Applied Math. (10)

The Theory of Electromagnetic Fields

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Academic year
2022/2023

Book data

College: Faculty of Education - Hurghada

Class: Four

Section: Math

Date of publication:

Number of pages: 102

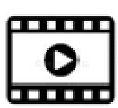
Author: Dr. Nasr Eldeen Faried Elansary

Icons used

Text to read and study

Activities and tasks

Questions for reflection and self-assessment



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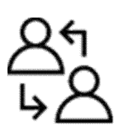

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Chapter 1 : Static Electricity

Introduction :

The documents dating back to before 600 BC indicate the availability of cognitive information on static electricity and the word static derived from the Greek word for the substance of electricity. The Greeks used to spend long hours kneading a piece of cloth with electrostatic material and noticing how this substance then attracted the small pieces, but the interest of the Greeks was focused on logic and philosophy and not on experimental science. Therefore, a long time elapsed before it became possible to prove that this phenomenon of attraction is not magic.

1- Coulomb's law:

The first person to conduct practical experiments was Dr. Colbert, a physician of the Queen of England, who declared in 1600 that this phenomenon is not limited to electricity, but rather to glass, wood, sulfur and other materials. Shortly thereafter, the French army engineer Coulomb conducted a number of advanced experiments using a special real torsional balance in order to find out the magnitude of the attractive force between two objects that each carry a static electric charge.

Coulomb's results are now known as Coulomb's law and bear a strong resemblance to Newton's law of universal gravitation, which was discovered 100 years earlier. Coulomb's law states that the force between two charged bodies separated by a large distance with respect to their volumes is proportional to the product of the two charges and inversely to the square of the distance between them. This force is considered a repulsive force for charges that have the same signal and an attractive force for charges of different signal.

Coulomb's law takes the following mathematical form:

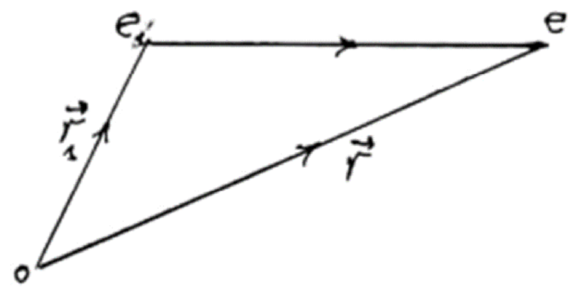


Fig. (1-1)

$$\vec{F} = \frac{e\, e_i}{\left| \vec{r} - \vec{r}_i \right|^3} (\vec{r} - \vec{r}_i) \ . \qquad (1)$$

Where $\vec{F}$ is the force acting on the charge ***e*** resulting from the presence of the charge e_i , $\vec{r}_i$ it is the vector of the position $\boldsymbol{e_1}$ with respect to the origin ***o***. In formula (1) we choose a constant of proportionality equal to unity.

If there ***N*** is a charges $e_1, e_2, \cdots , e_n$ then the force acting on the charge ***e*** becomes in the form:

$$\vec{F} = e \sum_{i=1}^{n} \frac{e_i}{\left| \vec{r} - \vec{r}_i \right|^3} (\vec{r} - \vec{r}_i) \ . \qquad (2)$$

The two formulas (1) and (2) can be generalized in the case of the continuous (uniform) distribution of charges, which distinguishes it by two standard functions in the position:

A - The volumetric density of the charge, which is the charge per unit of volume, is symbolized by the symbol $\rho (r')$.

B - The surface density of the charge, which is the charge per unit area, and it is symbolized by the symbol $\sigma(r')$. In this case, the force acting on the charge from the charged object can be placed in the form:

$$\vec{F} = e \iiint_V \frac{\rho(r')}{|\vec{r} - \vec{r}'|^3}(\vec{r} - \vec{r}')\, d\tau + e \oiint_S \frac{\sigma(r')}{|\vec{r} - \vec{r}'|^3}(\vec{r} - \vec{r}')\, dS$$

Where $\boldsymbol{V}$ is the volume of the object, $d\tau$ is the volume element, $\boldsymbol{S}$ is the surface of the body, dS is the surface element. The general mathematical picture of Coulomb's law of force on a charge, arising from a concentrated distribution of charges in addition to the previous two distributions is:

$$\vec{F} = e \iiint_V \frac{\rho(r')}{|\vec{r} - \vec{r}'|^3}(\vec{r} - \vec{r}')\, d\tau + e \oiint_S \frac{\sigma(r')}{|\vec{r} - \vec{r}'|^3}(\vec{r} - \vec{r}')\, dS + e \sum_{i=1}^{n} \frac{e_i}{|\vec{r} - \vec{r}_i|^3}(\vec{r} - \vec{r}_i) \quad (4)$$

2- The electric field:

It is a mathematical concept that we use to characterize the electric phenomenon, and it is a directional function in position, and the strength of the electric field at a point is defined as the force acting on a unit of positive charges if placed at that point. The general mathematical picture of the electric field strength is:

$$\vec{E} = \iiint_V \frac{\rho(r')}{|\vec{r} - \vec{r}'|^3}(\vec{r} - \vec{r}')\, d\tau +$$

$$\oiint_S \frac{\sigma(r')}{|\vec{r}-\vec{r}'|^3}(\vec{r}-\vec{r}')\,dS +$$

$$\sum_{i=1}^{n} \frac{e_i}{|\vec{r}-\vec{r}_i|^3}(\vec{r}-\vec{r}_i) \quad . \qquad (5)$$

By composing the vector product of the operator ∇ and the vector $\vec{E}$ (the rotation of the vector $\vec{E}$) given by equation (5), we find that:

$$\nabla \wedge \vec{E} = \vec{0} \quad . \qquad (6)$$

By using Stokes' theorem to convert the surface integral into linear integral, i.e.:

$$\iint_S (\nabla \wedge \vec{E}) \cdot d\vec{S} = \oint_c \vec{E} \cdot d\vec{l} \quad .$$

Where ***S*** a surface defined by the closed curve ***c***, $d\vec{l}$ the length vector element of the curve ***c***.

Then by substituting in equation (6), we find that:

$$\oint_c \vec{E} \cdot d\vec{l} = 0 \quad . \qquad (7)$$

That is, the vector $\vec{E}$ (electric field strength) represents a conservative field (i.e. conservative force).

3- Electric potential:

Sometimes it is difficult to find the electric field strength $\vec{E}$ using Coulomb's law (and in many cases very complex) due to the fact that the electric field strength vectors are of the type of directional fields arising from the distributions of charges, and in most cases it is necessary to make three integrals (one for each component). Of electric field compounds). The field analysis of its compounds increases the difficulty of the integration process in most cases. Therefore, it is desirable to find a scalar function and with one integral operation from which the electric field can be obtained.

This standard function is known as the voltage function and it is a function of the position, and since the electric field is a conservative force, the work done by the field to transfer the unit of positive charges from position $\boldsymbol{A}$ to position $\boldsymbol{B_0}$ does not depend on the path, but only on these two positions.

If $\boldsymbol{B_0}$ is a certain agreed fixed point, then the work done by the electric field to move the unit of positive charges from position $\boldsymbol{A}$ to the standard position $\boldsymbol{B_0}$ (this standard position is taken at infinity) is called with the energy of the positive charge voltage of the unit when placed at $\boldsymbol{A}$. Or, in short, the field voltage at $\boldsymbol{A}$. It is denoted by the symbol Φ_A , and since the work does not depend on the path between the two points, the standard function at the position is a single-valued function at any point in space. If we denote work as $\boldsymbol{W}$

So :

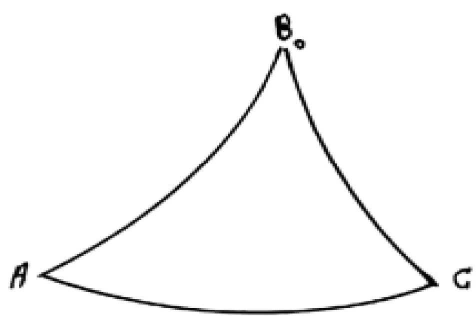


Fig. (1-2)

Accordingly,:

$$\Phi_A = W_{AB_o} \qquad \Phi_C = W_{CB_o}$$

$$W_{AC} = W_{AB_0} + W_{B_0c} = W_{AB_0} - W_{CB_0}$$

$$= \Phi_A - \Phi_C = -\left(\Phi_C - \Phi_A\right) \tag{8}$$

This means that the work done by the field to transfer the unit of positive charges is equal to the change in the voltage function between the two positions multiplied by a negative sign. Now suppose that the work is to transfer the unit of positive charges from the point $P(\vec{r})$ where the voltage function is at ϕ and the point $Q(\vec{r}+d\vec{r})$ and the voltage function $\Phi+d\Phi$ at it is dW where:

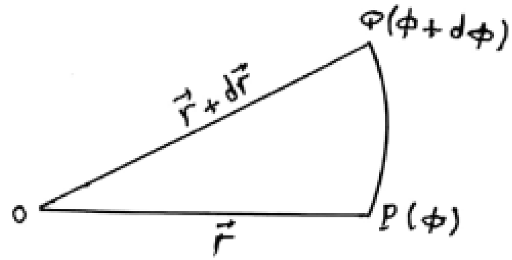


Fig. (1-3)

$$dW = -\left[\Phi + d\Phi - \Phi\right] = -d\Phi$$

$$dW = -\left[\Phi + d\Phi - \Phi\right] = -d\Phi$$

This work can be placed in the picture

$$dW = \vec{E} \bullet d\vec{r}$$

$$\vec{E}\,d\vec{r} = -d\Phi = -\left(\frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz\right)$$

$$= -\left(\frac{\partial\Phi}{\partial x}\vec{i} + \frac{\partial\Phi}{\partial y}\vec{j} + \frac{\partial\Phi}{\partial z}\vec{k}\right)\bullet\left(dx\vec{i} + dy\vec{j} + dz\vec{k}\right)$$

$$= -\left(\nabla\Phi\right)\bullet d\vec{r} \tag{9}$$

$$\vec{E} = -\nabla\Phi = grad\,\Phi \tag{10}$$

This form takes the Cartesian, cylindrical and sphical coordinates, respectively, the forms:

$$\left(E_x, E_y, E_z\right) = -\left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z}\right)$$

$$\left(E_\rho, E_\phi, E_z\right) = -\left(\frac{\partial \Phi}{\partial \rho}, \frac{1}{\rho}\frac{\partial \Phi}{\partial \phi}, \frac{\partial \Phi}{\partial z}\right)$$

$$\left(E_r, E_\theta, E_\phi\right) = -\left(\frac{\partial \Phi}{\partial r}, \frac{1}{r}\frac{\partial \Phi}{\partial \theta}, \frac{1}{r\sin}\frac{\partial \Phi}{\partial \phi}\right)$$

It is noted that relationship (10) achieves: $\nabla \wedge \vec{E} = \vec{0}$ It is also noted from the above that if there is a charge e placed in an electric field, the force acting on this charge becomes $e\vec{E}$ and the energy of the charge voltage is $e\Phi$

Example: Find the field and potential of a charge placed at the origin.

Solution: Suppose $\vec{r}$ the position of point P with respect to the charge so that the field at this point is:

$$\vec{E} = \frac{e}{r^3}\vec{r}$$

As for the potential function, it is given as:

$$\Phi = \int_P^\infty \vec{E} \bullet d\vec{r} = \int_P^\infty \frac{e}{r^3}\vec{r} \bullet d\vec{r} = \int_r^\infty \frac{e}{r^2}dr = \frac{e}{r}$$

Gaussian theory of flux: If N the electric flux emerging from the closed surface S of the electric field $\vec{E}$, then this flux is given by the formula:

$$N = \oint\!\!\oint_S \vec{E} \bullet d\vec{S} = 4\pi Q$$

Where Q is the total charge inside the surface.

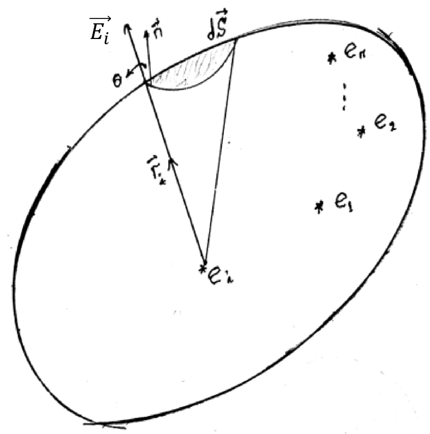


Fig. (1-4)

Proof: Assume that $\vec{E}_i$ the field strength created by the charge e_i at the point $P(\vec{r}_i)$ as shown in the figure. The previous surface integration can be illustrated as:

$$N = \oint\!\!\oint_S \vec{E} \bullet d\vec{S} = 4\pi Q$$

$$\oint\!\!\oint_S \vec{E} \bullet d\vec{S} = \oint\!\!\oint_S \left(\sum_{i=1}^{n} \frac{e_i}{r_i^3} \vec{r}_i \right) \bullet d\vec{S}$$

Where ω_i is the solid angle at charge e_i . And it is: $\omega_i = 0$ or $\omega_i = 4\pi$ when the charge is outside or inside the surface, respectively.

Accordingly, the previous surface integration becomes:

$$\oiint_S \vec{E} \bullet d\vec{S} = 4\pi Q \qquad (11)$$

If there were shipments e_i' distributed on the surface $\boldsymbol{S}$ in addition to the previous distribution. In this case, the stereoscopic angle ω_i' corresponds to the surface charge e_i'. Accordingly, the total flux resulting from surface charges and the internal distribution of the charges is as follows:

$$\oiint_S \vec{E} \bullet d\vec{S} = 4\pi Q + 2\pi Q'$$

Where Q' is the total charge on the surface $\boldsymbol{S}$. As for the uniform distribution of the shipment,

Assuming that the bulk density of the charge inside the surface is $\boldsymbol{\rho}$, the flux is in form:

$$\oiint_S \vec{E} \bullet d\vec{S} = 4\pi \iiint_V \rho d\tau \qquad (12)$$

Where $\boldsymbol{V}$ is the volume enclosed by the surface $\boldsymbol{S}$, is the volume element. If there is a uniform surface distribution with a surface density $\boldsymbol{\sigma}$ in addition to the previous uniform volume distribution, the flux takes the formula:

$$\oiint_S \vec{E} \bullet d\vec{S} = 4\pi \iiint_V \rho d\tau + 2\pi \oiint_S \sigma dS$$

Using the integral formula

$$\oiint_S \vec{E} \bullet d\vec{S} = \iiint_V \left(\nabla \bullet \vec{E}\right) d\tau$$

By substituting in equation (12), we find that:

$$\nabla \bullet \vec{E} = 4\pi\rho \qquad (13)$$

This equation is known as the differential formula for Gauss's law. If $\boldsymbol{\rho} = \mathbf{0}$ then:

$$\nabla \bullet \vec{E} = 0 \qquad (14)$$

That is, at the point of no charge, the electrostatic field divergence disappears.

Poisson's equation:

Putting $\vec{E} = -\nabla\Phi$ in (13) we get:

$$\nabla^2\Phi = -4\pi\rho \qquad (15)$$

This is called the Poisson equation, and it is a fundamental equation in electrophysiology.

Laplace Equation:

Putting $\boldsymbol{\rho} = \mathbf{0}$ in equation (15) we get:

$$\nabla^2\Phi = 0 \qquad (16)$$

This equation is called the Laplace equation, and it is one of the important equations in the branches of theoretical physics.

Power lines and power tubes:

The line of force is the curve whose tangent to it at any point on it fits with the direction of the electric field at that point. Assuming that $d\vec{\ell}$ it is the vector length component of the curve ℓ, then:

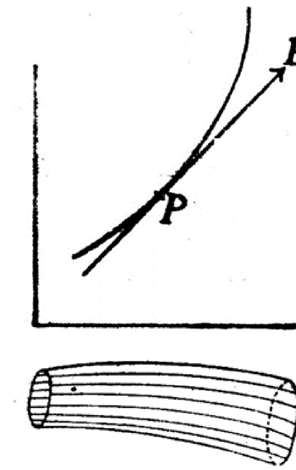


Fig. (1-5)

$$d\vec{\ell} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{E} = E_X\vec{i} + E_y\vec{j} + E_z\vec{k}$$

$$d\vec{\ell} = \lambda\vec{E}$$

Where $\vec{i}$, $\vec{j}$, $\vec{k}$ base unit vectors, and be:

$$\frac{dx}{E_X} = \frac{dy}{E_Y} = \frac{dz}{E_z} = \lambda$$

In general, at every point in space, one force line passes through, but when it is: $\vec{E} = \vec{0}$ the direction of the force line at this point is not specified, and such a point is called the break-even point. A bundle of force lines passing through a closed curve is called a power tube. The flux through any section of a power tube is called the intensity of the tube. Unit tube is that unit tensioned tube.

Notes :

(A) The iso voltage line cuts the power line $\vec{E}$ perpendicular to the orthogonal because the electric field is perpendicular to the iso voltage line.

(B) If a conductive object is charged with a charge, then this charge only settles and is distributed on the surface of the conductor, that is, there will be no charges inside this conductor.

(C) The conductor surface is the isoelectric surface.

Example (1): Find the potential and electric field strength of a capacitor consisting of two flat conductive plates of infinite length. One of them is connected to the ground while the pontential on the other plate at any point on it is equal Φ_1 and the distance between the two plates is equal ℓ

The solution: by choosing a group of cartesian axes ***z*** so that the axis is perpendicular to the plane of each plate. As the figure.

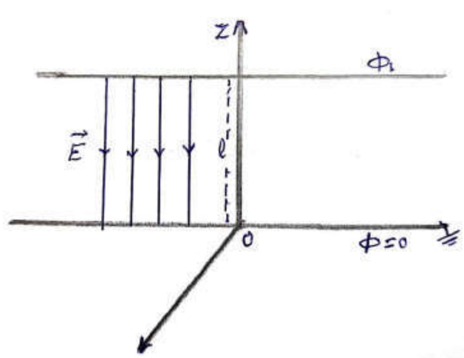


Fig. (1-6)

It is clear from the symmetry that the potential (and the electric field) is a function of the variable z, i.e. that $\Phi = \Phi(z)$ and the Laplace equation becomes:

$$\frac{d^2\Phi}{dz^2} = 0$$

Then by integrating, we find that:

$$\Phi = Az + B$$

Using the two conditions: $\Phi = 0$ when $z = 0$, $\Phi = \Phi_1$ when $z = \ell$ we find that: $A = \frac{\Phi_1}{\ell}$, $B = 0$, so the potential function takes the form:

$$\Phi = \frac{\Phi_1}{\ell} z$$

The strength of the electric field is given by:

$$\vec{E} = -\frac{d\Phi}{dz}\vec{k} = -\frac{\Phi_1}{\ell}\vec{k}$$

Where $\vec{k}$ the unit vector is in the direction of the axis z. It is evident that the electric field is a uniform field in the direction of the plate conducting the earth.

Example (2): Find the potential and electric field of a capacitor consisting of two cylinders infinite in length and shared in the axis, the inner cylinder radius ***a*** and the potential function at any point on it Φ_a, while the outer cylinder is connected to the ground and the radius ***b***.

The solution :

By choosing a group of cylindrical axes $(\rho \cdot \phi \cdot z)$ so that the axis ***z*** applies to the common axis of the two cylinders of the capacitor, it is clear that all points located on a cylinder of radius are ρ . Where $a\langle\rho\langle b$ similar for both the voltage and the electric field function. That is,

$\Phi = \Phi(\rho)$ the voltage function

Z

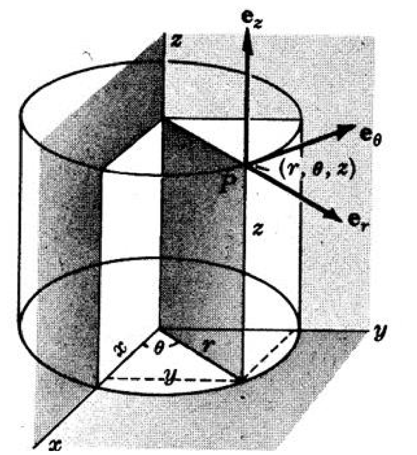


Fig. (1-7)

Laplace Equation in Cylindrical Coordinates:

$$\nabla^2\Phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi} + \frac{\partial^2\Phi}{\partial z^{\,2}} = 0$$

Which in this case becomes in the form :

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\Phi}{d\rho}\right) = 0$$

Then by integrating twice, we find that:

$$\Phi = A\ln\rho + B$$

Using the two conditions: $\Phi = \Phi_a$ when $\boldsymbol{\rho} = \mathbf{0}$, $\Phi = 0$ when $\boldsymbol{\rho} = \boldsymbol{b}$ we find that:

$$A = -\frac{\Phi_a}{\ln b - \ln a} \qquad B = \frac{\Phi_a \ln b}{\ln b - \ln a}$$

The field strength of the capacitor is given as:

$$\vec{E} = -\frac{d\Phi}{d\rho}\vec{e}_\rho = \frac{\Phi_a}{\rho(\ln b - \ln a)}\vec{e}_\rho$$

Where $\vec{e}_\rho$ the unit vector is in the direction of $\boldsymbol{\rho}$. It is evident that the field is inversely proportional to $\boldsymbol{\rho}$, and its direction is from the smaller cylinder to the largest cylinder.

Example (3): Find the electric potential and electric field of a capacitor consisting of two small spherical shells shared in the center. The inner radius of the sphere $\boldsymbol{a}$ and the potential function on it Φ_a. The outer sphere is connected to the ground and has a radius of $\boldsymbol{b}$.

The solution :

By choosing a set of spherical axes (r, θ, ϕ). Obviously, for all points on the surface of a sphere a radius $a < r < b$ .The voltage and electric field function are the same for these points, i.e.: $\Phi = \Phi(r)$

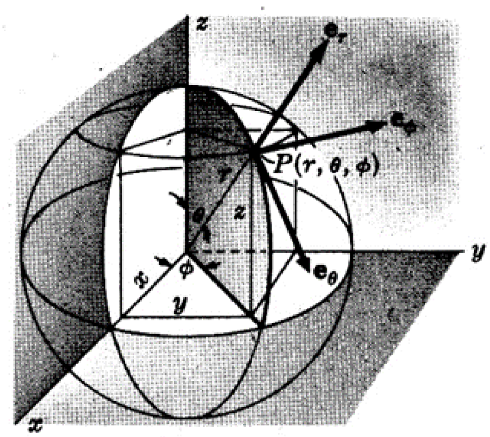


Fig. (1-8)

The Laplace equation for sphrical coordinates is:

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

Which in this case becomes:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 0$$

Complementary, we find that:

$$\Phi = -\frac{A}{r} + B$$

Using the two conditions: $\Phi = \Phi_a$ when $\boldsymbol{r = a}$, $\Phi = 0$ when $\boldsymbol{r = b}$. we find that :

$$A = -\frac{ab\Phi_a}{b-a} \qquad B = -\frac{a\Phi_a}{b-a}$$

Thus,:

$$\Phi = \frac{a\Phi_a}{b-a}\left(\frac{b}{r} - 1\right)$$

$$\vec{E} = -\frac{d\Phi}{dr}\vec{e}_r = \frac{ab\Phi_a}{r^2(b-a)}\vec{e}_r$$

That is, the electric field is inversely proportional to $\boldsymbol{r^2}$ and its direction from the minor to the major crust.

The electrical phenomenon of combinations of charge

1- A group of charges in a straight line:

Assume that $e_1, e_2, ..., e_n$ a group of charges focused on the straight line $\boldsymbol{\theta = 0}$ at the points. . $A_1, A_2, ..., A_n$ Suppose that point ***P*** is on the force line ℓ as shown in the figure. The total electric field $\vec{E}$ must be in the direction of the tangent to the force line through point ***P***. That is, the component perpendicular to the field on the tangent must vanish. That is,:

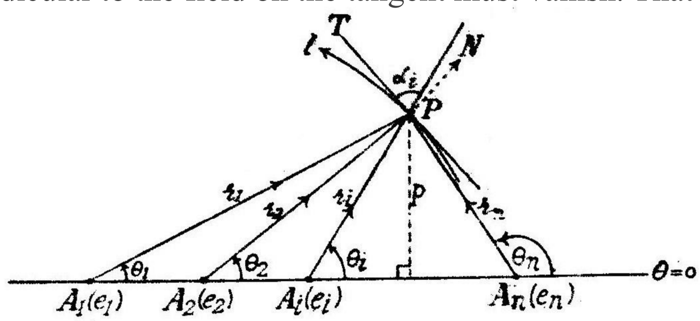


Fig. (1-9)

$$\frac{e_1}{r_1^2}\sin\alpha_1 + \frac{e_2}{r_2^2}\sin\alpha_2 + ... + \frac{e_n}{r_n^2}\sin\alpha_n = 0 \qquad (1)$$

Where α_i is the angle between $\vec{E}_i$ and the tangent at point of the force line ℓ , and where:

$\sin\alpha_i = r_i \dfrac{d\theta_i}{d\ell}$ Then equation (1) takes the form:

$$\frac{e_1}{r_1}d\theta_1 + \frac{e_2}{r_2}d\theta_2 + ... + \frac{e_n}{r_n}d\theta_n = 0 \qquad (2)$$

Using relationships:

$$r_1\sin\theta_1 = r_2\sin\theta_2 = ... = r_n\sin\theta_n = m \qquad (3)$$

And substituting in equation (2) we get:

$$e_1\sin\theta_1 d\theta_1 + e_2\sin\theta_2 d\theta_2 + ... + e_n\sin\theta_n d\theta_n = 0 \qquad (4)$$

By integrating this equation, we find that:

$$e_1 \cos\theta_1 + e_2 \cos\theta_2 + ... + e_n \cos\theta_n = cons. \qquad (5)$$

For all values of the constant we obtain the equations of the power lines of the previous charge distribution. Obviously, these lines lie on the surface of its axis ℓ . As for lines of equal voltage that cut the lines of forces perpendicular to each other:

$$\frac{e_1}{r_1} + \frac{e_2}{r_2} + ... + \frac{e_n}{r_n} = const.$$

That is,:

$$e_1 \sin\theta_1 + e_2 \sin\theta_2 + ... + e_n \sin\theta_n = const. \qquad (6)$$

For all values of the constant we obtain equations of isoelectric lines.

2- Electric field and electric potential of linear charges:

If linear charges are distributed in a continuous line on an infinitely long line, this distribution ***e*** is called a linear distribution of charge, and the charge ***e*** per unit of length is called linear density. If e has the same value at each point, the distribution is called the uniform distribution of the charge.

To find the strength of the electric field and the electric potential produced by a wire charged with a uniform charge, suppose that the wire is a cylinder of very small radius ***a*** and has a uniform linear charge. Now we imagine a cylinder of radius ***r*** coaxial with the previous cylinder. The total flux leaving the outer cylinder of length ***L*** is $2\pi rLE$ where the field strength magnitude is ***E***. Since the total internal charge is (***eL***), by applying the Gaussian flux theory we find that:

$$2\pi rLE = 4\pi(eL)$$

$$E = \frac{2e}{r} \quad (7)$$

$$E = -\frac{d\Phi}{dr} = \frac{2e}{r}$$

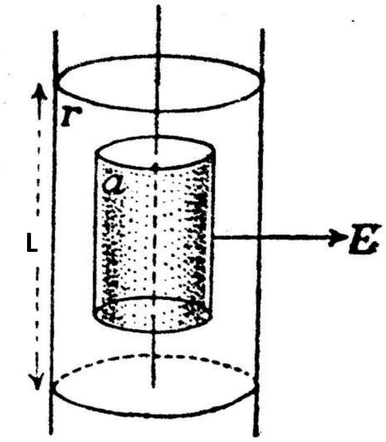


Fig. (1-10)

Then by integrating, we find that:

$$\Phi = const. - 2e\ln r \quad (8)$$

The two relationships (7) and (8) do not depend on the radius of the inner cylinder, and assuming that $a \to 0$, meaning that the inner cylinder devolved into a charged wire with a uniform linear charge, (7) and (8) are the equations for the electric field strength and the electric potential of the charged wire.

To find the forec lines of a group of infinitely long parallel wires charged with uniform charges: Suppose the linear density of the charge for these wires is $e_1, e_2, ..., e_n$ and suppose that the wires cut perpendicular to a plane in points $A_1, A_2, ..., A_n$ respectively, and suppose that is ***P*** a point on the line of force ℓ in this plane, and assume that: $A_1P, A_2P, ..., A_nP$ make the angles $\theta_1, \theta_2, ..., \theta_n$ With a fixed line in the plane $\boldsymbol{\theta = 0}$ as shown in the figure.

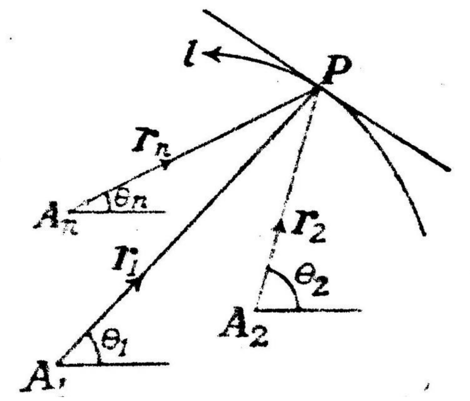


Fig. (1-11)

Since the total electric field component perpendicular to the tangent at point ***P*** of the curve ℓ must vanish, then:

$$\frac{2e_1}{r_1}\sin\alpha_1 + \frac{2e_2}{r_2}\sin\alpha_2 + ... + \frac{2e_n}{r_n}\sin\alpha_n = 0 \qquad (9)$$

Where $\boldsymbol{\alpha_i}$ is the angle between A_iP, and the tangent to the curve and where $\sin\alpha_i = r_i \frac{d\theta_i}{d\ell}$

$$e_1 d\theta_1 + e_2 d\theta_2 + ... + e_n d\theta_n = 0$$

Then by integrating, we find that:

$$e_1\theta_1 + e_2\theta_2 + ... + e_n\theta_n = const. \qquad (10)$$

For different constant values, equation (10) gives the equation of the plane force lines in the wire group charged parallel linear charges. As for the lines of equal potential at this level, they are required from the equation

$$e_1 \ln r_1 + e_2 \ln r_2 + ... + e_n \ln r_n = const. \qquad (11)$$

3-The electric doublet (or dipole):

It is two very large electrical charges $(+e),(-e)$ that are separated from each other by a very small distance $\delta\ell$. The electric doublet is characterized by a vector called the moment vector, and this vector is given by the mathematical formula: $\vec{M} = \lim_{\delta\ell \to \infty}(e\delta\ell)$

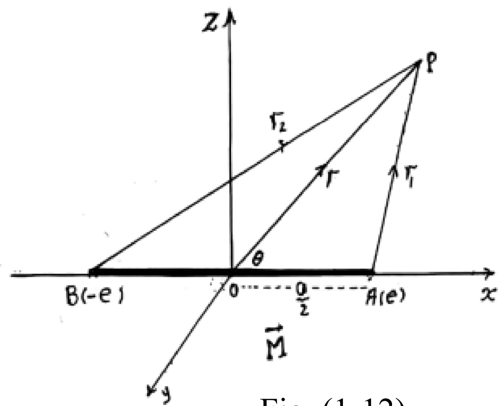


Fig. (1-12)

The vector connecting from negative charge to positive charge is called the doublet axis, and it is the same direction as the electric moment vector. The doublet potential at the point $P(\vec{r})$ can be shown as:

$$\Phi(P) = \frac{(+e)}{r_1} + \frac{(-e)}{r_2} = e\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \qquad (12)$$

By choosing a group of cartesian axes (x, y, z) so that the double axis applies to the axis $\boldsymbol{x}$ as shown in the figure. Then it can put:

$$r^2 = x^2 + y^2 + z^2, M = ae$$

Where $\boldsymbol{a}$ is the length of the electric doublet ($\boldsymbol{a}$ is very small). And also:

$$r_1 = \sqrt{\left(x - \frac{a}{2}\right)^2 + y^2 + z^2} = r\left(1 - \frac{ax}{r^2}\right)^{\frac{1}{2}} \qquad (13)$$

$$r_2 = \sqrt{\left(x - \frac{a}{2}\right)^2 + y^2 + z^2} = r\left(1 + \frac{ax}{r^2}\right)^{\frac{1}{2}} \qquad (14)$$

Then by substituting in (12), we find that the double voltage becomes:

$$\Phi(P) = \frac{e}{r}\left[\left(1 - \frac{ax}{r^2}\right)^{-\frac{1}{2}} - \left(1 + \frac{ax}{r^2}\right)^{-\frac{1}{2}}\right] = \frac{ae}{r^2}\frac{x}{r}$$

$$= \frac{M\cos\theta}{r^2} = \frac{\vec{M} \bullet \vec{r}}{r^3} \qquad (15)$$

Where $\boldsymbol{\theta}$ is the angle between the vectors: $\vec{r}, \vec{M}$. The electric field at point ***P*** is:

$$\vec{E} = -\nabla\Phi = -\nabla\left(\frac{\vec{M} \bullet \vec{r}}{r^3}\right) = -\frac{\vec{M}}{r^3} + 3\frac{\vec{M} \bullet \vec{r}}{r^5}\vec{r} \qquad (16)$$

The two polar components of the electric field require from two relationships:

$$E_r = -\frac{\partial\Phi}{\partial r} = \frac{2M\cos\theta}{r^3} \qquad (17)$$

$$E_\theta = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = \frac{M\sin\theta}{r^3} \qquad (18)$$

And the differential polar equation of force lines is given as:

$$\frac{dr}{E_r} = \frac{rd\theta}{E_\theta}$$

Then by substitution and integration we obtain the equation of the force lines with the formula:

$$r = c\sin^2\theta \qquad (19)$$

Example (4): Study the force lines of the two positive charges e_1, e_2 at the points A, B and. Then show that the tangent at infinity (the asymptote) of the line of force starting from e_1 with an angle of inclination α with BA makes the angle:

$$2\sin n^{-1}\left(\sqrt{\frac{e_1}{e_1+e_2}}\sin\frac{\alpha}{2}\right)$$

The solution :

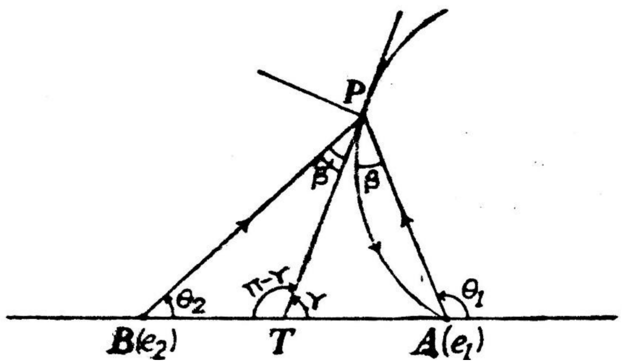


Fig. (1-13)

Suppose that **P** is a point on the force line that starts at A and ends at infinity. And suppose the line **BA.** The line is: $\theta = 0$. The equation of the force lines are:

$$e_1\cos\theta_1 + e_2\cos\theta_2 = C \qquad (1)$$

To find **C** for the force line that starts at A with an angle of inclination α and ends at infinity, we use the condition when $P \to A$, then: $\theta_2 = 0, \theta_1 = \alpha$ by substituting in the previous equation, we find that: $C = e_1\cos\alpha + e_2$ and the equation of the requird force line becomes:

$$e_1\cos\theta_1 + e_2\cos\theta_2 = e_1\cos\alpha + e_2 \qquad (2)$$

To find the slope of the tangent at infinity (which is the angle of inclination of the asymptote) we use the condition: when $P \to \infty$, then $\theta_1 = \theta_2 = \theta$ Then by substituting into equation (2):

$$(e_1+e_2)\cos\theta = e_1\cos\alpha + e_2$$

$$(e_1+e_2)\left(1-2\sin^2\frac{\theta}{2}\right)=e_1\left(1-2\sin^2\frac{\alpha}{2}\right)+e_2$$

$$(e_1+e_2)\sin^2\frac{\theta}{2}=e_1\sin^2\frac{\alpha}{2}$$

$$\sin\frac{\theta}{2}=\sqrt{\frac{e_1}{e_1+e_2}}\sin\frac{\alpha}{2}$$

$$\theta=2\sin^{-1}\left(\sqrt{\frac{e_1}{e_1+e_2}}\sin\frac{\alpha}{2}\right)$$

Exercises

1- Find the force lines of two parallel wires that are infinite in length, and the longitudinal density of the charge for them $-\boldsymbol{e},\boldsymbol{e}$. Also find the potential equal curves.

2- Three wires of infinite length, and the linear density of the charge is: **1, -2 , 1** unit charge. These wires are cut perpendicular to a plane at three points and on the same line: $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$, respectively, where $\boldsymbol{AB} = \boldsymbol{BC} = \boldsymbol{a}$. Prove that the equation of power lines is:

$$r^2=a^2\cos(2\theta+\alpha)\sec\alpha$$

Where is at the origin, $\boldsymbol{BC}$ is the angle measure $\boldsymbol{\theta}$, $\boldsymbol{\alpha a}$ is a parameter.

3- Three parallel and symmetric thin wires of uniform linear charge, and intersecting perpendicularly a plane at three points ***A*, *B*,*C*** which represent the vertices of an equilateral triangle and the length of its side $\sqrt{3c}$. Show that the polar equation for the voltage-equal curves plotted in the plane is as follows: $r^6+c^6-2r^3c^3\cos\theta=cons.$ Assuming the center of the triangle is the origin.

4 - Four parallel wires of infinite length, placed so that they cross a plane perpendicular to a plane in four points, which are the vertices of the square ***ABCD*** and the longitudinal density of the uniform charge is:

$\boldsymbol{e}$ when $\boldsymbol{A}$, $\boldsymbol{C}$, $-\boldsymbol{e}$ when $\boldsymbol{B}$, $\boldsymbol{C}$. If the side length of the square is $\boldsymbol{2a}$, prove that the potential Φ at the point ***P*** located inside the square takes the form: $\Phi=2er^2a^{-2}\cos 2\theta$ where (r,θ) the polar coordinates of the point are with respect to the center of the square.

Polarized dielectric materials

It has been found experimentally that some insulating materials such as mica and glass, if they are electrically affected by them, they polarize in the sense that every small element thereof turns into an electric doublet, and this phenomenon can be explained in two ways as follows:

(A) Atoms of any substance contain positive and negative charges. If the material is conductive then negative charges are free to move in the material under the influence of an electric field, resulting in an electric current flow. If the material is neutral and polarizable, then this flow does not occur, but the effective electric field displaces the charges in the atom slightly so that the charge is shifted in the direction of the effect field and the negative charge in the opposite direction, and thus the electric doublets in the polarized material appears in the direction of the electric field.

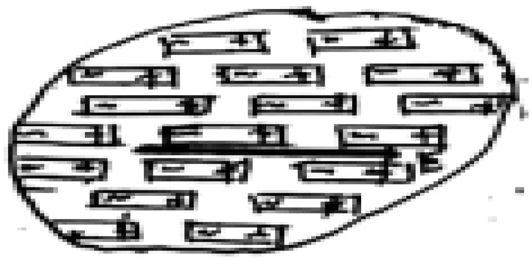


Fig. (1-14)

(B) It can be imagined that the molecules of this substance are mainly composed of electric doublets distributed in the substance randomly distributed so that each small element of it consists of a large number of these doublets and this element does not appear polarized because these doublets disappear each other, but if you put this substance in a field Electrification, it creates a regularity in the directions of these doublets and take the direction of the field, and the material becomes polarized.

Polarization vector

We now introduce a concept that characterizes the polarization intensity of the material, which is the polarization vector $\vec{P}$, which is defined as the electric doublet moment vector equivalent to the unit volume of the polarized material. If we have a volume element $d\tau$, it is equivalent to an electric doublet of its torque: $\vec{P}d\tau$. For many materials there is a linear relationship between the vector $\vec{P}$ and the vector of electric field strength $\vec{E}$ at any point within the polarized material. That is,:

$$\vec{P} = k\vec{E}$$

Where k is a constant dependent on the material and is called the polarizability coefficient. If the polarization vector is constant in magnitude and direction at all points of the polarized material, it is said that the polarization is uniform.

Poisson's rule of equivalent distribution

Suppose we have a polarized material of volume ***V*** and enclosed in the surface ***S*** and that $d\tau$ is a volume element of this body. This element is equivalent to an electric doublet vector torque $\vec{P}d\tau$ If ***q*** is a point outside the polarized body, then the electrical phenomenon arising from the body at point ***q*** is characterized by the field vector $\vec{E}$ and the potential function Φ, and they are estimated by the complementary effects of the electric doublet that make up the body.

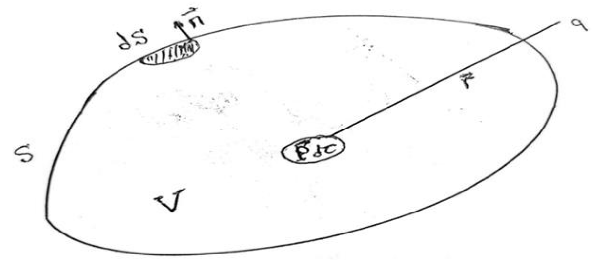


Fig. (1-15)

To find the potential function at point ***q***, suppose the potential due to the electric doublet $\vec{P}d\tau$ is:

$$d\Phi = \vec{P}d\tau \bullet \nabla\left(\frac{1}{r}\right)$$

The potential produced by the body at point ***q*** is:

$$\Phi_q = \iiint_V \vec{P} \bullet \nabla\left(\frac{1}{r}\right) d\tau = \iiint \left[\nabla \bullet \left(\frac{\vec{P}}{r}\right) - \frac{\nabla \bullet \vec{P}}{r}\right] d\tau$$

And be:

$$\Phi_q = \iiint_V \left(\frac{-\nabla \bullet \vec{P}}{r}\right) d\tau + \oiint_S \frac{\vec{P} \bullet d\vec{S}}{r} = \iiint_V \left(\frac{-\nabla \bullet \vec{P}}{r}\right) d\tau + \oiint_S \frac{P_n}{r} dS$$

This result shows that the polarized body is completely equivalent to the following electrical model:
(1) A group of charges distributed on the volume of an object and their volume density of the charge is:

$$\rho = -\nabla \bullet \vec{P}$$

(2) A group of charges distributed over the surface of an object whose surface charge density is:
$\sigma = P_n$ It is the component perpendicular to the polarization vector on the surface of a polarized object.
The previous two distributions are known as the equivalent Poisson distribution for the polarized body. It is clear that the total charge resulting from the two previous distributions is:

$$Q = \iiint_V \left(-\nabla \bullet \vec{P}\right) d\tau + \oiint_S P_n dS = -\iiint_V \left(\nabla \bullet \vec{P}\right) d\tau + \iiint_V \left(\nabla \bullet \vec{P}\right) d\tau = 0$$

As expected because the total charge inside the polarized body and on its surface must be zero. The electric field strength at point outside the polarized body is the force acting on the unit of positive charges at this point and satisfies the equations:

$$\vec{E} = -\nabla \Phi \quad , \nabla \wedge \vec{E} = \vec{0} \quad , \oint \vec{E} \bullet d\vec{\ell} = 0$$

The electric phenomenon inside a polarized body is also characterized by the electric field vector $\vec{E}$ and it satisfies the previous equations.

The electric displacement vector

At any point inside a polarized body the equivalent Poisson distribution is achieved. This means that the field $\vec{E}$ at any point inside a polarized body is related to the bulk density at this point (the density $-\nabla\bullet\vec{P}$) by Gaussian theory of flood. That is,:

$$\oiint_S \vec{E}\bullet d\vec{S} = 4\pi\iiint_V \left(-\nabla\bullet\vec{P}\right)d\tau$$

Where is the surface of the polarized object surrounding the volume. And by using the complementary relationship:

$$\oiint \vec{E}\bullet d\vec{S} = \iiint_V \left(\nabla\bullet\vec{E}\right)d\tau$$

Substituting in Gauss' flood theory, we obtain:

$$\nabla\bullet\left(\vec{E}+4\pi\vec{P}\right)=0$$

And now a new vector can be defined to distinguish the electrical phenomenon inside the polarized material, which can be placed in the image:

$$\vec{D}=\vec{E}+4\pi\vec{P}$$

It is called the electric displacement vector and it achieves the mathematical relationship: $\nabla\bullet\vec{D}=0$ This is true in the case of a polarized body only and is not charged with additional charges from the outside. Whereas $\vec{P}=k\vec{E}$, by substitution, the vector of electrical displacement takes the form:

$$\vec{D}=\vec{E}+4\pi k\vec{E}=K\vec{E} \quad ,K=1+4\pi k$$

And ***k*** is called the polarization constant. It is further evident that:

$$\nabla\bullet\left(K\vec{E}\right)=0 \quad , \quad \nabla\bullet\left(K\nabla\Phi\right)=0$$

If $\boldsymbol{k}$ does not depend on position, then the voltage function satisfies the Laplace equation: $\nabla^2\Phi = 0$

But if the polarized body is charged with additional free charges and its bulk density is $\boldsymbol{\rho}$, and by applying Gaussian flood theory, we find that:

$$\nabla \bullet \vec{E} = 4\pi\left(\rho - \nabla \bullet \vec{P}\right) \quad \Rightarrow \quad ,\nabla \bullet \left(\vec{E} + 4\pi\vec{P}\right) = 4\pi\rho$$

$$\nabla \bullet \vec{D} = 4\pi\rho \quad \Rightarrow \quad \nabla \bullet \left(K\,\nabla\Phi\right) = -4\pi\rho$$

If $\boldsymbol{k}$ does not depend on position, then the voltage function is fulfilled: $\nabla^2\Phi = -\frac{4\pi}{K}\rho$. this is the Poisson equation. In the case of space $\rho = 0$, $\boldsymbol{k}$= 1 and, $\vec{D} = \vec{E}$

Consequences:

1) If we imagine a closed surface drawn inside the material, it results from the above that:

A- If the material is not shipped with free shipment then:

$$\oiint_S \vec{D} \bullet d\vec{S} = \iiint \left(\nabla \bullet \vec{D}\right) d\tau = 0$$

B- If the material is charged with free bulk bulk density $\boldsymbol{\rho}$ then:

$$\oiint_S \vec{D} \bullet d\vec{S} = \iiint_V \left(\nabla \bullet \vec{D}\right) d\tau = 4\pi \iiint_V \rho\, d\tau$$

2) If the body is uniformly polarized, that is $\vec{P} = cons.$, the electric field inside the material is uniform, that is $\vec{E} = cons.$ it is achieved $\nabla \bullet \vec{E} = 0$ if there are no free charges, i.e. $\boldsymbol{p} = \mathbf{0}$ and the equivalent Poisson distribution converts to a surface distribution only, i.e.

$$\Phi = \oiint_S \frac{P_n}{r} dS$$

3) If the point at which we compute the voltage Φ outside the body that is not charged with free charges (i.e. $\boldsymbol{\rho} = \mathbf{0}$) is too far from the body then: $\nabla\left(\frac{1}{r}\right)$ it remains constant during the integration process, i.e. that:

$$\Phi = \nabla\left(\frac{1}{r}\right) \bullet \iiint_V \vec{P} d\tau = \vec{m} \bullet \nabla\left(\frac{1}{r}\right)$$

That is, in this case we consider the polarized body as if it were an electric double and its moment vector must be by the formula:

$$\vec{m} = \iiint_V \vec{P} d\tau$$

4) If we place a charge $+e$ in a substance with a constant polarization k, we enclose this charge with a ball of radius r and the center of the charge $+e$ and $\vec{P}$, $\vec{E}$, $\vec{D}$ at any point at a distance r from the charge, then:

$$\oiint_S \vec{D} \bullet d\vec{S} = 4\pi e \Rightarrow 4\pi r^2 D = 4\pi e \Rightarrow D = \frac{e}{r^2} \Rightarrow \vec{D} = \frac{e}{r^3}\vec{r}$$

The electric field should be: $\vec{E} = \frac{e}{Kr^3}\vec{r}$ The polarization intensity vector takes the picture:

$$\vec{P} = \frac{\vec{D} - \vec{E}}{4\pi} = \frac{(K-1)e}{4\pi r^3}\vec{r}$$

This means that if there is a charge $\boldsymbol{e'}$ at a distance $\boldsymbol{r}$ from the charge $+\boldsymbol{e}$, then the force acting on the charge $\boldsymbol{e'}$ is in the form: $\vec{F} = \frac{ee'}{Kr^3}\vec{r}$ and if there is no substance, $\vec{F} = \frac{ee'}{r^3}\vec{r}$ then, that is, the presence of the charge $+\boldsymbol{e}$ inside the substance decreases its amount from $+\boldsymbol{e}$ to. because $\boldsymbol{k} > \mathbf{1}$ can be understood from the Poisson distribution as follows: If we consider the charge $+\boldsymbol{e}$ in the form of a sphere of radius $\boldsymbol{b}$, the material after it polarizes is equivalent to a volumetric distribution of its density, then:

$$-\nabla\bullet\vec{P} = -\frac{(K-1)e}{4\pi K}\nabla\bullet\left(\frac{\vec{r}}{r^3}\right) = 0$$

The surface distribution has a surface density: $-P_{a=b}$ because $\vec{P}$ in the direction $\vec{r}$ of any perpendicular to the surface of the sphere, this means that the polarization of the material adds to the charge $+\boldsymbol{e}$ another charge of magnitude

$$q = \lim_{b\to 0} 4\pi b^2 P_{r=b} = \lim_{b\to 0} -4\pi b^2 \frac{(K-1)e}{4\pi K b^2} = -\frac{(K-1)e}{K}$$

The negative sign because the direction $\vec{P}$ is the direction $\vec{r}$ and the perpendicular direction on the surface (the surface of the sphere) is towards the outside in a direction $-\vec{r}$ and so the total charge is:

$$+e - \frac{(K-1)e}{K} = \frac{e}{K}$$

Surface conditions:

The surface conditions that must be met at the surface separating two polarized materials can be obtained by assuming that $\boldsymbol{S}$ is the separating surface between two materials (1) and (2) and their polarization, K_2 , K_1 respectively. We imagine a cylinder perpendicular to the surface $\boldsymbol{S}$ and its base area $\boldsymbol{A}$ and that the unit vector is perpendicular to the surface, and if there are free charges on the separating surface $\boldsymbol{S}$ and its surface density σ and assuming that the height of the cylinder is so small that an overflow $\vec{D}$ on the rotational surface can be neglected, we obtain:

$$D_{n2}A - D_{n1}A = 4\pi\sigma A \quad \Rightarrow \quad D_{n2} - D_{n1} = 4\pi\sigma$$

This means that the vertical component of the electric displacement vector at the charged separating surface is disconnected, and where

$$\vec{D} = K\vec{E} \ , \ \ E_n = \frac{\partial \Phi}{\partial n}$$

We find that:

$$K_2 E_{n2} - K_1 E_{n1} = 4\pi\sigma \ \ \Rightarrow \ \ K_2 \frac{\partial \Phi_2}{\partial n} - K_1 \frac{\partial \Phi_2}{\partial n} = -4\pi\sigma$$

If the surface is uncharged, i.e. $\boldsymbol{\sigma = 0}$, then:

$$D_{n2} - D_{n1} = 0 \ \ , \ \ K_2 E_{n2} - K_1 E_{n1} = 0 \ \ \Rightarrow \ \ K_2 \frac{\partial \Phi_2}{\partial n} = K_1 \frac{\partial \Phi_1}{\partial n}$$

As for the tangent component to the electric field vector in the direction of the separating surface, it is continuous, meaning that $E_{t2} = E_{t1}$ from it we find that the voltage is a continuous function at the surface, that is $\Phi_2 = \Phi_1$, if we assume that material (2) is a conductive substance, then $\vec{E}_2 = 0$ that is . $\vec{E}_{t2} = 0$, that is $E_{t1} = 0$, that is, the electric field in Article (1) is Just perpendicular to the surface ***S***:

$$E_1 = -\frac{4\pi\sigma}{K_1} \ \ , \ \ \sigma = +\frac{K_1}{4\pi} \cdot \frac{\partial \Phi}{\partial n} \ \ \ , \ \ \Phi_1 = \Phi_2$$

Application:

A sphere of radius ***a*** and constant polarization ***k*** is placed in a substance of constant polarity k_1 and extended to infinity, and the two materials are affected by an electric field $\vec{E}_0$ of uniform intensity. Calculate the voltage function at any point.

The solution: There are (1) and (2) inside the ball, $0 \le r \le a$ and (2) $a \le r \le \infty$ outside the ball. By taking the axis ***z*** in the direction of the uniform field $\vec{E}_0$ through the center of the sphere ***o***, then this axis is the axis of symmetry. If we choose the spherical polar coordinates to express the voltage function, which then depends on ***r***,***θ***, that is, $\Phi = \Phi(r,\theta)$ we will search for the two voltage functions: $\nabla^2\Phi_1 = 0$, $\nabla^2\Phi_2 = 0$ in the two regions (1) and (2) and each of them fulfills the conditions:

(A) $\Phi_1(r,\theta)$, $\Phi_2(r,\theta)$ When we write the Laplace equation in spherical polar coordinates.

(B) Φ_1 It is monovalent and finite when $r = \infty$, Φ_2 it is monovalent and finite when = 0.

(C) $\Phi_1 = \Phi_2$ when ***r*** =**a** for all values of the angle θ.

(D) $K_1 \dfrac{\partial \Phi_1}{\partial r} = K_2 \dfrac{\partial \Phi_2}{\partial r}$ When ***r*** =**a** for all values of the angle θ.

So we compound the two functions $\Phi_1 = \Phi_2$ from the functions:

$$\frac{C}{r} \ , \ \frac{A\cos\theta}{r^2} \ , \ Br\cos\theta \ , \ -E_0 r\cos\theta$$

Inside the ball $0 \le r \le a$ we choose $\Phi_2 = Br\cos\theta$.

Outside the ball $a \le r \prec \infty$, the potential must devolve to that uniform field at ***r*** =∞, and the field at far points is close to that of the electric duplex. So we choose

$$\Phi_1 = -E_0 r\cos\theta + \frac{A\cos\theta}{r^2}$$

Then by applying the two conditions (c) and (d) we get:

$$-E_0 a\cos\theta + \frac{A\cos\theta}{a^2} = Ba\cos\theta$$

$$K_1\left[-E_0\cos\theta - \frac{2A\cos\theta}{a^3}\right] = K_2 B\cos\theta$$

$$-E_0 + \frac{A}{a^3} = B \quad , \quad -E_0 - \frac{2A}{a^3} = \frac{K_2}{K_1}B$$

By solving these two equations, we find that:

$$A = \frac{K_2 - K_1}{K_2 + 2K_1}E_0 a^3 \quad , \quad B = \frac{-3K_1}{K_2 + 2K_1}E_0$$

Thus, it is:

$$\Phi_1 = -E_0 r\cos\theta + \frac{K_2 - K_1}{K_2 + 2K_1}\cdot\frac{E_0 a^3}{r^2}\cos\theta$$

$$\Phi_2 = \frac{-3K_1 E_0}{K_2 + 2K_1}\cdot r\cos\theta$$

Result (1): We can find a distribution of charges in space equivalent to the previous combination and give the same two voltage functions $\Phi_1 = \Phi_2$, from

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + 0 = 4\pi\rho$$

$$\left(E_{n1} - E_{N2}\right)_{r=a} = 4\pi\sigma$$

The two functions $\Phi_1 = \Phi_2$ fulfill the Laplace equation where $\boldsymbol{\rho = 0}$. In the two worlds (1), (2) we put $K_1 = 1 \quad K_2 = K$

$$\left(\frac{\partial\Phi_1}{\partial r} - \frac{\partial\Phi_2}{\partial r}\right)_{r=a} = -4\pi\sigma$$

$$-E_0\cos\theta - 2\frac{K-1}{K+2}\cdot E_0\cos\theta + \frac{3}{K+2}\cdot E_0\cos\theta = -4\pi\sigma$$

$$\Rightarrow \quad \sigma = \frac{3(K-1)}{4\pi(K+2)} \cdot E_0 \cos\theta$$

This means that a distribution of charges with a surface density $\boldsymbol{\sigma}$ on the surface of a sphere (radius $\boldsymbol{a}$) and constant polarization $\boldsymbol{k}$ placed in space gives the same electrical phenomenon (such as a sphere of radius **a** in a uniform electric field E_0).

Result (2): If the ball is a gap in material, $\boldsymbol{k_1}$ then:

$$\Phi_2 = \frac{-3K_1}{1+2K_1} \cdot E_0 r \cos\theta$$

Example (5): Calculate the equivalent Poisson distribution in the case of a sphere of radius $\boldsymbol{a}$ and polarized so that the polarization vector should be from: $\vec{P} = \alpha\vec{r}$ where $\boldsymbol{\alpha}$ is constant, $\vec{r}$ the position vector of a point with respect to the center of the sphere

The solution:

The bulk density of a charge is given by: $\rho = -\nabla \bullet \vec{P} = -\alpha \nabla \bullet \vec{r} = -3\alpha$

The surface density should be: $\sigma = \vec{P} \bullet \vec{n} = \alpha \vec{r} \bullet \frac{\vec{r}}{r} = \alpha r \quad \Rightarrow \quad \sigma = \alpha a$

Because $\boldsymbol{r}$ =**a** at the surface of the sphere. It is evident that the total charge:

$$Q = \iiint_V (-3\alpha)d\tau + \oiint_S (\alpha a)dS = (-3\alpha)\left(\frac{4}{3}\pi a^3\right) + (\alpha a)(4\pi a^2) = 0$$

Exercises

1 - a charge $+e$ placed in front of the plane surface of a substance of constant k and half-infinite polarization. Calculate the force between the charge and matter.

2 - A sphere of radius a is uniformly polarized and has $\vec{P}$ a polarization vector. Calculate the total electric field at the center resulting from the surface portion of the equivalent Poisson distribution.

3 - A thin rod with a cross-sectional area A is applied to the axis x from the origin to $x=L$. If it is known that the polarization was in the direction of the penis, it is necessary to: $\left|\vec{P}\right| = ax^2 + b$ Calculate the bulk density of the polarizing charge as well as the surface density of the charge at each end.

4 - Three concentric, thin spherical A , B , C shells of radii, a , b , c respectively. Where . $a < b < c$. Fill the space between A , B a substance of constant polarity k, and the space between B , C a substance with a constant polarization k'. The two crusts C , A reached the ground and the shell B was charged with a total charge Q. Prove that Q is divided between the inner and outer surfaces of the shell B with respect to: $\dfrac{Ka(c-b)}{K'c(b-a)}$

Electric currents:

When two conductors of the two voltage functions Φ_2 , Φ_1 are connected by a metal wire, then an electric flow occurs from the conductor with the highest voltage of the other conductor until the two voltages are equal.

Electric current:

The current intensity in a regular conductor is the rate of change of charge ***Q*** i.e. $I = \frac{dQ}{dt}$

Current density vector:

The rate of electric current flow through an element ***dS*** from a surface at a point $p(\vec{r})$ is: $\vec{j} \bullet d\vec{S}$ .where $\vec{j} = \vec{j}(\vec{r})$ Is the vector of current density.

Continuity equation:

The rate of flow of electric charge outside the surface ***S*** surrounding the volume ***V*** is: $\oiint_S \vec{j} \bullet d\vec{S}$ is equal to the rate of decrease in charge, and assuming that the surface is fixed, then:

$$\oiint_S \vec{j} \bullet d\vec{S} = -\frac{\partial}{\partial t}\iiint_V \rho \, d\tau$$

And using the integral relationship:

$$\oiint_S \vec{j} \bullet dS = \iiint_V \left(\nabla \bullet \vec{j}\right) d\tau$$

we find that :

$$\iiint_V \left(\nabla \bullet \vec{j} + \frac{\partial \rho}{\partial t}\right) d\tau = 0$$

$$\nabla \bullet \vec{j} + \frac{\partial \rho}{\partial t} = 0$$

This equation is called the coninuity equation. When $\frac{\partial \rho}{\partial t} = 0$ van : $\nabla \bullet \vec{j} = 0$.

By setting it $\nabla \bullet \vec{j} = -\frac{\partial \rho}{\partial t}$, it turns out that the divergent charge per second per unit volume is equal to the rate of decrease in the charge with respect to time per unit of volume at this point.

Chapter 2 : Magnetic Phenomenon

Introduction :

The phenomenon of magnetic field in space is determined by the presence of a vector known as a vector of magnetic field strength and a standard potential function Ω . The magnetic phenomenon can be treated in the same way that we treat the static electrical phenomenon.

Magnetic fields usually originate from two types of objects:

(A) By moving electric charges or electric currents.

(B) By means of magnetized objects (magnetic materials). It should be noted that there is a fundamental difference between the electric phenomenon and the magnetic phenomenon, as it is possible for electric charges to appear separately. (Positive or negative) As for the magnetic phenomenon, the magnetic poles appear in the form of correlative pairs (i.e. a positive magnetic pole associated with a negative magnetic pole or a north pole associated with a south pole)

Coulomb's reverse law:

For any two magnetic poles of intensity $\boldsymbol{P_1}$,$\boldsymbol{P_2}$ and separated by a distance $\boldsymbol{r}$, a force between them will appear directly proportional to the product $\boldsymbol{P_1 P_2}$ and inversely proportional to the square of the distance between the poles, that is:

$$\vec{F} \propto \frac{p_1 p_2}{r^3}\vec{r}$$

The constant of proportionality depends on the medium in which these two magnetic poles are located.

Magnetic Potential of Small Magnets:

A small magnet is called a magnetic double or magnetic bipole, and it consists of two large magnetic poles separated by a very small distance. The magnetic moment vector of a double is denoted by $\underset{M}{\rightarrow}$ and has a magnitude: $\boldsymbol{M = ma}$

Where is $\boldsymbol{m}$ the pole strength, $\boldsymbol{a}$ the length of the small magnet ($\boldsymbol{a}$ is very small).

The direction of the torque vector in the direction of the double axis (the line connecting from the negative pole to the positive pole) the magnetic potential produced by the the small magnet at the point $\boldsymbol{P}(\boldsymbol{r})$ is given by the formula:

$$\Omega(p) = \frac{m}{r_1} + \frac{-m}{r_2} = m\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

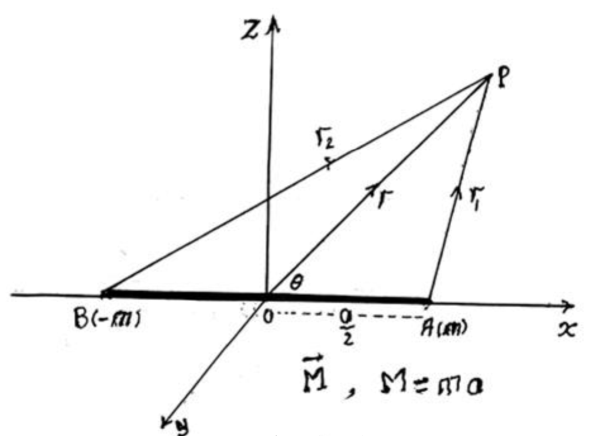


Fig. (2-1)

By choosing a group of cartesian axes so that the axis x applies to the axis of the small magnet as shown in the figure, and accordingly, it is possible to place:

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$r_1 = \sqrt{\left(x - \frac{a}{2}\right)^2 + y^2 + z^2} = r\left(1 - \frac{ax}{r^2}\right)^{\frac{1}{2}}$$

$$r_2 = \sqrt{\left(x + \frac{a}{2}\right)^2 + y^2 + z^2} = r\left(1 + \frac{ax}{r^2}\right)^{\frac{1}{2}}$$

By substitution, the potential function becomes:

$$\Omega(p) = \frac{m}{r}\left[\left(1 - \frac{ax}{r^2}\right)^{-\frac{1}{2}} - \left(1 + \frac{ax}{r^2}\right)^{-\frac{1}{2}}\right] = \frac{ma}{r^2} \cdot \frac{x}{r}$$

Assuming is $\boldsymbol{\theta}$ the angle between the two vectors $\underset{r}{\rightarrow}, \underset{M}{\rightarrow}$ then $\boldsymbol{cos\theta} = \frac{\boldsymbol{x}}{\boldsymbol{r}}$, and the potential function becomes:

$$\Omega(p) = \frac{M \cos\theta}{r^2} = \frac{\vec{M} \bullet \vec{r}}{r^3}$$

Magnetic field of a small magnet:

The magnetic field strength of a small magnet at point $\boldsymbol{p}$ is determined from the following relationship:

$$\vec{H} = -\nabla\Omega(p) = -\nabla\left(\frac{M \cos\theta}{r^2}\right)$$

Thus, the field component in the direction of increasing $\boldsymbol{r}$ is given as:

$$H_r = -\frac{\partial\Omega}{\partial r} = \frac{2M \cos\theta}{r^3}$$

The field component in the direction of increasing the angle $\boldsymbol{\theta}$ is given by:

$$H_\theta = -\frac{1}{r} \cdot \frac{\partial\Omega}{\partial\theta} = \frac{H \sin\theta}{r^3}$$

The magnetic field can also be placed with the formula:

$$\vec{H} = -\nabla\Omega = -\nabla\left(\frac{\vec{M} \bullet \vec{r}}{r^3}\right) = -\frac{1}{r^3}\nabla\left(\vec{M} \bullet \vec{r}\right) - \left(\vec{M} \bullet \vec{r}\right)\nabla\left(\frac{1}{r^3}\right)$$

$$= -\frac{\vec{M}}{r^3} + \frac{3\left(\vec{M} \bullet \vec{r}\right)}{r^3}\vec{r}$$

And be:

$$H_M = -\frac{M}{r^3} \quad , \quad H_r = \frac{3M \cos\theta}{r^3}$$

Magnetic materials:

Elements of these materials are small magnets. These tiny magnets are randomly distributed. If you put a material that can be magnetized in a magnetic field, then there is a modification in the small magnets so that the direction of the magnetic moment vector of each small magnet is in the direction of the magnetic field in which this material that can be magnetized is located, and then the material becomes magnetized.

To study such materials, we use a vector called the magnetization intensity vector, denoted by the symbol $\vec{I}$, and it is defined as the vector of the magnetic moment of the unit volumes of the magnetized material. There is a relationship (practically demonstrated for some magnetized materials) between the field strength $\vec{H}$ and the vector $\vec{I}$ of the magnetized material by the formula:

$$\vec{I} = k\vec{H}$$

Where is $\boldsymbol{k}$ a constant and is called the magnetic susceptibility parameter. Associates with the two vectors $\underset{I}{\rightarrow}$, $\underset{H}{\rightarrow}$, another vector $\underset{B}{\rightarrow}$

The vector of magnetic induction is called the relationship:

$$\vec{B} = \vec{H} + 4\pi\vec{I} = \mu\vec{H}$$

Where $\mu = 1 + 4\pi k$ it is called the magnetic permeability coefficient. The vector of magnetic induction achieves:

$$\nabla \bullet \vec{B} = 0$$

Vector potential:

From the relationship $\nabla \bullet \vec{B} = 0$, the vector of magnetic induction can be placed on the image:

$$\vec{B} = \nabla \wedge \vec{A}$$

The vector $\vec{A}$ is called the vector potential. If it $\vec{A}'$ is a vector potential then $\vec{A}' + \nabla\Psi$ it also represents a vector potential (since Ψ it is a scalar function), and gives the same vector of magnetic induction, because if it is $\vec{A} = \vec{A}' + \nabla\Psi$, then:

$$\nabla \wedge \vec{A} = \nabla \wedge \vec{A}'$$

Therefore, to determine the vector $\vec{A}$ from which the vector of magnetic induction is required to be solely determined, a condition (constraint) must be placed on the vector $\vec{A}$. From the relationship $\vec{A} = \vec{A}' + \nabla\Psi$ we find that: $\nabla \bullet \vec{A} = \nabla \bullet \vec{A}' + \nabla^2\Psi$ We will choose the standard function Ψ so that it fulfills the condition: $\nabla^2\Psi = -\nabla \bullet \vec{A}'$ then the vector of magnetic induction is given in the form: $\vec{B} = \nabla \wedge \vec{A}$ and the vector $\vec{A}$ fulfills the condition: $\nabla \bullet \vec{A} = 0$

Example (1):

Prove that the vector potential of a small magnet at a point is: $\vec{A} = \dfrac{\vec{M} \wedge \vec{r}}{r^3}$ Where $\vec{M}$ is the magnet moment vector of the magnet, $\vec{r}$ the point position vector with respect to the magnet.

The solution :

Assume that the axis of the small magnet is applied to the axis oz so that the torque vector of the small magnet is in the shape $\vec{M} = (0, 0, M)$, and the vector $\vec{A}$ becomes:

$$\vec{A} = \frac{\vec{M} \wedge \vec{r}}{r^3} = \left(\frac{-My}{r^3}, \frac{Mx}{r^3}, 0\right)$$

From it we find that:

$$\nabla \wedge \vec{A} = \left(\frac{3Mxz}{r^5}, \frac{3Myz}{r^5}, \frac{3Mz^2}{r^5} - \frac{M}{r^3}\right)$$

Where $\vec{M} \bullet \vec{r} = Mz$ as, this vector can be placed in the form:

$$\nabla \wedge \vec{A} = \left(\frac{3Mxz}{r^5}, \frac{3Myz}{r^5}, \frac{3Mz^2}{r^5} - \frac{M}{r^3} \right)$$

This vector represents the magnetic field strength of the small magnet at a point $p(r)$, and the vector $\vec{A}$ fulfills the condition :

$$\nabla \bullet \vec{A} = \frac{\partial}{\partial x}\left(\frac{-My}{r^3} \right) + \frac{\partial}{\partial y}\left(\frac{Mx}{r^3} \right) + 0 = 0$$

Therefore, the vector $\vec{A}$ represents the vector potential of the small magnet.

Example (2):

Prove that the vector potential of a constant magnetic field (where the magnetic field $\vec{H}$ is in the direction of the axis oz) can be put into the picture: $\vec{A} = \left[0, -aHz, (1-a)Hy \right]$ where a is constant.

The solution :

The rotation vector of a vector $\vec{A}$ is:

$$\nabla \wedge \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & -aHz & (1-a)Hy \end{vmatrix} = (H, 0, 0)$$

The vector $\vec{A}$ fulfills the condition:

$$\nabla \bullet \vec{A} = 0 + \frac{\partial}{\partial y}\left(-aHz\right) + \frac{\partial}{\partial z}\left(1-a\right)Hy = 0$$

From this we conclude that the vector $\vec{A}$ is the vector potential of the constant magnetic field $\vec{H}$ where $\vec{H} = \vec{B}$.

Example (3):

Using cylindrical coordinates, prove that if the magnetic field $\vec{H}$ is in the direction of the z axis, the vector potential takes the form:

$$\vec{A} = \left(0, \frac{1}{2}H\rho, 0\right)$$

The solution :

The rotation vector $\vec{A}$ of the given vector in cylindrical coordinates is as:

$$\nabla \wedge \vec{A} = \frac{1}{\rho}\begin{vmatrix} \vec{e}_\rho & \rho\vec{e}_\phi & \vec{e}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & \frac{1}{2}A\rho^2 & 0 \end{vmatrix} = \left(0, 0, H\right)$$

The vector $\vec{A}$ fulfills the condition:

$$\nabla \bullet \vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \phi}\left(\frac{1}{2} H \rho\right) = 0$$

From this it follows that the vector $\vec{A}$ represents the vector potential of a constant magnetic field in the direction of the axis oz where $\vec{B} = \vec{H}$

Example (4):

Prove that for a magnetic field $\vec{H}$ parallel to the axis of the angle θ in polar spherical coordinates, the vector potential can be positioned as:

$$\vec{A} = \left(0, 0, \frac{1}{2} Hr \sin\theta\right)$$

The solution :

The rotation vector of a vector $\vec{A}$ in spherical polar coordinates takes the shape:

$$\nabla \wedge \vec{A} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & r\sin\theta \vec{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & \frac{1}{2} Hr^2 \sin^2\theta \end{vmatrix}$$

$$= H\left(\cos\theta, -\sin\theta, 0\right)$$

The vector $\vec{A}$ fulfills the condition:

$$\nabla \bullet \vec{A} = 0 + 0 + \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi}\left(\frac{1}{2} Hr \sin\theta\right) = 0$$

Therefore, the vector $\nabla \wedge \vec{A}$ represents the strength of the magnetic field $\vec{H}$, and its first component is in the direction of increasing r and the second component is in the direction of decreasing angle. That is, the magnetic field is in the direction of the axis and is the axis of the angle $\boldsymbol{\theta}$.

The potential for regular magnetic shell:

We consider a crust of small thickness, and assume that n is the number of small magnetic positive poles per unit surface of this shell. Assume that $\vec{M}$ is the torque vector of each small magnet. Assume that the small magnets are uniformly distributed so that the positive poles are applied to one surface of the cortex while the negative poles are applied to the other surface of the cortex. To find the magnetic potential produced by the shell at point, we select the element from .the surface of the magnetic shell. Suppose r is the position of the point p with respect to $\boldsymbol{dS}$
The magnetic potential produced by the surface element at the point (placed on the side of the positive poles) is as:

$$d\Omega(p) = \frac{MndS \cos\theta}{r^2}$$

We draw a cone with the base element $\boldsymbol{dS}$ and the vertex of the point $\boldsymbol{p}$, so it is:

$$\frac{dS\cos\theta}{r^2} = d\omega$$

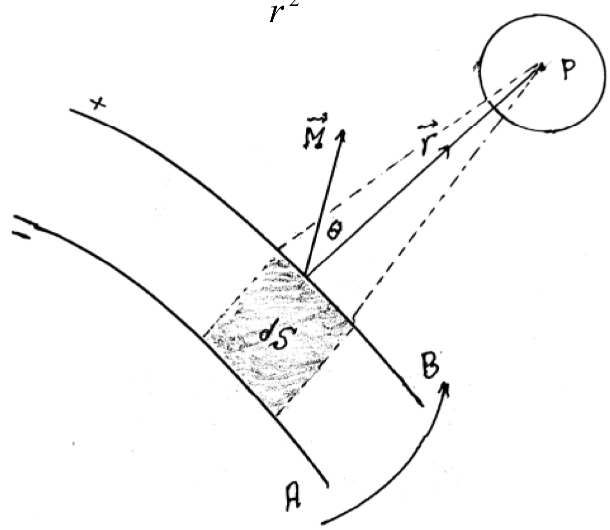


Fig. (2-2)

Where $\boldsymbol{d\omega}$ is the solid angle of the cone with its base $\boldsymbol{dS}$. Therefore, the total magnetic potential created by the shell at point $\boldsymbol{p}$ is given by the relation:

$$\Omega(p) = \Omega_+ = \iint_S d\Omega = Mn\iint_S d\omega = Mn\omega$$

Where $d\omega$ is the solid angle made by the shell at point $\boldsymbol{p}$. Putting $Mn = \Phi$ we find that:

$$\Omega_+ = \Phi\omega$$

Where Φ represents the magnetic moment of a unit surface (per unit area) or magnetic intensity. When the point is on the other side (i.e. on the negative poles side) then:

$$\Omega_- = -\Phi\omega$$

The work required to transfer the positive poles from the point (located on the surface of the negative poles) to the point p (located on the surface of the positive poles) is given by the relation:

$$W = \int_A^B \vec{F} \bullet d\vec{l} = \int_A^B \vec{B} \bullet d\vec{l} = \mu \int_A^B \vec{H} \bullet d\vec{l}$$

$$= \mu\left(\Omega_B - \Omega_A\right) = \mu\Phi\left(\omega_B + \omega_A\right)$$

$\vec{B} = \mu\vec{H}$ It is the vector of magnetic induction $d\vec{l}$, the displacement element, and the two solid angles ω_B , ω_A on the periphery of the magnetic shell. When the magnetic shell is of very small thickness, then point ***A*** applies roughly to point ***B***, and in this case:

$\omega_A + \omega_B = 2\pi$ The work then becomes as:

$$W = 2\pi\mu\Phi$$

Example (5):

A magnetic doublet of streng $\vec{M}$ is held in a regular magnetic field $\vec{H}$ so that the axis of the magnet is parallel to the magnetic field. Show that the resulting field will vanish on a circle or at two points. Find the ratio between the diameter of the circle and the distance between the two points.

The solution :

The resultant of the magnetic field $\vec{H}$ and the field arising from the dual at point $\boldsymbol{p}$ $(\boldsymbol{r})$ is:

$$\vec{H}_1 = \vec{H} - \frac{\vec{M}}{r^3} + \frac{3\vec{M} \bullet \vec{r}}{r^5}\vec{r}$$

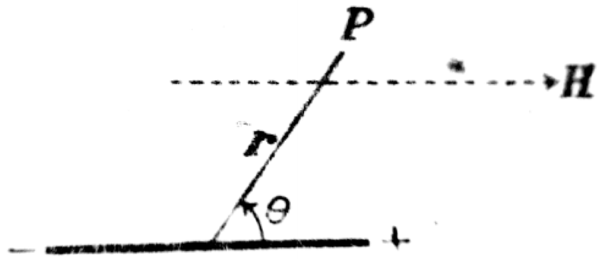


Fig. (2-3)

The two fields $\vec{H}$, $\vec{M}$ have the same direction, and therefore the field $\vec{H}_1$ is the sum of two vectors, one in the direction of the vector $\vec{H}$ and the other in the direction of the vector $\vec{r}$. The resultant vector $\vec{H}_1$ decays when it fades into a coefficient $\vec{r}$, $\vec{H}$, i.e. when it is:

$$H - \frac{M}{r^3} \quad , \quad \cos\theta = 0$$

Of which we find that: $r = \left(\frac{M}{H}\right)^{\frac{1}{3}}$ and r is perpendicular to $\vec{M}$. That is, the field $\vec{H}_1$ vanishes at all points of the circle of radius $\left(\frac{M}{H}\right)^{\frac{1}{3}}$, and this circle is in a perpendicular plane $\vec{M}$, meaning that the circle is perpendicular to the axis of the double, and on the other hand if $\vec{M}$ it is in an opposite direction to the vector $\vec{H}$, then it can be placed $\vec{M} = -\lambda\vec{H}$ where $\lambda \geq 0$, and the resulting field is then:

$$\vec{H}_1 = \left(1 + \frac{\lambda}{r^3}\right)\vec{H} - \frac{3\lambda\left(\vec{H} \bullet \vec{r}\right)}{r^5}\vec{r}$$

It is clear from this relationship that the coefficient $\vec{H}$ is not equal to zero, and it disappears $\vec{H}_1$ only when the vectors are parallel $\vec{H}$, $\vec{r}$. When it is $\vec{H}$, $\vec{r}$ in the same direction it will $\vec{H}_1$ vanish at the point (on the dual axis whose position is ***r*** which achieves):

$$\left(1 + \frac{\lambda}{r^3}\right)H - \frac{3\lambda H}{r^3} = 0$$

Among them $r^3 = 2\lambda$, we find that: $\vec{H}$, $\vec{r}$ When they are in opposite directions, the position of the point is determined by:

$$\left(1+\frac{\lambda}{r^3}\right)H-\frac{3\lambda H}{r^3}=0$$

That is $r^3 = 2\lambda$. That is, there are two points at which the field disappears $\vec{H}$. The distance between the two points is $2r = 2(2\lambda)^{\frac{1}{3}}$

. And the diameter of the perpendicular circle on the double axis is: $2r' = 2\left(\frac{M}{H}\right)^{\frac{1}{3}}$

That is: $2r = 2(2\lambda)^{\frac{1}{3}}$. The ratio between the diameter of the circle and the distance between the two points is:

$$\frac{2r'}{2r}=\frac{2(\lambda)^{\frac{1}{3}}}{2(2\lambda)^{\frac{1}{3}}}=\frac{1}{(2)^{\frac{1}{3}}}$$

Example (6):

Two small magnets of moments $\vec{M}$, $\vec{M}'$ are fixed at the vertices B , C of an equilateral ABC triangle so that the axis of the corresponding doublet angle bisected, and then placed a small magnet at ***A*** so that it rotated freely. Prove that the angle between the axis of the small magnet and the angle A bisector is:

$$\tan^{-1}\left(\frac{\sqrt{3}}{7}\bullet\frac{M-M'}{M+M'}\right)$$

The solution :

The double M , M' at two points B , C produces at the point the two magnetic fields:

$$\vec{H}=\left(H_r,H_\theta\right)\quad,\quad \vec{H}'=\left(H'_r,H'_\theta\right)$$

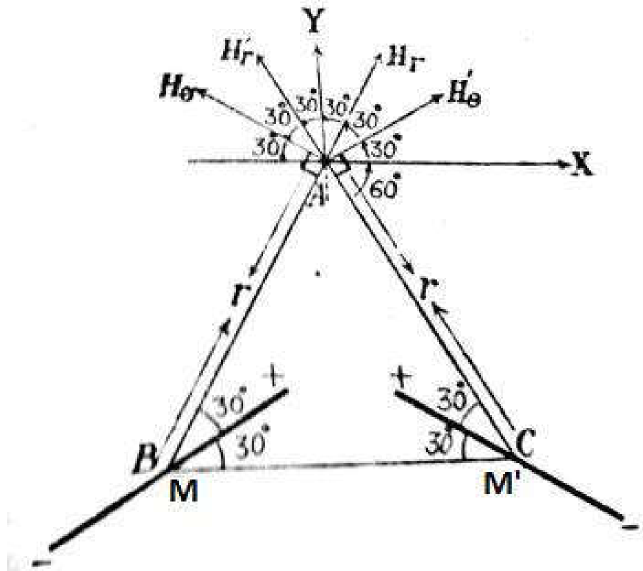


Fig. (2-4)

$$H_r = \frac{2M\ \cos 30^o}{r^3} = \frac{\sqrt{3}M}{r^3} \quad , \quad H_\theta = \frac{M\ \sin 30^o}{r^3} = \frac{M}{2r^3}$$

$$H_r' = \frac{2M'\cos 30^o}{r^3} = \frac{\sqrt{3}M'}{r^3} \quad , \quad H_\theta' = \frac{M'\sin 30^o}{r^3} = \frac{M'}{2r^3}$$

The components of the sum of the two fields $\vec{H}$, $\vec{H}'$ are X , Y where:

$$X = \left(H_r - H_r'\right)\cos 60^o + \left(H_\theta' - H_\theta\right)\cos 30^o = \frac{\sqrt{3}\left(M - M'\right)}{4r^3}$$

$$Y = \left(H_r + H_r'\right)\cos 30^o + \left(H_\theta + H_\theta'\right)\cos 60^o = \frac{7\left(M + M'\right)}{4r^3}$$

The line of force at $\boldsymbol{A}$ (the direction of the small magnet at $\boldsymbol{A}$ in equilibrium) is inclined on the bisector of the angle $\boldsymbol{A}$ at the angle θ where:

$$\tan\theta = \frac{X}{Y} = \frac{\sqrt{3}}{7} \bullet \frac{M - M'}{M + M'} \qquad \therefore \quad \theta = \tan^{-1}\left(\frac{\sqrt{3}}{7} \bullet \frac{M - M'}{M + M'}\right)$$

Example (7):

A magnetic crust of uniform intensity Φ bounded by a circular curve of radius $\boldsymbol{a}$. Find the magnetic field for this crust at a point on the axis of the shell that is from the center of the circle $\boldsymbol{z}$. Then find the mechanical force acting on a small magnet located on the axis of the shell (the axis of the small magnet applies to the axis of the shell $\boldsymbol{z}$).

The solution :

First, suppose that point $\boldsymbol{p}$ is away from point $\boldsymbol{o}$ by distance $\boldsymbol{z}$. The angle formed at point $\boldsymbol{p}$ by a right circular cone is:

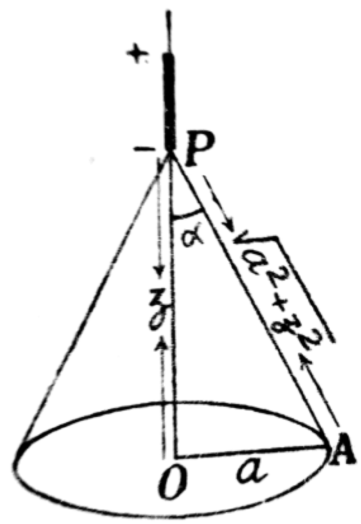


Fig. (2-5)

$$\omega = 2\pi\left(1-\cos\alpha\right)$$

The magnetic potential of the crust at ***p*** is:

$$\Omega\left(p\right) = \Phi\omega = 2\pi\Phi\left(1-\frac{z}{\sqrt{a^2+z^2}}\right)$$

The field strength of the crust when given as:

$$H = -\frac{\partial\Omega}{\partial z} = \frac{2\pi\Phi a^2}{\left(a^2+z^2\right)^{\frac{3}{2}}}$$

Second: Suppose that $\vec{M}$ the torque vector of the small magnet applied to the axis ***z*** is at ***p***, and assuming that the poles are two $\left(+m\right),\left(-m\right)$ distances from the point: $z+dz\ ,z$

The two forces acting on the magnetic poles are: $+m\left(\vec{H}+d\vec{H}\right), -m\vec{H}$ The force acting on the small magnet (the sum of the two forces) is:

$$F=-mH+m\left(H+dH\right)=mH=-\frac{6\pi\Phi a^2 z\left(mdz\right)}{\left(a^2+z^2\right)^{\frac{5}{2}}}=-\frac{6\pi\Phi a^2 Mz}{\left(a^2+z^2\right)^{\frac{5}{2}}}$$

And a negative sign means that the force acting is a force of attraction.

Example (8):

A regular magnetic crust (shell) is bounded by two concentric circules of center radii ***a,b***.

Show that it exerts no force at a point distance: $\frac{(ab)^{\frac{2}{3}}}{\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}}$ From its center along its axis, and the total work done in bringing a small magnet to this point from infinity is also zero.

The solution :

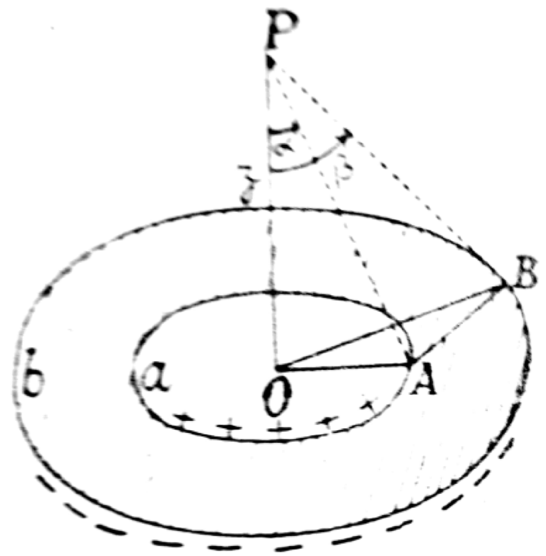


Fig. (2-6)

The magnetic potential of the shell at ***p*** is as:

$$\Omega = \Phi\left(\omega_b - \omega_a\right) = 2\pi\Phi\left\{\left(1-\cos\beta\right)-\left(1-\cos\alpha\right)\right\} = 2\pi\Phi\left(\cos\alpha - \cos\beta\right)$$

$$\cos\alpha = \frac{z}{\sqrt{z^2+a^2}} \quad , \quad \cos\beta = \frac{z}{\sqrt{z^2+b^2}}$$

The magnitude of the magnetic field at point ***p*** must be:

$$H = -\frac{\partial\Omega}{\partial z} = 2\pi\Phi\left\{\frac{b^2}{\left(b^2+z^2\right)^{\frac{3}{2}}} - \frac{a^2}{\left(a^2+z^2\right)^{\frac{3}{2}}}\right\}$$

The magnetic field $\vec{H}$ diminishes when the condition is met:

$$\frac{b^2}{\left(b^2+z^2\right)^{\frac{3}{2}}}=\frac{a^2}{\left(a^2+z^2\right)^{\frac{3}{2}}} \qquad \therefore a^4\left(b^2+z^2\right)^3=b^4\left(a^2+z^2\right)^3$$

Among them we find that:

$$z=z'=\frac{(ab)^{\frac{2}{3}}}{\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}}$$

That is, the field $\vec{H}$ diminishes at the point $\boldsymbol{p'}$ which is by the distance $\boldsymbol{z'}$ from the center $\boldsymbol{o}$.

The total work done by the magnetic field to move the tiny magnet from infinity to a certain point (representing the potential energy of the magnet) is given by the relation: $W=-\vec{M}\bullet\vec{H}$
And at point $\boldsymbol{p'}$ it is:

$$W=-\vec{M}\bullet\vec{H}'=0$$

Because $\vec{H}'=\vec{0}$ at the point $\boldsymbol{p'}$.

Chapter 3 : Varying Electromagnetic Fields

Introduction :

Now we are discussing the state of varing electromagnetic fields with time. We will present two new concepts: an electric field produced from a variable magnetic field, and this concept resulted from the experimental research of Michael Faraday. The second concept is a magnetic field that arises from an electric field that varies with time.

Faraday s Law

After Ørsted (1820 CE) showed that an electric current affected a compass needle. Ferday declared that if an electric current can produce a magnetic field, then the magnetic field must be able to produce an electric current.

Significance of fields It can be said that a magnetic field that varies with time produces an electric motive force (***e m f***) that creates a closed circuit current. An individual law is formulated mathematically in the formula:

$$emf = -\frac{dN}{dt} \qquad (1)$$

Where ***N*** is the total magnetic flux through the cross section of a magnetic circuit.

That:

$$N = \iint_S \vec{B}.d\vec{S} \qquad (2)$$

Where $\vec{B}$ magnetic induction. Relationship (1) shows that the electric motive is a scalar quantity.

This standard quantity is also known as:

$$emf = \oint \overrightarrow{E}.\, d\vec{l} \qquad (3)$$

Where $\underset{E}{\rightarrow}$ the electric field strength.

In general, the electric motive force changes if the path shape changes from equations (1) - (3), we find that:

$$\oint \overrightarrow{E}.\, d\vec{l} = -\frac{d}{dt}\iint_S \overrightarrow{B}.\, d\vec{S} \qquad (4)$$

We will consider here that the path is static, the equation (4) in the image:

$$\iint_S (\nabla \wedge \vec{E}).\, d\vec{S} = -\iint_S \frac{\partial \vec{B}}{\partial T}.\, d\vec{S} \qquad (5)$$

Applying Stokes' theorem, equation (5) takes the picture:

$$emf = \oint \overrightarrow{E}.\, d\vec{l} = -\iint \frac{\partial \vec{B}}{\partial T}.\, d\vec{S}$$

Since dS is an optional surface element, we obtain:

$$\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (6)$$

This equation represents one of Maxwell's equations. Equation (6) shows that a magnetic field, changing with time, creates an electric field. This electric field has the property of rotation and its linear integration around a generally closed path that is not equal to zero.

If the vector of magnetic induction $\vec{B}$ does not depend on time, then equations (5) and (6) translate respectively to the two electrostatic equations:

$$\oint \vec{E}.d\vec{l} = 0 \quad (7)$$

$$\nabla \wedge \vec{E} = \vec{0} \quad (8)$$

Displacement Current - Ampere's Circular Law:

Ampère's circular law in the case of magnetic fields that do not depend on time can be written in the mathematical form:

$$\nabla \wedge \vec{H} = \vec{J} \quad (9)$$

Where the strength of the magnetic field is the vector of the current density. In the case of magnetic field that varies with time, equation (9) is incorrect and this is evident because when we multiply both sides of equation (9), it is standard in the influence, that is,:

$$\nabla.\left(\nabla \wedge \vec{H}\right) = 0 = \nabla.\vec{J}$$

Which leads to the result: $\nabla \bullet \vec{J} = 0$, and this result conflicts with the equation of continiuty:

$$\nabla.\vec{J} = -\frac{\partial \rho}{\partial t}$$

Therefore, equation (9) is correct only if it is: $\frac{\partial \rho}{\partial t} = 0$ Suppose we add an unknown term $\vec{G}$ to the right-hand side of equation (9). Then (9) takes the picture:

$$\nabla \wedge \vec{H} = \vec{J} + \vec{G} \tag{9'}$$

Comparing this equation with the connection equation, we find that:

$$\nabla \bullet \vec{G} = \frac{\partial \rho}{\partial t}$$

Using the equation $\nabla \bullet \vec{D} = \rho$ we get the simplest solution to the vector $\vec{G}$ in the form:

$$\vec{G} = \frac{\partial \vec{D}}{\partial t}$$

Therefore, Ampère's Circular Law takes the following differential form:

$$\nabla \wedge \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{10}$$

Equation (10) was not inferred, but rather it is a mathematical form of Ampère's circular law that we obtained and does not contradict the connection equation. Equation (10) is also compatible with all other results and it is an acceptable equation as we usually do with any experimental law and the equation deduced from it. Equation (10) is another one of Maxwell's equations. The additional term on the right-hand side of equation (10) i.e. the term $\frac{\partial \vec{D}}{\partial t}$ has units of current density (ampere per square meter) and because it results from the time change of the displacement vector $\vec{D}$, it is called the displacement current density and denoted by the symbol $\vec{J}_d = \frac{\partial \vec{D}}{\partial t}$. As for the current density vector $\vec{J}$, it is the conduction current density $\sigma\vec{E}$ (which results from the movement of charges) as well as the load current $\rho\vec{v}$.

In the case of a non-conductive medium in which there is no volumetric charge density ($\rho = 0$), equation (10) yields the following simple form:

$$\nabla \wedge \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

The total displacement current passing through any surface ***S*** must integrate:

$$I = \iint_S \vec{J}_d . d\vec{S} = \iint_S \frac{\partial \vec{D}}{\partial t} . d\vec{S} \qquad (11)$$

We can obtain the time-varying form of Ampère's circular integral law by integrating equation (10) on the surface ***S***

$$\iint_S (\nabla \wedge \vec{H}) . d\vec{S} = \iint_S \vec{J} . d\vec{S} + \iint_S \frac{\partial \vec{D}}{\partial t} . d\vec{S} = I + I_d$$

Applying Stokes' theorem, the previous equation takes the formula:

$$\oint \overrightarrow{H} . d\vec{l} = I + I_d \qquad (12)$$

Maxwell's equations:

We had previously obtained two Maxwell's equations for time-varying fields in the two equations:

$$\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (13)$$

$$\nabla \wedge \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad (14)$$

The remaining two equations are unchanged from their time-changing images, namely:

$$\nabla . \vec{D} = \rho \qquad (15)$$

$$\nabla . \vec{B} = 0 \qquad (16)$$

The four preceding equations are Maxwell's equations that represent the basis for studying electromagnetic theory. They are partial differential equations that relate electric and magnetic fields to each other and their sources (charge and current density). Recognizing the integral forms of Maxwell's equations is usually easier in terms of the experimental laws from which these equations were obtained by a generalization process (experiments should deal with physical macroscopic quantities). So, the results of these experiments are expressed in complementary relationships.

We will now try to find the integral forms of the previous Maxwell's equations (13) - (16). By integrating equation (13) to a surface and applying Stokes' theorem, we obtain:

$$\oint \overrightarrow{E}.d\vec{l} = -\iint_S \frac{\partial \vec{B}}{\partial t}.d\vec{S} \tag{17}$$

This equation is called an odd law. By performing the same integral operation on equation (14), we find that:

$$\oint \overrightarrow{H}.d\vec{l} = I + \iint_S \frac{\partial \overrightarrow{D}}{\partial t}.d\vec{S} \tag{18}$$

This is called Ampère's Circular Law. By carrying out the volume integration of equation (15), considering that the total volume V is surrounded by the surface, we find that:

$$\iiint_V \left(\nabla.\overrightarrow{D}\right) d\tau = \iiint_V \rho d\tau$$

And using the Gaussian theorem to convert the scalar integral to a surface integration, the previous equation takes the formula:

$$\oiint_S \overrightarrow{D}.d\vec{S} = \iiint_V \rho d\tau \tag{19}$$

By conducting the same integration process prior to equation (16), we obtain:

$$\oiint_S \vec{B}.d\vec{S} = 0 \tag{20}$$

When using electrostatic units to measure $\boldsymbol{\rho}$ and vectors $\underset{\boldsymbol{D}}{\rightarrow}$, $\underset{\boldsymbol{E}}{\rightarrow}$, $\underset{\boldsymbol{J}}{\rightarrow}$, as well as using electromagnetic units to measure vectors $\underset{\boldsymbol{H}}{\rightarrow}$, $\underset{\boldsymbol{B}}{\rightarrow}$, Maxwell's equations for variable electromagnetic fields take the following differential images:

$$\nabla.\vec{D} = 4\pi\rho \tag{21}$$

$$\nabla.\vec{B} = 0 \tag{22}$$

$$\nabla \wedge \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{23}$$

$$\nabla \wedge \vec{H} = 4\pi\vec{J} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t} \tag{24}$$

Where c is the speed of light in space.

Electromagnetic potentials in Maxwell's equations:

From the equation $\nabla \bullet \vec{B} = 0$, the vector of magnetic induction $\vec{B}$ can be placed in the formula $\vec{B} = \nabla \wedge \vec{A}$. Where $\vec{A}$ it represents the vector potential. Substituting into Maxwell's equation (23) we get:

$$\nabla \wedge \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\nabla \wedge \vec{A}$$

Accordingly, it is possible to put:

$$\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -\nabla \Phi$$

That is, the electric field strength can be given as:

$$\vec{E} = -\nabla \Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \qquad (25)$$

Multiplying both sides of this scalar relationship by the operator ∇ using the equation $\nabla \bullet \vec{D} = 4\pi\rho$ where $\vec{D} = K\vec{E}$ we get:

$$\nabla^2 \Phi + \frac{1}{2}\frac{\partial}{\partial t}\nabla . \vec{A} = -\frac{4\pi}{K}\rho \qquad (26)$$

Where we assumed here that the ethotropic medium is homogeneous (where μ ,K are constant quantities). From Maxwell's equation (24) and using the relationships:

we find that :

$$\nabla \wedge (\nabla \wedge \vec{A}) = 4\pi\mu\vec{J} + \frac{\mu K}{c}\frac{\partial \vec{E}}{\partial t} \qquad (27)$$

And by using the following vector identity

$$\nabla \wedge (\nabla \wedge \vec{A}) = \nabla(\overline{\nabla . \vec{A}}) - \nabla^2 \vec{A}$$

Also, using equation (25), equation (27) takes the formula:

$$\nabla(\nabla\vec{A}) - \nabla^2\vec{A} = 4\pi\mu\vec{J} - \frac{\mu K}{c}\nabla\left(\frac{\partial\Phi}{\partial t}\right) - \frac{\mu K}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} \tag{28}$$

Then using the following condition:

$$\nabla\vec{A} + \frac{\mu K}{c}\frac{\partial\Phi}{\partial t} = 0 \tag{29}$$

Equation of guage invarience

Therefore, equations (26) and (28) become, respectively, in two forms:

$$\nabla^2\Phi - \frac{\mu K}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\frac{4\pi}{K}\rho \tag{30}$$

$$\nabla^2\vec{A} - \frac{\mu K}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -4\pi\mu\vec{J} \tag{31}$$

Equations (30) and (31) are the wave equations for the related scalar potential Φ and the vector potential $\vec{A}$

Example (1) :

Show that the function: $f\left(x,y,z\right)=\frac{1}{r}\rho\left(\xi,\eta,\zeta,t-\frac{r}{c}\right)$ satisfies the quation $\Box f=0$

, $\Box=\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}$, r Is the distance between the two points $\left(\xi,\eta,\zeta\right)$, $\left(x,y,z\right)$

c is the speed of light in space.

The solution :

To find $\nabla^2 f$, we assign first ∇f

$$\nabla f=\frac{1}{r}\nabla\rho+\rho\nabla\left(\frac{1}{r}\right)$$

Assume that $\rho'=\frac{\partial\rho}{\partial u}$ where $u=t-\frac{r}{c}$. So we get:

$$\nabla f=\frac{1}{r}\left(-\frac{\rho}{c}\nabla r\right)-\frac{\rho}{r^2}\nabla r=-\left(\frac{\rho'}{cr}+\frac{\rho}{r^2}\right)\nabla r$$

Since $\nabla r=\frac{\vec{r}}{r}$ any unit of vectors is in the direction of the vector $\vec{r}$, and therefore ∇f , the formula is taken:

$$\nabla f=-\left(\frac{\rho'}{cr^2}+\frac{\rho}{r^3}\right)\vec{r}$$

$$\nabla^2 f=\nabla.\nabla f=-\nabla\left(\frac{\rho'}{cr^2}+\frac{\rho}{r^3}\right)\vec{r}$$

$$= -\left[\left(\frac{\rho'}{cr^2}+\frac{\rho}{r^3}\right)\nabla\vec{r}+\vec{r}\nabla\left(\frac{\rho'}{cr^2}+\frac{\rho}{r^3}\right)\right]$$

$$= -\left[3\left(\frac{\rho'}{cr^2}+\frac{\rho}{r^3}\right)-\vec{r}\left(\frac{\rho''}{c^2r^2}+\frac{2\rho'}{cr^3}+\frac{3\rho}{r^4}+\frac{\rho'}{cr^3}\right)\frac{\vec{r}}{r}\right]=\frac{\rho''}{c^2r} \qquad (1)$$

The function ρ'' can be written with the formula:

$$\rho''=\frac{\partial^2\rho}{\partial u^2}=\frac{\partial^2\rho}{\partial t^2}=r\frac{\partial^2 f}{\partial t^2} \qquad (2)$$

By substituting ρ'' from equation (2) into equation (1), we find that:

$$\nabla^2 f=\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} \qquad \therefore\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)f=\square f=0 \qquad (3)$$

The differential operator $\square$ is called the D'Alembert operator. And equation (3) is called the . wave equation.

Example (2) :

Write Maxwell's equations for free space, and show that vector potential: $\vec{A}=\frac{f'(u)}{cr}\vec{k}$ represents a solution to these equations. Where $f(u)$ a function of the variable . $\vec{k}$, $r^2=x^2+y^2+z^2$, $u=t-\frac{r}{c}$ a unit vector in the direction of the axis z . Find the components of the electric field and the magnetic field, and show that the related scalar potential Φ is in the form:

$$\Phi=z\left(\frac{f'(u)}{cr^2}+\frac{f(u)}{r^3}\right)$$

The solution :

In the case of free space,: $K=1, \mu=1, \rho=0, \vec{J}=\vec{0}$ Therefore, Maxwell's equations in this case take the form:

$$\nabla \bullet \vec{E}=0 \quad , \quad \nabla \bullet \vec{B}=0$$

$$\nabla \wedge \vec{E}=-\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \quad , \quad \nabla \wedge \vec{B}=\frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

In the case of free space, the wave equation achieved by the vector potential takes the form:

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}=\vec{0}$$

We will now demonstrate that the vector potential given in the example satisfies the wave equation (1). To do so, we find the following quantities:

$$c\frac{\partial \vec{A}}{\partial x}=\frac{\partial}{\partial x}\left(\frac{f'(u)}{cr}\vec{k}\right)=\left[f'\cdot\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right)+\frac{f''}{r}\cdot\left(-\frac{1}{c}\frac{\partial r}{\partial x}\right)\right]\vec{k}$$

$$=-\left(\frac{xf'}{r^3}+\frac{xf'''}{cr^2}\right)\vec{k}$$

By performing the partial differentiation process again, we obtain:

$$c\frac{\partial^2 \vec{A}}{\partial x^2}=-\left[\frac{r^2-3x}{r^5}\cdot f'+\frac{r^2-3x^2}{cr^4}\cdot f''-\frac{x^2}{c^2r^3}\cdot f'''\right]\vec{k}$$

Likewise, one can find $c\frac{\partial^2 \vec{A}}{\partial y^2}$, $c\frac{\partial^2 \vec{A}}{\partial z^2}$ In two formats:

$$c\frac{\partial^2 \vec{A}}{\partial y^2} = -\left[\frac{r^2 - 3y^2}{r^5} \cdot f' + \frac{r^2 - 3y^2}{cr^4} \cdot f'' - \frac{y^2}{c^2 r^3} \cdot f'''\right]\vec{k} \qquad (3)$$

$$c\frac{\partial^2 \vec{A}}{\partial z^2} = -\left[\frac{r^2 - 3z^2}{r^5} \cdot f' + \frac{r^2 - 3z^2}{cr^4} \cdot f'' - \frac{z^2}{c^2 r^3} \cdot f'''\right]\vec{k} \qquad (4)$$

Summing equations (2), (3), (4) we get:

$$\frac{\partial^2 \vec{A}}{\partial x^2} + \frac{\partial^2 \vec{A}}{\partial y^2} + \frac{\partial^2 \vec{A}}{\partial z^2} = \nabla^2 \vec{A} = \frac{f'''}{c^3 r} \cdot \vec{k} \qquad (5)$$

and whereas :

$$\frac{\partial^2 \vec{A}}{\partial t^2} = \frac{f'''}{cr} \cdot \vec{k} \qquad (6)$$

By substituting from equation (6) into equation (5), we find that:

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \vec{0}$$

That is, the vector given in the example fulfills the wave equation (1), that is, it represents a solution to Maxwell's equations in free space. To find the components of the magnetic field strength $\vec{H}$, we use $\vec{H} = \vec{B}$ form:

$$\nabla \wedge \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{f'}{cr} \end{vmatrix} \qquad (7)$$

From equation (7), we find that the magnetic field components are:

$$H_x = -y\left(\frac{f'}{r^3}+\frac{f''}{cr^2}\right) \ , \ H_y = x\left(\frac{f'}{r^3}+\frac{f''}{cr^2}\right) \ , \ H_z = 0$$

To find the associated standard voltage function Φ we use the condition:

$$\nabla\vec{A}+\frac{1}{c}\frac{\partial\Phi}{\partial t}=0$$

$$\frac{\partial\Phi}{\partial t}=-c\nabla\vec{A}=-c\nabla\left(\frac{f'}{cr}\vec{k}\right)=-\frac{\partial}{\partial z}\left(\frac{f'}{r}\right)=z\left(\frac{f'}{r^3}+\frac{f''}{cr^2}\right)$$

By integrating with respect to time, we get:

$$\Phi = z\left(\frac{f}{r^3}+\frac{f'}{cr^2}\right)+F(x,y,z) \tag{8}$$

Where $F(x,y,z)$ is an optional and can be selected function equal to zero, so the related scalar potential takes the form:

$$\Phi = z\left(\frac{f}{r^3}+\frac{f'}{cr^2}\right)$$

The strength of the electric field $\vec{E}$ should be determined by the form:

$$\vec{E}=-\nabla\Phi-\frac{1}{c}\frac{\partial\vec{A}}{\partial t} \tag{9}$$

$$\frac{\partial\vec{A}}{\partial t}=\frac{f''}{cr} \tag{10}$$

$$\nabla\Phi = z\nabla\left(\frac{f}{r^3}+\frac{f'}{cr^2}\right)+\left(\frac{f}{r^3}+\frac{f'}{cr^2}\right)\vec{k}$$

$$= -z\left(\frac{3f}{r^4}+\frac{3f'}{cr^3}+\frac{f''}{c^2r^2}\right)\nabla r+\left(\frac{f}{r^3}+\frac{f'}{cr^2}\right)\vec{k} \tag{11}$$

Then by substituting equations (10) and (11) into equation (9), we get the electric field components as:

$$E_x = xz\left(\frac{3f}{r^5} + \frac{3f'}{cr^4} + \frac{f''}{c^2r^3}\right)$$

$$E_y = yz\left(\frac{3f}{r^5} + \frac{3f'}{cr^4} + \frac{f''}{c^2r^3}\right)$$

$$E_z = z^2\left(\frac{3f}{r^5} + \frac{3f'}{cr^4} + \frac{f''}{c^2r^3}\right) - \left(\frac{f}{r^3} + \frac{f'}{cr^2} + \frac{f''}{c^2r}\right)$$

Example (3):

Verify that the magnetic field: $\vec{H} = \frac{1}{c}\frac{\partial}{\partial t}\left[(\nabla\Phi)\wedge\vec{k}\right]$ and the electric field: $\vec{E} = -\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}\vec{k} + \frac{\partial}{\partial z}\nabla\Phi$ satisfy Maxwell's equations in free space. Where $\vec{k}$ the unit of vectors in the direction of the axis z , and the function Φ fulfills the wave equation:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0$$

The solution :

In the case of free space Maxwell's equations are:

$$\nabla\bullet\vec{H} = 0 \quad , \quad \nabla\bullet\vec{E} = 0$$

$$\nabla\wedge\vec{E} = -\frac{1}{c}\frac{\partial\vec{H}}{\partial t} \quad , \quad \nabla\wedge\vec{H} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

We will prove that the two vectors $\vec{E}$, $\vec{H}$ mentioned in the example satisfy these equations as follows:

.i

$$\nabla\vec{H} = \frac{1}{c}\frac{\partial}{\partial t}\nabla\{(\nabla\Phi)\wedge\vec{k}\} = \frac{1}{c}\frac{\partial}{\partial t}\{\vec{k}[\nabla\wedge(\nabla\Phi)] - (\nabla\Phi)\nabla\wedge\vec{k}\} = 0$$

.ii

$$\nabla\vec{E} = -\frac{1}{c^2}\nabla\left(\frac{\partial^2\Phi}{\partial t^2}\vec{k}\right) + \frac{\partial}{\partial z}\nabla(\nabla\Phi) = -\frac{1}{c^2}\frac{\partial}{\partial z}\left(\frac{\partial^2\Phi}{\partial t^2}\right) + \frac{\partial}{\partial z}\nabla^2\Phi$$

$$= \frac{\partial}{\partial z}\left(\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}\right) = 0$$

.iii

$$\nabla\wedge\vec{E} = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\nabla\wedge\left(\Phi\vec{k}\right)\right] + \frac{\partial}{\partial z}[\nabla\wedge(\nabla\Phi)] = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\nabla\wedge\left(\Phi\vec{k}\right)\right]$$

$$= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[(\nabla\Phi)\wedge\vec{k} + \Phi\left(\nabla\wedge\vec{k}\right)\right] = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[(\nabla\Phi)\wedge\vec{k}\right]$$

$$= -\frac{1}{c}\frac{\partial}{\partial t}\left\{\frac{1}{c}\frac{\partial}{\partial t}\left[(\nabla\Phi)\wedge\vec{k}\right]\right\} = -\frac{1}{c}\frac{\partial}{\partial t}\{\vec{H}\}$$

$$\therefore\quad \nabla\wedge\vec{E} = -\frac{1}{c}\frac{\partial\vec{H}}{\partial t}$$

.iv

$$\nabla \wedge \vec{H} = \frac{1}{c}\frac{\partial}{\partial t}\left\{\nabla \wedge \left[(\nabla \Phi) \wedge \vec{k}\right]\right\}$$

$$= \frac{1}{c}\frac{\partial}{\partial t}\left\{(k\nabla)(\nabla \Phi) - [(\nabla \Phi)\nabla]\vec{k} + (\nabla \Phi)\nabla \vec{k} - \vec{k}\nabla^2 \Phi\right\}$$

$$= \frac{1}{c}\frac{\partial}{\partial t}\left\{\frac{\partial}{\partial z}(\nabla \Phi) - \vec{k}\nabla^2 \Phi\right\} = \frac{1}{c}\frac{\partial}{\partial t}\left\{\vec{E} + \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2}\vec{k} - \vec{k}\nabla^2 \Phi\right\}$$

$$= \frac{1}{c}\frac{\partial}{\partial t}\left\{\vec{E} - \vec{k}\left(\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2}\right)\right\} \qquad \therefore \;\; \nabla \wedge \vec{H} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

Chapter 4 : Movement of Charged Particles

Introduction :

In this study, we will deal with the motions of charged particles, whether these particles are positively charged or negatively charged in an electric field, in a magnetic field, or in an electromagnetic field. And since the effect of these fields on the movement of charged particles is greater than the influence of the Newtonian forces of attraction between them, so we will neglect the forces of attraction compared to the forces of the electric field, magnetic field or electromagnetic field. We will assume that the mass of the charged particle is constant during motion.

Movement of a charged particle in an electrostatic field:

As we know, the electric field arises from a static electric charge, and this field is represented by a set of lines of force. The electrostatic field in each point of space strongly affects the unit positive charge if it is placed at that point, and this is known as the strength of the electric field $\vec{E}$, which is the position function. If a charged particle of mass m and charge q moves in a uniform electric field of intensity $\vec{E}$, then the force acting on this particle is determined by the magnitude and direction of Coulomb's law

$$\vec{F} = q\vec{E}$$

The acceleration of a particle $\vec{f}$ is given both magnitude and direction by Newton's law of motion. That is, the equation of motion for a charged particle in a uniform electric field is:

$$\vec{F} = m\vec{f} = q\vec{E} \qquad (1)$$

It is clear that the particle moves with a uniform acceleration that depends on the particle's mass $\boldsymbol{m}$, its charge $\boldsymbol{q}$, and the field strength $\vec{\boldsymbol{E}}$. and be $\vec{\boldsymbol{f}}$. In the direction of the field if the particle has a positive charge and in the opposite direction if the particle has a negative charge. It is noted that the movement of the particle is similar to the motion of a free projectile in the regular field of Earth's gravity near the surface of the Earth.

Movement of a charged particle in a uniform magnetic field:

The uniform magnetic field arises from a static magnetic pole or from the passage of a continuous electric current in a wire, and this field is represented by a set of lines of force. The magnetic field in each point of space strongly affects the unit of the positive poles stationary at that point. This is known as the magnetic field strength $\vec{\boldsymbol{H}}$, which is a function of position.

If a particle of mass $\boldsymbol{m}$ and charge $\boldsymbol{q}$ moves with a velocity of $\vec{\boldsymbol{v}}$ in a magnetic field of uniform intensity $\vec{\boldsymbol{H}}$. The force acting on this particle is determined by the magnitude and direction of Lorentz's law in the form:

$$\vec{F} = q\left(\vec{v} \wedge \vec{H}\right)$$

The acceleration of the particle $\vec{\boldsymbol{f}}$ is determined by the magnitude and direction of Newton's law of motion. That is, the equation of motion of a charged particle in a uniform magnetic field $\vec{\boldsymbol{H}}$ is:

$$\vec{F} = m\vec{f} = q\left(\vec{v} \wedge \vec{H}\right) \qquad (2)$$

Movement of a charged particle in a uniform electromagnetic field:

If a particle of mass $\boldsymbol{m}$ and charge $\boldsymbol{q}$ moves with velocity $\vec{\boldsymbol{v}}$ in two uniform fields, one of which has an electric field of $\vec{\boldsymbol{E}}$ and a magnetic field of intensity $\vec{\boldsymbol{H}}$. So the force arising from the two fields together is in the form:

$$\vec{F} = q\vec{E} + q\left(\vec{v} \wedge \vec{H}\right)$$

The acceleration of the particle is determined by the equation of motion:

$$\vec{F} = m\vec{f} = q\vec{E} + q\left(\vec{v} \wedge \vec{H}\right) \qquad (3)$$

Using Cartesian coordinates and unit vectors $\vec{i}, \vec{j}, \vec{k}$ in the direction of the orthogonal axes x, y, z, then: $\vec{E} = E_x\vec{i} + E_y\vec{j} + E_z\vec{k}$, $\vec{H} = H_x\vec{i} + H_y\vec{j} + H_z\vec{k}$, $\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$, $\vec{f} = f_x\vec{i} + f_y\vec{j} + f_z\vec{k}$. Then by substituting in the equations of motion, we get three differential equations that we solve by direct integration or by solving differential equations to get the parametric equations for the particle's trajectory:

$$x = x\left(t\right), y = y\left(t\right), z = z\left(t\right)$$

It is clear that there is a basic difference in the effect of electric and magnetic fields on charged particles, as the force that affects the direction perpendicular to the direction in which the body advances, cannot change the magnitude of the velocity of this particle (that is, the acceleration vector is always perpendicular to the velocity vector), and accordingly the kinetic energy remains for the unchanged particle. Therefore, the regular magnetic field (constant with time) is unable to transfer energy to the moving charge. On the other hand, the electric field affects a force that depends on the direction in which the particle is advancing, so the electric field causes a general transfer of energy between the field and the charged particle.

Charged particles:

Charged particles are of two types:

(a) Negatively charged particles: they are electrons and ions

negative. The electron is the smallest negative charge ever found. Electron constants :

Electron mass: $(m) = 9.107 X\, 10^{-31} \quad kg$

Electron charge: $(-e) = -1.602 X\, 10^{-19} \quad C$

(B) Positively charged particles: they are protons and positive ions. A proton has the same charge as an electron, but with an opposite sign. Proton constants:

Proton mass: $(m) = 1.67 X\, 10^{-27} \quad kg$

Proton charge: $(e) = 1.602 X\, 10^{-19} \quad C$

Example (1):

A particle with a positive charge of $\boldsymbol{4q}$ is held at the origin point $\boldsymbol{o}$ and a particle of mass $\boldsymbol{m}$ and charge of $\boldsymbol{2q}$ is placed at point $(\boldsymbol{a},\boldsymbol{0})$ and left to move from rest under the influence of the repulsive force between the two charges. Find the velocity of the particle when it is at a distance of $\boldsymbol{x}$ from $\boldsymbol{o}$ and the time of motion.

The solution :

The particle motion equation is:

$$m\ddot{x} = \frac{(2q)(4q)}{4\pi\varepsilon x^2}$$

$$\ddot{x} = \frac{2q^2}{m\pi\varepsilon x^2} \qquad (1)$$

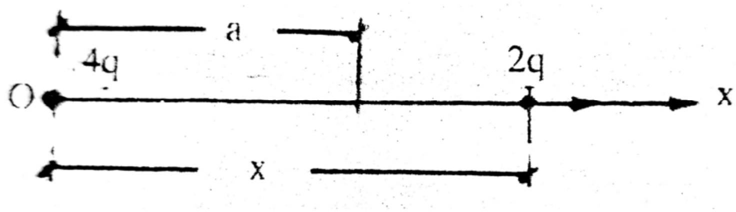


Fig. (4-1)

The initial conditions for movement are:

$$t = 0, \quad x = a, \quad \dot{x} = 0$$

$$\because \ddot{x} = v\frac{dv}{dx} = \frac{2q^2}{m\pi\varepsilon x^2}$$

Separating the variables and doing the integration, we get:

$$\int_0^v v\,dv = \frac{2q^2}{m\pi\varepsilon}\int_a^x \frac{dx}{x^2}$$

$$v = \pm\frac{2q}{a\sqrt{m\pi\varepsilon}}\sqrt{\frac{x-a}{x}}$$

And since it increases with time, we choose the positive sign, that is,

$$v = +\frac{2q}{a\sqrt{m\pi\varepsilon}}\sqrt{\frac{x-a}{x}} = \frac{dx}{dt}$$

Separating the variables and doing the integration, we get:

$$\int_0^t dt = \frac{a\sqrt{m\pi\varepsilon}}{2q}\int_a^x \sqrt{\frac{x}{x-a}}\,dx$$

Using the substitution: $\left(x = a\cosh^2 u\right)$, the value of the integration can be set in the form:

$$t = \frac{a\sqrt{m\pi\varepsilon}}{2q}\left\{\sqrt{x\left(x-a\right)} + a\cosh^{-1}\sqrt{\frac{x}{a}}\right\}$$

Example (2):

An electron of mass ***m*** and charge e of moving in a uniform electric field of intensity $\vec{E} = +E\vec{j}$. If the electron is ejected from the origin with an initial velocity of $\vec{v}_0 = U\vec{i} + V\vec{j}$. Find the parametric equations for the path (ignoring the weight of the particle).

The solution :

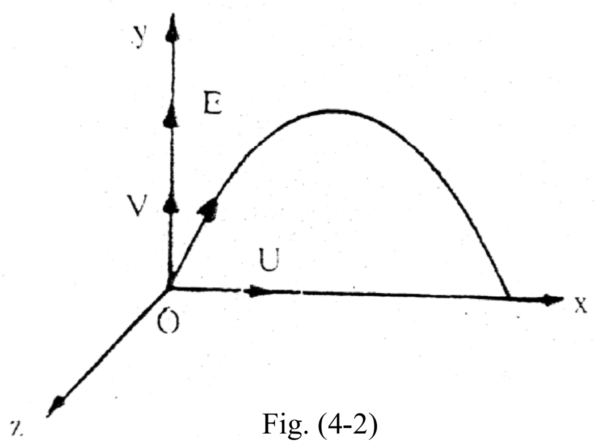


Fig. (4-2)

The equation for the directional motion of the particle is:

$$m\ddot{x}\,\vec{i} + m\ddot{y}\,\vec{j} + m\ddot{z}\vec{k} = -e\left(E\vec{j}\right)$$

$$\ddot{x} = 0, \quad \ddot{y} = -\frac{eE}{m}, \quad \ddot{z} = 0 \qquad (1)-(3)$$

The initial conditions for motion are: When:

$$\dot{x} = U, \dot{y} = V, \dot{z} = 0$$

$$x = 0, \quad y = 0, \quad z = 0$$

By solving the differential equations (3) - (1) and using the previous conditions, we get the parametric equations for the particle's trajectory in the form:

$$x = Ut, \quad y = Vt - \frac{eE}{2m}t^2, \quad z = 0 \qquad (4)-(6)$$

From equation (6) it is clear that the particle is moving in the plane and from equations (4), (5) and by deleting the parameter, we get the Cartesian equation for the path in the form:

$$y = \frac{V}{U}x - \frac{eE}{2mU^2}x^2$$

It is an equation of a parabola with a vertical axis that passes through the point of ejection (the origin) and is inverted downward.

Example (3):

An ejector shoots charged particles of mass $\boldsymbol{m}$ and charge $\boldsymbol{q}$ with a velocity $\boldsymbol{v}_\circ$ in the $\boldsymbol{yoz}$ plane and tilted at an angle of $\mathbf{45}^\circ$ to the $\boldsymbol{y}$ axis. A variable electric field is affected on the particle where $\vec{E} = E_0 \sin pt\vec{j} + E_0 \cos pt\vec{k}$, p, E_0 are constants. Prove that the motion of a particle is a planar motion in the $\boldsymbol{yoz}$ plane and can be represented by the motion of a point on a circle whose geometric center is moving at a constant speed.

The solution :

The equation of motion of a particle in an electric field

$$m\left(\ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k}\right) = q\left(E_0 \sin pt\vec{j} + E_0 \cos pt\vec{k}\right)$$

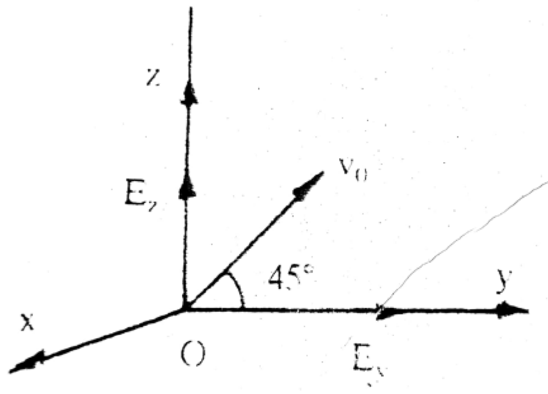


Fig. (4-3)

$$\ddot{x} = 0 \qquad (1)$$

$$\ddot{y} = \frac{qE_0}{m}\sin pt \qquad (2)$$

$$\ddot{z} = \frac{qE_0}{m}\cos pt \qquad (3)$$

And the initial conditions: when $t = 0$ are :

$$\dot{x} = 0, \quad \dot{y} = \frac{v_0}{\sqrt{2}}, \quad \dot{z} = \frac{v_0}{\sqrt{2}}$$

$$x = 0, \quad y = 0, \quad z = 0$$

By direct integration of the differential equations (3) - (1) and using the initial conditions, we get the parametric equations of the particle trajectory

$$x = 0 \qquad (4)$$

$$y = \left(\frac{v_0}{\sqrt{2}} + \frac{qE_0}{mp}\right)t - \frac{qE_0}{mp}\sin pt \qquad (5)$$

$$z = \left(\frac{v_0 t}{\sqrt{2}} + \frac{qE_0}{mp^2}\right) - \frac{qE_0}{mp^2}\cos pt \qquad (6)$$

From equation (4) it is clear that the particle moves in the ***yoz*** plane and from equations (5), (6) he obtained the Cartesian equation for the path of the particle in the form:

$$(y-y_1)^2+(z-z_1)^2=R^2 \qquad (7)$$

Where:

$$y_1=\left(\frac{v_0}{\sqrt{2}}+\frac{qE_0}{mp}\right)t \qquad \therefore \dot{y}_1=\frac{v_0}{\sqrt{2}}+\frac{qE_0}{mp}$$

$$z_1=\frac{v_0}{\sqrt{2}}t+\frac{qE_0}{mp^2} \qquad \dot{z}_1=\frac{v_0}{\sqrt{2}} \qquad R=\frac{qE_0}{mp^2}$$

Equation (7) shows that the motion of the particle can be represented by the motion of a point on a circle of radius ***R*** and centered at $O_1(y_1,z_1)$, and this center moves in the ***yoz*** plane with a uniform velocity of its components $\dot{y}_1,\dot{z}_1$, and the trajectory of the center is a straight line whose Cartesian equation (the relationship between y_1,z_1) is:

$$\frac{z_1-\frac{qE_0}{mp^2}}{\frac{v_0}{\sqrt{2}}}=\frac{y_1}{\frac{v_0}{\sqrt{2}}+\frac{qE_0}{mp}}$$

$$z_1=\frac{y_1}{1+\frac{qE_0\sqrt{2}}{mpv_0}}+\frac{qE_0}{mp^2}$$

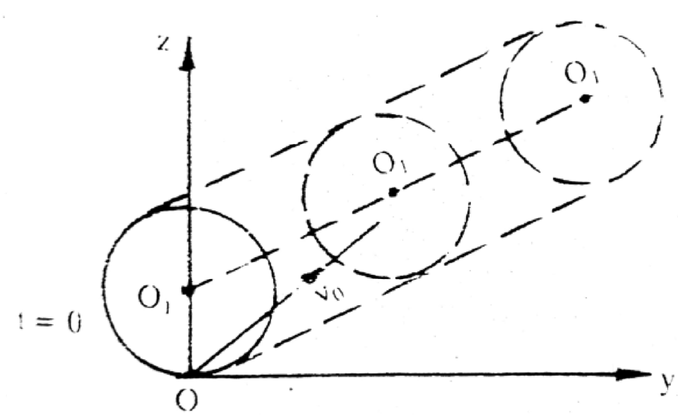


Fig. (4-4)

Example (4):

A cathode tube radiates electrons of $(-e)$ charge, mass m, and velocity $v_\circ$, placed in a uniform magnetic field $\vec{H} = H\vec{j}$. Determine the coordinates of the collision site of the electron with the screen.

The solution :

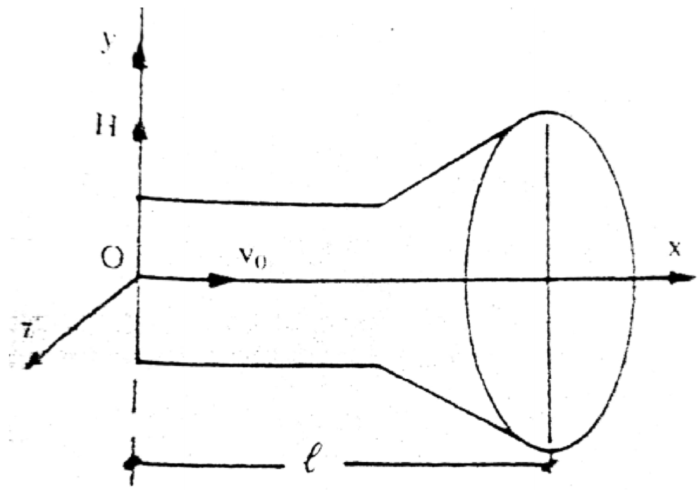


Fig. (4-5)

The particle motion equation is:

$$m\left(\ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k}\right) = (-e)\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & H & 0 \end{vmatrix}$$

$$\ddot{x} = \frac{eH}{m}\dot{z} \qquad (1)$$

$$\ddot{y} = 0 \qquad (2)$$

$$\ddot{z} = -\frac{eH}{m}\dot{x} \qquad (3)$$

The initial conditions are: When $t = 0$

$$\dot{x} = v_0, \;\; \dot{y} = 0, \;\; \dot{z} = 0$$

$$x = 0, \quad y = 0, \quad z = 0$$

By solving the differential equations (3) - (1) taking into account the initial conditions, we get the parametric equations of the particle's trajectory:

$$x = \frac{mv_0}{eH} \sin\left(\frac{eH}{m} t\right) \qquad (4)$$

$$y = 0 \qquad (5)$$

$$z = \frac{mv_0}{eH} \left\{\cos\left(\frac{eH}{m} t\right) - 1\right\} \qquad (6)$$

When the electron hits the screen of the tube, $\boldsymbol{x = L}$ and from equation (4) we get: The time of arrival of the particle to the screen as:

$$\therefore t_s = \frac{m}{eH} \sin^{-1}\left(\frac{LeH}{mv_0}\right)$$

$$L = \frac{mv_0}{eH} \sin\left(\frac{eH}{m} t_s\right)$$

Substituting in (5) and (6) we get:

$$y_s = 0$$

$$z_s = \frac{mv_0}{eH}\left\{\sqrt{1 - \left(\frac{LeH}{mv_0}\right)^2} - 1\right\}$$

That is, the coordinates of the location of the collision of the electron with the screen are: $(L, 0, z_s)$

.

Example (5):

A charged particle of mass ***m*** and charge ***q*** moves in a uniform magnetic field $\vec{H} = H\vec{k}$. If the particle is ejected from the origin ***o*** with an initial velocity $\vec{v}_0 = U\vec{i} + W\vec{k}$. Find the parametric equations for the particle's trajectory (ignoring the particle's weight).

The solution :

The equation of motion for the particle is:

$$M\left(\ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k}\right) = q\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & 0 & H \end{vmatrix}$$

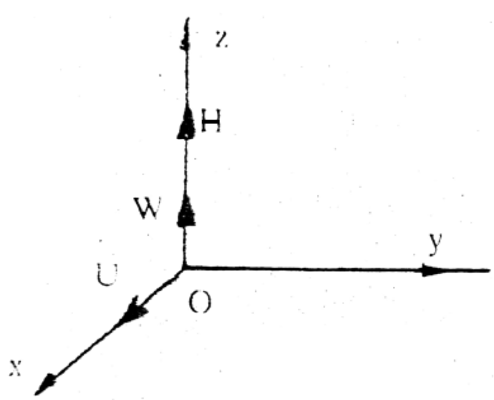


Fig. (4-6)

$$\ddot{x} = \frac{qH}{m}\dot{y} \qquad (1)$$

$$\ddot{y} = -\frac{qH}{m}\dot{x} \qquad (2)$$

$$\ddot{z} = 0 \qquad (3)$$

The initial conditions for motion are: When $\boldsymbol{t = 0}$

$$\dot{x} = U \qquad \dot{y} = 0 \qquad \dot{z} = W$$

$$x = 0 \qquad y = 0 \qquad z = 0$$

By solving the differential equations (3) - (1) with the use of the initial conditions, we get the parametric equations of the particle's trajectory.

$$x = \frac{mU}{qH} \sin\left(\frac{qH}{m}t\right) \tag{4}$$

$$y = \frac{mU}{qH} \left\{\cos\left(\frac{qH}{m}t\right) - 1\right\} \tag{5}$$

$$z = Wt \tag{6}$$

Assume that:

$$\Phi = \frac{qH}{m} \ , \ a = \frac{mU}{qH} \ , \ b = \frac{mW}{qH}$$

$$x = x' \ , \ y = y' - a \ , \ z = z'$$

The parametric equations of the particle are as follows:

$$x' = a \sin \Phi \tag{7}$$

$$y' = a \cos \Phi \tag{8}$$

$$z' = b\Phi \tag{9}$$

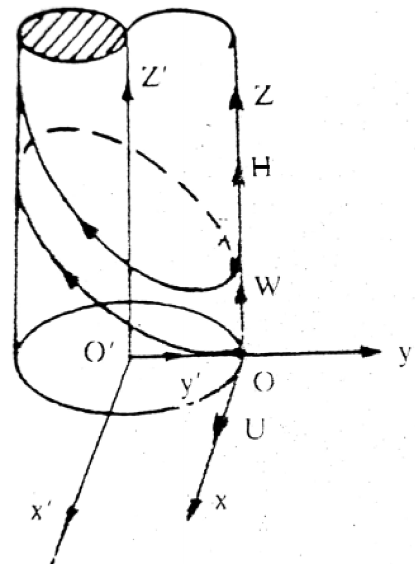


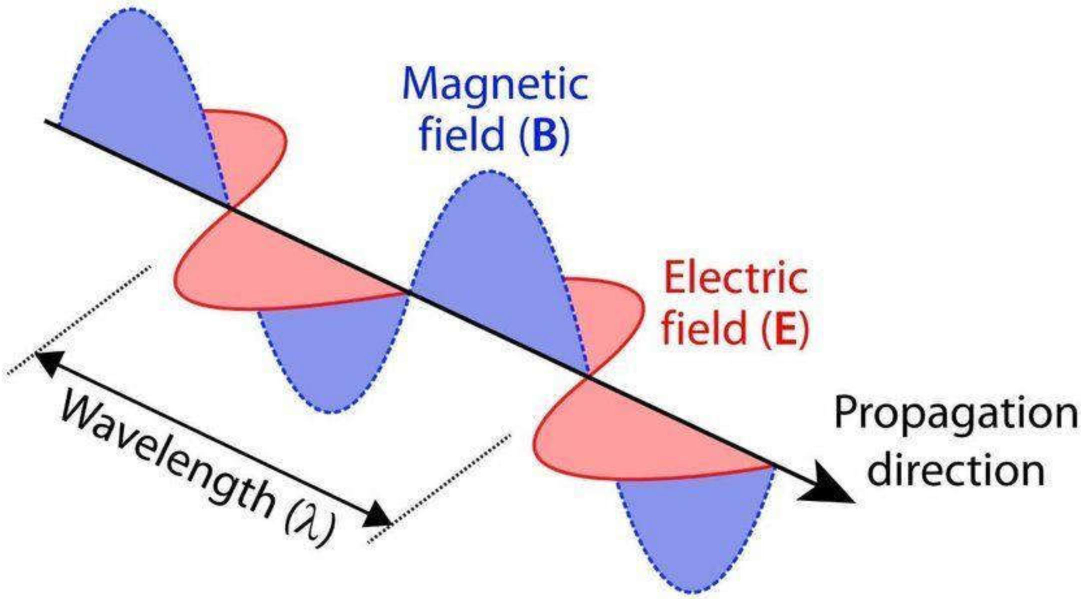
Fig. (4-7)

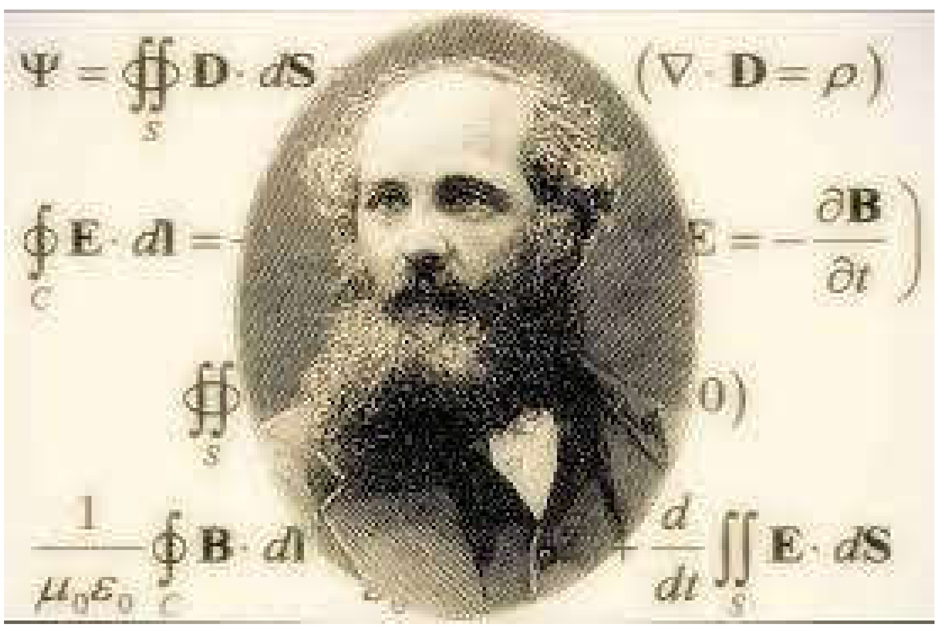
Equations (9) - (7) represent a circular helx (Burmese) curve located on the surface of a right circular cylinder with base radius $\boldsymbol{a}$, axis $\boldsymbol{z'}$ (the direction of the magnetic field) and the angle of the Burmese curve $\boldsymbol{\alpha}$ where:

$$\tan\alpha = \frac{b}{a} = \frac{W}{U}$$

And his step $\boldsymbol{p}$ is:

$$p = 2\pi a \tan\alpha = \frac{2\pi m W}{qH}$$


Magnetic field (B)
Electric field (E)
Propagation direction
Wavelength (λ)


$\Psi = \oiint_S \mathbf{D} \cdot d\mathbf{S}$
$(\nabla \cdot \mathbf{D} = \rho)$
$-\frac{\partial \mathbf{B}}{\partial t}$

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