محاضرات في بحتة (٥) "جبر عام"

لطلاب الفرقة الثانية – تعليم عام– شعبة الرياضيات بكلية التربية

"برنامج اللغة الانجليزية"

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THEORY OF EQUATIONS

1.0 Introduction

In this module, we will study about polynomial functions and various methods to find out the roots of polynomial equations. 'Solving equations' was an important problem from the beginning of study of Mathematics itself. The notion of complex numbers was first introduced because equations like $x^2 + 1 = 0$ has no solution in the set of real numbers. The "fundamental theorem of algebra" which states that every polynomial of degree ≥ 1 has at least one zero was first proved by the famous German Mathematician Karl Fredrich Gauss. We shall look at polynomials in detail and will discuss various methods for solving polynomial equations.

1.1. Polynomial Functions

Definition:

A function defined by

$f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$, *where* $a_o \neq 0$, n is a non negative integer and a_i (i = 0, 1….,n) are fixed complex numbers is called a **polynomial** of **degree** n in x. Then numbers $a_o, a_1, \ldots, a_n$ are called the **coefficients** of f.

If α is a complex number such that $f(\alpha) = 0$, then α is called **zero** of the polynomial.

1.1.1 Theorem (Fundamental Theorem of Algebra)

Every polynomial function of degree $n \geq 1$ has at least one zero.

Remark:

Fundamental theorem of algebra says that, if $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$,

where $a_0 \neq 0$ is the given polynomial of degree $n \geq 1$, then there exists a complex number α such that $a_0\alpha^n + a_1\alpha^{n-1} + \ldots + a_n = 0$.

We use the Fundamental Theorem of Algebra, to prove the following result.

1.1.2 Theorem

Every polynomial of degree n has n and only n zeroes.

Proof:

Let $f(x) = a_0x^n + a_1x^{n-1} + ... + a_n$, *where* $a_o \neq 0$, be a polynomial of degree $n \geq 1$.

By fundamental theorem of algebra, f(x) has at least one zero, let α_1 be that zero.

Then $(x - \alpha_1)$ is a factor of f(x).

Therefore, we can write:

f(x) = $(x - \alpha_1)Q_1(x)$, where $Q_1(x)$ is a polynomial function of degree n - 1.

If $n - 1 \geq 1$, again by Fundamental Theorem of Algebra, $Q_1(x)$ has at least one zero, say α_2.

Therefore, $f(x) = (x - \alpha_1)(x - \alpha_2)Q_2(x)$ *where* $Q_2(x)$ is a polynomial function of degree n - 2.

Repeating the above arguments, we get

$f(x) = (x - \alpha_1)(x - \alpha_2).....(x - \alpha_n)Q_n(x)$, where $Q_n(x)$ is a polynomial function of degree n - n = 0, i.e., $Q_n(x)$ is a constant.

Equating the coefficient of x^n on both sides of the above equation, we get

$Q_n(x) = a_o$.

Therefore, $f(x) = a_o(x - \alpha_1)(x - \alpha_2)....(x - \alpha_n)$.

If α is any number other than $\alpha_1, \alpha_2, ...\alpha_n$, then $f(x) \neq 0 \Rightarrow \alpha$ is not a zero of f(x).

Hence f(x) has n and only n zeros, namely $\alpha_1, \alpha_2 ...., \alpha_n$.

Note:

Let $f(x) = a_ox^n + a_1x^{n-1} + ... + a_n; a_o \neq 0$ be an n^{th} degree polynomial in x.

Then, $a_ox^n + a_1x^{n-1} + ... + a_n = 0$ --------- (1)

is called a **polynomial equation** in x of degree n.

A number α is called a **root** of the equation (1) if α is a zero of the polynomial f(x).

Hence every polynomial equation of degree n has n and only n roots.

Solved Problems

1. Solve $x^4 - 4x^2 + 8x + 35 = 0$, given $2 + i\sqrt{3}$ is a root.

Solution :

Given that $2 + i\sqrt{3}$ is a root of $x^4 - 4x^2 + 8x + 35 = 0$; since complex roots occurs in conjugate pairs $2 - i\sqrt{3}$ is also a root of it.

$\Rightarrow [x - (2 + i\sqrt{3})][x - (2 - i\sqrt{3})] = (x - 2)^2 + 3 = x^2 - 4x + 7$ is a factor of the given polynomial.

Dividing the given polynomial by this factor, we obtain the other factor as $x^2 + 4x + 5$.

The roots of $x^2 + 4x + 5 = 0$ are given by $\dfrac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i$.

Hence the roots of the given polynomial are $2 + i\sqrt{3},\ 2 - i\sqrt{3},\ -2 + i$ *and* $-2 - i$.

2. Solve $x^4 - 5x^3 + 4x^2 + 8x - 8 = 0$, given that one of the roots is $1 - \sqrt{5}$.

Solution:

Since quadratic surds occur in conjugate pairs as roots of a polynomial equation, $1 + \sqrt{5}$ is also a root of the given polynomial.

$\Rightarrow [x - (1 - \sqrt{5})][x - (1 + \sqrt{5})] = (x - 1)^2 - 5 = x^2 - 2x - 4$ is a factor.

Dividing the given polynomial by this factor, we obtain the other factor as $x^2 - 3x + 2$.

Also, $x^2 - 3x + 2 = (x - 2)(x - 1)$

Thus the roots of the given polynomial equation are $1 + \sqrt{5}, 1 - \sqrt{5}, 1, 2$.

3. Find a polynomial equation of the lowest degree with rational coefficients having $\sqrt{3}$ and $1 - 2i$ as two of its roots.

Solution:

Since quadratic surds occur in pairs as roots, $-\sqrt{3}$ is also a root.

Since complex roots occur in conjugate pairs, $1 + 2i$ is also a root of the required polynomial equation. Therefore the desired equation is given by

$$(x - \sqrt{3})\,(x + \sqrt{3})\,(x - (1 - 2i)\big(x - (1 + 2i)\big) = 0$$

$$\text{i.e., } x^4 - 2x^3 + 2x^2 + 6x - 15 = 0$$

4. Solve $4x^5 + x^3 + x^2 - 3x + 1 = 0$, given that it has rational roots.

Solution:

Let $f(x) = 4x^5 + x^3 + x^2 - 3x + 1$.

By theorem (1.1.5.), any rational root p/q (in its lowest terms) must satisfy the condition that, p is divisor of 1 and q is positive divisor of 4.

So the possible rational roots are $\pm 1, \pm \frac{1}{2}, \pm \frac{1}{4}$.

Note that $f(-1) = 0$, $f(\frac{1}{2}) = 0$. But $f(1) \neq 0$, $f(-\frac{1}{2}) \neq 0$, $f(\frac{1}{4}) \neq 0$ and $f(-1/4) \neq 0$.

Since $f(-1) = 0$ and $f(\frac{1}{2}) = 0$, we see that $(x + 1)$ and $(x - \frac{1}{2})$ are factors of the given polynomial. Also by factorizing , we find that

$$f(x) = (x - \tfrac{1}{2})(x + 1)(4x^3 - 2x^2 + 4x - 2)$$

Note that $x = \frac{1}{2}$ is a root of the third factor, if we divide $4x^3 - 2x^2 + 4x - 2$ by $x - \frac{1}{2}$, we obtain

$$f(x) = (x - \tfrac{1}{2})^2 (x + 1)(4x^2 + 4)$$
$$= 4(x - \tfrac{1}{2})^2 (x + 1)(x^2 + 1)$$

Hence the roots of $f(x) = 0$, are $\frac{1}{2}, \frac{1}{2}, -1, \pm i$.

5. Solve $x^3 - x^2 - 8x + 12 = 0$, given that has a double root.

Solution:

Let $f(x) = x^3 - x^2 - 8x + 12$

Differentiating, we obtain:

$$f^1(x) = 3x^2 - 2x - 8.$$

Since the multiple roots of $f(x) = 0$ are also the roots of $f^1(x) = 0$, the product of the factors corresponding to these roots will be the g.c.d of $f(x)$ and $f^1(x)$. Let us find the g.c.d of $f(x)$ and $f^1(x)$.

$3x$	$3x^2 - 2x - 8$	$x^3 - x^2 - 8x + 12$	
	$3x^2 - 6x$	3	
4	$4x - 8$	$3x^3 - 3x^2 - 24x + 36$	x
	$4x - 8$	$3x^3 - 2x^2 - 8x$	
0	0	$-x^2 - 16x + 36$	
		3	
		$-3x^2 - 48x + 108$	-1
		$-3x^2 + 2x + 8$	
		-50 $-50x + 100$	
		$x - 2$	

Therefore, g.c.d $= (x - 2)$

$\Rightarrow f(x)$ has a factor $(x - 2)^2$.

Also, $f(x) = (x - 2)^2 (x + 3)$

Thus the roots are 2, 2, - 3 .

6. Show that the equation $x^3 + qx + r = 0$ has two equal roots if $27r^2 + 4q^3 = 0$.

Solution:

Let $f(x) = x^3 + qx + r$ -------------- (1)

Differentiating, we obtain: $f^1(x) = 3x^2 + q$ ------------------ (2)

Given that $f(x) = 0$ has two equal roots, i.e., it has a double root, say α.

Then α is a root of both $f(x) = 0$ and $f^1(x) = 0$.

From the 2nd equation, we obtain $\alpha^2 = -q/3$

Now the first equation can be written as: $\alpha(\alpha^2 + q) + r = 0$

i.e., $\alpha(-q/3 + q) + r = 0 \quad \Rightarrow \quad \alpha = \frac{-3r}{2q}$

Squaring and simplifying, we obtain: $27r^2 + 4q^3 = 0$

Relation between the Roots and Coefficients of a Polynomial Equation

Consider the polynomial function $f(x) = a_o x^n + a_1 x^{n-1} + \ldots + a_n$, $a_0 \neq 0$

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of $f(x) = 0$.

Then we can write $f(x) = a_o(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$

Equating the two expressions for $f(x)$, we obtain:

$$a_o x^n + a_1 x^{n-1} + \ldots + a_n = a_o(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$$

Dividing both sides by a_0,

$$x^n + \left(\frac{a_1}{a_o}\right)x^{n-1} + \ldots + \left(\frac{a_n}{a_o}\right) = (x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$$

$$= x^n - S_1 x^{n-1} + S_2 x^{n-2} - \ldots + (-1)^n S_n$$

where S_r stands for the sum of the products of the roots $\alpha_1, \ldots, \alpha_n$ taken r at a time.

Comparing the coefficients on both sides , we see that

$$S_1 = \frac{-a_1}{a_o}, \quad S_2 = \frac{a_2}{a_o}, \ldots \quad S_n = (-1)^n \frac{a_n}{a_o}.$$

Special Cases

If α and β are the roots of $ax^2 + bx + c = 0$, $(a \neq 0)$, then $\alpha + \beta = \frac{-b}{a}$ *and* $\alpha\beta = \frac{c}{a}$

If α and β and γ are the roots of $ax^3 + bx^2 + cx + d = 0$, $(a \neq 0)$, then $\alpha + \beta + \gamma = \frac{-b}{a}$,

and $\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a}$ *and* $\alpha\beta\gamma = \frac{-d}{a}$.

Examples:

1. If the roots of the equation $x^3 + px^2 + qx + r = 0$ are in arithmetic progression, show that $2p^3 - 9pq + 27r = 0$.

Solution:

Let the roots of the given equation be $a - d$, a, $a + d$.

Then $S_1 = a - d + a + a + d = 3a = -p \Rightarrow a = \frac{-p}{3}$

Since a is a root, it satisfies the given polynomial

$$\Rightarrow \left(-p/3\right)^3 + p.\left(-p/3\right)^2 + q.\left(-p/3\right) + r = 0$$

On simplification, we obtain $2p^3 - 9pq + 27r = 0$.

2. Solve $27x^3 + 42x^2 - 28x - 8 = 0$, given that its roots are in geometric progression.

Solution:

Let the roots be $\frac{a}{r}, a, ar$

Then, $\frac{a}{r}.a.ar = a^3 = \frac{8}{27} \Rightarrow a = \frac{2}{3}$

Since $a = \frac{2}{3}$ is a root, $\left(x - \frac{2}{3}\right)$ is a factor. On division, the other factor of the polynomial is $27x^2 + 60x + 12$.

Its roots are $\frac{-60 \pm \sqrt{60^2 - 4 \times 27 \times 12}}{2 \times 27} = \frac{-2}{9}$ *or* -2

Hence the roots of the given polynomial eqution are $\frac{-2}{9}, -2, \frac{2}{3}$.

3. Solve the equation $15x^3 - 23x^2 + 9x - 1 = 0$ whose roots are in harmonic progression.

Solution:

[Recall that if a, b, c are in harmonic progression, then $^1/_a$, $^1/_b$, $^1/_c$ are in arithmetic progression and hence $b = \frac{2ac}{a+c}$]

Let α, β, γ be the roots of the given polynomial.

Then $\alpha\beta + \beta\gamma + \alpha\gamma = \frac{9}{15}$ ……….. (1)

$\alpha\beta\gamma = \frac{1}{15}$ ……… (2)

Since α, β, γ are in harmonic progression, $\beta = \frac{2\alpha\gamma}{\alpha + \gamma}$

$\Rightarrow \alpha\beta + \beta\gamma = 2\alpha\gamma$

Substitute in (1), $2\alpha\gamma + \alpha\gamma = \frac{.9}{15} \Rightarrow 3\alpha\gamma = \frac{9}{15}$

$\Rightarrow \alpha\gamma = \frac{3}{15}$.

Substitute in (2), we obtain $\frac{3}{15}\beta = \frac{1}{15}$

$\Rightarrow \beta = \frac{1}{3}$ is a root of the given polynomial.

Proceeding as in the above problem, we find that the roots are $\frac{1}{3}$, 1, $\frac{1}{5}$.

4. Show that the roots of the equation $ax^3 + bx^2 + cx + d = 0$ are in geometric progression, then $c^3a = b^3d$.

Solution:

Suppose the roots are $\frac{k}{r}$, k, kr

Then $\frac{k}{r}.k.kr = \frac{-d}{a}$

i.e., $k^3 = \frac{-d}{a}$

Since k is a root, it satisfies the polynomial equation,

$ak^3 + bk^2 + ck + d = 0$

$a\left(\frac{-d}{a}\right) + bk^2 + ck + d = 0$

$$\Rightarrow bk^2 + ck = 0 \quad \Rightarrow bk^2 = -ck$$

$$\Rightarrow (bk^2)^3 = (-ck)^3 \quad i.e., \quad b^3k^6 = -c^3k^3$$

$$\Rightarrow b^3\frac{d^2}{a^2} = -c^3\left(\frac{-d}{a}\right)$$

$$\Rightarrow \quad \frac{b^3 d}{a} = c^3 \quad \Rightarrow \quad b^3 d = c^3 a.$$

5. Solve the equation $x^3 - 9x^2 + 14x + 24 = 0$, given that two of whose roots are in the ratio 3: 2.

Solution:

Let the roots be $3\alpha, 2\alpha, \beta$

Then, $3\alpha + 2\alpha + \beta = 5\alpha + \beta = 9$ .... (1)

$3\alpha.2\alpha + 2\alpha.\beta + 3\alpha.\beta = 14$

i.e., $6\alpha^2 + 5\alpha\beta = 14$ .... (2)

and $3\alpha.2\alpha.\beta = 6\alpha^2\beta = -24$

$\Rightarrow \alpha^2\beta = -4$ .... (3)

From (1), $\beta = 9 - 5\alpha$. Substituting this in (2), we obtain

$$6\alpha^2 + 5\alpha(9 - 5\alpha) = 14$$

i.e., $19\alpha^2 - 45\alpha + 14 = 0$. On solving we get $\alpha = 2 \; or \; \frac{7}{19}$.

When $\alpha = \frac{7}{19}$, from (1), we get $\beta = \frac{136}{19}$. But these values do not satisfy (3).

So, $\alpha = 2$, then from (1), we get $\beta = -1$

Therefore, the roots are 4, 6, -1.

Symmetric Functions of the Roots

Consider the expressions like $\alpha^2 + \beta^2 + \gamma^2$, $(\beta - \gamma)^2 + (\gamma - \alpha)^2 + (\alpha - \beta)^2$, $(\beta + \gamma)\,(\gamma + \alpha)\,(\alpha - \beta)$. Each of these expressions is a function of α, β, γ with the property that if any two of α, β, γ are interchanged, the function remains unchanged.

Such functions are called **symmetric functions**.

2.3 Polynomial and Synthetic Division

What you should learn

- Use long division to divide polynomials by other polynomials.
- Use synthetic division to divide polynomials by binomials of the form $(x - k)$.
- Use the Remainder Theorem and the Factor Theorem.

Why you should learn it

Synthetic division can help you evaluate polynomial functions. For instance, in Exercise 73 on page 160, you will use synthetic division to determine the number of U.S. military personnel in 2008.

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Long Division of Polynomials

In this section, you will study two procedures for *dividing* polynomials. These procedures are especially valuable in factoring and finding the zeros of polynomial functions. To begin, suppose you are given the graph of

$$f(x) = 6x^3 - 19x^2 + 16x - 4.$$

Notice that a zero of f occurs at $x = 2$, as shown in Figure 2.27. Because $x = 2$ is a zero of f, you know that $(x - 2)$ is a factor of $f(x)$. This means that there exists a second-degree polynomial $q(x)$ such that

$$f(x) = (x - 2) \cdot q(x).$$

To find $q(x)$, you can use **long division,** as illustrated in Example 1.

Example 1 Long Division of Polynomials

Divide $6x^3 - 19x^2 + 16x - 4$ by $x - 2$, and use the result to factor the polynomial completely.

Solution

Think $\frac{6x^3}{x} = 6x^2$.

Think $\frac{-7x^2}{x} = -7x$.

Think $\frac{2x}{x} = 2$.

$$
\begin{array}{rll}
 & 6x^2 - 7x + 2 & \\
x - 2 \,\big)\! & 6x^3 - 19x^2 + 16x - 4 & \\
 & 6x^3 - 12x^2 & \text{Multiply: } 6x^2(x-2). \\
 & -7x^2 + 16x & \text{Subtract.} \\
 & -7x^2 + 14x & \text{Multiply: } -7x(x-2). \\
 & 2x - 4 & \text{Subtract.} \\
 & 2x - 4 & \text{Multiply: } 2(x-2). \\
 & 0 & \text{Subtract.}
\end{array}
$$

From this division, you can conclude that

$$6x^3 - 19x^2 + 16x - 4 = (x - 2)(6x^2 - 7x + 2)$$

and by factoring the quadratic $6x^2 - 7x + 2$, you have

$$6x^3 - 19x^2 + 16x - 4 = (x - 2)(2x - 1)(3x - 2).$$

Note that this factorization agrees with the graph shown in Figure 2.27 in that the three x-intercepts occur at $x = 2$, $x = \frac{1}{2}$, and $x = \frac{2}{3}$.

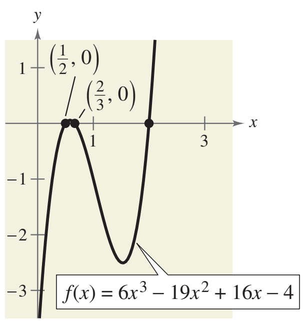


FIGURE 2.27

CHECKPOINT Now try Exercise 5.

Note that one of the many uses of polynomial division is to write a function as a sum of terms to find slant asymptotes (see Section 2.6). This is a skill that is also used frequently in calculus.

In Example 1, $x - 2$ is a factor of the polynomial $6x^3 - 19x^2 + 16x - 4$, and the long division process produces a remainder of zero. Often, long division will produce a nonzero remainder. For instance, if you divide $x^2 + 3x + 5$ by $x + 1$, you obtain the following.

$$
\begin{array}{rrl}
 & x + 2 & \leftarrow \text{Quotient} \\
\text{Divisor} \rightarrow \; x + 1 & \overline{)\,x^2 + 3x + 5} & \leftarrow \text{Dividend} \\
 & \underline{x^2 + x} & \\
 & 2x + 5 & \\
 & \underline{2x + 2} & \\
 & 3 & \leftarrow \text{Remainder}
\end{array}
$$

In fractional form, you can write this result as follows.

$$\underbrace{\frac{\overbrace{x^2 + 3x + 5}^{\text{Dividend}}}{\underbrace{x + 1}_{\text{Divisor}}}} = \overbrace{x + 2}^{\text{Quotient}} + \frac{\overset{\text{Remainder}}{3}}{\underbrace{x + 1}_{\text{Divisor}}}$$

Have students identify the dividend, divisor, quotient, and remainder when dividing polynomials.

This implies that

$$x^2 + 3x + 5 = (x + 1)(x + 2) + 3 \qquad \text{Multiply each side by } (x + 1).$$

which illustrates the following theorem, called the **Division Algorithm.**

The Division Algorithm

If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$\underset{\text{Dividend}}{f(x)} = \underset{\text{Divisor}}{d(x)}\,\underset{\text{Quotient}}{q(x)} + \underset{\text{Remainder}}{r(x)}$$

where $r(x) = 0$ *or* the degree of $r(x)$ is less than the degree of $d(x)$. If the remainder $r(x)$ is zero, $d(x)$ *divides evenly* into $f(x)$.

The Division Algorithm can also be written as

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.$$

In the Division Algorithm, the rational expression $f(x)/d(x)$ is **improper** because the degree of $f(x)$ is greater than or equal to the degree of $d(x)$. On the other hand, the rational expression $r(x)/d(x)$ is **proper** because the degree of $r(x)$ is less than the degree of $d(x)$.

Before you apply the Division Algorithm, follow these steps.

1. Write the dividend and divisor in descending powers of the variable.
2. Insert placeholders with zero coefficients for missing powers of the variable.

Example 2 Long Division of Polynomials

Divide $x^3 - 1$ by $x - 1$.

Solution

Because there is no x^2-term or x-term in the dividend, you need to line up the subtraction by using zero coefficients (or leaving spaces) for the missing terms.

$$
\begin{array}{rl}
 & x^2 + x + 1 \\
x - 1 \,\big) & x^3 + 0x^2 + 0x - 1 \\
 & \underline{x^3 - x^2} \\
 & x^2 + 0x \\
 & \underline{x^2 - x} \\
 & x - 1 \\
 & \underline{x - 1} \\
 & 0
\end{array}
$$

So, $x - 1$ divides evenly into $x^3 - 1$, and you can write

$$\frac{x^3 - 1}{x - 1} = x^2 + x + 1, \quad x \neq 1.$$

✓CHECKPOINT Now try Exercise 13.

You can check the result of Example 2 by multiplying.

$$(x - 1)(x^2 + x + 1) = x^3 + x^2 + x - x^2 - x - 1 = x^3 - 1$$

Example 3 Long Division of Polynomials

Divide $2x^4 + 4x^3 - 5x^2 + 3x - 2$ by $x^2 + 2x - 3$.

Solution

$$
\begin{array}{rl}
 & 2x^2 \qquad + 1 \\
x^2 + 2x - 3 \,\big) & 2x^4 + 4x^3 - 5x^2 + 3x - 2 \\
 & \underline{2x^4 + 4x^3 - 6x^2} \\
 & x^2 + 3x - 2 \\
 & \underline{x^2 + 2x - 3} \\
 & x + 1
\end{array}
$$

Note that the first subtraction eliminated two terms from the dividend. When this happens, the quotient skips a term. You can write the result as

Remind students that when division yields a remainder, it is important that they write the remainder term correctly.

$$\frac{2x^4 + 4x^3 - 5x^2 + 3x - 2}{x^2 + 2x - 3} = 2x^2 + 1 + \frac{x + 1}{x^2 + 2x - 3}.$$

✓CHECKPOINT Now try Exercise 15.

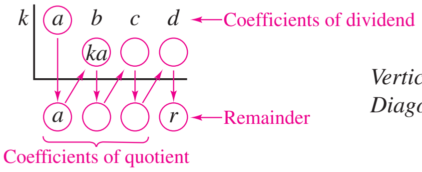
Point out to students that a graphing utility can be used to check the answer to a polynomial division problem. When students graph both the original polynomial division problem and the answer in the same viewing window, the graphs should coincide.

Synthetic Division

There is a nice shortcut for long division of polynomials when dividing by divisors of the form $x - k$. This shortcut is called **synthetic division.** The pattern for synthetic division of a cubic polynomial is summarized as follows. (The pattern for higher-degree polynomials is similar.)

Synthetic Division (for a Cubic Polynomial)

To divide $ax^3 + bx^2 + cx + d$ by $x - k$, use the following pattern.



Vertical pattern: Add terms.
Diagonal pattern: Multiply by k.

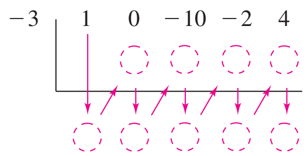
Synthetic division works only for divisors of the form $x - k$. [Remember that $x + k = x - (-k)$.] You cannot use synthetic division to divide a polynomial by a quadratic such as $x^2 - 3$.

Example 4 Using Synthetic Division

Use synthetic division to divide $x^4 - 10x^2 - 2x + 4$ by $x + 3$.

Solution

You should set up the array as follows. Note that a zero is included for the missing x^3-term in the dividend.



Then, use the synthetic division pattern by adding terms in columns and multiplying the results by -3.

Divisor: $x + 3$ Dividend: $x^4 - 10x^2 - 2x + 4$

-3	1	0	-10	-2	4
		-3	9	3	-3
	1	-3	-1	1	1

Remainder: 1

Quotient: $x^3 - 3x^2 - x + 1$

So, you have

$$\frac{x^4 - 10x^2 - 2x + 4}{x + 3} = x^3 - 3x^2 - x + 1 + \frac{1}{x + 3}.$$

CHECKPOINT Now try Exercise 19.

The Remainder and Factor Theorems

The remainder obtained in the synthetic division process has an important interpretation, as described in the **Remainder Theorem.**

The Remainder Theorem

If a polynomial $f(x)$ is divided by $x - k$, the remainder is

$$r = f(k).$$

For a proof of the Remainder Theorem, see Proofs in Mathematics on page 213.

The Remainder Theorem tells you that synthetic division can be used to evaluate a polynomial function. That is, to evaluate a polynomial function $f(x)$ when $x = k$, divide $f(x)$ by $x - k$. The remainder will be $f(k)$, as illustrated in Example 5.

Example 5 Using the Remainder Theorem

Use the Remainder Theorem to evaluate the following function at $x = -2$.

$$f(x) = 3x^3 + 8x^2 + 5x - 7$$

Solution

Using synthetic division, you obtain the following.

-2	3	8	5	-7
		-6	-4	-2
	3	2	1	-9

Because the remainder is $r = -9$, you can conclude that

$$f(-2) = -9. \qquad r = f(k)$$

This means that $(-2, -9)$ is a point on the graph of f. You can check this by substituting $x = -2$ in the original function.

Check

$$f(-2) = 3(-2)^3 + 8(-2)^2 + 5(-2) - 7$$
$$= 3(-8) + 8(4) - 10 - 7 = -9$$

CHECKPOINT Now try Exercise 45.

Additional Example

Use the Remainder Theorem to evaluate $f(x) = 4x^2 - 10x - 21$ when $x = 5$.

Solution

Using synthetic division, you obtain the following.

5	4	-10	-21
		20	50
	4	10	29

Because the remainder is 29, you can conclude that $f(5) = 29$.

Another important theorem is the **Factor Theorem,** stated below. This theorem states that you can test to see whether a polynomial has $(x - k)$ as a factor by evaluating the polynomial at $x = k$. If the result is 0, $(x - k)$ is a factor.

The Factor Theorem

A polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.

For a proof of the Factor Theorem, see Proofs in Mathematics on page 213.

Example 6 Factoring a Polynomial: Repeated Division

Show that $(x - 2)$ and $(x + 3)$ are factors of

$$f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18.$$

Then find the remaining factors of $f(x)$.

Solution

Using synthetic division with the factor $(x - 2)$, you obtain the following.

$$\begin{array}{r|rrrrr} 2 & 2 & 7 & -4 & -27 & -18 \\ & & 4 & 22 & 36 & 18 \\ \hline & 2 & 11 & 18 & 9 & 0 \end{array}$$

0 remainder, so $f(2) = 0$ and $(x - 2)$ is a factor.

Take the result of this division and perform synthetic division again using the factor $(x + 3)$.

$$\begin{array}{r|rrrr} -3 & 2 & 11 & 18 & 9 \\ & & -6 & -15 & -9 \\ \hline & 2 & 5 & 3 & 0 \end{array}$$

0 remainder, so $f(-3) = 0$ and $(x + 3)$ is a factor.

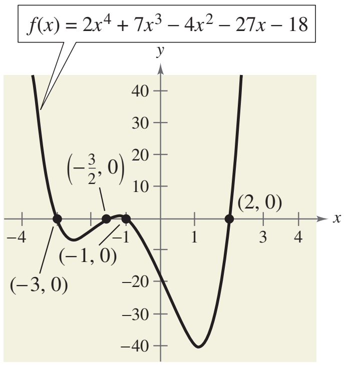


FIGURE 2.28

Because the resulting quadratic expression factors as

$$2x^2 + 5x + 3 = (2x + 3)(x + 1)$$

the complete factorization of $f(x)$ is

$$f(x) = (x - 2)(x + 3)(2x + 3)(x + 1).$$

Note that this factorization implies that f has four real zeros:

$$x = 2,\ x = -3,\ x = -\tfrac{3}{2},\ \text{and } x = -1.$$

This is confirmed by the graph of f, which is shown in Figure 2.28.

CHECKPOINT Now try Exercise 57.

Uses of the Remainder in Synthetic Division

The remainder r, obtained in the synthetic division of $f(x)$ by $x - k$, provides the following information.

1. The remainder r gives the value of f at $x = k$. That is, $r = f(k)$.

2. If $r = 0$, $(x - k)$ is a factor of $f(x)$.

3. If $r = 0$, $(k, 0)$ is an x-intercept of the graph of f.

Throughout this text, the importance of developing several problem-solving strategies is emphasized. In the exercises for this section, try using more than one strategy to solve several of the exercises. For instance, if you find that $x - k$ divides evenly into $f(x)$ (with no remainder), try sketching the graph of f. You should find that $(k, 0)$ is an x-intercept of the graph.

Activities

1. Divide using long division.
   $$\frac{4x^5 - x^3 + 2x^2 - x}{2x + 1}$$
   Answer: $2x^4 - x^3 + x - 1 + \dfrac{1}{2x + 1}$
2. Use synthetic division to determine if $(x + 3)$ is a factor of $f(x) = 3x^3 + 4x^2 - 18x - 3$.
   Answer: No, it is not.
3. Use the Remainder Theorem to evaluate $f(-3)$ for $f(x) = 2x^3 - 4x^2 + 1$.
   Answer: -89

2.3 Exercises

VOCABULARY CHECK:

1. Two forms of the Division Algorithm are shown below. Identify and label each term or function.

$$f(x) = d(x)q(x) + r(x) \qquad \frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

In Exercises 2–5, fill in the blanks.

2. The rational expression $p(x)/q(x)$ is called ________ if the degree of the numerator is greater than or equal to that of the denominator, and is called ________ if the degree of the numerator is less than that of the denominator.

3. An alternative method to long division of polynomials is called ________ ________, in which the divisor must be of the form $x - k$.

4. The ________ Theorem states that a polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.

5. The ________ Theorem states that if a polynomial $f(x)$ is divided by $x - k$, the remainder is $r = f(k)$.

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

Analytical Analysis **In Exercises 1 and 2, use long division to verify that $y_1 = y_2$.**

1. $y_1 = \dfrac{x^2}{x+2}, \quad y_2 = x - 2 + \dfrac{4}{x+2}$

2. $y_1 = \dfrac{x^4 - 3x^2 - 1}{x^2 + 5}, \quad y_2 = x^2 - 8 + \dfrac{39}{x^2 + 5}$

Graphical Analysis **In Exercises 3 and 4, (a) use a graphing utility to graph the two equations in the same viewing window, (b) use the graphs to verify that the expressions are equivalent, and (c) use long division to verify the results algebraically.**

3. $y_1 = \dfrac{x^5 - 3x^3}{x^2 + 1}, \quad y_2 = x^3 - 4x + \dfrac{4x}{x^2 + 1}$

4. $y_1 = \dfrac{x^3 - 2x^2 + 5}{x^2 + x + 1}, \quad y_2 = x - 3 + \dfrac{2(x + 4)}{x^2 + x + 1}$

In Exercises 5–18, use long division to divide.

5. $(2x^2 + 10x + 12) \div (x + 3)$

6. $(5x^2 - 17x - 12) \div (x - 4)$

7. $(4x^3 - 7x^2 - 11x + 5) \div (4x + 5)$

8. $(6x^3 - 16x^2 + 17x - 6) \div (3x - 2)$

9. $(x^4 + 5x^3 + 6x^2 - x - 2) \div (x + 2)$

10. $(x^3 + 4x^2 - 3x - 12) \div (x - 3)$

11. $(7x + 3) \div (x + 2)$

12. $(8x - 5) \div (2x + 1)$

13. $(6x^3 + 10x^2 + x + 8) \div (2x^2 + 1)$

14. $(x^3 - 9) \div (x^2 + 1)$

15. $(x^4 + 3x^2 + 1) \div (x^2 - 2x + 3)$

16. $(x^5 + 7) \div (x^3 - 1)$

17. $\dfrac{x^4}{(x-1)^3}$

18. $\dfrac{2x^3 - 4x^2 - 15x + 5}{(x-1)^2}$

In Exercises 19–36, use synthetic division to divide.

19. $(3x^3 - 17x^2 + 15x - 25) \div (x - 5)$

20. $(5x^3 + 18x^2 + 7x - 6) \div (x + 3)$

21. $(4x^3 - 9x + 8x^2 - 18) \div (x + 2)$

22. $(9x^3 - 16x - 18x^2 + 32) \div (x - 2)$

23. $(-x^3 + 75x - 250) \div (x + 10)$

24. $(3x^3 - 16x^2 - 72) \div (x - 6)$

25. $(5x^3 - 6x^2 + 8) \div (x - 4)$

26. $(5x^3 + 6x + 8) \div (x + 2)$

27. $\dfrac{10x^4 - 50x^3 - 800}{x - 6}$

28. $\dfrac{x^5 - 13x^4 - 120x + 80}{x + 3}$

29. $\dfrac{x^3 + 512}{x + 8}$

30. $\dfrac{x^3 - 729}{x - 9}$

31. $\dfrac{-3x^4}{x - 2}$

32. $\dfrac{-3x^4}{x + 2}$

33. $\dfrac{180x - x^4}{x - 6}$

34. $\dfrac{5 - 3x + 2x^2 - x^3}{x + 1}$

35. $\dfrac{4x^3 + 16x^2 - 23x - 15}{x + \frac{1}{2}}$

36. $\dfrac{3x^3 - 4x^2 + 5}{x - \frac{3}{2}}$

In Exercises 37–44, write the function in the form $f(x) = (x - k)q(x) + r$ for the given value of k, and demonstrate that $f(k) = r$.

	Function	*Value of k*
37.	$f(x) = x^3 - x^2 - 14x + 11$	$k = 4$
38.	$f(x) = x^3 - 5x^2 - 11x + 8$	$k = -2$

	Function	*Value of k*
39.	$f(x) = 15x^4 + 10x^3 - 6x^2 + 14$	$k = -\frac{2}{3}$
40.	$f(x) = 10x^3 - 22x^2 - 3x + 4$	$k = \frac{1}{5}$
41.	$f(x) = x^3 + 3x^2 - 2x - 14$	$k = \sqrt{2}$
42.	$f(x) = x^3 + 2x^2 - 5x - 4$	$k = -\sqrt{5}$
43.	$f(x) = -4x^3 + 6x^2 + 12x + 4$	$k = 1 - \sqrt{3}$
44.	$f(x) = -3x^3 + 8x^2 + 10x - 8$	$k = 2 + \sqrt{2}$

In Exercises 45–48, use synthetic division to find each function value. Verify your answers using another method.

45. $f(x) = 4x^3 - 13x + 10$

(a) $f(1)$ (b) $f(-2)$ (c) $f\left(\frac{1}{2}\right)$ (d) $f(8)$

46. $g(x) = x^6 - 4x^4 + 3x^2 + 2$

(a) $g(2)$ (b) $g(-4)$ (c) $g(3)$ (d) $g(-1)$

47. $h(x) = 3x^3 + 5x^2 - 10x + 1$

(a) $h(3)$ (b) $h\left(\frac{1}{3}\right)$ (c) $h(-2)$ (d) $h(-5)$

48. $f(x) = 0.4x^4 - 1.6x^3 + 0.7x^2 - 2$

(a) $f(1)$ (b) $f(-2)$ (c) $f(5)$ (d) $f(-10)$

In Exercises 49–56, use synthetic division to show that *x* is a solution of the third-degree polynomial equation, and use the result to factor the polynomial completely. List all real solutions of the equation.

	Polynomial Equation	*Value of x*
49.	$x^3 - 7x + 6 = 0$	$x = 2$
50.	$x^3 - 28x - 48 = 0$	$x = -4$
51.	$2x^3 - 15x^2 + 27x - 10 = 0$	$x = \frac{1}{2}$
52.	$48x^3 - 80x^2 + 41x - 6 = 0$	$x = \frac{2}{3}$
53.	$x^3 + 2x^2 - 3x - 6 = 0$	$x = \sqrt{3}$
54.	$x^3 + 2x^2 - 2x - 4 = 0$	$x = \sqrt{2}$
55.	$x^3 - 3x^2 + 2 = 0$	$x = 1 + \sqrt{3}$
56.	$x^3 - x^2 - 13x - 3 = 0$	$x = 2 - \sqrt{5}$

In Exercises 57–64, (a) verify the given factors of the function *f*, (b) find the remaining factors of *f*, (c) use your results to write the complete factorization of *f*, (d) list all real zeros of *f*, and (e) confirm your results by using a graphing utility to graph the function.

	Function	*Factors*
57.	$f(x) = 2x^3 + x^2 - 5x + 2$	$(x + 2), (x - 1)$
58.	$f(x) = 3x^3 + 2x^2 - 19x + 6$	$(x + 3), (x - 2)$
59.	$f(x) = x^4 - 4x^3 - 15x^2 + 58x - 40$	$(x - 5), (x + 4)$
60.	$f(x) = 8x^4 - 14x^3 - 71x^2 - 10x + 24$	$(x + 2), (x - 4)$
61.	$f(x) = 6x^3 + 41x^2 - 9x - 14$	$(2x + 1), (3x - 2)$
62.	$f(x) = 10x^3 - 11x^2 - 72x + 45$	$(2x + 5), (5x - 3)$
63.	$f(x) = 2x^3 - x^2 - 10x + 5$	$(2x - 1), \left(x + \sqrt{5}\right)$
64.	$f(x) = x^3 + 3x^2 - 48x - 144$	$\left(x + 4\sqrt{3}\right), (x + 3)$

Graphical Analysis **In Exercises 65–68, (a) use the *zero* or *root* feature of a graphing utility to approximate the zeros of the function accurate to three decimal places, (b) determine one of the exact zeros, and (c) use synthetic division to verify your result from part (b), and then factor the polynomial completely.**

65. $f(x) = x^3 - 2x^2 - 5x + 10$

66. $g(x) = x^3 - 4x^2 - 2x + 8$

67. $h(t) = t^3 - 2t^2 - 7t + 2$

68. $f(s) = s^3 - 12s^2 + 40s - 24$

In Exercises 69–72, simplify the rational expression by using long division or synthetic division.

69. $\dfrac{4x^3 - 8x^2 + x + 3}{2x - 3}$ **70.** $\dfrac{x^3 + x^2 - 64x - 64}{x + 8}$

71. $\dfrac{x^4 + 6x^3 + 11x^2 + 6x}{x^2 + 3x + 2}$ **72.** $\dfrac{x^4 + 9x^3 - 5x^2 - 36x + 4}{x^2 - 4}$

Model It

73. ***Data Analysis: Military Personnel*** The numbers M (in thousands) of United States military personnel on active duty for the years 1993 through 2003 are shown in the table, where t represents the year, with $t = 3$ corresponding to 1993. (Source: U.S. Department of Defense)

Year, t	Military personnel, M
3	1705
4	1611
5	1518
6	1472
7	1439
8	1407
9	1386
10	1384
11	1385
12	1412
13	1434

Model It (continued)

(a) Use a graphing utility to create a scatter plot of the data.

(b) Use the *regression* feature of the graphing utility to find a cubic model for the data. Graph the model in the same viewing window as the scatter plot.

(c) Use the model to create a table of estimated values of M. Compare the model with the original data.

(d) Use synthetic division to evaluate the model for the year 2008. Even though the model is relatively accurate for estimating the given data, would you use this model to predict the number of military personnel in the future? Explain.

74. ***Data Analysis: Cable Television*** The average monthly basic rates R (in dollars) for cable television in the United States for the years 1992 through 2002 are shown in the table, where t represents the year, with $t = 2$ corresponding to 1992. (Source: Kagan Research LLC)

Year, t	Basic rate, R
2	19.08
3	19.39
4	21.62
5	23.07
6	24.41
7	26.48
8	27.81
9	28.92
10	30.37
11	32.87
12	34.71

(a) Use a graphing utility to create a scatter plot of the data.

(b) Use the *regression* feature of the graphing utility to find a cubic model for the data. Then graph the model in the same viewing window as the scatter plot. Compare the model with the data.

(c) Use synthetic division to evaluate the model for the year 2008.

Synthesis

True or False? **In Exercises 75–77, determine whether the statement is true or false. Justify your answer.**

75. If $(7x + 4)$ is a factor of some polynomial function f, then $\frac{4}{7}$ is a zero of f.

76. $(2x - 1)$ is a factor of the polynomial

$$6x^6 + x^5 - 92x^4 + 45x^3 + 184x^2 + 4x - 48.$$

77. The rational expression

$$\frac{x^3 + 2x^2 - 13x + 10}{x^2 - 4x - 12}$$

is improper.

78. ***Exploration*** Use the form $f(x) = (x - k)q(x) + r$ to create a cubic function that (a) passes through the point $(2, 5)$ and rises to the right, and (b) passes through the point $(-3, 1)$ and falls to the right. (There are many correct answers.)

Think About It **In Exercises 79 and 80, perform the division by assuming that n is a positive integer.**

79. $\dfrac{x^{3n} + 9x^{2n} + 27x^n + 27}{x^n + 3}$

80. $\dfrac{x^{3n} - 3x^{2n} + 5x^n - 6}{x^n - 2}$

81. ***Writing*** Briefly explain what it means for a divisor to divide evenly into a dividend.

82. ***Writing*** Briefly explain how to check polynomial division, and justify your reasoning. Give an example.

Exploration **In Exercises 83 and 84, find the constant c such that the denominator will divide evenly into the numerator.**

83. $\dfrac{x^3 + 4x^2 - 3x + c}{x - 5}$

84. $\dfrac{x^5 - 2x^2 + x + c}{x + 2}$

Think About It **In Exercises 85 and 86, answer the questions about the division $f(x) \div (x - k)$, where $f(x) = (x + 3)^2(x - 3)(x + 1)^3$.**

85. What is the remainder when $k = -3$? Explain.

86. If it is necessary to find $f(2)$, is it easier to evaluate the function directly or to use synthetic division? Explain.

Skills Review

In Exercises 87–92, use any method to solve the quadratic equation.

87. $9x^2 - 25 = 0$

88. $16x^2 - 21 = 0$

89. $5x^2 - 3x - 14 = 0$

90. $8x^2 - 22x + 15 = 0$

91. $2x^2 + 6x + 3 = 0$

92. $x^2 + 3x - 3 = 0$

In Exercises 93–96, find a polynomial function that has the given zeros. (There are many correct answers.)

93. $0, 3, 4$

94. $-6, 1$

95. $-3, 1 + \sqrt{2}, 1 - \sqrt{2}$

96. $1, -2, 2 + \sqrt{3}, 2 - \sqrt{3}$

SECTION 2.3: LONG AND SYNTHETIC POLYNOMIAL DIVISION

PART A: LONG DIVISION

Ancient Example with Integers

$$\begin{array}{r} 2 \\ 4\overline{)\ 9} \\ -8 \\ \hline 1 \end{array}$$

We can say: $\frac{9}{4} = 2 + \frac{1}{4}$

By multiplying both sides by 4, this can be rewritten as:

$$9 = 4 \cdot 2 + 1$$

In general: $$\frac{\text{dividend}, f}{\text{divisor}, d} = (\text{quotient}, q) + \frac{(\text{remainder}, r)}{d}$$

where either:

$r = 0$ (in which case d divides evenly into f), or

$\frac{r}{d}$ is a positive proper fraction: i.e., $0 < r < d$

Technical Note: We assume f and d are positive integers, and q and r are nonnegative integers.

Technical Note: We typically assume $\frac{f}{d}$ is improper: i.e., $f \geq d$. Otherwise, there is no point in dividing this way.

Technical Note: Given f and d, q and r are unique by the Division Algorithm (really, it's a theorem).

By multiplying both sides by d, $\frac{f}{d} = q + \frac{r}{d}$ can be rewritten as:

$$f = d \cdot q + r$$

Now, we will perform polynomial division on $\frac{f(x)}{d(x)}$ so that we get:

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

where either:

$r(x)=0$, in which case $d(x)$ divides evenly into $f(x)$, or

$\frac{r(x)}{d(x)}$ is a proper rational expression: i.e., $\deg(r(x)) < \deg(d(x))$

Technical Note: We assume $f(x)$ and $d(x)$ are nonzero polynomials, and $q(x)$ and $r(x)$ are polynomials.

Technical Note: We assume $\frac{f(x)}{d(x)}$ is improper; i.e., $\deg(f(x)) \geq \deg(d(x))$. Otherwise, there is no point in dividing.

Technical Note: Given $f(x)$ and $d(x)$, $q(x)$ and $r(x)$ are unique by the Division Algorithm (really, it's a theorem).

By multiplying both sides by $d(x)$, $\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$ can be rewritten as:

$$f(x) = d(x) \cdot q(x) + r(x)$$

Example

Use Long Division to divide: $\dfrac{-5+3x^2+6x^3}{1+3x^2}$

Solution

Warning: First, write the N and the D in descending powers of x.

Warning: Insert "missing term placeholders" in the N (and perhaps even the D) with "0" coefficients. This helps you avoid errors. We get:

$$\frac{6x^3+3x^2+0x-5}{3x^2+0x+1}$$

Let's begin the Long Division:

$$3x^2+0x+1\,\overline{\big)\;6x^3+3x^2+0x-5}$$

The steps are similar to those for $4\,\overline{\big)\;9}$.

Think: How many "times" does the leading term of the divisor ($3x^2$) "go into" the leading term of the dividend ($6x^3$)? We get:
$\dfrac{6x^3}{3x^2}=2x$, which goes into the quotient.

$$\begin{array}{r}2x\\ 3x^2+0x+1\,\overline{\big)\;6x^3+3x^2+0x-5}\end{array}$$

Multiply the $2x$ by the divisor and write the product on the next line.

Warning: Line up like terms to avoid confusion!

$$\begin{array}{r}2x\\ 3x^2+0x+1\,\overline{\big)\;6x^3+3x^2+0x-5}\\ 6x^3+0x^2+2x\end{array}$$

Warning: We must **subtract** this product from the dividend. People have a much easier time adding than subtracting, so let's flip the sign on each term of the product, and add the result to the dividend. To avoid errors, we will cross out our product and do the sign flips on a separate line before adding.

Warning: Don't forget to bring down the -5.

$$\begin{array}{r|l} & 2x \\ \hline 3x^2+0x+1 & 6x^3+3x^2+0x-5 \\ & \cancel{6x^3+0x^2+2x} \\ & \underline{-6x^3-0x^2-2x} \\ & 3x^2-2x-5 \end{array}$$

We now treat the expression in blue above as our new dividend. Repeat the process.

$$\begin{array}{r|l} & 2x+1 \\ \hline 3x^2+0x+1 & 6x^3+3x^2+0x-5 \\ & \cancel{6x^3+0x^2+2x} \\ & \underline{-6x^3-0x^2-2x} \\ & 3x^2-2x-5 \end{array}$$

$$\begin{array}{r|l} & 2x+1 \\ \hline 3x^2+0x+1 & 6x^3+3x^2+0x-5 \\ & \cancel{6x^3+0x^2+2x} \\ & \underline{-6x^3-0x^2-2x} \\ & 3x^2-2x-5 \\ & 3x^2+0x+1 \end{array}$$

$$
\begin{array}{rl}
 & 2x+1 \\
3x^2+0x+1 & \overline{\big)\; 6x^3+3x^2+0x-5} \\
 & \cancel{6x^3+0x^2+2x} \\
 & \underline{-6x^3-0x^2-2x} \\
 & 3x^2-2x-5 \\
 & \cancel{3x^2+0x+1} \\
 & \underline{-3x^2-0x-1} \\
 & -2x-6
\end{array}
$$

We can now stop the process, because the degree of the new dividend is less than the degree of the divisor. The degree of $-2x-6$ is 1, which is less than the degree of $3x^2+0x+1$, which is 2. This guarantees that the fraction in our answer is a **proper** rational expression.

Our answer is of the form: $q(x)+\dfrac{r(x)}{d(x)}$

$$2x+1 \;+\; \frac{-2x-6}{3x^2+1}$$

If the leading coefficient of $r(x)$ is negative, then we factor a -1 out of it.

Answer: $\mathbf{2x+1 \;-\; \dfrac{2x+6}{3x^2+1}}$

Warning: Remember to flip **every** sign in the numerator.

Warning: If the N and the D of our fraction have any common factors aside from ± 1, they must be canceled out. Our fraction here is simplified as is.

PART B: SYNTHETIC DIVISION

There's a great short cut if the divisor is of the form $x-k$.

Example

Use Synthetic Division to divide: $\dfrac{2x^3-3x+5}{x+3}$.

Solution

The divisor is $x+3$, so $k=-3$.
Think: $x+3=x-(-3)$.
We will put -3 in a half-box in the upper left of the table below.

Make sure the N is written in standard form.
Write the coefficients in order along the first row of the table.
Write a "placeholder 0" if a term is missing.

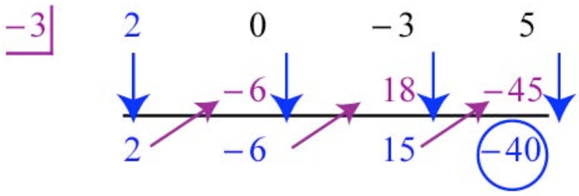
Bring down the first coefficient, the "2."

-3	2	0	-3	5
	↓			
	2			

The ↓ arrow tells us to add down the column and write the sum in the third row.

The ↗ arrow tells us to multiply the blue number by k (here, -3) and write the product one column to the right in the second row.

Circle the lower right number.



Since we are dividing a 3rd-degree dividend by a 1st-degree divisor, our answer begins with a 2nd-degree term.

The third (blue) row gives the coefficients of our quotient in descending powers of x. The circled number is our remainder, which we put over our divisor and factor out a -1 if appropriate.

Note: The remainder **must** be a **constant**, because the divisor is linear.

Answer: $\mathbf{2x^2 - 6x + 15 - \dfrac{40}{x+3}}$

Related Example

Express $f(x) = 2x^3 - 3x + 5$ in the following form:
$f(x) = d(x) \cdot q(x) + r$, where the divisor $d(x) = x + 3$.

Solution

We can work from our previous Answer. Multiply both sides by the divisor:

$$\frac{2x^3 - 3x + 5}{x+3} = 2x^2 - 6x + 15 - \frac{40}{x+3}$$
$$2x^3 - 3x + 5 = (x+3) \cdot (2x^2 - 6x + 15) - 40$$

Note: Synthetic Division works even if $k = 0$. What happens?

PART C: REMAINDER THEOREM

Remainder Theorem

If we are dividing a polynomial $f(x)$ by $x-k$, and if r is the remainder, then $f(k)=r$.

In our previous Examples, we get the following fact as a bonus.

$$f(-3) = -40$$

Synthetic Division therefore provides an efficient means of evaluating polynomial functions. (It may be much better than straight calculator button-pushing when dealing with polynomials of high degree.) We could have done the work in Part B if we had wanted to evaluate $f(-3)$, where $f(x)=2x^3-3x+5$.

Warning: Do **not** flip the sign of -3 when writing it in the half-box.
People get the "sign flip" idea when they work with polynomial division.

Technical Note: See the short Proof on p.192.

PART D: ZEROS, FACTORING, AND DIVISION

Recall from Section 2.2:

> Factor Theorem
>
> If $f(x)$ is a nonzero polynomial and k is a real number, then
> k is a zero of f $\Leftrightarrow$ $(x-k)$ is a factor of $f(x)$.

Technical Note: The Proof on p.192 uses the Remainder Theorem to prove this.

What happens if either Long or Synthetic polynomial division gives us a 0 remainder? Then, we can at least partially factor $f(x)$.

Example

Show that 2 is a zero of $f(x)=4x^3-5x^2-7x+2$.

Note: We saw this $f(x)$ in Section 2.2.

Note: In Section 2.5, we will discuss a trick for finding such a zero.

Factor $f(x)$ completely, and find **all** of its real zeros.

Solution

We will use Synthetic Division to show that 2 is a zero:

2	4	-5	-7	2
		8	6	-2
	4	3	-1	0

By the Remainder Theorem, $f(2)=0$, and so 2 is a zero.

By the Factor Theorem, $(x-2)$ must be a factor of $f(x)$.

> Technical Note: This can be seen from the form $f(x) = d(x)\cdot q(x) + r$. Since $r=0$ when $d(x)=x-2$, we have:
>
> $$f(x)=(x-2)\cdot q(x),$$
>
> where $q(x)$ is some (here, quadratic) polynomial.

We can find $q(x)$, the other (quadratic) factor, by using the last row of the table.

$$f(x)=(x-2)\cdot(4x^2+3x-1)$$

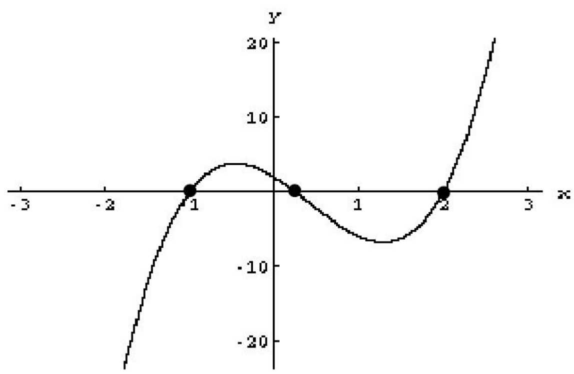
Factor $q(x)$ completely over the reals:

$$\boldsymbol{f(x)=(x-2)(4x-1)(x+1)}$$

The zeros of $f(x)$ are the zeros of these factors:

$$\mathbf{2,\ \frac{1}{4},\ -1}$$

Below is a graph of $f(x)=4x^3-5x^2-7x+2$. Where are the x-intercepts?



5.1 The Remainder and Factor Theorems; Synthetic Division

In this section you will learn to:
- understand the definition of a zero of a polynomial function
- use long and synthetic division to divide polynomials
- use the remainder theorem
- use the factor theorem

Example 1: Use long division to find the quotient and the remainder: $5593 \div 27$

<u>Steps for Long Division</u>:

1.

2.

3.

4.

Example 2: Use the "Steps for Long Division" to divide each of the polynomials below.

$x-5\overline{)\ x^2-2x-35}$

$\left(7-11x-3x^2+2x^3\right)\div(x-3)$

Example 3: Check your answer for the division problems in Example 2.

The Division Algorithm: If $f(x)$ and $d(x)$ are polynomials where $d(x)\neq 0$ and degree $d(x)$ < degree $f(x)$,

then

$\overline{)\qquad\qquad}$

$$f(x) \quad = \quad d(x) \quad \cdot \quad q(x) \quad + \quad r(x)$$

If $r(x) = 0$ then $d(x)$ and $q(x)$ are **factors** of $f(x)$.

Example 4: Perform the operation below. Write the remainder as a rational expression (remainder/divisor).

$$\frac{2x^5 - 8x^4 + 2x^3 + x^2}{2x^3 + 1}$$

Synthetic Division – Generally used for "short" division of polynomials when the divisor is in the form $x - c$. (Refer to page 506 in your textbook for more examples.)

Example 5: Use both long and short (synthetic) division to find the quotient and remainder for the problem below.

$$(2x^3 - 11x + 7) \div (x - 3)$$

Example 6: Divide $\frac{x^3 + 8}{x + 2}$ using synthetic division.

Example 7: Factor $x^3 + 8$ over the real numbers. (Hint: Refer to Example 6.)

Remainder Theorem	Factor Theorem
If the polynomial $f(x)$ is divided by $(x - c)$, then the remainder is $f(c)$.	Let $f(x)$ be a polynomial. If $f(c) = 0$, then $(x - c)$ is a factor of $f(x)$. If $(x - c)$ is a factor of $f(x)$, then $f(c) = 0$. If $(x-c)$ is a factor of $f(x)$ or if $f(c)=0$, then ***c*** is called a **zero** of $f(x)$.

Example 8: $f(x) = 3x^3 + 4x^2 - 5x + 7$. Find $f(-4)$ using

(a) synthetic division.

(b) the Remainder Theorem.

Example 9: Solve the equation $2x^3 - 3x^2 - 11x + 6 = 0$ given that -2 is a zero of $f(x) = 2x^3 - 3x^2 - 11x + 6$.

5.1 Homework Problems:

For Problems 1-5, use long division to find each quotient, $q(x)$, and remainder, $r(x)$.

1. $(x^2 - 2x - 15) \div (x - 5)$

2. $(x^3 + 5x^2 + 7x + 2) \div (x + 2)$

3. $(6x^3 + 7x^2 + 12x - 5) \div (3x - 1)$

4. $\dfrac{x^4 - 81}{x - 3}$

5. $\dfrac{18x^4 + 9x^3 + 3x^2}{3x^2 + 1}$

For Problems 6 – 11, divide using synthetic division.

6. $(2x^2 + x - 10) \div (x - 2)$

7. $(5x^3 - 6x^2 + 3x + 11) \div (x - 2)$

8. $(x^2 - 5x - 5x^3 + x^4) \div (5 + x)$

9. $\dfrac{x^7 + x^5 - 10x^3 + 12}{x + 2}$

10. $\dfrac{x^4 - 256}{x - 4}$

11. $\dfrac{x^5 - 2x^4 - x^3 + 3x^2 - x + 1}{x - 2}$

For Problems 12 – 16, use synthetic division and the Remainder Theorem to find the indicated function value.

12. $f(x) = x^3 - 7x^2 + 5x - 6;\ f(3)$

13. $f(x) = 4x^3 + 5x^2 - 6x - 4;\ f(-2)$

14. $f(x) = 2x^4 - 5x^3 - x^2 + 3x + 2;\ f\left(-\frac{1}{2}\right)$

15. $f(x) = 6x^4 + 10x^3 + 5x^2 + x + 1;\ f\left(-\frac{2}{3}\right)$

16. Use synthetic division to divide $f(x) = x^3 - 4x^2 + x + 6$ by $x + 1$. Use the result to find all zeros of *f*.

17. Solve the equation $2x^3 - 5x^2 + x + 2 = 0$ given that 2 is a zero of $f(x) = 2x^3 - 5x^2 + x + 2$.

18. Solve the equation $12x^3 + 16x^2 - 5x - 3 = 0$ given that $-\frac{3}{2}$ is a zero (root).

5.1 Homework Answers: **1.** $q(x) = x + 3$ **2.** $q(x) = x^2 + 3x + 1$ **3.** $q(x) = 2x^2 + 3x + 5$

4. $q(x) = x^3 + 3x^2 + 9x + 27$ **5.** $q(x) = 6x^2 + 3x - 1;\ \ r(x) = -3x + 1$ **6.** $q(x) = 2x + 5$

7. $q(x) = 5x^2 + 4x + 11;\ \ r(x) = 33$ **8.** $q(x) = x^3 - 10x^2 + 51x - 260;\ \ r(x) = 1300$

9. $q(x) = x^6 - 2x^5 + 5x^4 - 10x^3 + 10x^2 - 20x + 40;\ \ r(x) = -68$ **10.** $q(x) = x^3 + 4x^2 + 16x + 64$

11. $q(x) = x^4 - x^2 + x + 1;\ \ r(x) = 3$ **12.** -27 **13.** -4 **14.** 1 **15.** $\frac{7}{9}$

16. $x^2 - 5x + 6;\ \ x = -1, 2, 3$ **17.** $\left\{-\frac{1}{2}, 1, 2\right\}$ **18.** $\left\{-\frac{3}{2}, -\frac{1}{3}, \frac{1}{2}\right\}$

5.3 Roots of Polynomial Equations

In this section you will learn to:

- find zeros of polynomial equations
- solve polynomial equations with real and imaginary zeros
- find possible rational roots of polynomial equations
- understand properties of polynomial equatins
- use the Linear Factorization Theorem

Zeros of Polynomial Functions are the values of x for which $f(x) = 0$.

(Zero = Root = Solution = x-intercept (if the zero is a real number))

Example 1: Consider the polynomial that only has 3 and ½ as zeros.

(a) How many polynomials have such zeros?

(b) Find a polynomial that has a leading coefficient of 1 that has such zeros.

(c) Find a polynomial, with integer coefficients, that has such zeros.

If the same factor $(x - r)$ occurs k times, then the zero r is called a zero with **multiplicity *k*.**

Even Multiplicity → Graph **touches** x-axis and turns around.

Odd Multiplicity → Graph **crosses** x-axis.

Example 2: Find all of the (real) zeros for each of the polynomial functions below. Give the multiplicity of each zero and state whether the graph crosses the x-axis or touches (and turns at) the x-axis at each zero. Use this information and the Leading Coefficient Test to sketch a graph of each function

(a) $f(x) = x^3 + 2x^2 - 4x - 8$ (b) $f(x) = -x^4 + 4x^2$ (c) $g(x) = -x^4 + 4x^3 - 4x^2$

The Rational Zero Theorem: If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots a_2 x^2 + a_1 x + a_0$ has ***integer*** coefficients and $\frac{p}{q}$ (reduced to lowest terms) is a rational zero of f, then p is a factor of the constant term, a_0, and q is a factor of the leading coefficient, a_n.

Example 3: List all **possible** rational zeros of the polynomials below. (Refer to Rational Zero Theorem on
Page 1 of this handout.)

(a) $f(x) = -x^5 + 7x^2 - 12$ Possible Rational Zeros: ____________________________

(b) $p(x) = 6x^3 - 8x^2 - 8x + 8$ Possible Rational Zeros: ____________________________

Example 4: Find all zeros of $f(x) = 2x^3 - 5x^2 + x + 2$.

Example 5: Solve $x^4 - 8x^3 + 64x - 105 = 0$.

Linear Factorization Theorem:

If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots a_2 x^2 + a_1 x + a_0$, where n $\geq$ 1 and $a_n \neq 0$, then

$f(x) = a_n(x - c_1)(x - c_2)\dots(x - c_n)$, where $c_1, c_2, c_3, \dots c_n$ are complex numbers.

Example 6: Find all complex zeros of $f(x) = 2x^4 + 3x^3 + 3x - 2$, and then write the polynomial $f(x)$ as a **product of linear factors**.

$f(x) =$ ____

Properties of Polynomial Equations:

Given the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots a_2 x^2 + a_1 x + a_0$.

1. If a polynomial equation is of degree *n*, then counting multiple roots (multiplicities) separately, the equation has *n* roots.

2. If $a + bi$ is a root of a polynomial equation ($b \neq 0$), then the imaginary number $a - bi$ is also a root. In other words, imaginary roots, if they exist, occur in **conjugate pairs**.

Example 7: Find all zeros of $f(x) = x^4 - 4x^2 - 5$. (Hint: Use factoring techniques from Chapter 1.) Write $f(x)$ as a product of linear factors.

$f(x) =$ ____

Example 8: Find a third-degree polynomial function, $f(x)$, with real coefficients that has 4 and $2i$ as zeros and such that $f(-1) = 50$.

Step 1: Use the zeros to find the factors of $f(x)$.

Step 2: Write as a linear factorization, then expand/multiply.

Step 3: Use $f(-1) = 50$ to substitute values for x and $f(x)$.

Step 4: Solve for a_n.

Step 5: Substitute a_n into the equation for $f(x)$ and simplify.

Step 6: Use your calculator to check.

5.3 Homework Problems:

For Problems 1 – 4, use the Rational Zero Theorem to list all possible rational zeros for each function.

1. $f(x) = x^3 + 3x^2 - 6x - 8$
2. $f(x) = 2x^4 + 3x^3 - 11x^2 - 9x + 15$
3. $f(x) = 3x^4 - 11x^3 - 3x^2 - 6x + 8$
4. $f(x) = 4x^5 - 8x^4 - x + 2$

For Problems 5 – 8, find the zeros for the given functions.

5. $f(x) = x^3 - 2x^2 - 11x + 12$
6. $f(x) = 2x^3 - 5x^2 + x + 2$
7. $f(x) = 2x^3 + x^2 - 3x + 1$
8. $f(x) = x^3 - 4x^2 + 8x - 5$

For Problems 9 – 12, solve each of the given equations.

9. $x^3 - 2x^2 - 7x - 4 = 0$
10. $x^3 - 5x^2 + 17x - 13 = 0$
11. $2x^3 - 5x^2 - 6x + 4 = 0$
12. $x^4 - 2x^2 - 16x - 15 = 0$

For Problems 13-16, find an *nth* degree polynomial function, $f(x)$, with real coefficients that satisfies the given conditions.

13. $n = 3$; 1 and $5i$ are zeros; $f(-1) = -104$
14. $n = 4$; 2, -2, and i are zeros; $f(3) = -150$
15. $n = 3$; 6 and $-5 + 2i$ are zeros; $f(2) = -636$
16. $n = 4$; i and $3i$ are zeros; $f(-1) = 20$

5.3 Homework Answers: **1.** $\pm 1, \pm 2, \pm 4, \pm 8$ **2.** $\pm 1, \pm 3, \pm 5, \pm 15, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \pm \frac{15}{2}$

3. $\pm 1, \pm 2, \pm 4, \pm 8, \pm \frac{1}{2}, \pm \frac{2}{3}, \pm \frac{4}{3}, \pm \frac{8}{3}$ **4.** $\pm 1, \pm 2, \pm \frac{1}{2}, \pm \frac{1}{4}$ **5.** $-3, 1, 4$ **6.** $-\frac{1}{2}, 1, 2$

7. $\frac{1}{2}, \frac{-1 \pm \sqrt{5}}{2}$ **8.** $1, \frac{3 \pm i\sqrt{11}}{2}$ **9.** $\{-1, 4\}$ **10.** $\{1, 2 \pm 3i\}$ **11.** $\left\{\frac{1}{2}, 1 \pm \sqrt{5}\right\}$

12. $\{-1, 3, -1 \pm 2i\}$ **13.** $f(x) = 2x^3 - 2x^2 + 50x - 50$ **14.** $f(x) = -3x^4 + 9x^2 + 12$

15. $f(x) = 3x^3 + 12x^2 - 93x - 522$ **16.** $f(x) = x^4 + 10x^2 + 9$

Lecture 6
Solution Of Bi-quadratic Equations By Using Descartes Method

Let the given Bi-quadratic equation be

$$x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0.$$

Now to remove the second term, we have to diminish the roots by

$$h = -\frac{a_1}{a_0}$$

$\therefore$ $\quad y = x - h = x + \dfrac{a_1}{a_0}.$

or $\quad x = y - \dfrac{a_1}{a_0}$. Substituting the value of x in the given equation, the transformed equation becomes

$$a_0^4y^4 + 6a_0^2y^2(a_0a_1 - a_1^2) + 4a_0y(2a_1^2 - 3a_0a_1a_2 + a_0^2a_3) + (6a_0a_2a_1^2 - 3a_1^4 - 4a_0^2a_1a_3 + a_0^3a_4) = 0$$

$$a_0^4y^4 + 6a_0y^2(a_0a_1 - a_1^2) + 4a_0y(2a_1^2 - 3a_0a_1a_2 + a_0^2a_3) + [a_0^2(3a_2^2 - 4a_1a_3 + a_0a_4) - 3(a_0a_2 - a_1^2)^2] = 0$$

Putting $y = \dfrac{z}{a_0}, H = a_0a_2 - a_1^2, G = 2a_1^3 - 3a_0a_1a_2 + a_0^2a_2$ and $3a_2^2 - 4a_1a_3 + a_0a_4$, we get the transformed equation as

$$z^4 + 6Hz^2 + 4Gz + (a_0^2I - 3H^2) = 0, \tag{0.1}$$

where $\quad a_0x + a_1 = z.$

Let the quadratic factor of (0.1) be

$$z^2 + pz + q \qquad \text{and} \qquad z^2 - pz + q',$$

so that

$$(z^2 + pz + q)(z^2 - pz + q') = 0 \tag{0.2}$$

is the same as (0.1).

Comparing the coefficients of the left hand side of (0.1) and (0.2), we get

$$q' + q - p^2 = 6H$$

$$\text{or} \qquad q' + q = 6H + p^2 \tag{0.3}$$

$$p(q' - q) = 4G$$

$$\text{or} \qquad q' - q = \frac{4G}{p} \tag{0.4}$$

Also, $qq' = a_0^2I - 3H^2$ $\quad$ or $\quad$ $4qq' = 4a_0^2I - 12H^2$.

Since, we have

$$4qq' = (q' + q)^2 - (q' - q)^2 \tag{0.5}$$

$$\therefore \qquad 4a_0^2I - 12H^2 = p^4 + 12p^2H + 36H^2 - \frac{16G^2}{p^2}$$

$$\text{or} \qquad p^6 + 12p^4H + (48H^2 - 4a_0^2I)p^2 - 16G^2 = 0$$

Putting $p^2 = t$, we get

$$t^3 + 12Ht^2 + 4(12H^2 - a_0^2I)t - 16G^2 = 0 \tag{0.6}$$

Equation (0.6) is the called Descartes Resolvent. Now, we can find t from (0.6) and so also p.
From (0.3) and (0.4), we can find q and q'. Thus, the two quadratic equations $z^2 + pz + q = 0$ and $z^2 - pz + q' = 0$ can be solved.
Let z_1, z_2, z_3, z_4 be the roots of these equations, then corresponding to these we can find the four values of x from the relation

$$a_0 x + a_1 = z.$$

Example 1. *Solve*

$$x^4 - 6x^2 + 8x - 3 = 0. \tag{0.7}$$

Sol. Let the quadratic factors be

$$(x^2 + px + q)(x^2 - px + q') = 0 \tag{0.8}$$

$$\implies \quad x^4 - px^3 + q'x^2 + p^2x^2 + pq'x + qx^2 - pqx + qq' = 0$$

$$\implies \quad x^4 + (p + q' - p^2)x^2 + p(q' - q)x + qq' = 0 \tag{0.9}$$

Comparing the coefficients of the left hand side of (0.7) and (0.9), we get

$$q' + q - p^2 = -6$$

$$\text{or} \qquad q' + q = p^2 - 6 \tag{0.10}$$

$$p(q' - q) = 8$$

$$\text{or} \qquad q' - q = \frac{8}{p} \tag{0.11}$$

Also, $qq' = -3$
Since, we have

$$(q' + q)^2 - (q' - q)^2 = 4qq'$$

$$(p^2 - 6)^2 - \left(\frac{8}{p}\right)^2 = -12$$

$$\implies \quad p^6 + 36p^2 - 12p^4 - 64 + 12p^2 = 0$$

$$\implies \quad p^6 - 12p^4 + +48p^2 - 64 = 0$$

$$\implies \quad (p^2)^3 - 12(p^2)^2 + 48(p^2) - 64 = 0$$

Put $p^2 = t$, we get

$$t^3 - 12t^2 + 48t - 64 = 0 \tag{0.12}$$

test the cubic for numbers which are perfect squares i,e. $t = 1, 4, 9, 16, 25, \cdots$
$\therefore \quad t = 4$, satisfies equation (0.12).
So that $p^2 = t = 4 \implies \quad p^2 = 4 \quad \implies \quad p = 2$.
Also, from (0.11), we have

$$q' - q = \frac{8}{2}$$

$$q' - q = 4 \tag{0.13}$$

From (0.10), we get

$$\begin{aligned} q' + q &= 4 - 6 \\ \implies \quad q' + q &= -2 \end{aligned} \tag{0.14}$$

Adding (0.13) and (0.14), we get

$$\begin{aligned} 2q' &= 2 \\ \implies \quad q' &= 1 \end{aligned}$$

Put $q' = 1$ in (0.14), we have

$$\begin{aligned} q &= -2 - 1 \\ \implies \quad q &= -3. \end{aligned}$$

Thus substitute p, q, q' in (0.8), we get

$$\begin{aligned} (x^2 + 2x - 3)(x^2 - 2x + 1) &= 0 \\ \implies \quad (x^2 + 2x - 3) = 0 \quad \text{or} \quad (x^2 - 2x + 1) &= 0 \\ \implies \quad (x + 3)(x - 1) = 0 \quad \text{or} \quad (x - 1)(x - 1) &= 0 \\ \implies \quad x = 1, -3 \quad \text{or} \quad x &= 1, 1 \end{aligned}$$

Hence the four roots are $x = 1, 1, 1, -3$.

Example 2. *Solve* $x^4 - 4x^3 + 6x^2 + x - 10 = 0$.

Sol. we first remove the second term, so we diminish roots of the given equation by h, where

$$h = \frac{\text{Sum of roots}}{\text{Degree of equation}} = \frac{-\frac{-4}{1}}{4} = 1$$

$$\begin{array}{r|rrrrr}
1 & 1 & -4 & 6 & 1 & -10 \\
 & & 1 & -3 & 3 & 4 \\ \hline
 & 1 & -3 & 3 & 4 & \underline{|-6} \\
 & & 1 & -2 & 1 & \\ \hline
 & 1 & -2 & 1 & \underline{|5} & \\
 & & 1 & -1 & & \\ \hline
 & 1 & -1 & \underline{|0} & & \\
 & & 1 & & & \\ \hline
 & 1 & \underline{|0} & & & \\
 & 1 & & & &
\end{array}$$

$\therefore$ the transformed equation is

$$y^4 + 5y - 6 = 0 \tag{0.15}$$

Let the quadratic factors of the above equation be

$$(y^2 + py + q)(y^2 - py + q') = 0 \tag{0.16}$$

Comparing the coefficients of the left hand side of (0.15) and (0.16), we get

$$q' + q - p^2 = 0$$

$$\text{or} \qquad q' + q = p^2 \tag{0.17}$$

$$p(q' - q) = 5$$

$$\text{or} \qquad q' - q = \frac{5}{p} \tag{0.18}$$

Also, $qq' = -6$
Since, we have

$$(q' + q)^2 - (q' - q)^2 = 4qq'$$

$$(p^2)^2 - \left(\frac{5}{p}\right)^2 = -24$$

$$\implies \quad p^6 + 24p^2 - 25 = 0$$

$$\implies \quad (p^2)^3 + 24(p^2) - 25 = 0$$

Put $p^2 = t$, we get

$$t^3 + 24t^2 - 25 = 0 \tag{0.19}$$

test the cubic for numbers which are perfect squares i,e. $t = 1, 4, 9, 16, 25, \cdots$
$\therefore \quad t = 1$, satisfies equation (0.19).
So that $p^2 = t = 1 \implies \quad p^2 = 1 \quad \implies \quad p = 1$.
Also, from (0.17), we have

$$q' + q = 1 \tag{0.20}$$

$$q' - q = \frac{5}{1}$$

$$q' - q = 5 \tag{0.21}$$

Adding (0.20) and (0.21), we get

$$q' = 3$$

Put $q' = 3$ in (0.21), we have

$$q = -2$$

Thus substitute p, q, q' in (0.16), we get

$$(y^2 + y - 2)(y^2 - y + 3) = 0$$

Which gives $y = 1, 2, \dfrac{1 \pm i\sqrt{11}}{2}$.
Since $x = y + 1$, gives $x = 2, -1, 1, \dfrac{3 \pm i\sqrt{11}}{2}$ are the required roots.

Lecture 5
SOLUTION OF CUBIC EQUATIONS BY USING CARDEN'S METHOD

Let the cubic equation be

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0 \tag{0.1}$$

We first remove the second term of equation (0.1), so setting $h = -\dfrac{a_1}{a_0}$.
Put $y = x - h$, gives $y = x + \dfrac{a_1}{a_0}$, or $x = y - \dfrac{a_1}{a_0}$.
Substituting this value of x in (0.1), we get

$$a_0\left(y - \frac{a_1}{a_0}\right)^3 + 3a_1\left(y - \frac{a_1}{a_0}\right)^2 + 3a_2\left(y - \frac{a_1}{a_0}\right) + a_3 = 0$$

$$\text{or} \qquad a_0^3y^3 + 3a_0y(a_0a_2 - a_1^2) + (2a_1^3 - 3a_0a_1a_2 + a_0^2a_3) = 0$$

Putting $y = \dfrac{z}{a_0}$ in this equation, we get

$$z^3 + 3z(a_0a_2 - a_1^2) + (2a_1^3 - 3a_0a_1a_2 + a_0^2a_3) = 0$$

$$\text{or} \qquad z^3 + 3Hz + G = 0 \tag{0.2}$$

where $H = a_0a_2 - a_1^2$ and $G = 2a_1^3 - 3a_0a_1a_2 + a_0^2a_3$. Now to solve (0.2), we put

$$z = u + v.$$

Cubing it, we get

$$z^3 = (u + v)^3$$

$$\text{or} \qquad z^3 - 3uvz - (u^3 + v^3) = 0. \tag{0.3}$$

On equating coefficients of z and constant terms of (0.2) and (0.3), we have

$$u^3 + v^3 = -G$$

$$\text{and} \qquad uv = -H$$

$$\therefore \qquad u^3v^3 = -H^3.$$

Now a quadratic equation in t whose roots are u^3 and v^3 is

$$t^2 - (u^3 - v^3)t + u^3v^3 = 0$$

$$\text{or} \qquad t^2 - Gt + -H^3 = 0.$$

This gives
$$t = \frac{-G \pm \sqrt{G^2 + 4H^3}}{2},$$

which means that
$$u^3 = \frac{-G + \sqrt{G^2 + 4H^3}}{2}$$

and
$$v^3 = \frac{-G - \sqrt{G^2 + 4H^3}}{2}.$$

$$\therefore \qquad u = \left[\frac{-G + \sqrt{G^2 + 4H^3}}{2}\right]^{\frac{1}{3}}, \left[\frac{-G + \sqrt{G^2 + 4H^3}}{2}\right]^{\frac{1}{3}}\omega, \left[\frac{-G + \sqrt{G^2 + 4H^3}}{2}\right]^{\frac{1}{3}}\omega^2$$

$$\text{and} \qquad v = \left[\frac{-G - \sqrt{G^2 + 4H^3}}{2}\right]^{\frac{1}{3}}, \left[\frac{-G - \sqrt{G^2 + 4H^3}}{2}\right]^{\frac{1}{3}}\omega, \left[\frac{-G - \sqrt{G^2 + 4H^3}}{2}\right]^{\frac{1}{3}}\omega^2,$$

where ω, ω^2 the complex roots of unity.

Putting $\left[\dfrac{-G+\sqrt{G^2+4H^3}}{2}\right]^{\frac{1}{3}} = \alpha$ and $\left[\dfrac{-G-\sqrt{G^2+4H^3}}{2}\right]^{\frac{1}{3}} = \beta$, we obtain three values of z as

$$\alpha+\beta, \omega\alpha+\omega^2\beta, \omega^2\alpha+\omega\beta$$

and then the values of x can be obtained from

$$z = a_0x + a_1 \qquad \text{or} \qquad x = \frac{z-a_1}{a_0}.$$

Example 1. *Solve the cubic equation $x^3 - 6x - 9 = 0$, by Carden's method.*

Sol. The given cubic equation is

$$x^3 - 6x - 9 = 0. \tag{0.4}$$

Here 2^{nd} term is missing. Put $x = u + v \implies x^3 = (u+v)^3 = u^3 + v^3 + 3uv(u+v)$

$$\implies x^3 - 3uvx - (u^3+v^3) = 0 \tag{0.5}$$

Equating coefficients of (0.4) and (0.5), we get
$u^3 + v^3 = 9$ and $3uv = 6 \implies uv = 2 \implies u^3v^3 = 8.$
Now take u^3 and v^3 as roots of quadratic in t, we have

$$t^2 - (u^3+v^3)t + u^3v^3 = 0$$
$$\implies t^2 - 9t + 8 = 0$$
$$\implies (t-1)(t-8) = 0$$
$$\implies t = 1, 8$$
$$\therefore \quad u^3 = 1 \quad \text{and} \quad v^3 = 8$$
$$\implies u = 1 \quad \text{and} \quad v = 2$$
$$\therefore \quad x = 1 + 2 = 3,$$

is the one root. Now using synthetic division we have

3 |	1	0	−6	−9
		3	9	9
	1	3	3	|0

$\therefore \quad x^2 + 3x + 3 = 0$

$\implies x = \dfrac{-3 \pm \sqrt{9-12}}{2}$

$\implies x = \dfrac{-3 \pm \sqrt{-3}}{2}$

$\implies x = \dfrac{-3 \pm i\sqrt{3}}{2}$

Hence the roots are $x = 3, \dfrac{-3 \pm i\sqrt{3}}{2}$

Example 2. *Solve the cubic equation $x^3 + 3x^2 + 12x - 16 = 0$.*

Sol. Here second term is present, so we first remove this second term and hence diminish it by h where

$$h = \frac{\text{Sum of roots}}{\text{Degree of equation}} = \frac{-\frac{3}{1}}{3} = -1$$

$$\begin{array}{r|rrrr} -1 & 1 & 3 & 12 & -16 \\ & & -1 & -2 & -10 \\ \hline & 1 & 2 & 10 & |\underline{-26} \\ & & -1 & -1 & \\ \hline & 1 & 1 & |\underline{9} & \\ & & -1 & & \\ \hline & 1 & |\underline{0} & & \end{array}$$

$\therefore$ the transformed equation is

$$y^3 + 9y - 26 = 0 \tag{0.6}$$

Put $y = u + v$

$$\implies \quad y^3 - 3uvy - (u^3 + v^3) = 0 \tag{0.7}$$

Equating coefficients of (0.6) and (0.7), we get $u^3 + v^3 = 26$ and $3uv = -9 \implies uv = -3 \implies u^3v^3 = -27$.
Now take u^3 and v^3 as roots of quadratic in t, we have

$$t^2 - (u^3 + v^3)t + u^3v^3 = 0$$
$$\implies \quad t^2 - 26t - 27 = 0$$
$$\implies \quad (t+1)(t-27) = 0$$
$$\implies t = -1, 27$$
$$\therefore \quad u^3 = -1 \quad \text{and} \quad v^3 = 27$$
$$\implies \quad u = -1 \quad \text{and} \quad v = 3$$
$$\therefore \quad y = -1 + 3 = 2,$$

is the one root. Now using synthetic division we have

$$\begin{array}{r|rrrr} 2 & 1 & 0 & 9 & -26 \\ & & 2 & 4 & 26 \\ \hline & 1 & 2 & 13 & |\underline{0} \end{array}$$

$\therefore \quad y^2 + 2y + 13 = 0$
$\implies \quad y = \dfrac{-2 \pm \sqrt{4 - 52}}{2}$
$\implies y = \dfrac{-2 \pm \sqrt{-48}}{2}$
$\implies y = -1 \pm 2i\sqrt{3}$
Since $y = x - h \implies x = y + h = y - 1$
$\therefore \quad x = (2-1), -1 \pm 2i\sqrt{3} - 1$
$\implies \quad x = 1, -2 \pm 2i\sqrt{3}.$
Hence the roots are $x = 1, -2 \pm 2i\sqrt{3}$.

Nature Of The Roots Of The Cubic Equation

$$z^3 + 3Hz + G = 0$$

We know that the roots of the above cubic equation are given by $z = x + y$, where u^3 and v^3 are the roots of the quadratic equation $t^2 + Gt + H = 0$. The discriminant of this quadratic equation is $G^2 + 4H^3$. Therefore, we have the following three cases.

(i). $G^2 + 4H^3 > 0$.

In this case both u^3 and v^3 are real, so u and v are also real. Therefore, the three values of z are $u + v, u\omega + v\omega^2, u\omega^2 + v\omega$. The first root is real and the other two roots are complex.

Hence in this case one root is real and other two are complex.

(ii). $G^2 + 4H^3 = 0$.

In this case both u^3 and v^3 are real and equal. Hence, the three roots of the cubic equation are

$$\begin{aligned} u + v &= u + u = 2u \\ u\omega + v\omega^2 &= u(\omega + \omega^2) = -u \\ u\omega^2 + v\omega^2 &= u(\omega^2 + \omega) = -u. \end{aligned}$$

Thus in this case, all the roots are real and rational but two of them are equal.

(iii). $G^2 + 4H^3 < 0$.

In this case u^3 and v^3 are conjugate quantities and as such u and v are also conjugate complex quantities of the $a \pm ib$, where a and b are real.

Hence, the roots of the cubic equation are

$$\begin{aligned} u + v &= (a + ib) + (a - ib) = 2a \\ u\omega + v\omega^2 &= (a + ib)\omega + (a - ib)\omega^2 \\ &= a(\omega + \omega^2) + ib(\omega - \omega^2) \\ &= -a - b\sqrt{3} \qquad \text{using } 1 + \omega + \omega^2 = 0 \\ \text{and} \qquad u\omega^2 + v\omega^2 &= (a + ib)\omega^2 + (a - ib)\omega \\ &= a(\omega^2 + \omega) + ib(\omega^2 - \omega) \\ &= -a + b\sqrt{3} \qquad \text{using } 1 + \omega + \omega^2 = 0. \end{aligned}$$

Thus, in this case all the three roots are real and distinct but unequal.

Example 3. *Discuss nature of roots of the equation* $x^3 + 15x - 124 = 0$.

Sol. Comparing it with

$$z^3 + 3Hz + G = 0,$$

we get

$$H = 5, \quad G = -124 \qquad \therefore \quad G^2 + 4H^3 = (-124)^2 + 4(125) > 0.$$

By case (i), the given equation has one real and two complex roots.

Example 4. *Discuss nature of roots of the equation* $x^3 + 3x - 14 = 0$.

Sol. Comparing it with

$$z^3 + 3Hz + G = 0,$$

we get

$$H = 1, \quad G = -14 \qquad \therefore \quad G^2 + 4H^3 = (-14)^2 + 4(1) > 0.$$

Again by case (i), the given equation has one real and two complex roots.

Transcendental Equation

Equations which are not purely algebraic are called transcendental equations, that is, if $f(x)$ contains some other functions such as trigonometric, logarithmic, exponential etc. then $f(x) = 0$ is called a transcendental equation.

Examples (i) $2x + \log_{10}x - 5 = 0$

(ii) $x + \sin x + 2 = 0$

(iii) $e^x + \tan x + 3x = 10$

BISECTION (OR BOLZANO) METHOD

This method is based on the repeated application of the intermediate-value property. It is a lengthy and laborious method.

Let the given equation whose solution is to be found be

$$f(x) = 0 \tag{1}$$

where the function $f(x)$ is continuous on (a, b) and $f(a) f(b) < 0$. In order to find a root of (1) lying in the interval (a, b). We shall determine a very small interval (a_0, b_0) (by graphical method) in which $f(a_0) f(b_0) < 0$ and $f'(x)$ maintain the same sign in (a_0, b_0) so that there is only one real root of the equation $f(x) = 0$. Divide the interval in half and let

$$x_1 = \frac{a_0 + b_0}{2}$$

If $f(x) = 0$ then x_1 is a root of the equation. If $f(x_1) \neq 0$ then either $f(a_0) f(x_1) < 0$ of $f(b_0) f(x_1) <0$. If $f(a_0) f(x_1) < 0$ then the root of the equation lies in (a_0, x_1) otherwise the root of the equation lies in (x_1, b_1). We rename the interval in which the root lies as (a_1, b_1) so that

$$b_1 - a_1 = \frac{1}{2} (b_0 - a_0)$$

now we take $x_2 = \dfrac{a_1 + b_1}{2}$

If $f(x_2) = 0$ then x_2 is the root of $f(x) = 0$. If $f(x_2) \neq 0$ and $f(x_2) f(a_1) < 0$, then the root lies in (a_1, x_2). In which case we rename the interval as (a_2, b_2), otherwise (x_2, b_1) it is renamed as (a_2, b_2) where

$$a_2 - b_2 = \frac{1}{2^2} (b_0 - a_0)$$

$\Rightarrow$ the root lies between x_1 and x_2

$\therefore$ Third approximation is:

$$x_3 = \frac{1}{2}(x_1 + x_2) = 2.625$$

Then $$f(x_3) = (2.625)^3 - 4\,(2.625) - 9$$
$$= -\,1.4121 < 0 \text{ (negative)}$$

$\Rightarrow$ the root lies between x_2 and x_3

$\therefore$ Fourth approximation is

$$x_4 = \frac{1}{2}(x_2 + x_3) = 2.6875$$

Hence, the required approximate root is 2.6875.

Example 25. Using bisection method find a real root of the equation:

$$x^3 - 5x + 1 = 0.$$

Solution:

Let $$f(x) = x^3 - 5x + 1$$

Since $$f(0) = 1 > 0 \text{ (positive)}$$

and $$f(1) = -\,3 < 0 \text{ (negative)}$$

$\therefore$ $$f(0)\,f(1) < 0$$

$\Rightarrow$ a root lies between 0 and 1.

$$a_0 = 0, \quad b_0 = 1$$

$\therefore$ The first approximation of this root is

$$x_1 = \frac{1}{2}(a_0 + b_0) = \frac{1}{2}(0 + 1) = 0.5$$

Thus $f(x_1) = -\,1.375 < 0.$ (negative)

$\therefore$ a root lies between a_0 and x_1

$\therefore$ The second approximation of the root is

$$x_2 = \frac{1}{2}(a_0 + x_1) = \frac{1}{2}(0 + 0.5) = 0.25$$

Thus $f(x_2) = -\,0.234375 < 0$ (negative)

$\therefore$ The root lies between $a_0 = 0$ and $x_2 = 0.25$

$\therefore$ The third approximation of the root is

$$x_3 = \frac{1}{2}(a_0 + x_2) = \frac{1}{2}(0 + 0.25) = 0.125.$$

thus $f(x_3) = 0.37695 > 0$.

Since $f(x_3) > 0$ and $f(x_2) < 0$

$\therefore$ The root lies between $x_3 = 0.125$ and $x_2 = 0.25$

$\therefore$ The fourth approximation of the root is

$$x_4 = \frac{1}{2}(x_3 + x_2) = \frac{1}{2}(0.125 + 0.25) = 0.1875$$

Thus $f(x_4) = 0.06910 > 0$. Since $f(x_4) > 0$ and $f(x_2) < 0$, the root lies between $x_4 = 0.1875$ and $x_2 = 0.25$.

Hence the fifth approximation of the root is $x_5 = \frac{1}{2}(x_4 + x_2) = \frac{1}{2}(0.1875 + 0.25) = 0.21875$.

$\therefore$ 0.21875 is taken as an approximate value of the root of the given equation that lies between 0 and 1.

Example 26. Using bisection method compute one root of $e^x - 3x = 0$ correct to two decimal places, in the interval [1.5, 1.6]

Solution:

Let $f(x) = e^x - 3x$

Since $f(1.5) = -0.02 < 0$,

and $f(1.6) = 0.15 > 0$,

here $a_0 = 1.5,\ b_0 = 1.6$

$\therefore$ first approximation of the root is

$$x_1 = \frac{1}{2}(a_0 + b_0) = \frac{1}{2}(1.5 + 1.6) = 1.55$$

Thus $f(x_1) = f(1.55) = 0.06 > 0$

$\therefore$ a root lies between a_0 and x_1

$\therefore$ second approximation of the root is

$$x_2 = \frac{1}{2}(a_0 + x_1) = \frac{1}{2}(1.5 + 1.55) = 1.525$$

Thus, $f(x_2) = 0.02 > 0$

$\therefore$ The root lies between $a_0 = 1.5$ and $x_2 = 1.525$

$\therefore$ Third approximation of the root is

$$x_3 = \frac{1}{2}(a_0 + x_2) = \frac{1}{2}(1.5 + 1.525) = 1.5125$$

Thus $f(x_3) = 0.00056 > 0$

$\therefore$ a root lies between a_0 and x_3.

$\therefore$ Fourth approximation of the root is

$$x_4 = \frac{1}{2}(a_0 + x_3) = \frac{1}{2}(1.5 + 1.5125) = 1.5062$$

Thus $f(x_4) = -\ 0.0090 < 0$

$\therefore$ The root lies between $x_3 = 1.51125$ and

$x_4 = 1.5062$

$\therefore$ The fifth approximation of the root is

$$x_5 = \frac{1}{2}(x_3 + x_4) = \frac{1}{2}(1.5125 + 1.5062)$$

$$= 1.50935$$

Thus $f(x_5) = f(1.50935) = -\ 0.00426 < 0$

Check: $\therefore$	$x_4 = 1.5062,\ x_5 = 1.50935$
	$x_6 = 1.51092,\ f(x_6) = -\ 0.00184 < 0.$

In the 4th step a_n, b_n and x^{n+1} are equal upto ten decimal places. Thus, $x = 1.51$ is the root of $f(x) = 0$, correct to two decimal places.

Example 27. By using the bisection method, find an approximate root of the equation $\sin x = \dfrac{1}{x}$ that lies between $x = 1$ and $x = 1.5$ (measured in radians). Carry out computation upto 7th stage.

Solution: $f(x) = x \sin x - 1 = 0$

Since $f(1) = -\ 0.158529 < 0$

and $f(1.5) = 0.496242 > 0$

$\therefore$ There is a root x_1 between $a_0 = 1$ and $b_0 = 1.5$

$\therefore$ The first approximation is

$$x_1 = \frac{1}{2}(a_0 + b_0) = \frac{1}{2}(1 + 1.25) = 1.25$$

Thus, $f(x_1) = 0.186231 > 0.$

Since $f(1) < 0$ and $f(x_1) > 0$

Which is the required approximate root of the desired order of the root of the given equation that lies between 1 and 1.5.

Example 28. Find the root of tan $x + x = 0$ upto two decimal places, which lies between 2 and 2.1.

Solution:

Let $f(x) = \tan x + x$

Since $f(2) = -0.18 < 0$

and $f(2.1) = 0.39 > 0$

Thus, the root lies between 2.0 and 2.1

$\therefore$ $a_0 = 2,\ b_0 = 2.1$

$\therefore$ First approximation of the root is

$$x_1 = \frac{1}{2}(a_0 + b_0) = \frac{1}{2}(2 + 2.1) = 2.05$$

Thus, $f(x_1) = -0.12 > 0$

Similarly, we get the following successive approximation after computation,

$$\begin{aligned} x_2 &= 2.025, & f(x_2) &< 0 \\ x_3 &= 2.0375, & f(x_3) &> 0 \\ x_4 &= 2.03125, & f(x_4) &< 0 \\ x_5 &= 2.02812, & f(x_5) &< 0 \\ x_6 &= 2.02968, & f(x_6) &> 0 \\ x_7 &= 2.02890, & f(x_7) &> 0 \end{aligned}$$

In the fifth step, a_n, b_n and x_{n+1} are equal upto two decimal places.
$\therefore$ $x = 2.03$ is a root of $f(x) = 0$, correct to two decimal places.

Example 29. Find an approximate value of $\sqrt{3}$ correct to two decimal places, using bisection method.

Solution:

Let $x = \sqrt{3}$

$\Rightarrow$ $x_2 = 3$

$\Rightarrow$ $x_2 - 3 = 0$

Let $f(x) = x^2 - 3$

Since $f(1) = -2 < 0$

and $f(2) = 1 > 0$

$\therefore$ The root lies between $a_0 = 1$, $b_0 = 2$

$\therefore$ The first approximation of the root is

$$x_1 = \frac{1}{2}(a_0 + b_0) = \frac{1}{2}(1 + 2) = 1.5$$

Thus $f(x_1) = -0.75 < 0$, since $f(2) > 0$

$\therefore$ The root lies between $x_1 = 1.5$ and $b_0 = 2$

$$\therefore \qquad x_2 = \frac{1}{2}(x_1 + b_0) = \frac{1}{2}(1.5 + 2) = 1.75$$

Similarly we get the following approximation

$$\left.\begin{aligned} x_3 &= 1.625 \\ x_4 &= 1.6875 \\ x_5 &= 1.71875 \\ x_6 &= 1.734375 \end{aligned}\right\}$$ Since $\sqrt{3} = 1.7320508$, we take that x_6 is an approx value of $\sqrt{3}$ correct to two decimal places.

EXERCISES

1. Find a root of the following equations, using bisection method correct to three decimal places.
   (i) $x^3 - 4x - 9 = 0$ (Ans: 2.707)
   (ii) $x^3 - x - 11 = 0$ in (2, 3) which lies between 2 and 3 (Ans: 2.375)
   (iii) $x^4 - x - 10 = 0$ (Ans: 1.8125).
   (iv) $x^3 - 3x - 5 = 0$ (Ans: 2.28)
   (v) $x^3 - x^2 + x - 7 = 0$ in (2, 2.25) (Ans: 2.1049)
   (vi) $e^x = x + 2$ in (1, 1.4) (Ans: 1.146)
   (vii) $\tan x + \tan hx = 0$ in (1, 2) (Ans: 1.5707)
   (ix) $x + \log x = 5$ in (3.2, 4) (Ans: 3.692)
   (x) $\cos x - 1.3x = 0$ in (0, 1) (Ans: 0.625)
   (xi) $x^4 + 2x^2 - 16x + 5 = 0$ in (0, 0.5) (Ans: 0.3261)
   (xii) $x^3 - 5x - 4 = 0$ in (2, 3) (Ans: 2.5625)
   (xiii) $x^3 - x - 4 = 0$ in (1.5, 2) (Ans: 1.7964)
2. Find the roots correct to one decimal place using bisection method.
   (a) $x^2 - 4 \sin x = 0$ near 2 (Ans: 1.9)
   (b) $x - \frac{\pi}{2} - \sin x = 0$ near 2 (Ans: 2.3)
   (c) $x - 1 - \tan hx = 0$ near 2 (Ans: 2.0)

METHOD OF FALSE POSITION OR REGULA-FALSI METHOD

This method is a great improvement of the bisection method and is the oldest method of finding the real root of an equation $f(x) = 0$. For the equation $f(x) = 0$, having found x_0 and x_1 such that $y_0 = f(x_0)$ and $y_1 = f(x_1)$ are of opposite sign, that is, the graph of $y = f(x)$ crosses the x-axis between these points (Fig. 8). This indicates that a root lies between x_0 and x_1 and consequently $f(x_0)\, f(x_1) < 0$.

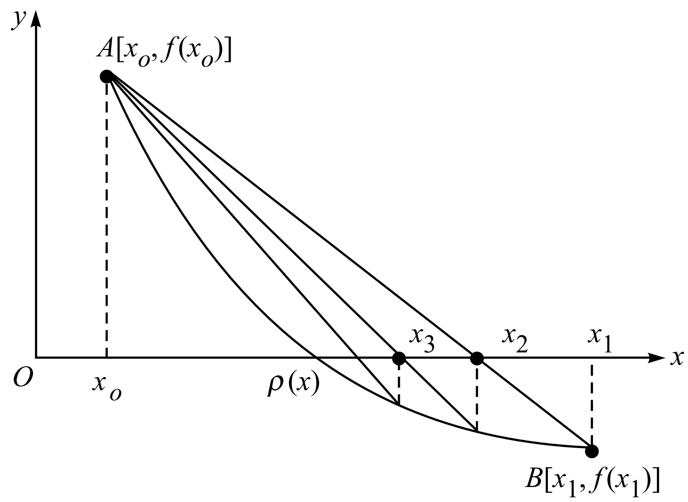


Fig. 8

Equation of the chord joining the points $A\ (x_0, f(x_0)]$ and $B[x_1, f(x_1)]$ is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

The method consists in replacing the curve AB by means of the chord AB and taking the point of intersection of the chord with the x-axis as an approximation to the root. So, the abscissa of the point where the chord cuts the x-axis ($y = 0$) is given by

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x)_0} f(x_0) \tag{1}$$

which is an approximation to the root. If we denote the first approximation by m_1 then, we may write (1) as

$$m_1 = x_2 = x_0 - \left[\frac{x_1 - x_0}{f_1 - f_0}\right] f_0 \tag{2a}$$

when $f_1 = f(x_1),\ f_0 = f(x_0)$

We can also write [2a] as:-

$$m_1 = x_2 = \left[\frac{x_0 f_1 - x_1 f_0}{f_1 - f_0}\right]$$ gives the first approximation (2b)

Find $f_2 = f(x_2)$, then either f_0 and f_2 are of opposite signs or f_2 and f_1 are of opposite signs. Suppose f_0 and f_1 are of opposite signs. Then employ the formula (2) with x_0 and x_2 to find the value x_3 of x. Thus, the second approximation of the root is and $$f_1 = f(x_1) = 4 > 0$$

$\therefore$ The root lies between $x_2 = 1.849$ and $x_1 = 2$

$\therefore$ The second approximation of the root is

$$m_2 = x_3 = x_2 - \left(\frac{x_1 - x_2}{f_1 - f_2}\right) f_2$$

$$= 1.8490 - \frac{2 - 1.849}{4 + 0.159} \times (-\ 0.159)$$

$$= 1.8548$$

Thus, $$f_3 = f(x_3) = f(1.8548) = -\ 0.019 < 0$$
and $$f_2 = f(x_2) = -\ 0.159 < 0$$

Hence, f_3 and f_2 are of the same sign. Hence, the root m does not lie between x_2 and x_3. But f_3 and f_1 are of opposite signs. Hence m lies between x_3 and x_1.

$\therefore$ The third approximation of the root is

$$m_3 = x_4 = x_3 - \left(\frac{x_1 - x_3}{f_1 - f_3}\right) f_3$$

$$= 1.8548 - \frac{2 - 1.8548}{4 + 0.019} \times (-\ 0.019)$$

$$= 1.8555$$

Example 31. Find the root of the equations $xe^x = \cos x$ using the regula-falsi method correct to four decimal places.

Solution:

Let us take $$f(x) = \cos x - xe^x$$
Since $$f(0) = 1,\ f(1) = \cos 1 - e = -\ 2.17798$$
Taking $$x_0 = 0,\ x_1 = 1$$
We find that $$f(0) = 1 > 0,\ f(1) = -\ 2.17798 < 0, \text{ or } f_0 > 0,\ f_1 < 0$$

Since f_0 and f_1 are of opposite signs, the first-order approximation of the root is

$$m_1 = x_2 = x_0 - \left(\frac{x_1 - x_0}{f_1 - f_0}\right) f_0$$

$$= 0 + \frac{1}{3.17798} \times 1$$

$$= 0.31647$$

We find that $f_2 = f(x_2) = f(0.31467) = 0.51987 > 0$ so that f_2 and f_1 are of opposite signs. Hence, the root m lies between x_2 and x_1 and the second order approximation of the root is

$$m_2 = x_3 = x_2 - \left(\frac{x_1 - x_2}{f_1 - f_2}\right) f_2$$

$$= 0.31467 + \frac{0.68533}{2.69785} \times 0.51987$$

$$= 0.44673$$

We find $f_3 = f(x_3) = f(0.44673) = 0.20356 > 0$
So that f_3 and f_2 are of the same sign.

Hence, the root m does not lie between x_2 and x_3. But f_3 and f_1 are of opposite signs. Hence, m lies between x_3 and x_1 and the third-order approximate value of the root is

$$m_3 = x_4 = x_3 - \left(\frac{x_1 - x_3}{f_1 - f_3}\right) f_3$$

$$= 0.44673 + \frac{0.55327}{2.38154} \times 0.20356$$

$$= 0.49402$$

Repeating the process, the successive approximations are:

$$m_4 = x_5 = 0.50995,\ m_7 = x_8 = 0.51748,$$
$$m_5 = x_6 = 0.51520,\ m_8 = x_9 = 0.51767,$$
$$m_6 = x_7 = 0.51692,\ m_9 = x_{10} = 0.51775 \text{ etc.}$$

Hence the root is 0.5177 correct to 4 decimal places.

Example 32. Find a real root of the equation $x^6 - x^4 - x^3 - 3 = 0$ by regula-falsi method in (1.5, 1.6) correct to three places of decimal.

Solution:

Let us take $f(x) = x^6 - x^4 - x^3 - 3$, and $x_0 = 1.5$, $x_1 = 1.6$

Since $f_0 = f(x_0) = f(1.5) = -0.047 < 0$

and $f_1 = f(x_1) = f(1.6) = 3.128 > 0.$

$\therefore$ f_0 and f_1 are of opposite signs, the equation $f(x) = 0$ has a root m between x_0 and x_1. The first-order approximation of this root is

$$m_1 = x_2 = x_0 - \left(\frac{x_1 - x_0}{f_1 - f_0}\right) f_0$$

$$= \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0}$$

We find that $f_2 = f(x_2) = f(0.8506) = -\ 0.00697$

$$m_2 = x_3 = x_2 - \left(\frac{x_1 - x_2}{f_1 - f_2}\right) f_2$$

$$= \frac{x_2 f_1 - x_1 f_2}{f_1 - f_2}$$

$$= \frac{(0.8506)(0.2136) - 0.9(-0.2196)}{0.2136 - (-0.2196)}$$

$$= \frac{0.36812}{0.4332} = 0.8506$$

We find that $f_3 = f(x_3) = f(0.8506) = -\ 0.00697 < 0$

$f_1 = f(x_1) = f(0.9) = 0.2136 > 0$

$$m_3 = x_4 = x_3 - \left(\frac{x_1 - x_3}{f_1 - f_3}\right) f_3$$

$$= \frac{x_3 f_1 - x_1 f_3}{f_1 - f_2} = \frac{0.8506 \times 0.2136 - 0.9(-0.00697)}{0.2136 - (-0.00697)}$$

$$= \frac{0.187972}{0.22057} = 0.8523.$$

We find that $f_4 = f(x_4) = f(0.8523) = -\ 0.00166 < 0$

$f_1 = f(x_1) = f(0.9) = 0.2136 > 0$

$$m_4 = x_5 = x_4 - \left(\frac{x_1 - x_4}{f_1 - f_4}\right) f_4$$

$$= \frac{x_4 f_1 - x_1 f_4}{f_1 - f_4}$$

$$= \frac{0.8523 \times (0.2136) - (0.9)(-0.00166)}{0.2136 - (-0.00166)}$$

$$= \frac{0.183545}{0.21526} = 0.8526.$$

$f(0.8526) = -\ 0.000023.$

Hence, the required root of the given equation is 0.8526, correct to four significant figures.

Example 34. By using the regula-falsi method, find the root, correct to 3 decimal places of the equation $x \log_{10}^{x} = 1.2$ that lies between 2 and 3.

Solution:

Given: $f(x) = 0$, where

$$f(x)\ x\log_{10}^{x} - 1.2$$

and $x_0 = 2$, $x_1 = 3$. Then $f_0 = f(x_0) = f(2) = -\ 0.59794 < 0$ and $f_1 = f(x_1)\ f(3) = 0.23136 > 0$.

Since f_0 and f_1 are of opposite signs, the equation $f(x) = 0$ has a root m between x_0 and x_1. The first approximation of this root is

$$m_1 = x_2 = x_0 - \left(\frac{x_1 - x_0}{f_1 - f_0}\right) f_0$$

$$= 2 - \frac{3 - 2}{0.23136 + 0.59794} \times (-0.59794)$$

$$= 2.721$$

Thus $f_2 = f(x_2) = f(2.721) = (2.721) = (2.721)\ \log_{10}\ (2.721) - 1.2 = -\ 0.0171 < 0$

$\therefore$ f_1 and f_2 are of opposite signs.

$\Rightarrow$ there is a root m between x_2 and x_1.

Hence, the second approximation of the root is

$$m_2 = x_3 = x_2 - \left(\frac{x_1 - x_2}{f_1 - f_2}\right) f_2$$

$$= 2.721 - \frac{3 - 2.721}{0.23136 + 0.0171} \times (-\ 0.0171)$$

$$= 2.74020$$

$\therefore$ $$f_3 = f(x_3) = 2.74020 \times \log_{10} 2.74020 - 1.2$$

$$= -\ 0.00039 < 0$$

and $$f_2 = -\ 0.0171 < 0$$

$\Rightarrow$ f_2 and f_3 are of the same sign.

$\Rightarrow$ the root 'm' does not lie between x_2 and x_3. But f_3 and f_1 are of opposite signs; hence 'm' lies between x_3 and x_1.

$\therefore$ the third approximation of the root is $m_3 = x_4 = x_3 - \left(\dfrac{x_1 - x_3}{f_1 - f_3}\right) f_3$

$$= 2.74020 - \frac{3 - 2.74020}{0.23136 + 0.00039} \times (-\ 0.00039)$$

$$= 2.74064$$

We find that $m_2 = 2.74020$ and $m_3 = 2.74064$ coincide upto 3 decimal places.

Hence $m_2 \approx m_6 \approx 2.740$ is the required root correct to three decimal places.

Example 35. Solve the equation $x^3 - 9x + 1 = 0$ by regula-falsi method.

Solution:

Let $$f(x) = x^3 - 9x + 1$$

Let us take $x_0 = 2$ and $x_1 = 3$

$\therefore$ $$f_0 = f(x_0) = f(2) = -9 < 0;$$
and $$f_1 = f(x_1) = f(3) = 1 > 0;$$

Hence, the root lies between 2 and 3.

$$m_1 = x_2 = x_0 - \left(\frac{x_1 - x_0}{f_1 - f_0}\right) f_0 \tag{1}$$

$$= 2 - \left(\frac{1}{1-(-9)}\right)(-9)$$

$$= 2 + \frac{9}{10} = 2.9$$

$\therefore$ $$f_2 = f(x_2) = f(2.9) = -0.711 < 0$$
Also,

$$f(x_2),\ f(x_1) = (-0.711 \times 1) = -0.711 < 0$$

Choosing x_2 and x_1,

$$m_2 = x_3 = x_2 - \left(\frac{x_1 - x_2}{f_1 - f_2}\right) f_2 = \frac{x_2 f_1 - x_1 f_2}{f_1 - f_2}$$

$$= \left[\frac{(2.9)(1) - 3(-0.711)}{1-(-0.711)}\right] = 2.9415$$

The process is continued unitil the value converges to 2.9428.

Example 36. Using regula-falsi method find a root of $f(x) = x^6 - x^4 - x^3 - 1 = 0$ in (1, 2). Carry out 3 iterations.

Solution:

Let $$f(x) = x^6 - x^4 - x^3 - 1; \quad \begin{cases} f(1) = -2 < 0 \\ f(2) = 39 > 0 \end{cases}$$

$\therefore$ take $x_0 = 1.5$ (between the approximates)
and $x_1 = 1$

First approximation: (m_1)

$$m_1 = x_2 = x_0 - \left(\frac{x_1 - x_0}{f_1 - f_0}\right) f_0$$