جامعة جنوب الوادى

كلية التربية بالغردقة

الفرقة الرابعة عام رياضيات Math

المادة : (pure 13) (Topology)

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topology Chapter 0 (1)

1. Sets

A set is any well defined list or collection of objects, and will be denoted by $A, B, C, \ldots$.
The objects are called its elements or members, and will be denoted by $a, b, c, x, y, \ldots$
The statement "p is an element of A" or, "p belongs to A" is written $p \in A$. The negation of $p \in A$ is written $p \notin A$.

Given two sets A and B we say that A is a subset of B (or B is a superset of A, or A is contained in B or B contains A) if every element of A is also an element of B and we write $A \subset B$. i.e $\forall x \in A \Longrightarrow x \in B$

or $x \notin B \Longrightarrow x \notin A$.

If $A \subset B$ with $A \neq B$, then we say that A is a proper subset of B and we write $A \subsetneqq B$.

If X is a set, the set of all subsets of X is called the power set of X and will be denoted by $P(X)$.

The complement of A in X denoted by A^c or $X - A$ is the set of such elements which are in X but not in A i.e.

$A^c = \{x \in X : x \notin A\}$.

If A and B are two sets then their union is ..e set

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

..e can define more generally the union of ..y family of sets. Let $\mathcal{A}$ be any class of ..ubsets of a universal set X i.e $\mathcal{A} \subset P(X)$. ..he union of the sets in $\mathcal{A}$, denoted by

$\cup\{A : A \in \mathcal{A}\} = \{x : x \in X \text{ and } x \in A \text{ for some } A \in \mathcal{A}\}$.

..t is also denoted by $\bigcup_{A \in \mathcal{A}} A$.

For an indexed class of subsets of X, say ..$\mathcal{A} = \{A_i : i \in I\}$ we write

$\cup\{A_i : i \in I\}$, $\bigcup_{i \in I} A_i$ or $\bigcup_i A_i$.

..nother important concept is that of intersection. $A \cap B = \{x : x \in A \text{ and } x \in B\}$ is the ..ntersection of the two sets A and B.

..More generally, if $\mathcal{A}$ is a family of sets, ..then its intersection denoted by $\cap \mathcal{A}$ is defined by the set: $\cap \mathcal{A} = \{x : x \in A \text{ for all } A \in \mathcal{A}\}$, ..r, if $\mathcal{A} = \{A_i : i \in I\}$, then

$$\cap \mathcal{A} = \bigcap_{i \in I} A_i = \bigcap_i A_i$$

If $A \cap B = \phi$ (the empty set), we say ..hat A and B are <u>disjont</u> sets.

Lemma 1.

If A is any set, and for each $p \in A$, let G_p be a subset of A such that $p \in G_p \subset A$, then
$A = \cup \{ G_p : p \in G_p \subset A \}$.

2. Functions (maps).

Let A and B be two subsets of X, we say that. $f : A \longrightarrow B$ is a function from A into B if f is a relation from A into B such that $\forall\, a \in A$ there exists a unique element $b \in B$ with $f(a) = b$.
A function $f : A \longrightarrow B$ is said to be one-to-one (or, 1-1) if distinct elements of A have distinct images i.e

$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2 \;\; \forall\, a_1, a_2 \in A$$

A function $f : A \longrightarrow B$ is said to be onto if every $b \in B$ is the image of some $a \in A$ i.e.

$$b \in B \Rightarrow \exists\, a \in A \text{ for which } f(a) = b$$

or,

$$f(A) = B$$

A function f is bijective iff f is 1-1 and onto.

The identity function $I_A : A \longrightarrow A$ is defined by:

$$I_A(a) = a \quad \forall\, a \in A$$

The constant function $f : X \longrightarrow Y$ is defined by:

$$f(x) = a \quad \forall\, x \in X$$

If $f: X \longrightarrow Y$, $g: Y \longrightarrow Z$ be a function, then the Composite function $g \circ f: X \longrightarrow Z$ defined by :

$$(g \circ f)(x) = g(f(x)).$$

If $f: A \longrightarrow B$, then $I_B \circ f = f = f \circ I_A$

If $f: A \longrightarrow B$ is bijective, then $f^{-1}: B \longrightarrow A$ exists and $f^{-1} \circ f = I_A$, $f \circ f^{-1} = I_B$

Lemma 1 : let $f: X \longrightarrow Y$ be a function. Then

(1) $A \subset B \Rightarrow f^{-1}(A) \subset f^{-1}(B)$, $f(A) \subset f(B)$

(2) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, $f(A \cup B) = f(A) \cup f(B)$.

(3) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$, $f(A \cap B) \subset f(A) \cap f(B)$

(4) $f^{-1}(A - B) = f^{-1}(A) - f^{-1}(B)$, $f(A - B) \supset f(A) - f(B)$

(5) $A \subset f^{-1} \circ f(A)$, $A \subset X$

(6) $f \circ f^{-1}(B) \subset B$, $B \subset Y$

(7) $f^{-1}(B^{c}) = (f^{-1}(B))^c$, $B \subset Y$

Note that if f is bijective, then :

(i) $f(A^c) = (f(A))^c$

(ii) $f^{-1} \circ f(A) = A$

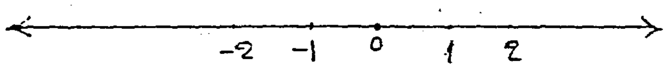
(iii) $f(A - B) = f(A) - f(B)$.

Chapter I
Topology of the real line and plan

1. Introduction:

(1) Real line :

The set of real numbers denoted by R. We assume that the reader is familiar with the geometric representation of R by means of points on a straight line.



Each point will represent a unique real number and each real number will be represented by a unique point.

(2) Intervals :

Let $a, b \in R$ with $a < b$.

(1) A set $[a,b] = \{x \in R : a \le x \le b\}$ is called closed interval

(2) A set $[a,b) = \{x \in R : a \le x < b\}$ is called a left closed interval

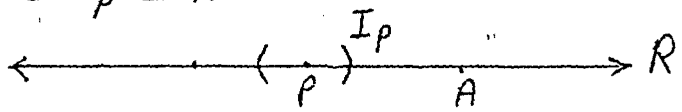
(3) A set $(a,b] = \{x \in R : a < x \le b\}$ is called a right closed interval

(4) A set $(a,b) = \{x \in R : a < x < b\}$ is called open interval

(5) The intervals $(-\infty, b)$, (a, ∞), $(-\infty, \infty)$ are called infinite intervals.

2. Open sets in R.

Definition 1. Let $A \subset R$. A point $p \in A$ is an interior point of A if $\exists$ open interval I_p containing p such that $p \in I_p \subset A$



Example 1:
Let $A = (0,1)$, for any $p \in A$ we have $p \in A = I_p \subset A$
So all points of A are interior points.

Example 2: Let $A = [1, 4)$
$1 \in A$, but 1 is not interior point of A. So all points of A except 1 are interior points.

Example 3: Let $A = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$.
No points of A is an interior point (why?)

Example 4. Let $A = [0, 1]$
All points of A except 0 and 1 are interior points.

Definition 2. Let $A \subset R$. The set A is said to be open if every point in A is an interior Point.

Example 1: An open interval $A = (a, b)$ is open set, because we may choose $A = I_p$ for each $p \in A$ and $p \in I_p \subset A$.

Example 2: The sets $A = [a, b]$, $A = [a, b)$ are not open sets.

Example 3: The set $A = (0, 1) \cup (3, 5)$ is open $\boxed{\text{set}}$

Remarks

(1) Every open interval is an open set

(2) Not every open set is an open interval (see Ex. 3)

Every open set is the union of open intervals (check)

(4) The intersection of two open intervals is open interval

(5) $R = \{x : -\infty < x < \infty\} = (-\infty, \infty)$ is open set

(6) ϕ is open set, since there is no point in ϕ which is not an interior point

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<u>Theorem 1</u>. Let $\mathcal{U} \subset P(R)$, be the set of all open sets in R. Then:

(1) $R, \phi \in \mathcal{U}$

(2) $\mathcal{U}$ is closed under arbitrary unions

(3) $\mathcal{U}$ is closed under finite intersections.

<u>Proof</u>.

(1) Since R and ϕ are open sets in R, then $R, \phi \in \mathcal{U}$

(2) Let $u_1, u_2, \dots \in \mathcal{U}$, we aim to prove that $\bigcup_{i=1}^{\infty} u_i \in \mathcal{U}$. Let $p \in \bigcup u_i \Rightarrow \exists\, u_k \in \mathcal{U}$ such that $p \in u_k \subset \bigcup_i u_i$. since $p \in u_k$ and u_k is open $\Rightarrow p$ is an interior point of $u_k \Rightarrow \exists\, I_p$ as open interval such that $p \in I_p \subset u_k$. so, $p \in I_p \subset u_k \subset \bigcup u_i$. Hence p is interior point of $\bigcup_i u_i$. Since p is arbitrary point, then p is interior point of $\bigcup_i u_i \Rightarrow \bigcup_i u_i$ is open set $\Rightarrow \bigcup_i u_i \in \mathcal{U}$.

(3) Let $u_1, u_2 \in \mathcal{U}$. we show that $u_1 \cap u_2 \in \mathcal{U}$.

Let $p \in u_1 \cap u_2 \Rightarrow p \in u_1$ and $p \in u_2$. Hence there exist open intervals I_p and I_p^* such that $p \in I_p \subset u_1$ and $p \in I_p^* \subset u_2$; Then $p \in I_p \cap I_p^* \subset u_1 \cap u_2$. since the intersection of open intervals is open interval, then every point $p \in \mathcal{U}_1 \cap \mathcal{U}_2$ is interior point. Hence, $\mathcal{U}_1 \cap \mathcal{U}_2 \in \mathcal{U}$.

Remark. The class $\mathcal{U}$ in Theorem (1) is said to be a topology, on R. The pair $(R, \mathcal{U})$ is called a topological space.

3. Closed sets in R:

Definition 1. A subset A of R is said to be closed set if its complement A^c is open.

Example 1: The set $A=[a,b]$ is closed set, since its complement $(-\infty, a) \cup (b, \infty)$, the union of two open sets, is open.

Example 2: The set $A = [0,1)$ is not closed, since $A^c = (-\infty, 0) \cup [1, \infty)$ is not open

Example 3: R and ϕ are closed sets

Example 4: The set of integer numbers Z is closed, since Z^c is the union of open intervals.

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Theorem 2: Let $\mathcal{F}$ be the set of all closed sets in R. Then:

(1) $R, \phi \in \mathcal{F}$

(2) The union of a finite number of sets in $\mathcal{F}$ belongs to $\mathcal{F}$.

(3) The intersection of any number of $\mathcal{F}$ belongs to $\mathcal{F}$.

Proof:

(1) Since R, ϕ are closed sets, then $R, \phi \in \mathcal{F}$.

(2) Let $F_1, F_2, \dots, F_n \in \mathcal{F}$ and $F = F_1 \cup F_2 \cup \dots \cup F_n$.

By De Morgan's law:

$$F^c = (F_1 \cup F_2 \cup \dots \cup F_n)^c = (F_1^c \cap F_2^c \cap \dots \cap F_n^c)$$

Then F^c is the intersection of a finite number of open sets F_i^c, $i = 1, 2, \dots, n$, and thus F^c is also open. Hence its complement $F^{cc} = F$ is closed.

(3) Let $\{F_i\}$ be a class of closed sets and let $F = \bigcap_i F_i$. By De Morgan's Law:

$$F^c = (\bigcap_i F_i)^c = \bigcup_i F_i^c$$

So, F^c is the union of open sets and, hence is open itself. Consequently, $F^{cc} = F$ is closed.

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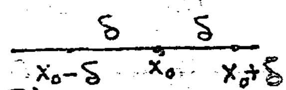
$\boxed{\text{II}}$

4. Continuity

Definition 1: A Function $f : R \to R$ is a continuous at a point $x_0 \in R$ if for every $\epsilon > 0$ $\exists$ $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow$ $|f(x) - f(x_0)| < \epsilon$.
we say that f is continuous if it is continuous at every point.

Remark. we may notice that

$$|x - x_0| < \delta \iff x_0 - \delta < x < x_0 + \delta$$
$$\iff x \in (x_0 - \delta, x_0 + \delta)$$



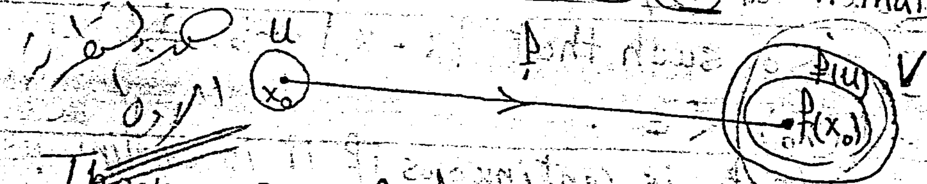
$$\iff x_0 \in (x_0 - \delta, x_0 + \delta)$$

So, if $x_0 \in (x_0 - \delta, x_0 + \delta) = u \Rightarrow f(x_0) \in f(u)$

$$|f(x) - f(x_0)| = |f(x_0) - f(x)| < \epsilon$$
$$\iff f(x_0) \in (f(x_0) - \epsilon, f(x_0) + \epsilon) = V$$

Then :
if $f(x_0) \in f(u) \Rightarrow f(x_0) \in V$ i.e. $f(u) \subset V$.
Hence, we can redefine the concept of continuity as follows :

Definition 2. A function $f: R \to R$ is continuous at x_0 if every open set V containing $f(x_0)$, $\exists$ an open set U containing x_0 such that $f(U) \subset V$



Theorem 3. A function $f: R \to R$ is continuous if and only if the inverse image of every open set is open

Proof: Let $f: R \to R$ be a continuous and V be an open set in R. we show that $f^{-1}(V)$ is open set. Let $p \in f^{-1}(V) \Rightarrow f(p) \in V$. By definition of continuity, $\exists$ an open set U_p containing p such that $f(U_p) \subset V$. Hence $U_p \subset f^{-1}(f(U_p)) \subset f^{-1}(V)$. So, $p \in U_p \subseteq f^{-1}(V)$. Therefore $f^{-1}(V) = \cup\{U_p : p \in f^{-1}(V)\}$ is the union of open sets i.e $f^{-1}(V)$ is open

Conversely, let V be an open set containing $f(p)$ i.e. $f(p) \in V$. Then $f^{-1}(V)$ is open set containing p. But $f(f^{-1}(V)) \subseteq V$. Hence f is continuous.

Example 1: The constant function is continous

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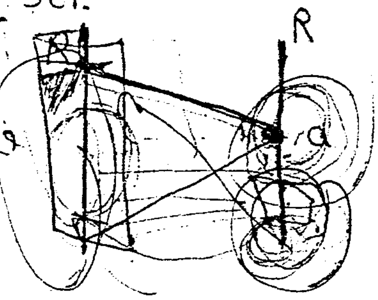
$x = f^{-1}(a)$

Let $f : R \longrightarrow R$, $\boxed{f(x)} = a \quad \forall \; x \in R$

Suppose G be any open set in R. Then :



$$f^{-1}(G) = \begin{cases} \phi & \text{if } a \notin G \\ R & \text{if } a \in G \end{cases}$$

So , $f^{-1}(G)$ is open

Example 2 The identity function is continuous

Let $f : R \longrightarrow R$, $f(x) = x$, $\forall \; x \in R$

Suppose G be any open set , then .

$f^{-1}(G) = G$

is also open

Example 3. Let $f : R \longrightarrow R$, $g : R \longrightarrow R$ are continuous functions . Then $g \circ f$ is also continuous

Let G be an open set in R , then

$$(g \circ f)^{-1}(G) = (f^{-1} \circ g^{-1})(G) = f^{-1}(g^{-1}(G)) .$$

since g is continuous , then $g^{-1}(G)$ is open , since f is continuous , then $f^{-1}(g^{-1}(G))$ is open Thus $(g \circ f)^{-1}(G)$ is open

(H)

Topology of the plane

Definition 1. An open disc D in the plane R^2 with center $p(a_1, b_1)$ and radius $\delta > 0$ is the set

$$D = \{ q \in R^2 : d(p, q) < \delta \}$$
$$= \{ (x, y) \in R^2 : (x - a_1)^2 + (y - b_1)^2 < \delta^2 \}$$

where $q \equiv (x, y)$

دور أو عمل

Remark. The open discs play a role in the topology of the plane R^2 that is analogous to the role of the open intervals in the topology of R. e.g.

(1) ~~The point~~ $p \in A \subset R^2$ is an interior point of A iff an open disc D containing p, $D \subset A$. i.e $p \in D \subset A$.

(2) A subset $A \subset R^2$ is an open set iff every point in A is an interior point of A.

(3) The set of all open sets in R^2 is called the usual topology on R^2.

There are theorems for R^2 analogous to theorems for R.

Chapter II

Topological Spaces

Inthischapterwegivethedefinitionofa topologic

In this chapter we give the definition of a topological space and a few examples of topological spaces. In the second section we give basic concepts (limit points, closed sets, closure, interior, ~~and~~ exterior and boundray of a set). In the third section we give three general constructions of topological spaces.

Definition 1 A topological space is a pair (X, τ) where X is a set and $\tau \subset P(X)$ is a family of subsets of X satisfying:

(1) $X, \phi \in \tau$

(2) τ is closed under arbitrary unions

(3) τ is closed under finite intersections

The family τ is said to be a topology on X.

Members of τ are called open in X or open subsets of X. open in X or open subse

Example 1 Let $X = \{a, b, c\}$. One can check the following:

$\tau_1 = \{X, \phi, \{a\}\}$, is a topology on X

$\tau_2 = \{X, \phi, \{a\}, \{a,b\}\}$ is a topology on X

$\tau_3 = \{X, \phi, \{a\}, \{b\}, \{b,c\}\}$ is not a topology on X
(since $\{a\} \cup \{b\} = \{a,b\} \notin \tau_3$)

$\tau_4 = \{X, \phi, \{a\}, \{b\}, \{a,c\}, \{b,c\}\}$ is not a topology on X (since $\{a,c\} \cap \{b,c\} = \{c\} \notin \tau_4$)

$\tau_5 = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}$
$= P(X)$ is a topology on X.

In the following we give ~~an~~ important examples of ~~a~~ topological spaces.

1. Discrete space

Let $X \neq \phi$ be any set and $\mathfrak{D} = P(X)$.
One can deduce that $\mathfrak{D}$ is a topology on X and this topology is called the discrete topology. Here every set is open, $(X, \mathfrak{D})$ is a discrete topological space (or, a discrete space).

2. Indiscrete Space

Let $X \neq \phi$, be any set and $\mathfrak{I} = \{X, \phi\}$. Then, $\mathfrak{I}$ is a topology on X and this topology is called the indiscrete topology and $(X, \mathfrak{I})$ is salled the indiscrete space.

3. Usual topological space

Let $X = R$ be the set of all real numbers and $\mathcal{U}$ be the class of open sets in R, that is, for each $p \in U$, $U \in \mathcal{U}$, $\exists\ \delta > 0$ such that $(p-\delta, p+\delta) \subset U$. Then $\mathcal{U}$ is a topology on R (see Chapter I) and this topology is called the usual topology, then $(R, \mathcal{U})$ is called the usual topological space.

4. Particular point topological space

Let X be a non-empty set and $p \in X$ be any point. The class $P = \{\phi, U \subset X : p \in U\}$ is a topology on X and is called particular point topology with a particular point p.

5. Cofinite topological space

Let X be any set. ~~A~~ subset A of X is said to be cofinite if its complement A^c is finite. Let $\mathcal{C}$ be the set of all cofinite subsets and the empty set, that is,

$\mathcal{C} = \{\phi, U \subset X : U^c \text{ is finite}\}$

We show that $\mathcal{C}$ is a topology on X.

(i) Since $X \subset X$ and $X^c = \phi \in \mathcal{C} \Rightarrow X, \phi \in \mathcal{C}$

(ii) Let $\{U_i : i \in I\} \subset \mathcal{C} \Rightarrow U_i^c$ is finite $\forall\ i \in I$.

$$\Rightarrow \left(\bigcap_{i\in I} U_i\right)^c = \left(\bigcup U_i^c\right)^c \text{ is finite}$$

$$\Rightarrow \bigcup_{i\in I} U_i \in \mathcal{C}$$

(iii) Let $U_1, U_2 \in \mathcal{C} \Rightarrow U_1^c$ and U_2^c are finite sets $\Rightarrow U_1^c \cup U_2^c$ is also finite. But,

$$U_1^c \cup U_2^c = (U_1 \cap U_2)^c$$

$$\Rightarrow (U_1 \cap U_2)^c \text{ is finite}$$

$$\Rightarrow U_1 \cap U_2 \in \mathcal{C}$$

Hence $\mathcal{C}$ is a topology on X and $(X, \mathcal{C})$ is said to be cofinite space.

Definition 2 Given two topologies τ_1 and τ_2 on X. We say that τ_1 is weaker (or coarser) than τ_2 or τ_2 is stronger (or finer) than τ_1 if $\tau_1 \subset \tau_2$. Note that $\tau_1 \subset \tau_2$ means that every open set in τ_1 is also open in τ_2.

It is clear that on any set the indiscrete Topology is the weakest of all ~~all~~ topologies on the same set, while the discrete topology is strongest of all topologies on the same set i.e if τ is a topology on X, then $\mathcal{J} \subset \tau \subset \mathcal{D}$.

تابع

Theorem 1: Let X be a set and $\{\tau_i : i \in I\}$ be an indexed family of topologies on X. Let $\tau = \bigcap_i \tau_i$. Then τ is a topology on X. It is weaker than each τ_i, $i \in I$

Proof: إثبات

(i) Clearly the empty set ϕ belongs to each τ_i since τ_i is a topology on X. and so $\phi \in \bigcap_i \tau_i$ i.e. ($\phi \in \tau$). Similarly $X \in \tau$.

(ii) Let $A_1, A_2, \ldots, A_n \in \tau$ and suppose $A = \bigcap_{j=1}^{n} A_j$. To show $A \in \tau$. Now, since $\tau = \bigcap_i \tau_i$, each $A_j \in \tau_i$. But τ_i, a topology on X, is closed under finite intersections so, $A \in \tau_i$ for each $i \in I$. Then $A \in \tau$.

(iii) Similar to (ii) and left to the reader.

ملاحظة **Remark**:

سؤال

If τ_1 and τ_2 are two topologies on X. IS $\tau_1 \cup \tau_2$ a topology on X ?! Now, if $\tau_1 \subset \tau_2$ or $\tau_2 \subset \tau_1$, then $\tau_1 \cup \tau_2 = \tau_2$ or τ_1 i.e., $\tau_1 \cup \tau_2$ is a topology. Generally $\tau_1 \cup \tau_2$ is not a topology on X, by the following example. Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}\}$ and $\tau_2 = \{X, \phi, \{b\}\}$, then $\tau_1 \cup \tau_2$ is not a topology on X, because $\{a\} \cup \{b\} = \{a, b\} \notin \tau_1 \cup \tau_2$

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2. Basic Concepts

(I) Accumulation (limit) points:

Definition 1. Let (X, τ) be a topological space and $A \subset X$. A point $p \in X$ is called limit point of A iff every open set containing p contains at least one point of A other than p. The set of all limit points of A is called the derived set of A, denoted by A'.

i.e.

$$p \in A' \Leftrightarrow \forall G \in \tau,\ p \in G,\ (G - \{p\}) \cap A \neq \phi$$

Example 1. Let $X = \{a, b, c, d, e\}$, $A = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$.

Find A'?

Solution:

$a \in X$ is not a limit point of A, because the open set $\{a\}$ containing a not contains a points of A different from a.

$b \in X$ is a limit point of A, since the open sets containing b are X and $\{b, c, d, e\}$ and each of them contains a point of A different from b.

$d, e \in X$ are limit points of A

$c \in X$ is not limit point of A.

Example 2. Let $(X, \mathfrak{I})$ and $A \subset X$. Find A'?

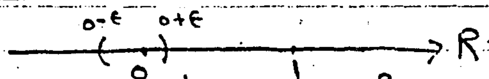
Solution: The set X is the only open set containing any point $p \in X$. So, $\forall\, p \in X$ we have $(X - \{p\}) \cap A \neq \phi$, if A contains two or more points $\Rightarrow A' = X$. But if $A = \{x\}$, then $(X - \{x\}) \cap A = \phi \Rightarrow x \notin A'$, while $(X - \{y\}) \cap A \neq \phi\ \forall\, y \neq x \Rightarrow A' = X - \{x\}$. Then, the derived set A' of a set A in indiscrete space is:

$$A' = \begin{cases} X & \text{if } A \text{ contains two or more points} \\ \{p\}^c & \text{if } A = \{p\} \end{cases}$$

Example 2. Let A be a subset of a discrete space X and $p \in X$ be any point. Since X is discrete topology, then $\{p\}$ is open set containing p. But $(\{p\} - \{p\}) \cap A = \phi \cap A = \phi \Rightarrow p \notin A'$, $\forall\, p \in X$. Therefore, $A' = \phi$.

Example 3: Let $(R, \mathcal{U})$ be the usual topological space. Find A', where $A = (0, 1)$ and $A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. Also, find Q' and Z'?

Solution



Let $x \in (0, 1)$ be any point. Since for any open set $G = (x - \epsilon, x + \epsilon)$, $(G - \{x\}) \cap (0, 1) \neq \phi \Rightarrow x \in A'$. On other hand let $(0 - \epsilon, 0 + \epsilon)$ be any open set containing 0. Since $((0 - \epsilon, 0 + \epsilon) - \{0\}) \cap A \neq \phi$, $\forall\, \epsilon > 0 \Rightarrow 0 \in A'$, also $1 \in A'$. Finally, let $x \notin (a, b)$, $x \neq a$, $x \neq b \Rightarrow |x - a| = \epsilon_1$, $|x - b| = \epsilon_2$, $\epsilon_1, \epsilon_2 > 0$. If we take $\epsilon = \min(\epsilon_1, \epsilon_2) \Rightarrow (x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}) \cap (a, b) = \phi$

$\Rightarrow x \notin A'$. Therefore $A' = [0,1]$.

- $A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. Every point $p \in A$ is not limit point (why?). The point 0 is a limit point of A, since any open set $G = (-a_1, a_1)$ containing 0 contains points of A other than 0. Finally, all points outside A except 0 is not limit point. So

$$A' = \{0\}$$

- The set Q is the set of rational numbers. Note that between any two real numbers there are infinite rational numbers, then every $p \in R$ is a limit point.

$$Q' = R$$

- $Z = \{0, \pm 1, \pm 2, \ldots\}$ we have $Z' = \phi$. since, let $0 < \epsilon < \frac{1}{2}$. Then $(1-\epsilon, 1+\epsilon)$ is open set containing 1, but $((1-\epsilon, 1+\epsilon) - \{1\}) \cap Z = \phi \Rightarrow 1 \notin Z'$. If $x \notin Z \Rightarrow x \neq 0, \pm 1, \pm 2, \ldots$. Take $|x-1| = \epsilon_1, \ldots\ldots |x-p| = \epsilon_p$ for $p \in Z \Rightarrow (x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}) \cap Z = \phi$, where $\epsilon = \min(\epsilon_1, \epsilon_2, \ldots)$

Therefore $Z' = \phi$

Theorem 2 Let (X, τ) be a topological space and A, B be two subsets of X. Then:

(1) $A \subset B \Rightarrow A' \subset B'$

(2) $(A \cup B)' = A' \cup B'$

(3) $(A \cap B)' \subset A' \cap B'$

Proof:

(1) Let $p \in A' \Rightarrow (G - \{p\}) \cap A \neq \phi, \forall$ G open set containing p. Since $A \subset B \Rightarrow (G - \{p\}) \cap B \neq \phi \Rightarrow p \in B'$. Hence, $A' \subset B'$.

2) Since $A \subset A \cup B$, $B \subset A \cup B$, by (1) we have

$A' \subset (A \cup B)'$, $B' \subset (A \cup B)' \Rightarrow$

$A' \cup B' \subset (A \cup B)'$ .... (i)

On other hand, let $x \notin A' \cup B' \Rightarrow x \notin A'$ and $x \notin B' \Rightarrow \exists$ open set G containing x such that $(G - \{x\}) \cap A = \phi$ and $(G - \{x\}) \cap B = \phi$. So, $[(G - \{x\}) \cap A] \cup [(G - \{x\}) \cap B] = \phi$. But $[(G - \{x\}) \cap A] \cup [(G - \{x\}) \cap B] = (G - \{x\}) \cap (A \cup B) = \phi$

$\Rightarrow x \notin (A \cup B)'$ Hence

$(A \cup B)' \subset A' \cup B'$ .... (ii)

From (i) and (ii) we have $(A \cup B)' = A' \cup B'$

3) Since $A \cap B \subset A$, $A \cap B \subset B \Rightarrow$

$(A \cap B)' \subset A'$ and $(A \cap B)' \subset B' \Rightarrow$

$(A \cap B)' \subset A' \cap B'$.

Remark: Let (X, τ) be a topological space, $A \subset X$ and $p \in X$. Then

(1) $p \notin A' \Rightarrow \exists G \in \tau, p \in G, (G - \{p\}) \cap A = \phi$

$\boxed{2}$ $p \notin A$, $p \in A' \Rightarrow \forall G \in \tau, p \in G, G \cap A \neq \phi$.

(3) $p \notin A$, $p \notin A' \Rightarrow \exists G \in \tau, p \in G, G \cap A = \phi$.

II Closed sets and Closure.

Definition 2: Let (X, τ) be a topological space. Then a subset F of X is said to be closed in X if its complement $F^c = X - F$ is open in X.

F^c

Example 1. Let $X = \{a, b, c, d, e\}$ and $\tau = \{X, \Phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$. Then the closed subsets of X are
$\Phi, X, \{b, c, d, e\}, \{a, b, e\}, \{a\}, \{b, e\}$

Remark: From the above example we deduce that there exists a subset of X which is open and closed like $\{b, c, d, e\}$ and a subsets of X which neither open nor closed like $\{a, b\}$.

Example 2

(1) In $(X, \mathcal{D})$ every subset of X is both open and closed

(2) In any infinite cofinite topological space $(X, \mathcal{C})$ every finite subset is closed and X is the only infinite closed set

Theorem 3: Let $\mathcal{F}$ be the family of all closed sets in a topological space (X, τ). Then $\mathcal{F}$ has the following properties: الخواص التالية

(1) $X, \Phi \in \mathcal{F}$ مغلقة تحت التقاطعات $\mathcal{F}$

(2) $\mathcal{F}$ is closed under arbitrary intersections

(3) $\mathcal{F}$ is closed under $\boxed{\text{finite}}$ unions

Conversely, given any set X and a family $\mathcal{F}$ of its subsets which satisfies these three properties, there exists a unique topology $\tau = \{U \subset X : U^c \in \mathcal{F}\}$ on X.

The proof is trivial.

Theorem 4: A subset A of a topological space (X,τ) is closed iff A contains each of its limit points. i.e.

$$A \text{ is closed} \Leftrightarrow A' \subset A$$

Proof: Suppose A is closed and let $p \notin A \Rightarrow p \in A^c$. But A^c is open and $A^c \cap A = \phi \Rightarrow p \notin A'$. Then $A' \subset A$.

Conversely, assume that $A' \subset A \Rightarrow \forall\, p \notin A$ we have $p \notin A' \Rightarrow \exists$ open set U_p containing p such that $U_p \cap A = \phi \Rightarrow p \in U_p \subset A^c$, $\forall\, p \notin A$. Then

$A^c = \cup\{U_p : p \in A^c\}$ which is the union of open sets. Then A^c is open $\Rightarrow A^{cc} = A$ is closed. //

Definition 3: Let (X,τ) be a topological space and $A \subset X$ the closure of A is the intersection of all closed sets which containing A and denoted by $\bar{A}$. i.e.

$$\bar{A} = \cap\{F \subset X : A \subset F \text{ and } F \text{ is closed}\}.$$

One may deduce that, $\bar{A}$ is closed set because it is the intersection of closed sets, and $\bar{A}$ is the smallest closed set containing A.

Example 3:

Let $X = \{a,b,c,d,e\}$ and

$$\tau = \{X, \phi, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d,e\}\}$$

Find $\overline{\{b\}}$, $\overline{\{a,b\}}$ and $\overline{\{c,d\}}$, $\overline{\{a,b,c\}}$

Solution:

The family of all closed subsets of X is:

closed sets $\mathcal{F} = \{\phi, X, \{b,c,d,e\}, \{a,b,e\}, \{b,e\}, \{a\}\}$

$\overline{\{b\}} = X \cap \{b,c,d,e\} \cap \{a,b,e\} \cap \{b,e\} = \{b,e\}$

$\overline{\{a,b\}} = X \cap \{a,b,e\} = \{a,b,e\}$

$\overline{\{c,d\}} = X \cap \{b,c,d,e\} = \{b,c,d,e\}$

$\overline{\{a,b,c\}} = X$

2) Let $(R, \mathcal{U})$ be the usual topological space and $A = (a,b)$, $B = [a,b)$

Find $\overline{A}$, $\overline{B}$?

Solution:

$\overline{A} = [a,b]$ and $\overline{B} = [a,b]$.

3) Let $(X, \mathcal{D})$ be a discrete space, $A \subset X$. Find $\overline{A}$?

Solution:

Since every subset of X is open and closed then

$\overline{A} = A$

4) Let $(X, \mathcal{I})$ be an indiscrete space and $A \subset X$. Find $\overline{A}$?

Solution:

All open and closed subsets of X are X and ϕ. Then

$$\overline{A} = \begin{cases} \phi & \text{if } A = \phi \\ X & \text{if } A \neq \phi \end{cases}$$

(5) If A is a subset of the infinite cofinite space $(X, \mathcal{C})$. Find $\bar{A}$?

Solution.

$\mathcal{C} = \{\phi, U \subset X : U^c \text{ is finite}\}$

The family of all closed sets in X is:

$\mathcal{F} = \{X, F : F \text{ is finite}\}$

so $A \subset X$

$$\bar{A} = \begin{cases} A & \text{if } A \text{ is finite} \\ X & \text{if } A \text{ is infinite} \end{cases}$$

[redacted], we give [redacted] properties of the closure of a [redacted].

Theorem 5. Let (X, τ) be a topological space and $A \subset X$. Then:

(1) $A \subseteq \bar{A}$

(2) $\bar{A}$ is the smallest closed set containing A i.e. if F is closed set containing A, then $A \subset \bar{A} \subset F$

(3) A is closed in X iff $\bar{A} = A$

(4) $\bar{A} = A \cup A'$

Proof. (1) and (2) follows from the definition.

(3) Let $A = \bar{A}$, since $\bar{A}$ is closed, then so is A.

Conversely, Let A be a closed set, by (2) $A \subset \bar{A} \subset A$.

then $A = \bar{A}$

(4) Since $\overline{A}$ is closed, then by theorem 4, $(\overline{A})' \subset \overline{A}$.
But $A \subset \overline{A}$, then $A' \subset (\overline{A})' \subset \overline{A}$. therefore

$A \subset \overline{A}$
$A' \subset \overline{A}$

$A \cup A' \subset \overline{A} \longrightarrow (1)$

Now, ~~we~~ prove that $\overline{A} \subset A \cup A'$.

Let $p \notin A \cup A' \Rightarrow p \notin A$ and $p \notin A' \Rightarrow \exists$ open set U containing p such that $U \cap A = \phi$. Since U is open and $U \cap A = \phi \Rightarrow \forall x \in U$, $x \notin A'$,

$\Rightarrow U \cap A' = \phi$

Therefore, $(U \cap A) \cup (U \cap A') = U \cap (A \cup A') = \phi$.

$\Rightarrow p \notin (A \cup A')'$. Then $(A \cup A')' \subset (A \cup A') \Rightarrow A \cup A'$ is closed set containing A, by (2) $\overline{A} \subset A \cup A' \longrightarrow (2)$

From (1) and (2) we have $\boxed{\overline{A} = A \cup A'}$

By theorem 5, we deduce that $p \in \overline{A}$ iff $p \in A$ or p is a limit point of A.

Theorem 6. Let (X, τ) be a topological space and $A \subset X$. Then

$$p \in \overline{A} \Leftrightarrow \forall U \in \tau, p \in U, U \cap A \neq \phi.$$

Proof. Let $p \in \overline{A}$. Since $\overline{A} = A \cup A' \Rightarrow p \in A$ or $p \in A'$. If $p \in A \Rightarrow U \cap A \neq \phi$ $\forall$ open set U containing p. If $p \in A' \Rightarrow (U - \{p\}) \cap A \neq \phi$ $\forall U \in \tau$, $p \in U$, since $\boxed{p \notin A} \Rightarrow U \cap A \neq \phi$.

Conversely, Let $U \cap A \neq \phi$ $\forall U \in \tau$, $p \in U \Rightarrow$ either $p \in A$ ~~or $p \notin A$~~.

If $p \in A \Rightarrow p \in \overline{A}$.

If $p \notin A$, by hypothesis $U \cap A \neq \phi \Rightarrow (U - \{p\}) \cap A \neq \phi$ $\forall U \in \tau$, $p \in U \Rightarrow p \in A'$ since $\overline{A} = A \cup A' \Rightarrow p \in \overline{A}$.

Example 4 : Let $(R, \mathcal{U})$ be the usual topological space and $Q \subset R$ be the set of rational numbers. show that $\overline{Q} = R$

Solution Assume that $x \in R$ be any point, since any open-set $(x-\delta, x+\delta)$ $\forall \delta > 0$ containing x contains infinite numbers of rational numbers $\Rightarrow$ $Q \cap (x-\delta, x+\delta) \neq \phi \Rightarrow x \in \overline{Q}$ $\forall x \in R$.

Thus

$\overline{Q} = R$.

Theorem 7 : Let A and B be two subsets of a topological space (X, τ). Then

(1) $\overline{\phi} = \phi$? (1) $\overline{\phi} = \cap \{F \subset X : \phi \subset F\}$ $= \cap \{\phi \subset X : \phi \subset \phi\}$ $= \phi$

(2) $A \subset \overline{A}$

(3) $A \subset B \Rightarrow \overline{A} \subset \overline{B}$

(4) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

(5) $\overline{\overline{A}} = \overline{A}$

defined

Proof (1) and (2) are immediate consequences of the definition and properties of closed sets.

(3) Since $A \subset B$, then $A' \subset B'$ (by theorem 2) and so $A \cup A' \subset B \cup B' \Rightarrow \overline{A} \subset \overline{B}$

(4) Since $A \subset A \cup B$ and $B \subset A \cup B$, then by (3),
$\overline{A} \subset \overline{A \cup B}$ and $\overline{B} \subset \overline{A \cup B}$, then
$\overline{A} \cup \overline{B} \subset \overline{A \cup B}$ (1).

Since $A \subset \overline{A}$, $B \subset \overline{B} \Rightarrow A \cup B \subset \overline{A} \cup \overline{B}$. But $\overline{A}$ and $\overline{B}$ are closed sets $\Rightarrow \overline{A} \cup \overline{B}$ is closed
$\Rightarrow \overline{A \cup B} \subset \overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$ (2)

From (1) and (2) we have $\overline{A \cup B} = \overline{A} \cup \overline{B}$

$\overline{A \cup B} \subset \overline{A \cup B}$

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(5) if $\bar{A} = F$ is closed set, then $\bar{F} = F \Rightarrow$
$\bar{\bar{A}} = \bar{A}$

Corollary Let A and B be two subsets of a topological space (X, τ). Then $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$.

Proof since $A \cap B \subset A, B \Rightarrow \overline{A \cap B} \subset \bar{A}$, $\overline{A \cap B} \subset \bar{B}$
$\Rightarrow \overline{A \cap B} \subset \bar{A} \cap \bar{B}$

The following example shows that
$$\overline{A \cap B} \neq \bar{A} \cap \bar{B}$$

Example 5 Let A and B be non-empty subsets of an indiscrete space such that $A \cap B = \Phi$.
Then $\overline{A \cap B} = \bar{\Phi} = \Phi$. But $\bar{A} = X$ and $\bar{B} = X$.
Therefore $\bar{A} \cap \bar{B} = X \cap X = X$. Hence $\overline{A \cap B} \neq \bar{A} \cap \bar{B}$.

Theorem 8 (Kuratowaski closure).
Let $c : P(X) \longrightarrow P(X)$ be any operator that assigns to each $A \in P(X)$ a subset $c(A) = \bar{A}$ of X such that the properties (1) - (5) in Theorem 7 are satisfied. Then the class $\tau = \{G : c(G^c) = G^c\}$ determines a unique topology on X.

1) $C(A) = \bar{A}$ (2) $C(\Phi) = \Phi$ (3) $A \subset B$ $C(A) \subset C(B)$ (4)

Definition 4. Let A be a subset of a topological space (X, τ). Then A is said to be dense subset in X iff
$\bar{A} = X$

Example 6 (1) The set Q is dense in R.

$\bar{Q} = R$

dense
$X = \bar{A}$

(2) A subset A of an indiscrete space is dense, because $\bar{A} = X$.

Theorem 9. A subset A of a topological space (X, τ) is dense in X iff $A \cap U \neq \phi$ $\forall U \in \tau$.

$A \cap U \neq \phi$ $\forall U \in \tau$

Proof. Let A is dense in $X \Rightarrow \bar{A} = X$. By theorem 6, $U \cap A \neq \phi$ $\forall p \in X$ and $p \in U \Rightarrow U \cap A \neq \phi$ $\forall U \in \tau$.

Conversely, Let $p \in X$ and U be any open set containing p such that $U \cap A \neq \phi$ $\forall U \in \tau$, $p \in U$ $\Rightarrow p \in \bar{A} \Rightarrow X \subset \bar{A}$. Therefore $\bar{A} = X$.

نقطة الداخلية

III. Interior and Exterior and boundary of a set.

Definition 5.

Let (X, τ) be a topological space and $A \subset X$.

(i) A point $p \in A$ is called an interior point of A if there exists open set $G \in \tau$ such that $p \in G \subset A$.
The set of all interior points of A denoted by A°

(ii) The exterior of A, denoted by ext(A) is the interior of the complement of A, i.e.

$$ext(A) = (X - A)^{\circ}$$

(iii) A point $p \in X$ is a boundary point of A if $p \notin A^{\circ}$ and $p \notin ext(A)$. The set of all boundary point of A denoted by b(A) i.e.

$$b(A) = X - [A^{\circ} \cup ext(A)]$$

A point $p \in X$ is bo

The set of

Example 1 Let (X, τ) be a topological space, where

$X = \{a, b, c, d, e\}$

$\tau = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$

and $A = \{b, c, d\}$. Find A°, $ext(A)$, $b(A)$?

Solution

~~$P \in G \subset A$~~

$A^\circ = \{c, d\}$, since $c, d \in \{c, d\} \subset A$

$A^c = \{a, e\}$, $(A^c)^\circ = \{a\} \Rightarrow ext(A) = \{a\}$

$b(A) = \{b, e\}$

Example 2

Let $(R, \mathcal{U})$ and $Q \subset R$. Find Q°, $ext(Q)$, $b(Q)$?

Solution:

Since every open interval of R contains both rational and irrational points. Then there are no interior or exterior points of Q i.e.

$Q^\circ = \phi$, $ext(Q) = \phi$ and $b(Q) = R$

Example 3

Let $(X, \mathcal{J})$ and $A \subset X$, $A \neq X$. Find A°, $ext(A)$ and $b(A)$?

Solution:

$A^\circ = \phi$ $ext(A) = \phi$, $b(A) = X$

Example 4 Let A be a subset of discrete space. Then

$A^\circ = A$, $ext(A) = A^c$, $b(A) = \phi$

Theorem 10. Let (X,τ) be a topological space and $A \subset X$. Then:

(1) $A^\circ = \cup\{G : G \in \tau \text{ and } G \subset A\}$

i.e. the interior of a set A is the union of all open subsets of A

(2) A° is open — bounded A

(3) A° is the largest open subset of A

(4) $A^\circ \subset A$

(5) A is open iff $A^\circ = A$

(6) $(A^\circ)^c = \overline{(A^c)}$ or $(A^c)^\circ = (\overline{A})^c$

(7) $b(A) = \overline{A} \cap \overline{A^c}$ — $b(A) = \overline{A} \cap \overline{A^c}$

Proof. (1) — (4) follows directly from the definition of the interior.

(5) If A is open then $A \subset A^\circ \subset A \Rightarrow A = A^\circ$.
If $A = A^\circ$ then A is open.

(6) Since $A^\circ = \cup\{E : E \in \tau \text{ and } E \subset A\}$.

$F = E^c$ — $E \subset A \Rightarrow A^c \subset E^c$. Thus

$$A^\circ = \cup\{F^c : F \text{ is closed}, A^c \subset F\}$$
$$= \cap\{F : F \text{ is closed}, A^c \subset F\}^c \qquad E^c = F$$
$$= (\overline{A^c})^c$$

or, if $p \in X - \overline{A} \Leftrightarrow x \notin \overline{A} \Leftrightarrow \exists$ open set U containing p such that $U \cap A = \phi \Leftrightarrow \exists$ open set U such that $p \in U \subset A^c \Leftrightarrow p \in (A^c)^\circ = (X - A)^\circ$.

Then, $X - A^\circ = \overline{X - A}$.

Also, $X - (\overline{X - A}) = (X - (X - A))^\circ = A^\circ \Rightarrow \overline{X - A} = X - A^\circ$.

topological space

Since

$\{A - \{p\}\} \cap A = \phi$

(7) $b(A) = \{p \in X : p \notin A^{\circ} \text{ and } p \notin (A^c)^{\circ}\}$

$= \{p \in X : p \notin (\overline{A^c})^c ,\ p \notin (\overline{A})^c\}$

$= \{p \in X : p \in \overline{A^c} ,\ p \in \overline{A}\}$

$= \overline{A^c} \cap \overline{A}$

Theorem 11. Let (X, τ) be a topological space and $A \subset X$. Then:

(1) $b(A)$ is closed.
(2) $b(A) = b(A^c)$
(3) $b(A) = \overline{A} - A^{\circ}$
(4) $b(A) \cap A^{\circ} = \phi$
(5) $\overline{A} = b(A) \cup A^{\circ}$
(6) The set $\{A^{\circ}, b(A), ext(A)\}$ is a partition of X.

Proof.

1) Since $b(A) = \overline{A} \cap \overline{A^c}$ is the intersection of two closed sets, then $b(A)$ is closed.

2) $b(A) = \overline{A} \cap \overline{A^c}$, then
$b(A^c) = \overline{A^c} \cap \overline{A} = b(A)$.

3) Since $\overline{A^c} = (A^{\circ})^c \Rightarrow b(A) = \overline{A} \cap \overline{X - A}$
$= \overline{A} \cap (X - A^{\circ}) = \overline{A} \cap X - \overline{A} \cap A^{\circ}$
$= \overline{A} - A^{\circ}$
because $A^{\circ} \subset \overline{A} \subset X$

(5) $b(A) \cup A^\circ = (\bar{A} - A^\circ) \cup A^\circ$, since $A^\circ \subset \bar{A}$, then $(\bar{A} - A^\circ) \cup A^\circ = \bar{A}$ i.e $b(A) \cup A^\circ = \bar{A}$

(4) $b(A) \cap A^\circ = (\bar{A} - A^\circ) \cap A^\circ = \phi$

(6) Since $b(A) \cap ext(A) = b(A) \cap (X - A^\circ)^\circ$

by (2) $= b(X-A) \cap (X - A^\circ)^\circ$

$= \phi$ (by 4)

since $A^\circ \cap ext(A) = \phi$ and $b(A) \cap A^\circ = \phi$.
Then the intersection of two sets A°, $ext(A)$, $b(A)$ are disjoint.

$$A^\circ \cup b(A) \cup ext(A) = [A^\circ \cup (\bar{A} - A^\circ)] \cup (X - \bar{A})$$
$$= \bar{A} \cup (X - \bar{A}) = X.$$

Hence the set $\{A^\circ, b(A), ext(A)\}$ is a partition of X.

Theorem 12.
Let (X, τ) be a topological space and A, B be two subsets of X. Then:

(1) $\phi^\circ = \phi$ — $A^\circ \subset A$, $B^\circ \subset B$ } → (1)
(2) $A^\circ \subset A$ — $A \cap B \subset A$, $B \subset B$ } → (2)
(3) $A \subset B \Rightarrow A^\circ \subset B^\circ$ — $A^\circ \subset A$
(4) $(A \cap B)^\circ = A^\circ \cap B^\circ$ — $B^\circ \subset B$ } (5)
(5) $(A^\circ)^\circ = A^\circ$ — $A^\circ \subset B^\circ$

Proof (1) and (2) by the definition.
3) Let $p \in A^\circ \Rightarrow \exists$ open set G such that $p \in G \subset A$.
since $A \subset B \Rightarrow p \in G \subset B$. Hence $A^\circ \subset B^\circ$

A Topological ... X

(4) Since $A \cap B \subset A$, $A \cap B \subset B$ By (2) we have
$(A\cap B)^\circ \subset A^\circ$, $(A\cap B)^\circ \subset B^\circ \Rightarrow$
$$(A\cap B)^\circ \subset A^\circ \cap B^\circ$$
Since $A^\circ \subset A$, $B^\circ \subset B \Rightarrow A^\circ \cap B^\circ \subset A\cap B$.
But $A^\circ \cap B^\circ$ is open set , then
$$(A^\circ \cap B^\circ)^\circ = A^\circ \cap B^\circ \subset (A\cap B)^\circ$$
Thus $(A\cap B)^\circ = A^\circ \cap B^\circ$

(5) Since $A^\circ = B$ is open , then
$B^\circ = B \Rightarrow (A^\circ)^\circ = B$.

Remark . Since $A \subset A\cup B$, $B \subset A \cup B \Rightarrow$
$A^\circ \subset (A\cup B)^\circ$, $B^\circ \subset (A\cup B)^\circ$
Therefor $A^\circ \cup B^\circ \subset (A\cup B)^\circ$,
The following example show that
$$(A\cup B)^\circ \neq A^\circ \cup B^\circ$$

Example
Let $(R, \mathcal{U})$ be the usual topological space and
$A = (1,2)$, $B = [2,5)$. Then
$A^\circ = (1,2)$, $B^\circ = (2,5)$, so
$A^\circ \cup B^\circ = (1,5) - \{2\}$
but, $(A\cup B)^\circ = (1,5)^\circ = (1,5)$. Then
$(A\cup B)^\circ \neq A^\circ \cup B^\circ$.

[redacted]

§3. General constructions of topological spaces.

(I) Neighbourhood and Neighbourhood system

Definition 1.

Let (X, τ) be a topological space, $p \in X$ and $N \subset X$. Then N is said to be a neighbourhood of p if there is an open set V such that $p \in V$ and $V \subset N$.

The set of all neighbourhoods (abbreviated to "nbds") is called the nbd system of p, denoted by $\mathcal{N}_p$.

Observe that the expression "N is nbd of p" equivalent to "p is interior point of N".

Example 1 Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{a,b\}, \{a,b,c\}\}$ be a topology on X. Then:

$\mathcal{N}_a = \{N \subset X : \exists\, U \in \tau,\ a \in U \subset N\}$

$= \{N \subset X : \{a\} \subseteq N\}$

$= \{X, \{a\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}\}$

$\mathcal{N}_b$; $\{N \subset X ; \{a,b\} \subset N$

$\mathcal{N}_b = \{X, \{a,b\}, \{a,b,c\}, \{a,b,d\}\}$

$\mathcal{N}_c = \{X, \{a,b,c\}\}$

$\mathcal{N}_d = \{X\}$

Note that the set N itself is not required to be open.

Example 2: (i) In $(X, \mathfrak{J})$, the set X is the only open set, then X is the only nbd of a point $p \in X$ i.e. $\mathcal{N}_p = \{X\}$

(ii) In $(X, \mathfrak{D})$, $\mathcal{N}_p = \{N \subset X : p \in N \subset X\}$

Theorem 1 A subset of a topological space is open iff it is a nbd of each of its points.

Proof: Let (X, τ) be a topological space and $A \subset X$.
First suppose A is open $\Rightarrow \forall\, p \in A$ $\exists$ open set $U \in \tau$ such that $p \in U \subset A \Rightarrow A$ is a nbd of a point p. Then A is a nbd of each of its points.
Conversely, suppose A is a nbd of its points. Then for each $p \in A$, $\exists\, U \in \tau$ such that $p \in U$ and $U \subset A$. Clearly, $A = \bigcup_{p \in A} U$. Since each U is open, then A is open.

Trivially if N is a nbd of a point p then so is any superset of N. It is also easy to show that the intersection of any two (and hence finitely many) nbds of a point is again a nbd of that point. The following theorem gives four properties of the nbd system $\mathcal{N}_p$ of a point p of a topological space X. These axioms may be used to define a topology on X.

Theorem 2 Let $\mathcal{N}_p$ be the nbd system of a point p in a topological space (X, τ). Then:

(1) $\mathcal{N}_p \neq \phi$ and p belongs to each member of $\mathcal{N}_p$ i.e $p \in N$ $\forall\, N \in \mathcal{N}_p$

(2) For any $N_1, N_2 \in \mathcal{N}_p \Rightarrow N_1 \cap N_2 \in \mathcal{N}_p$

(3) If $M \supset N$, $N \in \mathcal{N}_p \Rightarrow M \in \mathcal{N}_p$

(4) If $N \in \mathcal{N}_p$ then there exists $M \in \mathcal{N}_p$ such that $M \subset N$ and $M \in \mathcal{N}_q$ $\forall\, q \in M$.

Proof,

(1) Since عادام X is open set containing p $\forall p \in X \Rightarrow X \in \mathcal{N}_p$
$\Rightarrow \mathcal{N}_p \neq \phi$ and by definition $p \in N$ $\forall N \in \mathcal{N}_p$

(2) Let $N_1, N_2 \in \mathcal{N}_p \Rightarrow \exists$ two open sets U_1, U_2 such that $p \in U_1 \subset N_1$ and $p \in U_2 \subset N_2$. This implies that $p \in U_1 \cap U_2 \subset N_1 \cap N_2$ Since $U_1 \cap U_2$ is open $\Rightarrow$ $N_1 \cap N_2 \in \mathcal{N}_p$

(3) Let $M \supset N$ and $N \in \mathcal{N}_p \Rightarrow \exists$ open set U such that $p \in U \subset N$. Since $N \subset M \Rightarrow p \in U \subset M \Rightarrow$ M is a nbd of p i.e. $M \in \mathcal{N}_p$

(4) Let $N \in \mathcal{N}_p \Rightarrow N$ is a nbd of $p \Rightarrow \exists$ open set U such that $p \in U \subset N$. Since طالما U is open, then by theorem (1), U is a nbd of each of its points. So $U \in \mathcal{N}_q$ $\forall q \in U$.

Theorem 3. Let X be a set and suppose that for each $x \in X$, a non-empty family $\mathcal{N}_x$ of subsets of X, is given satisfying (1) - (4) in the theorem (2). Then there is a unique topology τ on X such that for each $x \in X$, $\mathcal{N}_x$ coincides تتوافق with the family of all nbds of x with respect to τ.

Proof: We let $\tau = \{U \subset X : U \in \mathcal{N}_x \text{ for all } x \in U\}$ and claim τ is a topology on X.

claim

ادعاء

(1) Clearly $\phi \in \tau$. To show that $X \in \tau$ note that for any $x \in X$, $X \in \mathcal{N}_x$ by (3). X is a superset of any member of $\mathcal{N}_x$.

(2) Property (2) in theorem (2) shows that τ is closed under finite intersections.

(3) property (3) in theorem (2) shows that τ is closed under arbitrary unions.

Theorem 4 : For a subset A of a topological space (X, τ) we have:

$$\bar{A} = \{ y \in X : N \in \mathcal{N}_y \Rightarrow N \cap A \neq \phi \}$$

Proof. Let $B = \{ y \in X : N \in \mathcal{N}_y \Rightarrow N \cap A \neq \phi \}$. We show that $\bar{A} = B$, or $B = A \cup A'$.

Let $y \in A \cup A'$. If $y \in A$ then every nbd of y meets A at least at point y (لان كل nbd موجودة y) and so $y \in B$. If $y \in A'$ then, by the definition of a limit point, every nbd of y contains a point of A and so $y \in B$. Thus $A \cup A' \subset B$.

Conversely let $y \in B$. If $y \notin A \cup A'$, then $y \notin \bar{A}$ and so $X - \bar{A}$ is a nbd of y which does not meet A, a contradicting that $y \in B$. So, $B \subset A \cup A'$. Thus $B = \bar{A}$.

(II) Subspaces and relative topologies

Definition

Let (X, τ) be a topological space and $A \subset X$. The class $\tau_A = \{A \cap U : U \in \tau\}$ is a topology on A and this topology is called the relative topology on A and the topological space (A, τ_A) is called the subspace of the space (X, τ). فضاء جزئي

One may notice that if U is open in X, then $A \cap U$ is open in A. Also, if V is open in A i.e. $V \in \tau_A \Rightarrow \exists\, U \in \tau$ such that $V = U \cap A$.

Now we verify that τ_A is a topology on A.

(1) Since $X, \phi \in \tau \Rightarrow A \cap X = A \in \tau_A$, $A \cap \phi = \phi \in \tau_A$

(2) Let $V_1, V_2 \in \tau_A \Rightarrow \exists\, U_1, U_2 \in \tau$ such that $V_1 = A \cap U_1$ and $V_2 = A \cap U_2 \Rightarrow$

$V_1 \cap V_2 = (A \cap U_1) \cap (A \cap U_2) = A \cap (U_1 \cap U_2)$

Since $U_1 \cap U_2 \in \tau \Rightarrow A \cap (U_1 \cap U_2) \in \tau_A \Rightarrow$

$V_1 \cap V_2 \in \tau_A$

3) Let $\{V_i : i \in I\} \subset \tau_A \Rightarrow \forall\, V_i \in \tau_A\ \exists\, U_i \in \tau$ such that $V_i = A \cap U_i \Rightarrow \bigcup_i V_i = \bigcup_i (A \cap U_i) = A \cap (\bigcup_i U_i)$

Since $\bigcup_i U_i \in \tau \Rightarrow A \cap (\bigcup_i U_i) \in \tau_A$ i.e. $\bigcup_i V_i \in \tau_A$

Hence τ_A is a topology on A

Example 1. Let $X = \{a, b, c, d, e\}$ with the topology $\tau = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$. If $A = \{a, d, e\}$ Then
$\tau_A = \{A, \phi, \{a\}, \{d\}, \{a, d\}, \{d, e\}\}$

Example 2 Any subspace of a discrete space is also discrete (A, D_A)

Example 3 Any subspace of an indiscrete space is indiscrete

Theorem Let A be a subspace of a space X. If $B \subset A$ is open (closed) in A and A is open (closed) in X, then B is open (closed) in X.

proof: Let A be an open subspace and B is open in A $\Rightarrow \exists$ open U in X such that $B = A \cap U$ $\Rightarrow$ B is the intersection of two open sets $\Rightarrow$ B is open in X.
Similarly, we can complete the proof.

i.e. if A is open in X and (A, τ_A) is a subspace of (X, τ). If $B \subset A \subset X$ is open in A, then B is open in X. Also, if B is closed in A and A is closed in X, then B is closed in X.
Note that a space (A, τ_A) is open if A is open and closed if A is cl.

الاساسيات

(III) Bases and subbases

Definition: Let (X, τ) be a topological space. A subfamily $\mathcal{B} \subset \tau$ is called a base for τ iff each open set is the union of members of $\mathcal{B}$

i.e.

$$\forall\ U \in \tau,\ \ U = \cup B_i,\ \text{where } B_i \in \mathcal{B}.$$

Example 1 The open intervals form a base for the usual topology on the real line

Example 2 Let $(X, \mathcal{D})$ be a discrete space. Then, $\mathcal{B} = \{\{p\} : p \in X\}$ is a base for $\mathcal{D}$.

Example 3: Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$, then $\mathcal{B} = \{\{a\}, \{b\}\}$ is not a base for τ, since $X \neq \{a\} \cup \{b\}$

Example 4: Let $X = \{a, b, c\}$. Then, the class $\mathcal{B} = \{\{a, b\}, \{b, c\}\}$ can not be a base for any topology τ on X, because

$\{a, b\}, \{b, c\} \in \tau \Longrightarrow \{a, b\} \cap \{b, c\} = \{b\} \in \tau$, but $\{b\}$ cannot written as the union of members of $\mathcal{B}$.

Theorem 1. Let (X, τ) be a topological space and $\mathcal{B} \subset \tau$. Then $\mathcal{B}$ is a base for τ iff for any $p \in X$ and any open set U containing p, there exists $B \in \mathcal{B}$ such that $p \in B$ and $B \subset U$.

i.e.

$\mathcal{B}$ is a base $\Longleftrightarrow$ $\forall\, U \in \tau$, $p \in U$, $\exists\, B \in \mathcal{B}$ such that $p \in B \subset U$.

Proof. Let $\mathcal{B}$ be a base for τ and $p \in U$. By definition 1, $U = \cup B_i$, where $B_i \in \mathcal{B}$ $\Longrightarrow$ $p \in \cup B_i \Longrightarrow \exists\, B_{i_0} \in \mathcal{B}$ such that $p \in B_{i_0} \subset U$.

Conversely, let $\forall\, U \in \tau$, $p \in U$, $\exists\, B_p \in \mathcal{B}$ such that $p \in B_p \subset U \xRightarrow{\text{By lemma (1)}} U = \bigcup_{p \in U} B_p$. Since $B_p \in \mathcal{B}$ $\Longrightarrow U$ is the union of members of $\mathcal{B}$. Hence $\mathcal{B}$ is a base for τ.

Theorem 2. Let X be a set and $\mathcal{B}$ a family of its subsets covering X i.e. $X = \cup B$, $B \in \mathcal{B}$. Then the following statements are equivalent:

(1) There exists a topology on X with $\mathcal{B}$ a base,

(2) For any $B_1, B_2 \in \mathcal{B} \Rightarrow B_1 \cap B_2$ can be expressed as the union of some members of $\mathcal{B}$.

Proof

(1) $\Rightarrow$ (2) Suppose there exists a topology τ on X for which $\mathcal{B}$ is a base. Let $B_1, B_2 \in \mathcal{B}$. Then $B_1, B_2 \in \tau \Rightarrow B_1 \cap B_2 \in \tau$, since τ is closed under finite intersections. So, by the definition of a base, $B_1 \cap B_2$ can be expressed as the union of some members of $\mathcal{B}$. $B_1 \cap B_2 = \cup B_i$ ($B_i \in \mathcal{B}$)

(2) $\Rightarrow$ (1) Let $\tau = \{ G \subset X : \forall x \in G, \exists B \in \mathcal{B}$ such that $x \in B$ and $B \subset G \}$. We show that τ is a topology on X. Clearly $\phi \in \tau$ while $X \in \tau$ since $\mathcal{B}$ is given to be a cover of X. (2) Let $G_1, G_2 \in \tau \Rightarrow G_1 = \cup B_i$, $G_2 = \cup B_j$ where $B_i, B_j \in \mathcal{B} \Rightarrow$ But $B_i \cap B_j = \cup B_m$

$G_1 \cap G_2 = (\cup B_i) \cap (\cup B_j) = \bigcup_{i,j} (B_i \cap B_j)$. But, by $B_m \in \mathcal{B}$ (2) we have $B_i \cap B_j \in \mathcal{B}$. Then $G_1 \cap G_2$ is the union of members of $\mathcal{B} \Rightarrow G_1 \cap G_2 \in \tau$.

Let $\{ U_i : i \in I \} \subset \tau \Rightarrow U_i$ is the union of members of $\mathcal{B} \Rightarrow \cup U_i$ is also the union of members of $\mathcal{B} \Rightarrow \cup U_i \in \tau$. Therefore τ is topology on X with $\mathcal{B}$ a base.

Definition 2 A family $\mathcal{S}$ of subsets of X is said to be a subbase for a topology τ on X if the family of all finite intersections of members of $\mathcal{S}$ is a base for τ.

Example 1 Let $(R, \mathcal{U})$ be the usual topological space and $\mathcal{S} = \{(a, \infty), (-\infty, b) : a, b \in R\}$. Then $\mathcal{S}$ is a subbase for τ, since

$$(a, b) = (a, \infty) \cap (-\infty, b)$$

and the family of open intervals is a base.

Example 2 Let $(X, \mathcal{D})$ be the discrete topology. Then,

$$\mathcal{S} = \{\{a, b\} : a, b \in X\}$$

is a subbase for $\mathcal{D}$.

مثلاً $X = \{a, b, c, d\}$

$\mathcal{S} = \{\{a, b\}, \{b, c\}, \{a, c\}, \{a, d\}, \{c, d\}, \{b, d\}\}$

$\cap =$

Chapter III

Continuity and Related Concepts

Definition 1. A mapping $f : (X, \tau_1) \longrightarrow (Y, \tau_2)$ is said to be continuous iff the inverse image of every open set in Y is open in X.

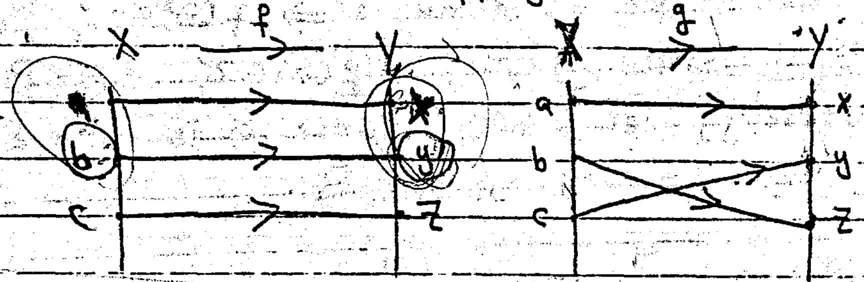
i.e

$$\forall\, V \in \tau_2 \Rightarrow f^{-1}(V) \in \tau_1$$

Example 1. Let $X = \{a, b, c\}$, $Y = \{x, y, z\}$

$\tau_1 = \{X, \phi, \{a\}, \{a, b\}\}$, $\tau_2 = \{Y, \phi, \{x\}, \{x, y\}\}$

and consider the mappings



Then f is continuous, but g is not continuous because $\{x, y\} \in \tau_2$, but $g^{-1}(\{x, y\}) = \{a, c\} \notin \tau_1$

Example 2 A constant mapping is continuous.

Example 3 The map $f : (X, \tau) \longrightarrow (Y, \mathcal{I})$ is continuous.

Example 4 The map $f : (X, \mathcal{D}) \longrightarrow (Y, \tau)$ is continuous.

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Example 5 If $f: X \longrightarrow Y$ is continuous, then the restriction map $f/_A : A \longrightarrow Y$, $A \subset X$ is also continuous. $f_A^{-1}(B) = f^{-1}(B) \cap A$

Let G be an open set in Y. Then $f^{-1}(G)$ is open in X. So,

$$f_A^{-1}(G) = A \cap f^{-1}(G)$$

by the definition of the relative topology. So $f_A^{-1}(G)$ is open in A i.e. f_A is continuous.

Theorem 1 Let $\mathcal{B}$ be a base for a topology τ_2 on Y. A mapping $f: (X, \tau_1) \longrightarrow (Y, \tau_2)$ is continuous iff $f^{-1}(B) \in \tau_1$ for each $B \in \mathcal{B}$.

Proof Let f be a continuous $\Rightarrow f^{-1}(B) \in \tau_1$, $\forall B \in \tau_2$. Since $\mathcal{B} \subset \tau_2 \Rightarrow f^{-1}(B) \in \tau_1 \ \forall B \in \mathcal{B}$.

Conversely, let $V \in \tau_2$. since $\mathcal{B}$ is a base for $\tau_2 \Rightarrow V = \cup B_i$, $B_i \in \mathcal{B}$

Now,

$$f^{-1}(V) = f^{-1}(\cup B_i) = \cup f^{-1}(B_i)$$

Since $f^{-1}(B_i) \in \tau_1 \ \forall i \Rightarrow \cup f^{-1}(B_i) \in \tau_1$

$\Rightarrow f^{-1}(V) \in \tau_1$ and so f is continuous.

The following theorem give an equivalent definition of the continuous mapping.

A

-3-

Theorem 2. Let $f: X \to Y$ be a mapping from a space X into a space Y. Then the following are equivalent:

(1) $\boxed{f \text{ is continuous}}$

(2) The inverse image of each closed set in Y is closed in X

(3) $f(\bar{A}) \subset \overline{f(A)}$, $\forall A \subset X$

(4) $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$, $\forall B \subset Y$

Proof

1) $\Rightarrow$ (2). Let F be a closed set in $Y \Rightarrow F^c$ is open in Y. Since f is continuous, then $f^{-1}(F^c)$ is open in X, but $f^{-1}(F^c) = (f^{-1}(F))^c \Rightarrow (f^{-1}(F))^c$ is open in X. Then $f^{-1}(F)$ is closed in X.

(2) $\Rightarrow$ (3)

Since $\overline{f(A)}$ is closed in Y, by (2) we have $f^{-1}(\overline{f(A)})$ is closed in X. Since $A \subset f^{-1}(f(A)) \Rightarrow$
$A \subset \bar{A} \subset f^{-1}(\overline{f(A)}) \Rightarrow f(\bar{A}) \subset f f^{-1}(\overline{f(A)}) \subset \overline{f(A)}$

(3) $\Rightarrow$ (4)

Let $B \subset Y \Rightarrow f^{-1}(B) = A \subset X$, by (3) we have
$f(\bar{A}) \subset \overline{f(A)} \Rightarrow f(\overline{f^{-1}(B)}) \subset \overline{f f^{-1}(B)} \subset \bar{B}$
$\Rightarrow \overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$

$\bar{B} \supseteq B$

$B \subset \bar{B}$

$f^{-1}(B) \subset f^{-1}(\bar{B})$

$\overline{f^{-1}(B)} \subset \overline{f^{-1}(\bar{B})}$

(4) $\Rightarrow$ (1)

Let $V \subset Y$ be open set in $Y \Rightarrow V^c$ is closed.
By (4), $\overline{f^{-1}(V^c)} \subset f^{-1}(\overline{V^c}) = f^{-1}(V^c)$. But
$f^{-1}(V^c) \subset \overline{f^{-1}(V^c)}$, then $f^{-1}(V^c)$ is closed.
Now,
$$f^{-1}(V^c) = (f^{-1}(V))^c$$
$\Rightarrow f^{-1}(V)$ is open. Therefore f is continuous.

Definition 2. A mapping $f: X \longrightarrow Y$ is called continuous at a point $p \in X$ iff the inverse image of each nbd of $f(p)$ is a nbd of p.

Example 6.

Let $X = \{a, b, c\}$, $Y = \{x, y, z\}$,
$\tau_1 = \{X, \phi, \{a\}\}$, $\tau_2 = \{Y, \phi, \{x\}, \{x, y\}\}$
and $f: X \longrightarrow Y$ be a mapping defined by:
$f(a) = x$, $f(b) = y$, $f(c) = z$.
Then f is continuous at a and c, but not continuous at b, because $\{x, y\}$ is a nbd of $f(b) = y$ and $f^{-1}(\{x, y\}) = \{a, b\}$ is not nbd of p.

Remark. A mapping $f: X \longrightarrow Y$ is continuous at $p \in X$ iff $\forall$ open set $\boxed{V}$ in Y, $f(p) \in V$ $\exists$ open set U in X such that $p \in U$, $U \subset f^{-1}(V)$.

[illegible]

Theorem 3: A mapping $f : X \to Y$ is continuous iff it is continuous at every point $\sim X$.

Proof: Assume that f is continuous and N is nbd of $f(p)$ in $Y \Rightarrow \exists$ open set V in Y such that $f(p) \in V$ and $V \subset N \Rightarrow p \in f^{-1}(V) \subset f^{-1}(N)$ since f is continuous $\Rightarrow f^{-1}(V)$ is open set in $X \Rightarrow f^{-1}(N)$ is a nbd of p.
i.e. f is continuous at p.

Conversely, let $V \subset Y$ be an open set and $p \in f^{-1}(V) \Rightarrow f(p) \in V \Rightarrow V$ is a nbd of $f(p)$ since f is continuous at every point $\Rightarrow f^{-1}(V)$ is nbd of p $\Rightarrow f^{-1}(V)$ is nbd of each of its points $\Rightarrow f^{-1}(V)$ is open set $\Rightarrow f$ is continuous.

Definition 2: A mapping $f : (X, \tau_1) \to (Y, \tau_2)$ is open if the image of every open set in X is open in Y.
i.e.

$$\forall\, U \in \tau_1 \Rightarrow f(U) \in \tau_2$$

Example 1 Every mapping $f : (X, \tau) \to (Y, \mathcal{D})$ from any space X into a discrete space Y is open, because for every $U \in \tau \Rightarrow f(U) \subset Y \Rightarrow f(U) \in \mathcal{D}$.
open But not continuous

Example 2 A mapping $f : R \to R$, defined by $f(x) = x^2$ is not open, because if $U = (-1, 1)$ is open, we have $f(U) = [0, 1)$ is not open.

[illegible]
[illegible]
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$F(X, D) \to (X, \tau)$

$\downarrow$ 6

Example 3. Give an example of real numbers $f : R \to R$ such that f is continuous, but not open:

Solution. Let $f : R \to R$ be a constant function, say $f(x) = a \ \forall x \in R$, $a \in R$.
Then f is continuous function (see Chapter I).
For any $A \subset R$, we have

$$f(A) = \{a\}$$

Hence f is not open function.

Theorem 4. Let $\mathcal{B}$ be a base for a topology τ_1 on a space X and $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping. Then f is open iff $f(B)$ is open in Y $\forall B \in \mathcal{B}$.

Proof. Let f be open $\Rightarrow f(U) \in \tau_2 \ \forall U \in \tau_1$.
Since $\mathcal{B}$ is a base for $\tau_1 \Rightarrow f(B) \in \tau_2 \ \forall B \in \mathcal{B}$.
Conversely, let $U \in \tau_1$. Since $\mathcal{B}$ is a base for τ_1
$\Rightarrow U = \cup B_i$, $B_i \in \mathcal{B} \Rightarrow f(U) = f(\cup B_i) = \cup f(B_i)$.
But $f(B_i) \in \tau_2 \ \forall B_i \in \mathcal{B} \Rightarrow \cup f(B_i) \in \tau_2 \Rightarrow f(U)$ is open $\Rightarrow f$ is open map.

Theorem 5. A mapping $f : X \to Y$ is open iff $f(A^\circ) \subset (f(A))^\circ$, $\forall A \subset X$.

Proof: Let f be open mapping since A° is open in X

- 7 - metric space

$A \subset B \Rightarrow A^\circ \subset B^\circ$

$\Rightarrow f(A^\circ)$ is open in $Y \Rightarrow f(A^\circ) = (f(A^\circ))^\circ$. But $A^\circ \subset A \Rightarrow f(A^\circ) \subset f(A) \Rightarrow (f(A^\circ))^\circ \subset (f(A))^\circ$. Then $f(A^\circ) \subset (f(A))^\circ \quad \forall A \subset X$.

Conversely, let U be open set in X. By hypothesis $f(U) = f(U^\circ) \subset (f(U))^\circ$. But $(f(U))^\circ \subset f(U)$. Then we have $f(U) = (f(U))^\circ$ i.e $f(U)$ is open, then f is open.

Definition: A mapping $f: X \longrightarrow Y$ is called closed if the image of every closed set in X is closed in Y

i.e $\forall F \in \mathcal{F} \Rightarrow f(F) \in \mathcal{F}^*$

where $\mathcal{F}$ and $\mathcal{F}^*$ are the families of closed sets in X and Y respectively.

Example 1 A constant function $f: R \longrightarrow R$ is closed.

$f(x) \quad R$

In general, mappings which are open need not be closed and vice versa.

Theorem 6 A mapping $f: X \to Y$ is closed iff $\overline{f(A)} \subset f(\overline{A})$, $\forall A \subset X$

Proof: Let f be a closed mapping and $A \subset X$. Then $\overline{A}$ is closed in X. Since f is closed $\Rightarrow$

$\bar{A} = A$ -8-

$f(\bar{A})$ is closed in Y. But $f(A) \subset f(\bar{A})$. Hence $\overline{f(A)} \subset \overline{f(\bar{A})} = f(\bar{A})$ closed

Conversely, let F be a closed set in X. By hypothesis $\overline{f(F)} \subset f(\bar{F}) = f(F) \Rightarrow \overline{f(F)} = f(F)$ $\Rightarrow f(F)$ is closed. Therefore f is closed.

Theorem 7 Let $f: X \to Y$ be a bijective mapping. Then, the following are equivalent:

(1) f is open

(2) f is closed

(3) f^{-1} is continuous

Proof:

(1) $\Rightarrow$ (2):

Let $F \subset X$ be a closed set $\Rightarrow F^c$ is open. By (1), $f(F^c)$ is open. Since f is bijective, we have $f(F^c) = (f(F))^c \Rightarrow (f(F))^c$ is open i.e $f(F)$ is closed. Then f is closed.

(2) $\Rightarrow$ (3): $f^{-1}: Y \to X$

Let $U \subset X$ be an open set $\Rightarrow U^c$ is closed in X. Since f is closed $\Rightarrow f(U^c)$ is closed. But, $f(U^c) = (f(U))^c$, because f is bijective. Then $(f(U))^c$ is closed $\Rightarrow f(U)$ is open. Since f is bijective $\Rightarrow f(U) = (f^{-1})^{-1}(U) \Rightarrow (f^{-1})^{-1}(U)$ is open in Y.

(3) ⇒ (1)

Let U be an open set in X. By (3) $(f^{-1})^{-1}(U)$ is open in Y. Since f is bijective, then $(f^{-1})^{-1}(U) = f(U) \Rightarrow f(U)$ is open in Y. Therefore f is open.

Definition 4. A mapping $f: X \longrightarrow Y$ is called homeomorphism if
(1) f is bijective
(2) f is continuous
(3) f^{-1} is continuous.

In this case two topological spaces X and Y are called homeomorphic or topologically equivalent.

Example 1. Let $f: (X, \mathcal{D}) \longrightarrow (Y, \mathcal{D})$ be a bijective mapping from a discrete space X onto a discrete space Y. Then, f is homeomorphism.

Example 2. Show that the real line R is homeomorphic to the open interval.

Solution. Let $f: X \longrightarrow R$, where $X = (-1, 1)$, defined by:

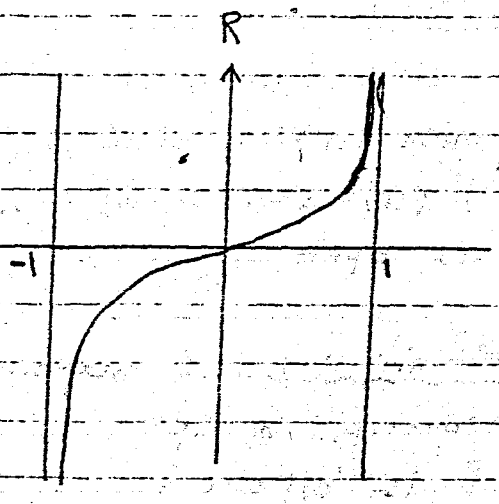
$$f(x) = \tan \frac{\pi}{2} x$$

(i) f is bijective
(ii) f is continuous.

(iii) f^{-1} is continuous

So, f is a homeomorphism

Hence X and R are homeomorphic



Theorem 8.

Let $f: X \longrightarrow Y$ be a bijective mapping. Then, the following are equivalent:

(1) f is homeomorphism

(2) f is continuous and open

(3) f is continuous and closed

(4) $f(\bar{A}) = \overline{f(A)}$, $\forall A \subset X$.

The proof follows directly from Theorems 2, 6, 7.

Definition 5 A property P of X is called a topological property (or, a topological invariant) if every topological space homeomorphic to X has also P.

Example(2) A closure is topological property, because for any homeomorphism $f: X \longrightarrow Y$ we have $f(\bar{A}) = \overline{f(A)}$ $\forall A \subset X$ (see theorem 8)

Example 2 Show that the length and the boundedness are not a topological properties

Solution: From example (2) in page 53, we see $X = (-1,1)$ and R are homeomorphic X and R have different lengths. Also X is bounded, but R is not bounded. Hence the length and boundedness are not topological property.

$\boxed{5.8}$

$$\int_0^1 \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$\int_0^1 t^{n-1}(1-t)^{n-1}\,dt$$

$$(m-1)(n-1) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Chapter IV

Metric Spaces

In this chapter we study spaces in which the topology is derived from a notion of the distance.

Definition 1: A metric space is a pair (X, d), where X is a set and d is a metric on X (or distance function), that is, a function $d: X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$ we have:

(1) $d(x,y) \geq 0$

(2) $d(x,y) = 0 \Leftrightarrow x = y$

(3) $d(x,y) = d(y,x)$ (symmetry)

(4) $d(x,y) \leq d(x,z) + d(z,y)$ (Triangle inequality)

Example 1: Let $X = \mathbb{R}$ be the set of real numbers with the usual metric defined by:

$$d(x,y) = |x-y| \quad \forall\ x, y \in \mathbb{R}$$

So, the properties (1) - (4) are known hold.
Hence $(\mathbb{R}, d)$ is a metric space which is called the usual metric space.

Example 2: Let $X = \mathbb{R}^2$, Eculidean plane الفراغ الاقليدي
if we take the set of order pairs of real numbers

$x = (x_1, y_1)$, $y = (x_2, y_2)$,
and the metric defined by:

$$d(x, y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Then d is a metric on the plane R^2.

Example 3. Let X be a set and $d: X \times X \longrightarrow R$,
defined by:

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Then d is a metric on X which is called the discrete (trivial) metric on X i.e. (X, d) is called the discrete metric space.

Example 4. Let $X = C[a,b]$ be the set of all continuous real valued function defined on $[a,b]$ with

$$d(f, g) = \max_{a \le x \le b} |f(x) - g(x)|$$

Then $(C[a,b], d)$ is a metric space, which is called function space.

Now, we study the topology induced by the metric. First we consider important type of subset of a given metric space.

Definition 2. Let (X, d) be a metric space. Given a point $p \in X$ and a real number $r > 0$. The set

$$S(p, r) = \{x \in X : d(x, p) < r\}$$

is called open sphere (or, open ball) with center p and radius r.

It is clear that

(i) $S(p, 0) = \phi$

(ii) If $r_1 < r_2 \Rightarrow S(p, r_1) \subset S(p, r_2)$.

Example 1. Let (X, d) be the usual metric space i.e.

$$d(x, y) = |x - y| \quad \forall\ x, y \in R$$

Take $p \in R$ and $r > 0$, then

$$\begin{aligned} S(p, r) &= \{x \in R : |p - x| < r\} \\ &= \{x \in R : -r < p - x < r\} \\ &= \{x \in R : -r < x - p < r\} \\ &= \{x \in R : p - r < x < p + r\} \\ &= (p - r, p + r) \end{aligned}$$

So, the open spheres in R is the set of all open intervals.

Example 2: Let (X, d) be the discrete metric i.e. $d(x, y) = 1$ if $x \neq y$ and 0 if $x = y$

If $p \in X$ and $0 < r \leq 1$, then

$$S(p, r) = \{x \in X : d(x, p) < r\}$$

$$= \begin{cases} \{p\} & \text{if } r \leq 1 \\ X & \text{if } r > 1 \end{cases}$$

Example 3. In $C[0,1]$ let $f_0 \in C[0,1]$

$$d(f, g) = \sup_{0 \leq x \leq 1} |f(x) - g(x)|$$

Given $r > 0$, then the open sphere $S(f_0, r)$ consists of all continuous function g which lie in the area bounded by $f_0 + r$, $f_0 - r$.

Definition 3. A subset M of a metric space (X, d) is said to be open if for each $x_0 \in M$ there is an $r > 0$ such that $S(x_0, r) \subset M$ (i.e. x_0 is an interior point of M). A subset F of X is said to be closed if its complement is open.

Example 1. The empty set ϕ and the whole space X are open.

Suppose that ϕ is not open. Then the statement البيان أو التصريح
" Every element of ϕ has a sphere about it contained in ϕ " is كاذب false.

$\Rightarrow \exists$ an element of ϕ such that ....

so,

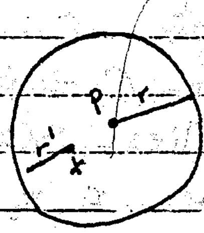
"There exists an element of ϕ" contradicts the fact that ϕ has no elements.

Hence ϕ must be open.

X is open since $x \in X \Rightarrow S(x,1) \subset X$.

Theorem 1 Every open sphere is open set.

Proof Let $S(p,r)$ be an open sphere, $p \in X$ where (X,d) is a metric space and $r > 0$.



Let $x \in S(p,r) \Rightarrow d(x,p) < r$.

put $r' = r - d(x,p) > 0$

It follows that $S(x,r') \subset S(p,r)$

Since,

$$\text{for } y \in S(x,r') \Rightarrow d(y,p) \leq d(y,x) + d(x,p)$$
$$< r' + d(x,p)$$
$$= r$$

$$\Rightarrow y \in S(p,r)$$

Thus for each point x in $S(p,r)$ there is a sphere $S(x,r')$ contained in $S(p,r)$.

Theorem 2 Let (X,d) be a metric space. Then

(1) ϕ and X are open sets.

(2) The union of any collection of open sets is open.

(3) The intersection of a finite number of open sets is open.

Proof

(1) obvious.

(2) If $y \in \cup G_\alpha$, then $y \in G_\alpha$ for some α, and so there exists open sphere $S(y,r)$ such that $S(y,r) \subset G_\alpha$. Hence $S(y,r) \subset \cup G_\alpha$ and so $\cup G_\alpha$ is open.

(3) Let G_1, G_2 be open sets. If $G_1 \cap G_2 = G$ is empty set use (1), then $G_1 \cap G_2$ is open. when $G \neq \phi$ take $x \in G$, so $x \in G_1$ and $x \in G_2$ implies $S(x,r_1) \subset G_1$ and $S(x,r_2) \subset G_2$ for some $r_1, r_2 > 0$. Clearly $S(x,r) \subset G$, where $r = \min(r_1, r_2)$.

Remark. One may deduce that the class of open spheres in a metric space (X,d) is a base for a topology on X.

Definition 4. The topology generated by the class of open spheres in a metric space (X,d) is called the metric topology (or, the topology induced by the metric d) and we denoted by $\tau(d)$.

7

Example 1 The usual metric on R induces the usual topology on R.

Example 2 Let $X \neq \phi$ and d be the trivial metric. For any $p \in X$, $r = \frac{1}{2}$, then

$S(p, \frac{1}{2}) = \{p\}$

Hence every singleton set is open, and so every set is open. Hence the set of open spheres in X contains all singletons in X. So $\tau(D)$ is the discrete topology.

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Remark: We have seen that any metric space induces a topology. The converse question arises. Given a topological space (X, τ), is there a metric d on X such that $\tau(d) = \tau$

Definition 5. Given a topological space (X, τ). We say that this space is metrizable if there exists a metric d on X such that $\tau(d) = \tau$

Example 1 Every discrete space is metrizable. Since the trivial metric on X induces the discrete metric i.e. $\tau(d) = \mathcal{D}$

Example 2 Every usual metric (R, d) induces a usual topology. Then $(R, \mathcal{U})$ is metrizable.

Definition 6. In a metric space (X, d). Define:

(1) The distance between a point $p \in X$ and a non-empty subset A of X is defined by:

$$d(p, A) = \inf\{d(p, a) : a \in A\}$$

(2) The distance between two non-empty subsets A and B of X is:

$$d(A, B) = \inf\{d(a, b) : a \in A \text{ and } b \in B\}$$

(3) The diameter of a non-empty subset A of X is:

$$d(A) = \sup\{d(a, a') : a, a' \in A\}$$

If the diameter of A is finite i.e. $d(A) < \infty$, then A is said to be bounded, if not, then A is said to be unbounded.

Since the topology of a metric is derived from a metric, one would correctly expect that the topological properties of X are related to the distance properties of X.

Theorem 3. The closure $\overline{A}$ of a subset A of a metric space (X, d) is the set of points whose distance from A is zero.

i.e.

$$\overline{A} = \{x \in X : d(x, A) = 0\}$$

Proof Let $x_0 \in \overline{A}$. If $x_0 \in A$, then $d(x_0, A) = 0$. suppose $x_0 \in A'$ and let $\epsilon > 0$ be given. Then there is a point $y \in A \cap S(x_0, \epsilon)$. Since $d(x_0, y) < \epsilon$ we must have $d(x_0, A) < \epsilon \Rightarrow d(x_0, A) = 0$.

Conversely, suppose $d(x_0, A) = 0$. If $x_0 \in A$, then $x_0 \in \overline{A}$. suppose $x_0 \notin \overline{A}$ and consider an open sphere $S(x_0, \epsilon)$ with centered at x_0. Since $d(x_0, A) = 0$, there is $y \in A$ with $d(x, y) < \epsilon$. Thus $y \in S(x_0, \epsilon)$ and $y \neq x_0$ since $x_0 \notin A$. It follows that x_0 is a limit point of A and then $x_0 \in \overline{A}$.

Remark From axiom (2) in a metric space (X, d). The point with zero distance from a singleton set $\{p\}$ is the point p itself i.e.

$$d(x, \{p\}) = 0 \Rightarrow x = \{p\}.$$

Hence, by theorem 3, every singleton sets $\{p\}$ in a metric space are closed. So, the finite unions of singleton sets i.e., finite sets, are closed.

Theorem 4

In a metric space all finite sets are closed.

The proof follows from theorem 3 and the above remark.

Example

Show that the indiscrete space $(X, \mathcal{J})$ where X consists of more than one point is not metrizable.

Solution. We know that X and ϕ are the only closed sets in an indiscrete space. But by theorem (4) all finite sets in a metric space are closed. Hence X and ϕ cannot be the only closed sets in a topology induced by any metric on X. Hence $(X, \mathcal{J})$ is not metrizable.

T_1 Axiom

Separation axiom

Space is T_1 →

[illegible] Space [illegible]

Hilbert [illegible]

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