محاضرات في الرياضيات "جبـر وهنـدسة الفراغية"

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THEORY OF EQUATIONS

1.0 Introduction

In this module, we will study about polynomial functions and various methods to find out the roots of polynomial equations. 'Solving equations' was an important problem from the beginning of study of Mathematics itself. The notion of complex numbers was first introduced because equations like $x^2 + 1 = 0$ has no solution in the set of real numbers. The "fundamental theorem of algebra" which states that every polynomial of degree ≥ 1 has at least one zero was first proved by the famous German Mathematician Karl Fredrich Gauss. We shall look at polynomials in detail and will discuss various methods for solving polynomial equations.

1.1. Polynomial Functions

Definition:

A function defined by

$$f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n, \quad where \quad a_o \neq 0, \text{ n is a non negative}$$

integer and a_i (i = 0, 1….,n) are fixed complex numbers is called a **polynomial** of degree n in x. Then numbers $a_o, a_1, \ldots, a_n$ are called the **coefficients** of f.

If α is a complex number such that $f(\alpha) = 0$, then α is called **zero** of the polynomial.

1.1.1 Theorem (Fundamental Theorem of Algebra)

Every polynomial function of degree $n \geq 1$ has at least one zero.

Remark:

Fundamental theorem of algebra says that, if $f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n$,

where $a_0 \neq 0$ is the given polynomial of degree $n \geq 1$, then there exists a complex number α such that $a_0 \alpha^n + a_1 \alpha^{n-1} + \ldots + a_n = 0$.

We use the Fundamental Theorem of Algebra, to prove the following result.

1.1.2 Theorem

Every polynomial of degree n has n and only n zeroes.

Proof:

Let $f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n$, where $a_o \neq 0$, be a polynomial of degree $n \geq 1$.

By fundamental theorem of algebra, f(x) has at least one zero, let α_1 be that zero.

Then $(x - \alpha_1)$ is a factor of f(x).

Therefore, we can write:

$f(x) = (x - \alpha_1)Q_1(x)$, where $Q_1(x)$ is a polynomial function of degree n - 1.

If n -1 ≥ 1, again by Fundamental Theorem of Algebra, $Q_1(x)$ has at least one zero,

say α_2.

Therefore, $f(x) = (x - \alpha_1)(x - \alpha_2)Q_2(x)$ where $Q_2(x)$ is a polynomial function of

degree n - 2.

Repeating the above arguments, we get

$f(x) = (x - \alpha_1)(x - \alpha_2)\dots(x - \alpha_n)Q_n(x)$, where $Q_n(x)$ is a polynomial function

of degree n - n = 0, i.e., $Q_n(x)$ is a constant.

Equating the coefficient of x^n on both sides of the above equation, we get

$Q_n(x) = a_o$.

Therefore, $f(x) = a_o(x - \alpha_1)(x - \alpha_2)\dots(x - \alpha_n)$.

If α is any number other than $\alpha_1, \alpha_2, \dots \alpha_n$, then $f(x) \neq 0 \Rightarrow \alpha$ is not a zero of f(x).

Hence f(x) has n and only n zeros, namely $\alpha_1, \alpha_2 \dots, \alpha_n$.

Note:

Let $f(x) = a_o x^n + a_1 x^{n-1} + \dots + a_n; a_o \neq 0$ be an nth degree polynomial in x.

Then, $a_o x^n + a_1 x^{n-1} + \dots + a_n = 0$ --------- (1)

is called a **polynomial equation** in x of degree n.

A number α is called a **root** of the equation (1) if α is a zero of the polynomial f(x).

Hence every polynomial equation of degree n has n and only n roots.

Solved Problems

1. Solve $x^4 - 4x^2 + 8x + 35 = 0$, given $2 + i\sqrt{3}$ is a root.

Solution :

Given that $2 + i\sqrt{3}$ is a root of $x^4 - 4x^2 + 8x + 35 = 0$; since complex roots occurs in conjugate pairs $2 - i\sqrt{3}$ is also a root of it.

$$\Rightarrow [x - (2 + i\sqrt{3})][x - (2 - i\sqrt{3})] = (x - 2)^2 + 3 = x^2 - 4x + 7$$ is a factor of the given polynomial.

Dividing the given polynomial by this factor, we obtain the other factor as $x^2 + 4x + 5$.

The roots of $x^2 + 4x + 5 = 0$ are given by $\dfrac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i$.

Hence the roots of the given polynomial are $2 + i\sqrt{3}, \ 2 - i\sqrt{3}, \ -2 + i \ and \ -2 - i$.

2. Solve $x^4 - 5x^3 + 4x^2 + 8x - 8 = 0$, given that one of the roots is $1 - \sqrt{5}$.

Solution:

Since quadratic surds occur in conjugate pairs as roots of a polynomial equation, $1 + \sqrt{5}$ is also a root of the given polynomial.

$$\Rightarrow [x - (1 - \sqrt{5})][x - (1 + \sqrt{5})] = (x - 1)^2 - 5 = x^2 - 2x - 4$$ is a factor.

Dividing the given polynomial by this factor, we obtain the other factor as $x^2 - 3x + 2$.

Also, $x^2 - 3x + 2 = (x - 2)(x - 1)$

Thus the roots of the given polynomial equation are $1 + \sqrt{5}, 1 - \sqrt{5}, 1, 2$.

3. Find a polynomial equation of the lowest degree with rational coefficients having $\sqrt{3}$ and $1 - 2i$ as two of its roots.

Solution:

Since quadratic surds occur in pairs as roots, $-\sqrt{3}$ is also a root.

Since complex roots occur in conjugate pairs, $1 + 2i$ is also a root of the required polynomial equation. Therefore the desired equation is given by

$$(x - \sqrt{3})(x + \sqrt{3})(x - (1 - 2i))(x - (1 + 2i)) = 0$$

$$\text{i.e., } x^4 - 2x^3 + 2x^2 + 6x - 15 = 0$$

4. Solve $4x^5 + x^3 + x^2 - 3x + 1 = 0$, given that it has rational roots.

Solution:

Let $f(x) = 4x^5 + x^3 + x^2 - 3x + 1$.

By theorem (1.1.5.), any rational root $\dfrac{p}{q}$ (in its lowest terms) must satisfy the condition that, p is divisor of 1 and q is positive divisor of 4.

So the possible rational roots are $\pm 1, \pm \frac{1}{2}, \pm \frac{1}{4}$.

Note that f(-1) = 0, f ($\frac{1}{2}$) = 0. But f(1) ≠ 0, f($-\frac{1}{2}$) ≠ 0, f($\frac{1}{4}$) ≠ 0 and f($-\frac{1}{4}$) ≠ 0.

Since f(-1) = 0 and f ($\frac{1}{2}$) = 0, we see that (x + 1) and (x − $\frac{1}{2}$) are factors of the given polynomial. Also by factorizing , we find that

$$f(x) = (x - \tfrac{1}{2})(x + 1)(4x^3 - 2x^2 + 4x - 2)$$

Note that x = $\frac{1}{2}$ is a root of the third factor, if we divide $4x^3 - 2x^2 + 4x - 2$ by x − $\frac{1}{2}$,

we obtain $\quad$ f(x) $\quad = (x - \tfrac{1}{2})^2 (x + 1)(4x^2 + 4)$

$$= 4(x - \tfrac{1}{2})^2 (x + 1)(x^2 + 1)$$

Hence the roots of f(x) = 0, are $\frac{1}{2}, \frac{1}{2}, -1, \pm i$.

5. Solve $x^3 - x^2 - 8x + 12 = 0$, given that has a double root.

Solution:

Let $\quad$ f(x) = $x^3 - x^2 - 8x + 12$

Differentiating, we obtain:

$$f^1(x) = 3x^2 - 2x - 8.$$

Since the multiple roots of f(x) = 0 are also the roots of $f^1(x)$ = 0, the product of the factors corresponding to these roots will be the g.c.d of f(x) and $f^1(x)$. Let us find the g.c.d of f(x) and $f^1(x)$.

3x	$3x^2 - 2x - 8$	$x^3 - x^2 - 8x + 12$	
	$3x^2 - 6x$	3	
4	$4x - 8$	$3x^3 - 3x^2 - 24x + 36$	x
	$4x - 8$	$3x^3 - 2x^2 - 8x$	
0	0	$- x^2 - 16x + 36$	
		3	
		$- 3x^2 - 48x + 108$	- 1
		$- 3x^2 + 2x + 8$	
		-50 $\mid$ $- 50x + 100$	
		$x - 2$	

Therefore, g.c.d = (x − 2)

$\Rightarrow$ f(x) has a factor $(x - 2)^2$.

Also, $f(x) = (x - 2)^2 (x + 3)$

      Thus the roots are   2, 2, - 3 .

6. Show that the equation $x^3 + qx + r = 0$ has two equal roots if $27r^2 + 4q^3 = 0$.

Solution:

    Let $f(x) = x^3 + qx + r$ -------------- (1)

Differentiating, we obtain: $f^1(x) = 3x^2 + q$ ------------------ (2)

Given that $f(x) = 0$ has two equal roots, i.e., it has a double root, say α.

Then α is a root of both $f(x) = 0$ and $f^1(x) = 0$.

From the 2nd equation, we obtain $\alpha^2 = -q/3$

Now the first equation can be written as: $\alpha (\alpha^2 + q) + r = 0$

       i.e., $\alpha (-q/3 + q) + r = 0$ $\Rightarrow$ $\alpha = \dfrac{-3r}{2q}$

Squaring and simplifying, we obtain: $27r^2 + 4q^3 = 0$

Relation between the Roots and Coefficients of a Polynomial Equation

      Consider the polynomial function $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$, $a_0 \neq 0$

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of $f(x) = 0$.

Then we can write $f(x) = a_o (x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_n)$

Equating the two expressions for $f(x)$, we obtain:

$$a_o x^n + a_1 x^{n-1} + \ldots + a_n = a_o (x - \alpha_1) (x - \alpha_2)\ldots(x - \alpha_n)$$

Dividing both sides by a_0,

$$x^n + \left(\frac{a_1}{a_o}\right)x^{n-1} + \ldots + \left(\frac{a_n}{a_o}\right) = (x - \alpha_1) (x - \alpha_2)\ldots(x - \alpha_n)$$

$$= x^n - S_1 x^{n-1} + S_2 x^{n-2} - \ldots + (-1)^n S_n$$

where S_r stands for the sum of the products of the roots $\alpha_1, \ldots, \alpha_n$ taken r at a time.

Comparing the coefficients on both sides , we see that

$$S_1 = \frac{-a_1}{a_o}, \quad S_2 = \frac{a_2}{a_o}, \ldots \quad S_n = (-1)^n \frac{a_n}{a_o}.$$

Special Cases

If α and β are the roots of $ax^2 + bx + c = 0$, $(a \neq 0)$, then $\alpha + \beta = \dfrac{-b}{a}$ and $\alpha\beta = \dfrac{c}{a}$

If α and β and γ are the roots of $ax^3 + bx^2 + cx + d = 0$, $(a \neq 0)$, then $\alpha + \beta + \gamma = \dfrac{-b}{a}$,

and $\alpha\beta + \beta\gamma + \alpha\gamma = \dfrac{c}{a}$ and $\alpha\beta\gamma = \dfrac{-d}{a}$.

Examples:

1. If the roots of the equation $x^3 + px^2 + qx + r = 0$ are in arithmetic progression, show that $2p^3 - 9\,pq + 27r = 0$.

Solution:

Let the roots of the given equation be $a - d$, a, $a + d$.

Then $S_1 = a - d + a + a + d = 3a = -p \implies a = \dfrac{-p}{3}$

Since a is a root, it satisfies the given polynomial

$$\implies \left(\dfrac{-p}{3}\right)^3 + p\left(\dfrac{-p}{3}\right)^2 + q\left(\dfrac{-p}{3}\right) + r = 0$$

On simplification, we obtain $2p^3 - 9pq + 27r = 0$.

2. Solve $27x^3 + 42x^2 - 28x - 8 = 0$, given that its roots are in geometric progression.

Solution:

Let the roots be $\dfrac{a}{r}$, a, ar

Then, $\dfrac{a}{r}.a.ar = a^3 = \dfrac{8}{27} \implies a = \dfrac{2}{3}$

Since $a = \dfrac{2}{3}$ is a root, $\left(x - \dfrac{2}{3}\right)$ is a factor. On division, the other factor of the polynomial is $27x^2 + 60x + 12$.

Its roots are $\dfrac{-60 \pm \sqrt{60^2 - 4 \times 27 \times 12}}{2 \times 27} = \dfrac{-2}{9}$ or -2

Hence the roots of the given polynomial eqution are $\dfrac{-2}{9}$, -2, $\dfrac{2}{3}$.

3. Solve the equation $15x^3 - 23x^2 + 9x - 1 = 0$ whose roots are in harmonic progression.

Solution:

[Recall that if a, b, c are in harmonic progression, then $^1/_a$, $^1/_b$, $^1/_c$ are in arithmetic progression and hence $b = \dfrac{2ac}{a+c}$]

Let α, β, γ be the roots of the given polynomial.

Then $\alpha\beta + \beta\gamma + \alpha\gamma = \dfrac{9}{15}$ .... (1)

$$\alpha\beta\gamma = \dfrac{1}{15} \qquad \text{.......... (2)}$$

Since α, β, γ are in harmonic progression, $\beta = \dfrac{2\alpha\gamma}{\alpha+\gamma}$

$$\Rightarrow \alpha\beta + \beta\gamma = 2\alpha\gamma$$

Substitute in (1), $2\alpha\gamma + \alpha\gamma = \dfrac{.9}{15} \Rightarrow 3\alpha\gamma = \dfrac{9}{15}$

$$\Rightarrow \alpha\gamma = \dfrac{3}{15}.$$

Substitute in (2), we obtain $\dfrac{3}{15}\beta = \dfrac{1}{15}$

$$\Rightarrow \beta = \dfrac{1}{3} \text{ is a root of the given polynomial.}$$

Proceeding as in the above problem, we find that the roots are $\dfrac{1}{3}$, 1, $\dfrac{1}{5}$.

4. Show that the roots of the equation $ax^3 + bx^2 + cx + d = 0$ are in geometric progression, then $c^3a = b^3d$.

Solution:

Suppose the roots are $\dfrac{k}{r}$, k, kr

Then $\dfrac{k}{r}.k.kr = \dfrac{-d}{a}$

i.e., $k^3 = \dfrac{-d}{a}$

Since k is a root, it satisfies the polynomial equation,

$$ak^3 + bk^2 + ck + d = 0$$

$$a\left(\dfrac{-d}{a}\right) + bk^2 + ck + d = 0$$

$$\Rightarrow bk^2 + ck = 0 \quad \Rightarrow bk^2 = -ck$$

$$\Rightarrow (bk^2)^3 = (-ck)^3 \quad i.e., \quad b^3 k^6 = -c^3 k^3$$

$$\Rightarrow b^3 \frac{d^2}{a^2} = -c^3 \left(\frac{-d}{a} \right)$$

$$\Rightarrow \frac{b^3 d}{a} = c^3 \quad \Rightarrow b^3 d = c^3 a.$$

5. Solve the equation $x^3 - 9x^2 + 14x + 24 = 0$, given that two of whose roots are in the ratio $3:2$.

Solution:

Let the roots be $3\alpha, 2\alpha, \beta$

Then, $\qquad 3\alpha + 2\alpha + \beta = 5\alpha + \beta = 9$ .... (1)

$$3\alpha.2\alpha + 2\alpha.\beta + 3\alpha.\beta = 14$$

i.e., $\quad 6\alpha^2 + 5\alpha\beta = 14$ .... (2)

and $\quad 3\alpha.2\alpha.\beta = 6\alpha^2 \beta = -24$

$$\Rightarrow \alpha^2 \beta = -4 \qquad \text{................. (3)}$$

From (1), $\beta = 9 - 5\alpha$. Substituting this in (2), we obtain

$$6\alpha^2 + 5\alpha(9 - 5\alpha) = 14$$

i.e., $\quad 19\alpha^2 - 45\alpha + 14 = 0$. On solving we get $\quad \alpha = 2 \;\; or \;\; \dfrac{7}{19}.$

When $\alpha = \dfrac{7}{19}$, from (1), we get $\beta = \dfrac{136}{19}$. But these values do not satisfy (3).

So, $\alpha = 2$, then from (1), we get $\beta = -1$

Therefore, the roots are 4, 6, -1.

Symmetric Functions of the Roots

Consider the expressions like $\alpha^2 + \beta^2 + \gamma^2, (\beta - \gamma)^2 + (\gamma - \alpha)^2 + (\alpha - \beta)^2$, $(\beta + \gamma)(\gamma + \alpha)(\alpha - \beta)$. Each of these expressions is a function of α, β, γ with the property that if any two of α, β, γ are interchanged, the function remains unchanged.

Such functions are called **symmetric functions**.

2.3 Polynomial and Synthetic Division

What you should learn

- Use long division to divide polynomials by other polynomials.
- Use synthetic division to divide polynomials by binomials of the form $(x - k)$.
- Use the Remainder Theorem and the Factor Theorem.

Why you should learn it

Synthetic division can help you evaluate polynomial functions. For instance, in Exercise 73 on page 160, you will use synthetic division to determine the number of U.S. military personnel in 2008.

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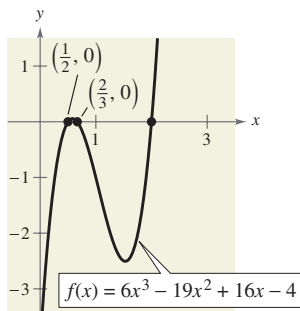
Long Division of Polynomials

In this section, you will study two procedures for *dividing* polynomials. These procedures are especially valuable in factoring and finding the zeros of polynomial functions. To begin, suppose you are given the graph of

$$f(x) = 6x^3 - 19x^2 + 16x - 4.$$

Notice that a zero of f occurs at $x = 2$, as shown in Figure 2.27. Because $x = 2$ is a zero of f, you know that $(x - 2)$ is a factor of $f(x)$. This means that there exists a second-degree polynomial $q(x)$ such that

$$f(x) = (x - 2) \cdot q(x).$$

To find $q(x)$, you can use **long division,** as illustrated in Example 1.

Example 1   Long Division of Polynomials

Divide $6x^3 - 19x^2 + 16x - 4$ by $x - 2$, and use the result to factor the polynomial completely.

Solution

Think $\dfrac{6x^3}{x} = 6x^2$.

Think $\dfrac{-7x^2}{x} = -7x$.

Think $\dfrac{2x}{x} = 2$.

$$
\begin{array}{r}
6x^2 - 7x + 2 \\
x - 2 \overline{) 6x^3 - 19x^2 + 16x - 4} \\
\underline{6x^3 - 12x^2} \\
-7x^2 + 16x \\
\underline{-7x^2 + 14x} \\
2x - 4 \\
\underline{2x - 4} \\
0
\end{array}
$$

Multiply: $6x^2(x - 2)$.
Subtract.
Multiply: $-7x(x - 2)$.
Subtract.
Multiply: $2(x - 2)$.
Subtract.

From this division, you can conclude that

$$6x^3 - 19x^2 + 16x - 4 = (x - 2)(6x^2 - 7x + 2)$$

and by factoring the quadratic $6x^2 - 7x + 2$, you have

$$6x^3 - 19x^2 + 16x - 4 = (x - 2)(2x - 1)(3x - 2).$$

Note that this factorization agrees with the graph shown in Figure 2.27 in that the three x-intercepts occur at $x = 2$, $x = \frac{1}{2}$, and $x = \frac{2}{3}$.

✔CHECKPOINT   Now try Exercise 5.

$f(x) = 6x^3 - 19x^2 + 16x - 4$

FIGURE 2.27

In Example 1, $x - 2$ is a factor of the polynomial $6x^3 - 19x^2 + 16x - 4$, and the long division process produces a remainder of zero. Often, long division will produce a nonzero remainder. For instance, if you divide $x^2 + 3x + 5$ by $x + 1$, you obtain the following.

$$
\begin{array}{r}
x + 2 \quad \longleftarrow \text{Quotient} \\
\text{Divisor} \longrightarrow x + 1 \overline{)\, x^2 + 3x + 5} \quad \longleftarrow \text{Dividend} \\
\underline{x^2 + x} \\
2x + 5 \\
\underline{2x + 2} \\
3 \quad \longleftarrow \text{Remainder}
\end{array}
$$

In fractional form, you can write this result as follows.

$$
\underbrace{\frac{\overbrace{x^2 + 3x + 5}^{\text{Dividend}}}{\underbrace{x + 1}_{\text{Divisor}}}} = \overbrace{x + 2}^{\text{Quotient}} + \frac{\overset{\text{Remainder}}{3}}{\underbrace{x + 1}_{\text{Divisor}}}
$$

This implies that

$$x^2 + 3x + 5 = (x + 1)(x + 2) + 3 \qquad \text{Multiply each side by } (x + 1).$$

which illustrates the following theorem, called the **Division Algorithm.**

The Division Algorithm

If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$\underset{\text{Dividend}}{f(x)} = \underset{\text{Divisor}}{d(x)} \underset{\text{Quotient}}{q(x)} + \underset{\text{Remainder}}{r(x)}$$

where $r(x) = 0$ *or* the degree of $r(x)$ is less than the degree of $d(x)$. If the remainder $r(x)$ is zero, $d(x)$ *divides evenly* into $f(x)$.

The Division Algorithm can also be written as

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.$$

In the Division Algorithm, the rational expression $f(x)/d(x)$ is **improper** because the degree of $f(x)$ is greater than or equal to the degree of $d(x)$. On the other hand, the rational expression $r(x)/d(x)$ is **proper** because the degree of $r(x)$ is less than the degree of $d(x)$.

Before you apply the Division Algorithm, follow these steps.

1. Write the dividend and divisor in descending powers of the variable.
2. Insert placeholders with zero coefficients for missing powers of the variable.

Example 2   **Long Division of Polynomials**

Divide $x^3 - 1$ by $x - 1$.

Solution

Because there is no x^2-term or x-term in the dividend, you need to line up the subtraction by using zero coefficients (or leaving spaces) for the missing terms.

$$
\begin{array}{r}
x^2 + x + 1 \\
x - 1 \overline{)\, x^3 + 0x^2 + 0x - 1} \\
\underline{x^3 - x^2} \\
x^2 + 0x \\
\underline{x^2 - x} \\
x - 1 \\
\underline{x - 1} \\
0
\end{array}
$$

So, $x - 1$ divides evenly into $x^3 - 1$, and you can write

$$\frac{x^3 - 1}{x - 1} = x^2 + x + 1, \quad x \neq 1.$$

✔CHECKPOINT   Now try Exercise 13.

You can check the result of Example 2 by multiplying.

$$(x - 1)(x^2 + x + 1) = x^3 + x^2 + x - x^2 - x - 1 = x^3 - 1$$

Example 3   **Long Division of Polynomials**

Divide $2x^4 + 4x^3 - 5x^2 + 3x - 2$ by $x^2 + 2x - 3$.

Solution

$$
\begin{array}{r}
2x^2 \qquad + 1 \\
x^2 + 2x - 3 \overline{)\, 2x^4 + 4x^3 - 5x^2 + 3x - 2} \\
\underline{2x^4 + 4x^3 - 6x^2} \\
x^2 + 3x - 2 \\
\underline{x^2 + 2x - 3} \\
x + 1
\end{array}
$$

Note that the first subtraction eliminated two terms from the dividend. When this happens, the quotient skips a term. You can write the result as

$$\frac{2x^4 + 4x^3 - 5x^2 + 3x - 2}{x^2 + 2x - 3} = 2x^2 + 1 + \frac{x + 1}{x^2 + 2x - 3}.$$

Remind students that when division yields a remainder, it is important that they write the remainder term correctly.
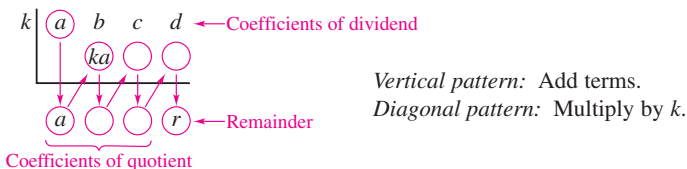
✔CHECKPOINT   Now try Exercise 15.
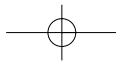
Synthetic Division

There is a nice shortcut for long division of polynomials when dividing by divisors of the form $x - k$. This shortcut is called **synthetic division.** The pattern for synthetic division of a cubic polynomial is summarized as follows. (The pattern for higher-degree polynomials is similar.)

> ### Synthetic Division (for a Cubic Polynomial)
>
> To divide $ax^3 + bx^2 + cx + d$ by $x - k$, use the following pattern.
>
> $k \; | \; a \quad b \quad c \quad d$ ←Coefficients of dividend
>
> ka
>
> $a \quad \bigcirc \quad \bigcirc \quad r$ ←Remainder
>
> Coefficients of quotient
>
> *Vertical pattern:* Add terms.
> *Diagonal pattern:* Multiply by k.

Synthetic division works only for divisors of the form $x - k$. [Remember that $x + k = x - (-k)$.] You cannot use synthetic division to divide a polynomial by a quadratic such as $x^2 - 3$.

Example 4   Using Synthetic Division

Use synthetic division to divide $x^4 - 10x^2 - 2x + 4$ by $x + 3$.

Solution

You should set up the array as follows. Note that a zero is included for the missing x^3-term in the dividend.
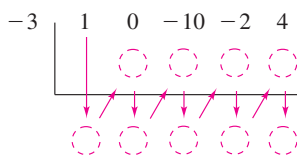
$$-3 \; | \; 1 \quad 0 \quad -10 \quad -2 \quad 4$$

Then, use the synthetic division pattern by adding terms in columns and multiplying the results by -3.

Divisor: $x + 3$ Dividend: $x^4 - 10x^2 - 2x + 4$

$$-3 \; | \; \begin{array}{ccccc} 1 & 0 & -10 & -2 & 4 \\ & -3 & 9 & 3 & -3 \\ \hline 1 & -3 & -1 & 1 & 1 \end{array}$$ ← Remainder: 1

Quotient: $x^3 - 3x^2 - x + 1$

So, you have

$$\frac{x^4 - 10x^2 - 2x + 4}{x + 3} = x^3 - 3x^2 - x + 1 + \frac{1}{x + 3}.$$

✔CHECKPOINT   Now try Exercise 19.
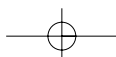
The Remainder and Factor Theorems

The remainder obtained in the synthetic division process has an important interpretation, as described in the **Remainder Theorem.**

The Remainder Theorem

If a polynomial $f(x)$ is divided by $x - k$, the remainder is

$$r = f(k).$$

For a proof of the Remainder Theorem, see Proofs in Mathematics on page 213.

The Remainder Theorem tells you that synthetic division can be used to evaluate a polynomial function. That is, to evaluate a polynomial function $f(x)$ when $x = k$, divide $f(x)$ by $x - k$. The remainder will be $f(k)$, as illustrated in Example 5.

Example 5   Using the Remainder Theorem

Use the Remainder Theorem to evaluate the following function at $x = -2$.

$$f(x) = 3x^3 + 8x^2 + 5x - 7$$

Solution

Using synthetic division, you obtain the following.

```
-2 |  3    8    5   -7
   |      -6   -4   -2
   ──────────────────────
      3    2    1   -9
```

Because the remainder is $r = -9$, you can conclude that

$$f(-2) = -9. \qquad r = f(k)$$

This means that $(-2, -9)$ is a point on the graph of f. You can check this by substituting $x = -2$ in the original function.

Check

$$f(-2) = 3(-2)^3 + 8(-2)^2 + 5(-2) - 7$$

$$= 3(-8) + 8(4) - 10 - 7 = -9$$

✔CHECKPOINT   Now try Exercise 45.

Additional Example

Use the Remainder Theorem to evaluate $f(x) = 4x^2 - 10x - 21$ when $x = 5$.

Solution

Using synthetic division, you obtain the following.

```
5 | 4   -10   -21
  |       20    50
  ──────────────────
    4    10    29
```

Because the remainder is 29, you can conclude that $f(5) = 29$.

Another important theorem is the **Factor Theorem,** stated below. This theorem states that you can test to see whether a polynomial has $(x - k)$ as a factor by evaluating the polynomial at $x = k$. If the result is 0, $(x - k)$ is a factor.

The Factor Theorem

A polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.

For a proof of the Factor Theorem, see Proofs in Mathematics on page 213.

Example 6   **Factoring a Polynomial: Repeated Division**

Show that $(x - 2)$ and $(x + 3)$ are factors of

$$f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18.$$

Then find the remaining factors of $f(x)$.

Solution

Using synthetic division with the factor $(x - 2)$, you obtain the following.

$$
\begin{array}{r|rrrrr}
2 & 2 & 7 & -4 & -27 & -18 \\
  &   & 4 & 22 & 36 & 18 \\
\hline
  & 2 & 11 & 18 & 9 & 0
\end{array}
$$

$\longrightarrow$ 0 remainder, so $f(2) = 0$ and $(x - 2)$ is a factor.

Take the result of this division and perform synthetic division again using the factor $(x + 3)$.

$$
\begin{array}{r|rrrr}
-3 & 2 & 11 & 18 & 9 \\
   &   & -6 & -15 & -9 \\
\hline
   & 2 & 5 & 3 & 0
\end{array}
$$

$\longrightarrow$ 0 remainder, so $f(-3) = 0$ and $(x + 3)$ is a factor.

Because the resulting quadratic expression factors as

$$2x^2 + 5x + 3 = (2x + 3)(x + 1)$$

the complete factorization of $f(x)$ is

$$f(x) = (x - 2)(x + 3)(2x + 3)(x + 1).$$

Note that this factorization implies that f has four real zeros:

$$x = 2, \ x = -3, \ x = -\tfrac{3}{2}, \text{ and } x = -1.$$

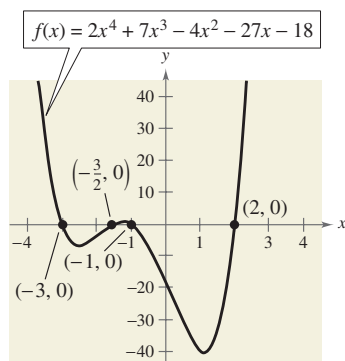This is confirmed by the graph of f, which is shown in Figure 2.28.

✓*CHECKPOINT*   Now try Exercise 57.

$f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18$

FIGURE 2.28

Uses of the Remainder in Synthetic Division

The remainder r, obtained in the synthetic division of $f(x)$ by $x - k$, provides the following information.

1. The remainder r gives the value of f at $x = k$. That is, $r = f(k)$.

2. If $r = 0$, $(x - k)$ is a factor of $f(x)$.

3. If $r = 0$, $(k, 0)$ is an x-intercept of the graph of f.

Throughout this text, the importance of developing several problem-solving strategies is emphasized. In the exercises for this section, try using more than one strategy to solve several of the exercises. For instance, if you find that $x - k$ divides evenly into $f(x)$ (with no remainder), try sketching the graph of f. You should find that $(k, 0)$ is an x-intercept of the graph.

Activities

1. Divide using long division.
    $$\frac{4x^5 - x^3 + 2x^2 - x}{2x + 1}$$
    Answer: $2x^4 - x^3 + x - 1 + \dfrac{1}{2x + 1}$

2. Use synthetic division to determine if $(x + 3)$ is a factor of
    $f(x) = 3x^3 + 4x^2 - 18x - 3.$
    Answer: No, it is not.

3. Use the Remainder Theorem to evaluate $f(-3)$ for $f(x) = 2x^3 - 4x^2 + 1.$
    Answer: -89

2.3 │ **Exercises**

VOCABULARY CHECK:

1. Two forms of the Division Algorithm are shown below. Identify and label each term or function.

$$f(x) = d(x)q(x) + r(x) \qquad \frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

In Exercises 2–5, fill in the blanks.

2. The rational expression $p(x)/q(x)$ is called _____ if the degree of the numerator is greater than or equal to that of the denominator, and is called _____ if the degree of the numerator is less than that of the denominator.

3. An alternative method to long division of polynomials is called _____ _____, in which the divisor must be of the form $x - k$.

4. The _____ Theorem states that a polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.

5. The _____ Theorem states that if a polynomial $f(x)$ is divided by $x - k$, the remainder is $r = f(k)$.

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

Analytical Analysis   **In Exercises 1 and 2, use long division to verify that $y_1 = y_2$.**

1. $y_1 = \dfrac{x^2}{x+2}$, $y_2 = x - 2 + \dfrac{4}{x+2}$

2. $y_1 = \dfrac{x^4 - 3x^2 - 1}{x^2 + 5}$, $y_2 = x^2 - 8 + \dfrac{39}{x^2 + 5}$

Graphical Analysis   **In Exercises 3 and 4, (a) use a graphing utility to graph the two equations in the same viewing window, (b) use the graphs to verify that the expressions are equivalent, and (c) use long division to verify the results algebraically.**

3. $y_1 = \dfrac{x^5 - 3x^3}{x^2 + 1}$, $y_2 = x^3 - 4x + \dfrac{4x}{x^2 + 1}$

4. $y_1 = \dfrac{x^3 - 2x^2 + 5}{x^2 + x + 1}$, $y_2 = x - 3 + \dfrac{2(x + 4)}{x^2 + x + 1}$

In Exercises 5–18, use long division to divide.

5. $(2x^2 + 10x + 12) \div (x + 3)$

6. $(5x^2 - 17x - 12) \div (x - 4)$

7. $(4x^3 - 7x^2 - 11x + 5) \div (4x + 5)$

8. $(6x^3 - 16x^2 + 17x - 6) \div (3x - 2)$

9. $(x^4 + 5x^3 + 6x^2 - x - 2) \div (x + 2)$

10. $(x^3 + 4x^2 - 3x - 12) \div (x - 3)$

11. $(7x + 3) \div (x + 2)$ **12.** $(8x - 5) \div (2x + 1)$

13. $(6x^3 + 10x^2 + x + 8) \div (2x^2 + 1)$

14. $(x^3 - 9) \div (x^2 + 1)$

15. $(x^4 + 3x^2 + 1) \div (x^2 - 2x + 3)$

16. $(x^5 + 7) \div (x^3 - 1)$

17. $\dfrac{x^4}{(x-1)^3}$

18. $\dfrac{2x^3 - 4x^2 - 15x + 5}{(x-1)^2}$

In Exercises 19–36, use synthetic division to divide.

19. $(3x^3 - 17x^2 + 15x - 25) \div (x - 5)$

20. $(5x^3 + 18x^2 + 7x - 6) \div (x + 3)$

21. $(4x^3 - 9x + 8x^2 - 18) \div (x + 2)$

22. $(9x^3 - 16x - 18x^2 + 32) \div (x - 2)$

23. $(-x^3 + 75x - 250) \div (x + 10)$

24. $(3x^3 - 16x^2 - 72) \div (x - 6)$

25. $(5x^3 - 6x^2 + 8) \div (x - 4)$

26. $(5x^3 + 6x + 8) \div (x + 2)$

27. $\dfrac{10x^4 - 50x^3 - 800}{x - 6}$

28. $\dfrac{x^5 - 13x^4 - 120x + 80}{x + 3}$

29. $\dfrac{x^3 + 512}{x + 8}$

30. $\dfrac{x^3 - 729}{x - 9}$

31. $\dfrac{-3x^4}{x - 2}$

32. $\dfrac{-3x^4}{x + 2}$

33. $\dfrac{180x - x^4}{x - 6}$

34. $\dfrac{5 - 3x + 2x^2 - x^3}{x + 1}$

35. $\dfrac{4x^3 + 16x^2 - 23x - 15}{x + \frac{1}{2}}$

36. $\dfrac{3x^3 - 4x^2 + 5}{x - \frac{3}{2}}$

In Exercises 37–44, write the function in the form $f(x) = (x - k)q(x) + r$ for the given value of k, and demonstrate that $f(k) = r$.

	Function	*Value of k*
37.	$f(x) = x^3 - x^2 - 14x + 11$	$k = 4$
38.	$f(x) = x^3 - 5x^2 - 11x + 8$	$k = -2$

Function	Value of k
39. $f(x) = 15x^4 + 10x^3 - 6x^2 + 14$	$k = -\frac{2}{3}$
40. $f(x) = 10x^3 - 22x^2 - 3x + 4$	$k = \frac{1}{5}$
41. $f(x) = x^3 + 3x^2 - 2x - 14$	$k = \sqrt{2}$
42. $f(x) = x^3 + 2x^2 - 5x - 4$	$k = -\sqrt{5}$
43. $f(x) = -4x^3 + 6x^2 + 12x + 4$	$k = 1 - \sqrt{3}$
44. $f(x) = -3x^3 + 8x^2 + 10x - 8$	$k = 2 + \sqrt{2}$

In Exercises 45–48, use synthetic division to find each function value. Verify your answers using another method.

45. $f(x) = 4x^3 - 13x + 10$

    (a) $f(1)$ (b) $f(-2)$ (c) $f\left(\frac{1}{2}\right)$ (d) $f(8)$

46. $g(x) = x^6 - 4x^4 + 3x^2 + 2$

    (a) $g(2)$ (b) $g(-4)$ (c) $g(3)$ (d) $g(-1)$

47. $h(x) = 3x^3 + 5x^2 - 10x + 1$

    (a) $h(3)$ (b) $h\left(\frac{1}{3}\right)$ (c) $h(-2)$ (d) $h(-5)$

48. $f(x) = 0.4x^4 - 1.6x^3 + 0.7x^2 - 2$

    (a) $f(1)$ (b) $f(-2)$ (c) $f(5)$ (d) $f(-10)$

In Exercises 49–56, use synthetic division to show that x is a solution of the third-degree polynomial equation, and use the result to factor the polynomial completely. List all real solutions of the equation.

Polynomial Equation	Value of x
49. $x^3 - 7x + 6 = 0$	$x = 2$
50. $x^3 - 28x - 48 = 0$	$x = -4$
51. $2x^3 - 15x^2 + 27x - 10 = 0$	$x = \frac{1}{2}$
52. $48x^3 - 80x^2 + 41x - 6 = 0$	$x = \frac{2}{3}$
53. $x^3 + 2x^2 - 3x - 6 = 0$	$x = \sqrt{3}$
54. $x^3 + 2x^2 - 2x - 4 = 0$	$x = \sqrt{2}$
55. $x^3 - 3x^2 + 2 = 0$	$x = 1 + \sqrt{3}$
56. $x^3 - x^2 - 13x - 3 = 0$	$x = 2 - \sqrt{5}$

In Exercises 57–64, (a) verify the given factors of the function f, (b) find the remaining factors of f, (c) use your results to write the complete factorization of f, (d) list all real zeros of f, and (e) confirm your results by using a graphing utility to graph the function.

Function	Factors
57. $f(x) = 2x^3 + x^2 - 5x + 2$	$(x + 2), (x - 1)$
58. $f(x) = 3x^3 + 2x^2 - 19x + 6$	$(x + 3), (x - 2)$
59. $f(x) = x^4 - 4x^3 - 15x^2$ $+ 58x - 40$	$(x - 5), (x + 4)$
60. $f(x) = 8x^4 - 14x^3 - 71x^2$ $- 10x + 24$	$(x + 2), (x - 4)$

Function	Factors
61. $f(x) = 6x^3 + 41x^2 - 9x - 14$	$(2x + 1), (3x - 2)$
62. $f(x) = 10x^3 - 11x^2 - 72x + 45$	$(2x + 5), (5x - 3)$
63. $f(x) = 2x^3 - x^2 - 10x + 5$	$(2x - 1), \left(x + \sqrt{5}\right)$
64. $f(x) = x^3 + 3x^2 - 48x - 144$	$\left(x + 4\sqrt{3}\right), (x + 3)$

Graphical Analysis **In Exercises 65–68, (a) use the *zero* or *root* feature of a graphing utility to approximate the zeros of the function accurate to three decimal places, (b) determine one of the exact zeros, and (c) use synthetic division to verify your result from part (b), and then factor the polynomial completely.**

65. $f(x) = x^3 - 2x^2 - 5x + 10$

66. $g(x) = x^3 - 4x^2 - 2x + 8$

67. $h(t) = t^3 - 2t^2 - 7t + 2$

68. $f(s) = s^3 - 12s^2 + 40s - 24$

In Exercises 69–72, simplify the rational expression by using long division or synthetic division.

69. $\dfrac{4x^3 - 8x^2 + x + 3}{2x - 3}$

70. $\dfrac{x^3 + x^2 - 64x - 64}{x + 8}$

71. $\dfrac{x^4 + 6x^3 + 11x^2 + 6x}{x^2 + 3x + 2}$

72. $\dfrac{x^4 + 9x^3 - 5x^2 - 36x + 4}{x^2 - 4}$

Model It

73. *Data Analysis: Military Personnel* The numbers M (in thousands) of United States military personnel on active duty for the years 1993 through 2003 are shown in the table, where t represents the year, with $t = 3$ corresponding to 1993. (Source: U.S. Department of Defense)

Year, t	Military personnel, M
3	1705
4	1611
5	1518
6	1472
7	1439
8	1407
9	1386
10	1384
11	1385
12	1412
13	1434

Model It (continued)

(a) Use a graphing utility to create a scatter plot of the data.

(b) Use the *regression* feature of the graphing utility to find a cubic model for the data. Graph the model in the same viewing window as the scatter plot.

(c) Use the model to create a table of estimated values of M. Compare the model with the original data.

(d) Use synthetic division to evaluate the model for the year 2008. Even though the model is relatively accurate for estimating the given data, would you use this model to predict the number of military personnel in the future? Explain.

74. *Data Analysis: Cable Television*   The average monthly basic rates R (in dollars) for cable television in the United States for the years 1992 through 2002 are shown in the table, where t represents the year, with $t = 2$ corresponding to 1992.   (Source: Kagan Research LLC)

Year, t	Basic rate, R
2	19.08
3	19.39
4	21.62
5	23.07
6	24.41
7	26.48
8	27.81
9	28.92
10	30.37
11	32.87
12	34.71

(a) Use a graphing utility to create a scatter plot of the data.

(b) Use the *regression* feature of the graphing utility to find a cubic model for the data. Then graph the model in the same viewing window as the scatter plot. Compare the model with the data.

(c) Use synthetic division to evaluate the model for the year 2008.

Synthesis

True or False?   In Exercises 75–77, determine whether the statement is true or false. Justify your answer.

75. If $(7x + 4)$ is a factor of some polynomial function f, then $\frac{4}{7}$ is a zero of f.

76. $(2x - 1)$ is a factor of the polynomial

$$6x^6 + x^5 - 92x^4 + 45x^3 + 184x^2 + 4x - 48.$$

77. The rational expression

$$\frac{x^3 + 2x^2 - 13x + 10}{x^2 - 4x - 12}$$

is improper.

78. *Exploration*   Use the form $f(x) = (x - k)q(x) + r$ to create a cubic function that (a) passes through the point $(2, 5)$ and rises to the right, and (b) passes through the point $(-3, 1)$ and falls to the right. (There are many correct answers.)

Think About It   In Exercises 79 and 80, perform the division by assuming that n is a positive integer.

79. $\dfrac{x^{3n} + 9x^{2n} + 27x^n + 27}{x^n + 3}$ **80.** $\dfrac{x^{3n} - 3x^{2n} + 5x^n - 6}{x^n - 2}$

81. *Writing*   Briefly explain what it means for a divisor to divide evenly into a dividend.

82. *Writing*   Briefly explain how to check polynomial division, and justify your reasoning. Give an example.

Exploration   In Exercises 83 and 84, find the constant c such that the denominator will divide evenly into the numerator.

83. $\dfrac{x^3 + 4x^2 - 3x + c}{x - 5}$ **84.** $\dfrac{x^5 - 2x^2 + x + c}{x + 2}$

Think About It   In Exercises 85 and 86, answer the questions about the division $f(x) \div (x - k)$, where $f(x) = (x + 3)^2(x - 3)(x + 1)^3$.

85. What is the remainder when $k = -3$? Explain.

86. If it is necessary to find $f(2)$, is it easier to evaluate the function directly or to use synthetic division? Explain.

Skills Review

In Exercises 87–92, use any method to solve the quadratic equation.

87. $9x^2 - 25 = 0$ **88.** $16x^2 - 21 = 0$

89. $5x^2 - 3x - 14 = 0$ **90.** $8x^2 - 22x + 15 = 0$

91. $2x^2 + 6x + 3 = 0$ **92.** $x^2 + 3x - 3 = 0$

In Exercises 93–96, find a polynomial function that has the given zeros. (There are many correct answers.)

93. $0, 3, 4$ **94.** $-6, 1$

95. $-3, 1 + \sqrt{2}, 1 - \sqrt{2}$ **96.** $1, -2, 2 + \sqrt{3}, 2 - \sqrt{3}$

SECTION 2.3: LONG AND SYNTHETIC POLYNOMIAL DIVISION

PART A: LONG DIVISION

Ancient Example with Integers

$$4 \overline{)9} \quad \begin{array}{c} 2 \\ \end{array}$$

$$\begin{array}{c} -8 \\ \hline 1 \end{array}$$

We can say: $\dfrac{9}{4} = 2 + \dfrac{1}{4}$

By multiplying both sides by 4, this can be rewritten as:

$$9 = 4 \cdot 2 + 1$$

In general: $\dfrac{\text{dividend}, f}{\text{divisor}, d} = \left(\text{quotient}, q\right) + \dfrac{\left(\text{remainder}, r\right)}{d}$

where either:

$r = 0$ (in which case d <u>divides evenly</u> into f), or

$\dfrac{r}{d}$ is a positive <u>proper</u> fraction: i.e., $0 < r < d$

Technical Note: We assume f and d are positive integers, and q and r are nonnegative integers.

Technical Note: We typically assume $\dfrac{f}{d}$ is <u>improper</u>: i.e., $f \geq d$. Otherwise, there is no point in dividing this way.

Technical Note: Given f and d, q and r are unique by the Division Algorithm (really, it's a theorem).

By multiplying both sides by d, $\dfrac{f}{d} = q + \dfrac{r}{d}$ can be rewritten as:

$$f = d \cdot q + r$$

Now, we will perform polynomial division on $\dfrac{f(x)}{d(x)}$ so that we get:

$$\frac{f(x)}{d(x)} \;=\; q(x) \;+\; \frac{r(x)}{d(x)}$$

where either:

$r(x) = 0$, in which case $d(x)$ <u>divides evenly</u> into $f(x)$, or

$\dfrac{r(x)}{d(x)}$ is a <u>proper</u> rational expression: i.e., $\deg\big(r(x)\big) < \deg\big(d(x)\big)$

<u>Technical Note</u>: We assume $f(x)$ and $d(x)$ are nonzero polynomials, and $q(x)$ and $r(x)$ are polynomials.

<u>Technical Note</u>: We assume $\dfrac{f(x)}{d(x)}$ is <u>improper</u>; i.e., $\deg\big(f(x)\big) \ge \deg\big(d(x)\big)$.
Otherwise, there is no point in dividing.

<u>Technical Note</u>: Given $f(x)$ and $d(x)$, $q(x)$ and $r(x)$ are unique by the Division Algorithm (really, it's a theorem).

By multiplying both sides by $d(x)$, $\dfrac{f(x)}{d(x)} = q(x) + \dfrac{r(x)}{d(x)}$ can be rewritten as:

$$f(x) \;=\; d(x) \cdot q(x) \;+\; r(x)$$

Example

Use Long Division to divide: $\dfrac{-5 + 3x^2 + 6x^3}{1 + 3x^2}$

Solution

Warning: First, write the N and the D in descending powers of x.

Warning: Insert "missing term placeholders" in the N (and perhaps even the D) with "0" coefficients. This helps you avoid errors. We get:

$$\frac{6x^3 + 3x^2 + 0x - 5}{3x^2 + 0x + 1}$$

Let's begin the Long Division:

$$3x^2 + 0x + 1 \overline{\smash{\big)}\ 6x^3 + 3x^2 + 0x - 5}$$

The steps are similar to those for $4\overline{\smash{\big)}\ 9}$.

Think: How many "times" does the leading term of the divisor $(3x^2)$ "go into" the leading term of the dividend $(6x^3)$? We get:
$\dfrac{6x^3}{3x^2} = 2x$, which goes into the quotient.

$$\begin{array}{r} 2x \\ 3x^2 + 0x + 1 \overline{\smash{\big)}\ 6x^3 + 3x^2 + 0x - 5} \end{array}$$

Multiply the $2x$ by the divisor and write the product on the next line.

Warning: Line up like terms to avoid confusion!

$$\begin{array}{r} 2x \\ 3x^2 + 0x + 1 \overline{\smash{\big)}\ 6x^3 + 3x^2 + 0x - 5} \\ 6x^3 + 0x^2 + 2x \end{array}$$

Warning: We must **subtract** this product from the dividend. People have a much easier time adding than subtracting, so let's flip the sign on each term of the product, and add the result to the dividend. To avoid errors, we will cross out our product and do the sign flips on a separate line before adding.

Warning: Don't forget to bring down the -5.

$$
\begin{array}{r}
2x \\
3x^2 + 0x + 1 \overline{\smash{)}\ 6x^3 + 3x^2 + 0x - 5} \\
\underline{\cancel{6x^3 + 0x^2 + 2x}} \\
\underline{-6x^3 - 0x^2 - 2x } \\
3x^2 - 2x - 5
\end{array}
$$

We now treat the expression in blue above as our new dividend. Repeat the process.

$$
\begin{array}{r}
2x + 1 \\
3x^2 + 0x + 1 \overline{\smash{)}\ 6x^3 + 3x^2 + 0x - 5} \\
\underline{\cancel{6x^3 + 0x^2 + 2x}} \\
\underline{-6x^3 - 0x^2 - 2x } \\
3x^2 - 2x - 5
\end{array}
$$

$$
\begin{array}{r}
2x + 1 \\
3x^2 + 0x + 1 \overline{\smash{)}\ 6x^3 + 3x^2 + 0x - 5} \\
\underline{\cancel{6x^3 + 0x^2 + 2x}} \\
\underline{-6x^3 - 0x^2 - 2x } \\
3x^2 - 2x - 5 \\
3x^2 + 0x + 1
\end{array}
$$

$$\begin{array}{r}
2x+1 \\
3x^2+0x+1 \overline{)\ 6x^3+3x^2+0x-5}
\end{array}$$

$$\begin{array}{r}
\underline{6x^3+0x^2+2x} \\
-6x^3-0x^2-2x \\
\hline
3x^2-2x-5 \\
\underline{3x^2+0x+1} \\
-3x^2-0x-1 \\
\hline
-2x-6
\end{array}$$

We can now stop the process, because the degree of the new dividend is less than the degree of the divisor. The degree of $-2x-6$ is 1, which is less than the degree of $3x^2+0x+1$, which is 2. This guarantees that the fraction in our answer is a **proper** rational expression.

Our answer is of the form: $\quad q(x)+\dfrac{r(x)}{d(x)}$

$$2x+1\ +\ \frac{-2x-6}{3x^2+1}$$

If the leading coefficient of $r(x)$ is negative, then we factor a -1 out of it.

Answer: $\quad \mathbf{2x+1\ -\ \dfrac{2x+6}{3x^2+1}}$

Warning: Remember to flip **every** sign in the numerator.

Warning: If the N and the D of our fraction have any common factors aside from ± 1, they must be canceled out. Our fraction here is simplified as is.

PART B: SYNTHETIC DIVISION

There's a great short cut if the divisor is of the form $x - k$.

Example

Use Synthetic Division to divide: $\dfrac{2x^3 - 3x + 5}{x + 3}$.

Solution

The divisor is $x + 3$, so $k = -3$.
Think: $x + 3 = x - \left(-3\right)$.
We will put -3 in a half-box in the upper left of the table below.

Make sure the N is written in standard form.
Write the coefficients in order along the first row of the table.
Write a "placeholder 0" if a term is missing.

Bring down the first coefficient, the "2."

The $\downarrow$ arrow tells us to add down the column and write the sum in the third row.

The $\nearrow$ arrow tells us to multiply the blue number by k (here, -3) and write the product one column to the right in the second row.

Circle the lower right number.

$$\begin{array}{r|rrrr}
-3 & 2 & 0 & -3 & 5 \\
& & -6 & 18 & -45 \\
\hline
& 2 & -6 & 15 & \boxed{-40}
\end{array}$$
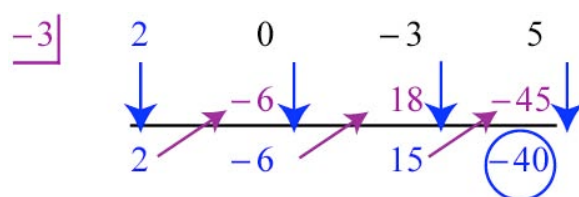
Since we are dividing a 3rd-degree dividend by a 1st-degree divisor, our answer begins with a 2nd-degree term.

The third (blue) row gives the coefficients of our quotient in descending powers of x. The circled number is our remainder, which we put over our divisor and factor out a -1 if appropriate.

Note: The remainder **must** be a **constant**, because the divisor is linear.

Answer: $\mathbf{2x^2 - 6x + 15} - \dfrac{\mathbf{40}}{x + 3}$

Related Example

Express $f(x) = 2x^3 - 3x + 5$ in the following form:

$f(x) = d(x) \cdot q(x) + r$, where the divisor $d(x) = x + 3$.

Solution

We can work from our previous Answer. Multiply both sides by the divisor:

$$\frac{2x^3 - 3x + 5}{x + 3} = 2x^2 - 6x + 15 - \frac{40}{x + 3}$$
$$2x^3 - 3x + 5 = (x + 3) \cdot (2x^2 - 6x + 15) - 40$$

Note: Synthetic Division works even if $k = 0$. What happens?

PART C: REMAINDER THEOREM

Remainder Theorem

If we are dividing a polynomial $f(x)$ by $x - k$, and if r is the remainder, then $f(k) = r$.

In our previous Examples, we get the following fact as a bonus.

$$f(-3) = \boxed{-40}$$

Synthetic Division therefore provides an efficient means of evaluating polynomial functions. (It may be much better than straight calculator button-pushing when dealing with polynomials of high degree.) We could have done the work in Part B if we had wanted to evaluate $f(-3)$, where $f(x) = 2x^3 - 3x + 5$.

Warning: Do **not** flip the sign of -3 when writing it in the half-box. People get the "sign flip" idea when they work with polynomial division.

Technical Note: See the short Proof on p.192.

PART D: ZEROS, FACTORING, AND DIVISION

Recall from Section 2.2:

Factor Theorem

If $f(x)$ is a nonzero polynomial and k is a real number, then

k is a zero of $f \Leftrightarrow (x-k)$ is a factor of $f(x)$.

Technical Note: The Proof on p.192 uses the Remainder Theorem to prove this.

What happens if either Long or Synthetic polynomial division gives us a 0 remainder? Then, we can at least partially factor $f(x)$.

Example

Show that 2 is a zero of $f(x) = 4x^3 - 5x^2 - 7x + 2$.

Note: We saw this $f(x)$ in Section 2.2.

Note: In Section 2.5, we will discuss a trick for finding such a zero.

Factor $f(x)$ completely, and find **all** of its real zeros.

Solution

We will use Synthetic Division to show that 2 is a zero:

By the Remainder Theorem, $f(2) = 0$, and so 2 is a zero.

By the Factor Theorem, $(x-2)$ must be a factor of $f(x)$.

> Technical Note: This can be seen from the form
> $f(x) = d(x) \cdot q(x) + r$. Since $r = 0$ when $d(x) = x - 2$, we have:
>
> $$f(x) = (x-2) \cdot q(x),$$
> where $q(x)$ is some (here, quadratic) polynomial.

We can find $q(x)$, the other (quadratic) factor, by using the last row of the table.

$$f(x) = (x-2) \cdot \left(4x^2 + 3x - 1\right)$$

Factor $q(x)$ completely over the reals:

$$f(x) = (x-2)(4x-1)(x+1)$$

The zeros of $f(x)$ are the zeros of these factors:

$$2, \frac{1}{4}, -1$$

Below is a graph of $f(x) = 4x^3 - 5x^2 - 7x + 2$. Where are the x-intercepts?
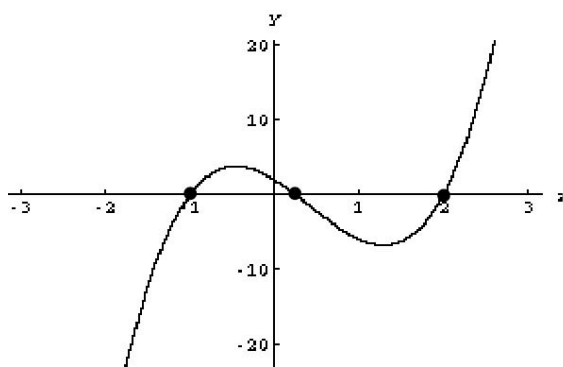
5.1 The Remainder and Factor Theorems; Synthetic Division

In this section you will learn to:
- understand the definition of a zero of a polynomial function
- use long and synthetic division to divide polynomials
- use the remainder theorem
- use the factor theorem

Example 1: Use long division to find the quotient and the remainder: $5593 \div 27$

      Steps for Long Division:

      1.

      2.

      3.

      4.

Example 2: Use the "Steps for Long Division" to divide each of the polynomials below.

$$x - 5 \overline{)\, x^2 - 2x - 35}$$
$$\left(7 - 11x - 3x^2 + 2x^3\right) \div (x - 3)$$

Example 3: Check your answer for the division problems in Example 2.

The Division Algorithm: If $f(x)$ and $d(x)$ are polynomials where $d(x) \neq 0$ and degree $d(x) <$ degree $f(x)$,

          then

$$f(x) \;=\; d(x) \;\cdot\; q(x) \;+\; r(x)$$

If $r(x) = 0$ then $d(x)$ and $q(x)$ are **factors** of $f(x)$.

Example 4: Perform the operation below. Write the remainder as a rational expression (remainder/divisor).

$$\frac{2x^5 - 8x^4 + 2x^3 + x^2}{2x^3 + 1}$$

Synthetic Division – Generally used for "short" division of polynomials when the divisor is in the form $x - c$. (Refer to page 506 in your textbook for more examples.)

Example 5: Use both long and short (synthetic) division to find the quotient and remainder for the problem below.

$$(2x^3 - 11x + 7) \div (x - 3)$$

Example 6: Divide $\dfrac{x^3 + 8}{x + 2}$ using synthetic division.

Example 7: Factor $x^3 + 8$ over the real numbers. (Hint: Refer to Example 6.)

Remainder Theorem	Factor Theorem
If the polynomial $f(x)$ is divided by $(x - c)$, then the remainder is $f(c)$.	Let $f(x)$ be a polynomial. If $f(c) = 0$, then $(x - c)$ is a factor of $f(x)$. If $(x - c)$ is a factor of $f(x)$, then $f(c) = 0$. If $(x - c)$ is a factor of $f(x)$ or if $f(c) = 0$, then c is called a **zero** of $f(x)$.

Example 8: $f(x) = 3x^3 + 4x^2 - 5x + 7$. Find $f(-4)$ using

    (a) synthetic division.   (b) the Remainder Theorem.

Example 9: Solve the equation $2x^3 - 3x^2 - 11x + 6 = 0$ given that -2 is a zero of $f(x) = 2x^3 - 3x^2 - 11x + 6$.

5.1 Homework Problems:

For Problems 1-5, use long division to find each quotient, $q(x)$, and remainder, $r(x)$.

1. $(x^2 - 2x - 15) \div (x - 5)$

   2. $(x^3 + 5x^2 + 7x + 2) \div (x + 2)$

3. $(6x^3 + 7x^2 + 12x - 5) \div (3x - 1)$ 4. $\dfrac{x^4 - 81}{x - 3}$ 5. $\dfrac{18x^4 + 9x^3 + 3x^2}{3x^2 + 1}$

For Problems 6 – 11, divide using synthetic division.

6. $(2x^2 + x - 10) \div (x - 2)$ 7. $(5x^3 - 6x^2 + 3x + 11) \div (x - 2)$ 8. $(x^2 - 5x - 5x^3 + x^4) \div (5 + x)$

9. $\dfrac{x^7 + x^5 - 10x^3 + 12}{x + 2}$ 10. $\dfrac{x^4 - 256}{x - 4}$ 11. $\dfrac{x^5 - 2x^4 - x^3 + 3x^2 - x + 1}{x - 2}$

For Problems 12 – 16, use synthetic division and the Remainder Theorem to find the indicated function value.

12. $f(x) = x^3 - 7x^2 + 5x - 6;\ \ f(3)$ 13. $f(x) = 4x^3 + 5x^2 - 6x - 4;\ \ f(-2)$

14. $f(x) = 2x^4 - 5x^3 - x^2 + 3x + 2;\ \ f\left(-\dfrac{1}{2}\right)$ 15. $f(x) = 6x^4 + 10x^3 + 5x^2 + x + 1;\ \ f\left(-\dfrac{2}{3}\right)$

16. Use synthetic division to divide $f(x) = x^3 - 4x^2 + x + 6$ by $x + 1$. Use the result to find all zeros of f.

17. Solve the equation $2x^3 - 5x^2 + x + 2 = 0$ given that 2 is a zero of $f(x) = 2x^3 - 5x^2 + x + 2$.

18. Solve the equation $12x^3 + 16x^2 - 5x - 3 = 0$ given that $-\dfrac{3}{2}$ is a zero (root).

5.1 Homework Answers: **1.** $q(x) = x + 3$ **2.** $q(x) = x^2 + 3x + 1$ **3.** $q(x) = 2x^2 + 3x + 5$

4. $q(x) = x^3 + 3x^2 + 9x + 27$ **5.** $q(x) = 6x^2 + 3x - 1;\ \ r(x) = -3x + 1$ **6.** $q(x) = 2x + 5$

7. $q(x) = 5x^2 + 4x + 11;\ \ r(x) = 33$ **8.** $q(x) = x^3 - 10x^2 + 51x - 260;\ \ r(x) = 1300$

9. $q(x) = x^6 - 2x^5 + 5x^4 - 10x^3 + 10x^2 - 20x + 40;\ \ r(x) = -68$ **10.** $q(x) = x^3 + 4x^2 + 16x + 64$

11. $q(x) = x^4 - x^2 + x + 1;\ \ r(x) = 3$ **12.** -27 **13.** -4 **14.** 1 **15.** $\dfrac{7}{9}$

16. $x^2 - 5x + 6;\ \ x = -1, 2, 3$ **17.** $\left\{-\dfrac{1}{2}, 1, 2\right\}$ **18.** $\left\{-\dfrac{3}{2}, -\dfrac{1}{3}, \dfrac{1}{2}\right\}$

5.3 Roots of Polynomial Equations

In this section you will learn to:
- find zeros of polynomial equations
- solve polynomial equations with real and imaginary zeros
- find possible rational roots of polynomial equations
- understand properties of polynomial equatins
- use the Linear Factorization Theorem

> **Zeros of Polynomial Functions** are the values of x for which $f(x) = 0$.
>
> (Zero = Root = Solution = x-intercept (if the zero is a real number))

Example 1: Consider the polynomial that only has 3 and ½ as zeros.

(a) How many polynomials have such zeros?

(b) Find a polynomial that has a leading
coefficient of 1 that has such zeros.

(c) Find a polynomial, with integer
coefficients, that has such zeros.

> If the same factor $(x - r)$ occurs k times, then the zero r is called a zero with **multiplicity k.**
>
> **Even Multiplicity** → Graph **touches** x-axis and turns around.
>
> **Odd Multiplicity** → Graph **crosses** x-axis.

Example 2: Find all of the (real) zeros for each of the polynomial functions below. Give the multiplicity of each zero and state whether the graph crosses the x-axis or touches (and turns at) the x-axis at each zero. Use this information and the Leading Coefficient Test to sketch a graph of each function

(a) $f(x) = x^3 + 2x^2 - 4x - 8$ (b) $f(x) = -x^4 + 4x^2$ (c) $g(x) = -x^4 + 4x^3 - 4x^2$

The Rational Zero Theorem: If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots a_2 x^2 + a_1 x + a_0$ has ***integer*** coefficients and $\dfrac{p}{q}$ (reduced to lowest terms) is a rational zero of f, then p is a factor of the constant term, a_0, and q is a factor of the leading coefficient, a_n.

Example 3: List all **possible** rational zeros of the polynomials below. (Refer to Rational Zero Theorem on
        Page 1 of this handout.)

(a) $f(x) = -x^5 + 7x^2 - 12$ Possible Rational Zeros: _____

(b) $p(x) = 6x^3 - 8x^2 - 8x + 8$ Possible Rational Zeros: _____

Example 4: Find all zeros of $f(x) = 2x^3 - 5x^2 + x + 2$.

Example 5: Solve $x^4 - 8x^3 + 64x - 105 = 0$.

Linear Factorization Theorem:

If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots a_2 x^2 + a_1 x + a_0$, where $n \geq 1$ and $a_n \neq 0$, then

$\qquad f(x) = a_n (x - c_1)(x - c_2) \ldots (x - c_n)$, where $c_1, c_2, c_3, \ldots c_n$ are complex numbers.

Example 6: Find all complex zeros of $f(x) = 2x^4 + 3x^3 + 3x - 2$, and then write the polynomial $f(x)$ as a **product of linear factors**.

$f(x) = $ _____

Properties of Polynomial Equations:

Given the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots a_2 x^2 + a_1 x + a_0$.

1. If a polynomial equation is of degree n, then counting multiple roots (multiplicities) separately, the equation has n roots.

2. If $a + bi$ is a root of a polynomial equation ($b \neq 0$), then the imaginary number $a - bi$ is also a root. In other words, imaginary roots, if they exist, occur in **conjugate pairs**.

Example 7: Find all zeros of $f(x) = x^4 - 4x^2 - 5$. (Hint: Use factoring techniques from Chapter 1.) Write $f(x)$ as a product of linear factors.

$f(x) = $ _____

Example 8: Find a third-degree polynomial function, $f(x)$, with real coefficients that has 4 and $2i$ as zeros and such that $f(-1) = 50$.

Step 1: Use the zeros to find the factors of $f(x)$.

Step 2: Write as a linear factorization, then expand/multiply.

Step 3: Use $f(-1) = 50$ to substitute values for x and $f(x)$.

Step 4: Solve for a_n.

Step 5: Substitute a_n into the equation for $f(x)$ and simplify.

Step 6: Use your calculator to check.

5.3 Homework Problems:

For Problems 1 – 4, use the Rational Zero Theorem to list all possible rational zeros for each function.

1. $f(x) = x^3 + 3x^2 - 6x - 8$

2. $f(x) = 2x^4 + 3x^3 - 11x^2 - 9x + 15$

3. $f(x) = 3x^4 - 11x^3 - 3x^2 - 6x + 8$

4. $f(x) = 4x^5 - 8x^4 - x + 2$

For Problems 5 – 8, find the zeros for the given functions.

5. $f(x) = x^3 - 2x^2 - 11x + 12$

6. $f(x) = 2x^3 - 5x^2 + x + 2$

7. $f(x) = 2x^3 + x^2 - 3x + 1$

8. $f(x) = x^3 - 4x^2 + 8x - 5$

For Problems 9 – 12, solve each of the given equations.

9. $x^3 - 2x^2 - 7x - 4 = 0$

10. $x^3 - 5x^2 + 17x - 13 = 0$

11. $2x^3 - 5x^2 - 6x + 4 = 0$

12. $x^4 - 2x^2 - 16x - 15 = 0$

For Problems 13-16, find an *nth* degree polynomial function, $f(x)$, with real coefficients that satisfies the given conditions.

13. $n = 3$; 1 and $5i$ are zeros; $f(-1) = -104$

14. $n = 4$; 2, -2, and i are zeros; $f(3) = -150$

15. $n = 3$; 6 and $-5 + 2i$ are zeros; $f(2) = -636$

16. $n = 4$; i and $3i$ are zeros; $f(-1) = 20$

5.3 Homework Answers: **1.** $\pm 1, \pm 2, \pm 4, \pm 8$ **2.** $\pm 1, \pm 3, \pm 5, \pm 15, \pm \dfrac{1}{2}, \pm \dfrac{3}{2}, \pm \dfrac{5}{2}, \pm \dfrac{15}{2}$

3. $\pm 1, \pm 2, \pm 4, \pm 8, \pm \dfrac{1}{2}, \pm \dfrac{2}{3}, \pm \dfrac{4}{3}, \pm \dfrac{8}{3}$ **4.** $\pm 1, \pm 2, \pm \dfrac{1}{2}, \pm \dfrac{1}{4}$ **5.** $-3, 1, 4$ **6.** $-\dfrac{1}{2}, 1, 2$

7. $\dfrac{1}{2}, \dfrac{-1 \pm \sqrt{5}}{2}$ **8.** $1, \dfrac{3 \pm i\sqrt{11}}{2}$ **9.** $\{-1, 4\}$ **10.** $\{1, 2 \pm 3i\}$ **11.** $\left\{ \dfrac{1}{2}, 1 \pm \sqrt{5} \right\}$

12. $\{-1, 3, -1 \pm 2i\}$ **13.** $f(x) = 2x^3 - 2x^2 + 50x - 50$ **14.** $f(x) = -3x^4 + 9x^2 + 12$

15. $f(x) = 3x^3 + 12x^2 - 93x - 522$ **16.** $f(x) = x^4 + 10x^2 + 9$

12.1   Three-Dimensional Coordinate Systems
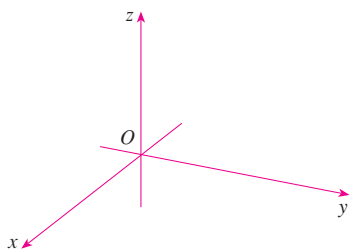
FIGURE 1
Coordinate axes
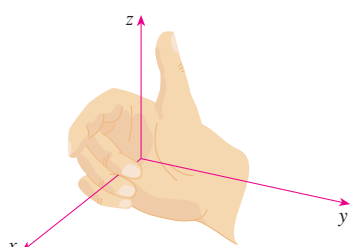
FIGURE 2
Right-hand rule

To locate a point in a plane, two numbers are necessary. We know that any point in the plane can be represented as an ordered pair (a, b) of real numbers, where a is the x-coordinate and b is the y-coordinate. For this reason, a plane is called two-dimensional. To locate a point in space, three numbers are required. We represent any point in space by an ordered triple (a, b, c) of real numbers.

In order to represent points in space, we first choose a fixed point O (the origin) and three directed lines through O that are perpendicular to each other, called the **coordinate axes** and labeled the x-axis, y-axis, and z-axis. Usually we think of the x- and y-axes as being horizontal and the z-axis as being vertical, and we draw the orientation of the axes as in Figure 1. The direction of the z-axis is determined by the **right-hand rule** as illustrated in Figure 2: If you curl the fingers of your right hand around the z-axis in the direction of a 90° counterclockwise rotation from the positive x-axis to the positive y-axis, then your thumb points in the positive direction of the z-axis.

The three coordinate axes determine the three **coordinate planes** illustrated in Figure 3(a). The xy-plane is the plane that contains the x- and y-axes; the yz-plane contains the y- and z-axes; the xz-plane contains the x- and z-axes. These three coordinate planes divide space into eight parts, called **octants**. The **first octant**, in the foreground, is determined by the positive axes.
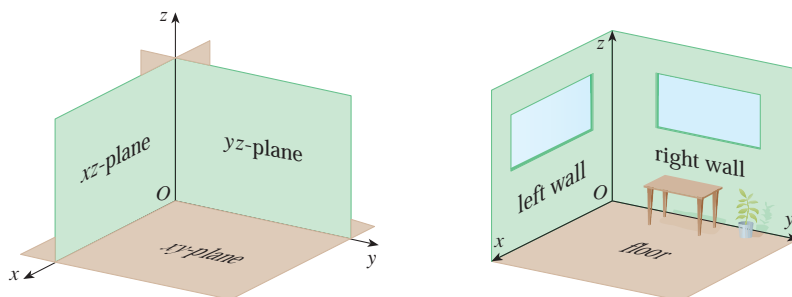
FIGURE 3   (a) Coordinate planes   (b)

Because many people have some difficulty visualizing diagrams of three-dimensional figures, you may find it helpful to do the following [see Figure 3(b)]. Look at any bottom corner of a room and call the corner the origin. The wall on your left is in the xz-plane, the wall on your right is in the yz-plane, and the floor is in the xy-plane. The x-axis runs along the intersection of the floor and the left wall. The y-axis runs along the intersection of the floor and the right wall. The z-axis runs up from the floor toward the ceiling along the intersection of the two walls. You are situated in the first octant, and you can now imagine seven other rooms situated in the other seven octants (three on the same floor and four on the floor below), all connected by the common corner point O.

Now if P is any point in space, let a be the (directed) distance from the yz-plane to P, let b be the distance from the xz-plane to P, and let c be the distance from the xy-plane to P. We represent the point P by the ordered triple (a, b, c) of real numbers and we call a, b, and c the **coordinates** of P; a is the x-coordinate, b is the y-coordinate, and c is the z-coordinate. Thus, to locate the point (a, b, c), we can start at the origin O and move a units along the x-axis, then b units parallel to the y-axis, and then c units parallel to the z-axis as in Figure 4.
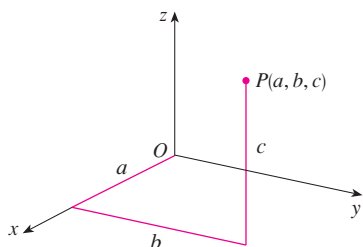

FIGURE 4

The point $P(a, b, c)$ determines a rectangular box as in Figure 5. If we drop a perpendicular from P to the xy-plane, we get a point Q with coordinates $(a, b, 0)$ called the **projection** of P onto the xy-plane. Similarly, $R(0, b, c)$ and $S(a, 0, c)$ are the projections of P onto the yz-plane and xz-plane, respectively.

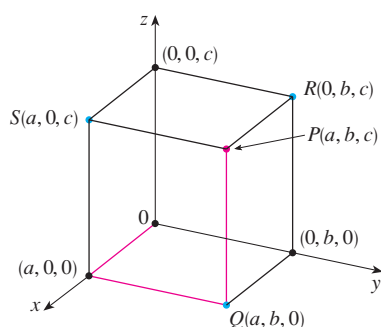As numerical illustrations, the points $(-4, 3, -5)$ and $(3, -2, -6)$ are plotted in Figure 6.
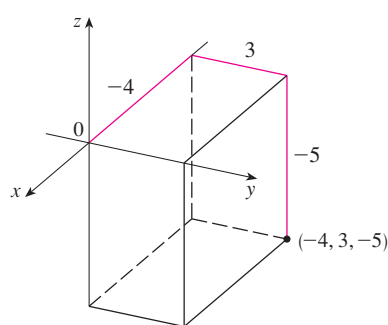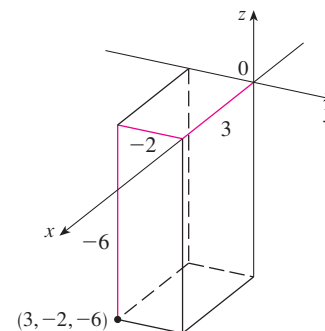

FIGURE 5

FIGURE 6

The Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$ is the set of all ordered triples of real numbers and is denoted by $\mathbb{R}^3$. We have given a one-to-one correspondence between points P in space and ordered triples (a, b, c) in $\mathbb{R}^3$. It is called a **three-dimensional rectangular coordinate system**. Notice that, in terms of coordinates, the first octant can be described as the set of points whose coordinates are all positive.

In two-dimensional analytic geometry, the graph of an equation involving x and y is a curve in $\mathbb{R}^2$. In three-dimensional analytic geometry, an equation in x, y, and z represents a *surface* in $\mathbb{R}^3$.

V   EXAMPLE 1   What surfaces in $\mathbb{R}^3$ are represented by the following equations?
(a) $z = 3$ (b) $y = 5$

SOLUTION
(a) The equation $z = 3$ represents the set $\{(x, y, z) \mid z = 3\}$, which is the set of all points in $\mathbb{R}^3$ whose z-coordinate is 3. This is the horizontal plane that is parallel to the xy-plane and three units above it as in Figure 7(a).
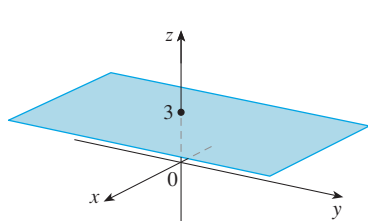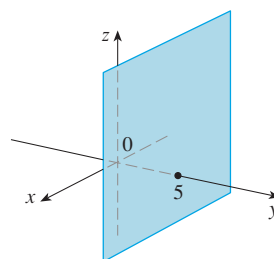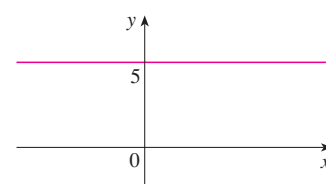

FIGURE 7   (a) $z = 3$, a plane in $\mathbb{R}^3$ (b) $y = 5$, a plane in $\mathbb{R}^3$ (c) $y = 5$, a line in $\mathbb{R}^2$

(b) The equation $y = 5$ represents the set of all points in $\mathbb{R}^3$ whose y-coordinate is 5. This is the vertical plane that is parallel to the xz-plane and five units to the right of it as in Figure 7(b).

NOTE When an equation is given, we must understand from the context whether it represents a curve in $\mathbb{R}^2$ or a surface in $\mathbb{R}^3$. In Example 1, $y = 5$ represents a plane in $\mathbb{R}^3$, but of course $y = 5$ can also represent a line in $\mathbb{R}^2$ if we are dealing with two-dimensional analytic geometry. See Figure 7(b) and (c).

In general, if k is a constant, then $x = k$ represents a plane parallel to the yz-plane, $y = k$ is a plane parallel to the xz-plane, and $z = k$ is a plane parallel to the xy-plane. In Figure 5, the faces of the rectangular box are formed by the three coordinate planes $x = 0$ (the yz-plane), $y = 0$ (the xz-plane), and $z = 0$ (the xy-plane), and the planes $x = a$, $y = b$, and $z = c$.

EXAMPLE 2
(a) Which points (x, y, z) satisfy the equations

$$x^2 + y^2 = 1 \qquad \text{and} \qquad z = 3$$

(b) What does the equation $x^2 + y^2 = 1$ represent as a surface in $\mathbb{R}^3$?

SOLUTION
(a) Because $z = 3$, the points lie in the horizontal plane $z = 3$ from Example 1(a). Because $x^2 + y^2 = 1$, the points lie on the circle with radius 1 and center on the z-axis. See Figure 8.
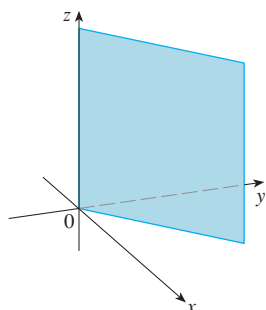
(b) Given that $x^2 + y^2 = 1$, with no restrictions on z, we see that the point (x, y, z) could lie on a circle in any horizontal plane $z = k$. So the surface $x^2 + y^2 = 1$ in $\mathbb{R}^3$ consists of all possible horizontal circles $x^2 + y^2 = 1$, $z = k$, and is therefore the circular cylinder with radius 1 whose axis is the z-axis. See Figure 9.
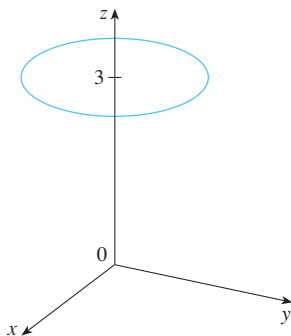
FIGURE 8
The circle $x^2 + y^2 = 1$, $z = 3$
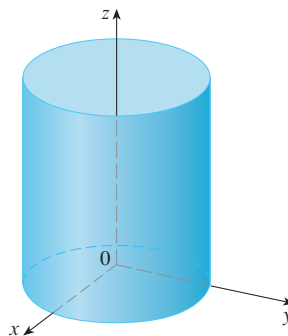
FIGURE 9
The cylinder $x^2 + y^2 = 1$

V **EXAMPLE 3** Describe and sketch the surface in $\mathbb{R}^3$ represented by the equation $y = x$.

SOLUTION The equation represents the set of all points in $\mathbb{R}^3$ whose x- and y-coordinates are equal, that is, $\{(x, x, z) \mid x \in \mathbb{R}, z \in \mathbb{R}\}$. This is a vertical plane that intersects the xy-plane in the line $y = x$, $z = 0$. The portion of this plane that lies in the first octant is sketched in Figure 10.

The familiar formula for the distance between two points in a plane is easily extended to the following three-dimensional formula.

FIGURE 10
The plane $y = x$

Distance Formula in Three Dimensions   The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$
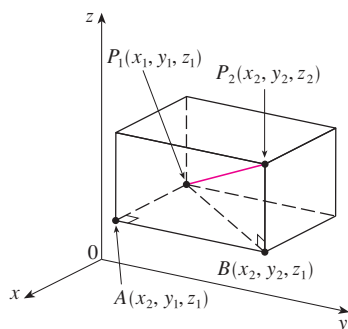

FIGURE 11

To see why this formula is true, we construct a rectangular box as in Figure 11, where P_1 and P_2 are opposite vertices and the faces of the box are parallel to the coordinate planes. If $A(x_2, y_1, z_1)$ and $B(x_2, y_2, z_1)$ are the vertices of the box indicated in the figure, then

$$|P_1A| = |x_2 - x_1| \qquad |AB| = |y_2 - y_1| \qquad |BP_2| = |z_2 - z_1|$$

Because triangles P_1BP_2 and P_1AB are both right-angled, two applications of the Pythagorean Theorem give

$$|P_1P_2|^2 = |P_1B|^2 + |BP_2|^2$$

and

$$|P_1B|^2 = |P_1A|^2 + |AB|^2$$

Combining these equations, we get

$$
\begin{aligned}
|P_1P_2|^2 &= |P_1A|^2 + |AB|^2 + |BP_2|^2 \\
&= |x_2 - x_1|^2 + |y_2 - y_1|^2 + |z_2 - z_1|^2 \\
&= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2
\end{aligned}
$$

Therefore

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

EXAMPLE 4 The distance from the point $P(2, -1, 7)$ to the point $Q(1, -3, 5)$ is

$$|PQ| = \sqrt{(1 - 2)^2 + (-3 + 1)^2 + (5 - 7)^2} = \sqrt{1 + 4 + 4} = 3 \qquad \blacksquare$$

V EXAMPLE 5 Find an equation of a sphere with radius r and center $C(h, k, l)$.

SOLUTION By definition, a sphere is the set of all points $P(x, y, z)$ whose distance from C is r. (See Figure 12.) Thus P is on the sphere if and only if $|PC| = r$. Squaring both sides, we have $|PC|^2 = r^2$ or

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2 \qquad \blacksquare$$

FIGURE 12

The result of Example 5 is worth remembering.

Equation of a Sphere An equation of a sphere with center $C(h, k, l)$ and radius r is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

In particular, if the center is the origin O, then an equation of the sphere is
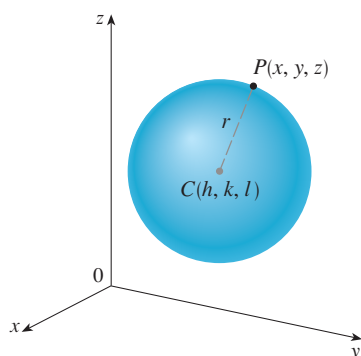
$$x^2 + y^2 + z^2 = r^2$$

EXAMPLE 6 Show that $x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$ is the equation of a sphere, and find its center and radius.

SOLUTION We can rewrite the given equation in the form of an equation of a sphere if we complete squares:

$$(x^2 + 4x + 4) + (y^2 - 6y + 9) + (z^2 + 2z + 1) = -6 + 4 + 9 + 1$$
$$(x + 2)^2 + (y - 3)^2 + (z + 1)^2 = 8$$

Comparing this equation with the standard form, we see that it is the equation of a sphere with center $(-2, 3, -1)$ and radius $\sqrt{8} = 2\sqrt{2}$.

EXAMPLE 7 What region in $\mathbb{R}^3$ is represented by the following inequalities?

$$1 \leqslant x^2 + y^2 + z^2 \leqslant 4 \qquad z \leqslant 0$$

SOLUTION The inequalities

$$1 \leqslant x^2 + y^2 + z^2 \leqslant 4$$

can be rewritten as

$$1 \leqslant \sqrt{x^2 + y^2 + z^2} \leqslant 2$$

so they represent the points (x, y, z) whose distance from the origin is at least 1 and at most 2. But we are also given that $z \leqslant 0$, so the points lie on or below the xy-plane. Thus the given inequalities represent the region that lies between (or on) the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ and beneath (or on) the xy-plane. It is sketched in Figure 13.
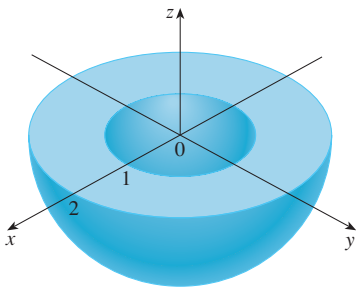

FIGURE 13

12.1 Exercises

1. Suppose you start at the origin, move along the x-axis a distance of 4 units in the positive direction, and then move downward a distance of 3 units. What are the coordinates of your position?

2. Sketch the points $(0, 5, 2)$, $(4, 0, -1)$, $(2, 4, 6)$, and $(1, -1, 2)$ on a single set of coordinate axes.

3. Which of the points $A(-4, 0, -1)$, $B(3, 1, -5)$, and $C(2, 4, 6)$ is closest to the yz-plane? Which point lies in the xz-plane?

4. What are the projections of the point $(2, 3, 5)$ on the xy-, yz-, and xz-planes? Draw a rectangular box with the origin and $(2, 3, 5)$ as opposite vertices and with its faces parallel to the coordinate planes. Label all vertices of the box. Find the length of the diagonal of the box.

5. Describe and sketch the surface in $\mathbb{R}^3$ represented by the equation $x + y = 2$.

6. (a) What does the equation $x = 4$ represent in $\mathbb{R}^2$? What does it represent in $\mathbb{R}^3$? Illustrate with sketches.
   (b) What does the equation $y = 3$ represent in $\mathbb{R}^3$? What does $z = 5$ represent? What does the pair of equations $y = 3$, $z = 5$ represent? In other words, describe the set of points (x, y, z) such that $y = 3$ and $z = 5$. Illustrate with a sketch.

7–8 Find the lengths of the sides of the triangle PQR. Is it a right triangle? Is it an isosceles triangle?

7. $P(3, -2, -3)$, $Q(7, 0, 1)$, $R(1, 2, 1)$

8. $P(2, -1, 0)$, $Q(4, 1, 1)$, $R(4, -5, 4)$

9. Determine whether the points lie on straight line.
   (a) $A(2, 4, 2)$, $B(3, 7, -2)$, $C(1, 3, 3)$
   (b) $D(0, -5, 5)$, $E(1, -2, 4)$, $F(3, 4, 2)$

10. Find the distance from $(4, -2, 6)$ to each of the following.
   (a) The xy-plane   (b) The yz-plane
   (c) The xz-plane   (d) The x-axis
   (e) The y-axis   (f) The z-axis

11. Find an equation of the sphere with center $(-3, 2, 5)$ and radius 4. What is the intersection of this sphere with the yz-plane?

12. Find an equation of the sphere with center $(2, -6, 4)$ and radius 5. Describe its intersection with each of the coordinate planes.

13. Find an equation of the sphere that passes through the point $(4, 3, -1)$ and has center $(3, 8, 1)$.

14. Find an equation of the sphere that passes through the origin and whose center is $(1, 2, 3)$.

15–18 Show that the equation represents a sphere, and find its center and radius.

15. $x^2 + y^2 + z^2 - 2x - 4y + 8z = 15$

16. $x^2 + y^2 + z^2 + 8x - 6y + 2z + 17 = 0$

17. $2x^2 + 2y^2 + 2z^2 = 8x - 24z + 1$

18. $3x^2 + 3y^2 + 3z^2 = 10 + 6y + 12z$

1. Homework Hints available at stewartcalculus.com

19. (a) Prove that the midpoint of the line segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

(b) Find the lengths of the medians of the triangle with vertices $A(1, 2, 3)$, $B(-2, 0, 5)$, and $C(4, 1, 5)$.

20. Find an equation of a sphere if one of its diameters has endpoints $(2, 1, 4)$ and $(4, 3, 10)$.

21. Find equations of the spheres with center $(2, -3, 6)$ that touch (a) the xy-plane, (b) the yz-plane, (c) the xz-plane.

22. Find an equation of the largest sphere with center $(5, 4, 9)$ that is contained in the first octant.

23–34 Describe in words the region of $\mathbb{R}^3$ represented by the equations or inequalities.

23. $x = 5$ **24.** $y = -2$

25. $y < 8$ **26.** $x \geqslant -3$

27. $0 \leqslant z \leqslant 6$ **28.** $z^2 = 1$

29. $x^2 + y^2 = 4, \quad z = -1$ **30.** $y^2 + z^2 = 16$

31. $x^2 + y^2 + z^2 \leqslant 3$ **32.** $x = z$

**33.** $x^2 + z^2 \leqslant 9$ **34.** $x^2 + y^2 + z^2 > 2z$

35–38 Write inequalities to describe the region.

35. The region between the yz-plane and the vertical plane $x = 5$

36. The solid cylinder that lies on or below the plane $z = 8$ and on or above the disk in the xy-plane with center the origin and radius 2

37. The region consisting of all points between (but not on) the spheres of radius r and R centered at the origin, where $r < R$

38. The solid upper hemisphere of the sphere of radius 2 centered at the origin

39. The figure shows a line L_1 in space and a second line L_2, which is the projection of L_1 on the xy-plane. (In other words,

the points on L_2 are directly beneath, or above, the points on L_1.)
(a) Find the coordinates of the point P on the line L_1.
(b) Locate on the diagram the points A, B, and C, where the line L_1 intersects the xy-plane, the yz-plane, and the xz-plane, respectively.
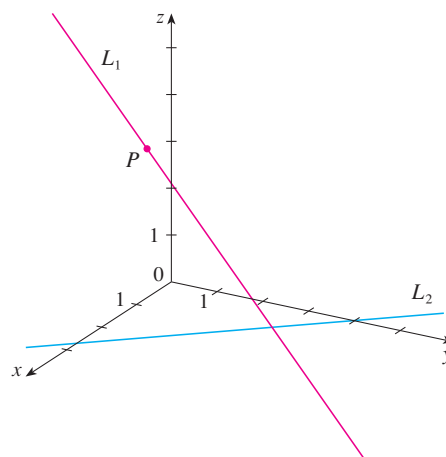
40. Consider the points P such that the distance from P to $A(-1, 5, 3)$ is twice the distance from P to $B(6, 2, -2)$. Show that the set of all such points is a sphere, and find its center and radius.

41. Find an equation of the set of all points equidistant from the points $A(-1, 5, 3)$ and $B(6, 2, -2)$. Describe the set.

42. Find the volume of the solid that lies inside both of the spheres

$$x^2 + y^2 + z^2 + 4x - 2y + 4z + 5 = 0$$

and $\qquad\qquad x^2 + y^2 + z^2 = 4$

43. Find the distance between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 4x + 4y + 4z - 11$.

44. Describe and sketch a solid with the following properties. When illuminated by rays parallel to the z-axis, its shadow is a circular disk. If the rays are parallel to the y-axis, its shadow is a square. If the rays are parallel to the x-axis, its shadow is an isosceles triangle.
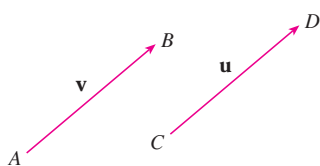

12.2 **Vectors**

FIGURE 1
Equivalent vectors

The term **vector** is used by scientists to indicate a quantity (such as displacement or velocity or force) that has both magnitude and direction. A vector is often represented by an arrow or a directed line segment. The length of the arrow represents the magnitude of the vector and the arrow points in the direction of the vector. We denote a vector by printing a letter in boldface (**v**) or by putting an arrow above the letter ($\vec{v}$).

For instance, suppose a particle moves along a line segment from point A to point B. The corresponding **displacement vector v**, shown in Figure 1, has **initial point** A (the tail) and **terminal point** B (the tip) and we indicate this by writing $\mathbf{v} = \overrightarrow{AB}$. Notice that the vec-

tor $\mathbf{u} = \overrightarrow{CD}$ has the same length and the same direction as $\mathbf{v}$ even though it is in a different position. We say that $\mathbf{u}$ and $\mathbf{v}$ are **equivalent** (or **equal**) and we write $\mathbf{u} = \mathbf{v}$. The **zero vector**, denoted by $\mathbf{0}$, has length 0. It is the only vector with no specific direction.

Combining Vectors

Suppose a particle moves from A to B, so its displacement vector is $\overrightarrow{AB}$. Then the particle changes direction and moves from B to C, with displacement vector $\overrightarrow{BC}$ as in Figure 2. The combined effect of these displacements is that the particle has moved from A to C. The resulting displacement vector $\overrightarrow{AC}$ is called the *sum* of $\overrightarrow{AB}$ and $\overrightarrow{BC}$ and we write

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

FIGURE 2

In general, if we start with vectors $\mathbf{u}$ and $\mathbf{v}$, we first move $\mathbf{v}$ so that its tail coincides with the tip of $\mathbf{u}$ and define the sum of $\mathbf{u}$ and $\mathbf{v}$ as follows.

> **Definition of Vector Addition**   If $\mathbf{u}$ and $\mathbf{v}$ are vectors positioned so the initial point of $\mathbf{v}$ is at the terminal point of $\mathbf{u}$, then the **sum $\mathbf{u} + \mathbf{v}$** is the vector from the initial point of $\mathbf{u}$ to the terminal point of $\mathbf{v}$.

The definition of vector addition is illustrated in Figure 3. You can see why this definition is sometimes called the **Triangle Law**.
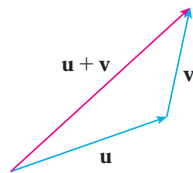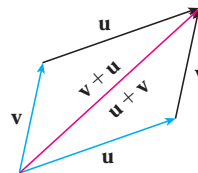

FIGURE 3 The Triangle Law

FIGURE 4 The Parallelogram Law

In Figure 4 we start with the same vectors $\mathbf{u}$ and $\mathbf{v}$ as in Figure 3 and draw another copy of $\mathbf{v}$ with the same initial point as $\mathbf{u}$. Completing the parallelogram, we see that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$. This also gives another way to construct the sum: If we place $\mathbf{u}$ and $\mathbf{v}$ so they start at the same point, then $\mathbf{u} + \mathbf{v}$ lies along the diagonal of the parallelogram with $\mathbf{u}$ and $\mathbf{v}$ as sides. (This is called the **Parallelogram Law**.)

V **EXAMPLE 1** Draw the sum of the vectors $\mathbf{a}$ and $\mathbf{b}$ shown in Figure 5.

SOLUTION First we translate $\mathbf{b}$ and place its tail at the tip of $\mathbf{a}$, being careful to draw a copy of $\mathbf{b}$ that has the same length and direction. Then we draw the vector $\mathbf{a} + \mathbf{b}$ [see Figure 6(a)] starting at the initial point of $\mathbf{a}$ and ending at the terminal point of the copy of $\mathbf{b}$.
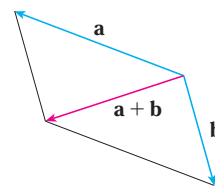
Alternatively, we could place $\mathbf{b}$ so it starts where $\mathbf{a}$ starts and construct $\mathbf{a} + \mathbf{b}$ by the Parallelogram Law as in Figure 6(b).
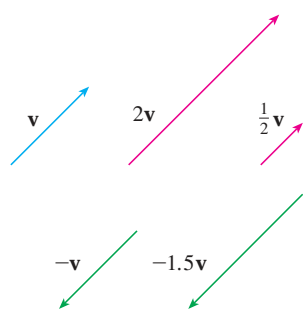

FIGURE 5
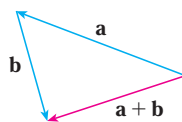
TEC Visual 12.2 shows how the Triangle and Parallelogram Laws work for various vectors $\mathbf{a}$ and $\mathbf{b}$.

FIGURE 6   (a)   (b)

It is possible to multiply a vector by a real number c. (In this context we call the real number c a **scalar** to distinguish it from a vector.) For instance, we want $2\mathbf{v}$ to be the same vector as $\mathbf{v} + \mathbf{v}$, which has the same direction as $\mathbf{v}$ but is twice as long. In general, we multiply a vector by a scalar as follows.

> **Definition of Scalar Multiplication** If c is a scalar and $\mathbf{v}$ is a vector, then the **scalar multiple** $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same as $\mathbf{v}$ if $c > 0$ and is opposite to $\mathbf{v}$ if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.

This definition is illustrated in Figure 7. We see that real numbers work like scaling factors here; that's why we call them scalars. Notice that two nonzero vectors are **parallel** if they are scalar multiples of one another. In particular, the vector $-\mathbf{v} = (-1)\mathbf{v}$ has the same length as $\mathbf{v}$ but points in the opposite direction. We call it the **negative** of $\mathbf{v}$.

By the **difference $\mathbf{u} - \mathbf{v}$** of two vectors we mean

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$
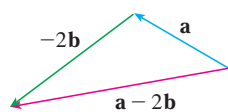
So we can construct $\mathbf{u} - \mathbf{v}$ by first drawing the negative of $\mathbf{v}$, $-\mathbf{v}$, and then adding it to $\mathbf{u}$ by the Parallelogram Law as in Figure 8(a). Alternatively, since $\mathbf{v} + (\mathbf{u} - \mathbf{v}) = \mathbf{u}$, the vector $\mathbf{u} - \mathbf{v}$, when added to $\mathbf{v}$, gives $\mathbf{u}$. So we could construct $\mathbf{u} - \mathbf{v}$ as in Figure 8(b) by means of the Triangle Law.

FIGURE 7
Scalar multiples of $\mathbf{v}$
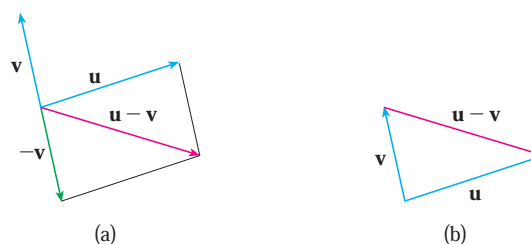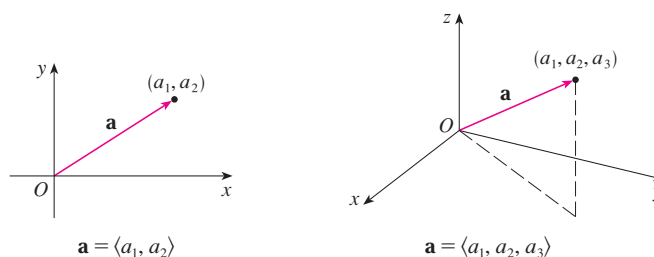

FIGURE 8
Drawing $\mathbf{u} - \mathbf{v}$

(a)　　　　　(b)

EXAMPLE 2 If $\mathbf{a}$ and $\mathbf{b}$ are the vectors shown in Figure 9, draw $\mathbf{a} - 2\mathbf{b}$.

SOLUTION We first draw the vector $-2\mathbf{b}$ pointing in the direction opposite to $\mathbf{b}$ and twice as long. We place it with its tail at the tip of $\mathbf{a}$ and then use the Triangle Law to draw $\mathbf{a} + (-2\mathbf{b})$ as in Figure 10. ▬
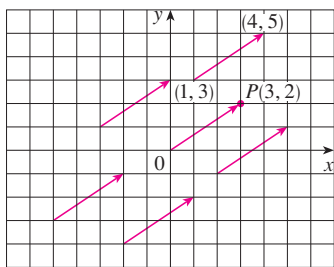

FIGURE 9

FIGURE 10

▬ **Components**

For some purposes it's best to introduce a coordinate system and treat vectors algebraically. If we place the initial point of a vector $\mathbf{a}$ at the origin of a rectangular coordinate system, then the terminal point of $\mathbf{a}$ has coordinates of the form (a_1, a_2) or (a_1, a_2, a_3), depending on whether our coordinate system is two- or three-dimensional (see Figure 11).
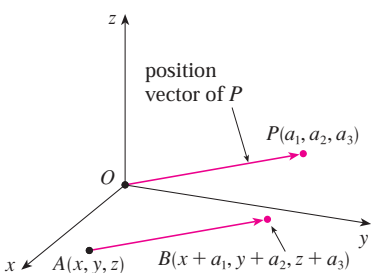

FIGURE 11

$\mathbf{a} = \langle a_1, a_2 \rangle$　　　　　$\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

FIGURE 12
Representations of the vector $\mathbf{a} = \langle 3, 2 \rangle$

FIGURE 13
Representations of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

These coordinates are called the **components** of $\mathbf{a}$ and we write

$$\mathbf{a} = \langle a_1, a_2 \rangle \qquad \text{or} \qquad \mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

We use the notation $\langle a_1, a_2 \rangle$ for the ordered pair that refers to a vector so as not to confuse it with the ordered pair (a_1, a_2) that refers to a point in the plane.

For instance, the vectors shown in Figure 12 are all equivalent to the vector $\overrightarrow{OP} = \langle 3, 2 \rangle$ whose terminal point is $P(3, 2)$. What they have in common is that the terminal point is reached from the initial point by a displacement of three units to the right and two upward. We can think of all these geometric vectors as **representations** of the algebraic vector $\mathbf{a} = \langle 3, 2 \rangle$. The particular representation $\overrightarrow{OP}$ from the origin to the point $P(3, 2)$ is called the **position vector** of the point P.

In three dimensions, the vector $\mathbf{a} = \overrightarrow{OP} = \langle a_1, a_2, a_3 \rangle$ is the **position vector** of the point $P(a_1, a_2, a_3)$. (See Figure 13.) Let's consider any other representation $\overrightarrow{AB}$ of $\mathbf{a}$, where the initial point is $A(x_1, y_1, z_1)$ and the terminal point is $B(x_2, y_2, z_2)$. Then we must have $x_1 + a_1 = x_2$, $y_1 + a_2 = y_2$, and $z_1 + a_3 = z_2$ and so $a_1 = x_2 - x_1$, $a_2 = y_2 - y_1$, and $a_3 = z_2 - z_1$. Thus we have the following result.

1 Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\mathbf{a}$ with representation $\overrightarrow{AB}$ is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

V EXAMPLE 3 Find the vector represented by the directed line segment with initial point $A(2, -3, 4)$ and terminal point $B(-2, 1, 1)$.

SOLUTION By $\boxed{1}$, the vector corresponding to $\overrightarrow{AB}$ is

$$\mathbf{a} = \langle -2 - 2, 1 - (-3), 1 - 4 \rangle = \langle -4, 4, -3 \rangle \qquad \blacksquare$$
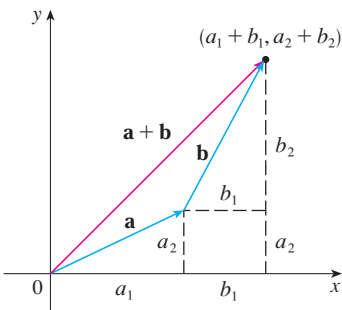
The **magnitude** or **length** of the vector $\mathbf{v}$ is the length of any of its representations and is denoted by the symbol $|\mathbf{v}|$ or $\|\mathbf{v}\|$. By using the distance formula to compute the length of a segment OP, we obtain the following formulas.

The length of the two-dimensional vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

FIGURE 14

How do we add vectors algebraically? Figure 14 shows that if $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then the sum is $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$, at least for the case where the components are positive. In other words, *to add algebraic vectors we add their components*. Similarly, *to subtract vectors we subtract components*. From the similar triangles in
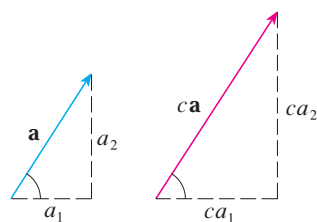
FIGURE 15

Figure 15 we see that the components of $c\mathbf{a}$ are ca_1 and ca_2. So *to multiply a vector by a scalar we multiply each component by that scalar.*

If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle \qquad \mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

Similarly, for three-dimensional vectors,

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

V EXAMPLE 4 If $\mathbf{a} = \langle 4, 0, 3 \rangle$ and $\mathbf{b} = \langle -2, 1, 5 \rangle$, find $|\mathbf{a}|$ and the vectors $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $3\mathbf{b}$, and $2\mathbf{a} + 5\mathbf{b}$.

SOLUTION

$$|\mathbf{a}| = \sqrt{4^2 + 0^2 + 3^2} = \sqrt{25} = 5$$

$$\mathbf{a} + \mathbf{b} = \langle 4, 0, 3 \rangle + \langle -2, 1, 5 \rangle$$

$$= \langle 4 + (-2), 0 + 1, 3 + 5 \rangle = \langle 2, 1, 8 \rangle$$

$$\mathbf{a} - \mathbf{b} = \langle 4, 0, 3 \rangle - \langle -2, 1, 5 \rangle$$

$$= \langle 4 - (-2), 0 - 1, 3 - 5 \rangle = \langle 6, -1, -2 \rangle$$

$$3\mathbf{b} = 3\langle -2, 1, 5 \rangle = \langle 3(-2), 3(1), 3(5) \rangle = \langle -6, 3, 15 \rangle$$

$$2\mathbf{a} + 5\mathbf{b} = 2\langle 4, 0, 3 \rangle + 5\langle -2, 1, 5 \rangle$$

$$= \langle 8, 0, 6 \rangle + \langle -10, 5, 25 \rangle = \langle -2, 5, 31 \rangle$$

We denote by V_2 the set of all two-dimensional vectors and by V_3 the set of all three-dimensional vectors. More generally, we will later need to consider the set V_n of all n-dimensional vectors. An n-dimensional vector is an ordered n-tuple:

$$\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle$$

where $a_1, a_2, \ldots, a_n$ are real numbers that are called the components of $\mathbf{a}$. Addition and scalar multiplication are defined in terms of components just as for the cases $n = 2$ and $n = 3$.

Vectors in n dimensions are used to list various quantities in an organized way. For instance, the components of a six-dimensional vector

$$\mathbf{p} = \langle p_1, p_2, p_3, p_4, p_5, p_6 \rangle$$

might represent the prices of six different ingredients required to make a particular product. Four-dimensional vectors $\langle x, y, z, t \rangle$ are used in relativity theory, where the first three components specify a position in space and the fourth represents time.

Properties of Vectors If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_n and c and d are scalars, then

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$	**2.** $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$	**4.** $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$	**6.** $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
7. $(cd)\mathbf{a} = c(d\mathbf{a})$	**8.** $1\mathbf{a} = \mathbf{a}$

These eight properties of vectors can be readily verified either geometrically or algebraically. For instance, Property 1 can be seen from Figure 4 (it's equivalent to the Parallelogram Law) or as follows for the case $n = 2$:

$$\mathbf{a} + \mathbf{b} = \langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle$$

$$= \langle b_1 + a_1, b_2 + a_2 \rangle = \langle b_1, b_2 \rangle + \langle a_1, a_2 \rangle$$
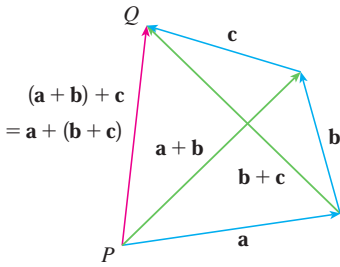
$$= \mathbf{b} + \mathbf{a}$$

We can see why Property 2 (the associative law) is true by looking at Figure 16 and applying the Triangle Law several times: The vector $\overrightarrow{PQ}$ is obtained either by first constructing $\mathbf{a} + \mathbf{b}$ and then adding $\mathbf{c}$ or by adding $\mathbf{a}$ to the vector $\mathbf{b} + \mathbf{c}$.

Three vectors in V_3 play a special role. Let

$$\mathbf{i} = \langle 1, 0, 0 \rangle \qquad \mathbf{j} = \langle 0, 1, 0 \rangle \qquad \mathbf{k} = \langle 0, 0, 1 \rangle$$

FIGURE 16

These vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are called the **standard basis vectors**. They have length 1 and point in the directions of the positive x-, y-, and z-axes. Similarly, in two dimensions we define $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$. (See Figure 17.)
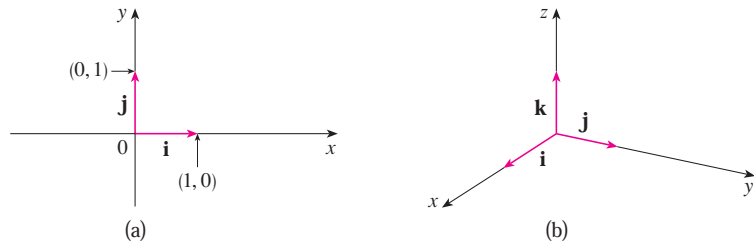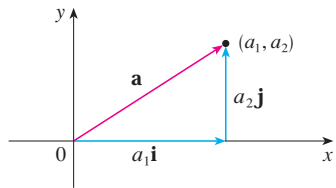

FIGURE 17
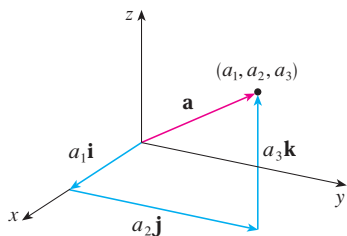Standard basis vectors in V_2 and V_3

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then we can write

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle$$

$$= a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle$$

$$\boxed{2} \qquad \mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$$

Thus any vector in V_3 can be expressed in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. For instance,

$$\langle 1, -2, 6 \rangle = \mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$$

Similarly, in two dimensions, we can write

$$\boxed{3} \qquad \mathbf{a} = \langle a_1, a_2 \rangle = a_1 \mathbf{i} + a_2 \mathbf{j}$$

(a) $\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j}$

(b) $\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$

FIGURE 18

See Figure 18 for the geometric interpretation of Equations 3 and 2 and compare with Figure 17.

EXAMPLE 5 If $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + 7\mathbf{k}$, express the vector $2\mathbf{a} + 3\mathbf{b}$ in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

SOLUTION Using Properties 1, 2, 5, 6, and 7 of vectors, we have

$$2\mathbf{a} + 3\mathbf{b} = 2(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) + 3(4\mathbf{i} + 7\mathbf{k})$$
$$= 2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k} + 12\mathbf{i} + 21\mathbf{k} = 14\mathbf{i} + 4\mathbf{j} + 15\mathbf{k}$$

A **unit vector** is a vector whose length is 1. For instance, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are all unit vectors. In general, if $\mathbf{a} \neq \mathbf{0}$, then the unit vector that has the same direction as $\mathbf{a}$ is

4
$$\mathbf{u} = \frac{1}{|\mathbf{a}|}\,\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

In order to verify this, we let $c = 1/|\mathbf{a}|$. Then $\mathbf{u} = c\mathbf{a}$ and c is a positive scalar, so $\mathbf{u}$ has the same direction as $\mathbf{a}$. Also

$$|\mathbf{u}| = |c\mathbf{a}| = |c|\,|\mathbf{a}| = \frac{1}{|\mathbf{a}|}\,|\mathbf{a}| = 1$$

EXAMPLE 6 Find the unit vector in the direction of the vector $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

SOLUTION The given vector has length

$$|2\mathbf{i} - \mathbf{j} - 2\mathbf{k}| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$$

so, by Equation 4, the unit vector with the same direction is

$$\tfrac{1}{3}(2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = \tfrac{2}{3}\mathbf{i} - \tfrac{1}{3}\mathbf{j} - \tfrac{2}{3}\mathbf{k}$$

Applications

Vectors are useful in many aspects of physics and engineering. In Chapter 13 we will see how they describe the velocity and acceleration of objects moving in space. Here we look at forces.

A force is represented by a vector because it has both a magnitude (measured in pounds or newtons) and a direction. If several forces are acting on an object, the **resultant force** experienced by the object is the vector sum of these forces.

EXAMPLE 7 A 100-lb weight hangs from two wires as shown in Figure 19. Find the tensions (forces) $\mathbf{T}_1$ and $\mathbf{T}_2$ in both wires and the magnitudes of the tensions.

SOLUTION We first express $\mathbf{T}_1$ and $\mathbf{T}_2$ in terms of their horizontal and vertical components. From Figure 20 we see that

5
$$\mathbf{T}_1 = -|\mathbf{T}_1|\cos 50°\,\mathbf{i} + |\mathbf{T}_1|\sin 50°\,\mathbf{j}$$

6
$$\mathbf{T}_2 = |\mathbf{T}_2|\cos 32°\,\mathbf{i} + |\mathbf{T}_2|\sin 32°\,\mathbf{j}$$

The resultant $\mathbf{T}_1 + \mathbf{T}_2$ of the tensions counterbalances the weight $\mathbf{w}$ and so we must have

$$\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{w} = 100\mathbf{j}$$

Thus

$$\left(-|\mathbf{T}_1|\cos 50° + |\mathbf{T}_2|\cos 32°\right)\mathbf{i} + \left(|\mathbf{T}_1|\sin 50° + |\mathbf{T}_2|\sin 32°\right)\mathbf{j} = 100\mathbf{j}$$

Gibbs

Josiah Willard Gibbs (1839–1903), a professor of mathematical physics at Yale College, published the first book on vectors, *Vector Analysis*, in 1881. More complicated objects, called quaternions, had earlier been invented by Hamilton as mathematical tools for describing space, but they weren't easy for scientists to use. Quaternions have a scalar part and a vector part. Gibb's idea was to use the vector part separately. Maxwell and Heaviside had similar ideas, but Gibb's approach has proved to be the most convenient way to study space.

FIGURE 19

FIGURE 20

Equating components, we get

$$-|\mathbf{T}_1|\cos 50° + |\mathbf{T}_2|\cos 32° = 0$$

$$|\mathbf{T}_1|\sin 50° + |\mathbf{T}_2|\sin 32° = 100$$

Solving the first of these equations for $|\mathbf{T}_2|$ and substituting into the second, we get

$$|\mathbf{T}_1|\sin 50° + \frac{|\mathbf{T}_1|\cos 50°}{\cos 32°}\sin 32° = 100$$

So the magnitudes of the tensions are

$$|\mathbf{T}_1| = \frac{100}{\sin 50° + \tan 32° \cos 50°} \approx 85.64 \text{ lb}$$

and

$$|\mathbf{T}_2| = \frac{|\mathbf{T}_1|\cos 50°}{\cos 32°} \approx 64.91 \text{ lb}$$

Substituting these values in $\boxed{5}$ and $\boxed{6}$, we obtain the tension vectors

$$\mathbf{T}_1 \approx -55.05\,\mathbf{i} + 65.60\,\mathbf{j} \qquad \mathbf{T}_2 \approx 55.05\,\mathbf{i} + 34.40\,\mathbf{j}$$

12.2   Exercises

1. Are the following quantities vectors or scalars? Explain.
  (a) The cost of a theater ticket
  (b) The current in a river
  (c) The initial flight path from Houston to Dallas
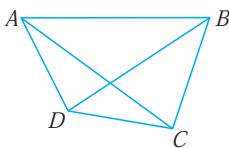 (d) The population of the world

2. What is the relationship between the point $(4, 7)$ and the vector $\langle 4, 7 \rangle$? Illustrate with a sketch.
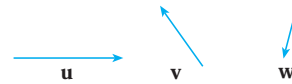
3. Name all the equal vectors in the parallelogram shown.
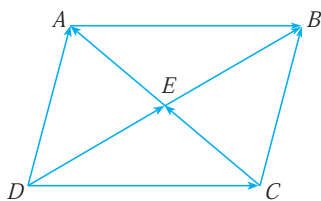
4. Write each combination of vectors as a single vector.
  (a) $\overrightarrow{AB} + \overrightarrow{BC}$ (b) $\overrightarrow{CD} + \overrightarrow{DB}$
  (c) $\overrightarrow{DB} - \overrightarrow{AB}$ (d) $\overrightarrow{DC} + \overrightarrow{CA} + \overrightarrow{AB}$

5. Copy the vectors in the figure and use them to draw the following vectors.
  (a) $\mathbf{u} + \mathbf{v}$ (b) $\mathbf{u} + \mathbf{w}$
  (c) $\mathbf{v} + \mathbf{w}$ (d) $\mathbf{u} - \mathbf{v}$
  (e) $\mathbf{v} + \mathbf{u} + \mathbf{w}$ (f) $\mathbf{u} - \mathbf{w} - \mathbf{v}$

6. Copy the vectors in the figure and use them to draw the following vectors.
  (a) $\mathbf{a} + \mathbf{b}$ (b) $\mathbf{a} - \mathbf{b}$
  (c) $\frac{1}{2}\mathbf{a}$ (d) $-3\mathbf{b}$
  (e) $\mathbf{a} + 2\mathbf{b}$ (f) $2\mathbf{b} - \mathbf{a}$

7. In the figure, the tip of $\mathbf{c}$ and the tail of $\mathbf{d}$ are both the midpoint of QR. Express $\mathbf{c}$ and $\mathbf{d}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
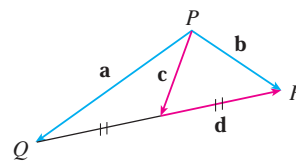
1. Homework Hints available at stewartcalculus.com

8. If the vectors in the figure satisfy $|\mathbf{u}| = |\mathbf{v}| = 1$ and $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$, what is $|\mathbf{w}|$?
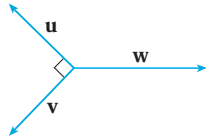
9–14 Find a vector $\mathbf{a}$ with representation given by the directed line segment $\overrightarrow{AB}$. Draw $\overrightarrow{AB}$ and the equivalent representation starting at the origin.

9. $A(-1, 1)$, $B(3, 2)$ **10.** $A(-4, -1)$, $B(1, 2)$

11. $A(-1, 3)$, $B(2, 2)$ **12.** $A(2, 1)$, $B(0, 6)$

13. $A(0, 3, 1)$, $B(2, 3, -1)$ **14.** $A(4, 0, -2)$, $B(4, 2, 1)$

15–18 Find the sum of the given vectors and illustrate geometrically.

15. $\langle -1, 4 \rangle$, $\langle 6, -2 \rangle$ **16.** $\langle 3, -1 \rangle$, $\langle -1, 5 \rangle$

17. $\langle 3, 0, 1 \rangle$, $\langle 0, 8, 0 \rangle$ **18.** $\langle 1, 3, -2 \rangle$, $\langle 0, 0, 6 \rangle$

19–22 Find $\mathbf{a} + \mathbf{b}$, $2\mathbf{a} + 3\mathbf{b}$, $|\mathbf{a}|$, and $|\mathbf{a} - \mathbf{b}|$.

19. $\mathbf{a} = \langle 5, -12 \rangle$, $\mathbf{b} = \langle -3, -6 \rangle$

20. $\mathbf{a} = 4\mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{i} - 2\mathbf{j}$

21. $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = -2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$

22. $\mathbf{a} = 2\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = 2\mathbf{j} - \mathbf{k}$

23–25 Find a unit vector that has the same direction as the given vector.

23. $-3\mathbf{i} + 7\mathbf{j}$ **24.** $\langle -4, 2, 4 \rangle$

**25.** $8\mathbf{i} - \mathbf{j} + 4\mathbf{k}$

26. Find a vector that has the same direction as $\langle -2, 4, 2 \rangle$ but has length 6.

**27–28** What is the angle between the given vector and the positive direction of the x-axis?

27. $\mathbf{i} + \sqrt{3}\,\mathbf{j}$ **28.** $8\mathbf{i} + 6\mathbf{j}$

29. If $\mathbf{v}$ lies in the first quadrant and makes an angle $\pi/3$ with the positive x-axis and $|\mathbf{v}| = 4$, find $\mathbf{v}$ in component form.

30. If a child pulls a sled through the snow on a level path with a force of 50 N exerted at an angle of $38°$ above the horizontal, find the horizontal and vertical components of the force.

31. A quarterback throws a football with angle of elevation $40°$ and speed 60 ft/s. Find the horizontal and vertical components of the velocity vector.

32–33 Find the magnitude of the resultant force and the angle it makes with the positive x-axis.

32.

33.
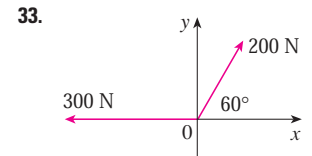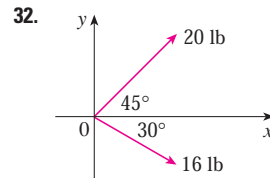
34. The magnitude of a velocity vector is called *speed.* Suppose that a wind is blowing from the direction N45°W at a speed of 50 km/h. (This means that the direction from which the wind blows is 45° west of the northerly direction.) A pilot is steering a plane in the direction N60°E at an airspeed (speed in still air) of 250 km/h. The *true course,* or *track,* of the plane is the direction of the resultant of the velocity vectors of the plane and the wind. The *ground speed* of the plane is the magnitude of the resultant. Find the true course and the ground speed of the plane.

35. A woman walks due west on the deck of a ship at 3 mi/h. The ship is moving north at a speed of 22 mi/h. Find the speed and direction of the woman relative to the surface of the water.

36. Ropes 3 m and 5 m in length are fastened to a holiday decoration that is suspended over a town square. The decoration has a mass of 5 kg. The ropes, fastened at different heights, make angles of 52° and 40° with the horizontal. Find the tension in each wire and the magnitude of each tension.
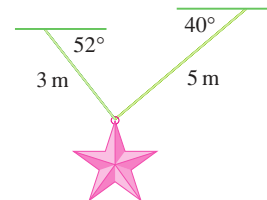
37. A clothesline is tied between two poles, 8 m apart. The line is quite taut and has negligible sag. When a wet shirt with a mass of 0.8 kg is hung at the middle of the line, the midpoint is pulled down 8 cm. Find the tension in each half of the clothesline.

38. The tension $\mathbf{T}$ at each end of the chain has magnitude 25 N (see the figure). What is the weight of the chain?
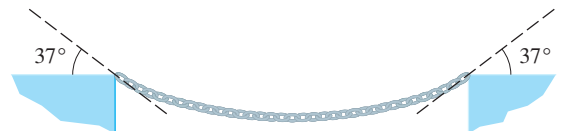
39. A boatman wants to cross a canal that is 3 km wide and wants to land at a point 2 km upstream from his starting point. The current in the canal flows at 3.5 km/h and the speed of his boat is 13 km/h.
(a) In what direction should he steer?
(b) How long will the trip take?

40. Three forces act on an object. Two of the forces are at an angle of 100° to each other and have magnitudes 25 N and 12 N. The third is perpendicular to the plane of these two forces and has magnitude 4 N. Calculate the magnitude of the force that would exactly counterbalance these three forces.

41. Find the unit vectors that are parallel to the tangent line to the parabola $y = x^2$ at the point $(2, 4)$.

42. (a) Find the unit vectors that are parallel to the tangent line to the curve $y = 2 \sin x$ at the point $(\pi/6, 1)$.
(b) Find the unit vectors that are perpendicular to the tangent line.
(c) Sketch the curve $y = 2 \sin x$ and the vectors in parts (a) and (b), all starting at $(\pi/6, 1)$.

43. If A, B, and C are the vertices of a triangle, find
$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}.$$

44. Let C be the point on the line segment AB that is twice as far from B as it is from A. If $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, and $\mathbf{c} = \overrightarrow{OC}$, show that $\mathbf{c} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$.

45. (a) Draw the vectors $\mathbf{a} = \langle 3, 2 \rangle$, $\mathbf{b} = \langle 2, -1 \rangle$, and $\mathbf{c} = \langle 7, 1 \rangle$.
(b) Show, by means of a sketch, that there are scalars s and t such that $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$.
(c) Use the sketch to estimate the values of s and t.
(d) Find the exact values of s and t.

46. Suppose that $\mathbf{a}$ and $\mathbf{b}$ are nonzero vectors that are not parallel and $\mathbf{c}$ is any vector in the plane determined by $\mathbf{a}$ and $\mathbf{b}$. Give a geometric argument to show that $\mathbf{c}$ can be written as $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$ for suitable scalars s and t. Then give an argument using components.

47. If $\mathbf{r} = \langle x, y, z \rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$, describe the set of all points (x, y, z) such that $|\mathbf{r} - \mathbf{r}_0| = 1$.

48. If $\mathbf{r} = \langle x, y \rangle$, $\mathbf{r}_1 = \langle x_1, y_1 \rangle$, and $\mathbf{r}_2 = \langle x_2, y_2 \rangle$, describe the set of all points (x, y) such that $|\mathbf{r} - \mathbf{r}_1| + |\mathbf{r} - \mathbf{r}_2| = k$, where $k > |\mathbf{r}_1 - \mathbf{r}_2|$.

49. Figure 16 gives a geometric demonstration of Property 2 of vectors. Use components to give an algebraic proof of this fact for the case $n = 2$.

50. Prove Property 5 of vectors algebraically for the case $n = 3$. Then use similar triangles to give a geometric proof.

51. Use vectors to prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

52. Suppose the three coordinate planes are all mirrored and a light ray given by the vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ first strikes the xz-plane, as shown in the figure. Use the fact that the angle of incidence equals the angle of reflection to show that the direction of the reflected ray is given by $\mathbf{b} = \langle a_1, -a_2, a_3 \rangle$. Deduce that, after being reflected by all three mutually perpendicular mirrors, the resulting ray is parallel to the initial ray. (American space scientists used this principle, together with laser beams and an array of corner mirrors on the moon, to calculate very precisely the distance from the earth to the moon.)
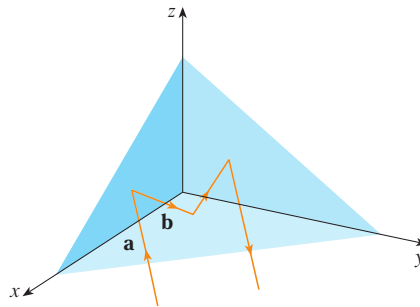
12.3 The Dot Product

So far we have added two vectors and multiplied a vector by a scalar. The question arises: Is it possible to multiply two vectors so that their product is a useful quantity? One such product is the dot product, whose definition follows. Another is the cross product, which is discussed in the next section.

> **1** **Definition** If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is the number $\mathbf{a} \cdot \mathbf{b}$ given by
> $$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Thus, to find the dot product of $\mathbf{a}$ and $\mathbf{b}$, we multiply corresponding components and add. The result is not a vector. It is a real number, that is, a scalar. For this reason, the dot product is sometimes called the **scalar product** (or **inner product**). Although Definition 1 is given for three-dimensional vectors, the dot product of two-dimensional vectors is defined in a similar fashion:

$$\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2$$

V **EXAMPLE 1**

$$\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle = 2(3) + 4(-1) = 2$$

$$\langle -1, 7, 4 \rangle \cdot \langle 6, 2, -\tfrac{1}{2} \rangle = (-1)(6) + 7(2) + 4\left(-\tfrac{1}{2}\right) = 6$$

$$(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \cdot (2\mathbf{j} - \mathbf{k}) = 1(0) + 2(2) + (-3)(-1) = 7$$ ▬

The dot product obeys many of the laws that hold for ordinary products of real numbers. These are stated in the following theorem.

> **2** **Properties of the Dot Product** If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_3 and c is a scalar, then
>
> 1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$ 2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
> 3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ 4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
> 5. $\mathbf{0} \cdot \mathbf{a} = 0$

These properties are easily proved using Definition 1. For instance, here are the proofs of Properties 1 and 3:

1. $\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2 = |\mathbf{a}|^2$

3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \langle a_1, a_2, a_3 \rangle \cdot \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle$

$$= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3)$$

$$= a_1 b_1 + a_1 c_1 + a_2 b_2 + a_2 c_2 + a_3 b_3 + a_3 c_3$$

$$= (a_1 b_1 + a_2 b_2 + a_3 b_3) + (a_1 c_1 + a_2 c_2 + a_3 c_3)$$

$$= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

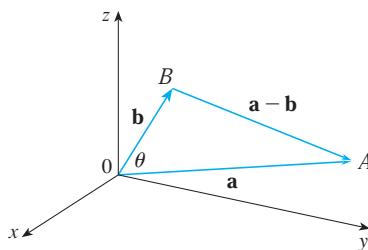The proofs of the remaining properties are left as exercises. ▬

The dot product $\mathbf{a} \cdot \mathbf{b}$ can be given a geometric interpretation in terms of the **angle** θ **between a and b**, which is defined to be the angle between the representations of $\mathbf{a}$ and $\mathbf{b}$ that start at the origin, where $0 \le \theta \le \pi$. In other words, θ is the angle between the line segments $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in Figure 1. Note that if $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors, then $\theta = 0$ or $\theta = \pi$.

The formula in the following theorem is used by physicists as the *definition* of the dot product.

FIGURE 1

> **3** **Theorem** If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then
>
> $$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

PROOF If we apply the Law of Cosines to triangle OAB in Figure 1, we get

4 $$|AB|^2 = |OA|^2 + |OB|^2 - 2|OA||OB| \cos \theta$$

(Observe that the Law of Cosines still applies in the limiting cases when $\theta = 0$ or π, or $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$.) But $|OA| = |\mathbf{a}|$, $|OB| = |\mathbf{b}|$, and $|AB| = |\mathbf{a} - \mathbf{b}|$, so Equation 4 becomes

5 $$|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos \theta$$

Using Properties 1, 2, and 3 of the dot product, we can rewrite the left side of this equation as follows:

$$|\mathbf{a} - \mathbf{b}|^2 = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$$
$$= \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b}$$
$$= |\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2$$

Therefore Equation 5 gives

$$|\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos \theta$$

Thus
$$-2\mathbf{a} \cdot \mathbf{b} = -2|\mathbf{a}||\mathbf{b}| \cos \theta$$

or
$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

EXAMPLE 2 If the vectors $\mathbf{a}$ and $\mathbf{b}$ have lengths 4 and 6, and the angle between them is $\pi/3$, find $\mathbf{a} \cdot \mathbf{b}$.

SOLUTION Using Theorem 3, we have

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\pi/3) = 4 \cdot 6 \cdot \tfrac{1}{2} = 12$$

The formula in Theorem 3 also enables us to find the angle between two vectors.

> **6 Corollary** If θ is the angle between the nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then
>
> $$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

V EXAMPLE 3 Find the angle between the vectors $\mathbf{a} = \langle 2, 2, -1 \rangle$ and $\mathbf{b} = \langle 5, -3, 2 \rangle$.

SOLUTION Since

$$|\mathbf{a}| = \sqrt{2^2 + 2^2 + (-1)^2} = 3 \qquad \text{and} \qquad |\mathbf{b}| = \sqrt{5^2 + (-3)^2 + 2^2} = \sqrt{38}$$

and since

$$\mathbf{a} \cdot \mathbf{b} = 2(5) + 2(-3) + (-1)(2) = 2$$

we have, from Corollary 6,

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{2}{3\sqrt{38}}$$

So the angle between $\mathbf{a}$ and $\mathbf{b}$ is

$$\theta = \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right) \approx 1.46 \quad \text{(or 84°)}$$

Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are called **perpendicular** or **orthogonal** if the angle between them is $\theta = \pi/2$. Then Theorem 3 gives

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\pi/2) = 0$$

and conversely if $\mathbf{a} \cdot \mathbf{b} = 0$, then $\cos \theta = 0$, so $\theta = \pi/2$. The zero vector $\mathbf{0}$ is considered to be perpendicular to all vectors. Therefore we have the following method for determining whether two vectors are orthogonal.
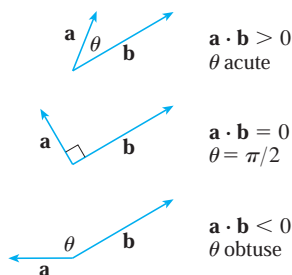
> **7**   Two vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

EXAMPLE 4   Show that $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is perpendicular to $5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.

SOLUTION   Since

$$(2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot (5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 2(5) + 2(-4) + (-1)(2) = 0$$

these vectors are perpendicular by $\boxed{7}$.

FIGURE 2

Because $\cos \theta > 0$ if $0 \leqslant \theta < \pi/2$ and $\cos \theta < 0$ if $\pi/2 < \theta \leqslant \pi$, we see that $\mathbf{a} \cdot \mathbf{b}$ is positive for $\theta < \pi/2$ and negative for $\theta > \pi/2$. We can think of $\mathbf{a} \cdot \mathbf{b}$ as measuring the extent to which $\mathbf{a}$ and $\mathbf{b}$ point in the same direction. The dot product $\mathbf{a} \cdot \mathbf{b}$ is positive if $\mathbf{a}$ and $\mathbf{b}$ point in the same general direction, 0 if they are perpendicular, and negative if they point in generally opposite directions (see Figure 2). In the extreme case where $\mathbf{a}$ and $\mathbf{b}$ point in exactly the same direction, we have $\theta = 0$, so $\cos \theta = 1$ and

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}|$$

If $\mathbf{a}$ and $\mathbf{b}$ point in exactly opposite directions, then $\theta = \pi$ and so $\cos \theta = -1$ and $\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}| |\mathbf{b}|$.

Direction Angles and Direction Cosines

The **direction angles** of a nonzero vector $\mathbf{a}$ are the angles α, β, and γ (in the interval $[0, \pi]$) that $\mathbf{a}$ makes with the positive x-, y-, and z-axes. (See Figure 3.)

The cosines of these direction angles, $\cos \alpha$, $\cos \beta$, and $\cos \gamma$, are called the **direction cosines** of the vector $\mathbf{a}$. Using Corollary 6 with $\mathbf{b}$ replaced by $\mathbf{i}$, we obtain

FIGURE 3

$$\boxed{8} \qquad \cos \alpha = \frac{\mathbf{a} \cdot \mathbf{i}}{|\mathbf{a}| |\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|}$$

(This can also be seen directly from Figure 3.)

Similarly, we also have

$$\boxed{9} \qquad \cos \beta = \frac{a_2}{|\mathbf{a}|} \qquad \cos \gamma = \frac{a_3}{|\mathbf{a}|}$$

By squaring the expressions in Equations 8 and 9 and adding, we see that

$$\boxed{10} \qquad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

We can also use Equations 8 and 9 to write

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = \langle |\mathbf{a}| \cos \alpha, |\mathbf{a}| \cos \beta, |\mathbf{a}| \cos \gamma \rangle$$
$$= |\mathbf{a}| \langle \cos \alpha, \cos \beta, \cos \gamma \rangle$$
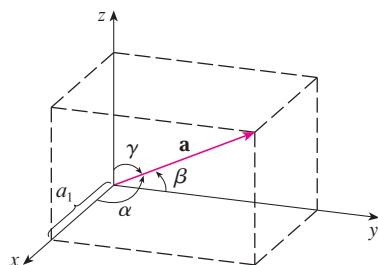
Therefore

$$\boxed{11} \qquad \frac{1}{|\mathbf{a}|}\,\mathbf{a} = \langle \cos\alpha,\, \cos\beta,\, \cos\gamma \rangle$$

which says that the direction cosines of $\mathbf{a}$ are the components of the unit vector in the direction of $\mathbf{a}$.

EXAMPLE 5   Find the direction angles of the vector $\mathbf{a} = \langle 1, 2, 3 \rangle$.

SOLUTION Since $|\mathbf{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$, Equations 8 and 9 give

$$\cos\alpha = \frac{1}{\sqrt{14}} \qquad \cos\beta = \frac{2}{\sqrt{14}} \qquad \cos\gamma = \frac{3}{\sqrt{14}}$$

and so

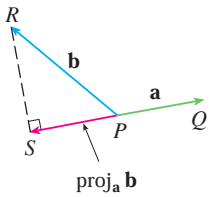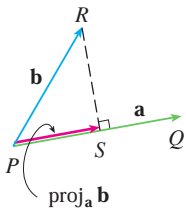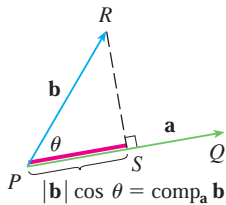$$\alpha = \cos^{-1}\!\left(\frac{1}{\sqrt{14}}\right) \approx 74° \qquad \beta = \cos^{-1}\!\left(\frac{2}{\sqrt{14}}\right) \approx 58° \qquad \gamma = \cos^{-1}\!\left(\frac{3}{\sqrt{14}}\right) \approx 37°$$

■ Projections

FIGURE 4
Vector projections

Figure 4 shows representations $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ with the same initial point P. If S is the foot of the perpendicular from R to the line containing $\overrightarrow{PQ}$, then the vector with representation $\overrightarrow{PS}$ is called the **vector projection** of $\mathbf{b}$ onto $\mathbf{a}$ and is denoted by $\text{proj}_{\mathbf{a}}\,\mathbf{b}$. (You can think of it as a shadow of $\mathbf{b}$).

The **scalar projection** of $\mathbf{b}$ onto $\mathbf{a}$ (also called the **component of b along a**) is defined to be the signed magnitude of the vector projection, which is the number $|\mathbf{b}|\cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. (See Figure 5.) This is denoted by $\text{comp}_{\mathbf{a}}\,\mathbf{b}$. Observe that it is negative if $\pi/2 < \theta \leq \pi$. The equation

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta = |\mathbf{a}|\big(|\mathbf{b}|\cos\theta\big)$$

shows that the dot product of $\mathbf{a}$ and $\mathbf{b}$ can be interpreted as the length of $\mathbf{a}$ times the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$. Since

$$|\mathbf{b}|\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{a}}{|\mathbf{a}|} \cdot \mathbf{b}$$

the component of $\mathbf{b}$ along $\mathbf{a}$ can be computed by taking the dot product of $\mathbf{b}$ with the unit vector in the direction of $\mathbf{a}$. We summarize these ideas as follows.

FIGURE 5
Scalar projection

Scalar projection of $\mathbf{b}$ onto $\mathbf{a}$: $\text{comp}_{\mathbf{a}}\,\mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$

Vector projection of $\mathbf{b}$ onto $\mathbf{a}$: $\text{proj}_{\mathbf{a}}\,\mathbf{b} = \left(\dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a}$

Notice that the vector projection is the scalar projection times the unit vector in the direction of $\mathbf{a}$.

V **EXAMPLE 6** Find the scalar projection and vector projection of $\mathbf{b} = \langle 1, 1, 2 \rangle$ onto $\mathbf{a} = \langle -2, 3, 1 \rangle$.

SOLUTION Since $|\mathbf{a}| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$, the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\text{comp}_\mathbf{a}\,\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{(-2)(1) + 3(1) + 1(2)}{\sqrt{14}} = \frac{3}{\sqrt{14}}$$

The vector projection is this scalar projection times the unit vector in the direction of $\mathbf{a}$:

$$\text{proj}_\mathbf{a}\,\mathbf{b} = \frac{3}{\sqrt{14}}\,\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3}{14}\,\mathbf{a} = \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle$$

One use of projections occurs in physics in calculating work. In Section 6.4 we defined the work done by a constant force F in moving an object through a distance d as $W = Fd$, but this applies only when the force is directed along the line of motion of the object. Suppose, however, that the constant force is a vector $\mathbf{F} = \overrightarrow{PR}$ pointing in some other direction, as in Figure 6. If the force moves the object from P to Q, then the **displacement vector** is $\mathbf{D} = \overrightarrow{PQ}$. The **work** done by this force is defined to be the product of the component of the force along $\mathbf{D}$ and the distance moved:
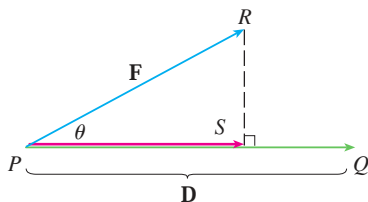
$$W = \left(|\mathbf{F}| \cos \theta \right) |\mathbf{D}|$$

FIGURE 6

But then, from Theorem 3, we have

12
$$W = |\mathbf{F}||\mathbf{D}| \cos \theta = \mathbf{F} \cdot \mathbf{D}$$

Thus the work done by a constant force $\mathbf{F}$ is the dot product $\mathbf{F} \cdot \mathbf{D}$, where $\mathbf{D}$ is the displacement vector.
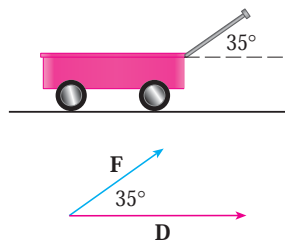
EXAMPLE 7 A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle of $35°$ above the horizontal. Find the work done by the force.

SOLUTION If $\mathbf{F}$ and $\mathbf{D}$ are the force and displacement vectors, as pictured in Figure 7, then the work done is

$$W = \mathbf{F} \cdot \mathbf{D} = |\mathbf{F}||\mathbf{D}| \cos 35°$$

$$= (70)(100) \cos 35° \approx 5734 \text{ N·m} = 5734 \text{ J}$$

FIGURE 7

EXAMPLE 8 A force is given by a vector $\mathbf{F} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and moves a particle from the point $P(2, 1, 0)$ to the point $Q(4, 6, 2)$. Find the work done.

SOLUTION The displacement vector is $\mathbf{D} = \overrightarrow{PQ} = \langle 2, 5, 2 \rangle$, so by Equation 12, the work done is

$$W = \mathbf{F} \cdot \mathbf{D} = \langle 3, 4, 5 \rangle \cdot \langle 2, 5, 2 \rangle$$

$$= 6 + 20 + 10 = 36$$

If the unit of length is meters and the magnitude of the force is measured in newtons, then the work done is 36 J.

12.3   **Exercises**

1. Which of the following expressions are meaningful? Which are meaningless? Explain.
   (a) $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$
   (b) $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$
   (c) $|\mathbf{a}|(\mathbf{b} \cdot \mathbf{c})$
   (d) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$
   (e) $\mathbf{a} \cdot \mathbf{b} + \mathbf{c}$
   (f) $|\mathbf{a}| \cdot (\mathbf{b} + \mathbf{c})$

2–10 Find $\mathbf{a} \cdot \mathbf{b}$.
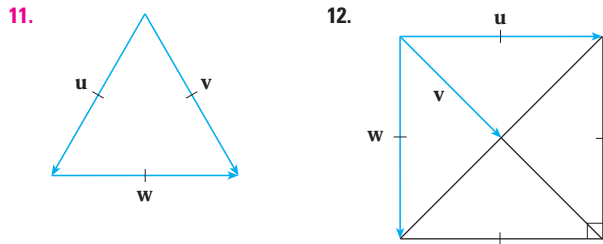
2. $\mathbf{a} = \langle -2, 3 \rangle$, $\mathbf{b} = \langle 0.7, 1.2 \rangle$

3. $\mathbf{a} = \langle -2, \frac{1}{3} \rangle$, $\mathbf{b} = \langle -5, 12 \rangle$

4. $\mathbf{a} = \langle 6, -2, 3 \rangle$, $\mathbf{b} = \langle 2, 5, -1 \rangle$

5. $\mathbf{a} = \langle 4, 1, \frac{1}{4} \rangle$, $\mathbf{b} = \langle 6, -3, -8 \rangle$

6. $\mathbf{a} = \langle p, -p, 2p \rangle$, $\mathbf{b} = \langle 2q, q, -q \rangle$

7. $\mathbf{a} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

8. $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{b} = 4\mathbf{i} + 5\mathbf{k}$

9. $|\mathbf{a}| = 6$, $|\mathbf{b}| = 5$,   the angle between $\mathbf{a}$ and $\mathbf{b}$ is $2\pi/3$

10. $|\mathbf{a}| = 3$, $|\mathbf{b}| = \sqrt{6}$,   the angle between $\mathbf{a}$ and $\mathbf{b}$ is $45°$

11–12 If $\mathbf{u}$ is a unit vector, find $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \cdot \mathbf{w}$.

11.

12.

13. (a) Show that $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$.
    (b) Show that $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$.

14. A street vendor sells a hamburgers, b hot dogs, and c soft drinks on a given day. He charges $2 for a hamburger, $1.50 for a hot dog, and $1 for a soft drink. If $\mathbf{A} = \langle a, b, c \rangle$ and $\mathbf{P} = \langle 2, 1.5, 1 \rangle$, what is the meaning of the dot product $\mathbf{A} \cdot \mathbf{P}$?

15–20 Find the angle between the vectors. (First find an exact expression and then approximate to the nearest degree.)

15. $\mathbf{a} = \langle 4, 3 \rangle$, $\mathbf{b} = \langle 2, -1 \rangle$

16. $\mathbf{a} = \langle -2, 5 \rangle$, $\mathbf{b} = \langle 5, 12 \rangle$

17. $\mathbf{a} = \langle 3, -1, 5 \rangle$, $\mathbf{b} = \langle -2, 4, 3 \rangle$

18. $\mathbf{a} = \langle 4, 0, 2 \rangle$, $\mathbf{b} = \langle 2, -1, 0 \rangle$

19. $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{k}$

20. $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = 4\mathbf{i} - 3\mathbf{k}$

21–22 Find, correct to the nearest degree, the three angles of the triangle with the given vertices.

21. $P(2, 0)$, $Q(0, 3)$, $R(3, 4)$

22. $A(1, 0, -1)$, $B(3, -2, 0)$, $C(1, 3, 3)$

23–24 Determine whether the given vectors are orthogonal, parallel, or neither.

23. (a) $\mathbf{a} = \langle -5, 3, 7 \rangle$, $\mathbf{b} = \langle 6, -8, 2 \rangle$
    (b) $\mathbf{a} = \langle 4, 6 \rangle$, $\mathbf{b} = \langle -3, 2 \rangle$
    (c) $\mathbf{a} = -\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$
    (d) $\mathbf{a} = 2\mathbf{i} + 6\mathbf{j} - 4\mathbf{k}$, $\mathbf{b} = -3\mathbf{i} - 9\mathbf{j} + 6\mathbf{k}$

24. (a) $\mathbf{u} = \langle -3, 9, 6 \rangle$, $\mathbf{v} = \langle 4, -12, -8 \rangle$
    (b) $\mathbf{u} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$
    (c) $\mathbf{u} = \langle a, b, c \rangle$, $\mathbf{v} = \langle -b, a, 0 \rangle$

25. Use vectors to decide whether the triangle with vertices $P(1, -3, -2)$, $Q(2, 0, -4)$, and $R(6, -2, -5)$ is right-angled.

26. Find the values of x such that the angle between the vectors $\langle 2, 1, -1 \rangle$, and $\langle 1, x, 0 \rangle$ is $45°$.

27. Find a unit vector that is orthogonal to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + \mathbf{k}$.

28. Find two unit vectors that make an angle of $60°$ with $\mathbf{v} = \langle 3, 4 \rangle$.

29–30 Find the acute angle between the lines.

29. $2x - y = 3$, $3x + y = 7$

30. $x + 2y = 7$, $5x - y = 2$

31–32 Find the acute angles between the curves at their points of intersection. (The angle between two curves is the angle between their tangent lines at the point of intersection.)

31. $y = x^2$, $y = x^3$

32. $y = \sin x$, $y = \cos x$, $0 \le x \le \pi/2$

33–37 Find the direction cosines and direction angles of the vector. (Give the direction angles correct to the nearest degree.)

33. $\langle 2, 1, 2 \rangle$

34. $\langle 6, 3, -2 \rangle$

35. $\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$

36. $\frac{1}{2}\mathbf{i} + \mathbf{j} + \mathbf{k}$

37. $\langle c, c, c \rangle$,   where $c > 0$

38. If a vector has direction angles $\alpha = \pi/4$ and $\beta = \pi/3$, find the third direction angle γ.

1. Homework Hints available at stewartcalculus.com

39–44 Find the scalar and vector projections of **b** onto **a**.

39. $\mathbf{a} = \langle -5, 12 \rangle$, $\mathbf{b} = \langle 4, 6 \rangle$

40. $\mathbf{a} = \langle 1, 4 \rangle$, $\mathbf{b} = \langle 2, 3 \rangle$

41. $\mathbf{a} = \langle 3, 6, -2 \rangle$, $\mathbf{b} = \langle 1, 2, 3 \rangle$

42. $\mathbf{a} = \langle -2, 3, -6 \rangle$, $\mathbf{b} = \langle 5, -1, 4 \rangle$

43. $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = \mathbf{j} + \frac{1}{2}\mathbf{k}$

44. $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

45. Show that the vector $\text{orth}_{\mathbf{a}}\, \mathbf{b} = \mathbf{b} - \text{proj}_{\mathbf{a}}\, \mathbf{b}$ is orthogonal to **a**. (It is called an **orthogonal projection** of **b**.)

46. For the vectors in Exercise 40, find $\text{orth}_{\mathbf{a}}\, \mathbf{b}$ and illustrate by drawing the vectors **a**, **b**, $\text{proj}_{\mathbf{a}}\, \mathbf{b}$, and $\text{orth}_{\mathbf{a}}\, \mathbf{b}$.

47. If $\mathbf{a} = \langle 3, 0, -1 \rangle$, find a vector **b** such that $\text{comp}_{\mathbf{a}}\, \mathbf{b} = 2$.

48. Suppose that **a** and **b** are nonzero vectors.
(a) Under what circumstances is $\text{comp}_{\mathbf{a}}\, \mathbf{b} = \text{comp}_{\mathbf{b}}\, \mathbf{a}$?
(b) Under what circumstances is $\text{proj}_{\mathbf{a}}\, \mathbf{b} = \text{proj}_{\mathbf{b}}\, \mathbf{a}$?

49. Find the work done by a force $\mathbf{F} = 8\mathbf{i} - 6\mathbf{j} + 9\mathbf{k}$ that moves an object from the point $(0, 10, 8)$ to the point $(6, 12, 20)$ along a straight line. The distance is measured in meters and the force in newtons.

50. A tow truck drags a stalled car along a road. The chain makes an angle of $30°$ with the road and the tension in the chain is 1500 N. How much work is done by the truck in pulling the car 1 km?

51. A sled is pulled along a level path through snow by a rope. A 30-lb force acting at an angle of $40°$ above the horizontal moves the sled 80 ft. Find the work done by the force.

52. A boat sails south with the help of a wind blowing in the direction S36°E with magnitude 400 lb. Find the work done by the wind as the boat moves 120 ft.

53. Use a scalar projection to show that the distance from a point $P_1(x_1, y_1)$ to the line $ax + by + c = 0$ is

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Use this formula to find the distance from the point $(-2, 3)$ to the line $3x - 4y + 5 = 0$.

54. If $\mathbf{r} = \langle x, y, z \rangle$, $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, show that the vector equation $(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0$ represents a sphere, and find its center and radius.

55. Find the angle between a diagonal of a cube and one of its edges.

56. Find the angle between a diagonal of a cube and a diagonal of one of its faces.

57. A molecule of methane, CH_4, is structured with the four hydrogen atoms at the vertices of a regular tetrahedron and the carbon atom at the centroid. The *bond angle* is the angle formed by the H—C—H combination; it is the angle between the lines that join the carbon atom to two of the hydrogen atoms. Show that the bond angle is about $109.5°$. [*Hint:* Take the vertices of the tetrahedron to be the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(1, 1, 1)$, as shown in the figure. Then the centroid is $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)$.]
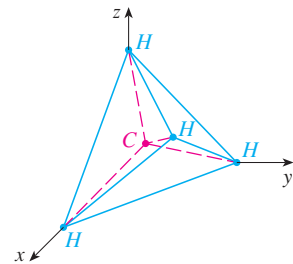
58. If $\mathbf{c} = |\mathbf{a}|\, \mathbf{b} + |\mathbf{b}|\, \mathbf{a}$, where **a**, **b**, and **c** are all nonzero vectors, show that **c** bisects the angle between **a** and **b**.

59. Prove Properties 2, 4, and 5 of the dot product (Theorem 2).

60. Suppose that all sides of a quadrilateral are equal in length and opposite sides are parallel. Use vector methods to show that the diagonals are perpendicular.

61. Use Theorem 3 to prove the Cauchy-Schwarz Inequality:

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}|\, |\mathbf{b}|$$

62. The Triangle Inequality for vectors is

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$$

(a) Give a geometric interpretation of the Triangle Inequality.
(b) Use the Cauchy-Schwarz Inequality from Exercise 61 to prove the Triangle Inequality. [*Hint:* Use the fact that $|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$ and use Property 3 of the dot product.]

63. The Parallelogram Law states that

$$|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$$

(a) Give a geometric interpretation of the Parallelogram Law.
(b) Prove the Parallelogram Law. (See the hint in Exercise 62.)

64. Show that if $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are orthogonal, then the vectors **u** and **v** must have the same length.

12.4   The Cross Product

Given two nonzero vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $b = \langle b_1, b_2, b_3 \rangle$, it is very useful to be able to find a nonzero vector $\mathbf{c}$ that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, as we will see in the next section and in Chapters 13 and 14. If $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ is such a vector, then $\mathbf{a} \cdot \mathbf{c} = 0$ and $\mathbf{b} \cdot \mathbf{c} = 0$ and so

$$\boxed{1} \qquad a_1 c_1 + a_2 c_2 + a_3 c_3 = 0$$

$$\boxed{2} \qquad b_1 c_1 + b_2 c_2 + b_3 c_3 = 0$$

To eliminate c_3 we multiply $\boxed{1}$ by b_3 and $\boxed{2}$ by a_3 and subtract:

$$\boxed{3} \qquad (a_1 b_3 - a_3 b_1) c_1 + (a_2 b_3 - a_3 b_2) c_2 = 0$$

Equation 3 has the form $p c_1 + q c_2 = 0$, for which an obvious solution is $c_1 = q$ and $c_2 = -p$. So a solution of $\boxed{3}$ is

$$c_1 = a_2 b_3 - a_3 b_2 \qquad\qquad c_2 = a_3 b_1 - a_1 b_3$$

Substituting these values into $\boxed{1}$ and $\boxed{2}$, we then get

$$c_3 = a_1 b_2 - a_2 b_1$$

This means that a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ is

$$\langle c_1, c_2, c_3 \rangle = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$$

The resulting vector is called the *cross product* of $\mathbf{a}$ and $\mathbf{b}$ and is denoted by $\mathbf{a} \times \mathbf{b}$.

Hamilton

The cross product was invented by the Irish mathematician Sir William Rowan Hamilton (1805–1865), who had created a precursor of vectors, called quaternions. When he was five years old Hamilton could read Latin, Greek, and Hebrew. At age eight he added French and Italian and when ten he could read Arabic and Sanskrit. At the age of 21, while still an undergraduate at Trinity College in Dublin, Hamilton was appointed Professor of Astronomy at the university and Royal Astronomer of Ireland!

> $\boxed{4}$ **Definition** If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is the vector
>
> $$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$$

Notice that the **cross product $\mathbf{a} \times \mathbf{b}$** of two vectors $\mathbf{a}$ and $\mathbf{b}$, unlike the dot product, is a vector. For this reason it is also called the **vector product**. Note that $\mathbf{a} \times \mathbf{b}$ is defined only when $\mathbf{a}$ and $\mathbf{b}$ are *three-dimensional* vectors.

In order to make Definition 4 easier to remember, we use the notation of determinants. A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For example,

$$\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix} = 2(4) - 1(-6) = 14$$

A **determinant of order 3** can be defined in terms of second-order determinants as follows:

$$\boxed{5} \qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Observe that each term on the right side of Equation 5 involves a number a_i in the first row of the determinant, and a_i is multiplied by the second-order determinant obtained from the left side by deleting the row and column in which a_i appears. Notice also the minus sign in the second term. For example,

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{vmatrix} = 1 \begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} - 2 \begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix} + (-1) \begin{vmatrix} 3 & 0 \\ -5 & 4 \end{vmatrix}$$

$$= 1(0 - 4) - 2(6 + 5) + (-1)(12 - 0) = -38$$

If we now rewrite Definition 4 using second-order determinants and the standard basis vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, we see that the cross product of the vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is

$$\boxed{6} \qquad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

In view of the similarity between Equations 5 and 6, we often write

$$\boxed{7} \qquad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Although the first row of the symbolic determinant in Equation 7 consists of vectors, if we expand it as if it were an ordinary determinant using the rule in Equation 5, we obtain Equation 6. The symbolic formula in Equation 7 is probably the easiest way of remembering and computing cross products.

V EXAMPLE 1 If $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, 7, -5 \rangle$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \mathbf{k}$$

$$= (-15 - 28)\mathbf{i} - (-5 - 8)\mathbf{j} + (7 - 6)\mathbf{k} = -43\mathbf{i} + 13\mathbf{j} + \mathbf{k}$$

V EXAMPLE 2 Show that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any vector $\mathbf{a}$ in V_3.

SOLUTION If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then

$$\mathbf{a} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= (a_2 a_3 - a_3 a_2)\mathbf{i} - (a_1 a_3 - a_3 a_1)\mathbf{j} + (a_1 a_2 - a_2 a_1)\mathbf{k}$$
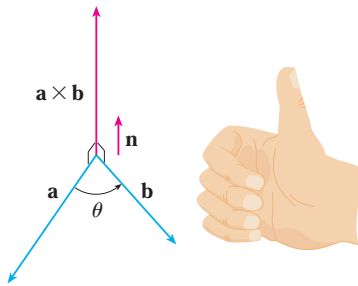
$$= 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$$

We constructed the cross product $\mathbf{a} \times \mathbf{b}$ so that it would be perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. This is one of the most important properties of a cross product, so let's emphasize and verify it in the following theorem and give a formal proof.

8 **Theorem** The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

PROOF In order to show that $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$, we compute their dot product as follows:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3$$

$$= a_1(a_2 b_3 - a_3 b_2) - a_2(a_1 b_3 - a_3 b_1) + a_3(a_1 b_2 - a_2 b_1)$$

$$= a_1 a_2 b_3 - a_1 b_2 a_3 - a_1 a_2 b_3 + b_1 a_2 a_3 + a_1 b_2 a_3 - b_1 a_2 a_3$$

$$= 0$$

A similar computation shows that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$. Therefore $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. ▮

If $\mathbf{a}$ and $\mathbf{b}$ are represented by directed line segments with the same initial point (as in Figure 1), then Theorem 8 says that the cross product $\mathbf{a} \times \mathbf{b}$ points in a direction perpendicular to the plane through $\mathbf{a}$ and $\mathbf{b}$. It turns out that the direction of $\mathbf{a} \times \mathbf{b}$ is given by the *right-hand rule:* If the fingers of your right hand curl in the direction of a rotation (through an angle less than $180°$) from $\mathbf{a}$ to $\mathbf{b}$, then your thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.

Now that we know the direction of the vector $\mathbf{a} \times \mathbf{b}$, the remaining thing we need to complete its geometric description is its length $|\mathbf{a} \times \mathbf{b}|$. This is given by the following theorem.

9 **Theorem** If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \leqslant \theta \leqslant \pi$), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

PROOF From the definitions of the cross product and length of a vector, we have

$$|\mathbf{a} \times \mathbf{b}|^2 = (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2$$

$$= a_2^2 b_3^2 - 2a_2 a_3 b_2 b_3 + a_3^2 b_2^2 + a_3^2 b_1^2 - 2a_1 a_3 b_1 b_3 + a_1^2 b_3^2$$

$$\quad + a_1^2 b_2^2 - 2a_1 a_2 b_1 b_2 + a_2^2 b_1^2$$

$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2$$

$$= |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$$

$$= |\mathbf{a}|^2 |\mathbf{b}|^2 - |\mathbf{a}|^2 |\mathbf{b}|^2 \cos^2\theta \qquad \text{(by Theorem 12.3.3)}$$

$$= |\mathbf{a}|^2 |\mathbf{b}|^2 (1 - \cos^2\theta)$$

$$= |\mathbf{a}|^2 |\mathbf{b}|^2 \sin^2\theta$$

Taking square roots and observing that $\sqrt{\sin^2\theta} = \sin \theta$ because $\sin \theta \geqslant 0$ when $0 \leqslant \theta \leqslant \pi$, we have

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta \qquad ▮$$

Since a vector is completely determined by its magnitude and direction, we can now say that $\mathbf{a} \times \mathbf{b}$ is the vector that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, whose orientation is deter-

---

$\mathbf{a} \times \mathbf{b}$

$\mathbf{n}$

$\mathbf{a}$ $\mathbf{b}$

θ

FIGURE 1
The right-hand rule gives
the direction of $\mathbf{a} \times \mathbf{b}$.

TEC Visual 12.4 shows how $\mathbf{a} \times \mathbf{b}$ changes as $\mathbf{b}$ changes.

Geometric characterization of $\mathbf{a} \times \mathbf{b}$

mined by the right-hand rule, and whose length is $|\mathbf{a}||\mathbf{b}|\sin\theta$. In fact, that is exactly how physicists *define* $\mathbf{a} \times \mathbf{b}$.

---

10 **Corollary** Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

PROOF Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\theta = 0$ or π. In either case $\sin\theta = 0$, so $|\mathbf{a} \times \mathbf{b}| = 0$ and therefore $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. ▬
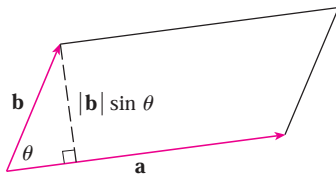
The geometric interpretation of Theorem 9 can be seen by looking at Figure 2. If $\mathbf{a}$ and $\mathbf{b}$ are represented by directed line segments with the same initial point, then they determine a parallelogram with base $|\mathbf{a}|$, altitude $|\mathbf{b}|\sin\theta$, and area

$$A = |\mathbf{a}|\big(|\mathbf{b}|\sin\theta\big) = |\mathbf{a} \times \mathbf{b}|$$

Thus we have the following way of interpreting the magnitude of a cross product.

The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

---

b $|\mathbf{b}|\sin\theta$

θ

a

FIGURE 2

EXAMPLE 3 Find a vector perpendicular to the plane that passes through the points $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

SOLUTION The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and is therefore perpendicular to the plane through P, Q, and R. We know from (12.2.1) that

$$\overrightarrow{PQ} = (-2 - 1)\mathbf{i} + (5 - 4)\mathbf{j} + (-1 - 6)\mathbf{k} = -3\mathbf{i} + \mathbf{j} - 7\mathbf{k}$$

$$\overrightarrow{PR} = (1 - 1)\mathbf{i} + (-1 - 4)\mathbf{j} + (1 - 6)\mathbf{k} = -5\mathbf{j} - 5\mathbf{k}$$

We compute the cross product of these vectors:

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix}$$

$$= (-5 - 35)\mathbf{i} - (15 - 0)\mathbf{j} + (15 - 0)\mathbf{k} = -40\mathbf{i} - 15\mathbf{j} + 15\mathbf{k}$$

So the vector $\langle -40, -15, 15 \rangle$ is perpendicular to the given plane. Any nonzero scalar multiple of this vector, such as $\langle -8, -3, 3 \rangle$, is also perpendicular to the plane. ▬

EXAMPLE 4 Find the area of the triangle with vertices $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

SOLUTION In Example 3 we computed that $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -40, -15, 15 \rangle$. The area of the parallelogram with adjacent sides PQ and PR is the length of this cross product:

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{(-40)^2 + (-15)^2 + 15^2} = 5\sqrt{82}$$

The area A of the triangle PQR is half the area of this parallelogram, that is, $\frac{5}{2}\sqrt{82}$. ▬

If we apply Theorems 8 and 9 to the standard basis vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ using $\theta = \pi/2$, we obtain

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

Observe that

$$\mathbf{i} \times \mathbf{j} \neq \mathbf{j} \times \mathbf{i}$$

⊘ Thus the cross product is not commutative. Also

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

whereas

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0}$$

⊘ So the associative law for multiplication does not usually hold; that is, in general,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

However, some of the usual laws of algebra *do* hold for cross products. The following theorem summarizes the properties of vector products.

---

11 Theorem If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

These properties can be proved by writing the vectors in terms of their components and using the definition of a cross product. We give the proof of Property 5 and leave the remaining proofs as exercises.

PROOF OF PROPERTY 5 If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$, then

$$
\begin{aligned}
\textbf{12} \qquad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= a_1(b_2 c_3 - b_3 c_2) + a_2(b_3 c_1 - b_1 c_3) + a_3(b_1 c_2 - b_2 c_1) \\
&= a_1 b_2 c_3 - a_1 b_3 c_2 + a_2 b_3 c_1 - a_2 b_1 c_3 + a_3 b_1 c_2 - a_3 b_2 c_1 \\
&= (a_2 b_3 - a_3 b_2)c_1 + (a_3 b_1 - a_1 b_3)c_2 + (a_1 b_2 - a_2 b_1)c_3 \\
&= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}
\end{aligned}
$$

Triple Products

The product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ that occurs in Property 5 is called the **scalar triple product** of the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. Notice from Equation 12 that we can write the scalar triple product as a determinant:

$$
\textbf{13} \qquad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}
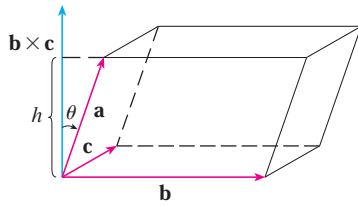$$

FIGURE 3

The geometric significance of the scalar triple product can be seen by considering the parallelepiped determined by the vectors **a**, **b**, and **c**. (See Figure 3.) The area of the base parallelogram is $A = |\mathbf{b} \times \mathbf{c}|$. If θ is the angle between **a** and $\mathbf{b} \times \mathbf{c}$, then the height h of the parallelepiped is $h = |\mathbf{a}||\cos\theta|$. (We must use $|\cos\theta|$ instead of $\cos\theta$ in case $\theta > \pi/2$.) Therefore the volume of the parallelepiped is

$$V = Ah = |\mathbf{b} \times \mathbf{c}||\mathbf{a}||\cos\theta| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

Thus we have proved the following formula.

---

14 The volume of the parallelepiped determined by the vectors **a**, **b**, and **c** is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

If we use the formula in **14** and discover that the volume of the parallelepiped determined by **a**, **b**, and **c** is 0, then the vectors must lie in the same plane; that is, they are **coplanar**.

V EXAMPLE 5 Use the scalar triple product to show that the vectors $\mathbf{a} = \langle 1, 4, -7 \rangle$, $\mathbf{b} = \langle 2, -1, 4 \rangle$, and $\mathbf{c} = \langle 0, -9, 18 \rangle$ are coplanar.

SOLUTION We use Equation 13 to compute their scalar triple product:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4 \begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} - 7 \begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix}$$

$$= 1(18) - 4(36) - 7(-18) = 0$$

Therefore, by **14**, the volume of the parallelepiped determined by **a**, **b**, and **c** is 0. This means that **a**, **b**, and **c** are coplanar. ∎

The product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ that occurs in Property 6 is called the **vector triple product** of **a**, **b**, and **c**. Property 6 will be used to derive Kepler's First Law of planetary motion in Chapter 13. Its proof is left as Exercise 50.

Torque

The idea of a cross product occurs often in physics. In particular, we consider a force **F** acting on a rigid body at a point given by a position vector **r**. (For instance, if we tighten a bolt by applying a force to a wrench as in Figure 4, we produce a turning effect.) The **torque** $\boldsymbol{\tau}$ (relative to the origin) is defined to be the cross product of the position and force vectors

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

and measures the tendency of the body to rotate about the origin. The direction of the torque vector indicates the axis of rotation. According to Theorem 9, the magnitude of the torque vector is

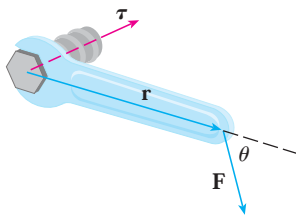$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin\theta$$
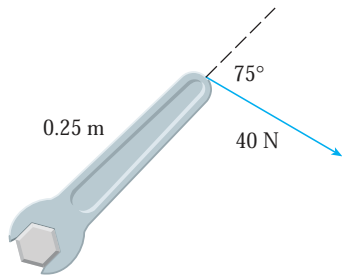

FIGURE 4

where θ is the angle between the position and force vectors. Observe that the only component of $\mathbf{F}$ that can cause a rotation is the one perpendicular to $\mathbf{r}$, that is, $|\mathbf{F}|\sin\theta$. The magnitude of the torque is equal to the area of the parallelogram determined by $\mathbf{r}$ and $\mathbf{F}$.

EXAMPLE 6   A bolt is tightened by applying a 40-N force to a 0.25-m wrench as shown in Figure 5. Find the magnitude of the torque about the center of the bolt.

SOLUTION   The magnitude of the torque vector is

$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}|\sin 75° = (0.25)(40)\sin 75°$$

$$= 10\sin 75° \approx 9.66 \text{ N·m}$$

If the bolt is right-threaded, then the torque vector itself is

$$\boldsymbol{\tau} = |\boldsymbol{\tau}|\,\mathbf{n} \approx 9.66\,\mathbf{n}$$

where $\mathbf{n}$ is a unit vector directed down into the page.

FIGURE 5

12.4 Exercises

1–7 Find the cross product $\mathbf{a} \times \mathbf{b}$ and verify that it is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

1. $\mathbf{a} = \langle 6, 0, -2 \rangle, \quad \mathbf{b} = \langle 0, 8, 0 \rangle$

2. $\mathbf{a} = \langle 1, 1, -1 \rangle, \quad \mathbf{b} = \langle 2, 4, 6 \rangle$

3. $\mathbf{a} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}, \quad \mathbf{b} = -\mathbf{i} + 5\mathbf{k}$

4. $\mathbf{a} = \mathbf{j} + 7\mathbf{k}, \quad \mathbf{b} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$

5. $\mathbf{a} = \mathbf{i} - \mathbf{j} - \mathbf{k}, \quad \mathbf{b} = \frac{1}{2}\mathbf{i} + \mathbf{j} + \frac{1}{2}\mathbf{k}$

6. $\mathbf{a} = t\mathbf{i} + \cos t\,\mathbf{j} + \sin t\,\mathbf{k}, \quad \mathbf{b} = \mathbf{i} - \sin t\,\mathbf{j} + \cos t\,\mathbf{k}$

7. $\mathbf{a} = \langle t, 1, 1/t \rangle, \quad \mathbf{b} = \langle t^2, t^2, 1 \rangle$

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8. If $\mathbf{a} = \mathbf{i} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{j} + \mathbf{k}$, find $\mathbf{a} \times \mathbf{b}$. Sketch $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a} \times \mathbf{b}$ as vectors starting at the origin.

9–12 Find the vector, not with determinants, but by using properties of cross products.

9. $(\mathbf{i} \times \mathbf{j}) \times \mathbf{k}$

10. $\mathbf{k} \times (\mathbf{i} - 2\mathbf{j})$

11. $(\mathbf{j} - \mathbf{k}) \times (\mathbf{k} - \mathbf{i})$

12. $(\mathbf{i} + \mathbf{j}) \times (\mathbf{i} - \mathbf{j})$

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13. State whether each expression is meaningful. If not, explain why. If so, state whether it is a vector or a scalar.
(a) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$
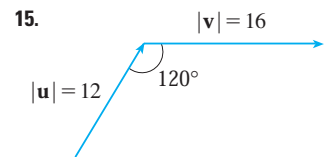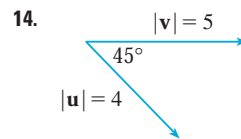(b) $\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$
(c) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$
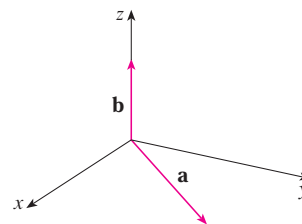(d) $\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$
(e) $(\mathbf{a} \cdot \mathbf{b}) \times (\mathbf{c} \cdot \mathbf{d})$
(f) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$

14–15 Find $|\mathbf{u} \times \mathbf{v}|$ and determine whether $\mathbf{u} \times \mathbf{v}$ is directed into the page or out of the page.

14.

15.

16. The figure shows a vector $\mathbf{a}$ in the xy-plane and a vector $\mathbf{b}$ in the direction of $\mathbf{k}$. Their lengths are $|\mathbf{a}| = 3$ and $|\mathbf{b}| = 2$.
(a) Find $|\mathbf{a} \times \mathbf{b}|$.
(b) Use the right-hand rule to decide whether the components of $\mathbf{a} \times \mathbf{b}$ are positive, negative, or 0.

17. If $\mathbf{a} = \langle 2, -1, 3 \rangle$ and $\mathbf{b} = \langle 4, 2, 1 \rangle$, find $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$.

18. If $\mathbf{a} = \langle 1, 0, 1 \rangle$, $\mathbf{b} = \langle 2, 1, -1 \rangle$, and $\mathbf{c} = \langle 0, 1, 3 \rangle$, show that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

19. Find two unit vectors orthogonal to both $\langle 3, 2, 1 \rangle$ and $\langle -1, 1, 0 \rangle$.

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1. Homework Hints available at stewartcalculus.com

20. Find two unit vectors orthogonal to both $\mathbf{j} - \mathbf{k}$ and $\mathbf{i} + \mathbf{j}$.

21. Show that $\mathbf{0} \times \mathbf{a} = \mathbf{0} = \mathbf{a} \times \mathbf{0}$ for any vector $\mathbf{a}$ in V_3.

22. Show that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$ for all vectors $\mathbf{a}$ and $\mathbf{b}$ in V_3.

23. Prove Property 1 of Theorem 11.

24. Prove Property 2 of Theorem 11.

25. Prove Property 3 of Theorem 11.

26. Prove Property 4 of Theorem 11.

27. Find the area of the parallelogram with vertices $A(-2, 1)$, $B(0, 4)$, $C(4, 2)$, and $D(2, -1)$.

28. Find the area of the parallelogram with vertices $K(1, 2, 3)$, $L(1, 3, 6)$, $M(3, 8, 6)$, and $N(3, 7, 3)$.

29–32 (a) Find a nonzero vector orthogonal to the plane through the points P, Q, and R, and (b) find the area of triangle PQR.

29. $P(1, 0, 1)$, $Q(-2, 1, 3)$, $R(4, 2, 5)$

30. $P(0, 0, -3)$, $Q(4, 2, 0)$, $R(3, 3, 1)$

31. $P(0, -2, 0)$, $Q(4, 1, -2)$, $R(5, 3, 1)$

**32.** $P(-1, 3, 1)$, $Q(0, 5, 2)$, $R(4, 3, -1)$

33–34 Find the volume of the parallelepiped determined by the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

33. $\mathbf{a} = \langle 1, 2, 3 \rangle$, $\mathbf{b} = \langle -1, 1, 2 \rangle$, $\mathbf{c} = \langle 2, 1, 4 \rangle$

34. $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{j} + \mathbf{k}$, $\mathbf{c} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

35–36 Find the volume of the parallelepiped with adjacent edges PQ, PR, and PS.

35. $P(-2, 1, 0)$, $Q(2, 3, 2)$, $R(1, 4, -1)$, $S(3, 6, 1)$

36. $P(3, 0, 1)$, $Q(-1, 2, 5)$, $R(5, 1, -1)$, $S(0, 4, 2)$

37. Use the scalar triple product to verify that the vectors $\mathbf{u} = \mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} - \mathbf{j}$, and $\mathbf{w} = 5\mathbf{i} + 9\mathbf{j} - 4\mathbf{k}$ are coplanar.

38. Use the scalar triple product to determine whether the points $A(1, 3, 2)$, $B(3, -1, 6)$, $C(5, 2, 0)$, and $D(3, 6, -4)$ lie in the same plane.

39. A bicycle pedal is pushed by a foot with a 60-N force as shown. The shaft of the pedal is 18 cm long. Find the magnitude of the torque about P.
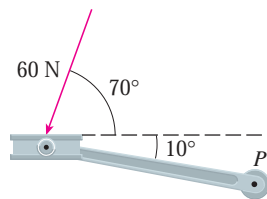
40. Find the magnitude of the torque about P if a 36-lb force is applied as shown.
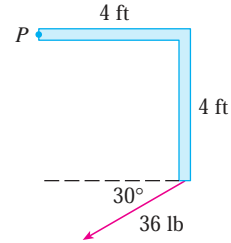
41. A wrench 30 cm long lies along the positive y-axis and grips a bolt at the origin. A force is applied in the direction $\langle 0, 3, -4 \rangle$ at the end of the wrench. Find the magnitude of the force needed to supply 100 N·m of torque to the bolt.

42. Let $\mathbf{v} = 5\mathbf{j}$ and let $\mathbf{u}$ be a vector with length 3 that starts at the origin and rotates in the xy-plane. Find the maximum and minimum values of the length of the vector $\mathbf{u} \times \mathbf{v}$. In what direction does $\mathbf{u} \times \mathbf{v}$ point?

43. If $\mathbf{a} \cdot \mathbf{b} = \sqrt{3}$ and $\mathbf{a} \times \mathbf{b} = \langle 1, 2, 2 \rangle$, find the angle between $\mathbf{a}$ and $\mathbf{b}$.

44. (a) Find all vectors $\mathbf{v}$ such that
$$\langle 1, 2, 1 \rangle \times \mathbf{v} = \langle 3, 1, -5 \rangle$$
(b) Explain why there is no vector $\mathbf{v}$ such that
$$\langle 1, 2, 1 \rangle \times \mathbf{v} = \langle 3, 1, 5 \rangle$$

45. (a) Let P be a point not on the line L that passes through the points Q and R. Show that the distance d from the point P to the line L is
$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|}$$
where $\mathbf{a} = \overrightarrow{QR}$ and $\mathbf{b} = \overrightarrow{QP}$.
(b) Use the formula in part (a) to find the distance from the point $P(1, 1, 1)$ to the line through $Q(0, 6, 8)$ and $R(-1, 4, 7)$.

46. (a) Let P be a point not on the plane that passes through the points Q, R, and S. Show that the distance d from P to the plane is
$$d = \frac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{a} \times \mathbf{b}|}$$
where $\mathbf{a} = \overrightarrow{QR}$, $\mathbf{b} = \overrightarrow{QS}$, and $\mathbf{c} = \overrightarrow{QP}$.
(b) Use the formula in part (a) to find the distance from the point $P(2, 1, 4)$ to the plane through the points $Q(1, 0, 0)$, $R(0, 2, 0)$, and $S(0, 0, 3)$.

47. Show that $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$.

48. If $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$, show that
$$\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$$

49. Prove that $(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2(\mathbf{a} \times \mathbf{b})$.

50. Prove Property 6 of Theorem 11, that is,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

51. Use Exercise 50 to prove that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$$

52. Prove that

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{d} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}$$

53. Suppose that $\mathbf{a} \neq \mathbf{0}$.
  (a) If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?
  (b) If $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?
  (c) If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ and $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?

54. If $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are noncoplanar vectors, let

$$\mathbf{k}_1 = \frac{\mathbf{v}_2 \times \mathbf{v}_3}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)} \qquad \mathbf{k}_2 = \frac{\mathbf{v}_3 \times \mathbf{v}_1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$$

$$\mathbf{k}_3 = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$$

(These vectors occur in the study of crystallography. Vectors of the form $n_1\mathbf{v}_1 + n_2\mathbf{v}_2 + n_3\mathbf{v}_3$, where each n_i is an integer, form a *lattice* for a crystal. Vectors written similarly in terms of $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$ form the *reciprocal lattice*.)
  (a) Show that $\mathbf{k}_i$ is perpendicular to $\mathbf{v}_j$ if $i \neq j$.
  (b) Show that $\mathbf{k}_i \cdot \mathbf{v}_i = 1$ for $i = 1, 2, 3$.
  (c) Show that $\mathbf{k}_1 \cdot (\mathbf{k}_2 \times \mathbf{k}_3) = \dfrac{1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$.

DISCOVERY PROJECT   **THE GEOMETRY OF A TETRAHEDRON**
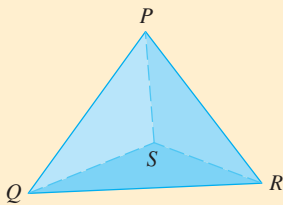
A tetrahedron is a solid with four vertices, P, Q, R, and S, and four triangular faces, as shown in the figure.

1. Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ be vectors with lengths equal to the areas of the faces opposite the vertices P, Q, R, and S, respectively, and directions perpendicular to the respective faces and pointing outward. Show that

$$\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$$

2. The volume V of a tetrahedron is one-third the distance from a vertex to the opposite face, times the area of that face.
  (a) Find a formula for the volume of a tetrahedron in terms of the coordinates of its vertices P, Q, R, and S.
  (b) Find the volume of the tetrahedron whose vertices are $P(1, 1, 1)$, $Q(1, 2, 3)$, $R(1, 1, 2)$, and $S(3, -1, 2)$.

3. Suppose the tetrahedron in the figure has a trirectangular vertex S. (This means that the three angles at S are all right angles.) Let A, B, and C be the areas of the three faces that meet at S, and let D be the area of the opposite face PQR. Using the result of Problem 1, or otherwise, show that

$$D^2 = A^2 + B^2 + C^2$$

(This is a three-dimensional version of the Pythagorean Theorem.)

12.5   **Equations of Lines and Planes**

A line in the xy-plane is determined when a point on the line and the direction of the line (its slope or angle of inclination) are given. The equation of the line can then be written using the point-slope form.

Likewise, a line L in three-dimensional space is determined when we know a point $P_0(x_0, y_0, z_0)$ on L and the direction of L. In three dimensions the direction of a line is conveniently described by a vector, so we let $\mathbf{v}$ be a vector parallel to L. Let $P(x, y, z)$ be an arbitrary point on L and let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of P_0 and P (that is, they have
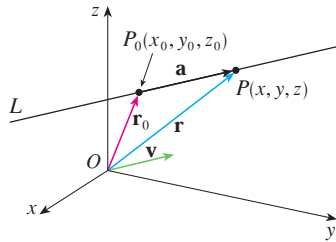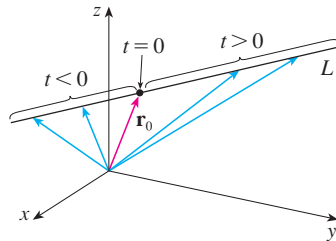
FIGURE 1

FIGURE 2

representations $\overrightarrow{OP_0}$ and $\overrightarrow{OP}$). If $\mathbf{a}$ is the vector with representation $\overrightarrow{P_0P}$, as in Figure 1, then the Triangle Law for vector addition gives $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}$. But, since $\mathbf{a}$ and $\mathbf{v}$ are parallel vectors, there is a scalar t such that $\mathbf{a} = t\mathbf{v}$. Thus

$$\boxed{1} \qquad \boxed{\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}}$$

which is a **vector equation** of L. Each value of the **parameter** t gives the position vector $\mathbf{r}$ of a point on L. In other words, as t varies, the line is traced out by the tip of the vector $\mathbf{r}$. As Figure 2 indicates, positive values of t correspond to points on L that lie on one side of P_0, whereas negative values of t correspond to points that lie on the other side of P_0.

If the vector $\mathbf{v}$ that gives the direction of the line L is written in component form as $\mathbf{v} = \langle a, b, c \rangle$, then we have $t\mathbf{v} = \langle ta, tb, tc \rangle$. We can also write $\mathbf{r} = \langle x, y, z \rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$, so the vector equation $\boxed{1}$ becomes

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

Two vectors are equal if and only if corresponding components are equal. Therefore we have the three scalar equations:

$$\boxed{2} \qquad \boxed{x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct}$$

where $t \in \mathbb{R}$. These equations are called **parametric equations** of the line L through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\mathbf{v} = \langle a, b, c \rangle$. Each value of the parameter t gives a point (x, y, z) on L.

Figure 3 shows the line L in Example 1 and its relation to the given point and to the vector that gives its direction.
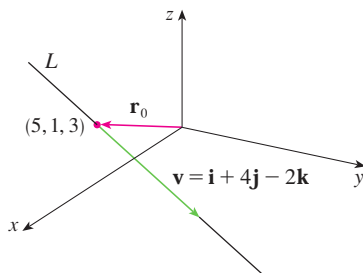

FIGURE 3

EXAMPLE 1
(a) Find a vector equation and parametric equations for the line that passes through the point $(5, 1, 3)$ and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.
(b) Find two other points on the line.

SOLUTION
(a) Here $\mathbf{r}_0 = \langle 5, 1, 3 \rangle = 5\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, so the vector equation $\boxed{1}$ becomes

$$\mathbf{r} = (5\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + t(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$$

or $\qquad\qquad\qquad \mathbf{r} = (5 + t)\mathbf{i} + (1 + 4t)\mathbf{j} + (3 - 2t)\mathbf{k}$

Parametric equations are

$$x = 5 + t \qquad y = 1 + 4t \qquad z = 3 - 2t$$

(b) Choosing the parameter value $t = 1$ gives $x = 6$, $y = 5$, and $z = 1$, so $(6, 5, 1)$ is a point on the line. Similarly, $t = -1$ gives the point $(4, -3, 5)$. ∎

The vector equation and parametric equations of a line are not unique. If we change the point or the parameter or choose a different parallel vector, then the equations change. For instance, if, instead of $(5, 1, 3)$, we choose the point $(6, 5, 1)$ in Example 1, then the parametric equations of the line become

$$x = 6 + t \qquad y = 5 + 4t \qquad z = 1 - 2t$$

Or, if we stay with the point (5, 1, 3) but choose the parallel vector $2\mathbf{i} + 8\mathbf{j} - 4\mathbf{k}$, we arrive at the equations

$$x = 5 + 2t \qquad y = 1 + 8t \qquad z = 3 - 4t$$

In general, if a vector $\mathbf{v} = \langle a, b, c \rangle$ is used to describe the direction of a line L, then the numbers a, b, and c are called **direction numbers** of L. Since any vector parallel to $\mathbf{v}$ could also be used, we see that any three numbers proportional to a, b, and c could also be used as a set of direction numbers for L.

Another way of describing a line L is to eliminate the parameter t from Equations 2. If none of a, b, or c is 0, we can solve each of these equations for t, equate the results, and obtain

3

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

These equations are called **symmetric equations** of L. Notice that the numbers a, b, and c that appear in the denominators of Equations 3 are direction numbers of L, that is, components of a vector parallel to L. If one of a, b, or c is 0, we can still eliminate t. For instance, if $a = 0$, we could write the equations of L as

$$x = x_0 \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

This means that L lies in the vertical plane $x = x_0$.

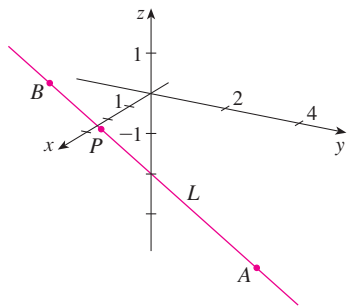Figure 4 shows the line L in Example 2 and the point P where it intersects the xy-plane.

FIGURE 4

EXAMPLE 2
(a) Find parametric equations and symmetric equations of the line that passes through the points $A(2, 4, -3)$ and $B(3, -1, 1)$.
(b) At what point does this line intersect the xy-plane?

SOLUTION
(a) We are not explicitly given a vector parallel to the line, but observe that the vector $\mathbf{v}$ with representation $\overrightarrow{AB}$ is parallel to the line and

$$\mathbf{v} = \langle 3 - 2, -1 - 4, 1 - (-3) \rangle = \langle 1, -5, 4 \rangle$$

Thus direction numbers are $a = 1$, $b = -5$, and $c = 4$. Taking the point $(2, 4, -3)$ as P_0, we see that parametric equations **2** are

$$x = 2 + t \qquad y = 4 - 5t \qquad z = -3 + 4t$$

and symmetric equations **3** are

$$\frac{x - 2}{1} = \frac{y - 4}{-5} = \frac{z + 3}{4}$$

(b) The line intersects the xy-plane when $z = 0$, so we put $z = 0$ in the symmetric equations and obtain

$$\frac{x - 2}{1} = \frac{y - 4}{-5} = \frac{3}{4}$$

This gives $x = \frac{11}{4}$ and $y = \frac{1}{4}$, so the line intersects the xy-plane at the point $\left(\frac{11}{4}, \frac{1}{4}, 0\right)$.

In general, the procedure of Example 2 shows that direction numbers of the line L through the points $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$ are $x_1 - x_0$, $y_1 - y_0$, and $z_1 - z_0$ and so symmetric equations of L are

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}$$

Often, we need a description, not of an entire line, but of just a line segment. How, for instance, could we describe the line segment AB in Example 2? If we put $t = 0$ in the parametric equations in Example 2(a), we get the point $(2, 4, -3)$ and if we put $t = 1$ we get $(3, -1, 1)$. So the line segment AB is described by the parametric equations

$$x = 2 + t \qquad y = 4 - 5t \qquad z = -3 + 4t \qquad 0 \leqslant t \leqslant 1$$

or by the corresponding vector equation

$$\mathbf{r}(t) = \langle 2 + t, 4 - 5t, -3 + 4t \rangle \qquad 0 \leqslant t \leqslant 1$$

In general, we know from Equation 1 that the vector equation of a line through the (tip of the) vector $\mathbf{r}_0$ in the direction of a vector $\mathbf{v}$ is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$. If the line also passes through (the tip of) $\mathbf{r}_1$, then we can take $\mathbf{v} = \mathbf{r}_1 - \mathbf{r}_0$ and so its vector equation is

$$\mathbf{r} = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1$$

The line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ is given by the parameter interval $0 \leqslant t \leqslant 1$.

$\boxed{4}$ The line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ is given by the vector equation

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \qquad 0 \leqslant t \leqslant 1$$

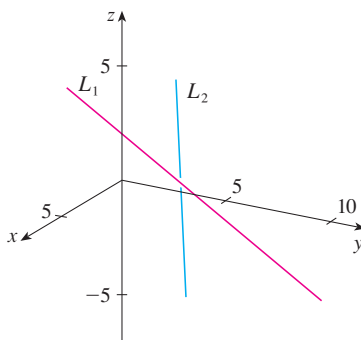The lines L_1 and L_2 in Example 3, shown in Figure 5, are skew lines.

FIGURE 5

V EXAMPLE 3 Show that the lines L_1 and L_2 with parametric equations

$$x = 1 + t \qquad y = -2 + 3t \qquad z = 4 - t$$
$$x = 2s \qquad y = 3 + s \qquad z = -3 + 4s$$

are **skew lines**; that is, they do not intersect and are not parallel (and therefore do not lie in the same plane).

SOLUTION The lines are not parallel because the corresponding vectors $\langle 1, 3, -1 \rangle$ and $\langle 2, 1, 4 \rangle$ are not parallel. (Their components are not proportional.) If L_1 and L_2 had a point of intersection, there would be values of t and s such that

$$1 + t = 2s$$
$$-2 + 3t = 3 + s$$
$$4 - t = -3 + 4s$$

But if we solve the first two equations, we get $t = \frac{11}{5}$ and $s = \frac{8}{5}$, and these values don't satisfy the third equation. Therefore there are no values of t and s that satisfy the three equations, so L_1 and L_2 do not intersect. Thus L_1 and L_2 are skew lines. ■

Planes

Although a line in space is determined by a point and a direction, a plane in space is more difficult to describe. A single vector parallel to a plane is not enough to convey the "direction" of the plane, but a vector perpendicular to the plane does completely specify
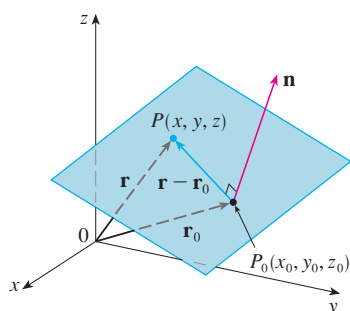
FIGURE 6

its direction. Thus a plane in space is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\mathbf{n}$ that is orthogonal to the plane. This orthogonal vector $\mathbf{n}$ is called a **normal vector**. Let $P(x, y, z)$ be an arbitrary point in the plane, and let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of P_0 and P. Then the vector $\mathbf{r} - \mathbf{r}_0$ is represented by $\overrightarrow{P_0P}$. (See Figure 6.) The normal vector $\mathbf{n}$ is orthogonal to every vector in the given plane. In particular, $\mathbf{n}$ is orthogonal to $\mathbf{r} - \mathbf{r}_0$ and so we have

$$\boxed{5} \qquad \boxed{\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0}$$

which can be rewritten as

$$\boxed{6} \qquad \boxed{\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0}$$

Either Equation 5 or Equation 6 is called a **vector equation of the plane**.

To obtain a scalar equation for the plane, we write $\mathbf{n} = \langle a, b, c \rangle$, $\mathbf{r} = \langle x, y, z \rangle$, and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$. Then the vector equation $\boxed{5}$ becomes

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

or

$$\boxed{7} \qquad \boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

Equation 7 is the **scalar equation of the plane through** $P_0(x_0, y_0, z_0)$ **with normal vector** $\mathbf{n} = \langle a, b, c \rangle$.

V **EXAMPLE 4** Find an equation of the plane through the point $(2, 4, -1)$ with normal vector $\mathbf{n} = \langle 2, 3, 4 \rangle$. Find the intercepts and sketch the plane.

SOLUTION Putting $a = 2$, $b = 3$, $c = 4$, $x_0 = 2$, $y_0 = 4$, and $z_0 = -1$ in Equation 7, we see that an equation of the plane is

$$2(x - 2) + 3(y - 4) + 4(z + 1) = 0$$

FIGURE 7

or

$$2x + 3y + 4z = 12$$

To find the x-intercept we set $y = z = 0$ in this equation and obtain $x = 6$. Similarly, the y-intercept is 4 and the z-intercept is 3. This enables us to sketch the portion of the plane that lies in the first octant (see Figure 7). ∎
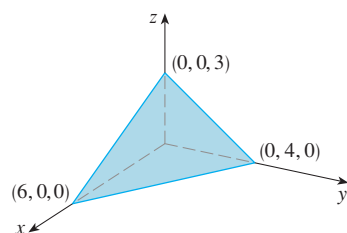
By collecting terms in Equation 7 as we did in Example 4, we can rewrite the equation of a plane as

$$\boxed{8} \qquad \boxed{ax + by + cz + d = 0}$$

where $d = -(ax_0 + by_0 + cz_0)$. Equation 8 is called a **linear equation** in x, y, and z. Conversely, it can be shown that if a, b, and c are not all 0, then the linear equation $\boxed{8}$ represents a plane with normal vector $\langle a, b, c \rangle$. (See Exercise 81.)

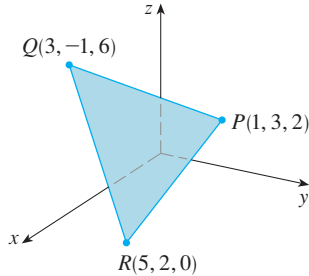Figure 8 shows the portion of the plane in Example 5 that is enclosed by triangle PQR.

FIGURE 8

EXAMPLE 5 Find an equation of the plane that passes through the points $P(1, 3, 2)$, $Q(3, -1, 6)$, and $R(5, 2, 0)$.

SOLUTION The vectors $\mathbf{a}$ and $\mathbf{b}$ corresponding to $\vec{PQ}$ and $\vec{PR}$ are

$$\mathbf{a} = \langle 2, -4, 4 \rangle \qquad \mathbf{b} = \langle 4, -1, -2 \rangle$$

Since both $\mathbf{a}$ and $\mathbf{b}$ lie in the plane, their cross product $\mathbf{a} \times \mathbf{b}$ is orthogonal to the plane and can be taken as the normal vector. Thus

$$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12\,\mathbf{i} + 20\,\mathbf{j} + 14\,\mathbf{k}$$

With the point $P(1, 3, 2)$ and the normal vector $\mathbf{n}$, an equation of the plane is

$$12(x - 1) + 20(y - 3) + 14(z - 2) = 0$$

or

$$6x + 10y + 7z = 50$$

EXAMPLE 6 Find the point at which the line with parametric equations $x = 2 + 3t$, $y = -4t$, $z = 5 + t$ intersects the plane $4x + 5y - 2z = 18$.

SOLUTION We substitute the expressions for x, y, and z from the parametric equations into the equation of the plane:

$$4(2 + 3t) + 5(-4t) - 2(5 + t) = 18$$

This simplifies to $-10t = 20$, so $t = -2$. Therefore the point of intersection occurs when the parameter value is $t = -2$. Then $x = 2 + 3(-2) = -4$, $y = -4(-2) = 8$, $z = 5 - 2 = 3$ and so the point of intersection is $(-4, 8, 3)$.
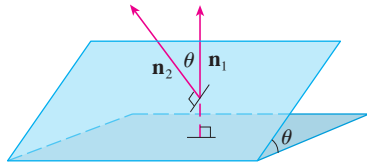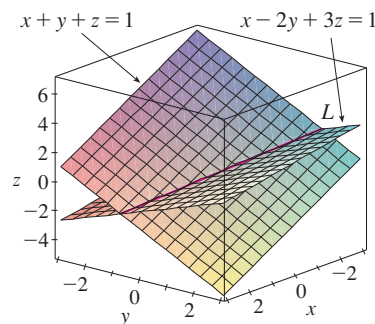
Two planes are **parallel** if their normal vectors are parallel. For instance, the planes $x + 2y - 3z = 4$ and $2x + 4y - 6z = 3$ are parallel because their normal vectors are $\mathbf{n}_1 = \langle 1, 2, -3 \rangle$ and $\mathbf{n}_2 = \langle 2, 4, -6 \rangle$ and $\mathbf{n}_2 = 2\mathbf{n}_1$. If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the acute angle between their normal vectors (see angle θ in Figure 9).

FIGURE 9

V EXAMPLE 7
(a) Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.
(b) Find symmetric equations for the line of intersection L of these two planes.

SOLUTION
(a) The normal vectors of these planes are

$$\mathbf{n}_1 = \langle 1, 1, 1 \rangle \qquad \mathbf{n}_2 = \langle 1, -2, 3 \rangle$$

and so, if θ is the angle between the planes, Corollary 12.3.6 gives

$$\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \frac{1(1) + 1(-2) + 1(3)}{\sqrt{1 + 1 + 1}\,\sqrt{1 + 4 + 9}} = \frac{2}{\sqrt{42}}$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right) \approx 72°$$

Figure 10 shows the planes in Example 7 and their line of intersection L.

FIGURE 10

(b) We first need to find a point on L. For instance, we can find the point where the line intersects the xy-plane by setting $z = 0$ in the equations of both planes. This gives the

equations $x + y = 1$ and $x - 2y = 1$, whose solution is $x = 1$, $y = 0$. So the point $(1, 0, 0)$ lies on L.

Now we observe that, since L lies in both planes, it is perpendicular to both of the normal vectors. Thus a vector $\mathbf{v}$ parallel to L is given by the cross product

Another way to find the line of intersection is to solve the equations of the planes for two of the variables in terms of the third, which can be taken as the parameter.
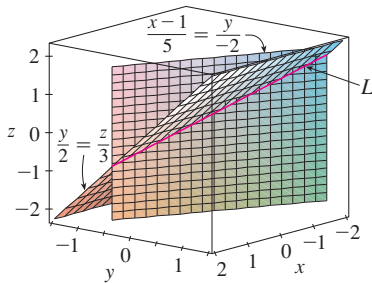
$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$$

and so the symmetric equations of L can be written as

$$\frac{x - 1}{5} = \frac{y}{-2} = \frac{z}{-3}$$

FIGURE 11

Figure 11 shows how the line L in Example 7 can also be regarded as the line of intersection of planes derived from its symmetric equations.

NOTE Since a linear equation in x, y, and z represents a plane and two nonparallel planes intersect in a line, it follows that two linear equations can represent a line. The points (x, y, z) that satisfy both $a_1 x + b_1 y + c_1 z + d_1 = 0$ and $a_2 x + b_2 y + c_2 z + d_2 = 0$ lie on both of these planes, and so the pair of linear equations represents the line of intersection of the planes (if they are not parallel). For instance, in Example 7 the line L was given as the line of intersection of the planes $x + y + z = 1$ and $x - 2y + 3z = 1$. The symmetric equations that we found for L could be written as

$$\frac{x - 1}{5} = \frac{y}{-2} \qquad \text{and} \qquad \frac{y}{-2} = \frac{z}{-3}$$

which is again a pair of linear equations. They exhibit L as the line of intersection of the planes $(x - 1)/5 = y/(-2)$ and $y/(-2) = z/(-3)$. (See Figure 11.)

In general, when we write the equations of a line in the symmetric form

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

we can regard the line as the line of intersection of the two planes

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} \qquad \text{and} \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

EXAMPLE 8   Find a formula for the distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$.

SOLUTION   Let $P_0(x_0, y_0, z_0)$ be any point in the given plane and let $\mathbf{b}$ be the vector corresponding to $\overrightarrow{P_0 P_1}$. Then

$$\mathbf{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$$

From Figure 12 you can see that the distance D from P_1 to the plane is equal to the absolute value of the scalar projection of $\mathbf{b}$ onto the normal vector $\mathbf{n} = \langle a, b, c \rangle$. (See Section 12.3.) Thus

FIGURE 12

$$D = |\text{comp}_{\mathbf{n}} \mathbf{b}| = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|}$$

$$= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

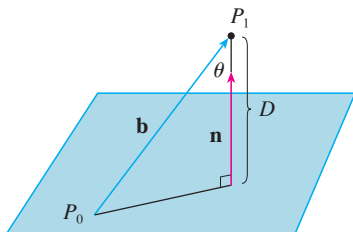$$= \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

Since P_0 lies in the plane, its coordinates satisfy the equation of the plane and so we have $ax_0 + by_0 + cz_0 + d = 0$. Thus the formula for D can be written as

9
$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

EXAMPLE 9 Find the distance between the parallel planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$.

SOLUTION First we note that the planes are parallel because their normal vectors $\langle 10, 2, -2 \rangle$ and $\langle 5, 1, -1 \rangle$ are parallel. To find the distance D between the planes, we choose any point on one plane and calculate its distance to the other plane. In particular, if we put $y = z = 0$ in the equation of the first plane, we get $10x = 5$ and so $\left(\frac{1}{2}, 0, 0\right)$ is a point in this plane. By Formula 9, the distance between $\left(\frac{1}{2}, 0, 0\right)$ and the plane $5x + y - z - 1 = 0$ is

$$D = \frac{\left|5\left(\frac{1}{2}\right) + 1(0) - 1(0) - 1\right|}{\sqrt{5^2 + 1^2 + (-1)^2}} = \frac{\frac{3}{2}}{3\sqrt{3}} = \frac{\sqrt{3}}{6}$$

So the distance between the planes is $\sqrt{3}/6$.

EXAMPLE 10 In Example 3 we showed that the lines

$$L_1: \quad x = 1 + t \qquad y = -2 + 3t \qquad z = 4 - t$$
$$L_2: \quad x = 2s \qquad y = 3 + s \qquad z = -3 + 4s$$

are skew. Find the distance between them.

SOLUTION Since the two lines L_1 and L_2 are skew, they can be viewed as lying on two parallel planes P_1 and P_2. The distance between L_1 and L_2 is the same as the distance between P_1 and P_2, which can be computed as in Example 9. The common normal vector to both planes must be orthogonal to both $\mathbf{v}_1 = \langle 1, 3, -1 \rangle$ (the direction of L_1) and $\mathbf{v}_2 = \langle 2, 1, 4 \rangle$ (the direction of L_2). So a normal vector is

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -1 \\ 2 & 1 & 4 \end{vmatrix} = 13\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}$$

If we put $s = 0$ in the equations of L_2, we get the point $(0, 3, -3)$ on L_2 and so an equation for P_2 is

$$13(x - 0) - 6(y - 3) - 5(z + 3) = 0 \qquad \text{or} \qquad 13x - 6y - 5z + 3 = 0$$

If we now set $t = 0$ in the equations for L_1, we get the point $(1, -2, 4)$ on P_1. So the distance between L_1 and L_2 is the same as the distance from $(1, -2, 4)$ to $13x - 6y - 5z + 3 = 0$. By Formula 9, this distance is

$$D = \frac{|13(1) - 6(-2) - 5(4) + 3|}{\sqrt{13^2 + (-6)^2 + (-5)^2}} = \frac{8}{\sqrt{230}} \approx 0.53$$

12.5   Exercises

1. Determine whether each statement is true or false.
(a) Two lines parallel to a third line are parallel.
(b) Two lines perpendicular to a third line are parallel.
(c) Two planes parallel to a third plane are parallel.
(d) Two planes perpendicular to a third plane are parallel.
(e) Two lines parallel to a plane are parallel.
(f) Two lines perpendicular to a plane are parallel.
(g) Two planes parallel to a line are parallel.
(h) Two planes perpendicular to a line are parallel.
(i) Two planes either intersect or are parallel.
(j) Two lines either intersect or are parallel.
(k) A plane and a line either intersect or are parallel.

2–5 Find a vector equation and parametric equations for the line.

2. The line through the point $(6, -5, 2)$ and parallel to the vector $\langle 1, 3, -\frac{2}{3} \rangle$

3. The line through the point $(2, 2.4, 3.5)$ and parallel to the vector $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

4. The line through the point $(0, 14, -10)$ and parallel to the line $x = -1 + 2t, y = 6 - 3t, z = 3 + 9t$

5. The line through the point $(1, 0, 6)$ and perpendicular to the plane $x + 3y + z = 5$

6–12 Find parametric equations and symmetric equations for the line.

6. The line through the origin and the point $(4, 3, -1)$

7. The line through the points $\left(0, \frac{1}{2}, 1\right)$ and $(2, 1, -3)$

8. The line through the points $(1.0, 2.4, 4.6)$ and $(2.6, 1.2, 0.3)$

9. The line through the points $(-8, 1, 4)$ and $(3, -2, 4)$

10. The line through $(2, 1, 0)$ and perpendicular to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \mathbf{k}$

11. The line through $(1, -1, 1)$ and parallel to the line $x + 2 = \frac{1}{2}y = z - 3$

12. The line of intersection of the planes $x + 2y + 3z = 1$ and $x - y + z = 1$

13. Is the line through $(-4, -6, 1)$ and $(-2, 0, -3)$ parallel to the line through $(10, 18, 4)$ and $(5, 3, 14)$?

14. Is the line through $(-2, 4, 0)$ and $(1, 1, 1)$ perpendicular to the line through $(2, 3, 4)$ and $(3, -1, -8)$?

15. (a) Find symmetric equations for the line that passes through the point $(1, -5, 6)$ and is parallel to the vector $\langle -1, 2, -3 \rangle$.
(b) Find the points in which the required line in part (a) intersects the coordinate planes.

16. (a) Find parametric equations for the line through $(2, 4, 6)$ that is perpendicular to the plane $x - y + 3z = 7$.
(b) In what points does this line intersect the coordinate planes?

17. Find a vector equation for the line segment from $(2, -1, 4)$ to $(4, 6, 1)$.

18. Find parametric equations for the line segment from $(10, 3, 1)$ to $(5, 6, -3)$.

19–22 Determine whether the lines L_1 and L_2 are parallel, skew, or intersecting. If they intersect, find the point of intersection.

19. L_1: $x = 3 + 2t$, $y = 4 - t$, $z = 1 + 3t$
L_2: $x = 1 + 4s$, $y = 3 - 2s$, $z = 4 + 5s$

20. L_1: $x = 5 - 12t$, $y = 3 + 9t$, $z = 1 - 3t$
L_2: $x = 3 + 8s$, $y = -6s$, $z = 7 + 2s$

21. L_1: $\dfrac{x-2}{1} = \dfrac{y-3}{-2} = \dfrac{z-1}{-3}$
L_2: $\dfrac{x-3}{1} = \dfrac{y+4}{3} = \dfrac{z-2}{-7}$

22. L_1: $\dfrac{x}{1} = \dfrac{y-1}{-1} = \dfrac{z-2}{3}$
L_2: $\dfrac{x-2}{2} = \dfrac{y-3}{-2} = \dfrac{z}{7}$

23–40 Find an equation of the plane.

23. The plane through the origin and perpendicular to the vector $\langle 1, -2, 5 \rangle$

24. The plane through the point $(5, 3, 5)$ and with normal vector $2\mathbf{i} + \mathbf{j} - \mathbf{k}$

25. The plane through the point $\left(-1, \frac{1}{2}, 3\right)$ and with normal vector $\mathbf{i} + 4\mathbf{j} + \mathbf{k}$

26. The plane through the point $(2, 0, 1)$ and perpendicular to the line $x = 3t, y = 2 - t, z = 3 + 4t$

27. The plane through the point $(1, -1, -1)$ and parallel to the plane $5x - y - z = 6$

28. The plane through the point $(2, 4, 6)$ and parallel to the plane $z = x + y$

29. The plane through the point $\left(1, \frac{1}{2}, \frac{1}{3}\right)$ and parallel to the plane $x + y + z = 0$

30. The plane that contains the line $x = 1 + t, y = 2 - t, z = 4 - 3t$ and is parallel to the plane $5x + 2y + z = 1$

31. The plane through the points $(0, 1, 1)$, $(1, 0, 1)$, and $(1, 1, 0)$

32. The plane through the origin and the points $(2, -4, 6)$ and $(5, 1, 3)$

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1. Homework Hints available at stewartcalculus.com