Lecture notes on numerical analysis II

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Preface

This lecture notes is devoted to draw the whole picture of the numerical analysis II, which is suitable for the student at the fourth year of under-graduation study. Here is the information of the course.

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Chapter 1

Initial-Value Problems

1.1 Introduction

It's well known that many differential equation, specially the nonlinear type, has no analytical solution, therefore the numerical methods arise for such cases.
In the current chapter, we will present some of those methods for the ordinary differential equation of order one that has the following form:

$$y' = \frac{dy}{dx} = f(x, y), \qquad y(a) = y_0, \text{ and } x \in [a, b]. \tag{1.1}$$

Equation (1.1), that has a given initial value, is well known as an **Initial value problem**. In this equation the function $f(x, y)$ in the right hand side has to be continuous function in its domain. Before we present the numerical methods for such type of equation, we shall present some preliminaries that has to be verified from the mathematical analysis point of view.

Definition 1.1.1 — Lipschitz condition. A function $f(x, y)$ is said to be Lipschitz in the variable y at a region I with $I = \{(x, y), a \leq x \leq b, c \leq y \leq d\}$, if there exist a constant $L > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2| \ \forall c \leq y_1 \leq y_2 \leq d.$$

Theorem 1.1.1 — Existence and uniqueness of the solution . The initial value problem

$$y' = f(x,y) \text{ on } [a,b], y(x_o) = y_0$$

has a unique solution in the interval $[a,b]$ if

1. $f(x,y)$ in continuous with respect to x, y
2. the function f in Lipschitz in the variable y.

The initial value problem

$$y' = f(x,y), y(x_o) = y_0$$

has a unique solution if $f(x,y)$ is differentiable with respect to y, and $|f_y(x,y)| \leq L$ in the region $I = \{(x,y), a \leq x \leq b, c \leq y \leq d\}$.

■ **Example 1.1** Verify that the following initial value problem

$$y' = \frac{1}{2}(x+y), \qquad y(0) = 1$$

has a unique solution. ■

Solution.

$$\begin{array}{l} f(x,y) = \frac{1}{2}(x+y) \\ f_y(x,y) = \frac{1}{2} \\ |f\,(x,y_1) - f\,(x,y_2)| \leq |f_y(x,y)\,(y_1 - y_2)| = \frac{1}{2}\,|y_1 - y_2| \end{array} \tag{1.2}$$

this means that $f(x,y)$ verifies the Lipschitz condition and it is a polynomial of order one, thus it's a continuous in x, y. Therefore, this initial value problem has a unique solution. ■

Ⓡ **Second order differential equation:** The second order differential equation with two initial conditions can be converted to two equations from the first order, for instance,

$$\begin{array}{l} y'' - xy' - x^2y^2 = x^3 \\ y(0) = 1, y'(0) = 2 \end{array}$$

using $y' = z$, then, we can rewritten that equation in the following system

$$\begin{aligned}&y' = z\\&z' = xz + x^2y^2 + x^3\\&y(0) = 1, z(0) = 2\end{aligned}$$

equivalent to the following form,

$$\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} z \\ xz + xz + x^3 \end{pmatrix}, \begin{pmatrix} y(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

This remark could be generalized to n order differential equation with n initial conditions in the form

$$\begin{aligned}&a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)\\&y^{(m)}(0) = y_0^{(m)}, m = 0, 1, 2, \cdots, n-1\end{aligned}$$

where, $a_o, a_1, \ldots, a_n$ are functions of x, y only. The resulting system will be n equations from the first order as

$$Y' = F(x, Y), Y(0) = Y_0$$

1.2 The numerical solution for the initial value problem

Using the different numerical method, we are able to find an approximate value for the function $y(x)$ at the points $x_1, x_2, x_3, \ldots, x_n$ which divided the interval $[a, b]$ into equal partitions. During this course, we will present the numerical methods for solving a system of first order differential equation as well as the higher order system of Ordinary Differential Equations(ODEs).
The known methods that is used to solve the ODEs could be classified into two main type, namely

- One-step methods
- Multi-step methods

In the one-step methods, the value of the function at a point is determined using only it's value at the previous point. On the other hand, for the multi-step methods it is calculated using the value of the function at many points that which are known from the previous steps.
Some of those methods(one-step methods)that used to solve the first order ODEs are:

- Picard method
- Taylor method
- Runge-Kutta method

1.3 Single-step methods or One-step methods

As its mentioned above, in those method the value of the functions is estimated at a point using only its value at the previous point, thus using only one value to estimate the value of the function at another point.

1.3.1 Picard method

One of the one-step method that is used to solve the ODEs of first order and this method depends on the integration of the function as we will see later.
Let $y' = f(x, y)$ with the initial condition is $y(x_0) = y_0$ and we need to find the value of the function at $x_0 + h$ i.e. $y(x_0 + h)$ such that

$$y' = f(x, y), \qquad y(x_0) = y_0 \tag{1.3}$$

integrating the above equation from x_0 to x, we have

$$\int_{x_0}^{x} \frac{dy}{dx} dx = \int_{x_0}^{x} f(x, y) dx$$

$$y = y_0 + \int_{x_o}^{x} f(x, y) dx \tag{1.4}$$

then, the first approximation y_1 for y can be obtained by substituting y_0 instead of y in the right hand side of the last equation, i.e.,

$$y_1 = y_0 + \int_{x_o}^{x} f(x, y_0)dx, \tag{1.5}$$

the second approximation y_2, can be obtained also by substituting y_1 instead of y in the right hand side of equation (1.4), i.e.,

$$y_2 = y_0 + \int_{x_o}^{x} f(x, y_1)dx, \tag{1.6}$$

continuing with a similar way, then we cab obtained the following repeated relations

$$y_{n+1} = y(x_0) + \int_{x_o}^{x} f(x, y_n(x))dx, \tag{1.7}$$

and this repeated relation can be stopped whenever the following condition holds

$$|y_{n+1} - y_n| \leq \epsilon, \tag{1.8}$$

where, ϵ is a small positive constant.

■ **Example 1.2** Using Picard method, find an approximate value of y at $x = 0.2$ if

$$y' = x - y, \qquad y(0) = 1$$

■

Solution.

$$\begin{aligned} y_{n+1} &= y(x_0) + \int_{x_o}^{x} f(x, y_n(x))dx \\ &= y(x_0) + \int_{x_o}^{x} (x - y_n))dx \\ &= 1 + \int_{x_o}^{x} (x - y_n)dx, \qquad n = 0, 1, 2, \ldots \end{aligned}$$

$$y_1(x) = 1 + \int_0^x (x-1)dx = 1 - x + \frac{x^2}{2}$$

$$y_2(x) = 1 + \int_0^x \left[x - \left(1 - x + \frac{x^2}{2}\right)\right]dx = 1 - x + x^2 - \frac{x^3}{6}$$

$$y_3(x) = 1 + \int_0^x \left[x - \left(1 - x + x^2 - \frac{x^3}{6}\right)\right]dx = 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{24}$$

$$\begin{aligned} y_4(x) &= 1 + \int_0^x \left[x - \left(1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{24}\right)\right]dx \\ &= 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{120} \end{aligned}$$

$$y_5(x) = 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{60} + \frac{x^6}{720}$$

at $x = 0.2$, we have

$$y_0 = 1, \qquad y_1 = 0.2, \qquad y_2 = 0.83867, \qquad y_3 = 0.83740,$$

$$y_4 = 0.83746, \qquad y_5 = 0.83746,$$

thus,

$$y(0.2) = 0.83746$$

■

■ **Example 1.3** Using Picard method, find the solution of the following initial value problem

$$y' = y, \qquad y(0) = 1,$$

(Note that, the analytical solution is $y(x) = e^x$) ■

Solution.

$$\begin{aligned} y_{n+1} &= y(x_0) + \int_{x_o}^{x} f(x, y_n(x))dx \\ &= y_0 + \int_{x_o}^{x} y_n dx \end{aligned}$$

$$y_1(x) = 1 + \int_0^x dx = 1 + x$$

$$y_2(x) = 1 + \int_0^x (1 + x)dx = 1 + x + \frac{x^2}{2}$$

$$\cdots\cdots\cdots\cdots$$

$$y_n(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}$$

■

■ **Example 1.4** Using Picard method, find the solution of the following initial value problem

$$\frac{dy}{dx} = xe^y, \qquad y(0) = 0,$$

then, find $y(0.1), y(0.2), y(1)$
(Note that, the analytical solution is $y(x) = -\ln\left[1 - \frac{x^2}{2}\right]$) ■

Solution.

$$\begin{aligned} y_{n+1} &= y(x_0) + \int_{x_o}^{x} f(x, y_n(x))dx \\ &= y_0 + \int_0^x x e^{y_n} dx \\ &= 0 + \int_0^x x e^{y_n} dx, \qquad n = 0, 1, 2, \ldots \end{aligned}$$

$$y_1(x) = 0 + \int_0^x x e^0 dx = \frac{x^2}{2}$$

$$y_2(x) = 0 + \int_0^x \left[x\left(e^{\frac{x^2}{2}}\right)\right] dx = e^{\frac{x^2}{2}} - 1$$

$$y(x) = e^{\frac{x^2}{2}} - 1$$

$$\begin{aligned} y(0.1) &= 0.0050125 \\ y(0.1) &= 0.0202013 \\ y(1) &= 0.6487213 \end{aligned}$$

■

■ **Example 1.5** Using Picard method, find an approximate value for y at $x = 0.1, 0.2, 0.3$ assuming that

$$\frac{dy}{dx} = 1 + xy, \qquad y(0) = 1$$

■

Solution. we use the repeated relations for the Picard which are

$$
\begin{aligned}
y_{n+1} &= y_0 + \int_{x_o}^{x} f(x, y_n(x))dx \\
y_1(x) &= y_0 + \int_{x_0}^{x} f(x, y_0)dx = 1 + \int_{x_0}^{x} (1 + xy_0)dx \\
&= 1 + \int_{0}^{x} (1 + x)dx = 1 + x + \frac{x^2}{2} \\
y_2(x) &= y_0 + \int_{x_0}^{x} f(x, y_1)dx \\
&= 1 + \int_{0}^{x} (1 + xy_1)dx \\
&= 1 + \int_{0}^{x} \Big[1 + x\big(1 + x + \frac{x^2}{2}\big)\Big]dx \\
&= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} \\
y_3(x) &= y_0 + \int_{x_0}^{x} f(x, y_2)dx \\
&= 1 + \int_{0}^{x} (1 + xy_2)dx \\
&= 1 + \int_{0}^{x} \Big[1 + x\big(1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}\big)\Big]dx \\
&= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48} \\
y_4(x) &= y_0 + \int_{x_0}^{x} f(x, y_3)dx \\
&= 1 + \int_{0}^{x} (1 + xy_3)dx \\
&= 1 + \int_{0}^{x} \Big[1 + x\big(1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}\big)\Big]dx \\
&= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48} + \frac{x^7}{120} + \frac{x^8}{284}
\end{aligned}
$$

- First, in order to obtain the solution at $x = 0.1$, we put $x = 0.1$ in the above relations, then we have

$$y_1 = 1.105, \qquad y_2 = 1.1053458, \qquad y_3 = 1.3551897, \qquad y_4 = 1.355192$$

- Similarly, in order to obtain the solution at $x = 0.2$, we put $x = 0.2$ in the above relations, then we have

$$y_1 = 1.22, \qquad y_2 = 1.2228667, \qquad y_3 = 1.2228894, \qquad y_4 = 1.2228895$$

thus, $y(0.2) = 1.223$
- Also, in order to obtain the solution at $x = 0.3$, we put $x = 0.3$ in the above relations, then we have

$$y_1 = 1.345, \qquad y_2 = 1.35550125, \qquad y_3 = 1.3551897, \qquad y_4 = 1.355192$$

thus, $y(0.3) = 1.355$ ■

(R) **(Disadvantage of this method)** Due to the integration that exist in this method it is considered non practical method. Also, it might be difficult to perform a programming code for it.

1.3.2 Taylor serious method

This method depends in the derivatives of the function. Suppose that $y(x)$ is a solution for equation (1.1), then $y(x)$ can be written using Taylor expansion around the point $x = x_0$ as follows

$$y(x) = y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!}y_0'' + \cdots + \frac{(x - x_0)^n}{n!}y_0^{(n)} + R_{n+1}$$

where,

$$R_{n+1} = \frac{(x - x_0)^{n+1}}{(n+1)!}y^{(n+1)}(\eta), \qquad \eta \in (x_0, x)$$

putting $h = (x - x_0)$ then we can rewrite $y(x)$ as

$$y(x) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \cdots + \frac{h^n}{n!}y_0^{(n)} + R_{n+1} \qquad (1.9)$$

with

$$R_{n+1} = \frac{h^{n+1}}{(n+1)!} y^{(n+1)}(\eta), \qquad \eta \in (x_0, x_0 + h)$$

Now, in order to obtain the solution we have to determine the following derivatives

$$y'(x_0), y''(x_0), y'''(x_0)$$

that can by performed as

$$\begin{aligned} y'(x) = f(x,y) \Rightarrow y''^{(x)} &= f'(x,y) \\ &= f_x(x,y) + f_y(x,y)y' \\ &= f_x(x,y) + f_y(x,y)f \end{aligned} \tag{1.10}$$

similarly, for all the other higher order derivatives. thus all the derivatives is going to be a function of $f(x,y)$ and the derivatives of $f(x,y)$. Now, from (1.10) into (1.9), we have

$$\begin{aligned} y(x_0+h) =& y_0 + hf_0 + \frac{h^2}{2!}(f_x + f_y f)_{(x_0,y_0)} + \\ & \frac{h^3}{3!}(f_{xx} + 2f_{xy} + f_{yy}f^2 + f_x f_y + f_y^2 f)_{(x_0,y_0)} + \dots \end{aligned} \tag{1.11}$$

and the error in this case takes the following form

$$\text{Error} = \frac{h^{n+1} y^{(n+1)(\eta)}}{(n+1)!}, \quad 0 < \eta < h$$

The following are the needed steps for performing the current method:

- First: to obtain $y(x_1)$, we have to compute the following derivatives; $y'(x_0), y''(x_0), y'''(x_0), \dots$ such that
  - y' is $f(x,y)$ from the ODE,
  - y'' can be obtained by by performing the derivative of y' with respect to x,
  -y''' can be obtained by by performing the derivative of y'' with respect to x and so on - This should be done each time with substituting x with x_0 , thus we can write the following

  $$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \dots$$

  Doing so means that we have calculated $y(x_1)$ $(x_1 = x_0 + h)$.

- Second: to obtain $y(x_2)$, we have to compute the following derivatives; $y'(x_1), y''(x_1), y'''(x_1), \ldots$. Therefore, we can write the following such that

$$y_2 = y(x_2) = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \ldots$$

where, $x_2 = x_1 + h$

- Third: to obtain $y(x_3)$, we have to compute the following derivatives; $y'(x_2), y''(x_2), y'''(x_2), \ldots$. Therefore, we can write the following such that

$$y_3 = y(x_3) = y_2 + hy_2' + \frac{h^2}{2!}y_2'' + \frac{h^3}{3!}y_2''' + \ldots$$

where, $x_3 = x_2 + h$

- Finally: we can easily repeat the above steps several times till we obtain a value for $y_n = y(x_n)$ at the points $n = 0, 1, 2, 3, \ldots, x_n = x_0 + nh$, and we have

$$y_n = y(x_n) = y_{n-1} + hy_{n-1}' + \frac{h^2}{2!}y_{n-1}'' + \frac{h^3}{3!}y_{n-1}''' + \ldots$$

■ **Example 1.6** Using Taylor method to find the solution of the following ODEs;

$$\frac{dy}{dx} = x - y, \qquad y(0) = 1, \qquad h = 0.2$$

■

Solution. It's easily to write;

$$\begin{aligned}
y &= y(x) & y(0) &= 1 \\
y' &= f(x, y) = x - y & y'(0) &= -1 \\
y'' &= 1 - y' & y''(0) &= 2 \\
y''' &= -y'' & y'''(0) &= -2 \\
y^{iv} &= -y''' & y^{iv}(0) &= 2 \\
y^{v} &= -y^{ive} & y^{v}(0) &= -2
\end{aligned} \tag{1.12}$$

then, substituting from (1.12) in the following relations

$$y_1 = y(x_1) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \dots$$

leads to

$$\begin{aligned} y(0.2) = y_1 = & 1 + (0.2)(-1) + \frac{((0.2)^2)}{2!}(2) + \frac{(0.2)^3}{3!}(-2) \\ & + \frac{(0.2)^4}{4!}(2) + \frac{(0.2)^5}{5!}(-2) + \dots \end{aligned} \tag{1.13}$$

thus, $y(0.2) = y_1 = 0.83746$ ■

■ **Example 1.7** Find the solution of the following ODE

$$\frac{dy}{dx} = x + y, \quad y(0) = 2$$

then, find $y(0.1), y(0.2)$. ■

Solution.

$$\begin{aligned} & y = y(x), && y'(x) = x + y, && y'' = 1 + y', \\ & y'''(x) = y'', && y^{iv} = y''', && y^{v} = y^{iv}, \ \dots \end{aligned} \tag{1.14}$$

<u>First:</u> In order to calculate $y(0.1)$, we plug in $x = 0.1$ in the right hand side of relations (1.14), then

$$\begin{aligned} & y(0) = 2, && y'(0) = 0 + 2, && y''(0) = 1 + 2 = 3, \\ & y'''(0) = 3, && y^{iv}(0) = 3, && y^{v}(0) = 3, \dots \end{aligned} \tag{1.15}$$

thus,

$$y(x_1) = y_1 = y_0 + hy'(0) + \frac{h^2}{2!}y''(0) + \frac{h^3}{3!}y'''(0) + \frac{h^4}{4!}y^{iv}(0) + \frac{h^5}{5!}y^{v}(0) + \dots,$$

$$h = x_1 - x_0 = 0.1 - 0 = 0.1$$

$$y_1 = y(0.1) = 2 + (0.1)(2) + \frac{(0.1)^2}{2!}(3) + \frac{(0.1)^3}{3!}(3) + \frac{(0.1)^4}{4!}(3) + \frac{(0.1)^5}{5!}(3)$$

$$y_1 = y(0.1) = 2.2$$

Second: in order to calculate $y(0.2)$, we plug in $x = 0.2$ in the right hand side of relations (1.14), then

$$\begin{aligned} &y(0.1) = 2.2, \qquad y'(0.1) = 0.1 + 2.2, \qquad y''(0.1) = 1 + 2.3 = 3.3, \\ &y'''(0.1) = 3.3, \qquad y^{iv}(0.1) = 3.3, \qquad y^{v}(0.1) = 3.3, \ldots \end{aligned} \tag{1.16}$$

thus,

$$y(x_2) = y_2 = y(x_1) + hy'(x_1) + \frac{h^2}{2!}y''(x_1) + \frac{h^3}{3!}y'''(x_1) + \frac{h^4}{4!}y^{iv}(x_1) + \frac{h^5}{5!}y^{v}(x_1) + \ldots,$$

$$h = x_2 - x_1 = 0.2 - 0.1 = 0.1$$

$$y_2 = y(0.2) = 2.2 + (0.1)(2.3) + \frac{(0.1)^2}{2!}(3.3) + \frac{(0.1)^3}{3!}(3.3) + \frac{(0.1)^4}{4!}(3.3) + \frac{(0.1)^5}{5!}(3.3)$$

$$y_2 = y(0.2) = 2.21551275$$

■

■ **Example 1.8** Using Taylor method, find the solution for the following ODE

$$\frac{dy}{dx} = x^2 + y^2, \quad y(0) = 1.$$

■

$$\begin{aligned} &f(x, y) = x^2 + y^2,\ x_0 = 0,\ y_0 = 1, \\ &y = y(x), && y(0) = 1, \\ &y' = f(x, y) = x^2 + y^2 && y'(0) = 1 \\ &y'' = 2x + 2yy' && y''(0) = 2 \\ &y''' = 2 + 2yy'' + 2(y')^2 && y'''(0) = 8 \end{aligned} \tag{1.17}$$

then using

$$y(x) = y_0 + (x - x_0)y'(x_o) + \frac{(x - x_0)^2}{2!}y''(x_0) + \ldots,$$

we conclude that

$$y(x) = 1 + x + x^2 + \frac{8}{3!}x^3$$

(R) **(Disadvantage of this method)** It is clear that this method is non practical method due to the various differentiations that one have to compute during the solution. There for we present here some other methods that we can practically deal with.

1.3.3 Normal Euler method

This method is driven from Taylor method assuming that $h << 1$ in the Taylor expansion. Assuming so leads to the possibility of taking only three terms in the Taylor expansion, thus,

$$y(x) = y(x_0+h) = y(x_0)+hy'(x_0)+\frac{h^2}{2!}y''(\xi), \; x_0 < \xi < x_0+h \quad (1.18)$$

The third term in the above equation represents the error in the method and it becomes very small whenever h is small enough, thus,

$$\text{Error} = E = \frac{y''(\xi)h^2}{2} = O(h^2) \quad (1.19)$$

Equation (1.18) represents the solution at a point $x = x_0 + h$ with the given solution at $x = x_0$ i.e., $y(x_0)$ is given as an initial value. Similarly, we can find the solution at $x = x_0 + 2h$ and repeating this steps we can find also the solution at $x = x_0 + (n-1)h$. Thus, the normal Euler can take the following form;

$$y_{n+1} = y_n + hy'_n + O(h^2)$$

Also, since

$$y'_n = f(x_n, y_n),$$

then, Euler formula can be rewritten as

$$y_{n+1} = y_n + hf(x_n, y_n), \qquad E = \frac{h^2}{2}y''(\xi), \quad x_n < \xi < x_{n+1} \quad (1.20)$$

■ **Example 1.9** Find the solution of the following ODE

$$\frac{dy}{dx} = x + y, \quad y(0) = 1, \text{in the interval } [0, 0.1] \text{ taking } h = 0.02.$$

■

Solution. Using the repeated relations (1.20)

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$$y(0.02) = y_1 = y_0 + hf(x_0, y_0) = 1 + (0.02)(0 + 1) = 1.02$$

$$y(0.04) = y_2 = y_1 + hf(x_1, x_1) = 1.02 + (0.02)(0.02 + 1.02) = 1.0408$$

$$y(0.06) = y_3 = y_2 + hf(x_2, x_2) = 1.0408 + (0.02)(0.04 + 1.0408) = 1.0624$$

$$y(0.08) = y_4 = y_3 + hf(x_3, x_3) = 1.0048$$

$$y(0.1) = y_5 = y_4 + hf(x_4, x_4) = 1.1081$$

■

R The analytical solution for the ODE in the previous example at $x = 0.1$ is 1.1103, hence the numerical error is

$$E = 1.1103 - 1.1081 = 0.0022$$

1.3.4 A modified Euler method

The modified Euler method is driven also from Taylor serious with an extra term compare to the normal Euler method, i.e.,

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2} y''_n \tag{1.21}$$

Since, $y''_n = \frac{y'_{n+1} - y'_n}{h}$ (from the usual definition of the first derivative of a function). Substituting in (1.21) for the value of y''_n, we have

$$\begin{aligned} y_{n+1} &= y_n + hy^prime_n + \frac{h^2}{2}\Big(\frac{y'_{n+1} - y'_n}{h}\Big) \\ &= y_n + h\Big(y'_n + \frac{1}{2}y'_{n+1} - \frac{1}{2}y'_n\Big) \\ &= y_n + \frac{h}{2}(y'_n + y'_{n+1}) \end{aligned} \tag{1.22}$$

Hence, the final form of the modified Euler method is

$$y_{n+1} = y_n + \frac{h(y'_n + y'_{n+1})}{2}, \tag{1.23}$$

where, $y'_n = f(_n, y_n),\ y'_{n+1} = f(x_{n+1}, y_{n+1})$

(R) Determining y'_{n+1}, that appears in the right hand side of equation (1.23), depends on the value of y_{n+1}, that is still unknown, therefore, the steps of for solving such case using the modified Euler method are

- Determine y_{n+1}, using the normal Euler method.
- Use the previous value to compute the value of y'_{n+1} such that

$$y'_{n+1} = f(x_{n+1}, y_{n+1})$$

- Substitute for the values of y_n, y'_n, y'_{n+1} in the right hand side of equation (1.23) in order to obtain a value for y_{n+1} which is now obtained by the modified Euler method, thus,this method is called Predictor Corrector method, i.e.,

$$\begin{aligned} y_1^{(P)} &= y_0 + hy'_0 = y_0 + hf(x_0, y_0) \\ y_1^{(C)} &= y_0 + \tfrac{h}{2}(y'_0 + y'_1) = y_0 + \tfrac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(P)}\right)\right] \end{aligned}$$

■ **Example 1.10** Find the numerical solution of the following ODEs

$$\frac{dy}{dx} = x^2 + y,\ \ y(0) = 1$$

at $x = 0.2$, considering $h = 0.1$, using the modified Euler method. ■

Solution.

$$\begin{aligned} y_1^{(P)} &= y_0 + hf(x_0, y_0) = 1 + (0.1)(0+1) = 1.1 \\ y_1^{(C)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(P)}\right)\right] \\ &= 1 + \frac{0.1}{2}\left\{(0+1) + \left[(0.1)^2 + 1.1\right]\right\} = 1.1055 \\ y(0.1) &= 1.1055 \end{aligned}$$

$$
\begin{aligned}
y_2^{(P)} &= y_1 + hf\left(x_1, y_1^{(C)}\right) \\
&= 1.1055 + (0.1)\left[(0.1)^2 + 1.1055\right] \\
&= 1.22605 \\
y_2^{(C)} &= y_1 + \frac{h}{2}\left[f\left(x_1, y_1\right) + f\left(x_2, y_2^{(P)}\right)\right] \\
&= 1.1055 + \frac{0.1}{2}\left\{\left[(0.1)^2 + 1.1055\right] + \left[(0.2)^2 + 1.22605\right]\right\} \\
&= 1.224577 \\
y_2^{(C)} &= 1.224577
\end{aligned}
$$

■

■ **Example 1.11** Using the modified Euler method, find the solution of the following ODEs

$$\frac{dy}{dx} = x + y, \;\; y(0) = 1$$

at $x = 0.04$, considering $h = 0.02$. ■

Solution.

$$
\begin{aligned}
y_1^{(P)} &= y_0 + hf\left(x_0, y_0\right) = 1 + (0.02)(0 + 1) = 1.02 \\
y_1^{(C)} &= y_0 + \frac{h}{2}\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(P)}\right)\right] \\
&= 1 + \frac{0.02}{2}\{(0 + 1) + [(0.02) + 1.02]\} = 1.0204 \\
y(0.02) &= 1.0204
\end{aligned}
$$

$$
\begin{aligned}
y_2^{(P)} &= y_1 + hf\left(x_1, y_1^{(C)}\right) \\
&= 1.0204 + 0.02(0.02 + 1.0204) = 1.041208 \\
&= 1.041208 \\
y_2^{(C)} &= y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2^{(P)}\right)\right] \\
&= 1.0204 + \frac{0.02}{2}\{[(0.02) + 1.0204] + [(0.04) + 1.041208]\} \\
&= 1.0416 \\
y_2^{(C)} &= 1.0416
\end{aligned}
$$

■

■ **Example 1.12** Use the modified Euler method to find the solution of the following ODEs

$$y' = x + y, \; y(0) = 2$$

for obtaining the value of $y(0.2)$ using the step size $h = 0.025$. ■

Solution. First, obtaining y_{n+1}, which means $y(0.2)$, using the modified Euler method. we apply the following repeated relations

$$
\begin{aligned}
&y_{n+1} = y_n + hf(x_n, y_n) \\
&y(0.025) = y_1 = y_0 + hf(x_0, y_0) \\
&\qquad = 2 + (0.025)[(0)(4)] = 2 \\
&y(0.05) = y_2 = y_1 + hf(x_1, y_1) \\
&\qquad = 2 + (0.025)[(0)(4)] = 2 \\
&y(0.075) = y_3 = y_2 + hf(x_2, y_2) \\
&\qquad = 2 + (0.025)[(0)(4)] = 2 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&y(0.100) = y(0.125) = y(0.150) = y(0.175) = y(0.200) = 2 \\
&y_{n+1} = y(0.2) = 2, \\
&y'_{n+1} = -(x_{n+1})\left(y_{n+1}^2\right) = -(0.2)(4) = -0.8
\end{aligned}
\tag{1.24}
$$

Now, we use the relation of the modified Euler equation, that is

$$y_{n+1} = y_n + \tfrac{h\left(y'_n + y'_{n+1}\right)}{2}, y_n = y(0.175) = 2$$
$$y'_n = -(0.175)(4) = -0.700$$
$$y_{n+1} = y(0.2) = -0.7 + \tfrac{0.025(-0.7-0.8)}{2} = -0.7187$$

■

1.3.5 Runge-Kutta method

It's one of the most important methods for solving the differential equations, which can be driven using Taylor expansion and the order of this method depends on how many terms are considered from the Taylor expansion, thus we have the following types of the method

Runge-Kutta method of second order (RK2)

It is used to obtain the solution of a differential equation of the form

$$\frac{dy}{dx} = f(x,y),\ y(x_0) = y_0, \tag{1.25}$$

and it can be driven as follows; assume

$$\begin{aligned} y_{n+1} &= y_n + ak_1 + bk_2, \\ k_1 &= hf(x_n, y_n), \\ k_2 &= hf(x_n + \alpha h, y_n + \beta k_1), \end{aligned} \tag{1.26}$$

where, a, b, α, β are constants that can be determined with the following way;
-Using the Taylor expansion for eq. (1.25) at a point x_n, we have

$$y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2!} f\prime(x_n, y_n) + O(h^3), \tag{1.27}$$

where,

$$f\prime(x_n, y_n) = \frac{df_n}{dx} = \left(f_x + f_y \frac{dy}{dx}\right)_n = (f_x + f_y f)_n$$

Now, substituting in the above equation about the value of $f\prime(x_n, y_n)$, we have

$$y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2!}(f_x + f_y f)_n + O(h^3), \qquad (1.28)$$

the term k_2 which is used in RK2 can be rewritten in the following form (using Taylor expansion for a two variable function)

$$\begin{aligned} k_2 &= hf\left(x_n + \alpha h, y_n + \beta k_1\right) \\ &= hf\left(x_n, y_n\right) + \alpha h^2 f_x\left(x_n, y_n\right) + \beta h k_1 f_y\left(x_n, y_n\right) \\ &= h\left(f_n + \alpha h f_x + \beta k_1 f_y\right)_n = h\left(f_n + \alpha h f_x + \beta h f_y f\right)_n \text{ (since } k_1 = hf) \end{aligned}$$

Substituting in (1.25) for the value of k_2, we have

$$y_{n+1} = y_n + ahf\left(x_n, y_n\right) + bh\left(f + \alpha h f_x + \beta h f_y\right)_n,$$

which can be rewritten as,

$$y_{n+1} = y_n + (a+b)hf\left(x_n, y_n\right) + h^2\left(\alpha b f_x + \beta b f f_y\right)_n,$$

Thus,

$$a + b = 1,\ \alpha b = \frac{1}{2},\ \beta b = \frac{1}{2}$$

This equation has three relations in four variables, therefore the solution of is infinite number in which one can pick any value for one of the variables to get the other three variables. Also, this equation can be rewritten in the following form

$$b(\alpha - \beta) = 0,\ b \neq 0 \implies \alpha - \beta = 0 \implies \alpha = \beta$$

Now,

- choose $\alpha = \beta = \frac{1}{2}$, leads to $a = 0, b = 1$, which is incorrect as we should have $(a \neq 0)$.
- choose $\alpha = \beta = 1$, leads to $a = b = \frac{1}{2}$

then substituting for the values of a, b, α, β,then we achieve to our goal i.e.,

$$\begin{aligned} y_{n+1} &= y_n + ak_1 + bk_2, \text{ with} \\ k_1 &= hf(x_n, y_n), \\ k_2 &= hf(x_n + \alpha h, y_n + \beta k_1), \end{aligned} \tag{1.29}$$

that is RK2.

■ **Example 1.13** Use RK2 method to find the solution of the following ODEs

$$\frac{dy}{dx} = x^2 + y^2, \;\; y(2) = -1$$

at $x = 2.3$ using the step size $h = 0.1$. ■

Solution.

$$f(x, y) = x^2 + y^2$$

$$y_1 = y_0 + \frac{k_1 + k_2}{2},$$

$$k_1 = hf(x_0, y_0) = hf(2, -1) = (0.1)(4 + 1) = 0.5$$

$$k_2 = hf(x_0+h, y_0+k_1) = hf(2+0.1, -1+0.5) = hf(2.1, -0.5) = 0.466$$

$$y_1 = -1 + \frac{1}{2}(0.5 + 0.466) = -0.517$$

$$y_2 = y_1 + \frac{k_1 + k_2}{2},$$

$$\begin{aligned} k_1 &= hf\,(x_1, y_1) \\ &= hf\,(x_1, y_1) = hf_1 = hf(2.1, -0.517) \\ &= (0.1)\left[(2.1)^2 + (-0.517)^2\right] = 0.468, \\ k_2 &= hf(x_1 + h, y_1 + k_1) \\ &= (0.1)\left[(2.2)^2 + (-0.049)^2\right] = 0.484 \end{aligned}$$

$$
\begin{aligned}
y_2 &= -0.517 + \frac{1}{2}(0.468 + 0.484) = -0.041 \\
y_3 &= y_2 + \frac{k_1 + k_2}{2} \\
k_1 &= hf(x_2, y_2) = hf_2 = hf(2.2, -0.041) \\
&= (0.1)\left[(2.2)^2 + (-0.041)^2\right] = 0.484, \\
k_2 &= hf(2.2 + 0.1, -0.041 + 0.484) = hf(2.3, 0.443) \\
&= (0.1)\left[(2.3)^2 + (0.443)^2\right] = 0.548 \\
y_3 &= -0.041 + \frac{1}{2}(0.484 + 0.548) = 0.475
\end{aligned}
$$

■

■ **Example 1.14** Let

$$\frac{dy}{dx} = x^2 - y, \; y(0) = 1$$

find $y(0.1), y(0.2)$ using RK2. ■

Solution.

$$f(x, y) = x^2 - y$$

$$x_0 = 0, y_0 = 1 \Longrightarrow f(x_0, y_0) = -1,$$

Now, the RK2 method is

$$k_1 = hf(x_0, y_0) = ()0.1(0 - 1) = -0.1$$

$$k_2 = hf(x_0 + h, y_0 + k_1) = hf(0.1, 0.9) = (0.1)((0.1)^2 - 0.9) = -0.089$$

$$K = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(-0.1 + 0.089) = -0.0945$$

$$y_1 = y(0.1) = y_0 + K = 1 - 0.0945 = 0.9055$$

For computing $y(0.2)$ we take $(x_1, y_1) = (0.1, 0.9055)$ instead of (x_0, y_0), then we repeat the method again

$$
\begin{aligned}
k_1 &= hf\,(x_1, y_1) \\
&= h(x_1^2 - y_1) = (0.1)\left[(0.1)^2 - 0.905\right] = -0.08955, \\
k_2 &= hf(x_0 + h, y_0 + k_1) \\
&= (0.1)\left[(0.2)^2 - 0.81595\right] = 0.077595
\end{aligned}
$$

$K = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(-0.08955 - 0.077595) = -0.0835725$
$y_2 = y(0.2) = y_1 + k = 0.9055 - 0.0835725 = 0.821975$

■

Rung-Kutta of fourth order (RK4)

This method is considered one of the most popular method as its is more accurate compare to the Rung-Kutta of second order method. This ,method could be driven in a similar way to that of RK2 increasing. It takes the following form

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where,

$$\begin{aligned} k_1 &= hf(x_n, y_n), \\ k_2 &= hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_1), \\ k_3 &= hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_2), \\ k_4 &= hf(x_n + h, y_n + k_3), \end{aligned}$$

■ **Example 1.15** Use the RK4 in order to solve the following ODE

$$\frac{dy}{dx} = x + y,\ y(0) = 1$$

at $x = 0.1$, using,$h = 0.1$. ■

Solution.

$$k_1 = hf\left(x_n, y_n\right) = hf(0,1) = 0.1(0+1) = 0.1$$

$$k_2 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right)$$

$$= hf\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\right)$$

$$= hf(0.05, 1.05)$$

$$= 0.1(0.05 + 1.05) = 0.11$$

$$k_3 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2\right)$$

$$= 0.1 \tag{1.30}$$

$$= 0.1 \quad 0.05 + 1.055 = 0.11050$$

$$k_4 = hf\left(x_n + h, y_n + k_3\right)$$

$$= 0.1f(0.1, 1.11050) = 0.12105$$

$$y_{n+1} = y_n + \frac{1}{6}k_1 + 2k_2 + 2k_3 + k_4$$

$$y(0.1) = 1.0 + \frac{1}{6}(0.1) + 0.22 + 0.221 + 0.1205$$

$$= 1.11034$$

■

■ **Example 1.16** From the following ODE

$$\frac{dy}{dx} = x^2 - y,\ \ y(0) = 1$$

find $y(0.1)$, $y(0.2)$, using RK4. ■

Solution.

$$k_1 = hf(x_n, y_n) = hf(0, 1) = 0.1(0 - 1) = -0.1$$

$$k_2 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right)$$

$$= hf(0.05, 0.98)$$

$$= 0.1(0.05^2 - 0.95) = 0.09475$$

$$k_3 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2\right)$$

$$= 0.1f(0.05^2, 0.952625) = -0.0950125$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

$$= 0.1f(0.1^2 - 0.0950125) = 0.0894987$$

$$K = \frac{1}{6}k_1 + 2k_1 + 2k_2 + k_3$$

$$= \frac{1}{6}[-0.1 + 2 - 0.09475$$

$$+ 2 - 0.0950125 - 0.0894987]$$

$$= -0.0948372$$

$$y_1 = y(0.1) = y_o + K = 1 - 0.0948372 = 0.9051627.$$

Now, to compute $y(0.2)$ we take $(x_1, y_1) = (0.1, 0.9051627)$ instead of (x_0, y_0) and repeat the method to get the following

$$k_1 = hf\left(x_1, y_1\right) = hf(0.1, 0.9051627)$$

$$= 0.1\left[0.1^2 - 0.9051627\right] = -0.0895162$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = hf(0.15, 0.8604046)$$

$$= 0.1\left[0.15^2 - 0.8604046\right] = -0.837904$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = hf(0.15, 0.8632674)$$

$$= 0.1\left[0.15^2 - 0.8632674\right] = -0.0840767$$

$$k_4 = hf \quad x_1 + h, y_1 + k_3 = hf(0.2, 0.8210859)$$

$$= 0.1\left[0.2^2 - 0.8210859\right] = -0.0781085$$

$$K = \frac{1}{6}k_1 + 2k_1 + 2k_2 + k_3$$

$$= \frac{1}{6}[-0.0895162 + 2 - 0.0837904$$

$$+ 2 - 0.0840767 - 0.0781085]$$

$$= -0.0838931$$

$$y_2 = y(0.2) = y_1 + K$$

$$= 0.9051627 - 0.0838931$$

$$= 0.08212695$$

■

■ **Example 1.17** Suppose we have the following ODE

$$\frac{dy}{dx} = x^2 - y, \; y(0) = 1,$$

find $y(0.1), \; y(0.2)$, using RK2. ■

Solution.

$$
\begin{aligned}
k_1 &= hf\,(x_0, y_0) = 0.1[0-1] = -0.1 \\
k_2 &= hf\,(x_0 + h, y_0 + k_1) \\
&= hf0.1, 0.9 = 0.1\left[0.1^2 - 0.9\right] \\
&= -0.089 \\
K &= \frac{1}{2}k_1 + k_2 = \frac{1}{2} - 0.1 + 0.089 = -0.0945 \\
y_1 &= y(0.1) = y_0 + k = 1 - 0.0945 = 0.9055
\end{aligned}
$$

Then, to compute $y(0.2)$, we take $(x_1, y_1) = (0.1, 0.9055)$ instead of (x_0, y_0) and repeat the method to get the following

$$
\begin{aligned}
k_1 &= hf\,(x_1, y_1) = hx_1^2 - y_1 \\
&= 0.1\left[0.1^2 - 0.905\right] = -0.08955 \\
k_2 &= hf\,(x_0 + h, y_0 + k_1) \\
&= hf0.2, 0.81595 = 0.1\left[0.2^2 - 0.81595\right] \\
&= -0.077595 \\
K &= \frac{1}{2}k_1 + k_2 = \frac{1}{2} - 0.08955 - 0.077595 = -0.0835725 \\
y_2 &= y(0.2) = y_1 + k = 0.9055 - 0.0835725 = 0.821975
\end{aligned}
$$

■

■ **Example 1.18** Use RK2 to solve the following ODE

$$\frac{dy}{dx} = y - x, \; y(0) = 2,$$

at $x = 0.2$, using $h = 0.1$. ■

Solution.

$$k_1 = hf\,(x_0, y_0) = hf(0,2)$$

$$= 0.2[2 - 0] = 0.4$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = hf(0.1, 2.2)$$

$$= 0.2[2.2 - 0.1] = 0.42$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = hf(0.1, 2.21)$$

$$= 0.2[2.21 - 0.1] = 0.422$$

$$k_4 = hf \quad x_0 + h, y_0 + k_3 = hf(0.2, 2.422)$$

$$= 0.2[2.422 - 0.2] = 0.4644$$

$$y(0.2) = y_0 + \frac{1}{6}k_1 + 2k_2 + 2k_3 + k_4$$

$$= 2 + [0.4 + 20.42 + 20.422 + 0.4644]$$

$$= 2.4247266$$

■

Exercise 1.1 Use RK4 to find the values of $y(0.1), y(0.2), y(0.3)$ of the following ODE

$$\frac{dy}{dx} = xy + y^2,\ y(0) = 1,$$

Chapter 2

Numerical solution for systems of ordinary differential equation

2.1 Solving differential systems of first order

The general form of system of ordinary differential equation from the first order is

$$\left.\begin{aligned} y_1' &= f_1\left(x, y_1, y_2, \cdots, y_n\right) \\ y_2' &= f_2\left(x, y_1, y_2, \cdots, y_n\right) \\ &\vdots \qquad \vdots \\ y_n' &= f_n\left(x, y_1, y_2, \cdots, y_n\right) \\ \text{with,} \\ y_1(x_0) &= \alpha_1, y_2(x_0) = \alpha_2, \ldots, y_n(x_0) = \alpha_n \end{aligned}\right\} \tag{2.1}$$

All the methods mentioned in the previous chapter for solving an equation from the initial value problem type can be used to solve system of ordinary differential equation as in (2.1). We are going to show how those methods can be extended to solve a system of ODEs. During our discussion, we are going to focus our attention to a system of two equations and in order to make the picture more clear we will use alternative notations as;

$$y' = f(x, y, z), z' = \phi(x, y, z)$$

be a system formed of two equations with the following two initial conditions

$$y(x_0) = y_0, z(x_0) = z_0$$

2.1.1 Picard method

Suppose

$$\begin{cases} y' = f(x, y, z) \\ z' = \phi(x, y, z) \\ \text{with the initial condition} \\ y(x_0) = y_0, \ z(x_0) = z_0 \end{cases} \tag{2.2}$$

the first approximation y_1, z_1 can be obtained in a similar way to that of the one differential equation, i.e.,

$$\begin{array}{l} y_1 = y_0 + \int_{x_0}^{x} f\left(x, y_0, z_0\right) dx \\ z_1 = z_0 + \int_{x_0}^{x} \phi\left(x, y_0, z_0\right) dx, \end{array}$$

the second approximation is

$$\begin{array}{l} y_2 = y_0 + \int_{x_0}^{x} f\left(x, y_1, z_1\right) dx \\ z_2 = z_0 + \int_{x_0}^{x} \phi\left(x, y_1, z_1\right) dx \end{array}$$

and, so on

■ **Example 2.1** Use Picard method to find an approximate vale for y, z to solve

$$\frac{dy}{dx} = z, \ \frac{dz}{dx} = x^3(y + z)$$

with the initial conditions $y(0) = 1, z(0) = \frac{1}{2}$. ■

Solution. Since,

$$\begin{array}{l} \frac{dy}{dx} = f(x, y, z) = z \\ \frac{dz}{dx} = \phi(x, y, z) = x^3(y + z) \\ y = y_0 + \int_{x_0}^{x} f(x, y, z) dx \\ z = z_0 + \int_{x_0}^{x} \phi(x, y, z) dx \end{array}$$

The first approximation is

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0, z_0)\, dx = 1 + \int_0^x (1/2) dx = 1 + \tfrac{x}{2}$$

$$z_1 = z_0 + \int_{x_0}^{x} \phi(x, y_0, z_0)\, dx = \tfrac{1}{2} + \int_0^x x^3 \left(1 + \tfrac{1}{2}\right) dx$$
$$= \tfrac{1}{2} + \tfrac{3x^4}{8},$$

the second approximation is

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1, z_1)\, dx = 1 + \int_0^x \left(\frac{1}{2} + \frac{3x^4}{8}\right) dx$$
$$= 1 + \frac{x}{2} + \frac{3x^4}{40}$$

$$z_2 = z_0 + \int_{x_0}^{x} \phi(x, y_1, z_1)\, dx = \frac{1}{2} + \int_0^x x^3 \left(1 + \frac{x}{2} + \frac{1}{2} + \frac{3x^4}{8}\right) dx$$
$$= \frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64},$$

and, the third approximation is

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2, z_2)\, dx = 1 + \int_0^x \left(\frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64}\right) dx$$
$$= 1 + \frac{x}{2} + \frac{3x^4}{40} + \frac{x^6}{60} + \frac{x^9}{192}$$

$$z_3 = z_0 + \int_{x_0}^{x} \phi(x, y_2, z_2)\, dx$$
$$= \frac{1}{2} + \int_0^x x^3 \left(1 + \frac{x}{2} + \frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64}\right) dx$$
$$= \frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64} + \frac{7x^9}{360} + \frac{x^{12}}{256}$$

therefore, at $x = 0.1$ we have

$$y_1 = 1.05, \qquad y_2 = 1.500008, \qquad y_3 = 1.500008$$
$$z_1 = 0.5000375, \qquad z_2 = 0.5000385, \qquad z_3 = 0.5000385$$

■

2.1.2 Taylor method

Let $y(x), z(x)$ be the solution of the system (2.1), then by Taylor expansion of $y(x), z(x)$ around the point $x = x_0$, we have

$$\begin{cases} y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots \\ \\ z_1 = z_0 + hz_0' + \frac{h^2}{2!}z_0'' + \frac{h^3}{3!}z_0''' + \cdots \end{cases} \tag{2.3}$$

in order to obtain the solution, we have to determine the values of $y_0', y_0'', y_0''', \ldots$, also the values of $z_0', z_0'', z_0''', \ldots$, which can be done by differentiating $y' = f(x, y, z), z' = \phi(x, y, z)$ with respect to x, then substituting in (2.3), we have y_1, z_1 in the first step.
Similarly, in the second step we have

$$\begin{aligned} y_2 &= y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \cdots \\ \\ z_2 &= z_1 + hz_1' + \frac{h^2}{2!}z_1'' + \frac{h^3}{3!}z_1''' + \cdots \end{aligned} \tag{2.4}$$

where, y_1, z_1 and all its derivatives we obtained n the previous step. Repeating this, we will be able to obtain the values for the other steps

■ **Example 2.2** Using Taylor method, find the solution for

$$\begin{aligned} \tfrac{dy}{dx} &= x + z, y(0) = 2 \\ \tfrac{dz}{dx} &= x - y^2, \ z(0) = 1 \end{aligned}$$

at the point $x = 0.2$ with $h = 0.1$. ■

Solution. Since,

$$y' = x + z, \quad y(0) = 2$$

$$z' = x - y^2, \quad z(0) = 1$$

we can evaluate the following derivatives

$$\begin{aligned} y' &= x + z \\ y'' &= 1 + z' \\ y''' &= z'' \end{aligned}$$

$$\begin{aligned} z' &= x - y^2 \\ z'' &= 1 - 2yy' \\ z''' &= -2\left[yy'' + y'^2\right] \end{aligned}$$

then, we use Taylor series to obtain y_1, z_1 as

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots$$
$$z_1 = z_0 + hz_0' + \frac{h^2}{2!}z_0'' + \frac{h^3}{3!}z_0''' + \cdots$$

at

$$x_0 = 0, y_0 = 2, \quad z_0 = 1, h = 0.1$$

we get,

$$\begin{array}{ll} y_0' = x_0 + z_0 = 1, & z_0' = x_0 - y_0^2 = -4 \\ y_0'' = 1 + z_0' = 1 - 4 = -3, & z_0'' = 1 - 2y_0y_0' = 1 - 2(2)(1) = -3 \\ y_0''' = z_0'' = -3, & z_0''' = -2\left[y_0y_0'' + y_0'^2\right] = -2\left[2(-3) + 1^2\right] = 10 \end{array}$$

substituting with those values in the Taylor series we get

$$\begin{aligned} y_1 &= 2 + (0.1)(1) + \frac{(0.1)^2}{2!}(-3) + \frac{(0.1)^3}{3!}(-3) + \cdots \\ &= 2 + 0.1 - 0.015 - 0.0005 = 2.0845 \end{aligned}$$

$$\begin{aligned} z_1 &= 1 + (0.1)(-4) + \frac{(0.1)^2}{2!}(-3) + \frac{(0.1)^3}{3!}(10) + \cdots \\ &= 1 - 0.4 - 0.015 + 0.001667 = 0.5867 \\ y(0.1) &= 2.0845 \\ z(0.1) &= 0.5867 \end{aligned}$$

Similarly, for obtaining $y(0.2), z(0.2)$, we can write

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \cdots$$
$$z_2 = z_1 + hz_1' + \frac{h^2}{2!}z_1'' + \frac{h^3}{3!}z_1''' + \cdots$$

at,

$$x_1 = 0.1, \quad y_1 = 2.0845, \quad z_1 = 0.5867$$

we get,

$$\begin{array}{ll} y_1' = x_1 + z_1 = 0.06867, & z_1' = x_1 - y_1^2 = -4.2451403 \\ y_1'' = 1 + z_1' = -3.2451403, & z_1'' = 1 - 2y_1y_1' = -1.8628523 \\ y_1''' = z_1'' = -1.8628523, & z_1''' = -2\left[y_1y_1'' + y_1'^2\right] = 12.585876 \end{array}$$

thus,

$$\begin{aligned} y_2 &= 2.0845 + (0.1)(0.6867) + \frac{(0.1)^2}{2!}(-3.2451403) \\ &+ \frac{(0.1)^3}{3!}(-1.8628523) + \cdots \\ &= 2.1366338 \end{aligned}$$

$$\begin{aligned} z_2 &= 0.5867 + (0.1)(-4.2451403) + \frac{(0.1)^2}{2!}(-1.8628523) \\ &+ \frac{(0.1)^3}{3!}(12.585876) + \cdots \\ &= 0.1549693 \end{aligned}$$

■

2.1.3 Runge-kutta method

Let,

$$\frac{dy}{dx} = f_1(x, y, z), \ \frac{dz}{dx} = f_2(x, y, z)$$

with the initial conditions

$$y(x_0) = y_0, \ z(x_0) = z_0$$

The solution of the previous system using RK2, takes the following form

$$\begin{aligned} y_{n+1} &= y_n + \tfrac{1}{2}(k_1 + k_2) \\ z_{n+1} &= z_n + \tfrac{1}{2}(l_1 + l_2) \end{aligned}$$

where,

$$\begin{aligned} k_1 &= hf_1(x, y, z), & l_1 &= hf_2(x, y, z) \\ k_2 &= hf_1(x + h, y + k_1, z + l_1), & l_2 &= hf_2(x + h, y + k_1, z + l_1) \end{aligned}$$

The solution of the previous system using RK4, takes the following form

$$\begin{aligned} y_{n+1} &= y_n + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ z_{n+1} &= z_n + \tfrac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) \end{aligned}$$

where,

$$k_1 = hf(x, y, z),\ l = hf_2(x, y, z)$$

$$k_2 = hf_1\left(x + \tfrac{h}{2}, y + \tfrac{k_1}{2}, z + \tfrac{l_1}{2}\right)$$

$$l_2 = hf_2\left(x + \tfrac{h}{2}, y + \tfrac{k_1}{2}, z + \tfrac{l_1}{2}\right)$$

$$k_3 = hf_1\left(x + \tfrac{h}{2}, y + \tfrac{k_2}{2}, z + \tfrac{l_2}{2}\right),$$

$$l_3 = hf_2\left(x + \tfrac{h}{2}, y + \tfrac{k_2}{2}, z + \tfrac{l_2}{2}\right)$$

$$k_4 = hf_1\left(x + h, y + k_3, z + l_3\right)$$

$$l_4 = hf_2\left(x + h, y + k_3, z + l_3\right)$$

■ **Example 2.3** Using Rung-Kutta 4th find the solution for

$$\tfrac{dy}{dx} = yz + x, y(0) = 1$$

$$\tfrac{dz}{dx} = xz + y,\ z(0) = -1$$

and then find $y(0.2), z(0.2)$ ■

Solution. since

$$f_1(x, y, z) = yz + x, \quad f_2(x, y, z) = xz + y$$

$$x_0 = 0, y_0 = 1, z_0 = -1$$

$$k_1 = hf_1\left(x_0, y_0, z_0\right) = (0.1)[(1)(-1) + 0] = -0.1$$

$$l_1 = hf_2\left(x_0, y_0, z_0\right) = (0.1)[(0)(-1) + 1] = 0.1$$

$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = hf_1(0.05, 0.95, -0.95)$$

$$= (0.1)[(0.95)(-0.95) + 0.05] = -0.08525$$

$$l_2 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = hf_2(0.05, 0.95, -0.95)$$

$$= (0.1)[(0.05)(-0.95) + 0.95] = 0.09025$$

$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$= hf_1(0.05, 0.957375, -0.954875)$$

$$= (0.1)[(0.957375)(-0.954875) + 0.05] = -0.0864173$$

$$l_3 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$=hf_2(0.05, 0.957375, -0.954875)$$

$$=(0.1)[(0.05)(-0.954875) + 0.957375] = -0.0909631$$

$$k_4 = hf_1\left(x + h, y + k_3, z + l_3\right)$$

$$= hf_1(0.1, 0.9135827, -0.9090369)$$

$$= (0.1)[(0.9135827)(-0.9090369) + 0.1]$$

$$= -0.073048$$

$$l_4 = hf_2\left(x + h, y + k_3, z + l_3\right)$$

$$= hf_2(0.1, 0.9135827, -0.9090369)$$

$$= (0.1)[(0.1)(-0.9090369) + 0.9135827]$$

$$= 0.822679$$

$$k = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$= \frac{1}{6}[0.1 + 2(-0.08525) + 2(-0.0864173) - 0.073048]$$

$$= -0.0860637$$

$$l = \frac{1}{6}\left(l_1 + 2l_2 + 2l_3 + l_4\right)$$

$$= \frac{1}{6}[0.1 + 2(0.09025) + 2(0.0909631) - 0.0822679]$$

$$= -0.0907823$$

$$y_1 = y(0.1) = y_0 + k = 1 - 0.0860637 = 0.9139363$$

$$z_1 = z(0.1) = z_0 + l = -1 + 0.0907823 = -0.9092176$$

$$x_1 = 0.1, y_1 = 0.9139363, z_1 = -0.9092176$$

and, to get $y(0.2), z(0.2)$, we perform the following

$$k_1 = hf_1\left(x_1, y_1, z_1\right) = h\left(y_1 z_1 + x_1\right) = -0.0730966$$

$$l_1 = hf_2\left(x_1, y_1, z_1\right) = h\left(x_1 z_1 + y_1\right) = -0.08230145$$

$$k_2 = hf_1\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}, z_1 + \frac{l_1}{2}\right)$$

$$= hf_1(0.15, 0.877388, -0.8680669)$$

$$= (0.1)[(0.877388)(-0.8680669) + 0.15] = -0.0611631$$

$$l_2 = hf_2\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}, z_1 + \frac{l_1}{2}\right)$$

$$= hf_2(0.15, 0.877388, -0.8680669)$$

$$= (0.1)[(0.15)(-0.8680669) + 0.877388] = 0.0747177$$

$$k_3 = hf_1\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}, z_1 + \frac{l_2}{2}\right)$$

$$= hf_1(0.15, 0.8833547, -0.8718587)$$

$$= (0.1)[(0.8833547)(-0.8718587) + 0.15] = -0.062016$$

$$l_3 = hf_2\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}, z_1 + \frac{l_2}{2}\right)$$

$$= hf_2(0.15, 0.8833547, -0.8718587)$$

$$= (0.1)[(0.15)(-0.8718587) + 0.8833547] = 0.0750851$$

$$k_4 = hf_1\left(x + h, y + k_3, z + l_3\right)$$

$$= hf_1(0.2, 0.8519203, -0.8341324)$$

$$= (0.1)[(0.8519203)(-0.8341324) + 0.2]$$

$$= -0.0510614$$

$$l_4 = hf_2\left(x + h, y + k_3, z + l_3\right)$$

$$= hf_2(0.2, 0.8519203, -0.8341324)$$

$$= (0.1)[(0.2)(-0.8341324) + 0.8519203]$$

$$= 0.0685093$$

$$k = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$= \frac{1}{6}[-0.0730966 + 2(-0.0611631)$$

$$+ 2(-0.062016) - 0.0510614]$$

$$= -0.0617527$$

$$l = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

$$= \frac{1}{6}[0.08230145 + 2(-0.0747177)$$

$$+ 2(0.0750851) + 0.0685093]$$

$$= 0.0750693$$

$$y_2 = y(0.2) = y_1 + k = 0.9139363 - 0.0617527$$

$$= 0.8521836$$

$$z_2 = z(0.2) = z_1 + l = -0.9092176 + 0.0750693$$

$$= -0.8341482$$

■

2.2 Ordinary differential equation of higher order

The generalized form of ordinary differential equation of n order is

$$y^{(n)} = f(x, y, y', y'', y''', \dots, y^{(n-1)}) \tag{2.5}$$

and the initial values are

$$y(x_0) = \alpha_0, y'(x_0) = \alpha_1, y''(x_0) = \alpha_2, \dots, y^{(n-1)}(x_0) = \alpha_{n-1}.$$

This equation could be solved after converting it into a system of ordinary differential equation of first order that had been discussed before.
In order to convert equation (2.5) into a system of ordinary differential

equation of first order, we let

$$\begin{aligned} y_1 &= y, \\ y_2 &= y' \\ y_3 &= y'' \\ &\vdots \\ y_n &= y^{(n-1)}, \end{aligned} \tag{2.6}$$

differentiating this system, we have

$$\begin{aligned} y_1' &= y' = y_2, \\ y_2' &= y'' = y_3 \\ y_3' &= y''' = y_4 \\ &\vdots \\ y_n' &= y^{(n)} = f(x, y_1, y_2, y_3, y_4, \dots, y_n), \end{aligned} \tag{2.7}$$

This means that high order differential equation has been converted into a system of first order. Here, it will be enough to solve a second order differential equation using the previous mentioned methods.

2.2.1 Picard method for solving a second order differential equation

Consider the second order ordinary differential equation

$$y'' = f(x, y, y') \tag{2.8}$$

with the initial conditions

$$y(x_0) = y_0 = \alpha_0, y'(x_0) = \alpha_1$$

we write this equation in a form of system of first order which can be done by letting

$$y' = z, \qquad z' = y'' = f(x, y, z)$$

■ **Example 2.4** Using Picard method, find the solution of the following second order differential equation

$$\begin{aligned} &y'' + 2xy' + y = 0 \\ &y(0) = 0.5,\ y'(0) = 0.1 \end{aligned} \tag{2.9}$$

at $x = 0.1$. ■

Solution. let

$$y' = z \Rightarrow y'' = z' = \frac{dz}{dx}$$

thus, eq. (2.9) reads

$$\frac{dz}{dx} + 2xz + y = 0 \ \Rightarrow \frac{dz}{dx} = -(2xz + y)$$

This means that eq. (2.9) can be rewritten in the follow system form

$$\begin{aligned} y' &= z, \\ z' &= -(2xz + y) \end{aligned}$$

with the following initial conditions

$$y(0) = y_0 = 0.5, z(0) = z_0 = 0.1$$

let

$$y' = f(x, y, z) = z, \qquad z' = \phi(x, y, z) = -(2xz + y)$$

Using Picard method, we get

$$\begin{aligned} y &= y_o + \textstyle\int_{x_0}^{x} f(x, y, z)dx \\ z &= z_0 + \textstyle\int_{x_0}^{x} \phi(x, y, z)dx \end{aligned}$$

The first approximation reads

$$\begin{aligned} y_1 &= y_0 + \int_{x_0}^{x} f\left(x, y_0, z_0\right) dx \\ &= 0.5 + \int_{x_0}^{x} z_0 dx = 0.5 + \int_{x_0}^{x} (0.1) dx \\ &= 0.5 + (0.1)x \\ z_1 &= z_0 + \int_{x_0}^{x} \phi\left(x, y_0, z_0\right) dx \\ &= 0.1 - \int_{x_0}^{x} \left(2xz_0 + y_0\right) dx = 0.1 - \int_{x_0}^{x} (0.2x + 0.5) dx \\ &= 0.1 - (0.5)x - (0.1)x^2 \end{aligned}$$

the second approximation is

$$\begin{aligned} y_2 &= y_0 + \int_{x_0}^{x} f\left(x, y_1, z_1\right) dx \\ &= 0.5 + \int_{x_0}^{x} z_1 dx = 0.5 + \int_{x_0}^{x} \left(0.1 - (0.5)x - (0.1)x^2\right) dx \\ &= 0.5 + (0.1)x - \frac{(0.5)x^2}{2} - \frac{(0.1)x^3}{3} \\ z_2 &= z_0 + \int_{x_0}^{x} \phi\left(x, y_1, z_1\right) dx \\ &= 0.1 - \int_{x_0}^{x} \left(2xz_1 + y_1\right) dx \\ &= 0.1 - \int_{x_0}^{x} \left[\left(2x\left(0.1 - 0.5x - 0.1x^2\right) + (0.5 + 0.1x)\right]\right. dx \\ &= 0.1 - (0.5)x - \frac{(0.3)x^2}{2} - \frac{(2.5)x^3}{6} + \frac{(0.2)x^4}{4} \end{aligned}$$

and, the third approximation is

$$\begin{aligned}
y_3 &= y_0 + \int_{x_0}^{x} f(x, y_2, z_2)\, dx \\
&= 0.5 + \int_{x_0}^{x} z_2 dx = 0.5 + \int_{x_0}^{x} \left[0.1 - 0.5x + \frac{0.3}{2}x^2 - \frac{2.5}{6}x^3 + \frac{0.1}{4}x^4\right] dx \\
&= 0.5 + (0.1)x - \frac{(0.5)x^2}{2} - \frac{(0.1)x^3}{3} + \frac{x^4}{12} + \frac{(0.1)x^5}{10} \\
z_3 &= z_0 + \int_{x_0}^{x} \phi(x, y_2, z_2)\, dx \\
&= 0.1 - \int_{x_0}^{x} (2xz_2 + y_2)\, dx \\
&= 0.1 - (0.5)x - \frac{(0.3)x^2}{2} - \frac{(2.5)x^3}{6} + \frac{(0.2)x^4}{4} + \frac{2x^5}{15} + \frac{(0.1)x^6}{6}
\end{aligned}$$

Now, at $x = 0.1$, we have

$$y_1 = 0.51,\ y_2 = 0.50746667,\ y_3 = 0.50745933,$$

Thus, $y(0.1) = 0.5075$. ■

2.2.2 Taylor method

Suppose we have the following second order differential equation

$$y'' = f(x, y, y')$$

with the initial conditions

$$y(x_0) = y_0 = \alpha_0, y'(x_0) = \alpha_1$$

this equation can be converted into

$$\begin{aligned}
&y' = z, \\
&z' = f(x, y, z) \Rightarrow y'' = z' = f(x, y, z)
\end{aligned}$$

with the initial conditions

$$y(x_0) = y_0, y'(x_0) = z_0$$

Now, using Taylor expansion for the last two equation, we have

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!}z_0'' + \frac{h^3}{3!}z_0''' + \dots$$

$$\begin{aligned} y_1 &= y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \dots \\ &= y_0 + hz_0 + \frac{h^2}{2!}z_0' + \frac{h^3}{3!}z_0'' + \dots \end{aligned}$$

where, z_0', z_0'', z_0''' can be obtained be differentiating the second equation of the system.
With a similar way, we can get the second approximation of y_2, z_2 as

$$\begin{aligned} z_2 &= z_1 + hz_1' + \frac{h^2}{2!}z_1'' + \frac{h^3}{3!}z_1''' + \cdots \\ y_2 &= y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \cdots \\ &= y_1 + hz_1 + \frac{h^2}{2!}z_1' + \frac{h^3}{3!}z_1'' + \cdots \end{aligned}$$

where, y_1, z_1 are know at this stage from the previous iterations. Finally, using the same manner, we can get approximate values for the other intervals.

■ **Example 2.5** Using Taylor expansion at $x = 0.1, 0.2$, find the solution of the following second order differential equation

$$\begin{aligned} & y'' - x(y')^2 + y^2 = 0 \\ & y(0) = 1, \;\; y'(0) = 0 \end{aligned} \tag{2.10}$$

at $x = 0.1$. ■

Solution. Putting

$$y' = z \Rightarrow y'' = z'$$

Therefore, the differential equation takes the following form

$$\begin{cases} y' = z \\ z' = xz^2 - y^2 \end{cases} \tag{2.11}$$

with the initial conditions

$$\begin{aligned} y(0) = y_0 = 1 \\ z(0) = z_0 = 0 \end{aligned} \tag{2.12}$$

Using Taylor expansion

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!}z_0'' + \frac{h^3}{3!}z_0''' + \cdots$$
$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} + \cdots$$

from the first equation, we have

$$\begin{aligned} z' &= xz^2 - y^2, & y'' &= z' \\ z'' &= z^2 + 2xzz' - 2yy', & y''' &= z'' \\ z''' &= 2zz' + 2\left[xPz' + x\left(z'\right)^2 + zz'\right] & & \\ &- 2\left[yy'' + \left(y'\right)^2\right], & y^{iv} &= z''' \end{aligned}$$

thus,

$$z_0' = x_0 z_0^2 - y_0^2 = (0)(0)^2 - (1)^2 = -1$$
$$\begin{aligned} z_0'' &= z_0^2 + 2x_0 z_0 z_0' - 2y_0 y_0' \\ &= (0)^2 + 2(0)(0)(-1) - 2(1)(0) = 0 \end{aligned}$$
$$\begin{aligned} z_0''' &= 2z_0 z_0' + 2\left[x_0 z_0 z_0' + x_0\left(z_0'\right)^2 + z_0 z_0'\right] \\ &- 2\left[y_0 y_0'' + \left(y_0'\right)^2\right] \\ &= 2(0)(-1) + 2\left[(0)(0)(-1) + (0)(-1)^2 + (0)(-1)\right] \\ &- 2\left[(1)(-1) + (0)^2\right] = 2 \end{aligned}$$

substituting into the two equations of the system, we get

$$\begin{aligned} z_1 &= 0 + (0.1)(-1) + \frac{(0.1)^2}{2!}(0) + \frac{(0.1)^3}{3!}(2) + \cdots \\ &= -0.0997 \end{aligned}$$
$$\begin{aligned} y_1 &= y(0.1) = 1 + (0.1)(0) + \frac{(0.1)^2}{2!}(-1) + \frac{(0.1)^3}{3!}(0) + \frac{(0.1)^4}{4!}(2) + \cdots \\ &= 0.9950083 \approx 0.995 \end{aligned}$$

$$y_2 = y(0.2) = y_1 + hy_1' + \frac{h^2}{2!}y_1' + \frac{h^3}{3!}y_1''' + \frac{h^4}{4!}y_1^{iv} + \cdots$$

$$= y_1 + hz_1 + \frac{h^2}{2!}z_1' + \frac{h^3}{3!}z_1'' + \frac{h^4}{4!}z_1''' + \cdots$$

thus,

$$y_1 = 0.995, \quad z_1 = -0.0997 \tag{2.13}$$

$$z_1' = x_1 z_1^2 - y_1^2 = (0.1)(-0.0997) - (0.995)^2$$

$$= -0.9890309$$

$$z_1'' = z_1^2 + 2x_1 z_1 z_1' - 2y_1 y_1' = -0.1687416$$

then,

$$y_2 = 0.995 + \frac{(0.1)}{1!}(-0.0997) + \frac{(0.1)^2}{2!}(-0.9890309)$$

$$+ \frac{(0.1)^3}{3!}(-0.1687416) + \cdots = 0.9801129 \approx 0.9801$$

$$z_2 = z_1 + \frac{h}{1!}z_1' + \frac{h^2}{2!}z_1'' + \frac{h^3}{3!}z_1''' + \cdots$$

$$= -0.0997 + \frac{(0.1)}{1!}(-0.0997) + \frac{(0.1)^2}{2!}(-0.9890309)$$

$$+ \frac{(0.1)^3}{3!}(-0.1687416) = -0.1145871$$

■

2.2.3 Runge-Kutta

Suppose we have the following second order differential equation

$$y'' = f(x, y, y')$$

with the initial conditions

$$y(x_0) = y_0 = \alpha_0, y'(x_0) = \alpha_1$$

let

$$y' = z \Rightarrow y'' = z'$$

this equation now is converted into two equations from the first order as

$$\begin{aligned} &y' = z = f_1(x, y, z) \\ &y'' = z' = f_2(x, y, z) \\ &y(x_0) = y_0,\ z(x_0) = z_0 \end{aligned}$$

that can be solved numerically using Rung-Kutta method.

■ **Example 2.6** Using Runge-Kutta of fourth order method (RK4), find the solution of the following second order differential equation

$$\begin{aligned} &y'' = xy' - y \\ &y(0) = 3,\ y'(0) = 0 \end{aligned} \tag{2.14}$$

at $x = 0.1$. ■

Solution. Suppose

$$\begin{aligned} &y' = z = f_1(x, y, z) \\ &z' = xz - y = f_2(x, y, z) \\ &y(0) = 3, z(0) = 0 \end{aligned}$$

here,

$$x_0 = 0, y_0 = 3, z_0 = 0$$

Using RK4

$$\begin{aligned} k_1 &= hf_1(x_0, y_0, z_0) = h(z_0) = (0.1)(0) = 0 \\ l_1 &= hf_2(x_0, y_0, z_0) = h(x_0 z_0 - y_0) \\ &= (0.1)[(0)(0) - 3] = -0.3 \\ k_2 &= hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = hf_1(0.05, 3, -0.15) \\ &= (0.1)(-0.15) = -0.015 \\ l_2 &= hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = hf_2(0.05, 3, -0.15) \\ &= (0.1)[(0.05)(-0.15) - 3] = 0.030075 \end{aligned}$$

$$
\begin{aligned}
k_3 &= hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) \\
&= hf_1(0.05, 2.9925, -0.150375) \\
&= (0.1)(-0.150375) = -0.0150375 \\
l_3 &= hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) \\
&= hf_2(0.05, 2.9925, -0.150375) \\
&= (0.1)[(0.05)(-0.150375) - 2.9925] = -0.03000018 \\
k_4 &= hf_1\left(x + h, y + k_3, z + l_3\right) \\
&= hf_1(0.1, 2.9849624, -0.3000018) \\
&= (0.1)(-0.3000018) = -0.03000018 \\
l_4 &= hf_2\left(x + h, y + k_3, z + l_3\right) \\
&= hf_2(0.1, 2.9849624, -0.3000018) \\
&= (0.1)[(0.1)(-0.3000018) - 2.9849624] \\
&= -0.3014962 \\
&= \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) \\
&= \frac{1}{6}[0 + 2(-0.015) + 2(-0.0150375) - 0.03000018] \\
&= -0.0150125 \\
l &= \frac{1}{6}\left(l_1 + 2l_2 + 2l_3 + l_4\right) \\
&= \frac{1}{6}[-0.3 + 2(-0.30075) + 2(-0.3000018) - 0.3014962] \\
&= -0.3004999 \\
y_1 &= y(0.1) = y_0 + k = 3 - 0.0150125 = 2.9849875 \\
z_1 &= z(0.1) = z_0 + l = 0 - 0.3004999 = -0.3004999
\end{aligned}
$$

■

Chapter 3

Multi-step methods

3.1 Introduction

In the previous chapters, we have studied the one-step methods which require the information of the solution at only one point, say; $x = x_0$, to obtain the value of the solution at $x = x_{n+1}$. On the other hand, the multi-step methods require the information of the solution at many points to obtain the final solution and those methods need the computation of $y(x), y'(x)$ at the points $x_0, x_1, x_2, \ldots, x_n$. Moreover, they depend on the integration of the differential equation.

3.2 Adam's Bashforth method

This method is used to solve the differential equation of the following form

$$y' = f(x, y), \qquad y(x_0) = y_0 \tag{3.1}$$

by integrating the two sides of the above equation from x_n to x_{n+1}, we have

$$\int_{x_n}^{x_{n+1}} dy = \int_{x_n}^{x_{n+1}} f(x, y)dx$$

or,

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f(x, y)dx$$

in order to perform the integration of the right hand side of the above equation, we approximate the function $f(x, y)$ in the form of a polynomial of second order using the Newton backward difference form, i.e.,

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} \Big[f_n + q\nabla f_n + \frac{q(q+1)}{2!}\nabla^2 f_n + \frac{q(q+1)(q+2)}{3!}\nabla^3 f_n + \ldots \Big] dx$$

using the following change of variables

$$x = x_n + qh \Rightarrow dx = hdq$$
$$x = x_n \Rightarrow q = 0,$$
$$x = x_{n+1} \Rightarrow q = 1, (\text{since } x_{n+1} - x_n = h)$$

then the previous integration reads

$$y_{n+1} = y_n + h\int_0^1 \Big[f_n + q\nabla f_n + \frac{q(q+1)}{2!}\nabla^2 f_n + \frac{q(q+1)(q+2)}{3!}\nabla^3 f_n + \ldots \Big] dq$$

$$y_{n+1} = y_n + h\Big[qf_n + \frac{q}{2}\nabla f_n + \frac{(q^3/3) + (q^2/2)}{2!}\nabla^2 f_n \Big]_0^1$$

from which, we get

$$y_{n+1} = y_n + h\Big[f_n + \frac{1}{2}\nabla f_n + \frac{5}{12}\nabla^2 f_n]$$

then, substituting for $\nabla f_n, \nabla^2 f_n$, we have

$$\nabla f_n = f_n - f_{n-1}$$
$$\nabla^2 f_n = f_n - 2f_{n-1} + f_{n-2}$$
$$y_{n+1} = y_n + h\left[f_n + \tfrac{1}{2}(f_n - f_{n-1}) + \tfrac{5}{12}(f_n - 2f_{n-1} + f_{n-2}) \right]$$
$$y_{n+1} = y_n + \tfrac{h}{12}(23f_n - 16f_{n-1} + 5f_{n-2}), n \geq 2$$

this equation represents the Adam's Bashforth method for solving a differential equation of first order at a certain point.

■ **Example 3.1** Using Adam's Bashforth method, find the solution of the following differential equation

$$y' = y^2, \; y(0) = 1, \; h = 0.1 \tag{3.2}$$

then, find $y(0.3)$. ■

Solution. The Adam's Bashforth method of order three is

$$y_{n+1} = y_n + \frac{h}{12}(23f_n - 16f_{n-1} + 5f_{n-2}), \qquad n \geq 2$$

this means that we need to know the value of the function at three constituting points, one of those needed values can be obtained from the initial condition while the other two values can be computed using one of the one-step methods.
In this example, we choose the Taylor method as a one-step method, i.e.,

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2!}y''_n + \frac{h^3}{3!}y'''_n + \ldots$$

where,

$$\begin{aligned}
y'_n &= -y_n^2 \\
y''_n &= -2y_n y'_n = -2y_n\left(-y_n^2\right) = 2y_n^3 \\
y''_n &= 6y_n^2 y'_n = 6y_n^2\left(-y_n^2\right) = -6y_n^4 \\
\therefore y_{n+1} &= y_n + h\left(-y_n^2\right) + \tfrac{h^2}{2!}\left(2y_n^3\right) + \tfrac{h^3}{3!}\left(-6y_n^4\right) + \cdots \\
y_1 &= y_0 - hy_0^2 + h^2y_0^3 - h^3y_0^4 \\
&= 1 - (0.1)(1)^2 + (0.1)^2(1)^3 - (0.1)^3(1)^4 = 0.909 \\
y'_0 &= -y_1^2 \Rightarrow y'_1 = -(0.909)^2 = -0.826281 \\
\therefore f_1 &= -0.826281
\end{aligned}$$

$$
\begin{aligned}
y_2 &= y_1 - hy_1^2 + h^2 y_1^3 - h^3 y_1^4 \\
&= 0.909 - (0.1)(0.909)^2 + (0.1)^2(0.909)^3 - (0.1)^3(0.909)^4 \\
&= 0.833200055 \\
\therefore y_2' &= -y_2^2 \Rightarrow y_2' = -(0.833200055)^2 \\
&= -0.69422233 \\
\therefore f_2 &= -0.69422233
\end{aligned}
$$

Now, using Adam's Bashforth method, we have

$$
\begin{aligned}
y_3 &= y_2 + \frac{0.1}{12}\left(23f_2 - 16f_1 + 5f_0\right) \\
&= 0.83300054 + \frac{0.1}{12}[23(-0.69422233) \\
&\quad - 16(-0.826281) + 5(-1)] = 0.7686449074
\end{aligned}
$$

■

3.3 Adam's Maulton method

This method is one of the multi-step method and its difference compare to the Adam's Bashforth method is that it is an implicit method i.e., the expected method is corrected in the same step before moving to the next step.
Consider the following differential equation

$$y' = f(x, y), \qquad y(x_0) = y_0$$

Then, integrating the above equation from x_n to x_{n+1} leads to

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f(x, y)dx$$

and, in order to integrate the right hand side of that equation, we approximate the function $f(x, y)$ as a polynomial using Newton formula of backward interpolation.

$$
\begin{aligned}
y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} \Bigg[& f_{n+1} + q\nabla f_{n+1} + \frac{q(q+1)}{2!}\nabla^2 f_{n+1} \\
& + \frac{q(q+1)(q+2)}{3!}\nabla^3 f_{n+1} + \cdots \Bigg] dx
\end{aligned}
$$

Now, using the following relation

$$x = x_{n+1} + qh \Rightarrow dx = hdq$$
$$x = x_n \Rightarrow q = -1$$
$$x = x_{n+1} \Rightarrow q = 0, (\text{ since } x_{n+1} - x_n = h)$$

we get,

$$y_{n+1} = y_n + h \int_{-1}^{0} \Big[f_{n+1} + q\nabla f_{n+1} + \frac{q(q+1)}{2!}\nabla^2 f_{n+1} + \frac{q(q+1)(q+2)}{3!}\nabla^3 f_{n+1} + \cdots \Big] dq$$

Performing the previous integration, we have

$$y_{n+1} = y_n + h \left[q f_{n+1} + \frac{q^2}{2}\nabla f_{n+1} + \frac{(q^3/3) + (q^2/2)}{2!}\nabla^2 f_{n+1} \right]_{-1}^{0}.$$

Substituting the valued of $\nabla f_{n+1}, \nabla^2 f_{n+1}$

$$\nabla f_{n+1} = f_{n+1} - f_n$$
$$\nabla^2 f_{n+1} = f_{n+1} - 2f_n + f_{n-1}$$

we get,

$$y_{n+1} = y_n + h\Big[f_{n+1} - \frac{1}{2}(f_{n+1} - f_n) - \frac{1}{12}(f_{n+1} - 2f_n + f_{n-1})\Big]$$

which concludes the following formula

$$y_{n+1} = y_n + \frac{h}{12}\Big[5f_{n+1} + 8f_n - f_{n-1}\Big], \quad n \geq 1 \tag{3.3}$$

that is the Adam's Maulton method.

■ **Example 3.2** Using Adam's Maulton method, find $y(0.4)$ for the following differential equation

$$y' = x + y, \ y(0) = 1, \ h = 0.1 \tag{3.4}$$

■

Solution. In order to determine $y(0.4)$, using Adam's Maulton method, by eq. (3.3)

$$y_4 = y_3 + \frac{h}{12}\Big[5f_4 + 8f_3 - f_2\Big]$$

and, to determine f_4, it is required to use an explicit method; let's say Adam's Bashforth method i.e.,

$$y_4 = y_3 + \frac{h}{12}(23f_3 - 16f_2 + 5f_1)$$

also,

$$y_3 = y_2 + \frac{h}{12}(23f_2 - 16f_1 + 5f_0)$$

the question now is to obtain f_1 and f_2, that can be obtained with the help of one-step method, for instance, RK4

$$\begin{aligned}
&y_1 = y_0 + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)\\
&k_1 = hf(x_0, y_0) = h[x_0 + y_0] = (0.1)(1) = 0.1\\
&k_2 = hf\left(x_0 + \tfrac{h}{2}, y_0 + \tfrac{k_1}{2}\right) = (0.1)f(0.05, 1.05)\\
&\quad = (0.1)[0.05 + 1.05] = 0.11\\
&k_3 = hf\left(x_0 + \tfrac{h}{2}, y_0 + \tfrac{k_2}{2}\right) = hf(0.05, 1.055)\\
&\quad = (0.1)[0.05 + 1.055] = 0.11050\\
&k_4 = hf\left(x_0 + h, y_0 + k_3\right) = hf(0.1, 1.1105)\\
&\quad = (0.1)[0.1 + 1.1105] = 0.12105
\end{aligned}$$

thus,

$$\begin{aligned}
y_1 =& y_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)\\
&= 1.0 + \frac{1}{6}[0.1 + 0.22 + 0.221 + 0.12105]\\
&= 1.11034
\end{aligned}$$

Similarly, we can use RK4 again to obtain $y_2 = 1.2428$

$$\begin{aligned}
&y' = f(x, y) = x + y\\
&f_1 = x_1 + y_1 = 0.1 + 1.1034\\
&\quad = 1.21034
\end{aligned}$$

substituting, f_1, f_2, we obtain the value of y_3

$$\begin{aligned} y_3 &= y_2 + \frac{h}{12}(23f_2 - 16f_1 + 5f_0) \\ &= 1.2428 + \frac{0.1}{12}[23(1.4428) - 16(1.21034) + 5(1)] \\ &= 1.399624667 \\ f_3 &= x_3 + y_3 = 0.3 + 1.399624667 \\ &= 1.699625 \end{aligned}$$

then, substituting for f_3, y_3 we have

$$\begin{aligned} y_4^{(P)} &= y_3 + \frac{h}{12}(23f_3 - 16f_2 + 5f_1) \\ &= 1.39962447 + \frac{0.1}{12}[23(1.699635) - 16(1.4428) + 5(1.21034)] \\ &= 1.583443599 \\ f_4 &= x_4 + y_4^{(P)} = 0.4 + 1.583443899 \\ &= 1.98344 \end{aligned}$$

then, we have

$$\begin{aligned} y_4^{(C)} &= 1.399624667 + \frac{0.1}{12}[5(1.98344) + 8(1.699625) - 1.4425] \\ &= 1.58385045 \end{aligned}$$

R We can obtain the value of y_3, using RK4 instead of using Adam's Bashforth method.

■

3.4 Milne's method

One of the multi-step method and it is different from the previous methods in the following issues; (1) The expected value at a certain step is corrected before moving to the next step and, (2) It's required to know the values of the function $f(x, y)$ at four constitutive points i.e., we need to know y at $x_n, x_{n-1}, x_{n-2}, x_{n-3}$ to evaluate y at x_{n+1}. Consider the following differential equation

$$y' = f(x, y),\ y(x_0) = y_0 \tag{3.5}$$

integrating this equation from x_{n-3} to x_{n+1}, we get

$$\int_{x_{n-3}}^{x_{n+1}} dy = \int_{x_{n-3}}^{x_{n+1}} f(x, y)dx$$

As in Adam's method, we approximate $f(x, y)$ by a polynomial of second order using Newton formula for backward interpolation, then we can write,

$$y_{n+1} - y_{n-3} = \int_{x_{n-3}}^{x_{n+1}} \Big(f_n + q\nabla f_n + \frac{q(q+1)}{2!}\nabla^2 f_n + E \Big)$$

where,

$$E = \frac{q(q+1)(q+2)}{3!} h^3 f^{(3)}(\xi),\ x_{n-3} \leq \xi \leq x_{n+1}$$

using the following relation

$$\begin{aligned} &x = x_{n+1} + qh \Rightarrow dx = hdq \\ &x = x_{n-3} \Rightarrow q = -3 \\ &x = x_{n+1} \Rightarrow q = 1, (\text{ since } x_{n+1} - x_n = h) \end{aligned}$$

we get,

$$y_{n+1} = y_{n-3} + h\int_{-3}^{1} \left[f_n + q\nabla f_n + \frac{q(q+1)}{2!}\nabla^2 f_{n+1} + E \right] dq$$

performing the integration for the variable q, we get

$$y_{n+1} = y_{n-3} + 4h\Big(f_n - \nabla f_n + \frac{2}{3}\nabla^2 f_n \Big) + O(h^5)$$

Substitution for the vales of $\nabla f_n, \nabla^2 f_n$, we have

$$y_{n+1} = y_{n-3} + \frac{4h}{3}\Big(2f_n - f_{n-1} + 2f_{n-2} \Big) + O(h^5)$$

Note that, the value of y_{n+1} obtained from the above equation is called the predicted vale which denoted by $y_{n+1}^{(P)}$ and in order to correct or enhance this value, we may use Simpson rule for integration. Integrate (3.5) from x_{n-1} to x_{n+1} and change the limits of the integration as done before, we have

$$y_{n+1} = y_{n-1} + h \int_{-1}^{1} \left[f_n + q\Delta f_n + \frac{q(q+1)}{2!}\Delta^2 f_n + \cdots \right] dq$$

substituting for Δf_n and $\Delta^2 f_n$, we have

$$y_{n+1} = y_{n-1} + \frac{h}{3}\Big(f_{n-1} + 4f_n + f_{n+1}\Big) + O(h^5)$$

which is called the corrected value and is denoted by $y_{n+1}^{(C)}$.

(R) For the purpose of applying the above method, it's required to know four values of the function and in case of they are not known, we may use any method of the one-step methods.

■ **Example 3.3** let

$$\frac{dy}{dx} = \frac{1}{x+y},$$
$$y(0) = 2,\ y(0.2) = 2.0933,\ y(0.4) = 2.1755,\ y(0.6) = 2.2493$$

find $y(0.8)$ using Milne's method. ■

Solution.

$$y_{n+1}^{(P)} = y_{n-3} + \frac{4h}{3}\left(2y'_n - y'_{n-1} + 2y'_{n-2}\right)$$

since,

$$x_0 = 0,\ x_1 = 0.2,\ x_3 = 0.6,\ h = 0.2,$$

$$y_0 = 2,\ y_1 = 2.0933,\ y_2 = 2.1755,\ y_3 = 2.2493$$

Now, we have

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left(2y'_3 - y'_2 + 2y'_1\right)$$

and,

$$y_1' = \frac{1}{x_1 + y_1} = \frac{1}{0.2 + 2.0933} = 0.4360528$$

$$y_2' = \frac{1}{x_2 + y_2} = \frac{1}{0.4 + 2.1755} = 0.3882741$$

$$y_3' = \frac{1}{x_3 + y_3} = \frac{1}{0.6 + 2.2493} = 0.3509633$$

thus,

$$\begin{aligned} y_4^{(P)} &= 2 + \frac{4(0.2)}{3}\left(2(0.3509633) - (0.3882741) + 2(0.4360528)\right) \\ &= 2.3162022 \end{aligned}$$

Now, for the corrected values, we have

$$y_{n+1}^{(C)} = y_{n-1} + \frac{h}{3}\left(y_{n-1}' + 4y_n' + y_{n+1}'\right)$$

for the current case, we have $n = 3$ i.e.,

$$y_4^{(C)} = y_2 + \frac{h}{3}\left(y_2' + 4y_3' + y_4'\right)$$

and,

$$\begin{aligned} y_4^{(P)} &= 2.3162022, x_4 = 0.8 \\ y_4' &= \frac{1}{x_4 + y_4^{(P)}} = \frac{1}{0.8 + 2.3162022} = 0.3209034 \end{aligned} \tag{3.6}$$

thus,

$$\begin{aligned} y_4^{(C)} &= 2.1755 + \frac{0.2}{3}[0.3882741 + 4(0.3509633) + 0.3209034] \\ &= 2.3163687 \\ \therefore\ y(0.8) &= y_4 = 2.3164 \end{aligned}$$

■

■ **Example 3.4** Find the solution of the following differential

$$\frac{dy}{dx} = (x + y)y,\ y(0) = 1,\ h = 0.1 \tag{3.7}$$

using Milne's method to obtain $y(0.4)$. compute y at $x = 0.1, 0.2, 0.3$ using RK4 ■

Solution. First, we compute $y(0.1)$, $y(0.2)$ $y(0.3)$ using RK4, this computations are left to the reader, which lead to

$$y(0.1) = 1.11689,\ y(0.2) = 1.27739,\ y(0.3) = 1.50412,$$

$$\begin{array}{ll} x_0 = 0, & y_0 = 1 \\ x_1 = 0.1, & y_1 = 1.11689 \\ x_2 = 0.2, & y_2 = 1.27739 \\ x_3 = 0.3, & y_3 = 1.1.50412 \end{array} \tag{3.8}$$

Since,

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left(2y_3' - y_2' + 2y_1'\right)$$

and,

$$\begin{aligned} y' &= (x+y)y \\ y_1' &= (x_1 + y_1)\, y_1 = (0.1 + 1.11689)(1.11689) = 1.3591323 \\ y_2' &= (x_2 + y_2)\, y_2 = (0.2 + 1.27739)(1.27739) = 1.8872032 \\ y_3' &= (x_3 + y_3)\, y_3 = (0.3 + 1.50412)(1.50412) = 2.713613 \end{aligned} \tag{3.9}$$

Thus,

$$y_4^{(P)} = 1 + \frac{4(0.1)}{3}\left(2(2.713613) - 1.8872032 + 2(1.3591323)\right) = 1.8344383$$

Now, the corrected value reads,

$$y_4^{(C)} = y_2 + \frac{h}{3}\left(y_2' + 4y_3' + y_4'\right)$$

$$y_4' = (x_4 + y_4^{(P)})y_4^{(P)} = (0.4 + 1.8344383)(1.8344383) = 4.0989392$$

Thus,

$$y_4^{(C)} = 1.27739 + \frac{(0.1)}{3}\left(1.8872.32 + 4(2.713613) + 4.0989392\right) = 1.8387431$$

■

■ **Example 3.5** Find for the following differential equation

$$\frac{dy}{dx} = (x+y),\ y(0) = 1,\ h = 0.1 \tag{3.10}$$

the value of $y(0.5)$. ■

Solution. Since Milne's method for the predicted value reads

$$y_{n+1}^{(P)} = y_{n-3} + \frac{4h}{3}\left(2y'_n - y'_{n-1} + 2y'_{n-2}\right)$$

we have to determine the value of y at four points, so we use RK for this purpose, and we have the following results,

x	y	$y' = f(x, y) = x + y$
0	$y_{n-3} = 1$	$f_{n-3} = 1$
0.1	$y_{n-2} = 1.11$	$f_{n-2} = 1.210$
0.2	$y_{n-1} = 1.242$	$f_{n-1} = 1.442$
0.3	$y_n = 1.399$	$f_n = 1.699$

Therefore,

$$y_4^{(P)} = 1 + \frac{4(0.1)}{3}\left[2(1.699) - (1.442) + 2(1.210)\right] = 1.58364$$

Now, to compute $y_{n+1}^{(C)}$ we need to find f_{n+1}

$$f_{n+1} = f(x_{n+1}, y_{n+1}^{(P)}) = f(0.4, 1.584) = 1.984$$

and since,

$$y_4^{(C)} = y_2 + \frac{h}{3}(y'_2 + 4y'_3 + y'_4)$$

we have,

$$y_4^{(C)} = 1.242 + \frac{(0.1)}{3}[1.984 + 4(1.699) + 1.442] = 1.58364$$

Note that, $y_{n+1}^{(P)}$, $y_{n+1}^{(C)}$ have the same value i.e. there is no enhancement in the value of y. Now, we have the values of f ready and we do not have to use RK again. Thus,

$$y_{n+1}^{(P)} = y(0.5) = 2.29742$$

$$y_{n+1}^{(C)} = y(0.5) = 2.29742$$

■

Chapter 4

Boundary Value Problems

This chapter is devoted for the following items:

4.1 The Finite Difference Method for Linear Problems

4.2 Solution of the Discretized Problem

Boundary Value Problems

The Finite Difference Method for Linear Problems

In the previous chapters, we have considered the **initial value problems** for ordinary differential equations which has the following form

$y'(t) = f(t, y) \quad , t \geq a$

$y(a) = \alpha$

In many problems, however, there will be conditions on the solution given at more than one point. For a single first order equation $y'(t) = f(t, y)$, data at one point completely determines the solution so that if conditions at more than one point are given, either higher order equations or systems of equations must be treated.

Consider the second-order equation

$y''(t) = f(t, y, y'), 0 \leq t \leq 1$ (1)

With the boundary conditions:

$y(0) = \alpha, y(1) = \beta$ (2)

Equations (1) and (2) define a two-point **boundary value problem**.

[MCQ]The problem:[$y''(t) = f(t, y, y'), 0 \leq t \leq 1$] is… for ordinary differential equations. boundary value problem**- initial value problems-…**
[MCQ]The problem:[$y'(t) = f(t, y) \quad , t \geq a, y(a) = \alpha, y(0) = \alpha, y(1) = \beta$] is… for ordinary differential equations. **boundary value problem-** initial value problems-…

If the function *f* of Eq. (1) is nonlinear in either y(t) or y'(t), the **boundary value problem** is nonlinear. Nonlinear boundary value problems are more difficult to solve, and we shall not consider them.

In this chapter we treat only linear problems, in which Eq. (1) may be written in the form

$y''(t) = b(t)y'(t) + c(t)y(t) + d(t), 0 \leq t \leq 1$ (3)

where b, c, and d are given functions of t. The boundary conditions that we consider first will be of form (2). Later, we shall treat other types of boundary conditions.

Equations (3) and (2) define a linear two-point boundary-value problem for the unknown function y, and our task is to develop procedures to approximate the solution. We will assume that the problem has a unique solution that is at least two times continuously differentiable. We first consider the special case of (3) in which b(x) = 0, so we have the following example:

Example

Consider the following boundary value problem

$\boldsymbol{y''(t) = c(t)y(t) + d(t), 0 \leq t \leq 1}$

with the conditions,

$\boldsymbol{y(0) = \alpha, y(1) = \beta}$

Use finite difference approximation to obtain y'' (t).

Obtain the resulting tridaigonal system. Find the coefficient matrix when c(t)=0.

Solution:

We will assume that c(t) > 0 for 0 < t < 1; this is a sufficient condition for the problem (4), (2) to have a unique solution.

To begin the numerical solution we divide the interval [0,1] into a number of equal subintervals of length h, as shown in Figure 3.1.

To obtain numerical solution for this problem , we divide the interval [0,1] into n+1 sub-interval using the points
$t_i = t_0 + ih = ih\,;t_0 = 0\,, h = \frac{1}{n+1}$, i= 1,2,…..,n+1

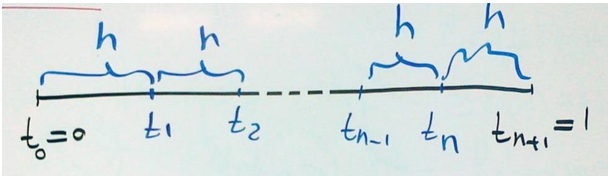


Figure 1: Grid Points

Using difference method to approximate $\boldsymbol{y''(t)}$

$$\boldsymbol{y''(t_i)} = \frac{\boldsymbol{y(t_{i+1})} - 2\boldsymbol{y(t_i)} + \boldsymbol{y(t_{i-1})}}{h^2} \quad (4)$$

Where $\quad t_{i+1} = t_i + h$

First , we write the given problem at $\quad t = t_i$

$$y''(t_i) = c(t_i)y(t_i) + d(t_i), \qquad i = 1,2,\dots,n \quad (5)$$

$$\mathrm{y}(t_0) = \mathrm{y}(0) = \alpha \qquad ,\mathrm{y}(t_{n+1}) = \mathrm{y}(1) = \beta \quad (6)$$

substituting from (4), Eq (5) becomes

$$\frac{y(t_{i+1}) - 2y(t_i) + y(t_{i-1})}{h^2} = c(t_i)y(t_i) + d(t_i) \quad (7)$$

For simplicity , we write $\quad y(t_i) = y_i$

$$\boldsymbol{\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = c_i y_i + d_i \qquad ,i = 1,2,\dots,n} \quad \textbf{(8)}$$

$$\boldsymbol{y_0 = \alpha, \qquad y_{n+1} = \beta} \quad \textbf{(9)}$$

For (8) multiplying on $\boldsymbol{h^2}$

$$\boldsymbol{y_{i+1} - 2y_i + y_{i-1} = h^2 c_i y_i + h^2 d_i}$$

$$\boldsymbol{y_{i+1} - 2y_i - h^2 c_i y_i + y_{i-1} = h^2 d_i}$$

This equation can be rearranged to have the following scheme for all he values of i

$$y_{i+1} - (2 + h^2c_i)y_i + y_{i-1} = h^2d_i\,, \quad i = 1, 2, \dots, n \tag{10}$$

$$y_0 = \alpha \quad , y_{n+1} = \beta \tag{9}$$

$$\left.\begin{aligned}
i = 1 \qquad & y_2 - \left(2 + h^2c_1\right)y_1 + y_0 = h^2d_1 \\
i = 2 \qquad & y_3 - \left(2 + h^2c_2\right)y_2 + y_1 = h^2d_2 \\
& \quad \vdots \\
i = n-1 \qquad & y_n - \left(2 + h^2c_{n-1}\right)y_{n-1} + y_{n-2} = h^2d_{n-1} \\
i = n \qquad & y_{n+1} - (2 + h^2c_n)y_n + y_{n-1} = h^2d_n
\end{aligned}\right\} \tag{11}$$

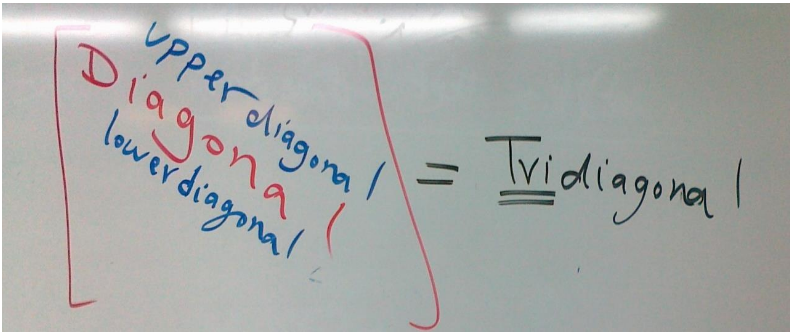
$$\left.\begin{aligned}
y_2 - \left(2 + h^2c_1\right)y_1 + \alpha &= h^2d_1 \\
y_3 - \left(2 + h^2c_2\right)y_2 + y_1 &= h^2d_2 \\
&\ \vdots \\
y_n - \left(2 + h^2c_{n-1}\right)y_{n-1} + y_{n-2} &= h^2d_{n-1} \\
\beta - (2 + h^2c_n)y_n + y_{n-1} &= h^2d_n
\end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned}
y_2 - \left(2 + h^2c_1\right)y_1 &= h^2d_1 - \alpha \\
y_3 - \left(2 + h^2c_2\right)y_2 + y_1 &= h^2d_2 \\
&\ \vdots \\
y_n - \left(2 + h^2c_{n-1}\right)y_{n-1} + y_{n-2} &= h^2d_{n-1} \\
-\left(2 + h^2c_n\right)y_n + y_{n-1} &= h^2d_n - \beta
\end{aligned}\right\} \tag{13}$$

In matrix form AY=B

$$\begin{bmatrix}
-(2+h^2c_2) & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -(2+h^2c_2) & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & \vdots & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & -(2+h^2c_2) & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -(2+h^2c_2)
\end{bmatrix}
\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix}
=
\begin{bmatrix} h^2d_1 - \alpha \\ h^2d_2 \\ \vdots \\ h^2d_{n-1} \\ h^2d_n - \beta \end{bmatrix} \tag{14}$$

The coefficient matrix A is tridiagonal Eq.4 is the resulting tridiagonal system which we must solve to obtain the numerical solution.



$$\begin{bmatrix} \boldsymbol{-(2+h^2c_2)} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \boldsymbol{-(2+h^2c_2)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & & & \cdot & & & & \\ & & & \cdot & & & & \\ & & & \cdot & & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{-(2+h^2c_2)} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \boldsymbol{-(2+h^2c_2)} \end{bmatrix}$$

When c(t) = 0 in the given problem the the coefficient matrix A is

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & \cdot & & & & \\ & & & \cdot & & & & \\ & & & \cdot & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

This is an important matrix which arises in many contexts, as we shall see. Matrices of the form (3.1.9) or (3.1.10) are called *tridiagonal* since only the three main diagonals of the matrix have non-zero elements. Tridiagonal matrices arise in a variety of applications in addition to the two-point boundary value problems of this chapter.

[MCQ]The resulting algebraic system of applying difference methods in approximating BVP of ODE's is
Tridiagonal-diagonal-…

Example
Consider the boundary value problem
$\boldsymbol{y''(t) = 2\,, 0 \le t \le 1}$ **(1)**
With the conditions,
$\boldsymbol{y(0) = 0, y(1) = 1}$ **(2)**

Use n=3 with difference approximation to y'' (t).
Obtain the resulting tridaigonal system.

Answer:

Consider the BVP $\boldsymbol{y''(t) = 2\ ,\ 0 \le t \le 1}$ **(1)**

$\boldsymbol{y(0) = 0, y(1) = 1}$ **(2)**

$t_i = 0 + ih = ih\ ;\ h = \dfrac{1}{n+1}$

With n=3➔ h = $\frac{1}{4}$: step between points

$t_0 = 0$	$t_1 = \frac{1}{4}$	$t_2 = \frac{1}{2}$	$t_3 = \frac{3}{4}$	$t_4 = 1$
$y_0 = 0$	$y_1 = ?$	$y_2 = ?$	$y_3 = ?$	$y_4 = 1$

0   0.25   0.5   0.75   1

$$y''(t) = 2$$

by integration

$$y'(t) = 2\int dt + c = 2t + c$$

by integration

$$y(t) = 2\int t dt + ct + d$$

$y(t) = t^2 + ct + d$

$y(0) = 0$ ⟹ 0 = 0 + 0 + d ⟹ d = 0

$y(1) = 1$ ⟹ 1 = 1 + c ⟹ c = 0

Hence the exact solution

$y(t) = t^2$

$$h = \frac{b-a}{n+1} = \frac{1-0}{4} = \frac{1}{4}$$

$y''(t) = 2$ ➔ $y''(t_i) = \frac{y(t_{i+1}) - 2y(t_i) + y(t_{i-1})}{h^2}$ (3)

$\frac{y(t_{i+1}) - 2y(t_i) + y(t_{i-1})}{h^2} = 2$, h = $\frac{1}{4}$

So

$$16\ [y(t_{i+1}) - 2y(t_i) + y(t_{i-1})] = 2$$

Subsitute in the given Eq.(1) , divide by 16

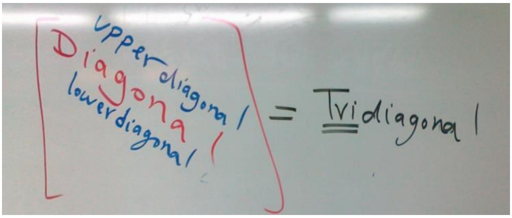
$$y_{i+1} - 2y_i + y_{i-1} = \frac{2}{16} = \frac{1}{8}$$

$$y_{i+1} - 2y_i + y_{i-1} = \frac{1}{8} \quad ;i = 1, 2,3 \qquad (4)$$

The resulting tridiagonal system is

$$
\left.\begin{aligned}
y_2 - 2y_1 &= \frac{1}{8} && (i)\\
y_3 - 2y_2 + y_1 &= \frac{1}{8} && (ii)\\
1 - 2y_3 + y_2 &= \frac{1}{8} && (iii)
\end{aligned}\right\}
$$

$$
\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}
$$



[MCQ]The resulting algebraic system of applying difference methods in approximating BVP of ODE's is
Tridiagonal-diagonal-…

[MCQ]The resulting algebraic system of applying difference approximation to $y''(t)$ and backward formula for $y'(t)$ for the BVP:$\boldsymbol{y''(t) = 2\ , 0 \leq t \leq 1,}\ y(0) = 0, y(1) = 1$. with $h = \frac{1}{4}$ is
$\boldsymbol{a_{13}y_3 + a_{12}y_2 + a_{11}y_1} = \mathrm{b}_1, \boldsymbol{a_{23}y_3 + a_{22}y_2 + a_{21}y_1} = \mathrm{b}_2, \boldsymbol{a_{33}y_3 + a_{32}y_2 + a_{31}y_1} = \mathrm{b}_3$
Answer the following 9 questions:

1) $\boldsymbol{a_{13}} = \left[\boldsymbol{0, 1, -2, \frac{1}{8}}\right]$ (2) $\boldsymbol{a_{12}} = \left[\boldsymbol{0, 1, -2, \frac{1}{8}}\right]$ (3) $\boldsymbol{a_{11}} = \left[\boldsymbol{0, 1, -2, \frac{1}{8}}\right]$
4) $\boldsymbol{a_{23}} = \left[\boldsymbol{0, 1, -2, \frac{1}{8}}\right]$ (5) $\boldsymbol{a_{22}} = \left[\boldsymbol{0, 1, -2, \frac{1}{8}}\right]$ (6) $\boldsymbol{a_{21}} = \left[\boldsymbol{0, 1, -2, \frac{1}{8}}\right]$
7) $\boldsymbol{a_{33}} = \left[\boldsymbol{0, 1, -2, \frac{1}{8}}\right]$ (8) $\boldsymbol{a_{32}} = \left[\boldsymbol{0, 1, -2, \frac{1}{8}}\right]$ (9) $\boldsymbol{a_{31}} = \left[\boldsymbol{0, 1, -2, \frac{1}{8}}\right]$

This is the resulting system of the equation which defines the unknowns y_1, y_2 and y_3

Solution of the Discretized Problem

In the previous section we saw that the use of finite difference discretization of the two-point boundary value problem C.1.3) led to a system of linear equations. The exact form of this system depends on the boundary conditions, but in all the cases we considered, except periodic boundary conditions, the system was of the tridiagonal form

$$
\begin{bmatrix}
a_{11} & a_{12} & & & \\
a_{21} & a_{22} & a_{23} & & \\
 & a_{32} & \ddots & \ddots & \\
 & & \ddots & & a_{n-1,n} \\
 & & & a_{n,n-1} & a_{nn}
\end{bmatrix}
\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}
=
\begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}. \qquad (3.2.1)
$$

Gaussian Elimination
We will solve the system (3.2.1) by the Gaussian elimination method. This method, along with several variants, will be considered in detail for general linear systems in the next chapter

[Q] Use **Gaussian Elimination** method to obtain the numerical solution for the resulting system of the previous problem

$$y_2 - 2y_1 = \frac{1}{8} \qquad (i)$$

$$y_3 - 2y_2 + y_1 = \frac{1}{8} \qquad (ii)$$

$$1 - 2y_3 + y_2 = \frac{1}{8} \qquad (iii)$$

To solve this system

2(i) + (ii) ⟹ Eliminating y_2

2(i) ⟹ $2y_2 - 4y_1 = \frac{2}{8}$

(ii) ⟹ $y_3 - 2y_2 + y_1 = \frac{1}{8}$

$y_3 - 3y_1 = \frac{3}{8}$ ⟶ (iv)

(ii) + 2(iii) ⟹ Eliminating y_2

2(iii) ⟹ $\frac{16}{8} - 4y_3 + 2y_2 = \frac{2}{8}$

(ii) ⟹ $y_3 - 2y_2 + y_1 = \frac{1}{8}$

$-3y_3 + y_1 = \frac{-13}{8}$ ⟶ (v)

$$y_3 - 3y_1 = \frac{3}{8} \qquad \text{(iv)}$$

$$-3y_3 + y_1 = \frac{-13}{8} \qquad \text{(v)}$$

3(v) + (iv) ⟹ Eliminating y_3

3(v) ⟹ $-9y_3 + 3y_1 = \frac{-39}{8}$

(ii) ⟹ $y_3 - 3y_1 = \frac{3}{8}$

$-8y_3 = \frac{-36}{8} = \frac{18}{4} = \frac{9}{2}$

$$y_3 = \frac{-9}{-2(8)} = \frac{9}{16}$$

In (v)

$y_1 = \frac{-13}{8} + 3y_3 = \frac{-13}{8} + 3(\frac{9}{16}) = \frac{-26}{16} + \frac{27}{16} = \frac{1}{16}$

$y_0 = 0 = y(t_0) = y(0)$

$y_1 = \frac{1}{16} \cong y(t_1) = y\left(\frac{1}{4}\right)$

$y_2 = \frac{1}{4} \cong y(t_2) = y\left(\frac{1}{2}\right)$

$y_3 = \frac{9}{16} \cong y(t_3) = y\left(\frac{1}{2}\right)$

$y_4 = 1 = y(t_4) = y(1)$

Exact solution

So a= b = 0

$y_e(t) = t^2$

t	y_a	y_e	Error
0	0	0	0
$\frac{1}{4}$	$y_1 = \frac{1}{16}$	$\frac{1}{16}$	0
$\frac{1}{2}$	$y_2 = \frac{1}{4}$	$\frac{1}{4}$	0
$\frac{3}{4}$	$y_3 = \frac{9}{16}$	$\frac{9}{16}$	0
1	1	1	0

[MCQ]Consider the application of Gaussian Elimination method for solving the resulting algebraic system of applying difference approximation tocetain BVP:

$$y_2 - 2y_1 = \frac{1}{8}, y_3 - 2y_2 + y_1 = \frac{1}{8}, 1 - 2y_3 + y_2 = \frac{1}{8}.$$

Answer the following 3 questions:

1) $y_1 = \left[0, \frac{1}{16}, \frac{1}{4}, \frac{9}{16}\right]$ (2) $y_2 = \left[0, \frac{1}{16}, \frac{1}{4}, \frac{9}{16}\right]$ (3) $y_3 = \left[0, \frac{1}{16}, \frac{1}{4}, \frac{9}{16}\right]$

Home Work

[1] Consider the boundary value problem

$$y''(t) + y'(t) = 2(1 + t), 0 \le t \le 1 \quad (1)$$

With the conditions,

$$y(0) = 0, y(1) = 1 \quad (2)$$

Use $h = \frac{1}{4}$ with difference approximation to y'' (t) and forward formula for y'. Obtain the resulting tridaigonal system.

[1](b) Use Gaussian Elimination method to obtain the numerical solution for the resulting system of the previous problem. If you know that the exact solution is $y(t) = t^2$ then obtain the numerical error.

[2a] Consider the boundary value problem

$$y''(t) + ty'(t) - 2y(t) = 2$$

With the conditions,

$$y(0) = 0, y(1) = 1$$

Use n=3 with difference approximation of y'' (t) and central difference approximation of y' (t). Obtain the resulting tridaigonal system.

[2b] Use Gaussian Elimination method to obtain the numerical solution for the resulting system of the previous problem. If you know that the exact solution is $y(t) = t^2$ then obtain the numerical error.

[3a] Consider the boundary value problem

$$y''(t) + 5ty'(t) - 3y(t) = 7t^2 + 2$$

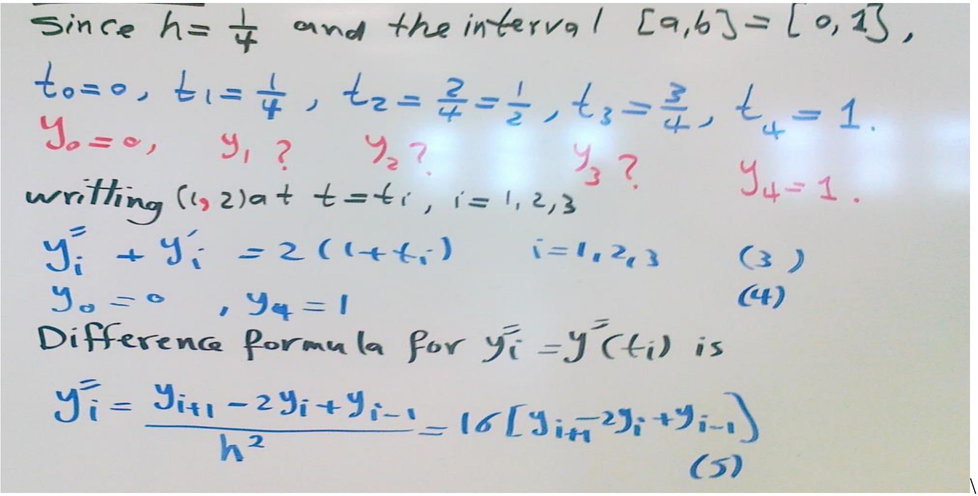
With the conditions,

$$y(0) = 0, y(1) = 1$$

Use n=3 with difference approximation to y'' (t) and y' (t). Obtain the resulting tridaigonal system.

[3b] Use Gaussian Elimination method to obtain the numerical solution for the resulting system of the previous problem. If you know that the exact solution is $y(t) = t^2$ then obtain the numerical error.

<u>Answer of [1a]</u>



\

backward formula

$$y_i' = \frac{y_i - y_{i-1}}{h} = 4[y_i - y_{i-1}] \qquad (6)$$

Inserting (5), (6) in (3),

$$16[y_{i+1} - 2y_i + y_{i-1}] + 4[y_i - y_{i-1}] = 2(1 + t_i)$$
$$16y_{i+1} - 32y_i + 16y_{i-1} + 4y_i - 4y_{i-1} = (2 + 2t_i)$$
$$16y_{i+1} + (4y_i - 32y_i) + (16y_{i-1} - 4y_{i-1}) = (2 + 2t_i)$$

$$16y_{i+1} - 28y_i + 12y_{i-1} = 2 + 2t_i \quad , i = 1, 2, 3 \qquad \mathbf{(7)}$$
$$i = 1 ==> 16y_2 - 28y_1 + 12y_0 = 2 + 2t_1 = 2 + 2 * \frac{1}{4}$$
$$i = 2 ==> 16y_3 - 28y_2 + 12y_1 = 2 + 2 * \frac{2}{4}$$
$$i = 3 ==> 16y_4 - 28y_3 + 12y_2 = 2 + 2 * \frac{3}{4}$$

y_0=0, $y_4 = 1$

$$16y_2 - 28y_1 = 2 + 2t_1 = 2\frac{1}{2}$$
$$16y_3 - 28y_2 + 12y_1 = 3$$
$$-28y_3 + 12y_2 = 3\frac{1}{2} - 16$$

the resulting tridaigonal system **is**

$$16y_2 - 28y_1 = \frac{5}{2}$$
$$16y_3 - 28y_2 + 12y_1 = 3$$
$$-28y_3 + 12y_2 = -\frac{25}{2}$$

[MCQ]The resulting algebraic system of applying difference approximation to $y''(t)$ and backward formula for $y'(t)$ for the BVP: $y''(t) + y'(t) = 2(1+t)\,, 0 \le t \le 1,\ y(0) = 0, y(1) = 1$. with $h = \frac{1}{4}$ is

$\boldsymbol{a_{13}y_3} + \boldsymbol{a_{12}y_2} + \boldsymbol{a_{11}y_1} = b_1, \boldsymbol{a_{23}y_3} + \boldsymbol{a_{22}y_2} + \boldsymbol{a_{21}y_1} = b_2, \boldsymbol{a_{33}y_3} + \boldsymbol{a_{32}y_2} + \boldsymbol{a_{31}y_1} = b_3$

Answer the following 9 questions:

1) $\boldsymbol{a_{13}} = [0, \mathbf{16}, -\mathbf{28}, \mathbf{12}]$	(2) $\boldsymbol{a_{12}} = [\mathbf{0}, 16, -\mathbf{28}, \mathbf{12}]$	(3) $\boldsymbol{a_{11}} = [\mathbf{0}, \mathbf{16}, -28, \mathbf{12}]$
4) $\boldsymbol{a_{23}} = [\mathbf{0}, 16, -\mathbf{28}, \mathbf{12}]$	(5) $\boldsymbol{a_{22}} = [\mathbf{0}, \mathbf{16}, -28, \mathbf{12}]$	(6) $\boldsymbol{a_{21}} = [\mathbf{0}, \mathbf{16}, -\mathbf{28}, 12]$
7) $\boldsymbol{a_{33}} = [\mathbf{0}, \mathbf{16}, -28, \mathbf{12}]$	(8) $\boldsymbol{a_{32}} = [\mathbf{0}, \mathbf{16}, -\mathbf{28}, 12]$	(9) $\boldsymbol{a_{31}} = [0, \mathbf{16}, -\mathbf{28}, \mathbf{12}]$

Answer of [3a]

$y''(t) + 5t\,y'(t) - 3y(t) = 7t^2 + 2$

0 — 0.25 — 0.5 — .75 — 1

t_0 — t_1 — t_2 — t_3 — t_4

$h = \frac{b-a}{n+1} = \frac{1}{4}$

$y_0 = 0,\ y_1 = ?\quad y_2 = ?,\ y_3 = ?,\ y_4 = 1$

$y''_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = 16(y_{i+1} - 2y_i + y_{i-1})$

$y'_i = \frac{y_{i+1} - y_i}{h} = 4(y_{i+1} - y_i)$

$y''_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = 16(y_{i+1} - 2y_i + y_{i-1})$

$y'_i = \frac{y_{i+1} - y_i}{h} = 4(y_{i+1} - y_i)$

$i=3$

$y_0 =$

$\therefore 16(y_{i+1} - 2y_i + y_{i-1}) + 5t_i(4(y_{i+1} - y_i)) - 3y_i = 7t_i^2 + 2$

$\therefore 16y_{i+1} - 32y_i + 16y_{i-1} + 5t_i(4y_{i+1} - 4y_i) - 3y_i = 7t_i^2 + 2$

$\therefore 16y_{i+1} - 32y_i + 16y_{i-1} + 20t_i y_{i+1} - 20t_i y_i - 3y_i = 7t_i^2 + 2$

$\therefore 16y_{i+1} - 35y_i + 16y_{i-1} + 20t_i y_{i+1} - 20t_i y_i = 7t_i^2 + 2$

$i=1 \quad , t_1 = 0.25$

$\therefore 16y_2 - 35y_1 + 16y_0 + 20\times(0.25)y_2 - 20\times(0.25)y_1 = 7(0.25)^2 + 2$

$\therefore 16y_2 - 35y_1 + 16\times(0) + 5y_2 - 5y_1 = \frac{39}{16}$

$21y_2 - 40y_1 = \frac{39}{16} \Rightarrow 1$

$i=2, \quad t_2 = .5$

$i=2, \quad t_2 = .5$

$16y_3 - 35y_2 + 16y_1 + 20\times(.5)y_3 - 20\times(.5)y_2 = 7\times(.5)^2 + 2$

~~$16y_3 - 35y_2 + 16y_1 + 5y_3 - 5y_2 = \frac{15}{4}$~~

~~$20y_3 - 35$~~ $40y_2$

$16y_3 - 35y_2 + 16y_1 + 10y_3 - 10y_2 = \frac{15}{4}$

$26y_3 - 45y_2 + 16y_1 = \frac{15}{4} \Rightarrow 2$

at $i=3, \quad t=0.75$

$16y_3 - 35y_2 + 16y_1 + 10y_3 - 10y_2 = \frac{15}{4}$

$26y_3 - 45y_2 + 16y_1 = \frac{15}{4} \Rightarrow 2$

at $i=3, \quad t=0.75$

$16y_4 - 35y_3 + 16y_2 + 20\times(.75)y_4 - 20\times(.75)y_3 = 7\times(.75)^2 + 2$

$16y_4 - 35y_3 + 16y_2 + 15y_4 - 15y_3 = \frac{95}{16}$

$\therefore y_4 = 1$

$\therefore 31\times1$ ~~$-35y_3$~~ $-50y_3 + 16y_2 = \frac{95}{16}$

$16y_2 - 50y_3 = \frac{95}{16} - 31 = -\frac{401}{16} \Rightarrow 3$

resulting tridagional system

$$\begin{bmatrix} -40 & 21 & 0 \\ 16 & -45 & 26 \\ 0 & 16 & -50 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \frac{39}{16} \\ \frac{15}{4} \\ -\frac{401}{16} \end{bmatrix}$$

Answer of [3b]

الجزءالثانى من الحل هوحساب y_1, y_2, y_3

باستعمال الالةالحاسبة

$12y_2 - 40y_1 = \frac{39}{16} \Rightarrow 1$

$26y_3 - 45y_2 + 16y_1 = \frac{15}{4} \Rightarrow 2$

$16y_2 - 50y_3 = -\frac{401}{16}$

$y_1 = -0,018$

$y_2 = 0,142$

$y_3 = 0,546$

t	y	y_e	error
0	0 ~~0,018~~	0	-0
,25	~~0,142~~ 0,018	1/16	0,044
0,5	0,142	1/4	0,108
0,75	0,546	9/16	0,016
1	1	1	0

Chapter 5

Theory of Approximation

In this Chapter, we will cover the following sections:

5.1 Best Approximation

5.2 Least Squares Approximation for continuous functions

5.3 Discrete Least Squares Approximation

5.4 Weighted Least Squares Approximation

5.5 Orthogonal Polynomials

5.6 Trigonometric Polynomial Approximation

1. Best Approximation

To obtain an efficiency, we want to find the 'best' possible approximation of a given degree n. Therefore, we introduce the following,

$$\rho_n(f) = \min_{\deg(p)\le n} E(p) = \min_{\deg(p)\le n}\left[\max_{a\le x\le b}|f(x)-p(x)|\right] \quad (1)$$

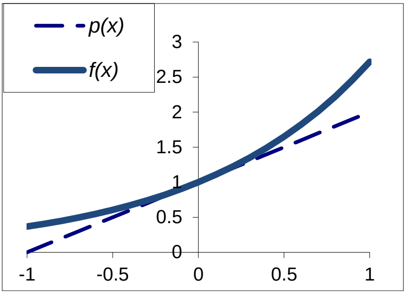
The number $\rho_n(f)$ will be the smallest possible uniform error, or **minimax error**, when approximating $f(x)$ by polynomials of degree at most n. If there is a polynomial giving this smallest error, we denote it by $m_n(x)$; thus

$$E(m_n) = \rho_n(f) \quad (2)$$

Example

Let $f(x) = e^x$ on $[-1, 1]$. In the following table, we give the values of $E(t_n)$, $t_n(x)$ the Taylor polynomial of degree n for e^x about $x = 0$, and $E(m_n)$.

Consider graphically how we can improve on the Taylor polynomial $p_1(x) = 1 + x$



Linear Taylor approximations to e^x

Error in linear minimax approximation to e^x

x	$f(x)$	$P_1(x)$	Abs(error)
-1.0	0.368	0.000	0.368
-0.8	0.449	0.200	0.249
-0.6	0.549	0.400	0.149
-0.4	0.670	0.600	0.070
-0.2	0.819	0.800	0.019
0.0	1.000	1.000	0.000
0.2	1.221	1.200	0.021
0.4	1.492	1.400	0.092
0.6	1.822	1.600	0.222
0.8	2.226	1.800	0.426
1.0	2.718	2.000	0.718

A few look of the last column we see that:

$$E(p_1) = \max_{-1\le x\le 1} |f(x) - p(x)| = 0.718 \qquad (*)$$

When n=2,

$$p_2(x) = 1 + x + \tfrac{1}{2}x^2$$

So, we have the following table:

x	$f(x)$	$P_2(x)$	Abs(error)
-1.0	0.368	0.500	0.132
-0.8	0.449	0.520	0.071
-0.6	0.549	0.580	0.031
-0.4	0.670	0.680	0.010
-0.2	0.819	0.820	0.001
0.0	1.000	1.000	0.000
0.2	1.221	1.220	0.001
0.4	1.492	1.480	0.012
0.6	1.822	1.780	0.042
0.8	2.226	2.120	0.106
1.0	2.718	2.500	0.218

A few look of the last column we see that:

$$E(p_2) = \max_{-1\le x\le 1} |f(x) - p(x)| = 0.218 \qquad (**)$$

Using Eq. (*) and (**), we conclude

$$\rho_2(f) = \min_{\deg(p)\le 2} [E(p_1), E(p_2)] = \min_{\deg(p)\le 2} [0.718, 0.218] = 0.218$$

Hence the best approximation for the two cases is

$$p_2(x) = 1 + x + \tfrac{1}{2}x^2$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \frac{x^n}{n!}$$

Accuracy of the Minimax approximation.

Example.

Define the Minimax error bound. Then use it to bound the error when Minmax error measurement is used for $f(x) = e^x$ for $-1 \le x \le 1$ for $n \ge 1$. Does the bound converge to zero as $n \to \infty$.

Answer

$$\rho_n(f) = \frac{|(b-a)|^{n+1}}{(n+1)!2^{2n+1}} \max_{a\le x\le b} \left| f^{(n+1)}(x) \right| \qquad (1)$$

$$\rho_n(f) = \frac{2^{n+1}}{(n+1)!2^{2n+1}} e \qquad (*)$$

$$\lim_{n\to\infty} \rho_n(f) = \lim_{n\to\infty} \frac{e}{(n+1)!2^n} = \frac{e}{\infty} = 0$$

So the bound converges to zero as $n \to \infty$

[Homework] Find the n-order derivative for all common functions $f(x)$.

2. Least Squares Approximation for continuous functions

Another approach to approximating a function $f(x)$ on an interval a $\leq x \leq$ b is to seek an approximation p(x) with a small 'average error' over the interval of approximation. A convenient definition of the average error of the approximation is given by

$$E(p;f) \equiv \left[\frac{1}{b-a}\int_a^b [f(x)-p(x)]^2 dx\right]^{\frac{1}{2}} \quad \text{(2a)}$$

This is also called the root-mean-square-error (denoted subsequently by RMSE) in the approximation of $f(x)$ by p(x).

To Approximate $f(x)$ we choose p(x) to minimize E(p, f)

$$E(p,f) = Min \int_a^b [f(x)-p(x)]^2 dx \quad \text{(2b)}$$

thus dispensing with the square root and multiplying fraction (although the minimums are generally different). The minimizing of (2a) is called the least squares approximation problem.

Example 3

Let $f(x) = e^x$, let $p(x) = \alpha_0 + \alpha_1 x$, where α_0, α_1 unknowns. Approximate $f(x)$ over $[-1,1]$ using LSA.

Solution:

To approximate $f(x)$,

$$E(p,f) = Min \int_a^b [f(x)-p(x)]^2 dx \quad \text{(2b)}$$

$$g(\alpha_0,\alpha_1) \equiv \int_{-1}^{1}\left[e^x - \alpha_0 - \alpha_1 x\right]^2 dx \quad (5)$$

g is a function in the two variables α_0, α_1.

To find its minimum, solve the system

$$\frac{\partial g}{\partial \alpha_0} = 0, \quad \frac{\partial g}{\partial \alpha_1} = 0$$

It is simpler to return to (5) to differentiate, obtaining

$$g(\alpha_0,\alpha_1) \equiv \int_{-1}^{1}\left[e^x - \alpha_0 - \alpha_1 x\right]^2 dx \quad (5)$$

$$2\int_{-1}^{1}\left[e^x - \alpha_0 - \alpha_1 x\right](-1)dx = 0 \qquad 2\int_{-1}^{1}\left[e^x - \alpha_0 - \alpha_1 x\right](-x)dx = 0$$

This simplifies to

$$e^x - \left(\alpha_0 x + \frac{1}{2}\alpha_1 x^2\right)\Bigg|_{-1}^{1} = 0, \quad \int_{-1}^{1} x e^x - \left(\frac{1}{2}\alpha_0 x^2 - \frac{1}{3}\alpha_1 x^3\right)\Bigg|_{-1}^{1} = 0$$

But $\int xe^{x} = xe^{x} - \int e^{x}$

$$e^{1} - e^{-1} - \left\{ \alpha_0 (1+1) + \frac{1}{2} \alpha_1 \left[(1)^2 - (-1)^2 \right] \right\} = 0$$

$$2\alpha_0 = e - e^{-1} \qquad \frac{2}{3}\alpha_1 = 2e^{-1}$$

So $\alpha_1 = 3e^{-1} \approx 1.1036$

Similarly, $\alpha_0 = \dfrac{e - e^{-1}}{2} \approx 1.1752$

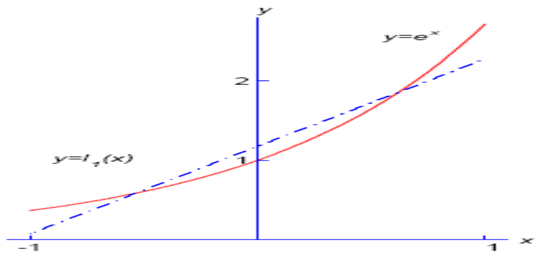
Using these values for α_0 and α_1, we denote the resulting linear approximation by

$$p_1(x) = \alpha_0 + \alpha_1 x = 1.1752 + 1.1036x$$

It is called the best linear approximation to e^x in the sense of least squares approximation.

For the error,

$$\max_{-1 \le x \le 1} \left| e^{x} - p_1(x) \right| \approx 0.439$$



The linear least squares approximation to e^x

<u>THE GENERAL CASE</u>

[Q]Approximate $f(x)$ on [0, 1], and let $n \ge 0$. Seek $p(x)$ to minimize the least squares error.
Answer:
Write

$$p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \ldots + \alpha_n x^n$$

Then

$$g(\alpha_0, \alpha_1, \ldots, \alpha_n) = \int_0^1 [f(x) - p(x)]^2 \, dx$$

To find coefficients $\alpha_0, \alpha_1, \ldots, \alpha_n$ to minimize this integral. The integral $g(\alpha_0, \alpha_1, \ldots, \alpha_n)$ is a quadratic polynomial in the n + 1 variables $\alpha_0, \alpha_1, \ldots, \alpha_n$.

To minimize $g(\alpha_0, \alpha_1, \ldots, \alpha_n)$, use the conditions

$$\frac{\partial g}{\partial \alpha_i} = 0, \quad i = 0,1,\ldots,n$$

This yields a set of n+1 equations that must be satisfied by a minimizing set $\alpha_0, \alpha_1, \ldots, \alpha_n$ for g.

Manipulating this set of conditions leads to a simultaneous linear system.

To better understand the form of the linear system, consider the special case of [a, b] = [0, 1].

$$g(\alpha_0,\alpha_1,\ldots,\alpha_n) = \int_0^1 \left[f(x) - \left(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \ldots + \alpha_n x^n \right) \right]^2 dx$$

Differentiating g with respect to each α_i, we obtain

$$2\int_0^1 \left[f(x) - \left(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \ldots + \alpha_n x^n \right) \right](-1)dx = 0$$

$$2\int_0^1 \left[f(x) - \left(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \ldots + \alpha_n x^n \right) \right](-x)dx = 0 \qquad (*)$$

.
.
.

$$2\int_0^1 \left[f(x) - \left(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \ldots + \alpha_n x^n \right) \right](-x^n)dx = 0$$

Then the linear system is

$$\sum_{j=0}^{n} \frac{\alpha_j}{i + j + 1} = \int_0^1 x^i f(x)dx, \quad i = 0,1,\ldots,n \qquad (**)$$

Which can be written as

$$A_{n\times n}\alpha_{n\times 1} = B_{n\times 1}$$

Example 4

Find least squares approximation from degree 2 for the function

$$f(x) = sin\pi x$$

In the interval [0,1] then evaluate the uniform error

Solution: We set $p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$ So using (*) we get

$$g(\alpha_0,\alpha_1,\alpha_2) = \int_0^1 \left[f(x) - \left(\alpha_0 + \alpha_1 x + \alpha_2 x^2 \right) \right]^2 dx$$

$$2\int_0^1 \left[f(x) - \left(\alpha_0 + \alpha_1 x + \alpha_2 x^2 \right) \right](-1)dx = 0$$

$$2\int_0^1 \left[f(x) - \left(\alpha_0 + \alpha_1 x + \alpha_2 x^2 \right) \right] (-x)\, dx = 0 \qquad (*)$$

$$2\int_0^1 \left[f(x) - \left(\alpha_0 + \alpha_1 x + \alpha_2 x^2 \right) \right] (-x^2)\, dx = 0$$

So

$$2\int_0^1 \left[\sin(\pi x) - \alpha_0 - \alpha_1 x - \alpha_2 x^2 \right] (-1)\, dx = 0$$

$$2\int_0^1 \left[\sin(\pi x) - \alpha_0 - \alpha_1 x - \alpha_2 x^2 \right] (-x)\, dx = 0$$

$$2\int_0^1 \left[\sin(\pi x) - \alpha_0 - \alpha_1 x - \alpha_2 x^2 \right] (-x^2)\, dx = 0$$

Which become

$$\int_0^1 \left[\sin(\pi x) - \alpha_0 - \alpha_1 x - \alpha_2 x^2 \right] dx = 0$$

$$\int_0^1 \left[x \sin(\pi x) - \alpha_0 x - \alpha_1 x^2 - \alpha_2 x^3 \right] dx = 0$$

$$\int_0^1 \left[x^2 \sin(\pi x) - \alpha_0 x^2 - \alpha_1 x^3 - \alpha_2 x^4 \right] dx = 0$$

Or

$$\left[\tfrac{\alpha_0}{1} x + \tfrac{\alpha_1}{2} x^2 + \tfrac{\alpha_2}{3} x^3 \right]_0^1 = \int_0^1 \sin(\pi x)\, dx$$

$$\left[\tfrac{\alpha_0}{2} x^2 + \tfrac{\alpha_1}{3} x^3 + \tfrac{\alpha_2}{4} x^4 \right]_0^1 = \int_0^1 x \sin(\pi x)\, dx$$

$$\left[\tfrac{\alpha_0}{3} x^3 + \tfrac{\alpha_1}{4} x^4 + \tfrac{\alpha_2}{5} x^5 \right]_0^1 = \int_0^1 x^2 \sin(\pi x)\, dx$$

Then

$$\tfrac{\alpha_0}{1} + \tfrac{\alpha_1}{2} + \tfrac{\alpha_2}{3} = \int_0^1 \sin(\pi x)\, dx$$

$$\tfrac{\alpha_0}{2} + \tfrac{\alpha_1}{3} + \tfrac{\alpha_2}{4} = \int_0^1 x \sin(\pi x)\, dx$$

$$\tfrac{\alpha_0}{3} + \tfrac{\alpha_1}{4} + \tfrac{\alpha_2}{5} = \int_0^1 x^2 \sin(\pi x)\, dx$$

evaluating integrals, we obtain

$$\int_0^1 \sin(\pi x)\, dx = \tfrac{-1}{\pi} \cos(\pi x) \Big|_0^1 = \tfrac{-1}{\pi} \left(-1 - 1 \right) = \tfrac{2}{\pi}$$

$$\int_0^1 x\sin(\pi x)dx = \tfrac{-1}{\pi}x\cos(\pi x)\ \Big|_0^1 + \tfrac{1}{\pi}\int_0^1 \cos(\pi x)dx = \tfrac{-1}{\pi}(-1-0) + \tfrac{1}{\pi}(\sin(\pi x))\ \Big|_0^1$$

$$= \tfrac{1}{\pi} + \tfrac{1}{\pi}(0-0) = \tfrac{1}{\pi}$$

$$\int_0^1 x^2\sin(\pi x)dx = \tfrac{-1}{\pi}x^2\cos(\pi x)\ \Big|_0^1 + \tfrac{2}{\pi}\int_0^1 x\cos(\pi x)dx$$

$$= \tfrac{-1}{\pi}(-1-0) + \tfrac{2}{\pi}\left[\tfrac{-1}{\pi}x\sin(\pi x)\ \Big|_0^1 - \tfrac{1}{\pi}\int_0^1 \sin(\pi x)dx\right]$$

$$= \tfrac{1}{\pi} + \tfrac{2}{\pi}\left[\tfrac{-1}{\pi}(0-0) - \tfrac{1}{\pi}\int_0^1 \sin(\pi x)dx\right]$$

$$= \tfrac{1}{\pi} + \tfrac{2}{\pi}\left[-\tfrac{1}{\pi}\tfrac{2}{\pi}\right] = \tfrac{1}{\pi} + \tfrac{2}{\pi}\left[-\tfrac{2}{\pi^2}\right] = \tfrac{1}{\pi} - \tfrac{4}{\pi^3} = \tfrac{\pi^2-4}{\pi^3}$$

So the linear system become

$$\frac{\alpha_0}{1} + \frac{\alpha_1}{2} + \frac{\alpha_2}{3} = \frac{2}{\pi}$$

$$\frac{\alpha_0}{2} + \frac{\alpha_1}{3} + \frac{\alpha_2}{4} = \frac{1}{\pi}$$

$$\frac{\alpha_0}{3} + \frac{\alpha_1}{4} + \frac{\alpha_2}{5} = \frac{\pi^2-4}{\pi^3}$$

Solving them we get

$$\alpha_2 = \tfrac{60}{\pi^3}\left[\pi^2 - 12\right] = -4.12251$$

$$\alpha_1 = -\alpha_2$$

$$\alpha_0 = \tfrac{2}{\pi} - \tfrac{\alpha_1}{2} - \tfrac{\alpha_2}{3} = -0.050465$$

Hence

$$P_2(x) = -0.050465 + 4.122251x - 4.122251x^2$$

We have the table

x	$f(x) = \sin(\pi x)$	$P_2(x)$	Abs(error)
0.0	0.000	-0.050	0.050
0.2	0.588	0.609	0.021
0.4	0.951	0.939	0.012
0.6	0.951	0.939	0.012
0.8	0.588	0.609	0.021
1.0	0.000	-0.050	0.050

The uniform error is

$$E(p_2) = \max_{0\le x\le 1}|f(x) - p(x)| = 0.050$$

[Q]Find least squares approximation from degree 2 for the function $f(x) = e^x$ In the interval [0,1] .

Discrete Least Squares Approximation

To approximate $f(x)$ by a polynomial $P(x)$ at some points x_i, $i = 1,2,\dots,m$, we solve:

$$\text{Min } E = \sum_{i=1}^{m}\left[f_i - P(x_i)\right]^2$$

[TF]The Discrete Least Squares Approximation formula for approximating $f(x)$ by a polynomial $P(x)$ at some points x_i, $i = 1,2,\dots,m$ is Min $E = \sum_{i=1}^{m}\left[f_i - P(x_i)\right]^2$

Example

Use Least Squares Approximation *method with polynomial of degree 3, to approximate solution for the BVP*

$$\boldsymbol{y''(t) = 2\,, 0 \le t \le 1} \qquad \textbf{(1)}$$

$$\boldsymbol{y(0) = 0, y(1) = 1} \qquad \textbf{(2)}$$

Compare between the *approximate solution* and the exact solution ($y(t) = t^2$).

==

Answer:

Let

$$\tilde{y}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \qquad (3)$$

Using (2) $\boldsymbol{y(0) = 0}$,

$$0 = a_0$$

Again using (2) $\boldsymbol{y(1) = 1}$

$$1 = a_1 + a_2 + a_3 \rightarrow a_3 = 1 - a_1 - a_2$$

Then the approximate solution (3) become

$$\tilde{y}(t) = a_1 t + a_2 t^2 + (1 - a_1 - a_2)t^3 \qquad (4)$$

Differentiation of (4) twice, we have

$$\tilde{y}'(t) = a_1 + 2a_2 t + 3(1 - a_1 - a_2)t^2 \qquad (5)$$

$$\tilde{y}''(t) = 2a_2 + 6(1 - a_1 - a_2)t \qquad (6)$$

We define the error in approximating the differential problem (1)

$\boldsymbol{y''(t) = 2}$

Is

$$R(t) = y''(t) - 2 \qquad (7)$$

R should be zero. If not, and indeed in the case of approximation R is often not zero. Minimizing R will reduce the error as can as we could.

Using (6), Eq. (7) becomes:

$$R(t) = 2a_2 + 6(1 - a_1 - a_2)t - 2 \tag{7}$$

To Apply the LS rule , we evaluate $R(t)$ at 2 points inside the interval $[0, 1]$, namely, $t_1 = \frac{1}{3}, t_2 = \frac{2}{3}$.
So

$R\left(\frac{1}{3}\right) = 2a_2 + 6(1 - a_1 - a_2)\frac{1}{3} - 2, R\left(\frac{2}{3}\right) = 2a_2 + 6(1 - a_1 - a_2)\frac{2}{3} - 2$

Or

$R\left(\frac{1}{3}\right) = 2a_2 + 2(1 - a_1 - a_2) - 2, R\left(\frac{2}{3}\right) = 2a_2 + 4(1 - a_1 - a_2) - 2$

$R\left(\frac{1}{3}\right) = 2(1 - a_1) - 2, \qquad R\left(\frac{2}{3}\right) = -2a_2 + 4(1 - a_1) - 2$

Min $g(a_1, a_2) = R\left(\frac{1}{3}\right)^2 + R\left(\frac{2}{3}\right)^2 =$

$$[2(1 - a_1) - 2]^2 + [-2a_2 + 4(1 - a_1) - 2]^2$$

$$\frac{\partial g}{\partial a_1} = 0, \frac{\partial g}{\partial a_2} = 0$$

$$2[2(1 - a_1) - 2] \quad (-2) + 2[-2a_2 + 4(1 - a_1) - 2] \quad (-4) = 0$$

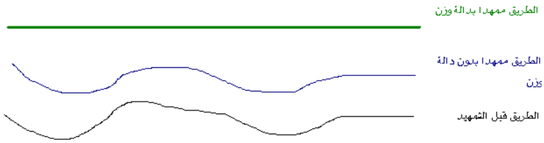
$$0 + 2[-2a_2 + 4(1 - a_1) - 2] \quad (-2) = 0$$

Home Work>>>>>

Weighted Least Squares Approximation

weight function

The integrable function $w(x)$ in an interval I is said to be weight function if $w(x) \geq 0$ for some $x \in I$. $w(x)$ must be nonzero positive value ($w(x)>0$) at least in some parts of I.



شكل 9-5: تطبيق فكرة دالة الوزن على رصف الطرق

Weighted Least Squares Approximation

To approximate $f(x)$ on I=[a, b]. we seek $p(x)$ to minimize the least squares error. i.e.

$$Min\ E(p) = \int_a^b w(x)[f(x) - p(x)]^2 dx$$

Where w(x) is the weight function.

An example of set of Polynomials:

$$P_0(x) = 1, \quad P_1(x) = x, P_2(x) = 1 + 2x + x^2$$

Orthogonal Polynomials

Definition:

The set of function $\{\phi_k\}_{k=0}^{n}$ are said to be orthogonal with weight function $w(x)$ in [a, b] if

$$\int_a^b w(x)\phi_j(x)\phi_k(x)\,dx = \alpha_j \delta_{kj}$$

Where the Kronecker function δ_{kj} is defined by

$$\delta_{kj} = \begin{cases} 0, j \neq k \\ 1, j = k \end{cases}$$

Furthermore $\{\phi_k\}_{k=0}^{n}$ are said to be orthonormal if they are orthogonal and

$$\alpha_j = 1,\ j = 0,1,\dots,n$$

Function representation

Theorem 1:

let $f(x)$ is approximated by

$$p_n(x) = \sum_{k=0}^{n} a_k \phi_k(x), \qquad (1)$$

where $\phi_k(x), k = 01, \dots, n$, are orthogonal with weight function $w(x)$ in[a, b].

Then $a_k(x), k = 01, \dots, n.$ are defined by

$$a_j = \frac{1}{\alpha_j}\int_a^b w(x) f(x)\phi_j(x)\,dx$$

Proof: The Least Squares Approximation with weight function $w(x)$ is

$$Min\,E(p) = \int_a^b w(x)\left[f(x) - p_n(x)\right]^2 dx$$

Using $p_n(x) = \sum_{k=0}^{n} a_k \phi_k(x)$

$$Min\,E(p) = \int_a^b w(x)\left[f(x) - \sum_{k=0}^{n} a_k \phi_k(x)\right]^2 dx$$

The necessary conditions for E(p) to attain its minimum are

$$\frac{\partial E(p)}{\partial a_j} = 0, \qquad j = 0,1,\dots,n$$

$$2\int_a^b w(x)\left[f(x) - \sum_{k=0}^{n} a_k \phi_k(x)\right](-\phi_j(x))dx = 0$$

$$\int_a^b w(x)\left[f(x) - \sum_{k=0}^{n} a_k \phi_k(x)\right]\phi_j(x)dx = 0$$

$$\int_a^b w(x) f(x)\phi_j(x)dx - \int_a^b w(x)\sum_{k=0}^{n} a_k \phi_k(x)\phi_j(x)dx = 0$$

$$\int_a^b w(x) f(x)\phi_j(x)dx = \sum_{k=0}^{n} a_k \int_a^b w(x)\phi_k(x)\phi_j(x)dx$$

Since $\phi_k(x), k = 01, \ldots, n$ are orthogonal

$$\int_a^b w(x) f(x)\phi_j(x)dx = a_j \alpha_j$$

Hence

$$a_j = \frac{1}{\alpha_j}\int_a^b w(x) f(x)\phi_j(x)dx$$

<u>Examples of orthogonal polynomials:</u>

1-Legendre polynomials

The usual notation for the k[th] -degree Legendre polynomial is $P_k(x)$ and corresponds to the normalization $P_k(1) = 1$.

In terms of the $P_k(x)$, the recurrence relation is

$$P_{k+1}(x) = \frac{(2k+1)}{(k+1)} x P_k(x) - \frac{k}{(k+1)} P_{k-1}(x), \quad k = 1,2,\ldots$$

$P_0(x) = 1$, $P_1(x) = x$

To obtain the first 4 members of the polynomials $P_k(x)$,

$P_0(x) = 1$, $P_1(x) = x$

$k = 1$ ➔

$P_2(x) = \frac{3}{2} x P_1(x) - \frac{1}{2} P_0(x) = \frac{3}{2} x(x) - \frac{1}{2}(1) = \frac{3}{2}x^2 - \frac{1}{2}$.

$k = 2$ ➔

$$P_3(x) = \frac{5}{3} x P_2(x) - \frac{2}{3} P_1(x) = \frac{5}{3} x\left[\frac{3}{2}x^2 - \frac{1}{2}\right] - \frac{2}{3}x = \frac{5}{2}x^3 - \frac{3}{2}x$$

$k = 3$ ➔

$$P_4(x) = \frac{7}{4} x P_3(x) - \frac{3}{4} P_2(x) = \frac{7}{4} x\left[\frac{5}{2}x^3 - \frac{3}{2}x\right] - \frac{3}{4}\left[\frac{3}{2}x^2 - \frac{1}{2}\right]$$
$$= \frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8}$$

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Thus

$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$,

$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$, $P_4(x) = \frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8}$,

The Legendre polynomials are orthogonal with respect to the weight function $w(x) = 1$:

$$\int_{-1}^{1} P_j(x) P_k(x) dx = \begin{cases} 0, & j \neq k \\ \frac{2}{2j+1} = \alpha_j, & j = k \end{cases}$$

2-Chebyshev polynomials

$$T_n(x) = 2x T_{n-1}(x) - T_{n-2}(x),$$

$$\text{where } T_0(x) = 1 \text{ and } T_1(x) = x.$$

The Chebyshev polynomials are orthogonal with respect to the weight function $w(x) = (1-x^2)^{\frac{-1}{2}}$:

$$(Chebyshev) \qquad \int_{-1}^{1} T_n(x)T_m(x)\,\frac{dx}{\sqrt{1-x^2}} = 0,$$

$$(Chebyshev) \qquad \int_{-1}^{1} T_n^2(x)\,\frac{dx}{\sqrt{1-x^2}} = \begin{cases} \pi & \text{if } n = 0, \\ \frac{\pi}{2} & \text{if } n > 0, \end{cases}$$

Gram-Schmidt Process

Theorem 1: Let $\{\psi_k(x)\}_{k=1}^{n}$ are linearly independent functions, defined in the interval $[a, b]$. Then, we can construct from it the orthogonal /orthonormal set of polynomials by:
$\phi_1(x) = c_{11}\psi_1(x)$,
$\phi_2(x) = c_{21}\psi_1(x) + c_{22}\psi_2(x)$,
$\phi_3(x) = c_{31}\psi_1(x) + c_{32}\psi_2(x) + c_{33}\psi_3(x)$,
…
Where, the constants $\{c_{ij}\}_{i,j=1}^{n}$ can evaluated by applying the orthogonal/orthonormal property.

Example:
Consider the set of functions:

$$\{\psi_k(x)\}_{k=1}^{n} = \{1, x, x^2, x^3, \dots\}$$

defined in the interval $[-1,1]$. Construct from it an orthonormal set of polynomials $\{\phi_k(x)\}_{k=1}^{n}$ with the weight function $w(x) = 1$, making use of Gram-Schmidt Process.
Solution:
The first three orthonormal of polynomials $\{\phi_k(x)\}_{k=1}^{3}$ are defined by
$\phi_1(x) = c_{11}$,
$\phi_2(x) = c_{21} + c_{22}x$
$\phi_3(x) = c_{31} + c_{32}x + c_{33}x^2$
Where, the constants $\{c_{ij}\}_{i,j=1}^{n}$ can evaluated by applying the orthonormal property.

$$\int_{-1}^{1} w(x)\phi_j(x)\phi_k(x)\,dx = \delta_{kj} = \begin{cases} 0, j \neq k \\ 1, j = k \end{cases}$$

To obtain c_{11}, we apply only the orthonormal property when $j = k = 1$:
$\int_{-1}^{1} \phi_1(x)\phi_1(x)dx = 1 \Rightarrow \int_{-1}^{1} (c_{11})^2 dx = 1 \Rightarrow 2(c_{11})^2 = 1$
So $c_{11} = \pm\frac{1}{\sqrt{2}}$, and thus

$$\phi_1(x) = c_{11} = \pm\frac{1}{\sqrt{2}}$$

Now, for the second polynomial $\phi_2(x) = c_{21} + c_{22}x$, we apply only the orthonormal property as follows:
$\int_{-1}^{1} \phi_2(x)\phi_2(x)dx = 1$, $\qquad \int_{-1}^{1} \phi_1(x)\phi_2(x)dx = 0$
$\int_{-1}^{1} (c_{21} + c_{22}x)^2 dx = 1$, $\qquad \pm\frac{1}{\sqrt{2}}\int_{-1}^{1} (c_{21} + c_{22}x)dx = 0$

The first integration gives

$\int_{-1}^{1}[(c_{21})^2 + 2c_{21}c_{22}x + (c_{22})^2x^2]dx =$

$[(c_{21})^2x + c_{21}c_{22}x^2 + \frac{1}{3}(c_{22})^2x^3]_{-1}^{1} = 2(c_{21})^2 + 0 + \frac{2}{3}(c_{22})^2 = 1$

The second integration gives

$$\int_{-1}^{1}\left(xc_{21} + \frac{1}{2}c_{22}x^2\right)dx = [xc_{21} + \frac{1}{2}c_{22}x^2]_{-1}^{1} = 2c_{21} + 0 = 0$$

This gives $c_{21} = 0$ and $c_{22} = \pm\sqrt{\frac{3}{2}}$. Thus

$$\phi_2(x) = \pm\sqrt{\frac{3}{2}}x$$

Now, for the third polynomial $\phi_3(x) = c_{31} + c_{32}x + c_{33}x^2$, we apply only the orthonormal property as follows:

$\int_{-1}^{1}\phi_3(x)\phi_3(x)dx = 1$, $\int_{-1}^{1}\phi_1(x)\phi_3(x)dx = 0$, $\int_{-1}^{1}\phi_2(x)\phi_3(x)dx = 0$

These integrations implies

$$\int_{-1}^{1}[c_{31} + c_{32}x + c_{33}x^2]^2dx = 1$$

$$\pm\frac{1}{\sqrt{2}}\int_{-1}^{1}[c_{31} + c_{32}x + c_{33}x^2]dx = 0$$

$$\pm\sqrt{\frac{3}{2}}\int_{-1}^{1}[c_{31} + c_{32}x + c_{33}x^2]dx = 0$$

Solving these three equations the constants c_{31}, c_{32}, c_{33} can be obtained and so $\phi_3(x) = c_{31} + c_{32}x + c_{33}x^2$ is defined.
(left for student as homework).

[HomeWork] Consider the set of functions:

$$\{\psi_k(x)\}_{k=1}^{n} = \{1, x, x^2, x^3, \ldots\}$$

defined in the interval [0,1]. Construct from it an orthonormal set of polynomials $\{\phi_k(x)\}_{k=1}^{n}$ with the weight function $w(x) = 1$, making use of Gram-Schmidt Process.

<u>Gram-Schmidt Process(b)</u>

We can define an orthogonal set $\{\varphi_k\}_{k=0}^{n}$ of polynomials with weight function $w(x)$ in $[a,b]$ as follows:

$$\varphi_0(x) = 1, \varphi_1(x) = x - B_1, B_1 = \frac{\int_a^b xw(x)[\varphi_0(x)]^2}{\int_a^b w(x)[\varphi_0(x)]^2},$$

and the sequence

$$\phi_k(x) = (x - B_k)\phi_{k-1}(x) - C_k\phi_{k-2}(x),\ k \ge 2,$$

$$B_k = \frac{\int_a^b xw(x)[\phi_{k-1}(x)]^2dx}{\int_a^b w(x)[\phi_{k-1}(x)]^2dx}, \quad C_k = \frac{\int_a^b xw(x)\phi_{k-1}(x)\phi_{k-2}(x)dx}{[illegible]}$$

Example(Home Work)
(a)Use Gram-Schmidt Process to construct an orthogonal polynomial in [-1,1] with weight function $w(x)=1$

(b) use the constructed polynomial in (a) to approximate f(x)=e^x in[-1,1]

Answer:

(a)

$$B_1=\frac{\int_a^b xw(x)[\phi_0(x)]^2dx}{\int_a^b w(x)[\phi_0(x)]^2dx}=\frac{\int_{-1}^1 x[1]^2dx}{\int_{-1}^1[1]^2dx}=\frac{\int_{-1}^1 x\ dx}{\int_{-1}^1 dx}=0$$

$$\therefore\ \phi_1=x$$

k=2

$$\phi_2=(x-B_2)\phi_1(x)-C_2\phi_0(x)$$

$$B_2=\frac{\int_a^b xw(x)[\phi_1(x)]^2dx}{\int_a^b w(x)[\phi_1(x)]^2dx}=\frac{\int_{-1}^1 x[x]^2dx}{\int_{-1}^1[x]^2dx}=\frac{\int_{-1}^1 x^3dx}{\int_{-1}^1 x^2dx}=0$$

$$C_2=\frac{\int_a^b xw(x)\phi_1(x)\phi_0(x)dx}{\int_a^b w(x)[\phi_0(x)]^2dx}=\frac{\int_{-1}^1 x^2dx}{\int_{-1}^1[1]^2dx}=\frac{\frac{2}{3}}{2}=\frac{1}{3}$$

$$B_3=0,C_3=\tfrac{4}{15}\Rightarrow\ \ \phi_3=x^3-\tfrac{3}{5}x$$

$$\phi_4=x^4-\tfrac{6}{7}x^2+\tfrac{3}{35}\ ,\ \phi_5=x^5-\tfrac{10}{9}x^3+\tfrac{5}{21}x\ ,.....$$

(b)

$$f(x)\cong P_n(x)=\sum_{k=0}^{n}a_k\phi_k(x)$$

$$a_k=\frac{1}{\alpha_k}\int_a^b w(x)\phi_k(x)f(x)dx$$

And from orthogonal property, $\alpha_k=\int_a^b w(x)\left[\phi_k(x)\right]^2dx$

$$k=0\Rightarrow$$

$$\alpha_0=\int_{-1}^1\left[\phi_0(x)\right]^2dx=\int_{-1}^1 dx=x\Big|_{-1}^1=1-(-1)=2$$

$$a_0=\frac{1}{2}\int_{-1}^1\phi_0(x)f(x)dx=\frac{1}{2}\int_{-1}^1 e^x dx=\frac{1}{2}e^x\Big|_{-1}^1$$

$$=\frac{1}{2}\left[e^{1}-e^{-1}\right]=1.1752$$

$k=1 \Rightarrow$

$$\alpha_1=\int_{-1}^{1}\left[\phi_1(x)\right]^2 dx=\int_{-1}^{1}x^2 dx=\left.\frac{x^3}{3}\right|_{-1}^{1}=\frac{1}{2}-\left(-\frac{1}{3}\right)=\frac{2}{3}$$

$$a_1=\frac{3}{2}\int_{-1}^{1}\phi_1(x)f(x)dx=\frac{3}{2}\int_{-1}^{1}xe^{x}dx=\frac{3}{2}\left[xe^{x}-e^{x}\right]_{-1}^{1}$$

$$=\frac{3}{2}\left[(e-e)-\left(-e^{-1}-e^{-1}\right)\right]=+3e^{-1}=+1.1036$$

$k=2 \Rightarrow$

$$\alpha_2=\int_{-1}^{1}\left[\phi_2(x)\right]^2 dx=\int_{-1}^{1}\left(x^2-\frac{1}{3}\right)^2 dx$$

$$=\int_{-1}^{1}\left[x^4-\frac{2}{3}x^2+\frac{1}{9}\right]dx$$

$$=\left[\frac{1}{5}x^5-\frac{2}{9}x^3+\frac{1}{9}x\right]_{-1}^{1}=\left[\frac{1}{5}-\frac{2}{9}-\frac{1}{9}\right]=\frac{8}{45}$$

$$=0.1778$$

$$a_2=\frac{1}{\alpha_2}\int_{-1}^{1}\phi_2 f(x)dx=\frac{45}{8}\int_{-1}^{1}\left(x^2-\frac{1}{3}\right)e^{x}dx$$

$$=\frac{45}{8}\left[x^2e^{x}-2xe^{x}-\frac{7e^{x}}{3}\right]_{-1}^{1}=\frac{45}{8}\left(\frac{2}{3}e-\frac{14}{3}e^{-1}\right)=-0.5288$$

$$\therefore f(x)\cong a_0\phi_0(x)+a_1\phi_1(x)+a_2\phi_2(x)$$

$$=1.1752\phi_0-1.1036\phi_1-0.528\phi_2(x)$$

$$=1.1752(1)-1.1036x-0.528\left(x^2-\tfrac{1}{3}\right)$$

$$=0.192-1.1036x+0.528x^2$$

Home Work :

Use Gram-Schmidt Process(b) to construct an orthogonal polynomial in [0,1] with weight function $w(x)=1$

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Use Gram-Schmidt Process(b) to construct an orthogonal polynomial in [-1,1] with weight function w(x)=??

&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&

[1]: Use *Galerkin method with* Legendre polynomial *of degree 3, to approximate solution for the BVP*

$$\boldsymbol{y''(t) = 2\,, -1 \leq t \leq 1} \qquad \textbf{(1)}$$

$$\boldsymbol{y(-1) = 1, y(1) = 1} \qquad \textbf{(2)}$$

Compare between the *approximate solution* and the exact solution($y(t) = t^2$).

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Answer:

Let

$$\tilde{y}(t) = a_0P_0(t) + a_1P_1(t) + a_2P_2(t) + a_3P_3(t) \qquad (3)$$

Using (2) $\boldsymbol{y(-1) = 1}$,

$$0 = a_0P_0(\mathbf{-1}) + a_1P_1(\mathbf{-1}) + a_2P_2(\mathbf{-1}) + a_3P_3(\mathbf{-1})$$

$P_0(x) = 1\,, P_1(x) = x, P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}, \; P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$
$P_0(\mathbf{-1}) = 1\,, P_1(\mathbf{-1}) = -1, P_2(\mathbf{-1}) = 1, \; P_3(\mathbf{-1}) = -1$

$$0 = a_0 - a_1 + a_2 - a_3$$

$$a_3 = a_0 - a_1 + a_2 \qquad (4)$$

Again using (2) $\boldsymbol{y(1) = 1}$

$$1 = a_0P_0(1) + a_1P_1(1) + a_2P_2(1) + a_3P_3(1)$$

$P_0(x) = 1\,, P_1(x) = x, P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}, \; P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$
$P_0(\mathbf{1}) = 1\,, P_1(\mathbf{1}) = 1, P_2(\mathbf{1}) = 1, \; P_3(\mathbf{1}) = 1$

$$1 = a_0 + a_1 + a_2 + a_3$$

Using (4), $1 = a_0 + a_1 + a_2 + a_0 - a_1 + a_2 = 2a_0 + 2a_2$

$$a_2 = \frac{1}{2} - a_0 \qquad (5)$$

Substituting from (4)-(5), then the approximate solution (3) become

$$\tilde{y}(t) = a_0P_0(t) + a_1P_1(t) + \left[\frac{1}{2} - a_0\right]P_2(t) + [a_0 - a_1 + a_2]P_3(t) \qquad (6)$$

Differentiation of (4) twice, we have

$$\tilde{y}'(t) = a_0P'_0(t) + a_1P'_1(t) + \left[\frac{1}{2} - a_0\right]P'_2(t) + [a_0 - a_1 + a_2]P'_3(t)$$

$P_0(x) = 1\,, P_1(x) = x, P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}, \; P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$
$P'_0(x) = 0\,, P'_1(x) = 1, P'_2(x) = 3x, \; P'_3(x) = 3(\frac{5}{2})x^2 - \frac{3}{2}$
$P'_1(x) = P_0(x), P'_2(x) = 3P_1(x), \; P'_3(x) = 5\left(\frac{3}{2}\right)x^2 - \frac{5}{2} + 1 = 5P_2(x) + 1$

$$\tilde{y}'(t) = 0 + a_1P_0(x) + \left[\frac{1}{2} - a_0\right]3P_1(x) + [a_0 - a_1 + a_2][5P_2(x) + 1]$$

$$\tilde{y}''(t) = a_1 P'_0(x) + 3\left[\frac{1}{2} - a_0\right] P'_1(x) + [a_0 - a_1 + a_2][5P'_2(x) + 1]$$

$$\tilde{y}''(t) = 0 + 3\left[\frac{1}{2} - a_0\right] + [a_0 - a_1 + a_2][5(3)P_1(x) + 1]$$

$$\tilde{y}''(t) = 3\left[\frac{1}{2} - a_0\right] + [a_0 - a_1 + a_2][15P_1(x) + 1]$$

$$\tilde{y}''^{(t)} = 3\left[\frac{1}{2} - a_0\right] + 15[a_0 - a_1 + a_2]P_1(x) + [a_0 - a_1 + a_2]$$

$$\tilde{y}''(t) = 15[a_0 - a_1 + a_2]P_1(x) + \left[-2a_0 - a_1 + a_2 + \frac{3}{2}\right]$$

We define the error in approximating the differential problem (1)

$\boldsymbol{y''(t) = 2}$

Is

$$R(t) = y''(t) - 2 = 15[a_0 - a_1 + a_2]P_1(x) + \left[-2a_0 - a_1 + a_2 + \frac{3}{2}\right] - 2$$

$$R(t) = 15[a_0 - a_1 + a_2]P_1(x) + \left[-2a_0 - a_1 + a_2 - \frac{1}{2}\right]$$

Galerkin method obtain the unknowns by

$$\int_a^b w(t)R(t)\,P_j(t)dt = 0, j = 0,1,2$$

$w(t) = 1$

$$\int_a^b \left(15[a_0 - a_1 + a_2]P_1(x) + \left[-2a_0 - a_1 + a_2 - \frac{1}{2}\right]\right) P_j(t)dt = 0, j = 0,1,2$$

<u>Trigonometric Polynomial Approximation</u>

Trigonometric Polynomial consists of all linear combination of the set $\{\phi_0, \phi_1, \ldots, \phi_{2n-1}\}$, where

$$\phi_0(x) = \frac{1}{\sqrt{2\pi}}$$

$$\phi_k(x) = \frac{1}{\sqrt{\pi}} \cos kx \quad \text{for each } k = 1, 2, \ldots, n$$

$$\phi_{n+k}(x) = \frac{1}{\sqrt{\pi}} \sin kx \quad \text{for each } k = 1, 2, \ldots, n-1$$ Which are orthonormal in $[-\pi, \pi]$

Theorem: If $f(x)$ If is approximated by trigonometric polynomial, by

نظرية:إذا تم تقريب الدالة $f \in C[-\pi,\pi]$ باستخدام تقريب كثيرات الحدود الثلاثية المثلثية

$$S_n(x)=\sum_{k=0}^{2n-1} a_k \phi_k(x)$$

Then

فإن

$$a_k=\int_{-\pi}^{\pi} f(x)\phi_k(x)dx \quad for\,each \quad k=0,1,\ldots,2n-1$$

مثال:

Example:

Consider that the function

إذا تم تقريب الدالة

$$f(x)=|x| \quad \text{for} \quad -\pi<x<\pi$$

is approximated by trigonometric polynomial. Obtain the unknown coefficients.

باستخدام تقريب كثيرات الحدود الثلاثية المثلثية أوجد صيغ المعاملات

Answer:

$$a_0=\int_{-\pi}^{\pi}|x|\frac{1}{\sqrt{2\pi}}dx=-\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{0}xdx+\frac{1}{\sqrt{2\pi}}\int_{0}^{\pi}xdx$$

$$=\frac{2}{\sqrt{2\pi}}\int_{0}^{\pi}xdx=\frac{\sqrt{2}\pi^2}{2\sqrt{\pi}}$$

$$=\sqrt{2}\pi^{\frac{3}{2}}$$

$$a_k=\frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi}|x|\cos kx dx=\frac{2}{\sqrt{\pi}}\int_{0}^{\pi}x\cos kx dx$$

$$=\frac{2}{\sqrt{\pi}k^2}\left[(-1)^k-1\right], \quad \text{for each} \quad k=1,2,\ldots,n$$

صيغ المعاملات بعد الحد رقم n سوف نسميها b_k:

$$b_k=a_{n+k} \quad \text{for} \quad k=1,2,\ldots,n-1$$

$$b_k=\frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi}|x|\sin kx dx=\frac{2\sqrt{\pi}}{k}(-1)^{k+1} \quad \text{for each } k=1,2,..,n-1$$

وبناءاً على ذلك فإن

$$S_n(x)=\frac{\pi}{2}+\frac{2}{\pi}\sum_{k=0}^{n}\frac{(-1)^k-1}{k^2}\cos kx+2\sum_{k=1}^{n-1}\frac{(-1)^{k+1}}{k}\sin kx$$

HomeWork

مثال:

أوجد تقريب كثيرات الحدود الثلاثية المثلثية $S_n(x)$ للدالة

Consider that the function

إذا تم تقريب الدالة

$$f(x)=\begin{cases}0 & if\ -\pi<x\le 0\\ 1 & if\ 0<x\le\pi\end{cases}$$

is approximated by trigonometric polynomial. Obtain the unknown coefficients.

الحل:

إذا تم تقريب الدالة المعطاة باستخدام تقريب كثيرات الحدود الثلاثية المثلثية

$$S_n(x)=\sum_{k=0}^{2n-1}a_k\phi_k(x)=a_0+\sum_{k=0}^{n}a_k\phi_k+\sum_{k=0}^{n-1}b_k\phi_k$$

حيث

$$\phi_0(x)=\frac{1}{\sqrt{2\pi}}$$

$$\phi_k(x)=\frac{1}{\sqrt{\pi}}\cos kx \quad \text{for each } k=1,2,\dots,n$$

$$\phi_{n+k}(x)=\frac{1}{\sqrt{\pi}}\sin kx \quad \text{for each } k=1,2,\dots,n-1$$

فإن صيغ المعاملات تتعين من

$$a_k=\int_{-\pi}^{\pi}f(x)\phi_k(x)dx,\quad k=0,1,\dots,2n-1$$

من تعريف الدالة (1) فإن

$$a_0=\int_{-\pi}^{\pi}f(x)\frac{1}{2\sqrt{\pi}}dx=\int_{-\pi}^{0}\frac{0}{\sqrt{2\pi}}dx+\int_{0}^{\pi}\frac{dx}{\sqrt{2\pi}}$$

$$=0+\frac{x\big|_0^{\pi}}{\sqrt{2\pi}}=\frac{\pi}{\sqrt{2\pi}}$$

$$=\frac{1}{\sqrt{\pi}}\int_{-\pi}^{0}0\cos kx\,dx+\frac{1}{\sqrt{\pi}}\int_{0}^{\pi}\cos kx\,dx$$

$$=0+\frac{1}{\sqrt{\pi}}\frac{-\sin kx\big|_0^{\pi}}{k}=\frac{-1}{\sqrt{\pi}k}\left[\sin k\pi-\sin 0\right]=0$$

$$b_0 = \frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} f\left(x\right)\sin kx dx$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\pi}^{0} 0*\sin kx dx + \frac{1}{\sqrt{\pi}}\int_{0}^{\pi} 1*\sin kx$$

$$= 0 + \frac{1}{\sqrt{\pi}}\cos kx \Big|_0^{\pi} = \frac{1}{\sqrt{\pi}k}\left[\cos k\pi - \cos 0\right]$$

$$= \frac{\left(-1\right)^k - 1}{\sqrt{\pi}k}$$

$$S_n = a_0\phi_0 + \sum_{k=0}^{n} a_k\phi_k + \sum_{k=0}^{n-1} b_k\phi_k$$

$$= \frac{n}{\sqrt{2n}}\frac{1}{\sqrt{2n}} + \sum \frac{\left[\left(-1\right)^k - 1\right]}{\sqrt{n}k}\frac{1}{\sqrt{n}}\sin kx$$

$$= \frac{n}{\sqrt{2n}} + \sum \frac{\left[\left(-1\right)^k - 1\right]}{nk}\sin kx = \frac{1}{2} + \frac{1}{n}\sum_{k=0}^{n-1}\frac{\left(-1\right)^k - 1}{k}\sin kx$$

Exercises

[1] Test which of the following pairs of functions are orthogonal /orthonormal in the given interval:
(a) $\sin x$, $\cos x$, $[-1,1]$.
(b) $1, x$, $[-1,1]$.
[2] Test if the following set of functions are orthogonal /orthonormal in the given interval:
(a) $\left\{\sin\frac{m\pi x}{\tau}\right\}_{m=1}^{M}$, $[-1,1]$.
(b) $\{1, x, x^2, \ldots\}$, $[-1,1]$.

$$\int_{-\tau}^{\tau} \boldsymbol{\cos}\left(\frac{k\pi x}{\tau}\right) \boldsymbol{dx} = \frac{\tau}{k\pi}\left[\boldsymbol{\sin}\left(\frac{k\pi x}{\tau}\right)\right]_{-\tau}^{\tau} = \frac{\tau}{k\pi}\left[\boldsymbol{\sin}\left(\frac{k\pi\tau}{\tau}\right) - \boldsymbol{\sin}\left(-\frac{k\pi\tau}{\tau}\right)\right]$$
$$= \frac{\tau}{k\pi}\left[\boldsymbol{\sin}(k\pi) + \boldsymbol{\sin}(k\pi)\right] = \frac{\tau}{k\pi}\left[\boldsymbol{0} + \boldsymbol{0}\right] = \boldsymbol{0}$$

[1] Test which of the following pairs of functions are orthogonal /orthonormal in the given interval:
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$$\int_{-\tau}^{\tau} \boldsymbol{cos}\left(\frac{k\pi x}{\tau}\right) \boldsymbol{dx} \;= \frac{\tau}{k\pi}\left[\boldsymbol{sin}\left(\frac{k\pi x}{\tau}\right)\right]_{-\tau}^{\tau} = \frac{\tau}{k\pi}\left[\boldsymbol{sin}\left(\frac{k\pi\tau}{\tau}\right) - \boldsymbol{sin}\left(-\frac{k\pi\tau}{\tau}\right)\right]$$
$$= \frac{\tau}{k\pi}[\boldsymbol{sin}(k\pi) + \boldsymbol{sin}(k\pi)] \quad = \frac{\tau}{k\pi}[\mathbf{0} + \mathbf{0}] \quad = \mathbf{0}$$

[Exercise 1]The following recurrence relation generates the Legendre polynomials.

$$P_{k+1}(x) = \frac{(2k+1)}{(k+1)} x P_k(x) - \frac{k}{(k+1)} P_{k-1}(x), \;\; k = 1,2, \ldots$$

starting with

$P_0(x) = 1$ and $P_1(x) = x$.

(a) Obtain the first five members of the polynomials $\mathrm{P}_k\ (x)$,

(b) obtain the value of $\mathrm{P}_4\ (0.5)$.

(c) if $f(x) = \sum_{k=o}^{4} \alpha_k \mathrm{P}_k\ (x)$, with $\alpha = [1,0,0,1,0]$. Obtain the expression for $f(x)$, and the value of $f(0.5)$.

[Exercise 2]The following recurrence relation generates the Chebyshev polynomials.

$$T_n(x) \;=\; 2x\,T_{n-1}(x) \;-\; T_{n-2}(x)\,,$$

starting with

$$T_0(x) = 1 \;\text{ and }\; T_1(x) = x.$$

(a) Obtain the first five members of the polynomials $\mathrm{T}_k\ (x), k = 0,1, \ldots ,4$

(b) obtain the value of $\mathrm{P}_4\ (0.75)$.

(c) if $f(x) = \sum_{k=o}^{4} \alpha_k \mathrm{T}_k\ (x)$, with $\alpha = [0,1,1,0,1]$. Obtain the expression for $f(x)$, and the value of $f(-0.5)$.

[Exercise 1]The following recurrence relation generates the Legendre polynomials.

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Chapter 6

Spline Interpolation

Spline Interpolation_(linear)

[Q11] Obtain linear spline function that interpolate the given data: $f(0) = 0$

, $f(1) = 1$ $f(2) = 2$ and satisfies the conditions $f'(0) = 1$, $f'(2) = 1$

Solution:

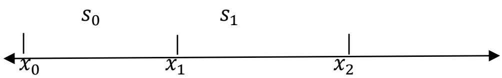
Consider the data : $\{x_i \ , f(x_i)\}_{i=0}^{n}$ is given

i	x_i	$f(x)$
0	0	0
1	1	1
2	2	2

To obtain an interpolated function for this data, we construct , firstly the linear spline function .

$$s_j(x) = a_j + b_j(x - x_j) \ , \text{j=0,1.} \qquad (1)$$

in each subinterval.



$s_0 = a_0 + b_0(x - x_0) = a_0 + b_0 x \qquad (2)$

Since $x_0 = 0$, similarly, since $x_1 = 1$, we have

$s_1(x) = a_1 + b_1(x - x_1) = a_1 + b_1(x - 1) \quad (3)$

we have 4 unknowns a_0 , a_1 , b_0 , b_1 to be determined

Spline function must satisfy:

$s_j(x_j) = f(x_j) \qquad (4)$

بالتطبيق في المعادلتين (2) و (3)

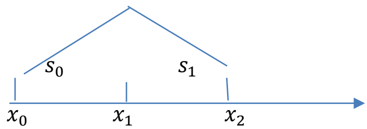
$s_0(x_0) = f(x_0) \rightarrow \ a_0 + b_0 x0 = 0 \ \rightarrow \ a_0 = 0$

$s_1(x_1) = f(x_1) \rightarrow \ a_1 + b_1(x1 - 1) = 1 \ \rightarrow \ a_1 = 1$

$$s_{j+1}(x_{j+1}) = s_j(x_{j+1}) \qquad (5)$$

$s_1(x_1) = s_0(x_1)$

Condition (5) is applied to ensure that the resulting spline function is continuous



$s_1(x_1) = s_0(x_1)$

بالتطبيق في المعادلتين (2) و (3)

$s_0 = a_0 + b_0 x \quad (2) \qquad s_1(x) = a_1 + b_1(x-1) \quad (3) \;,\; x_1 = 1$

$a_0 + b_0 = a_1$

$0 + b_0 = 1$

$b_0 = 1$

One more condition must be given for define spline interpolation that's

$$s_0{}'(x_0) = f'(x_0) \;, \qquad s_1{}'(x_2) = f'(x_2) \qquad (6)$$

From the given condition:

$s_0{}'(0) = 1 \;,\; s_1{}'(2) = 1$

بالتعويض في (2) و (3) بعد التفاضل

Differentiation (2,3)

$s_0{}' = b_0 = 1 \,, \qquad s_1{}' = b_1 = 1$

Hence the spline function that interpolate the given data is

$s_0 = a_0 + b_0(x - x_0) = a_0 + b_0 x \qquad s_1(x) = a_1 + b_1(x - x_1) = a_1 + b_1(x-1)$

$s_0(x) = x$

$s_1(x) = 1 + (x-1) = x$

Which can be written as:

$$s(x) = \begin{cases} s_0 = x \,, & 0 \le x \le 1 \\ s_1 = x, & 1 \le x \le 2 \end{cases} \Rightarrow \qquad s(x) = \begin{cases} x \,, & 0 \le x \le 1 \\ x, & 1 \le x \le 2 \end{cases}$$

The exact solution for this problem is :

$\boldsymbol{f(x) = x}$

The resulting interpolation is equal the exact solution.

$\boldsymbol{x}$	$s(x)$	$\boldsymbol{f(x)}$	Absolute error
0.0		**0.0**	
0.2		**0.2**	
0.4		**0.4**	
2.0			

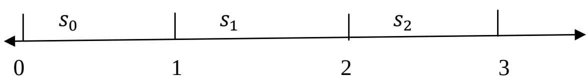
[Q12] Obtain quadratic spline function that interpolates the given data:

$f(0) = 0$, $f(1) = 1$, $f(2) = 2, f(3) = 3$ and satisfies the condition $f'(0) = 1$, $f'(3) = 1$

Solution :

The spline function (quadratic) is

$$s_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 \qquad j = 0,1,2, \qquad (1)$$



$s_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2$

$$s_0(x) = a_0 + b_0 x + c_0 x^2 \qquad (2)$$

$s_1(x) = a_1 + b_1(x - x_1) + c_1(x - x_1)^2$

$$s_1(x) = a_1 + b_1(x - 1) + c_1(x - 1)^2 \qquad (3)$$

$s_2(x) = a_2 + b_2(x - x_2) + c_2(x - x_2)^2$

$$s_2(x) = a_2 + b_2(x - 2) + c_2(x - 2)^2 \qquad (4)$$

The first condition spline function must satisfying

$$s_j(x_j) = f(x_j)$$

$s_0(x_0) = f(x_0)$

$s_1(x_1) = f(x_1)$

$s_2(x_2) = f(x_2)$

بالتطبيق في المعادلتين (2) و (3) و (4)

$s_0(x) = a_0 + b_0 x + c_0 x^2, \; f(x_0) = 0$

$s_0(x_0) = f(x_0)$ بالتعويض في (2) $a_0 = 0$

$s_1(x) = a_1 + b_1(x-1) + c_1(x-1)^2 \;\; , f(x_1) = 1$

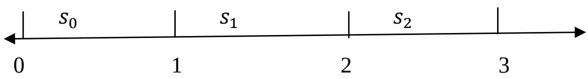
$s_1(x_1) = f(x_1)$ بالتعويض في (3) $a_1 = 1$

$s_2(x) = a_2 + b_2(x-2) + c_2(x-2)^2 \;\; , f(x_2) = 2$

$s_2(x_2) = f(x_2) \rightarrow a_2 = 2$

The second condition to ensure continues spline function at entire points

$$s_{j+1}(x_{j+1}) = s_j(x_{j+1})$$



$s_1(x_1) = s_0(x_1) \qquad (5)$

$s_1(1) = s_0(1)$

$s_0(x) = a_0 + b_0 x + c_0 x^2 \qquad s_1(x) = a_1 + b_1(x-1) + c_1(x-1)^2$

$a_0 + b_0 + c_0 = a_1 \rightarrow 0 + b_0 + c_0 = 1 \rightarrow b_0 + c_0 = 1 \qquad (6)$

$s_1(2) = s_2(2)$

$s_1(x) = a_1 + b_1(x-1) + c_1(x-1)^2 \;\; s_2(x) = a_2 + b_2(x-2) + c_2(x-2)^2 \;\;,$

$a_1 + b_1 + c_1 = a_2 \rightarrow \;\; 1 + b_1 + c_1 = 2 \;\; \rightarrow b_1 + c_1 = 1 \; (7)$

The given condition is at the end points of the interval

$f'(0) = 1,$ So:

$s_0(x) = a_0 + b_0 x + c_0 x^2$ ➔ $s'_0(x) = b_0 + 2c_0 x$■ ➔ $s'_0(0) = 1$ ➔

$$b_0 = 1$$

==========

$$f'(3) = 1$$

$$s_2(x) = a_2 + b_2(x-2) + c_2(x-2)^2$$

$s_2{}' = b_2 + 2c_2(x-2)$

$s_2{}'(3) = b_2 + 2c_2(3-2) = 1$

$= b_2 + 2c_2 = 1 \quad (8)$

بالتعويض من b_0 في (6)

$b_0 + c_0 = 1 \quad ==> 1 + c_0 = 1$ ➔ $c_0 = 0$

since the spline function is quadratic one more condition must be added to the linear spline case

we must insure the continuity of first derivative of spline function at

$s_0{}'(x_1) = s_1{}'(x_1)$

$s_0{}' = b_0 + 2c_0 x \quad , \quad x = x_1 = 1$

$s_1{}' = b_1 + 2c_1(x-1) \quad , \quad x = x_1 = 1$

$b_1 = b_0 + 2c_1$

$b_1 = 1 + 2c_1 \quad (9)$

(7), (9) must be solved to obtain b_1 , c_1

$b_1 + c_1 = 1$ (7)

So :

$2b_1 = 2 \Rightarrow b_1 = 1$

بالتعويض في (9)

$1 = 1 + 2c_1 \Rightarrow c_1 = 0$

There are 4 more unknowns, and 1 more condition, that is

$s_1{}'(x_2) = s_2{}'(x_2), x_2 = 2$

$s_1(x) = a_1 + b_1(x-1) + c_1(x-1)^2$ ➔ $s'_1(x) = b_1 + 2c_1(x-1)$

$$s'_1(2) = b_1 + 2c_1(2-1) \quad = b_1 + 2c_1 \quad (*)$$

$$s_2(x) = a_2 + b_2(x-2) + c_2(x-2)^2 \rightarrow s'_2(x) = b_2 + 2c_2(x-2)$$

$$s'_2(2) = b_2 + 2c_2(2-2) \quad = b_2 \quad (**)$$

$$s_1'(x_2) = s_2'(x_2), x_2 = 2 \rightarrow b_1 + 2c_1 = b_2 \rightarrow b_2 = 1 \ (9)$$

Together with $b_2 + 2c_2 = 1$ (7) $\rightarrow c_2 = 0$

Hence the quadratic spline function that approximate the given data is

$$s_0(x) = a_0 + b_0 x + c_0 x^2 \qquad (2)$$

$$s_1(x) = a_1 + b_1(x-1) + c_1(x-1)^2 \qquad (3)$$

$$s_2(x) = a_2 + b_2(x-2) + c_2(x-2)^2 \qquad (4)$$

$$s_0(x) = a_0 + b_0(x) + c_0(x)^2 = x$$

$$s_1(x) = a_1 + b_1(x-1) + c_1(x-1)^2 = 1 + x - 1 = x$$

$$s_2(x) = a_2 + b_2(x-2) + c_2(x-2)^2 = 2 + (x-2) = x$$

Which is the exact solution

==

[HW_Q13] Obtain quadratic spline function that interpolate the given data:

$f(0) = 0$, $f(1) = 1$, $f(2) = 4$

And the additional condition $f'(0) = 0$, $f'(3) = 6$

With the exact solution = x^2

==

Chapter 7

Eigen value Problem

Eigen value Problem

Use usual analytic solution to obtain all of the eigenvalues and the corresponding eigenvectors of the matrix , given by

$$A = \begin{bmatrix} 8 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 13 \end{bmatrix}$$

Then Write the algorithm of your favorite numerical method to approximate the largest eigenvalue of the matrix and the corresponding eigenvectors.

The eigenvalues can be found by expanding $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ and finding the roots of the resulting nth-order polynomial, which is called the *characteristic equation*. This procedure is illustrated in the following discussion.

Consider the dynamic spring-mass problem specified by Eq. (2.9):

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \begin{bmatrix} (8-\lambda) & -2 & -2 \\ -2 & (4-\lambda) & -2 \\ -2 & -2 & (13-\lambda) \end{bmatrix} \mathbf{x} = 0 \tag{2.28}$$

The characteristic equation, $|\mathbf{A} - \lambda\mathbf{I}| = 0$, is

$$(8-\lambda)\,[(4-\lambda)(13-\lambda) - 4] - (-2)\,[(-2)(13-\lambda) - 4] + (-2)[4 + 2(4-\lambda)] = 0 \tag{2.29}$$

$$\lambda^3 - 25\lambda^2 + 176\lambda - 300 = 0 \tag{2.30}$$

The eigenvalues are

$$\lambda = 13.870585, \quad 8.620434, \quad 2.508981 \tag{2.31}$$

$$\boxed{(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \begin{bmatrix} (8-\lambda) & -2 & -2 \\ -2 & (4-\lambda) & -2 \\ -2 & -2 & (13-\lambda) \end{bmatrix} \mathbf{x} = 0$$

$$(8-\lambda)X_1 - 2X_2 - 2X_3 = 0 \tag{2.9a}$$

$$-2X_1 + (4-\lambda)X_2 - 2X_3 = 0 \tag{2.9b}$$

$$-2X_1 - 2X_2 + (13-\lambda)X_3 = 0 \tag{2.9c}$$

The eigenvectors corresponding to λ_1 to λ_3 are determined as follows. For each eigenvalue $\lambda_i (i = 1, 2, 3)$, find the amplitudes X_2 and X_3 relative to the amplitude X_1 by letting $X_1 = 1.0$. Any two of the three equations given by Eq. (2.9) can be used to solve for X_2 and X_3 with $X_1 = 1.0$. From Eqs. (2.9a) and (2.9c),

$$(8 - \lambda)X_1 - 2X_2 - 2X_3 = 0 \quad (2.33a)$$

$$-2X_1 - 2X_2 + (13 - \lambda)X_3 = 0 \quad (2.33b)$$

$$8 - \lambda - 2x_2 - 2x_3 = 0$$

$$-2 - 2x_2 + (13 - \lambda)x_3 = 0$$

Solving Eqs. (2.33a) and (2.33b) for X_3 and substituting that result in Eq. (2.33a) yields

$$X_3 = \frac{(10 - \lambda)}{15} - \lambda \quad \text{and} \quad X_2 = \frac{(8 - \lambda)}{2} - X_3 \quad (2.33c)$$

Substituting λ_1 to λ_3 into Eq. (2.33c) yields:

For $\lambda_1 = 13.870586$:

$$\mathbf{X}_1 = [1.000000 \quad 0.491779 \quad -3.427072] \quad (2.34a)$$

For $\lambda_2 = 8.620434$:

$$\mathbf{X}_2 = [1.000000 \quad -0.526465 \quad 0.216247] \quad (2.34b)$$

For $\lambda_3 = 2.508981$:

$$\mathbf{X}_3 = [1.000000 \quad 2.145797 \quad 0.599712] \quad (2.34c)$$

Definition 7.1 A **vector norm** on $\mathbb{R}^n$ is a function, $\|\cdot\|$, from $\mathbb{R}^n$ into $\mathbb{R}$ with the following properties:

(i) $\|\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$,

(ii) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$,

(iii) $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for all $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$,

(iv) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. ∎

Definition 7.2 The l_2 and l_∞ norms for the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ are defined by

$$\|\mathbf{x}\|_2 = \left\{\sum_{i=1}^{n} x_i^2\right\}^{1/2} \quad \text{and} \quad \|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

The l_2 norm is called the **Euclidean norm** of the vector $\mathbf{x}$ since it represents the usual notion of distance from the origin in case $\mathbf{x}$ is in $\mathbb{R}^1 \equiv \mathbb{R}$, $\mathbb{R}^2$, or $\mathbb{R}^3$. For example, the l_2 norm of the vector $\mathbf{x} = (x_1, x_2, x_3)^t$ gives the length of the straight line joining the points $(0, 0, 0)$ and (x_1, x_2, x_3). Figure 7.1 shows the boundary of those vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ that have l_2 norm less than 1. Figure 7.2 is a similar illustration for the l_∞ norm.

Definition

If A is a square matrix n on n , the characteristic polynomial of A is defined by

$$p(\lambda) = \det(A - \lambda I).$$

It is not difficult to show that p is an nth-degree polynomial and, consequently, has at most n distinct zeros, If λ is a zero of p, then, since det(A — λ I) = 0, Theorem 6.16 in Section 6.4 implies that the linear system defined by (A — λ l)x = 0 has a solution with x ≠ 0.

Definition

if p is the characteristic polynomial of the matrix A, the zeros of p are eigenvalues, or characteristic values, of the matrix A. If λ is an eigenvalue of A and x ≠ 0 satisfies

(A — λ l)x = 0, then x is an eigenvector, or characteristic vector, of A corresponding to

the eigenvalue λ. ■

Definition

The spectral radius ρ (A) of a matrix A is defined by ρ (A) = max | λ |, where λ is an eigenvalue of A.

Example:

$$A = \begin{bmatrix} 8 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 13 \end{bmatrix}$$

$\lambda_1 = 13.87, \lambda_2 = 8.62, \lambda_3 = 2.5$

ρ(A) = max{13.87, 8.62, 2.5 }=13.87

Theorem : If A is an n x n matrix, then

(i) $\|A\|_2 = [\rho(A^t A)]^{1/2}$,

(ii) $\rho(A) \leq \|A\|$, **for any natural norm** $\|\cdot\|$.

EXAMPLE 2

If $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$ Then obtain $||A||_2$

$$A^t A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 6 & 4 \\ -1 & 4 & 5 \end{bmatrix}.$$

To calculate $\rho(A^t A)$ we need the eigenvalues of $A^t A$. If

$$\begin{aligned} 0 &= \det(A^t A - \lambda I) \\ &= \det \begin{bmatrix} 3-\lambda & 2 & -1 \\ 2 & 6-\lambda & 4 \\ -1 & 4 & 5-\lambda \end{bmatrix} \\ &= -\lambda^3 + 14\lambda^2 - 42\lambda = -\lambda(\lambda^2 - 14\lambda + 42), \end{aligned}$$

then

$$\lambda = 0 \quad \text{or} \quad \lambda = 7 \pm \sqrt{7},$$

so

$$||A||_2 = \sqrt{\rho(A^t A)} = \sqrt{\max\{0, 7-\sqrt{7}, 7+\sqrt{7}\}} = \sqrt{7+\sqrt{7}} \approx 3.106. \quad \blacksquare$$

Definition

We call an nxn matrix A convergent if

$$\lim_{k\to\infty} (A^k)_{ij} = 0, \quad \text{for each } i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, n.$$

EXAMPLE 3

Let $A = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$. Is A is convergent matrix?

$$AA = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Computing powers of A, we obtain:

$$A^2 = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}, \quad A^3 = \begin{bmatrix} \frac{1}{8} & 0 \\ \frac{3}{16} & \frac{1}{8} \end{bmatrix}, \quad A^4 = \begin{bmatrix} \frac{1}{16} & 0 \\ \frac{1}{8} & \frac{1}{16} \end{bmatrix},$$

and, in general,

$$A^k = \begin{bmatrix} (\frac{1}{2})^k & 0 \\ \frac{k}{2^{k+1}} & (\frac{1}{2})^k \end{bmatrix}.$$

Since

$$\lim_{k \to \infty} \left(\frac{1}{2}\right)^k = 0 \quad \text{and} \quad \lim_{k \to \infty} \frac{k}{2^{k+1}} = 0,$$

A is a convergent matrix.

بتفاضل البسط وحده والمقام وحده مع الأخذ في الاعتبار أن

إذا كانت $y = a^x$ فإن $\frac{dy}{dx} = a^x \ln a$

Theorem

The following statements are equivalent.

(i) A is a convergent matrix,

(ii) , for all natural norms,

(iv) ρ (A) < 1.

(V) , for every .

Approximating Eigenvalues

The Power Method

The Power method is an iterative technique used to determine the dominant eigenvalue of a matrix—that is, the eigenvalue with the largest magnitude. By modifying the method slightly, it can also used to determine other eigenvalues. One useful feature of the Power method is that it produces not only an eigenvalue, but an associated eigenvector. In fact, the Power method is often applied to find an eigenvector for an eigenvalue that is determined by some other means.

The Eigen value problem is: Find λ and X which satisfy:

$AX = \lambda X$Or

$$|A - \lambda I|X = 0$$

2.3.1. The Direct Power Method

When the largest (in absolute value) eigenvalue of $\mathbf{A}$ is distinct, its value can be found using an iterative technique called the *direct power method*. The procedure is as follows:

1. Assume a trial value $\mathbf{x}^{(0)}$ for the eigenvector $\mathbf{x}$. Choose one component of $\mathbf{x}$ to be unity. Designate that component as the *unity* component.
2. Perform the matrix multiplication:

$$\mathbf{A}\mathbf{x}^{(0)} = \mathbf{y}^{(1)} \tag{2.37}$$

3. Scale $\mathbf{y}^{(1)}$ so that the unity component remains unity:

$$\mathbf{y}^{(1)} = \lambda^{(1)}\mathbf{x}^{(1)} \tag{2.38}$$

4. Repeat steps 2 and 3 with $\mathbf{x} = \mathbf{x}^{(1)}$. Iterate to convergence. At convergence, the value λ is the largest (in absolute value) eigenvalue of $\mathbf{A}$, and the vector $\mathbf{x}$ is the corresponding eigenvector (scaled to unity on the unity component).

The general algorithm for the power method is as follows:

$$\boxed{\mathbf{A}\mathbf{x}^{(k)} = \mathbf{y}^{(k+1)} = \lambda^{(k+1)}\mathbf{x}^{(k+1)}} \tag{2.39}$$

When the iterations indicate that the unity component could be zero, a different unity component must be chosen. The method is slow to converge when the magnitudes (in absolute value) of the largest eigenvalues are nearly the same. When the largest eigenvalues are of equal magnitude, the power method, as described, fails.

Example 2.1. The direct power method.

Find the largest ,in absolute value, eigenvalue and the corresponding eigenvector of the matrix (to two digits of accuracy), given by

$$A = \begin{bmatrix} 8 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 13 \end{bmatrix}$$

Answer

Assume $\mathbf{x}^{(0)T} = [1.0 \quad 1.0 \quad 1.0]$.

Scale the third component x_3 to unity.

Then apply Eq.

$$\boxed{\mathbf{Ax}^{(k)} = \mathbf{y}^{(k+1)} = \lambda^{(k+1)}\mathbf{x}^{(k+1)}}$$

$$Ax^{(0)} = y^{(1)}$$

$y^{(1)} = Ax^{(0)}$

$$\mathbf{Ax}^{(0)} = \begin{bmatrix} 8 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 13 \end{bmatrix} \begin{bmatrix} 1.0 \\ 1.0 \\ 1.0 \end{bmatrix} = \begin{bmatrix} 4.00 \\ 0.00 \\ 9.00 \end{bmatrix}$$

$$y^{(1)} = \begin{bmatrix} 4 \\ 0 \\ 9 \end{bmatrix}$$

$\lambda^{(1)} = 9.00$

$$y^{(1)} = \lambda^{(1)} x^{(1)}$$

$$\frac{1}{\lambda^{(1)}} y^{(1)} = x^{(1)}$$

$$\mathbf{x}^{(1)} = \begin{bmatrix} 0.444444 \\ 0.000000 \\ 1.000000 \end{bmatrix}$$

$$\boxed{\mathbf{A}\mathbf{x}^{(k)} = \mathbf{y}^{(k+1)} = \lambda^{(k+1)}\mathbf{x}^{(k+1)}}$$

$$\mathbf{A}\mathbf{x}^{(1)} = \begin{bmatrix} 8 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 13 \end{bmatrix} \begin{bmatrix} 0.444444 \\ 0.000000 \\ 1.000000 \end{bmatrix} = \begin{bmatrix} 1.555555 \\ -2.888888 \\ 12.111111 \end{bmatrix}$$

$$\lambda^{(2)} = 12.111111$$

$$\mathbf{x}^{(2)} = \begin{bmatrix} 0.128440 \\ -0.238532 \\ 1.000000 \end{bmatrix}$$

Table 2.1. The Power Method

k	λ	x_1	x_2	x_3
0		1.000000	1.000000	1.000000
1	9.000000	0.444444	0.000000	1.000000
2	12.111111	0.128440	−0.238532	1.000000
3	13.220183	−0.037474	−0.242887	1.000000
4	13.560722	−0.133770	−0.213602	1.000000
5	13.694744	−0.192991	−0.188895	1.000000
...	...	...	...	...
29	13.870583	−0.291793	−0.143499	1.000000
30	13.870584	−0.291794	−0.143499	1.000000

Homework:

The required answer is up to k=5

HomeWork: The direct power method.

Find the largest ,in absolute value, eigenvalue and the corresponding eigenvector of the matrix (to five iterations), given by

$$A = \begin{bmatrix} -4 & 14 & 0 \\ -5 & 13 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

Let $\mathbf{x}^{(0)} = (1, 1, 1)^t$, then

$$\mathbf{y}^{(1)} = A\mathbf{x}^{(0)} = (10, 8, 1)^t,$$

$$\mathbf{x}^{(1)} = \frac{\mathbf{y}^{(1)}}{10} = (1, 0.8, 0.1)^t.$$

…

Answer: $(1, 0.714316, -0.249895)$ 6.000837

Find the largest eigenvalue and the corresponding eigenvector of the following matrix to two digits of accuracy. Take first component be the unity component.

$$A = \begin{bmatrix} -4 & 14 & 0 \\ -5 & 13 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix}$$

Step 1 Set $k = 1$;
$\mathbf{x} = \mathbf{x}/\|\mathbf{x}\|_2$.

Step 2 While ($k \le N$) do Steps 3–8.

Step 3 Set $\mathbf{y} = A\mathbf{x}$.

Step 4 Set $\mu = \mathbf{x}^t\mathbf{y}$.

Step 5 If $\|\mathbf{y}\|_2 = 0$, then OUTPUT ('Eigenvector', $\mathbf{x}$);
OUTPUT ('A has eigenvalue 0, select new vector $\mathbf{x}$ and restart');
STOP.

Step 6 Set $ERR = \left\| \mathbf{x} - \frac{\mathbf{y}}{\|\mathbf{y}\|_2} \right\|_2$;
$\mathbf{x} = \mathbf{y}/\|\mathbf{y}\|_2$.

Step 7 If $ERR < TOL$ then OUTPUT (μ, $\mathbf{x}$);
(*The procedure was successful.*)
STOP.

Step 8 Set $k = k + 1$.

Step 9 OUTPUT ('Maximum number of iterations exceeded');
(*The procedure was unsuccessful.*)
STOP.

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix}$$

is symmetric with eigenvalues $\lambda_1 = 6, \lambda_2 = 3$, and $\lambda_3 = 1$. Table 9.2 lists the results the Power method, and the results in Table 9.3 come from the Symmetric Power method, assuming in each case that $\mathbf{y}^{(0)} = \mathbf{x}^{(0)} = (1, 0, 0)^t$. Notice the significant improvement that the Symmetric Power method provides. The approximations to the eigenvectors produced in the Power method converge to $(1, -1, 1)^t$, a vector with $\|(1, -1, 1)^t\|_\infty = 1$. In the Symmetric Power method, the convergence is to the parallel vector $(\sqrt{3}/3, -\sqrt{3}/3, \sqrt{3}/3)^t$, with $\|(\sqrt{3}/3, -\sqrt{3}/3, \sqrt{3}/3)^t\|_2 = 1$. ∎

Table 9.3

m	$(\mathbf{y}^{(m)})^t$	$\mu^{(m)}$	$\hat{\mu}^{(m)}$	$(\mathbf{x}^{(m)})^t$ with $\|\mathbf{x}^{(m)}\|_2 = 1$
0	(1, 0, 0)			(1, 0, 0)
1	(4, −1, 1)	4	7	(0.942809, −0.235702, 0.235702)
2	(4.242641, −2.121320, 2.121320	5	6.047619	(0.816497, −0.408248, 0.408248)
3	(4.082483, −2.857738, 2.857738)	5.666667	6.002932	(0.710669, −0.497468, 0.497468)
4	(3.837613, −3.198011, 3.198011)	5.909091	6.000183	(0.646997, −0.539164, 0.539164)
5	(3.666314, −3.342816, 3.342816)	5.976744	6.000012	(0.612836, −0.558763, 0.558763)
6	(3.568871, −3.406650, 3.406650)	5.994152	6.000000	(0.595247, −0.568190, 0.568190)
7	(3.517370, −3.436200, 3.436200)	5.998536	6.000000	(0.586336, −0.572805, 0.572805)
8	(3.490952, −3.450359, 3.450359)	5.999634		(0.581852, −0.575086, 0.575086)
9	(3.477580, −3.457283, 3.457283)	5.999908		(0.579603, −0.576220, 0.576220)
10	(3.470854, −3.460706, 3.460706)	5.999977		(0.578477, −0.576786, 0.576786)

$\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\|A\mathbf{x} - \lambda\mathbf{x}\|_2 < \varepsilon$

$$\min_{1 \le j \le n} |\lambda_j - \lambda| < \varepsilon.$$