - 1).$$

□

## 4.10 Transfer Function and Impulse Response Function of a Linear System

Many science and engineering systems are described by initial value problems that are governed by linear ordinary differential equations. In general, a linear system is governed by an $n$th order linear ordinary differential equation with constant coefficients in the form

$$L(D)[x(t)] \equiv a_n\, x^{(n)}(t) + a_{n-1}\, x^{(n-1)}(t) + \ldots + a_0\, x(t) = f(t)\,, \tag{4.10.1}$$

where $a_n$, $a_{n-1}$, ..., $a_0$ are real constants with $a_n \neq 0$ and the initial conditions are

$$x(0) = x_0, \quad x'(0) = x_1, \quad \ldots, \quad x^{(n-1)}(0) = x_{n-1}. \tag{4.10.2}$$

The solution, $x(t)$ of the system (4.10.1)–(4.10.2) is called the *output* or the *response function*, and the given $f(t)$ is called the *input function* (or *driving function*) of time $t$.

The *transfer function* $\overline{h}(s)$ of a linear system is defined as the ratio of the Laplace transform of the output function $x(t)$ to the Laplace transform of the input function $f(t)$, under the assumption that all initial conditions are zero.

More generally, however, the Laplace transform of the system (4.10.1)–(4.10.2) gives

$$\begin{aligned} a_n & \left[s^n\, \overline{x}(s) - s^{n-1} x(0) - \ldots - x^{(n-1)}(0)\right] \\ & + a_{n-1}\left[s^{n-1}\, \overline{x}(s) - s^{n-2}\, x(0) - \ldots - x^{(n-2)}\right] \\ & + \ldots + a_1\left[s\, \overline{x}(s) - x(0)\right] + a_0\, \overline{x}(s) = \overline{f}(s)\,. \end{aligned} \tag{4.10.3}$$

Or, equivalently,

$$\left(a_n\, s^n + a_{n-1}\, s^{n-1} + \ldots + a_0\right) \overline{x}(s) = \overline{f}(s) + \overline{g}(s)\,,$$

or,

$$\overline{p_n}(s)\, \overline{x}(s) = \overline{f}(s) + \overline{g}(s)\,, \tag{4.10.4}$$

where

$$\overline{p_n}(s) = \left(a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0\right) \tag{4.10.5}$$

is a polynomial of degree $n$, $\overline{g}(s)$ is a polynomial of degree less than or equal to $(n-1)$ consisting of the various products of the coefficients $a_r$ $(r = 1, 2, \ldots, n)$ and the given initial conditions $x_0$, $x_1$, ..., $x_{n-1}$.

The *transfer function* (or *system function*) is denoted by $\overline{h}(s)$ and defined by

$$\overline{h}(s) = \frac{1}{\overline{p_n}(s)} = \frac{1}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0}. \tag{4.10.6}$$

Consequently, equation (4.10.4) becomes

$$\overline{x}(s) = \frac{\overline{f}(s)}{\overline{p_n}(s)} + \frac{\overline{g}(s)}{\overline{p_n}(s)} = \overline{h}(s)\left[\overline{f}(s) + \overline{g}(s)\right]. \tag{4.10.7}$$

The inverse Laplace transform of (4.10.7) provides the response function $x(t)$ of the system which is the superposition of two responses as follows:

$$\begin{aligned} x(t) &= \mathscr{L}^{-1}\left\{\overline{h}(s)\,\overline{g}(s)\right\} + \mathscr{L}^{-1}\left\{\overline{h}(s)\,\overline{f}(s)\right\} && (4.10.8) \\ &= \int_0^t h(t-\tau)\,g(\tau)\,d\tau + \int_0^t h(t-\tau)\,f(\tau)\,d\tau && (4.10.9) \\ &= x_0(t) + x_1(t), && (4.10.10) \end{aligned}$$

where

$$x_0(t) = \mathscr{L}^{-1}\left\{\overline{h}(s)\,\overline{g}(s)\right\}, \quad x_1(t) = \mathscr{L}^{-1}\left\{\overline{h}(s)\,\overline{f}(s)\right\},$$

and

$$h(t) = \mathscr{L}^{-1}\left\{\overline{h}(s)\right\} = \mathscr{L}^{-1}\left\{\frac{1}{\overline{p_n}(s)}\right\}, \tag{4.10.11}$$

are often called the *impulse response function* of the linear system.

If the input is $f(t) \equiv 0$, the solution of the problem is $x_0(t)$, which is called the *zero-input response* of the system. On the other hand, $x_1(t)$ is the output due to the input $f(t)$ and is called the *zero-state response* of the system. If all initial conditions are zero, that is, $x_0 = x_1 = \ldots = x_{n-1} = 0$, then $\overline{g}(s) = 0$ and so, the unique solution of the nonhomogeneous equation (4.10.1) is $x_1(t)$.

For example, $h(t) = \mathscr{L}^{-1}\left\{\overline{h}(s)\right\}$ describes the solution for a mass-spring system when it is struck by a hammer. For an electric circuit, the function $\overline{z}(s) = \left[s\,\overline{h}(s)\right]^{-1}$ is called the *impendence* of the circuit.

The polynomial $\overline{p_n}(s) = \left(a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0\right)$ in $s$ of degree $n$ is called the *characteristic polynomial* of the system, and $\overline{p_n}(s) = 0$ is called the *characteristic equation* of the system. Since the coefficients of $\overline{p_n}(s)$ are real, it follows that roots of the characteristic equation are all real or, if complex, they must occur in complex conjugate pairs. If $\overline{h}(s)$ is expressed in partial fractions, the system is said to be *stable* provided all roots of the characteristic equation have negative real parts. From a physical point of view, when every root of $\overline{p_n}(s) = 0$ has a negative real part, any bounded input to a system that is stable will lead to an output that is also bounded for all time $t$.

We close this section by adding the following examples:

***Example 4.10.1***
Find the transfer function for each of the following linear systems. Determine the order of each system and find which is stable.

$$\text{(a)} \qquad L\,\frac{dI}{dt} + R\,I + \frac{1}{C}\int_0^t I\,(\tau)\,d\tau = E\,(t)\,, \tag{4.10.12}$$

$$\text{(b)} \qquad x''\,(t) + 2\,x'\,(t) + 5\,x\,(t) = 3f'\,(t) + 2f\,(t)\,, \tag{4.10.13}$$

$$\text{(c)} \qquad x'''\,(t) + x''\,(t) + 3\,x'\,(t) - 5x\,(t) = 6f''\,(t) - 13f'\,(t) + 6f\,(t)\,,$$

where $L$, $R$, and $C$ are constants. (4.10.14)

(a) This current equation is solved in Example 4.2.13. The Laplace transformed equation with the zero initial condition is given by

$$\left(Ls + R + \frac{1}{Cs}\right)\overline{I}\,(s) = \overline{E}\,(s)$$

so that the transfer equation is

$$\overline{h}\,(s) = \frac{1}{\left(Ls + R + \frac{1}{Cs}\right)} = \frac{1}{L}\,\frac{s}{\left(s^2 + \frac{R}{L}\,s + \frac{1}{CL}\right)}.$$

The system is of order 2 and its characteristic equation is

$$s^2 + \frac{R}{L}\,s + \frac{1}{CL} = 0.$$

Or,

$$(s+k)^2 + n^2 = 0$$

where

$$k = \frac{R}{2L}, \qquad n^2 = \frac{1}{CL} - \frac{R^2}{4L^2}.$$

The roots of the characteristic equation are complex and they are $s = -k \pm i\,n$ with the negative real part. So, the system is stable.

(b) We take the Laplace transform of the equation (4.10.13) with zero initial conditions so that

$$\left(s^2 + 2s + 5\right)\overline{x}\,(s) = (3s+2)\,\overline{f}\,(s)\,.$$

Thus, the transfer function is

$$\overline{h}\,(s) = \frac{\overline{x}\,(s)}{\overline{f}\,(s)} = \frac{(3s+2)}{s^2 + 2s + 5}.$$

The system is of order 2 and its characteristic equation is

$$s^2 + 2s + 5 = 0$$

with complex roots $s = -1 \pm 2i$. Since the real part of these roots is negative, the system is stable.
(c) Similarly,

$$\bar{h}(s) = \frac{\bar{x}(s)}{\bar{f}(s)} = \frac{\left(6s^2 - 13s + 6\right)}{(s^3 + s^2 + 3s - 5)}.$$

The system is of order 3 and its characteristic equation is

$$s^3 + s^2 + 3s - 5 = 0$$

with roots $s_1 = 1,\ s_2,\ s_3 = -1 \pm 2i$. Since the real parts of all roots are not negative, the system is *unstable*. ▯

***Example 4.10.2***
Find the transfer function, the impulse response function, and the solution of a linear system described by

$$x''(t) + 2a\,x'(t) + \left(a^2 + 4\right) x(t) = f(t) \qquad (4.10.15)$$

$$x(0) = 1, \qquad x'(0) = -a. \qquad (4.10.16ab)$$

According to formula (4.10.4), the transfer function of this system is

$$\bar{h}(s) = \frac{1}{(s^2 + 2as + a^2 + 4)} = \frac{1}{(s+a)^2 + 2^2}.$$

The inverse Laplace transform of the transform function $\bar{h}(s)$ is the impulse response function

$$\bar{h}(t) = \mathscr{L}^{-1}\left\{\bar{h}(s)\right\} = \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{2}{(s^2+a)^2 + 2^2}\right\} = \frac{1}{2}\,e^{-at}\sin 2t. \qquad (4.10.17)$$

Solving the homogeneous initial value problem gives

$$x_0(t) = e^{-at}\cos(2t). \qquad (4.10.18)$$

The solution of the problem (4.10.15)–(4.10.16ab) is

$$\begin{aligned} x(t) &= x_0(t) + h(t) * f(t) \\ &= e^{-at}\cos(2t) + \int_0^t e^{-at} f(t-\tau)\sin 2\tau\, d\tau. \end{aligned} \qquad (4.10.19)$$

▯

***Example 4.10.3***
Consider a linear system governed by the differential equation

$$a_2\, x''(t) + a_1\, x'(t) + a_0\, x(t) = H(t)\,, \tag{4.10.20}$$

where $H(t)$ is the Heaviside unit step function.

Derive Duhamel's formulas

$$\text{(a)} \quad x(t) = \int_0^t A'(t-\tau)\, f(\tau)\, d\tau, \tag{4.10.21}$$

$$\text{(b)} \quad x(t) = \int_0^t A(\tau)\, f'(t-\tau)\, d\tau \, + A(t)\, f(0)\,. \tag{4.10.22}$$

The transfer function for this system (4.10.20) is

$$\overline{h}(s) = \frac{\overline{x}(s)}{\overline{f}(s)} = s\, \overline{x}(s)\,. \tag{4.10.23}$$

Or,

$$\overline{x}(s) = \frac{\overline{h}(s)}{s}. \tag{4.10.24}$$

The output function in this special case is called the *indicial admittance* and is denoted by $A(t)$ so that

$$\overline{A}(s) = \frac{\overline{h}(s)}{s}. \tag{4.10.25}$$

We next derive Duhamel's formulas. We have, from (4.10.7) with $\overline{g}(s) = 0$,

$$\overline{x}(s) = s\left[\frac{\overline{h}(s)}{s}\right]\overline{f}(s) = s\,\overline{A}(s)\,\overline{f}(s)\,. \tag{4.10.26}$$

Using the convolution theorem gives

$$\begin{aligned} x(t) &= \mathscr{L}^{-1}\left\{s\,\overline{A}(s)\cdot\overline{f}(s)\right\} \\ &= \int_0^t A'(t-\tau)\, f(\tau)\, d\tau = \frac{d}{dt}\int_0^t A(\tau)\, f(t-\tau)\, d\tau \end{aligned}$$

which is, by Leibniz's rule,

$$= \int_0^t A(\tau)\, f'(t-\tau)\, d\tau \, + \, A(t)\, f(0)\,,$$

where the initial conditions $A(0) = A'(0) = 0$ are used. $\square$

## 4.11 Exercises

1. Using the Laplace transform, solve the following initial value problems

   (a) $\dfrac{dx}{dt} + ax = e^{-bt}$, $t > 0$, $a \neq b$ with $x(0) = 0$.

   (b) $\dfrac{dx}{dt} - x = t^2$, $t > 0$, $x(0) = 0$.

   (c) $\dfrac{dx}{dt} + 2x = \cos t$, $t > 0$, $x(0) = 1$.

   (d) $\dfrac{dx}{dt} - 2x = 4$, $t > 0$, $x(0) = 0$.

2. Solve the initial value problem for the radioactive decay of an element

$$\frac{dx}{dt} = -kx, \;\; (k > 0), \;\; t > 0, \;\; x(0) = x_0.$$

   Prove that the half-life time $T$ of the element, which is defined as the time taken for half a given amount of the element to decay, is

$$T = \frac{1}{k}\log 2.$$

3. Find the solutions of the following systems of equations with the initial data:

   (a) $\dfrac{dx}{dt} = x - 2y, \quad \dfrac{dy}{dt} = y - 2x, \quad x(0) = 1, \quad y(0) = 0.$

   (b) $\dfrac{dx_1}{dt} = x_1 + 2x_2 + t, \quad \dfrac{dx_2}{dt} = x_2 + 2x_1 + t; \quad x_1(0) = 2, \quad x_2(0) = 4.$

   (c) $\dfrac{dx}{dt} = 6x - 7y + 4z, \quad \dfrac{dy}{dt} = 3x - 4y + 2z, \quad \dfrac{dz}{dt} = -5x + 5y - 3z,$
   with $x(0) = 5, \quad y(0) = z(0) = 0.$

   (d) $\dfrac{dx}{dt} = 2x - 3y, \quad \dfrac{dy}{dt} = y - 2x; \quad x(0) = 2, \quad y(0) = 1.$

   (e) $\dfrac{dx}{dt} + x = y, \quad \dfrac{dy}{dt} - y = x, \quad x(0) = y(0) = 1.$

   (f) $\dfrac{dx}{dt} + \dfrac{dy}{dt} + x = 0, \quad \dfrac{dx}{dt} + 2\,\dfrac{dy}{dt} - x = e^{-at}, \quad x(0) = y(0) = 1.$

4. Solve the matrix differential system

$$\frac{dx}{dt} = Ax \text{ with } x(0) = \begin{pmatrix} x_0 \\ 0 \end{pmatrix},$$

   where $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ and $A = \begin{pmatrix} -3 & -2 \\ 3 & 2 \end{pmatrix}$.

5. Find the solution of the autonomous system described by

$$\frac{dx}{dt} = x, \quad \frac{dy}{dt} = x + 2y \quad \text{with} \quad x(0) = x_0, \quad y(0) = y_0.$$

6. Solve the differential systems

(a)

$$\left.\begin{aligned} \frac{d^2x}{dt^2} - 2k\frac{dy}{dt} + lx = 0 \\ \frac{d^2y}{dt^2} + 2k\frac{dx}{dt} + ly = 0 \end{aligned}\right\} \quad t > 0$$

with the initial conditions

$$x(0) = a, \quad \dot{x}(0) = 0; \quad y(0) = 0, \quad \dot{y}(0) = \upsilon,$$

where $k, l, a$, and $\upsilon$ are constants.

(b)

$$\left.\begin{aligned} \frac{d^2x}{dt^2} = y - 2\,x \\ \frac{d^2y}{dt^2} = x - 2\,y \end{aligned}\right\} \quad t > 0$$

with the initial conditions

$$x(0) = y(0) = 1, \quad \text{and} \quad \dot{x}(0) = \dot{y}(0) = 0.$$

7. The glucose concentration in the blood during continuous intravenous injection of glucose is $C(t)$, which is in excess of the initial value at the start of the infusion. The function $C(t)$ satisfies the initial value problem

$$\frac{dC}{dt} + k\,C = \frac{\alpha}{V}, \quad t > 0, \quad C(0) = 0,$$

where $k$ is the constant velocity of elimination, $\alpha$ is the rate of infusion (in mg/min), and $V$ is the volume in which glucose is distributed. Solve this problem.

8. The blood is pumped into the aorta by the contraction of the heart. The pressure $p(t)$ in the aorta satisfies the initial value problem

$$\frac{dp}{dt} + \frac{c}{k}\,p = cA\,\sin\omega t, \quad t > 0; \quad p(0) = p_0$$

where $c, k, A$, and $p_0$ are constants. Solve this initial value problem.

9. The zero-order chemical reaction satisfies the initial value problem

$$\frac{dc}{dt} = -k_0, \quad t > 0, \quad \text{with} \quad c = c_0 \quad \text{at} \quad t = 0$$

where $k_0$ is a positive constant and $c(t)$ is the concentration of a reacting substance at time $t$. Show that

$$c(t) = c_0 - k_0\, t.$$

10. Solve the equation governing the first order chemical reaction

$$\frac{dc}{dt} = -k_1 c \quad \text{with} \quad c(t) = c_0 \quad \text{at} \quad t = 0 \quad (k_1 > 0).$$

11. Obtain the solutions of the systems of differential equations governing the consecutive chemical reactions of the first order

$$\frac{dc_1}{dt} = -k_1 c_1, \quad \frac{dc_2}{dt} = k_1 c_1 - k_2 c_2, \quad \frac{dc_3}{dt} = k_2 c_2, \quad t > 0,$$

with the initial conditions

$$c_1(0) = c_1, \quad c_2(0) = c_3(0) = 0,$$

where $c_1(t)$ is the concentration of a substance $A$ at time $t$, which breaks down to form a new substance $A_2$ with concentration $c_2(t)$, and $c_3(t)$ is the concentration of a new element originated from $A_2$.

12. Solve the following initial value problems

(a) $\ddot{x} + \omega^2 x = \cos nt, \quad (\omega \neq n) \qquad x(0) = 1, \quad \dot{x}(0) = 0.$

(b) $\ddot{x} + x = \sin 2t, \quad x(0) = \dot{x}(0) = 0.$

(c) $\dfrac{d^3x}{dt^3} + \dfrac{d^2x}{dt^2} = 3e^{-4t}$, $x(0) = 0$, $\dot{x}(0) = -1$, $\ddot{x}(0) = 1$.

(d) $\dfrac{d^4x}{dt^4} = 16x$, $x(t) = \ddot{x}(t) = 0$, $\dot{x}(t) = \dddot{x}(t) = 1$ at $t = 0$.

(e) $(D^4 + 2D^3 - D^2 - 2D + 10)x(t) = 0$, $t > 0$,
$x(0) = -1$, $\dot{x}(0) = 3$, $\ddot{x}(0) = -1$, $\dddot{x}(0) = 4$.

(f) $\dfrac{d^2x}{dt^2} + b\,\dfrac{dx}{dt} = \delta(t-a)$, $x(0) = \alpha$, $\dot{x}(0) = \beta$.

(g) $C\,\dfrac{d^2v}{dt^2} + \dfrac{1}{R}\dfrac{dv}{dt} + \dfrac{v}{L} = \dfrac{di}{dt}$, $v(0) = \dot{v}(0) = 0$; $i(t) = H(t-1) - H(t)$,
where $R$, $L$, and $C$ are constants.

(h) $\dfrac{d^2x}{dt^2} + 2\,t\,\dfrac{dx}{dt} - 4\,x = 2, \quad x(0) = 0 = \dot{x}(0).$

(i) $\frac{d^2x}{dt^2} - 2\,a\,\frac{dx}{dt} + a^2\,x = t - (t-a)\,H(t-a) - a\,H(t-a),$

$x(0) = 0 = \dot{x}(0).$

13. Solve the following systems of equations:

(a) $\ddot{x} - 2\dot{y} - x = 0,\ \ddot{y} + 2\dot{x} - y = 0,$

$x(t) = y(t) = 0, \quad \dot{x}(t) = \dot{y}(t) = 1$ at $t = 0.$

(b) $\ddot{x}_1 + 3\dot{x}_1 - 2x_1 + \dot{x}_2 - 3x_2 = 2e^{-t}, \qquad 2\dot{x}_1 - x_1 + \dot{x}_2 - 2x_2 = 0,$

with $x_1(0) = \dot{x}_1(0) = 0$ and $x_2(0) = 4.$

14. With the aid of the Laplace transform, investigate the motion of a particle governed by the equations of motion $\ddot{x} - \omega\dot{y} = 0, \ddot{y} + \omega\dot{x} = \omega^2 a$ and the initial conditions $x(0) = y(0) = \dot{x}(0) = \dot{y}(0) = 0.$

15. Show that the solution of the equation

$$\frac{d^2y}{dx^2} + (a+b)\frac{dy}{dx} + aby = e^{-ax},\ x > 0$$

with the initial data $y(x) = \frac{1}{a^2}$ and $\frac{dy}{dx} = 0$ at $x = 0$ is

$$y(x) = \frac{1}{a^2(a-b)}(ae^{-bx} - be^{-ax} - xa^2e^{-ax}) + \frac{e^{-bx} - e^{-ax}}{(a-b)^2}.$$

16. The motion of an electron of charge $-e$ in a static electric field $\mathbf{E} = (E, 0, 0)$ and a static magnetic field $\mathbf{H} = (0, 0, H)$ is governed by the vector equation

$$m\ddot{\mathbf{r}} = -e\,\mathbf{E} + \frac{e}{c}\,(\dot{\mathbf{r}} \times \mathbf{H}), \quad t > 0,$$

with zero initial velocity and displacement ($\mathbf{r} = \dot{\mathbf{r}} = \mathbf{0}$ at $t = 0$) where $\mathbf{r} = (x, y, z)$ and $c$ is the velocity of light. Show that the displacement fields are

$$x(t) = \frac{eE}{m\omega^2}\,(\cos\omega t - 1), \quad y(t) = \frac{eE}{m\omega^2}\,(\sin\omega t - \omega t), \quad z(t) = 0,$$

where $\omega = \frac{eH}{mc}$. Hence, calculate the velocity field.

17. An electron of mass $m$ and charge $-e$ is acted on by a periodic electric field $E\sin\omega_0 t$ along the $x$-axis and a constant magnetic field $H$ along the $z$-axis. Initially, the electron is emitted at the origin with zero velocity. With the same $\omega$ as given in exercise 16, show that

$$x(t) = \frac{eE}{m\omega(\omega^2 - \omega_0^2)}\,(\omega_0\ \sin\,\omega t - \omega\ \sin\,\omega_0 t)\,,$$

$$y(t) = \frac{eE}{m\omega\,(\omega^2 - \omega_0^2)\,\omega_0}\,\left\{(\omega^2 - \omega_0^2) + \left(\omega_0^2\cos\,\omega t - \omega^2\cos\,\omega_0 t\right)\right\}.$$

18. The stress-strain relation and equation of motion for a viscoelastic rod in the absence of external force are

$$\frac{\partial e}{\partial t} = \frac{1}{E}\frac{\partial \sigma}{\partial t} + \frac{\sigma}{\eta}, \qquad \frac{\partial \sigma}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2},$$

where $e$ is the strain, $\eta$ is the coefficient of viscosity, and the displacement $u(x,t)$ is related to the strain by $e = \dfrac{\partial u}{\partial x}$. Prove that the stress $\sigma(x,t)$ satisfies the equation

$$\frac{\partial^2 \sigma}{\partial x^2} - \frac{\rho}{\eta}\frac{\partial \sigma}{\partial t} = \frac{1}{c^2}\frac{\partial^2 \sigma}{\partial t^2}.$$

Show that the stress distribution in a semi-infinite viscoelastic rod subject to the boundary and initial conditions

$$\dot{u}(0,t) = UH(t), \quad \sigma(x,t) \to 0 \quad \text{as} \quad x \to \infty,$$
$$\sigma(x,0) = 0, \quad \dot{u}(x,0) = 0, \quad \text{for} \quad 0 < x < \infty,$$

is given by

$$\sigma(x,t) = -U\rho\, c \exp\left(-\frac{Et}{2\eta}\right) I_0\left[\frac{E}{2\eta}\left(t^2 - \frac{x^2}{c^2}\right)^{1/2}\right] H\left(t - \frac{x}{c}\right).$$

19. An elastic string is stretched between $x = 0$ and $x = \ell$ and is initially at rest in the equilibrium position. Find the Laplace transform solution for the displacement subject to the boundary conditions $y(0,t) = f(t)$ and $y(l,t) = 0$, $t > 0$.

20. The end $x = 0$ of a semi-infinite submarine cable is maintained at a potential $V_0 H(t)$. If the cable has no initial current and potential, determine the potential $V(x,t)$ at a point $x$ and at time $t$.

21. A semi-infinite lossless transmission line has no initial current or potential. A time-dependent electromagnetic force, $V_0(t)H(t)$ is applied at the end $x = 0$. Find the potential $V(x,t)$. Hence, determine the potential for cases (i) $V_0(t) = V_0 = \text{constant}$, and (ii) $V_0(t) = V_0 \cos \omega t$.

22. Solve the Blasius problem of an unsteady boundary layer flow in a semi-infinite body of viscous fluid enclosed by an infinite horizontal disk at $z = 0$. The governing equation and the boundary and initial conditions are

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \nu \frac{\partial^2 u}{\partial z^2}, \quad z > 0, \quad t > 0,\\
u(z,t) &= Ut \quad \text{on } z = 0, \quad t > 0,\\
u(z,t) &\to 0 \quad \text{as } z \to \infty, \quad t > 0,\\
u(z,t) &= 0 \quad \text{at } t \le 0, \quad z > 0.
\end{aligned}$$

Explain the significance of the solution.

23. Obtain the solution of the Stokes-Ekman problem of an unsteady boundary layer flow in a semi-infinite body of viscous fluid bounded by an infinite horizontal disk at $z=0$, when both the fluid and the disk rotate with a uniform angular velocity $\Omega$ about the $z$-axis. The governing boundary layer equation, the boundary and the initial conditions are

$$\begin{aligned}
&\frac{\partial q}{\partial t} + 2\Omega i q = \nu \frac{\partial^2 q}{\partial z^2}, \qquad z>0,\\
&q(z,t) = a e^{i\omega t} + b e^{-i\omega t} \quad \text{on} \quad z=0,\ t>0,\\
&q(z,t) \to 0 \quad \text{as} \quad z\to\infty,\ t>0,\\
&q(z,t) = 0 \quad \text{at} \quad t\le 0 \quad \text{for all} \quad z>0,
\end{aligned}$$

where $q=u+iv$, $\omega$ is the frequency of oscillations of the disk and $a, b$ are complex constants. Hence, deduce the steady-state solution and determine the structure of the associated boundary layers.

24. Show that, when $\omega=0$ in exercise 23, the steady flow field is given by

$$q(z,t) \sim (a+b)\exp\left\{\left(-\frac{2i\Omega}{\nu}\right)^{1/2} z\right\}.$$

Hence, determine the thickness of the Ekman layer.

25. Solve the following integral and integro-differential equations:

(a) $f(t) = \sin 2t + \int_0^t f(t-\tau)\sin\tau\, d\tau.$

(b) $f(t) = \frac{t}{2}\sin t + \int_0^t f(\tau)\sin(t-\tau)\, d\tau.$

(c) $\int_0^t f(\tau) J_0[a(t-\tau)]\, d\tau = \sin at.$

(d) $f(t) = \sin t + \int_0^t f(\tau)\sin\{2(t-\tau)\}\, d\tau.$

(e) $f(t) = t^2 + \int_0^t f'(t-\tau)\exp(-a\tau)\, d\tau, \quad f(0)=0.$

(f) $x(t) = 1 + a^2 \int_0^t (t-\tau)\, x(\tau)\, d\tau.$

(g) $x(t) = t + \frac{1}{a} \int_0^t (t-\tau)^3 \, x(\tau)\, d\tau.$

26. Prove that the solution of the integro-differential equation

$$f(t) = \frac{2}{\sqrt{\pi}} \left[ \sqrt{t} + \sqrt{a} \int_0^t (t-\tau)^{1/2} f'(\tau)\, d\tau \right], \quad f(0) = 0$$

is

$$f(t) = \frac{e^{at}}{\sqrt{a}} [1 + erf \sqrt{at}] - \frac{1}{\sqrt{a}}.$$

27. Solve the integro-differential equations

(a) $\frac{d^2x}{dt^2} = \exp(-2t) - \int_0^t \exp\{-2(t-\tau)\} \left(\frac{dx}{d\tau}\right) d\tau, \quad x(0) = 0 \text{ and } \dot{x}(0) = 0.$

(b) $\frac{dx}{dt} = \int_0^t x(\tau) \cos(t-\tau) d\tau, \qquad x(0) = 1.$

28. Using the Laplace transform, evaluate the following integrals:

(a) $\int_0^\infty \frac{\sin tx}{x(x^2+a^2)} dx, \ (a, t > 0),$ (b) $\int_0^\infty \frac{\sin tx}{x} dx,$

(c) $\int_{-\infty}^\infty \frac{\cos tx}{x^2+a^2} dx, \ (a, t > 0),$ (d) $\int_{-\infty}^\infty \frac{x \sin xt}{x^2+a^2} dx, \ (a, t > 0),$

(e) $\int_0^\infty \exp(-tx^2) dx, \quad t > 0,$ (f) $\int_0^\infty \cos(tx^2)\, dx.$

29. Show that

(a) $\int_0^\infty e^{-ax} \left( \frac{\cos px - \cos qx}{x} \right) dx = \frac{1}{2} \log \left( \frac{a^2+q^2}{a^2+p^2} \right), \ (a > 0).$

(b) $\int_0^\infty e^{-ax} \left( \frac{\sin qx - \sin px}{x} \right) dx = \tan^{-1}\left(\frac{q}{a}\right) - \tan^{-1}\left(\frac{p}{a}\right), \quad a > 0.$

30. Establish the following results:

(a) $\int_{-\infty}^\infty \frac{\cos tx\, dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2} \left( \frac{e^{-bt}}{b} - \frac{e^{-at}}{a} \right), \quad a, b, t > 0.$

(b) $\displaystyle\int_0^\infty \frac{\sin(\pi\, tx)}{x(1+x^2)}\, dx = \frac{\pi}{2}(1-e^{-\pi t}), \quad t>0.$

(c) $\displaystyle\int_0^\infty \cos(tu^2)\, du = \int_0^\infty \sin(tu^2)\, du = \frac{1}{2}\left(\frac{\pi}{2t}\right)^{1/2}, \quad t>0.$

31. In Example 4.5.1(i), write the solution when the point load is applied at the mid point of the beam.

32. A uniform horizontal beam of length $2\ell$ is clamped at the end $x=0$ and freely supported at $x=2\ell$. It carries a distributed load of constant value $W$ in $\frac{\ell}{2}<x<\frac{3\ell}{2}$ and zero elsewhere. Obtain the deflection of the beam which satisfies the boundary value problem

$$EI\,\frac{d^4y}{dx^4} = W\left[H\left(x-\frac{\ell}{2}\right) - H\left(x-\frac{3\ell}{2}\right)\right], \quad 0<x<2\ell,$$
$$y(0)=0=y'(0), \quad y''(2\ell)=0=y'''(2\ell).$$

33. Solve exercise 32 if the beam carries a constant distributed load $W$ per unit length in $0<x<\ell$ and zero in $\ell<x<2\ell$. Find the bending moment and shear at $x=\frac{\ell}{2}$.

34. A horizontal cantilever beam of length $2\ell$ is deflected under the combined effect of its own constant weight $W$ and a point load of magnitude $P$ located at the midpoint. Obtain the deflection of the beam which satisfies the boundary value problem

$$EI\,\frac{d^4y}{dx^4} = W[H(x)-H(x-2\ell)] + P\,\delta(x-\ell), \quad 0<x<2\ell,$$
$$y(0)=0=y'(0), \quad y''(2\ell)=0=y'''(2\ell).$$

Find the bending moment and shear at $x=\frac{\ell}{2}$.

35. Using the Laplace transform, solve the following difference equations:

(a) $\Delta u_n - 2u_n = 0, \quad u_0 = 1,$

(b) $\Delta^2 u_n - 2u_{n+1} + 3u_n = 0, \quad u_0 = 0 \quad \text{and} \quad u_1 = 1,$

(c) $u_{n+2} - 4u_{n+1} + 4u_n = 0, \quad u_0 = 1 \quad \text{and} \quad u_1 = 4,$

(d) $u_{n+2} - 5u_{n+1} + 6u_n = 0, \quad u_0 = 1 \quad \text{and} \quad u_1 = 4,$

(e) $\Delta^2 u_n + 3u_n = 0, \quad u_0 = 0, \quad u_1 = 1,$

(f) $u_{n+2} - 4u_{n+1} + 3u_n = 0,$

(g) $u_{n+2} - 9u_n = 0, \quad u_0 = 1 \text{ and } u_1 = 3,$

(h) $\Delta u_n - (a-1)u_n = 0, \quad u_0 = \text{constant}.$

36. Show that the solution of the difference equation

$$u_{n+2} + 4u_{n+1} + u_n = 0, \quad \text{with} \quad u_0 = 0 \quad \text{and} \quad u_1 = 1,$$

is

$$u_n = \frac{1}{2\sqrt{3}}\left[\left(\sqrt{3}-2\right)^n + (-1)^{n+1}\left(2+\sqrt{3}\right)^n\right].$$

37. Show that the solution of the differential-difference equation

$$\dot{u}(t) - u(t-1) = 2, \quad u(0) = 0$$

is

$$u(t) = 2\left[t - \frac{(t-1)^2}{2!} + \frac{(t-2)^3}{3!} + \cdots + \frac{(t-n)^{n+1}}{(n+1)!}\right], \quad t > n.$$

38. Obtain the solution of the differential-difference equation

$$\dot{u} = u(t-1), \quad u(0) = 1, \quad 0 < t < \infty \text{ with } u(t) = 1 \text{ when } -1 \le t < 0.$$

39. Use the Laplace transform to solve the initial-boundary value problem

$$\begin{aligned}
&u_{tt} - u_{xx} = k^2\, u_{xxtt}, \quad 0 < x < \infty, \quad t > 0, \\
&u(x,0) = 0, \quad \left(\frac{\partial u}{\partial x}\right)_{t=0} = 0, \quad \text{for} \quad x > 0, \\
&u(x,t) \to 0 \quad \text{as} \quad x \to \infty, \quad t > 0, \\
&u(0,t) = 1 \quad \text{for} \quad t > 0.
\end{aligned}$$

Hence, show that

$$\left(\frac{\partial u}{\partial x}\right)_{x=0} = -\frac{1}{k}\, J_0\left(\frac{t}{k}\right).$$

40. Solve the telegraph equation

$$\begin{aligned}
u_{tt} - c^2 u_{xx} + 2a u_t = 0, \quad -\infty < x < \infty, \quad t > 0, \\
u(x,0) = 0, \quad u_t(x,0) = g(x).
\end{aligned}$$

41. Use the joint Laplace and Fourier transform to solve Example 2.12.3 in Chapter 2.

42. Use the Laplace transform to solve the initial-boundary value problem

$$\begin{aligned}
u_t = c^2 u_{xx}, \quad 0 < x < a, \quad t > 0, \\
u(x,0) = x + \sin\left(\frac{3\pi x}{a}\right) \quad \text{for} \quad 0 < x < a, \\
u(0,t) = 0 = u(a,t) \quad \text{for} \quad t > 0.
\end{aligned}$$

43. Solve the diffusion equation

$$u_t = k u_{xx}, \quad -a < x < a, \quad t > 0,$$
$$u(x,0) = 1 \quad \text{for} \quad -a < x < a,$$
$$u(-a,t) = 0 = u(a,t) \quad \text{for} \quad t > 0.$$

44. Use the joint Laplace and Fourier transform to solve the initial value problem for water waves which satisfies (see Debnath, 1994, p. 92)

$$\nabla^2\phi = \phi_{xx} + \phi_{zz} = 0, \quad -\infty < z < 0, \quad -\infty < x < \infty, \quad t > 0$$
$$\left.\begin{aligned} \phi_z &= \eta_t \\ \phi_t + g\eta &= -\frac{P}{\rho} p(x) e^{i\omega t} \end{aligned}\right\} \text{ on } \quad z = 0, \quad t > 0,$$
$$\phi(x,z,0) = 0 = \eta(x,0) \quad \text{for all } x \text{ and } z,$$

where $P$ and $\rho$ are constants.

45. Show that

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{a_n}{\sqrt{n^2+x^2}} = \int_0^{\infty} b(t) J_0(xt)\, dt$, where $b(t)$ is given by (4.9.3).

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2-a^2} = \frac{1}{2a^2}(1 - \pi a \cot \pi a).$

46. Show that

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{(n^2-a^2)} = \frac{1}{2a^2}\left[1 - \frac{\pi a \cos ax}{\sin a\pi}\right].$

(b) $\displaystyle\sum_{n=1}^{\infty} \log\left(1 + \frac{a^2}{n^2}\right) = \log\left(\frac{\sinh \pi a}{\pi a}\right).$

47. (a) If $f(t) = 1$ in Example 4.3.3, show that

$$u(x,t) = \frac{x}{\sqrt{4\pi\kappa}} \int_o^t \tau^{-\frac{3}{2}} \exp\left(-\frac{x^2}{4\kappa\tau}\right) d\tau = u_0(x,t) \qquad \text{(say)}$$

(b) Hence or otherwise derive the *Duhamel's formula* from (4.3.16):

$$u(x,t) = \int_o^t f(t-\tau)\left(\frac{\partial u_0}{\partial \tau}\right) d\tau,$$
$$\text{where} \quad \frac{\partial u_0}{\partial t} = \frac{x}{\sqrt{4\pi\kappa}} \tau^{-\frac{3}{2}} \exp\left(-\frac{x^2}{4\kappa t}\right).$$

48. Consider a progressive plane wave solution that propagates to the right with the phase velocity $\left(\frac{\omega}{k}\right)$ of the telegraph equation (4.3.55)

 (a) Derive the dispersion relation

$$\omega^2 + i\,(p+q)\,\omega - \left(c^2k^2 + p\,q\right) = 0.$$

 (b) If $4p\,q \neq (p+q)^2$, show that the plane wave solution is given by

$$u(x,t) = A\exp\left[-\frac{1}{2}\,(p+q)\,t\right]\exp\left[i(kx \pm \sigma t)\right],$$
$$\text{where} \quad \sigma = \frac{1}{2}\sqrt{4c^2k^2 + 4p\,q - (p+q)^2}.$$

 (c) If $4p\,q = (p+q)^2$, show that the plane wave solution is given by

$$u(x,t) = A\exp\left[-\frac{1}{2}\,(p+q)\,t\right]\exp\left[ik(x \pm ct)\right].$$

Explain the physical significance of the solutions given in cases (b) and (c).

49. (a) Use the substitution $v(x,t) = \exp\left[\frac{1}{2}\,(p+q)\,t\right]u(x,t)$ into (4.3.55) to show that $v(x,t)$ satisfies the wave equation

$$v_{tt} - c^2\,v_{xx} = \frac{1}{4}\,(p-q)^2\,v.$$

 (b) Show that the undistorted wave solution exists if $p = q$ and that a progressive wave of the form $\exp(-at)f(x \pm ct)$ propagates in either direction where $f$ is an arbitrary twice differentiable function of its argument.

50. (a) Use the joint Laplace and Fourier transform to solve the inhomogeneous diffusion problem

$$\begin{aligned} u_t - \kappa\,u_{xx} &= q(x,t) \quad x \in \mathbb{R}, \quad t > 0, \\ u(x,0) &= f(x), \quad \text{for all } x \in \mathbb{R}. \end{aligned}$$

 (b) Solve the initial-boundary value problem for the diffusion equation

$$\begin{aligned} u_t - \kappa\,u_{xx} &= 0, \quad 0 < x < l, \quad t > 0 \\ u(x,0) &= 0, \quad \text{and} \quad u(0,t) = 1 = u(l,t). \end{aligned}$$

51. Use the Laplace transform to solve for the small displacement $y(x,\,t)$ of a semi-infinite string fixed at $x = 0$ under the action of gravity $g$ that

satisfies the wave equation and the initial-boundary conditions

$$\begin{aligned}\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} &= -g, \quad 0 < x < \infty, \quad t > 0,\\ y(x, 0) &= 0 = y_t(x, 0), \quad x \geq 0,\\ \frac{\partial y}{\partial x} &\to 0 \quad \text{as} \quad \text{x} \to \infty.\end{aligned}$$

52. Use the Laplace transform to solve the boundary layer equation (4.3.97) subject to the boundary and initial conditions

$$\begin{aligned}u(z, t) &= U_0\, f(t), \quad \text{on z} = 0, \quad \text{t} > 0,\\ u(z, t) &\to 0 \quad \text{as} \quad z \to \infty, \quad t > 0,\\ u(z, t) &\to 0 \quad \text{at} \quad t \leq 0 \quad \text{for all } z > 0.\end{aligned}$$

Consider the special case where $f(t) = \sin \omega t$.

53. Find the transfer function, the impulse response function and a formula for the solution of the following systems:

(a) $x''(t) + 2\,x'(t) + 5\,x(t) = f(t), \quad x(0) = 2,\ x'(0) = -2.$

(b) $x''(t) - 2\,x'(t) + 5\,x(t) = f(t), \quad x(0) = 0,\ x'(0) = 2.$

(c) $x''(t) + 9\,x'(t) = f(t), \quad x(0) = 2,\ x'(0) = -3.$

(d) $x''(t) - 2\,x'(t) + 5\,x(t) = f(t), \quad x(0) = x_0,\ x'(0) = x_1.$

54. Determine the transfer function for each of the following systems. Obtain the order of each system and find which is stable.

(a) $x''(t) + 2\,x'(t) + 2\,x(t) = 3\,f'(t) + 2\,f(t).$

(b) $4\,x''(t) + 16\,x'(t) + 25\,x(t) = 2\,f'(t) + 3\,f(t).$

(c) $36\,x''(t) + 12\,x'(t) + 37\,x(t) = 2\,f''(t) + f'(t) - 6\,f(t).$

(d) $x''(t) - 6\,x'(t) + 10\,x(t) = 2\,f'(t) + 5\,f(t).$

55. Examine the stability of a system for real constants $a$ and $b$ with zero initial data

$$x'''(t) - a\,x''(t) + b^2\,x'(t) - ab^2 x(t) = f(t),$$

where $x(t)$ is the output corresponding to input $f(t)$.

Discuss three cases: (a) $a > 0$, (b) $a \leq 0,\ \ b \neq 0$, (c) $a \neq 0,\ \ b = 0$.

# 8

## *Mellin Transforms and Their Applications*

> "One cannot understand ... the universality of laws of nature, the relationship of things, without an understanding of mathematics. There is no other way to do it."
>
> Richard P. Feynman

> "The research worker, in his efforts to express the fundamental laws of Nature in mathematical form, should strive mainly for mathematical beauty. He should take simplicity into consideration in a subordinate way to beauty. ... It often happens that the requirements of simplicity and beauty are the same, but where they clash the latter must take precedence."
>
> Paul Dirac

## 8.1 Introduction

This chapter deals with the theory and applications of the Mellin transform. We derive the Mellin transform and its inverse from the complex Fourier transform. This is followed by several examples and the basic operational properties of Mellin transforms. We discuss several applications of Mellin transforms to boundary value problems and to summation of infinite series. The Weyl transform and the Weyl fractional derivatives with examples are also included.

Historically, Riemann (1876) first recognized the *Mellin transform* in his famous memoir on prime numbers. Its explicit formulation was given by C-ahen (1894). Almost simultaneously, Mellin (1896, 1902) gave an elaborate discussion of the Mellin transform and its inversion formula.

## 8.2 Definition of the Mellin Transform and Examples

We derive the Mellin transform and its inverse from the complex Fourier transform and its inverse, which are defined respectively by

$$\mathscr{F}\{g(\xi)\} = G(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik\xi} g(\xi) d\xi, \tag{8.2.1}$$

$$\mathscr{F}^{-1}\{G(k)\} = g(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik\xi} G(k) dk. \tag{8.2.2}$$

Making the changes of variables $\exp(\xi) = x$ and $ik = c - p$, where $c$ is a constant, in results (8.2.1) and (8.2.2) we obtain

$$G(ip - ic) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} x^{p-c-1} g(\log x) dx, \tag{8.2.3}$$

$$g(\log x) = \frac{1}{\sqrt{2\pi}} \int_{c-i\infty}^{c+i\infty} x^{c-p} G(ip - ic) dp. \tag{8.2.4}$$

We now write $\frac{1}{\sqrt{2\pi}} x^{-c} g(\log x) \equiv f(x)$ and $G(ip - ic) \equiv \tilde{f}(p)$ to define the *Mellin transform* of $f(x)$ and the *inverse Mellin transform* as

$$\mathscr{M}\{f(x)\} = \tilde{f}(p) = \int_{0}^{\infty} x^{p-1} f(x) dx, \tag{8.2.5}$$

$$\mathscr{M}^{-1}\{\tilde{f}(p)\} = f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-p} \tilde{f}(p) dp, \tag{8.2.6}$$

where $f(x)$ is a real valued function defined on $(0, \infty)$ and the Mellin transform variable $p$ is a complex number. Sometimes, the Mellin transform of $f(x)$ is denoted explicitly by $\tilde{f}(p) = \mathscr{M}[f(x), p]$. Obviously, $\mathscr{M}$ and $\mathscr{M}^{-1}$ are linear integral operators.

***Example 8.2.1***

(a) If $f(x) = e^{-nx}$, where $n > 0$, then

$$\mathscr{M}\{e^{-nx}\} = \tilde{f}(p) = \int_{0}^{\infty} x^{p-1} e^{-nx} dx,$$

which is, by putting $nx = t$,

$$= \frac{1}{n^p}\int_0^\infty t^{p-1}e^{-t}dt = \frac{\Gamma(p)}{n^p}. \tag{8.2.7}$$

(b) If $f(x) = \dfrac{1}{1+x}$, then

$$\mathscr{M}\left\{\frac{1}{1+x}\right\} = \tilde{f}(p) = \int_0^\infty x^{p-1}\cdot\frac{dx}{1+x},$$

which is, by substituting $x = \dfrac{t}{1-t}$ or $t = \dfrac{x}{1+x}$,

$$= \int_0^1 t^{p-1}(1-t)^{(1-p)-1}dt = B(p, 1-p) = \Gamma(p)\Gamma(1-p),$$

which is, by a well-known result for the gamma function,

$$= \pi\operatorname{cosec}(p\pi), \qquad 0 < \operatorname{Re}(p) < 1. \tag{8.2.8}$$

(c) If $f(x) = (e^x - 1)^{-1}$, then

$$\mathscr{M}\left\{\frac{1}{e^x-1}\right\} = \tilde{f}(p) = \int_0^\infty x^{p-1}\frac{1}{e^x-1}dx,$$

which is, by using $\displaystyle\sum_{n=0}^\infty e^{-nx} = \frac{1}{1-e^{-x}}$ and hence, $\displaystyle\sum_{n=1}^\infty e^{-nx} = \frac{1}{e^x-1}$,

$$= \sum_{n=1}^\infty \int_0^\infty x^{p-1}e^{-nx}dx = \sum_{n=1}^\infty \frac{\Gamma(p)}{n^p} = \Gamma(p)\zeta(p), \tag{8.2.9}$$

where $\zeta(p) = \displaystyle\sum_{n=1}^\infty \frac{1}{n^p}$, $(\operatorname{Re} p > 1)$ is the famous *Riemann zeta function.*

(d) If $f(x) = \dfrac{2}{e^{2x}-1}$, then

$$\mathscr{M}\left\{\frac{2}{e^{2x}-1}\right\} = \tilde{f}(p) = 2\int_0^\infty x^{p-1}\frac{dx}{e^{2x}-1} = 2\sum_{n=1}^\infty \int_0^\infty x^{p-1}e^{-2nx}dx$$

$$= 2\sum_{n=1}^\infty \frac{\Gamma(p)}{(2n)^p} = 2^{1-p}\,\Gamma(p)\sum_{n=1}^\infty \frac{1}{n^p} = 2^{1-p}\,\Gamma(p)\zeta(p). \tag{8.2.10}$$

(e) If $f(x) = \dfrac{1}{e^x+1}$, then

$$\mathscr{M}\left\{\frac{1}{e^x+1}\right\} = (1-2^{1-p})\Gamma(p)\,\zeta(p). \tag{8.2.11}$$

This follows from the result

$$\left[\frac{1}{e^x-1} - \frac{1}{e^x+1}\right] = \frac{2}{e^{2x}-1}$$

combined with (8.2.9) and (8.2.10).

(f) If $f(x) = \dfrac{1}{(1+x)^n}$, then

$$\mathscr{M}\left\{\frac{1}{(1+x)^n}\right\} = \int_0^\infty x^{p-1}(1+x)^{-n}\,dx,$$

which is, by putting $x = \dfrac{t}{1-t}$ or $t = \dfrac{x}{1+x}$,

$$= \int_0^1 t^{p-1}(1-t)^{n-p-1}\,dt$$

$$= B(p, n-p) = \frac{\Gamma(p)\Gamma(n-p)}{\Gamma(n)}, \tag{8.2.12}$$

where $B(p,q)$ is the standard beta function.

Hence,

$$\mathscr{M}^{-1}\{\Gamma(p)\Gamma(n-p)\} = \frac{\Gamma(n)}{(1+x)^n}.$$

(g) Find the Mellin transform of $\cos kx$ and $\sin kx$.

It follows from Example 8.2.1(a) that

$$\mathscr{M}\left[e^{-ikx}\right] = \frac{\Gamma(p)}{(ik)^p} = \frac{\Gamma(p)}{k^p}\left(\cos\frac{p\pi}{2} - i\sin\frac{p\pi}{2}\right).$$

Separating real and imaginary parts, we find

$$\mathscr{M}\left[\cos kx\right] = k^{-p}\,\Gamma(p)\cos\left(\frac{\pi p}{2}\right), \tag{8.2.13}$$

$$\mathscr{M}\left[\sin kx\right] = k^{-p}\,\Gamma(p)\sin\left(\frac{\pi p}{2}\right). \tag{8.2.14}$$

These results can be used to calculate the Fourier cosine and Fourier sine transforms of $x^{p-1}$. Result (8.2.13) can be written as

$$\int_0^\infty x^{p-1}\cos kx\,dx = \frac{\Gamma(p)}{k^p}\cos\left(\frac{\pi p}{2}\right).$$

Or, equivalently,

$$\mathscr{F}_c\left\{\sqrt{\frac{\pi}{2}}\,x^{p-1}\right\} = \frac{\Gamma(p)}{k^p}\cos\left(\frac{\pi p}{2}\right).$$

Or,

$$\mathscr{F}_c\left\{x^{p-1}\right\} = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(p)}{k^p}\cos\left(\frac{\pi p}{2}\right). \tag{8.2.15}$$

Similarly,

$$\mathscr{F}_s\left\{x^{p-1}\right\} = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(p)}{k^p}\sin\left(\frac{\pi p}{2}\right). \tag{8.2.16}$$

□

## 8.3 Basic Operational Properties of Mellin Transforms

If $\mathscr{M}\{f(x)\} = \tilde{f}(p)$, then the following operational properties hold:

(a) (*Scaling Property*).

$$\mathscr{M}\left\{f(ax)\right\} = a^{-p}\tilde{f}(p), \;\; a > 0. \tag{8.3.1}$$

**PROOF** By definition, we have,

$$\mathscr{M}\left\{f(ax)\right\} = \int_0^\infty x^{p-1} f(ax)dx,$$

which is, by substituting $ax = t$,

$$= \frac{1}{a^p}\int_0^\infty t^{p-1} f(t)dt = \frac{\tilde{f}(p)}{a^p}.$$

∎

(b) (*Shifting Property*).

$$\mathscr{M}\left[x^a f(x)\right] = \tilde{f}(p+a). \tag{8.3.2}$$

Its proof follows from the definition.

(c)

$$\mathscr{M}\left\{f(x^a)\right\} = \frac{1}{a}\,\tilde{f}\left(\frac{p}{a}\right), \tag{8.3.3}$$

$$\mathscr{M}\left\{\frac{1}{x}f\left(\frac{1}{x}\right)\right\}=\tilde{f}(1-p), \tag{8.3.4}$$

$$\mathscr{M}\left\{(\log x)^n f(x)\right\}=\frac{d^n}{dp^n}\tilde{f}(p), \quad n=1,2,3,\ldots. \tag{8.3.5}$$

The proofs of (8.3.3) and (8.3.4) are easy and hence, left to the reader.

Result (8.3.5) can easily be proved by using the result

$$\frac{d}{dp}x^{p-1}=(\log x)x^{p-1}. \tag{8.3.6}$$

(d) (*Mellin Transforms of Derivatives*).

$$\mathscr{M}\left[f'(x)\right]=-(p-1)\tilde{f}(p-1), \tag{8.3.7}$$

provided $[x^{p-1}f(x)]$ vanishes as $x\to 0$ and as $x\to\infty$.

$$\mathscr{M}\left[f''(x)\right]=(p-1)(p-2)\tilde{f}(p-2). \tag{8.3.8}$$

More generally,

$$\begin{aligned}\mathscr{M}\left[f^{(n)}(x)\right]&=(-1)^n\frac{\Gamma(p)}{\Gamma(p-n)}\tilde{f}(p-n)\\&=(-1)^n\frac{\Gamma(p)}{\Gamma(p-n)}\,\mathscr{M}\left[f(x),p-n\right],\end{aligned} \tag{8.3.9}$$

provided $x^{p-r-1}f^{(r)}(x)=0$ as $x\to 0$ for $r=0,1,2,\ldots,(n-1)$.

**PROOF** We have, by definition,

$$\mathscr{M}\left[f'(x)\right]=\int_0^\infty x^{p-1}f'(x)\,dx,$$

which is, integrating by parts,

$$\begin{aligned}&=\left[x^{p-1}f(x)\right]_0^\infty-(p-1)\int_0^\infty x^{p-2}f(x)\,dx\\&=-(p-1)\tilde{f}(p-1).\end{aligned}$$

∎

The proofs of (8.3.8) and (8.3.9) are similar and left to the reader.

(e) If $\mathscr{M}\{f(x)\}=\tilde{f}(p)$, then

$$\mathscr{M}\left\{xf'(x)\right\}=-p\tilde{f}(p), \tag{8.3.10}$$

provided $x^p f(x)$ vanishes at $x=0$ and as $x\to\infty$.

$$\mathscr{M}\{x^2 f''(x)\} = (-1)^2 p(p+1)\tilde{f}(p). \tag{8.3.11}$$

More generally,

$$\mathscr{M}\{x^n f^{(n)}(x)\} = (-1)^n \frac{\Gamma(p+n)}{\Gamma(p)}\tilde{f}(p). \tag{8.3.12}$$

**PROOF** We have, by definition,

$$\mathscr{M}\{xf'(x)\} = \int_0^\infty x^p f'(x)dx,$$

which is, integrating by parts,

$$= [x^p f(x)]_0^\infty - p\int_0^\infty x^{p-1} f(x)dx = -p\tilde{f}(p).$$

∎

Similar arguments can be used to prove results (8.3.11) and (8.3.12).

(f) (*Mellin Transforms of Differential Operators*).
If $\mathscr{M}\{f(x)\} = \tilde{f}(p)$, then

$$\mathscr{M}\left[\left(x\frac{d}{dx}\right)^2 f(x)\right] = \mathscr{M}\left[x^2 f''(x) + xf'(x)\right] = (-1)^2\, p^2\tilde{f}(p), \tag{8.3.13}$$

and more generally,

$$\mathscr{M}\left[\left(x\frac{d}{dx}\right)^n f(x)\right] = (-1)^n p^n \tilde{f}(p). \tag{8.3.14}$$

**PROOF** We have, by definition,

$$\begin{aligned}
\mathscr{M}\left[\left(x\frac{d}{dx}\right)^2 f(x)\right] &= \mathscr{M}\left[x^2 f''(x) + x\, f'(x)\right] \\
&= \mathscr{M}\left[x^2 f''(x)\right] + \mathscr{M}\left[x\, f'(x)\right] \\
&= -p\tilde{f}(p) + p(p+1)\tilde{f}(p) \qquad \text{by (8.3.10) and (8.3.11)} \\
&= (-1)^2\, p^2\, \tilde{f}(p).
\end{aligned}$$

∎

Similar arguments can be used to prove the general result (8.3.14).

(g) (*Mellin Transforms of Integrals*).

$$\mathscr{M}\left\{\int_0^x f(t)dt\right\} = -\frac{1}{p}\,\tilde{f}(p+1). \tag{8.3.15}$$

In general,

$$\mathscr{M}\{I_n\,f(x)\} = \mathscr{M}\left\{\int_0^x I_{n-1}f(t)dt\right\} = (-1)^n\frac{\Gamma(p)}{\Gamma(p+n)}\tilde{f}(p+n), \tag{8.3.16}$$

where $I_n\,f(x)$ is the $n$th repeated integral of $f(x)$ defined by

$$I_n f(x) = \int_0^x I_{n-1}f(t)dt. \tag{8.3.17}$$

**PROOF** We write

$$F(x) = \int_0^x f(t)dt$$

so that $F'(x) = f(x)$ with $F(0) = 0$. Application of (8.3.7) with $F(x)$ as defined gives

$$\mathscr{M}\,\{f(x) = F'(x), p\} = -(p-1)\mathscr{M}\left\{\int_0^x f(t)dt,\, p-1\right\},$$

which is, replacing $p$ by $p+1$,

$$\mathscr{M}\left\{\int_0^x f(t)\,dt,\, p\right\} = -\frac{1}{p}\mathscr{M}\,\{f(x), p+1\} = -\frac{1}{p}\,\tilde{f}(p+1).$$

An argument similar to this can be used to prove (8.3.16). ∎

(h) (*Convolution Type Theorems*).
If $\mathscr{M}\,\{f(x)\} = \tilde{f}(p)$ and $\mathscr{M}\{g(x)\} = \tilde{g}(p)$, then

$$\mathscr{M}\,[f(x) * g(x)] = \mathscr{M}\left[\int_0^\infty f(\xi)\,g\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi}\right] = \tilde{f}(p)\tilde{g}(p), \tag{8.3.18}$$

$$\mathscr{M}\,[f(x)\circ g(x)] = \mathscr{M}\left[\int_0^\infty f(x\xi)\,g(\xi)d\xi\right] = \tilde{f}(p)\tilde{g}(1-p). \tag{8.3.19}$$

**PROOF** We have, by definition,

$$\begin{aligned}
\mathscr{M}\left[f(x)*g(x)\right] &= \mathscr{M}\left[\int_0^\infty f(\xi)\,g\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi}\right] \\
&= \int_0^\infty x^{p-1}dx \int_0^\infty f(\xi)\,g\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi} \\
&= \int_0^\infty f(\xi)\,\frac{d\xi}{\xi}\int_0^\infty x^{p-1}g\left(\frac{x}{\xi}\right)\,dx, \quad \left(\frac{x}{\xi}=\eta\right), \\
&= \int_0^\infty f(\xi)\,\frac{d\xi}{\xi}\int_0^\infty (\xi\eta)^{p-1}g(\eta)\,\xi\,d\eta \\
&= \int_0^\infty \xi^{p-1}f(\xi)d\xi \int_0^\infty \eta^{p-1}g(\eta)d\eta = \tilde{f}(p)\tilde{g}(p).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mathscr{M}\left[f(x)\circ g(x)\right] &= \mathscr{M}\left[\int_0^\infty f(x\xi)\,g(\xi)d\xi\right] \\
&= \int_0^\infty x^{p-1}dx \int_0^\infty f(x\xi)\,g(\xi)d\xi, \quad (x\xi=\eta), \\
&= \int_0^\infty g(\xi)d\xi \int_0^\infty \eta^{p-1}\xi^{1-p}f(\eta)\frac{d\eta}{\xi} \\
&= \int_0^\infty \xi^{1-p-1}g(\xi)d\xi \int_0^\infty \eta^{p-1}f(\eta)d\eta = \tilde{g}(1-p)\tilde{f}(p).
\end{aligned}$$

∎

Note that, in this case, the operation $\circ$ is not commutative.
Clearly, putting $x=s$,

$$\mathscr{M}^{-1}\{\tilde{f}(1-p)\tilde{g}(p)\} = \int_0^\infty g(st)f(t)dt.$$

Putting $g(t)=e^{-t}$ and $\tilde{g}(p)=\Gamma(p)$, we obtain the Laplace transform of $f(t)$

$$\mathscr{M}^{-1}\left\{\tilde{f}(1-p)\Gamma(p)\right\} = \int_0^\infty e^{-st}f(t)dt = \mathscr{L}\left\{f(t)\right\} = \bar{f}(s). \tag{8.3.20}$$

(i) (*Parseval's Type Property*).
If $\mathscr{M}\{f(x)\} = \tilde{f}(p)$ and $\mathscr{M}\{g(x)\} = \tilde{g}(p)$, then

$$\mathscr{M}[f(x)g(x)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(s)\tilde{g}(p-s)ds. \tag{8.3.21}$$

Or, equivalently,

$$\int_0^\infty x^{p-1} f(x)g(x)dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(s)\tilde{g}(p-s)ds. \tag{8.3.22}$$

In particular, when $p = 1$, we obtain the *Parseval formula* for the Mellin transform,

$$\int_0^\infty f(x)g(x)dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(s)\tilde{g}(1-s)ds. \tag{8.3.23}$$

**PROOF** By definition, we have

$$\begin{aligned}
\mathscr{M}[f(x)g(x)] &= \int_0^\infty x^{p-1} f(x)g(x)dx \\
&= \frac{1}{2\pi i}\int_0^\infty x^{p-1} g(x)dx \int_{c-i\infty}^{c+i\infty} x^{-s}\tilde{f}(s)ds \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(s)ds \int_0^\infty x^{p-s-1} g(x)dx \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(s)\tilde{g}(p-s)ds.
\end{aligned}$$

When $p = 1$, the above result becomes (8.3.23). ∎

## 8.4 Applications of Mellin Transforms

***Example 8.4.1***
Obtain the solution of the boundary value problem

$$x^2 u_{xx} + x u_x + u_{yy} = 0, \qquad 0 \le x < \infty, \qquad 0 < y < 1 \tag{8.4.1}$$

$$u(x,0) = 0, \quad u(x,1) = \left\{ \begin{array}{ll} A, & 0 \le x \le 1 \\ 0, & x > 1 \end{array} \right\}, \tag{8.4.2}$$

where $A$ is a constant.

We apply the Mellin transform of $u(x, y)$ with respect to $x$ defined by

$$\tilde{u}(p,y) = \int_0^\infty x^{p-1} u(x,y)\, dx$$

to reduce the given system into the form

$$\tilde{u}_{yy} + p^2 \tilde{u} = 0, \quad 0 < y < 1$$

$$\tilde{u}(p,0) = 0, \quad \tilde{u}(p,1) = A \int_0^1 x^{p-1} dx = \frac{A}{p}.$$

The solution of the transformed problem is

$$\tilde{u}(p,y) = \frac{A}{p} \frac{\sin\, py}{\sin\, p}, \qquad 0 < \operatorname{Re} p < 1.$$

The inverse Mellin transform gives

$$u(x,y) = \frac{A}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{-p}}{p} \frac{\sin\, py}{\sin\, p}\, dp, \tag{8.4.3}$$

where $\tilde{u}(p,y)$ is analytic in the vertical strip $0 < \operatorname{Re}(p) = c < \pi$. The integrand of (8.4.3) has simple poles at $p = n\pi$, $n = 1, 2, 3, \ldots$ which lie inside a semicircular contour in the right half plane. Evaluating (8.4.3) by theory of residues gives the solution for $x > 1$ as

$$u(x,y) = \frac{A}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (-1)^n\, x^{-n\pi} \sin\, n\pi y. \tag{8.4.4}$$

□

***Example 8.4.2***
(*Potential in an Infinite Wedge*). Find the potential $\phi(r,\theta)$ that satisfies the Laplace equation

$$r^2\phi_{rr} + r\phi_r + \phi_{\theta\theta} = 0 \tag{8.4.5}$$

in an infinite wedge $0 < r < \infty$, $-\alpha < \theta < \alpha$ as shown in Figure 8.1 with the boundary conditions

$$\phi(r,\alpha) = f(r), \;\; \phi(r,\,-\alpha) = g(r) \quad 0 \le r < \infty, \tag{8.4.6ab}$$

$$\phi(r,\theta) \to 0 \;\text{ as }\; r \to \infty \;\text{ for all }\; \theta \;\text{ in }\; -\alpha < \theta < \alpha. \tag{8.4.7}$$

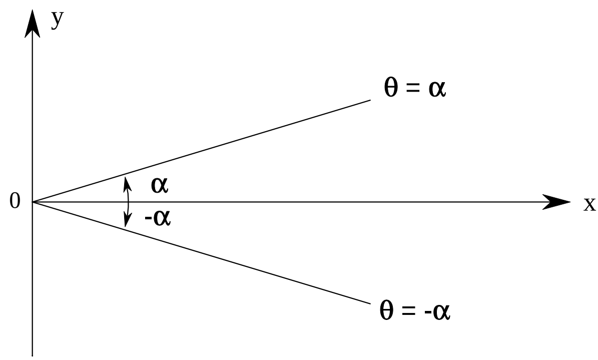


**Figure 8.1** An infinite wedge.

We apply the Mellin transform of the potential $\phi(r,\theta)$ defined by

$$\mathscr{M}\,[\phi(r,\theta)] = \tilde{\phi}(p,\theta) = \int_0^\infty r^{p-1}\phi(r,\theta)\,dr$$

to the differential system (8.4.5)–(8.4.7) to obtain

$$\frac{d^2\tilde{\phi}}{d\theta^2} + p^2\tilde{\phi} = 0, \tag{8.4.8}$$

$$\tilde{\phi}(p,\alpha) = \tilde{f}(p), \qquad \tilde{\phi}(p,\,-\alpha) = \tilde{g}(p). \tag{8.4.9ab}$$

The general solution of the transformed equation is

$$\tilde{\phi}(p,\theta) = A\cos p\theta + B\,\sin p\theta, \tag{8.4.10}$$

where $A$ and $B$ are functions of $p$ and $\alpha$. The boundary conditions (8.4.9ab) determine $A$ and $B$, which satisfy

$$A \cos p\alpha + B \sin p\alpha = \tilde{f}(p),$$
$$A \cos p\alpha - B \sin p\alpha = \tilde{g}(p).$$

These give $A = \dfrac{\tilde{f}(p) + \tilde{g}(p)}{2\cos p\alpha}, \quad B = \dfrac{\tilde{f}(p) - \tilde{g}(p)}{2\sin p\alpha}.$

Thus, solution (8.4.10) becomes

$$\begin{aligned}\tilde{\phi}(p,\theta) &= \tilde{f}(p).\frac{\sin p(\alpha+\theta)}{\sin(2p\alpha)} + \tilde{g}(p)\frac{\sin p(\alpha-\theta)}{\sin(2p\alpha)} \\ &= \tilde{f}(p)\tilde{h}(p,\alpha+\theta) + \tilde{g}(p)\tilde{h}(p,\alpha-\theta),\end{aligned} \tag{8.4.11}$$

where

$$\tilde{h}(p,\theta) = \frac{\sin p\theta}{\sin(2p\alpha)}.$$

Or, equivalently,

$$h(r,\theta) = \mathscr{M}^{-1}\left\{\frac{\sin p\theta}{\sin 2p\alpha}\right\} = \left(\frac{1}{2\alpha}\right)\frac{r^n \sin n\theta}{(1+2r^n\cos n\theta + r^{2n})}, \tag{8.4.12}$$

where

$$n = \frac{\pi}{2\alpha} \quad \text{or,} \quad 2\alpha = \frac{\pi}{n}.$$

Application of the inverse Mellin transform to (8.4.11) gives

$$\phi(r,\theta) = \mathscr{M}^{-1}\left\{\tilde{f}(p)\tilde{h}(p,\alpha+\theta)\right\} + \mathscr{M}^{-1}\left\{\tilde{g}(p)\tilde{h}(p,\alpha-\theta)\right\},$$

which is, by the convolution property (8.3.18),

$$\begin{aligned}\phi(r,\theta) = \frac{r^n\cos n\theta}{2\alpha}&\left[\int_0^\infty \frac{\xi^{n-1}f(\xi)d\xi}{\xi^{2n} - 2(r\xi)^n \sin n\theta + r^{2n}}\right. \\ &\left.+\int_0^\infty \frac{\xi^{n-1}g(\xi)d\xi}{\xi^{2n} + 2(r\xi)^n \sin n\theta + r^{2n}}\right], \quad |\alpha| < \frac{\pi}{2n}.\end{aligned} \tag{8.4.13}$$

This is the formal solution of the problem.

In particular, when $f(r) = g(r)$, solution (8.4.11) becomes

$$\tilde{\phi}(p,\theta) = \tilde{f}(p)\,\frac{\cos p\theta}{\cos p\alpha} = \tilde{f}(p)\tilde{h}(p,\theta), \tag{8.4.14}$$

where

$$\tilde{h}(p,\theta) = \frac{\cos p\theta}{\cos p\alpha} = \mathscr{M}\{h(r,\theta)\}.$$

Application of the inverse Mellin transform to (8.4.14) combined with the convolution property (8.3.18) yields the solution

$$\phi(r,\theta) = \int_0^\infty f(\xi)\, h\left(\frac{r}{\xi},\theta\right)\frac{d\xi}{\xi}, \tag{8.4.15}$$

where

$$h(r,\theta) = \mathscr{M}^{-1}\left\{\frac{\cos p\theta}{\cos p\alpha}\right\} = \left(\frac{r^n}{\alpha}\right)\frac{(1+r^{2n})\cos(n\theta)}{(1+2r^{2n}\cos 2n\theta + r^{2n})}, \tag{8.4.16}$$

and $n = \dfrac{\pi}{2\alpha}$. ▯

Some applications of the Mellin transform to boundary value problems are given by Sneddon (1951) and Tranter (1966).

***Example 8.4.3***
Solve the integral equation

$$\int_0^\infty f(\xi)\, k(x\xi)d\xi = g(x), \qquad x > 0. \tag{8.4.17}$$

Application of the Mellin transform with respect to $x$ to equation (8.4.17) combined with (8.3.19) gives

$$\tilde{f}(1-p)\tilde{k}(p) = \tilde{g}(p),$$

which gives, replacing $p$ by $1-p$,

$$\tilde{f}(p) = \tilde{g}(1-p)\tilde{h}(p),$$

where

$$\tilde{h}(p) = \frac{1}{\tilde{k}(1-p)}.$$

The inverse Mellin transform combined with (8.3.19) leads to the solution

$$f(x) = \mathscr{M}^{-1}\left\{\tilde{g}(1-p)\tilde{h}(p)\right\} = \int_0^\infty g(\xi)h(x\xi)d\xi, \tag{8.4.18}$$

provided $h(x) = \mathscr{M}^{-1}\left\{\tilde{h}(p)\right\}$ exists. Thus, the problem is formally solved.

If, in particular, $\tilde{h}(p) = \tilde{k}(p)$, then the solution of (8.4.18) becomes

$$f(x) = \int_0^\infty g(\xi)\, k(x\xi)d\xi, \tag{8.4.19}$$

provided $\tilde{k}(p)\tilde{k}(1-p)=1$. ▯

***Example 8.4.4***
Solve the integral equation

$$\int_0^\infty f(\xi)\, g\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi} = h(x), \tag{8.4.20}$$

where $f(x)$ is unknown and $g(x)$ and $h(x)$ are given functions.

Applications of the Mellin transform with respect to $x$ gives

$$\tilde{f}(p) = \tilde{h}(p)\tilde{k}(p), \qquad \tilde{k}(p) = \frac{1}{\tilde{g}(p)}.$$

Inversion, by the convolution property (8.3.18), gives the solution

$$f(x) = \mathscr{M}^{-1}\left\{\tilde{h}(p)\tilde{k}(p)\right\} = \int_0^\infty h(\xi)\, k\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi}. \tag{8.4.21}$$

▯

## 8.5 Mellin Transforms of the Weyl Fractional Integral and the Weyl Fractional Derivative

***DEFINITION 8.5.1*** *The Mellin transform of the Weyl fractional integral of $f(x)$ is defined by*

$$W^{-\alpha}[f(x)] = \frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1} f(t)dt, \quad 0 < \operatorname{Re}\alpha < 1, \quad x > 0. \tag{8.5.1}$$

*Often ${}_xW_\infty^{-\alpha}$ is used instead of $W^{-\alpha}$ to indicate the limits to integration. Result (8.5.1) can be interpreted as the Weyl transform of $f(t)$, defined by*

$$W^{-\alpha}[f(t)] = F(x,\alpha) = \frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1} f(t)dt. \tag{8.5.2}$$

We first give some simple examples of the Weyl transform.

If $f(t)=\exp(-at)$, Re $a>0$, then the Weyl transform of $f(t)$ is given by

$$W^{-\alpha}[\exp(-at)]=\frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}\exp(-at)dt,$$

which is, by the change of variable $t-x=y$,

$$=\frac{e^{-ax}}{\Gamma(\alpha)}\int_0^\infty y^{\alpha-1}\exp(-ay)dy$$

which is, by letting $ay=t$,

$$W^{-\alpha}[f(t)]=\frac{e^{-ax}}{a^\alpha}\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}dt=\frac{e^{-ax}}{a^\alpha}. \tag{8.5.3}$$

Similarly, it can be shown that

$$W^{-\alpha}[t^{-\mu}]=\frac{\Gamma(\mu-\alpha)}{\Gamma(\mu)}\,x^{\alpha-\mu}, \qquad 0<\operatorname{Re}\alpha<\operatorname{Re}\mu. \tag{8.5.4}$$

Making reference to Gradshteyn and Ryzhik (2000, p. 424), we obtain

$$W^{-\alpha}[\sin at] = a^{-\alpha}\sin\left(ax+\frac{\pi\alpha}{2}\right), \tag{8.5.5}$$

$$W^{-\alpha}[\cos at] = a^{-\alpha}\cos\left(ax+\frac{\pi\alpha}{2}\right), \tag{8.5.6}$$

where $0<\operatorname{Re}\alpha<1$ and $a>0$.

It can be shown that, for any two positive numbers $\alpha$ and $\beta$, the Weyl fractional integral satisfies the laws of exponents

$$W^{-\alpha}[W^{-\beta}f(x)]=W^{-(\beta+\alpha)}[f(x)]=W^{-\beta}[W^{-\alpha}f(x)]. \tag{8.5.7}$$

Invoking a change of variable $t-x=y$ in (8.5.1), we obtain

$$W^{-\alpha}[f(x)]=\frac{1}{\Gamma(\alpha)}\int_0^\infty y^{\alpha-1}f(x+y)dy. \tag{8.5.8}$$

We next differentiate (8.5.8) to obtain, $D=\dfrac{d}{dx}$,

$$\begin{aligned}D[W^{-\alpha}f(x)] &= \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\frac{\partial}{\partial x}f(x+t)dt\\ &=\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}Df(x+t)dt\\ &=W^{-\alpha}[Df(x)].\end{aligned} \tag{8.5.9}$$

A similar argument leads to a more general result

$$D^n[W^{-\alpha} f(x)] = W^{-\alpha}[D^n f(x)], \tag{8.5.10}$$

where $n$ is a positive integer.

Or, symbolically,

$$D^n W^{-\alpha} = W^{-\alpha} D^n. \tag{8.5.11}$$

We now calculate the Mellin transform of the Weyl fractional integral by putting $h(t) = t^\alpha f(t)$ and $g\left(\frac{x}{t}\right) = \frac{1}{\Gamma(\alpha)}\left(1 - \frac{x}{t}\right)^{\alpha-1} H\left(1 - \frac{x}{t}\right)$, where $H\left(1 - \frac{x}{t}\right)$ is the Heaviside unit step function so that (8.5.1) becomes

$$F(x,\alpha) = \int_0^\infty h(t)\, g\left(\frac{x}{t}\right) \frac{dt}{t}, \tag{8.5.12}$$

which is, by the convolution property (8.3.18),

$$\tilde{F}(p,\alpha) = \tilde{h}(p)\tilde{g}(p),$$

where

$$\tilde{h}(p) = \mathscr{M}\{x^\alpha f(x)\} = \tilde{f}(p+\alpha),$$

and

$$\begin{aligned}\tilde{g}(p) &= \mathscr{M}\left\{\frac{1}{\Gamma(\alpha)}(1-x)^{\alpha-1} H(1-x)\right\}\\ &= \frac{1}{\Gamma(\alpha)}\int_0^1 x^{p-1}(1-x)^{\alpha-1}dx = \frac{B(p,\alpha)}{\Gamma(\alpha)} = \frac{\Gamma(p)}{\Gamma(p+\alpha)}.\end{aligned}$$

Consequently,

$$\tilde{F}(p,\alpha) = \mathscr{M}\left[W^{-\alpha} f(x), p\right] = \frac{\Gamma(p)}{\Gamma(p+\alpha)}\,\tilde{f}(p+\alpha). \tag{8.5.13}$$

It is important to note that this result is an obvious extension of result 7(b) in Exercise 8.8

***DEFINITION 8.5.2*** *If $\beta$ is a positive number and $n$ is the smallest integer greater than $\beta$ such that $n - \beta = \alpha > 0$, the Weyl fractional derivative of a function $f(x)$ is defined by*

$$\begin{aligned}W^\beta[f(x)] &= E^n\, W^{-(n-\beta)}[f(x)]\\ &= \frac{(-1)^n}{\Gamma(n-\beta)}\frac{d^n}{dx^n}\int_x^\infty (t-x)^{n-\beta-1} f(t)dt,\end{aligned} \tag{8.5.14}$$

where $E = -D$.

Or, symbolically,

$$W^{\beta} = E^n W^{-\alpha} = E^n W^{-(n-\beta)}. \tag{8.5.15}$$

It can be shown that, for any $\beta$,

$$W^{-\beta} W^{\beta} = I = W^{\beta} W^{-\beta}. \tag{8.5.16}$$

And, for any $\beta$ and $\gamma$, the Weyl fractional derivative satisfies the laws of exponents

$$W^{\beta}[W^{\gamma} f(x)] = W^{\beta+\gamma}[f(x)] = W^{\gamma}[W^{\beta} f(x)]. \tag{8.5.17}$$

We now calculate the Weyl fractional derivative of some elementary functions.

If $f(x) = \exp(-ax)$, $a > 0$, then the definition (8.5.14) gives

$$W^{\beta} e^{-ax} = E^n [W^{-(n-\beta)} e^{-ax}]. \tag{8.5.18}$$

Writing $n - \beta = \alpha > 0$ and using (8.5.3) yields

$$\begin{aligned} W^{\beta} e^{-ax} &= E^n [W^{-\alpha} e^{-ax}] = E^n [a^{-\alpha} e^{-ax}] \\ &= a^{-\alpha}(a^n e^{-ax}) = a^{\beta} e^{-ax}. \end{aligned} \tag{8.5.19}$$

Replacing $\beta$ by $-\alpha$ in (8.5.19) leads to result (8.5.3) as expected.

Similarly, we obtain

$$W^{\beta} x^{-\mu} = \frac{\Gamma(\beta+\mu)}{\Gamma(\mu)}\, x^{-(\beta+\mu)}. \tag{8.5.20}$$

It is easy to see that

$$W^{\beta}(\cos ax) = E[W^{-(1-\beta)} \cos ax],$$

which is, by (8.5.6),

$$= a^{\beta} \cos\left(ax - \frac{1}{2}\pi\beta\right). \tag{8.5.21}$$

Similarly,

$$W^{\beta}(\sin ax) = a^{\beta} \sin\left(ax - \frac{1}{2}\pi\beta\right), \tag{8.5.22}$$

provided $\alpha$ and $\beta$ lie between 0 and 1.

If $\beta$ is replaced by $-\alpha$, result (8.5.20)–(8.5.22) reduce to (8.5.4)–(8.5.6) respectively.

Finally, we calculate the Mellin transform of the Weyl fractional derivative with the help of (8.3.9) and find

$$\begin{aligned} \mathscr{M}[W^{\beta} f(x)] &= \mathscr{M}[E^n W^{-(n-\beta)} f(x)] = (-1)^n \mathscr{M}[D^n W^{-(n-\beta)} f(x)] \\ &= \frac{\Gamma(p)}{\Gamma(p-n)} \mathscr{M}[W^{-(n-\beta)} f(x), p-n], \end{aligned}$$

which is, by result (8.5.13),

$$
\begin{aligned}
&= \frac{\Gamma(p)}{\Gamma(p-n)} \cdot \frac{\Gamma(p-n)}{\Gamma(p-\beta)}\, \tilde{f}(p-\beta) \\
&= \frac{\Gamma(p)}{\Gamma(p-\beta)}\, \mathscr{M}[f(x), p-\beta] \\
&= \frac{\Gamma(p)}{\Gamma(p-\beta)}\, \tilde{f}(p-\beta). \qquad (8.5.23)
\end{aligned}
$$

***Example 8.5.1***
(*The Fourier Transform of the Weyl Fractional Integral*).

$$
\mathscr{F}\{W^{-\alpha} f(x)\} = \exp\left(-\frac{\pi i \alpha}{2}\right) k^{-\alpha} \mathscr{F}\{f(x)\}. \qquad (8.5.24)
$$

We have, by definition,

$$
\begin{aligned}
\mathscr{F}\{W^{-\alpha} f(x)\} &= \frac{1}{\sqrt{2\pi}} \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\infty} e^{-ikx} dx \int_{x}^{\infty} (t-x)^{\alpha-1} f(t) dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) dt \cdot \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} \exp(-ikx)(t-x)^{\alpha-1} dx.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\mathscr{F}\{W^{-\alpha} f(x)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikt} f(t) dt \cdot \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} e^{ik\tau} \tau^{\alpha-1} d\tau, \quad (t-x=\tau) \\
&= \mathscr{F}\{f(x)\} \frac{1}{\Gamma(\alpha)} \mathscr{M}\{e^{ik\tau}\} \\
&= \exp\left(-\frac{\pi i \alpha}{2}\right) k^{-\alpha} \mathscr{F}\{f(x)\}.
\end{aligned}
$$

In the limit as $\alpha \to 0$

$$
\lim_{\alpha \to 0} \mathscr{F}\{W^{-\alpha} f(x)\} = \mathscr{F}\{f(x)\}.
$$

This implies that

$$
W^{0}\{f(x)\} = f(x).
$$

We conclude this section by proving a general property of the Riemann-Liouville fractional integral operator $D^{-\alpha}$, and the Weyl fractional integral operator $W^{-\alpha}$. It follows from the definition (6.2.1) that $D^{-\alpha}f(t)$ can be expressed as the convolution

$$D^{-\alpha}f(x) = g_\alpha(t) * f(t), \tag{8.5.25}$$

where

$$g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \ \ t > 0.$$

Similarly, $W^{-\alpha}f(x)$ can also be written in terms of the convolution

$$W^{-\alpha}f(x) = g_\alpha(-x) * f(x). \tag{8.5.26}$$

Then, under suitable conditions,

$$\mathscr{M}\left[D^{-\alpha}f(x)\right] = \frac{\Gamma(1-\alpha-p)}{\Gamma(1-p)}\,\tilde{f}(p+\alpha), \tag{8.5.27}$$

$$\mathscr{M}\left[W^{-\alpha}f(x)\right] = \frac{\Gamma(p)}{\Gamma(\alpha+p)}\,\tilde{f}(p+\alpha). \tag{8.5.28}$$

Finally, a formal computation gives

$$\begin{aligned}\int_0^\infty \{D^{-\alpha}f(x)\}g(x)dx &= \frac{1}{\Gamma(\alpha)}\int_0^\infty g(x)dx\int_0^x (x-t)^{\alpha-1}f(t)dt \\ &= \int_0^\infty f(t)dt\cdot\frac{1}{\Gamma(\alpha)}\int_t^\infty (x-t)^{\alpha-1}g(x)dx \\ &= \int_0^\infty f(t)\left[W^{-\alpha}g(t)\right]dt,\end{aligned}$$

which is, using the inner product notation,

$$\langle D^{-\alpha}f,\, g\rangle = \langle f,\, W^{-\alpha}g\rangle. \tag{8.5.29}$$

This show that $D^{-\alpha}$ and $W^{-\alpha}$ behave like adjoint operators. Obviously, this result can be used to define fractional integrals of distributions. This result is taken from Debnath and Grum (1988). □

## 8.6 Application of Mellin Transforms to Summation of Series

In this section we discuss a method of summation of series that is particularly associated with the work of Macfarlane (1949).

***THEOREM 8.6.1***
If $\mathscr{M}\{f(x)\} = \tilde{f}(p)$, then

$$\sum_{n=0}^{\infty} f(n+a) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{f}(p)\,\xi(p,a)dp, \tag{8.6.1}$$

where $\xi(p,a)$ is the *Hurwitz zeta function* defined by

$$\xi(p,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^p}, \qquad 0 \le a \le 1,\ \mathrm{Re}(p) > 1. \tag{8.6.2}$$

**PROOF** If follows from the inverse Mellin transform that

$$f(n+a) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{f}(p)(n+a)^{-p}\,dp. \tag{8.6.3}$$

Summing this over all $n$ gives

$$\sum_{n=0}^{\infty} f(n+a) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{f}(p)\,\xi(p,a)\,dp.$$

This completes the proof.

Similarly, the scaling property (8.3.1) gives

$$f(nx) = \mathscr{M}^{-1}\{n^{-p}\,\tilde{f}(p)\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-p}\,n^{-p}\tilde{f}(p)dp.$$

Thus,

$$\sum_{n=1}^{\infty} f(nx) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-p}\tilde{f}(p)\,\zeta(p)dp = \mathscr{M}^{-1}\{\tilde{f}(p)\,\zeta(p)\}, \tag{8.6.4}$$

where $\zeta(p) = \sum_{n=1}^{\infty} n^{-p}$ is the *Riemann zeta function.*

When $x = 1$, result (8.6.4) reduces to

$$\sum_{n=1}^{\infty} f(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{f}(p)\,\zeta(p)dp. \tag{8.6.5}$$

This can be obtained from (8.6.1) when $a = 0$. ∎

***Example 8.6.1***
Show that
$$\sum_{n=1}^{\infty}(-1)^{n-1}n^{-p} = (1-2^{1-p})\,\zeta(p). \tag{8.6.6}$$

Using Example 8.2.1(a), we can write the left-hand side of (8.6.6) multiplied by $t^n$ as

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n-1}n^{-p}t^n &= \sum_{n=1}^{\infty}(-1)^{n-1}t^n \cdot \frac{1}{\Gamma(p)}\int_0^{\infty} x^{p-1}e^{-nx}dx \\ &= \frac{1}{\Gamma(p)}\int_0^{\infty} x^{p-1}dx \sum_{n=1}^{\infty}(-1)^{n-1}t^{nx}e^{-nx} \\ &= \frac{1}{\Gamma(p)}\int_0^{\infty} x^{p-1}\cdot\frac{te^{-x}}{1+te^{-x}}\cdot dx \\ &= \frac{1}{\Gamma(p)}\int_0^{\infty} x^{p-1}\cdot\frac{t}{e^x+t}\,dx.\end{aligned}$$

In the limit as $t \to 1$, the above result gives

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n-1}n^{-p} &= \frac{1}{\Gamma(p)}\int_0^{\infty} x^{p-1}\frac{1}{e^x+1}\,dx \\ &= \frac{1}{\Gamma(p)}\mathscr{M}\left\{\frac{1}{e^x+1}\right\} = (1-2^{1-p})\,\zeta(p),\end{aligned}$$

in which result (8.2.11) is used. ▯

***Example 8.6.2***
Show that
$$\sum_{n=1}^{\infty}\left(\frac{\sin an}{n}\right) = \frac{1}{2}(\pi - a), \qquad 0 < a < 2\pi. \tag{8.6.7}$$

The Mellin transform of $f(x) = \left(\dfrac{\sin ax}{x}\right)$ gives

$$\begin{aligned}\mathscr{M}\left[\frac{\sin ax}{x}\right] &= \int_0^{\infty} x^{p-2}\sin ax\,dx \\ &= \mathscr{F}_s\left\{\sqrt{\frac{\pi}{2}}\,x^{p-2}\right\} \\ &= -\frac{\Gamma(p-1)}{a^{p-1}}\cos\left(\frac{\pi p}{2}\right).\end{aligned}$$

Substituting this result into (8.6.5) gives

$$\sum_{n=1}^{\infty}\left(\frac{\sin an}{n}\right)=-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(p-1)}{a^{p-1}}\,\zeta(p)\,\cos\left(\frac{\pi p}{2}\right)\,dp. \tag{8.6.8}$$

We next use the well-known functional equation for the zeta function

$$(2\pi)^{p}\,\zeta(1-p)=2\Gamma(p)\,\zeta(p)\cos\left(\frac{\pi p}{2}\right) \tag{8.6.9}$$

in the integrand of (8.6.8) to obtain

$$\sum_{n=1}^{\infty}\left(\frac{\sin an}{n}\right)=-\frac{a}{2}\cdot\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{2\pi}{a}\right)^{p}\frac{\zeta(1-p)}{p-1}\,dp.$$

The integral has two simple poles at $p=0$ and $p=1$ with residues 1 and $-\pi/a$, respectively, and the complex integral is evaluated by calculating the residues at these poles. Thus, the sum of the series is

$$\sum_{n=1}^{\infty}\left(\frac{\sin an}{n}\right)=\frac{1}{2}(\pi-a).$$

□

## 8.7 Generalized Mellin Transforms

In order to extend the applicability of the classical Mellin transform, Naylor (1963) generalized the method of Mellin integral transforms. This generalized Mellin transform is useful for finding solutions of boundary value problems in regions bounded by the natural coordinate surfaces of a spherical or cylindrical coordinate system. They can be used to solve boundary value problems in finite regions or in infinite regions bounded internally.

The *generalized Mellin transform* of a function $f(r)$ defined in $a<r<\infty$ is introduced by the integral

$$\mathscr{M}_{-}\{f(r)\}=F_{-}(p)=\int_{a}^{\infty}\left(r^{p-1}-\frac{a^{2p}}{r^{p+1}}\right)f(r)\,dr. \tag{8.7.1}$$

The inverse transform is given by

$$\mathscr{M}_{-}^{-1}\{F_{-}(p)\}=f(r)=\frac{1}{2\pi i}\int_{L}r^{-p}\,F(p)\,dp, \quad r>a, \tag{8.7.2}$$

where $L$ is the line Re $p = c$, and $F(p)$ is analytic in the strip $|\text{Re}(p)| = |c| < \gamma$.

By integrating by parts, we can show that

$$\mathscr{M}_{-}\left[r^2 \frac{\partial^2 f}{\partial r^2} + r \frac{\partial f}{\partial r}\right] = p^2 F_{-}(p) + 2\, p\, a^p f(a), \tag{8.7.3}$$

provided $f(r)$ is appropriately behaved at infinity. More precisely,

$$\lim_{r\to\infty} \left[(r^p - a^{2p} r^{-p}) r f_r - p(r^p + a^{2p} r^{-p}) f\right] = 0. \tag{8.7.4}$$

Obviously, this generalized transform seems to be very useful for finding the solution of boundary value problems in which $f(r)$ is prescribed on the internal boundary at $r = a$.

On the other hand, if the derivative of $f(r)$ is prescribed at $r = a$, it is convenient to define the associated integral transform by

$$\mathscr{M}_{+}[f(r)] = F_{+}(p) = \int_a^\infty \left(r^{p-1} + \frac{a^{2p}}{r^{p+1}}\right) f(r)\, dr, \quad |\text{Re}(p)| < r, \tag{8.7.5}$$

and its inverse given by

$$\mathscr{M}_{+}^{-1}[f(p)] = f(r) = \frac{1}{2\pi i} \int_L r^{-p} F_{+}(p) dp, \quad r > a. \tag{8.7.6}$$

In this case, we can show by integration by parts that

$$\mathscr{M}_{+}\left[r^2 \frac{\partial^2 f}{\partial r^2} + r \frac{\partial f}{\partial r}\right] = p^2 F_{+}(p) - 2\, a^{p+1} f'(a), \tag{8.7.7}$$

where $f'(r)$ exists at $r = a$.

***THEOREM 8.7.1***

(*Convolution*). If $\mathscr{M}_{+}\{f(r)\} = F_{+}(p)$, and $\mathscr{M}_{+}\{g(r)\} = G_{+}(p)$, then

$$\mathscr{M}_{+}\{f(r)\, g(r)\} = \frac{1}{2\pi i} \int_L F_{+}(\xi) G_{+}(p-\xi)\, d\xi. \tag{8.7.8}$$

Or, equivalently,

$$f(r) g(r) = \mathscr{M}_{+}^{-1}\left[\frac{1}{2\pi i} \int_L F_{+}(\xi)\, G_{+}(p-\xi) d\xi\right]. \tag{8.7.9}$$

**PROOF** We assume that $F_+(p)$ and $G_+(p)$ are analytic in some strip $|\mathrm{Re}(p)| < \gamma$. Then

$$\mathscr{M}_+\{f(r)\,g(r)\} = \int_a^\infty \left(r^{p-1} + \frac{a^{2p}}{r^{p+1}}\right) f(r)g(r)dr$$

$$= \int_a^\infty r^{p-1} f(r)g(r)dr + \int_a^\infty \frac{a^{2p}}{r^{p+1}} f(r)g(r)dr. \quad (8.7.10)$$

$$= \frac{1}{2\pi i}\int_L F_+(\xi)d\xi \int_a^\infty r^{p-\xi-1} g(r)dr$$

$$+\frac{1}{2\pi}\int_a^\infty \frac{a^{2p}}{r^{p+1}}\, g(r)dr \int_L r^{-\xi} F_+(\xi)\, d\xi. \quad (8.7.11)$$

Replacing $\xi$ by $-\xi$ in the first integral term and using $F_+(\xi) = a^{2\xi} F_+(-\xi)$, which follows from the definition (8.7.5), we obtain

$$\int_L r^{-\xi}\, F_+(\xi)d\xi = \int_L r^{\xi}\, a^{-2\xi}\, F_+(\xi)d\xi. \quad (8.7.12)$$

The path of integration $L$, $\mathrm{Re}(\xi) = c$, becomes $\mathrm{Re}(\xi) = -c$, but these paths can be reconciled if $F(\xi)$ tends to zero for large $\mathrm{Im}(\xi)$.

In view of (8.7.11), we have rewritten

$$\int_a^\infty \frac{a^{2p}}{r^{p+1}} f(r)\, g(r)dr = \frac{1}{2\pi i}\int_L F_+(\xi)d\xi \int_a^\infty \frac{a^{2p-2\xi}}{r^{p-\xi+1}}\, g(r)\, dr. \quad (8.7.13)$$

This result is used to rewrite (8.7.10) as

$$\mathscr{M}_+\{f(r)g(r)\} = \int_a^\infty \left(r^{p-1} + \frac{a^{2p}}{r^{p+1}}\right) f(r)g(r)dr$$

$$= \int_a^\infty r^{p-1} f(r)\, g(r)\, dr + \int_a^\infty \frac{a^{2p}}{r^{p+1}} f(r)\, g(r)\, dr$$

$$= \frac{1}{2\pi i}\int_L F_+(\xi)\, d\xi \int_a^\infty r^{p-\xi-1} g(r)\, dr$$

$$+\frac{1}{2\pi i}\int_L F_+(\xi)d\xi \int_a^\infty \frac{a^{2p-2\xi}}{r^{p-\xi+1}}\, g(r)\, dr$$

$$= \frac{1}{2\pi i}\int_L F_+(\xi)\, G_+(p-\xi)\, d\xi.$$

This completes the proof. ▮

If the range of integration is finite, then we define the *generalized finite Mellin transform* by

$$\mathscr{M}_{-}^{a}\{f(r)\} = F_{-}^{a}(p) = \int_{0}^{a} \left(r^{p-1} - \frac{a^{2p}}{r^{p+1}}\right) f(r)dr, \tag{8.7.14}$$

where $\text{Re}\, p < \gamma$.

The corresponding inverse transform is given by

$$f(r) = -\frac{1}{2\pi i}\int_{L} \left(\frac{r}{a^2}\right)^{p} F_{-}^{a}(p)dp, \quad 0 < r < a,$$

which is, by replacing $p$ by $-p$ and using $F_{-}^{a}(-p) = -a^{-2p} F_{-}^{a}(p)$,

$$= \frac{1}{2\pi i}\int_{L} r^{-p} F_{-}^{a}(p)dp, \quad 0 < r < a, \tag{8.7.15}$$

where the path $L$ is Re $p = -c$ with $|c| < \gamma$.

It is easy to verify the result

$$\begin{aligned}\mathscr{M}_{-}^{a}\{r^2 f_{rr} + r f_{-r}\} &= \int_{0}^{a} \left(r^{p-1} - \frac{a^{2p}}{r^{p+1}}\right)\{r^2 f_{rr} + r f_r\} dr \\ &= p^2 F_{-}^{a}(p) - 2\, p\, a^p\, f(a).\end{aligned} \tag{8.7.16}$$

This is a useful result for applications.

Similarly, we define the generalized finite Mellin transform-pair by

$$\mathscr{M}_{+}^{a}\{f(r)\} = F_{+}^{a}(p) = \int_{0}^{a} \left(r^{p-1} + \frac{a^{2p}}{r^{p+1}}\right) f(r)\, dr, \tag{8.7.17}$$

$$f(r) = \left(\mathscr{M}_{+}^{a}\right)^{-1} \left[F_{+}^{a}(p)\right] = \frac{1}{2\pi i}\int_{L} r^{-p} F_{+}^{a}(p)\, dp, \tag{8.7.18}$$

where $|\text{Re}\, p| < \gamma$.

For this finite transform, we can also prove

$$\begin{aligned}\mathscr{M}_{+}^{a}\left[r^2 f_{rr} + r\ f_r\right] &= \int_{0}^{a} \left(r^{p-1} + \frac{a^{2p}}{r^{p+1}}\right)\left(r^2 f_{rr} + r\ f_r\right) dr \\ &= p^2 F_{+}^{a}(p) + 2\, a^{p-1}\ f'(a).\end{aligned} \tag{8.7.19}$$

This result also seems to be useful for applications. The reader is referred to Naylor (1963) for applications of the above results to boundary value problems.

## 8.8 Exercises

1. Find the Mellin transform of each of the following functions:

(a) $f(x) = H(a-x)$, $a > 0$,

(b) $f(x) = x^m e^{-nx}$, $\quad m, n > 0$,

(c) $f(x) = \dfrac{1}{1+x^2}$,

(d) $f(x) = J_0^2(x)$,

(e) $f(x) = x^z H(x - x_0)$,

(f) $f(x) = [H(x - x_0) - H(x)]x^z$,

(g) $f(x) = Ei(x)$,

(h) $f(x) = e^x Ei(x)$,

where the exponential integral is defined by

$$Ei(x) = \int_x^\infty t^{-1} e^{-t}\, dt = \int_1^\infty \xi^{-1} e^{-\xi x}\, d\xi.$$

2. Derive the Mellin transform-pairs from the bilateral Laplace transform and its inverse given by

$$\bar{g}(p) = \int_{-\infty}^{\infty} e^{-pt} g(t) dt, \qquad g(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}\, \bar{g}(p) dp.$$

3. Show that

$$\mathscr{M}\left[\frac{1}{e^x + e^{-x}}\right] = \Gamma(p)\, L(p),$$

where $L(p) = \dfrac{1}{1^p} - \dfrac{1}{3^p} + \dfrac{1}{5^p} - \cdots$ is the *Dirichlet L-function.*

4. Show that

$$\mathscr{M}\left\{\frac{1}{(1+ax)^n}\right\} = \frac{\Gamma(p)\Gamma(n-p)}{a^p\, \Gamma(n)}.$$

5. Show that

$$\mathscr{M}\left\{x^{-n} J_n(ax)\right\} = \frac{1}{2}\left(\frac{a}{2}\right)^{n-p} \frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(n - \frac{p}{2} + 1\right)}, \quad a > 0,\ n > -\frac{1}{2}.$$

6. Show that

(a) $\mathscr{M}^{-1}\left[\cos\left(\dfrac{\pi p}{2}\right)\Gamma(p)\, \tilde{f}(1-p)\right] = \mathscr{F}_c\left\{\sqrt{\dfrac{\pi}{2}}\, f(x)\right\}$,

(b) $\mathscr{M}^{-1}\left[\sin\left(\frac{\pi p}{2}\right)\Gamma(p)\,\tilde{f}(1-p)\right] = \mathscr{F}_s\left\{\sqrt{\frac{\pi}{2}}\,f(x)\right\}.$

7. If $I_n^\infty f(x)$ denotes the $n$th repeated integral of $f(x)$ defined by

$$I_n^\infty f(x) = \int_x^\infty I_{n-1}^\infty f(t)dt,$$

show that

(a) $\mathscr{M}\left[\int_x^\infty f(t)dt,\, p\right] = \frac{1}{p}\,\tilde{f}(p+1),$

(b) $\mathscr{M}\left[I_n^\infty\; f(x)\right] = \frac{\Gamma(p)}{\Gamma(p+n)}\;\tilde{f}(p+n).$

8. Show that the integral equation

$$f(x) = h(x) + \int_0^\infty g(x\xi)\; f(\xi)\; d\xi$$

has the formal solution

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[\frac{\tilde{h}(p)+\tilde{g}(p)\,\tilde{h}(1-p)}{1-\tilde{g}(p)\,\tilde{g}(1-p)}\right]x^{-p}\,dp.$$

9. Find the solution of the Laplace integral equation

$$\int_0^\infty e^{-x\xi}\; f(\xi)\; d\xi = \frac{1}{(1+x)^n}.$$

10. Show that the integral equation

$$f(x) = h(x) + \int_0^\infty f(\xi)\; g\left(\frac{x}{\xi}\right)\;\frac{d\xi}{\xi}$$

has the formal solution

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{-p}\,\tilde{h}(p)}{1-\tilde{g}(p)}\,dp.$$

11. Show that the solution of the integral equation

$$f(x) = e^{-ax} + \int_0^\infty \exp\left(-\frac{x}{\xi}\right)\ f(\xi)\,\frac{d\xi}{\xi}$$

is

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (ax)^{-p}\left\{\frac{\Gamma(p)}{1-\Gamma(p)}\right\} dp.$$

12. Assuming (see Harrington, 1967)

$$\mathscr{M}\left[f(re^{i\theta})\right] = \int_0^\infty r^{p-1} f(re^{i\theta})\ dr, \quad p \text{ is real},$$

and putting $re^{i\theta} = \xi$, $\mathscr{M}\{f(\xi)\} = F(p)$ show that

(a) $\mathscr{M}\,[f(re^{i\theta});\ r \to p] = \exp(-ip\theta)\,F(p)$.

Hence, deduce

(b) $\mathscr{M}^{-1}\{F(p)\ \cos\ p\theta\} = \text{Re}[f(re^{i\theta})]$,

(c) $\mathscr{M}^{-1}\{F(p)\ \sin\ p\theta\} = -\text{Im}[f(re^{i\theta})]$.

13. (a) If $\mathscr{M}\,[\exp(-r)] = \Gamma(p)$, show that

$$\mathscr{M}\left[\exp(-re^{i\theta})\right] = \Gamma(p)\,e^{-ip\theta},$$

(b) If $\mathscr{M}\,[\log(1+r)] = \dfrac{\pi}{p\sin \pi p}$, then show that

$$\mathscr{M}\left[\text{Re}\ \log\left(1+re^{i\theta}\right)\right] = \frac{\pi\cos\ p\theta}{p\sin\pi p}.$$

14. Use $\mathscr{M}^{-1}\left\{\dfrac{\pi}{\sin p\pi}\right\} = \dfrac{1}{1+x} = f(x)$, and Exercises 12(b) and 12(c), respectively, to show that

(a) $\mathscr{M}^{-1}\left\{\dfrac{\pi\cos p\theta}{\sin p\pi};\ p \to r\right\} = \dfrac{1 + r\cos\theta}{1+2r\cos\theta + r^2}$,

(b) $\mathscr{M}^{-1}\left\{\dfrac{\pi\sin p\theta}{\sin p\pi};\ p \to r\right\} = \dfrac{r\sin\theta}{1+2r\cos\theta + r^2}$.

15. Find the inverse Mellin transforms of

(a) $\Gamma(p)\cos p\theta$, where $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$, (b) $\Gamma(p)\sin p\theta$.

16. Obtain the solution of Example 8.4.2 with the boundary data

    (a) $\phi(r,\,\alpha) = \phi(r,\,-\alpha) = H(a-r)$.

    (b) Solve equation (8.4.5) in $0 < r < \infty$, $0 < \theta < \alpha$ with the boundary conditions $\phi(r,\,0) = 0$ and $\phi(r,\,\alpha) = f(r)$.

17. Show that

    (a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos kn}{n^2} = \left[\frac{k^2}{4} - \frac{\pi k}{2} + \frac{\pi^2}{6}\right]$, and (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

18. If $f(x) = \sum_{n=1}^{\infty} a_n e^{-nx}$, show that

$$\mathscr{M}\{f(x)\} = \tilde{f}(p) = \Gamma(p)\, g(p),$$

where $g(p) = \displaystyle\sum_{n=1}^{\infty} a_n\, n^{-p}$ is the Dirichlet series.
If $a_n = 1$ for all $n$, derive

$$\tilde{f}(p) = \Gamma(p)\,\zeta(p).$$

Show that

$$\mathscr{M}\left\{\frac{\exp(-ax)}{1-e^{-x}}\right\} = \Gamma(p)\,\xi(p,a).$$

19. Show that

    (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = (1-2^{1-p})\,\zeta(p)$.

    Hence, deduce

    (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$, (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4} = \left(\frac{7}{8}\right)\frac{\pi^4}{90}$.

20. Find the sum of the following series

    (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \cos kn$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin kn$.

21. Show that the solution of the boundary value problem

$$r^2\phi_{rr} + r\phi_r + \phi_{\theta\theta} = 0, \quad 0 < r < \infty,\ 0 < \theta < \pi$$

$$\phi(r,0) = \phi(r,\pi) = f(r),$$

is

$$\phi(r,\theta)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} r^{-p}\,\frac{\tilde{f}(p)\,\cos\left\{p\left(\theta-\frac{\pi}{2}\right)\right\}\,dp}{\cos\left(\frac{\pi p}{2}\right)}.$$

22. Evaluate

$$\sum_{n=1}^{\infty}\frac{\cos an}{n^3}=\frac{1}{12}(a^3-3\pi a^2+2\pi^2 a).$$

23. Prove the following results:

(a) $\mathscr{M}\left[\int_0^\infty \xi^n f(x\xi)\,g(\xi)d\xi\right]=\tilde{f}(p)\,\tilde{g}(1+n-p),$

(b) $\mathscr{M}\left[\int_0^\infty \xi^n f\left(\frac{x}{\xi}\right)\,g(\xi)d\xi\right]=\tilde{f}(p)\,\tilde{g}(p+n+1).$

24. Show that

(a) $W^{-\alpha}[e^{-x}]=e^{-x},\quad \alpha>0,$

(b) $W^{\frac{1}{2}}\left[\frac{1}{\sqrt{x}}\exp\left(-\sqrt{x}\right)\right]=\frac{K_1(\sqrt{x})}{\sqrt{\pi x}},\quad x>0,$

where $K_1(x)$ is the modified Bessel function of the second kind and order one.

25. (a) Show that the integral (Wong, 1989, pp. 186–187)

$$I(x)=\int_0^{\pi/2} J_\nu^2(x\cos\theta)\,d\theta,\qquad \nu>-\frac{1}{2},$$

can be written as a Mellin convolution

$$I(x)=\int_0^\infty f(x\xi)\,g(\xi)\,d\xi,$$

where

$$f(\xi)=J_\nu^2(\xi)\ \text{ and }\ \mathrm{g}(\xi)=\begin{Bmatrix}(1-\xi^2)^{-\frac{1}{2}}, & 0<\xi<1\\ 0, & \xi\geq 1\end{Bmatrix}.$$

(b) Prove that the integration contour in the Parseval identity

$$I(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-p}\,\tilde{f}(p)\,\tilde{g}(1-p)\,dp,\qquad -2\nu<c<1,$$

cannot be shifted to the right beyond the vertical line $\text{Re}\,p=2$.

26. If $f(x) = \int_0^\infty \exp(-x^2t^2) \cdot \frac{\sin t}{t^2} J_1(t)dt$, show that

$$\mathscr{M}\{f(x)\} = \frac{\Gamma\left(p+\frac{3}{2}\right)\Gamma\left(\frac{1-p}{2}\right)}{p\,\Gamma(p+3)}.$$

27. Prove the following relations to the Laplace and the Fourier transforms:

 (a) $\mathscr{M}[f(x), p] = \mathscr{L}[f(e^{-t}), p]$,
 (b) $\mathscr{M}[f(x); a+i\omega] = \mathscr{F}[f(e^{-t})e^{-at}; \omega]$,

 where $\mathscr{L}$ is the two-sided Laplace transform and $\mathscr{F}$ is the Fourier transform without the factor $(2\pi)^{-\frac{1}{2}}$.

28. Prove the following properties of convolution:

 (a) $f * g = g * f$,
 (b) $(f * g) * h = f * (g * h)$,
 (c) $f(x) * \delta(x-1) = f(x)$,
 (d) $\delta(x-a) * f(x) = a^{-1} f\left(\frac{x}{a}\right)$,
 (e) $\delta^n(n-1) * f(x) = \left(\frac{d}{dx}\right)^n (x^n f(x))$,
 (f) $\left(x\frac{d}{dx}\right)^n (f * g) = \left[\left(x\frac{d}{dx}\right)^n f\right] * g = f * \left[\left(x\frac{d}{dx}\right)^n g\right]$.

29. If $\mathscr{M}\{f(r,\theta)\} = \tilde{f}(p,\theta)$ and $\nabla^2 f(r,\theta) = f_{rr} + \frac{1}{r}f_r + \frac{1}{r^2}f_{\theta\theta}$, show that

$$\mathscr{M}\left\{\nabla^2 f(r,\theta)\right\} = \left[\frac{d^2}{d\theta^2} + (p-2)^2\right]\tilde{f}(p-2,\theta).$$