جامعة جنوب الوادى

كلية التربية بالغردقة

الفرقة الثالثة عام رياضيات Math

المادة : (5) Applied (Mathematical Method)

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الفصل الدراسى الأول 2023-2024

The first chapter

مسائل القيم الحدية

Boundary Value Problems

Data on the partial deferential equations:

The partial deferential equations is a equation contains on an unknown function of two or more variables, as well as on the various derivatives of the function with respect to these variables.

Equation rank:

It is a higher order derived in the equation.

An example of this is the equation $\frac{\partial^2 u}{\partial x \partial y} = 2x - y$: It is of the second order.

Solve the equation:

It is any function that satisfies the equation. The general solution contains a number of optional functions equal to the rank of the equation. The specific solution is a solution that can be obtained from the general solution after specifying the optional functions.

Example:

$$u = x^2 y - \tfrac{1}{2} x y^2 + F(x) + G(y)$$

Where F , G are optional functions - this function is a general solution to the equation

$$\frac{\partial^2 u}{\partial x \partial y} = 2x - y$$

Because it contains two optional functions $F(x), G(y)$, so if we consider

$$F(x) = 2\sin x \quad , \quad G(y) = 3y^4 - 5$$

For example, we get the specific solution

$$u = x^2 y - (1/2) x y^2 + 2\sin x + 3y^4 - 5$$

Boundary value issue:

It is a solution to a partial differential equation on the condition that the solution fulfills certain conditions called Boundary Conditions

Linear P.D.E.:

The general formula for a second-order linear partial differential equation in two variables is:

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + F\, u = G$$

In general, where $A, B, \ldots\ldots\ldots\ldots, G$ are functions in x, y , but they do not include u . An equation to which this formula does not apply is called nonlinear, and if $G = 0$, it is called homogeneous, otherwise it is called non-homogeneous.

The equation is classified as elliptic, hyperbolic, or parabolic depending on whether the amount $B^2 - 4AC$ is less than, greater than, or equal to zero.

<u>Some famous equations:</u>

<u>(1)Vibrating String:</u>

If the string is in equilibrium, aligned with an x-axis.

His movement was always completed

at one plane, which was the plane xy.

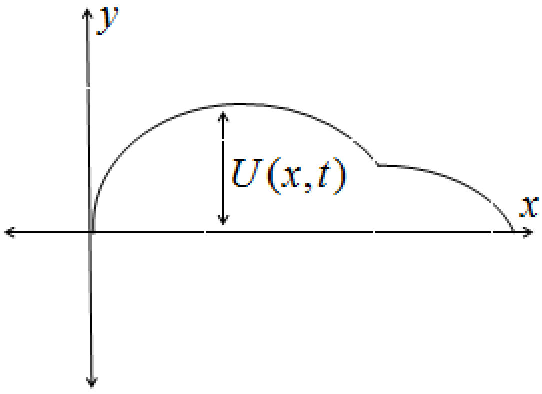
If we assume that $u(x,t)$ is the

displacement of all points of the

string from the equilibrium position

at any time, then the partial equation

to which this function is subject is:



$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

Where a^2 is a natural constant depends on the tension in the string and the longitudinal density of the string material.

As for the vibrating membrane, its equation is:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

Where $u(x,y,t)$ is the displacement of at any time $.t$

The oscillations that occur in an elastic body are subject to the following equation:

The oscillations that occur in an elastic body are subject to the following equation:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

Where $u(x,y,z,t)$ is the displacement of any point in the body from its equilibrium position.

2) Heat conduction equation:

The equation is $\frac{\partial u}{\partial t} = K\,\nabla^2 u$ where $u(x,y,z,t)$ is the temperature for any point at any time and where K is a natural constant that depends on the conductivity coefficient and specific heat, where $\nabla^2 u$ is the Laplacian whose formula is

in Cartesian coordinates they are:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

3) Laplace Equation:

The equation is $\nabla^2 u = 0$ and this is the equation of heat conduction in the steady state and that after sufficient time has passed for it to become $\frac{\partial u}{\partial t} = 0$. So this equation expresses the electric potential in the static field at the points which has no electrical charges.

And if it is required to solve this equation in an area Q and the value of u is given on the borders of this area, then the problem is called (Dirichelt).

Laplacians in different coordinates:

In cylindrical coordinates

$$\nabla^2 u = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2}$$

where $\rho \ge 0 \quad , \quad 0 \le \phi < 2\pi \quad , \quad -\infty < z < \infty$

In spherical coordinates

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}$$

where $\rho \ge 0 \quad , \quad 0 \le \phi < 2\pi \quad , \quad -\infty < z < \infty$

$$r \ge 0 \quad , \quad 0 \le \theta < \pi \quad , \quad 0 \le \phi < 2\pi$$

Theories in solving partial differential equations:

Theorem (1): It is called the principle of addition... (Superposition Principle)

This theorem states that if functions $u_1 , u_2 ,........, u_n$ are different solutions to a linear partial differential equation and $c_1 , c_2 ,........ , c_n$ are constants, then the function $u = c_1 u_1 + c_2 u_2 + ... + c_n u_n$ is a solution to the same equation.

Theorem:(2) : The general solution of a non-homogeneous linear equation is obtained by adding a special solution for the non-homogeneous linear equation to the general solution for the homogeneous linear equation.

Separating variables:

The method of separating variables depends on the assumption that there is a solution to the equation that is in the form of the product of the functions of each of them for one variable. After obtaining the specific solutions, the principle of addition is applied to find the required solution.

Examples

(1) Write the boundary conditions for a string of length L, given that:

A - The ends $x=0$, $x=L$ are fixed.

B - The initial form of the string is given by the function $f(x)$ /

C - The initial distribution of velocities is given by the function $g(x)$.

D - The displacement of any point at any time is limited.

The solution

$a)$ $y(0,t)=0$, $y(L,t)=0$

$b)$ $y(x,0)=f(x)$, $0<x<L$

$c)$ $y_t(x,0)=g(x)$

$d)$ $|y(x,t)|<M$

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Exercises

1) Write the boundary conditions for a string of length L, noting that:

A- The end $x=0$ moves so that its movement is given by the function $G(t)$.

B- The end $x=L$ is unrestrained and free to move.

2) A thin metal rod placed on an x-axis and its two ends $x = 0$, $x = L$ are thermally insulated sides.

Write the boundary condition in the following cases:

A- Both ends are fixed at zero degrees.

B- The first end is at zero degrees and the second is insulated, knowing that $f(x)$ is a function of the initial heat distribution.

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Separation of Variables

Example (1) Find the solution to the boundary problem:

$$\frac{\partial U}{\partial t} = 2\frac{\partial^2 U}{\partial x^2} \quad , \quad 0 < x < 3$$

With the conditions:

$$U(0,t) = U(3,t) = 0 \quad , \quad |U(x,t)| < M.$$

$$U(x,0) = 5\sin 4\pi x - 3\sin 8\pi x + 2\sin 10\pi x.$$

then suggest the natural meaning of this problem.

The solution

Assume that

$$U(x,t) = X(x) \,.\, T(t)$$

Substituting into the equation results:

$$\therefore\ X\,T' = 2\,X''\,T$$

Dividing by $2\ X\ T$ results in:

$$\frac{X''}{X} = \frac{T'}{2T}$$

The right side of this relationship is a function of the variable t , however it is equal to the left side, which does not depend on t. So we are faced with a function in the variable t that does not depend on t, and therefore this function must be equal to a constant amount.

The same statement applies to the left side, which is a function of x and is equal to the right side, which does not depend on x. Therefore, the left side is equal to the same constant that the right side is equal to.

Hence it results that

$$\frac{X''}{X} = \mu \quad , \quad \frac{T'}{2T} = \mu$$

Where μ is an unknown constant, and we will then learn how to determine its value.

In our problem that we are now discussing, we will set the constant $\mu = -\lambda^2$, that is:

$$X'' + \lambda^2 X = 0 \quad , \quad T' + 2\lambda^2 T = 0$$

We will understand the reason for our situation $\mu = -\lambda^2$ shortly. Now we can say that we are faced with two ordinary differential equations.

It is as if solving one partial differential equation was reduced by separating the variables to solving two ordinary differential equations, and this is of course much easier than solving the partial equation.

The reason for our situation $\mu = -\lambda^2$ is as follows:

The general thing is that when we solve a partial equation, we are faced with a general solution from which we want to reach a specific solution that fulfills the required conditions. The solution is a comprehensive solution that suits all circumstances, and therefore we are supposed to remove from the general solution everything that would not be consistent with our conditions in the issue. Therefore, the constant μ from the point of view of the general solution has no restrictions on its choice, and accordingly it can be negative or positive. Therefore, if we had written the second equation in T as $T' - 2\mu T = 0$, it would have been the solution to be $T = e^{2\mu t}$.

Here we find ourselves faced with two situations:

First: If μ is negative, then the function T is a decreasing function, this meaning that its value decreases as the value t increases.

Second: If μ is positive, we get a function that increases with time and we reach an infinite value.

But one of our conditions in this matter is that $|U(x,t)| < M$:

That is, U must be limited, regardless of the value of time.

$$X = A_1 \cos\lambda x + B_1 \sin\lambda x \quad , \quad T = C_1\, e^{-2\lambda^2 t}$$

Therefore, the solution that is in it μ positive does not suit this particular issue. Therefore, we must remove from the general solution what would lead to such a solution, so the constant must be negative.

As for why we put $\mu = -\lambda^2$, it is to ensure that the constant is negative. We put a negative sign in front of a square expression λ^2 in order to ensure that λ^2 a positive expression is always preceded by a negative sign, and thus we ensure that the constant is always negative.

Therefore, the solution of the two equations after this procedure is:

$$X = A_1 \cos\lambda x + B_1 \sin\lambda x \quad , \quad T = C_1\, e^{-2\lambda^2 t}$$

Hence:

$$U(x,t) = C_1 e^{-2\lambda^2 t} (A_1 \cos\lambda x + B_1 \sin\lambda x)$$

That is:

$$U(x,t) = e^{-2\lambda^2 t} (A \cos\lambda x + B \sin\lambda x)$$

It now remains to determine the values of the constants A, B, λ by applying the rest of the conditions.

But : $U(0,t) = 0 = A e^{-2\lambda^2 t}$

This results in that $A = 0$ .and then

Applying the other condition: $U(3,t) = 0 = B e^{-2\lambda^2 t} \sin 3\lambda$

Here we cannot say $B = 0$ otherwise the solution is $U(x,t) = 0$

This is what is called a trivial solution that fulfills the equation and its conditions without having any natural meaning.

So the only possibility is that $\sin 3\lambda = 0$ results in:

$$3\lambda = \pi \quad , \quad \lambda = \frac{m\pi}{3}$$

Where $m = 1, 2, 3, \ldots\ldots\ldots\ldots\ldots\ldots$

With this, we have not completely determined, but we know that .it cannot take any value except one of the set of values $\frac{m\pi}{3}$

$$\therefore U(x,t) = B e^{\frac{-2m^2\pi^2 t}{9}} \sin\frac{m\pi x}{3}$$

It is clear that every value of the number m gives us a solution to the equation, and the general solution is obtained from the principle of addition.

Since the values of m their number are infinite, then the number of special solutions is infinite, so we are supposed to say that:

$$U(x,t) = \sum_{m=1}^{\infty} B_m \; e^{\frac{-2m^2\pi^2 t}{9}} \sin\frac{m\pi x}{3}$$

But the last condition in the problem includes only three terms, each of which is a sine function, so it suffices to say:

$$U(x,t) = B_1 \; e^{\frac{-2m_1^2\pi^2 t}{9}} \sin\frac{m_1\pi x}{3} + B_2 \; e^{\frac{-2m_2^2\pi^2 t}{9}} \sin\frac{m_2\pi x}{3} + B_3 \; e^{\frac{-2m_3^2\pi^2 t}{9}} \sin\frac{m_3\pi x}{3}$$

Applying the last condition:

$$U(x,0) = 5\sin 4\pi x - 3\sin 8\pi x + 2\sin 10\pi x$$

$$= B_1 \sin\frac{m_1\pi x}{3} + B_2 \sin\frac{m_2\pi x}{3} + B_3 \sin\frac{m_3\pi x}{3}$$

That is

$B_1 = 5$, $B_2 = -3$, $B_3 = 2$

$m_1 = 12$, $m_2 = 24$, $m_3 = 30$

Thus the solution is:

$$U(x,t) = 5\; e^{-32\pi^2 t} \sin 4\pi x - 3\; e^{-128\pi^2 t} \sin 8\pi x + 2\; e^{-200\pi^2 t} \sin 10\pi x$$

The natural meaning of the issue:

This problem expresses the thermal conductivity in a thin rod three units long, with its ends at zero degrees, and the initial thermal distribution is:

$$U(x,0) = 5\sin 4\pi x - 3\sin 8\pi x + 2\sin 10\pi x$$

The solution we obtained gives the temperature distribution at any moment.

Example (2)Solve the boundary problem

$$\frac{\partial U}{\partial t} = 2\frac{\partial^2 U}{\partial x^2} \quad , \quad 0 < x < 3$$

$$U(0,t) = U(3,t) = 0 \quad , \quad U(x,0) = f(x) \quad , \quad |U(x,t)| < M.$$

<u>The solution</u>

This problem is the same as the previous problem, except that the initial distribution function in the previous problem consists of three terms of sine functions. However, in this problem, the initial distribution is given by a function $f(x)$ that may include an unlimited number of sine functions, and it may not itself have a direct relationship with the sine functions, and to solve them, we follow the previous steps until we reach:

$$U(x,t) = \sum_{m=1}^{\infty} B_m \; e^{\frac{-2m^2\pi^2 t}{9}} \sin\frac{m\pi x}{3}$$

Applying the last condition we find that:

$$U(x,0) = f(x) = \sum_{m=1}^{\infty} B_m \sin\frac{m\pi x}{3}$$

Here it is necessary for us to search for a way that enables us to find values of B_m with information of $f(x)$.

Here we need what is called Fourier analysis, and so the goal of this example is to explain to us the reason for using this method, which we will discuss in the next chapter. This method will be similar to the same method used in the next chapter to determine c_n

Exercises

1) Solve the following boundary problem:

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial y} \quad , \quad U(0,y) = 8\,e^{-3y}$$

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2) Solve the previous exercise provided that:

$$U(0,y) = 8\,e^{-3y} + 4\,e^{-5y}$$

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3)Find the solution to the following marginal problem and explain its natural meaning:

$$\frac{\partial^2 Y}{\partial t^2} = 16\,\frac{\partial^2 Y}{\partial x^2} \quad , \quad 0 < x < 2$$

$$Y(0,t) = 0 \quad , \quad Y(2,t) = 0 \quad , \quad Y(x,0) = 6\sin \pi x - 3\sin 4\pi x,$$

$$Y_t(x,0) = 0 \quad , \quad |Y(x,t)| < M.$$

The second chapter

Fourier Series ..... متسلسلة فورير

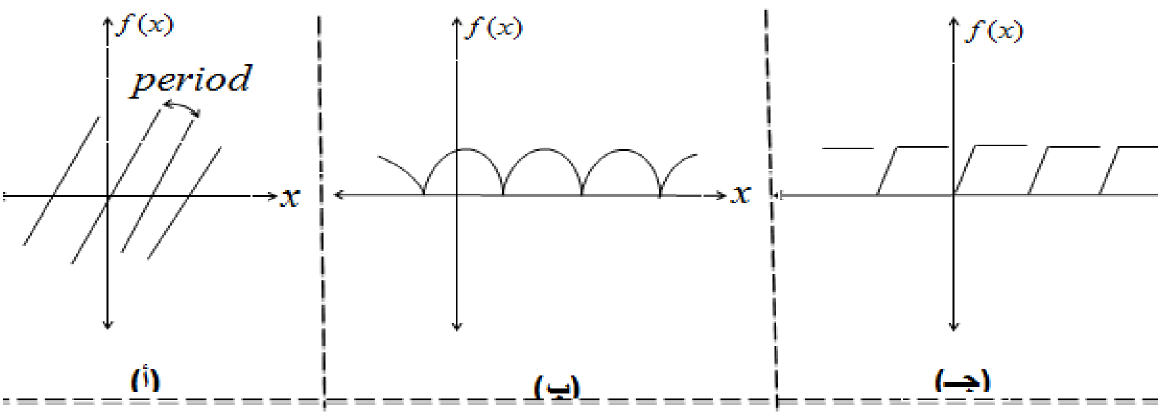
We have seen in the last chapter that the function $f(x)$ can be expressed as a sum of some sine and cosine functions, and since these functions are periodic functions then $f(x)$ must be periodic

Definition:

$f(x)$ is called a periodic functions if $f(x+p)=f(x)$, where p is a positive constant and the least value of it is called the period of the function.

For example $\sin x$ its period is 2π , $\tan x$ its period is π , and $\cos nx$ its period is $\frac{2\pi}{n}$.

The following figures are another examples for periodic functions.



Orthogonal Functions

In case of the vectors we say that the tow vectors A, B are orthogonal if $\underline{A}\bullet\underline{B}=A_1B_1+A_2B_2+A_3B_3=0$, and based on that it can be say that the two functions $A(x), B(x)$ are orthogonal on the interval (a,b) if $\int_a^b A(y)B(x)dx = 0$.

Also it can be say that A is a unit vector if $\underline{A} \bullet \underline{A} = A^2 = 1$, and so in case of functions it can say that the function $A(x)$ is a normal function if $\int_a^b A^2(x)dx = 1$.

Examples for the Orthogonal Functions

Example (1):Prove that

$$(a) \quad \int_{-L}^{L} \sin \frac{k\pi x}{L}\, dx = \int_{-L}^{L} \cos \frac{k\pi x}{L}\, dx = 0 \quad ; \quad k = 1, 2, 3, ...$$

$$(b) \quad \int_{-L}^{L} \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L}\, dx = \quad = \int_{-L}^{L} \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L}\, dx$$

$$= \begin{cases} 0 & : \quad m \neq n \\ L & : \quad m = n \end{cases}$$

$$(c) \quad \int_{-L}^{L} \sin \frac{m\pi x}{L} \cos \frac{m\pi x}{L}\, dx = 0$$

The solution

$$(a) \quad \int_{-L}^{L} \sin \frac{k\pi x}{L}\, dx = -\frac{L}{k\pi} \cos \frac{k\pi x}{L} \Big]_{-L}^{L}$$

$$= -\frac{L}{k\pi} \left(\cos k\pi - \cos(-k\pi) \right) = 0$$

And by the same method we can prove that

$$\int_{-L}^{L} \cos \frac{k\pi x}{L}\, dx = 0$$

(b) if $\underline{m \neq n}$ then

$$(b)\ \int_{-L}^{L} \cos\frac{m\pi x}{L}\ \cos\frac{n\pi x}{L}\ dx = \tfrac{1}{2}\left[\int_{-L}^{L}\cos\frac{(m-n)\pi x}{L}\,dx + \int_{-L}^{L}\cos\frac{(m+n)\pi x}{L}\,dx\right] = 0$$

And by the same method we can prove that if $\underline{m=n}$ then

$$\int_{-L}^{L}\cos^2\left(\frac{n\pi x}{L}\right)dx = \tfrac{1}{2}\int_{-L}^{L}\left(1+\cos\frac{2n\pi x}{L}\right)dx = L$$

$$\int_{-L}^{L}\sin^2\left(\frac{n\pi x}{L}\right)dx = \tfrac{1}{2}\int_{-L}^{L}\left(1-\cos\frac{2n\pi x}{L}\right)dx = L$$

$$(c)\ \int_{-L}^{L}\sin\frac{m\pi x}{L}\ \cos\frac{n\pi x}{L}\ dx = \tfrac{1}{2}\left[\int_{-L}^{L}\sin\frac{(m-n)\pi x}{L}\,dx + \int_{-L}^{L}\sin\frac{(m+n)\pi x}{L}\,dx\right] = 0$$

Generally if $\phi_k(x)$ where $k=1,2,.....$ are some functions has the following properties

$$\int_a^b \phi_m(x)\phi_n(x)dx = 0 \qquad for\ \ m \neq n$$

$$\int_a^b \{\phi_n(x)\}^2 dx = 1 \qquad for\ \ m = n$$

Then it can be say that these functions are Orthonormal, and which can be written in an another way as

$$\int_a^b \phi_m(x)\phi_n(x)dx = \delta_{mn}$$

Where

$$\delta_{mn} = \begin{cases} 0 & : m \neq n \\ 1 & : m = n \end{cases}$$

δ_{mn} is called Kronecker Symbol, and example for this kind of these functions we see that the group $\phi_m(x) = \sqrt{\frac{2}{\pi}}.\sin m\pi x$ are Orthonormal functions on the interval $(0,\pi)$. you can verify for that.

Orthogonality with respect to weight Functions

If $\int_a^b \psi_m(x)\psi_n(x)w(x)dx = \delta_{mn}$ where $w(x) > 0$ is a known function , then it can be say that the group $\{\psi_n(x)\}$ are Orthonormal functions with respect to the weight function $w(x)$, and in this case it can be considered that the group $\phi_m(x) = \sqrt{w(x)}\psi_m(x)$ are Orthonormal functions with respect to a weight function $\phi_k(x)$ as

The expansion by Orthonormal Functions:

As in the case of the vector A which can be written in term of the Orthonormal vectors $\underline{i}, \underline{j}, \underline{k}$ in the form $\underline{A} = c_1\underline{i} + c_2\underline{j} + c_3\underline{k}$,

We can write $f(x)$ as an expansion of the Orthonormal Functions in the form

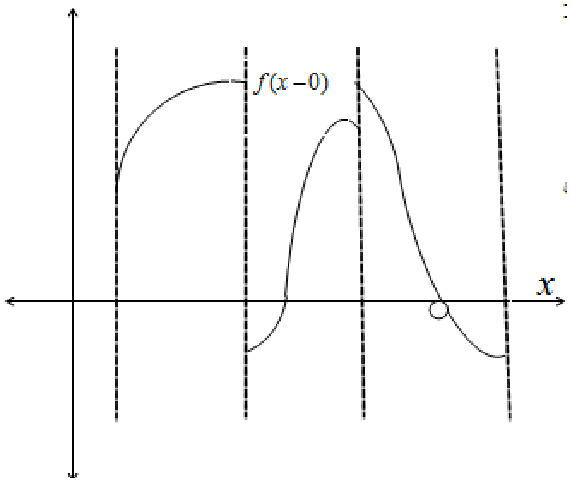
$$f(x)=\sum_{n-1}^{\infty}c_n\phi_n(x)$$

This can be considered as a generalized for Fourier theorem which can be disused later where the coefficients c_m can be determined from

$$c_m=\int_a^b f(x)\phi_m(x)dx$$

Piecewise Continuous Functions:

The function is called a pricewise continuous function on some interval if:



(1) The interval can be divided into a finite number of subintervals such that the function itself is continuous on each of it.

(2) The limits at the points of discontinuity are finite values.

The Fourier series:

If the function $f(x)$ is defined on the interval $(-L,L)$ and periodic with period $2L$, then its Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

where the coefficients a_n, b_n can be determined from the following formulas:

The proof:

Here we shall use the idea of the orthogonality of functions which mentioned in the Fourier series and has been proved in the last example .

(a) By taking the integral from $-L$ to L for both sides we get

$$\int_{-L}^{L} f(x)\, dx = \frac{a_0}{2}\int_{-L}^{L} dx + \sum_{n=1}^{\infty}\left[a_n \int_{-L}^{L} \cos\frac{n\pi x}{L}\, dx + b_n \int_{-L}^{L} \sin\frac{n\pi x}{L}\, dx \right]$$

$$= \frac{a_0}{2} \,.\, 2L + 0 = a_0 L$$

Thus $\quad a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\, dx$

Now multiplying both sides by $\cos\frac{m\pi x}{L}$ and integrate we get

$$\int_{-L}^{L} f(x) \cos\frac{m\pi x}{L}\, dx = \frac{a_0}{2}\int_{-L}^{L} \cos\frac{m\pi x}{L}\, dx$$

$$+ \sum_{n=1}^{\infty} a_n \int_{-L}^{L} \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\, dx$$

$$+ \sum_{n=1}^{\infty} b_n \int_{-L}^{L} \sin\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\, dx$$

$$= 0 + a_m L + 0$$

$$\text{Thus } a_m = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{m\pi x}{L} \, dx \qquad ; m = 1, 2, 3, \ldots\ldots\ldots\ldots$$

(b) multiplying both sides by $\sin \frac{m\pi x}{L}$ and integrate we get

$$\int_{-L}^{L} f(x) \sin \frac{m\pi x}{L} \, dx = \frac{a_0}{2} \int_{-L}^{L} \sin \frac{m\pi x}{L} \, dx + \sum_{n=1}^{\infty} a_n \int_{-L}^{L} \cos \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx$$

$$+ \sum_{n=1}^{\infty} b_n \int_{-L}^{L} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx = 0 + 0 + b_m L$$

$$\text{Thus} \quad b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{m\pi x}{L} \, dx$$

Note:

We can take the limits of integrals from any point to another point such that the distance between them is equal to value of the period

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The Dirichlet conditions for convergence of Fourier series:

The Fourier series will converge to the function $f(x)$ if:

(a) The function $f(x)$ is defined and single-valued except possibly at a finite number of points on the interval $(-L, L)$ this condition necessary but not sufficient.

(b) $f(x)$ is periodic function with period $2L$

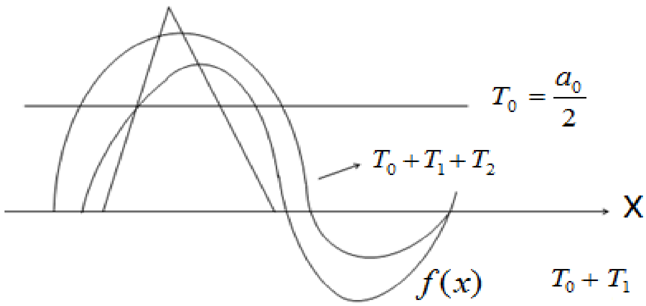
(c) $f(x)$, $f'(x)$ are piecewise continuous.

If these conditions are satisfied then the series (1) with the coefficients (2) will convergent to the function $f(x)$ at the points of continuity and to $\dfrac{f(x+0)+f(x-0)}{2}$ at the points of discontinuity.

We can note the convergent of Fourier series to the function $f(x)$ from the following figure:

We suppose that $f(x) = T_0 + T_1 + T_2 + ...$ where

$$T_0 = \frac{a_0}{2} \quad , \quad T_1 = a_1 \cos\frac{\pi x}{L} \quad , \quad T_2 = a_2 \cos\frac{2\pi x}{L} \quad , \quad ...$$



<u>Odd & Even Functions</u>

The $f(x)$ is called odd if $f(-x) = -f(x)$, examples for that

$$x^3 \ , \ x^5 - 3x^3 + 2x \ , \ \sin x \ , \ \tan 3x$$

While $f(x)$ is called even if $f(-x) = f(x)$, examples for that

$$x^4 \ , \ 2x^2 + 5 \ , \ \cos x \ , \ e^x + e^{-x}$$

<u>Half –Range Series:</u>

If the function is odd then cosine terms will vanish, whil If the function is even then sine terms will vanish, and in these cases the coefficients will take the following formulas:

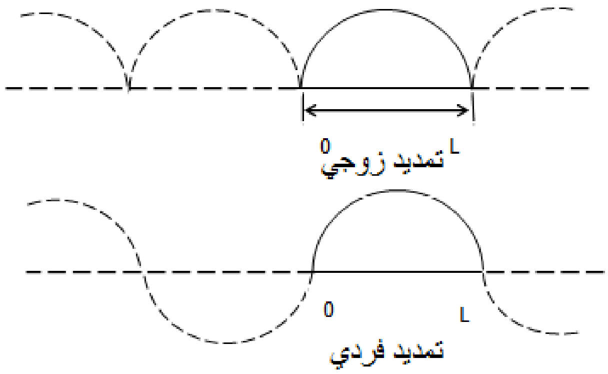
$$a_n = 0 \quad , \quad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx \qquad \text{odd}$$

$$b_n = 0 \quad , \quad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx \qquad \text{even}$$

The benefit of half range of Fourier series:

There are some functions that are not periodic at all. Rather, they may be defined over a period and undefined outside this period. However, we can find a Fourier series (half the period) for them

For example, we assume that a rod has length L at both ends $x = 0 \quad , \quad x = L$



and that $f(x)$ is the load distribution

function on the interval $(0, L)$

That is, $f(x)$ is defined on the

Interval $(0, L)$ and is not defined

Outside this interval. In this case we

Can make a hypothetical extension

or odd according to our desire, as in the figure

0n the interval $(-L, 0)$, the extension can be even

Then we also consider that the function repeats periodically before and after the period $(-L, L)$. Then we can find the half-range series for the periodic extended function that we created, and let us assume that this function is $F(x)$.

It is clear that the values of the periodic extended function $F(x)$ agree with the values of the non-periodic function $f(x)$ in the interval $(0, L)$, and therefore $F(x)$ can be taken as representative of the function $f(x)$ in the interval $(0, L)$ without paying attention to the values of the function $F(x)$ outside this period.

Complex Fourier Series:

Because of the equality $e^{i\theta} = \cos\theta + i\sin\theta$ the Fourier series take the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{i\frac{n\pi x}{L}} \qquad ; \quad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-i\frac{n\pi x}{L}}\, dx$$

Double Fourier Series:

We can generalize the Fourier series for a function with one variable to find the Fourier series for a function with two variables , in this case the double sine Fourier series is

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} \sin\frac{m\pi x}{L_1} \sin\frac{n\pi y}{L_2}$$

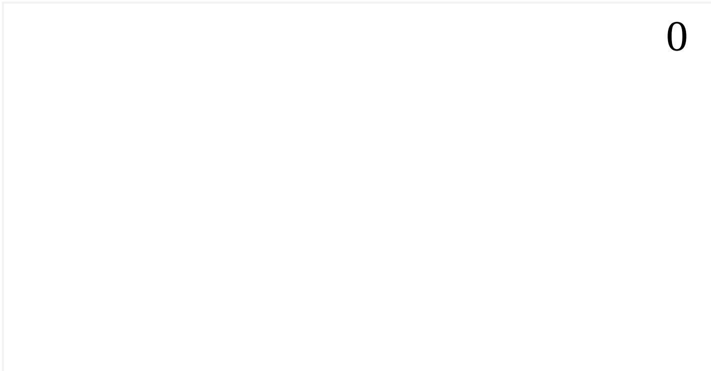
There are similar formulas for the double cosine Fourier series or the mixed Fourier series. Also it can be generalize this idea fo three variables or more.

Example (1): Find the Fourier series for the periodic function on a period as $f(x) = x^2 \qquad ; \quad 0 < x < 2\pi$

The solution

The function is shown

in the figure



to the Fourier series we should find the coefficients a_0 , a_n , b_n as follows

$$a_0 = \frac{1}{L} \int_{c}^{c+2L} f(x)\ dx = \frac{1}{\pi} \int_{0}^{2\pi} x^2\ dx = \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{L} \int_{c}^{c+2L} f(x) \cos \frac{n\pi x}{L}\ dx$$

$$= \frac{1}{\pi} \int_{0}^{2\pi} x^2 \cos \frac{n\pi x}{\pi}\ dx = \frac{1}{\pi} \int_{0}^{2\pi} x^2 \cos nx\ dx$$

$$= \frac{1}{\pi} \left[x^2 \frac{\sin nx}{n} - 2x \left(\frac{-\cos nx}{n^2} \right) + 2 \left(\frac{-\sin nx}{n^3} \right) \right] \Bigg|_{0}^{2\pi} = \frac{4}{n^2} \quad ; \quad n \neq 0$$

$$b_n = \frac{1}{L} \int_{c}^{c+2L} f(x) \sin \frac{n\pi x}{L}\ dx$$

$$= \frac{1}{\pi} \int_{0}^{2\pi} x^2 \sin \frac{n\pi x}{\pi}\ dx = \frac{1}{\pi} \int_{0}^{2\pi} x^2 \sin nx\ dx$$

$$= \frac{1}{\pi} \left[x^2 \left(\frac{-\cos nx}{n} \right) - 2x \left(\frac{-\sin nx}{n^2} \right) + 2 \left(\frac{\cos nx}{n^3} \right) \right] \Bigg|_{0}^{2\pi} = \frac{-4\pi}{n}$$

So the Fourier series for this periodic function $f(x) = x^2$, takes the form

$$f(x) = x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \left(\frac{4}{n^2} \cos nx - \frac{4\pi}{n} \sin nx \right)$$

$$= \frac{4\pi^2}{3} + 4 \cos x + \cos 2x + \frac{4}{9} \cos 3x + ...$$

$$- 4\pi \sin x - 2\pi \sin 2x - \frac{4\pi}{3} \sin 3x - ....$$

Example (2): Show that the Fourier series of the even function does not contain the sine functions and its coefficients are

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx$$

The solution

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx = \frac{1}{L} \int_{-L}^{0} f(x) \sin \frac{n\pi x}{L} \, dx + \frac{1}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} \, dx \quad \text{(أ)}$$

والآن نتناول التكامل الأول ونرمز له بالرمز I_1 ونعتبر التحويل $x = -u$.

Let $x = -u$ in the first integral I_1 , then we have

$$I_1 = \frac{1}{L} \int_{-L}^{0} f(-u) \sin(\frac{-n\pi u}{L}) \, d(-u) = \frac{-1}{L} \int_{0}^{L} f(u) \sin \frac{n\pi u}{L} \, du$$

$$= \frac{-1}{L} \int_{0}^{L} f(-x) \sin(\frac{-n\pi x}{L}) \, d(-x) = \frac{-1}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} \, dx$$

It is clear that this integral is equal and opposite to the second integral, then $b_n = 0$.

This means that the Fourier series of the even function does not contain the sine functions.

Now

$$a_n = \frac{1}{L} \int_{-L}^{0} f(x) \cos \frac{n\pi x}{L} \, dx + \frac{1}{L} \int_{0}^{L} f(x) \cos \frac{n\pi x}{L} \, dx$$

Let $x = -u$ in the first integral I_1 , then we see that its value is totally equal to the second integral, that is

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx$$

Example (3): Find the Fourier series for the periodic function on a period as $f(x) = \frac{5}{\pi} x \quad , -\pi < x < \pi$

The solution

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{5}{\pi} x dx = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos \frac{n\pi x}{\pi} dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{5}{\pi} x \cos nx dx$$

By taking the integral by parts we have

$$a_n = \frac{5}{\pi^2} \left\{ \left[x \frac{\sin x}{n} \right]_{-\pi}^{\pi} - \frac{1}{n} \int_{-\pi}^{\pi} \sin \mathrm{nx} \, dx \right\}$$

$$= \frac{5}{\pi^2} \left\{ \left[x \frac{\sin x}{n} \right]_{-\pi}^{\pi} + \frac{1}{n} \times 0 \right\}$$

$$= \frac{5}{\pi^2} \{0 - 0\} = 0$$

It is clear that

$$a_0 = 0 \,\&\, a_n = 0$$

And this true because that the function is odd so it must get this result without proceed these integral.

Here we calculate only b_n as

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin \frac{n\pi x}{\pi} dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{5}{\pi} x \sin nx dx$$

By taking the integral by parts we have

$$b_n = \frac{5}{\pi^2}\left\{\left[x\left(\frac{-\cos x}{n}\right)\right]_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}\cos nx\, dx\right\}$$

$$= \frac{5}{\pi^2}\left\{\frac{-\pi\cos n\pi}{n}\frac{-\pi\cos n\pi}{n} + 0\right\}$$

$$= \frac{-10}{\pi}\frac{\cos n\pi}{n}$$

Therefore we can write the coefficients b_n as

$$b_n = \begin{cases} \dfrac{-10}{n\pi} & , n\, even \\ \dfrac{10}{n\pi} & , n\, odd \end{cases}$$

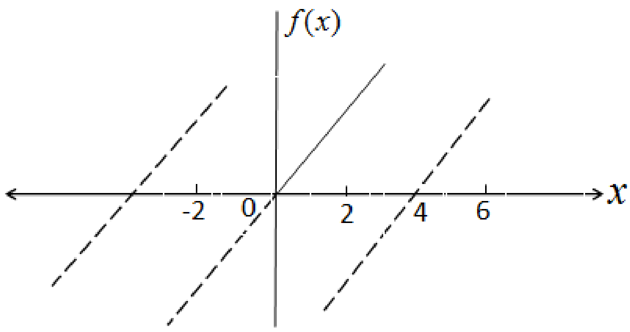
Example (4): Find the half Fourier series (a) the sine (b) the cosine for the function $f(x) = x \quad ; \quad 0 < x < 2$

The solution

(a) The figure shows the real function

And the function after the expanding

It to be periodic odd function



Now since the function is odd

Then $a_n = 0 \;\Rightarrow\; a_0 = 0$

And

$$b_n = \frac{2}{L}\int_0^L f(x)\,\sin\frac{n\pi x}{L}\,dx = \frac{2}{2}\int_0^2 x\,\sin\frac{n\pi x}{2}\,dx = \frac{4}{n\pi}\cos n\pi$$

Thus the Fourier series is

$$f(x) = \sum_{n=1}^{\infty} \frac{-4 \cos n\pi}{n} \sin \frac{n\pi x}{2}$$

$$= \frac{4}{\pi} \left(\sin \frac{\pi x}{2} - \frac{1}{2} \sin \frac{2\pi x}{2} + \frac{1}{3} \sin \frac{3\pi x}{2} \ldots\ldots\ldots\ldots\ldots \right)$$

And

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx = \frac{2}{2} \int_0^2 x \sin \frac{n\pi x}{2}\, dx = \frac{4}{n\pi} \cos n\pi$$

Thus the Fourier series is

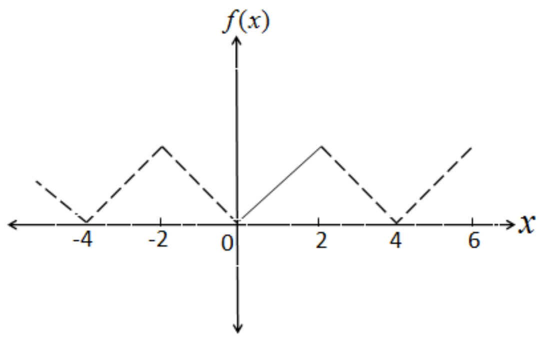
$$f(x) = \sum_{n=1}^{\infty} \frac{-4 \cos n\pi}{n} \sin \frac{n\pi x}{2}$$

$$= \frac{4}{\pi} \left(\sin \frac{\pi x}{2} - \frac{1}{2} \sin \frac{2\pi x}{2} + \frac{1}{3} \sin \frac{3\pi x}{2} \ldots\ldots\ldots\ldots\ldots \right)$$

(b)The figure shows the real function

And the function after the expanding

It to be periodic even function



Now since the function is even then

$$b_n = 0 \quad , \quad a_0 = \int_0^2 x\, dx = 2$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx = \frac{2}{2} \int_0^2 x \cos \frac{n\pi x}{2}\, dx = \frac{-4\ (\cos n\pi\ -1)}{n^2 \pi^2} \quad , \quad n \neq 0$$

Thus the Fourier series is

$$f(x) = \sum_{n=1}^{\infty} \frac{-4\cos n\pi}{n} \sin\frac{n\pi x}{2}$$

$$= \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} \ldots\ldots\ldots\ldots\ldots \right)$$

Note that the second series convergences faster than the first series.

Example (5): In case of the double Fourier series

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} \sin\frac{m\pi x}{L_1} \sin\frac{n\pi y}{L_2}$$

Find the formula of calculating the coefficients B_{mn}

The solution

If we consider y as a parameter, then we can write the series in the form

$$f(x,y) = \sum_{m=1}^{\infty} C_m \sin\frac{m\pi x}{L_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

where

$$C_m = \sum_{n=1}^{\infty} B_{mn} \sin\frac{n\pi y}{L_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

In this case C_m will be function of y.

If we consider the series (2) as an expansion for the function of C_m then

$$B_{mn} = \frac{2}{L_2}\int_0^{L_2} C_m \sin\frac{n\pi y}{L_2}\, dy$$

Also from (1) we get

$$C_m = \frac{2}{L_1} \int_0^{L_1} f(x,y) \sin \frac{m\pi x}{L_1} \, dx$$

That is

$$B_{mn} = \frac{4}{L_1 L_2} \int_0^{L_1} \int_0^{L_2} f(x,y) \sin \frac{m\pi x}{L_1} \sin \frac{n\pi x}{L_2} \, dxdy$$

Exercises

(1)Find the coefficients of the periodic function

$$f(x) = \begin{cases} 0 & : \quad -5 < x < 0 \\ 3 & : \quad 0 < x < 5 \end{cases}$$

Then draw its figure and find its Fourier series, then redefine the function at the points $x = -5$, $x = 0$, $x = 5$ such that the series will converge to the function in the interval $-5 \le x \le 5$

(2)Find the half cosine Fourier series for the following

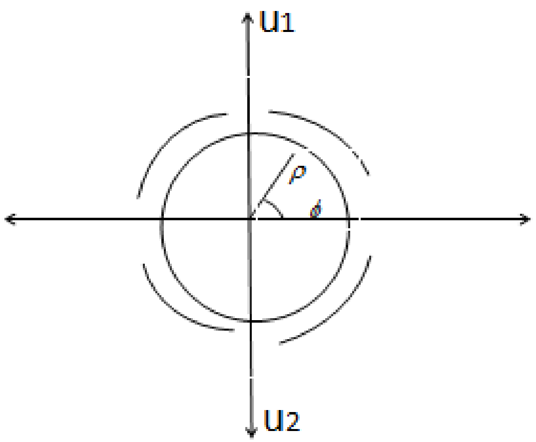
Applications for Fourier Analyses:

Example (1): A thin metal disk with a diameter of one unit, insulated on both sides, with half of its edge kept at a constant temperature u_1 and the other half at a constant temperature u_2 . Find the heat distribution in the steady state.

The solution

We use the polar coordinates

The equation of the heat distribution in the steady state which is Laplace equation $\nabla^2 u = 0$, that is



$$\nabla^2 u = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} = 0$$

where the boundary conditions are

$$u(1,\phi) = \begin{cases} u_1 & : \quad 0 < \phi < \pi \\ u_2 & : \quad \pi < \phi < 2\pi \end{cases} \qquad , \quad \left| u(\rho,\phi) \right| < M$$

$$u(1,\phi) = \begin{cases} u_1 & : \quad 0 < \phi < \pi \\ u_2 & : \quad \pi < \phi < 2\pi \end{cases}$$

To separate the variables, we assume that

$$u(\rho,\phi) = P(\rho)\ \Phi(\phi)$$

By substituting into the differential equation and dividing by $P\,\Phi$, we get:

$$\frac{\rho^2 P'' + \rho P'}{P} = \frac{-\Phi''}{\Phi} = \lambda^2$$

Therefore

$$\Phi'' + \lambda^2 \Phi = 0 \qquad , \qquad \rho^2 P'' + \rho P' - \lambda^2 P = 0$$

Note: the constant has been chosen to be positive such that each side after separation variables will equal to λ^2 not $-\lambda^2$ as in another example before. Here the causes not the condition $\left| u(\rho,\phi) \right| < M$, but the equation $\Phi'' + \lambda^2 \Phi = 0$ must contains the sine and cosine functions because the solution $\Phi(\phi)$ of it is periodic.

We return to the tow ordinary deferential, the first its solution will be

$$\Phi(\phi) = A_1 \cos\lambda\phi + B_1 \sin\lambda\phi$$

While the second of Euler type its special solution are ρ^{λ} , $\rho^{-\lambda}$, and its general solution is

$$P(\rho) = A_2\, \rho^{\lambda} + B_2\, \rho^{-\lambda}$$

Here comes the role of the condition that the function be finite. It is clear that each of the two limits ρ^{λ} , $\rho^{-\lambda}$ does not reach an infinite value when ρ it reaches its maximum value at the circumference of the circle, where the two halves of the diameter are a finite value. But the problem appears at the center of the disk $\rho = 0$, where the second term $\rho^{-\lambda} = \frac{1}{\rho^{\lambda}}$ reaches an infinite value.

Therefore, this must be $B_2 = 0$, this is the result of applying the condition that the function be finite everywhere. Also - as we pointed out - the solution must be in a periodic manner, and its period must also be 2π specifically. So then λ it must be in the form.

$$\lambda = m \quad , \quad m = 0, 1 , 2 , 3 , ...$$

And the conclusion is

$$u(\rho , \phi) = \rho^{m} (A_m \cos m\phi + B_m \sin m\phi)$$

By applying the addition principle it result

$$u(1 , \phi) = \frac{A_0}{2} + \sum_{m=1}^{\infty} (A_m \cos m\phi + B_m \sin m\phi)$$

By applying the boundary condition, it results that

$$u(1 , \phi) = \frac{A_0}{2} + \sum_{m=1}^{\infty} (A_m \cos m\phi + B_m \sin m\phi)$$

According to Fourier's theorem, we get the coefficients A_m , B_m as follows:

$$A_m = \frac{1}{\pi} \int_0^{2\pi} u(1 , \phi) \cos m\phi \; d\phi$$

$$= \frac{1}{\pi} \int_0^{\pi} u_1 \cos m\phi \; d\phi + \frac{1}{\pi} \int_{\pi}^{2\pi} u_2 \cos m\phi \; d\phi$$

The resultis

$$A_m = \begin{cases} 0 & : \quad m > 0 \\ u_1 + u_2 & : \quad m = 0 \end{cases}$$

. That is, all transactions A_m progress except $A_0 = u_1 + u_2$.

:The other set of transactions B_m is obtained from

$$B_m = \frac{1}{\pi} \int_0^{\pi} u_1 \sin m\phi \; d\phi + \frac{1}{\pi} \int_{\pi}^{2\pi} u_2 \sin m\phi \; d\phi$$

$$= \frac{(u_1 - u_2)(1 - \cos m\pi)}{m\pi}$$

Then the temperature distribution in the steady state becomes as follows:

$$u(\rho, \phi) = \frac{u_1 + u_2}{2} + \sum_{m=1}^{\infty} \frac{(u_1 - u_2)(1 - \cos m\pi)\rho^m}{m\pi} \sin m\phi$$

$$= \frac{u_1 + u_2}{2} + \frac{2(u_1 - u_2)}{\pi} \left(\rho \sin\phi + \frac{1}{3} \rho^3 \sin 3\phi + \ldots\ldots\ldots\ldots \right)$$

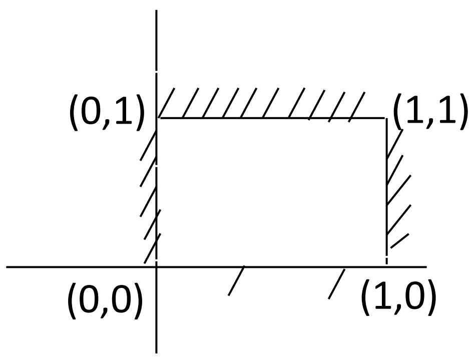
Example (1): A thin plate in the form of a square with one side length, with both faces insulated, and all its edges preserved in a degree Zero... If the initial temperature distribution is known, find the general distribution

<u>The solution</u>

المعادلة التفاضلية هـــــــي :

$$\frac{\partial u}{\partial t} = K \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

والشروط هــــي :



$/ / \quad / / / \quad /$ $\qquad\qquad$ $|u(x, y)| < M ,$

$$u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0$$
$$u(x, y, 0) = f(x, y)$$

حيث $f(x,y)$ هي دالة التوزيع الحراري الابتدائي المعلومة

لفصل المتغيرات افترض أن : $u = X(x)\ Y(y)\ T(t)$

The deferential equation is

$$\frac{\partial u}{\partial t} = K \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

The conditions are

$$u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0$$
$$u(x, y, 0) = f(x, y)$$
$$|u(x, y)| < M ,$$

Where $f(x,y)$ is the known initial heat distribution function

To separate the variables, assume that $u = X(x)\ Y(y)\ T(t)$

After substituting into the equation and dividing by $K\ X\ Y\ T$

we get $\dfrac{T'}{KT} = \dfrac{X''}{X} + \dfrac{Y''}{Y} = -\lambda^2$

That is $T' + K\lambda^2 T = 0$, $\dfrac{X''}{X} + \dfrac{Y''}{Y} = -\lambda^2$

Thus, we have applied the first condition for the finite function. The second equation also separates the variables, so we get

$$\frac{X''}{X} = -\frac{Y''}{Y} - \lambda^2 = -\mu^2$$

Where $-\mu^2$ is another optional constant. It was chosen so that it is negative. However, if we had chosen it as positive and then applied the remaining conditions, we would have obtained the trivial solution. The student can verify this.

That is $\quad X'' + \mu^2 X = 0 \quad , \quad Y'' + (\lambda^2 - \mu^2)\, Y = 0$

The solutions of these equations are:

$$X = a_1 \cos \mu x + b_1 \sin \mu x \ ,$$

$$Y = a_2 \cos \sqrt{\lambda^2 - \mu^2}\; y + b_2 \sin \sqrt{\lambda^2 - \mu^2}\; y \ ,$$

$$T = a_3\, e^{-K\lambda^2 t}$$

Then

$$u(x, y, t) = e^{-K\lambda^2 t}\,(a_1 \cos \mu x + a_2 \sin \mu x)$$
$$(a_2 \cos \sqrt{\lambda^2 - \mu^2}\; y + b_2 \sin \sqrt{\lambda^2 - \mu^2}\; y)$$

Where we have integrated a_3 to the rest of the constants.

Applying the condition $u(0, y, t) = 0$ we get $a_1 = 0$.

Applying the condition $u(x, 0, t) = 0$ we get $a_2 = 0$.

That is $u(x,y,t) = B e^{-K\lambda^2 t} \sin \mu x \ \sin \sqrt{\lambda^2 - \mu^2}\ y)$

Applying the condition $u(1, y, t) = 0$ we get $\mu = m\pi$.

Applying the condition $u(x,1,t) = 0$ we get $\sqrt{\lambda^2 - \mu^2} = n\pi$.

That is: $\lambda = \sqrt{m^2 + n^2}$

By applying the principle of addition, we obtain the general solution:

$$u(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \ e^{-K(m^2+n^2)\pi^2 t} \sin m\pi x \ \sin n\pi y$$

Applying the condition $u(x,y,0) = f(x,y)$ we get:

$$f(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin m\pi x \ \sin n\pi y$$

It is a double Fourier series whose coefficients are:

$$B_{mn} = 4 \int_0^1 \int_0^1 f(x,y) \sin m\pi x \ \sin n\pi y \ dx\, dy$$

If $f(x,y)$ were a known function, we could calculate this integral, obtain the numerical values of the coefficients B_{mn}, and then substitute them into the final formula of the function $u(x,y,t)$.

Exercises

(1) Find the solution to the problem of thermal conductivity of a thin rod, where

$$\frac{\partial u}{\partial t} = 2 \frac{\partial^2 u}{\partial x^2} \quad , \quad 0 < x < 3$$

Such that $u(0,t) = u(3,t) = 0$, and Note that the initial temperature $25\ ^\circ C$ for all points of the rod.

(2) A thin rod with insulated sides and insulated at both ends $x = 0$, $x = L$. If the primary heat distribution is given by the function $f(x)$ so find the general distribution.

(3) state Solution to Laplace's equation in two dimensions: A thin square plate of unit side length, three of which are preserved on Zero degrees and the fourth side is at a degree μ. Find the steady- heat distribution.

The third chapter

Fourier Integral&Transform ..... تكامـــــــل وتحويل فوريــــــــر

The needs to Fourier Integral:

In the previous chapter we study periodic functions with finite period, but if $L \to \infty$ and if the interval of the function is unbounded, then the Fourier series tends to Fourier integral.

Fourier transform techniques have been widely used to solve problems involving semi-infinite or totally infinite range of the variables or unbounded regions. In order to deal with such problems, it is necessary to generalize Fourier series to include infinite intervals and to introduce the concept of Fourier integral. In this chapterwe deal wi, th Fourier integral representations and Fourier transform, with some applications to Diffusion, Wave and Laplace equations.

Fourier Integral:

We have seen before that the periodic function $f(x)$ with period $2L$ which satisfy Dirichlet conditions in the interval $(-L, L)$ can be represented in the form of Fourier series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \qquad (1)$$

Where

$$\left.\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \end{aligned}\right\} \qquad (2)$$

That series can be extended to some non –periodic functions also, provided the integral of modulus of such a function satisfies the condition

$$\int_{-\infty}^{\infty} |f(x)|\,dx < M \qquad (M \text{ محدود})$$

Substituting equation (2) in equation (1), we get

$$f(x)=\frac{1}{2L}\int_{-L}^{L}f(u)du+\sum_{n=1}^{\infty}\left[\frac{1}{L}\left[\int_{-L}^{L}f(u)\left\{\cos\frac{n\pi u}{L}\cos\frac{n\pi x}{L}+\sin\frac{n\pi u}{L}\sin\frac{n\pi x}{L}\right\}du\right]\right]$$

which after interchanging the order of summation and integration can be reduced to the form

$$f(x)=\frac{1}{2L}\int_{-L}^{L}f(u)du+\frac{1}{L}\int_{-L}^{L}f(u)\sum_{n=1}^{\infty}\cos\frac{n\pi(u-x)}{L}du \quad (3)$$

Further, if we assume that the function $f(x)$ is absolutely integrable , and allowing $L\to\infty$,i.e., $\int_{-\infty}^{\infty}|f(x)|dx<\infty$, we get

$$\lim_{L\to\infty}\int_{-L}^{L}f(u)du=0 \quad (4)$$

In the remaining part of the infinite sum of equation (3), if we set $\Delta s=\pi/L$, the equation reduces to

$$f(x)=\lim_{\Delta s\to 0}\frac{1}{\pi}\int_{-\pi/\Delta s}^{\pi/\Delta s}f(u)\sum_{n=1}^{\infty}\cos\{n\Delta s(u-x)\}\Delta s du \quad (5)$$

As $L\to\infty$ & $\Delta s\to 0$, implying that Δs is small positive number and the points $n\Delta s$ are equally spaced along the axis of s , then the series (5) under the integral can be approximated to

$$\int_{0}^{\infty}\cos\{s(u-x)\}ds=0 \quad as \;\; \Delta s\to 0$$

Thus equation (5) can be rewritten as

$$f(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(u)\int_{0}^{\infty}\cos\{s(u-x)\}ds\,du \quad (6)$$

Or in the form

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(u)\cos\{s(u-x)\}\,ds\,du \qquad (7)$$

The equation (7) with some kind of treatments can be reduced to the form

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(u)\{\cos\alpha u\cos\alpha x+\sin\alpha u\sin\alpha x\}\,du\,d\alpha$$

If we put

$$\left.\begin{aligned} A(\alpha)&=\frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\alpha u\,du \\ B(\alpha)&=\frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin\alpha u\,du \end{aligned}\right\} \qquad (8)$$

Then the equation (7) takes the form

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}\left[A(\alpha)\cos\alpha x+B(\alpha)\sin\alpha x\right]du\,d\alpha \qquad (9)$$

The equation (9) with the equations (8) is called Fourier Integral.

Fourier Integral convergent to the values of the function $f(x)$ at the points of continuity, and to the average of the values of the function at the points of discontinuity as in the case of Fourier series.

Different formulas for Fourier Integral:

Substituting equation (8) in equation (9), then Fourier Integral takes the form

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(u)\cos\alpha(x-u)\,du\,d\alpha \qquad (10)$$

which can be rewritten in the form

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\,e^{i\alpha(x-u)}\,du\,d\alpha \qquad (11)$$

and by redistribute the amount $e^{i\alpha(x-u)}=e^{i\alpha x}\,e^{-i\alpha u}$, then the equation (11), becomes

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha x}\,d\alpha\int_{-\infty}^{\infty} f(u)\,e^{-i\alpha u}\,du \qquad (12)$$

If the function $f(x)$ is odd, then the equation (12) takes the form

$$f(x)=\frac{2}{\pi}\int_{0}^{\infty}\sin\alpha x\,d\alpha\int_{0}^{\infty} f(u)\,\sin\alpha u\,du \qquad (13)$$

while if the function $f(x)$ is even, then the equation (12) takes the form

Fourier transform:

The (12) can be rewritten in the form

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\,e^{i\alpha x}\,d\alpha \qquad (15)$$

where

$$F(\alpha)=\int_{-\infty}^{\infty} f(u)\,e^{-i\alpha u}\,du \qquad (16)$$

The function $F(\alpha)$ is called Fourier transform for the function $f(x)$ and can be written in the form

$$F(\alpha) = \Im\{f(x)\} = \int_{-\infty}^{\infty} f(u)\, e^{-i\alpha u} du \qquad (17)$$

while the function $f(x)$ is called the inverse Fourier transform for the function $F(\alpha)$ and can be written in the form

$$f(x) = \Im^{-1}\{F(\alpha)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\, e^{i\alpha x} d\alpha \qquad (18)$$

Each of the functions $e^{-i\alpha u}$ & $e^{i\alpha x}$ is called kernel of the transform.

In some circumstances the amount $\frac{1}{2\pi}$ in the equation (18) can be redistribute equally between the two equations (17) & (18) in the form

$\frac{1}{\sqrt{2\pi}}$, $\frac{1}{\sqrt{2\pi}}$, or put it in the equation (17) instead of equation (18) .

The tow equations (17) & (18) are called The Fourier transform pair.

<u>Sone and Cosine Fourier transform:</u>

If the function $f(x)$ is odd, then The Fourier transform pair (17) & (18) becomes

$$\left.\begin{aligned} F_s(\alpha) &= \Im\{f(x)\} = 2\int_0^{\infty} f(u) \sin \alpha u du \\ f_s(x) &= \Im^{-1}\{F(\alpha)\} = \frac{1}{\pi}\int_0^{\infty} F(\alpha) \sin \alpha x \, d\alpha \end{aligned}\right\} \qquad (19)$$

while if the function $f(x)$ is even, then The Fourier transform pair (17) & (18) becomes

$$\left.\begin{aligned} F_c(\alpha) &= \Im\{f(x)\} = 2\int_0^{\infty} f(u)\cos\alpha u\, du \\ f_c(x) &= \Im^{-1}\{F(\alpha)\} = \frac{1}{\pi}\int_0^{\infty} F(\alpha)\cos\alpha x\, d\alpha \end{aligned}\right\} \quad (20)$$

Properties of Fourier transform:

(1) Linearity Property:

If $F_1(\alpha)\,\&\,F_2(\alpha)$ are the Fourier transform of $f_1(x)\,\&\,f_2(x)$ respectively then

$$F(\alpha) = \Im\{c_1 f_1(x) + c_2 f_2(x)\} = c_1\Im\{f_1(x)\} + c_2\Im\{f_2(x)\}$$
$$= c_1 F_1(\alpha) + c_2 F_2(\alpha)$$

where $c_1\,\&\,c_2$ are constants.

The proof

$$\Im\{c_1 f_1(x) + c_2 f_2(x)\} = \int_{-\infty}^{\infty} e^{-i\alpha u}\left(c_1 f_1(u) + c_2 f_2(u)\right)du$$

$$= c_1\int_{-\infty}^{\infty} f_1(u)e^{-i\alpha u}du + c_2\int_{-\infty}^{\infty} f_2(u)e^{-i\alpha u}du = c_1 F_1(\alpha) + c_2 F_2(\alpha)$$

(2) Shifting Property:

If $F(\alpha)$ is the Fourier transform of $f(x)$ then Fourier transform of the function $f(x-a), a = const.$ is $e^{i\alpha a}F(\alpha)$, i.e.,

$$\Im\{f(x-a)\} = e^{-i\alpha a}F(\alpha)$$

The proof

From the definition of Fourier transform then the equation (17)

$$F(\alpha) = \Im\{f(x)\} = \int_{-\infty}^{\infty} f(u)\, e^{-i\alpha u} du \qquad (17)$$

and therefore

$$F(\alpha) = \Im\{f(x-a)\} = \int_{-\infty}^{\infty} f(u-a)\, e^{-i\alpha u} du$$

and by put $u - a = v \Rightarrow du = dv$ we get

$$\Im\{f(x-a)\} = \int_{-\infty}^{\infty} f(v)\, e^{-i\alpha(v+a)} dv$$

$$= e^{-i\alpha a} \int_{-\infty}^{\infty} f(v) e^{-i\alpha v}\, dv = e^{-i\alpha a} F(\alpha)$$

(3) Change of scale Property:

If $F(\alpha)$ is the Fourier transform of $f(x)$ then Fourier transform of the function $f(ax), a = const.$ is $\frac{1}{a} F\left(\frac{\alpha}{a}\right)$, i.e., $\Im\{f(ax)\} = \frac{1}{a} F\left(\frac{\alpha}{a}\right)$

The proof

From the definition of Fourier transform then the equation (17)

$$F(\alpha) = \Im\{f(x)\} = \int_{-\infty}^{\infty} f(u)\, e^{-i\alpha u} du \qquad (17)$$

and therefore

$$\Im\{f(ax)\} = \int_{-\infty}^{\infty} f(au)\, e^{-i\alpha u} du$$

and by put $au = v \Rightarrow du = dv/a$ we get

$$\Im\{f(ax)\} = \int_{-\infty}^{\infty} f(v)\, e^{-i\frac{\alpha}{a}v}\, dv$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} f(v) e^{-i\frac{\alpha}{a}v}\, dv = \frac{1}{a} F\left(\frac{\alpha}{a}\right)$$

(4)Modulation Property:

If $F(\alpha)$ is the Fourier transform of $f(x)$ then Fourier transform of the function $f(ax), a = const.$ is $\frac{1}{a}F\left(\frac{\alpha}{a}\right)$, i.e., $\Im\{f(ax)\} = \frac{1}{a}F\left(\frac{\alpha}{a}\right)$

The proof

From the definition of Fourier transform then the equation (17)

$$F(\alpha) = \Im\{f(x)\} = \int_{-\infty}^{\infty} f(u)\, e^{-i\alpha u} du \qquad (17)$$

and therefore

$$\Im\{f(ax)\} = \int_{-\infty}^{\infty} f(au)\, e^{-i\alpha u} du$$

and by put $au = v \Rightarrow du = dv/a$ we get

$$\Im\{f(ax)\} = \int_{-\infty}^{\infty} f(v)\, e^{-i\frac{\alpha}{a}v}\, dv$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} f(v) e^{-i\frac{\alpha}{a}v}\, dv = \frac{1}{a} F\left(\frac{\alpha}{a}\right)$$

If $F(\alpha)$ is the Fourier transform of $f(x)$ then Fourier transform of the function $f(x)\cos ax, a = const.$ is $\frac{1}{2}\left[F(\alpha - a) + F(\alpha + a)\right]$, i.e.,

$$\Im\{f(x)\cos ax\} = \frac{1}{2}\left[F(\alpha - a) + F(\alpha + a)\right]$$

(5 Differentiation Property:

If $F(\alpha)$ is the Fourier transform of $f(x)$ and its first $(r-1)$ derivatives are continuous and if its $r\underline{th}$ derivative is piecewise continuous then Fourier transform of the function $f^{(r)}(x)$ is $(-i\alpha)^r F(\alpha)$, i.e.,

$\Im\left\{f^{(r)}(x)\right\} = (-i\alpha)^r F(\alpha) \quad , r = 0,1,2,......$

The proof

From the definition of Fourier transform then the equation (17)

$$F(\alpha) = \Im\{f(x)\} = \int_{-\infty}^{\infty} f(u)\, e^{-i\alpha u} du \qquad (17)$$

and therefore

$$\Im\left\{f^{(r)}(x)\right\} = \int_{-\infty}^{\infty} f^{(r)}(u)\, e^{-i\alpha u} du = F^{(r)}(\alpha)$$ مثـ لا

Integrating by parts, we get

$$\int_{-\infty}^{\infty} f^{(r)}(u)\, e^{-i\alpha u} du = \left| f^{(r-1)}(x)(i\alpha) e^{-i\alpha u} \right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f^{(r-1)}(x)(-i\alpha) e^{-i\alpha u} du$$

If we assume that $f^{(r-1)}(x) \to 0 \, as \; x \to \pm\infty$, we may write the above result in the form

$$F^{(r)}(\alpha) = -(i\alpha) F^{(r-1)}(\alpha) = -(i\alpha)^2 F^{(r-2)}(\alpha) = ... = -(i\alpha)^r F(\alpha)$$

Hence

$$F^{(r)}(\alpha) = -(i\alpha)^r F(\alpha)$$

and therefore

$$\Im\left\{f^{(r)}(x)\right\} = -(i\alpha)^r F(\alpha)$$

(6) Convolution Theorem:

If $F(\alpha)$ & $G(\alpha)$ are the Fourier transform of $f(x)$ & $g(x)$ respectively then the product of $F(\alpha)$ & $G(\alpha)$ is Fourier transform of the convolution of $f(x)$ & $g(x)$

,i.e., $\Im\{f * g\} = \Im\{f\}\Im\{g\}$

where is the convolution of the two functions is defined as

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(u) g(x-u) du$$

Now

$$\Im\{f(x) * g(x)\} = \Im\left\{\int_{-\infty}^{\infty} f(u) g(x-u) du\right\} = \int_{-\infty}^{\infty} e^{-i\alpha x}\left[\int_{-\infty}^{\infty} f(u) g(x-u) du\right] dx$$

Which can be rewritten in the form

$$\Im\{f(x) * g(x)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u) e^{-i\alpha x} g(x-u) du\, dx$$

Since $f(x)$ & $g(x)$ are absolutely integrable , the order of integration can be interchanged and, therefore

$$\Im\{f(x) * g(x)\} = \int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} g(x-u) e^{-i\alpha x} dx\right] du$$

And by multiplication with the amount $e^{i\alpha u}$ and dividing on it, then the last equation becomes

$$\Im\{f(x) * g(x)\} = \int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} g(x-u) e^{-i\alpha(x-u)} e^{-i\alpha u} dx\right] du$$

Let $x - u = y \Rightarrow dx = dy$, we get

$$\Im\{f(x)*g(x)\}=\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} g(y)e^{-i\alpha y}\, e^{-i\alpha x} dy\right]du$$

Which can be rearrange it in the form

$$\Im\{f(x)*g(x)\}=\int_{-\infty}^{\infty} f(u)e^{-i\alpha u}du \int_{-\infty}^{\infty} g(y)e^{-i\alpha y}\, dx = F(\alpha)G(\alpha)$$

Hence the theorem is proved.

It can verify that

$$f(x)*g(x)=g(x)*f(x)$$
$$f(x)*[g(x)*h(x)]=[f(x)*g(x)]*h(x)$$
$$f(x)*[g(x)+h(x)]=[f(x)*g(x)]+f(x)*h(x)$$

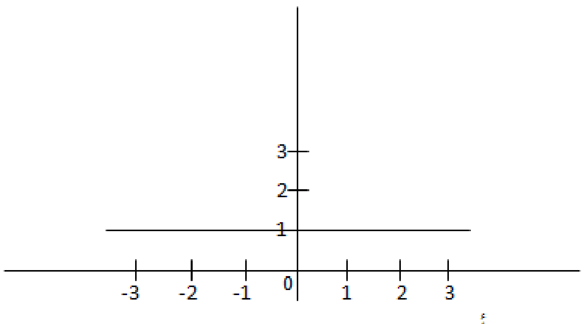
Examples

Example(1):Find the Fourier transform of the function $f(x)=\begin{cases}1 & :|x|<a\\ 0 & :|x|>a\end{cases}$

And hence evaluate $(i)\ \int_{-\infty}^{\infty}\frac{\sin\alpha a\cos\alpha x}{\alpha}d\alpha$, $(ii)\ \int_{0}^{\infty}\frac{\sin u}{u}du$

The solution

The curve of the function



$f(x)$ takes the form

From the definition of Fourier transform

$$F(\alpha)=\Im\{f(x)\}=\int_{-\infty}^{\infty} f(u)e^{-i\alpha u}du$$

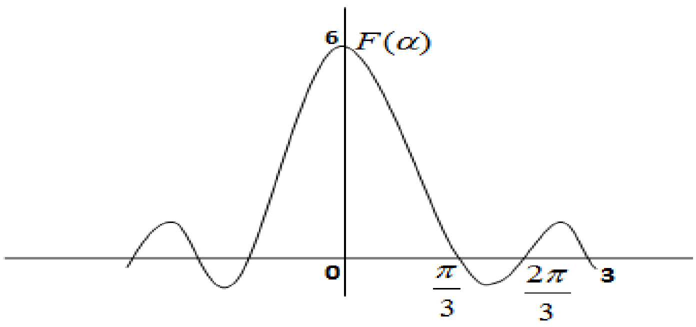
and by substituting about the function $f(x)$, we get

$$F(\alpha)=\Im\{f(x)\}=\int_{-a}^{a} e^{-i\alpha u}du=\left|\frac{e^{-i\alpha u}}{-i\alpha}\right|_{-a}^{a}=\left(\frac{e^{i\alpha a}-e^{-i\alpha a}}{i\alpha}\right)$$
$$=\frac{2\sin\alpha a}{\alpha}$$

We note that
$F(0)\to 2a \ \ as \ \ \alpha\to 0$

And the curve of the transform function

at $a=3$ takes the front figure



Now

$$f(x)=\Im^{-1}\{F(\alpha)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\alpha)e^{i\alpha x}d\alpha=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin\alpha a}{\alpha}e^{i\alpha x}d\alpha$$

$$\therefore f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin\alpha a}{\alpha}(\cos\alpha x+i\sin\alpha x)d\alpha$$
$$=\frac{1}{\pi}\left[\int_{-\infty}^{\infty}\frac{\sin\alpha a}{\alpha}\cos\alpha x d\alpha+i\int_{-\infty}^{\infty}\frac{\sin\alpha a}{\alpha}\sin\alpha x \ d\alpha\right]$$

and since $\sin\alpha a$ & α are odd functions then the function under the second integration will be odd and so the second integration tends to zero and the result becomes

$$\therefore f(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\alpha a}{\alpha}\cos\alpha x\, d\alpha$$

$$\therefore \int_{-\infty}^{\infty}\frac{\sin\alpha a\cos\alpha x}{\alpha}d\alpha=\pi\begin{cases}1 & :|x|<a\\ 0 & :|x|>a\end{cases}=\begin{cases}\pi & :|x|<a\\ \frac{\pi}{2} & :|x|=a\\ 0 & :|x|>a\end{cases}$$

And that the result of the first integration (i)

If we put $x=0$ & $a=1$ in the last integration (i), where it is clear that $x<1$

so we chose the first value in (i), that is

$$(ii)\quad \int_{-\infty}^{\infty}\frac{\sin u}{u}du=\pi \quad\Rightarrow \int_{0}^{\infty}\frac{\sin u}{u}du=\frac{\pi}{2}$$

Example(2):Find the Fourier transform of the function $f(x)=e^{-x^2/2}$

The solution

From the definition of Fourier transform

$$F(\alpha)=\int_{-\infty}^{\infty}e^{-u^2/2}e^{-i\alpha u}du$$

if we multiply and dividing by the amount $e^{-\alpha^2/2}$ inside the last integral the result becomes

$$F(\alpha)=\int_{-\infty}^{\infty} e^{-u^2/2} e^{-i\alpha u} e^{\alpha^2/2} e^{-\alpha^2/2} du = \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(u^2+2i\alpha u-\alpha^2\right)} e^{-\alpha^2/2} du$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(u^2+2i\alpha u+(i\alpha)^2\right)} e^{\alpha^2/2} du = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(u+i\alpha)^2} e^{-\alpha^2/2} du$$

Let $\dfrac{u+i\alpha}{\sqrt{2}}=t \Rightarrow du=\sqrt{2}dt$, then the transform will be

$$F(\alpha)=\sqrt{2}e^{-\alpha^2/2} \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{2}\, e^{-\alpha^2/2} \sqrt{\pi} = \sqrt{2\pi}\, e^{-\alpha^2/2}$$

Example(3):Find the Fourier transform of the function $f(x)=e^{a|x|}$

<u>The solution</u>

From the definition of Fourier transform

$$F(\alpha)=\int_{-\infty}^{\infty} e^{a|x|} e^{-i\alpha u} du$$

where $|x|=\begin{cases} -x & : x<0 \\ x & : x\geq 0 \end{cases}$ then the result becomes

$$F(\alpha) = \int_{-\infty}^{0} e^{-au} e^{-i\alpha u} du + \int_{0}^{\infty} e^{au} e^{-i\alpha u} du$$

$$= \int_{-\infty}^{0} e^{-(a+i\alpha)u} du + \int_{0}^{\infty} e^{(a-i\alpha)u} du = \left(\frac{1}{-(a+i\alpha)} + \frac{1}{a-i\alpha} \right)$$

$$= \frac{-2\alpha}{a^2+\alpha^2}$$

Example(4):Find the cosine and sine Fourier transform of the function $f(x) = e^{-bx}$

And hence evaluate $(i) \int_{0}^{\infty} \frac{\cos \alpha x}{\alpha^2 + b^2} d\alpha$, $(ii) \int_{0}^{\infty} \frac{\alpha \sin \alpha x}{\alpha^2 + b^2} du$

<u>The solution</u>

From the definition of the cosine and sine Fourier transform we get

$$F_c(\alpha) = \Im\{f(x)\} = 2\int_{0}^{\infty} f(u) \cos \alpha u \, du = 2\int_{0}^{\infty} e^{-bu} \cos \alpha u \, du$$

$$F_s(\alpha) = \Im\{f(x)\} = 2\int_{0}^{\infty} f(u) \sin \alpha u du = 2\int_{0}^{\infty} e^{-bu} \sin \alpha u du$$

Let $I_1 = \int_{0}^{\infty} e^{-bu} \cos \alpha u \, du$ & $I_2 = \int_{0}^{\infty} e^{-bu} \sin \alpha u \, du$

Integrating by parts, we have

$$I_1 = \left(-\frac{1}{b} e^{-bu} \cos \alpha u \right)_0^\infty - \frac{\alpha}{b} \int_0^\infty e^{-bu} \sin \alpha u \, du = \frac{1}{b} - \frac{\alpha}{b} I_2 \qquad (1)$$

$$I_2 = \left(-\frac{1}{b} e^{-bu} \sin \alpha u \right)_0^\infty - \frac{\alpha}{b} \int_0^\infty e^{-bu} \cos \alpha u \, du = \frac{\alpha}{b} I_1 \qquad (2)$$

Solving equations (1) & (2) we obtain

$$I_1 = = \frac{b}{\alpha^2 + b^2} \qquad \& \quad I_2 = \frac{\alpha}{\alpha^2 + b^2}$$

Hence $\quad F_c(\alpha) = 2\left(\frac{b}{\alpha^2 + b^2} \right) \qquad \& \quad F_s(\alpha) = 2\left(\frac{\alpha}{\alpha^2 + b^2} \right)$

And by find the invers Fourier transform, we get

$$f(x) = \mathfrak{I}^{-1}\{F(\alpha)\} = \frac{2}{\pi} \int_0^\infty F_c(\alpha) \cos \alpha x \, d\alpha = \frac{2}{\pi} \int_0^\infty \frac{b}{\alpha^2 + b^2} \cos \alpha x \, d\alpha$$

$$\therefore \quad \int_0^\infty \frac{b}{\alpha^2 + b^2} \cos \alpha x \, d\alpha = \frac{\pi}{2} f(x)$$

Therefore $\quad \int_0^\infty \frac{\cos \alpha x}{\alpha^2 + b^2} \, d\alpha = \frac{\pi}{2b} f(x) = \frac{\pi}{2b} e^{-bx}$

And this is the first integral required.

Similarly, it can be shown that the second integral is

$$\int_0^\infty \frac{\alpha \sin \alpha x}{\alpha^2 + b^2} du = \frac{\pi}{2} e^{-bx}$$

Example(5):Find the sine Fourier transform of the function $f(x) = \begin{cases} 0 & : 0 < x < a \\ x & : \mathrm{a} \le x \le b \\ 0 & : \quad x > b \end{cases}$

The solution

From the definition of the cosine and sine Fourier transform we get

$$F_s(\alpha) = \Im\{f(x)\} = 2\int_0^\infty f(u)\sin \alpha u\, du = 2\int_a^b x \sin \alpha u\, du$$

$$= 2\left[\left(-\frac{u\cos\alpha u}{\alpha}\right)_a^b + \frac{1}{\alpha}\int_a^b \cos \alpha u\, du\right]$$

$$= 2\left(\frac{\mathrm{a}\cos\alpha a - b\cos\alpha a}{\alpha} + \frac{\mathrm{a}\sin\alpha b - \sin\alpha a}{\alpha^2}\right)$$

<u>The solution</u>

From the definition of the cosine and sine Fourier transform we get

$$f(x) = \Im^{-1}\{F(\alpha)\} = \frac{2}{\pi}\int_0^\infty F_s(\alpha)\sin\alpha x\, d\alpha = \frac{2}{\pi}\int_0^\infty \frac{\alpha}{1+\alpha^2}\sin\alpha x\, d\alpha$$

$$= \frac{2}{\pi}\int_0^\infty \frac{(\alpha^2+1)-1}{\alpha(1+\alpha^2)}\sin\alpha x\, d\alpha = \frac{2}{\pi}\left[\int_0^\infty \frac{\sin\alpha x}{\alpha}\, d\alpha - \int_0^\infty \frac{\sin\alpha x}{\alpha(1+\alpha^2)}\, d\alpha\right]$$

But $$\int_0^\infty \frac{\sin\alpha x}{\alpha} d\alpha = \frac{\pi}{2}$$

Hence $$f(x) = \frac{2}{\pi}\left[\frac{\pi}{2} - 2\int_0^\infty \frac{\sin\alpha x}{\alpha(1+\alpha^2)}\, d\alpha\right] \qquad (1)$$

And by differentiation w.r.t. x , we get

$$\frac{df}{dx} = \frac{2}{\pi}\left[-\int_0^\infty \frac{\cos\alpha x}{(1+\alpha^2)}\, d\alpha\right] \qquad (2)$$

And by differentiation w.r.t. x another once, we get

$$\frac{d^2 f}{dx^2} = \frac{2}{\pi}\int_0^\infty \frac{\alpha \sin \alpha x}{\left(1+\alpha^2\right)}\, d\alpha \qquad (3)$$

Subtracting equation (1) from equation (3), it is clear that

$$\frac{d^2 f}{dx^2} - f = \frac{2}{\pi}\left[\int_0^\infty \frac{\sin \alpha x}{\alpha}\, d\alpha - \frac{\pi}{2}\right] = 0$$

This is an ordinary linear differential of second order with constant coefficients, its solution is found to be

$$f = c_1 e^{x} + c_2 e^{-x} \qquad (4)$$

Therefore

$$\frac{df}{dx} = c_1 e^{x} - c_2 e^{-x}$$

When $x = 0 \Rightarrow f(0) = 1$, this is from equation (1), and from equation (2) will be

$$\frac{df}{dx} = \frac{2}{\pi}\left[-\int_0^\infty \frac{1}{\left(1+\alpha^2\right)}\, d\alpha\right] = -\frac{2}{\pi}\left[\tan^{-1}\alpha\right]_0^\infty = -\frac{2}{\pi}\left[\frac{\pi}{2}\right] = -1$$

Also from equation (4), Using these results, we get

$$c_1 + c_2 = 1 \quad \& \quad c_1 - c_2 = -1$$

And by solving these equations we get $c_1 = 0 \quad \& \quad c_2 = 1$

Therefore

$$f(x) = e^{-x}$$

Example(7):This example is an application for the convolution theorem

Solve the following integral equation $y(x)=g(x)+\int_{-\infty}^{\infty} y(u)r(x-u)du$

where $r(x)$, $g(x)$ are given functions.

The solution

By taking the Fourier transform for both sides and using the convolution theorem we get

$$y(\alpha)=G(\alpha)+Y(\alpha)R(\alpha)$$

where $R(\alpha)$, $G(\alpha)$ are now known function because they are the Fourier transform of the given functions, then we have the result

$$y(\alpha)=\frac{G(\alpha)}{1-R(\alpha)}$$

It is clear that $Y(\alpha)$ is a known Fourier transform of the unknown function $y(x)$ which can be get it by taking the inverse Fourier transform as

$$y(x)=\Im^{-1}\left\{\frac{G(\alpha)}{1-R(\alpha)}\right\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{G(\alpha)}{1-R(\alpha)}e^{i\alpha x}d\alpha$$

Example(8):This is another example of application for the convolution theorem

Solve the following integral equation $\int_{-\infty}^{\infty}\frac{y(u)\,du}{(x-u)^2+a^2}=\frac{1}{x^2+b^2}$.

The solution

By taking the Fourier transform for R.H.S

$$\Im\left\{\frac{1}{x^2+b^2}\right\}=\int_{-\infty}^{\infty}\frac{e^{-i\alpha x}}{x^2+b^2}dx$$

$$= 2\int_0^\infty \frac{\cos\alpha x}{x^2+b^2}dx$$

This integral has been given in example (4) before, and as a result

$$\Im\left\{\frac{1}{x^2+b^2}\right\} = \frac{\pi}{b}e^{-b\alpha} \qquad (1)$$

get so also and we

$$\Im\left\{\frac{1}{x^2+b^2}\right\} = \frac{\pi}{a}e^{-a\alpha} \qquad (2)$$

Now by taking the Fourier transform for both sides of the given equation and using the convolution theorem we get

$$\Im\{y\}\ \Im\left\{\frac{1}{x^2+a^2}\right\} = \Im\left\{\frac{1}{x^2+b^2}\right\} \qquad (3)$$

Substituting from (1) , (2) in (3) we get

$$Y(\alpha)\frac{\pi}{a}e^{-a\alpha} = \frac{\pi}{b}e^{-b\alpha} \qquad \text{or} \qquad Y(\alpha) = \frac{a}{b}e^{-(b-a)\alpha}$$

Therefore

$$y(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha x}Y(\alpha)d\alpha$$

$$= \frac{a}{b\pi}\int_0^\infty e^{-(b-a)\alpha}\cos\alpha x d\alpha$$

$$= \frac{a(b-a)}{b\pi\left[x^2 + (b-a)^2\right]}$$

where we use the relation $e^{i\alpha x} = \cos\alpha x + i\sin\alpha x$ and the second term is tends to zero where the inside function is odd, and change the boundary of the integration from zero to ∞ instead of from $-\infty$ to ∞

of the two functions

$f(x) = e^{-ax}$ & $g(x) = e^{-bx}$, clarify that

$$\int_0^\infty \frac{d\alpha}{\left(a^2+\alpha^2\right)\left(b^2+\alpha^2\right)} = \frac{2}{\pi ab\,(a+b)} \quad a\,\&\,b > 0$$

The solution

The cosine Fourier transform takes the form

$$F_c(\alpha) = \Im\{f(x)\} = 2\int_0^\infty f(u)\cos\alpha u\,du = 2\int_0^\infty e^{-au}\cos\alpha u\,du$$

$$= 2\left(\frac{a}{a^2+\alpha^2}\right)$$

This result has been given in example (4) before, and as a result

Similarly the cosine Fourier transform of the second function takes the form

$$G_c(\alpha) = \Im\{G(x)\} = \int_0^\infty G(u)\cos\alpha u\,du = \int_0^\infty e^{-bu}\cos\alpha u\,du$$

$$= \left(\frac{b}{b^2+\alpha^2}\right)$$

Therefore

$$\int_0^\infty F_c(\alpha)G(\alpha)d\alpha = \int_0^\infty F_c(\alpha)d\alpha \int_0^\infty g(u)\cos\alpha u du$$

$$= 4\int_0^\infty g(u)du \int_0^\infty F_c(\alpha)d\alpha \cos\alpha u d\alpha = \frac{2}{\pi}\int_0^\infty f(u)g(u)du$$

That is $$\int_0^\infty F_c(\alpha)G(\alpha)d\alpha = \frac{2}{\pi}\int_0^\infty f(u)g(u)du$$

Substituting about $f(x) = e^{-ax}$ & $g(x) = e^{-bx}$) we get

$$\int_0^\infty F_c(\alpha)G(\alpha)d\alpha = \frac{2}{\pi}\int_0^\infty f(u)g(u)du = \frac{2}{\pi}\int_0^\infty e^{-au} e^{-bu} du = \frac{2}{\pi}\int_0^\infty e^{-(a+b)u} du = \frac{2}{\pi}\left(\frac{1}{a+b}\right)$$

Therefore $$\int_0^\infty \frac{abd\alpha}{(a^2+\alpha^2)(b^2+\alpha^2)} = \frac{2}{\pi(a+b)} \quad a \& b > 0$$

That is $$\int_0^\infty \frac{d\alpha}{(a^2+\alpha^2)(b^2+\alpha^2)} = \frac{2}{\pi ab(a+b)} \quad a \& b > 0$$

Exercises

(1) Clarify that the two equations (3) , (4) in Fourier integral are equal.

(2)If the cosine Fourier transform of the function $f(x)$ is $F_c(\alpha) = \alpha^n e^{-a\alpha}$

Then find $f(x)$

(2) Solve the following integral equation

$$\int_0^{\infty} f(x)\sin \alpha x dx = \begin{cases} 1-\alpha : 0 \le \alpha \le 1 \\ 0 : \alpha > 1 \end{cases}$$

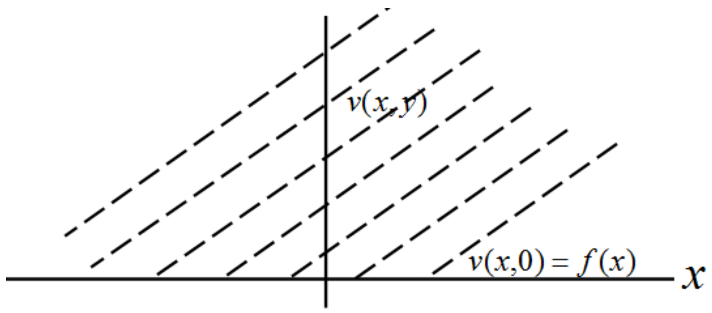
Application for Fourier integral:

Example(1):Find the limited solution of Laplace equation $\nabla^2 v = 0$, for the half plane $y > 0$, where v is given by the function $f(x)$ at the x axis .

The solution

The boundary value problem is given by

$$|v(x,y)| < M \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0, \quad v(x,0) = f(x),$$



For the separation of the variables let $v = XY$

Then T $\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2$

$$X'' + \lambda^2 X = 0 \quad , \quad Y'' - \lambda^2 Y = 0$$

$$v(x, y) = (a_1 \cos\lambda x + b_1 \sin\lambda x)(a_2 e^{\lambda y} + b_2 e^{-\lambda y})$$

And because that $v(x, y)$ is must to be bounded then $a_2 = 0$

Then $,v(x, y) = e^{-\lambda y}(A\cos\lambda x + B\sin\lambda x)$

And because there is no conditions on the parameter λ , then by **apply the** addition principle we get

$$v(x, y) = \int_0^{\infty} e^{-\lambda y}\left[A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x\right]d\lambda$$

And since $v(x,0) = f(x)$, then

$$f(x) = \int_0^{\infty}\left[A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x\right]d\lambda$$

From Fourier integral we have:

$$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\lambda u du$$

$$B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin\lambda u du$$

Then

$$v(x, y) = \frac{1}{\pi}\int_{\lambda=0}^{\infty}\int_{u=-\infty}^{\infty} e^{-\lambda y} f(u)\cos\lambda(x-u) du d\lambda$$

Example(2):Show that the result of the above example can be reduced to

$$v(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{yf(u)}{y^2+(u-x)^2}\,du$$

The solution

$$v(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(u)\left[\int_{0}^{\infty}e^{-\lambda y}\cos\lambda(x-u)d\lambda\right]du$$

Then by taking the internal integral, we find

$$\int_{0}^{\infty}e^{-\lambda y}\cos\lambda(x-u)d\lambda=\frac{y}{y^2+(u-x)^2}$$

$$\therefore v(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{yf(u)}{y^2+(u-x)^2}du$$

Example(3):Use the Fourier transform to solve the following boundary value problem

$$\frac{\partial u}{\partial t}=k\frac{\partial u}{\partial x^2}\ ,u(x,0)=f(x)\ ,|u(x,t)<u|\ ,-\infty<x<\infty$$

Then explain the physical meaning of that problem.

The solution

By taking Fourier transform for the problem we get

$$\frac{d}{dt}\Im\{u\}=-k\alpha^2\Im\{u\}$$

Here we used the differentiation property number(5)

And this is an ordinary differential equation for the function $\Im\{u\}$, its solution is:

$$\Im\{u\} = c(\alpha)e^{-k\alpha^2 t}$$

If we put $t = 0$, then $\Im\{u(x,0)\} = \Im\{f(x)\} = c(\alpha)$

That is $$\Im\{u\} = \Im\{f\}.e^{-k\alpha^2 t} \qquad (1)$$

So to apply the convolution theorem, then we should write the second factor in the R,H,S. in the form of an integral ,where there a known integral which is

$$\int_0^\infty e^{-Mx^2} \cos \beta x dx = \frac{1}{2}\sqrt{\frac{\pi}{M}} e^{\frac{-\beta^2}{4M}}$$

From which $$e^{-k\alpha^2 t} = 2\sqrt{\frac{1}{4k\pi t}} \int_0^\infty e^{-\frac{x^2}{4kt}} \cos \alpha x dx = 2\int_{-\infty}^{\infty} \sqrt{\frac{1}{4k\pi t}} e^{-\frac{x^2}{4kt}} \cos \alpha x dx$$

Then $$e^{-k\alpha^2 t} = \Gamma\left\{\sqrt{\frac{1}{4k\pi t}} e^{-\frac{x^2}{4kt}}\right\} \qquad (2$$

from (1) ,(2) we get

$$\Im\{u\} = \Im\{f\}.\Im\left\{\sqrt{\frac{1}{4k\pi t}} e^{-\frac{x^2}{4kt}}\right\}$$

It clear that the convolution theorem can be applied to give

$$u(x,t) = f(x) * \sqrt{\frac{1}{4k\pi t}} e^{-\frac{x^2}{4kt}}$$

$$= \int_{-\infty}^{\infty} f(w)\sqrt{\frac{1}{4k\pi t}}\, e^{-\frac{(x-w)^2}{4kt}}\, dw$$

This result can be simplified, if we make the following transform

$$\frac{(w-x)^2}{4kt} = z^2 \qquad \text{Or} \qquad z = \frac{x-w}{2\sqrt{kt}}$$

Then the result is

$$\therefore u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} f(x - 2z\sqrt{kt})dz$$

the physical meaning of that problem is the heat conduction in a thin infinite length bar.

The fourth chapter

Laplace Transform ..... تحاويـــــــل لابـــــــلاس

In *Laplase'trnsform* we can take s as a real variable but in some circumstances may be taken as a complex variable.

So *The Laplase'trnsform* can be written as:

$$\ell\{F(t)\} = f(s) = \int_0^\infty F(t)e^{-st}dt$$

We can said That $f(s)$ is existence if the integral $\int_0^\infty F(t)e^{-st}\,dt$ is convergence if some certain conditions can be satisfied for the function $F(t)$ which we will be discuss later.

Definition : Let $F(t)$ be a continuous and single-valued function of real variable t defined for all $0 < t < \infty$, and is of exponential order. Then *The Laplase'trnsform* of $F(t)$ is defined as a function $f(s)$ Denoted by the integral :

$$L\{F(t)\} = \int_0^\infty F(t)e^{-st}\,dt = f(s) \qquad (1)$$

Over that range of values of s for which the integral exists. Here s is a parameter, real or complex. Obviously, $L\{F(t)\}$ is a function of s

$$L\{F(t)\} = f(s) \qquad \& \quad L^{-1}\{f(s)\} = F(t)$$ Thus:

where L is the operator which transform $F(t)$ into $f(s)$,is called *The Laplase'trnsform* operator , and L^{-1} is *The Inverse Laplase'trnsform* Operator.

The Laplase' trnsform belongs to the family of "*integral transform*".

An integral transform $f(s)$ of the function $F(t)$ is defined by an integral of the form:

$$\int_a^b K(s,t)F(t)dt = f(s)$$

Where $K(s,t)$, a function of tow variables s & t, is called the kernel of the integral transform. Here the kernel of *The Laplase' trnsform* is e^{-st} .we show before the kernel of Fourier transform

In the following schedule we rote *The Laplase' trnsform* for some elementary functions which we shall prove through the examples.

$F(t)$	$F(t) = f(s)$
1	$\frac{1}{s}$, $s > 0$
t	$\frac{1}{s^2}$, $s > 0$
t^n	$\frac{n!}{s^{n+1}}$, $s > 0$
e^{at}	$\frac{1}{s-a}$, $s > a$
$\sin at$	$\frac{a}{s^2+a^2}$, $s > 0$
$\cos at$	$\frac{s}{s^2+a^2}$, $s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}$, $s > \lvert a \rvert$
$\cosh at$	$\frac{s}{s^2-a^2}$, $s > \lvert a \rvert$

Example(1):Find *The Laplase'trnsform* for the following functions:

$1-0\ ,2-1,3-t\ ,4-t^{2},4-t^{2},5-t^{n}\ ,6-e^{at}\ ,7-e^{-at}$

The solution

$$1-\ :L\{0\}=\int_{0}^{\infty}0\,e^{-st}\,dt=0$$

$$2-\ :L\{1\}=\int_{0}^{\infty}(1)e^{-st}\,dt=\left|\frac{e^{-st}}{-s}\right|_{0}^{\infty}=\frac{1}{s}\qquad ,s>0$$

$$3-\ :L\{t\}=\int_{0}^{\infty}(t)e^{-st}\,dt=\int_{0}^{\infty}td\left(\frac{e^{-st}}{-s}\right)=t\left(\frac{e^{-st}}{-s}\right)_{0}^{\infty}-\int_{0}^{\infty}\frac{e^{-st}}{-s}dt==\frac{1}{s^{2}}\qquad ,s>0$$

$$4-\ :L\{t^{2}\}=\int_{0}^{\infty}(t^{2})e^{-st}\,dt=\int_{0}^{\infty}t^{2}d\left(\frac{e^{-st}}{-s}\right)=.....=\frac{2!}{s^{3}}\qquad ,s>0$$

$$5-\ :L\{t^{n}\}=\int_{0}^{\infty}t^{n}d\left(\frac{e^{-st}}{-s}\right)=.....=\frac{n!}{s^{n+1}}\qquad ,s>0$$

In the tow cases $4\ \&\ 5$ was the same as in the case 1 with rebate n of times .

$$6-\ :L\{e^{at}\}=\int_{0}^{\infty}e^{at}\,e^{-st}d\,t=\int_{0}^{\infty}e^{-(s-a)t}\,d\,t=\left|\frac{e^{-(s-a)t}}{-(s-a)}\right|_{0}^{\infty}=\frac{1}{s-a}\qquad ,s>a$$

$$7-\ :L\{e^{-at}\}=\int_{0}^{\infty}e^{-at}\,e^{-st}d\,t=\int_{0}^{\infty}e^{-(s+a)t}\,d\,t=\left|\frac{e^{-(s+a)t}}{-(s+a)}\right|_{0}^{\infty}=\frac{1}{s+a}\qquad ,s>a$$

Example(2):Find *The Laplase'trnsform* for the following functions:

$1-\cos at\qquad ,\qquad 2-\sin st$

The solution

$$1- \quad : L\left\{\cos t\right\} = \int_0^\infty \cos t \; e^{-st}\, dt = \operatorname{Re} \int_0^\infty e^{iat}\; e^{-st}\, dt = \operatorname{Re} L\left\{e^{iat}\right\}$$

$$= \operatorname{Re} \frac{1}{s-ia} = \operatorname{Re} \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} \;,\; s>0$$

$$2- \quad : L\left\{\sin t\right\} = \int_0^\infty \sin t \; e^{-st}\, dt = \operatorname{Im} \int_0^\infty e^{iat}\; e^{-st}\, dt = \operatorname{Im} L\left\{e^{iat}\right\}$$

$$= \operatorname{Im} \frac{1}{s-ia} = \operatorname{Im} \frac{s+ia}{s^2+a^2} = \frac{a}{s^2+a^2} \;,\; s>0$$

Example(2):Find *The Laplase trnsform* for the following functions:

$1-\cosh at$, $2-\sinh at$

The solution

By using the result of the previous example we get:

$$1- \quad : L\left\{\cosh t\right\} = L\left\{\frac{e^{at}+e^{-at}}{2}\right\} = \frac{1}{2}\left[L\left\{e^{at}\right\} + L\left\{e^{-at}\right\}\right] = \frac{s}{s^2-a^2} \;, s>|a|$$

$$----2-: L\left\{\sinh t\right\} = L\left\{\frac{e^{at}-e^{-at}}{2}\right\} = \frac{1}{2}\left[L\left\{e^{at}\right\} - L\left\{e^{-at}\right\}\right] = \frac{a}{s^2-a^2} \;, s>|a|$$

Properties of Laplace transform:

(1) Linearity property:

If $c_1 \;\& c_2$ are any tow constants and $f_1(s) \;\& f_2(s)$ are *the Laplce transform* respectively of $F_1(t) \;\& F_2(t)$, then:

$$L\left\{c_1F_1(t) \;\& c_2F_2(t)\right\} = c_1L\left\{F_1(t)\right\} + c_2L\left\{F_2(t)\right\} = c_1f_1(s) + c_2f_2(s)$$

The proof

$$L\left\{c_1F_1(t)+c_2F_2(t)\right\}=\int_0^\infty e^{-st}\left(c_1F_1(t)+c_2F_2(t)\right)dt$$

$$=c_1\int_0^\infty F_1(t)e^{-st}dt+c_2\int_0^\infty F_2(t)e^{-st}dt=c_1f_1(s)+c_2f_2(s)$$

Example:

$$\ell\left\{4t^2-3\cos 2t+5e^{-t}\right\}=4\ell\left\{t^2\right\}-3\ell\left\{\cos 2t\right\}+5\ell\left\{c^{-t}\right\}$$

$$=4\left(\frac{2}{s^3}\right)-3\left(\frac{s}{s^2+4}\right)+5\left(\frac{1}{s+1}\right)$$

$$=\frac{8}{s^3}-\frac{3s}{s^2+4}+\frac{5}{s+1}$$

(2) First Shifting Property:

If $f(s)$ is *the Laplce transform* of $F(t)$, then *Laplce transform* of $e^{at}F(t)$ is $f(s-a)$,i.e., : $L\left\{e^{at}F(t)\right\}=f(s-a)$

The proof

$$\ell\left\{F(t)\right\}=\int_0^\infty e^{-st}F(t)dt=f(s)$$

$$\ell\left\{e^{at}F(t)\right\}=\int_0^\infty e^{-(s-a)t}F(t)dt=f(s-a)$$

Example: $L\left\{e^{-t}\cos 2t\right\}=\dfrac{s+1}{(s+1)^2+4}=\dfrac{s+1}{s^2+2s+5}$

(3) Second Shifting Property:

If $f(s)$ is *the Laplce transform* of $F(t)$, then *Laplce transform* of $G(t)$ is $e^{-as}f(s)$,where $G(t)=\begin{cases}F(t-a) & :t>a\\ 0 & :t<a\end{cases}$ i.e., :

$L\left\{G(t)\right\}=e^{-as}f(s)$.

<u>The proof</u>

$$\ell\{G(t)\}=\int_0^\infty e^{-st}G(t)dt=\int_0^a e^{-st}G(t)dt+\int_a^\infty e^{-st}G(t)dt$$

$$=0+\int_{t=a}^\infty e^{-st}F(t-a)dt$$

Then by using the transform $u=t-a$, we get:

$$\ell\{G(t)\}=\int_{u=0}^\infty e^{-s(u+a)}F(u)du=e^{-as}\int_0^\infty e^{-su}F(u)du=e^{-as}f(s)$$

Example:

If $\ell\{t^3\} = \dfrac{3!}{s^4} = \dfrac{6}{s^4}$ then

$$L\{G(t)\} = \frac{6e^{-2s}}{s^4} \; where \; G(t) = \begin{cases} (t-2)^3 & : t > 2 \\ 0 & : t < 2 \end{cases}$$

(4) Change of scale Property:

If $f(s)$ is *the Laplce transform* of $F(t)$, then *Laplce transform* of $F(at)$ is $\frac{1}{a} f\left(\frac{s}{a}\right)$,i.e., : $L\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$

<u>The proof</u>

$$\ell\{F(at)\} = \frac{1}{a} f(\frac{s}{a}) \quad \text{فــإن} \quad \ell\{F(t)\} = f(s) \quad \text{اذا كان}$$

$$L\{F(at)\} = \int_0^\infty F(at) e^{-st} dt = \int_0^\infty F(u) e^{-su/a} du/a$$

$$= \frac{1}{a} \int_0^\infty F(u) e^{-\left(\frac{s}{a}\right)u} du = \frac{1}{a} f\left(\frac{s}{a}\right)$$

$$\ell\{F(t)\} = \int_0^\infty e^{-st} F(t) dt = f(s)$$

$$\ell\{e^{at} F(t)\} = \int_0^\infty e^{-(s-a)t} F(t) dt = f(s-a)$$

Example:

If $\ell\{\sin t\} = \dfrac{1}{s^2+1}$ then $\ell\{\sin 3t\} = \dfrac{1}{3}\dfrac{1}{(\frac{s}{3})^2+1} = \dfrac{3}{s^2+9}$

(5) The differentiation Property:

If $f(s)$ is *the Laplce transform* of $F(t)$, then *Laplce transform* of $F^{(n)}(t)$ is :

$$L\left\{F^{(n)}(t)\right\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F^1(0)....-sF^{n-2}(0) - F^{n-1}(0)$$

<u>The proof</u>

From the definition of Laplace transform we find that:

$$L\left\{F^{'}(t)\right\} = \int_0^\infty F^{'}(t)e^{-st}dt = \left|e^{-st}F(t)\right|_0^\infty + s\int_0^\infty e^{-st}F(t)dt$$

$$= -F(0) + sf(s) = sf(s) - F(0)$$

Similarly it can be shown that

$$L\left\{F^{(2)}(t)\right\} = \int_0^\infty F^{(2)}(t)e^{-st}dt = s\,L\left\{F^{'}(t)\right\} - F^{'}(0) = s\left[sf(s) - F(0)\right] - F^{'}(0)$$

$$= s^2 f(s) - sF(0) - F^{'}(0)$$

Thus in general

$$L\left\{F^{(n)}(t)\right\} = s^{(n)}f(s) - s^{(n-1)}F(0) - s^{(n-2)}F^{'}(0)....-sF^{(n-2)}(0) - F^{(n-1)}(0)$$

Example: If

$$\ell\{\cos 3t\} = \frac{s}{s^2+9} \quad \text{then} \quad \ell\{-3\sin 3t\} = s(\frac{s}{s^2+9}) - 1 = \frac{-9}{s^2+9}$$

$$\ell\{-3\sin 3t\} = s(\frac{s}{s^2+9}) - 1 = \frac{-9}{s^2+9}$$

$$\ell\{F^n(t)\} = s^2 f(s) - sF(0) - F'(0)$$

$$\ell\{F^{(n)}(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - ..... - sF^{(n-2)}(0) - F^{(n-1)}(0)$$

6) The Integration Property:

If $f\left(s\right)$ is *the Laplce transform* of $F\left(t\right)$, then *Laplce transform* of

is : $L\left\{\int_0^t F\left(u\right)du\right\} = \frac{f\left(s\right)}{s}$:

<u>**The proof**</u>

By supposing That $G(u) = \int_0^t F(u)du$ then it result that

$$G'(t) = F(t), G(0) = 0$$

and by taking Laplace transform of both sides it follows that :

$$\ell\{F(t)\} = \ell\{G'(t)\} = s\ell\{G\} - G(0) = s\ell\{G(t)\}$$

$$\therefore \ell\left\{\int_0^t F(u)du\right\} = \ell\{G\} = \frac{f(s)}{s}$$

Example: If $\ell\{\sin 2t\} = \dfrac{2}{s^2+4}$ then $\ell\left\{\int_0^t \sin 2u du\right\} = \dfrac{2}{s(s^2+4)}$

And to satisfying from that we see that

$$F(t) = \int_0^t \sin 2u du = \frac{1}{2}(1-\cos 2t)$$

$$\therefore \ell\{F(t)\} = \frac{1}{2}[\ell\{1\} - \ell\{\cos 2t\}] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] = \frac{2}{s(s^2+4)}$$

$$F(t) = \int_0^t \sin 2u du = \frac{1}{2}(1-\cos 2t)$$

$$\therefore \ell\{F(t)\} = \frac{1}{2}[\ell\{1\} - \ell\{\cos 2t\}] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] = \frac{2}{s(s^2+4)}$$

7) Multiplication by power of t :

If $f(s)$ is *the Laplce transform* of $F(t)$, then :

$$L\left\{\int_0^t t^n F(u) du\right\} = (-1)^n \frac{d^n f(s)}{ds^n} = (-1)^n f^{(n)}(s) \quad :$$

The proof

From the definition of Laplace transform we find that:

$$f(s) = L\{F'(t)\} = \int_0^\infty F'(t) e^{-st} dt$$

Hence

$$\frac{d}{ds} f(s) = L\{F'(t)\} = \frac{d}{ds}\left[\int_0^\infty F'(t)e^{-st}dt\right]$$

Interchanging the operations of differentiation and integration for which we assume that the necessary are satisfied, and since there are two variables $s\ \&\ t$, we use the notation of partial differentiation and obtain:

$$\frac{df(s)}{ds} = \int_0^\infty \frac{\partial}{\partial s}\{e^{-st}F(t)dt\} = -\int_0^\infty e^{-st}tF(t)dt = -L\{tF(t)\}$$

Therefore

$$L\{tF(t)\} = -\frac{df(s)}{ds}$$

By repeated application of the above result, it can be shown that

$$L\left\{\int_0^t t^n F(u)du\right\} = (-1)^n \frac{d^n f(s)}{ds^n} = (-1)^n f^{(n)}(s)$$

$$\frac{df(s)}{ds} = \int_0^\infty \frac{\partial}{\partial s}\{e^{-st}F(t)dt\} = -\int_0^\infty e^{-st}tF(t)dt = -L\{tF(t)\}$$

$$L\{tF(t)\} = -\frac{df(s)}{ds}$$

Example: If $\ell\{e^{2t}\} = \frac{1}{s-2}$ then

$$\ell\{te^{2t}\} = -\frac{d}{ds}\left(\frac{1}{s-2}\right) = \frac{1}{(s-2)^2}$$

Also $\ell\{t^2e^{2t}\} = \frac{d^2}{ds^2}\left(\frac{1}{s-2}\right) = \frac{2}{(s-2)^3}$

8) Division by of t :

If $f(s)$ is *the Laplce transform* of $F(t)$, then :

$$L\left\{\frac{F(u)}{t}\right\}=f(s)=\int_s^{\infty} f(u)du \quad :$$

<u>The proof</u>

From the definition of Laplace transform we find that:

$$f(s)=L\left\{F^{'}(t)\right\}=\int_0^{\infty} F^{'}(t)e^{-st}dt$$

Integrating the above equation with respect to s *between the* limts s & ∞

$$\int_s^{\infty} f(s)=\int_s^{\infty} f(s)\left[\int_0^{\infty} F^{'}(t)e^{-st}dt\right]=\int_0^{\infty} F^{'}(t)\left[\int_s^{\infty} e^{-st}ds\right]dt \qquad \frac{F^{'}(t)}{t}$$

$$=\int_0^{\infty} F^{'}(t)\left[\frac{e^{-st}}{-t}\right]_s^{\infty} dt=\int_0^{\infty}\frac{F^{'}(t)}{t}e^{-st}dt=L\left\{\frac{F^{'}(t)}{t}\right\}$$

$$\therefore\ L\left\{\frac{F^{'}(t)}{t}\right\}=\int_s^{\infty} f(s)$$

Note: In applying this rule, should be careful. Since $F^{'}(t)/t$ may have an infinite discontinuity at $t=0$, it may not be integrable . If $F^{'}(t)/t$ is not integrable, then its Laplace transform does not exit. For example, at $t=0$, the function $\sin t/t$ does not have an infinite discontinuity, while the the function $\cos t/t$ has an infinite discontinuity.

Example: Since $\ell\{\sin t\}=\dfrac{1}{s^2+1}$ and $\lim\limits_{t\to 0}\dfrac{\sin t}{t}==1$ then

$$\ell\left\{\frac{\sin t}{t}\right\} = \int_{s}^{\infty} \frac{du}{u^2 \oplus 1} = \tan^{-1}(\frac{1}{s})$$

9) Transform of periodic function:

If $F(t)$ is a periodic function with period T, then

If $f(s)$ is *the Laplce transform* of $F(t)$, then :

$$L\{F(t)\} = f(s) = \int_{0}^{T} F(t)\, e^{-st}\, du \Big/ \left(1-e^{-sT}\right) \quad :$$

<u>**The proof**</u>

From the definition of Laplace transform we find that:

$$f(s) = L\{F'(t)\} = \int_{0}^{\infty} F'(t) e^{-st} dt = \int_{0}^{T} e^{-st} F(t) dt + \int_{T}^{\infty} e^{-st} F(t) dt$$

If we substitute $t = u + T$ in the second integral on the right-hand side and write $dt = du$, we have

$$f(s) = L\{F(t)\} = \int_{0}^{T} e^{-st} F(t) dt + \int_{0}^{\infty} e^{-s(u+T)} F(u+T) du$$

$$= \int_{0}^{T} e^{-st} F(t) dt + e^{-sT} \int_{T}^{\infty} e^{-su} F(u) du$$

$$= \int_{0}^{T} e^{-st} F(t) dt + e^{-sT} f(s)$$

Rearranging,

we get

$$\left(1-e^{-sT}\right)\ f\ (s) = \int_0^T F(T)e^{-st}\,dt$$

Therefore

$$f\ (s) = \int_0^T F(T)e^{-st}\,dt \Big/ \left(1-e^{-sT}\right)$$

Hence it is the result.

Example: Find the Laplace transform the of periodic function

$F(T) = \dfrac{t}{T}$ of period $0 < t < T$

<u>The solution</u>

$$f\ (s) = \frac{1}{1-e^{-sT}}\int_0^T \frac{t}{T}e^{-st}\,dt = \frac{1}{T\left(1-e^{-sT}\right)}\int_0^T te^{-st}\,dt$$

$$= \frac{1}{T\left(1-e^{-sT}\right)}\int_0^T te^{-st}\,dt = \frac{1}{T\left(1-e^{-sT}\right)}\int_0^T t\,d\left(\frac{e^{-st}}{-s}\right)$$

$$= \frac{1}{T\left(1-e^{-sT}\right)}\int_0^T te^{-st}\,dt = \frac{1}{T\left(1-e^{-sT}\right)}\int_0^T t\,d\left(\frac{e^{-st}}{-s}\right)$$

$$= \frac{1}{T\left(1-e^{-sT}\right)}\left[\left(\frac{te^{-st}}{-s}\right)_0^T + \frac{1}{s}\int_0^T e^{-st}\,dt\right]$$

$$= \frac{1}{T\left(1-e^{-sT}\right)}\left[\frac{Te^{-st}}{-s} - \frac{1}{s^2}\left(e^{-sT}-1\right)\right] = \frac{1}{s^2T} - \frac{e^{-st}}{s\left(1-e^{-sT}\right)}$$

$$\therefore f\ (s) = \frac{1}{s^2T} - \frac{e^{-st}}{s\left(1-e^{-sT}\right)}$$

10) The function of transformation at the infinity:

If $L\{F(T)\} = f(s)$, then $\lim_{s\to\infty} f(s) = 0$

$$\therefore sf(s) = 1 + \frac{s}{(s+1)^2+1} + \frac{s^2+s}{(s+1)^2+1} = 1 + \frac{s^2+2s}{s^2+2s+2}$$

$$\therefore \lim_{s\to\infty} sf(s) = \lim_{s\to\infty} 1 + \lim_{s\to\infty} \frac{s^2+2s}{s^2+2s+2} = 1 + \lim_{s\to\infty} \frac{1+2/s}{1+2/s+2/s^2} = 1+1 = 2$$

11) Initial –value Theorem :

If $F(T)$ & $F'(T)$ are Laplace transformable and $L\{F(T)\} = f(s)$, then the behavior of $F(t)$ in the neighborhood of $t \to 0$ corresponds to the behavior of $sf(s)$ in the neighborhood of $s \to \infty$. Mathematically

$$\lim_{t\to 0} F(t) = \lim_{s\to\infty} sf(s)$$

<u>The proof</u>

From the property of derivative, we have

$$L\{F'(t)\} = sf(s) - F(0)$$

Taking the limit $s \to \infty$ on both sides, we get

$$\lim_{s\to\infty} \int_0^\infty e^{-st} F'(t)\,dt = \lim_{s\to\infty} sf(s) - \lim_{s\to\infty} F(0)$$

Since s is independent of t , we can take the limit before integrating the lift-hand side of the last equation, thus getting

$$\lim_{s\to\infty}\int_0^\infty e^{-st}F'(t)dt=\int_0^\infty\left[\lim_{s\to\infty}e^{-st}F'(t)\right]dt=0$$

And the equation becomes

$$\lim_{s\to\infty} sf(s)=F(0)=\lim_{s\to\infty}F(0)$$

Hence the result.

Example: Verify the initial value theorem for the function

$$F(T)=1+e^{-t}(\sin t+\cos t)$$

<u>The solution</u>

بإجراء التحويل لهذه الدالة نحصل على:

By taking Laplace transform we have

$$f(s)=L\left\{1+e^{-t}(\sin t+\cos t)\right\}=L\{1\}+L\left\{e^{-t}(\sin t+\cos t)\right\}$$

$$=\frac{1}{s}+\frac{1}{(s+1)^2+1}+\frac{s+1}{(s+1)^2+1}$$

هنا تم استخدام الجدول وخاصية الأزاحة الأولى

Here we used both of the schedule and the first shifting property

$$\therefore sf(s)=1+\frac{s}{(s+1)^2+1}+\frac{s^2+s}{(s+1)^2+1}=1+\frac{s^2+2s}{s^2+2s+2}$$

$$\therefore \lim_{s\to\infty} sf(s)=\lim_{s\to\infty}1+\lim_{s\to\infty}\frac{s^2+2s}{s^2+2s+2}=1+\lim_{s\to\infty}\frac{1+2/s}{1+2/s+2/s^2}=1+1=2$$

ولكن

but

$$\lim_{t\to 0}F(t)=F(0)=1+1=2$$

$$\therefore \lim_{s\to\infty} sf(s)=\lim_{t\to 0}F(t)$$

Hence the result.

12) Final–value Theorem :

If $F(T)$ & $F^{'}(T)$ are Laplace transformable and $L\{F(T)\}=f(s)$, then the behavior of $F(t)$ in the neighborhood of $t \to \infty$ corresponds to the behavior of $sf(s)$ in the neighborhood of $s \to 0$. Mathematically

$\underset{t\to\infty}{Lim}\,F(t)=\underset{s\to 0}{Lim}\,sf(s)$

The proof

From the property of derivative, we have

$$L\{F^{/}(t)\}=sf(s)-F(0)$$

Taking the limit $S \to 0$ on both sides, we get

$$\lim_{s\to 0}\int_0^\infty e^{-st}F^{'}(t)dt=\lim_{s\to 0}sf(s)-\lim_{s\to 0}F(0)$$

Since s is independent of t , we can take the limit before integrating the lift-hand side of the last equation, thus getting

$$\lim_{s\to 0}\int_0^\infty e^{-st}F^{'}(t)dt=\int_0^\infty\left[\lim_{s\to 0}e^{-st}F^{'}(t)\right]dt=\int_0^\infty\left[\lim_{s\to 0}F^{'}(t)\right]dt$$

$$=\left[F(t)\right]_0^\infty=\lim_{t\to\infty}F(t)-F(0)$$

Using the result in the above equation, we get

$$\lim_{s\to 0}sf(s)=\lim_{t\to\infty}F(0) \qquad \lim_{s\to\infty}sf(s)=F(0)=\lim_{s\to\infty}F(0)$$

Hence the result

We can apply the example in the last property for this property

11) generalization of the Initial –value Theorem :

If $\lim_{t\to 0}\frac{F(t)}{G(t)}=1$ then the value of the function $F(t)$ will be corresponding to the values of $G(t)$, this as t is small and this can be represented as

$$t\to 0 \quad \text{as} \quad F(t)\approx G(t)$$

Similarly if $\lim_{s\to\infty}\frac{f(s)}{g(s)}=1$ then for the biggest values of s the function $f(s)$ will be corresponding to the values of $g(s)$, and this can be represented as

$$f(s)\approx g(s) \text{ as } s\to\infty$$

Therefore we can reformulate the Initial –value Theorem as

If $\quad F(t)\approx G(t) \quad$ as $\quad t\to 0$

then $\quad f(s)\approx g(s) \quad$ as $\quad s\to\infty$

where $\quad f(s)=\ell\{F(t)\} \quad , \quad g(s)=\ell\{g(t)\}$

12) generalization of the Final –value Theorem :

If $\quad t\to\infty \quad$ as $\quad F(t)\approx G(t)$

Then $\quad f(s)\approx g(s) \quad$ as $\quad s\to 0$

where $\quad f(s)=\ell\{F(t)\} \quad , \quad g(s)=\ell\{g(t)\}$

Method for finding Laplace transform :

(1) The role of Laplace transform $f(s)=\int_0^\infty e^{-st}F(t)dt$

(2)The series method where $F(t)$ can represented as

$$F(t)=a_0+a_1t+a_2t^2+..........=\sum_{n=0}^{\infty}a_nt^n$$

(3)Using the properties of Laplace transform .

(4) Using the schedule.

<u>Examples and exercises</u>

1) Find $\ell\{F(t)\}$ where $F(t)=\begin{cases}5: & 0<t<3\\ 0: & t>3\end{cases}$

<u>The solution</u>

$$F(t)=\begin{cases}5 & :0<t<3\\ \\ 0 & :t>3\end{cases}$$

$$\ell\{F(t)\}=\int_0^\infty e^{-st}F(t)dt=\int_0^3 5e^{-st}dt+\int_3^\infty 0.e^{-st}dt$$

$$\ell\{F(t)\}=5\frac{e^{-st}}{-s}\Big|_0^3=\frac{5(1-e^{-3s})}{s}$$

2) Find $\ell\{F(t)\}$ where $\ell\{4e^{5t}+6t^3-3\sin 4t+2\cos 2t\}$

3) Find $\ell\{F(t)\}$ for the following

3

$\ell\{t^2e^{3t}\}$, $\ell\{e^{-2t}\sin 4t\}$, $\ell\{e^{3t}\cosh 5t\}$, $\ell\{e^{-2t}(3\cos 6t-5\sin 6t)\}$

<u>**The solution**</u>

(3

$$\ell\{t^2\}=\frac{2}{s^3},\quad then\ \ell\{t^2e^{3t}\}=\frac{2}{(s-a)^3}\quad ,then\ \ell\{\sin 4t\}=\frac{4}{s^2+16}\quad ,$$

$$\ell\{e^{-2t}\sin 4t\}=\frac{4}{(s+2)^2+16}\quad ,then\ \ell\{\cosh 5t\}=\frac{s}{s^2-25}\quad ,$$

$$\ell\{e^{4t}\cosh 5t\}=\frac{s-4}{(s-4)^2-25},$$

حل اخـــر:

$$\ell\{\cosh 5t\}=\frac{s}{s^2-25},$$

$$\ell\{e^{4t}\cosh 5t\}=\ell\left\{e^{4t}\left(\frac{e^{5t}+e^{-5t}}{2}\right)\right\}$$

$$=\frac{1}{2}\ell\{e^{9t}\}+\frac{1}{2}\ell\{e^{-t}\}=\frac{1}{2}\left(\frac{1}{s-9}+\frac{1}{s+1}\right)$$

$$=\frac{s-4}{s^2-8s-9}=\frac{s-4}{(s-4)^2-25}$$

$$\ell\{3\cos 6t - 5\sin 6t\} = 3(\frac{s}{s^2+36}) - 5(\frac{6}{s^2+36}) = \frac{3s-30}{s^2+36}$$

$$\therefore \ell\{e^{-2t}(3\cos 6t - 5\sin 6t)\} = \frac{3(s+2)-30}{(s+2)^2+36}$$

4) Find $\ell\{F(t)\}$ for

$$F(t) = \begin{cases} \cos(t - \frac{2\pi}{3}) : t > \frac{2\pi}{3} \\ 0 \quad : t < \frac{2\pi}{3} \end{cases}$$

<u>The solution</u>

$$F(t) = \begin{cases} \cos(t - \frac{2\pi}{3}) & : t > \frac{2\pi}{3} \\ 0 & : t < \frac{2\pi}{3} \end{cases}$$

$$\ell\{F(t)\} = \int_0^{2\pi/3} 0.e^{-st}dt + \int_{2\pi/3}^{\infty} e^{-st}\cos(t - \frac{2\pi}{3})dt$$

$$= \int_{u+0}^{\infty} e^{-s(u+\frac{2\pi}{3})}\cos u du = e^{\frac{-2\pi s}{3}} \int_{u+0}^{\infty} e^{-su}\cos u du = \frac{se^{-\frac{2\pi s}{3}}}{s^2+1}$$

This solution has been done by the direct method but we can solve by using the second shifting property as

$$\ell\{\cos t\} = \frac{s}{s^2+1} \therefore \ell\{F(t)\} = \frac{se^{-\frac{2\pi s}{3}}}{s^2+1}$$

5) If $\ell\left\{\frac{\sin t}{t}\right\} = \tan^{-1}(\frac{1}{s})$ then find $\ell\{F(t)\}$

6) Find $\ell\left\{\int_0^t \frac{\sin u}{u} du\right\}$

7) Find a) $\ell\{t \sin at\}$ b) $\ell\{t^2 \cos at\}$

8) Prove that $\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$

9) Draw the curve of the period function which is 2π ,

then find $\ell\{F(t)\}$

10) Verify the initial value theorem and the final value theorem for the function $F(t) = 3se^{-2t}$.

The fifth chapter

تحويـــلات لابلاس العكسي

The Inverse Laplace Transform

If $f(s) = L\{F(t)\}$ then the function $F(t)$ is called $inverse\ laplace\ transform$ of the function $f(s)$ and can be written as $F(t) = L^{-1}\{f(s)\}$, an example for this is

$$F(t) = L^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}$$

Generally if there exist a schedule for $laplace\ transform$ then it can be another schedule for $inverse\ laplace\ transform$ **,** also we can find some properties and rules for $inverse\ laplace\ transform$ as in the flowing**:**

(1) Linearity property:

If c_1 &c_2 are constants and $F_1(t)$ &$F_2(t)$ are $invers\ Laplace\ trans.$ for $f_1(s)$ &$f_2(s)$ respectively , then:

$$L^{-1}\{c_1 f_1(s) + c_2 f_2(s)\} \quad = c_1 L^{-1}\{f_1(s)\} + c_2 L^{-1}\{f_2(s)\}$$

Example:

$$L^{-1}\left\{\frac{4}{s-2}-\frac{3s}{s^2+16}+\frac{5}{s^2+4}\right\}$$

$$= 4L^{-1}\left\{\frac{1}{s-2}\right\} - 3L^{-1}\left\{\frac{s}{s^2+16}\right\} + 5L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= 4e^{2t} - 3\cos 4t + 5\sin 2t$$

(1) <u>First shifting Property</u>

$L^{-1}\{f(s)\} = F(t)$ then $L^{-1}\{f(s+a)\} = e^{at}F(t)$

<u>The proof</u>

Since $L\,\{F(t)\} = f(s)$ then $L^{-1}\{f(s)\} = F(t)$

Recalling the first shifting property of $Laplace\ trans.$,we find that

$L\,\{e^{at}F(t)\} = f(s+a)$ then $L^{-1}\{f(s+a)\} = e^{at}F(t)$

Thus that is the required .

Example: Since $L^{-1}\left\{\frac{1}{s^2+4}\right\} = \sin 2t$ then

$$L^{-1}\left\{\frac{1}{s^2-2s+5}\right\} = L^{-1}\left\{\frac{1}{(s-1)^2+4}\right\} = \frac{1}{2}e^{t}\sin 2t$$

<u>(3)Second shifting Property</u>

$L^{-1}\{f(s)\} = F(t)$ then $L^{-1}\{e^{-as}f(s)\} = \begin{cases} F(t-a) & t > a \\ 0 & t < a \end{cases}$

Example: Since $L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$ then

$$L^{-1}\left\{\frac{e^{-\pi s/3}}{s^2+1}\right\} = \begin{cases} sin\left(t-\frac{\pi}{3}\right) & t > \frac{\pi}{3} \\ 0 & t < \frac{\pi}{3} \end{cases}$$

(4)Change of scale Property

$L^{-1}\{f(s)\} \quad = F(t)$ then $\quad L^{-1}\{f(as)\} = \; \frac{1}{a} F\left(\frac{t}{a}\right)$

The proof

Since $L\,\{F(t)\} = f(s)$ then $\quad f(s) = \int_0^\infty F(t)\,dt$

Therefore

$$f(as) = \int_0^\infty F(t)e^{-ast}\,dt$$

Let $\alpha t = x \rightarrow dt = dx/\alpha$,then we get

$$f(as) = \frac{1}{\alpha}\int_0^\infty F\left(\frac{x}{a}\right)e^{-sx}\,dx = \frac{1}{a}F\left(\frac{x}{a}\right)$$

$$\therefore \quad L^{-1}\{f(as)\} = \frac{1}{a}F\left(\frac{t}{a}\right)$$

Example: Since $L^{-1}\left\{\frac{s}{s^2+16}\right\} = \cos\ 4t$

$$L^{-1}\left\{\frac{2s}{(2S)^2+16}\right\} = \frac{1}{2}\cos\ \frac{4t}{2} = \frac{1}{2}\cos\ 2T$$

(5)Multiplication by power of :

If $L^{-1}\{f(s)\} = F(t)$ then $L^{-1}\left\{\frac{d^n}{ds^n}f(s)\right\} = (-1)^n t^n F(t)$

The proof

From the definition of Multiplication by power of t if $L\{F(t)\} = f(s)$ then $L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$,then we get

$$L^{-1}\left\{\frac{d^n}{ds^n} f(s)\right\} = (-1)^n t^n F(t)$$

Example: Since $L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$ then

$$L^{-1}\left\{\frac{-2s}{(s^2+1)^2}\right\} = -\sin t$$

(6)Dividing by t

If $L\{F(t)\} = f(s)$ then $L\left\{\frac{F(t)}{t}\right\} = \int_o^\infty f(u)\,du$

This leads to $L^{-1}\left\{\int_o^\infty f(u)\,du\right\} = \frac{F(t)}{t}$

Example: Since $L^{-1}\left\{\frac{1}{s(s+1)}\right\} = L^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\} = 1 - e^{-t}$ then

$$L^{-1}\left\{\int_0^\infty \left(\frac{1}{u} - \frac{1}{u+1}\right) du\right\} = L^{-1}\left\{lin^{\left(1+\frac{1}{s}\right)}\right\} = \frac{1-e^{-t}}{t}$$

(7)Integration property

If $L^{-1}\{f(s)\} = \frac{F(t)}{t}$ then $L^{-1}\left\{\int_o^\infty f(u)\,du\right\} = \frac{F(t)}{t}$

The proof

Let $G(t) = \int_0^t F(u)du \quad \therefore G(0) = 0 \;\; \& G^{/}(t) = F(t)$

Also $L\{G^{/}(t)\} = s\,L\{G(t)\} - G(0) = s\,L\{G(t)\}$

That is $L\{G(t)\}=\frac{f(s)}{s}$, therefore $L^{-1}\left\{\frac{f(s)}{s}\right\}=G(t)=\int_0^t F(u)du$

This result can be generalized to show that

$$L^{-1}\left\{\frac{f(s)}{s^n}\right\}=G(t)=\int_0^t F(u)du=\int_0^t\int_0^t\int_0^t........\int_0^t F(u)du$$

Example: Since $\ell^{-1}\left\{\frac{1}{s^2+4}\right\}=\frac{1}{2}\sin 2t$ then

$$\ell^{-1}\left\{\frac{1}{s(s^2+1)}\right\}=\int_0^t\frac{1}{2}\sin 2u du=\frac{1}{2}(1-\cos 2t)$$

(8)Differentiation property

If $\ell^{-1}\{f(s)\}=F(t)$ and $F(0)=0$ then $\ell^{-1}\{sf(s)\}=F'(t)$

And if $F(0)\neq 0$ then $\ell^{-1}\{sf(s)-F(0)\}=F'(t)$

Example: Since $\ell^{-1}\left\{\frac{1}{s^2+1}\right\}=\sin t$ and $\sin 0=0$ then

$$\ell^{-1}\left\{\frac{s}{s^2+1}\right\}=\frac{d}{dt}\sin t=\cos t$$

(9) Convolution Theorem

If $\ell^{-1}\{f(s)\}=F(t)$ and $\ell^{-1}\{g(s)\}=G(t)$ then

$$\ell^{-1}\{f(s).g(s)\}=\int_0^t F(u)G(t-u)du=F*G$$

The integral is called the convolution of $F(t)$ & $G(t)$ and is denoted by $F(t)*G(t)$

The proof

From the definition of Laplace transform we have

$$f(s)g(s)=\left[\int_0^\infty F(v)e^{-sv}dv\right]\left[\int_0^\infty G(u)e^{-su}du\right]=\int_0^\infty\int_0^\infty F(v)G(u)e^{-s(v+u)}dvdu$$

$$=\int_0^\infty G(u)\left[\int_0^\infty F(v)e^{-s(v+u)}dv\right]du$$

Let $u+v=t \Rightarrow dt=dv$ in the inner integral, then

$$f(s)g(s)=\int_0^\infty G(u)\left[\int_u^\infty F(t-u)e^{-st}dt\right]du$$

Change the order of integration we get

$$=\int_0^\infty\left[\int_0^t F(t-u)G(u)du\right]dt$$

$$=L\left[\int_0^t F(t-u)G(u)du\right]$$

$$\therefore f(s)g(s)=L\left[\int_0^t F(t-u)G(u)du\right]$$

And from this we get

$$L^{-1}\left\{\int_0^t F(t-u)G(u)du\right\}=L^{-1}\left\{F(t)*G(u)\right\}=f(s)g(s)$$

Example: Since $\ell^{-1}\left\{\frac{1}{s-1}\right\}=e^t$ and $\ell^{-1}\left\{\frac{1}{s-2}\right\}=e^{2t}$ $\sin 0=0$

then

$$\ell^{-1}\left\{\frac{1}{(s-1)(s-2)}\right\}=\int_0^t e^u e^{2(t-u)}du=\int_0^t e^{2t}e^{-u}du$$

(10) Heaviside Expansion Theorem:

Let $P(s)$ and $Q(s)$ be two polynomials where the degree of $P(s)$ is lower than that of $Q(s)$ which has n of different roots they are $\alpha_k, k = 1,2,....., n$, then

$$\ell^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q(\alpha_k)} e^{\alpha_k t}$$

The proof

We can write the ratio in the theorem as

$$\frac{P(s)}{Q(s)} = \frac{P(s)}{\prod_{k=1}^{n}(s-\alpha_k)} = \frac{A_1}{(s-\alpha_1)} + \frac{A_2}{(s-\alpha_2)} + ... + \frac{A_n}{(s-\alpha_n)}$$

This was done by using partial fraction. Multiplying both sides by $(s-\alpha_k)$ and taking the limit as $s \to \alpha_k$, we obtain the coefficients

$$A_k = \lim_{s\to\alpha_k} \frac{F(s)(s-\alpha_k)}{G(s)} = F(\alpha_k) \lim_{s\to\alpha_k} \frac{(s-\alpha_k)}{G(s)}$$

And since this limit takes the indeterminate form $\frac{0}{0}$, then by using L"Hospital" role , we get $A_k = F(\alpha_k) \lim_{s\to\alpha_k} \frac{1}{G'(s)} = \frac{F(\alpha_k)}{G'(\alpha_k)}$

Hence

$$\frac{P(s)}{Q(s)} = \frac{F(\alpha_1)}{G'(\alpha_1)} \frac{1}{(s-\alpha_1)} + \frac{F(\alpha_2)}{G'(\alpha_2)} \frac{1}{(s-\alpha_2)} + ... + \frac{F(\alpha_n)}{G'(\alpha_n)} \frac{1}{(s-\alpha_n)}$$

Thus

$$L^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \frac{F(\alpha_1)}{G'(\alpha_1)} L^{-1}\left\{\frac{1}{(s-\alpha_1)}\right\} + ... + \frac{F(\alpha_n)}{G'(\alpha_n)} L^{-1}\left\{\frac{1}{(s-\alpha_n)}\right\}$$

$$L^{-1}\left\{\frac{P(s)}{Q(s)}\right\}=\frac{F(\alpha_1)}{G^{/}(\alpha_1)}e^{\alpha_1 t}+\ldots+\frac{F(\alpha_n)}{G^{/}(\alpha_n)}e^{\alpha_n t}=\sum_{k=1}^{n}\frac{F(\alpha_k)}{G^{/}(\alpha_k)}e^{\alpha_k t}$$

Example: Find the inverse Laplace transform for the function $\frac{s^2+1}{s^3+3s^2+2s}$ by using Heaviside Expansion

<u>The solution</u>

Put $p(s)=s^2+1$ $\&Q(s)=s^3+3s^2+2s=s(s+1)(s+2)$

It is clear that $Q(s)$ has three different roots $0,-1,-2$ and the degree of $P(s)$ is lower than that of $Q(s)$, then by using Heaviside Expansion, we get:

$$L^{-1}\left\{\frac{P(s)}{Q(s)}\right\}=L^{-1}\left\{\frac{s^2+1}{s^3+3s^2+2s}\right\}=\frac{F(0)}{G^{/}(0)}e^{0t}+\frac{F(-1)}{G^{/}(-1)}e^{-t}+\frac{F(-2)}{G^{/}(-2)}e^{-2t}$$

$$L^{-1}\left\{\frac{s^2+1}{s^3+3s^2+2s}\right\}=\frac{1}{2}-2e^{-t}+\frac{5}{2}e^{-2t}$$

Method for finding Laplace transform :

(1) Using the schedule.

(2)Using the properties of Laplace transform as we will see in the examples

(3) The series method where can represented as

$$f(s)=\frac{a_0}{s}+\frac{a_1}{s^2}+\frac{a_2}{s^3}+..........$$

Then $\ell^{-1}\{f(s)\}=F(t)=a_0+\frac{a_1}{t}+\frac{a_2t^2}{2!}+\frac{a_3t^3}{3!}+..........$

(4) Heaviside Expansion Theorem.

(5)The partial fraction method, as in the following example

$$L^{-1}\left\{\frac{s^2+1}{s^3+3s^2+2s}\right\}=\frac{A_1}{s}+\frac{A_2}{s+1}+\frac{A_3}{s+2}=\frac{1}{2}L^{-1}\left\{\frac{1}{s}\right\}-2L^{-1}\left\{\frac{1}{s+1}\right\}+\frac{5}{2}L^{-1}\left\{\frac{1}{s+2}\right\}$$

Note: If we can solve any example by Heaviside theorem we can solve it by the partial fraction method also but the inverse is not always correct.

.(6)The formula of the complex inverse transform

Examples

Introduction: The number $n!$ is a function for the number n, and this function is considered as a special case from the Jama function $\Gamma(x)$. And at that time while $n!$ is defined for the positive integer numbers only then $\Gamma(x)$ is defined for the integer and the ratio numbers which are positive and negative. If x is positive integer then $\Gamma(x+1)=n!$, and in case $x=\frac{1}{2}$ then $\Gamma(\frac{1}{2})=\sqrt{\pi}$. But if n is positive integer then $n!=n(n-1)(n-2)$ ....2.1, and in the general case $\Gamma(x)$ there is a similar formula as in following example

$$\Gamma(\frac{9}{2})=\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\ \Gamma(\frac{1}{2})=\frac{105}{16}\sqrt{\pi}$$

(1) Find each of the following

(i) $\ell^{-1}\left\{\frac{5s+4}{s^3}-\frac{2s-18}{s^2+9}+\frac{24-30\sqrt{s}}{s^4}\right\}$

The solution

$$\ell^{-1}\{.......\}=\ell^{-1}\left\{\frac{5}{s^2}+\frac{4}{s^3}-\frac{2s}{s^2+9}+\frac{18}{s^2+9}+\frac{24}{s^4}-\frac{30}{s^{7/2}}\right\}$$

$$=5t+4(\frac{t^2}{2!})-2\cos 3t+18(\frac{1}{3}\sin 3t)+24\frac{t^3}{3!}-30(\frac{t^{\frac{5}{2}}}{\Gamma(\frac{7}{2})})$$

$$=5t+2t^2-2\cos 3t+6\sin 3t+4t^3-\frac{16}{\sqrt{\pi}}t^{5/2}$$

(ii) $\ell^{-1}\left\{\dfrac{6}{2s-3}-\dfrac{3+4s}{9s^2-16}+\dfrac{8-6s}{16s^2+9}\right\}$

<u>The solution</u>

$$\ell^{-1}\{.......\}=\ell^{-1}\left\{\frac{3}{s-3/2}-\frac{1}{3}\left(\frac{1}{s^2-16/9}\right)-\frac{4}{9}\left(\frac{s}{s^2-16/9}\right)+\frac{1}{2}\left(\frac{1}{s^2+9/16}\right)-\frac{3}{8}\left(\frac{s}{s^2+9/16}\right)\right\}$$

$$=3e^{3t/2}-\frac{1}{4}\sinh\frac{4}{3}t-\frac{4}{9}\cosh\frac{4}{3}t+\frac{2}{3}\sin\frac{3}{4}t-\frac{3}{8}\cos\frac{3}{4}t$$

<u>(2)</u> by using the convolution theorem find each of

(i) $\ell^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\}$

<u>The solution</u>

$$\ell^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}=\ell^{-1}\left\{\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right\}$$ -

$$\ell^{-1}\left\{\frac{s}{s^2+a^2}\right\}=\cos at\quad,\quad \ell^{-1}\left\{\frac{1}{s^2+a^2}\right\}=\frac{\sin at}{a}$$ ولكن

$$\therefore\ \ell^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}=\frac{1}{a}\int_0^t \cos au\,\sin a(t-u)\,du$$

$$=\frac{1}{a}\int_0^t \cos au(\sin at\cos au-\cos at\sin au)\,du$$

$$=\frac{1}{a}\sin at\int_0^t \cos^2 au\,du-\frac{1}{a}\cos at\int_0^t \sin au\cos au\,du$$

$$= \frac{1}{a}\ \sin at \int_0^t \frac{1+\cos 2au}{2}\, du - \frac{1}{a}\ \cos at \int_0^t \frac{\sin 2au}{2}\, du$$

$$= \frac{1}{a}\ \sin at\ \left(\frac{t}{2} + \frac{\sin 2at}{4a}\right) - \frac{1}{a}\ \cos at \left(\frac{1-\cos 2at}{4a}\right)$$

$$= \frac{1}{a}\ \sin at\ \left(\frac{t}{2} + \frac{\sin at \cos at}{2a}\right) - \frac{1}{a}\ \cos at \left(\frac{\sin^2 at}{2a}\right)$$

$$= \frac{t\ \sin\ at}{2a}$$

(ii) $\ell^{-1}\left\{\frac{1}{s^2 (s+1)^2}\right\}$

<u>The solution</u>

$$\ell^{-1}\left\{\frac{1}{s^2}\right\} = t \qquad , \ell^{-1}\left\{\frac{1}{(s+1)^2}\right\} = t\ \ e^{-t}$$ -

$$\therefore \quad \ell^{-1}\left\{\frac{1}{s^2\ (s+1)^2}\right\} = \int_0^t u\ e^{-u}\ (t-u)\ du\ = \int_0^t e^{-u}\ (ut-u^2)\ du$$

وبعد إجراء التكامل بالتجزئ نحصل على :

$$\ell^{-1}\left\{\frac{1}{s^2\ (s+1)^2}\right\} = t\ e^{-t} + 2\ e^{-t} + t - 2$$

<u>(3)</u> by using the Heaviside expansion find each of

$$\ell^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\}$$

<u>The solution</u>

$$p(s) = 3s+1 \quad , Q(s) = s^3 - s^2 + s - 1 \quad , Q^{'}(s) = 3s^2 - 2s + 1$$

$$\alpha_1 = 1 \quad , \quad \alpha_2 = i \quad , \quad \alpha_3 = -i$$

$$\ell^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\} = \frac{P(1)}{Q^{'}(1)}\ e^t \ + \frac{P(i)}{Q^{'}(i)}\ e^{it} \ + \frac{P(-i)}{Q^{'}(-i)}\ e^{-it}$$

$$= \frac{4}{2}\ e^t + \frac{3i+1}{-2-2i}\ e^{it} + \frac{-3i+1}{-2+2i}\ e^{-it}$$

$$= 2e^t + (-1 - \frac{1}{2}\ i)\ (\cos t + i \sin t) + (-1 + \frac{1}{2}\ i)\ (\cos t - i \sin t)$$

$$= 2e^t - 2\cos t + \sin t$$

<u>Exercises</u>

(1) Find each of the following

(i) $\ell^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\}$ (ii) $\ell^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\}$

(iii) $\ell^{-1}\left\{\frac{4s+12}{s^2+8s+16}\right\}$ (iv) $\ell^{-1}\left\{\frac{1}{\sqrt{2s+3}}\right\}$

(2) If $\ell^{-1}\left\{\frac{e^{-\frac{1}{s}}}{s^{\frac{1}{2}}}\right\} = \frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$ then find $\ell^{-1}\left\{\frac{e^{-\frac{a}{s}}}{s^{\frac{1}{2}}}\right\}$

(3) By using the property $\ell^{-1}\{f^{(n)}(s)\} = (-1)^n t^n F(t)$ find

$$\ell^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$$

(4) Find $\ell^{-1}\left\{\ln\left(1+\frac{1}{s^2}\right)\right\}$

(5))By using the property $\ell^{-1}\left\{\frac{f(s)}{s}\right\}=\int_0^t F(u)\,du$ find

$$\ell^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\}$$

(6) If $\ell^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}=\frac{1}{2}t\sin t$ then find $\ell^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$

(7) By using the ratio fraction find each of the following

(i) $\ell^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\}$ (ii) $\ell^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$

(iii) $\ell^{-1}\left\{\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right\}$ (iv) $\ell^{-1}\left\{\frac{3s+1}{(s-1`)(s^2+1)}\right\}$

(v) $\ell^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\}$

The six chapter

تطبيق تحويـــــــــل لابـــــــــلاس فى حل المعادلات التفاضلية العادية

Laplace transform A**pplication for solution of Ordinary differential equation with the constant coefficients**

Laplace transform will be used for solving this kind of equations .for example if it required to solve the linear equation of second order

$$\frac{d^2Y}{dt^2}+\alpha\frac{dY}{dt}+\beta Y=F(t)\quad ,or\ Y''+\alpha Y+'\beta Y=F(t)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

Where α,β are constants with the boundary conditions

$$Y(0)=A\quad ,\quad Y'(0)=B\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Where A, B are known constants. Then by taking Laplace transform for both sides of equation (1) and applying the conditions (2) we shall get an algebraic equation contain the function $y(s)=\ell\{Y(t)\}$. then we want to find the inverse Laplace transform for the function $y(s)$.

This method can be applied for solving equations with order bigger than second order as we shall see in some examples.

The ordinary differential equations with variable coefficients

Laplace transform will be used for solving some equations of this kind especially when its boundaries of the form $t^mY^{(n)}(t)$, where the Laplace transform for this boundary is $(-1)^m\frac{d^m}{ds^m}\ell\{Y^{(n)}(t)\}$, as we shall see later in the exercises.

The simultaneous differential equations

Laplace transform will be used for solving two or more of simultaneous differential equations as it will be clear from some examples.

The partial differential equations

Laplace transform will be used for solving the partial differential equations of this kind under boundary conditions as we shall see later in the examples.

Examples and exercises

(1) Find the solution of the following equation under the mentioned conditions

$$Y'' + Y = t \quad , \quad Y(0) = 1 \quad , \quad Y'(0) = -2$$

(2)Find the solution of the following equation under the mentioned conditions

$$Y'' - 3Y' + 2Y = 4e^{2t} \quad , \quad Y(0) = -3 \quad , \quad Y'(0) = 5$$

The solution

By taking Laplace transform for both sides and substituting by the boundary conditions we get

$$(s^2 y - 3s - 5) - 3(sy + 3) + + 2y = \frac{4}{s-2}$$

$$(s^2 - 3s + 2)\ y + 3s - 14 = \frac{4}{s-2}$$

$$y = \frac{4}{(s^2 - 3s + 2)(s-2)} + \frac{14 - 3s}{(s^2 - 3s + 2)} = \frac{-3s^2 + 20s - 24}{(s-1)(s-2)^2}$$

$$y = \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$$

$$Y = -7e^t + 4e^{2t} + 4te^{2t}$$

(3)Find the solution of the following equation under the mentioned conditions

$$Y'' + 2Y' + 5Y = e^{-t} \sin t \quad , \quad Y(0)=0 \quad , \quad Y'(0)=1$$

(4)Find the solution of the following equation under the mentioned conditions

$$Y''' - 3Y'' + 3Y' - Y = t^2 \, e^t \quad , \quad Y(0)=1 \quad , \quad Y'(0)=0 \quad , Y''(0)=-2$$

(5)Find the solution of the following equation under the mentioned conditions

$$Y'' + 9Y = \cos 2t \quad , \quad Y(0)=1 \quad , \quad Y(\pi/2)=-1$$

The solution

By taking Laplace transform for both sides and substituting by the boundary conditions and since $Y'(0)$ is unknown so we shall suppose that is equal to constant C we get

$$(s^2 y - sY(0) - Y'(0)) + 9y = \frac{s}{s^2+4}$$

$$(s^2+9)y - s - c = \frac{s}{s^2+4}$$

$$y = \frac{s+c}{s^2+9} + \frac{s}{(s^2+9)(s^2+4)}$$

$$y=\frac{s}{s^2+9}+\frac{c}{s^2+9}+\frac{s}{5\,(s^2+4)}-\frac{s}{5(s^2+9)}$$

$$y=\frac{4}{5}(\frac{s}{s^2+9})+\frac{c}{s^2+9}+-\frac{s}{5(s^2+4)}$$

$$Y=\frac{4}{5}\ \cos 3t\ +\ \frac{c}{3}\ \sin 3t\ +\ \frac{1}{5}\ \cos 2t$$

To determine the constant c we use the condition $Y(\frac{\pi}{2})=-1$ to be $c=\frac{12}{5}$ and from that we get

$$Y=\frac{4}{5}\ \cos 3t\ +\ \frac{4}{5}\sin 3t\ +\frac{1}{5}\ \cos 2t$$

(6)Find the solution of the following equation under the mentioned conditions

$$Y''+a^2Y=F(t)\quad ,\quad Y(0)=1\quad ,\ Y'(0)=-2$$

<u>The solution</u>

By taking Laplace transform for both sides and substituting by the boundary conditions we get

$$(s^2y-sY(0)-Y'(0))+a^2\ y=\ell\{F(t)\}=f(s)$$

$$(s^2y-s+2)+a^2\ y=\ f(s)$$

$$y=\frac{s-2}{s^2+a^2}+\frac{f(s)}{s^2+a^2}$$

$$Y=\cos\ at-\frac{2}{a}\ \sin at+F(t)*\frac{\sin at}{a}$$

$$Y=\cos\ at-\frac{2}{a}\ \sin at+\frac{1}{a}\int_0^t F(u)\sin a(t-u)\,du$$

(7)Find the solution of the following equation under the mentioned conditions

$$tY'' + 2Y' + tY = 0 \quad , \quad Y(0) = 1 \quad , \quad Y(\pi) = 0$$

The solution

By taking Laplace transform for both sides and substituting by the boundary conditions we get

since $Y'(0)$ is unknown so we shall suppose that is equal to constant C we get

$$-\frac{d}{ds}(s^2 y - sY(0) - Y'(0)) + 2(sy - Y(0)) - \frac{dy}{ds} = 0$$

$$-s^2 y' - 2sy + 1 + 2sy - 2 - y' = 0$$

$$-(s^2+1)y' - 1 = 0 \quad or \quad y' = \frac{-1}{s^2+1}$$

And the solution of this equation is

$$y = -\tan^{-1} s + A$$

But from the final value property we find that

$$y \to 0 \quad as \quad s \to \infty$$

And the result of that we get $A = \frac{\pi}{2}$

$$y = \frac{\pi}{2} - \tan^{-1} s = \tan^{-1}\frac{1}{s} = \int_s^\infty \frac{du}{u^2+1} = \ell\left\{\frac{\sin t}{t}\right\}$$

It is clear that this function satisfies the condition : $Y(\pi) = 0$, so this is the required.

(8)Find the solution of the following simultaneous equations under the mentioned conditions

$$\frac{dX}{dt} = 2X - 3Y \qquad , \qquad X(0) = 8$$

$$\frac{dY}{dt} = Y \quad - 2X \qquad , \qquad Y(0) = 3$$

The solution

By taking Laplace transform for both sides in the two equations and substituting by the boundary conditions we get

Suppose that $\ell\{X\} = x \quad , \quad \ell\{Y\} = y$, then

$$s\,x - 8 = 2x - 3 \qquad or \quad (s-2)x + 3y = 8$$
$$s\,y - 3 = y \;\; - 2 \qquad or \;\; 2x + (s-1)y = 3$$

وبحل هاتين المعادلتين الجبريتين بالنسبة إلى y , x ينتج أن :

By solving these equations we get

$$x = \frac{\begin{vmatrix} 8 & 3 \\ 3 & s-1 \end{vmatrix}}{\begin{vmatrix} s-2 & 3 \\ 2 & s-1 \end{vmatrix}} = \frac{8s-17}{s^2 - 3s - 4} = \frac{8s-17}{(s+1)(s-4)} = \frac{5}{s+1} + \frac{3}{s-4}$$

$$y = \frac{\begin{vmatrix} s-2 & 8 \\ 2 & 3 \end{vmatrix}}{\begin{vmatrix} s-2 & 3 \\ 2 & s-1 \end{vmatrix}} = \frac{3s-22}{s^2 - 3s - 4} = \frac{3s-22}{(s+1)(s-4)} = \frac{5}{s+1} - \frac{2}{s-4}$$

$$\therefore \; X = \ell^{-1}\{x\} = 3e^{-t} + 3e^{4t}$$

$$Y = \ell^{-1}\{y\} = 5e^{-t} - 2e^{4t}$$

(9)Find the solution of the following simultaneous equations under the mentioned conditions

$$\frac{d^2X}{dt^2}+\frac{dY}{dt}+3X=15\,e^{-t} \qquad X(0)=35 \quad , \; X'(0)=-48$$

$$\frac{d^2Y}{dt^2}-4\frac{dX}{dt}+3Y=15\sin 2t \qquad Y(0)=27 \qquad Y'(0)=-55$$

<u>تطبيقات على المعادلات التفاضلية الجزئية</u>

يستخدم تحويل لابلاس كثيرا فى حل المعادلات التفاضلية الجزئية الخاضعة لشروط حدية ----- - أى فى حل المسائل الحدية كما سنرى فى الأمثلة .

نفرض أن $U(x,t)$ دالة فى المتغيرين x,t حيث $t\geq 0$, $0\leq x\leq b$

ونفرض أن $u(x,s)$

هى تحويل لابلاس للدالة $U(x,t)$ بالنسبة إلى t أى أن :

$$u(x,s)=\ell\{U(x,t)\}=\int_0^\infty U(x,t)\;e^{-st}\;dt$$

بالنسبة للمشتقات الجزئية للدالة $U(x,t)$ فإن :

$$\ell\left\{\frac{\partial U}{\partial t}\right\}=\int_0^\infty \frac{\partial U}{\partial t}\,e^{-st}\,dt$$

$$=-s\,e^{-st}\,U(x,t)\,\Big|_0^\infty+s\int_0^\infty U(x,t)\,e^{-st}dt$$

$$=s\,u(x,s)-sU(x,0)$$

كذل

$$\ell\left\{\frac{\partial U}{\partial x}\right\}=\int_0^\infty \frac{\partial U}{\partial x}\,e^{-st}\,dt \;=\frac{d}{dt}\int_0^\infty U(x,t)\,e^{-st}\,dt \;=\frac{dU}{dx}$$

وبنفس الطريقة يمكن إثبات أن :

$$\ell\left\{\frac{\partial^2 u}{\partial t^2}\right\} = s^2\, u(x,s) - s\, U(x.0) - U_t(x,0)$$

كذلك $$\ell\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \frac{d^2 u}{dx^2}$$

<u>أمثـــــلة وتماريــــــن</u>

1) أوجد حل المعادلة الأتية مع الشروط المذكورة :

$$\frac{\partial U}{\partial x} = 2\,\frac{\partial U}{\partial t} - U \qquad U(x,0) = 6\, e^{-3x} \qquad , \qquad |U(x,t)| < M$$

2) أوجد حل المعادلة الأتية مع الشروط المذكورة:

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \qquad U(x,0) = 3\,\sin 2\pi x \qquad , \qquad U(0,t) = 0 \quad , U(1,t) = 0 \quad , 0 < x < 1$$

1) أوجد حل المعادلة الأتية مع الشروط المذكورة:

$$\frac{\partial^2 Y}{\partial t^2} - 4\,\frac{\partial^2 Y}{\partial x^2} + Y = 16x + 20\sin x$$

<u>بالشـــروط:</u>

$$Y(0,t) = 0 \quad , Y(\pi,t) = 16\,\pi \quad , Y_t(x,0) = 0$$

$$Y(x,0) = 16\,x + 12\sin 2x - 8\sin 3x$$

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<u>حل التمرين (3)</u>

بأخذ تحويل لابلاس ينتج أن :

$$s^2 y - sY(x,0) - Y_t(x,0) - 4\,\frac{d^2 y}{dx^2} + y = \frac{16x}{s} + \frac{20\sin x}{s} \qquad (1)$$

وبالتعويض من $Y(x,0) \quad ,Y_t(x,0)$ ينتج أن :

$$s^2 y-16s x-12\sin 2x+8 s \sin 3x-4\frac{d^2y}{dx^2}+y=\frac{16x}{s}+\frac{20\sin x}{s}$$

والتى يمكن وضعها فى صورة معادلة تفاضلية عادية غير متجانسة بالشكل:

$$4\frac{d^2y}{dx^2}-(s^2+1)y=-16s x-12s\sin x-8\sin 3x-\frac{16}{s}x-\frac{20\sin x}{s} \qquad (2)$$

والتى تحقق الشروط :

$$Y(0,t)=0 \quad ,Y(\pi,t)=16\pi \qquad (3)$$

الحل الخاص للمعادلة (2) يكون على الصورة :

$$y_p=a x+b\sin x+c\sin 2x+d\sin 3x$$

وبالتعويض من هذا فى المعادلة (1) نحصل على الثوابت a,b,c,d ويصبح الحل الخاص هو :

$$y_p=\frac{16x}{s}+\frac{20\sin x}{s(s^2+5)}+\frac{12\ s\sin 2x}{s^2+17}-\frac{8s\sin 3x}{s^2+37} \qquad (4)$$

والحل التكميلى (أى حل المعادلة المتجانسة هو):

(5)

$$y_c=c_1 e^{-\frac{1}{2}\sqrt{s^2+1}\ x}+c_2 e^{\frac{1}{2}\sqrt{s^2+1}\ x}$$

أما الحل العام فهو :

$$y=y_p+y_c \qquad (6)$$

وبالتعويض من الشروط (3) فى المعادلة (6)ينتج أن :

$$c_1+c_2=0 \qquad ,c_1 e^{-\frac{1}{2}\sqrt{s^2+1}}+c_2 e^{\frac{1}{2}\sqrt{s^2+1}\ x}=0$$

ومن هذا نجد أن: .... $c_1=0 \quad ,c_2=0$

$$\therefore\ y = \frac{16x}{s} + \frac{20\sin x}{s(s^2+5)} + \frac{12s\ \sin 2x}{s^2\ +17} - \frac{8s\sin 3x}{s^2+37}$$

وبإجراء التحويل العكسى ينتج أن :

$$Y(x,t) = 16x + 4\sin x(1-\cos\sqrt{5}\ t) + 12\sin 2x\cos\sqrt{17}\,t - 8\sin 3x\cos\sqrt{37}\,t$$

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