Lectures in

Complex Analysis

CHAPTER I

Complex Numbers and Functions

One of the advantages of dealing with the real numbers instead of the rational numbers is that certain equations which do not have any solutions in the rational numbers have a solution in real numbers. For instance, $x^2 = 2$ is such an equation. However, we also know some equations having no solution in real numbers, for instance $x^2 = -1$, or $x^2 = -2$. We define a new kind of number where such equations have solutions. The new kind of numbers will be called **complex** numbers.

I, §1. DEFINITION

The **complex numbers** are a set of objects which can be added and multiplied, the sum and product of two complex numbers being also a complex number, and satisfy the following conditions.

1. Every real number is a complex number, and if α, β are real numbers, then their sum and product as complex numbers are the same as their sum and product as real numbers.
2. There is a complex number denoted by i such that $i^2 = -1$.
3. Every complex number can be written uniquely in the form $a + bi$ where a, b are real numbers.
4. The ordinary laws of arithmetic concerning addition and multiplication are satisfied. We list these laws:

   If α, β, γ are complex numbers, then $(\alpha\beta)\gamma = \alpha(\beta\gamma)$, and

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma).$$

We have $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$, and $(\beta + \gamma)\alpha = \beta\alpha + \gamma\alpha$.

We have $\alpha\beta = \beta\alpha$, and $\alpha + \beta = \beta + \alpha$.

If 1 is the real number one, then $1\alpha = \alpha$.

If 0 is the real number zero, then $0\alpha = 0$.

We have $\alpha + (-1)\alpha = 0$.

We shall now draw consequences of these properties. With each complex number $a + bi$, we associate the point (a, b) in the plane. Let $\alpha = a_1 + a_2 i$ and $\beta = b_1 + b_2 i$ be two complex numbers. Then

$$\alpha + \beta = a_1 + b_1 + (a_2 + b_2)i.$$

Hence addition of complex numbers is carried out "componentwise". For example, $(2 + 3i) + (-1 + 5i) = 1 + 8i$.

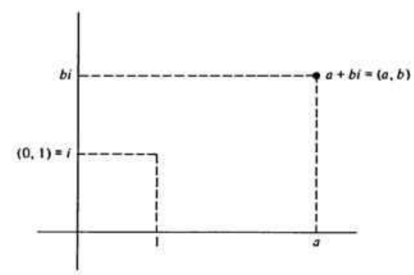

Figure 1

In multiplying complex numbers, we use the rule $i^2 = -1$ to simplify a product and to put it in the form $a + bi$. For instance, let $\alpha = 2 + 3i$ and $\beta = 1 - i$. Then

$$\begin{aligned}\alpha\beta = (2 + 3i)(1 - i) &= 2(1 - i) + 3i(1 - i)\\ &= 2 - 2i + 3i - 3i^2\\ &= 2 + i - 3(-1)\\ &= 2 + 3 + i\\ &= 5 + i.\end{aligned}$$

Let $\alpha = a + bi$ be a complex number. We define $\bar{\alpha}$ to be $a - bi$. Thus if $\alpha = 2 + 3i$, then $\bar{\alpha} = 2 - 3i$. The complex number $\bar{\alpha}$ is called the

conjugate of α. We see at once that

$$\alpha\bar{\alpha} = a^2 + b^2.$$

With the vector interpretation of complex numbers, we see that $\alpha\bar{\alpha}$ is the square of the distance of the point (a, b) from the origin.

We now have one more important property of complex numbers, which will allow us to divide by complex numbers other than 0.

If $\alpha = a + bi$ is a complex number $\neq 0$, and if we let

$$\lambda = \frac{\bar{\alpha}}{a^2 + b^2}$$

then $\alpha\lambda = \lambda\alpha = 1$.

The proof of this property is an immediate consequence of the law of multiplication of complex numbers, because

$$\alpha \frac{\bar{\alpha}}{a^2 + b^2} = \frac{\alpha\bar{\alpha}}{a^2 + b^2} = 1.$$

The number λ above is called the **inverse** of α, and is denoted by α^{-1} or $1/\alpha$. If α, β are complex numbers, we often write β/α instead of $\alpha^{-1}\beta$ (or $\beta\alpha^{-1}$), just as we did with real numbers. We see that we can divide by complex numbers $\neq 0$.

Example. To find the inverse of $(1 + i)$ we note that the conjugate of $1 + i$ is $1 - i$ and that $(1 + i)(1 - i) = 2$. Hence

$$(1 + i)^{-1} = \frac{1 - i}{2}.$$

Theorem 1.1. *Let α, β be complex numbers. Then*

$$\overline{\alpha\beta} = \bar{\alpha}\bar{\beta}, \qquad \overline{\alpha + \beta} = \bar{\alpha} + \bar{\beta}, \qquad \bar{\bar{\alpha}} = \alpha.$$

Proof. The proofs follow immediately from the definitions of addition, multiplication, and the complex conjugate. We leave them as exercises (Exercises 3 and 4).

Let $\alpha = a + bi$ be a complex number, where a, b are real. We shall call a the **real part** of α, and denote it by $\text{Re}(\alpha)$. Thus

$$\alpha + \bar{\alpha} = 2a = 2\,\text{Re}(\alpha).$$

The real number b is called the **imaginary part** of α, and denoted by $\mathrm{Im}(\alpha)$.

We define the **absolute value** of a complex number $\alpha = a_1 + ia_2$ (where a_1, a_2 are real) to be

$$|\alpha| = \sqrt{a_1^2 + a_2^2}.$$

If we think of α as a point in the plane (a_1, a_2), then $|\alpha|$ is the length of the line segment from the origin to α. In terms of the absolute value, we can write

$$\alpha^{-1} = \frac{\bar{\alpha}}{|\alpha|^2}$$

provided $\alpha \neq 0$. Indeed, we observe that $|\alpha|^2 = \alpha\bar{\alpha}$.

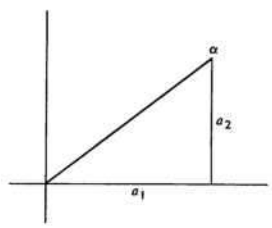

Figure 2

If $\alpha = a_1 + ia_2$, we note that

$$|\alpha| = |\bar{\alpha}|$$

because $(-a_2)^2 = a_2^2$, so $\sqrt{a_1^2 + a_2^2} = \sqrt{a_1^2 + (-a_2)^2}$.

Theorem 1.2. *The absolute value of a complex number satisfies the following properties. If α, β are complex numbers, then*

$$|\alpha\beta| = |\alpha||\beta|,$$
$$|\alpha + \beta| \leqq |\alpha| + |\beta|.$$

Proof. We have

$$|\alpha\beta|^2 = \alpha\beta\overline{\alpha\beta} = \alpha\bar{\alpha}\beta\bar{\beta} = |\alpha|^2|\beta|^2.$$

Taking the square root, we conclude that $|\alpha||\beta| = |\alpha\beta|$, thus proving the first assertion. As for the second, we have

$$\begin{aligned}|\alpha + \beta|^2 = (\alpha + \beta)(\overline{\alpha + \beta}) &= (\alpha + \beta)(\bar{\alpha} + \bar{\beta}) \\ &= \alpha\bar{\alpha} + \beta\bar{\alpha} + \alpha\bar{\beta} + \beta\bar{\beta} \\ &= |\alpha|^2 + 2\,\mathrm{Re}(\beta\bar{\alpha}) + |\beta|^2\end{aligned}$$

because $\alpha\bar{\beta} = \overline{\beta\bar{\alpha}}$. However, we have

$$2\,\mathrm{Re}(\beta\bar{\alpha}) \leqq 2|\beta\bar{\alpha}|$$

because the real part of a complex number is $\leqq$ its absolute value. Hence

$$\begin{aligned}|\alpha + \beta|^2 &\leqq |\alpha|^2 + 2|\beta\bar{\alpha}| + |\beta|^2 \\ &\leqq |\alpha|^2 + 2|\beta||\alpha| + |\beta|^2 \\ &= (|\alpha| + |\beta|)^2.\end{aligned}$$

Taking the square root yields the second assertion of the theorem.

The inequality

$$|\alpha + \beta| \leqq |\alpha| + |\beta|$$

is called the **triangle inequality**. It also applies to a sum of several terms. If $z_1, \ldots, z_n$ are complex numbers then we have

$$|z_1 + \cdots + z_n| \leqq |z_1| + \cdots + |z_n|.$$

Also observe that for any complex number z, we have

$$|-z| = |z|.$$

Proof?

I, §1. EXERCISES

1. Express the following complex numbers in the form $x + iy$, where x, y are real numbers.
   (a) $(-1 + 3i)^{-1}$
   (b) $(1 + i)(1 - i)$
   (c) $(1 + i)i(2 - i)$
   (d) $(i - 1)(2 - i)$
   (e) $(7 + \pi i)(\pi + i)$
   (f) $(2i + 1)\pi i$
   (g) $(\sqrt{2}i)(\pi + 3i)$
   (h) $(i + 1)(i - 2)(i + 3)$

2. Express the following complex numbers in the form $x + iy$, where x, y are real numbers.

(a) $(1+i)^{-1}$ (b) $\frac{1}{3+i}$ (c) $\frac{2+i}{2-i}$ (d) $\frac{1}{2-i}$

(e) $\frac{1+i}{i}$ (f) $\frac{i}{1+i}$ (g) $\frac{2i}{3-i}$ (h) $\frac{1}{-1+i}$

3. Let α be a complex number $\neq 0$. What is the absolute value of $\alpha/\bar{\alpha}$? What is $\bar{\bar{\alpha}}$?

4. Let α, β be two complex numbers. Show that $\overline{\alpha\beta} = \bar{\alpha}\bar{\beta}$ and that

$$\overline{\alpha + \beta} = \bar{\alpha} + \bar{\beta}.$$

5. Justify the assertion made in the proof of Theorem 1.2, that the real part of a complex number is $\leqq$ its absolute value.

6. If $\alpha = a + ib$ with a, b real, then b is called the **imaginary part** of α and we write $b = \text{Im}(\alpha)$. Show that $\alpha - \bar{\alpha} = 2i\,\text{Im}(\alpha)$. Show that

$$\text{Im}(\alpha) \leqq |\text{Im}(\alpha)| \leqq |\alpha|.$$

7. Find the real and imaginary parts of $(1+i)^{100}$.

8. Prove that for any two complex numbers z, w we have:
   (a) $|z| \leqq |z - w| + |w|$
   (b) $|z| - |w| \leqq |z - w|$
   (c) $|z| - |w| \leqq |z + w|$

9. Let $\alpha = a + ib$ and $z = x + iy$. Let c be real > 0. Transform the condition

$$|z - \alpha| = c$$

into an equation involving only x, y, a, b, and c, and describe in a simple way what geometric figure is represented by this equation.

10. Describe geometrically the sets of points z satisfying the following conditions.
    (a) $|z - i + 3| = 5$ (b) $|z - i + 3| > 5$
    (c) $|z - i + 3| \leqq 5$ (d) $|z + 2i| \leqq 1$
    (e) $\text{Im}\, z > 0$ (f) $\text{Im}\, z \geqq 0$
    (g) $\text{Re}\, z > 0$ (h) $\text{Re}\, z \geqq 0$

I, §2. POLAR FORM

Let $(x, y) = x + iy$ be a complex number. We know that any point in the plane can be represented by polar coordinates (r, θ). We shall now see how to write our complex number in terms of such polar coordinates.

Let θ be a real number. We define the expression $e^{i\theta}$ to be

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Thus $e^{i\theta}$ is a complex number.

For example, if $\theta = \pi$, then $e^{i\pi} = -1$. Also, $e^{2\pi i} = 1$, and $e^{i\pi/2} = i$. Furthermore, $e^{i(\theta+2\pi)} = e^{i\theta}$ for any real θ.

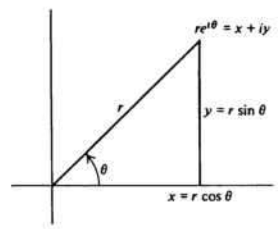

Figure 3

Let x, y be real numbers and $x + iy$ a complex number. Let

$$r = \sqrt{x^2 + y^2}.$$

If (r, θ) are the polar coordinates of the point (x, y) in the plane, then

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta.$$

Hence

$$x + iy = r \cos \theta + ir \sin \theta = re^{i\theta}.$$

The expression $re^{i\theta}$ is called the **polar form** of the complex number $x + iy$. The number θ is sometimes called the **angle**, or **argument** of z, and we write

$$\theta = \arg z.$$

The most important property of this polar form is given in Theorem 2.1. It will allow us to have a very good geometric interpretation for the product of two complex numbers.

Theorem 2.1. *Let θ, φ be two real numbers. Then*

$$e^{i\theta + i\varphi} = e^{i\theta}e^{i\varphi}.$$

Proof. By definition, we have

$$e^{i\theta+i\varphi} = e^{i(\theta+\varphi)} = \cos(\theta + \varphi) + i \sin(\theta + \varphi).$$

Using the addition formulas for sine and cosine, we see that the preceding expression is equal to

$$\cos\theta\cos\varphi - \sin\theta\sin\varphi + i(\sin\theta\cos\varphi + \sin\varphi\cos\theta).$$

This is exactly the same expression as the one we obtain by multiplying out

$$(\cos\theta + i\sin\theta)(\cos\varphi + i\sin\varphi).$$

Our theorem is proved.

Theorem 2.1 justifies our notation, by showing that the exponential of complex numbers satisfies the same formal rule as the exponential of real numbers.

Let $\alpha = a_1 + ia_2$ be a complex number. We define e^α to be

$$e^{a_1}e^{ia_2}.$$

For instance, let $\alpha = 2 + 3i$. Then $e^\alpha = e^2e^{3i}$.

Theorem 2.2. *Let α, β be complex numbers. Then*

$$e^{\alpha+\beta} = e^\alpha e^\beta.$$

Proof. Let $\alpha = a_1 + ia_2$ and $\beta = b_1 + ib_2$. Then

$$e^{\alpha+\beta} = e^{(a_1+b_1)+i(a_2+b_2)} = e^{a_1+b_1}e^{i(a_2+b_2)}$$
$$= e^{a_1}e^{b_1}e^{ia_2+ib_2}.$$

Using Theorem 2.1, we see that this last expression is equal to

$$e^{a_1}e^{b_1}e^{ia_2}e^{ib_2} = e^{a_1}e^{ia_2}e^{b_1}e^{ib_2}.$$

By definition, this is equal to $e^\alpha e^\beta$, thereby proving our theorem.

Theorem 2.2 is very useful in dealing with complex numbers. We shall now consider several examples to illustrate it.

Example 1. Find a complex number whose square is $4e^{i\pi/2}$.

Let $z = 2e^{i\pi/4}$. Using the rule for exponentials, we see that $z^2 = 4e^{i\pi/2}$.

Example 2. Let n be a positive integer. Find a complex number w such that $w^n = e^{i\pi/2}$.

It is clear that the complex number $w = e^{i\pi/2n}$ satisfies our requirement. In other words, we may express Theorem 2.2 as follows:

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ be two complex numbers. To find the product $z_1 z_2$, we multiply the absolute values and add the angles. Thus

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}.$$

In many cases, this way of visualizing the product of complex numbers is more useful than that coming out of the definition.

Warning. We have not touched on the logarithm. As in calculus, we want to say that $e^z = w$ if and only if $z = \log w$. Since $e^{2\pi ik} = 1$ for all integers k, it follows that the inverse function $z = \log w$ is defined only up to the addition of an integer multiple of $2\pi i$. We shall study the logarithm more closely in Chapter II, §3, Chapter II, §5, and Chapter III, §6.

I, §2. EXERCISES

1. Put the following complex numbers in polar form.
   (a) $1 + i$ (b) $1 + i\sqrt{2}$ (c) -3 (d) $4i$
   (e) $1 - i\sqrt{2}$ (f) $-5i$ (g) -7 (h) $-1 - i$

2. Put the following complex numbers in the ordinary form $x + iy$.
   (a) $e^{3i\pi}$ (b) $e^{2i\pi/3}$ (c) $3e^{i\pi/4}$ (d) $\pi e^{-i\pi/3}$
   (e) $e^{2\pi i/6}$ (f) $e^{-i\pi/2}$ (g) $e^{-i\pi}$ (h) $e^{-5i\pi/4}$

3. Let α be a complex number $\neq 0$. Show that there are two distinct complex numbers whose square is α.

4. Let $a + bi$ be a complex number. Find real numbers x, y such that

$$(x + iy)^2 = a + bi,$$

   expressing x, y in terms of a and b.

5. Plot all the complex numbers z such that $z^n = 1$ on a sheet of graph paper, for $n = 2$, 3, 4, and 5.

6. Let α be a complex number $\neq 0$. Let n be a positive integer. Show that there are n distinct complex numbers z such that $z^n = \alpha$. Write these complex numbers in polar form.

7. Find the real and imaginary parts of $i^{1/4}$, taking the fourth root such that its angle lies between 0 and $\pi/2$.

8. (a) Describe all complex numbers z such that $e^z = 1$.
   (b) Let w be a complex number. Let α be a complex number such that $e^\alpha = w$. Describe all complex numbers z such that $e^z = w$.

9. If $e^z = e^w$, show that there is an integer k such that $z = w + 2\pi ki$.

10. (a) If θ is real, show that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

(b) For arbitrary complex z, suppose we define $\cos z$ and $\sin z$ by replacing θ with z in the above formula. Show that the only values of z for which $\cos z = 0$ and $\sin z = 0$ are the usual real values from trigonometry.

11. Prove that for any complex number $z \neq 1$ we have

$$1 + z + \cdots + z^n = \frac{z^{n+1} - 1}{z - 1}.$$

12. Using the preceding exercise, and taking real parts, prove:

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin[(n + \frac{1}{2})\theta]}{2\sin\frac{\theta}{2}}$$

for $0 < \theta < 2\pi$.

13. Let z, w be two complex numbers such that $\bar{z}w \neq 1$. Prove that

$$\left|\frac{z - w}{1 - \bar{z}w}\right| < 1 \quad \text{if } |z| < 1 \text{ and } |w| < 1,$$

$$\left|\frac{z - w}{1 - \bar{z}w}\right| = 1 \quad \text{if } |z| = 1 \text{ or } |w| = 1.$$

(There are many ways of doing this. One way is as follows. First check that you may assume that z is real, say $z = r$. For the first inequality you are reduced to proving

$$(r - w)(r - \bar{w}) < (1 - rw)(1 - r\bar{w}).$$

Expand both sides and make cancellations to simplify the problem.)

I, §3. COMPLEX VALUED FUNCTIONS

Let S be a set of complex numbers. An association which to each element of S associates a complex number is called a **complex valued function**, or a **function** for short. We denote such a function by symbols like

$$f: S \to \mathbf{C}.$$

If z is an element of S, we write the association of the **value** $f(z)$ to z by the special arrow

$$z \mapsto f(z).$$

We can write

$$f(z) = u(z) + iv(z),$$

where $u(z)$ and $v(z)$ are real numbers, and thus

$$z \mapsto u(z), \qquad z \mapsto v(z)$$

are real valued functions. We call u the **real part** of f, and v the **imaginary part** of f.

We shall usually write

$$z = x + iy,$$

where x, y are real. Then the values of the function f can be written in the form

$$f(z) = f(x + iy) = u(x, y) + iv(x, y),$$

viewing u, v as functions of the two real variables x and y.

Example. For the function

$$f(z) = x^3y + i \sin(x + y),$$

we have the real part,

$$u(x, y) = x^3y,$$

and the imaginary part,

$$v(x, y) = \sin(x + y).$$

Example. The most important examples of complex functions are the power functions. Let n be a positive integer. Let

$$f(z) = z^n.$$

Then in polar coordinates, we can write $z = re^{i\theta}$, and therefore

$$f(z) = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta).$$

For this function, the real part is $r^n \cos n\theta$, and the imaginary part is $r^n \sin n\theta$.

Let $\bar{D}$ be the closed disc of radius 1 centered at the origin in **C**. In other words, $\bar{D}$ is the set of complex numbers z such that $|z| \leqq 1$. If z is an element of $\bar{D}$, then z^n is also an element of $\bar{D}$, and so $z \mapsto z^n$ maps $\bar{D}$ into itself. Let S be the sector of complex numbers $re^{i\theta}$ such that

$$0 \leqq \theta \leqq 2\pi/n,$$

as shown on Fig. 4.

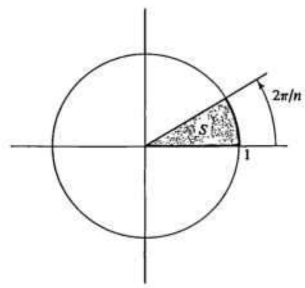

Figure 4

The function of a real variable

$$r \mapsto r^n$$

maps the unit interval $[0, 1]$ onto itself. The function

$$\theta \mapsto n\theta$$

maps the interval

$$[0, 2\pi/n] \rightarrow [0, 2\pi].$$

In this way, we see that the function $f(z) = z^n$ maps the sector S onto the full disc of all numbers

$$w = te^{i\varphi},$$

with $0 \leqq t \leqq 1$ and $0 \leqq \varphi \leqq 2\pi$. We may say that the power function wraps the sector around the disc.

We could give a similar argument with other sectors of angle $2\pi/n$

as shown on Fig. 5. Thus we see that $z \mapsto z^n$ wraps the disc n times around.

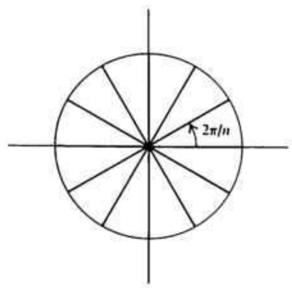

Figure 5

Given a complex number $z = re^{i\theta}$, you should have done Exercise 6 of the preceding section, or at least thought about it. For future reference, we now give the answer explicitly. We want to describe all complex numbers w such that $w^n = z$. Write

$$w = te^{i\varphi}.$$

Then

$$w^n = t^n e^{in\varphi}, \qquad 0 \leqq t.$$

If $w^n = z$, then $t^n = r$, and there is a unique real number $t \geqq 0$ such that $t^n = r$. On the other hand, we must also have

$$e^{in\varphi} = e^{i\theta},$$

which is equivalent with

$$in\varphi = i\theta + 2\pi ik,$$

where k is some integer. Thus we can solve for φ and get

$$\varphi = \frac{\theta}{n} + \frac{2\pi k}{n}.$$

The numbers

$$w_k = e^{i\theta/n}e^{2\pi ik/n}, \qquad k = 0, 1, \ldots, n-1$$

are all distinct, and are drawn on Fig. 6. These numbers w_k may be described pictorially as those points on the circle which are the vertices of a regular polygon with n sides inscribed in the unit circle, with one vertex being at the point $e^{i\theta/n}$.

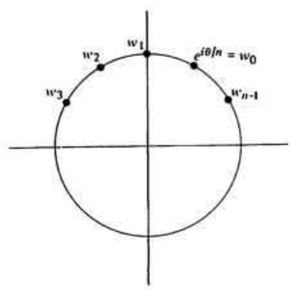

Figure 6

Each complex number

$$\zeta^k = e^{2\pi i k/n}$$

is called a **root of unity**, in fact, an n-th root of unity, because its n-th power is 1, namely

$$(\zeta^k)^n = e^{2\pi i k n/n} = e^{2\pi i k} = 1.$$

The points w_k are just the product of $e^{i\theta/n}$ with all the n-th roots of unity,

$$w_k = e^{i\theta/n}\zeta^k.$$

One of the major results of the theory of complex variables is to reduce the study of certain functions, including most of the common functions you can think of (like exponentials, logs, sine, cosine) to power series, which can be approximated by polynomials. Thus the power function is in some sense the unique basic function out of which the others are constructed. For this reason it was essential to get a good intuition of the power function. We postpone discussing the geometric aspects of the other functions to Chapters VII and VIII, except for some simple exercises.

I, §3. EXERCISES

1. Let $f(z) = 1/z$. Describe what f does to the inside and outside of the unit circle, and also what it does to points on the unit circle. This map is called **inversion** through the unit circle.

2. Let $f(z) = 1/\bar{z}$. Describe f in the same manner as in Exercise 1. This map is called **reflection** through the unit circle.

3. Let $f(z) = e^{2\pi i z}$. Describe the image under f of the set shaded in Fig. 7, consisting of those points $x + iy$ with $-\frac{1}{2} \leqq x \leqq \frac{1}{2}$ and $y \geqq B$.

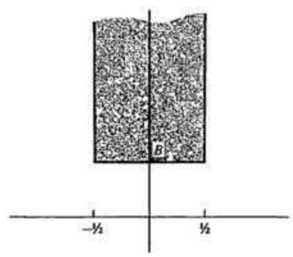

Figure 7

4. Let $f(z) = e^z$. Describe the image under f of the following sets:
   (a) The set of $z = x + iy$ such that $x \leqq 1$ and $0 \leqq y \leqq \pi$.
   (b) The set of $z = x + iy$ such that $0 \leqq y \leqq \pi$ (no condition on x).

I, §4. LIMITS AND COMPACT SETS

Let α be a complex number. By the **open disc** of radius $r > 0$ centered at α we mean the set of complex numbers z such that

$$|z - \alpha| < r.$$

For the **closed** disc, we use the condition $|z - \alpha| \leqq r$ instead. We shall deal only with the open disc unless otherwise specified, and thus speak simply of the **disc**, denoted by $D(\alpha, r)$. The closed disc is denoted by $\bar{D}(\alpha, r)$.

Let U be a subset of the complex plane. We say that U is **open** if for every point α in U there is a disc $D(\alpha, r)$ centered at α, and of some radius $r > 0$ such that this disc $D(\alpha, r)$ is contained in U. We have illustrated an open set in Fig. 8.

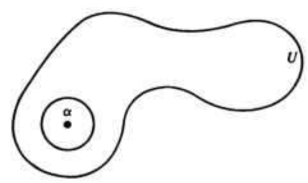

Figure 8

Note that the radius r of the disc depends on the point α. As α comes closer to the boundary of U, the radius of the disc will be smaller.

Examples of Open Sets. The first quadrant, consisting of all numbers $z = x + iy$ with $x > 0$ and $y > 0$ is open, and drawn on Fig. 9(a).

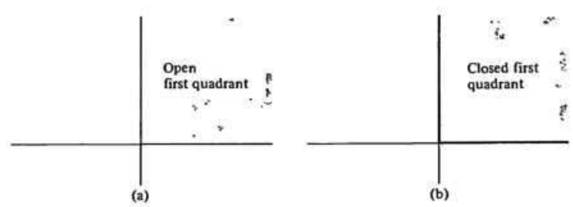

Figure 9

On the other hand, the set consisting of the first quadrant and the vertical and horizontal axes as on Fig. 9(b) is not open.

The **upper half plane** by definition is the set of complex numbers

$$z = x + iy$$

with $y > 0$. It is an open set.

Let S be a subset of the plane. A **boundary point** of S is a point α such that **every** disc $D(\alpha, r)$ centered at α and of radius $r > 0$ contains both points of S and points not in S. In the closed first quadrant of Fig. 9(b), the points on the x-axis with $x \geqq 0$ and on the y-axis with $y \geqq 0$ are boundary points of the quadrant.

A point α is said to be **adherent** to S if **every** disc $D(\alpha, r)$ with $r > 0$ contains **some** element of S. A point α is said to be an **interior** point of S if **there exists** a disc $D(\alpha, r)$ which is contained in S. Thus an adherent point can be a boundary point or an interior point of S. A set is called

closed if it contains all its boundary points. The complement of a closed set is then open.

The **closure** of a set S is defined to be the union of S and all its boundary points. We denote the closure by $\bar{S}$.

A set S is said to be **bounded** if there exists a number $C > 0$ such that

$$|z| \leqq C \quad \text{for all } z \text{ in } S.$$

For instance, the set in Fig. 10 is bounded. The first quadrant is not bounded.

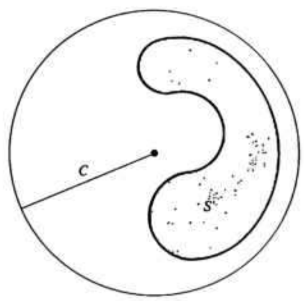

Figure 10

The upper half plane is not bounded. The condition for boundedness means that the set is contained in the closed disc of radius C, as shown on Fig. 10.

Let f be a function on S, and let α be an adherent point of S. Let w be a complex number. We say that

$$w = \lim_{\substack{z \to \alpha \\ z \in S}} f(z)$$

if the following condition is satisfied. Given $\epsilon > 0$ there exists $\delta > 0$ such that if $z \in S$ and $|z - \alpha| < \delta$, then

$$|f(z) - w| < \epsilon.$$

We usually omit the symbols $z \in S$ under the limit sign, and write merely

$$\lim_{z \to \alpha} f(z).$$

In some applications $\alpha \in S$ and in some applications, $\alpha \notin S$.

Let $\alpha \in S$. We say that f is **continuous** at α if

$$\lim_{z \to \alpha} f(z) = f(\alpha).$$

These definitions are completely analogous to those which you should have had in some analysis or advanced calculus course, so we don't spend much time on them. As usual, we have the rules for limits of sums, products, quotients as in calculus.

If $\{z_n\}$ $(n = 1, 2, \ldots)$ is a sequence of complex numbers, then we say that

$$w = \lim_{n \to \infty} z_n$$

if the following condition is satisfied:

Given $\epsilon > 0$ there exists an integer N such that if $n \geqq N$, then

$$|z_n - w| < \epsilon.$$

Let S be the set of fractions $1/n$, with $n = 1, 2, \ldots$. Let $f(1/n) = z_n$. Then

$$\lim_{n \to \infty} z_n = w \quad \text{if and only if} \quad \lim_{\substack{z \to 0 \\ z \in S}} f(z) = w.$$

Thus basic properties of limits for $n \to \infty$ are reduced to similar properties for functions. Note that in this case, the number 0 is not an element of S.

A sequence $\{z_n\}$ is said to be a **Cauchy sequence** if, given ϵ, there exists N such that if $m, n \geqq N$, then

$$|z_n - z_m| < \epsilon.$$

Write

$$z_n = x_n + iy_n.$$

Since

$$|z_n - z_m| = \sqrt{(x_n - x_m)^2 + (y_n - y_m)^2},$$

and

$$|x_n - x_m| \leqq |z_n - z_m|, \qquad |y_n - y_m| \leqq |z_n - z_m|,$$

we conclude that $\{z_n\}$ is Cauchy if and only if the sequences $\{x_n\}$ and $\{y_n\}$ of real and imaginary parts are also Cauchy. Since we know that real Cauchy sequences converge (i.e. have limits), we conclude that complex Cauchy sequences also converge.

We note that all the usual theorems about limits hold for complex numbers: Limits of sums, limits of products, limits of quotients, limits of composite functions. The proofs which you had in advanced calculus hold without change in the present context. It is then usually easy to compute limits.

Example. Find the limit

$$\lim_{n\to\infty} \frac{nz}{1+nz}$$

for any complex number z.

If $z = 0$, it is clear that the limit is 0. Suppose $z \neq 0$. Then the quotient whose limit we are supposed to find can be written

$$\frac{nz}{1+nz} = \frac{z}{\dfrac{1}{n} + z}.$$

But

$$\lim_{n\to\infty} \left(\frac{1}{n} + z\right) = z.$$

Hence the limit of the quotient is $z/z = 1$.

Compact Sets

We shall now go through the basic results concerning compact sets. Let S be a set of complex numbers. Let $\{z_n\}$ be a sequence in S. By a **point of accumulation of** $\{z_n\}$ we mean a complex number v such that given ϵ (always assumed > 0) there exist infinitely many integers n such that

$$|z_n - v| < \epsilon.$$

We may say that given an open set U containing v, there exist infinitely many n such that $z_n \in U$.

Similarly we define the notion of **point of accumulation** of an infinite set S. It is a complex number v such that given an open set U containing v, there exist infinitely many elements of S lying in U. In particular, a point of accumulation of S is adherent to S.

We assume that the reader is acquainted with the **Weierstrass–Bolzano theorem** about sets of real numbers: *If S is an infinite bounded set of real numbers, then S has a point of accumulation.*

We define a set of complex numbers S to be **compact** if every sequence of elements of S has a point of accumulation in S. This property is equivalent to the following properties, which could be taken as alternate definitions:

(a) Every infinite subset of S has a point of accumulation in S.

(b) Every sequence of elements of S has a convergent subsequence whose limit is in S.

We leave the proof of the equivalence between the three possible definitions to the reader.

Theorem 4.1. *A set of complex numbers is compact if and only if it is closed and bounded.*

Proof. Assume that S is compact. If S is not bounded, for each positive integer n there exists $z_n \in S$ such that

$$|z_n| > n.$$

Then the sequence $\{z_n\}$ does not have a point of accumulation. Indeed, if v is a point of accumulation, pick $m > 2|v|$, and note that $|v| > 0$. Then

$$|z_m - v| \geqq |z_m| - |v| \geqq m - |v| > |v|.$$

This contradicts the fact that for infinitely many m we must have z_m close to v. Hence S is bounded. To show S is closed, let v be in its closure. Given n, there exists $z_n \in S$ such that

$$|z_n - v| < 1/n.$$

The sequence $\{z_n\}$ converges to v, and has a subsequence converging to a limit in S because S is assumed compact. This limit must be v, whence $v \in S$ and S is closed.

Conversely, assume that S is closed and bounded, and let B be a bound, so $|z| \leqq B$ for all $z \in S$. If we write

$$z = x + iy,$$

then $|x| \leqq B$ and $|y| \leqq B$. Let $\{z_n\}$ be a sequence in S, and write

$$z_n = x_n + iy_n.$$

There is a subsequence $\{z_{n_1}\}$ such that $\{x_{n_1}\}$ converges to a real number a, and there is a sub-subsequence $\{z_{n_2}\}$ such that $(y_{n_2}\}$ converges to a real number b. Then

$$\{z_{n_2} = x_{n_2} + iy_{n_2}\}$$

converges to $a + ib$, and S is compact. This proves the theorem.

Theorem 4.2. *Let S be a compact set and let $S_1 \supset S_2 \supset \cdots$ be a sequence of non-empty closed subsets such that $S_n \supset S_{n+1}$. Then the intersection of all S_n for all $n = 1, 2, \ldots$ is not empty.*

Proof. Let $z_n \in S_n$. The sequence $\{z_n\}$ has a point of accumulation in S. Call it v. Then v is also a point of accumulation for each subsequence $\{z_k\}$ with $k \geqq n$, and hence lies in the closure of S_n for each n. But S_n is assumed closed, and hence $v \in S_n$ for all n. This proves the theorem.

Theorem 4.3. *Let S be a compact set of complex numbers, and let f be a continuous function on S. Then the image of f is compact.*

Proof. Let $\{w_n\}$ be a sequence in the image of f, so that

$$w_n = f(z_n) \qquad \text{for} \qquad z_n \in S.$$

The sequence $\{z_n\}$ has a convergent subsequence $\{z_{n_k}\}$, with a limit v in S. Since f is continuous, we have

$$\lim_{k\to\infty} w_{n_k} = \lim_{k\to\infty} f(z_{n_k}) = f(v).$$

Hence the given sequence $\{w_n\}$ has a subsequence which converges in $f(S)$. This proves that $f(S)$ is compact.

Theorem 4.4. *Let S be a compact set of complex numbers, and let*

$$f: S \to \mathbf{R}$$

be a continuous function. Then f has a maximum on S, that is, there exists $v \in S$ such that $f(z) \leqq f(v)$ for all $z \in S$.

Proof. By Theorem 4.3, we know that $f(S)$ is closed and bounded. Let b be its least upper bound. Then b is adherent to $f(S)$, whence in $f(S)$ because $f(S)$ is closed. So there is some $v \in S$ such that $f(v) = b$. This proves the theorem.

Remarks. In practice, one deals with a continuous function $f: S \to \mathbf{C}$ and one applies Theorem 4.4 to the absolute value of f, which is also continuous (composite of two continuous functions).

Theorem 4.5. *Let S be a compact set, and let f be a continuous function on S. Then f is uniformly continuous, i.e. given ϵ there exists δ such that whenever $z, w \in S$ and $|z - w| < \delta$, then $|f(z) - f(w)| < \epsilon$.*

Proof. Suppose the assertion of the theorem is false. Then there exists ϵ, and for each n there exists a pair of elements $z_n, w_n \in S$ such that

$$|z_n - w_n| < 1/n \qquad \text{but} \qquad |f(z_n) - f(w_n)| > \epsilon.$$

There is an infinite subset J_1 of positive integers and some $v \in S$ such that $z_n \to v$ for $n \to \infty$ and $n \in J_1$. There is an infinite subset J_2 of J_1 and $u \in S$ such that $w_n \to u$ for $n \to \infty$ and $n \in J_2$. Then, taking the limit for $n \to \infty$ and $n \in J_2$ we obtain $|u - v| = 0$ and $u = v$ because

$$|v - u| \leqq |v - z_n| + |z_n - w_n| + |w_n - u|.$$

Hence $f(v) - f(u) = 0$. Furthermore,

$$|f(z_n) - f(w_n)| \leqq |f(z_n) - f(v)| + |f(v) - f(u)| + |f(u) - f(w_n)|.$$

Again taking the limit as $n \to \infty$ and $n \in J_2$, we conclude that

$$f(z_n) - f(w_n)$$

approaches 0. This contradicts the assumption that

$$|f(z_n) - f(w_n)| > \epsilon,$$

and proves the theorem.

Let A, B be two sets of complex numbers. By the **distance** between them, denoted by $d(A, B)$, we mean

$$d(A, B) = \text{g.l.b.}|z - w|,$$

where the greatest lower bound g.l.b. is taken over all elements $z \in A$ and $w \in B$. If B consists of one point, we also write $d(A, w)$ instead of $d(A, B)$.

We shall leave the next two results as easy exercises.

Theorem 4.6. *Let S be a closed set of complex numbers, and let v be a complex number. There exists a point $w \in S$ such that*

$$d(S, v) = |w - v|.$$

[*Hint*: Let E be a closed disc of some suitable radius, centered at v, and consider the function $z \mapsto |z - v|$ for $z \in S \cap E$.]

Theorem 4.7. *Let K be a compact set of complex numbers, and let S be a closed set. There exist elements $z_0 \in K$ and $w_0 \in S$ such that*

$$d(K, S) = |z_0 - w_0|.$$

[*Hint*: Consider the function $z \mapsto d(S, z)$ for $z \in K$.]

Theorem 4.8. *Let S be compact. Let r be a real number > 0. There exists a finite number of open discs of radius r whose union contains S.*

Proof. Suppose this is false. Let $z_1 \in S$ and let D_1 be the open disc of radius r centered at z_1. Then D_1 does not contain S, and there is some $z_2 \in S$, $z_2 \neq z_1$. Proceeding inductively, suppose we have found open discs $D_1, \ldots, D_n$ of radius r centered at points $z_1, \ldots, z_n$, respectively, such that z_{k+1} does not lie in $D_1 \cup \cdots \cup D_k$. We can then find z_{n+1} which does not lie in $D_1 \cup \cdots \cup D_n$, and we let D_{n+1} be the disc of radius r centered at z_{n+1}. Let v be a point of accumulation of the sequence $\{z_n\}$. By definition, there exist positive integers m, k with $k > m$ such that

$$|z_k - v| < r/2 \quad \text{and} \quad |z_m - v| < r/2.$$

Then $|z_k - z_m| < r$ and this contradicts the property of our sequence $\{z_n\}$ because z_k lies in the disc D_m. This proves the theorem.

Let S be a set of complex numbers, and let I be some set. Suppose that for each $i \in I$ we are given an open set U_i. We denote this association by $\{U_i\}_{i \in I}$, and call it a **family of open sets**. The **union** of the family is the set U consisting of all z such that $z \in U_i$ for some $i \in I$. We say that the family **covers** S if S is contained in this union, that is, every $z \in S$ is contained in some U_i. We then say that the family $\{U_i\}_{i \in I}$ is an **open covering** of S. If J is a subset of I, we call the family $\{U_j\}_{j \in J}$ a **subfamily**, and if it covers S also, we call it a **subcovering** of S. In particular, if

$$U_{i_1}, \ldots, U_{i_n}$$

is a finite number of the open sets U_i, we say that it is a **finite subcovering** of S if S is contained in the finite union

$$U_{i_1} \cup \cdots \cup U_{i_n}.$$

Theorem 4.9. *Let S be a compact set, and let $\{U_i\}_{i \in I}$ be an open covering of S. Then there exists a finite subcovering, that is, a finite number of open sets $U_{i_1}, \ldots, U_{i_n}$ whose union covers S.*

Proof. By Theorem 4.8, for each n there exists a finite number of open discs of radius $1/n$ which cover S. Suppose that there is no finite subcovering of S by open sets U_i. Then for each n there exists one of the open discs D_n from the preceding finite number such that $D_n \cap S$ is not covered by any finite number of open sets U_i. Let $z_n \in D_n \cap S$, and let w be a point of accumulation of the sequence $\{z_n\}$. For some index i_0 we have $w \in U_{i_0}$. By definition, U_{i_0} contains an open disc D of radius $r > 0$ centered at w. Let N be so large that $2/N < r$. There exists $n > N$ such that

$$|z_n - w| \leq 1/N.$$

Any point of D_n is then at a distance $\leq 2/N$ from w, and hence D_n is contained in D, and thus contained in U_{i_0}. This contradicts the hypothesis made on D_n, and proves the theorem.

I, §4. EXERCISES

1. Let α be a complex number of absolute value < 1. What is $\lim_{n\to\infty} \alpha^n$? Proof?
2. If $|\alpha| > 1$, does $\lim_{n\to\infty} \alpha^n$ exist? Why?
3. Show that for any complex number $z \neq 1$, we have

$$1 + z + \cdots + z^n = \frac{z^{n+1} - 1}{z - 1}.$$

If $|z| < 1$, show that

$$\lim_{n\to\infty} (1 + z + \cdots + z^n) = \frac{1}{1 - z}.$$

4. Let f be the function defined by

$$f(z) = \lim_{n\to\infty} \frac{1}{1 + n^2 z}.$$

Show that f is the characteristic function of the set $\{0\}$, that is, $f(0) = 1$, and $f(z) = 0$ if $z \neq 0$.

5. For $|z| \neq 1$ show that the following limit exists:

$$f(z) = \lim_{n\to\infty} \left(\frac{z^n - 1}{z^n + 1}\right).$$

Is it possible to define $f(z)$ when $|z| = 1$ in such a way to make f continuous?

6. Let

$$f(z) = \lim_{n\to\infty} \frac{z^n}{1 + z^n}.$$

(a) What is the domain of definition of f, that is, for which complex numbers z does the limit exist?
(b) Give explicitly the values of $f(z)$ for the various z in the domain of f.

7. Show that the series

$$\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1 - z^n)(1 - z^{n+1})}$$

converges to $1/(1-z)^2$ for $|z| < 1$ and to $1/z(1-z)^2$ for $|z| > 1$. Prove that the convergence is uniform for $|z| \leqq c < 1$ in the first case, and $|z| \geqq b > 1$ in the second. [*Hint*: Multiply and divide each term by $1 - z$, and do a partial fraction decomposition, getting a telescoping effect.]

I, §5. COMPLEX DIFFERENTIABILITY

In studying differentiable functions of a real variable, we took such functions defined on intervals. For complex variables, we have to select domains of definition in an analogous manner.

Let U be an open set, and let z be a point of U. Let f be a function on U. We say that f is **complex differentiable** at z if the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists. This limit is denoted by $f'(z)$ or df/dz.

In this section, differentiable will always mean complex differentiable.

The usual proofs of a first course in calculus concerning basic properties of differentiability are valid for complex differentiability. We shall run through them again.

We note that if f is differentiable at z then f is continuous at z because

$$\lim_{h \to 0} \big(f(z+h) - f(z)\big) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} h$$

and since the limit of a product is the product of the limits, the limit on the right-hand side is equal to 0.

We let f, g be functions defined on the open set U. We assume that f, g are differentiable at z.

Sum. *The sum $f + g$ is differentiable at z, and*

$$(f + g)'(z) = f'(z) + g'(z).$$

Proof. This is immediate from the theorem that the limit of a sum is the sum of the limits.

Product. *The product fg is differentiable at z, and*

$$(fg)'(z) = f'(z)g(z) + f(z)g'(z).$$

Proof. To determine the limit of the Newton quotient

$$\frac{f(z+h)g(z+h) - f(z)g(z)}{h}$$

we write the numerator in the form

$$f(z+h)g(z+h) - f(z)g(z+h) + f(z)g(z+h) - f(z)g(z).$$

Then the Newton quotient is equal to a sum

$$\frac{f(z+h) - f(z)}{h}g(z+h) + f(z)\frac{g(z+h) - g(z)}{h}.$$

Taking the limits yields the formula.

Quotient. *If* $g(z) \neq 0$, *then the quotient* f/g *is differentiable at* z, *and*

$$(f/g)'(z) = \frac{g(z)f'(z) - f(z)g'(z)}{g(z)^2}.$$

Proof. This is again proved as in ordinary calculus. We first prove the differentiability of the quotient function $1/g$. We have

$$\frac{\dfrac{1}{g(z+h)} - \dfrac{1}{g(z)}}{h} = -\frac{g(z+h) - g(z)}{h}\frac{1}{g(z+h)g(z)}$$

Taking the limit yields

$$-\frac{1}{g(z)^2}g'(z),$$

which is the usual value. The general formula for a quotient is obtained from this by writing

$$f/g = f \cdot 1/g,$$

and using the rules for the derivative of a product, and the derivative of $1/g$.

Examples. As in ordinary calculus, from the formula for a product and induction, we see that for any positive integer n,

$$\frac{dz^n}{dz} = nz^{n-1}.$$

The rule for a quotient also shows that this formula remains valid when n is a negative integer.

The derivative of $z^2/(2z - 1)$ is

$$\frac{(2z-1)2z - 2z^2}{(2z-1)^2}.$$

This formula is valid for any complex number z such that $2z - 1 \neq 0$. More generally, let

$$f(z) = P(z)/Q(z),$$

where P, Q are polynomials. Then f is differentiable at any point z where $Q(z) \neq 0$.

Last comes the chain rule. Let U, V be open sets in $\mathbf{C}$, and let

$$f\colon U \to V \qquad \text{and} \qquad g\colon V \to \mathbf{C}$$

be functions, such that the image of f is contained in V. Then we can form the composite function $g \circ f$ such that

$$(g \circ f)(z) = g\big(f(z)\big).$$

Chain Rule. *Let $w = f(z)$. Assume that f is differentiable at z, and g is differentiable at w. Then $g \circ f$ is differentiable at z, and*

$$(g \circ f)'(z) = g'\big(f(z)\big)f'(z).$$

Proof. Again the proof is the same as in calculus, and depends on expressing differentiability by an equivalent property not involving denominators, as follows.

Suppose that f is differentiable at z, and let

$$\varphi(h) = \frac{f(z+h) - f(z)}{h} - f'(z).$$

Then

$$f(z+h) - f(z) = f'(z)h + h\varphi(h), \tag{1}$$

and

$$\lim_{h \to 0} \varphi(h) = 0. \tag{2}$$

Furthermore, even though φ is at first defined only for sufficiently small h and $h \neq 0$, we may also define $\varphi(0) = 0$, and formula (1) remains valid for $h = 0$.

Conversely, suppose that there exists a function φ defined for sufficiently small h and a number a such that

$$f(z + h) - f(z) = ah + h\varphi(h) \tag{1'}$$

and

$$\lim_{h \to 0} \varphi(h) = 0. \tag{2}$$

Dividing by h in formula (1') and taking the limit as $h \to 0$, we see that the limit exists and is equal to a. Thus $f'(z)$ exists and is equal to a. Hence the existence of a function φ satisfying (1'), (2) is equivalent to differentiability.

We apply this to a proof of the chain rule. Let $w = f(z)$, and

$$k = f(z + h) - f(z),$$

so that

$$g\big(f(z + h)\big) - g\big(f(z)\big) = g(w + k) - g(w).$$

There exists a function $\psi(k)$ such that $\lim_{k \to 0} \psi(k) = 0$ and

$$\begin{aligned} g(w + k) - g(w) &= g'(w)k + k\psi(k) \\ &= g'(w)\big(f(z + h) - f(z)\big) + \big(f(z + h) - f(z)\big)\psi(k). \end{aligned}$$

Dividing by h yields

$$\frac{g \circ f(z + h) - g \circ f(z)}{h} = g'(w)\frac{f(z + h) - f(z)}{h} + \frac{f(z + h) - f(z)}{h}\psi(k).$$

As $h \to 0$, we note that $k \to 0$ also by the continuity of f, whence $\psi(k) \to 0$ by assumption. Taking the limit of this last expression as $h \to 0$ proves the chain rule.

A function f defined on an open set U is said to be **differentiable** if it is differentiable at every point. We then also say that f is **holomorphic** on U. The word holomorphic is usually used in order not to have to specify *complex* differentiability as distinguished from real differentiability.

In line with general terminology, a holomorphic function

$$f: U \to V$$

from an open set into another is called a **holomorphic isomorphism** if there exists a holomorphic function

$$g: V \to U$$

such that g is the inverse of f, that is,

$$g \circ f = \mathrm{id}_U \quad \text{and} \quad f \circ g = \mathrm{id}_V.$$

A holomorphic isomorphism of U with itself is called a holomorphic **automorphism**. In the next chapter we discuss this notion in connection with functions defined by power series.

I, §6. THE CAUCHY–RIEMANN EQUATIONS

Let f be a function on an open set U, and write f in terms of its real and imaginary parts,

$$f(x + iy) = u(x, y) + iv(x, y).$$

It is reasonable to ask what the condition of differentiability means in terms of u and v. We shall analyze this situation in detail in Chapter VIII, but both for the sake of tradition, and because there is some need psychologically to see right away what the answer is, we derive the equivalent conditions on u, v for f to be holomorphic.

At a fixed z, let $f'(z) = a + bi$. Let $w = h + ik$, with h, k real. Suppose that

$$f(z + w) - f(z) = f'(z)w + \sigma(w)w,$$

where

$$\lim_{w \to 0} \sigma(w) = 0.$$

Then

$$f'(z)w = (a + bi)(h + ki) = ah - bk + i(bh + ak).$$

On the other hand, let

$$F: U \to \mathbf{R}^2$$

be the map (often called vector field) such that

$$F(x, y) = \big(u(x, y), v(x, y)\big).$$

We call F the (real) **vector field associated with** f. Then

$$F(x + h, y + k) - F(x, y) = (ah - bk, bh + ak) + \sigma_1(h, k)h + \sigma_2(h, k)k,$$

where $\sigma_1(h, k)$, $\sigma_2(h, k)$ are functions tending to 0 as (h, k) tends to 0. Hence if we assume that f is holomorphic, we conclude that F is differentiable in the sense of real variables, and that its derivative is represented by the (Jacobian) matrix

$$J_F(x, y) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}.$$

This shows that

$$\boxed{f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}}$$

and

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.}$$

These are called the **Cauchy–Riemann equations.**

Conversely, let $u(x, y)$ and $v(x, y)$ be two functions satisfying the Cauchy–Riemann equations, and continuously differentiable in the sense of real functions. Define

$$f(z) = f(x + iy) = u(x, y) + iv(x, y).$$

Then it is immediately verified by reversing the above steps that f is complex-differentiable, i.e. holomorphic.

The Jacobian determinant Δ_F of the associated vector field F is

$$\Delta_F = a^2 + b^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2.$$

Hence $\Delta_F \geqq 0$, and is $\neq 0$ if and only if $f'(z) \neq 0$. We have

$$\boxed{\Delta_F(x, y) = |f'(z)|^2.}$$

We now drop these considerations until Chapter VIII.

The study of the real part of a holomorphic function and its relation with the function itself will be carried out more substantially in Chapter VIII. It is important, and much of that chapter depends only on elementary facts. However, the most important part of complex analysis at the present level lies in the power series aspects and the immediate applications of Cauchy's theorem. The real part plays no role in these matters. Thus we do not wish to interrupt the straightforward flow of the book now towards these topics.

However, the reader may read immediately the more elementary parts §1 and §2 of Chapter VIII, which can be understood already at this point.

I, §6. EXERCISE

1. Prove in detail that if u, v satisfy the Cauchy–Riemann equations, then the function

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

is holomorphic.

I, §7. ANGLES UNDER HOLOMORPHIC MAPS

In this section, we give a simple geometric property of holomorphic maps. Roughly speaking, they preserve angles. We make this precise as follows.

Let U be an open set in C and let

$$\gamma: [a, b] \to U$$

be a curve in U, so we write

$$\gamma(t) = x(t) + iy(t).$$

We assume that γ is differentiable, so its derivative is given by

$$\gamma'(t) = x'(t) + iy'(t).$$

Let $f: U \to \mathbf{C}$ be holomorphic. We let the reader verify the chain rule

$$\frac{d}{dt} f(\gamma(t)) = f'(\gamma(t))\gamma'(t).$$

We interpret $\gamma'(t)$ as a vector in the direction of a tangent vector at the point $\gamma(t)$. This derivative $\gamma'(t)$, if not 0, defines the direction of the curve at the point.

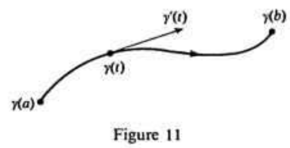

Figure 11

Consider two curves γ and η passing through the same point z_0. Say

$$z_0 = \gamma(t_0) = \eta(t_1).$$

Then the tangent vectors $\gamma'(t_0)$ and $\eta'(t_1)$ determine an angle θ which is defined to be the **angle between the curves**.

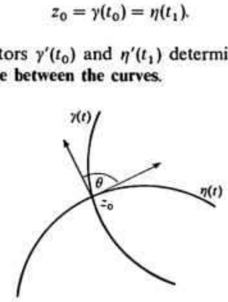

Figure 12

Applying f, the curves $f \circ \gamma$ and $f \circ \eta$ pass through the point $f(z_0)$, and by the chain rule, tangent vectors of these image curves are

$$f'(z_0)\gamma'(t_0) \quad \text{and} \quad f'(z_0)\eta'(t_1).$$

Theorem 7.1. *If $f'(z_0) \neq 0$ then the angle between the curves γ, η at z_0 is the same as the angle between the curves $f \circ \gamma$, $f \circ \eta$ at $f(z_0)$.*

Proof. Geometrically speaking, the tangent vectors under f are changed by multiplication with $f'(z_0)$, which can be represented in polar coordinates as a dilation and a rotation, so preserves the angles.

We shall now give a more formal argument, dealing with the cosine and sine of angles.

Let z, w be complex numbers,

$$z = a + bi \quad \text{and} \quad w = c + di,$$

where a, b, c, d are real. Then

$$z\bar{w} = ac + bd + i(bc - ad).$$

Define the **scalar product**

$$\langle z, w \rangle = \operatorname{Re}(z\bar{w}). \tag{1}$$

Then $\langle z, w \rangle$ is the ordinary scalar product of the vectors (a, b) and (c, d) in $\mathbf{R}^2$. Let $\theta(z, w)$ be the angle between z and w. Then

$$\cos \theta(z, w) = \frac{\langle z, w \rangle}{|z||w|}. \tag{2}$$

Since $\sin \theta = \cos\left(\theta - \frac{\pi}{2}\right)$, we can define

$$\sin \theta(z, w) = \frac{\langle z, -iw \rangle}{|z||w|}. \tag{3}$$

This gives us the desired precise formulas for the cosine and sine of an angle, which determine the angle.

Let $f'(z_0) = \alpha$. Then

$$\langle \alpha z, \alpha w \rangle = \operatorname{Re}(\alpha z \bar{\alpha} \bar{w}) = \alpha\bar{\alpha} \operatorname{Re}(z\bar{w}) = |\alpha|^2 \langle z, w \rangle \tag{4}$$

because $\alpha\bar{\alpha} = |\alpha|^2$ is real. It follows immediately from the above formulas that

$$\cos \theta(\alpha z, \alpha w) = \cos \theta(z, w) \quad \text{and} \quad \sin \theta(\alpha z, \alpha w) = \sin \theta(z, w). \tag{5}$$

This proves the theorem.

A map which preserves angles is called **conformal**. Thus we can say that a holomorphic map with non-zero derivative is conformal. The complex conjugate of a holomorphic map also preserves angles, if we disregard the orientation of an angle.

In Chapter VII, we shall consider holomorphic maps which have inverse holomorphic maps, and therefore such that their derivatives are never equal to 0. The theorem proved in this section gives additional geometric information concerning the nature of such maps. But the emphasis of the theorem in this section is local, whereas the emphasis in Chapter VII will be global. The word "conformal", however, has become a code word for this kind of map, even in the global case, which explains the title of Chapter VII. The reader will notice that the local property of preserving angles is irrelevant for the global arguments given in Chapter VII, having to do with inverse mappings. Thus in Chapter VII, we shall use a terminology which emphasizes the invertibility, namely the terminology of isomorphisms and automorphisms.

In this terminology, we can say that a holomorphic isomorphism is conformal. The converse is false in general. For instance, let U be the open set obtained by deleting the origin from the complex numbers. The function

$$f\colon U \to U \qquad \text{given by} \qquad z \mapsto z^2$$

has everywhere non-zero derivative in U, but it does not admit an inverse function. This function f is definitely conformal. The invertibility is true locally, however. See Theorem 5.1 of Chapter II.

CHAPTER II

Power Series

So far, we have given only rational functions as examples of holomorphic functions. We shall study other ways of defining such functions. One of the principal ways will be by means of power series. Thus we shall see that the series

$$1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

converges for all z to define a function which is equal to e^z. Similarly, we shall extend the values of $\sin z$ and $\cos z$ by their usual series to complex valued functions of a complex variable, and we shall see that they have similar properties to the functions of a real variable which you already know.

First we shall learn to manipulate power series formally. In elementary calculus courses, we derived Taylor's formula with an error term. Here we are concerned with the full power series. In a way, we pick up where calculus left off. We study systematically sums, products, inverses, and composition of power series, and then relate the formal operations with questions of convergence.

II, §1. FORMAL POWER SERIES

We select at first a neutral letter, say T. In writing a formal power series

$$\sum_{n=0}^{\infty} a_n T^n = a_0 + a_1 T + a_2 T^2 + \cdots$$

what is essential are its "coefficients" $a_0, a_1, a_2, \ldots$ which we shall take as complex numbers. Thus the above series may be defined as the function

$$n \mapsto a_n$$

from the integers $\geqq 0$ to the complex numbers.

We could use other letters besides T, writing

$$f(T) = \sum a_n T^n,$$
$$f(r) = \sum a_n r^n,$$
$$f(z) = \sum a_n z^n.$$

In such notation, f does not denote a function, but a formal expression.

Also, as a matter of notation, we write a single term

$$a_n T^n$$

to denote the power series such that $a_k = 0$ if $k \neq n$. For instance, we would write

$$5T^3$$

for the power series

$$0 + 0T + 0T^2 + 5T^3 + 0T^4 + \cdots,$$

such that $a_3 = 5$ and $a_k = 0$ if $k \neq 3$.

By definition, we call a_0 the **constant term** of f.

If

$$f = \sum a_n T^n \quad \text{and} \quad g = \sum b_n T^n$$

are such formal power series, we define their **sum** to be

$$f + g = \sum c_n T^n, \quad \text{where} \quad c_n = a_n + b_n.$$

We define their **product** to be

$$fg = \sum d_n T^n, \quad \text{where} \quad d_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

The sum and product are therefore defined just as for polynomials. The first few terms of the product can be written as

$$fg = a_0 b_0 + (a_0 b_1 + a_1 b_0)T + (a_0 b_2 + a_1 b_1 + a_2 b_0)T^2 + \cdots.$$

If α is a complex number, we define

$$\alpha f = \sum (\alpha a_n) T^n$$

to be the power series whose n-th coefficient is αa_n. Thus we can multiply power series by numbers.

Just as for polynomials, one verifies that the sum and product are associative, commutative, and distributive. Thus in particular, if f, g, h are power series, then

$$f(g + h) = fg + fh \quad \text{(distributivity).}$$

We omit the proof, which is just elementary algebra.

The **zero power series** is the series such that $a_n = 0$ for all integers $n \geqq 0$.

Suppose a power series is of the form

$$f = a_r T^r + a_{r+1} T^{r+1} + \cdots,$$

and $a_r \neq 0$. Thus r is the smallest integer n such that $a_n \neq 0$. Then we call r the **order** of f, and write

$$r = \text{ord } f.$$

If ord $g = s$, so that

$$g = b_s T^s + b_{s+1} T^{s+1} + \cdots,$$

and $b_s \neq 0$, then by definition,

$$fg = a_r b_s T^{r+s} + \text{higher terms},$$

and $a_r b_s \neq 0$. Hence

$$\boxed{\text{ord } fg = \text{ord } f + \text{ord } g.}$$

A power series has order 0 if and only if it starts with a non-zero constant term. For instance, the geometric series

$$1 + T + T^2 + T^3 + \cdots$$

has order 0.

Let $f = \sum a_n T^n$ be a power series. We say that $g = \sum b_n T^n$ is an

inverse for f if

$$fg = 1.$$

In view of the relation for orders which we just mentioned, we note that if an inverse exists, then we must have

$$\text{ord } f = \text{ord } g = 0.$$

In other words, both f and g start with non-zero constant terms. The converse is true:

If f has a non-zero constant term, then f has an inverse as a power series.

Proof. Considering $a_0^{-1}f$ instead of f, we are reduced to the case when the constant term is equal to 1. We first note that the old geometric series gives us a formal inverse,

$$\frac{1}{1-r} = 1 + r + r^2 + \cdots.$$

Written multiplicatively, this amounts to

$$\begin{aligned}(1-r)(1+r+r^2+\cdots) &= 1 + r + r^2 + \cdots - r(1 + r + r^2 + \cdots)\\ &= 1 + r + r^2 + \cdots - r - r^2 - \cdots\\ &= 1.\end{aligned}$$

Next, write

$$f = 1 - h, \qquad \text{where} \quad h = -(a_1 T + a_2 T^2 + \cdots).$$

To find the inverse $(1 - h)^{-1}$ is now easy, namely it is the power series

$$(*) \qquad \varphi = 1 + h + h^2 + h^3 + \cdots.$$

We have to verify that this makes sense. Any finite sum

$$1 + h + h^2 + \cdots + h^m$$

makes sense because we have defined sums and products of power series. Observe that the order of h^n is at least n, because h^n is of the form

$$(-1)^n a_1^n T^n + \text{higher terms.}$$

Thus in the above sum (∗), if $m > n$, then the term h^m has all coefficients of order $\leqq n$ equal to 0. Thus we may define the n-th coefficient of φ to be the n-th coefficient of the finite sum

$$1 + h + h^2 + \cdots + h^n.$$

It is then easy to verify that

$$(1 - h)\varphi = (1 - h)(1 + h + h^2 + h^3 + \cdots)$$

is equal to

$$1 + \text{a power series of arbitrarily high order,}$$

and consequently is equal to 1. Hence we have found the desired inverse for f.

Example. Let

$$\cos T = 1 - \frac{T^2}{2!} + \frac{T^4}{4!} - \cdots$$

be the formal power series whose coefficients are the same as for the Taylor expansion of the ordinary cosine function in elementary calculus. We want to write down the first few terms of its (formal) inverse,

$$\frac{1}{\cos T}.$$

Up to terms of order 4, these will be the same as

$$\begin{aligned}\frac{1}{1 - \left(\frac{T^2}{2!} - \frac{T^4}{4!} + \cdots\right)} &= 1 + \left(\frac{T^2}{2!} - \frac{T^4}{4!} + \cdots\right) \\ &\quad + \left(\frac{T^2}{2!} - \frac{T^4}{4!} + \cdots\right)^2 + \cdots \\ &= 1 + \frac{T^2}{2!} - \frac{T^4}{4!} + \cdots + \frac{T^4}{(2!)^2} + \cdots \\ &= 1 + \frac{1}{2}T^2 + \left(\frac{-1}{24} + \frac{1}{4}\right)T^4 + \text{higher terms.}\end{aligned}$$

This gives us the coefficients of $1/\cos T$ up to order 4.

The substitution of h in the geometric series used to find an inverse can be generalized. Let

$$f = \sum a_n T^n$$

be a power series, and let

$$h(T) = c_1 T + \cdots$$

be a power series *whose constant term is* 0, so ord $h \geqq 1$. Then we may **"substitute"** h in f to define a power series $f \circ h$ or $f(h)$, by

$$(f \circ h)(T) = f(h(T)) = f \circ h = a_0 + a_1 h + a_2 h^2 + a_3 h^3 + \cdots$$

in a natural way. Indeed, the finite sums

$$a_0 + a_1 h + \cdots + a_n h^n$$

are defined by the ordinary sum and product of power series. If $m > n$, then $a_m h^m$ has order $> n$; in other words, it is a power series starting with non-zero terms of order $> n$. Consequently we can define the power series $f \circ h$ as that series whose n-th coefficient is the n-th coefficient of

$$a_0 + a_1 h + \cdots + a_n h^n.$$

This composition of power series, like addition and multiplication, can therefore be computed by working only with polynomials. In fact, it is useful to discuss this approximation by polynomials a little more systematically.

We say that two power series $f = \sum a_n T^n$ and $g = \sum b_n T^n$ are **congruent mod** T^N and write $f \equiv g \pmod{T^N}$ if

$$a_n = b_n \quad \text{for} \quad n = 0, \ldots, N - 1.$$

This means that the terms of order $\leqq N - 1$ coincide for the two power series. Given the power series f, there is a unique polynomial $P(T)$ of degree $\leqq N - 1$ such that

$$f(T) \equiv P(T) \pmod{T^N},$$

namely the polynomial

$$P(T) = a_0 + a_1 T + \cdots + a_{N-1} T^{N-1}.$$

If $f_1 \equiv f_2$ *and* $g_1 \equiv g_2 \pmod{T^N}$, *then*

$$f_1 + g_1 \equiv f_2 + g_2 \quad \text{and} \quad f_1 g_1 \equiv f_2 g_2 \pmod{T^N}.$$

If h_1, h_2 *are power series with zero constant term, and*

$$h_1 \equiv h_2 \pmod{T^N},$$

then

$$f_1(h_1(T)) \equiv f_2(h_2(T)) \pmod{T^N}.$$

Proof. We leave the sum and product to the reader. Let us look at the proof for the composition $f_1 \circ h_1$ and $f_2 \circ h_2$. First suppose h has zero constant term. Let P_1, P_2 be the polynomials of degree $N-1$ such that

$$f_1 \equiv P_1 \quad \text{and} \quad f_2 \equiv P_2 \pmod{T^N}.$$

Then by hypothesis, $P_1 = P_2 = P$ is the same polynomial, and

$$f_1(h) \equiv P_1(h) = P_2(h) \equiv f_2(h) \pmod{T^N}.$$

Next let Q be the polynomial of degree $N-1$ such that

$$h_1(T) \equiv h_2(T) \equiv Q(T) \pmod{T^N}.$$

Write $P = a_0 + a_1 T + \cdots + a_{N-1} T^{N-1}$. Then

$$\begin{aligned} P(h_1) &= a_0 + a_1 h_1 + \cdots + a_{N-1} h_1^{N-1} \\ &\equiv a_0 + a_1 Q + \cdots + a_{N-1} Q^{N-1} \\ &\equiv a_0 + a_1 h_2 + \cdots + a_{N-1} h_2^{N-1} \\ &\equiv P(h_2) \pmod{T^N}. \end{aligned}$$

This proves the desired property, that $f_1 \circ h_1 \equiv f_2 \circ h_2 \pmod{T^N}$.

With these rules we can compute the coefficients of various operations between power series by reducing the computations to polynomial operations, which amount to high-school algebra. Indeed, two power series f, g are equal if and only if

$$f \equiv g \pmod{T^N}$$

for every positive integer N. Verifying that $f \equiv g \pmod{T^N}$ can be done by working entirely with polynomials of degree $< N$.

If f_1, f_2 *are power series, then*

$$(f_1 + f_2)(h) = f_1(h) + f_2(h),$$

$$(f_1 f_2)(h) = f_1(h) f_2(h), \qquad \textit{and} \qquad (f_1/f_2)(h) = f_1(h)/f_2(h)$$

if $\operatorname{ord} f_2 = 0$. *If* g, h *have constant terms equal to* 0, *then*

$$f(g(h)) = (f \circ g)(h).$$

Proof. In each case, the proof is obtained by reducing the statement to the polynomial case, and seeing that the required properties hold for polynomials, which is standard. For instance, for the associativity of composition, given a positive integer N, let P, Q, R be polynomials of degree $\leqq N - 1$ such that

$$f \equiv P, \qquad g \equiv Q, \qquad h \equiv R \pmod{T^N}.$$

The ordinary theory of polynomials shows that

$$P(Q(R)) = (P \circ Q)(R).$$

The left-hand side is congruent to $f(g(h))$, and the right-hand side is congruent to $(f \circ g)(h) \pmod{T^N}$ by the properties which have already been proved. Hence

$$f(g(h)) \equiv (f \circ g)(h) \pmod{T^N}.$$

This is true for each N, whence $f(g(h)) = (f \circ g)(h)$, as desired.

In applications it is useful to consider power series which have a finite number of terms involving $1/z$, and this amounts also to considering arbitrary quotients of power series as follows.

Just as fractions m/n are formed with integers m, n and $n \neq 0$, we can form quotients

$$f/g = f(T)/g(T)$$

of power series such that $g \neq 0$. Two such quotients f/g and f_1/g_1 are regarded as equal if and only if

$$f g_1 = f_1 g,$$

which is exactly the condition under which we regard two rational numbers m/n and m_1/n_1 as equal. We have then defined for power series all the operations of arithmetic.

Let

$$f(T) = a_m T^m + a_{m+1} T^{m+1} + \cdots = \sum_{n \geqq m} a_n T^n$$

be a power series with $a_m \neq 0$. We may then write f in the form

$$f = a_m T^m (1 + h(T)),$$

where $h(T)$ has zero constant term. Consequently $1/f$ has the form

$$1/f = \frac{1}{a_m T^m} \frac{1}{1 + h(T)}.$$

We know how to invert $1 + h(T)$, say

$$(1 + h(T))^{-1} = 1 + b_1 T + b_2 T^2 + \cdots.$$

Then $1/f(T)$ has the shape

$$1/f = a_m^{-1} \frac{1}{T^m} + a_m^{-1} b_1 \frac{1}{T^{m-1}} + \cdots.$$

It is a power series with a finite number of terms having negative powers of T. In this manner, one sees that an arbitrary quotient can always be expressed as a power series of the form

$$\begin{aligned} f/g &= \frac{c_{-m}}{T^m} + \frac{c_{-m+1}}{T^{m-1}} + \cdots + c_0 + c_1 T + c_2 T^2 + \cdots \\ &= \sum_{n \geqq -m} c_n T^n. \end{aligned}$$

If $c_{-m} \neq 0$, then we call $-m$ the **order** of f/g. It is again verified as for power series without negative terms that if

$$\varphi = f/g \quad \text{and} \quad \varphi_1 = f_1/g_1,$$

then

$$\operatorname{ord} \varphi\varphi_1 = \operatorname{ord} \varphi + \operatorname{ord} \varphi_1.$$

Example. Find the terms of order $\leqq 3$ in the power series for $1/\sin T$. By definition,

$$\begin{aligned} \sin T &= T - T^3/3! + T^5/5! - \cdots \\ &= T(1 - T^2/3! + T^4/5! - \cdots). \end{aligned}$$

Hence

$$\frac{1}{\sin T} = \frac{1}{T}\frac{1}{1 - T^2/3! + T^4/5! + \cdots}$$

$$= \frac{1}{T}(1 + T^2/3! - T^4/5! + (T^2/3!)^2 + \text{higher terms}$$

$$= \frac{1}{T} + \frac{1}{3!}T + \left(\frac{1}{(3!)^2} - \frac{1}{5!}\right)T^3 + \text{higher terms.}$$

This does what we wanted.

II, §1. EXERCISES

We shall write the formal power series in terms of z because that's the way they arise in practice. The series for $\sin z$, $\cos z$, e^z, etc. are to be viewed as formal series.

1. Give the terms of order $\leqq 3$ in the power series:

(a) $e^z \sin z$ (b) $(\sin z)(\cos z)$ (c) $\dfrac{e^z - 1}{z}$

(d) $\dfrac{e^z - \cos z}{z}$ (e) $\dfrac{1}{\cos z}$ (f) $\dfrac{\cos z}{\sin z}$

(g) $\dfrac{\sin z}{\cos z}$ (h) $e^z/\sin z$

2. Let $f(z) = \sum a_n z^n$. Define $f(-z) = \sum a_n(-z)^n = \sum a_n(-1)^n z^n$. We define $f(z)$ to be **even** if $a_n = 0$ for n odd. We define $f(z)$ to be **odd** if $a_n = 0$ for n even. Verify that f is even if and only if $f(-z) = f(z)$ and f is odd if and only if $f(-z) = -f(z)$.

3. Define the **Bernoulli numbers** B_n by the power series

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$

Prove the recursion formula

$$\frac{B_0}{n!\,0!} + \frac{B_1}{(n-1)!\,1!} + \cdots + \frac{B_{n-1}}{1!\,(n-1)!} = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

Then $B_0 = 1$. Compute B_1, B_2, B_3, B_4. Show that $B_n = 0$ if n is odd $\neq 1$.

4. Show that

$$\frac{z}{2}\frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}} = \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} z^{2n}.$$

Replace z by $2\pi iz$ to show that

$$\pi z \cot \pi z = \sum_{n=0}^{\infty} (-1)^n \frac{(2\pi)^{2n}}{(2n)!} B_{2n} z^{2n}.$$

5. Express the power series for $\tan z$, $z/\sin z$, $z \cot z$, in terms of Bernoulli numbers.

6. **(Difference Equations).** Given complex numbers a_0, a_1, u_1, u_2 define a_n for $n \geqq 2$ by

$$a_n = u_1 a_{n-1} + u_2 a_{n-2}.$$

If we have a factorization

$$T^2 - u_1 T - u_2 = (T - \alpha)(T - \alpha'), \quad \text{and} \quad \alpha \neq \alpha',$$

show that the numbers a_n are given by

$$a_n = A\alpha^n + B\alpha'^n$$

with suitable A, B. Find A, B in terms of a_0, a_1, α, α'. Consider the power series

$$F(T) = \sum_{n=0}^{\infty} a_n T^n.$$

Show that it represents a rational function, and give its partial fraction decomposition.

7. More generally, let $a_0, \ldots, a_{r-1}$ be given complex numbers. Let $u_1, \ldots, u_r$ be complex number such that the polynomial

$$P(T) = T^r - (u_1 T^{r-1} + \cdots + u_r)$$

has distinct roots $\alpha_1, \ldots, \alpha_r$. Define a_n for $n \geqq r$ by

$$a_n = u_1 a_{n-1} + \cdots + u_r a_{n-r}.$$

Show that there exist numbers $A_1, \ldots, A_r$ such that for all n,

$$a_n = A_1 \alpha_1^n + \cdots + A_r \alpha_r^n.$$

II, §2. CONVERGENT POWER SERIES

We first recall some terminology about series of complex numbers.

Let $\{z_n\}$ be a sequence of complex numbers. Consider the series

$$\sum_{n=1}^{\infty} z_n.$$

We define the **partial sum**

$$s_n = \sum_{k=1}^{n} z_k = z_1 + z_2 + \cdots + z_n.$$

We say that the series **converges** if there is some w such that

$$\lim_{n\to\infty} s_n = w$$

exists, in which case we say that w is equal to the **sum of the series**, that is,

$$w = \sum_{n=1}^{\infty} z_n.$$

If $A = \sum \alpha_n$ and $B = \sum \beta_n$ are two convergent series, with partial sums

$$s_n = \sum_{k=1}^{n} \alpha_k \qquad \text{and} \qquad t_n = \sum_{k=1}^{n} \beta_k,$$

then the sum and product converge. Namely,

$$A + B = \sum (\alpha_n + \beta_n)$$
$$AB = \lim_{n\to\infty} s_n t_n$$

Let $\{c_n\}$ be a series of real numbers $c_n \geqq 0$. If the partial sums

$$\sum_{k=1}^{n} c_k$$

are bounded, we recall from calculus that the series converges, and that the least upper bound of these partial sums is the limit.

Let $\sum \alpha_n$ be a series of complex numbers. We shall say that this series **converges absolutely** if the real positive series

$$\sum |\alpha_n|$$

converges. If a series converges absolutely, then it converges. Indeed, let

$$s_n = \sum_{k=1}^{n} \alpha_k$$

be the partial sums. Then for $m \leqq n$ we have

$$s_n - s_m = \alpha_{m+1} + \cdots + \alpha_n$$

whence

$$|s_n - s_m| \leqq \sum_{k=m+1}^{n} |\alpha_k|.$$

Assuming absolute convergence, given ϵ there exists N such that if n, $m \geqq N$, then the right-hand side of this last expression is $< \epsilon$, thereby proving that the partial sums form a Cauchy sequence, and hence that the series converges.

We have the usual test for convergence:

Let $\sum c_n$ be a series of real numbers $\geqq 0$ which converges. If $|\alpha_n| \leqq c_n$ for all n, then the series $\sum \alpha_n$ converges absolutely.

Proof. The partial sums

$$\sum_{k=1}^{n} c_k$$

are bounded by assumption, whence the partial sums

$$\sum_{k=1}^{n} |\alpha_k| \leqq \sum_{k=1}^{n} c_k$$

are also bounded, and the absolute convergence follows.

In the sequel we shall also assume some standard facts about absolutely convergent series, namely:

(i) *If a series $\sum \alpha_n$ is absolutely convergent, then the series obtained by any rearrangement of the terms is also absolutely convergent, and converges to the same limit.*

(ii) *If a double series*

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} \alpha_{mn}\right)$$

is absolutely convergent, then the order of summation can be interchanged, and the series so obtained is absolutely convergent, and converges to the same value.

The proof is easily obtained by considering approximating partial sums (finite sums), and estimating the tail ends. We omit it.

We shall now consider series of functions, and deal with questions of uniformity.

Let S be a set, and f a bounded function on S. Then we define the

sup norm

$$\|f\|_S = \|f\| = \sup_{z \in S} |f(z)|,$$

where sup means least upper bound. It is a norm in the sense that for two functions f, g we have $\|f + g\| \leqq \|f\| + \|g\|$, and for any number c we have $\|cf\| = |c|\,\|f\|$. Also $f = 0$ if and only if $\|f\| = 0$.

Let $\{f_n\}$ $(n = 1, 2, \ldots)$ be a sequence of functions on S. We shall say that this sequence **converges uniformly** on S if there exists a function f on S satisfying the following properties. Given ϵ, there exists N such that if $n \geqq N$, then

$$\|f_n - f\| < \epsilon.$$

We say that $\{f_n\}$ is a **Cauchy sequence** (for the sup norm), if given ϵ, there exists N such that if $m, n \geqq N$, then

$$\|f_n - f_m\| < \epsilon.$$

In this case, for each $z \in S$, the sequence of complex numbers

$$\{f_n(z)\}$$

converges, because for each $z \in S$, we have the inequality

$$|f_n(z) - f_m(z)| \leqq \|f_n - f_m\|.$$

Theorem 2.1. *If a sequence $\{f_n\}$ of functions on S is Cauchy, then it converges uniformly.*

Proof. For each $z \in S$, let

$$f(z) = \lim_{n \to \infty} f_n(z).$$

Given ϵ, there exists N such that if $m, n \geqq N$, then

$$|f_n(z) - f_m(z)| < \epsilon, \qquad \text{for all} \quad z \in S.$$

Let $n \geqq N$. Given $z \in S$ select $m \geqq N$ sufficiently large (depending on z) such that

$$|f(z) - f_m(z)| < \epsilon.$$

Then

$$\begin{aligned} |f(z) - f_n(z)| &\leqq |f(z) - f_m(z)| + |f_m(z) - f_n(z)| \\ &< \epsilon + \|f_m - f_n\| \\ &< 2\epsilon. \end{aligned}$$

This is true for any z, and therefore $\|f - f_n\| < 2\epsilon$, which proves the theorem.

Remark. It is immediately seen that if the functions f_n in the theorem are bounded, then the limiting function f is also bounded.

Consider a series of functions, $\sum f_n$. Let

$$s_n = \sum_{k=1}^{n} f_k = f_1 + f_2 + \cdots + f_n$$

be the partial sum. We say that the **series converges uniformly** if the sequence of partial sums $\{s_n\}$ converges uniformly.

A series $\sum f_n$ is said to converge **absolutely** if for each $z \in S$ the series

$$\sum |f_n(z)|$$

converges.

The next theorem is sometimes called the **comparison test**.

Theorem 2.2. *Let $\{c_n\}$ be a sequence of real numbers $\geqq 0$, and assume that*

$$\sum c_n$$

converges. Let $\{f_n\}$ be a sequence of functions on S such that $\|f_n\| \leqq c_n$ for all n. Then $\sum f_n$ converges uniformly and absolutely.

Proof. Say $m \leqq n$. We have an estimate for the difference of partial sums,

$$\|s_n - s_m\| \leqq \sum_{k=m+1}^{n} \|f_k\| \leqq \sum_{k=m+1}^{n} c_k.$$

The assumption that $\sum c_k$ converges implies at once the uniform convergence of the partial sums. The argument also shows that the convergence is absolute.

Theorem 2.3. *Let S be a set of complex numbers, and let $\{f_n\}$ be a sequence of continuous functions on S. If this sequence converges uniformly, then the limit function f is also continuous.*

Proof. You should already have seen this theorem some time during a calculus course. We reproduce the proof for convenience. Let $\alpha \in S$. Select n so large that $\|f - f_n\| < \epsilon$. For this choice of n, using the continuity of f_n at α, select δ such that whenever $|z - \alpha| < \delta$ we have

$$|f_n(z) - f_n(\alpha)| < \epsilon.$$

Then

$$|f(z) - f(\alpha)| \leqq |f(z) - f_n(z)| + |f_n(z) - f_n(\alpha)| + |f_n(\alpha) - f(\alpha)|.$$

The first and third term on the right are bounded by $\|f - f_n\| < \epsilon$. The middle term is $< \epsilon$. Hence

$$|f(z) - f(\alpha)| < 3\epsilon,$$

and our theorem is proved.

We now consider the power series, where the functions f_n are

$$f_n(z) = a_n z^n,$$

with complex numbers a_n.

Theorem 2.4. *Let $\{a_n\}$ be a sequence of complex numbers, and let r be a number > 0 such that the series*

$$\sum |a_n| r^n$$

converges. Then the series $\sum a_n z^n$ converges absolutely and uniformly for $|z| \leqq r$.

Proof. Special case of the comparison test.

Example. For any $r > 0$, the series

$$\sum z^n/n!$$

converges absolutely and uniformly for $|z| \leqq r$. Indeed, let

$$c_n = r^n/n!.$$

Then

$$\frac{c_{n+1}}{c_n} = \frac{r^{n+1}}{(n+1)!}\frac{n!}{r^n} = \frac{r}{n+1}.$$

Take $n \geqq 2r$. Then the right-hand side is $\leqq 1/2$. Hence for all n sufficiently large, we have

$$c_{n+1} \leqq \tfrac{1}{2}c_n.$$

Therefore there exists some positive integer n_0 such that

$$c_n \leqq C/2^{n-n_0},$$

for some constant C and all $n \geqq n_0$. We may therefore compare our series with a geometric series to get the absolute and uniform convergence.

The series

$$\exp(z) = \sum_{n=0}^{\infty} z^n/n!$$

therefore defines a continuous function for all values of z. Similarly, the series

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

and

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

converge absolutely and uniformly for all $|z| \leqq r$. They give extensions of the sine and cosine functions to the complex numbers. We shall see later that $\exp(z) = e^z$ as defined in Chapter I, and that these series define the unique analytic functions which coincide with the usual exponential, sine, and cosine functions, respectively, when z is real.

Theorem 2.5. *Let $\sum a_n z^n$ be a power series. If it does not converge absolutely for all z, then there exists a number r such that the series converges absolutely for $|z| < r$ and does not converge absolutely for $|z| > r$.*

Proof. Suppose that the series does not converge absolutely for all z. Let r be the least upper bound of those numbers $s \geqq 0$ such that

$$\sum |a_n| s^n$$

converges. Then $\sum |a_n||z|^n$ diverges if $|z| > r$, and converges if $|z| < r$ by the comparison test, so our assertion is obvious.

The number r in Theorem 2.5 is called the **radius of convergence** of the power series. If the power series converges absolutely for all z, then we say that its **radius of convergence** is **infinity**. When the radius of convergence is 0, then the series converges absolutely only for $z = 0$.

If a power series has a non-zero radius of convergence, then it is called a **convergent** power series. If D is a disc centered at the origin and contained in the disc $D(0, r)$, where r is the radius of convergence, then we say that the power series **converges on** D.

The radius of convergence can be determined in terms of the coefficients. Let t_n be a sequence of real numbers. We recall that a **point of accumulation** of this sequence is a number t such that, given ϵ, there exist infinitely many indices n such that

$$|t_n - t| < \epsilon.$$

In other words, infinitely many points of the sequence lie in a given interval centered at t. An elementary property of real numbers asserts that every bounded sequence has a point of accumulation (Weierstrass–Balzano theorem).

Assume now that $\{t_n\}$ is a bounded sequence. Let S be the set of points of accumulation, so that S looks like Fig. 1.

Figure 1

We define the **limit superior**, lim sup, of the sequence to be

$$\lambda = \limsup t_n = \text{least upper bound of } S.$$

Then the reader will verify at once that λ is itself a point of accumulation of the sequence, and is therefore the largest such point. Furthermore, λ has the following properties:

Given ϵ, there exist only finitely many n such that $t_n \geqq \lambda + \epsilon$. There exist infinitely many n such that

$$t_n \geqq \lambda - \epsilon.$$

Proof. If there were infinitely many n such that $t_n \geqq \lambda + \epsilon$, then the sequence $\{t_n\}$ would have a point of accumulation

$$\geqq \lambda + \epsilon > \lambda,$$

contrary to assumption. On the other hand, since λ itself is a point of accumulation, given the ϵ-interval about λ, there have to be infinitely many n such that t_n lies in this ϵ-interval, thus proving the second assertion.

We leave it to the reader to verify that *if a number λ has the above properties, then it is the* lim sup *of the sequence.*

For convenience, if $\{t_n\}$ is not bounded from above, we define its lim sup to be infinity, written ∞.

As an exercise, you should prove:

Let $\{t_n\}$, $\{s_n\}$ be sequences of real numbers $\geqq 0$. Let

$$t = \limsup t_n \quad \text{and} \quad s = \limsup s_n.$$

Then

$$\limsup(t_n + s_n) \leqq t + s.$$

If $t \neq 0$, then

$$\limsup(t_n s_n) \leqq ts.$$

If $\lim_{n\to\infty} t_n$ exists, then $t = \lim_{n\to\infty} t_n$.

This last statement says that if the sequence has an ordinary limit, then that limit is the lim sup of the sequence.

The second statement is often applied in case one sequence has a lim sup, and the other sequence has a limit $\neq 0$. The hypothesis $t \neq 0$ is made only to allow the possibility that $s = \infty$, in which case ts is understood to be ∞. If $s \neq \infty$, and $t \neq \infty$, and $\lim t_n$ exists, then it is true unrestrictedly that

$$\limsup(t_n s_n) = ts.$$

Theorem 2.6. *Let $\sum a_n z^n$ be a power series, and let r be its radius of convergence. Then*

$$\frac{1}{r} = \limsup |a_n|^{1/n}.$$

If $r = 0$, this relation is to be interpreted as meaning that the sequence $\{|a_n|^{1/n}\}$ is not bounded. If $r = \infty$, it is to be interpreted as meaning that $\limsup |a_n|^{1/n} = 0$.

Proof. Let $t = \limsup |a_n|^{1/n}$. Suppose first that $t \neq 0, \infty$. Given $\epsilon > 0$, there exist only a finite number of n such that $|a_n|^{1/n} \geqq t + \epsilon$. Thus for all but a finite number of n, we have

$$|a_n| \leqq (t + \epsilon)^n,$$

whence the series $\sum a_n z^n$ converges absolutely if $|z| < 1/(t + \epsilon)$, by comparison with the geometric series. Therefore the radius of convergence r satisfies $r \geqq 1/(t + \epsilon)$ for every $\epsilon > 0$, whence $r \geqq 1/t$.

Conversely, given ϵ there exist infinitely many n such that $|a_n|^{1/n} \geqq t - \epsilon$, and therefore

$$|a_n| \geqq (t - \epsilon)^n.$$

Hence the series $\sum a_n z^n$ does not converge if $|z| = 1/(t - \epsilon)$, because its n-th term does not even tend to 0. Therefore the radius of convergence r satisfies $r \leqq 1/(t - \epsilon)$ for every $\epsilon > 0$, whence $r \leqq 1/t$. This concludes the proof in case $t \neq 0, \infty$.

The case when $t = 0$ or ∞ will be left to the reader. The above arguments apply, even with some simplifications.

Corollary 2.7. *If* $\lim |a_n|^{1/n} = t$ *exists, then* $r = 1/t$.

Proof. If the limit exists, then t is the only point of accumulation of the sequence $|a_n|^{1/n}$, and the theorem states that $t = 1/r$.

Corollary 2.8. *Suppose that* $\sum a_n z^n$ *has a radius of convergence greater than* 0. *Then there exists a positive number C such that if* $A > 1/r$ *then*

$$|a_n| \leqq CA^n \quad \textit{for all } n.$$

Proof. Let $s = 1/A$ so $0 < s < r$ at the beginning of the proof of the theorem.

In the next examples, we shall use a weak form of **Stirling's formula**, namely

$$n! = n^n e^{-n} u_n \qquad \text{where} \quad \lim u_n^{1/n} = 1.$$

You can prove this estimate by comparing the integral

$$\int_1^n \log x \, dx = n \log n - n + 1$$

with the upper and lower Riemann sums on the interval $[1, n]$, using the partition consisting of the integers from 1 to n. This is a very simple exercise in calculus. Exponentiating the inequalities given by the Riemann sums yields the weak form of Stirling's formula.

Let $\{a_n\}$, $\{b_n\}$ be two sequences of positive numbers. We shall write

$$a_n \equiv b_n \qquad \text{for} \quad n \to \infty$$

if for each n there exists a positive real number u_n such that $\lim u_n^{1/n} = 1$, and $a_n = b_n u_n$. If $\lim a_n^{1/n}$ exists, and $a_n \equiv b_n$, then $\lim b_n^{1/n}$ exists and is equal to $\lim a_n^{1/n}$. We can use this result in the following examples.

Example. The radius of convergence of the series $\sum n!\, z^n$ is 0. Indeed, we have $n! \equiv n^n e^{-n}$ and $(n!)^{1/n}$ is unbounded as $n \to \infty$.

Example. The radius of convergence of $\sum (1/n!) z^n$ is infinity, because $1/n! \equiv e^n/n^n$ so $(1/n!)^{1/n} \to 0$ as $n \to \infty$.

Example. The radius of convergence of $\sum (n!/n^n) z^n$ is e, because $n!/n^n \equiv e^{-n}$, so $\lim (n!/n^n)^{1/n} = e^{-1}$.

Ratio Test. *Let $\{a_n\}$ be a sequence of positive numbers, and assume that $\lim a_{n+1}/a_n = A \geqq 0$. Then $\lim a_n^{1/n} = A$ also.*

Proof. Suppose first $A > 0$ for simplicity. Given $\epsilon > 0$, let n_0 be such that $A - \epsilon \leqq a_{n+1}/a_n \leqq A + \epsilon$ if $n \geqq n_0$. Without loss of generality, we can assume $\epsilon < A$ so $A - \epsilon > 0$. Write

$$a_n = a_1 \prod_{k=1}^{n_0 - 1} \frac{a_{k+1}}{a_k} \prod_{k=n_0}^{n} \frac{a_{k+1}}{a_k}.$$

Then by induction, there exist constants $C_1(\epsilon)$ and $C_2(\epsilon)$ such that

$$C_1(\epsilon)(A - \epsilon)^{n - n_0} \leqq a_n \leqq C_2(\epsilon)(A + \epsilon)^{n - n_0},$$

Put $C_1'(\epsilon) = C_1(\epsilon)(A - \epsilon)^{-n_0}$ and $C_2'(\epsilon) = C_2(\epsilon)(A + \epsilon)^{n_0}$. Then

$$C_1'(\epsilon)^{1/n}(A - \epsilon) \leqq a_n^{1/n} \leqq C_2'(\epsilon)^{1/n}(A + \epsilon).$$

There exists $N \geqq n_0$ such that for $n \geqq N$ we have

$$C_1'(\epsilon)^{1/n} = 1 + \delta_1(n) \quad \text{where } |\delta_1(n)| \leqq \epsilon/(A - \epsilon),$$

and similarly $C_2'(\epsilon)^{1/n} = 1 + \delta_2(n)$ with $|\delta_2(n)| \leqq \epsilon/(A + \epsilon)$. Then

$$A - \epsilon + \delta_1(n)(A - \epsilon) \leqq a_n^{1/n} \leqq A + \epsilon + \delta_2(n)(A + \epsilon).$$

This shows that $|a_n^{1/n} - A| \leqq 2\epsilon$, and concludes the proof of the ratio test when $A > 0$. If $A = 0$, one can simply replace the terms on the left of the inequalities by 0 throughout.

Example (The Binomial Series). Let α be any complex number $\neq 0$. Define the **binomial coefficients** as usual,

$$\binom{\alpha}{n} = \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{n!},$$

and the **binomial series**

$$(1+T)^\alpha = B_\alpha(T) = \sum_{n=0}^{\infty} \binom{\alpha}{n} T^n.$$

By convention,

$$\binom{\alpha}{0} = 1.$$

The radius of convergence of the binomial series is 1 if α is not equal to an integer $\geqq 0$.

Proof. Under the stated assumption, none of the coefficients a_n are 0, and we have

$$|a_{n+1}/a_n| = \left|\frac{\alpha - n}{n+1}\right|.$$

The limit is 1 as $n \to \infty$, so we can apply the ratio test.

Warning. Let r be the radius of convergence of the series $f(z)$. *Nothing* has been said about possible convergence if $|z| = r$. Many cases can occur concerning convergence or non-convergence on this circle. See Exercises 6 and 8 for example.

II, §2. EXERCISES

1. Let $|\alpha| < 1$. Express the sum of the geometric series $\sum_{n=1}^{\infty} \alpha^n$ in its usual simple form.

2. Let r be a real number, $0 \leqq r < 1$. Show that the series

$$\sum_{n=0}^{\infty} r^n e^{in\theta} \quad \text{and} \quad \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$$

converge (θ is real). Express these series in simple terms using the usual formula for a geometric series.

3. Show that the usual power series for $\log(1+z)$ or $\log(1-z)$ from elementary calculus converges absolutely for $|z| < 1$.

4. Determine the radius of convergence for the following power series.

(a) $\sum n^n z^n$	(b) $\sum z^n/n^n$
(c) $\sum 2^n z^n$	(d) $\sum (\log n)^2 z^n$
(e) $\sum 2^{-n} z^n$	(f) $\sum n^2 z^n$
(g) $\sum \frac{n!}{n^n} z^n$	(h) $\sum \frac{(n!)^3}{(3n)!} z^n$

5. Let $f(z) = \sum a_n z^n$ have radius of convergence $r > 0$. Show that the following series have the same radius of convergence:
(a) $\sum n a_n z^n$ (b) $\sum n^2 a_n z^n$
(c) $\sum n^d a_n z^n$ for any positive integer d (d) $\sum_{n \geqq 1} n a_n z^{n-1}$

6. Give an example of a power series whose radius of convergence is 1, and such that the corresponding function is continuous on the closed unit disc. [*Hint*: Try $\sum z^n/n^2$.]

7. Let a, b be two complex numbers, and assume that b is not equal to any integer $\leqq 0$. Show that the radius of convergence of the series

$$\sum \frac{a(a+1)\cdots(a+n)}{b(b+1)\cdots(b+n)} z^n$$

is at least 1. Show that this radius can be ∞ in some cases.

8. Let $\{a_n\}$ be a decreasing sequence of positive numbers approaching 0. Prove that the power series $\sum a_n z^n$ is uniformly convergent on the domain of z such that

$$|z| \leqq 1 \quad \text{and} \quad |z - 1| \geqq \delta,$$

where $\delta > 0$. [*Hint*: For this problem and the next, use summation by parts, see Appendix, §1.]

9. **(Abel's Theorem).** Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $\geqq 1$. Assume that the series $\sum_{n=0}^{\infty} a_n$ converges. Let $0 \leqq x < 1$. Prove that

$$\lim_{x \to 1} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n.$$

Remark. This result amounts to proving an interchange of limits. If

$$s_n(x) = \sum_{k=1}^{n} a_k x^k,$$

then one wants to prove that

$$\lim_{n \to \infty} \lim_{x \to 1} s_n(x) = \lim_{x \to 1} \lim_{n \to \infty} s_n(x).$$

Cf. Theorem 3.5 of Chapter VII in my *Undergraduate Analysis*, Springer-Verlag, 1983.

10. Let $\sum a_n z^n$ and $\sum b_n z^n$ be two power series, with radius of convergence r and s, respectively. What can you say about the radius of convergence of the series:
(a) $\sum (a_n + b_n) z^n$ (b) $\sum a_n b_n z^n$?

11. Let α, β be complex numbers with $|\alpha| < |\beta|$. Let

$$f(z) = \sum (3\alpha^n - 5\beta^n) z^n.$$

Determine the radius of convergence of $f(z)$.

12. Let $\{a_n\}$ be the sequence of real numbers defined by the conditions:

$$a_0 = 1,\ a_1 = 2, \quad \text{and} \quad a_n = a_{n-1} + a_{n-2} \quad \text{for} \quad n \geqq 2.$$

Determine the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n z^n.$$

[*Hint*: What is the general solution of a difference equation? Cf. Exercise 6 of §1.]

13. More generally, let u_1, u_2 be complex numbers such that the polynomial

$$P(T) = T^2 - u_1 T - u_2 = (T - \alpha_1)(T - \alpha_2)$$

has two distinct roots with $|\alpha_1| < |\alpha_2|$. Let a_0, a_1 be given, and let

$$a_n = u_1 a_{n-1} + u_2 a_{n-2} \quad \text{for} \quad n \geqq 2.$$

What is the radius of convergence of the series $\sum a_n T^n$?

II, §3. RELATIONS BETWEEN FORMAL AND CONVERGENT SERIES

Sums and Products

Let $f = f(T)$ and $g = g(T)$ be formal power series. We may form their formal product and sum, $f + g$ and fg. If f converges absolutely for some complex number z, then we have the value $f(z)$, and similarly for $g(z)$.

Theorem 3.1. *If f, g are power series which converge absolutely on the disc $D(0, r)$, then $f + g$ and fg also converge absolutely on this disc. If α is a complex number, αf converges absolutely on this disc, and we have*

$$(f + g)(z) = f(z) + g(z), \qquad (fg)(z) = f(z)g(z),$$
$$(\alpha f)(z) = \alpha \cdot f(z)$$

for all z in the disc.

Proof. We give the proof for the product, which is the hardest. Let

$$f = \sum a_n T^n \quad \text{and} \quad g = b_n T^n,$$

so that

$$fg = \sum c_n T^n, \qquad \text{where} \quad c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Let $0 < s < r$. We know that there exists a positive number C such that for all n,

$$|a_n| \leqq C/s^n \qquad \text{and} \qquad |b_n| \leqq C/s^n.$$

Then

$$|c_n| \leqq \sum_{k=0}^{n} |a_k b_{n-k}| \leqq (n+1)\frac{C}{s^k}\frac{C}{s^{n-k}} = \frac{(n+1)C^2}{s^n}.$$

Therefore

$$|c_n|^{1/n} \leqq \frac{(n+1)^{1/n} C^{2/n}}{s}.$$

But $\lim_{n\to\infty} (n+1)^{1/n} C^{1/n} = 1$. Hence

$$\limsup |c_n|^{1/n} \leqq 1/s.$$

This is true for every $s < r$. It follows that $\limsup |c_n|^{1/n} \leqq 1/r$, thereby proving that the formal product converges absolutely on the same disc. We have also shown that the series of positive terms

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} |a_k||b_{n-k}||z|^n$$

converges.

Let

$$f_N(T) = a_0 + a_1 T + \cdots + a_N T^N,$$

and similarly, let $g_N(T)$ be the polynomial consisting of the terms of order $\leqq N$ in the power series for g. Then

$$f(z) = \lim_N f_N(z) \qquad \text{and} \qquad g(z) = \lim_N g_N(z).$$

Furthermore,

$$|(fg)_N(z) - f_N(z)g_N(z)| \leqq \sum_{n=N+1}^{\infty} \sum_{k=0}^{n} |a_k||b_{n-k}||z|^n.$$

In view of the convergence proved above, for N sufficiently large the right-hand side is arbitrarily small, and hence

$$f(z)g(z) = \lim_N f_N(z)g_N(z) = (fg)(z),$$

thereby proving the theorem for the product.

The previous theorem shows that a formal power series determines a function on the disc of absolute convergence. We can raise the question: If two formal power series f, g give rise to the same function on some neighborhood of 0, are they equal as formal power series? Subtracting g from f, this amounts to asking: If a power series determines the zero function on some disc centered at the origin, is it the zero series, i.e. are all its coefficients equal to 0? The answer is yes. In fact, more is true.

Theorem 3.2.

(a) *Let $f(T) = \sum a_n T^n$ be a non-constant power series, having a non-zero radius of convergence. If $f(0) = 0$, then there exists $s > 0$ such that $f(z) \neq 0$ for all z with $|z| \leqq s$, and $z \neq 0$.*

(b) *Let $f(T) = \sum a_n T^n$ and $g(T) = \sum b_n T^n$ be two convergent power series. Suppose that $f(x) = g(x)$ for all points x in an infinite set having 0 as a point of accumulation. Then $f(T) = g(T)$, that is $a_n = b_n$ for all n.*

Proof. We can write

$$\begin{aligned} f(z) &= a_m z^m + \text{higher terms}, \qquad \text{and } a_m \neq 0 \\ &= a_m z^m(1 + b_1 z + b_2 z^2 + \cdots) \\ &= a_m z^m(1 + h(z)), \end{aligned}$$

where $h(z) = b_1 z + b_2 z^2 + \cdots$ is a power series having a non-zero radius of convergence, and zero constant term. For all sufficiently small $|z|$, the value $|h(z)|$ is small, and hence

$$1 + h(z) \neq 0.$$

If $z \neq 0$, then $a_m z^m \neq 0$. This proves the first part of the theorem.

For part (b), let $h(t) = f(T) - g(T) = \sum (a_n - b_n) T^n$. We have $h(x) = 0$ for an infinite set of points x having 0 as point of accumulation. By part (a), this implies that $h(T)$ is the zero power series, so $a_n = b_n$ for all n, thus proving the theorem.

Example. There exists at most one convergent power series $f(T) = \sum a_n T^n$ such that for some interval $[-\epsilon, \epsilon]$ we have $f(x) = e^x$ for all x in $[-\epsilon, \epsilon]$. This proves the uniqueness of any power series extension of the exponential function to all complex numbers. Similarly, one has the uniqueness of the power series extending the sine and cosine functions.

Furthermore, let $\exp(z) = \sum z^n/n!$. Then

$$\exp(iz) = \sum (iz)^n/n!.$$

Summing over even n and odd n, we find that

$$\exp(iz) = C(z) + iS(z),$$

where $C(z)$ and $S(z)$ are the power series for the cosine and sine of z respectively. Hence $e^{i\theta}$ for real θ coincides with $\exp(i\theta)$ as given by the power series expansion.

Quite generally, if $g(T)$, $h(T)$ are power series with 0 constant term, then

$$\exp\big(g(T) + h(T)\big) = \big(\exp g(T)\big)\big(\exp h(T)\big).$$

Proof. On one hand, by definition,

$$\exp\big(g(T) + h(T)\big) = \sum_{n=0}^{\infty} \frac{\big(g(T) + h(T)\big)^n}{n!}$$

and on the other hand,

$$\big(\exp g(T)\big)\big(\exp h(T)\big) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{g(T)^k h(T)^{n-k}}{k!(n-k)!}$$

$$= \sum_{n=0}^{\infty} \frac{\big(g(T) + h(T)\big)^n}{n!} \quad \text{qed.}$$

In particular, for complex numbers z, w we have

$$\exp(z + w) = (\exp z)(\exp w),$$

because we can apply the above identity to $g(T) = zT$ and $h(T) = wT$, and then substitute $T = 1$. Thus we see that the exponential function e^z defined in Chapter I has the same values as the function defined by the usual power series $\exp(z)$. From now on, we make no distinction between e^z and $\exp(z)$.

Theorem 3.2 also allows us to conclude that any polynomial relation between the elementary functions which have a convergent Taylor expansion at the origin also holds for the extension of these functions as complex power series.

Example. We can now conclude that $\sin^2 z + \cos^2 z = 1$, where $\sin z = S(z)$, $\cos z = C(z)$ are defined by the usual power series. Indeed, the power series $S(z)^2 + C(z)^2$ has infinite radius of convergence, and has value 1 for all real z. Theorem 3.2 implies that there is at most one series having this property, and that is the series 1, as desired. It would be disagreeable to show directly that the formal power series for the sine and cosine satisfy this relation. It is easier to do it through elementary calculus as above.

Example. Let m be a positive integer. We have seen in §2 that the binomial series

$$B(z) = \sum \binom{\alpha}{n} z^n$$

with $\alpha = 1/m$ has a radius of convergence $\geqq 1$, and thus converges absolutely for $|z| < 1$. By elementary calculus, we have

$$B(x)^m = 1 + x$$

when x is real, and $|x| < 1$ (or even when $|x|$ is sufficiently small). Therefore $B(T)^m$ is the unique formal power series such that

$$B(x)^m = 1 + x$$

for all sufficiently small real x, and therefore we conclude that

$$B(T)^m = 1 + T.$$

In this manner, we see that we can take m-th roots

$$(1 + z)^{1/m}$$

by the binomial series when $|z| < 1$.

Quotients

In our discussion of formal power series, besides the polynomial relations, we dealt with quotients and also composition of series. We still have to relate these to the convergent case. It will be convenient to introduce a simple notation to estimate power series.

Let $f(T) = \sum a_n T^n$ be a power series. Let

$$\varphi(T) = \sum c_n T^n$$

be a power series with **real coefficients** $c_n \geqq 0$. We say that f is **dominated by** φ, and write

$$f \prec \varphi \quad \text{or} \quad f(T) \prec \varphi(T),$$

if $|a_n| \leqq c_n$ for all n. It is clear that if φ, ψ are power series with real coefficients $\geqq 0$ and if

$$f \prec \varphi, \qquad g \prec \psi,$$

then

$$f + g \prec \varphi + \psi \quad \text{and} \quad fg \prec \varphi\psi.$$

Theorem 3.3. *Suppose that f has a non-zero radius of convergence, and non-zero constant term. Let g be the formal power series which is inverse to f, that is, $fg = 1$. Then g also has a non-zero radius of convergence.*

Proof. Multiplying f by some constant, we may assume without loss of generality that the constant term is 1, so we write

$$f = 1 + a_1 T + a_2 T^2 + \cdots = 1 - h(T),$$

where $h(T)$ has constant term equal to 0. By Corollary 2.8, we know that there exists a number $A > 0$ such that

$$|a_n| \leqq A^n, \qquad n \geqq 1.$$

(We can take $C = 1$ by picking A large enough.) Then

$$\frac{1}{f(T)} = \frac{1}{1 - h(T)} = 1 + h(T) + h(T)^2 + \cdots.$$

But

$$h(T) \prec \sum_{n=1}^{\infty} A^n T^n = \frac{AT}{1 - AT}.$$

Therefore $1/f(T) = g(T)$ satisfies

$$g(T) \prec 1 + \frac{AT}{1 - AT} + \frac{(AT)^2}{(1 - AT)^2} + \cdots = \frac{1}{1 - \dfrac{AT}{1 - AT}}.$$

But

$$\frac{1}{1 - \dfrac{AT}{1 - AT}} = (1 - AT)(1 + 2AT + (2AT)^2 + \cdots)$$

$$\prec (1 + AT)(1 + 2AT + (2AT)^2 + \cdots).$$

Therefore $g(T)$ is dominated by a product of power series having non-zero radius of convergence, whence $g(T)$ itself a non-zero radius of convergence, as was to be shown.

Composition of Series

Theorem 3.4. *Let*

$$f(z) = \sum_{n \geqq 0} a_n z^n \qquad \text{and} \qquad h(z) = \sum_{n \geqq 1} b_n z^n$$

be convergent power series, and assume that the constant term of h is 0. Assume that $f(z)$ is absolutely convergent for $|z| \leqq r$, with $r > 0$, and that $s > 0$ is a number such that

$$\sum |b_n| s^n \leqq r.$$

Let $g = f(h)$ be the formal power series obtained by composition,

$$g(T) = \sum_{n \geqq 0} a_n \left(\sum_{k=1}^{\infty} b_k T^k \right)^n.$$

Then g converges absolutely for $|z| \leqq s$, and for such z,

$$g(z) = f(h(z)).$$

Proof. Let $g(T) = \sum c_n T^n$. Then $g(T)$ is dominated by the series

$$g(T) \prec \sum_{n=0}^{\infty} |a_n| \left(\sum_{k=1}^{\infty} |b_k| T^k \right)^n$$

and by hypothesis, the series on the right converges absolutely for $|z| \leqq s$, so $g(z)$ converges absolutely for $|z| \leqq s$. Let

$$f_N(T) = a_0 + a_1 T + \cdots + a_{N-1} T^{N-1}$$

be the polynomial of degree $\leqq N - 1$ beginning the power series f. Then

$$f(h(T)) - f_N(h(T)) \prec \sum_{n=N}^{\infty} |a_n| \left(\sum_{k=1}^{\infty} |b_k| T^k \right)^n,$$

and $f(h(T)) = g(T)$ by definition. By the absolute convergence we conclude: Given ϵ, there exists N_0 such that if $N \geqq N_0$ and $|z| \leqq s$, then

$$|g(z) - f_N(h(z))| < \epsilon.$$

Since the polynomials f_N converge uniformly to the function f on the closed disc of radius r, we can pick N_0 sufficiently large so that for

$N \geqq N_0$ we have

$$|f_N(h(z)) - f(h(z))| < \epsilon.$$

This proves that

$$|g(z) - f(h(z))| < 2\epsilon,$$

for every ϵ, whence $g(z) - f(h(z)) = 0$, thereby proving the theorem.

Example. Let m be a positive integer, and let $h(z)$ be a convergent power series with zero constant term. Then we can form the m-th root

$$(1 + h(z))^{1/m}$$

by the binomial expansion, and this m-th root is a convergent power series whose m-th power is $1 + h(z)$.

Example. Define

$$f(w) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{w^n}{n}.$$

Readers should immediately recognize that the series on the right is the usual series of calculus for $\log(1 + w)$ when $w = x$ and x is real. This series converges absolutely for $|w| < 1$. We can therefore define $\log z$ for $|z - 1| < 1$ by

$$\log z = f(z - 1).$$

We leave it as Exercise 1 to verify that $\exp \log z = z$.

II, §3. EXERCISES

1. (a) Use the above definition of $\log z$ for $|z - 1| < 1$ to prove that $\exp \log z = z$. [*Hint*: What are the values on the left when $z = x$ is real?]

   (b) Let $z_0 \neq 0$. Let α be any complex number such that $\exp(\alpha) = z_0$. For $|z - z_0| < |z_0|$ define

   $$\log z = f\left(\frac{z}{z_0} - 1\right) + \alpha = f\left(\frac{z - z_0}{z_0}\right) + \alpha.$$

   Prove that $\exp \log z = z$ for $|z - z_0| < |z_0|$.

   Warning. The above definitions in parts (a) and (b) may differ by a constant. Since you should have proved that $\exp \log z = z$ in both cases, and since $\exp(w_1) = \exp(w_2)$ if and only if there exists an integer k such that $w_1 = w_2 + 2\pi ik$, it follows that if we denote the two logs by $\log_1$ and $\log_2$, respectively, then $\log_1(z) = \log_2(z) + 2\pi ik$.

2. (a) Let $\exp(T) = \sum_{n=0}^{\infty} T^n/n!$ and $\log(1+T) = \sum_{k=1}^{\infty} (-1)^{k-1} T^k/k$. Show that

$$\exp \log(1+T) = 1 + T \quad \text{and} \quad \log \exp(T) = T.$$

(b) Let $h_1(T)$ and $h_2(T)$ be formal power series with 0 constant terms. Prove that $\log\big((1+h_1(T))(1+h_2(T))\big) = \log\big(1+h_1(T)\big) + \log\big(1+h_2(T)\big)$.

(c) For complex numbers α, β show that $\log(1+T)^\alpha = \alpha \log(1+T)$ and

$$(1+T)^\alpha (1+T)^\beta = (1+T)^{\alpha+\beta}.$$

3. Prove that for all complex z we have

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

4. Show that the only complex numbers z such that $\sin z = 0$ are $z = k\pi$, where k is an integer. State and prove a similar statement for $\cos z$.

5. Find the power series expansion of $f(z) = 1/(z+1)(z+2)$, and find the radius of convergence.

6. The **Legendre polynomials** can be defined as the coefficients $P_n(\alpha)$ of the series expansion of

$$f(z) = \frac{1}{(1 - 2\alpha z + z^2)^{1/2}}$$
$$= 1 + P_1(\alpha)z + P_2(\alpha)z^2 + \cdots + P_n(\alpha)z^n + \cdots.$$

Calculate the first four Legendre polynomials.

II, §4. ANALYTIC FUNCTIONS

So far we have looked at power series expansions at the origin. Let f be a function defined in some neighborhood of a point z_0. We say that f is **analytic** at z_0 if there exists a power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

and some $r > 0$ such that the series converges absolutely for $|z - z_0| < r$, and such that for such z, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Suppose f is a function on an open set U. We say that f is **analytic on** U if f is analytic at every point of U.

In the light of the uniqueness theorem for power series, Theorem 3.2, we see that the above power series expressing f in some neighborhood of z_0 is uniquely determined. We have

$$f(z_0) = 0 \qquad \text{if and only if} \qquad a_0 = 0.$$

A point z_0 such that $f(z_0) = 0$ is called a **zero** of f. Instead of saying that f is analytic at z_0, we also say that f **has a power series expansion at** z_0 (meaning that the values of $f(z)$ for z near z_0 are given by an absolutely convergent power series as above).

If S is an arbitrary set, not necessarily open, it is useful to make the convention that a function is **analytic** on S if it is the restriction of an analytic function on an open set containing S. This is useful, for instance, when S is a closed disc.

The theorem concerning sums, products, quotients and composites of convergent power series now immediately imply:

If f, g are analytic on U, so are $f + g$, fg. Also f/g is analytic on the open subset of $z \in U$ such that $g(z) \neq 0$.

If $g\colon U \to V$ is analytic and $f\colon V \to \mathbf{C}$ is analytic, then $f \circ g$ is analytic.

For this last assertion, we note that if $z_0 \in U$ and $g(z_0) = w_0$, so

$$g(z) = w_0 + \sum_{n \geqq 1} b_n(z - z_0)^n \quad \text{and} \quad f(w) = \sum_{n \geqq 0} a_n(w - w_0)^n$$

for w near w_0, then $g(z) - w_0$ is represented by a power series $h(z - z_0)$ without constant term, so that Theorem 3.4 applies: We can "substitute"

$$f(g(z)) = \sum a_n(g(z) - w_0)^n$$

to get the power series representation for $f(g(z))$ in a neighborhood of z_0.

The next theorem, although easy to prove, requires being stated. It gives us in practice a way of finding a power series expansion for a function at a point.

Theorem 4.1. *Let $f(z) = \sum a_n z^n$ be a power series whose radius of convergence is r. Then f is analytic on the open disc $D(0, r)$.*

Proof. We have to show that f has a power series expansion at an arbitrary point z_0 of the disc, so $|z_0| < r$. Let $s > 0$ be such that

$|z_0| + s < r$. We shall see that f can be represented by a convergent power series at z_0, converging absolutely on a disc of radius s centered at z_0.

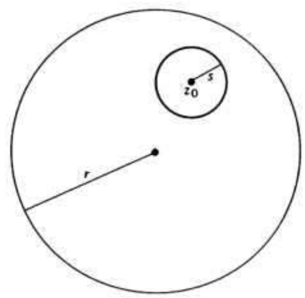

Figure 2

We write

$$z = z_0 + (z - z_0)$$

so that

$$z^n = (z_0 + (z - z_0))^n.$$

Then

$$f(z) = \sum_{n=0}^{\infty} a_n \left(\sum_{k=0}^{n} \binom{n}{k} z_0^{n-k}(z - z_0)^k \right).$$

If $|z - z_0| < s$ then $|z_0| + |z - z_0| < r$, and hence the series

$$\sum_{n=0}^{\infty} |a_n|(|z_0| + |z - z_0|)^n = \sum_{n=0}^{\infty} |a_n| \left[\sum_{k=0}^{n} \binom{n}{k} |z_0|^{n-k} |z - z_0|^k \right]$$

converges. Then we can interchange the order of summation, to get

$$f(z) = \sum_{k=0}^{\infty} \left[\sum_{n=k}^{\infty} a_n \binom{n}{k} z_0^{n-k} \right] (z - z_0)^k,$$

which converges absolutely also, as was to be shown.

Example. Let us find the terms of order $\leqq 3$ in the power series expansion of the function

$$f(z) = z^2/(z + 2)$$

at the point $z_0 = 1$. We write

$$z = 1 + (z-1), \qquad z + 2 = 3 + (z-1).$$

Let $\equiv$ denote congruence $\bmod (z-1)^4$ (so disregard terms of order > 3). Then

$$z^2 = 1 + 2(z-1) + (z-1)^2$$

$$z + 2 = 3\left(1 + \frac{1}{3}(z-1)\right)$$

$$\frac{1}{z+2} = \frac{1}{3}\frac{1}{1 + \frac{1}{3}(z-1)}$$

$$= \frac{1}{3}\left(1 - \frac{1}{3}(z-1) + \frac{1}{3^2}(z-1)^2 - \frac{1}{3^3}(z-1)^3 + \cdots\right).$$

Hence

$$\frac{z^2}{z+2} \equiv (1 + 2(z-1) + (z-1)^2)$$

$$\times \frac{1}{3}\left(1 - \frac{1}{3}(z-1) + \frac{1}{3^2}(z-1)^2 - \frac{1}{3^3}(z-1)^3\right)$$

$$= \frac{1}{3}\left[1 + \frac{5}{3}(z-1) + \left(\frac{1}{3} + \frac{1}{3^2}\right)(z-1)^2 + \left(-\frac{1}{3} + \frac{2}{3^2} - \frac{1}{3^3}\right)(z-1)^3\right].$$

These are the desired terms of the expansion.

Remark. Making a translation, the theorem shows that if f has a power series expansion on a disc $D(z_0, r)$, that is,

$$f(z) = \sum a_n(z - z_0)^n$$

for $|z - z_0| < r$, then f is analytic on this disc.

II, §4. EXERCISES

1. Find the terms of order $\leqq 3$ in the power series expansion of the function $f(z) = z^2/(z-2)$ at $z = 1$.

2. Find the terms of order $\leqq 3$ in the power series expansion of the function $f(z) = (z-2)/(z+3)(z+2)$ at $z = 1$.

II, §5. DIFFERENTIATION OF POWER SERIES

Let $D(0, r)$ be a disc of radius $r > 0$. A function f on the disc for which there exists a power series $\sum a_n z^n$ having a radius of convergence $\geqq r$ and such that

$$f(z) = \sum a_n z^n$$

for all z in the disc is said to **admit a power series expansion on this disc.** We shall now see that such a function is holomorphic, and that its derivative is given by the "obvious" power series.

Indeed, define the **formal derived series** to be

$$\sum n a_n z^{n-1} = a_1 + 2a_2 z + 3a_3 z^2 + \cdots.$$

Theorem 5.1. *If $f(z) = \sum a_n z^n$ has radius of convergence r, then:*

(i) *The series $\sum n a_n z^{n-1}$ has the same radius of convergence.*
(ii) *The function f is holomorphic on $D(0, r)$, and its derivative is equal to $\sum n a_n z^{n-1}$.*

Proof. By Theorem 2.6, we have

$$\limsup |a_n|^{1/n} = 1/r.$$

But

$$\limsup |n a_n|^{1/n} = \limsup n^{1/n} |a_n|^{1/n}.$$

Since $\lim n^{1/n} = 1$, the sequences

$$|n a_n|^{1/n} \quad \text{and} \quad |a_n|^{1/n}$$

have the same lim sup, and therefore the series $\sum a_n z^n$ and $\sum n a_n z^n$ have the same radius of convergence. Then

$$\sum n a_n z^{n-1} \quad \text{and} \quad \sum n a_n z^n$$

converge absolutely for the same values of z, so the first part of the theorem is proved.

As to the second, let $|z| < r$, and $\delta > 0$ be such that $|z| + \delta < r$. We consider complex numbers h such that

$$|h| < \delta.$$

We have

$$\begin{aligned} f(z + h) &= \sum a_n (z + h)^n \\ &= \sum a_n \left(z^n + n z^{n-1} h + h^2 P_n(z, h)\right), \end{aligned}$$

where $P_n(z, h)$ is a polynomial in z and h, with positive integer coefficients, in fact

$$P_n(z, h) = \sum_{k=2}^{n} \binom{n}{k} h^{k-2} z^{n-k}.$$

Note that we have the estimate:

$$|P_n(z, h)| \leqq \sum_{k=2}^{n} \binom{n}{k} \delta^{k-2} |z|^{n-k} = P_n(|z|, \delta).$$

Subtracting series, we find

$$f(z + h) - f(z) - \sum n a_n z^{n-1} h = h^2 \sum a_n P_n(z, h),$$

and since the series on the left is absolutely convergent, so is the series on the right. We divide by h to get

$$\frac{f(z + h) - f(z)}{h} - \sum n a_n z^{n-1} = h \sum a_n P_n(z, h).$$

For $|h| < \delta$, we have the estimate

$$\begin{aligned} |\sum a_n P_n(z, h)| &\leqq \sum |a_n| |P_n(z, h)| \\ &\leqq \sum |a_n| P_n(|z|, \delta). \end{aligned}$$

This last expression is fixed, independent of h. Hence

$$|h \sum a_n P_n(z, h)| \leqq |h| \sum |a_n| P_n(|z|, \delta).$$

As h approaches 0, the right-hand side approaches 0, and therefore

$$\lim_{h \to 0} |h \sum a_n P_n(z, h)| = 0.$$

This proves that f is differentiable, and that its derivative at z is given by the series $\sum n a_n z^{n-1}$, as was to be shown.

Remark. Conversely, we shall see after Cauchy's theorem that a function which is differentiable admits power series expansion—a very remarkable fact, characteristic of complex differentiability.

From the theorem, we see that the k-th derivative of f is given by the series

$$f^{(k)}(z) = k!\, a_k + h_k(z),$$

where h_k is a power series without constant term. Therefore we obtain the standard expression for the coefficients of the power series in terms of the derivatives, namely

$$\boxed{a_n = \frac{f^{(n)}(0)}{n!}.}$$

If we deal with the expansion at a point z_0, namely

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots,$$

then we find

$$\boxed{a_n = \frac{f^{(n)}(z_0)}{n!}.}$$

It is utterly trivial that the formally integrated series

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1}$$

has radius of convergence at least r, because its coefficients are smaller in absolute value than the coefficients of f. Since the derivative of this integrated series is exactly the series for f, it follows from Theorem 6.1 that the integrated series has the same radius of convergence as f.

Let f be a function on an open set U. If g is a holomorphic function on U such that $g' = f$, then g is called a **primitive** for f. We see that a function which has a power series expansion on a disc always has a primitive on that disc. In other words, an analytic function has a local primitive at every point.

Example. The function $1/z$ is analytic on the open set U consisting of the plane from which the origin has been deleted. Indeed, for $z_0 \neq 0$, we have the power series expansion

$$\begin{aligned}\frac{1}{z} = \frac{1}{z_0 + z - z_0} &= \frac{1}{z_0} \frac{1}{(1 + (z - z_0)/z_0)} \\ &= \frac{1}{z_0}\left(1 - \frac{1}{z_0}(z - z_0) + \cdots\right)\end{aligned}$$

converging on some disc $|z - z_0| < r$. Hence $1/z$ has a primitive on such a disc, and this primitive may be called $\log z$.

II, §5. EXERCISES

In Exercises 1 through 5, also determine the radius of convergence of the given series.

1. Let
$$f(z) = \sum \frac{z^{2n}}{(2n)!}$$
Prove that $f''(z) = f(z)$.

2. Let
$$f(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(n!)^2}.$$
Prove that
$$z^2 f''(z) + zf'(z) = 4z^2 f(z).$$

3. Let
$$f(z) = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots.$$
Show that $f'(z) = 1/(z^2 + 1)$.

4. Let
$$J(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{z}{2}\right)^{2n}.$$
Prove that
$$z^2 J''(z) + zJ'(z) + z^2 J(z) = 0.$$

5. For any positive integer k, let
$$J_k(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+k)!} \left(\frac{z}{2}\right)^{2n+k}.$$
Prove that
$$z^2 J_k''(z) + zJ_k'(z) + (z^2 - k^2)J_k(z) = 0.$$

6. (a) For $|z - 1| < 1$, show that the derivative of the function
$$\log z = \log\big(1 + (z - 1)\big) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(z-1)^n}{n}$$
is $1/z$.

(b) Let $z_0 \neq 0$. For $|z - z_0| < 1$, define $f(z) = \sum (-1)^{n-1}\big((z - z_0)/z_0\big)^n/n$. Show that $f'(z) = 1/z$.

II, §6. THE INVERSE AND OPEN MAPPING THEOREMS

Let f be an analytic function on an open set U, and let $f(U) = V$. We shall say that f is an **analytic isomorphism** if V is open and there exists an analytic function

$$g: V \to U$$

such that $f \circ g = \mathrm{id}_V$ and $g \circ f = \mathrm{id}_U$, in other words, f and g are inverse functions to each other.

We say that f is a **local analytic isomorphism** or is **locally invertible** at a point z_0 if there exists an open set U containing z_0 such that f is an analytic isomorphism on U.

Remark. The word "inverse" is used in two senses: the sense of §1, when we consider the reciprocal $1/f$ of a function f, and in the current sense, which may be called the **composition inverse**, i.e. an inverse for the composite of mappings. The context makes clear which is meant. In this section, we mean the composition inverse.

Theorem 6.1.

(a) *Let $f(T) = a_1 T +$ higher terms be a formal power series with $a_1 \neq 0$. Then there exists a unique power series $g(T)$ such that $f(g(T)) = T$. This power series also satisfies $g(f(T)) = T$.*

(b) *If f is a convergent power series, so is g.*

(c) *Let f be an analytic function on an open set U containing z_0. Suppose that $f'(z_0) \neq 0$. Then f is a local analytic isomorphism at z_0.*

Proof. We first deal with the formal power series problem (a), and we find first a formal inverse for $f(T)$. For convenience of notation below we write $f(T)$ in the form

$$f(T) = a_1 T - \sum_{n=2}^{\infty} a_n T^n.$$

We seek a power series

$$g(T) = \sum_{n=1}^{\infty} b_n T^n$$

such that

$$f(g(T)) = T.$$

The solution to this problem is given by solving the equations in terms of the coefficients of the power series

$$a_1 g(T) - a_2 g(T)^2 - \cdots = T.$$

These equations are of the form

$$a_1 b_n - P_n(a_2, \ldots, a_n, b_1, \ldots, b_{n-1}) = 0, \quad \text{and} \quad a_1 b_1 = 1 \quad \text{for } n = 1,$$

where P_n is a polynomial with positive integer coefficients (generalized binomial coefficients). In fact, one sees at once that

$$P_n(a_2, \ldots, a_n, b_1, \ldots, b_{n-1})$$
$$= a_2 P_{2,n}(b_1, \ldots, b_{n-1}) + \cdots + a_n P_{n,n}(b_1, \ldots, b_{n-1}),$$

where again $P_{k,n}$ is a polynomial with positive coefficients. In this manner we can solve recursively for the coefficients

$$b_1, b_2, \ldots$$

since b_n appears linearly with coefficient $a_1 \neq 0$ in these equations, and the other terms do not contain b_n. This shows that a formal inverse exists and is uniquely determined.

Next we prove that $g(f(T)) = T$. By what we have proved already, there exists a power series $h(T) = c_1 T +$ higher terms with $c_1 \neq 0$ such that $g(h(T)) = T$. Then using $f(g(T)) = T$ and $g(h(T)) = T$, we obtain:

$$g(f(T)) = g(f(g(h(T)))) = g(h(T)) = T,$$

which proves the desired formal relation.

Assume next that f is convergent.

We must now show that $g(z)$ is absolutely convergent on some disc. To simplify the number of symbols used, we assume that $a_1 = 1$. This loses no generality, because if we find a convergent inverse power series for $a_1^{-1} f(z)$, we immediately get the convergent inverse power series for $f(z)$ itself.

Let

$$f^*(T) = T - \sum_{n \geq 2} a_n^* T^n$$

be a power series with a_n^* real $\geqq 0$ such that $|a_n| \leqq a_n^*$ for all n. Let $\varphi(T)$ be the formal inverse of $f^*(T)$, so

$$\varphi(T) = \sum_{n \geqq 1} c_n T^n, \qquad c_1 = 1.$$

Then we have

$$c_n - P_n(a_2^*, \ldots, a_n^*, c_1, \ldots, c_{n-1}) = 0$$

with those same polynomials P_n as before. By induction, it therefore follows that c_n is real $\geqq 0$, and also that

$$|b_n| \leqq c_n,$$

since $b_n = P_n(a_2, \ldots, b_{n-1})$. It suffices therefore to pick the series f^* so that it has an easily computed formal inverse φ which is easily verified to have a positive radius of convergence.

It is now a simple matter to carry out this idea, and we pick for f^* a geometric series. There exists $A > 0$ such that for all n we have

$$|a_n| \leqq A^n.$$

(We can omit a constant C in front of A^n by picking A sufficiently large.) Then

$$f^*(T) = T - \sum_{n \geqq 2} A^n T^n = T - \frac{A^2T^2}{1 - AT}.$$

The power series $\varphi(T)$ is such that $f^*(\varphi(T)) = T$, namely

$$\varphi(T) - \frac{A^2\varphi(T)^2}{1 - A\varphi(T)} = T,$$

which is equivalent with the quadratic equation

$$(A^2 + A)\varphi(T)^2 - (1 + AT)\varphi(T) + T = 0.$$

This equation has the solution

$$\varphi(T) = \frac{1 + AT - \sqrt{(1 + AT)^2 - 4T(A^2 + A)}}{2(A^2 + A)}.$$

The expression under the radical sign is of the form

$$(1 + AT)^2\left(1 - \frac{4T(A^2 + A)}{(1 + AT)^2}\right)$$

and its square root is given by

$$(1 + AT)\left(1 - \frac{4T(A^2 + A)}{(1 + AT)^2}\right)^{1/2}.$$

We use the binomial expansion to find the square root of a series of the form $1 + h(T)$ when $h(T)$ has zero constant term. It is now clear that $\varphi(T)$ is obtained by composition of convergent power series, and hence has a non-zero radius of convergence. This proves that the power series $g(T)$ also converges.

Finally, for (c), suppose first that $z_0 = 0$ and $f(z_0) = 0$, so f is analytic on an open set containing 0. This means that f has a convergent power series expansion at 0, so we view f as being defined on its open disc of convergence

$$f: D \to \mathbf{C}.$$

Let V_0 be an open disc centered at 0 such that V_0 is contained in the disc of convergence of g, and such that $g(V_0) \subset D$. Such a neighborhood of 0 exists simply because g is continuous. Let $U_0 = f^{-1}(V_0)$ be the set of all $z \in D$ such that $f(z) \subset V_0$. Let

$$f_0: U_0 \to V_0$$

be the restriction of f to U_0. We claim that f_0 is an analytic isomorphism. Note that $g(V_0) \subset U_0$ because for $w \in V_0$ we have $f(g(w)) = w$ by Theorem 3.4, so we consider the restriction g_0 of g to V_0 as mapping

$$g_0: V_0 \to U_0.$$

Again by Theorem 3.4, for $z \in U_0$ we have $g_0(f_0(z)) = z$, which proves that f_0 and g_0 are inverse to each other, and concludes the proof of Theorem 6.1(c) in case $z_0 = 0$ and $f(z_0) = 0$.

The general case is reduced to the above case by translation, as one says. Indeed, for an arbitrary f, with $f(z) = \sum a_n(z - z_0)^n$, change variables and let

$$w = z - z_0, \qquad F(w) = f(z) - f(z_0) = \sum_{n=1}^{\infty} a_n w^n.$$

Then we may apply the previous special case to F and find a local inverse G for F. Let $w_0 = f(z_0)$, and let

$$g(w) = G(w - w_0) + z_0.$$

Then g is a local inverse for f, thus finishing the proof of Theorem 6.1.

There are (at least) four ways of proving the inverse function theorem.

1. The way we have just gone through, by estimating the formal inverse to show that it converges.

2. Reproducing the real variable proof for real functions of class C^1. By the contraction principle, (shrinking lemma), one first shows that the map is locally surjective, and one constructs a local inverse, which is shown to be differentiable, and whose derivative satisfies, for $w = f(z)$, the relation

$$g'(w) = 1/f'(z).$$

The reader should be able to copy the proof from any standard book on analysis, (certainly from my *Undergraduate Analysis* [La 83]).

3. Assuming the theorem for C^∞ real functions. One can show (and we shall do so later when we discuss the real aspects of an analytic function) that an analytic function is C^∞, as a function of (x, y), writing

$$z = x + iy.$$

The hypothesis $f'(z_0) \neq 0$ (namely $a_1 \neq 0$) is then seen to amount to the property that the Jacobian of the real function of two variables has non-zero determinant, whence f has a C^∞ inverse locally by the real theorem. It is then an easy matter to show by the chain rule that this inverse satisfies the Cauchy–Riemann equations, and is therefore holomorphic, whence analytic by the theory which follows Cauchy's theorem.

4. Giving an argument based on more complex function theory, and carried out in Theorem 1.7 of Chapter VI.

All four methods are important, and are used in various contexts in analysis, both of functions of one variable, and functions of several variables.

Let U be an open set and let f be a function on U. We say that f is an **open mapping** if for every open subset U' of U the image $f(U')$ is open.

Theorem 6.1 shows that the particular type of function considered there, i.e. with non-zero first coefficient in the power series expansion, is locally open. We shall now consider arbitrary analytic functions, first at the origin.

Let

$$f(z) = \sum a_n z^n$$

be a convergent non-constant power series, and let $m = \operatorname{ord} f$, so that

$$\begin{aligned} f(z) &= a_m z^m + \text{higher terms}, \qquad a_m \neq 0. \\ &= a_m z^m(1 + h(z)), \end{aligned}$$

where $h(z)$ is convergent, and has zero constant term. Let a be a complex number such that $a^m = a_m$. Then we can write $f(z)$ in the form

$$f(z) = \big(az(1 + h_1(z))\big)^m,$$

where $h_1(z)$ is a convergent power series with zero constant term, obtained from the binomial expansion

$$(1 + h(z))^{1/m} = 1 + h_1(z),$$

and

$$f_1(z) = az\big(1 + h_1(z)\big) = az + azh_1(z)$$

is a power series whose coefficient of z is $a \neq 0$. Theorem 6.1 therefore applies to $f_1(z)$, which is therefore locally open at the origin. We have

$$f(z) = f_1(z)^m.$$

Let U be an open disc centered at the origin on which f_1 converges. Then $f_1(U)$ contains an open disc V. The image of V under the map

$$w \mapsto w^m$$

is a disc. Hence $f(U)$ contains an open disc centered at the origin.

Theorem 6.2. *Let f be analytic on an open set U, and assume that for each point of U, f is not constant on a given neighborhood of that point. Then f is an open mapping.*

Proof. We apply the preceding discussion to the power series expansion of f at a point of U, so the proof is obvious in the light of what we have already done.

The construction in fact yielded the following statement which it is worthwhile extracting as a theorem.

Theorem 6.3. *Let f be analytic at a point z_0,*

$$f(z) = a_0 + \sum_{n=m}^{\infty} a_n(z - z_0)^n,$$

with $m \geqq 1$ and $a_m \neq 0$. Then there exists a local analytic isomorphism φ at 0 such that

$$f(z) = a_0 + \varphi(z - z_0)^m.$$

We interpret Theorem 6.3 as follows. Let

$$\psi: U \to V$$

be an analytic isomorphism. We write $w = \psi(z)$. We may view ψ as a change of coordinates, from the coordinate z to the coordinate w. In Theorem 6.3 we may therefore write

$$f(z) = a_0 + w^m,$$

where $w = \varphi(z - z_0)$. The expansion for f in terms of the coordinate w is therefore much simpler than in terms of the coordinate z.

We also get a criterion for a function to have an analytic inverse on a whole open set.

Theorem 6.4. *Let f be analytic on an open set U, and assume that f is injective. Let $V = f(U)$ be its image. Then $f: U \to V$ is an analytic isomorphism, and $f'(z) \neq 0$ for all $z \in U$.*

Proof. The function f between U and V is bijective, so we can define an inverse mapping $g: V \to U$. Let z_0 be a point of U, and let the power series expansion of f at z_0 be as in Theorem 6.3. If $m > 1$ then we see that f cannot be injective, because the m-th power function in a neighborhood of the origin is not injective (it wraps the disc m times around). Hence $m = 1$, and Theorem 6.1 now shows that the inverse function g is analytic at $f(z_0)$. This proves the theorem.

Example 1. Let $f(z) = 3 - 5z +$ higher terms. Then $f(0) = 3$, and

$$f'(0) = a_1 = -5 \neq 0.$$

Hence f is a local analytic isomorphism, or locally invertible, at 0.

Example 2. Let $f(z) = 2 - 2z + z^2$. We want to determine whether f is locally invertible at $z = 1$. We write the power series expansion of f at 1, namely

$$f(z) = 1 + (z - 1)^2 = 1 + a_2(z - 1)^2.$$

Here we have $a_1 = 0$. Hence f is not locally invertible at $z = 1$.

Example 3. Let $f(z) = \cos z$. Determine whether f is locally invertible at $z = 0$. In this case,

$$f(z) = 1 - \frac{z^2}{2} + \text{higher terms},$$

so $a_1 = 0$ and f is not locally invertible.

Example 4. Let $f(z) = z^3$. Then $f'(z) = 3z^2$ and $f'(0) = 0$. Thus f is not locally invertible at 0. On the other hand, $f'(z) \neq 0$ if $z \neq 0$. Hence if $z_0 \neq 0$ then f is locally invertible at z_0. However, let U be the open set obtained by deleting the origin from **C**. Then f is not invertible on U. (Why?)

II, §6. EXERCISES

Determine which of the following functions are local analytic isomorphism at the given point. Give the reason for your answer.

1. $f(z) = e^z$ at $z = 0$.
2. $f(z) = \sin(z^2)$ at $z = 0$.
3. $f(z) = (z - 1)/(z - 2)$ at $z = 1$.
4. $f(z) = (\sin z)^2$ at $z = 0$.
5. $f(z) = \cos z$ at $z = \pi$.
6. **Linear Differential Equations.** Prove:

   Theorem. *Let $a_0(z), \dots, a_k(z)$ be analytic functions in a neighborhood of 0. Assume that $a_0(0) \neq 0$. Given numbers $c_0, \dots, c_{k-1}$, there exists a unique analytic function f at 0 such that*

   $$D^n f(0) = c_n \qquad \text{for} \quad n = 0, \dots, k - 1$$

   and such that

   $$a_0(z)D^k f(z) + a_1(z)D^{k-1}f(z) + \cdots + a_k(z)f(z) = 0.$$

   [*Hint*: First you may assume $a_0(z) = 1$ (why?). Then solve for f by a formal power series. Then prove this formal series converges.]

7. **Ordinary Differential Equations.** Prove:

   Theorem. *Let g be analytic at 0. There exists a unique analytic function f at 0 satisfying*

   $$f(0) = 0, \qquad \text{and} \qquad f'(z) = g(f(z)).$$

   [*Hint*: Again find a formal solution, and then prove that it converges.]

[*Note*: You will find the above two problems worked out in the Appendix, §3, but please try to do them first before looking up the solutions.]

II, §7. THE LOCAL MAXIMUM MODULUS PRINCIPLE

This principle is an immediate application of the open mapping theorem, and so we give it here, to emphasize its direct dependence with the preceding section. On the other hand, we wait for a later chapter for less basic applications mostly for psychological reasons. We want to alternate the formal operations with power series and the techniques which will arise from Cauchy's theorem. The later chapter could logically be read almost in its entirety after the present section, however.

We say that a function f is **locally constant** at a point z_0 if there exists an open set D (or a disc) containing z_0 such that f is constant on D.

Theorem 7.1. *Let f be analytic on an open set U. Let $z_0 \in U$ be a maximum for $|f|$, that is,*

$$|f(z_0)| \geqq |f(z)|, \qquad \textit{for all} \quad z \in U.$$

Then f is locally constant at z_0.

Proof. The function f has a power series expansion at z_0,

$$f(z) = a_0 + a_1(z - z_0) + \cdots.$$

If f is not the constant $a_0 = f(z_0)$, then by Theorem 6.2 we know that f is an open mapping in a neighborhood of z_0, and therefore the image of f contains a disc $D(a_0, s)$ of radius $s > 0$, centered at a_0. Hence the set of numbers $|f(z)|$, for z in a neighborhood of z_0, contains an open interval around a_0, so $|f(z)| > |f(z_0)|$ for some z. Hence

$$|f(z_0)| = |a_0|$$

cannot be a maximum for f, a contradiction which proves the theorem.

Corollary 7.2. *Let f be analytic on an open set U, and let $z_0 \in U$ be a maximum for the real part* $\text{Re } f$, *that is,*

$$\text{Re } f(z_0) \geqq \text{Re } f(z), \qquad \textit{for all} \quad z \in U.$$

Then f is locally constant at z_0.

Proof. The function $e^{f(z)}$ is analytic on U, and if

$$f(z) = u(z) + iv(z)$$

is the expression of f in terms of its real and imaginary parts, then

$$|e^{f(z)}| = e^{u(z)}.$$

Hence a maximum for $\text{Re } f$ is also a maximum for $|e^{f(z)}|$, and the corollary follows from the theorem.

The theorem is often applied when f is analytic on an open set U and is continuous at the boundary of U. Then a maximum for $|f(z)|$ necessarily occurs on the boundary of U. For this one needs that U is connected, and the relevant form of the theorem will be proved as Theorem 1.3 of the next chapter.

We shall give here one more example of the power of the maximum modulus principle, and postpone to a later chapter some of the other applications.

Theorem 7.3. *Let*

$$f(z) = a_0 + a_1 z + \cdots + a_d z^d$$

be a polynomial, not constant, and say $a_d \neq 0$. Then f has some complex zero, i.e. a number z_0 such that $f(z_0) = 0$.

Proof. Suppose otherwise, so that $1/f(z)$ is defined for all z, and defines an analytic function. Writing

$$f(z) = a_d z^d \left(\frac{a_0}{a_d z^d} + \frac{a_1 z}{a_d z^d} + \cdots + 1 \right),$$

one sees that

$$\lim_{|z| \to \infty} 1/f(z) = 0.$$

Let α be some complex number such that $f(\alpha) \neq 0$. Pick a positive number R large enough such that $|\alpha| < R$, and if $|z| \geqq R$, then

$$\frac{1}{|f(z)|} < \frac{1}{|f(\alpha)|}.$$

Let S be the closed disc of radius R centered at the origin. Then S is closed and bounded, and $1/|f(z)|$ is continuous on S, whence has a maximum on S, say at z_0. By construction, this point z_0 cannot be on the boundary of the disc, and must be an interior point. By the maximum modulus principle, we conclude that $1/f(z)$ is locally constant at z_0. This is obviously impossible since f itself is not locally constant, say from the expansion

$$f(z) = b_0 + b_1(z - z_0) + \cdots + b_d(z - z_0)^d,$$

with suitable coefficients $b_0, \ldots, b_d$ and $b_d \neq 0$. This proves the theorem.

CHAPTER III

Cauchy's Theorem, First Part

III, §1. HOLOMORPHIC FUNCTIONS ON CONNECTED SETS

Let $[a, b]$ be a closed interval of real numbers. By a **curve** γ (defined on this interval) we mean a function

$$\gamma: [a, b] \to \mathbf{C}$$

which we assume to be of class C^1.

Figure 1

We recall what this means. We write

$$\gamma(t) = \gamma_1(t) + i\gamma_2(t),$$

where γ_1 is the real part of γ, and γ_2 is its imaginary part. For instance, the curve

$$\gamma(\theta) = \cos\theta + i\sin\theta, \qquad 0 \leqq \theta \leqq 2\pi,$$

is the unit circle. **Of class** C^1 means that the functions $\gamma_1(t)$, $\gamma_2(t)$ have continuous derivatives in the ordinary sense of calculus. We have drawn a curve in Fig. 1. Thus a curve is a parametrized curve. We call $\gamma(a)$ the **beginning point**, and $\gamma(b)$ the **end point** of the curve. By a **point on the curve** we mean a point w such that $w = \gamma(t)$ for some t in the interval of definition of γ.

We define the derivative $\gamma'(t)$ in the obvious way, namely

$$\gamma'(t) = \gamma_1'(t) + i\gamma_2'(t).$$

It is easily verified as usual that the rules for the derivative of a sum, product, quotient, and chain rule are valid in this case, and we leave this as an exercise. In fact, prove systematically the following statements:

Let $F: [a, b] \to \mathbf{C}$ and $G: [a, b] \to \mathbf{C}$ be complex valued differentiable functions, defined on the same interval. Then:

$$(F + G)' = F' + G',$$
$$(FG)' = FG' + F'G,$$
$$(F/G)' = (GF' - FG')/G^2$$

(this quotient rule being valid only on the set where $G(t) \neq 0$).

Let $\psi: [c, d] \to [a, b]$ be a differentiable function. Then $\gamma \circ \psi$ is differentiable, and

$$(\gamma \circ \psi)'(t) = \gamma'(\psi(t))\psi'(t),$$

as illustrated on Fig. 2(i).

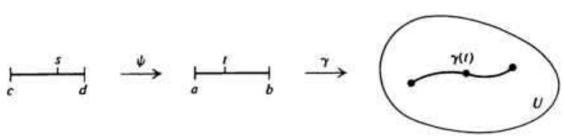

Figure 2(i)

Finally suppose γ is a curve in an open set U and

$$f: U \to \mathbf{C}$$

is a holomorphic function. Then the composite $f \circ \gamma$ is differentiable (as a function of the real variable t) and

$$(f \circ \gamma)'(t) = f'(\gamma(t))\gamma'(t),$$

as illustrated on Fig. 2(ii).

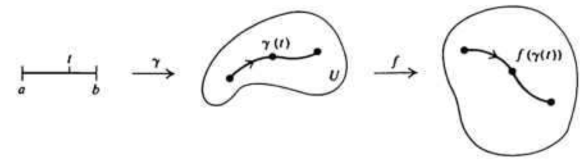

Figure 2(ii)

It is technically convenient to deal with a generalization of curves. By a **path** we shall mean a sequence of curves,

$$\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$$

(so each curve γ_j is C^1) such that the end point of γ_j is equal to the beginning point of γ_{j+1}. If γ_j is defined on the interval $[a_j, b_j]$, this means that

$$\gamma_j(b_j) = \gamma_{j+1}(a_{j+1}).$$

We have drawn a path on Fig. 3, where z_j is the end point of γ_j. We call $\gamma_1(a_1)$ the **beginning point** of γ, and $\gamma_n(b_n)$ the **end point** of γ. The path is said to **lie in an open set** U if each curve γ_j lies in U, i.e. for each t, the point $\gamma_j(t)$ lies in U.

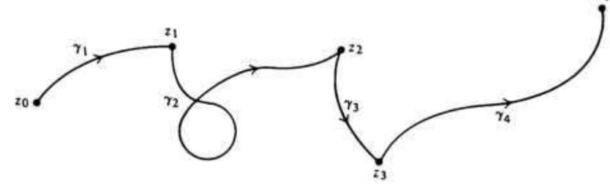

Figure 3

We define an open set U to be **connected** if given two points α and β in U, there exists a path $\{\gamma_1, \ldots, \gamma_n\}$ in U such that α is the beginning point of γ_1 and β is the end point of γ_n; in other words, if there exists a path in U which joins α to β. In Fig. 4 we have drawn an open set which is not connected. In Fig. 5 we have drawn a connected open set.

(The definition of connected applies of course equally well to a set which is not necessarily open. It is usually called **pathwise connected**, but for open sets, this coincides with another possible definition. See the appendix of this section.)

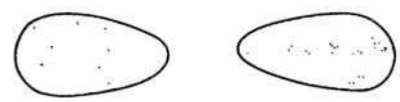

Figure 4

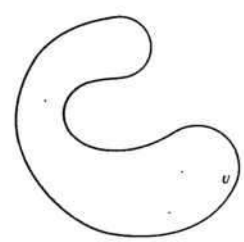

Figure 5

Theorem 1.1. *Let U be a connected open set, and let f be a holomorphic function on U. If $f' = 0$ then f is constant.*

Proof. Let α, β be two points in U, and suppose first that γ is a curve joining α to β, so that

$$\gamma(a) = \alpha \quad \text{and} \quad \gamma(b) = \beta.$$

The function

$$t \mapsto f(\gamma(t))$$

is differentiable, and by the chain rule, its derivative is

$$f'(\gamma(t))\gamma'(t) = 0.$$

Hence this function is constant, and therefore

$$f(\alpha) = f(\gamma(a)) = f(\gamma(b)) = f(\beta).$$

Next, suppose that $\gamma = \{\gamma_1, \ldots, \gamma_n\}$ is a path joining α to β, and let z_j be the end point of γ_j, putting

$$z_0 = \alpha, \qquad z_n = \beta.$$

By what we have just proved, we have

$$f(\alpha) = f(z_0) = f(z_1) = f(z_2) = \cdots = f(z_n) = f(\beta),$$

thereby proving the theorem.

If f is a function on an open set U and g is a holomorphic function on U such that $g' = f$, then we call g a **primitive** of f on U. Theorem 1.1 says that on a connected open set, a primitive of f is uniquely determined up to a constant, i.e. if g_1 and g_2 are two primitives, then $g_1 - g_2$ is constant, because the derivative of $g_1 - g_2$ is equal to 0.

In what follows we shall attempt to get primitives by integration. On the other hand, primitives can also be written down directly.

Example. For each integer $n \neq -1$, the function $f(z) = z^n$ has the usual primitive

$$\frac{z^{n+1}}{n+1}.$$

Let S be a set of points, and let $z_0 \in S$. We say that z_0 is **isolated in** S if there exists a disc $D(z_0, r)$ of some radius $r > 0$ such that $D(z_0, r)$ does not contain any point of S other than z_0. We say that S is **discrete** if every point of S is isolated.

Theorem 1.2. *Let U be a connected open set.*

(i) *If f is analytic on U and not constant, then the set of zeros of f on U is discrete.*

(ii) *Let f, g be analytic on U. Let S be a set of points in U which is not discrete (so some point of S is not isolated). Assume that $f(z) = g(z)$ for all z in S. Then $f = g$ on U.*

Proof. We observe that (ii) follows from (i). It suffices to consider the difference $f - g$. Therefore we set about to prove (i). We know from Theorem 3.2 of the preceding chapter that either f is locally constant and equal to 0 in the neighborhood of a zero z_0, or z_0 is an isolated zero.

Suppose that f is equal to 0 in the neighborhood of some point z_0. We have to prove that $f(z) = 0$ for all $z \in U$. Let S be the set of points z such that f is equal to 0 in a neighborhood of z. Then S is open. By

Theorem 1.6 below, it will suffice to prove that S is closed in U. Let z_1 be a point in the closure of S in U. Since f is continuous, it follows that $f(z_1) = 0$. If z_1 is not in S, then there exist points of S arbitrarily close to z_1, and by Theorem 3.2 of the preceding chapter, it follows that f is locally equal to 0 in a neighborhood of z_1. Hence in fact $z_1 \in S$, so S is closed in U. This concludes the proof.

Remarks. The argument using open and closed subsets of U applies in very general situations, and shows how to get a global statement on a connected set U knowing only a local property as in Theorem 3.2 of the preceding chapter.

It will be proved in Chapter V, §1, that a function is holomorphic if and only if it is analytic. Thus Theorem 1.2 will also apply to holomorphic functions.

The second part of Theorem 1.2 will be used later in the study of analytic continuation, but we make some comments here in anticipation. Let f be an analytic function defined on an open set U and let g be an analytic function defined on an open set V. Suppose that U and V have a non-empty intersection, as illustrated on Fig. 6. If U, V are connected, and if $f(z) = g(z)$ for all $z \in U \cap V$, i.e. if f and g are equal on the intersection $U \cap V$, then Theorem 1.2 tells us that g is the only possible analytic function on V having this property. In the applications, we shall be interested in extending the domain of definition of an analytic function f, and Theorem 1.2 guarantees the uniqueness of the extended function. We say that g is the **analytic continuation** of f to V.

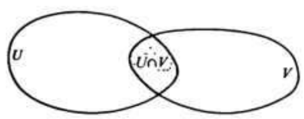

Figure 6

It is also appropriate here to formulate the global version of the **maximum modulus principle**.

Theorem 1.3. *Let U be a connected open set, and let f be an analytic function on U. If $z_0 \in U$ is a maximum point for $|f|$, that is*

$$|f(z_0)| \geqq |f(z)|$$

for all $z \in U$, then f is constant on U.

Proof. By Theorem 6.1 of the preceding chapter, we know that f is locally constant at z_0. Therefore f is constant on U by Theorem 1.2(ii) (compare the constant function and f). This concludes the proof.

Corollary 1.4. *Let U be a connected open set and U^c its closure. Let f be a continuous function on U^c, analytic and non-constant on U. If z_0 is a maximum for f on U^c, that is, $|f(z_0)| \geqq |f(z)|$ for all $z \in U^c$, then z_0 lies on the boundary of U^c.*

Proof. This comes from a direct application of Theorem 1.3.

Remark. If U^c is closed and bounded, then a continuous function has a maximum on U^c, so a maximum for f always exists in Corollary 1.4.

Appendix: Connectedness

The purpose of this appendix is to put together a couple of statements describing connectedness in various terms. Essentially we want to prove that two possible definitions of connectedness are equivalent. For purposes of this appendix, we use the words pathwise connected for the notion we have already defined. Let U be an open set in the complex numbers. We say that U is **topologically connected** if U cannot be expressed as a union $U = V \cup W$, where V, W are open, non-empty, and disjoint. We start with what amounts to a remark. Let S be a subset of U. We say that S is **closed in** U if given $z \in U$ and z in the closure of S, then $z \in S$.

Lemma 1.5. *Let S be a subset of an open set U. Then S is closed in U if and only if the complement of S in U is open, that is, $U - S$ is open. In particular, if S is both open and closed in U, then $U - S$ is also open and closed in U.*

Proof. Exercise 1.

Theorem 1.6. *Let U be an open set. Then U is pathwise connected if and only if U is topologically connected.*

Proof of Theorem 1.6. Assume that U is pathwise connected. We want to prove that U is topologically connected. Suppose not. Then $U = V \cup W$ where V, W are non-empty and open. Let $z_1 \in V$ and $z_2 \in W$. By assumption there exists a path $\gamma: [a, b] \to U$ such that $\gamma(a) = z_1$ and $\gamma(b) = z_2$. Let T be the set of $t \in [a, b]$ such that $\gamma(t) \in V$. Then T is not empty because $a \in T$, and T is bounded by b. Let c be the least upper bound of T. Then $c \neq b$. By definition of an upper bound, there exists a sequence of real numbers t_n, with $c < t_n \leqq b$ such that $\gamma(t_n) \in W$, and t_n converges to c. Since γ is continuous, it follows that $\gamma(c) = \lim \gamma(t_n)$, and since W is closed in U, it follows that $\gamma(c) \in W$. On the other hand, by definition of a least upper bound, there exists a sequence of real numbers s_n with $a \leqq s_n \leqq c$ such that s_n converges to c, and $\gamma(s_n) \in V$. Since γ is continuous, it follows that $\gamma(c) = \lim \gamma(s_n)$, and since V is closed in U, it follows that $\gamma(c) \in V$, which is a contradiction proving that U is topologically connected.

Conversely, assume U is topologically connected. We want to prove that U is pathwise connected. Let $z_0 \in U$. Let V be the set of points in U which can be joined to z_0 by a path in U. Then V is open. Indeed, suppose that there is a path in U joining z_0 to z_1. Since U is open, there exists a disc $D(z_1, r)$ of radius $r > 0$ contained in U. Then every element of this disc can be joined to z_1 by a line segment in the disc, and can therefore be joined to z_0 by a path in U, so V is open. We assert further that V is closed. To see this, let $\{z_n\}$ be a sequence in V converging to a point u in U. Since U is open, there exists a disc $D(u, r)$ of radius $r > 0$ contained in U. For some n the point z_n lies in $D(u, r)$. Then there is a line segment in $D(u, r)$ joining u and z_n, and so u can be joined by a path to z_0. This proves that V is closed. Hence V is both open and closed, and by assumption, $V = U$. This proves that U is pathwise connected, and concludes the proof of Theorem 1.6.

Warning. The equivalence of the two notions of connectedness for open sets may not be valid for other types of sets. For instance, consider the set consisting of the horizontal positive x-axis, together with vertical segments of length 1 above the points $1, 1/2, 1/3, \ldots, 1/n, \ldots$ and also above 0. Now delete the origin. The remaining set is topologically connected but not pathwise connected. Draw the picture! Also compare with inaccessible points as in Chapter X, §4.

III, §1. EXERCISES

1. Prove Lemma 1.5.

2. Let U be a bounded open connected set, $\{f_n\}$ a sequence of continuous functions on the closure of U, analytic on U. Assume that $\{f_n\}$ converges uniformly on the boundary of U. Prove that $\{f_n\}$ converges uniformly on U.

3. Let $a_1, \ldots, a_n$ be points on the unit circle. Prove that there exists a point z on the unit circle so that the product of the distances from z to the a_j is at least 1. (You may use the maximum principle.)

III, §2. INTEGRALS OVER PATHS

Let $F: [a, b] \to \mathbf{C}$ be a continuous function.

Write F in terms of its real and imaginary parts, say

$$F(t) = u(t) + iv(t).$$

Define the **indefinite integral** by

$$\int F(t)\,dt = \int u(t)\,dt + i\int v(t)\,dt.$$

Verify that integration by parts is valid (assuming that F' and G' exist and are continuous), namely

$$\int F(t)G'(t)\,dt = F(t)G(t) - \int G(t)F'(t)\,dt.$$

(The proof is the same as in ordinary calculus, from the derivative of a product.)

We define the **integral of** F over $[a, b]$ to be

$$\int_a^b F(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$

Thus the integral is defined in terms of the ordinary integrals of the real functions u and v. Consequently, by the fundamental theorem of calculus *the function*

$$t \mapsto \int_a^t F(s)\,ds$$

is differentiable, and its derivative is $F(t)$, because this assertion is true if we replace F by u and v, respectively.

Using simple properties of the integral of real-valued functions, one has the inequality

$$\left|\int_a^b F(t)\,dt\right| \leqq \int_a^b |F(t)|\,dt.$$

Work it out as Exercise 11.

Let f be a continuous function on an open set U, and suppose that γ is a curve in U, meaning that all values $\gamma(t)$ lie in U for $a \leqq t \leqq b$. We define the **integral of f along** γ to be

$$\int_\gamma f = \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

This is also frequently written

$$\int_\gamma f(z)\,dz.$$

Example 1. Let $f(z) = 1/z$. Let $\gamma(\theta) = e^{i\theta}$. Then

$$\gamma'(\theta) = ie^{i\theta}.$$

We want to find the value of the integral of f over the circle,

$$\int_\gamma \frac{1}{z}\,dz$$

so $0 \leqq \theta \leqq 2\pi$. By definition, this integral is equal to

$$\int_0^{2\pi} \frac{1}{e^{i\theta}} ie^{i\theta}\,d\theta = i\int_0^{2\pi} d\theta = 2\pi i.$$

As in calculus, we have defined the integral over parametrized curves. In practice, we sometimes describe a curve without giving an explicit parametrization. The context should always make it clear what is meant. Furthermore, one can also easily see that the integral is independent of the parametrization, in the following manner

Let

$$g\colon [a, b] \to [c, d]$$

be a C^1 function, such that $g(a) = c$, $g(b) = d$, and let

$$\psi\colon [c, d] \to \mathbf{C}$$

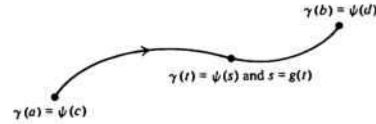

Figure 7

be a curve. Then we may form the composed curve

$$\gamma(t) = \psi(g(t)).$$

We find:

$$\begin{aligned}\int_\gamma f &= \int_a^b f(\gamma(t))\gamma'(t)\,dt \\ &= \int_a^b f(\psi(g(t)))\psi'(g(t))g'(t)\,dt \\ &= \int_c^d f(\psi(s))\psi'(s)\,ds \\ &= \int_\psi f.\end{aligned}$$

Thus the integral of f along the curve is independent of the parametrization.

If $\gamma = \{\gamma_1, \ldots, \gamma_n\}$ is a path, then we define

$$\int_\gamma f = \sum_{i=1}^{n} \int_{\gamma_i} f,$$

to be the sum of the integrals of f over each curve γ_i of the path.

Theorem 2.1. *Let f be continuous on an open set U, and suppose that f has a primitive g, that is, g is holomorphic and $g' = f$. Let α, β be two points of U, and let γ be a path in U joining α to β. Then*

$$\int_\gamma f = g(\beta) - g(\alpha),$$

and in particular, this integral depends only on the beginning and end point of the path. It is independent of the path itself.

Proof. Assume first that the path is a curve. Then

$$\int_\gamma f(z)\,dz = \int_a^b g'(\gamma(t))\gamma'(t)\,dt.$$

By the chain rule, the expression under the integral sign is the derivative

$$\frac{d}{dt}g(\gamma(t)).$$

Hence by ordinary calculus, the integral is equal to

$$g(\gamma(t))\Big|_a^b = g(\gamma(b)) - g(\gamma(a)),$$

which proves the theorem in this case. In general, if the path consists of curves $\gamma_1, \ldots, \gamma_n$, and z_j is the end point of γ_j, then by the case we have just settled, we find

$$\begin{aligned}\int_\gamma f &= g(z_1) - g(z_0) + g(z_2) - g(z_1) + \cdots + g(z_n) - g(z_{n-1}) \\ &= g(z_n) - g(z_0),\end{aligned}$$

which proves the theorem.

Example 2. Let $f(z) = z^3$. Then f has a primitive, $g(z) = z^4/4$. Hence the integral of f from $2 + 3i$ to $1 - i$ over *any path* is equal to

$$\frac{(1-i)^4}{4} - \frac{(2+3i)^4}{4}.$$

Example 3. Let $f(z) = e^z$. Find the integral of f from 1 to $i\pi$ taken over a line segment. Here again $f'(z) = f(z)$, so f has a primitive. Thus the integral is independent of the path and equal to $e^{i\pi} - e^1 = -1 - e$.

By a **closed path**, we mean a path whose beginning point is equal to its end point. We may now give an important example of the theorem:

If f is a continuous function on U admitting a holomorphic primitive g, and γ is any closed path in U, then

$$\int_\gamma f = 0.$$

Example 4. Let $f(z) = z^n$, where n is an integer $\neq -1$. Then for any closed path γ (or any closed path not passing through the origin if n is negative), we have

$$\int_\gamma z^n\,dz = 0.$$

This is true because z^n has the primitive $z^{n+1}/(n+1)$. [When n is negative, we have to assume that the closed path does not pass through the origin, because the function is then not defined at the origin.]

Putting this together with Example 1, we have the following tabulation. Let C_R be the circle of radius R centered at the origin oriented counterclockwise. Let n be an integer. Then:

$$\int_{C_R} z^n\,dz = \begin{cases} 0 & \text{if } n \neq -1, \\ 2\pi i & \text{if } n = -1. \end{cases}$$

Of course, in Example 1 we did the computation when $R = 1$, but you can check that one gets the same value for arbitrary R. In the exercises, you can check similar values for the integral around a circle centered around any point z_0.

We shall see later that holomorphic functions are analytic. In that case, in the domain of convergence a power series

$$\sum a_n(z - z_0)^n$$

can be integrated term by term, and thus integrals of holomorphic functions are reduced to integrals of polynomials. This is the reason why there is no need here to give further examples.

Theorem 2.2. *Let U be a connected open set, and let f be a continuous function on U. If the integral of f along any closed path in U is equal to 0, then f has a primitive g on U, that is, a function g which is holomorphic such that $g' = f$.*

Proof. Pick a point z_0 in U and define

$$g(z) = \int_{z_0}^{z} f,$$

where the integral is taken along any path from z_0 to z in U. If γ, η are two such paths, and η^- is the reverse path of η (cf. Exercise 9), then $\{\gamma, \eta^-\}$ is a closed path, and by Exercise 9 we know that

$$\int_\gamma f = \int_\eta f.$$

Therefore the integral defining g is independent of the path from z_0 to z, and defines the function. We have

$$\frac{g(z + h) - g(z)}{h} = \frac{1}{h}\int_{z}^{z+h} f(\zeta)\,d\zeta,$$

and the integral from z to $z + h$ can be taken along a segment in U from z to $z + h$. Write

$$f(\zeta) = f(z) + \varphi(\zeta),$$

where $\lim_{\zeta \to z} \varphi(\zeta) = 0$ (this can be done by the continuity of f at z). Then

$$\begin{aligned} \frac{1}{h}\int_z^{z+h} f(\zeta)\, d\zeta &= \frac{1}{h}\int_z^{z+h} f(z)\, d\zeta + \frac{1}{h}\int_z^{z+h} \varphi(\zeta)\, d\zeta \\ &= f(z) + \frac{1}{h}\int_z^{z+h} \varphi(\zeta)\, d\zeta. \end{aligned}$$

The length of the interval from z to $z + h$ is $|h|$. Hence the integral on the right is estimated by (see below, Theorem 2.3)

$$\frac{1}{|h|}|h| \max |\varphi(\zeta)|,$$

where the max is taken for ζ on the interval. This max tends to 0 as $h \to 0$, and this proves the theorem.

Remarks. The reader should recognize Theorems 2.1 and 2.2 as being the exact analogues for (complex) differentiable functions of the standard theorems of advanced calculus concerning the relation between the existence of a primitive (potential function for a vector field), and the independence of the integral (of a vector field) from the path. We shall see later that a holomorphic function is infinitely complex differentiable, and therefore that f itself is analytic.

Let γ be a curve, $\gamma: [a, b] \to \mathbf{C}$, assumed of class C^1 as always. The **speed** is defined as usual to be $|\gamma'(t)|$, and the **length** $L(\gamma)$ is defined to be the integral of the speed,

$$L(\gamma) = \int_a^b |\gamma'(t)|\, dt.$$

If $\gamma = \{\gamma_1, \ldots, \gamma_n\}$ is a path, then by definition

$$L(\gamma) = \sum_{i=1}^{n} L(\gamma_i).$$

Let f be a bounded function on a set S. We let $\|f\|$ be the **sup norm**, written $\|f\|_S$ if the reference to S needs to be made for clarity, so that

$$\|f\| = \sup_{z \in S} |f(z)|$$

is the least upper bound of the values $|f(z)|$ for $z \in S$.

Let f be continuous on an open set U. By standard results of elementary real analysis, Theorem 4.3 of Chapter I, the image of a curve or a path γ is closed and bounded, i.e. compact. If the curve is in U, then the function

$$t \mapsto f(\gamma(t))$$

is continuous, and hence f is bounded on the image of γ. By the compactness of the image of γ, we can always find an open subset of U containing γ, on which f is bounded. If γ is defined on $[a, b]$, we let

$$\|f\|_\gamma = \max_{t \in [a,b]} |f(\gamma(t))|.$$

Theorem 2.3. *Let f be a continuous function on U. Let γ be a path in U. Then*

$$\left|\int_\gamma f\right| \leqq \|f\|_\gamma L(\gamma).$$

Proof. If γ is a curve, then

$$\begin{aligned}\left|\int_\gamma f\right| &= \left|\int_a^b f(\gamma(t))\gamma'(t)\,dt\right| \\ &\leqq \int_a^b |f(\gamma(t))||\gamma'(t)|\,dt \\ &\leqq \|f\|_\gamma L(\gamma),\end{aligned}$$

as was to be shown. The statement for a path follows by taking an appropriate sum.

Theorem 2.4. *Let $\{f_n\}$ be a sequence of continuous functions on U, converging uniformly to a function f. Then*

$$\lim \int_\gamma f_n = \int_\gamma f.$$

If $\sum f_n$ is a series of continuous functions converging uniformly on U, *then*

$$\int_\gamma \sum f_n = \sum \int_\gamma f_n.$$

Proof. The first assertion is immediate from the inequality.

$$\left|\int_\gamma f_n - \int_\gamma f\right| \leqq \int_\gamma |f_n - f| \leqq \|f_n - f\| L(\gamma).$$

The second follows from the first because uniform convergence of a series is defined in terms of the uniform convergence of its partial sums,

$$s_n = f_1 + \cdots + f_n.$$

This proves the theorem.

Example 5. Let f be analytic on an open set containing the closed disc $\bar{D}(0, R)$ of radius R centered at the origin, except possibly at the origin. Suppose f has a power series expansion

$$f(z) = \frac{a_{-m}}{z^m} + \cdots + \frac{a_{-1}}{z} + a_1 z + \cdots + a_n z^n + \cdots$$

possibly with negative terms, such that the series with non-negative terms

$$\sum_{n=0}^{\infty} a_n z^n$$

has a radius of convergence $> R$. Let C_R be the circle of radius R centered at the origin. Then

$$\int_{C_R} f(z)\, dz = 2\pi i a_{-1}.$$

This is a special case of Theorem 2.4 and Example 4, by letting

$$f_n(z) = \sum_{k=-m}^{n} a_k z^k.$$

Each f_n is a finite sum, so the integral of f_n is the sum of the integrals of the individual terms, which were evaluated in Example 4.

III, §2. EXERCISES

1. (a) Given an arbitrary point z_0, let C be a circle of radius $r > 0$ centered at z_0, oriented counterclockwise. Find the integral

$$\int_C (z - z_0)^n \, dz$$

for all integers n, positive or negative.

(b) Suppose f has a power series expansion

$$f(z) = \sum_{k=-m}^{\infty} a_k (z - z_0)^k,$$

which is absolutely convergent on a disc of radius $> R$ centered at z_0. Let C_R be the circle of radius R centered at z_0. Find the integral

$$\int_{C_R} f(z) \, dz.$$

2. Find the integral of $f(z) = e^z$ from -3 to 3 taken along a semicircle. Is this integral different from the integral taken over the line segment between the two points?

3. Sketch the following curves with $0 \leqq t \leqq 1$.
   (a) $\gamma(t) = 1 + it$
   (b) $\gamma(t) = e^{-\pi i t}$
   (c) $\gamma(t) = e^{\pi i t}$
   (d) $\gamma(t) = 1 + it + t^2$

4. Find the integral of each one of the following functions over each one of the curves in Exercise 3.
   (a) $f(z) = z^3$
   (b) $f(z) = \bar{z}$
   (c) $f(z) = 1/z$

5. Find the integral

$$\int_\gamma z e^{z^2} \, dz$$

   (a) from the point i to the point $-i + 2$, taken along a straight line segment, and
   (b) from 0 to $1 + i$ along the parabola $y = x^2$.

6. Find the integral

$$\int_\gamma \sin z \, dz$$

from the origin to the point $1 + i$, taken along the parabola

$$y = x^2.$$

7. Let σ be a vertical segment, say parametrized by

$$\sigma(t) = z_0 + itc, \qquad -1 \leqq t \leqq 1,$$

where z_0 is a fixed complex number, and c is a fixed real number > 0. (Draw the picture.) Let $\alpha = z_0 + x$ and $\alpha' = z_0 - x$, where x is real positive. Find

$$\lim_{x \to 0} \int_\sigma \left(\frac{1}{z - \alpha} - \frac{1}{z - \alpha'} \right) dz.$$

(Draw the picture.) *Warning*: The answer is not 0!

8. Let $x > 0$. Find the limit:

$$\lim_{B \to \infty} \int_{-B}^{B} \left(\frac{1}{t + ix} - \frac{1}{t - ix} \right) dt.$$

9. Let $\gamma\colon [a, b] \to \mathbf{C}$ be a curve. Define the **reverse** or opposite curve to be

$$\gamma^-\colon [a, b] \to \mathbf{C}$$

such that $\gamma^-(t) = \gamma(a + b - t)$. Show that

$$\int_{\gamma^-} F = -\int_\gamma F.$$

10. Let $[a, b]$ and $[c, d]$ be two intervals (not reduced to a point). Show that there is a function $g(t) = rt + s$ such that g is strictly increasing, $g(a) = c$ and $g(b) = d$. Thus a curve can be parametrized by any given interval.

11. Let F be a continuous complex-valued function on the interval $[a, b]$. Prove that

$$\left| \int_a^b F(t)\, dt \right| \leqq \int_a^b |F(t)|\, dt.$$

[*Hint*: Let $P = [a = a_0, a_1, \ldots, a_n = b]$ be a partition of $[a, b]$. From the definition of integrals with Riemann sums, the integral

$$\int_a^b F(t)\, dt \quad \text{is approximated by the Riemann sum} \quad \sum_{k=0}^{n-1} F(a_k)(a_{k+1} - a_k)$$

whenever $\max(a_{k+1} - a_k)$ is small, and

$$\int_a^b |F(t)|\, dt \quad \text{is approximated by} \quad \sum_{k=0}^{n-1} |F(a_k)|(a_{k+1} - a_k).$$

The proof is concluded by using the triangle inequality.]

III, §3. LOCAL PRIMITIVE FOR A HOLOMORPHIC FUNCTION

Let U be a connected open set, and let f be holomorphic on U. Let $z_0 \in U$. We want to define a primitive for f on some open disc centered at z_0, i.e. locally at z_0. The natural way is to define such a primitive by an integral,

$$g(z) = \int_{z_0}^{z} f(\zeta)\, d\zeta,$$

taken along some path from z_0 to z. However, the integral may depend on the path.

It turns out that we may define g locally by using only a special type of path. Indeed, suppose U is a disc centered at z_0. Let $z \in U$. We select for a path from z_0 to z the edges of a rectangle as shown on Fig. 8.

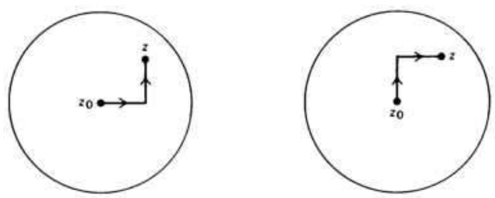

Figure 8

We then have restricted our choice of path to two possible choices as shown. We shall see that we get the same value for the integrals in the two cases. It will be shown afterwards that the integral then gives us a primitive.

By a **rectangle** R we shall mean a rectangle whose sides are vertical or horizontal, and R is meant as the set of points inside and on the boundary of the rectangle, so R is assumed to be closed. The path describing the boundary of the rectangle taken counterclockwise will be also called the **boundary of the rectangle**, and will be denoted by

$$\partial R.$$

If S is an arbitrary set of points, we say that a function f is **holomorphic on** S if it is holomorphic on some open set containing S.

Theorem 3.1 (Goursat). *Let R be a rectangle, and let f be a function holomorphic on R. Then*

$$\int_{\partial R} f = 0.$$

Proof. Decompose the rectangle into four rectangles by bisecting the sides, as shown on Fig. 9.

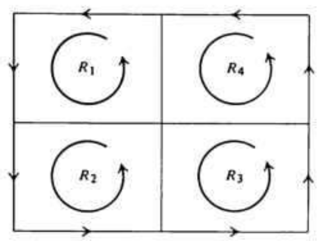

Figure 9

Then

$$\int_{\partial R} f = \sum_{i=1}^{4} \int_{\partial R_i} f.$$

Consequently,

$$\left| \int_{\partial R} f \right| \leqq \sum_{i=1}^{4} \left| \int_{\partial R_i} f \right|,$$

and there is one rectangle, say $R^{(1)}$, among R_1, R_2, R_3, R_4 such that

$$\left| \int_{\partial R^{(1)}} f \right| \geqq \frac{1}{4} \left| \int_{\partial R} f \right|.$$

Next we decompose $R^{(1)}$ into four rectangles, again bisecting the sides of $R^{(1)}$ as shown on Fig. 10.

For one of the four rectangles thus obtained, say $R^{(2)}$, we have the similar inequality

$$\left| \int_{\partial R^{(2)}} f \right| \geqq \frac{1}{4} \left| \int_{\partial R^{(1)}} f \right|.$$

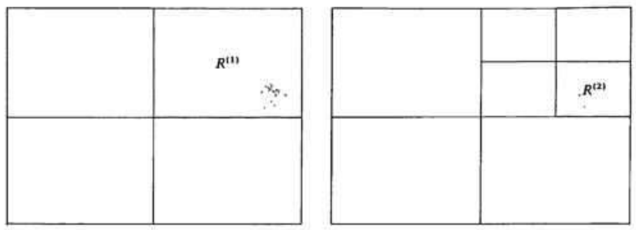

Figure 10

We continue in this way, to obtain a sequence of rectangles

$$R^{(1)} \supset R^{(2)} \supset R^{(3)} \supset \cdots$$

such that

$$\left|\int_{\partial R^{(n+1)}} f\right| \geqq \frac{1}{4}\left|\int_{\partial R^{(n)}} f\right|.$$

Then

$$\left|\int_{\partial R^{(n)}} f\right| \geqq \frac{1}{4^n}\left|\int_{\partial R} f\right|.$$

On the other hand, let L_n be the length of $\partial R^{(n)}$. Then

$$L_{n+1} = \frac{1}{2} L_n$$

so that by induction,

$$L_n = \frac{1}{2^n} L_0,$$

where $L_0 =$ length of ∂R.

We contend that the intersection

$$\bigcap_{n=1}^{\infty} R^{(n)}$$

consists of a single point z_0. Since the diameter of $R^{(n)}$ tends to 0 as n becomes large, it is immediate that there is at most one point in the intersection. Let α_n be the center of $R^{(n)}$. Then the sequence $\{\alpha_n\}$ is a

Cauchy sequence, because given ϵ, let N be such that the diameter of $R^{(N)}$ is less than ϵ. If $n, m \geqq N$, then α_n, α_m lie in $R^{(N)}$ and so

$$|\alpha_n - \alpha_m| \leqq \operatorname{diam} R^{(N)} < \epsilon.$$

Let $z_0 = \lim \alpha_n$. Then z_0 lies in each rectangle, because each rectangle is closed. Hence z_0 lies in the intersection of the rectangles $R^{(N)}$ for $N = 1, 2, \ldots$, as desired. (See also Theorem 4.2 of Chapter I.)

Since f is differentiable at z_0, there is a disc V centered at z_0 such that for all $z \in V$ we have

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + (z - z_0)h(z),$$

where

$$\lim_{z \to z_0} h(z) = 0.$$

If n is sufficiently large, then $R^{(n)}$ is contained in V, and then

$$\int_{\partial R^{(n)}} f(z)\, dz = \int_{\partial R^{(n)}} f(z_0)\, dz + f'(z_0) \int_{\partial R^{(n)}} (z - z_0)\, dz$$
$$+ \int_{\partial R^{(n)}} (z - z_0)h(z)\, dz.$$

By Example 4 of §2, we know that the first two integrals on the right of this equality sign are 0. Hence

$$\int_{\partial R^{(n)}} f = \int_{\partial R^{(n)}} (z - z_0)h(z)\, dz,$$

and we obtain the inequalities

$$\frac{1}{4^n}\left|\int_{\partial R} f\right| \leqq \left|\int_{\partial R^{(n)}} f\right| \leqq \left|\int_{\partial R^{(n)}} (z - z_0)h(z)\, dz\right|$$
$$\leqq \frac{1}{2^n} L_0 \operatorname{diam} R^{(n)} \sup|h(z)|,$$

where the sup is taken for all $z \in R^{(n)}$. But $\operatorname{diam} R^{(n)} = (1/2^n) \operatorname{diam} R$. This yields

$$\left|\int_{\partial R} f\right| \leqq L_0 \operatorname{diam} R \sup|h(z)|.$$

The right-hand side tends to 0 as n becomes large, and consequently

$$\int_{\partial R} f = 0,$$

as was to be shown.

We carry out the program outlined at the beginning of the section to find a primitive locally.

Theorem 3.2. *Let U be a disc centered at a point z_0. Let f be continuous on U, and assume that for each rectangle R contained in U we have*

$$\int_{\partial R} f = 0.$$

For each point z_1 in the disc, define

$$g(z_1) = \int_{z_0}^{z_1} f,$$

where the integral is taken along the sides of a rectangle R whose opposite vertices are z_0 and z_1. Then g is holomorphic on U and is a primitive for f, namely

$$g'(z) = f(z).$$

Proof. We have

$$g(z_1 + h) - g(z_1) = \int_{z_1}^{z_1+h} f(z)\,dz$$

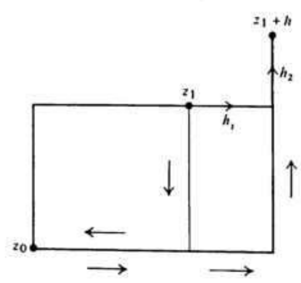

Figure 11

The integral between z_1 and $z_1 + h$ is taken over the bottom side h_1 and vertical side h_2 of the rectangle shown in Fig. 11. Since f is continuous at z_1, there exists a function $\psi(z)$ such that

$$\lim_{z \to z_1} \psi(z) = 0$$

and

$$f(z) = f(z_1) + \psi(z).$$

Then

$$\begin{aligned} g(z_1 + h) - g(z_1) &= \int_{z_1}^{z_1+h} f(z_1)\,dz + \int_{z_1}^{z_1+h} \psi(z)\,dz \\ &= hf(z_1) + \int_{z_1}^{z_1+h} \psi(z)\,dz. \end{aligned}$$

We divide by h and take the limit as $h \to 0$. The length of the path from z_1 to $z_1 + h$ is bounded by $|h_1| + |h_2|$. Hence we get a bound

$$\left| \frac{1}{h} \int_{z_1}^{z_1+h} \psi(z)\,dz \right| \leqq \frac{1}{|h|}(|h_1| + |h_2|) \sup|\psi(z)|,$$

where the sup is taken for z on the path of integration. The expression on the right therefore tends to 0 as $h \to 0$. Hence

$$\lim_{h \to 0} \frac{g(z_1 + h) - g(z_1)}{h} = f(z_1),$$

as was to be shown.

Knowing that a primitive for f exists on a disc U centered at z_0, we can now conclude that the integral of f along any path between z_0 and z in U is independent of the path, according to Theorem 2.1, and we find:

Theorem 3.3. *Let U be a disc and suppose that f is holomorphic on U. Then f has a primitive on U, and the integral of f along any closed path in U is 0.*

Remark. In Theorem 7.2 we shall prove that a holomorphic function is analytic. Applying this result to the function g in Theorem 3.2, we shall conclude that the function f in Theorem 3.2 is analytic. See Theorem 7.7.

III, §4. ANOTHER DESCRIPTION OF THE INTEGRAL ALONG A PATH

Knowing the existence of a local primitive for a holomorphic function allows us to describe its integral along a path in a way which makes no use of the differentiability of the path, and would apply to a continuous path as well. We start with curves.

Lemma 4.1. *Let $\gamma\colon [a, b] \to U$ be a continuous curve in an open set U. Then there is some positive number $r > 0$ such that every point on the curve lies at distance $\geqq r$ from the complement of U.*

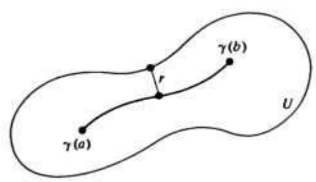

Figure 12

Proof. The image of γ is compact. Consider the function

$$\varphi(t) = \min_{w} |\gamma(t) - w|,$$

where the minimum is taken for all w in the complement of U. This minimum exists because it suffices to consider w lying inside some big circle. Then $\varphi(t)$ is easily verified to be a continuous function of t, whence φ has a minimum on $[a, b]$, and this minimum cannot be 0 because U is open. This proves our assertion.

Let $P = [a_0, \ldots, a_n]$ be a partition of the interval $[a, b]$. We also write P in the form

$$a = a_0 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_n = b.$$

Let $\{D_0, \ldots, D_n\}$ be a sequence of discs. We shall say that this sequence of discs is **connected by the curve along the partition** if D_i contains the image $\gamma([a_i, a_{i+1}])$. The following figure illustrates this.

One can always find a partition and such a connected sequence of discs. Indeed, let $\epsilon > 0$ be a positive number such that $\epsilon < r/2$ where r is as in Lemma 4.1. Since γ is uniformly continuous, there exists δ such

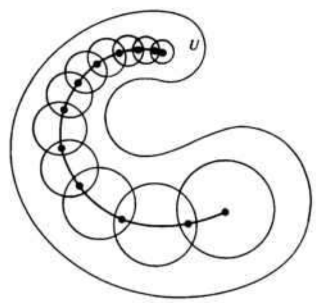

Figure 13

that if t, $s \in [a, b]$ and $|t - s| < \delta$, then $|\gamma(t) - \gamma(s)| < \epsilon$. We select an integer n and a partition P such that each interval $[a_i, a_{i+1}]$ has length $< \delta$. Then the image $\gamma([a_i, a_{i+1}])$ lies in a disc D_i centered at $\gamma(a_i)$ of radius ϵ, and this disc is contained in U.

Let f be holomorphic on U. Let $\gamma_i: [a_i, a_{i+1}] \to U$ be the restriction of γ to the smaller interval $[a_i, a_{i+1}]$. Then

$$\int_\gamma f = \sum_{i=0}^{n-1} \int_{\gamma_i} f.$$

Let $\gamma(a_i) = z_i$, and let g_i be a primitive of f on the disc D_i. If each γ_i is of class C^1 then we find:

$$\boxed{\int_\gamma f = \sum_{i=0}^{n-1} [g_i(z_{i+1}) - g_i(z_i)].}$$

Thus even though f may not have a primitive g on the whole open set U, its integral can nevertheless be expressed in terms of local primitives by decomposing the curve as a sum of sufficiently smaller curves. The same formula then applies to a path.

This procedure allows us to define the **integral of f along any continuous curve**; we do not need to assume any differentiability property of the curve. We need only apply the above procedure, but then we must show that the expression

$$\sum_{i=0}^{n-1} [g_i(z_{i+1}) - g_i(z_i)]$$

is independent of the choice of partition of the interval $[a, b]$ and of the choices of the discs D_i containing $\gamma([a_i, a_{i+1}])$. Then this sum can be taken as the definition of the integral

$$\int_\gamma f.$$

The reader interested only in applications may omit the following considerations. First we state formally this independence, repeating the construction.

Lemma 4.2. *Let $\gamma: [a, b] \to U$ be a continuous curve. Let*

$$a_0 = a \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_n = b$$

be a partition of $[a, b]$ such that the image $\gamma([a_i, a_{i+1}])$ is contained in a disc D_i, and D_i is contained in U. Let f be holomorphic on U and let g_i be a primitive of f on D_i.

Let $z_i = \gamma(a_i)$. Then the sum

$$\sum_{i=0}^{n-1} [g_i(z_{i+1}) - g_i(z_i)]$$

is independent of the choices of partitions, discs D_i, and primitives g_i on D_i subject to the stated conditions.

Proof. First let us work with the given partition, but let B_i be another disc containing the image $\gamma([a_i, a_{i+1}])$, and B_i contained in U. Let h_i be a primitive of f on B_i. Then both g_i, h_i are primitives of f on the intersection $B_i \cap D_i$, which is open and connected. Hence there exists a constant C_i such that $g_i = h_i + C_i$ on $B_i \cap D_i$. Therefore the differences are equal:

$$g_i(z_{i+1}) - g_i(z_i) = h_i(z_{i+1}) - h_i(z_i).$$

Thus we have proved that given the partition, the value of the sum is independent of the choices of primitives and choices of discs.

Given two partitions, we can always find a common refinement, as in elementary calculus. Recall that a partition

$$Q = [b_0, \ldots, b_m]$$

is called a **refinement** of the partition P if every point of P is among the points of Q, that is if each a_j is equal to some b_i. Two partitions always have a common refinement, which we obtain by inserting all the points of one partition into the other. Furthermore, we can obtain a refinement of a partition by inserting one point at a time. Thus it suffices to prove that if the partition Q is a refinement of the partition P obtained by inserting one point, then Lemma 4.2 is valid in this case. So we can suppose that Q is obtained by inserting some point c in some interval $[a_k, a_{k+1}]$ for some k, that is Q is the partition

$$[a_0, \ldots, a_k, c, a_{k+1}, \ldots, a_n].$$

We have already shown that given a partition, the value of the sum as in the statement of the lemma is independent of the choice of discs and primitives as described in the lemma. Hence for this new partition Q, we can take the same discs D_i for all the old intervals $[a_i, a_{i+1}]$ when $i \neq k$, and we take the disc D_k for the intervals $[a_k, c]$ and $[c, a_{k+1}]$. Similarly, we take the primitive g_i on D_i as before, and g_k on D_k. Then the sum with respect to the new partition is the same as for the old one, except that the single term

$$g_k(z_{k+1}) - g_k(z_k)$$

is now replaced by two terms

$$g_k(z_{k+1}) - g_k(\gamma(c)) + g_k(\gamma(c)) - g_k(z_k).$$

This does not change the value, and concludes the proof of Lemma 4.2.

For any continuous path $\gamma: [a, b] \to U$ we may thus **define**

$$\int_\gamma f = \sum_{i=0}^{n-1} [g_i(\gamma(a_{i+1})) - g_i(\gamma(a_i))]$$

for any partition $[a_0, a_1, \ldots, a_n]$ of $[a, b]$ such that $\gamma([a_i, a_{i+1}])$ is contained in a disc D_i, $D_i \subset U$, and g_i is a primitive of f on D_i. We have just proved that the expression on the right-hand side is independent of the choices made, and we had seen previously that if γ is piecewise C^1 then the expression on the right-hand side gives the same value as the definition used in §2. It is often convenient to have the additional flexibility provided by arbitrary continuous paths.

Remark. The technique of propagating discs along a curve will again be used in the chapter on holomorphic continuation along a curve.

As an application, we shall now see that if two paths lie "close together", and have the same beginning point and the same end point, then the integrals of f along the two paths have the same value. We must define precisely what we mean by "close together". After a reparametrization, we may assume that the two paths are defined over the same interval $[a, b]$. We say that they are **close together** if there exists a partition

$$a = a_0 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_n = b,$$

and for each $i = 0, \ldots, n-1$ there exists a disc D_i contained in U such that the images of each segment $[a_i, a_{i+1}]$ under the two paths γ, η are contained in D_i, that is,

$$\gamma([a_i, a_{i+1}]) \subset D_i \quad \text{and} \quad \eta([a_i, a_{i+1}]) \subset D_i.$$

Lemma 4.3. *Let γ, η be two continuous paths in an open set U, and assume that they have the same beginning point and the same end point. Assume also that they are close together. Let f be holomorphic on U. Then*

$$\int_\gamma f = \int_\eta f.$$

Proof. We suppose that the paths are defined on the same interval $[a, b]$, and we choose a partition and discs D_i as above. Let g_i be a primitive of f on D_i. Let

$$z_i = \gamma(a_i) \quad \text{and} \quad w_i = \eta(a_i).$$

We illustrate the paths and their partition in Fig. 14.

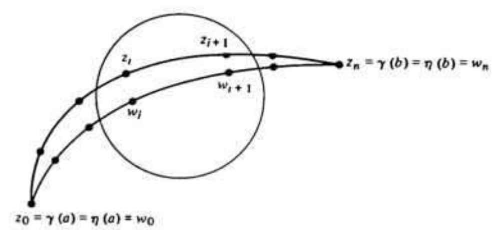

Figure 14

But g_{i+1} and g_i are primitives of f on the connected open set $D_{i+1} \cap D_i$, so $g_{i+1} - g_i$ is constant on $D_{i+1} \cap D_i$. But $D_{i+1} \cap D_i$ contains z_{i+1} and w_{i+1}. Consequently

$$g_{i+1}(z_{i+1}) - g_{i+1}(w_{i+1}) = g_i(z_{i+1}) - g_i(w_{i+1}).$$

Then we find

$$\int_\gamma f - \int_\eta f = \sum_{i=0}^{n-1} [g_i(z_{i+1}) - g_i(z_i) - (g_i(w_{i+1}) - g_i(w_i))]$$

$$= \sum_{i=0}^{n-1} [(g_i(z_{i+1}) - g_i(w_{i+1})) - (g_i(z_i) - g_i(w_i))]$$

$$= g_{n-1}(z_n) - g_{n-1}(w_n) - (g_0(z_0) - g_0(w_0))$$
$$= 0,$$

because the two paths have the same beginning point $z_0 = w_0$, and the same end point $z_n = w_n$. This proves the lemmas.

One can also formulate an analogous lemma for closed paths.

Lemma 4.4. *Let γ, η be closed continuous paths in the open set U, say defined on the same interval $[a, b]$. Assume that they are close together. Let f be holomorphic on U. Then*

$$\int_\gamma f = \int_\eta f.$$

Proof. The proof is the same as above, except that the reason why we find 0 in the last step is now slightly different. Since the paths are closed, we have

$$z_0 = z_n \quad \text{and} \quad w_0 = w_n,$$

as illustrated in Fig. 15. The two primitives g_{n-1} and g_0 differ by a constant on some disc contained in U and containing z_0, w_0. Hence the last expression obtained in the proof of Lemma 4.3 is again equal to 0, as was to be shown.

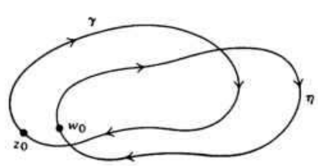

Figure 15

III, §5. THE HOMOTOPY FORM OF CAUCHY'S THEOREM

Let γ, η be two paths in an open set U. After a reparametrization if necessary, we assume that they are defined over the same interval $[a, b]$. We shall say that γ is **homotopic** to η if there exists a continuous function

$$\psi: [a, b] \times [c, d] \to U$$

defined on a rectangle $[a, b] \times [c, d]$, such that

$$\psi(t, c) = \gamma(t) \quad \text{and} \quad \psi(t, d) = \eta(t)$$

for all $t \in [a, b]$.

For each number s in the interval $[c, d]$, we may view the function ψ_s such that

$$\psi_s(t) = \psi(t, s)$$

as a continuous curve, defined on $[a, b]$, and we may view the family of continuous curves ψ_s as a deformation of the path γ to the path η. The picture is drawn on Fig. 16. The paths have been drawn with the same end points because that's what we are going to use in practice. Formally, we say that the homotopy ψ **leaves the end points fixed** if we have

$$\psi(a, s) = \gamma(a) \quad \text{and} \quad \psi(b, s) = \gamma(b)$$

for all values of s in $[c, d]$. *In the sequel it will be always understood that when we speak of a homotopy of paths having the same end points, then the homotopy leaves the end points fixed.*

Similarly, when we speak of a homotopy of closed paths, **we assume always that each path ψ_s is a closed path.** These additional requirements are now regarded as part of the definition of homotopy and will not be repeated each time.

Theorem 5.1. *Let γ, η be paths in an open set U having the same beginning point and the same end point. Assume that they are homotopic in U. Let f be holomorphic on U. Then*

$$\int_\gamma f = \int_\eta f.$$

Theorem 5.2. *Let γ, η be closed paths in U, and assume that they are*

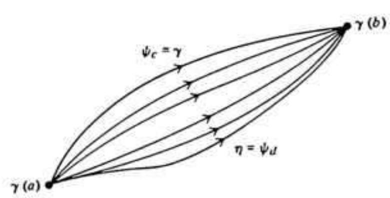

Figure 16

homotopic in U. Let f be holomorphic on U. Then

$$\int_\gamma f = \int_\eta f.$$

In particular, if γ is homotopic to a point in U, then

$$\int_\gamma f = 0.$$

Either of these statements may be viewed as a form of Cauchy's theorem. We prove Theorem 5.2 in detail, and leave Theorem 5.1 to the reader; the proof is entirely similar using Lemma 4.3 instead of Lemma 4.4 from the preceding section. The idea is that the homotopy gives us a finite sequence of paths close to each other in the sense of these lemmas, so that the integral of f over each successive path is unchanged.

The formal proof runs as follows. Let

$$\psi\colon [a, b] \times [c, d] \to U$$

be the homotopy. The image of ψ is compact, and hence has distance > 0 from the complement of U. By uniform continuity we can therefore find partitions

$$a = a_0 \leqq a_1 \leqq \cdots \leqq a_n = b,$$
$$c = c_0 \leqq c_1 \leqq \cdots \leqq c_m = d$$

of these intervals, such that if

$$S_{ij} = \text{small rectangle } [a_i, a_{i+1}] \times [c_j, c_{j+1}]$$

then the image $\psi(S_{ij})$ is contained in a disc D_{ij} which is itself contained in U. Let ψ_j be the continuous curve defined by

$$\psi_j(t) = \psi(t, c_j), \qquad j = 0, \ldots, m.$$

Then the continuous curves ψ_j, ψ_{j+1} are close together, and we can apply the lemma of the preceding section to conclude that

$$\int_{\psi_j} f = \int_{\psi_{j+1}} f.$$

Since $\psi_0 = \gamma$ and $\psi_m = \eta$, we see that the theorem is proved.

Remark. It is usually not difficult, although sometimes it is tedious, to exhibit a homotopy between continuous curves. Most of the time, one can achieve this homotopy by simple formulas when the curves are given explicitly.

Example. Let z, w be two points in the complex numbers. The segment between z, w, denoted by $[z, w]$, is the set of points

$$z + t(w - z), \qquad 0 \leqq t \leqq 1,$$

or equivalently,

$$(1 - t)z + tw, \qquad 0 \leqq t \leqq 1.$$

A set S of complex numbers is called **convex**, if, whenever z, $w \in S$, then the segment $[z, w]$ is also contained in S. We observe that a disc and a rectangle are convex.

Lemma 5.3. *Let S be a convex set, and let γ, η be continuous closed curves in S. Then γ, η are homotopic in S.*

Proof. We define

$$\psi(t, s) = s\gamma(t) + (1 - s)\eta(t).$$

It is immediately verified that each curve ψ_s defined by $\psi_s(t) = \psi(t, s)$ is a closed curve, and ψ is continuous. Also

$$\psi(t, 0) = \eta(t) \qquad \text{and} \qquad \psi(t, 1) = \gamma(t),$$

so the curves are homotopic. Note that the homotopy is given by a linear function, so if γ, η are smooth curves, that is C^1 curves, then each curve ψ_s is also of class C^1.

We say that an open set U is **simply connected** if it is connected and if every closed path in U is homotopic to a point. By Lemma 5.3, a convex open set is simply connected. Other examples of simply connected open sets will be given in the exercises. Simply connected open sets will be used in an essential way in the next section.

Remark. The technique used in this section, propagating along curves, will again be used in the theory of analytic continuation in Chapter XI, §1, which actually could be read immediately as a continuation of this section.

III, §5. EXERCISES

1. A set S is called **star-shaped** if there exists a point z_0 in S such that the line segment between z_0 and any point z in S is contained in S. Prove that a star-shaped set is simply connected, that is, every closed path is homotopic to a point.

2. Let U be the open set obtained from $\mathbf{C}$ by deleting the set of real numbers $\geqq 0$. Prove that U is simply connected.

3. Let V be the open set obtained from $\mathbf{C}$ by deleting the set of real numbers $\leqq 0$. Prove that V is simply connected.

4. (a) Let U be a simply connected open set and let f be an analytic function on U. Is $f(U)$ simply connected?
   (b) Let H be the upper half-plane, that is, the set of complex numbers $z = x + iy$ such that $y > 0$. Let $f(z) = e^{2\pi i z}$. What is the image $f(H)$? Is $f(H)$ simply connected?

III, §6. EXISTENCE OF GLOBAL PRIMITIVES. DEFINITION OF THE LOGARITHM

In §3 we constructed locally a primitive for a holomorphic function by integrating. We now have the means of constructing primitives for a much wider class of open sets.

Theorem 6.1. *Let f be holomorphic on a simply connected open set U. Let $z_0 \in U$. For any point $z \in U$ the integral*

$$g(z) = \int_{z_0}^{z} f(\zeta)\, d\zeta$$

is independent of the path in U from z_0 to z, and g is a primitive for f, namely $g'(z) = f(z)$.

Proof. Let γ_1, γ_2 be two paths in U from z_0 to z. Let γ_2^- be the reverse path of γ_2, from z to z_0. Then

$$\gamma = \{\gamma_1, \gamma_2^-\}$$

is a closed path, and by the first form of Cauchy's theorem,

$$\int_{\gamma_1} f + \int_{\gamma_2^-} f = \int_{\gamma} f = 0.$$

Since the integral of f over γ_2^- is the negative of the integral of f over γ_2, we have proved the first assertion.

As to the second, to prove the differentiability of g at a point z_1, if z is near z_1, then we may select a path from z_0 to z by passing through z_1, that is

$$g(z) = g(z_1) + \int_{z_1}^{z} f,$$

and we have already seen that this latter integral defines a local primitive for f in a neighborhood of z_1. Hence

$$g'(z) = f(z),$$

as desired.

Example. Let U be the plane from which a ray starting from the origin has been deleted. Then U is simply connected.

Proof. Let γ be any closed path in U. For simplicity, suppose the ray is the negative x-axis, as on Fig. 17. Then the path may be described in terms of polar coordinates,

$$\gamma(t) = r(t)e^{i\theta(t)}, \qquad a \leqq t \leqq b,$$

with $-\pi < \theta(t) < \pi$. We define the homotopy by

$$\psi(t, u) = r\big(ua + (1 - u)t\big)e^{i\theta(t)(1-u)}, \qquad 0 \leqq u \leqq 1.$$

Geometrically, we are folding back the angle towards 0, and we are contracting the distance $r(t)$ towards $r(a)$. It is clear that ψ has the desired property.

Remark. You could also note that the open set U is star-shaped (proof?), and so if you did Exercise 1 of §5, you don't need the above argument to show that U is simply connected.

Example (Definition of the Logarithm). Let U be a simply connected open set not containing 0. Pick a point $z_0 \in U$. Let w_0 be a complex

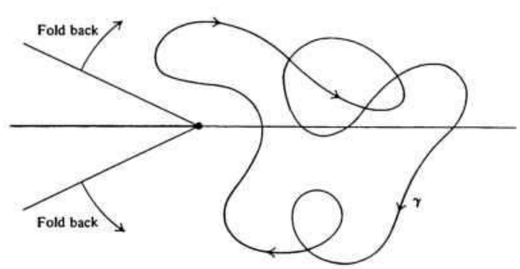

Figure 17

number such that

$$e^{w_0} = z_0.$$

(Any two such numbers differ by an integral multiple of $2\pi i$.) Define

$$\log z = w_0 + \int_{z_0}^{z} \frac{1}{\zeta} d\zeta.$$

Then $\log z$ (which depends on the choice of z_0 and w_0 only) is a primitive for $1/z$ on U, and any other primitive differs from this one by a constant.

Let $L_0(1 + w) = \sum (-1)^{n-1} w^n/n$ be the usual power series for the log in a neighborhood of 1. If z is near z_0, then the function

$$F(z) = w_0 + L_0(1 + (z - z_0)/z_0)$$

defines an analytic function. By Exercise 6 of Chapter II, §5, we have $F'(z) = 1/z$. Hence there exists a constant K such that for all z near z_0 we have $\log z = F(z) + K$. Since both $\log z_0 = w_0$ and $F(z_0) = w_0$, it follows that $K = 0$, so

$$\log z = F(z) \qquad \text{for } z \text{ near } z_0.$$

Consequently, by Exercise 1 of Chapter II, §3, we find that

$$e^{\log z} = z \qquad \text{for } z \text{ near } z_0.$$

Furthermore, given $z_1 \in U$, we have

$$\int_{z_0}^{z} = \int_{z_0}^{z_1} + \int_{z_1}^{z},$$

so by a similar argument, we see that $\log z$ is analytic on U. The two analytic functions $e^{\log z}$ and z are equal near z_0. Since U is connected, they are equal on U by Theorem 1.2(ii), and the equation $e^{\log z} = z$ remains valid for all $z \in U$.

If $L(z)$ is a primitive for $1/z$ on U such that $e^{L(z)} = z$, then there exists an integer k such that

$$L(z) = \log z + 2\pi i k.$$

Indeed, if we let $g(z) = L(z) - \log z$, then $e^{g(z)} = 1$, so $g(z) = 2\pi i k$ for some integer k.

Example. Let V be the open set obtained by deleting the negative real axis from $\mathbf{C}$, and write a complex number $z \in V$ in the form

$$z = re^{i\theta} \quad \text{with} \quad -\pi < \theta < \pi.$$

We can select some $z_0 \in V$ with

$$z_0 = r_0 e^{i\theta_0}.$$

For a positive real number r we let $\log r$ be the usual real logarithm, and we let

$$\log z_0 = \log r_0 + i\theta_0.$$

Then V is simply connected, and for all $z \in V$ we have

$$\log z = \log r + i\theta \quad \text{with} \quad -\pi < \theta < \pi.$$

For a numerical example, we have

$$1 - i = re^{i\theta} = \sqrt{2}e^{i(-\pi/4)},$$

so

$$\log(1 - i) = \tfrac{1}{2}\log 2 - \frac{i\pi}{4}.$$

Example. On the other hand, let U be the open set obtained by deleting the positive real axis from $\mathbf{C}$, i.e. $U = \mathbf{C} - \mathbf{R}_{\geq 0}$. Take $0 < \theta < 2\pi$. For this determination of the logarithm, let us find $\log(1 - i)$. We write

$$1 - i = re^{i\theta} = \sqrt{2}e^{i7\pi/4}.$$

Then

$$\log(1 - i) = \tfrac{1}{2}\log 2 + \frac{i7\pi}{4}.$$

We see concretely how the values of the logarithm depend on the choice of open set and the choice of a range for the angle.

Definition of z^α for any complex α. By using the logarithm, we can define z under the following conditions.

Let U be simply connected not containing 0. Let α be a complex number $\neq 0$. Fix a determination of the log on U. With respect to this determination, we define

$$z^\alpha = e^{\alpha \log z}.$$

Then z^α is analytic on U.

Example. Let U be the open set obtained by deleting the positive real axis from the complex plane. We define the log to have the values

$$\log re^{i\theta} = \log r + i\theta,$$

where $0 < \theta < 2\pi$. This is also called a **principal value** for the log in that open set. Then

$$\log i = i\pi/2 \quad \text{and} \quad \log(-i) = 3\pi i/2.$$

In this case,

$$i^i = e^{i \log i} = e^{\pi i^2/2} = e^{-\pi/2}.$$

Definition of $\log f(z)$. Let U be a simply connected open set and let f be an analytic function on U such that $f(z) \neq 0$ for all $z \in U$. We want to define $\log f(z)$. If we had this logarithm, obeying the same formalism as in ordinary calculus, then we should have

$$\frac{d}{dz} \log f(z) = \frac{1}{f(z)} f'(z) = \frac{f'(z)}{f(z)}.$$

Conversely, this suggests the correct definition. Select a point $z_0 \in U$. Let w_0 be a complex number such that $\exp(w_0) = f(z_0)$. Since f is assumed to be without zeros on U, the function f'/f is analytic on U. Therefore we can define an analytic function L_f on U by the integral

$$L_f(z) = w_0 + \int_{z_0}^{z} \frac{f'(\zeta)}{f(\zeta)} \, d\zeta.$$

The function L_f depends on the choice of z_0 and w_0, and we shall determine the extent of this dependence in a moment. The integral can be taken along any path in U from z_0 to z because U is assumed to be simply connected. From the definition, we get the derivative

$$L_f'(z) = f'(z)/f(z).$$

This derivative is independent of the choice of z_0 and w_0, so choosing a different z_0 and w_0 changes L_f at most by an additive constant which we shall prove is an integral multiple of $2\pi i$. We claim that

$$\exp L_f(z) = f(z).$$

To prove this formula, abbreviate $L_f(z)$ by $L(z)$, and differentiate

$e^{-L(z)}f(z)$. We find:

$$\begin{aligned}\frac{d}{dz}e^{-L(z)}f(z) &= e^{-L(z)}(-L'(z))f(z) + e^{-L(z)}f'(z)\\ &= e^{-L(z)}\left(-\frac{f'(z)}{f(z)}f(z) + f'(z)\right)\\ &= 0.\end{aligned}$$

Therefore $e^{-L(z)}f(z)$ is constant on U since U is connected. By definition

$$e^{-L(z_0)}f(z_0) = e^{-w_0}f(z_0) = 1,$$

so the constant is 1, and we have proved that $\exp L_f(z) = f(z)$ for $z \in U$.

If we change the choice of z_0 and w_0 such that $e^{w_0} = f(z_0)$, then the new value for $L_f(z)$ which we obtain is simply

$$\log f(z) + 2\pi ik \quad \text{for some integer } k,$$

because the exponential of both values gives $f(z)$.

Remark. The integral for $\log f(z)$ which we wrote down cannot be written in the form

$$\int_{f(z_0)}^{f(z)} \frac{1}{\zeta}\,d\zeta,$$

because even though U is simply connected, the image $f(U)$ may not be simply connected, as you can see in Exercise 7. Of course, if $\gamma: [a, b] \to U$ is a path from z_0 to z, then we may form the composite path $f \circ \gamma: [a, b] \to \mathbf{C}$. Then we could take the integral

$$\int_{f(z_0),\, f\circ\gamma}^{f(z)} \frac{1}{\zeta}\,d\zeta$$

along the path $f \circ \gamma$. In this case, by the chain rule,

$$\begin{aligned}\int_{f(z_0),\, f\circ\gamma}^{f(z)} \frac{1}{\zeta}\,d\zeta &= \int_{f(\gamma(a)),\, f\circ\gamma}^{f(\gamma(b))} \frac{1}{\zeta}\,d\zeta\\ &= \int_a^b \frac{1}{f(\gamma(t))} f'(\gamma(t))\gamma'(t)\,dt\\ &= \int_{z_0,\gamma}^{z} \frac{f'(\zeta)}{f(\zeta)}\,d\zeta,\end{aligned}$$

which is the integral that was used to define $L_f(z)$.

III, §6. EXERCISES

1. Compute the following values when the log is defined by its principal value on the open set U equal to the plane with the positive real axis deleted.
 (a) $\log i$ (b) $\log(-i)$ (c) $\log(-1+i)$
 (d) i^i (e) $(-i)^i$ (f) $(-1)^i$
 (g) $(-1)^{-i}$ (h) $\log(-1-i)$

2. Compute the values of the same expressions as in Exercise 1 (except (f) and (g)) when the open set consists of the plane from which the negative real axis has been deleted. Then take $-\pi < \theta < \pi$.

3. Let U be the plane with the negative real axis deleted. Let $y > 0$. Find the limit
$$\lim_{y \to 0} [\log(a + iy) - \log(a - iy)]$$
where $a > 0$, and also where $a < 0$.

4. Let U be the plane with the positive real axis deleted. Find the limit
$$\lim_{y \to 0} [\log(a + iy) - \log(a - iy)]$$
where $a < 0$, and also where $a > 0$.

5. Over what kind of open sets could you define an analytic function $z^{1/3}$, or more generally $z^{1/n}$ for any positive integer n? Give examples, taking the open set to be as "large" as possible.

6. Let U be a simply connected open set. Let f be analytic on U and assume that $f(z) \neq 0$ for all $z \in U$. Show that there exists an analytic function g on U such that $g^2 = f$. Does this last assertion remain true if 2 is replaced by an arbitrary positive integer n?

7. Let U be the upper half plane, consisting of all complex numbers $z = x + iy$ with $y > 0$. Let $\varphi(z) = e^{2\pi i z}$. Prove that $\varphi(U)$ is the open unit disc from the origin has been deleted.

8. Let U be the open set obtained by deleting 0 and the negative real axis from the complex numbers. For an integer $m \geqq 1$ define
$$L_{-m}(z) = \left(\log z - \left(1 + \frac{1}{2} + \cdots + \frac{1}{m}\right)\right)\frac{z^m}{m!}.$$
Show that $L'_{-m}(z) = L_{-m+1}(z)$, and that $L'_{-1}(z) = \log z$. Thus L_{-m} is an m-fold integral of the logarithm.

III, §7. THE LOCAL CAUCHY FORMULA

We shall next give an application of the homotopy Theorem 5.2 to prove that a holomorphic function is analytic. The property of being analytic is local: it means that a function has a power series expansion at every point (absolutely convergent on a disc of positive radius centered at the point).

Theorem 7.1 (Local Cauchy Formula). *Let $\bar{D}$ be a closed disc of positive radius, and let f be holomorphic on $\bar{D}$ (that is, on an open disc U containing $\bar{D}$). Let γ be the circle which is the boundary of $\bar{D}$. Then for every $z_0 \in D$ we have*

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz.$$

Proof. Let C_r be the circle of radius r centered at z_0, as illustrated on Fig. 18.

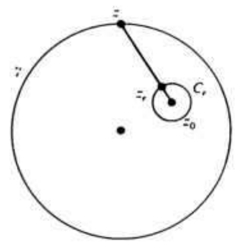

Figure 18

Then for small r, γ and C_r are homotopic. The idea for constructing the homotopy is to shrink γ toward C_r along the rays emanating from z_0. The formula can easily be given. Let z_r be the point of intersection of a line through z and z_0 with the circle of radius r, as shown on Fig. 18. Then

$$z_r = z_0 + r\frac{z - z_0}{|z - z_0|}.$$

Let $\gamma(t)$ $(0 \leqq t \leqq 2\pi)$ parametrize the circle γ. Substituting $\gamma(t)$ for z we obtain

$$\gamma(t)_r = z_0 + r\frac{\gamma(t) - z_0}{|\gamma(t) - z_0|}.$$

Now define the homotopy by letting

$$h(t, u) = u\gamma(t)_r + (1 - u)\gamma(t) \qquad \text{for} \quad 0 \leqq u \leqq 1.$$

Let U be an open disc containing $\bar{D}$, and let U_0 be the open set obtained by removing z_0 from U. Then $h(t, u) \in U_0$, that is, z_0 does not lie in the image of h, because the segment between z and the point z_r lies entirely outside the open disc of radius r centered at z_0. Thus γ is homotopic to C_r in U_0.

Let

$$g(z) = \frac{f(z) - f(z_0)}{z - z_0}$$

for $z \in D$ and $z \neq z_0$. Then g is holomorphic on the open set U_0. By Theorem 5.2, we get

$$\int_\gamma g(z)\,dz = \int_{C_r} g(z)\,dz.$$

Since f is differentiable at z_0, it follows that g is bounded in a neighborhood of z_0. Let B be a bound, so let $|g(z)| \leqq B$ for all z sufficiently close to z_0. Then for r sufficiently small we get

$$\left| \int_{C_r} g(z)\,dz \right| \leqq B(\text{length of } C_r) = B2\pi r,$$

and the right side approaches 0 as r approaches 0. Hence we conclude that

$$\int_\gamma g(z)\,dz = 0.$$

But then

$$\begin{aligned} \int_\gamma \frac{f(z)}{z - z_0}\,dz = \int_\gamma \frac{f(z_0)}{z - z_0}\,dz &= f(z_0) \int_\gamma \frac{1}{z - z_0}\,dz. \\ &= f(z_0) \int_{C_r} \frac{1}{z - z_0}\,dz \\ &= f(z_0)2\pi i. \end{aligned}$$

This proves the theorem.

Theorem 7.2. *Let f be holomorphic on an open set U. Then f is analytic on U.*

Proof. We must show that f has a power series expansion at every point z_0 of U. Because U is open, for each $z_0 \in U$ there is some $R > 0$ such that the closed disc $\bar{D}(z_0, R)$ centered at z_0 and of radius R is contained in U. We are therefore reduced to proving the following theorem, which will give us even more information concerning the power series expansion of f at z_0.

Theorem 7.3. *Let f be holomorphic on a closed disc $\bar{D}(z_0, R)$, $R > 0$. Let C_R be the circle bounding the disc. Then f has a power series expansion*

$$f(z) = \sum a_n(z - z_0)^n$$

whose coefficients a_n are given by the formula:

$$a_n = \frac{1}{n!} f^{(n)}(z_0) = \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta.$$

Furthermore, if $\|f\|_R$ denotes the sup norm *of f on the circle C_R, then we have the estimate*

$$|a_n| \leqq \|f\|_R / R^n.$$

In particular, the radius of convergence of the series is $\geqq R$.

Proof. By Theorem 7.1, for all z inside the circle C_R, we have

$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

Let $0 < s < R$. Let $D(z_0, s)$ be the disc of radius s centered at z_0. We shall see that f has a power series expansion on this disc. We write

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{1}{\zeta - z_0} \left(\frac{1}{1 - \dfrac{z - z_0}{\zeta - z_0}} \right)$$

$$= \frac{1}{\zeta - z_0} \left(1 + \frac{z - z_0}{\zeta - z_0} + \left(\frac{z - z_0}{\zeta - z_0} \right)^2 + \cdots \right).$$

This geometric series converges absolutely and uniformly for $|z - z_0| \leqq s$ because

$$\left| \frac{z - z_0}{\zeta - z_0} \right| \leqq s/R < 1.$$

The function f is bounded on γ. By Theorem 2.4 of Chapter III, we can therefore integrate term by term, and we find

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta \cdot (z - z_0)^n$$

$$= \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,d\zeta.$$

This proves that f is analytic, and gives us the coefficients of its power series expansion on the disc of radius R.

In particular, we now see that **a function is analytic if and only if it is holomorphic**. The two words will be used interchangeably from now on.

There remains only to estimate the integral to get an estimate for the coefficients. The estimate is taken as usual, equal to the product of the sup norm of the expression under the integral sign, and the length of the curve which is $2\pi R$. For all ζ on the circle, we have

$$|\zeta - z_0| = R,$$

so the desired estimate falls out. Taking the n-th root of $|a_n|$, we conclude at once that the radius of convergence is at least R.

Remark. From the statement about the radius of convergence in Theorem 7.3 we now see that if R is the radius of convergence of a power series, then its analytic function does not extend to a disc of radius $> R$; otherwise the given power series would have a larger radius of convergence, and would represent this analytic function on the bigger disc. For example, let

$$f(z) = e^z/(z - 1).$$

Then f is analytic except at $z = 1$. From the theorem, we conclude:

The radius of convergence of the power series for f at the origin is 1.

The radius of convergence of the power series for f at 2 is 1.

The radius of convergence of the power series for f at 5 is 4.

The radius of convergence of the power series for f at -3 is 4.

A function f is called **entire** if it is holomorphic on all of **C**. We also conclude from the above remark and the theorem that if a function is entire, then its power series converges for all $z \in \mathbf{C}$, in other words the radius of convergence is ∞.

Corollary 7.4. *Let f be an entire function, and let $\|f\|_R$ be its* sup norm *on the circle of radius R. Suppose that there exists a constant C and a positive integer k such that*

$$\|f\|_R \leqq CR^k$$

for arbitrarily large R. Then f is a polynomial of degree $\leqq k$.

Proof. Exercise 3, but we carry out one important special case explicitly:

Theorem 7.5 (Liouville's Theorem). *A bounded entire function is constant.*

Proof. If f is bounded, then $\|f\|_R$ is bounded for all R. In the preceding theorem, we let R tend to infinity, and conclude that the coefficients are all equal to 0 if $n \geqq 1$. This proves Liouville's theorem.

We have already proved that a polynomial always has a root in the complex numbers. We give here the more usual proof as a corollary of Liouville's theorem.

Corollary 7.6. *A polynomial over the complex numbers which does not have a root in* $\mathbf{C}$ *is constant.*

Proof. Let $f(z)$ be a non-constant polynomial,

$$f(z) = a_n z^n + \cdots + a_0,$$

with $a_n \neq 0$. Suppose that $f(z) \neq 0$ for all z. Then the function

$$g(z) = 1/f(z)$$

is defined for all z and analytic on $\mathbf{C}$. On the other hand, writing

$$f(z) = a_n z^n(1 + b_1/z + \cdots + b_n/z^n)$$

with appropriate constants $b_1, \ldots, b_n$ we see that $|f(z)|$ is large when $|z|$ is large, and hence that $|g(z)| \to 0$ as $|z| \to \infty$. For sufficiently large radius R, $|g(z)|$ is small for z outside the closed disc of radius R, and $|g(z)|$ has a maximum on this disc since the disc is compact. Hence g is a bounded entire function, and therefore constant by Liouville's theorem. This is obviously a contradiction, proving that f must have a zero somewhere in $\mathbf{C}$.

We end this section by pointing out that the main argument of Theorem 7.3 can be used essentially unchanged to define an analytic function and its derivatives by means of an integral, as follows.

Theorem 7.7. *Let γ be a path in an open set U and let g be a continuous function on γ (i.e. on the image $\gamma([a, b])$ if γ is defined on $[a, b]$). If z is not on γ, define*

$$f(z) = \int_\gamma \frac{g(\zeta)}{\zeta - z}\, d\zeta.$$

Then f is analytic on the complement of γ in U, and its derivatives are given by

$$f^{(n)}(z) = n! \int_\gamma \frac{g(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta.$$

Proof. Let $z_0 \in U$ and z_0 not on γ. Since the image of γ is compact, there is a minimum distance between z_0 and points on γ. Select $0 < R < \text{dist}(z_0, \gamma)$, and take R also small enough that the closed disc $\bar{D}(z_0, R)$ is contained in U. Now we are essentially in the situation of Theorem 7.3. We may repeat the arguments of the proof. We select $0 < s < R$, and we simply replace f by g inside the integral sign. We expand $1/(\zeta - z)$ by means of the geometric series, and proceed without any further change to see that f has a power series expansion $f = \sum a_n(z - z_0)^n$, where now the coefficients a_n are given by

$$a_n = \int_\gamma \frac{g(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta.$$

We know from Chapter II, §5 that $a_n = f^{(n)}(z_0)/n!$, which gives us the proof of Theorem 7.7.

There is also another way of looking at Theorem 7.7. Indeed, from the formula for f, it is natural to think that one can differentiate with respect to z under the integral sign. This differentiation will be justified in Theorem A3, §6, Chapter VIII, which the reader may wish to look at now. Then one gets the integral formula also for the derivatives.

From Theorem 7.7 we obtain a bound for the derivative of an analytic function in terms of the function itself. This is of course completely different from what happens for real differentiable functions.

Corollary 7.8. *Let f be analytic on a closed disc $\bar{D}(z_0, R)$, $R > 0$. Let $0 < R_1 < R$. Denote by $\|f\|_R$ the sup norm of f on the circle of radius R. Then for $z \in \bar{D}(z_0, R_1)$ we have*

$$|f^{(n)}(z)| \leqq \frac{n!\,R}{(R - R_1)^{n+1}} \|f\|_R.$$

Proof. This is immediate by using Theorem 7.1, and putting $g = f$ inside the integral, with a factor of $1/2\pi i$ in front. The factor R in the numerator comes from the length of the circle in the integral. The 2π in the denominator cancels the 2π in the numerator, coming from the formula for the length of the circle.

Note that if R_1 is close to R, then the denominator may be correspondingly large. On the other hand, suppose $R_1 = R/2$. Then the estimate reads

$$|f^{(n)}(z)| \leqq \frac{n!\,2^{n+1}}{R^n}\|f\|_R,$$

which is thus entirely in terms of f, n, and R.

Finally we return to reconsider Theorem 3.2 in light of the fact that a holomorphic function is analytic.

Theorem 7.9 (Morera's Theorem). *Let U be an open set in $\mathbf{C}$ and let f be continuous on U. Assume that the integral of f along the boundary of every closed rectangle contained in U is 0. Then f is analytic.*

Proof. By Theorem 3.2, we know that f has a local primitive g at every point on U, and hence that g is holomorphic. By Theorem 7.2, we conclude that g is analytic, and hence that $g' = f$ is analytic, as was to be shown.

We have now come to the end of a chain of ideas linking complex differentiability and power series expansions. The next two chapters treat different applications, and can be read in any order, but we have to project the book in a totally ordered way on the page axis, so we have to choose an order for them. The next chapter will study more systematically a global version of Cauchy's formula and winding numbers, which amounts to studying the relation between an integral and the winding number which we already encountered in some way via the logarithm. After that in Chapter V, we return to analytic considerations and estimates.

III, §7. EXERCISES

1. Find the integrals over the unit circle γ:

(a) $\displaystyle\int_\gamma \frac{\cos z}{z}\,dz$ (b) $\displaystyle\int_\gamma \frac{\sin z}{z}\,dz$ (c) $\displaystyle\int_\gamma \frac{\cos(z^2)}{z}\,dz$

2. Write out completely the proof of Theorem 7.6 to see that all the steps in the proof of Theorem 7.3 apply.

3. Prove Corollary 7.4.

CHAPTER IV

Winding Numbers and Cauchy's Theorem

We wish to give a general global criterion when the integral of a holomorphic function along a closed path is 0. In practice, we meet two types of properties of paths: (1) properties of homotopy, and (2) properties having to do with integration, relating to the number of times a curve "winds" around a point, as we already saw when we evaluated the integral

$$\int \frac{1}{\zeta - z}\, d\zeta$$

along a circle centered at z. These properties are of course related, but they also exist independently of each other, so we now consider those conditions on a closed path γ when

$$\int_\gamma f = 0$$

for all holomorphic functions f, and also describe what the value of this integral may be if not 0.

We shall give two proofs for the global version of Cauchy's theorem. Artin's proof depends only on Goursat's theorem for the integral of a holomorphic function around a rectangle, and a self-contained topological lemma, having only to do with paths and not holomorphic functions. Dixon's proof uses some of the applications to holomorphic functions which bypass the topological considerations.

In this chapter, paths are again assumed to be piecewise C^1, and curves are again C^1.

IV, §1. THE WINDING NUMBER

In an example of Chapter III, §2, we found that

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z}\,dz = 1,$$

if γ is a circle around the origin, oriented counterclockwise. It is therefore reasonable to define for any closed path γ its **winding number with respect to a point** α to be

$$W(\gamma, \alpha) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-\alpha}\,dz,$$

provided the path does not pass through α. If γ is a curve defined on an interval $[a, b]$, then this integral can be written in the form

$$\int_\gamma \frac{1}{z-\alpha}\,dz = \int_a^b \frac{\gamma'(t)}{\gamma(t)-\alpha}\,dt.$$

Intuitively, the integral of $1/(z-\alpha)$ should be called $\log(z-\alpha)$, but it depends on the path. Later, we shall analyze this situation more closely, but for the moment, we need only the definition above without dealing with the log formally, although the interpretation in terms of the log is suggestive.

The definition of the winding number would be improper if the following lemma were not true.

Lemma 1.1. *If γ is a closed path, then $W(\gamma, \alpha)$ is an integer.*

Proof. Let $\gamma = \{\gamma_1, \ldots, \gamma_n\}$ where each γ_i is a curve defined on an interval $[a_i, b_i]$. After a reparametrization of each curve if necessary, we may assume without loss of generality that $b_i = a_{i+1}$ for $i = 1, \ldots, n-1$. Then γ is defined and continuous on an interval $[a, b]$, where $a = a_1$, $b = b_n$, and γ is differentiable on each open interval $]a_i, b_i[$, (at the end points, γ is merely right and left differentiable). Let

$$F(t) = \int_a^t \frac{\gamma'(t)}{\gamma(t)-\alpha}\,dt.$$

Then F is continuous on $[a, b]$ and differentiable for $t \neq a_i, b_i$. Its derivative is

$$F'(t) = \frac{\gamma'(t)}{\gamma(t)-\alpha}.$$

(Intuitively, $F(t) = \log(\gamma(t) - \alpha)$ except for the dependence of path and a constant of integration, but this suggests our next step.) We compute the derivative of another function:

$$\frac{d}{dt} e^{-F(t)}(\gamma(t) - \alpha) = e^{-F(t)}\gamma'(t) - F'(t)e^{-F(t)}(\gamma(t) - \alpha) = 0.$$

Hence there is a constant C such that $e^{-F(t)}(\gamma(t) - \alpha) = C$, so

$$\gamma(t) - \alpha = Ce^{F(t)}.$$

Since γ is a closed path, we have $\gamma(a) = \gamma(b)$, and

$$Ce^{F(b)} = \gamma(b) - \alpha = \gamma(a) - \alpha = Ce^{F(a)}.$$

Since $\gamma(a) - \alpha \neq 0$ we conclude that $C \neq 0$, so that

$$e^{F(a)} = e^{F(b)}.$$

Hence there is an integer k such that

$$F(b) = F(a) + 2\pi ik.$$

But $F(a) = 0$, so $F(b) = 2\pi ik$, thereby proving the lemma.

The winding number of the curve in Fig. 1 with respect to α is equal to 2.

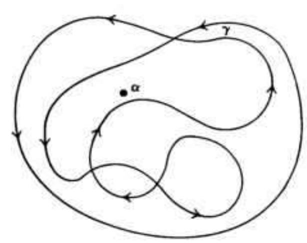

Figure 1

Lemma 1.2. *Let γ be a path. Then the function of α defined by*

$$\alpha \mapsto \int_\gamma \frac{1}{z - \alpha}\, dz$$

for α not on the path, is a continuous function of α.

Proof. Given α_0 not on the path, we have to see that

$$\int_\gamma \left(\frac{1}{z-\alpha} - \frac{1}{z-\alpha_0}\right) dz$$

tends to 0 as α tends to α_0. This integral is estimated as follows. The function $t \mapsto |\alpha_0 - \gamma(t)|$ is continuous and not 0, hence it has a minimum, the minimum distance between α_0 and the path, say

$$\min_t |\alpha_0 - \gamma(t)| = s.$$

If α is sufficiently close to α_0, then $|\alpha - \gamma(t)| \geqq s/2$, as illustrated in Fig. 2.

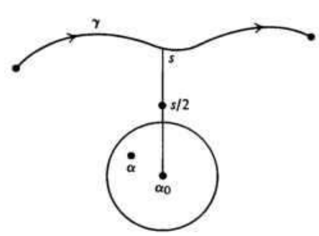

Figure 2

We have

$$\frac{1}{z-\alpha} - \frac{1}{z-\alpha_0} = \frac{\alpha-\alpha_0}{(z-\alpha)(z-\alpha_0)}$$

whence the estimate

$$\left|\frac{1}{z-\alpha} - \frac{1}{z-\alpha_0}\right| \leqq \frac{1}{s^2/4}|\alpha-\alpha_0|.$$

Consequently, we get

$$\left|\int_\gamma \left(\frac{1}{z-\alpha} - \frac{1}{z-\alpha_0}\right) dz\right| \leqq \frac{1}{s^2/4}|\alpha-\alpha_0| L(\gamma).$$

The right-hand side tends to 0 as α tends to α_0, and the continuity is proved.

Lemma 1.3. *Let γ be a closed path. Let S be a connected set not intersecting γ. Then the function*

$$\alpha \mapsto \frac{1}{2\pi i} \int_\gamma \frac{1}{z - \alpha}\, dz$$

is constant for α in S. If S is not bounded, then this constant is 0.

Proof. We know from Lemma 1.1 that the integral is the winding number, and is therefore an integer. If a function takes its values in the integers, and is continuous, then it is constant on any curve, and consequently constant on a connected set. If S is not bounded, then for α arbitrarily large, the integrand has arbitrarily small absolute value, that is,

$$\frac{1}{|z - \alpha|}$$

is arbitrarily small, and estimating the integral shows that it must be equal to 0, as desired.

Example. Let U be the open set in Fig. 3. Then the set of points not in U consists of two connected components, one inside U and the other unbounded. Let γ be the closed curve shown in the figure, and let α_1 be the point inside γ, whereas α_2 is the point outside U, in the unbounded connected region. Then

$$W(\gamma, \alpha_1) = 1, \qquad \text{but} \qquad W(\gamma, \alpha_2) = 0.$$

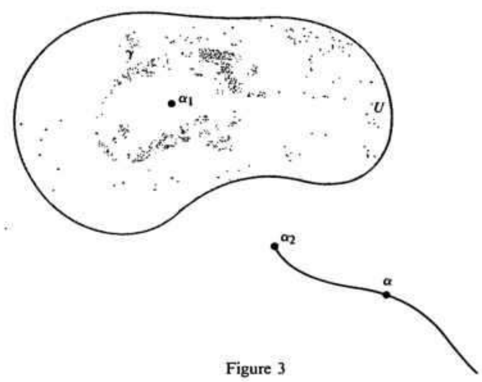

Figure 3

We have drawn a curve extending from α_2 towards infinity, such that $W(\gamma, \alpha) = 0$ for α on this curve, according to the argument of Lemma 1.3.

IV, §2. THE GLOBAL CAUCHY THEOREM

Let U be an open set. Let γ be a closed path in U. We want to give conditions that

$$\int_\gamma f = 0$$

for every holomorphic function f on U. We already know from the example of a winding circle that if the path winds around some point outside of U (in this example, the center of the circle), then definitely we can find functions whose integral is not equal to 0, and even with the special functions

$$f(z) = \frac{1}{z - \alpha},$$

where α is a point not in U. The remarkable fact about Cauchy's theorem is that it will tell us this is the only obstruction possible to having

$$\int_\gamma f = 0$$

for all possible functions f. In other words, the functions

$$\frac{1}{z - \alpha}, \qquad \alpha \notin U,$$

suffice to determine the behavior of $\int_\gamma f$ for all possible functions. With this in mind, we want to give a name to those closed paths in U having the property that they do not wind around points in the complement of U. The name we choose is homologous to 0, for historical reasons. Thus formally, we say that a closed path γ in U is **homologous to 0 in U** if

$$\int_\gamma \frac{1}{z - \alpha}\, dz = 0$$

for every point α not in U, or in other words, more briefly,

$$W(\gamma, \alpha) = 0$$

for every such point.

Similarly, let γ, η be closed paths in U. We say that they are **homologous in** U if

$$W(\gamma, \alpha) = W(\eta, \alpha)$$

for every point α in the complement of U. It will also follow from Cauchy's theorem that if γ and η are homologous, then

$$\int_\gamma f = \int_\eta f$$

for all holomorphic functions f on U.

Theorem 2.1.

(i) *If γ, η are closed paths in U and are homotopic, then they are homologous.*
(ii) *If γ, η are closed paths in U and are close together then they are homologous.*

Proof. The first statement follows from Theorem 5.2 of the preceding chapter because the function $1/(z - \alpha)$ is analytic on U for $\alpha \notin U$. The second statement is a special case of Lemma 4.4 of the preceding chapter.

Next we draw some examples of homologous paths.

In Fig. 4, the curves γ and η are **homologous**. Indeed, if α is a point inside the curves, then the winding number is 1, and if α is a point in the connected part going to infinity, then the winding number is 0.

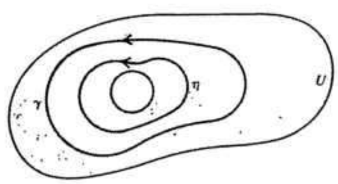

Figure 4

In Fig. 5 the path indicated is supposed to go around the top hole counterclockwise once, then around the bottom hole counterclockwise once, then around the top in the opposite direction, and then around the bottom in the opposite direction. This path is homologous to 0, but not homotopic to a point.

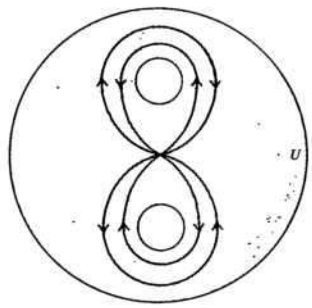

Figure 5

In Fig. 6, we are dealing with a simple closed curve, whose inside is contained in U, and the figure is intended to show that γ can be deformed to a point, so that γ is homologous to 0.

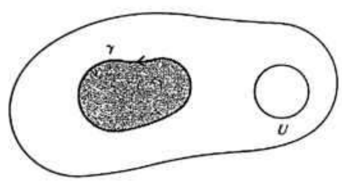

Figure 6

Given an open set U, we wish to determine in a simple way those closed paths which are not homologous to 0. For instance, the open set U might be as in Fig. 7, with three holes in it, at points z_1, z_2, z_3, so these points are assumed not to be in U.

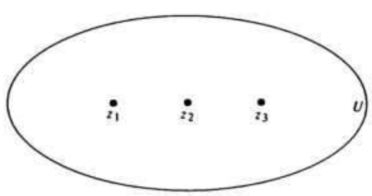

Figure 7

Let γ be a closed path in U, and let f be holomorphic on U. We illustrate γ in Fig. 8.

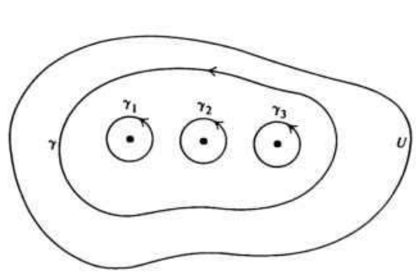

Figure 8

In that figure, we see that γ winds around the three points, and winds once. Let γ_1, γ_2, γ_3 be small circles centered at z_1, z_2, z_3 respectively, and oriented counterclockwise, as shown on Fig. 8. Then it is reasonable to expect that

$$\int_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f + \int_{\gamma_3} f.$$

This will in fact be proved after Cauchy's theorem. We observe that taking γ_1, γ_2, γ_3 together does not constitute a "path" in the sense we have used that word, because, for instance, they form a disconnected set. However, it is convenient to have a terminology for a formal sum like $\gamma_1 + \gamma_2 + \gamma_3$, and to give it a name η, so that we can write

$$\int_\gamma f = \int_\eta f.$$

The name that is standard is the name **chain**. Thus let, in general, $\gamma_1, \ldots, \gamma_n$ be curves, and let $m_1, \ldots, m_n$ be integers which need not be positive. A formal sum

$$\gamma = m_1\gamma_1 + \cdots + m_n\gamma_n = \sum_{i=1}^{n} m_i\gamma_i$$

will be called a **chain**. If each curve γ_i is a curve in an open set U, we call γ a **chain in** U. We say that the chain is **closed** if it is a finite sum of closed paths. If γ is a chain as above, we define

$$\int_\gamma f = \sum m_i \int_{\gamma_i} f.$$

If $\gamma = \sum m_i \gamma_i$ is a closed chain, where each γ_i is a closed path, then its winding number with respect to a point α not on the chain is defined as before,

$$W(\gamma, \alpha) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - \alpha}\, dz.$$

If γ, η are closed chains in U, then we have

$$W(\gamma + \eta, \alpha) = W(\gamma, \alpha) + W(\eta, \alpha).$$

We say that γ is **homologous to** η in U, and write $\gamma \sim \eta$, if

$$W(\gamma, \alpha) = W(\eta, \alpha)$$

for every point $\alpha \notin U$. We say that γ is **homologous to** 0 **in** U and write $\gamma \sim 0$ if

$$W(\gamma, \alpha) = 0$$

for every point $\alpha \notin U$.

Example. Let γ be the curve illustrated in Fig. 9, and let U be the plane from which three points z_1, z_2, z_3 have been deleted. Let γ_1, γ_2, γ_3 be small circles centered at z_1, z_2, z_3 respectively, oriented counterclockwise. Then it will be shown after Cauchy's theorem that

$$\gamma \sim \gamma_1 + 2\gamma_2 + 2\gamma_3,$$

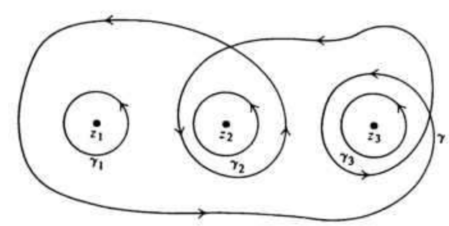

Figure 9

so that for any function f holomorphic on U, we have

$$\int_\gamma f = \int_{\gamma_1} f + 2\int_{\gamma_2} f + 2\int_{\gamma_3} f.$$

The above discussion and definition of chain provided motivation for what follows. We now go back to the formal development, and state the global version of Cauchy's theorem.

Theorem 2.2 (Cauchy's Theorem). *Let γ be a closed chain in an open set U, and assume that γ is homologous to 0 in U. Let f be holomorphic in U. Then*

$$\int_\gamma f = 0.$$

A proof will be given in the next section. Observe that all we shall need of the holomorphic property is the existence of a primitive locally at every point of U, which was proved in the preceding chapter.

Corollary 2.3. *If γ, η are closed chains in U and γ, η are homologous in U, then*

$$\int_\gamma f = \int_\eta f.$$

Proof. Apply Cauchy's theorem to the closed chain $\gamma - \eta$.

Before giving the proof of Cauchy's theorem, we state two important applications, showing how one reduces integrals along complicated paths to integrals over small circles.

Theorem 2.4.

(a) *Let U be an open set and γ a closed chain in U such that γ is homologous to 0 in U. Let $z_1, \ldots, z_n$ be a finite number of distinct points of U. Let γ_i $(i = 1, \ldots, n)$ be the boundary of a closed disc $\overline{D}_i$ contained in U, containing z_i, and oriented counterclockwise. We assume that $\overline{D}_i$ does not intersect $\overline{D}_j$ if $i \neq j$. Let*

$$m_i = W(\gamma, z_i).$$

Let U^* be the set obtained by deleting $z_1, \ldots, z_n$ from U. Then γ is homologous to $\sum m_i\gamma_i$ in U^*.

(b) *Let f be holomorphic on U^*. Then*

$$\int_\gamma f = \sum_{i=1}^{n} m_i \int_{\gamma_i} f.$$

Proof. Let $C = \gamma - \sum m_i\gamma_i$. Let α be a point outside U. Then

$$W(C, \alpha) = W(\gamma, \alpha) - \sum m_i W(\gamma_i, \alpha) = 0$$

because α is outside every small circle γ_i. If $\alpha = z_k$ for some k, then $W(\gamma_i, z_k) = 1$ if $i = k$ and 0 if $i \neq k$ by Lemma 1.3. Hence

$$W(C, z_k) = W(\gamma, z_k) - m_k = 0.$$

This proves that C is homologous to 0 in U^*. We apply Theorem 2.2 to conclude the proof.

The theorem is illustrated in Fig. 10. We have

$$\gamma \sim -\gamma_1 - 2\gamma_2 - \gamma_3 - 2\gamma_4,$$

and

$$\int_\gamma f = -\int_{\gamma_1} f - 2\int_{\gamma_2} f - \int_{\gamma_3} f - 2\int_{\gamma_4} f.$$

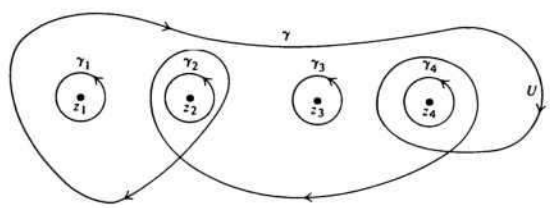

Figure 10

The theorem will be applied in many cases when U is a disc, say centered at the origin, and γ is a circle in U. Then certainly γ is homotopic to a point in U, and therefore homologous to 0 in U. Let $z_1, \ldots, z_n$ be points inside the circle, as on Fig. 11. Then Theorem 2.4 tells us that

$$\boxed{\int_\gamma f = \sum_{i=1}^{n} \int_{C_i} f,}$$

where C_i is a small circle around z_i. (Circles throughout are assumed oriented counterclockwise unless otherwise specified.)

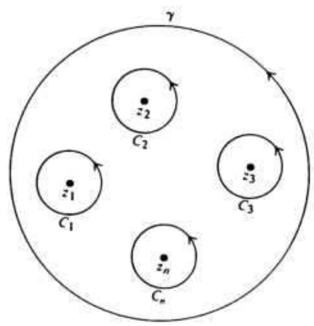

Figure 11

In Example 5 of Chapter III, §2, we gave explicitly the values of the integrals around small circles in terms of the power series expansion of f around the points $z_1, \ldots, z_n$. We may also state the global version of **Cauchy's formula**.

Theorem 2.5 (Cauchy's Formula). *Let γ be a closed chain in U, homologous to 0 in U. Let f be analytic on U, let z_0 be in U and not on γ. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\,dz = W(\gamma, z_0)f(z_0).$$

Proof. We base this proof on Theorems 2.2 and 2.4. An independent proof will be given below. By assumption, in a neighborhood of z_0, we have a power series expansion

$$f(z) = a_0 + a_1(z - z_0) + \text{higher terms}, \qquad \text{with } a_0 = f(z_0).$$

Let C_r be the circle of radius r centered at z_0 for a small value of r. By Theorem 2.4, the integral over γ can be replaced by the integral over C_r times the appropriate winding number, that is

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\,dz = W(\gamma, z_0)\frac{1}{2\pi i}\int_{C_r} \sum_{n=0}^{\infty} a_n(z - z_0)^{n-1}\,dz = W(\gamma, z_0)a_0,$$

because we can integrate term by term by Theorem 2.4 of Chapter III, and we can apply Example 5 or Exercise 1 of Chapter III, §2, to conclude the proof.

Example. Using Theorem 2.5, we find the integral

$$\int_\gamma \frac{e^z}{z}\,dz$$

taken over a path γ not passing through the origin, and having winding number 1 with respect to 0, that is, $W(\gamma, 0) = 1$. We let $U = \mathbf{C}$. Then γ is homologous to 0 in U, and in fact γ is homotopic to a point. Hence Theorem 2.5 applies by letting $z_0 = 0$, and we find

$$\int_\gamma \frac{e^z}{z}\,dz = 2\pi i e^0 = 2\pi i.$$

Remark 1. We have shown that Theorem 2.2 (Cauchy's theorem) implies Theorem 2.5 (Cauchy's formula). Conversely, it is easily seen that Cauchy's formula implies Cauchy's theorem. Namely, we let z_0 be a point in U not on γ, and we let

$$F(z) = (z - z_0)f(z).$$

Applying Cauchy's formula to F yields

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \frac{1}{2\pi i}\int_\gamma \frac{F(z)}{z - z_0}\,dz = F(z_0)W(\gamma, z_0) = 0,$$

as desired.

Remark 2. In older texts, Cauchy's theorem is usually stated for the integral over a simple closed curve, in the following form:

Let U be an open set, f holomorphic on U and let γ be a simple closed curve whose interior is contained in U. Then

$$\int_\gamma f = 0.$$

It was realized for a long time that it is rather hard to prove that a simple closed curve decomposes the plane into two regions, its interior and exterior. It is not even easy to define what is meant by "interior" or "exterior" a priori. In fact, the theorem would be that the plane from which one deletes the curve consists of two connected sets. For all points in one of the sets the winding number with respect to the curve is 1, and for all points in the other, the winding number is 0. In any case, these general results are irrelevant in the applications. Indeed, both in theoretical work and in practical applications, the statement of Cauchy's theorem as we gave it is quite efficient. In special cases, it is usually immediate to define the "interior" and "exterior" having the above property, for instance for circles or rectangles. One can apply Theorem 2.2 without appealing to any complicated result about general closed curves.

Dixon's Proof of Theorem 2.5 (Cauchy's Formula)

The proof we gave of Theorem 2.5 was based on Theorem 2.2 via Theorem 2.4. We shall now reproduce Dixon's proof of Theorem 2.5, which is direct, and is based only on Cauchy's formula for a circle and Liouville's theorem. Those results were proved in Chapter III, §7. Dixon's proof goes as follows.

We define a function g on $U \times U$ by:

$$g(z, w) = \begin{cases} \dfrac{f(w) - f(z)}{w - z} & \text{if } w \neq z, \\ f'(z) & \text{if } w = z. \end{cases}$$

For each w, the function $z \mapsto g(z, w)$ is analytic. Furthermore, g is continuous on $U \times U$. This is obvious for points off the diagonal, and if (z_0, z_0) is on the diagonal, then for (z, w) close to (z_0, z_0)

$$g(z, w) - g(z_0, z_0) = \frac{1}{w - z} \int_z^w [f'(\zeta) - f'(z_0)]\, d\zeta.$$

The integral can be taken along the line segment from z to w. Estimating the right-hand side, we see that $1/|w - z|$ cancels the length of the interval, and the expression under the integral sign tends to 0 by the continuity of f', as (z, w) approaches (z_0, z_0). Thus g is continuous.

Let V be the open set of complex numbers z not on γ such that $W(\gamma, z) = 0$. By the hypothesis of Cauchy's theorem, we know that V contains the complement of U. Hence $\mathbf{C} = U \cup V$. We now define a function h on $\mathbf{C}$ by two integrals:

$$h(z) = \frac{1}{2\pi i} \int_\gamma g(z, w)\, dw \quad \text{if} \quad z \in U,$$

$$h(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z}\, dw \quad \text{if} \quad z \in V.$$

We note that for $z \in U \cap V$, the two definitions of h coincide. We shall prove that h is a bounded entire function, whence constant by Liouville's theorem, whence equal to 0 by letting z tend to infinity for $z \in V$, and using the definition of h. It is then clear that for $z \in U$ the first integral being zero immediately implies Cauchy's formula

$$\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw = f(z)W(\gamma, z).$$

We have already seen in Remark 1 that Cauchy's formula implies Cauchy's theorem.

There remains therefore to prove that h is an analytic function and is bounded. We first prove that h is analytic. It is immediate that h is analytic on V. Hence it suffices to prove that h is analytic on U. So let $z_0 \in U$. From the uniform continuity of g on compact subsets of $U \times U$ it follows at once that h is continuous. To prove that h is analytic, by Theorem 3.2 of Chapter III, and the fact that a holomorphic function is analytic, it suffices to prove that in some disc centered at z_0, the integral of h around the boundary of any rectangle contained in the disc is 0. But we have

$$\begin{aligned}\int_{\partial R} h(z)\,dz &= \frac{1}{2\pi i}\int_{\partial R}\int_\gamma g(z, w)\,dw\,dz \\ &= \frac{1}{2\pi i}\int_\gamma\int_{\partial R} g(z, w)\,dz\,dw.\end{aligned}$$

Since for each w, the function $z \mapsto g(z, w)$ is analytic, we obtain the value 0, thereby concluding the proof that h is analytic.

As for the boundedness, suppose that z lies outside a large circle. Then

$$\int_\gamma \frac{f(z)}{w-z}\,dw = f(z)\int_\gamma \frac{1}{w-z}\,dw = 0$$

because the winding number of γ with respect to z is 0 by Lemma 1.3 of Chapter IV. Furthermore, if $|z| \to \infty$ then

$$\int_\gamma \frac{f(w)}{w-z}\,dw \to 0.$$

It follows that h is bounded outside a large circle, whence bounded since h is analytic. This concludes the proof.

IV, §2. EXERCISES

1. (a) Show that the association $f \mapsto f'/f$ (where f is holomorphic) sends products to sums.
   (b) If $P(z) = (z - a_1)\cdots(z - a_n)$, where $a_1, \ldots, a_n$ are the roots, what is P'/P?
   (c) Let γ be a closed path such that none of the roots of P lie on γ. Show that

$$\frac{1}{2\pi i}\int_\gamma (P'/P)(z)\,dz = W(\gamma, a_1) + \cdots + W(\gamma, a_n).$$

2. Let $f(z) = (z - z_0)^m h(z)$, where h is analytic on an open set U, and $h(z) \neq 0$ for all $z \in U$. Let γ be a closed path homologous to 0 in U, and such that z_0 does not lie on γ. Prove that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = W(\gamma, z_0)m.$$

3. Let U be a simply connected open set and let $z_1, \ldots, z_n$ be points of U. Let $U^* = U - \{z_1, \ldots, z_n\}$ be the set obtained from U by deleting the points $z_1, \ldots, z_n$. Let f be analytic on U^*. Let γ_k be a small circle centered at z_k and let

$$a_k = \frac{1}{2\pi i}\int_{\gamma_k} f(\zeta)\,d\zeta.$$

   Let $h(z) = f(z) - \sum a_k/(z - z_k)$. Prove that there exists an analytic function H on U^* such that $H' = h$.

Note. The train of thought of the above exercises will be pursued systematically in Chapter VI, Theorem 1.5.

IV, §3. ARTIN'S PROOF

In this section we prove Theorem 2.2 by making greater use of topological considerations. We reduce Theorem 2.2 to a theorem which involves only the winding number, and not the holomorphic function f, and we state this result as Theorem 3.2. The application to the holomorphic function will then be immediate by applying some results of Chapter III. We have already found that integrating along sides of a rectangle works better than over arbitrary curves. We pursue this idea. A path will be said to be **rectangular** if every curve of the path is either a horizontal segment or a vertical segment. We shall see that every path is homologous with a rectangular path, and in fact we prove:

Lemma 3.1. *Let γ be a path in an open set U. Then there exists a rectangular path η with the same end points, and such that γ, η are close together in U in the sense of Chapter* III, §4. *In particular, γ and η are* *homologous in U, and for any holomorphic function f on U we have*

$$\int_\gamma f = \int_\eta f.$$

Proof. Suppose γ is defined on an interval $[a, b]$. We take a partition of the interval,

$$a = a_0 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_n = b$$

such that the image of each small interval

$$\gamma([a_i, a_{i+1}])$$

is contained in a disc D_i on which f has a primitive. Then we replace the curve γ on the interval $[a_i, a_{i+1}]$ by the rectangular curve drawn on Fig. 12. This proves the lemma.

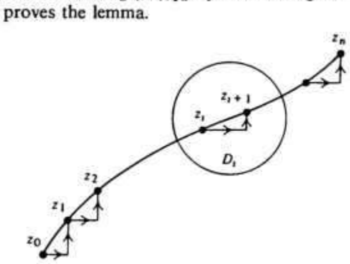

Figure 12

In the figure, we let $z_i = \gamma(a_i)$.

If γ is a closed path, then it is clear that the rectangular path constructed in the lemma is also a closed path, looking like this:

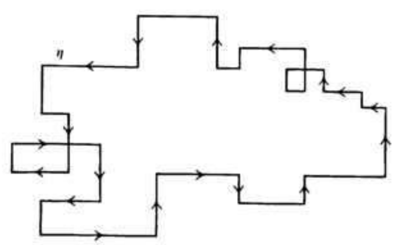

Figure 13

The lemma reduces the proof of Cauchy's theorem to the case when γ is a rectangular closed chain. We shall now reduce Cauchy's theorem to the case of rectangles by stating and proving a theorem having nothing to do with holomorphic functions. We need a little more terminology.

Let γ be a curve in an open set U, defined on an interval $[a, b]$. Let

$$a = a_0 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_n = b$$

be a partition of the interval. Let

$$\gamma_i: [a_i, a_{i+1}] \to U$$

be the restriction of γ to the smaller interval $[a_i, a_{i+1}]$. Then we agree to call the chain

$$\gamma_1 + \gamma_2 + \cdots + \gamma_n$$

a **subdivision** of γ. Furthermore, if η_i is obtained from γ_i by another parametrization, we again agree to call the chain

$$\eta_1 + \eta_2 + \cdots + \eta_n$$

a **subdivision of** γ. For any practical purposes, the chains γ and

$$\eta_1 + \eta_2 + \cdots + \eta_n$$

do not differ from each other. In Fig. 14 we illustrate such a chain γ and a subdivision $\eta_1 + \eta_2 + \eta_3 + \eta_4$.

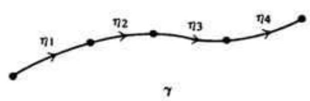

Figure 14

Similarly, if $\gamma = \sum m_i\gamma_i$ is a chain, and $\{\eta_{ij}\}$ is a subdivision of γ_i, we call

$$\sum_i \sum_j m_i \eta_{ij}$$

a **subdivision of** γ.

Theorem 3.2. *Let γ be a rectangular closed chain in U, and assume that γ is homologous to 0 in U, i.e.*

$$W(\gamma, \alpha) = 0$$

for every point α not in U. Then there exist rectangles $R_1, \ldots, R_N$

contained in U, such that if ∂R_i is the boundary of R_i oriented counterclockwise, then a subdivision of γ is equal to

$$\sum_{i=1}^{N} m_i \cdot \partial R_i$$

for some integers m_i.

Lemma 3.1 and Theorem 3.2 make Cauchy's Theorem 2.2 obvious because we know that for any holomorphic function f on U, we have

$$\int_{\partial R_i} f = 0$$

by Goursat's theorem. Hence the integral of f over the subdivision of γ is also equal to 0, whence the integral of f over γ is also equal to 0.

We now prove the theorem. Given the rectangular chain γ, we draw all vertical and horizontal lines passing through the sides of the chain, as illustrated on Fig. 15.

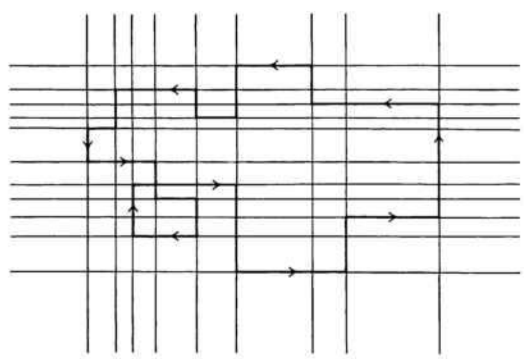

Figure 15

Then these vertical and horizontal lines decompose the plane into rectangles, and rectangular regions extending to infinity in the vertical and horizontal direction. Let R_i be one of the rectangles, and let α_i be a point inside R_i. Let

$$m_i = W(\gamma, \alpha_i).$$

For some rectangles we have $m_i = 0$, and for some rectangles, we have $m_i \neq 0$. We let $R_1, \ldots, R_N$ be those rectangles such that $m_1, \ldots, m_N$ are not 0, and we let ∂R_i be the boundary of R_i for $i = 1, \ldots, N$, oriented counterclockwise. We shall prove the following two assertions:

1. *Every rectangle R_i such that $m_i \neq 0$ is contained in U.*
2. *Some subdivision of γ is equal to*

$$\sum_{i=1}^{N} m_i \partial R_i.$$

This will prove the desired theorem.

Assertion 1. By assumption, α_i must be in U, because $W(\gamma, \alpha) = 0$ for every point α outside of U. Since the winding number is constant on connected sets, it is constant on the interior of R_i, hence $\neq 0$, and the interior of R_i is contained in U. If a boundary point of R_i is on γ, then it is in U. If a boundary point of R_i is not on γ, then the winding number with respect to γ is defined, and is equal to $m_i \neq 0$ by continuity (Lemma 3.2). This proves that the whole rectangle R_i, including its boundary, is contained in U, and proves the first assertion.

Assertion 2. We now replace γ by an appropriate subdivision. The vertical and horizontal lines cut γ in various points. We can then find a subdivision η of γ such that every curve occurring in η is some side of a rectangle, or the finite side of one of the infinite rectangular regions. The subdivision η is the sum of such sides, taken with appropriate multiplicities. If a finite side of an infinite rectangle occurs in the subdivision, after inserting one more horizontal or vertical line, we may assume that this side is also the side of a finite rectangle in the grid. Thus without loss of generality, we may assume that every side of the subdivision is also the side of one of the finite rectangles in the grid formed by the horizontal and vertical lines.

It will now suffice to prove that

$$\eta = \sum m_i \partial R_i.$$

Suppose $\eta - \sum m_i \partial R_i$ is not the 0 chain. Then it contains some horizontal or vertical segment σ, so that we can write

$$\eta - \sum m_i \partial R_i = m\sigma + C^*,$$

where m is an integer, and C^* is a chain of vertical and horizontal segments other than σ. Then σ is the side of a finite rectangle R_k. We take σ with the orientation arising from the counterclockwise orientation of the boundary of the rectangle R_k. Then the closed chain

$$C = \eta - \sum m_i \partial R_i - m\partial R_k$$

does not contain σ. Let α_k be a point interior to R_k, and let α' be a point near σ but on the opposite side from α_k, as shown on the figure.

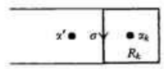

Figure 16

Since $\eta - \sum m_i \partial R_i - m\partial R_k$ does not contain σ, the points α_k and α' are connected by a line segment which does not intersect C. Therefore

$$W(C, \alpha_k) = W(C, \alpha').$$

But $W(\eta, \alpha_k) = m_k$ and $W(\partial R_i, \alpha_k) = 0$ unless $i = k$, in which case $W(\partial R_k, \alpha_k) = 1$. Similarly, if α' is inside some finite rectangle R_j, so $\alpha' = \alpha_j$, we have

$$W(\partial R_k, \alpha_j) = \begin{cases} 0 & \text{if } j \neq k, \\ 1 & \text{if } j = k. \end{cases}$$

If α' is in an infinite rectangle, then $W(\partial R_k, \alpha') = 0$. Hence:

$$W(C, \alpha_k) = W\left(\eta - \sum m_i \partial R_i - m\partial R_k, \alpha_k\right) = m_k - m_k - m = -m;$$

$$W(C, \alpha') = W\left(\eta - \sum m_i \partial R_i - m\partial R_k, \alpha'\right) = 0.$$

This proves that $m = 0$, and concludes the proof that $\eta - \sum m_i \partial R_i = 0$.

CHAPTER V

Applications of Cauchy's Integral Formula

In this chapter, we return to the ideas of Theorem 7.3 of Chapter III, which we interrupted to discuss some topological considerations about winding numbers. We come back to analysis. We shall give various applications of the fact that the derivative of an analytic function can be expressed as an integral. This is completely different from real analysis, where the derivative of a real function often is less differentiable than the function itself. In complex analysis, one can exploit the phenomenon in various ways. For instance, in real analysis, a uniform limit of a sequence of differentiable functions may be only continuous. However, in complex analysis, we shall see that a uniform limit of analytic functions is analytic.

We shall also study a point where a function is analytic near the point, but not necessarily at the point itself. Such points are the isolated singular points of the function, and the behavior of the function can be described rather accurately near these points.

V, §1. UNIFORM LIMITS OF ANALYTIC FUNCTIONS

We first prove a general theorem that the uniform limit of analytic functions is analytic. This will allow us to define analytic functions by uniformly convergent series, and we shall give several examples, in text and in the exercises.

Theorem 1.1. *Let $\{f_n\}$ be a sequence of holomorphic functions on an open set U. Assume that for each compact subset K of U the sequence converges uniformly on K, and let the limit function be f. Then f is holomorphic.*

Proof. Let $z_0 \in U$, and let $\overline{D}_R$ be a closed disc of radius R centered at z_0 and contained in U. Then the sequence $\{f_n\}$ converges uniformly on $\overline{D}_R$. Let C_R be the circle which is the boundary of $\overline{D}_R$. Let $\overline{D}_{R/2}$ be the closed disc of radius $R/2$ centered at z_0. Then for $z \in \overline{D}_{R/2}$ we have

$$f_n(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f_n(\zeta)}{\zeta - z}\, d\zeta,$$

and $|\zeta - z| \geqq R/2$. Since $\{f_n\}$ converges uniformly, for $|z - z_0| \leqq R/2$, we get

$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

By Theorem 7.7 of Chapter III it follows that f is holomorphic on a neighborhood of z_0. Since this is true for every z_0 in U, we have proved what we wanted.

Theorem 1.2. *Let $\{f_n\}$ be a sequence of analytic functions on an open set U, converging uniformly on every compact subset K of U to a function f. Then the sequence of derivatives $\{f_n'\}$ converges uniformly on every compact subset K, and $\lim f_n' = f'$.*

Proof. The proof will be left as an exercise to the reader. [*Hint*: Cover the compact set with a finite number of closed discs contained in U, and of sufficiently small radius. Cauchy's formula expresses the derivative f_n' as an integral, and one can argue as in the previous theorem.]

Example. Let

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}.$$

We shall prove that this function is holomorphic for $\operatorname{Re} z > 1$. Each term

$$f_n(z) = n^{-z} = e^{-z \log n}$$

is an entire function. Let $z = x + iy$. We have

$$|e^{-z \log n}| = |e^{-x \log n} e^{-iy \log n}| = n^{-x}.$$

Let $c > 1$. For $x \geqq c$ we have $|n^{-z}| \leqq n^{-c}$ and the series

$$\sum_{n=1}^{\infty} \frac{1}{n^c}$$

converges for $c > 1$. Hence the series $\sum f_n(z)$ converges uniformly and absolutely for $\operatorname{Re} z \geqq c$, and therefore defines a holomorphic function for $\operatorname{Re} z > c$. This is true for every $c > 1$, and hence f is holomorphic for $\operatorname{Re} z > 1$.

In the same example, we have

$$f_n'(z) = \frac{-\log n}{n^z}.$$

By Theorem 1.2, it follows that

$$f'(z) = \sum_{n=1}^{\infty} \frac{-\log n}{n^z}$$

in this same region.

V, §1. EXERCISES

1. Let f be analytic on an open set U, let $z_0 \in U$ and $f'(z_0) \neq 0$. Show that

$$\frac{2\pi i}{f'(z_0)} = \int_C \frac{1}{f(z) - f(z_0)}\,dz,$$

where C is a small circle centered at z_0.

2. Weierstrass' theorem for a real interval $[a, b]$ states that a continuous function can be uniformly approximated by polynomials. Is this conclusion still true for the closed unit disc, i.e. can every continuous function on the disc be uniformly approximated by polynomials?

3. Let $a > 0$. Show that each of the following series represents a holomorphic function:

(a) $\displaystyle\sum_{n=1}^{\infty} e^{-an^2z}$ for $\operatorname{Re} z > 0$;

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{-anz}}{(a+n)^2}$ for $\operatorname{Re} z > 0$;

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(a+n)^z}$ for $\operatorname{Re} z > 1$.

4. Show that each of the two series converges uniformly on each closed disc $|z| \leqq c$ with $0 < c < 1$:

$$\sum_{n=1}^{\infty} \frac{nz^n}{1 - z^n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{z^n}{(1 - z^n)^2}.$$

5. Prove that the two series in Exercise 4 are actually equal. [*Hint*: Write each one in a double series and reverse the order of summation.]

6. **Dirichlet Series.** Let $\{a_n\}$ be a sequence of complex numbers. Show that the series $\sum a_n/n^s$, if it converges absolutely for some complex s, converges absolutely in a right half-plane $\text{Re}(s) > \sigma_0$, and uniformly in $\text{Re}(s) > \sigma_0 + \varepsilon$ for every $\epsilon > 0$. Show that the series defines an analytic function in this half plane. The number σ_0 is called the **abscissa of convergence**.

The next exercises give expressions and estimates for an analytic function in terms of integrals.

7. Let f be analytic on a closed disc $\bar{D}$ of radius $b > 0$, centered at z_0. Show that

$$\frac{1}{\pi b^2}\iint_D f(x+iy)\,dy\,dx = f(z_0).$$

[*Hint*: Use polar coordinates and Cauchy's formula. Without loss of generality, you may assume that $z_0 = 0$. Why?]

8. Let D be the unit disc and let S be the unit square, that is, the set of complex numbers z such that $0 < \text{Re}(z) < 1$ and $0 < \text{Im}(z) < 1$. Let $f: D \to S$ be an analytic isomorphism such that $f(0) = (1+i)/2$. Let u, v be the real and imaginary parts of f respectively. Compute the integral

$$\iint_D \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dx\,dy.$$

9. (a) Let f be an analytic isomorphism on the unit disc D, and let

$$f(z) = \sum_{n=1}^{\infty} a_n z^n$$

be its power series expansion. Prove that

$$\text{area } f(D) = \pi \sum_{n=1}^{\infty} n|a_n|^2.$$

(b) Suppose that f is an analytic isomorphism on the closed unit disc $\bar{D}$, and that $|f(z)| \geqq 1$ if $|z| = 1$, and $f(0) = 0$. Prove that area $f(D) \geqq \pi$.

10. Let f be analytic on the unit disc D and assume that $\iint_D |f|^2\,dx\,dy$ exists. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Prove that

$$\frac{1}{2\pi}\iint_D |f(z)|^2\,dx\,dy = \sum_{n=0}^{\infty} |a_n|^2/(2n+2).$$

For the next exercise, recall that a norm $\|\ \|$ on a space of functions associates to each function f a real number $\geqq 0$, satisfying the following conditions:

N 1. We have $\|f\| = 0$ if and only if $f = 0$.

N 2. If c is a complex number, then $\|cf\| = |c|\,\|f\|$.

N 3. $\|f + g\| \leqq \|f\| + \|g\|$.

11. Let A be the closure of a bounded open set in the plane. Let f, g be continuous functions on A. Define their scalar product

$$\langle f, g\rangle = \iint_A f(z)\overline{g(z)}\, dy\, dx$$

and define the associated L^2-**norm** by its square,

$$\|f\|_2^2 = \iint_A |f(z)|^2\, dy\, dx.$$

Show that $\|f\|_2$ does define a norm. Prove the **Schwarz inequality**

$$|\langle f, g\rangle| \leqq \|f\|_2 \|g\|_2.$$

On the other hand, define

$$\|f\|_1 = \iint_A |f(z)|\, dy\, dx.$$

Show that $f \mapsto \|f\|_1$ is a norm on the space of continuous functions on A, called the L^1-**norm**. This is just preliminary. Prove:

(a) Let $0 < s < R$. Prove that there exist constants C_1, C_2 having the following property. If f is analytic on a closed disc $\bar{D}$ of radius R, then

$$\|f\|_s \leqq C_1 \|f\|_{1,R} \leqq C_2 \|f\|_{2,R},$$

where $\|\ \|_s$ is the sup norm on the closed disc of radius s, and the L^1, L^2 norms refer to the integral over the disc of radius R.

(b) Let $\{f_n\}$ be a sequence of holomorphic functions on an open set U, and assume that this sequence is L^2-Cauchy. Show that it converges uniformly on compact subsets of U.

12. Let U, V be open discs centered at the origin. Let $f = f(z, w)$ be a *continuous* function on the product $U \times V$, such that for each w the function $z \mapsto f(z, w)$ and for each z the function $w \mapsto f(z, w)$ are analytic on U and V, respectively. Show that f has a power series expansion

$$f(z, w) = \sum a_{mn} z^m w^n$$

which converges absolutely and uniformly for $|z| \leqq r$ and $|w| \leqq r$, for some positive number r. [*Hint*: Apply Cauchy's formula for derivatives twice, with respect to the two variables to get an estimate for the coefficients a_{mn}.] Generalize to several variables instead of two variables.

Note. This exercise is really quite trivial, although it is not generally realized that it is so. *The point is that the function f is assumed to be continuous.* If that assumption is not made, the situation becomes much more difficult to handle, and the result is known as Hartogs' theorem. In practice, continuity is indeed satisfied.

V, §2. LAURENT SERIES

By a **Laurent series**, we mean a series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n.$$

Let A be a set of complex numbers. We say that the Laurent series **converges absolutely** (resp. uniformly) on A if the two series

$$f^+(z) = \sum_{n \geqq 0} a_n z^n \qquad \text{and} \qquad f^-(z) = \sum_{n<0} a_n z^n$$

converge absolutely (resp. uniformly) on A. If that is the case, then $f(z)$ is regarded as the sum,

$$f(z) = f^+(z) + f^-(z).$$

Let r, R be positive numbers with $0 \leqq r < R$. We shall consider the annulus A consisting of all complex numbers z such that

$$r \leqq |z| \leqq R.$$

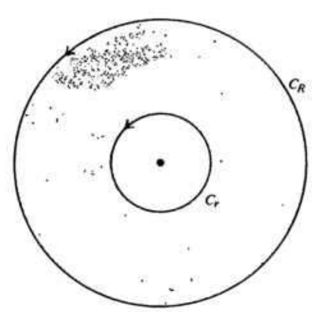

Figure 1

Theorem 2.1. *Let A be the above annulus, and let f be a holomorphic function on A. Let $r < s < S < R$. Then f has a Laurent expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

which converges absolutely and uniformly on $s \leqq |z| \leqq S$. Let C_R and C_r be the circles of radius R and r, respectively. Then the coefficients a_n are obtained by the usual formula:

$$a_n = \frac{1}{2\pi i}\int_{C_R} \frac{f(\zeta)}{\zeta^{n+1}}\,d\zeta \qquad \text{if} \quad n \geqq 0,$$

$$a_n = \frac{1}{2\pi i}\int_{C_r} \frac{f(\zeta)}{\zeta^{n+1}}\,d\zeta \qquad \text{if} \quad n < 0.$$

Proof. For some $\epsilon > 0$ we may assume (by the definition of what it means for f to be holomorphic on the closed annulus) that f is holomorphic on the open annulus U of complex numbers z such that

$$r - \epsilon < |z| < R + \epsilon.$$

The chain $C_R - C_r$ is homologous to 0 on U, because if a point lies in the outer part then its winding number is zero by the usual Lemma 1.3 of Chapter IV, and if the point lies in the disc inside the annulus, then its winding number is 0. Cauchy's formula then implies that for z in the annulus,

$$f(z) = \frac{1}{2\pi i}\int_{C_R} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\int_{C_r} \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

We may now prove the theorem. The first integral is handled just as in the ordinary case of the derivation of Cauchy's formula, and the second is handled in a similar manner as follows. We write

$$\zeta - z = -z\left(1 - \frac{\zeta}{z}\right).$$

Then

$$\left|\frac{\zeta}{z}\right| \leqq r/s < 1,$$

so the geometric series converges,

$$\frac{1}{z}\frac{1}{1 - \zeta/z} = \frac{1}{z}\left(1 + \frac{\zeta}{z} + \left(\frac{\zeta}{z}\right)^2 + \cdots\right).$$

We can then integrate term by term, and the desired expansion falls out. To show the uniqueness of the coefficients, integrate the series $\sum a_n s^n e^{in\theta}$ against $e^{-ik\theta}$ for a given integer k, term by term from 0 to 2π. All terms drop out except for $n = k$, showing that the k-th coefficient is determined by f.

An example of a function with a Laurent series with infinitely many negative terms is given by $e^{1/z}$, that is, by substituting $1/z$ in the ordinary exponential series.

If an annulus is centered at a point z_0, then one obtains a Laurent series at z_0 of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n.$$

Example. We want to find the Laurent series for

$$f(z) = \frac{1}{z(z-1)}$$

for $0 < |z| < 1$. We write f in partial fractions:

$$f(z) = \frac{1}{z-1} - \frac{1}{z}.$$

Then for one term we get the geometric series,

$$\frac{1}{z-1} = -\frac{1}{1-z} = -(1 + z + z^2 + \cdots)$$

whence

$$f(z) = -\frac{1}{z} - 1 - z - z^2 - \cdots.$$

On the other hand, suppose we want the Laurent series for $|z| > 1$. Then we write

$$\frac{1}{z-1} = \frac{1}{z}\left(\frac{1}{1-1/z}\right) = \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right)$$

whence

$$f(z) = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots.$$

V, §2. EXERCISES

1. Prove that the Laurent series can be differentiated term by term in the usual manner to give the derivative of f on the annulus.

2. Let f be holomorphic on the annulus A, defined by $0 < r \leqq |z| \leqq R$.

Prove that there exist functions f_1, f_2 such that f_1 is holomorphic for $|z| \leqq R$, f_2 is holomorphic for $|z| \geqq r$ and

$$f = f_1 + f_2$$

on the annulus.

3. Is there a polynomial $P(z)$ such that $P(z)e^{1/z}$ is an entire function? Justify your answer. What is the Laurent expansion of $e^{1/z}$ for $|z| \neq 0$?

4. Expand the function

$$f(z) = \frac{z}{1 + z^3}$$

(a) in a series of positive powers of z, and
(b) in a series of negative powers of z.
In each case, specify the region in which the expansion is valid.

5. Give the Laurent expansions for the following functions:
(a) $z/(z + 2)$ for $|z| > 2$
(b) $\sin 1/z$ for $z \neq 0$
(c) $\cos 1/z$ for $z \neq 0$
(d) $\dfrac{1}{(z-3)}$ for $|z| > 3$

6. Prove the following expansions:

(a) $e^z = e + e \sum_{n=1}^{\infty} \dfrac{1}{n!}(z-1)^n$

(b) $1/z = \sum_{n=0}^{\infty} (-1)^n (z-1)^n$ for $|z - 1| < 1$

(c) $1/z^2 = 1 + \sum_{n=1}^{\infty} (n+1)(z+1)^n$ for $|z + 1| < 1$

7. Expand (a) $\cos z$, (b) $\sin z$ in a power series about $\pi/2$.

8. Let $f(z) = \dfrac{1}{(z-1)(z-2)}$. Find the Laurent series for f:
(a) In the disc $|z| < 1$.
(b) In the annulus $1 < |z| < 2$.
(c) In the region $2 < |z|$.

9. Find the Laurent series for $(z + 1)/(z - 1)$ in the region (a) $|z| < 1$; (b) $|z| > 1$.

10. Find the Laurent series for $1/z^2(1 - z)$ in the regions:
(a) $0 < |z| < 1$; (b) $|z| > 1$.

11. Find the power series expansion of

$$f(z) = \frac{1}{1 + z^2}$$

around the point $z = 1$, and find the radius of convergence of this series.

12. Find the Laurent expansion of

$$f(z) = \frac{1}{(z-1)^2(z+1)^2}$$

for $1 < |z| < 2$.

13. Obtain the first four terms of the Laurent series expansion of

$$f(z) = \frac{e^z}{z(z^2+1)}$$

valid for $0 < |z| < 1$.

*14. Assume that f is analytic in the upper half plane, and that f is periodic of period 1. Show that f has an expansion of the form

$$f = \sum_{-\infty}^{\infty} c_n e^{2\pi i n z},$$

where

$$c_n = \int_0^1 f(x+iy)e^{-2\pi i n(x+iy)}\,dx,$$

for any value of $y > 0$. [*Hint*: Show that there is an analytic function f^* on a disc from which the origin is deleted such that

$$f^*(e^{2\pi i z}) = f(z).$$

What is the Laurent series for f^*? Abbreviate $q = e^{2\pi i z}$.

*15. Assumptions being as in Exercise 14, suppose in addition that there exists $y_0 > 0$ such that $f(z) = f(x+iy)$ is bounded in the domain $y \geqq y_0$. Prove that the coefficients c_n are equal to 0 for $n < 0$. Is the converse true? Proof?

V, §3. ISOLATED SINGULARITIES

Let z_0 be a complex number and let D be an open disc centered at z_0. Let U be the open set obtained by removing z_0 from D. A function f which is analytic on U is said to have an **isolated singularity** at z_0. We suppose this is the case.

Removable Singularities

Theorem 3.1. *If f is bounded in some neighborhood of z_0, then one can define $f(z_0)$ in a unique way such that the function is also analytic at z_0.*

Proof. Say $z_0 = 0$. By §2, we know that f has a Laurent expansion

$$f(z) = \sum_{n \geqq 0} a_n z^n + \sum_{n<0} a_n z^n$$

for $0 < |z| < R$. We have to show $a_n = 0$ if $n < 0$. Let $n = -m$ with $m > 0$. We have

$$a_{-m} = \frac{1}{2\pi i} \int_{C_r} f(\zeta)\zeta^{m-1}\, d\zeta,$$

for any circle C_r of small radius r. Since f is assumed bounded near 0 it follows that the right-hand side tends to 0 as r tends to 0, whence $a_{-m} = 0$, as was to be shown. (The uniqueness is clear by continuity.)

In the case of Theorem 3.1 it is customary to say that z_0 is a **removable singularity**.

Poles

Suppose the Laurent expansion of f in the neighborhood of a singularity z_0 has only a finite number of negative terms,

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \cdots + a_0 + a_1(z - z_0) + \cdots,$$

and $a_{-m} \neq 0$. Then f is said to have a **pole of order** (or multiplicity m) at z_0. However, we still say that the **order of f at z_0 is** $-m$, that is,

$$\operatorname{ord}_{z_0} f = -m,$$

because we want the formula

$$\operatorname{ord}_{z_0}(fg) = \operatorname{ord}_{z_0} f + \operatorname{ord}_{z_0} g$$

to be true. This situation is characterized as follows:

f has a pole of order m at z_0 if and only if $f(z)(z - z_0)^m$ is holomorphic at z_0 and has no zero at z_0.

The proof is immediate and is left to the reader.

If g is holomorphic at z_0 and $g(z_0) \neq 0$, then the function f defined by

$$f(z) = (z - z_0)^{-m} g(z)$$

in a neighborhood of z_0 from which z_0 is deleted, has a pole of order m. We abide by the convention that a pole is a zero of negative order.

A pole of order 1 is said to be a **simple pole**.

Examples. The function $1/z$ has a simple pole at the origin.

The function $1/\sin z$ has a simple pole at the origin. This comes from the power series expansion, since

$$\sin z = z(1 + \text{higher terms}),$$

and

$$\frac{1}{\sin z} = \frac{1}{z}(1 + \text{higher terms})$$

by inverting the series $1/(1-h) = 1 + h + h^2 + \cdots$ for $|h| < 1$.

Let f be defined on an open set U except at a discrete set of points S which are poles. Then we say that f is **meromorphic** on U. If z_0 is such a point, then there exists an integer m such that $(z - z_0)^m f(z)$ is holomorphic in a neighborhood of z_0. Thus f is the quotient of two holomorphic functions in the neighborhood of a point. We say that f is **meromorphic at a point** z_0 if f is meromorphic on some open set U containing z_0.

Example. Let $P(z)$ be a polynomial. Then $f(z) = 1/P(z)$ is a meromorphic function. This is immediately seen by factoring $P(z)$ into linear factors.

Example. A meromorphic function can be defined often by a uniformly convergent series. For instance, let

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{z^2 - n^2}.$$

We claim that f is meromorphic on $\mathbf{C}$ and has simple poles at the integers, but is holomorphic elsewhere.

We prove that f has these properties inside every disc of radius R centered at the origin. Let $R > 0$ and let $N > 2R$. Write

$$f(z) = g(z) + h(z),$$

where

$$g(z) = \frac{1}{z} + \sum_{n=1}^{N} \frac{z}{z^2 - n^2} \quad \text{and} \quad h(z) = \sum_{n=N+1}^{\infty} \frac{z}{z^2 - n^2}.$$

Then g is a rational function, and is therefore meromorphic on **C**. Furthermore, from its expression as a finite sum, we see that g has simple poles at the integers n such that $|n| \leqq N$.

For the infinite series defining h, we apply Theorem 1.1 and prove that the series is uniformly convergent. For $|z| < R$ we have the estimate

$$\left|\frac{z}{z^2 - n^2}\right| \leqq \frac{R}{n^2 - R^2} = \frac{1}{n^2}\frac{R}{1 - (R/n)^2}.$$

The denominator satisfies

$$1 - (R/n)^2 \geqq \tfrac{3}{4}$$

for $n > N > 2R$. Hence

$$\left|\frac{z}{z^2 - n^2}\right| \leqq \frac{4R}{3n^2} \qquad \text{for} \quad n \geqq 2R.$$

Therefore the series for h converges uniformly in the disc $|z| < R$, and h is holomorphic in this disc. This proves the desired assertion.

Essential Singularities

If the Laurent series has a infinite number of negative terms, then we say that z_0 is an **essential singularity** of f.

Example. The function $f(z) = e^{1/z}$ has an essential singularity at $z = 0$ because its Laurent series is

$$\sum_{n=0}^{\infty} \frac{1}{z^n n!}.$$

Theorem 3.2 (Casorati–Weierstrass). *Let* 0 *be an essential singularity of the function* f, *and let* D *be a disc centered at* 0 *on which* f *is holomorphic except at* 0. *Let* U *be the complement of* 0 *in* D. *Then* $f(U)$ *is dense in the complex numbers. In other words, the values of* f *on* U *come arbitrarily close to any complex number.*

Proof. Suppose the theorem is false. There exists a complex number α and a positive number $s > 0$ such that

$$|f(z) - \alpha| > s \qquad \text{for all } z \in U.$$

The function

$$g(z) = \frac{1}{f(z) - \alpha}$$

is then holomorphic on U, and bounded on the disc D. Hence 0 is a removable singularity of g, and g may be extended to a holomorphic function on all of D. It then follows that $1/g(z)$ has at most a pole at 0, which means that $f(z) - \alpha$ has at most a pole, contradicting the hypothesis that $f(z)$ has an essential singularity (infinitely many terms of negative order in its Laurent series). This proves the theorem.

Actually, it was proved by Picard that f not only comes arbitrarily close to every complex number, but takes on every complex value except possibly one. The function $e^{1/z}$ omits the value 0, so it is necessary to allow for this one omission. See Chapter XI, §3 and Chapter XII, §2.

We recall that an **analytic isomorphism**

$$f: U \to V$$

from one open set to another is an analytic function such that there exists another analytic function

$$g: V \to U$$

satisfying

$$f \circ g = \mathrm{id}_V \qquad \text{and} \qquad g \circ f = \mathrm{id}_U,$$

where id is the identity function. An **analytic automorphism** of U is an analytic isomorphism of U with itself.

Using the Casorati–Weierstrass theorem, we shall prove:

Theorem 3.3. *The only analytic automorphisms of* $\mathbf{C}$ *are the functions of the form* $f(z) = az + b$, *where* a, b *are constants,* $a \neq 0$.

Proof. Let f be an analytic automorphism of $\mathbf{C}$. After making a translation by $-f(0)$, we may assume without loss of generality that $f(0) = 0$. We then have to prove that $f(z) = az$ for some constant a. Let

$$h(z) = f(1/z) \qquad \text{for} \quad z \neq 0.$$

Then h is defined for all complex numbers except for the origin. We first prove that h cannot have an essential singularity at 0. Since f is a local analytic isomorphism at 0, f gives a bijection between an open neighborhood of 0 with some open neighborhood of 0. Since f is also an analytic isomorphism of **C**, it follows that there exists $\delta > 0$ and $c > 0$ such that if $|w| > 1/\delta$ then $|f(w)| > c$. Let $z = 1/w$ or $w = 1/z$. Then $|h(z)| > c$ for $|z| < \delta$. If 0 is an essential singularity, this contradicts the Casorati–Weierstrass theorem.

Let $f(z) = \sum a_n z^n$ so $h(z) = \sum a_n(1/z)^n$. Since 0 is not an essential singularity of h, it follows that the series for h, hence for f, has only a finite number of terms, and

$$f(z) = a_0 + a_1 z + \cdots + a_N z^N$$

is a polynomial of degree N for some N. If f has two distinct roots, then f cannot be injective, contradicting the fact that f has an inverse function. Hence f has only one root, and

$$f(z) = a(z - z_0)^N$$

for some z_0. If $N > 1$, it is then clear that f is not injective so we must have $N = 1$, and the theorem is proved.

V, §3. EXERCISES

1. Show that the following series define a meromorphic function on **C** and determine the set of poles, and their orders.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+z)}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{z^2+n^2}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(z+n)^2}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin nz}{n!(z^2+n^2)}$ (e) $\displaystyle\frac{1}{z} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left[\frac{1}{z-n} + \frac{1}{n}\right]$

2. Show that the function

$$f(z) = \sum_{n=1}^{\infty} \frac{z^2}{n^2 z^2 + 8}$$

is defined and continuous for the real values of z. Determine the region of the complex plane in which this function is analytic. Determine its poles.

3. Show that the series

$$\sum_{n=1}^{\infty} \left(\frac{z+i}{z-i}\right)^n$$

defines an analytic function on a disc of radius 1 centered at $-i$.

4. Let $\{z_n\}$ be a sequence of distinct complex numbers such that

$$\sum \frac{1}{|z_n|^3} \text{ converges.}$$

Prove that the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{(z-z_n)^2}-\frac{1}{z_n^2}\right)$$

defines a meromorphic function on **C**. Where are the poles of this function?

5. Let f be meromorphic on **C** but not entire. Let $g(z) = e^{f(z)}$. Show that g is not meromorphic on **C**.

6. Let f be a non-constant entire function, i.e. a function analytic on all of **C**. Show that the image of f is dense in **C**.

7. Let f be meromorphic on an open set U. Let

$$\varphi : V \to U$$

be an analytic isomorphism. Suppose $\varphi(z_0) = w_0$, and f has order n at w_0. Show that $f \circ \varphi$ has order n at z_0. In other words, the order is invariant under analytic isomorphisms. [Here n is a positive or negative integer.]

8. A meromorphic function f is said to be **periodic** with period w if $f(z+w) = f(z)$ for all $z \in \mathbf{C}$. Let f be a meromorphic function, and suppose f is periodic with three periods w_1, w_2, w_3 which are linearly independent over the rational numbers. Prove that f is constant. [*Hint*: Prove that there exist elements w which are integral linear combinations of w_1, w_2, w_3 and arbitrarily small in absolute value.] The exponential function is an example of a singly periodic function. Examples of doubly periodic functions will be given in Chapter XIV.

9. Let f be meromorphic on **C**, and suppose

$$\lim_{|z|\to\infty} |f(z)| = \infty.$$

Prove that f is a rational function. (You cannot assume as given that f has only a finite number of poles.)

10. **(The Riemann Sphere).** Let S be the union of **C** and a single other point denoted by ∞, and called **infinity**. Let f be a function on S. Let $t = 1/z$, and define

$$g(t) = f(1/t)$$

for $t \neq 0$, ∞. We say that f has an **isolated singularity** (resp. is **meromorphic** resp. is **holomorphic**) **at infinity** if g has an isolated singularity (resp. is meromorphic, resp. is holomorphic) at 0. The order of g at 0 will also be called the **order of f at infinity**. If g has a removable singularity at 0, and so can be defined as a holomorphic function in a neighborhood of 0, then we say that f is **holomorphic at infinity**.

We say that f is meromorphic on S if f is meromorphic on **C** and is also meromorphic at infinity. We say that f is holomorphic on S if f is holomorphic on **C** and is also holomorphic at infinity.

Prove:

The only meromorphic functions on S are the rational functions, that is, quotients of polynomials. The only holomorphic functions on S are the constants. If f is holomorphic on **C** *and has a pole at infinity, then f is a polynomial.*

In this last case, how would you describe the order of f at infinity in terms of the polynomial?

11. Let f be a meromorphic function on the Riemann sphere, so a rational function by Exercise 10. Prove that

$$\sum_P \operatorname{ord}_P f = 0,$$

where the sum is taken over all points P which are either points of **C**, or $P = \infty$.

12. Let P_i $(i = 1, \ldots, r)$ be points of **C** or ∞, and let m_i be integers such that

$$\sum_{i=1}^{r} m_i = 0.$$

Prove that there exists a meromorphic function f on the Riemann sphere such that

$$\operatorname{ord}_{P_i} f = m_i \qquad \text{for} \quad i = 1, \ldots, r$$

and $\operatorname{ord}_P f = 0$ if $P \neq P_i$.

CHAPTER VI

Calculus of Residues

We have established all the theorems needed to compute integrals of analytic functions in terms of their power series expansions. We first give the general statements covering this situation, and then apply them to examples.

VI, §1. THE RESIDUE FORMULA

Let

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$$

have a Laurent expansion at a point z_0. We call a_{-1} the **residue** of f at z_0, and write

$$a_{-1} = \operatorname{Res}_{z_0} f.$$

Theorem 1.1. *Let z_0 be an isolated singularity of f, and let C be a small circle oriented counterclockwise, centered at z_0 such that f is holomorphic on C and its interior, except possibly at z_0. Then*

$$\int_C f(\zeta)\, d\zeta = 2\pi i a_{-1} = 2\pi i \operatorname{Res}_{z_0} f.$$

Proof. Since the series for $f(\zeta)$ converges uniformly and absolutely for ζ on the circle, we may integrate it term by term. The integral of

$(\zeta - z_0)^n$ over the circle is equal to 0 for all values of n except possibly when $n = -1$, in which case we know that the value is $2\pi i$, cf. Examples 1 and 4 of Chapter III, §2. This proves the theorem.

From this local result, we may then deduce a global result for more general paths, by using the reduction of Theorem 2.4, Chapter IV.

Theorem 1.2 (Residue Formula). *Let U be an open set, and γ a closed chain in U such that γ is homologous to 0 in U. Let f be analytic on U except at a finite number of points $z_1, \ldots, z_n$. Let $m_i = W(\gamma, z_i)$. Then*

$$\int_\gamma f = 2\pi\sqrt{-1} \sum_{i=1}^{n} m_i \cdot \operatorname{Res}_{z_i} f.$$

Proof. Immediate by plugging Theorem 1.1 in the above mentioned theorem of Chapter IV.

Theorem 1.2 is used most often when U is simply connected, in which case every closed path is homologous to 0 in U, and the hypothesis on γ need not be mentioned explicitly. In the applications, U will be a disc, or the inside of a rectangle, where the simple connectedness is obvious.

Remark. The notation $\sqrt{-1}$ is the standard device used when we don't want to confuse the complex number i with an index i.

We shall give examples how to find residues.

A pole of a function f is said to be **simple** if it is of order 1, in which case the power series expansion of f is of type

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + \text{higher terms},$$

and $a_{-1} \neq 0$.

Lemma 1.3.

(a) *Let f have a simple pole at z_0, and let g be holomorphic at z_0. Then*

$$\operatorname{Res}_{z_0}(fg) = g(z_0) \operatorname{Res}_{z_0}(f).$$

(b) *Suppose $f(z_0) = 0$ but $f'(z_0) \neq 0$. Then $1/f$ has a pole of order 1 at z_0 and the residue of $1/f$ at z_0 is $1/f'(z_0)$.*

Proof. (a) Let $f(z) = a_{-1}/(z - z_0)$ + higher terms. Say $z_0 = 0$ for simplicity of notation. We have

$$f(z)g(z) = \left(\frac{a_{-1}}{z} + \cdots\right)(b_0 + b_1 z + \cdots)$$

$$= \frac{a_{-1}b_0}{z} + \text{higher terms,}$$

so our assertion is clear.

(b) Let $f(z_0) = 0$ but $f'(z_0) \neq 0$. Then $f(z) = a_1(z - z_0)$ + higher terms, and $a_1 \neq 0$. Say $z_0 = 0$ for simplicity. Then

$$f(z) = a_1 z(1 + h) \qquad \text{with} \quad \text{ord}\, h \geqq 1,$$

so

$$\frac{1}{f(z)} = \frac{1}{a_1 z}(1 - h + h^2 - \cdots) = \frac{1}{a_1 z} + \text{higher terms,}$$

so $\text{res}(1/f) = a_1^{-1} = 1/f'(0)$, as was to be shown.

Remark. Part (a) of the lemma merely repeats what you should have seen before, to make this chapter more systematic.

Example. We give an example for part (b) of the lemma. Let $f(z) = \sin z$. Then f has a simple zero at $z = \pi$, because $f'(z) = \cos z$ and $f'(\pi) = -1 \neq 0$. Hence $1/f(z)$ has a simple pole at $z = \pi$, and

$$\text{res}_\pi \frac{1}{\sin z} = \frac{1}{\cos \pi} = -1.$$

Example. Find the residue of $f(z) = \dfrac{z^2}{z^2 - 1}$ at $z = 1$.

To do this, we write

$$f(z) = \frac{z^2}{(z + 1)(z - 1)}.$$

Note that $g(z) = z^2/(z + 1)$ is holomorphic at 1, and that the residue of $1/(z - 1)$ is 1. Hence

$$\text{Res}_1 f = g(1) = 1/2.$$

Example. Find the residue of $(\sin z)/z^2$ at $z = 0$. We have

$$\frac{\sin z}{z^2} = \frac{1}{z^2}\left(z - \frac{z^3}{3!} + \cdots\right)$$
$$= \frac{1}{z} + \text{higher terms.}$$

Hence the desired residue is 1.

Example. Find the residue of $f(z) = \dfrac{z^2}{(z + 1)(z - 1)^2}$ at $z = 1$.

We note that the function

$$g(z) = \frac{z^2}{z + 1}$$

is holomorphic at $z = 1$, and has an expansion of type

$$g(z) = b_0 + b_1(z - 1) + \text{higher terms.}$$

Then

$$f(z) = \frac{g(z)}{(z - 1)^2} = \frac{b_0}{(z - 1)^2} + \frac{b_1}{(z - 1)} + \cdots$$

and therefore the residue of f at 1 is b_1, which we must now find. We write $z = 1 + (z - 1)$, so that

$$\frac{z^2}{z + 1} = \frac{1 + 2(z - 1) + (z + 1)^2}{2(1 + \frac{1}{2}(z - 1))}.$$

Inverting by the geometric series gives

$$\frac{z^2}{z + 1} = \frac{1}{2}\left(1 + \frac{3}{2}(z - 1) + \cdots\right).$$

Therefore

$$f(z) = \frac{1}{2(z - 1)^2} + \frac{3/4}{z - 1} + \cdots$$

whence $\text{Res}_1 f = 3/4$.

Example. Let C be a circle centered at 1, of radius 1. Let

$$f(z) = \frac{z^2}{(z+1)(z-1)^2}.$$

Find $\int_C f$.

The function f has only two singularities, at 1 and -1, and the circle is contained in a disc of radius > 1, centered at 1, on which f is holomorphic except at $z = 1$. Hence the residue formula and the preceding example give us

$$\int_C f = 2\pi i \cdot \tfrac{3}{4}.$$

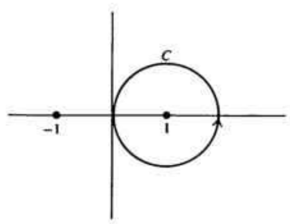

Figure 1

If C is the boundary of the rectangle as shown on Fig. 2, then we also find

$$\int_C f = 2\pi i \cdot \tfrac{3}{4}.$$

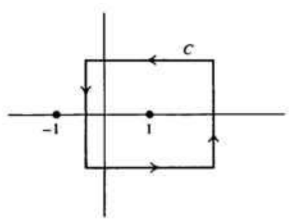

Figure 2

Example. Let $f(z) = z^2 - 2z + 3$. Let C be a rectangle as shown on Fig. 3, oriented clockwise. Find

$$\int_C \frac{1}{f(z)}\,dz.$$

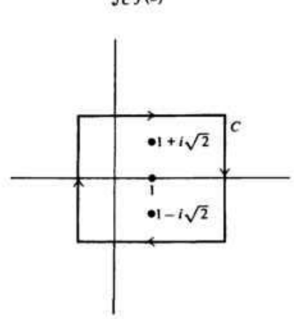

Figure 3

The roots of $f(z)$ are found by the quadratic formula to be

$$\frac{2 \pm \sqrt{-8}}{2}$$

and so are $z_1 = 1 + i\sqrt{2}$ and $z_2 = 1 - i\sqrt{2}$. The rectangle goes around these two points, in the clockwise direction. The residue of $1/f(z)$ at z_1 is $1/(z_1 - z_2)$ because f has a simple pole at z_1. The residue of $1/f(z)$ at z_2 is $1/(z_2 - z_1)$ for the same reason. The desired integral is equal to

$$-2\pi i(\text{sum of the residues}) = 0.$$

Example. Let f be the same function as in the preceding example, but now find the integral of $1/f$ over the rectangle as shown on Fig. 4. The rectangle is oriented clockwise. In this case, we have seen that the residue at $1 - i\sqrt{2}$ is

$$\frac{1}{z_2 - z_1} = \frac{1}{-2i\sqrt{2}}.$$

Therefore the integral over the rectangle is equal to

$$-2\pi i(\text{residue}) = -2\pi i/(-2i\sqrt{2}) = \pi/\sqrt{2}.$$

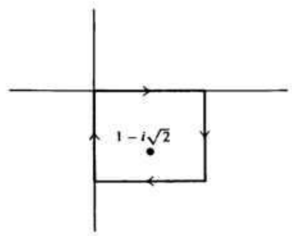

Figure 4

Next we give an example which has theoretical significance, besides computational significance.

Example. Let f have a power series expansion with only a finite number of negative terms (so at most a pole), say at the origin,

$$f(z) = a_m z^m + \text{higher terms}, \qquad a_m \neq 0,$$

and m may be positive or negative. Then we can write

$$f(z) = a_m z^m(1 + h(z)),$$

where $h(z)$ is a power series with zero constant term. For any two functions f, g we know the derivative of the product,

$$(fg)' = f'g + fg',$$

so that dividing by fg yields

$$\boxed{\frac{(fg)'}{fg} = \frac{f'}{f} + \frac{g'}{g}.}$$

Therefore we find for $f(z) = (a_m z^m)(1 + h(z))$,

$$\frac{f'(z)}{f(z)} = \frac{m}{z} + \frac{h'(z)}{1 + h(z)}$$

and $h'(z)/(1 + h(z))$ is holomorphic at 0. Consequently, we get:

Lemma 1.4. *Let f be meromorphic at 0. Then*

$$\operatorname{Res}_0 f'/f = \operatorname{ord}_0 f$$

and for any point z_0 where f has at most a pole,

$$\operatorname{Res}_{z_0} f'/f = \operatorname{ord}_{z_0} f.$$

Theorem 1.5. *Let γ be a closed chain in U, homologous to 0 in U. Let f be meromorphic on U, with only a finite number of zeros and poles, say at the points $z_1, \ldots, z_n$, none of which lie on γ. Let $m_i = W(\gamma, z_i)$. Then*

$$\int_\gamma f'/f = 2\pi\sqrt{-1} \sum m_i \operatorname{ord}_{z_i} f.$$

Proof. This is immediate by plugging the statement of the lemma into the residue formula.

In applications, γ is frequently equal to a circle C, or a rectangle, and the points $z_1, \ldots, z_n$ are inside C. Suppose that the zeros of f inside C are

$$a_1, \ldots, a_r$$

and the poles are

$$b_1, \ldots, b_s.$$

Then in the case,

$$\boxed{\int_C f'/f = 2\pi\sqrt{-1}\left(\sum_{i=1}^{r} \operatorname{ord}_{a_i} f - \sum_{j=1}^{s} \operatorname{mul}_{b_j} f\right).}$$

We follow our convention whereby the multiplicity of a pole is the negative of the order of f at the pole, so that

$$\operatorname{mul}_{b_j} f = -\operatorname{ord}_{b_j} f$$

by definition.

If one counts zeros and poles with their multiplicities, one may rephrase the above formula in the more suggestive fashion:

Let C be a simple closed curve, and let f be meromorphic on C and its interior. Assume that f has no zero or pole on C. Then

$$\boxed{\int_C f'/f = 2\pi i \text{ (number of zeros} - \text{number of poles),}}$$

where number of zeros = number of zeros of f in the interior of C, and number of poles = number of poles of f in the interior of C.

Of course, we have not proved that a simple closed curve has an "interior". The theorem is applied in practice only when the curve is so explicitly given (as with a circle or rectangle) that it is clear what "interior" is meant.

Besides, one can (not so artificially) formalize what is needed of the notion of "interior" so that one can use the standard language. Let γ be a closed path. We say that γ has an **interior** if $W(\gamma, \alpha) = 0$ or 1 for every complex number α which does not lie on γ. Then the set of points α such that $W(\gamma, \alpha) = 1$ will be called the **interior** of γ. It's that simple.

Theorem 1.6 (Rouché's Theorem). *Let γ be a closed path homologous to 0 in U and assume that γ has an interior. Let f, g be analytic on U, and*

$$|f(z) - g(z)| < |f(z)|$$

for z on γ. Then f and g have the same number of zeros in the interior of γ.

Proof. Note that the assumption implies automatically that f, g have no zero on γ. We have

$$\left|\frac{g(z)}{f(z)} - 1\right| < 1$$

for z on γ. Then the values of the function g/f are contained in the open disc with center 1 and radius 1. Let $F = g/f$. Then $F \circ \gamma$ is a closed path contained in that disc, and therefore

$$W(F \circ \gamma, 0) = 0$$

because 0 lies outside the disc. If γ is defined on $[a, b]$ then

$$\begin{aligned} 0 = W(F \circ \gamma, 0) &= \int_{F \circ \gamma} \frac{1}{z}\, dz = \int_a^b \frac{F'(\gamma(t))}{F(\gamma(t))} \gamma'(t)\, dt \\ &= \int_\gamma F'/F \\ &= \int_\gamma g'/g - f'/f. \end{aligned}$$

What we want now follows from Theorem 1.5, as desired.

Example. Let $P(z) = z^8 - 5z^3 + z - 2$. We want to find the number of roots of this polynomial inside the unit circle. Let

$$f(z) = -5z^3.$$

For $|z| = 1$ it is immediate that

$$|f(z) - P(z)| = |-z^8 - z + 2| < |f(z)| = 5.$$

Hence f and P have the same number of zeros inside the unit circle, and this number is clearly equal to 3. (Remember, you have to count multiplicities, and the equation

$$5z^3 = 0$$

has one zero with multiplicity 3.)

We shall use Rouché's theorem to give an alternative treatment of the inverse function theorem, not depending on solving for an inverse power series as was done in Chapter II, §5.

Theorem 1.7. *Let f be analytic in a neighborhood of a point z_0, and assume $f'(z_0) \neq 0$. Then f is a local analytic isomorphism at z_0.*

Proof. Making translations, we may assume without loss of generality that $z_0 = 0$ and $f(z_0) = 0$, so that

$$f(z) = \sum_{n=m}^{\infty} a_n z^n,$$

and $m \geqq 1$. Since $f'(0) = a_1$ we have $m = 1$ and $a_1 \neq 0$. Dividing by a_1 we may assume without loss of generality that $a_1 = 1$. Thus f has the power series expansion

$$f(z) = z + h(z),$$

where $h(z)$ is divisible by z^2. In particular, if we restrict the values of z to some sufficiently small disc around 0, then there is a constant $K > 0$ such that

$$|f(z) - z| \leqq K|z|^2.$$

Let C_r be the circle of radius r, and let $|\alpha| < r/2$. Let r be sufficiently small, and let

$$f_\alpha(z) = f(z) - \alpha \quad \text{and} \quad g_\alpha(z) = z - \alpha.$$

We have the inequality

$$|f_\alpha(z) - g_\alpha(z)| = |f(z) - z| \leqq K|z|^2.$$

If z is on C_r, that is $|z| = r$, then

$$K|z|^2 = Kr^2 < |z - \alpha| = |g_\alpha(z)|$$

because $|z - \alpha| > r/2$ and $Kr^2 < r/2$ (for instance, taking $r < 1/2K$). By Rouché's theorem, f_α and g_α have the same number of zeros inside C_r, and since g_α has exactly one zero, it follows that f_α has exactly one zero. This means that the equation

$$f(z) = \alpha$$

has exactly one solution inside C_r if $|\alpha| < r/2$.

Let U be the set of points z inside C_r such that

$$|f(z)| < r/2.$$

Then U is open because f is continuous, and we have just shown that

$$f\colon U \to D(0, r/2)$$

is a bijection of U with the disc of radius $r/2$. The argument we have given also shows that f is an open mapping, and hence the inverse function

$$\varphi\colon D(0, r/2) \to U$$

is continuous. There remains only to prove that φ is analytic. As in freshman calculus, we write the Newton quotient

$$\frac{\varphi(w) - \varphi(w_1)}{w - w_1} = \frac{z - z_1}{f(z) - f(z_1)}.$$

Fix w_1 with $|w_1| < r/2$, and let w approach w_1. Since φ is continuous, it must be that $z = \varphi(w)$ approaches $z_1 = \varphi(w_1)$. Thus the right-hand side approaches

$$1/f'(z_1),$$

provided we took r so small that $f'(z_1) \neq 0$ for all z_1 inside the circle of radius r, which is possible by the continuity of f' and the fact that $f'(0) \neq 0$. This proves that φ is holomorphic, whence analytic and concludes the proof of the theorem.

Residues of Differentials

Let $f(T) = \sum a_n T^n$ be a power series with a finite number of negative terms. We defined the **residue** to be a_{-1}. This was convenient for a number of applications, but in some sense so far it constituted an incomplete treatment of the situation with residues because this definition did not take into account the chain rule when computing integrals by means of residues. We shall now fill the remaining gap.

Let U be an open set in **C**. We define a **meromorphic differential** on U to be an expression of the form

$$\omega = f(z)\,dz$$

where f is meromorphic on U. Let $z_0 \in U$. Then f has a power series expansion at z_0, say

$$(1) \qquad f(z) = \sum_{n=m}^{\infty} a_n(z - z_0)^n = f_0(z - z_0).$$

Often one wants to make a change of coordinate. Thus we define a function w to be a **local coordinate** at z_0 if w has a zero of order 1 at z_0. Then w is a local isomorphism at z_0, and there is a power series h such that

$$(2) \qquad z - z_0 = h(w) = c_1 w + \text{higher terms} \qquad \text{with} \quad c_1 \neq 0.$$

Then substituting (2) in (1) we obtain

$$(3) \qquad f(z)\,dz = f_0(h(w))h'(w)\,dw = g(w)\,dw,$$

where $g(w) = f_0(h(w))h'(w)$ also has a power series expansion in terms of w. We denote the coefficients of this power series by b_n, so that

$$(4) \qquad g(w) = \sum_{n=m}^{\infty} b_n w^n.$$

Since $h(w)$ has order 1, $h'(0) = c_1 \neq 0$, it follows that the power series for g also has order m. Of course, the coefficients for the power series of g seem to be complicated expressions in the coefficients for the power series for f. However, it turns out that the really important coefficient, namely the residue b_{-1}, has a remarkable invariance property, stated in the next theorem.

Theorem 1.8. *Let w be a local coordinate at z_0. Let ω be a meromorphic differential in a neighborhood of z_0, and write $\omega = f(z)\,dz =$*

$g(w)\,dw$, where f, g are meromorphic functions, with the power series expansions as in (1) *and* (4) *above. Then the residues of the power series for f and g are equal, that is*

$$b_{-1} = a_{-1}.$$

Proof. Let γ be a small circle around z_0 in the z-plane. Let $w = \varphi(z)$. Then

$$b_{-1} = \frac{1}{2\pi i}\int_{\varphi\circ\gamma} g(w)\,dw = \frac{1}{2\pi i}\int_{\gamma} f(z)\,dz = a_{-1},$$

which proves the theorem.

In light of Theorem 1.8, we define the residue of a meromorphic differential $f(z)\,dz$ at a point z_0 as follows. We let w be a local coordinate at z_0. (Thus w may be $z - z_0$, but there are plenty of other local coordinates.) We write the differential as a power series in w,

$$\omega = g(w)\,dw \qquad \text{with} \qquad g(w) = \sum b_n w^n,$$

and we define the **residue of the differential** to be

$$\operatorname{res}_{z_0}(\omega) = b_{-1}.$$

This value b_{-1} is independent of the choice of coordinate at z_0. Using residues of differentials rather than residues of power series will be especially appropriate when the change of variables formula enters into consideration, for example in Exercises 35 and 36 below, when we deal with residues "at infinity" using the change of coordinate $w = 1/z$.

Remark. We could have defined a meromorphic differential on U also as an expression of the form $f\,dg$ where f and g are meromorphic. If w is a local coordinate at z_0, then both f and g have power series expansions in terms of w, so

$$f\,dg = f(w)\frac{dg}{dw}\,dw.$$

However, if U is an open set and f is a meromorphic function on U, not constant, then note that $d\log f$ is a meromorphic differential on U, because

$$d\log f(z) = \frac{f'(z)}{f(z)}\,dz.$$

Even though $\log f$ itself is not well defined on U, because of the ambiguity arising from the constant of integration, taking the derivative eliminates this constant, so that the differential itself is well defined.

VI, §1. EXERCISES

Find the residues of the following functions at 0.

1. $(z^2 + 1)/z$
2. $(z^2 + 3z - 5)/z^3$
3. $z^3/(z - 1)(z^4 + 2)$
4. $(2z + 1)/z(z^3 - 5)$
5. $(\sin z)/z^4$
6. $(\sin z)/z^5$
7. $(\sin z)/z^6$
8. $(\sin z)/z^7$
9. e^z/z
10. e^z/z^2
11. e^z/z^3
12. e^z/z^4
13. $z^{-2}\log(1 + z)$
14. $e^z/\sin z$

Find the residues of the following functions at 1.

15. $1/(z^2 - 1)(z + 2)$
16. $(z^3 - 1)(z + 2)/(z^4 - 1)^2$
17. Factor the polynomial $z^n - 1$ into factors of degree 1. Find the residue at 1 of $1/(z^n - 1)$.
18. Let $z_1, \ldots, z_n$ be distinct complex numbers. Let C be a circle around z_1 such that C and its interior do not contain z_j for $j > 1$. Let

$$f(z) = (z - z_1)(z - z_2)\cdots(z - z_n).$$

Find

$$\int_C \frac{1}{f(z)}\,dz.$$

19. Find the residue at i of $1/(z^4 - 1)$. Find the integral

$$\int_C \frac{1}{(z^4 - 1)}\,dz$$

where C is a circle of radius 1/2 centered at i.

20. (a) Find the integral

$$\int_C \frac{1}{z^2 - 3z + 5}\,dz,$$

where C is a rectangle oriented clockwise, as shown on the figure.

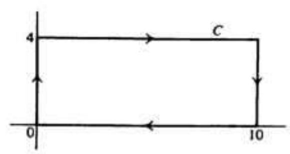

Figure 5

(b) Find the integral $\int_C 1/(z^2 + z + 1)\, dz$ over the same C.

(c) Find the integral $\int_C 1/(z^2 - z + 1)\, dz$ over this same C.

21. (a) Let $z_1, \ldots, z_n$ be distinct complex numbers. Determine explicitly the partial fraction decomposition (i.e. the numbers a_i):

$$\frac{1}{(z - z_1)\cdots(z - z_n)} = \frac{a_1}{z - z_1} + \cdots + \frac{a_n}{z - z_n}.$$

(b) Let $P(z)$ be a polynomial of degree $\leqq n - 1$, and let $a_1, \ldots, a_n$ be distinct complex numbers. Assume that there is a partial fraction decomposition of the form

$$\frac{P(z)}{(z - a_1)\cdots(z - a_n)} = \frac{c_1}{z - a_1} + \cdots + \frac{c_n}{z - a_n}.$$

Prove that

$$c_1 = \frac{P(a_1)}{(a_1 - a_2)\cdots(a_1 - a_n)},$$

and similarly for the other coefficients c_j.

22. Let f be analytic on an open disc centered at a point z_0, except at the point itself where f has a simple pole with residue equal to an integer n. Show that there is an analytic function g on the disc such that $f = g'/g$, and

$$g(z) = (z - z_0)^n h(z), \qquad \text{where } h \text{ is analytic and} \quad h(z_0) \neq 0.$$

(To make life simpler, you may assume $z_0 = 0$.)

23. (a) Let f be a function which is analytic on the upper half plane, and on the real line. Assume that there exist numbers $B > 0$ and $c > 0$ such that

$$|f(\zeta)| \leqq \frac{B}{|\zeta|^c}$$

for all ζ. Prove that for any z in the upper half plane, we have the integral formula

$$\boxed{f(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(t)}{t - z}\,dt.}$$

[*Hint*: Consider the integral over the path shown on the figure, and take the limit as $R \to \infty$.]

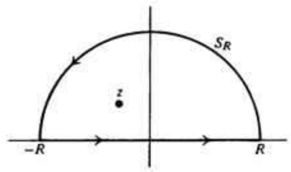

Figure 6

The path consists of the segment from $-R$ to R on the real axis, and the semicircle S_R as shown.

(b) By using a path similar to the previous one, but slightly raised over the real axis, and taking a limit, prove that the formula is still true if instead of assuming that f is analytic on the real line, one merely assumes that f is continuous on the line, but otherwise satisfies the same hypotheses as before.

24. Determine the poles and find the residues of the following functions.

(a) $1/\sin z$ (b) $1/(1 - e^z)$ (c) $z/(1 - \cos z)$.

25. Show that

$$\int_{|z|=1} \frac{\cos e^{-z}}{z^2}\,dz = 2\pi i \cdot \sin 1.$$

26. Find the integrals, where C is the circle of radius 8 centered at the origin.

(a) $\displaystyle\int_C \frac{1}{\sin z}\,dz$ (b) $\displaystyle\int_C \frac{1}{1 - \cos z}\,dz$

(c) $\displaystyle\int_C \frac{1 + z}{1 - e^z}\,dz$ (d) $\displaystyle\int_C \tan z\,dz$

(e) $\displaystyle\int_C \frac{1 + z}{1 - \sin z}\,dz$

27. Let f be holomorphic on and inside the unit circle, $|z| \leqq 1$, except for a pole of order 1 at a point z_0 on the circle. Let $f = \sum a_n z^n$ be the power series for f on the open disc. Prove that

$$\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = z_0.$$

28. Let a be real > 1. Prove that the equation $ze^{a-z} = 1$ has a single solution with $|z| \leqq 1$, which is real and positive.

29. Let U be a connected open set, and let D be an open disc whose closure is contained in U. Let f be analytic on U and not constant. Assume that the absolute value $|f|$ is constant on the boundary of D. Prove that f has at least one zero in D. [*Hint*: Consider $g(z) = f(z) - f(z_0)$ with $z_0 \in D$.]

30. Let f be a function analytic inside and on the unit circle. Suppose that $|f(z) - z| < |z|$ on the unit circle.
    (a) Show that $|f'(1/2)| \leqq 8$.
    (b) Show that f has precisely one zero inside of the unit circle.

31. Determine the number of zeros of the polynomial

$$z^{87} + 36z^{57} + 71z^4 + z^3 - z + 1$$

inside the circle
    (a) of radius 1,
    (b) of radius 2, centered at the origin.
    (c) Determine the number of zeros of the polynomial

$$2z^5 - 6z^2 + z + 1 = 0$$

in the annulus $1 \leqq |z| \leqq 2$.

32. Let f, h be analytic on the closed disc of radius R, and assume that $f(z) \neq 0$ for z on the circle of radius R. Prove that there exists $\epsilon > 0$ such that $f(z)$ and $f(z) + \epsilon h(z)$ have the same number of zeros inside the circle of radius R. Loosely speaking, we may say that f and a small perturbation of f have the same number of zeros inside the circle.

33. Let $f(z) = a_n z^n + \cdots + a_0$ be a polynomial with $a_n \neq 0$. Use Rouché's theorem to show that $f(z)$ and $a_n z^n$ have the same number of zeros in a disc of radius R for R sufficiently large.

34. (a) Let f be analytic on the closed unit disc. Assume that $|f(z)| = 1$ if $|z| = 1$, and f is not constant. Prove that the image of f contains the unit disc.
    (b) Let f be analytic on the closed unit disc $\bar{D}$. Assume that there exists some point $z_0 \in D$ such that $|f(z_0)| < 1$, and that $|f(z)| \geqq 1$ if $|z| = 1$. Prove that $f(D)$ contains the unit disc.

35. Let $P_n(z) = \sum_{k=0}^{n} z^k/k!$. Given R, prove that P_n has no zeros in the disc of radius R for all n sufficiently large.

36. Let $z_1, \ldots, z_n$ be distinct complex numbers contained in the disc $|z| < R$. Let f be analytic on the closed disc $\bar{D}(0, R)$. Let

$$Q(z) = (z - z_1) \cdots (z - z_n).$$

Prove that

$$P(z) = \frac{1}{2\pi i} \int_{C_R} f(\zeta) \frac{1 - Q(z)/Q(\zeta)}{\zeta - z} \, d\zeta$$

is a polynomial of degree $n - 1$ having the same value as f at the points $z_1, \ldots, z_n$.

37. Let f be analytic on **C** with the exception of a finite number of isolated singularities which may be poles. Define the **residue at infinity**

$$\mathbf{res}_\infty f(z)\, dz = -\frac{1}{2\pi i} \int_{|z|=R} f(z)\, dz$$

for R so large that f has no singularities in $|z| \geqq R$.

(a) Show that $\text{res}_\infty f(z)\, dz$ is independent of R.

(b) Show that the sum of the residues of f at all singularities and the residue at infinity is equal to 0.

38. **Cauchy's Residue Formula on the Riemann Sphere.** Recall Exercise 2 of Chapter V, §3 on the Riemann sphere. By a (meromorphic) **differential** ω on the Riemann sphere S, we mean an expression of the form

$$\omega = f(z)\, dz,$$

where f is a rational function. For any point $z_0 \in \mathbf{C}$ the **residue** of ω at z_0 is defined to be the usual residue of $f(z)\, dz$ at z_0. For the point ∞, we write $t = 1/z$,

$$dt = -\frac{1}{z^2}\, dz \quad \text{and} \quad dz = -\frac{1}{t^2}\, dt,$$

so we write

$$\omega = f(1/t)\left(-\frac{1}{t^2}\right) dt = -\frac{1}{t^2} f(1/t)\, dt.$$

The **residue of** ω **at infinity** is then defined to be the residue of $-\frac{1}{t^2} f(1/t)\, dt$ at $t = 0$. Prove:

(a) $\sum$ residues $\omega = 0$, if the sum is taken over all points of **C** and also infinity.

(b) Let γ be a circle of radius R centered at the origin in **C**. If R is sufficiently large, show that

$$\frac{1}{2\pi i} \int_\gamma f(z)\, dz = -\text{residue of } f(z)\, dz \text{ at infinity}.$$

(Instead of a circle, you can also take a simple closed curve such that all the poles of f in $\mathbf{C}$ lie in its interior.)

(c) If R is arbitrary, and f has no pole on the circle, show that

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = -\sum \text{residues of } f(z)\,dz \text{ outside the circle, including the residue at } \infty.$$

[**Note**: In dealing with (a) and (b), you can either find a direct algebraic proof of (a), as in Exercise 38 and deduce (b) from it, or you can prove (b) directly, using a change of variables $t = 1/z$, and then deduce (a) from (b). You probably should carry both ideas out completely to understand fully what's going on.]

39. (a) Let $P(z)$ be a polynomial. Show directly from the power series expansions of $P(z)\,dz$ that $P(z)\,dz$ has 0 residue in $\mathbf{C}$ and at infinity.

(b) Let α be a complex number. Show that $dz/(z-\alpha)$ has residue -1 at infinity.

(c) Let m be an integer $\geqq 2$. Show that $dz/(z-\alpha)^m$ has residue 0 at infinity and at all complex numbers.

(d) Let $f(z)$ be a rational function. The theorem concerning the **partial fraction decomposition** of f states that f has an expression

$$f(z) = \sum_{i=1}^{r}\sum_{m=1}^{m_i}\frac{a_{ij}}{(z-\alpha_i)^m} + P(z)$$

where $\alpha_1, \ldots, \alpha_r$ are the roots of the denominator of f, a_{ij} are constants, and P is some polynomial. Using this theorem, give a direct (algebraic) proof of Exercise 37(a).

40. Let $a, b \in \mathbf{C}$ with $|a|$ and $|b| < R$. Let C_R be the circle of radius R. Evaluate

$$\int_{C_R}\frac{z\,dz}{\sqrt{(z-a)(z-b)}}.$$

The square root is chosen so that the integrand is continuous for $|z| > R$ and has limit 1 as $|z| \to \infty$.

VI, §2. EVALUATION OF DEFINITE INTEGRALS

Let $f(x)$ be a continuous function of a real variable x. We want to compute

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{A\to\infty}\int_{-A}^{0} f(x)\,dx + \lim_{B\to\infty}\int_0^B f(x)\,dx.$$

We shall use the following method. We let γ be the closed path as indicated on Fig. 7, consisting of a segment on the real line, and a semicircle.

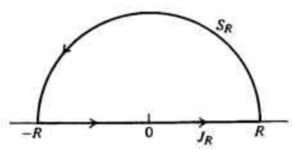

Figure 7

We suppose that $f(x)$ is the restriction to the line of a function f on the upper half plane, meromorphic and having only a finite number of poles. We let J_R be the segment from $-R$ to R, and let S_R be the semicircle. If we can prove that

$$\lim_{R\to\infty} \int_{S_R} f = 0$$

then by the residue formula, we obtain

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \text{residues of } f \text{ in the upper half plane.}}$$

For this method to work, it suffices to know that $f(z)$ goes sufficiently fast to 0 when $|z|$ becomes large, so that the integral over the semicircle tends to 0 as the radius R becomes large. It is easy to state conditions under which this is true.

Theorem 2.1. *Suppose that there exists a number $B > 0$ such that for all $|z|$ sufficiently large, we have*

$$|f(z)| \leqq B/|z|^2.$$

Then

$$\lim_{R\to\infty} \int_{S_R} f = 0$$

and the above formula is valid.

Proof. The integral is estimated by the sup norm of f, which is B/R^2 by assumption, multiplied by the length of the semicircle, which is πR. Since $\pi B/R$ tends to 0 as $R \to \infty$, our theorem is proved.

Remark. We really did not need an R^2, only R^{1+a} for some $a > 0$, so the theorem could be correspondingly strengthened.

Example. Let us compute

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx.$$

The function $1/(z^4+1)$ is meromorphic on $\mathbf{C}$, and its poles are at the zeros of z^4+1, that is the solutions of $z^4 = -1$, which are

$$e^{\pi ik/4}, \qquad k = 1, -1, 3, -3.$$

Let $f(z) = z^4 + 1$. Since $f'(z) = 4z^3 \neq 0$ unless $z = 0$, we conclude that all the zeros of f are simple. The two zeros in the upper half plane are

$$z_1 = e^{\pi i/4} \qquad \text{and} \qquad z_2 = e^{3\pi i/4}.$$

The residues of $1/f(z)$ at these points are $1/f'(z_1)$, $1/f'(z_2)$, respectively, by Lemma 1.3(b), and

$$f'(z_1) = 4z_1^3 = 4e^{3\pi i/4}, \qquad f'(z_2) = 4z_2^3 = 4e^{\pi i/4}.$$

The estimate

$$\left|\frac{1}{z^4+1}\right| \leqq B/R^4$$

is satisfied for some constant B when $|z| = R$. Hence the theorem applies, and

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx &= 2\pi i(\tfrac{1}{4}e^{-3\pi i/4} + \tfrac{1}{4}e^{-\pi i/4}) \\
&= \frac{\pi i}{2} e^{-\pi i/4}(e^{-2\pi i/4} + 1) \\
&= \frac{\pi i}{2}\left(\frac{1-i}{\sqrt{2}}\right)(1-i) \\
&= \frac{\pi}{\sqrt{2}}.
\end{aligned}$$

The estimate for $1/(z^4+1)$ on a circle of radius R presented no subtlety. We give an example where the estimate takes into account a different phenomenon, and a different path. The fact that the integral over the part going to infinity like the semicircle tends to 0 will be due to a more conditional convergence, and the evaluation of an integral explicitly.

Fourier Transforms

Integrals of the form discussed in the next examples are called **Fourier transforms**, and the technique shows how to evaluate them.

Theorem 2.2. *Let f be meromorphic on* **C**, *having only a finite number of poles, not lying on the real axis. Suppose that there is a constant K such that*

$$|f(z)| \leqq K/|z|$$

for all sufficiently large $|z|$. Let $a > 0$. Then

$$\int_{-\infty}^{\infty} f(x)e^{iax}\,dx = 2\pi i \sum \text{residues of } e^{iaz}f(z) \text{ in the upper half plane.}$$

Proof. For simplicity, take $a = 1$. We integrate over any rectangle as shown on Fig. 8, taking $T = A + B$. Taking A, $B > 0$ sufficiently large, it suffices to prove that the integral over the three sides other than the bottom side tend to 0 as A, B tend to infinity.

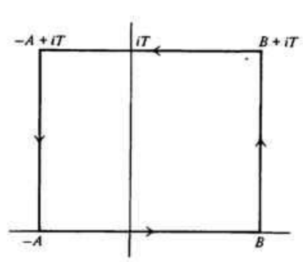

Figure 8

Note that

$$e^{iz} = e^{i(x+iy)} = e^{ix}e^{-y}.$$

In absolute value this is e^{-y}, and tends to 0 rapidly as y tends to infinity. We show that the integral over the top tends to 0. Parametrizing the top by $x + iT$, with $-A \leqq x \leqq B$, we find

$$-\int_{\text{top}} e^{iz}f(z)\,dz = \int_{-A}^{B} e^{ix}e^{-T}f(x + iT)\,dx$$

and in absolute value, this is less than

$$e^{-T}\int_{-A}^{B}|f(x+iT)|\,dx \leqq e^{-T}\frac{K}{T}(A+B).$$

Having picked $T = A + B$ shows that this integral becomes small as A, B become large, as desired.

For the right-hand side, we pick the parametrization

$$B + iy, \qquad \text{with} \quad 0 \leqq y \leqq T,$$

and we find that the right-hand side integral is bounded by

$$\left|\int_0^T e^{iB}e^{-y}f(B+iy)\,dy\right| \leqq \frac{K}{B}\int_0^T e^{-y}\,dy = \frac{K}{B}(1-e^{-T}),$$

which tends to 0 as B becomes large. A similar estimate shows that the integral over the left side tends to 0, and proves what we wanted.

Next we show an adjustment of the above techniques when the function may have some singularity on the real axis. We do this by an example.

Example. Let us compute

$$I = \int_0^\infty \frac{\sin x}{x}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx.$$
$$= \frac{1}{2i}\lim_{\epsilon\to 0}\left[\int_{-\infty}^{-\epsilon}\frac{e^{ix}}{x}\,dx + \int_{\epsilon}^{\infty}\frac{e^{ix}}{x}\,dx\right].$$

Note that the integral I converges, although not absolutely. It is an oscillatory integral. The estimate for convergence comes from integration by parts, and is left to the reader. We can then use the technique of complex analysis to evaluate the integral. We use the closed path C as shown on Fig. 9. To compute such an integral, one has to show that both limits exist, and then one can deal with the more symmetric expression

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx.$$

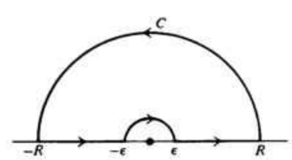

Figure 9

Let $S(\epsilon)$ be the small semicircle from ϵ to $-\epsilon$, oriented counterclockwise, and let $S(R)$ be the big semicircle from R to $-R$ similarly oriented. The function e^{iz}/z has no pole inside C, and consequently

$$0 = \int_C e^{iz}/z\,dz = \int_{S(R)} + \int_{-R}^{-\epsilon} - \int_{S(\epsilon)} + \int_{\epsilon}^{R} e^{iz}/z\,dz.$$

Hence

$$\int_{-R}^{-\epsilon} + \int_{\epsilon}^{R} e^{iz}/z\,dz = \int_{S(\epsilon)} e^{iz}/z\,dz - \int_{S(R)} e^{iz}/z\,dz$$
$$= I_{S(\epsilon)} - I_{S(R)}.$$

We now assert that

$$\lim_{R\to\infty} I_{S(R)} = 0.$$

Proof. We have for $z = R(\cos\theta + i\sin\theta)$,

$$I_{S(R)} = \int_0^\pi \frac{e^{iR\cos\theta}e^{-R\sin\theta}}{Re^{i\theta}} Rie^{i\theta}\,d\theta$$

so that

$$|I_{S(R)}| \leqq \int_0^\pi e^{-R\sin\theta}\,d\theta = 2\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta.$$

But if $0 \leqq \theta \leqq \pi/2$, then $\sin\theta \geqq 2\theta/\pi$ (any similar estimate would do), and hence

$$|I_{S(R)}| \leqq 2\int_0^{\pi/2} e^{-2R\theta/\pi}\,d\theta$$
$$= \frac{\pi}{R}(1 - e^{-R})$$

by freshman calculus. This proves our assertion.

There remains to evaluate the limit of $I_{S(\epsilon)}$ as $\epsilon \to 0$. We state this as a general lemma.

Lemma. *Let g have a simple pole at* 0. *Then*

$$\lim_{\epsilon\to 0} \int_{S(\epsilon)} g(z)\,dz = \pi i\,\mathrm{Res}_0(g).$$

Proof. Write

$$g(z) = \frac{a}{z} + h(z),$$

where h is holomorphic at 0. Then the integral of h over $S(\epsilon)$ approaches 0 as $\epsilon \to 0$ because the length of $S(\epsilon)$ approaches 0 and h is bounded near the origin. A direct integration of a/z shows that the integral of a/z over the semicircle is equal to $\pi i a$. This proves the lemma.

We may therefore put everything together to find the value

$$\boxed{\int_0^\infty \frac{\sin x}{x}\, dx = \pi/2.}$$

Trigonometric Integrals

We wish to evaluate an integral of the form

$$\int_0^{2\pi} Q(\cos\theta, \sin\theta)\, d\theta,$$

where Q is a rational function of two variables, $Q = Q(x, y)$, which we assume is continuous on the unit circle. Since

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i},$$

we see that these expressions are equal to

$$\frac{z + 1/z}{2} \quad \text{and} \quad \frac{z - 1/z}{2i},$$

respectively, when z lies on the unit circle, $z = e^{i\theta}$. It is therefore natural to consider the function

$$f(z) = \frac{Q\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right)}{iz}$$

(the denominator iz is put there for a purpose which will become apparent in a moment). This function f is a rational function of z, and in view of our assumption on Q, it has no pole on the unit circle.

Theorem 2.3. *Let $Q(x, y)$ be a rational function which is continuous when $x^2 + y^2 = 1$. Let $f(z)$ be as above. Then*

$$\int_0^{2\pi} Q(\cos\theta, \sin\theta)\, d\theta = 2\pi i \sum \textit{residues of } f \textit{ inside the unit circle.}$$

Proof. Let C be the unit circle. Then

$$\int_C f(z)\, dz = 2\pi i \sum \text{residues of } f \text{ inside the circle.}$$

On the other hand, by definition the integral on the left is equal to

$$\int_0^{2\pi} f(e^{i\theta}) ie^{i\theta}\, d\theta = \int_0^{2\pi} Q(\cos\theta, \sin\theta)\, d\theta,$$

as desired. [The term iz in the denominator of f was introduced to cancel $ie^{i\theta}$ at this point.]

Example. Let us compute the integral

$$I = \int_0^{2\pi} \frac{1}{a + \sin\theta}\, d\theta$$

where a is real > 1. By the theorem,

$$I = 2\pi \sum \text{residues of } \frac{2i}{z^2 + 2iaz - 1} \text{ inside circle.}$$

The only pole inside the circle is at

$$z_0 = -ia + i\sqrt{a^2 - 1}$$

and the residue is

$$\frac{i}{z_0 + ia} = \frac{1}{\sqrt{a^2 - 1}}.$$

Consequently,

$$I = \frac{2\pi}{\sqrt{a^2 - 1}}.$$

Mellin Transforms

We give a final example introducing new complications. Integrals of type

$$\int_0^\infty f(x)x^a \frac{dx}{x}$$

are called **Mellin transforms** (they can be viewed as functions of a). We wish to show how to evaluate them. *We assume that $f(z)$ is analytic on* **C** *except for a finite number of poles, none of which lies on the positive real axis $0 < x$, and we also assume that a is not an integer.* Then under appropriate conditions on the behavior of f near 0, and when x becomes large, we can show that the following formula holds:

$$\int_0^\infty f(x)x^a \frac{dx}{x} = -\frac{\pi e^{-\pi i a}}{\sin \pi a} \sum \text{residues of } f(z)z^{a-1} \text{ at the poles of } f, \text{ excluding the residue at } 0.$$

We comment right away on what we mean by z^{a-1}, namely z^{a-1} is defined as

$$z^{a-1} = e^{(a-1)\log z},$$

where the log is defined on the simply connected set equal to the plane from which the axis $x \geqq 0$ has been deleted. We take the value for the log such that if $z = re^{i\theta}$ and $0 < \theta < 2\pi$, then

$$\log z = \log r + i\theta.$$

Then, for instance,

$$\log i = \pi i/2 \quad \text{and} \quad \log(-i) = 3\pi i/2.$$

Precise sufficient conditions under which the formula is true are given in the next theorem. They involve suitable estimates for the function f near 0 and infinity.

Theorem 2.4. *The formula given for the integral*

$$\int_0^\infty f(x)x^a \frac{dx}{x}, \quad \textit{with} \quad a > 0,$$

is valid under the following conditions:

1. *There exists a number $b > a$ such that*

$$|f(z)| \ll 1/|z|^b \quad \textit{for} \quad |z| \to \infty.$$

2. *There exists a number b' with $0 < b' < a$ such that*

$$|f(z)| \ll 1/|z|^{b'} \quad \textit{for} \quad |z| \to 0.$$

The symbol $\ll$ means that the left-hand side is less than or equal to some constant times the right-hand side.

For definiteness, we carry out the arguments on a concrete example, and let the reader verify that the arguments work under the conditions stated in Theorem 2.4.

Example. We shall evaluate for $0 < a < 2$:

$$\int_0^\infty \frac{1}{x^2+1} x^a \frac{dx}{x} = \frac{\pi \cos a\pi/2}{\sin a\pi}.$$

We choose the closed path C as on Fig. 10. Then C consists of two line segments L^+ and L^-, and two pieces of semicircles $S(R)$ and $-S(\epsilon)$, if we take $S(\epsilon)$ oriented in counterclockwise direction. The angle φ which the two segments L^+ and L^- make with the positive real axis will tend to 0.

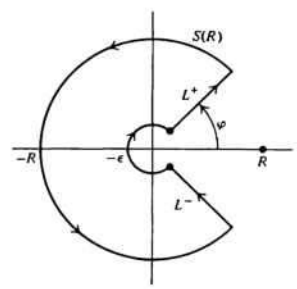

Figure 10

We let

$$g(z) = \frac{1}{z^2+1} z^{a-1}.$$

Then $g(z)$ has only simple poles at $z = i$ and $z = -i$, where the residues are found to be:

$$\text{at } i\text{: } \frac{1}{2i} e^{(a-1)\log i} = \frac{1}{2i} e^{(a-1)\pi i/2},$$

$$\text{at } -i\text{: } -\frac{1}{2i} e^{(a-1)\log(-i)} = -\frac{1}{2i} e^{(a-1)3\pi i/2}.$$

The sum of the residues inside C is therefore equal to

$$\frac{1}{2i}\left(e^{(a-1)\pi i/2} - e^{(a-1)3\pi i/2}\right) = -e^{a\pi i}\cos(a\pi/2),$$

after observing that $e^{\pi i/2} = i$, $e^{-3\pi i/2} = i$, and factoring out $e^{a\pi i}$ from the sum.

The residue formula yields

$$2\pi i \sum \text{residues} = I_{S(R)} + I_{L^-} - I_{S(\epsilon)} + I_{L^+},$$

where I_X denotes the integral of $f(z)$ over the path X. We shall prove:

The integrals $I_{S(R)}$ and $I_{S(\epsilon)}$ tend to 0 as R becomes large and ϵ becomes small, independently of the angle φ.

Proof. First estimate the integral over $S(R)$. When comparing functions of R, it is useful to use the following notation. Let $F(R)$ and $G(R)$ be functions of R, and assume that $G(R)$ is > 0 for all R sufficiently large. We write

$$F(R) \ll G(R) \qquad (\text{for } R \to \infty)$$

if there exists a constant K such that

$$|F(R)| \leqq KG(R)$$

for all R sufficiently large.

With this notation, using $z^{a-1} = e^{(a-1)\log z}$, and

$$|\log z| \leqq \log R + \theta \leqq \log R + 2\pi,$$

we find

$$|z^{a-1}| = |e^{(a-1)\log z}| \ll R^{a-1}.$$

Consequently from $|1/(z^2+1)| \ll 1/R^2$ for $|z| = R$, we find

$$\left|\int_{S(R)} f(z)z^{a-1}\,dz\right| \ll 2\pi R\frac{1}{R^2}\max|z^{a-1}| \ll R^a/R^2.$$

Since we assumed that $a < 2$, the quotient R^a/R^2 approaches 0 as R becomes large, as desired. The estimate is independent of φ.

We use a similar estimating notation for functions of ϵ,

$$F(\epsilon) \ll G(\epsilon) \qquad (\text{for } \epsilon \to 0)$$

if there exists a constant K such that

$$|F(\epsilon)| \leqq KG(\epsilon)$$

for all $\epsilon > 0$ sufficiently small. With this notation, for $|z| = \epsilon$, we have

$$|z^{a-1}| = |e^{(a-1)\log z}| \ll \epsilon^{a-1}.$$

Hence

$$\left|\int_{S(\epsilon)} f(z)z^{a-1}\,dz\right| \ll 2\pi\epsilon\,\epsilon^{a-1} \ll \epsilon^a.$$

Again since we assumed that $a > 0$, the right-hand side approaches 0 and ϵ tends to 0, as desired. The estimate is independent of φ.

There remains to analyze the sums of the integrals over L^+ and L^-. We parametrize L^+ by

$$z(r) = re^{i\varphi}, \qquad \epsilon \leqq r \leqq R,$$

so that $\log z(r) = \log r + i\varphi$. Then

$$\begin{aligned}\int_{L^+} f(z)e^{(a-1)\log z}\,dz &= \int_\epsilon^R f(re^{i\varphi})e^{(a-1)(\log r + i\varphi)}e^{i\varphi}\,dr \\ &= \int_\epsilon^R f(re^{i\varphi})e^{(a-1)i\varphi}e^{i\varphi}r^{a-1}\,dr \\ &\to \int_\epsilon^R f(x)x^{a-1}\,dx \qquad \text{as} \quad \varphi \to 0.\end{aligned}$$

On the other hand, $-L^-$ is parametrized by

$$z(r) = re^{i(2\pi - \varphi)}, \qquad \epsilon \leqq r \leqq R,$$

and $\log z(r) = \log r + i(2\pi - \varphi)$. Consequently,

$$\int_{L^-} f(z)e^{(a-1)\log z}\,dz = -\int_\epsilon^R f(re^{-i\varphi})r^{a-1}e^{(a-1)(2\pi-i\varphi)}e^{i(2\pi-i\varphi)}\,dr$$
$$= -\int_\epsilon^R f(re^{i\varphi})r^{a-1}e^{ai(2\pi-\varphi)}\,dr$$
$$\to -\int_\epsilon^R f(x)x^{a-1}e^{2\pi ia}\,dx \qquad \text{as} \quad \varphi \to 0.$$

Hence as $\varphi \to 0$, we find

$$\int_{L^+} + \int_{L^-} f(z)z^{a-1}\,dz \to \int_\epsilon^R f(x)x^{a-1}(1 - e^{2\pi ia})\,dx$$
$$= e^{\pi ia}\int_\epsilon^R f(x)x^{a-1}(e^{-\pi ia} - e^{\pi ia})\,dx$$
$$= 2ie^{\pi ia}\sin \pi a \int_\epsilon^R f(x)x^{a-1}\,dx.$$

Let $C = C(R, \epsilon, \varphi)$ denote the path of integration. We obtain

$$\int_{C(R,\epsilon,\varphi)} f(z)z^{a-1}\,dz = 2\pi i \sum \text{residues of } f(z)z^{a-1} \text{ except at } 0$$
$$= I_{S(R,\varphi)} + I_{S(\epsilon,\varphi)} + E(R, \epsilon, \varphi)$$
$$+ 2ie^{\pi ia}\sin \pi a \int_\epsilon^R f(x)x^{a-1}\,dx.$$

The expression $E(R, \epsilon, \varphi)$ denotes a term which goes to 0 as φ goes to 0, and we have put subscripts on the integrals along the arcs of the circle to show that they depend on R, ϵ, φ. We divide by $2ie^{\pi ia}\sin \pi a$, and let φ tend to 0. Then $E(R, \epsilon, \varphi)$ approaches 0. Consequently,

$$\int_\epsilon^R f(x)x^{a-1}\,dx - \frac{\pi e^{-\pi ia}}{\sin \pi a}\sum = \lim_{\varphi\to 0} \frac{I_{S(R,\varphi)} + I_{S(\epsilon,\varphi)}}{2ie^{\pi ia}\sin \pi a}.$$

The right-hand side has been seen to be uniformly small, independently of φ, and tends to 0 when $R \to \infty$ and $\epsilon \to 0$. Taking the limits as $R \to \infty$ and $\epsilon \to 0$ proves what we wanted.

Finally, we observe that in situations of contour integrals as we just considered, it is often the practice to draw the limit contour as in Fig. 11.

It is then understood that the value for the log when integrating over the segment from ϵ to R from left to right, and the value for the log when integrating over the segment from R to ϵ, are different, arising from the analytic expressions for the log with values $\theta = 0$ for the first and $\theta = 2\pi$ for the second.

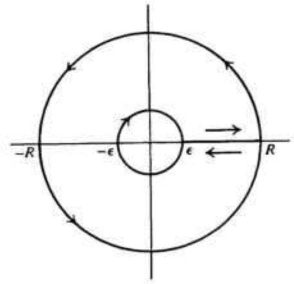

Figure 11

For the Mellin transform of the gamma function, which provides an interesting special concrete case of the considerations of this section, see Exercise 7 of Chapter XV, §2.

VI, §2. EXERCISES

Find the following integrals.

1. (a) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^6+1}\,dx = 2\pi/3$

   (b) Show that for a positive integer $n \geqq 2$,

$$\int_0^\infty \frac{1}{1+x^n}\,dx = \frac{\pi/n}{\sin \pi/n}.$$

   [*Hint*: Try the path from 0 to R, then R to $Re^{2\pi i/n}$, then back to 0, or apply a general theorem.]

2. (a) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx = \pi\sqrt{2}/2$ (b) $\displaystyle\int_0^\infty \frac{x^2}{x^6+1}\,dx = \pi/6$

3. Show that

$$\int_{-\infty}^{\infty} \frac{x-1}{x^5-1}\,dx = \frac{4\pi}{5}\sin\frac{2\pi}{5}.$$

4. Evaluate

$$\int_\gamma \frac{e^{-z^2}}{z^2}\,dz,$$

where γ is:

(a) the square with vertices $1+i$, $-1+i$, $-1-i$, $1-i$.

(b) the ellipse defined by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

(The answer is 0 in both cases.)

5. (a) $\displaystyle\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,dx = \pi e^{-a}$ if $a > 0$

(b) For any real number $a > 0$,

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,dx = \pi e^{-a}/a.$$

[*Hint*: This is the real part of the integral obtained by replacing $\cos x$ by e^{ix}.]

6. Let $a, b > 0$. Let $T \geqq 2b$. Show that

$$\left|\frac{1}{2\pi i}\int_{-T}^{T} \frac{e^{iaz}}{z-ib}\,dz - e^{-ba}\right| \leqq \frac{1}{Ta}(1-e^{-Ta}) + e^{-Ta}.$$

Formulate a similar estimate when $a < 0$.

7. Let $c > 0$ and $a > 0$. Taking the integral over the vertical line, prove that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{a^z}{z}\,dz = \begin{cases} 0 & \text{if } a < 1, \\ \frac{1}{2} & \text{if } a = 1, \\ 1 & \text{if } a > 1. \end{cases}$$

If $a = 1$, the integral is to be interpreted as the limit

$$\int_{c-i\infty}^{c+i\infty} = \lim_{T\to\infty}\int_{c-iT}^{c+iT}$$

[*Hint*: If $a > 1$, integrate around a rectangle with corners $c - Ai$, $c + Bi$, $-X + Bi$, $-X - Ai$, and let $X \to \infty$. If $a < 1$, replace $-X$ by X.]

8. (a) Show that for $a > 0$ we have

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)^2}\,dx = \frac{\pi(1+a)}{2a^3e^a}.$$

(b) Show that for $a > b > 0$ we have

$$\int_0^\infty \frac{\cos x}{(x^2 + a^2)(x^2 + b^2)}\,dx = \frac{\pi}{a^2 - b^2}\left(\frac{1}{be^b} - \frac{1}{ae^a}\right).$$

9. $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \pi/2$. [*Hint*: Consider the integral of $(1 - e^{2ix})/x^2$.]

10. $\int_{-\infty}^\infty \frac{\cos x}{a^2 - x^2}\,dx = \frac{\pi \sin a}{a}$ for $a > 0$. The integral is meant to be interpreted as the limit:

$$\lim_{B\to\infty}\lim_{\delta\to 0}\int_{-B}^{-a-\delta} + \int_{-a+\delta}^{a-\delta} + \int_{a+\delta}^{B}.$$

11. $\int_{-\infty}^\infty \frac{\cos x}{e^x + e^{-x}}\,dx = \frac{\pi}{e^{\pi/2} + e^{-\pi/2}}$. Use the indicated contour:

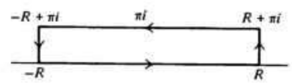

Figure 12

12. $\int_0^\infty \frac{x \sin x}{x^2 + a^2}\,dx = \frac{1}{2}\pi e^{-a}$ if $a > 0$.

13. $\int_{-\infty}^\infty \frac{e^{ax}}{e^x + 1}\,dx = \frac{\pi}{\sin \pi a}$ for $0 < a < 1$.

14. (a) $\int_0^\infty \frac{(\log x)^2}{1 + x^2}\,dx = \pi^3/8$. Use the contour

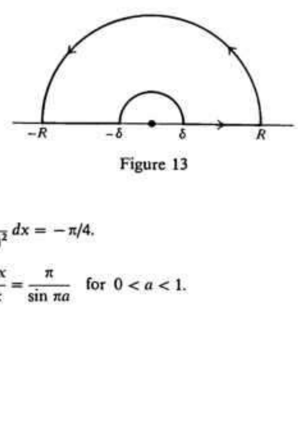

Figure 13

(b) $\int_0^\infty \frac{\log x}{(x^2 + 1)^2}\,dx = -\pi/4$.

15. (a) $\int_0^\infty \frac{x^a}{1 + x}\frac{dx}{x} = \frac{\pi}{\sin \pi a}$ for $0 < a < 1$.

(b) $\displaystyle\int_0^\infty \frac{x^a}{1+x^3}\frac{dx}{x} = \frac{\pi}{3\sin(\pi a/3)}$ for $0 < a < 3$.

16. Let f be a continuous function, and suppose that the integral

$$\int_0^\infty f(x)x^a \frac{dx}{x}$$

is absolutely convergent. Show that it is equal to the integral

$$\int_{-\infty}^\infty f(e^t)e^{at}\,dt.$$

If we put $g(t) = f(e^t)$, this shows that a Mellin transform is essentially a Fourier transform, up to a change of variable.

17. $\displaystyle\int_0^{2\pi} \frac{1}{1+a^2-2a\cos\theta}\,d\theta = \frac{2\pi}{1-a^2}$ if $0 < a < 1$. The answer comes out the negative of that if $a > 1$.

18. $\displaystyle\int_0^\pi \frac{1}{1+\sin^2\theta}\,d\theta = \pi/\sqrt{2}$.

19. $\displaystyle\int_0^\pi \frac{1}{3+2\cos\theta}\,d\theta = \pi/\sqrt{5}$.

20. $\displaystyle\int_0^\pi \frac{a\,d\theta}{a^2+\sin^2\theta} = \int_0^{2\pi} \frac{a\,d\theta}{1+2a^2-\cos\theta} = \frac{\pi}{\sqrt{1+a^2}}$.

21. $\displaystyle\int_0^{\pi/2} \frac{1}{(a+\sin^2\theta)^2}\,d\theta = \frac{\pi(2a+1)}{4(a^2+a)^{3/2}}$ for $a > 0$.

22. $\displaystyle\int_0^{2\pi} \frac{1}{2-\sin\theta}\,d\theta = 2\pi/\sqrt{3}$.

23. $\displaystyle\int_0^{2\pi} \frac{1}{(a+b\cos\theta)^2}\,d\theta = \frac{2\pi a}{(a^2-b^2)^{3/2}}$ for $0 < b < a$.

24. Let n be an even integer. Find

$$\int_0^{2\pi} (\cos\theta)^n\,d\theta$$

by the method of residues.