Math101-General Mathematics

No.	Title
1	Lecturer
2	Why we study Mathematics
3	Preliminaries
4	Functions
5	Properties of Functions
6	Limits
7	Limits Examples
8	Continuity
9	Derivatives
10	Derivatives-B
11	Roll's theorem and the mean value theorem
12	Exponential functions
13	Integration
14	Definite integral
15	Application-1: Maximum and Minimum Values
16	Application-2: arc length

First Course in Mathematics

Prof. Mahmoud Bakhet

E-mail:
mmansour6@sci.svu.edu.eg

Why we study Mathematics?

Learning math is good for your brain

Practically every career uses math in some way.

Math is all around us and helps us understand the world better

Math is a universal language

Important of Mathematics in Biology and Geology

Why we study Mathematics?

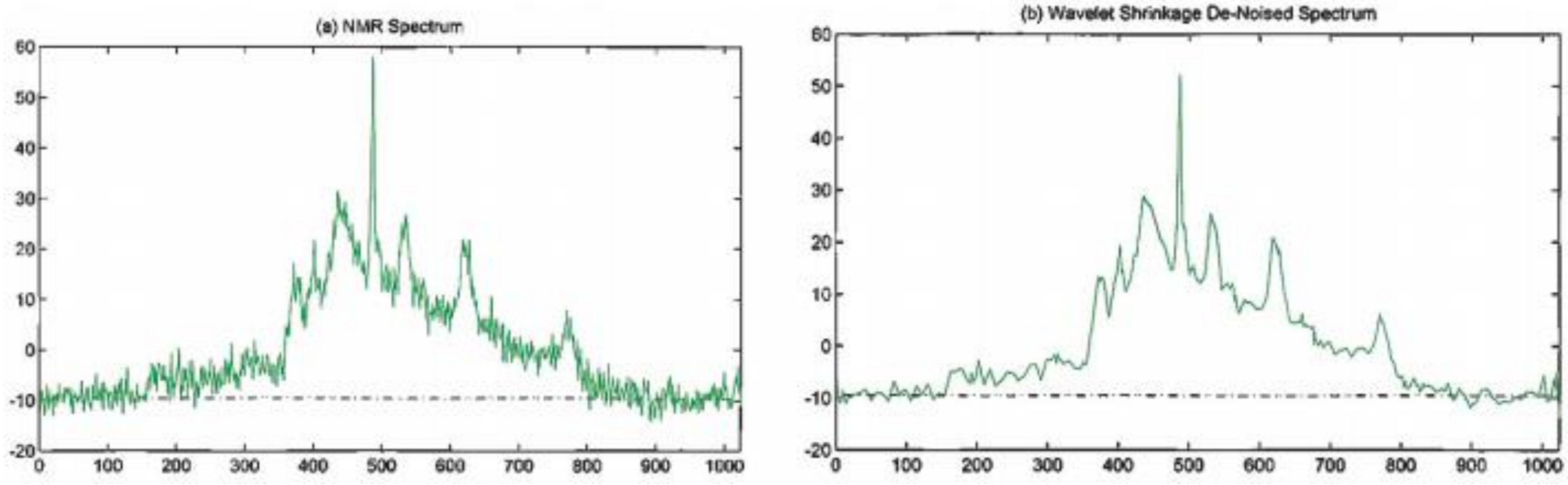
Original

Reconstruction

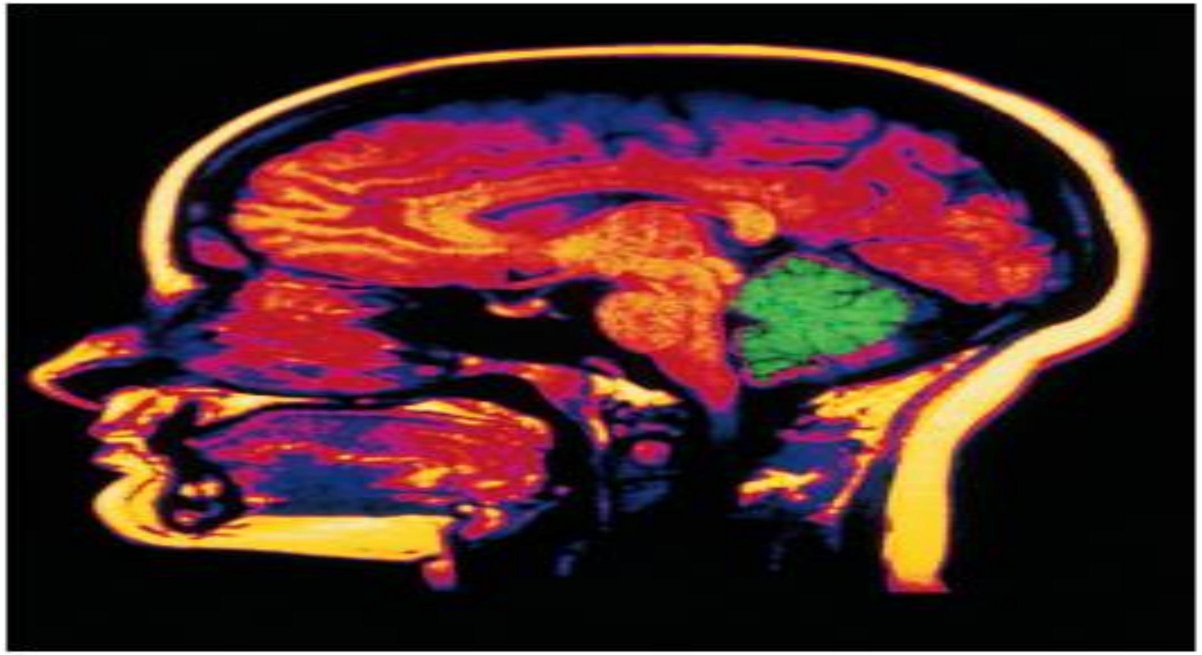
FBI Fingerprint Compression — The U.S. Federal Bureau of Investigation began collecting fingerprints and handprints in 1924 and now has more than 30 million such prints in its files, all of which are being digitized for storage on computer. It takes about 0.6 megabyte of storage space to record a fingerprint and 6 megabytes to record a pair of handprints, so that digitizing the current FBI archive would result in about 200×10^{12} bytes of data to be stored, which is the capacity of roughly 138 million floppy disks. At today's prices for computer equipment, storage media, and labor, this would cost roughly 200 million dollars. To reduce this cost, the FBI's Criminal Justice Information Service Division began working in 1993 with the National Institute of Standards, the Los Alamos National Laboratory, and several other groups to devise compression methods for reducing the storage space. These methods, which are based on wavelets, are proving to be highly successful. Figure 1 is a good example—the image on the left is an original thumbprint and the one on the right is a mathematical reconstruction from a 26:1 data compression.

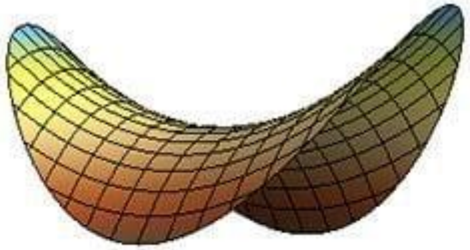
Original Reconstruction

Removing Noise from Data — In fields ranging from planetary science to molecular spectroscopy, scientists are faced with the problem of recovering a true signal from incomplete or noisy data. For example, weak signals from deep space probes are often so overwhelmed with background noise that the signal itself is barely detectable, yet the signal must be used to produce a photograph or provide other information. Researchers at Stanford University and elsewhere have been working for several years on using wavelet methods to filter out such noise. For example, Figure   shows a signal from a medical imaging signal that has been cleaned up (de-noised) using wavelets.



Medical Imaging and DNA Structure — Advances in *nuclear magnetic resonance* (NMR) have made it possible to determine the structure of biological macromolecules, study DNA replication, and determine how proteins act as enzymes and antibodies. Related advances in *magnetic resonance imaging* (MRI) have made it possible to view internal human tissue without invasive surgery and to provide real-time images during surgical procedures (Figure 5). High-quality NMR and MRI would not be possible without mathematical discoveries that have occurred within the last decade.





$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1$$

Who said that mathematics is not used in practical life? There is a use that you can eat, or you may already have.

The unique shape of the **Pringles** is no accident. In 1956, chemist Fredric J. Baur at Procter & Gamble was commissioned to develop a new type of potato chips, after frequent customer complaints about them breaking and greasing in their packages. The man spent two years of his life solving this problem, and ended up choosing the famous horse saddle shape as his design for the potato, and cylinders as their container.By the way, this shape is known in mathematics as a "hyperbolic paraboloid."Every time you open a pack of those potato chips to find they're all intact, remember it's thanks to the math.

Preliminaries

The real systems
Inequalities
The rectangular coordinate system
The straight line

Math 101: General mathematics- 3Credit (Lecture 2h/w+ tutorial 2h/W)

Contents: Preliminaries the real number system, Inequalities, Absolute values, The rectangular coordinate system, The straight line. Functions and limits: Functions and their graphs, Operations on functions, The trigonometric functions, Limit of functions, Continuity of functions. The derivative: Definition of derivative rules for finding derivative, Derivatives of trigonometric functions, The chain rule, Derivatives of exponential and logarithm functions, Higher-order derivatives. Applications of the derivatives: Maxima and minima, The mean theorem. Integral: Antiderivatives (Indefinite integrals), The definite integral, Applications of the definite integral.

References:

Edwin J. Purcell and Dale Varberg. (1984). Calculus with analytic geometry.

The real systems

Real Numbers are just numbers like: 1 12.38 −0.8625 $\frac{7}{9}$ **π** (pi) 198

In fact:

Nearly any number you can think of is a Real Number

Real Numbers include:

- ✓ Whole Numbers (like 0, 1, 2, 3, 4, etc)
- ✓ Rational Numbers (like 3/4, 0.125, 0.333..., 1.1, etc)
- ✓ Irrational Numbers (like **π**, √2, etc)

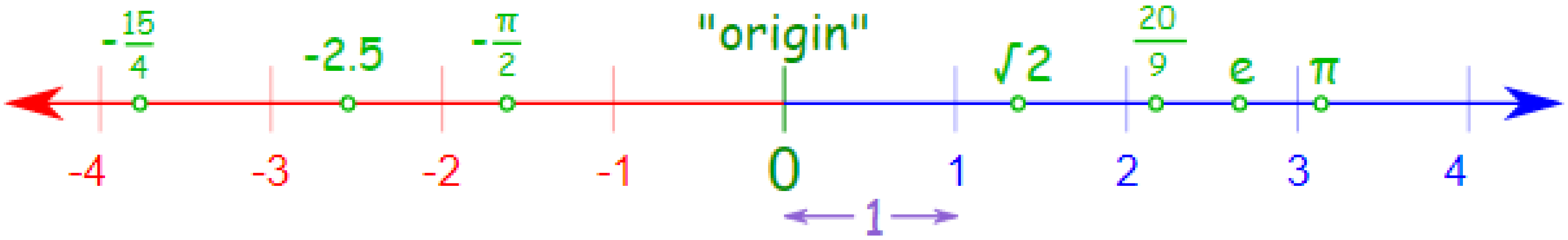
So ... what is NOT a Real Number?

- ❌ Imaginary Numbers like $\sqrt{-1}$ (the square root of minus 1) are not Real Numbers
- ❌ Infinity ∞ is not a Real Number

The Real Number Line

The Real Number Line is like a geometric line.

A point is chosen on the line to be the **"origin"**. Points to the right are positive, and points to the left are negative.



A distance is chosen to be "1", then whole numbers are marked off: {1,2,3,...}, and also in the negative direction: {...,−3,−2,−1}

Any point on the line is a Real Number:

- The numbers could be whole (like 7)
- or rational (like 20/9)
- or irrational (like π)

But we won't find Infinity, or an Imaginary Number.

Any Number of Digits

A Real Number can have any number of digits either side of the decimal point

- 120.
- 0.12345
- 12.5509
- 0.000 000 0001

There can be an infinite number of digits, such as $\frac{2}{3} = 0.6666\ldots$

Why are they called "Real" Numbers?

Because they are not Imaginary Numbers

The Real Numbers had no name before Imaginary Numbers were thought of. They got called "Real" because they were not Imaginary. That is the actual answer!

Inequalities

Symbol	Words	Example
>	greater than	$x + 3 > 2$
<	less than	$7x < 28$
≥	greater than or equal to	$5 \geq x - 1$
≤	less than or equal to	$2y + 1 \leq 7$

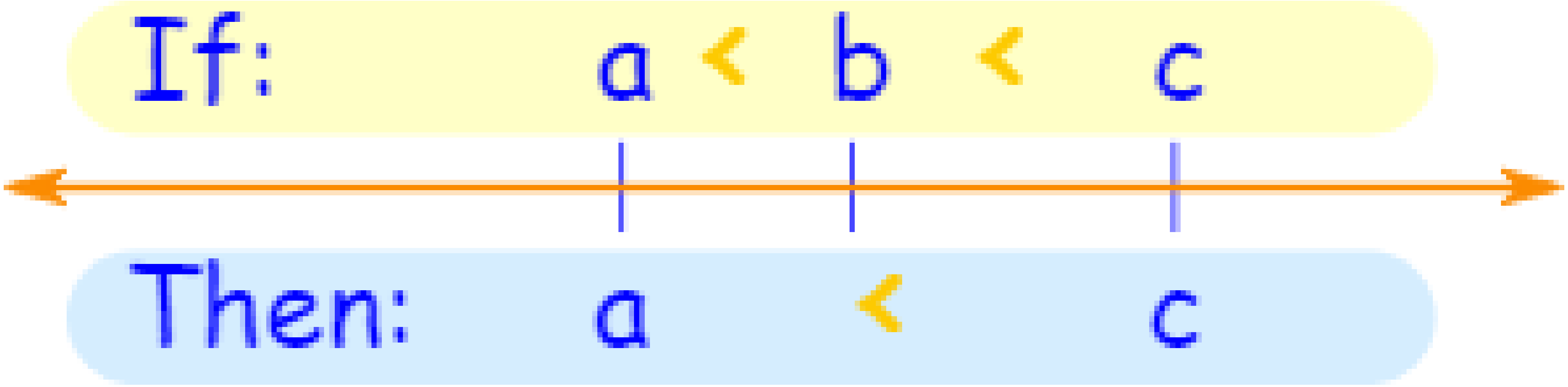
We must also pay attention to the **direction of the inequality**.

Properties of Inequalities

Transitive Property

When we link up inequalities in order, we can "jump over" the middle inequality.

If a < b **and** b < c, then a < c



Reversal Property

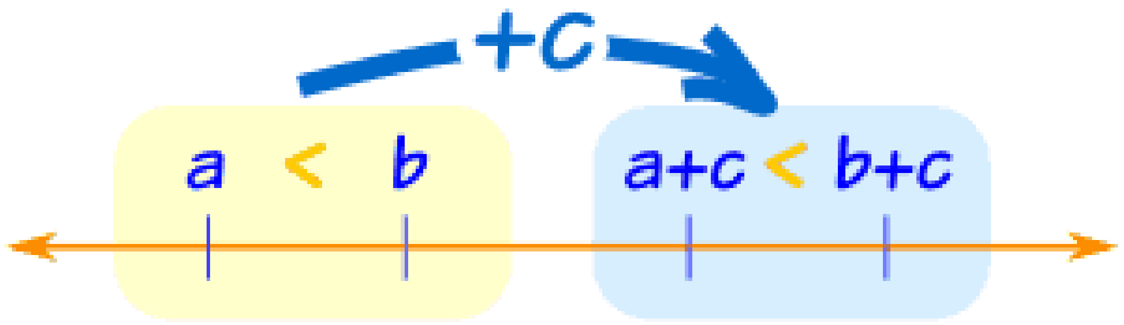
We can swap **a** and **b** over, if we make sure the symbol still "points at" the smaller value.

- If $a > b$ then $b < a$
- If $a < b$ then $b > a$

Addition and Subtraction

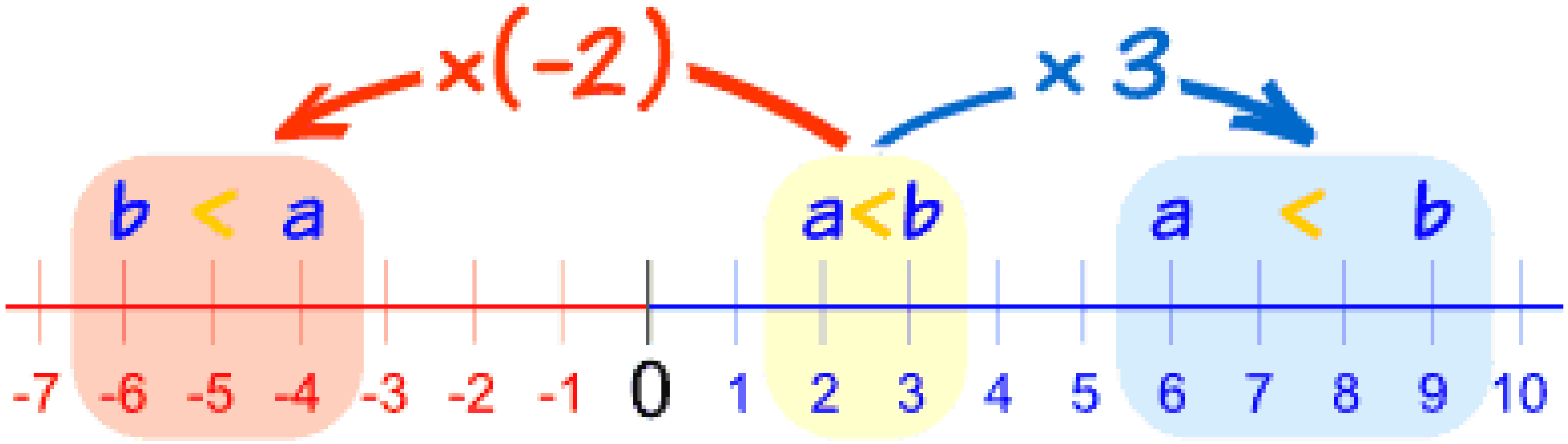
Adding **c** to both sides of an inequality just **shifts everything along**, and the inequality stays the same.

If $a < b$, then $a \mathbf{+ c} < b \mathbf{+ c}$



Multiplication and Division

When we multiply both a and b by a **positive number**, the inequality **stays the same**.
But when we multiply both a and b by a **negative number**, the inequality **swaps over**!



Notice that **a<b** becomes **b<a** after multiplying by (-2)
But the inequality stays the same when multiplying by +3

Here are the rules:

- If a < b, and **c is positive**, then **ac < bc**
- If a < b, and **c is negative**, then **ac > bc** (inequality swaps over!)

Additive Inverse

As we just saw, putting minuses in front of a and b **changes the direction** of the inequality. This is called the "Additive Inverse":

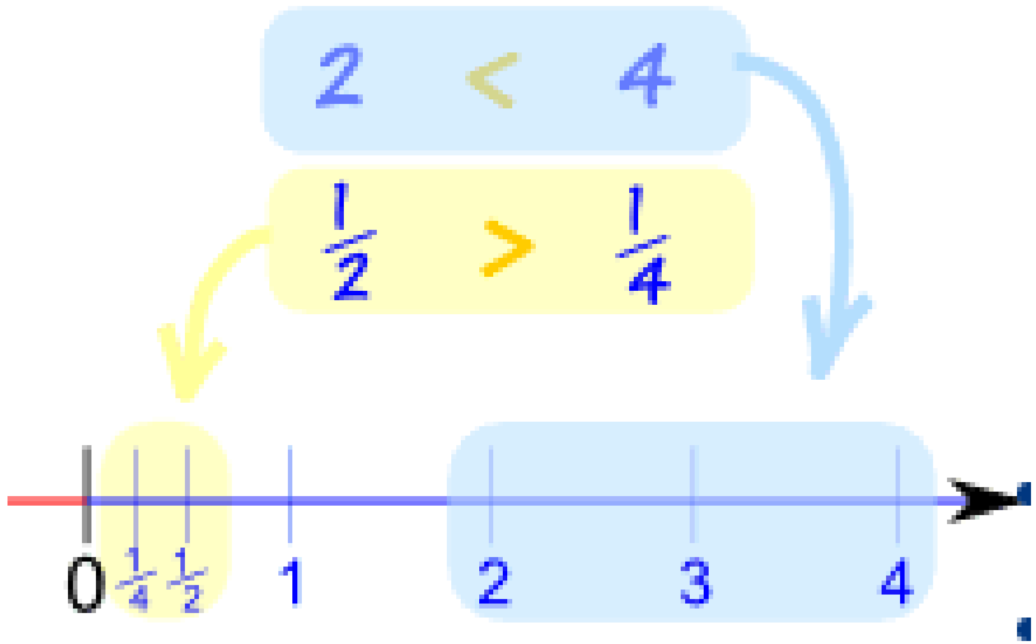
- If $a < b$ then $-a > -b$
- If $a > b$ then $-a < -b$

Multiplicative Inverse

Taking the reciprocal (1/value) of both a and b **can change the direction** of the inequality.

When a and b are **both positive** or **both negative**:

- If $a < b$ then $1/a > 1/b$
- If $a > b$ then $1/a < 1/b$



The straight line

A line has length but has no width or thickness.

A line may be straight, curved, or a combination of these.

A line is designated by the capital letters of any two of its points or by a small letter, thus:

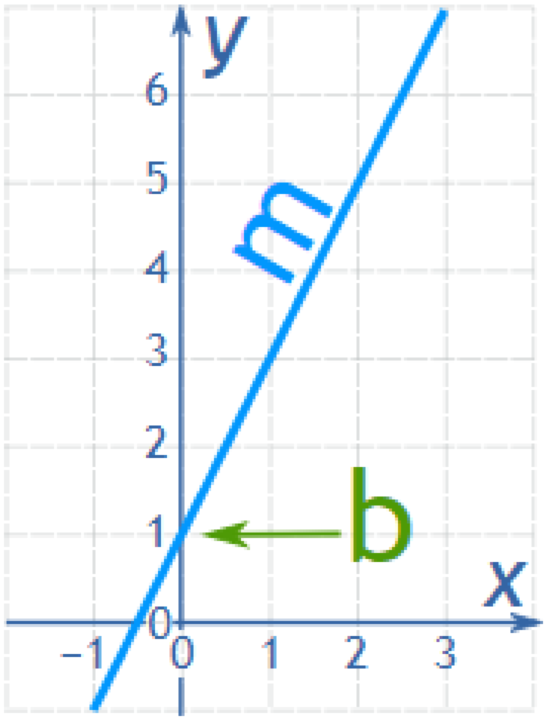
, or $\overleftrightarrow{AB}$.

Equation of a Straight Line

The equation of a straight line is usually written this way:

$y = mx + b$

What does it stand for?



$y \; = \; mx + b$

Slope or Gradient

y value when **x=0** (*see Y Intercept*)

y = how far up

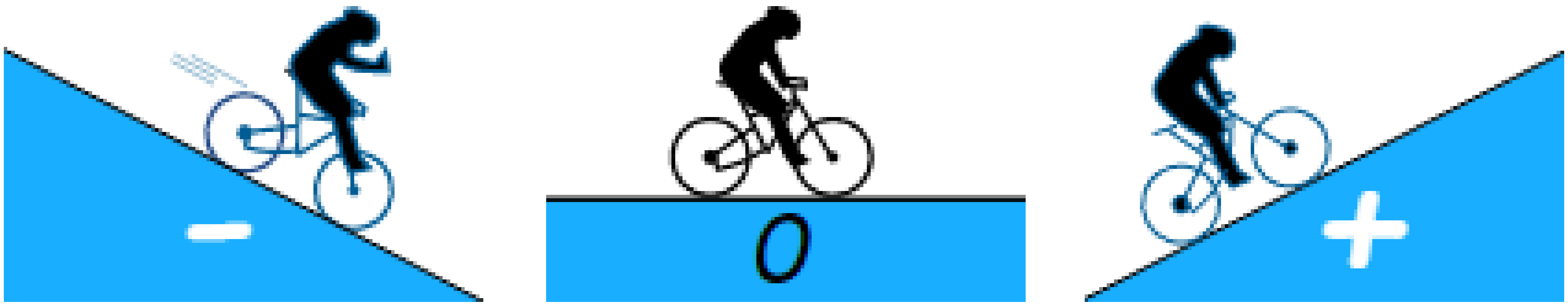
x = how far along

m = Slope or Gradient

b = value of **y** when **x=0**

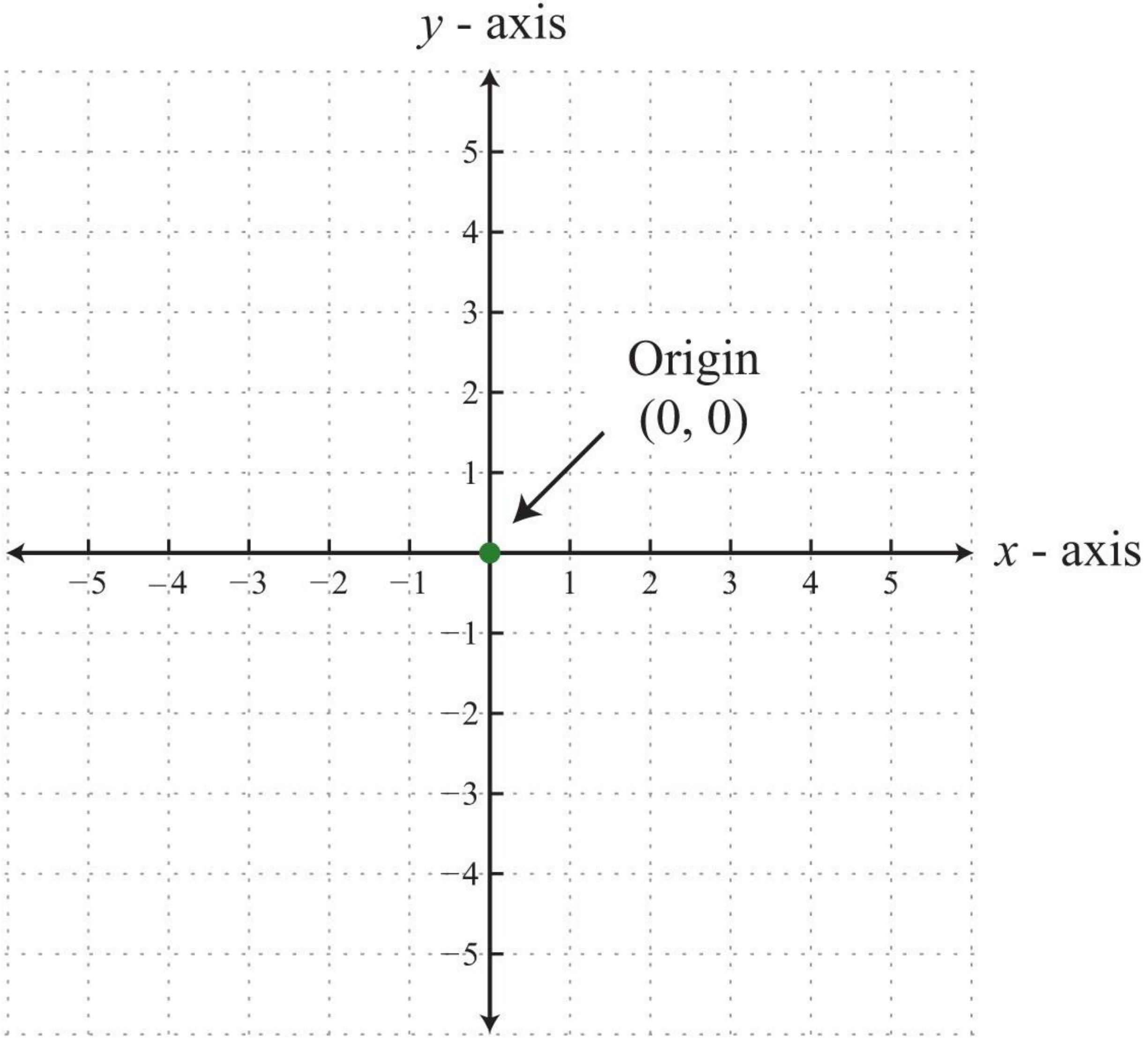
Positive or Negative Slope?

Going from left-to-right, the cyclist has to **P**ush on a **P**ositive Slope:

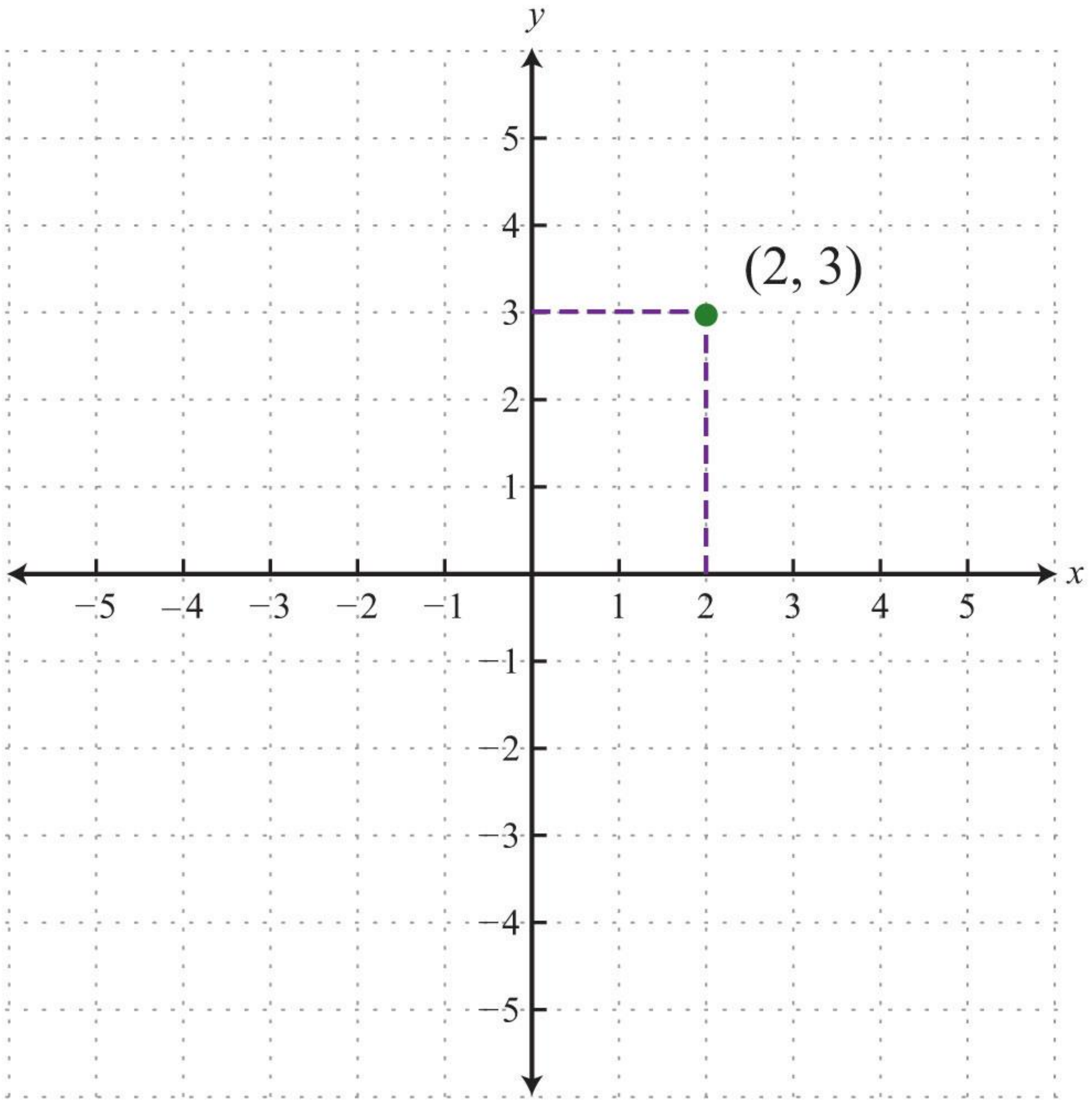


The rectangular coordinate system

The **rectangular coordinate system** consists of two real number lines that intersect at a right angle. The horizontal number line is called the **x-axis**, and the vertical number line is called the **y-axis**. These two number lines define a flat surface called a **plane**, and each point on this plane is associated with an **ordered pair** of real numbers (x, y). The first number is called the x-**coordinate**, and the second number is called the y-**coordinate**. The intersection of the two axes is known as the **origin**, which corresponds to the point (0, 0).

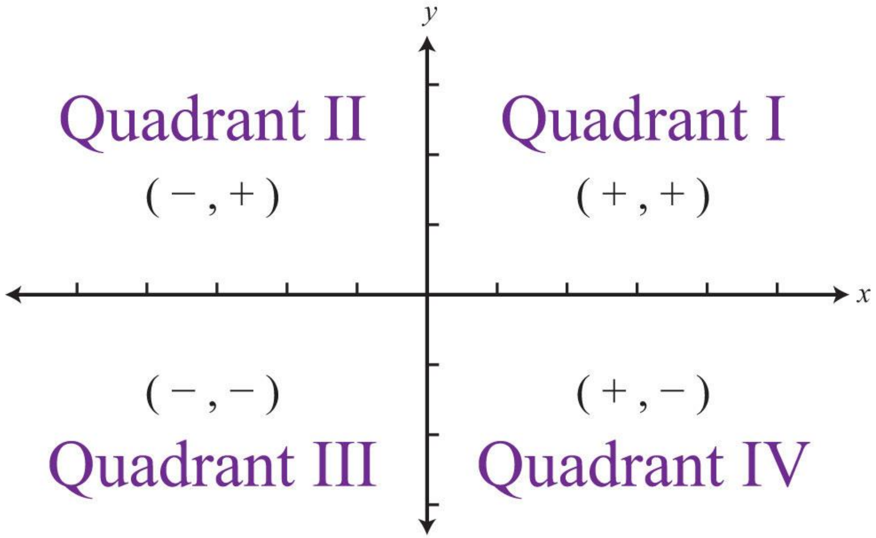


An ordered pair (x, y) represents the position of a point relative to the origin. The x-coordinate represents a position to the right of the origin if it is positive and to the left of the origin if it is negative. The y-coordinate represents a position above the origin if it is positive and below the origin if it is negative. Using this system, every position (point) in the plane is uniquely identified. For example, the pair (2, 3) denotes the position relative to the origin as shown:



This system is often called the **Cartesian coordinate system**, named after the French mathematician René Descartes (1596–1650).

The x- and y-axes break the plane into four regions called **quadrants**, named using roman numerals I, II, III, and IV, as pictured. In quadrant I, both coordinates are positive. In quadrant II, the x-coordinate is negative and the y-coordinate is positive. In quadrant III, both coordinates are negative. In quadrant IV, the x-coordinate is positive and the y-coordinate is negative.



1

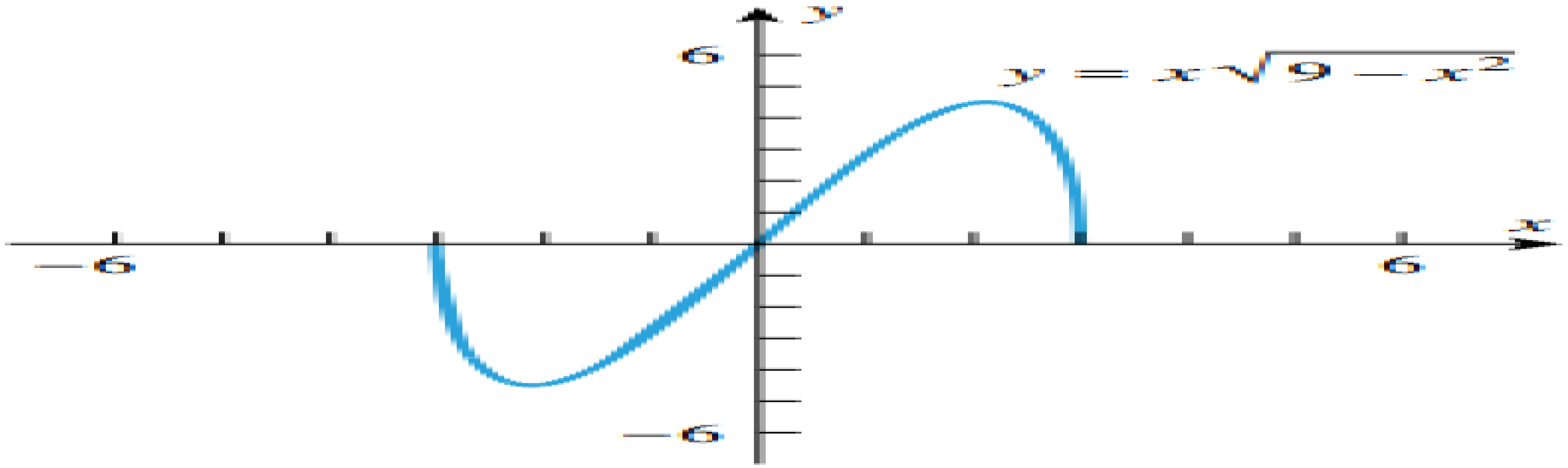
FUNCTIONS

One of the important themes in calculus is the analysis of relationships between physical or mathematical quantities. Such relationships can be described in terms of graphs, formulas, numerical data, or words. In this chapter we will develop the concept of a function, which is the basic idea that underlies almost all mathematical and physical relationships, regardless of the form in which they are expressed. We will study properties of some of the most basic functions that occur in calculus.

Graphs can be used to describe mathematical equations as well as physical data. For example, consider the equation

$$y = x\sqrt{9 - x^2}$$

x	-3	-2	-1	0	1	2	3
y	0	$-2\sqrt{5} \approx -4.47214$	$-2\sqrt{2} \approx -2.82843$	0	$2\sqrt{2} \approx 2.82843$	$2\sqrt{5} \approx 4.47214$	0



Some illustrative examples.

- The area A of a circle depends on its radius r by the equation $A = pr^2$, so we say that A is a function of r.
- Volume of a sphere depends on its radius by the equation $V = \frac{4}{3}pr^3$.
- Surface area of a cube depends on the length of its side by the equation

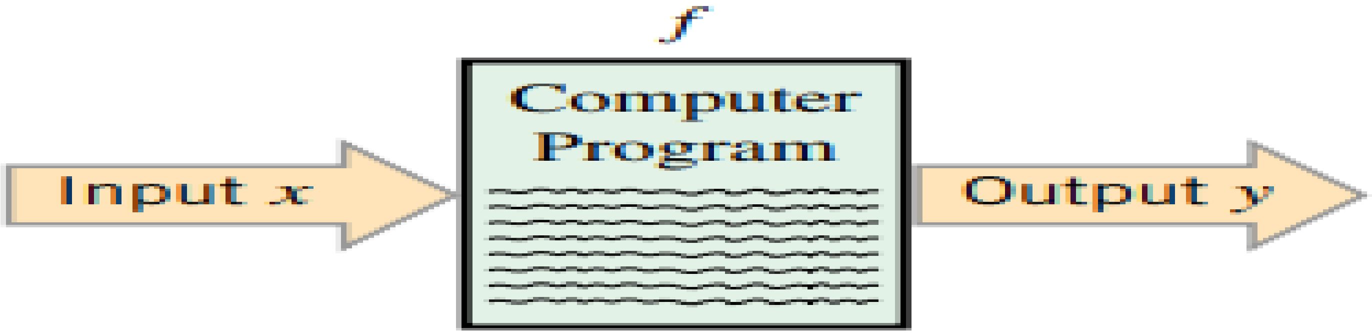
  $S = 6x^2$
- The velocity A of a ball falling freely in the Earth's gravitational field increases with time A until it hits the ground, so we say that A is a function of A.

FUNCTIONS

Tables, graphs, and equations provide three methods for describing how one quantity depends on another—numerical, visual, and algebraic. The fundamental importance of this idea was recognized by Leibniz in 1673 when he coined the term *function* to describe the dependence of one quantity on another.

Definition 1.

If a variable y depends on a variable x in such a way that each value of x determines exactly one value of y, then we say that y **is a function of** x.



Definition 2.

A function f is a rule that associates a unique output with each input. If the input is denoted by x , then the output is denoted by $f(x)$ (read "f of x ").

INDEPENDENT AND DEPENDENT VARIABLES

a function f is a rule that associates a unique output $f(x)$ with each input x. This output is sometimes called the ***value*** of f at x or the ***image*** of x under f. Sometimes we will want to denote the output by a single letter, say y, and write

$$y = f(x)$$

This equation expresses y as a function of x; the variable x is called the ***independent variable*** (or ***argument***) of f, and the variable y is called the ***dependent variable*** of f.

Example (1)

For $f(x) = x^2 - 2x$, find and simplify

(a) $f(4)$, (b) $f(4+h)$, (c) $f(4+h) - f(4)$

(d) $[f(4+h) - f(4)]/h$, where $h > 0$.

Solution

$$f(4) = 4^2 - 2(4) = 16 - 8 = 8$$

$$\begin{aligned} f(4+h) &= (4+h)^2 - 2(4+h) \\ &= (16 + 8h + h^2) - (8 + 2h) \\ &= 8 + 6h + h^2 \end{aligned}$$

$$\begin{aligned} f(4+h) - f(4) &= 8 + 6h + h^2 - 8 \\ &= 6h + h^2 \end{aligned}$$

$$[f(4+h) - f(4)]/h = (6h + h^2)/h = 6 + h$$

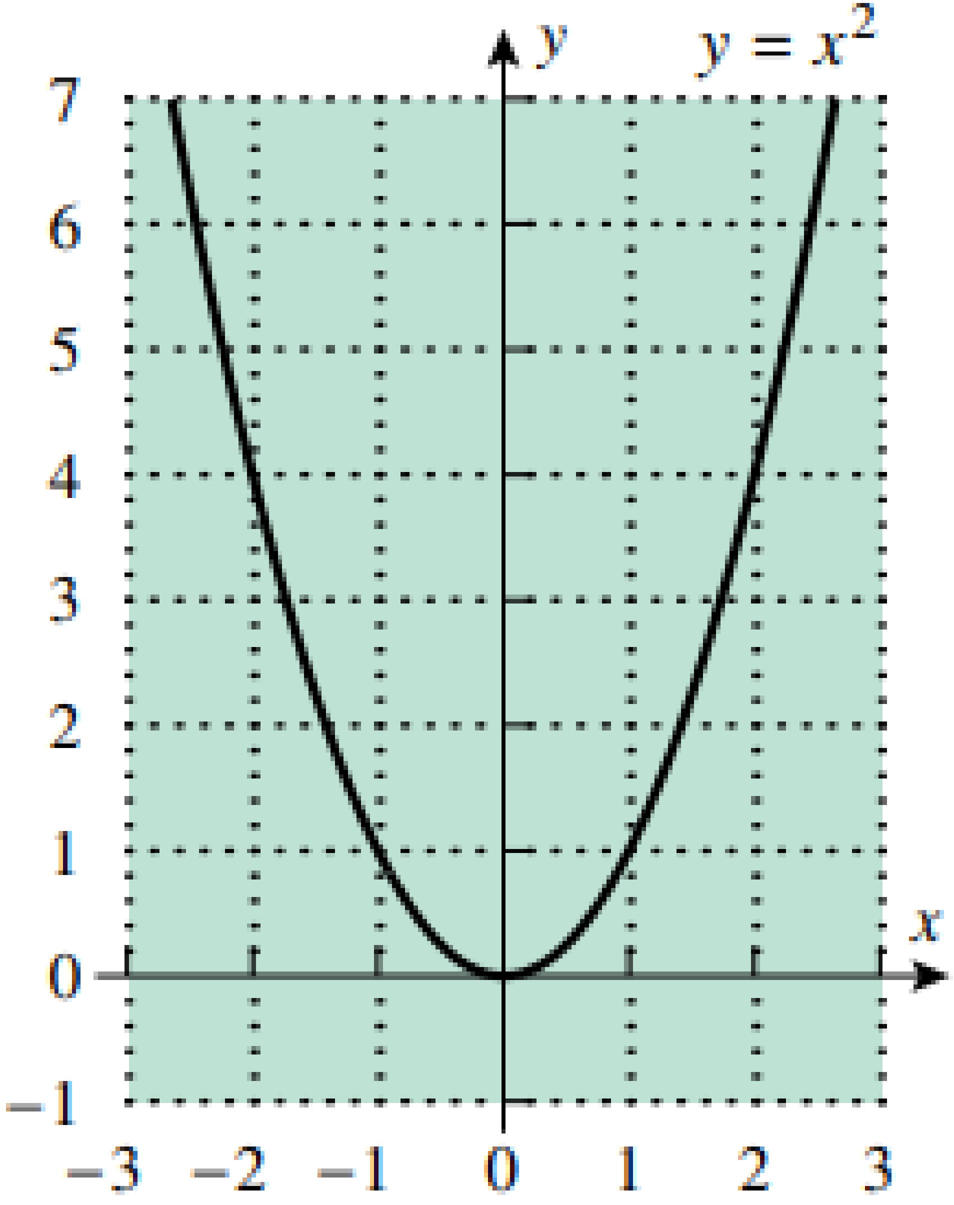
DOMAIN AND RANGE

If $y = f(x)$ then the set of all possible inputs (x -values) is called the domain of f , and the set of outputs (y -values) that result when x varies over the domain is called the range of f .

For example,

$$y = x^2 \quad \text{and} \quad y = x^2, \quad x \geq 2$$

In the first equation there is no restriction on x, so we may assume that any real value of x is an allowable input. Thus, the equation defines a function $f(x) = x^2$ with domain $-\infty < x < +\infty$. In the second equation, the inequality $x \geq 2$ restricts the allowable inputs to be greater than or equal to 2, so the equation defines a function $g(x) = x^2, x \geq 2$ with domain $2 \leq x < +\infty$.



y
$y = x^2$
x
7
6
5
4
3
2
1
0
−1
−3
−2
−1
0
1
2
3

THE ABSOLUTE VALUE FUNCTION

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

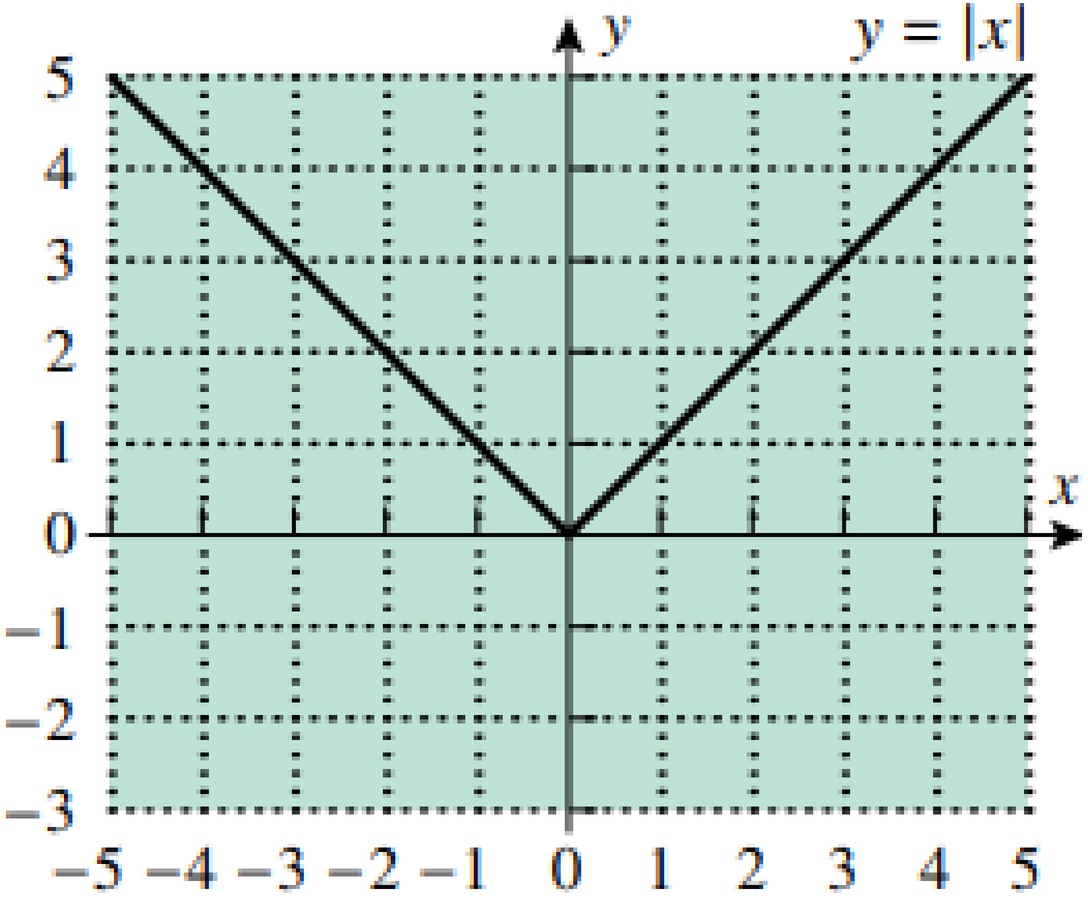
If a and b are real numbers, then

(a) $|-a| = |a|$ A number and its negative have the same absolute value.

(b) $|ab| = |a|\,|b|$ The absolute value of a product is the product of the absolute values.

(c) $|a/b| = |a|/|b|$ The absolute value of a ratio is the ratio of the absolute values.

(d) $|a+b| \leq |a| + |b|$ The ***triangle inequality***



y
y = |x|
x
5
4
3
2
1
0
−1
−2
−3
−5 −4 −3 −2 −1 0 1 2 3 4 5

Example (2)

Find the domain of :

(a) $f(x) = x^3$ (b) $f(x) = \dfrac{1}{(x-1)(x-3)}$

(c) $f(x) = \tan x$ (d) $f(x) = \sqrt{x^2 - 5x + 6}$

Solution

(a) The function f has real values for all real x, so its domain is the interval $(-\infty, \infty)$.

(b) The function f has real values for all real x, except $x = 1$ and $x = 3$, where divisions by zero occur. Thus, the domain is

$(-\infty, 1) \cup (1,3) \cup (3, \infty)$.

(c) $f(x) = \tan x$ (d) $f(x) = \sqrt{x^2 - 5x + 6}$

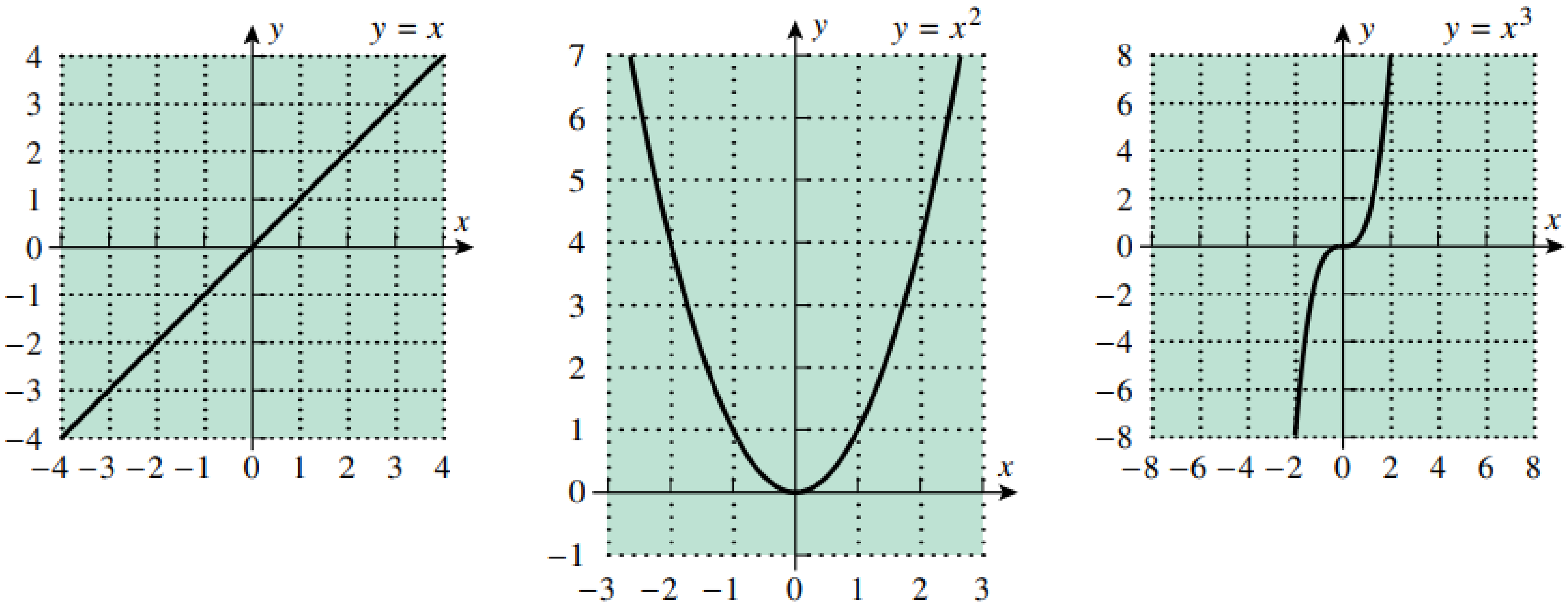
(c) Since $f(x) = \tan x = \dfrac{\sin x}{\cos x}$, the function f has real values except where $\cos x = 0$, and this occurs when x is an odd integer multiple of $\dfrac{p}{2}$. Thus, the domain consists of all real numbers except $x = \dfrac{p}{2}, \dfrac{3p}{2}, \dfrac{5p}{2}, \ldots$

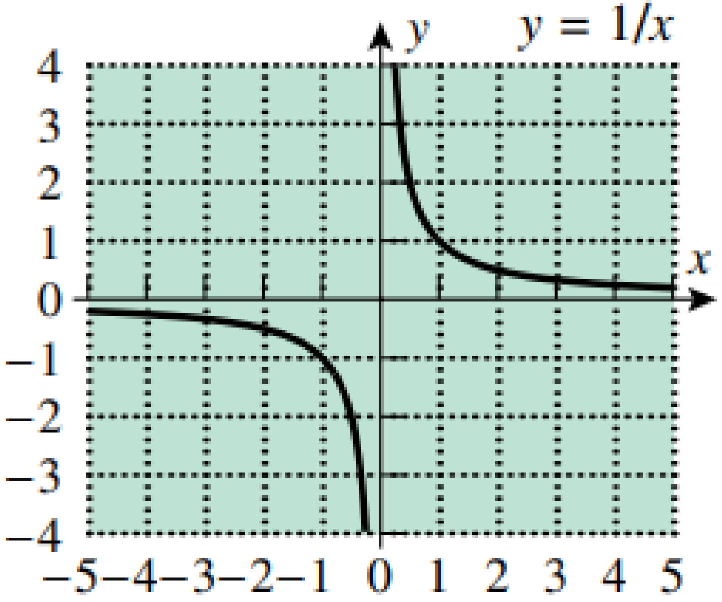
(d) The function f has real values, except when the expression inside the radical is negative. Thus the domain consists of all real numbers x such that $x^2 - 5x + 6 = (x - 3)(x - 2)$ ³ 0. This inequality is satisfied if $x = 2$ or $x = 3$, so the natural domain of f is $(-\infty, 2] \cup [3, \infty)$.

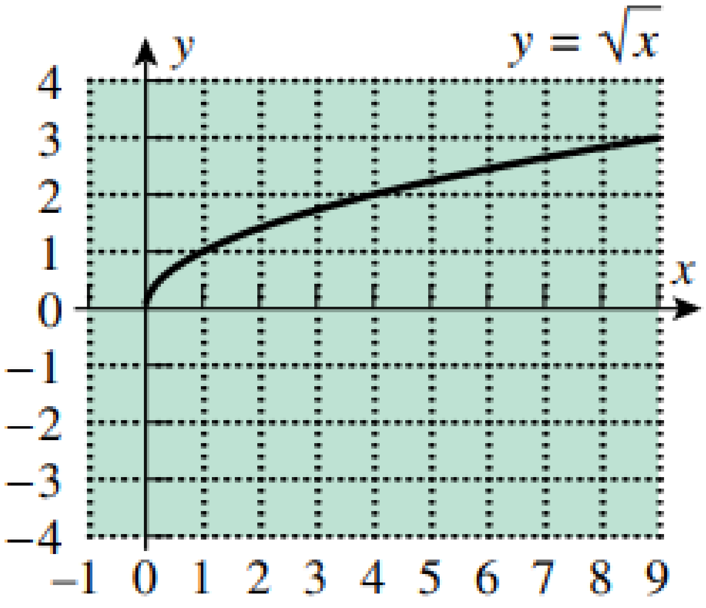
PROPERTIES OF FUNCTIONS

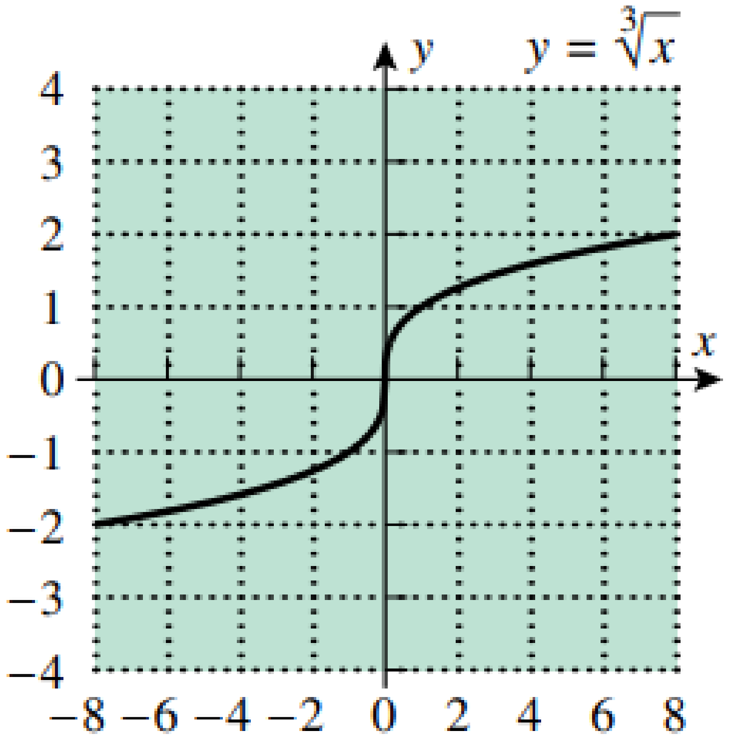
GRAPHS OF FUNCTIONS

If f is a real-valued function of a real variable, then the ***graph*** of f in the xy-plane is defined to be the graph of the equation $y = f(x)$. For example, the graph of the function $f(x) = x$ is the graph of the equation $y = x$, shown in Figure That figure also shows the graphs

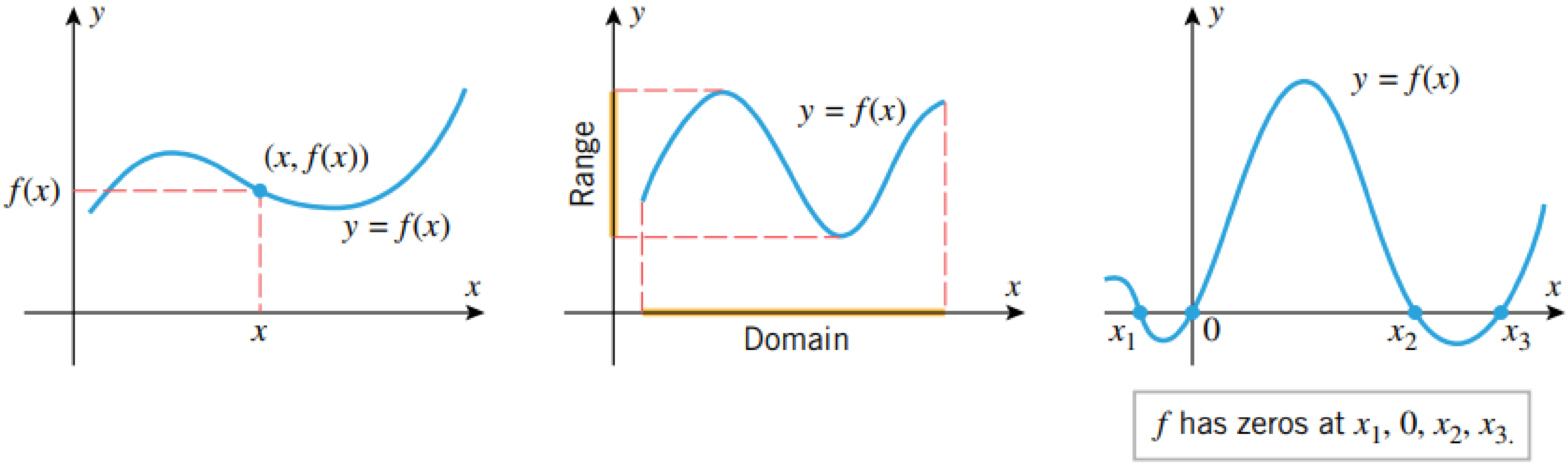



y = 1/x
y
x


y = √x
y
x


y = ∛x
y
x

Graphs can provide useful visual information about a function. For example, because the graph of a function f in the xy-plane consists of all points whose coordinates satisfy the equation $y = f(x)$, the points on the graph of f are of the form $(x, f(x))$; hence each y-coordinate is the value of f at the x-coordinate (Figure *a*). Pictures of the domain and range of f can be obtained by projecting the graph of f onto the coordinate axes (Figure *b*). The values of x for which $f(x) = 0$ are the x-coordinates of the points where the graph of f intersects the x-axis (Figure *c*); these values of x are called the ***zeros*** of f, the ***roots*** of $f(x) = 0$, or the ***x-intercepts*** of $y = f(x)$.



(*a*) (*b*) (*c*)

THE NATURAL DOMAIN

DEFINITION. If a real-valued function of a real variable is defined by a formula, and if no domain is stated explicitly, then it is to be understood that the domain consists of all real numbers for which the formula yields a real value. This is called the ***natural domain*** of the function.

Example Find the natural domain of

(a) $f(x) = x^3$

(b) $f(x) = 1/[(x-1)(x-3)]$

(c) $f(x) = \tan x$

(d) $f(x) = \sqrt{x^2 - 5x + 6}$

(a) $f(x) = x^3$

The function f has real values for all real x, so its natural domain is the interval $(-\infty, +\infty)$.

(b) $f(x) = 1/[(x-1)(x-3)]$

The function f has real values for all real x, except $x = 1$ and $x = 3$, where divisions by zero occur. Thus, the natural domain is

$$\{x : x \neq 1 \text{ and } x \neq 3\} = (-\infty, 1) \cup (1, 3) \cup (3, +\infty)$$

(c) $f(x) = \tan x$

Since $f(x) = \tan x = \sin x / \cos x$, the function f has real values except where $\cos x = 0$, and this occurs when x is an odd integer multiple of $\pi/2$. Thus, the natural domain consists of all real numbers except

$$x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$$

(d) $f(x) = \sqrt{x^2 - 5x + 6}$

The function f has real values, except when the expression inside the radical is negative. Thus the natural domain consists of all real numbers x such that

$$x^2 - 5x + 6 = (x - 3)(x - 2) \geq 0$$

This inequality is satisfied if $x \leq 2$ or $x \geq 3$ (verify), so the natural domain of f is

$$(-\infty, 2] \cup [3, +\infty)$$

Even and Odd Functions

(i) A function is an even function if for every x in the domain of f
$f(-x) = f(x)$.

(ii) A function is an odd function if for every x in the domain of f
$f(-x) = -f(x)$.

Example

I. A polynomial function of the following form is an even function:

$$f(x) = a_0 + a_1 x^2 + a_2 x^4 + \cdots + a_n x^{2n}$$

Observe that the power of x in each term is an even integer.

II. We have that $\cos(-x) = \cos x$ for all x. Thus, the cosine function is an even function.

III. A constant function is always even (why?).

Example

I. It can be easily verified that the functions $f(x) = x$ and $f(x) = x^3$ are odd functions. In fact, any polynomial function in which the power of each term is an odd integer is an odd function.

Period of a Periodic Function

If a function f is periodic, then the smallest $p > 0$, if it exists such that $f(x+p) = f(x)$ for all x , is called the period of the function.

Obviously, the period of the sine and cosine functions is 2π . It can be shown that the period of the tangent function (and that of the cotangent function) is π .

Remark

Aperiodic function may not have a period. Note that a constant function f is periodic as $f(x+p) = f(x) =$ constant for all $p > 0$, however, there is no smallest $p > 0$ for which the relation holds. Hence, there is no period of this function, though it is periodic by definition.

Algebraic operation on functions

Functions are not numbers. But, just as two numbers a and b can be added to produce a new number $(a + b)$, two functions f and g can be added to produce a new function $(f + g)$. This is just one of the several operations on functions.

(a) Sums, Differences, Products and Quotients of Functions

Let f and g be functions. We define the sum $f + g$, the difference $f - g$, and the product $f.g$ to be the functions whose domains consist of all those numbers that are common in the domains of both f and g and whose rules are given by

$(f + g)(x) = f(x) + g(x)$

$(f - g)(x) = f(x) - g(x)$

$(f.g)(x) = f(x).g(x)$.

In each case, the domain is consisting of those values of x for which both $f(x)$ and $g(x)$ are defined.

Next, because division by 0 is excluded, we give the definition of quotient of two functions separately as follows: The quotient $\frac{f}{g}$ is the function whose domain consists of all numbers x in the domains of both $f(x)$ and $g(x)$ for which $g(x) \neq 0$, and whose rule is given by

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)},\ g(x) \neq 0$$

Algebraic Functions and Their Combinations

Polynomial Function:

Any function, that can be obtained from the constant functions and the identity function by using the operations of addition, subtraction, and multiplication, is called a polynomial function. This amounts to say that "$f(x)$" is a polynomial function, if it is of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$$

where $a_n, a_{n-1}, \dots, a_2, a_1, a_0$ are real numbers, $a_n \neq 0$, and n is a nonnegative integer. If the coefficient $a_n \neq 0$, then "n" (in x^n), the nonnegative integral exponent of x, is called the degree of the polynomial. Obviously, the degree of constant functions is zero.

Constant Function:

A function of the form $f(x) = a$, where "a" is a nonzero real number and is called a constant function. The range of a constant function consists of only one nonzero number.

I. Linear Function: Polynomials of degree 1 are called linear functions. They are of the form $f(x) = a_1x + a_0$, with $a_1 \neq 0$. Note that, the identity function [$f(x) = x$] is a particular linear function.

II. $f(x) = a_2x^2 + a_1x + a_0$ is a second degree polynomial, called a quadratic function. If the degree of the polynomial is 3, the function is called a cubic function.

III. Rational Functions: Quotients of polynomials are called rational functions. Examples are as follows:

$$f(x) = \frac{1}{x^2},\ f(x) = x^3 + \sqrt{5}x\ ;\ f(x) = \frac{x^3 - 2x + p}{x - \sqrt{2}}$$

$$f(x) = \frac{x^2 + x - 2}{x^2 + 5x - 6}.$$

Example

Let $f(x) = \dfrac{x^2 + x - 2}{x^2 + 5x - 6}$. Find the domain of f.

Solution

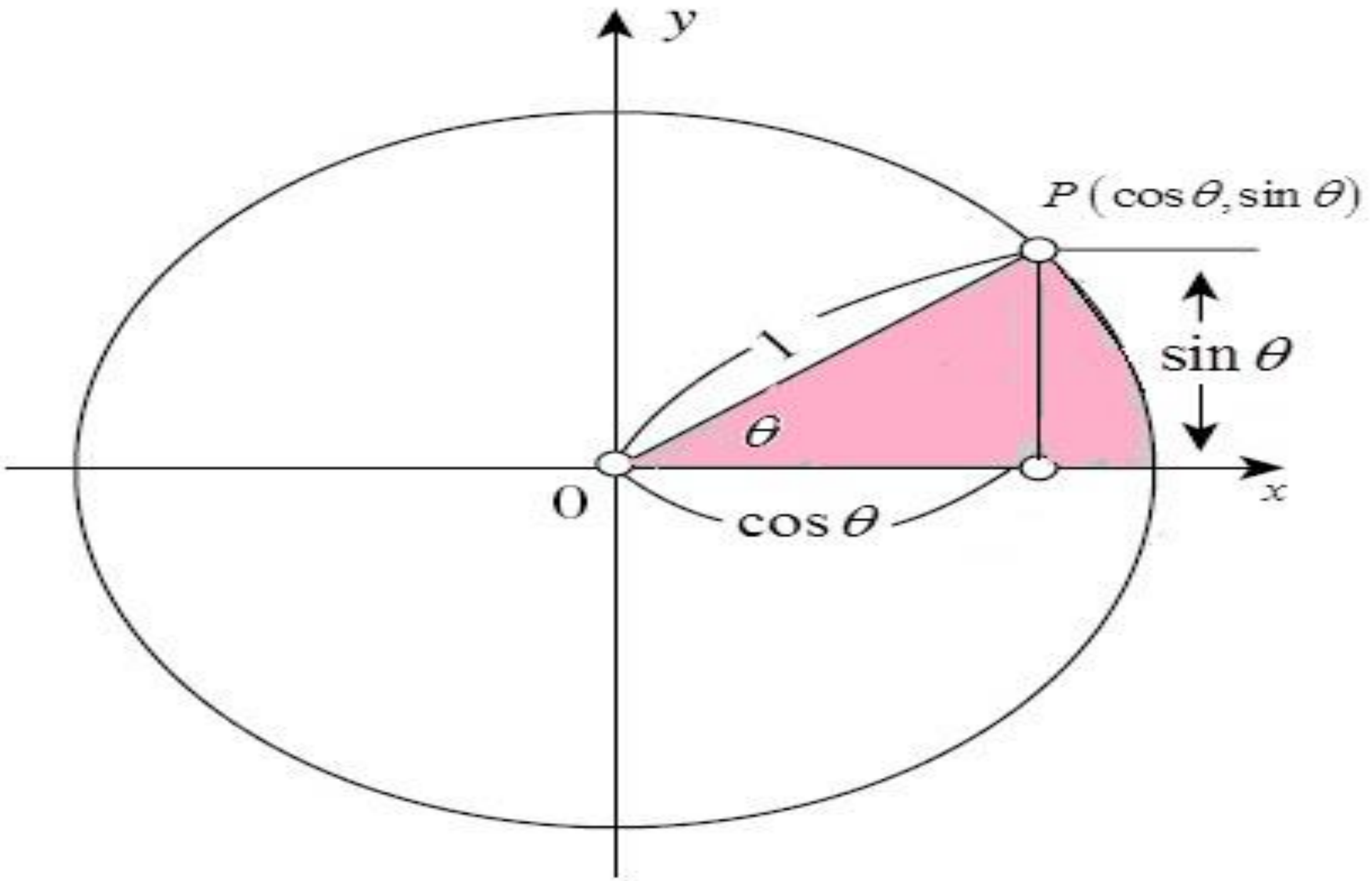
We have $x^2 + 5x - 6 = (x - 1)(x + 6)$. Therefore, the denominator is 0 for $x = 1$ and $x = -6$. Thus, the domain of f consists of all numbers except 1 and -6.

Non-algebraic Functions and Their Combinations

Trigonometric functions

Let a point $p(x, y)$ moves along a circle perimeter with radius $r = 1$ and q is the angle that the revolving line OP makes with the x-axis (see figure 1.16). Then, we can define the **sine** and **cosine** functions of q by:

$$\sin q = \frac{x}{r}, \ \cos q = \frac{y}{r}$$



$$x^2 + y^2 = 1$$

There are four other basic trigonometric functions that are defined in terms of $\sin q$ and $\cos q$, we define

$$\tan q = \frac{\sin q}{\cos q}, \quad \cot q = \frac{\cos q}{\sin q}$$

$$\sec q = \frac{1}{\cos q}, \quad \operatorname{co\,sec} q = \frac{1}{\sin q}$$

The values of these functions can be quickly computed from the corresponding values of $\sin q$ and $\cos q$.

Properties of trigonometric functions

1. Sine function

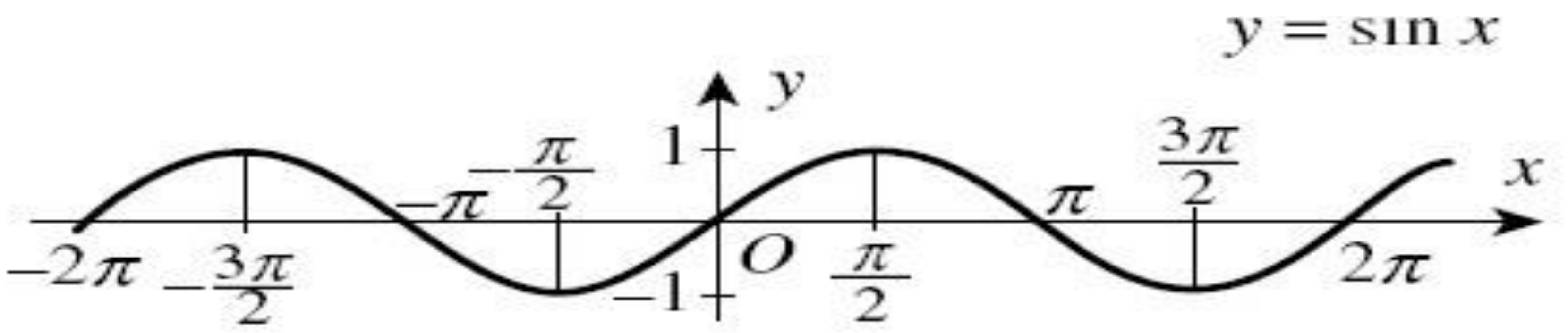
Sine function has the following properties

Its domain is R and its range is $[-1, 1]$

It is periodic function with period $2p$, that is

$$\sin(q + 2p) = \sin q.$$

It is odd function, that is, $\sin(-x) = -\sin x$.



2. Cosine function

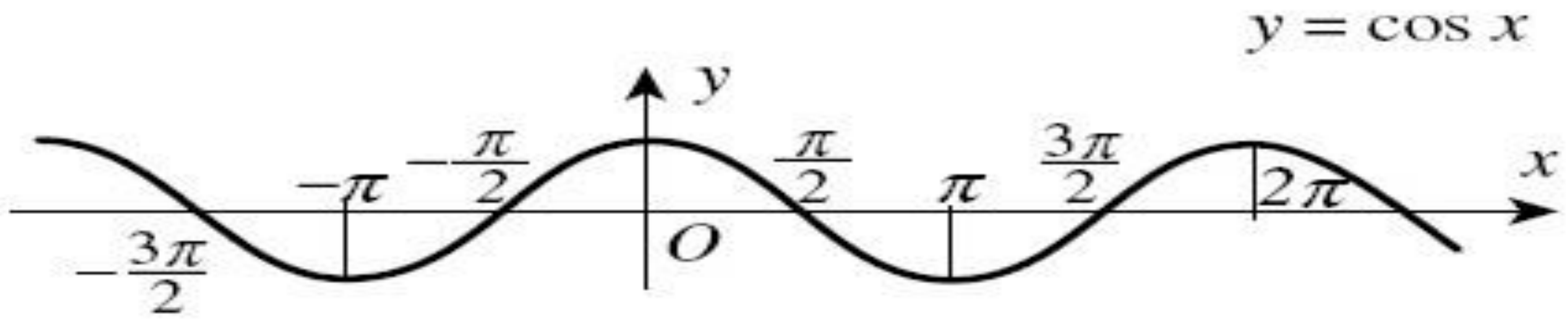
Cosine function has the following properties

Its domain is R and its range is $[-1, 1]$

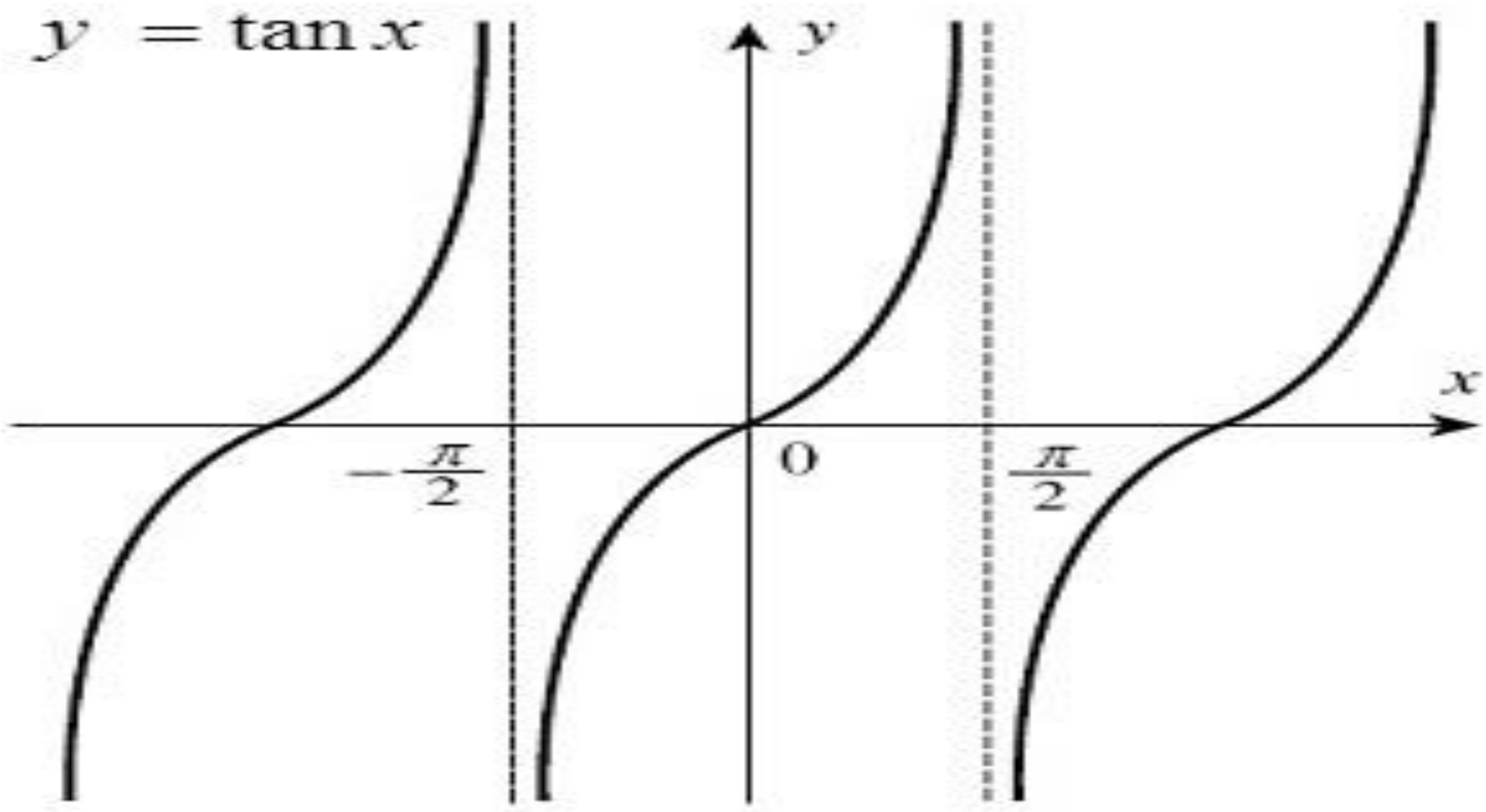
It is periodic function with period $2p$, that is

$\cos(x + 2p) = \cos x$.

It is even function, that is, $\cos(-x) = \cos x$.



Tangent function

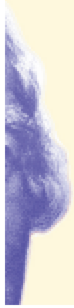


Cosecant function **Secant function** **Cotangent function**

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$

Trigonometric Identities

1. $\sin^2 x + \cos^2 x = 1$
2. $1 + \tan^2 x = \sec^2 x$
3. $1 + \cot^2 x = \operatorname{cosec}^2 x$
4. $\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x$
5. $\cos(x \pm y) = \cos x \cos y \text{ m } \sin x \sin y$
6. $\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \text{ m } \tan x \tan y}$



2 LIMITS AND CONTINUITY

LIMITS (AN INFORMAL VIEW). If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but not equal to a), then we write

$$\lim_{x \to a} f(x) = L$$

which is read "the limit of $f(x)$ as x approaches a is L."

Example the limit

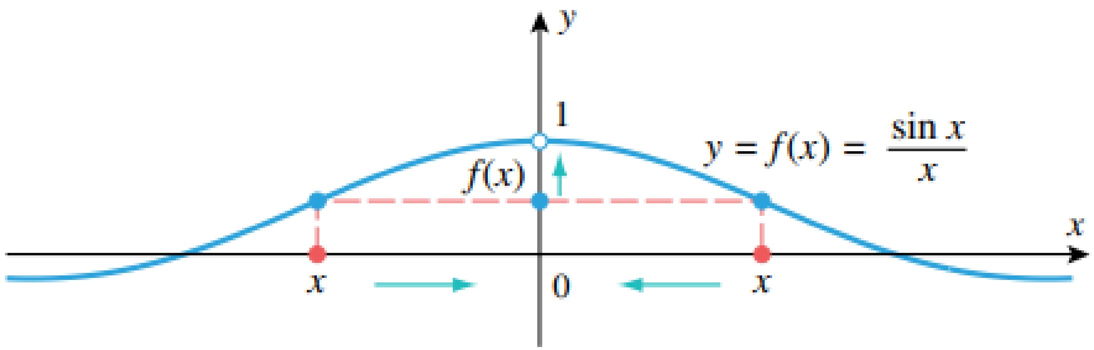
$$\lim_{x \to 0} \frac{x}{\sqrt{x+1}-1}$$

$$\left(\frac{x}{\sqrt{x+1}-1}\right)\left(\frac{\sqrt{x+1}-1}{\sqrt{x+1}-1}\right)=\frac{x(\sqrt{x+1}+1)}{x+1-1}=\frac{x(\sqrt{x+1}+1)}{x}=\sqrt{x+1}+1$$

$$\lim_{x \to 0} \frac{x}{\sqrt{x+1}-1}=2$$

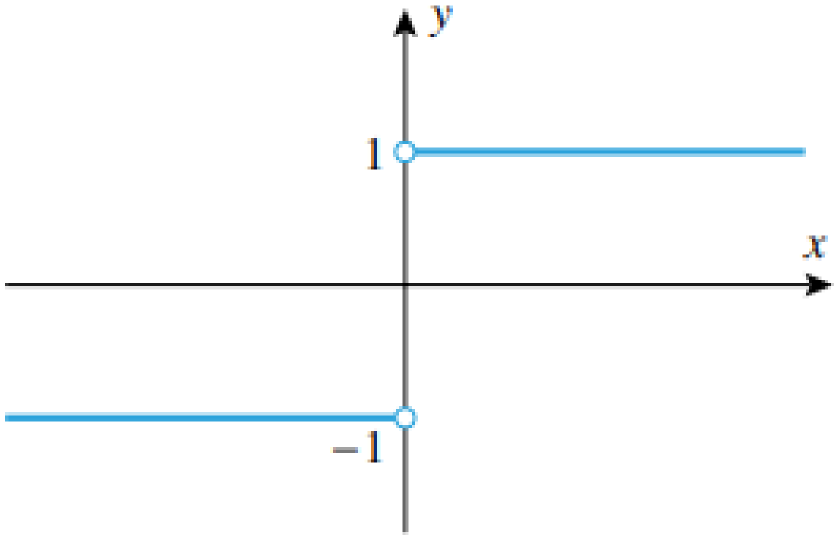
$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

x (RADIANS)	$y = \frac{\sin x}{x}$
±1.0	0.84147
±0.9	0.87036
±0.8	0.89670
±0.7	0.92031
±0.6	0.94107
±0.5	0.95885
±0.4	0.97355
±0.3	0.98507
±0.2	0.99335
±0.1	0.99833
±0.01	0.99998



As x approaches 0 from the left or right, $f(x)$ approaches 1.

ONE-SIDED LIMITS



$y = \frac{|x|}{x}$

$$\lim_{x \to 0^+} \frac{|x|}{x} = 1 \quad \text{and} \quad \lim_{x \to 0^-} \frac{|x|}{x} = -1$$

ONE-SIDED LIMITS . If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but greater than a), then we write

$$\lim_{x \to a^+} f(x) = L$$

which is read "the limit of $f(x)$ as x approaches a from the right is L." Similarly, if the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but less than a), then we write

$$\lim_{x \to a^-} f(x) = L$$

which is read "the limit of $f(x)$ as x approaches a from the left is L."

COMPUTING LIMITS

SOME BASIC LIMITS

THEOREM. *Let a and k be real numbers.*

$$\lim_{x \to a} k = k \qquad \lim_{x \to a} x = a$$

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty \qquad \lim_{x \to 0^+} \frac{1}{x} = +\infty$$

example,

$$\lim_{x \to -25} 3 = 3, \qquad \lim_{x \to 0} 3 = 3, \qquad \lim_{x \to \pi} 3 = 3$$

Example 2 Find $\lim_{x \to 2} \dfrac{5x^3 + 4}{x - 3}$.

Solution.

$$\lim_{x \to 2} \frac{5x^3 + 4}{x - 3} = \frac{\lim_{x \to 2} (5x^3 + 4)}{\lim_{x \to 2} (x - 3)}$$

$$= \frac{5 \cdot 2^3 + 4}{2 - 3} = -44$$

Example 3 Find

(a) $\lim_{x \to 4^-} \dfrac{2 - x}{(x - 4)(x + 2)}$ (b) $\lim_{x \to 4^+} \dfrac{2 - x}{(x - 4)(x + 2)}$ (c) $\lim_{x \to 4} \dfrac{2 - x}{(x - 4)(x + 2)}$

$$\lim_{x \to 4} \frac{2 - x}{(x - 4)(x + 2)} = -\infty$$

COMPUTING LIMITS: END BEHAVIOR

THEOREM. *Suppose that*

$$\lim_{x \to +\infty} f(x) = L_1 \quad \textit{and} \quad \lim_{x \to +\infty} g(x) = L_2$$

That is, the limits exist and have values L_1 *and* L_2*, respectively. Then,*

(*a*) $\lim_{x \to +\infty} [f(x) + g(x)] = \lim_{x \to +\infty} f(x) + \lim_{x \to +\infty} g(x) = L_1 + L_2$

(*b*) $\lim_{x \to +\infty} [f(x) - g(x)] = \lim_{x \to +\infty} f(x) - \lim_{x \to +\infty} g(x) = L_1 - L_2$

(*c*) $\lim_{x \to +\infty} [f(x)g(x)] = \left(\lim_{x \to +\infty} f(x)\right)\left(\lim_{x \to +\infty} g(x)\right) = L_1 L_2$

(*d*) $\lim_{x \to +\infty} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to +\infty} f(x)}{\lim_{x \to +\infty} g(x)} = \dfrac{L_1}{L_2}$, *provided* $L_2 \neq 0$

(*e*) $\lim_{x \to +\infty} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to +\infty} f(x)} = \sqrt[n]{L_1}$, *provided* $L_1 > 0$ *if* n *is even.*

Moreover, these statements are also true if $x \to -\infty$.

REMARK. As in the remark following Theorem results (a) and (c) can be extended to sums or products of any finite number of functions. In particular, for any positive integer n,

$$\lim_{x \to +\infty} (f(x))^n = \left(\lim_{x \to +\infty} f(x) \right)^n \qquad \lim_{x \to -\infty} (f(x))^n = \left(\lim_{x \to -\infty} f(x) \right)^n$$

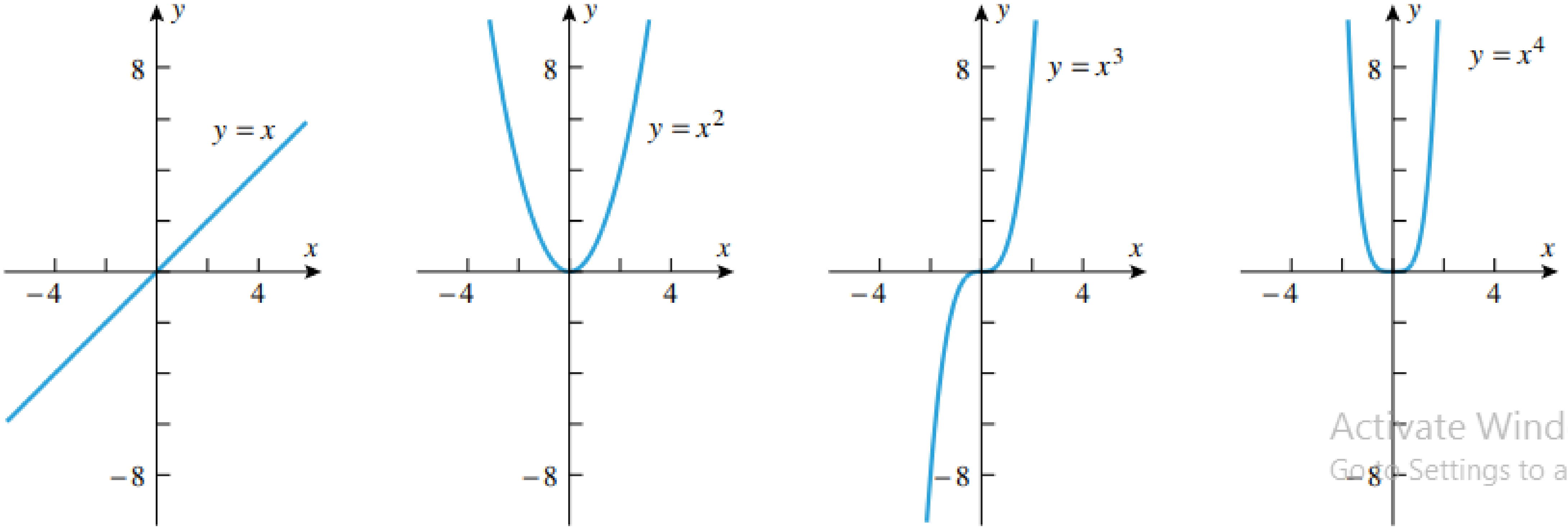
Also, since $\lim_{x \to +\infty}(1/x) = 0$, if n is a positive integer, then

$$\lim_{x \to +\infty} \frac{1}{x^n} = \left(\lim_{x \to +\infty} \frac{1}{x} \right)^n = 0 \qquad \lim_{x \to -\infty} \frac{1}{x^n} = \left(\lim_{x \to -\infty} \frac{1}{x} \right)^n = 0$$

LIMITS OF x^n AS $x \to \pm\infty$

$$\lim_{x \to +\infty} x^n = +\infty, \quad n = 1, 2, 3, \ldots$$

$$\lim_{x \to -\infty} x^n = \begin{cases} -\infty, & n = 1, 3, 5, \ldots \\ +\infty, & n = 2, 4, 6, \ldots \end{cases}$$



LIMITS OF POLYNOMIALS AS $x \to \pm\infty$

$$\lim_{x \to -\infty} (c_0 + c_1 x + \cdots + c_n x^n) = \lim_{x \to -\infty} c_n x^n$$

$$\lim_{x \to +\infty} (c_0 + c_1 x + \cdots + c_n x^n) = \lim_{x \to +\infty} c_n x^n$$

Example

$$\lim_{x \to -\infty} (7x^5 - 4x^3 + 2x - 9) = \lim_{x \to -\infty} 7x^5 = -\infty$$

$$\lim_{x \to -\infty} (-4x^8 + 17x^3 - 5x + 1) = \lim_{x \to -\infty} -4x^8 = -\infty$$

LIMITS OF RATIONAL FUNCTIONS AS $x \to \pm\infty$

Example Find $\lim_{x \to +\infty} \frac{3x+5}{6x-8}$.

Solution. Divide the numerator and denominator by the highest power of x that occurs in the denominator; that is, $x^1 = x$. We obtain

$$\lim_{x \to +\infty} \frac{3x+5}{6x-8} = \lim_{x \to +\infty} \frac{x(3+5/x)}{x(6-8/x)} = \lim_{x \to +\infty} \frac{3+5/x}{6-8/x} = \frac{\lim_{x \to +\infty}(3+5/x)}{\lim_{x \to +\infty}(6-8/x)}$$

$$= \frac{\lim_{x \to +\infty} 3 + \lim_{x \to +\infty} 5/x}{\lim_{x \to +\infty} 6 - \lim_{x \to +\infty} 8/x} = \frac{3 + 5\lim_{x \to +\infty} 1/x}{6 - 8\lim_{x \to +\infty} 1/x}$$

$$= \frac{3+(5 \cdot 0)}{6-(8 \cdot 0)} = \frac{1}{2}$$

◀

Example 4 Find

(a) $\lim_{x \to -\infty} \dfrac{4x^2 - x}{2x^3 - 5}$

Solution (a). Divide the numerator and denominator by the highest power of x that occurs in the denominator, namely x^3. We obtain

$$\lim_{x \to -\infty} \frac{4x^2 - x}{2x^3 - 5} = \lim_{x \to -\infty} \frac{x^3(4/x - 1/x^2)}{x^3(2 - 5/x^3)} = \lim_{x \to -\infty} \frac{4/x - 1/x^2}{2 - 5/x^3}$$

$$= \frac{\lim\limits_{x \to -\infty} (4/x - 1/x^2)}{\lim\limits_{x \to -\infty} (2 - 5/x^3)} = \frac{(4 \cdot 0) - 0}{2 - (5 \cdot 0)} = \frac{0}{2} = 0$$

(b) $\lim_{x \to -\infty} \dfrac{5x^3 - 2x^2 + 1}{3x + 5}$

Solution (b). Divide the numerator and denominator by x to obtain

$$\lim_{x \to -\infty} \frac{5x^3 - 2x^2 + 1}{3x + 5} = \lim_{x \to -\infty} \frac{5x^2 - 2x + 1/x}{3 + 5/x} = +\infty$$

where the final step is justified by the fact that

$$5x^2 - 2x \to +\infty, \quad \frac{1}{x} \to 0, \quad \text{and} \quad 3 + \frac{5}{x} \to 3$$

as $x \to -\infty$. ◀

LIMITS INVOLVING RADICALS

Example 5 Find $\lim_{x \to +\infty} \sqrt[3]{\frac{3x+5}{6x-8}}$.

Solution.

$$\lim_{x \to +\infty} \sqrt[3]{\frac{3x+5}{6x-8}} = \sqrt[3]{\lim_{x \to +\infty} \frac{3x+5}{6x-8}}$$

$$= \sqrt[3]{\frac{1}{2}}$$

Example 3

Example 6 Find

$$\text{(a)} \quad \lim_{x \to +\infty} \frac{\sqrt{x^2+2}}{3x-6}$$

Solution (a). As $x \to +\infty$, the values of x under consideration are positive, so we can replace $|x|$ by x where helpful. We obtain

$$\lim_{x \to +\infty} \frac{\sqrt{x^2+2}}{3x-6} = \lim_{x \to +\infty} \frac{\sqrt{x^2+2}/|x|}{(3x-6)/|x|} = \lim_{x \to +\infty} \frac{\sqrt{x^2+2}/\sqrt{x^2}}{(3x-6)/x}$$

$$= \lim_{x \to +\infty} \frac{\sqrt{1+2/x^2}}{3-6/x} = \frac{\lim_{x \to +\infty} \sqrt{1+2/x^2}}{\lim_{x \to +\infty} (3-6/x)}$$

$$= \frac{\sqrt{\lim_{x \to +\infty} (1+2/x^2)}}{\lim_{x \to +\infty} (3-6/x)} = \frac{\sqrt{\left(\lim_{x \to +\infty} 1\right) + \left(2 \lim_{x \to +\infty} 1/x^2\right)}}{\left(\lim_{x \to +\infty} 3\right) - \left(6 \lim_{x \to +\infty} 1/x\right)}$$

$$= \frac{\sqrt{1+(2 \cdot 0)}}{3-(6 \cdot 0)} = \frac{1}{3}$$

(b) $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{x^2+2}}{3x-6}$

Solution (b). As $x \to -\infty$, the values of x under consideration are negative, so we can replace $|x|$ by $-x$ where helpful. We obtain

$$\lim_{x\to-\infty}\frac{\sqrt{x^2+2}}{3x-6}=\lim_{x\to-\infty}\frac{\sqrt{x^2+2}/|x|}{(3x-6)/|x|}=\lim_{x\to-\infty}\frac{\sqrt{x^2+2}/\sqrt{x^2}}{(3x-6)/(-x)}$$

$$=\lim_{x\to-\infty}\frac{\sqrt{1+2/x^2}}{-3+6/x}=-\frac{1}{3}$$

◀

Continuity

DEFINITION A function is **continuous at a number *a*** if

1. $f(a)$ is defined (that is, a is in the domain of f)
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

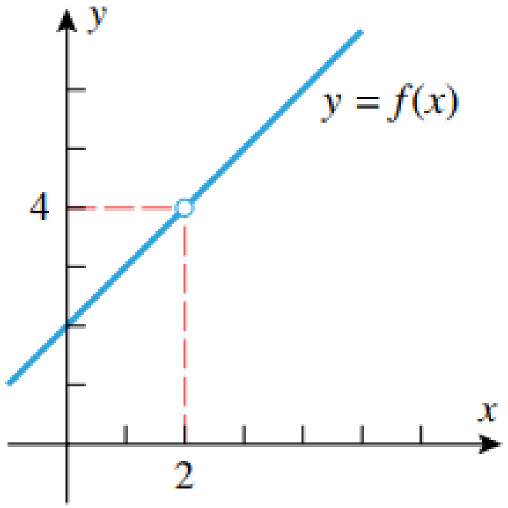
THEOREM. *Polynomials are continuous everywhere.*

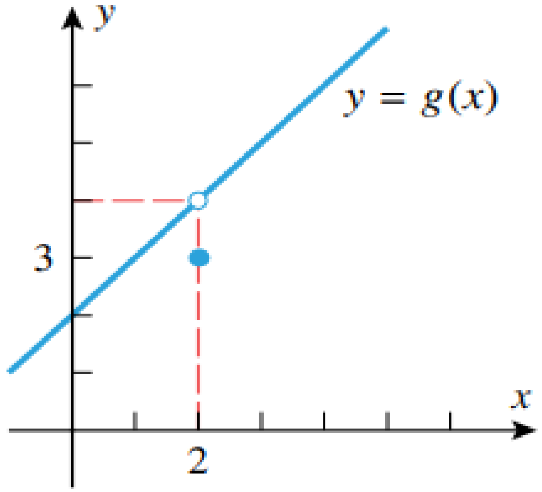
THEOREM. *A rational function is continuous at every number where the denominator is nonzero.*

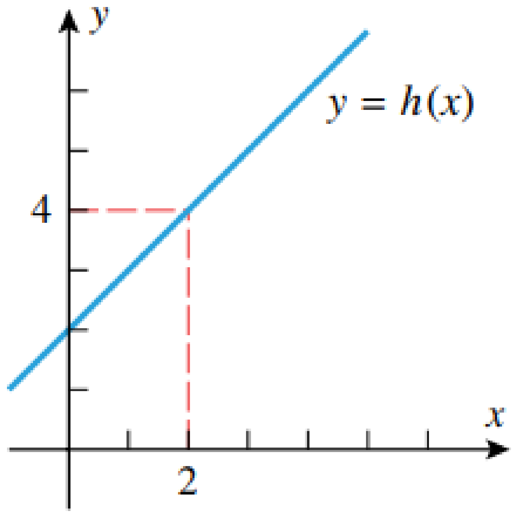
Example Determine whether the following functions are continuous at $x = 2$.

$$f(x) = \frac{x^2 - 4}{x - 2}, \qquad g(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 3, & x = 2, \end{cases} \qquad h(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} g(x) = \lim_{x \to 2} h(x) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} (x + 2) = 4$$

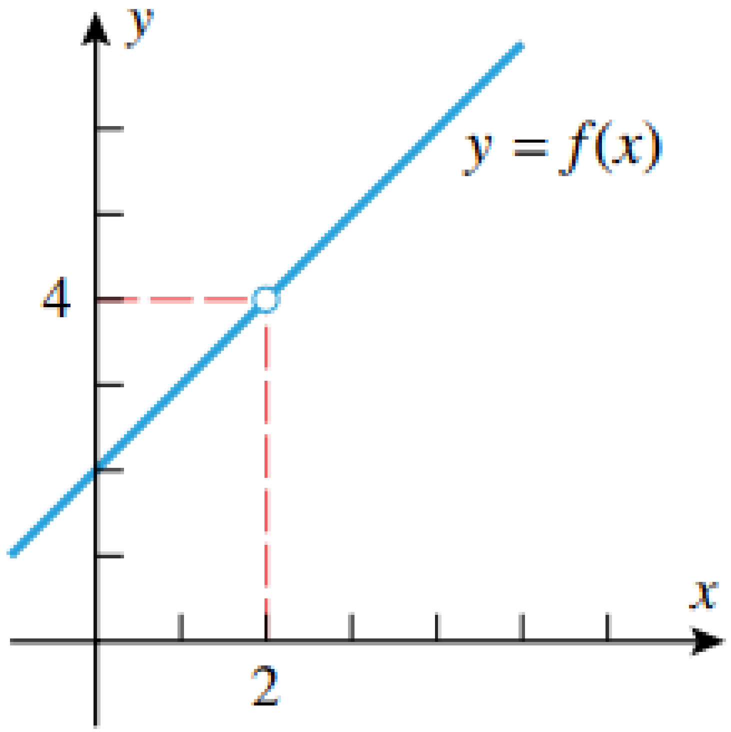






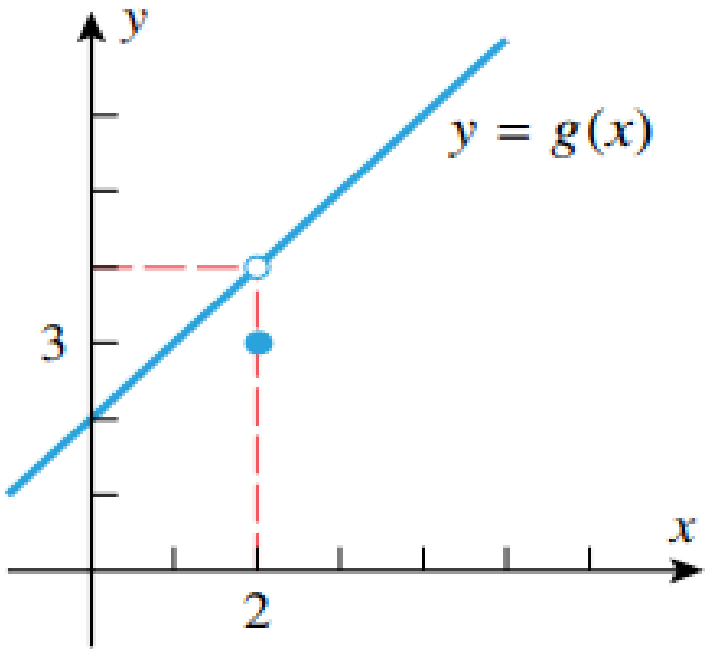
$$f(x) = \frac{x^2 - 4}{x - 2},$$

$$g(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 3, & x = 2, \end{cases}$$



The function f is undefined at $x = 2$,

and hence is not continuous at $x = 2$

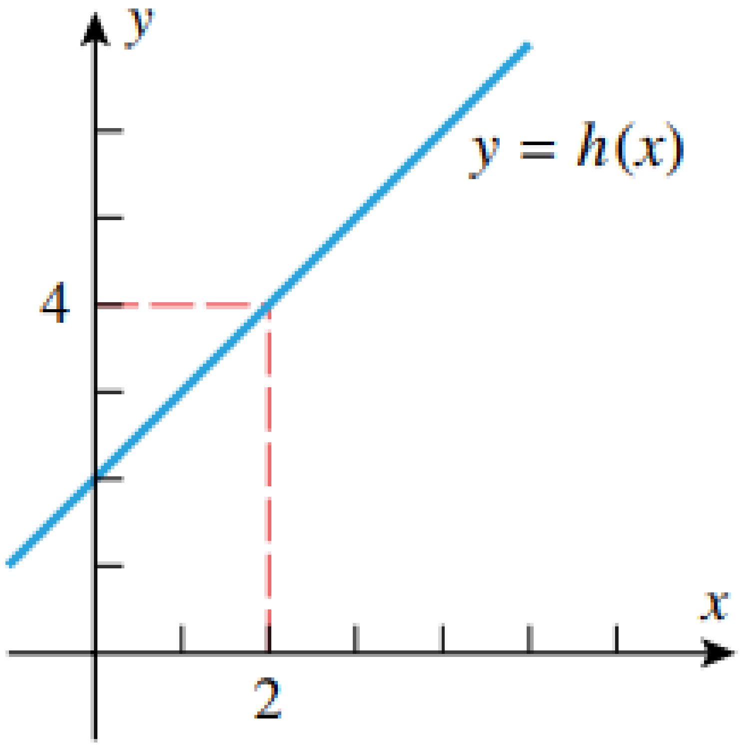


The function g is defined at $x = 2$,

but its value there is $g(2) = 3$,

hence, g is also not continuous at $x = 2$

$$h(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$$



The value of the function h at $x = 2$ is $h(2) = 4$,

h is continuous at $x = 2$

1. $f(a)$ is defined (that is, a is in the domain of f)
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

EXAMPLE Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$

between 1 and 2

SOLUTION Let

$$f(x) = 4x^3 - 6x^2 + 3x - 2.$$

We are looking for a solution of the given equation, that is,
a number ***c*** between 1 and 2 such that

$$f(c) = 0.$$

$$f(1) = 4 - 6 + 3 - 2 = -1 < 0$$

$$f(2) = 32 - 24 + 6 - 2 = 12 > 0$$

Thus $f(1) < 0 < f(2)$;

Now *f* (x) is continuous since it is a polynomial,

So is, a number ***c*** between 1 and 2 such that there is a root at it

SOME PROPERTIES OF CONTINUOUS FUNCTIONS

THEOREM. *If the functions f and g are continuous at c, then*

(a) $f+g$ *is continuous at* c.

(b) $f-g$ *is continuous at* c.

(c) fg *is continuous at* c.

(d) f/g *is continuous at* c *if* $g(c) \neq 0$ *and has a discontinuity at* c *if* $g(c) = 0$.

THEOREM. *A rational function is continuous at every number where the denominator is nonzero.*

Example 3 For what values of x is there a hole or a gap in the graph of

$$y = \frac{x^2 - 9}{x^2 - 5x + 6}?$$

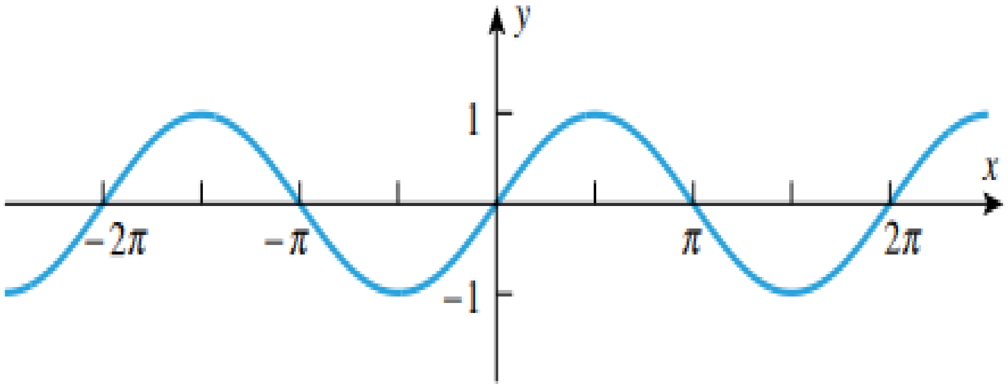
Solution. The function being graphed is a rational function, and hence is continuous at every number where the denominator is nonzero. Solving the equation

$$x^2 - 5x + 6 = 0$$

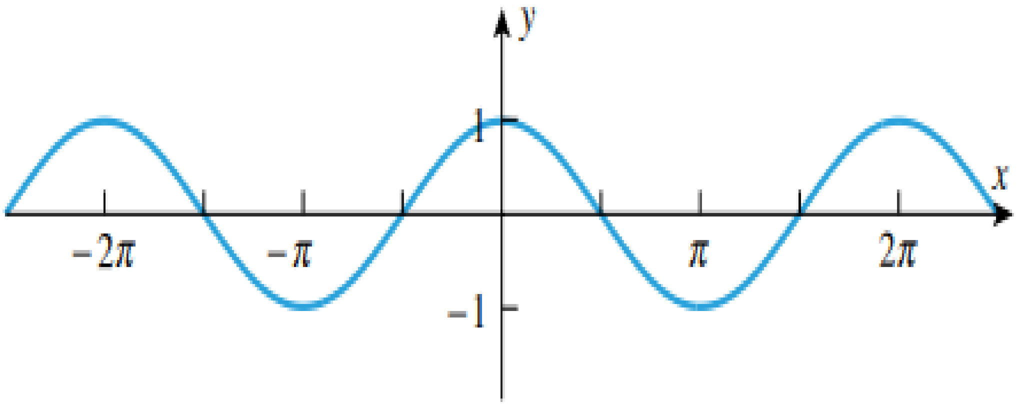
yields discontinuities at $x = 2$ and at $x = 3$. ◀

LIMITS AND CONTINUITY OF TRIGONOMETRIC FUNCTIONS

$$\lim_{x \to c} \sin x = \sin c \quad \text{and} \quad \lim_{x \to c} \cos x = \cos c$$



y = sin x



y = cos x

THEOREM. *If c is any number in the natural domain of the stated trigonometric function, then*

$$\lim_{x \to c} \sin x = \sin c \qquad \lim_{x \to c} \cos x = \cos c \qquad \lim_{x \to c} \tan x = \tan c$$

$$\lim_{x \to c} \csc x = \csc c \qquad \lim_{x \to c} \sec x = \sec c \qquad \lim_{x \to c} \cot x = \cot c$$

Example 1 Find the limit

$$\lim_{x \to 1} \cos\left(\frac{x^2 - 1}{x - 1}\right)$$

Solution. Recall from the last section that since the cosine function is continuous everywhere,

$$\lim_{x \to 1} \cos(g(x)) = \cos(\lim_{x \to 1} g(x))$$

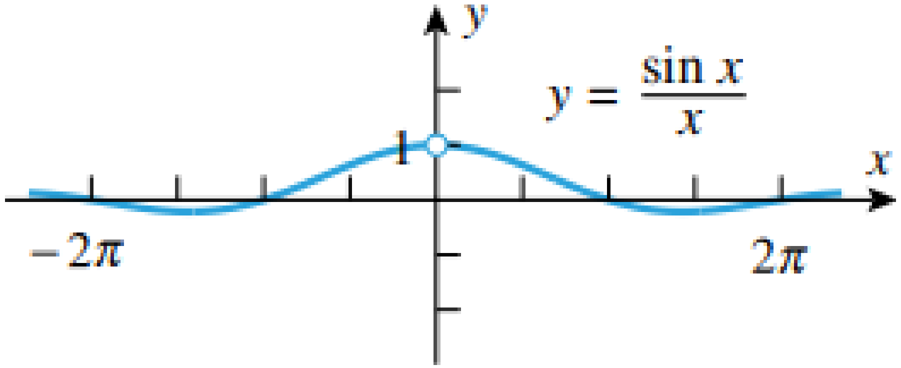
provided $\lim_{x \to 1} g(x)$ exists. Thus,

$$\lim_{x \to 1} \cos\left(\frac{x^2 - 1}{x - 1}\right) = \lim_{x \to 1} \cos(x + 1) = \cos\left(\lim_{x \to 1} (x + 1)\right) = \cos 2 \qquad \blacktriangleleft$$

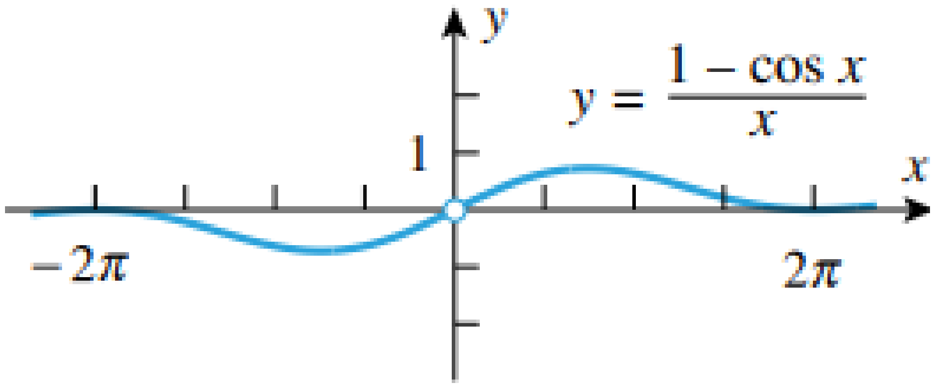
THEOREM.

(a) $$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

(b) $$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$



$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$



$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

***Proof* (*b*).** For this proof we will use the limit in part (*a*), the continuity of the sine function, and the trigonometric identity $\sin^2 x = 1 - \cos^2 x$. We obtain

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \left[\frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x}\right] = \lim_{x\to 0} \frac{\sin^2 x}{(1+\cos x)x}$$

$$= \left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{x\to 0} \frac{\sin x}{1+\cos x}\right) = (1)\left(\frac{0}{1+1}\right) = 0 \qquad \blacksquare$$

Example 2 Find

(a) $\displaystyle\lim_{x\to 0} \frac{\tan x}{x}$ (b) $\displaystyle\lim_{\theta\to 0} \frac{\sin 2\theta}{\theta}$ (c) $\displaystyle\lim_{x\to 0} \frac{\sin 3x}{\sin 5x}$

Solution (a).

$$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \left(\frac{\sin x}{x} \cdot \frac{1}{\cos x} \right) = (1)(1) = 1$$

Solution (b). The trick is to multiply and divide by 2, which will make the denominator the same as the argument of the sine function [just as in Theorem 2.6.3(*a*)]:

$$\lim_{\theta \to 0} \frac{\sin 2\theta}{\theta} = \lim_{\theta \to 0} 2 \cdot \frac{\sin 2\theta}{2\theta} = 2 \lim_{\theta \to 0} \frac{\sin 2\theta}{2\theta}$$

Now make the substitution $x = 2\theta$, and use the fact that $x \to 0$ as $\theta \to 0$. This yields

$$\lim_{\theta \to 0} \frac{\sin 2\theta}{\theta} = 2 \lim_{\theta \to 0} \frac{\sin 2\theta}{2\theta} = 2 \lim_{x \to 0} \frac{\sin x}{x} = 2(1) = 2$$

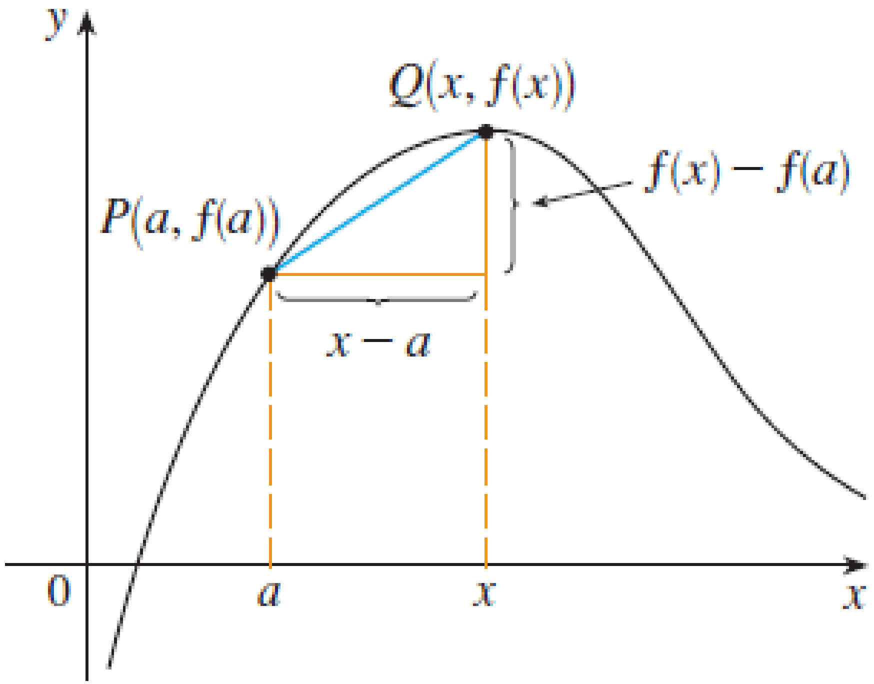
Solution (c).

$$\lim_{x \to 0} \frac{\sin 3x}{\sin 5x} = \lim_{x \to 0} \frac{\dfrac{\sin 3x}{x}}{\dfrac{\sin 5x}{x}} = \lim_{x \to 0} \frac{3 \cdot \dfrac{\sin 3x}{3x}}{5 \cdot \dfrac{\sin 5x}{5x}} = \frac{3 \cdot 1}{5 \cdot 1} = \frac{3}{5}$$

Derivatives and rates of change

DEFINITION The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$



DEFINITION The **derivative of a function** f **at a number** a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

EXAMPLE Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a.

$$
\begin{aligned}
f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h \to 0} \frac{[(a+h)^2 - 8(a+h) + 9] - [a^2 - 8a + 9]}{h} \\
&= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h} \\
&= \lim_{h \to 0} \frac{2ah + h^2 - 8h}{h} = \lim_{h \to 0} (2a + h - 8) \\
&= 2a - 8
\end{aligned}
$$

DERIVATIVE NOTATION

$$\frac{d}{dx}[f(x)] = f'(x)$$

$$\left.\frac{d}{dx}[f(x)]\right|_{x=x_0} = f'(x_0)$$

Find the derivative with respect to x of $f(x) = x^3 - x$.

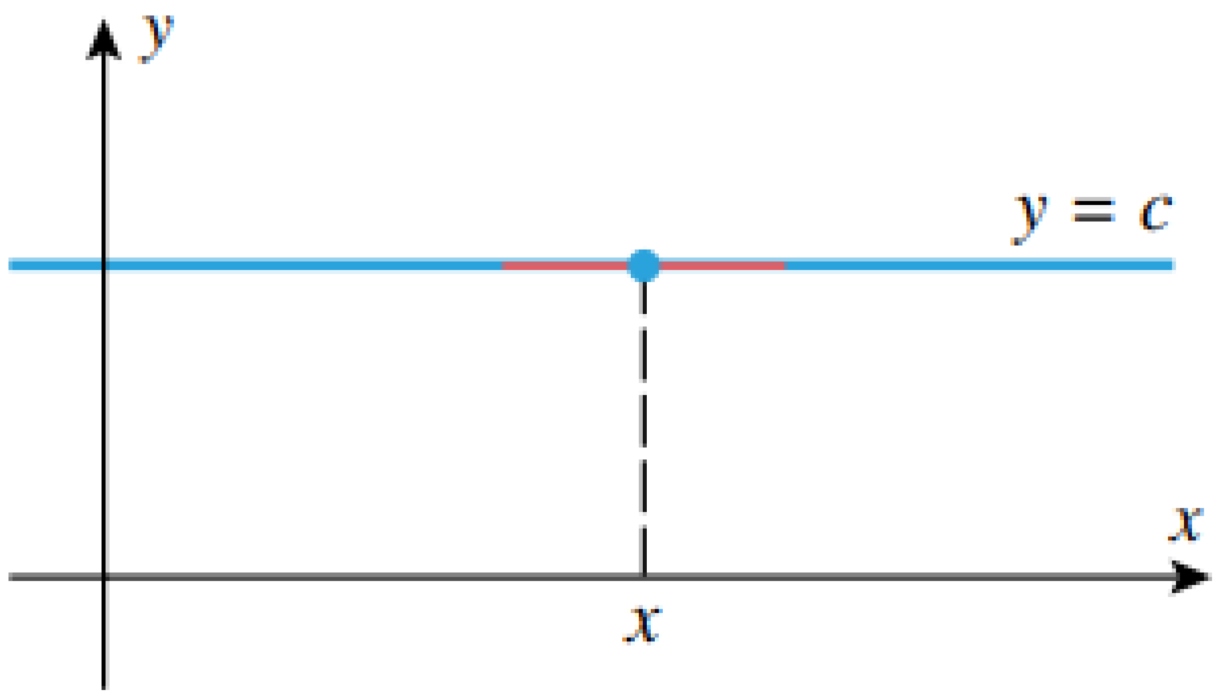
$$f'(x) = 3x^2 - 1$$

$$\frac{d}{dx}[x^3 - x]\bigg|_{x=1} = 3(1^2) - 1 = 2,$$

TECHNIQUES OF DIFFERENTIATION

THEOREM. *The derivative of a constant function is* 0; *that is, if c is any real number, then*

$$\frac{d}{dx}[c] = 0$$



Example $\frac{d}{dx}[5] = 0$

THEOREM (***The Power Rule***).

If n is any integer, then

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

Example

$$\frac{d}{dx}[x^5] = 5x^4, \quad \frac{d}{dx}[x] = 1 \cdot x^0 = 1, \quad \frac{d}{dx}[x^{12}] = 12x^{11}$$

$$\frac{d}{dx}[x^{-9}] = -9x^{-9-1} = -9x^{-10}$$

$$\frac{d}{dx}\left[\frac{1}{x}\right] = \frac{d}{dx}[x^{-1}] = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

THEOREM. *If f is differentiable at x and c is any real number, then cf is also differentiable at x and*

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

Example

$$\frac{d}{dx}[4x^8] = 4\frac{d}{dx}[x^8] = 4[8x^7] = 32x^7$$

$$\frac{d}{dx}[-x^{12}] = (-1)\frac{d}{dx}[x^{12}] = -12x^{11}$$

$$\frac{d}{dx}\left[\frac{x}{\pi}\right] = \frac{1}{\pi}\frac{d}{dx}[x] = \frac{1}{\pi}$$

THEOREM. *If f and g are differentiable at x, then so are $f+g$ and $f-g$ and*

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$$

Example

$$\frac{d}{dx}[x^4+x^2] = \frac{d}{dx}[x^4] + \frac{d}{dx}[x^2] = 4x^3+2x$$

$$\frac{d}{dx}[6x^{11}-9] = \frac{d}{dx}[6x^{11}] - \frac{d}{dx}[9] = 66x^{10}-0 = 66x^{10}$$

THEOREM (*The Product Rule*).

If f and g are differentiable at x, then so is the product $f \cdot g$, and

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

The product rule can be written in function notation as

$$(f \cdot g)' = f \cdot g' + g \cdot f'$$

Example Find dy/dx if $y = (4x^2 - 1)(7x^3 + x)$.

Method I. (*Using the Product Rule*)

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[(4x^2 - 1)(7x^3 + x)] \\ &= (4x^2 - 1)\frac{d}{dx}[7x^3 + x] + (7x^3 + x)\frac{d}{dx}[4x^2 - 1] \\ &= (4x^2 - 1)(21x^2 + 1) + (7x^3 + x)(8x) = 140x^4 - 9x^2 - 1\end{aligned}$$

Method II. (*Multiplying First*)

$$y = (4x^2 - 1)(7x^3 + x) = 28x^5 - 3x^3 - x$$

Thus,

$$\frac{dy}{dx} = \frac{d}{dx}[28x^5 - 3x^3 - x] = 140x^4 - 9x^2 - 1$$

THEOREM (***The Quotient Rule***).

If f and g are differentiable at x and $g(x) \neq 0$,

then f/g is differentiable at x and

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

The quotient rule can be written in function notation as

$$\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2}$$

Example

$$f(x) = \frac{x^2 - 1}{x^4 + 1}.$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x^2 - 1}{x^4 + 1}\right] = \frac{(x^4 + 1)\frac{d}{dx}[x^2 - 1] - (x^2 - 1)\frac{d}{dx}[x^4 + 1]}{(x^4 + 1)^2}$$

$$= \frac{(x^4 + 1)(2x) - (x^2 - 1)(4x^3)}{(x^4 + 1)^2}$$

$$= \frac{-2x^5 + 4x^3 + 2x}{(x^4 + 1)^2} = -\frac{2x(x^4 - 2x^2 - 1)}{(x^4 + 1)^2}$$

HIGHER DERIVATIVES

$$f', \quad f'' = (f')', \quad f''' = (f'')', \quad f^{(4)} = (f''')', \quad f^{(5)} = (f^{(4)})', \ldots$$

Example If $f(x) = 3x^4 - 2x^3 + x^2 - 4x + 2$, then

$$\begin{aligned}
f'(x) &= 12x^3 - 6x^2 + 2x - 4 \\
f''(x) &= 36x^2 - 12x + 2 \\
f'''(x) &= 72x - 12 \\
f^{(4)}(x) &= 72 \\
f^{(5)}(x) &= 0
\end{aligned}$$

Successive derivatives can also be denoted as follows:

$$f'(x) = \frac{d}{dx}[f(x)]$$

$$f''(x) = \frac{d}{dx}\left[\frac{d}{dx}[f(x)]\right] = \frac{d^2}{dx^2}[f(x)]$$

$$f'''(x) = \frac{d}{dx}\left[\frac{d^2}{dx^2}[f(x)]\right] = \frac{d^3}{dx^3}[f(x)]$$

$$\vdots \qquad \vdots$$

In general, we write

$$f^{(n)}(x) = \frac{d^n}{dx^n}[f(x)]$$

$$\frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \frac{d^3y}{dx^3}, \quad \frac{d^4y}{dx^4}, \ldots, \frac{d^ny}{dx^n}, \ldots$$

or more briefly,

$$y', \quad y'', \quad y''', \quad y^{(4)}, \ldots, y^{(n)}, \ldots$$

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

Example Find $f'(x)$ if $f(x) = x^2 \tan x$.

$$f'(x) = x^2 \cdot \frac{d}{dx}[\tan x] + \tan x \cdot \frac{d}{dx}[x^2] = x^2 \sec^2 x + 2x \tan x$$

Example Find dy/dx if $y = \dfrac{\sin x}{1 + \cos x}$.

$$\frac{dy}{dx} = \frac{(1 + \cos x) \cdot \dfrac{d}{dx}[\sin x] - \sin x \cdot \dfrac{d}{dx}[1 + \cos x]}{(1 + \cos x)^2}$$

$$= \frac{(1 + \cos x)(\cos x) - (\sin x)(-\sin x)}{(1 + \cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1 + \cos x)^2} = \frac{\cos x + 1}{(1 + \cos x)^2} = \frac{1}{1 + \cos x}$$

Example Find $y''(\pi/4)$ if $y(x) = \sec x$.

$$
\begin{aligned}
y'(x) &= \sec x \tan x \\
y''(x) &= \sec x \cdot \frac{d}{dx}[\tan x] + \tan x \cdot \frac{d}{dx}[\sec x] \\
&= \sec x \cdot \sec^2 x + \tan x \cdot \sec x \tan x \\
&= \sec^3 x + \sec x \tan^2 x
\end{aligned}
$$

Thus,

$$
\begin{aligned}
y''(\pi/4) &= \sec^3(\pi/4) + \sec(\pi/4)\tan^2(\pi/4) \\
&= (\sqrt{2})^3 + (\sqrt{2})(1)^2 = 3\sqrt{2}
\end{aligned}
$$

DERIVATIVE OF THE NATURAL EXPONENTIAL FUNCTION

$$\frac{d}{dx}(e^x) = e^x$$

EXAMPLE If $f(x) = e^x - x$, find f' and f''.

$$f'(x) = \frac{d}{dx}(e^x - x) = \frac{d}{dx}(e^x) - \frac{d}{dx}(x) = e^x - 1$$

$$f''(x) = \frac{d}{dx}(e^x - 1) = \frac{d}{dx}(e^x) - \frac{d}{dx}(1) = e^x$$

The chain rule

THEOREM (**The Chain Rule**). *If g is differentiable at x and f is differentiable at $g(x)$, then the composition $f \circ g$ is differentiable at x. Moreover,*

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

Alternatively, if

$$y = f(g(x)) \quad \text{and} \quad u = g(x)$$

then $y = f(u)$ and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{d}{dx}[f(g(x))] = \underbrace{f'(g(x))}_{\text{Derivative of the outside evaluated at the inside}} \cdot \underbrace{g'(x)}_{\text{Derivative of the inside}}$$

$$\frac{d}{dx}[f(g(x))] = \underbrace{f'(g(x))}_{\text{Derivative of the outside evaluated at the inside}} \cdot \underbrace{g'(x)}_{\text{Derivative of the inside}}$$

For example,

$$\frac{d}{dx}[\cos(x^2+9)] = \underbrace{-\sin(x^2+9)}_{\text{Derivative of the outside evaluated at the inside}} \cdot \underbrace{2x}_{\text{Derivative of the inside}}$$

$$\frac{d}{dx}[f(g(x))] = \underbrace{f'(g(x))}_{\text{Derivative of the outside evaluated at the inside}} \cdot \underbrace{g'(x)}_{\text{Derivative of the inside}}$$

$$\frac{d}{dx}[\tan^2 x] = \frac{d}{dx}[(\tan x)^2] = \underbrace{(2\tan x)}_{\text{Derivative of the outside evaluated at the inside}} \cdot \underbrace{(\sec^2 x)}_{\text{Derivative of the inside}} = 2\tan x \sec^2 x$$

IMPLICIT DIFFERENTIATION

An equation of the form $y = f(x)$ is said to define y ***explicitly*** as a function of x because the variable y appears alone on one side of the equation.

$$f(x) = \frac{x-1}{x+1}$$

$$x^2 + y^2 = 1 \qquad\qquad yx + y + 1 = x$$

define y ***implicitly*** as a function of x.

Example 1 Use implicit differentiation to find dy/dx if $5y^2 + \sin y = x^2$.

$$\frac{d}{dx}[5y^2 + \sin y] = \frac{d}{dx}[x^2]$$

$$5\frac{d}{dx}[y^2] + \frac{d}{dx}[\sin y] = 2x$$

$$5\left(2y\frac{dy}{dx}\right) + (\cos y)\frac{dy}{dx} = 2x$$

The chain rule was used here because y is a function of x.

$$10y\frac{dy}{dx} + (\cos y)\frac{dy}{dx} = 2x$$

Solving for dy/dx we obtain $$\frac{dy}{dx} = \frac{2x}{10y + \cos y}$$

Example Find the slopes of the curve $y^2 - x + 1 = 0$ at the points $(2, -1)$ and $(2, 1)$.

$$\frac{d}{dx}[y^2 - x + 1] = \frac{d}{dx}[0]$$

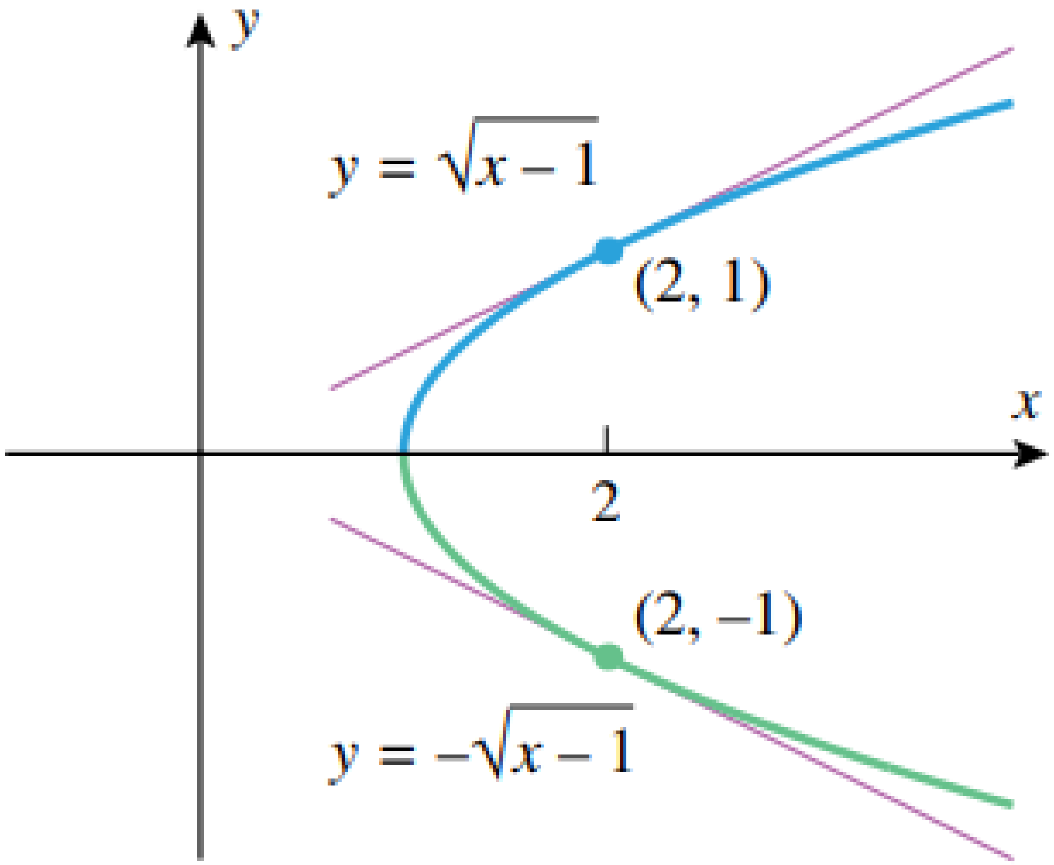
$$\frac{d}{dx}[y^2] - \frac{d}{dx}[x] + \frac{d}{dx}[1] = \frac{d}{dx}[0]$$

$$2y\frac{dy}{dx} - 1 = 0$$

$$\frac{dy}{dx} = \frac{1}{2y}$$

At $(2, -1)$ we have $y = -1$, and at $(2, 1)$ we have $y = 1$, so the slopes of the curve at those points are

$$\left.\frac{dy}{dx}\right|_{\substack{x=2\\y=-1}} = -\frac{1}{2} \quad \text{and} \quad \left.\frac{dy}{dx}\right|_{\substack{x=2\\y=1}} = \frac{1}{2}$$

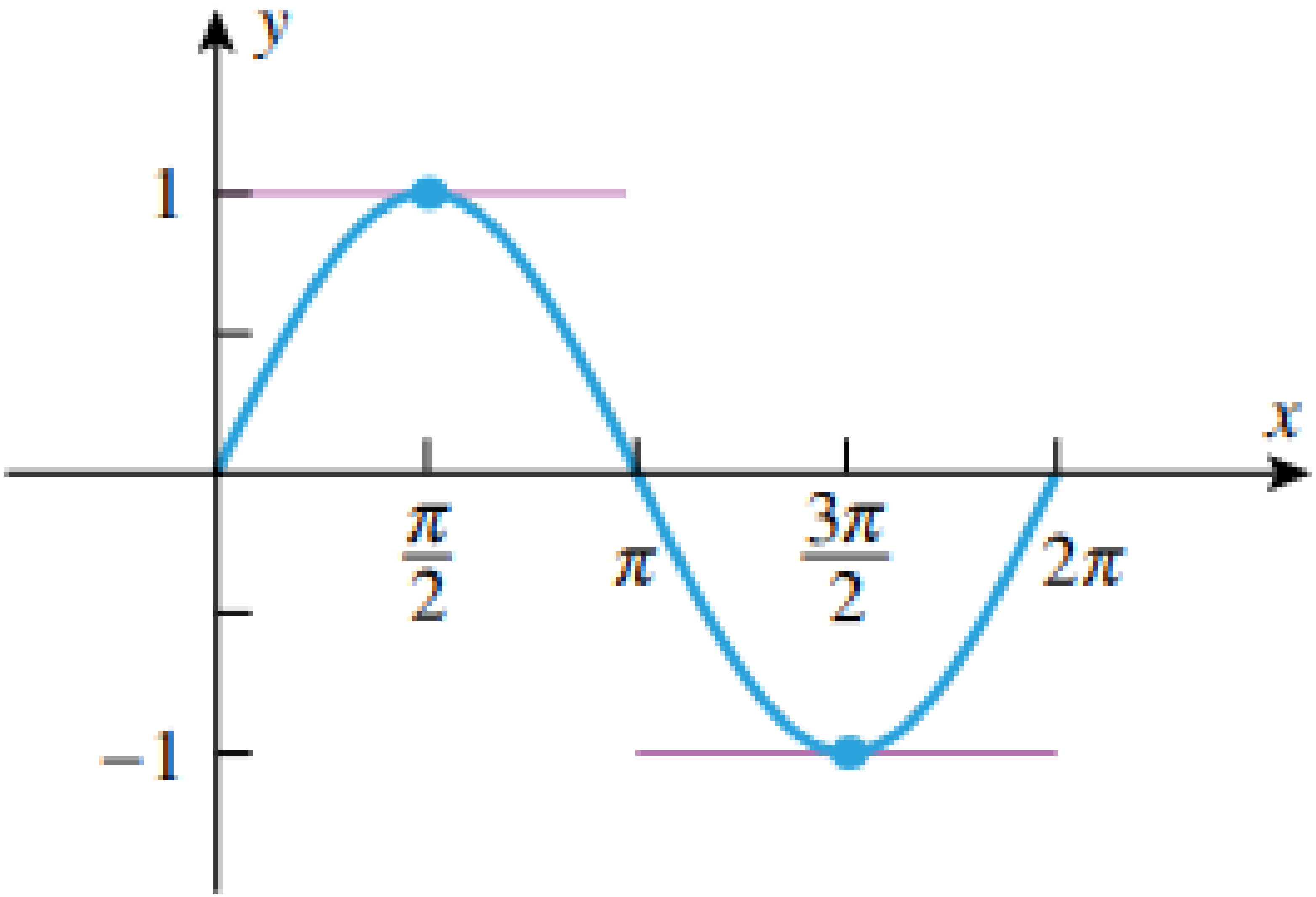


Roll’s theorem and the mean value theorem

THEOREM (***Rolle’s Theorem***). *Let f be differentiable on (a, b) and continuous on $[a, b]$. If $f(a) = f(b) = 0$, then there is at least one number c in (a, b) such that $f'(c) = 0$.*

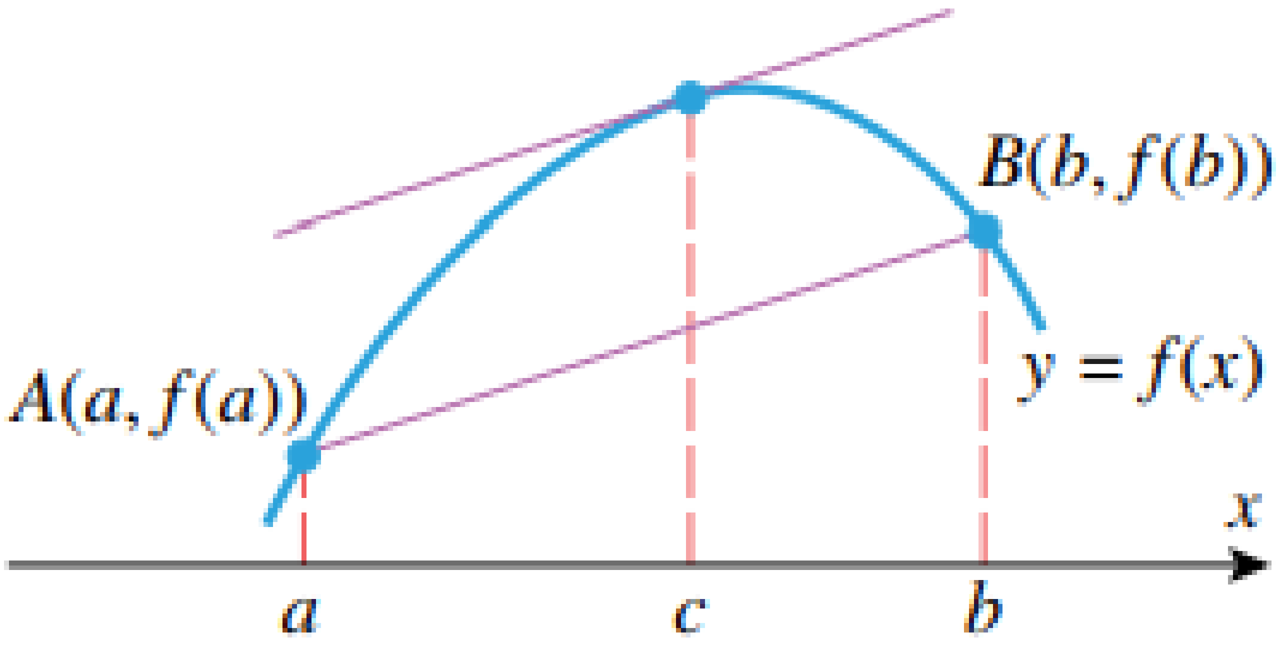
Example The function $f(x) = \sin x$ has roots at $x = 0$ and $x = 2\pi$. Verify the hypotheses and conclusion of Rolle’s Theorem for $f(x) = \sin x$ on $[0, 2\pi]$.

Solution. Since f is continuous and differentiable everywhere, it is differentiable on $(0, 2\pi)$ and continuous on $[0, 2\pi]$. Thus, Rolle’s Theorem guarantees that there is at least one number c in the interval $(0, 2\pi)$ such that $f'(c) = 0$. Since $f'(x) = \cos x$, we can find c by solving the equation $\cos c = 0$ on the interval $(0, 2\pi)$. This yields two values for c, namely $c_1 = \pi/2$ and $c_2 = 3\pi/2$ (Figure 4.8.2).


y
1
x
π/2
π
3π/2
2π
−1

THEOREM (***Mean-Value Theorem***). *Let f be differentiable on (a, b) and continuous on $[a, b]$. Then there is at least one number c in (a, b) such that*

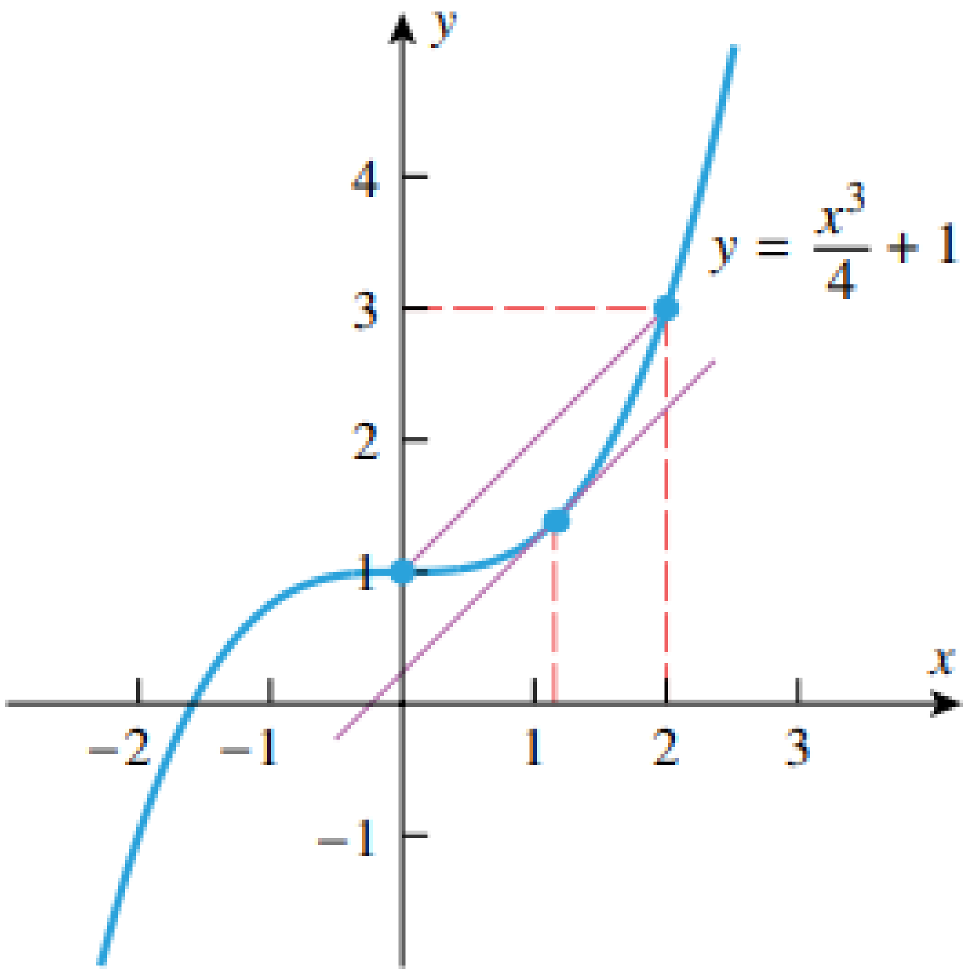
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Example

(a) Generate the graph of $f(x) = (x^3/4) + 1$ over the interval $[0, 2]$, and use it to determine the number of tangent lines to the graph of f over the interval $(0, 2)$ that are parallel to the secant line joining the endpoints of the graph.

(b) Show that f satisfies the hypotheses of the Mean-Value Theorem on the interval $[0, 2]$, and find all values of c in the interval $(0, 2)$ whose existence is guaranteed by the Mean-Value Theorem. Confirm that these values of c are consistent with your graph in part (a).



y
4
3
2
1
−1
x
−2
−1
1
2
3
$y = \frac{x^3}{4} + 1$

Solution (a). The graph of f in Figure suggests that there is only one tangent line over the interval (0, 2) that is parallel to the secant line joining the endpoints.

Solution (b). The function f is continuous and differentiable everywhere because it is a polynomial. In particular, f is continuous on [0, 2] and differentiable on (0, 2), so the hypotheses of the Mean-Value Theorem are satisfied with $a = 0$ and $b = 2$. But

$$f(a) = f(0) = 1, \quad f(b) = f(2) = 3$$

$$f'(x) = \frac{3x^2}{4}, \qquad f'(c) = \frac{3c^2}{4}$$

so in this case Equation (1) becomes

$$\frac{3c^2}{4} = \frac{3-1}{2-0} \quad \text{or} \quad 3c^2 = 4$$

which has the two solutions $c = \pm 2/\sqrt{3} \approx \pm 1.15$. However, only the positive solution lies in the interval (0, 2); this value of c is consistent with Figure

In general, an **exponential function** is a function of the form

$$f(x) = a^x$$

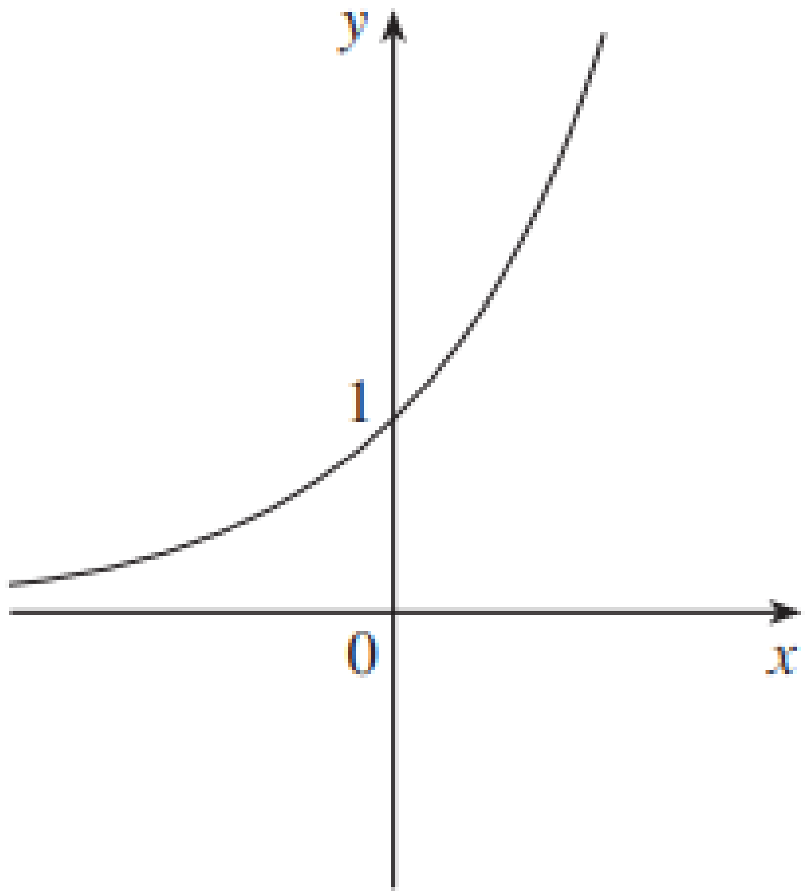
where a is a positive constant. Let's recall what this means.

If $x = n$, a positive integer, then

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

LAWS OF EXPONENTS If a and b are positive numbers and x and y are any real numbers, then

1. $a^{x+y} = a^x a^y$ **2.** $a^{x-y} = \frac{a^x}{a^y}$ **3.** $(a^x)^y = a^{xy}$ **4.** $(ab)^x = a^x b^x$



$y = e^x$

INVERSE FUNCTIONS AND LOGARITHMS

the formulation of an inverse function given by

$$f^{-1}(x) = y \iff f(y) = x$$

$$\log_a x = y \iff a^y = x$$

$$\log_a(a^x) = x \quad \text{for every } x \in \mathbb{R}$$

$$a^{\log_a x} = x \quad \text{for every } x > 0$$

LAWS OF LOGARITHMS If x and y are positive numbers, then

1. $\log_a(xy) = \log_a x + \log_a y$

2. $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$

3. $\log_a(x^r) = r\log_a x$ (where r is any real number)

the **natural logarithm** and has a special notation:

$$\log_e x = \ln x$$

$$\ln x = y \iff e^y = x$$

$$\ln e = 1$$

$$\ln(e^x) = x \qquad x \in \mathbb{R}$$

$$e^{\ln x} = x \qquad x > 0$$

$$\log_a x = \frac{\ln x}{\ln a}$$

INVERSE TRIGONOMETRIC FUNCTIONS

$$f^{-1}(x) = y \iff f(y) = x$$

$$\sin^{-1}x = y \iff \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

EXAMPLE Evaluate (a) $\sin^{-1}\left(\frac{1}{2}\right)$ and (b) $\tan\left(\arcsin \frac{1}{3}\right)$.

SOLUTION

(a) We have

$$\sin^{-1}\left(\tfrac{1}{2}\right) = \frac{\pi}{6}$$

because $\sin(\pi/6) = \frac{1}{2}$ and $\pi/6$ lies between $-\pi/2$ and $\pi/2$.

(b) Let $\theta = \arcsin \frac{1}{3}$, so $\sin \theta = \frac{1}{3}$. Then we can draw a right triangle with angle θ as in Figure 19 and deduce from the Pythagorean Theorem that the third side has length $\sqrt{9 - 1} = 2\sqrt{2}$. This enables us to read from the triangle that

$$\tan\left(\arcsin \tfrac{1}{3}\right) = \tan \theta = \frac{1}{2\sqrt{2}}$$

DERIVATIVE OF THE NATURAL EXPONENTIAL FUNCTION

$$\frac{d}{dx}(e^x) = e^x$$

DERIVATIVES OF LOGARITHMIC FUNCTIONS

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

EXAMPLE Differentiate $y = \ln(x^3 + 1)$.

SOLUTION To use the Chain Rule, we let $u = x^3 + 1$. Then $y = \ln u$, so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}\frac{du}{dx} = \frac{1}{x^3 + 1}(3x^2) = \frac{3x^2}{x^3 + 1}$$

DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1 + x^2}$$

$$\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1 + x^2}$$

EXAMPLE Find $\frac{d}{dx} \ln(\sin x)$.

SOLUTION

$$\frac{d}{dx} \ln(\sin x) = \frac{1}{\sin x} \frac{d}{dx} (\sin x) = \frac{1}{\sin x} \cos x = \cot x$$

EXAMPLE Differentiate $f(x) = \sqrt{\ln x}$.

SOLUTION This time the logarithm is the inner function, so the Chain Rule gives

$$f'(x) = \tfrac{1}{2}(\ln x)^{-1/2} \frac{d}{dx} (\ln x) = \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$$

LOGARITHMIC DIFFERENTIATION

EXAMPLE Differentiate $y = x^{\sqrt{x}}$.

SOLUTION 1 Using logarithmic differentiation, we have

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x}\ln x$$

$$\frac{y'}{y} = \sqrt{x}\cdot\frac{1}{x} + (\ln x)\frac{1}{2\sqrt{x}}$$

$$y' = y\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right)$$

SOLUTION 2 Another method is to write $x^{\sqrt{x}} = (e^{\ln x})^{\sqrt{x}}$:

$$\frac{d}{dx}\left(x^{\sqrt{x}}\right) = \frac{d}{dx}\left(e^{\sqrt{x}\ln x}\right) = e^{\sqrt{x}\ln x}\frac{d}{dx}\left(\sqrt{x}\ln x\right)$$

$$= x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right) \qquad \text{(as in Solution 1)}$$

THE NUMBER e AS A LIMIT

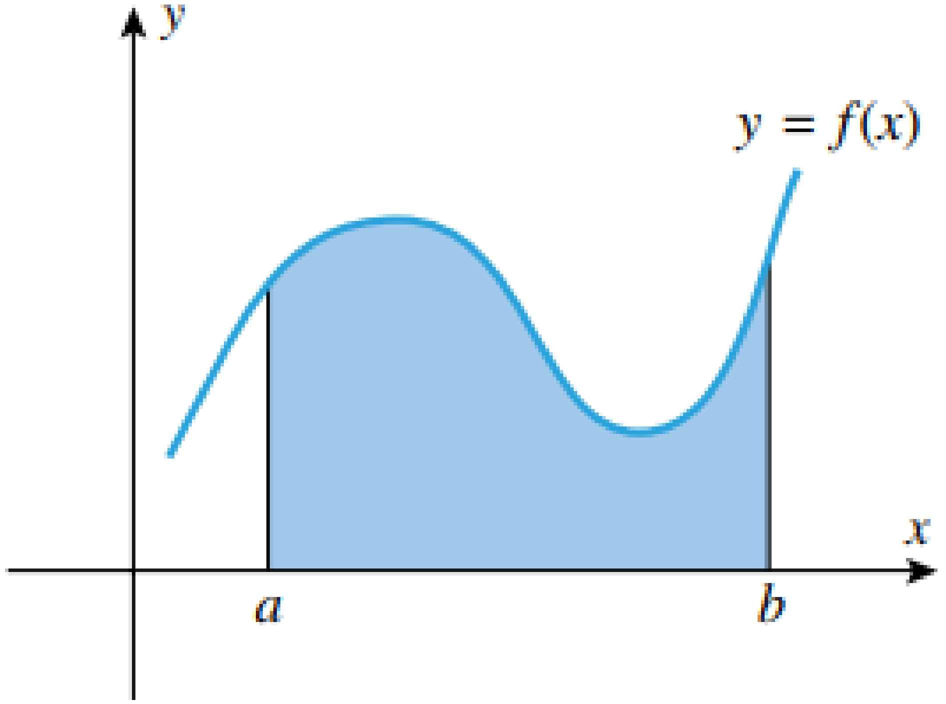
$$e = \lim_{x \to 0} (1 + x)^{1/x}$$

If we put $n = 1/x$, then $n \to \infty$ as $x \to 0^+$ and so an alternative expression

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

INTEGRATION

THE AREA PROBLEM. Given a function f that is continuous and nonnegative on an interval $[a, b]$, find the area between the graph of f and the interval $[a, b]$ on the x-axis



THE INDEFINITE INTEGRAL

DEFINITION. A function F is called an ***antiderivative*** of a function f on a given interval I if $F'(x) = f(x)$ for all x in the interval.

For example, the function $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$

$$F'(x) = \frac{d}{dx}\left[\tfrac{1}{3}x^3\right] = x^2 = f(x)$$

However, $F(x) = \frac{1}{3}x^3$ is not the only antiderivative of f on this interval. If we add any constant C to $\frac{1}{3}x^3$, then the function $G(x) = \frac{1}{3}x^3 + C$ is also an antiderivative of f , since

$$G'(x) = \frac{d}{dx}\left[\tfrac{1}{3}x^3 + C\right] = x^2 + 0 = f(x)$$

The process of finding antiderivatives is called ***antidifferentiation*** or ***integration***. Thus, if

$$\frac{d}{dx}[F(x)] = f(x) \tag{1}$$

then ***integrating*** (or ***antidifferentiating***) the function $f(x)$ produces an antiderivative of the form $F(x) + C$. To emphasize this process, Equation (1) is recast using ***integral notation***,

$$\int f(x)\,dx = F(x) + C \tag{2}$$

where C is understood to represent an arbitrary constant. It is important to note that (1) and (2) are just different notations to express the same fact. For example,

$$\int x^2\,dx = \tfrac{1}{3}x^3 + C \quad \text{is equivalent to} \quad \frac{d}{dx}\left[\tfrac{1}{3}x^3\right] = x^2$$

The differential symbol, dx, in the differentiation and antidifferentiation operations

$$\frac{d}{dx}[\ ] \quad \text{and} \quad \int [\ ]\,dx$$

$$\frac{d}{dt}[F(t)] = f(t) \quad \text{and} \quad \int f(t)\,dt = F(t) + C$$

are equivalent statements.

DERIVATIVE FORMULA	EQUIVALENT INTEGRATION FORMULA
$\frac{d}{dx}[x^3] = 3x^2$	$\int 3x^2\,dx = x^3 + C$
$\frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}}$	$\int \frac{1}{2\sqrt{x}}\,dx = \sqrt{x} + C$
$\frac{d}{dt}[\tan t] = \sec^2 t$	$\int \sec^2 t\,dt = \tan t + C$
$\frac{d}{du}[u^{3/2}] = \frac{3}{2}u^{1/2}$	$\int \frac{3}{2}u^{1/2}\,du = u^{3/2} + C$

$$\int x^2\,dx = \frac{x^3}{3} + C$$

$$\int x^3\,dx = \frac{x^4}{4} + C$$

$$\int \frac{1}{x^5}\,dx = \int x^{-5}\,dx = \frac{x^{-5+1}}{-5+1} + C = -\frac{1}{4x^4} + C$$

$$\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \tfrac{2}{3}x^{\frac{3}{2}} + C$$

DIFFERENTIATION FORMULA	INTEGRATION FORMULA
1. $\frac{d}{dx}[x] = 1$	$\int dx = x + C$
2. $\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r \quad (r \neq -1)$	$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$
3. $\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\,dx = \sin x + C$
4. $\frac{d}{dx}[-\cos x] = \sin x$	$\int \sin x\,dx = -\cos x + C$
5. $\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
6. $\frac{d}{dx}[-\cot x] = \csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
7. $\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$
8. $\frac{d}{dx}[-\csc x] = \csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$

PROPERTIES OF THE INDEFINITE INTEGRAL

THEOREM. *Suppose that $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$, respectively, and that c is a constant. Then:*

(a) *A constant factor can be moved through an integral sign; that is,*

$$\int cf(x)\,dx = cF(x) + C$$

(b) *An antiderivative of a sum is the sum of the antiderivatives; that is,*

$$\int [f(x) + g(x)]\,dx = F(x) + G(x) + C$$

(c) *An antiderivative of a difference is the difference of the antiderivatives; that is,*

$$\int [f(x) - g(x)]\,dx = F(x) - G(x) + C$$

Example Evaluate

(a) $\int 4\cos x\,dx$ (b) $\int (x + x^2)\,dx$

Solution (a). Since $F(x) = \sin x$ is an antiderivative for $f(x) = \cos x$ obtain

$$\int 4\cos x\,dx = 4\int \cos x\,dx = 4\sin x + C$$

Solution (b).

$$\int (x + x^2)\,dx = \int x\,dx + \int x^2\,dx = \frac{x^2}{2} + \frac{x^3}{3} + C$$

Example Evaluate

(a) $\displaystyle\int \frac{\cos x}{\sin^2 x}\,dx$ (b) $\displaystyle\int \frac{t^2 - 2t^4}{t^4}\,dt$

Solution (a).

$$\int \frac{\cos x}{\sin^2 x}\,dx = \int \frac{1}{\sin x}\,\frac{\cos x}{\sin x}\,dx = \int \csc x \cot x\,dx = -\csc x + C$$

Solution (b).

$$\int \frac{t^2 - 2t^4}{t^4}\,dt = \int \left(\frac{1}{t^2} - 2\right)dt = \int (t^{-2} - 2)\,dt$$

$$= \frac{t^{-1}}{-1} - 2t + C = -\frac{1}{t} - 2t + C$$

INTEGRATION BY SUBSTITUTION

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x)$$

which we can write in integral form as

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C$$

or since F is an antiderivative of f,

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C$$

Example 1 Evaluate $\int (x^2+1)^{50} \cdot 2x\,dx$.

Solution. let $u = x^2+1$, then $du/dx = 2x$, $du = 2x\,dx$. Thus,

$$\int (x^2+1)^{50} \cdot 2x\,dx = \int u^{50}\,du = \frac{u^{51}}{51} + C = \frac{(x^2+1)^{51}}{51} + C$$

Example 2

$$\int \sin(x+9)\,dx = \int \sin u\,du = -\cos u + C = -\cos(x+9) + C$$

$$u = x + 9$$
$$du = 1 \cdot dx = dx$$

$$\int (x-8)^{23}\,dx = \int u^{23}\,du = \frac{u^{24}}{24} + C = \frac{(x-8)^{24}}{24} + C$$

$$u = x - 8$$
$$du = 1 \cdot dx = dx$$

Example 3 Evaluate $\int \cos 5x\, dx$.

Solution.

$$\int \cos 5x\, dx = \int (\cos u) \cdot \frac{1}{5} du = \frac{1}{5} \int \cos u\, du = \frac{1}{5} \sin u + C = \frac{1}{5} \sin 5x + C$$

$u = 5x$
$du = 5\, dx$ or $dx = \frac{1}{5}\, du$

Example 4

$$\int \frac{dx}{\left(\frac{1}{3}x - 8\right)^5} = \int \frac{3\,du}{u^5} = 3\int u^{-5}\,du = -\frac{3}{4}u^{-4} + C = -\frac{3}{4}\left(\frac{1}{3}x - 8\right)^{-4} + C$$

$$u = \tfrac{1}{3}x - 8$$
$$du = \tfrac{1}{3}\,dx \text{ or } dx = 3\,du$$

Example 5

$$\int \left(\frac{1}{x^2} + \sec^2 \pi x\right) dx = \int \frac{dx}{x^2} + \int \sec^2 \pi x \, dx = -\frac{1}{x} + \int \sec^2 \pi x \, dx$$

$$= -\frac{1}{x} + \frac{1}{\pi} \int \sec^2 u \, du$$

$u = \pi x$
$du = \pi \, dx$ or $dx = \frac{1}{\pi} \, du$

$$= -\frac{1}{x} + \frac{1}{\pi} \tan u + C = -\frac{1}{x} + \frac{1}{\pi} \tan \pi x + C$$

Example 6 Evaluate $\int \sin^2 x \cos x \, dx$.

Solution. If we let $u = \sin x$, then

$$\frac{du}{dx} = \cos x, \quad \text{so} \quad du = \cos x \, dx$$

Thus,

$$\int \sin^2 x \cos x \, dx = \int u^2 \, du = \frac{u^3}{3} + C = \frac{\sin^3 x}{3} + C$$

Example 7 Evaluate $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$.

Solution. If we let $u = \sqrt{x}$, then

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}, \quad \text{so} \quad du = \frac{1}{2\sqrt{x}}\,dx \quad \text{or} \quad 2\,du = \frac{1}{\sqrt{x}}\,dx$$

Thus,

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \int 2\cos u\,du = 2\int \cos u\,du = 2\sin u + C = 2\sin\sqrt{x} + C$$

Example 8 Evaluate $\int t^4 \sqrt[3]{3-5t^5}\,dt$.

Solution.

$$\int t^4 \sqrt[3]{3-5t^5}\,dt = -\frac{1}{25}\int \sqrt[3]{u}\,du = -\frac{1}{25}\int u^{1/3}\,du$$

$u = 3 - 5t^5$
$du = -25t^4\,dt$ or $-\frac{1}{25}\,du = t^4\,dt$

$$= -\frac{1}{25}\frac{u^{4/3}}{4/3} + C = -\frac{3}{100}\left(3-5t^5\right)^{4/3} + C$$

Example 9 Evaluate $\int x^2\sqrt{x-1}\,dx$.

Solution. Let

$$u = x - 1 \quad \text{so that} \quad du = dx$$

From the first equality in (4)

$$x^2 = (u+1)^2 = u^2 + 2u + 1$$

so that

$$\int x^2\sqrt{x-1}\,dx = \int (u^2+2u+1)\sqrt{u}\,du = \int (u^{5/2} + 2u^{3/2} + u^{1/2})\,du$$

$$= \tfrac{2}{7}u^{7/2} + \tfrac{4}{5}u^{5/2} + \tfrac{2}{3}u^{3/2} + C$$

$$= \tfrac{2}{7}(x-1)^{7/2} + \tfrac{4}{5}(x-1)^{5/2} + \tfrac{2}{3}(x-1)^{3/2} + C$$

Example 10 Evaluate $\int \cos^3 x \, dx$.

Solution. The only compositions in the integrand that suggest themselves are

$$\cos^3 x = (\cos x)^3 \quad \text{and} \quad \cos^2 x = (\cos x)^2$$

However, neither the substitution $u = \cos x$ nor the substitution $u = \cos^2 x$ work (verify). Following the suggestion in Step 1(c), we write

$$\int \cos^3 x \, dx = \int \cos^2 x \cos x \, dx$$

and solve the equation $du = \cos x \, dx$ for $u = \sin x$. Since $\sin^2 x + \cos^2 x = 1$, we then have

$$\int \cos^3 x \, dx = \int \cos^2 x \cos x \, dx = \int (1 - \sin^2 x) \cos x \, dx = \int (1 - u^2) \, du$$

$$= u - \frac{u^3}{3} + C = \sin x - \frac{1}{3} \sin^3 x + C$$

$$\int f(x)dx = F(x) + c$$

$$\int_a^b f(x)dx = F(b) - F(a)$$

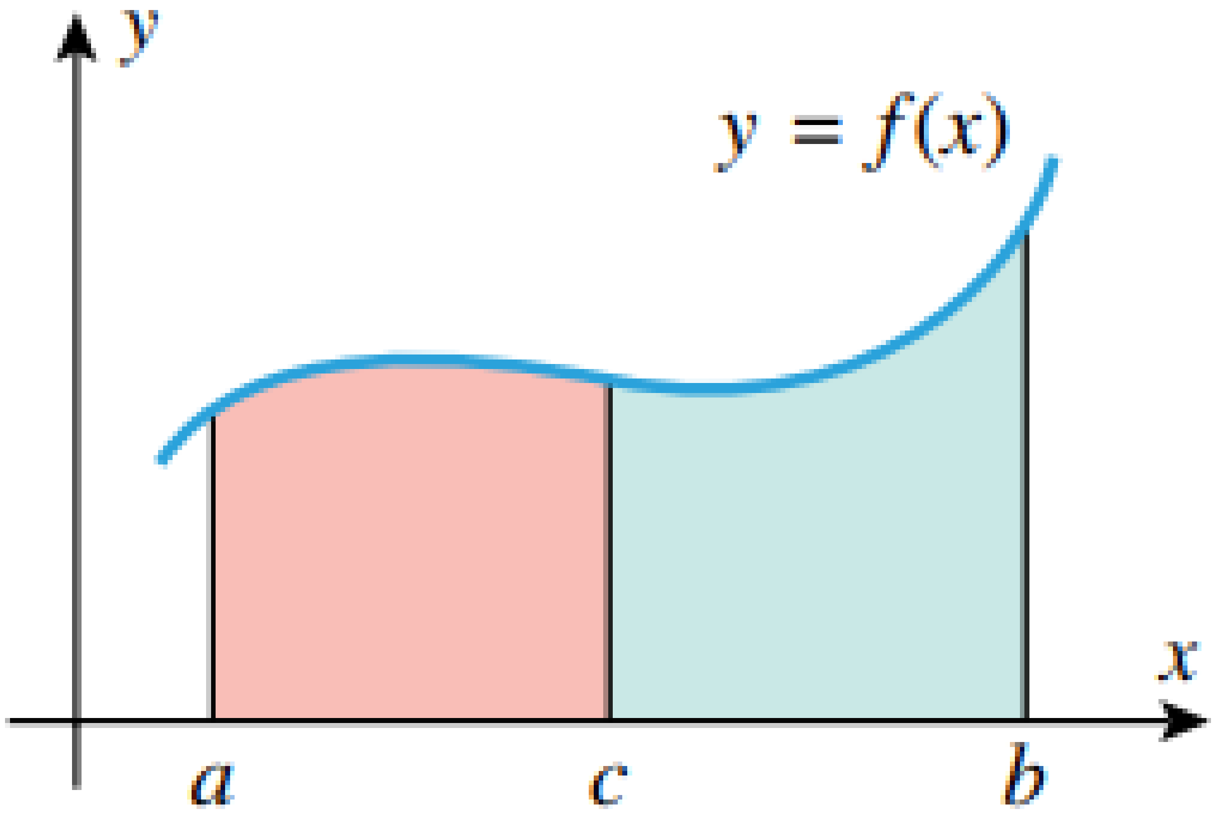
DEFINITION.

(a) If a is in the domain of f, we define

$$\int_a^a f(x)\,dx = 0$$

(b) If f is integrable on $[a, b]$, then we define

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$



$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Example Evaluate $\int_{-1}^{2} |x|\,dx$.

Solution. Since $|x| = x$ when $x \geq 0$ and $|x| = -x$ when $x \leq 0$,

$$\begin{aligned}\int_{-1}^{2} |x|\,dx &= \int_{-1}^{0} |x|\,dx + \int_{0}^{2} |x|\,dx \\ &= \int_{-1}^{0} (-x)\,dx + \int_{0}^{2} x\,dx \\ &= -\frac{x^2}{2}\Big]_{-1}^{0} + \frac{x^2}{2}\Big]_{0}^{2} = \frac{1}{2} + 2 = \frac{5}{2}\end{aligned}$$

Example 1 Evaluate $\int_1^2 x\,dx$.

$$\int_1^2 x\,dx = \frac{1}{2}x^2\Big]_1^2 = \frac{1}{2}(2)^2 - \frac{1}{2}(1)^2 = 2 - \frac{1}{2} = \frac{3}{2}$$

Example

$$\int_1^9 \sqrt{x}\,dx = \int \sqrt{x}\,dx\Big]_1^9 = \int x^{1/2}\,dx\Big]_1^9 = \frac{2}{3}x^{3/2}\Big]_1^9 = \frac{2}{3}(27 - 1) = \frac{52}{3}$$

$$\int_4^9 x^2\sqrt{x}\,dx = \int_4^9 x^{5/2}\,dx = \tfrac{2}{7}x^{7/2}\Big]_4^9 = \tfrac{2}{7}(2187 - 128) = \tfrac{4118}{7} = 588\tfrac{2}{7}$$

$$\int_0^{\pi/2} \frac{\sin x}{5}\,dx = -\frac{\cos x}{5}\Big]_0^{\pi/2} = -\frac{1}{5}\left[\cos\left(\frac{\pi}{2}\right) - \cos 0\right] = -\frac{1}{5}[0 - 1] = \frac{1}{5}$$

$$\int_0^{\pi/3} \sec^2 x\,dx = \tan x\Big]_0^{\pi/3} = \tan\left(\frac{\pi}{3}\right) - \tan 0 = \sqrt{3} - 0 = \sqrt{3}$$

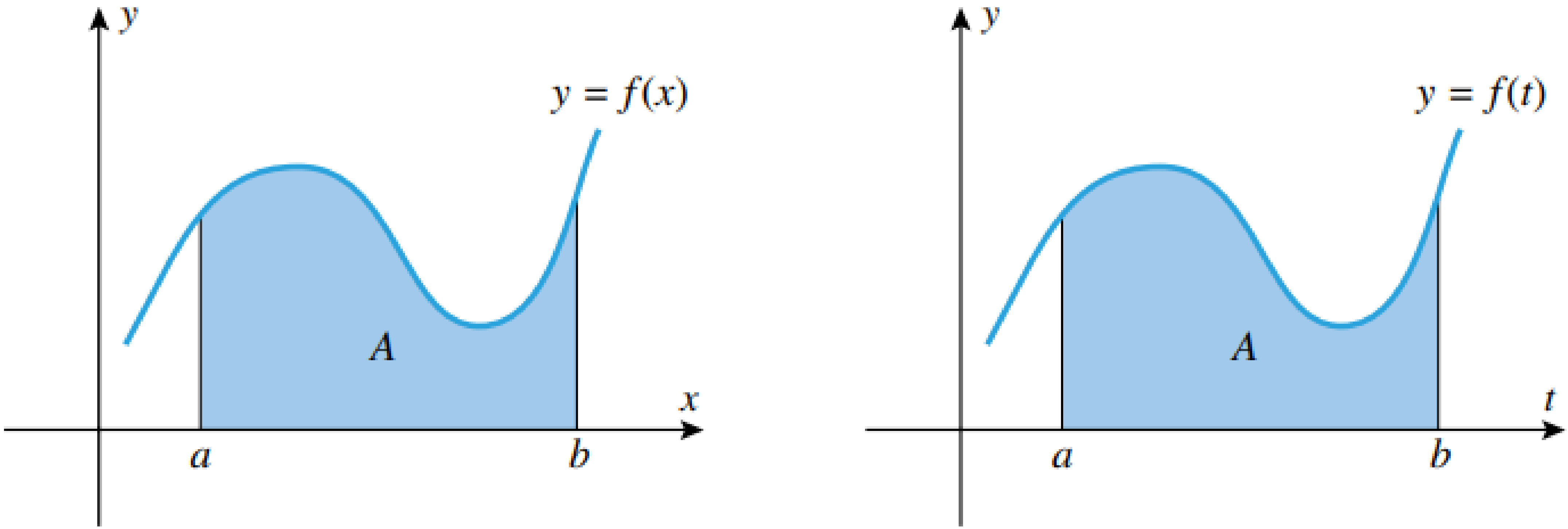
$$\int_{-\pi/4}^{\pi/4} \sec x \tan x\,dx = \sec x\Big]_{-\pi/4}^{\pi/4} = \sec\left(\frac{\pi}{4}\right) - \sec\left(-\frac{\pi}{4}\right) = \sqrt{2} - \sqrt{2} = 0$$

$$\int_1^1 x^2\,dx = \frac{x^3}{3}\Big]_1^1 = \frac{1}{3} - \frac{1}{3} = 0$$

$$\int_4^0 x\,dx = \frac{x^2}{2}\Big]_4^0 = \frac{0}{2} - \frac{16}{2} = -8$$

$$\int_4^0 x\,dx = -\int_0^4 x\,dx = -\frac{x^2}{2}\Big]_0^4 = -\left[\frac{16}{2} - \frac{0}{2}\right] = -8$$

DUMMY VARIABLES



$$A = \int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

THEOREM (***The Fundamental Theorem of Calculus, Part 2***). *If f is continuous on an interval I, then f has an antiderivative on I. In particular, if a is any number in I, then the function F defined by*

$$F(x) = \int_a^x f(t)\,dt$$

is an antiderivative of f on I; that is, $F'(x) = f(x)$ for each x in I, or in an alternative notation

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

Example Find

$$\frac{d}{dx}\left[\int_1^x t^3\,dt\right]$$

Example Find

$$\frac{d}{dx}\left[\int_1^x t^3\,dt\right]$$

$$\int_1^x t^3\,dt = \frac{t^4}{4}\Bigg]_{t=1}^{x} = \frac{x^4}{4} - \frac{1}{4}, \quad \frac{d}{dx}\left[\frac{x^4}{4} - \frac{1}{4}\right] = x^3$$

evaluate $\int_0^2 x(x^2+1)^3\,dx$.

Solution by If we let

$$u = x^2 + 1 \quad \text{so that} \quad du = 2x\,dx$$

then we obtain

$$\int x(x^2+1)^3\,dx = \frac{1}{2}\int u^3\,du = \frac{u^4}{8} + C = \frac{(x^2+1)^4}{8} + C$$

Thus,

$$\int_0^2 x(x^2+1)^3\,dx = \left[\int x(x^2+1)^3\,dx\right]_{x=0}^2 = \frac{(x^2+1)^4}{8}\Bigg]_{x=0}^2$$

$$= \frac{625}{8} - \frac{1}{8} = 78$$

Example Evaluate

(a) $\int_0^{\pi/8} \sin^5 2x \cos 2x \, dx$ (b) $\int_2^5 (2x-5)(x-3)^9 \, dx$

Solution (a). Let

$$u = \sin 2x \quad \text{so that} \quad du = 2\cos 2x \, dx \quad \left(\text{or } \tfrac{1}{2}\, du = \cos 2x \, dx\right)$$

With this substitution,

$$\text{if} \quad x = 0, \quad u = \sin(0) = 0$$

$$\text{if} \quad x = \pi/8, \quad u = \sin(\pi/4) = 1/\sqrt{2}$$

so

$$\int_0^{\pi/8} \sin^5 2x \cos 2x \, dx = \frac{1}{2}\int_0^{1/\sqrt{2}} u^5 \, du = \frac{1}{2} \cdot \frac{u^6}{6}\bigg]_{u=0}^{1/\sqrt{2}}$$

$$= \frac{1}{2}\left[\frac{1}{6(\sqrt{2})^6} - 0\right] = \frac{1}{96}$$

Solution (b). Let

$$u = x - 3 \quad \text{so that} \quad du = dx$$

This leaves a factor of $2x + 5$ unresolved in the integrand. However,

$$x = u + 3, \quad \text{so} \quad 2x - 5 = 2(u + 3) - 5 = 2u + 1$$

With this substitution,

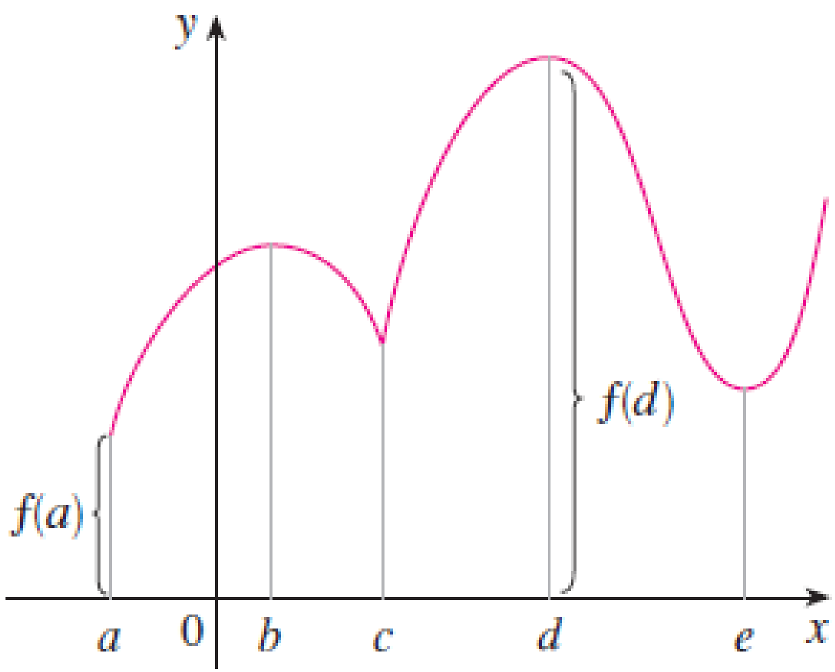
$$\text{if} \quad x = 2, \quad u = 2 - 3 = -1$$
$$\text{if} \quad x = 5, \quad u = 5 - 3 = 2$$

so

$$\int_2^5 (2x - 5)(x - 3)^9\, dx = \int_{-1}^2 (2u + 1)u^9\, du = \int_{-1}^2 (2u^{10} + u^9)\, du$$
$$= \left[\frac{2u^{11}}{11} + \frac{u^{10}}{10}\right]_{u=-1}^2 = \left(\frac{2^{12}}{11} + \frac{2^{10}}{10}\right) - \left(-\frac{2}{11} + \frac{1}{10}\right)$$
$$= \frac{52{,}233}{110} = 474\frac{93}{110}$$

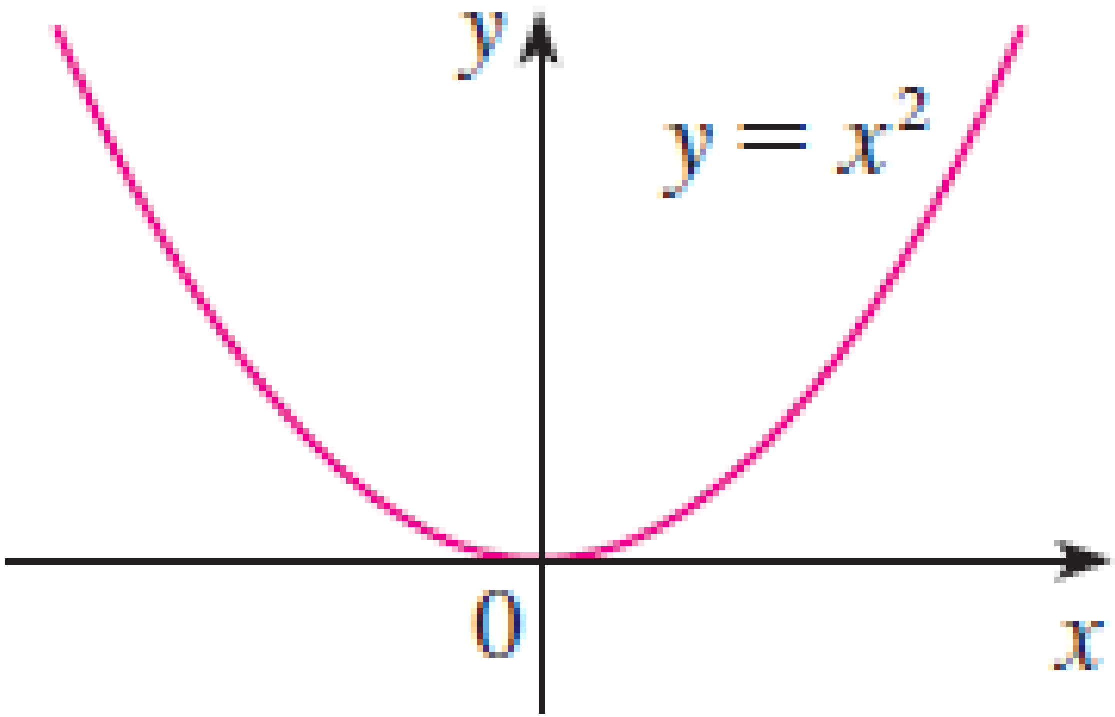
Maximum and Minimum Values

1 DEFINITION A function f has an **absolute maximum** (or **global maximum**) at c if $f(c) \geqslant f(x)$ for all x in D, where D is the domain of f. The number $f(c)$ is called the **maximum value** of f on D. Similarly, f has an **absolute minimum** at c if $f(c) \leqslant f(x)$ for all x in D and the number $f(c)$ is called the **minimum value** of f on D. The maximum and minimum values of f are called the **extreme values** of f.

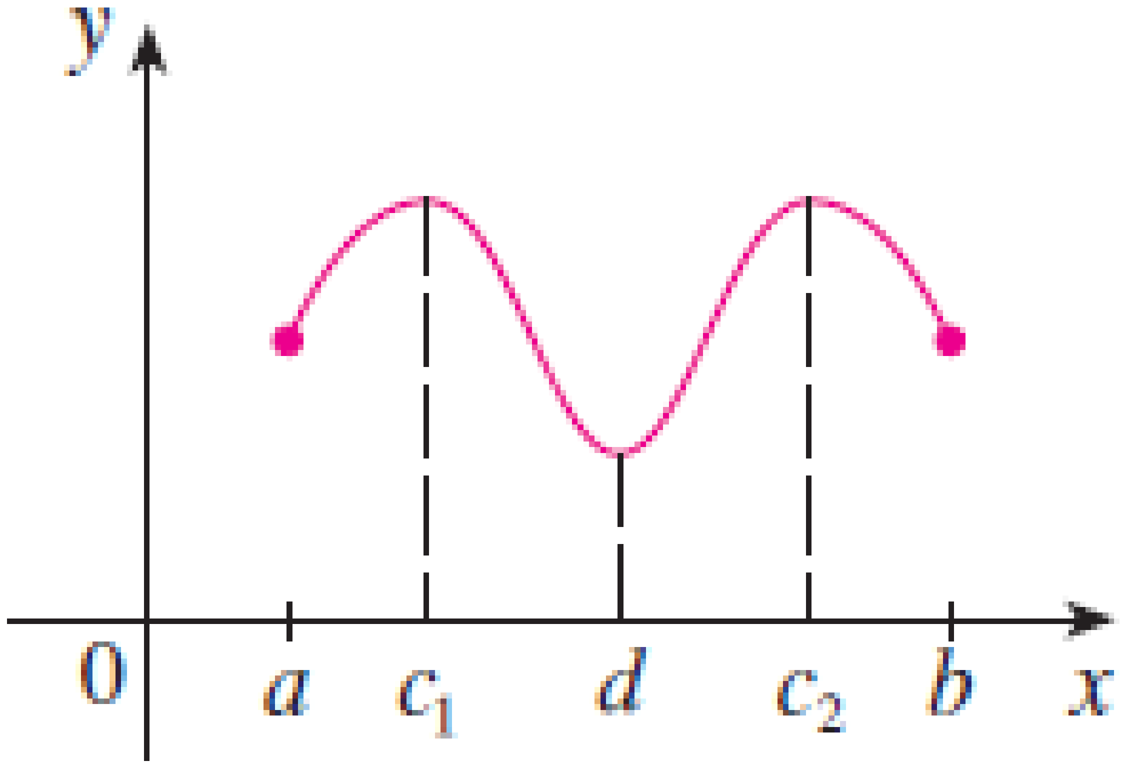


EXAMPLES

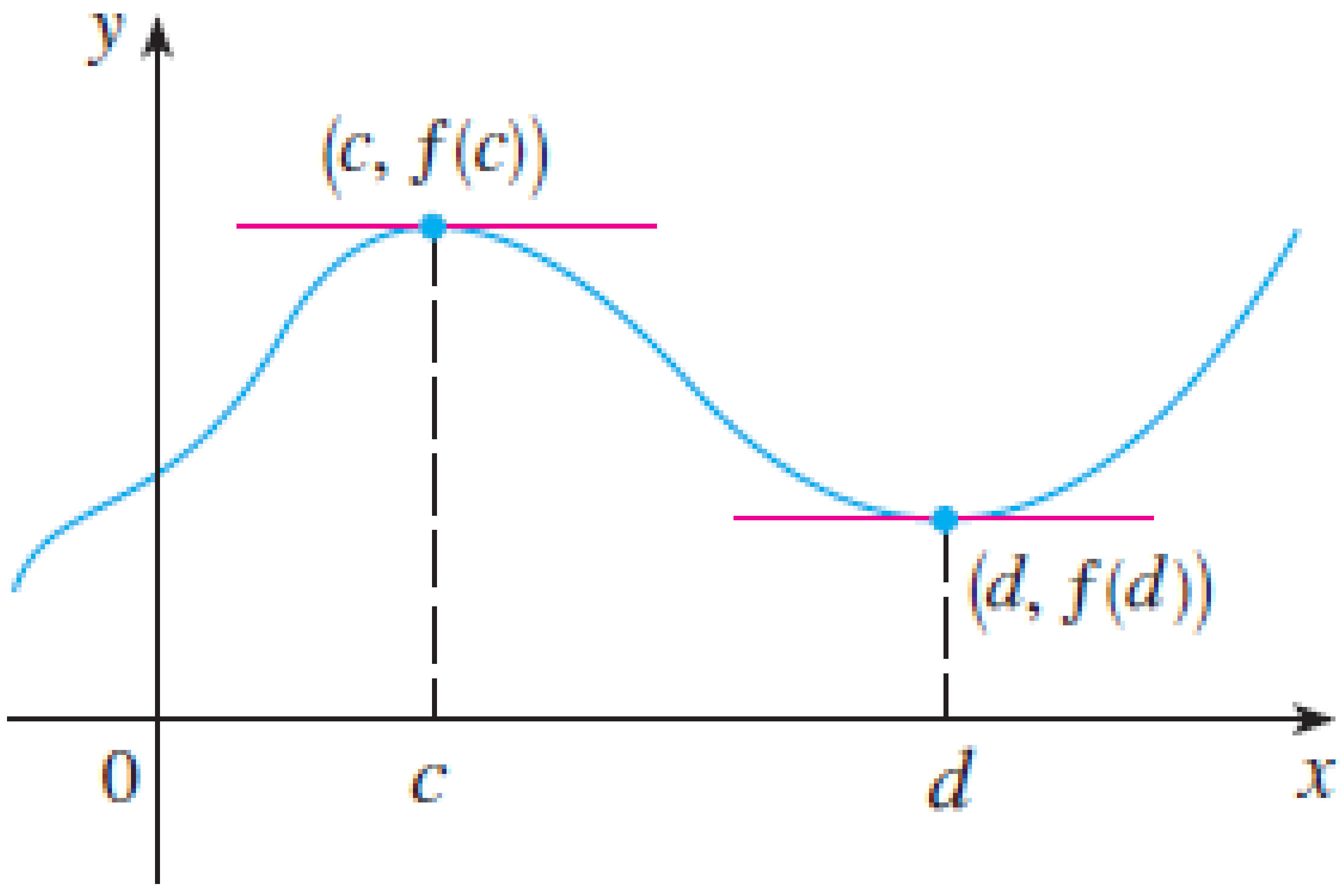
If $f(x) = x^2$, then $f(x) \geqslant f(0)$ because $x^2 \geqslant 0$ for all x. Therefore $f(0) = 0$ is the absolute (and local) minimum value of f. This corresponds to the fact that the origin is the lowest point on the parabola $y = x^2$.



THE EXTREME VALUE THEOREM If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.



FERMAT'S THEOREM If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.



271stwart

A **critical number**

DEFINITION A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

V EXAMPLE Find the critical numbers of $f(x) = x^{3/5}(4 - x)$.

EXAMPLE Find the critical numbers of $f(x) = x^{3/5}(4 - x)$.

SOLUTION The Product Rule gives

$$f'(x) = x^{3/5}(-1) + (4 - x)\left(\tfrac{3}{5}x^{-2/5}\right) = -x^{3/5} + \frac{3(4 - x)}{5x^{2/5}}$$

$$= \frac{-5x + 3(4 - x)}{5x^{2/5}} = \frac{12 - 8x}{5x^{2/5}}$$

[The same result could be obtained by first writing $f(x) = 4x^{3/5} - x^{8/5}$.] Therefore $f'(x) = 0$ if $12 - 8x = 0$, that is, $x = \frac{3}{2}$, and $f'(x)$ does not exist when $x = 0$. Thus the critical numbers are $\frac{3}{2}$ and 0. □

If f has a local maximum or minimum at c, then c is a critical number of f.

To find an absolute maximum or minimum of a continuous function on a closed interval, we note that either it is local [in which case it occurs at a critical number by (7)] or it occurs at an endpoint of the interval. Thus the following three-step procedure always works.

THE CLOSED INTERVAL METHOD To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

EXAMPLE Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 1 \qquad -\tfrac{1}{2} \leqslant x \leqslant 4$$

SOLUTION Since f is continuous on $[-\frac{1}{2}, 4]$, we can use the Closed Interval Method:

$$f(x) = x^3 - 3x^2 + 1$$

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

Since $f'(x)$ exists for all x, the only critical numbers of f occur when $f'(x) = 0$, that is, $x = 0$ or $x = 2$. Notice that each of these critical numbers lies in the interval $(-\frac{1}{2}, 4)$. The values of f at these critical numbers are

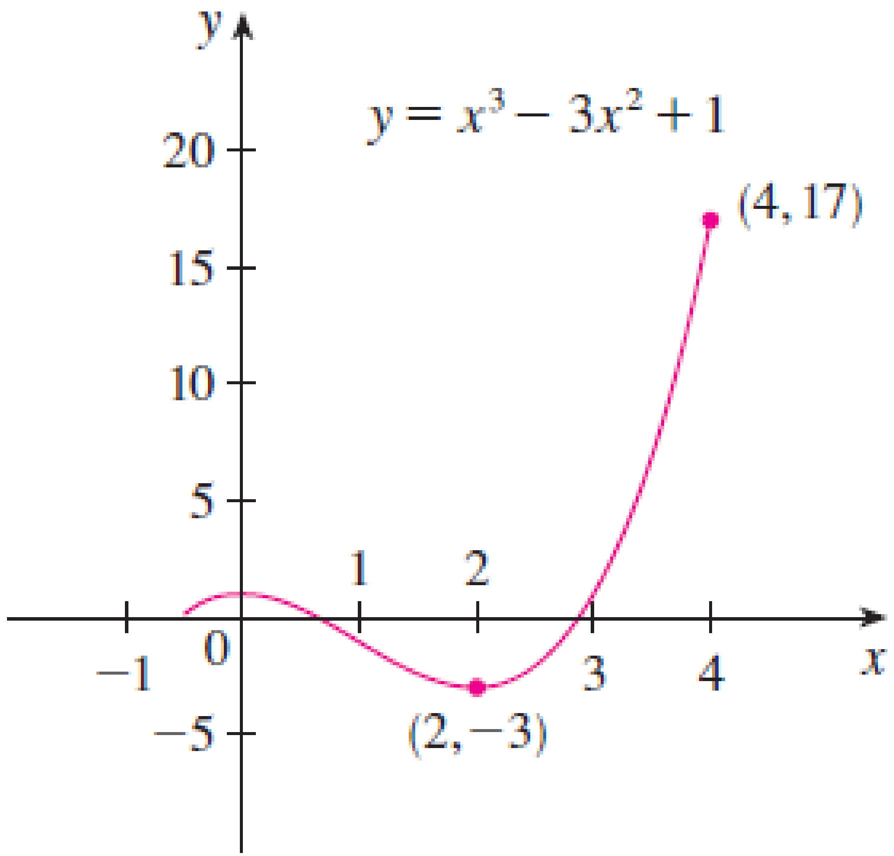
$$f(0) = 1 \qquad f(2) = -3$$

The values of f at the endpoints of the interval are

$$f(-\tfrac{1}{2}) = \tfrac{1}{8} \qquad f(4) = 17$$

Comparing these four numbers, we see that the absolute maximum value is $f(4) = 17$ and the absolute minimum value is $f(2) = -3$.

Note that in this example the absolute maximum occurs at an endpoint, whereas the absolute minimum occurs at a critical number. The graph of f is sketched in Figure 12. ☐



EXAMPLE The Hubble Space Telescope was deployed on April 24, 1990, by the space shuttle *Discovery*. A model for the velocity of the shuttle during this mission, from liftoff at $t = 0$ until the solid rocket boosters were jettisoned at $t = 126$ s, is given by

$$v(t) = 0.001302t^3 - 0.09029t^2 + 23.61t - 3.083$$

(in feet per second). Using this model, estimate the absolute maximum and minimum values of the *acceleration* of the shuttle between liftoff and the jettisoning of the boosters.

SOLUTION We are asked for the extreme values not of the given velocity function, but rather of the acceleration function. So we first need to differentiate to find the acceleration:

$$a(t) = v'(t) = \frac{d}{dt}(0.001302t^3 - 0.09029t^2 + 23.61t - 3.083)$$

$$= 0.003906t^2 - 0.18058t + 23.61$$

We now apply the Closed Interval Method to the continuous function a on the interval $0 \leqslant t \leqslant 126$. Its derivative is

$$a'(t) = 0.007812t - 0.18058$$

The only critical number occurs when $a'(t) = 0$:

$$t_1 = \frac{0.18058}{0.007812} \approx 23.12$$

Evaluating $a(t)$ at the critical number and at the endpoints, we have

$$a(0) = 23.61 \qquad a(t_1) \approx 21.52 \qquad a(126) \approx 62.87$$

So the maximum acceleration is about 62.87 ft/s^2 and the minimum acceleration is about 21.52 ft/s^2. □