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Faculty of science-Qena

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Analytical Mechanics: Lecture Notes

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Introduction

In theoretical physics and mathematical physics, analytical mechanics, or theoretical mechanics is a collection of closely related alternative formulations of classical mechanics. It was developed by many scientists and mathematicians during the 18th century and onward, after Newtonian mechanics. Since Newtonian mechanics considers vector quantities of motion, particularly accelerations, momenta, forces, of the constituents of the system, an alternative name for the mechanics governed by Newton's laws and Euler's laws is vectorial mechanics.

By contrast, analytical mechanics uses scalar properties of motion representing the system as a whole—usually its total kinetic energy and potential energy—not Newton's vectorial forces of individual particles. A scalar is a quantity, whereas a vector is represented by quantity and direction. The equations of motion are derived from the scalar quantity by some underlying principle about the scalar's variation.

Analytical mechanics takes advantage of a system's constraints to solve problems. The constraints limit the degrees of freedom the system can have, and can be used to reduce the number of coordinates needed to solve for the motion. The formalism is well suited to arbitrary choices of coordinates, known in the context as generalized coordinates. The kinetic and potential energies of the system are expressed using these generalized coordinates or momenta, and the equations of motion can be readily set up, thus analytical mechanics allows numerous mechanical problems to be solved with greater efficiency than fully vectorial methods. It does not always work for non-conservative forces or dissipative forces like friction, in which case one may revert to Newtonian mechanics.

Two dominant branches of analytical mechanics are Lagrangian mechanics (using generalized coordinates and corresponding generalized velocities in configuration space) and Hamiltonian mechanics (using coordinates and corresponding momenta in phase space). Both formulations are equivalent by a Legendre transformation on the generalized coordinates, velocities and momenta, therefore both contain the same information for describing the dynamics of a system. There are other formulations such as Hamilton–Jacobi theory, Routhian mechanics, and Appell's equation of motion. All equations of motion for particles and fields, in any formalism, can be derived from the widely applicable result called the principle of least action. One result is Noether's theorem, a statement which connects conservation laws to their associated symmetries.

Analytical mechanics does not introduce new physics and is not more general than Newtonian mechanics. Rather it is a collection of equivalent formalisms which have broad application. In fact the same principles and formalisms can be used in relativistic mechanics and general relativity, and with some modifications, quantum mechanics and quantum field theory. Analytical mechanics is used widely, from fundamental physics to applied mathematics, particularly chaos theory.

The methods of analytical mechanics apply to discrete particles, each with a finite number of degrees of freedom. They can be modified to describe continuous fields or fluids, which have infinite degrees of freedom. The definitions and equations have a close analogy with those of mechanics.

Dynamical system

It is a system of particles moving under the influence of a set of forces and these particles may be separate from each other or connected.

Generalized Coordinates

In two-dimensions the positions of a point can be specified either by its rectangular coordinates (x, y) or by its polar coordinates. There are other possibilities such as confocal conical coordinates that might be less familiar. In three dimensions there are the options of rectangular coordinates (x, y, z), or cylindrical coordinates (ρ, ϕ, z) or spherical coordinates (r, θ, ϕ) or again there may be others that may be of use for specialized purposes (inclined coordinates in crystallography, for example, come to mind). The state of a molecule might be described by a number of parameters, such as the bond lengths and the angles between the bonds, and these may be varying periodically with time as the molecule vibrates and twists, and these bonds lengths and bond angles constitute a set of coordinates which describe the molecule. We are not going to think about any particular sort of coordinate system or set of coordinates. Rather, we are going to think about generalized coordinates, which may be lengths or angles or various combinations of them. We shall call these coordinates ($q_1, q_2, q_3, \ldots\ldots\ldots\ldots, q_n$). If we are thinking of a single particle in three-dimensional space, there will be three of them, which could be rectangular, or cylindrical, or spherical. If there were N particles, we would need $3N$ coordinates to describe the system – unless there were some constraints on the system.

With each generalized coordinate q_α is associated a generalized force $\vec{F}_i$.

A generalized force need not always be dimensionally equivalent to a force. For example, if a generalized coordinate is an angle, the corresponding generalized force will be a torque.

A set of parameters that describes the configuration of a system with respect to some reference configuration and written symbolically as $q_1, q_2, q_3, ..., q_n$ $(q_\alpha, \alpha = 1, 2, 3.........., n)$.

Generalized Velocities

The first derivatives with respect to time of the generalized coordinates of a particle is called generalized velocities and are written as $\dot{q}_1, \dot{q}_2, \dot{q}_3, \dot{q}_4, ..., (\dot{q}_\alpha = \frac{\partial q_\alpha}{\partial t})$.

It is noted that it is not required that the units of the general velocity be (length / time).

$\alpha = 1, 2, 3, 4, ..., n$. n the number of coordinates.

Generalized Accelerations

The second derivatives with respect to time of the generalized coordinates of a particle is called generalized Accelerations and are written as $\ddot{q}_1, \ddot{q}_2, \ddot{q}_3, \ddot{q}_4, ..., (\ddot{q}_\alpha = \frac{\partial^2 q_\alpha}{\partial t^2})$.

Constraints

Motion of particle not always remains free but often is subjected to given conditions. These conditions are called constraints.

Types of the Constraints

Holonomic Constrains: Expressible in terms of equation involving coordinates and time (may or may not present), $f(q_1, q_2, q_3, ..., q_n, t) = 0$, where q_i are the instantaneous coordinates

- Differential (kinematical) Constrains

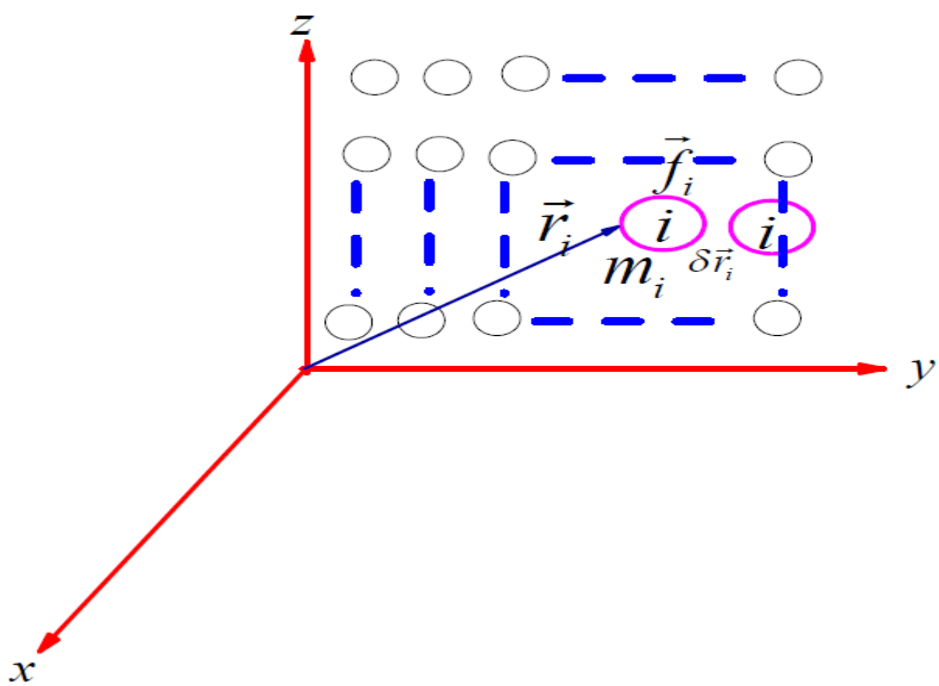
- Geometric Constrains

- Holonomic Constraint is a typical constraint condition that is involved if the position vector of the particle is considered along the time-taken.

But for non-holonomic it means that the position vector of the particle is not considered along the time taken.

<u>Generalized forces</u>

Suppose that we have a mechanical system that contains a number N of particles, and select one of these particles, let it be a number i, which has a mass m_i, its position is $\vec{r}_i$ and it is affected by the force $\vec{F}_i$, then it moves a displacement $\delta \vec{r}_i$.



So, we can write the work in the form

$$\delta W = \sum_{i=1}^{N} \vec{F}_i \cdot \delta \vec{r}_i \tag{1}$$

But

$$\delta \vec{r}_i = \sum_{\alpha=1}^{n} \frac{\partial \vec{r}_i}{\partial q_\alpha} \delta q_\alpha \tag{2}$$

Substituting from Eq. (2) into Eq. (1), we get

$$\delta W = \sum_{i=1}^{N} \vec{F}_i \cdot \sum_{\alpha=1}^{n} \frac{\partial \vec{r}_i}{\partial q_\alpha} \delta q_\alpha = \sum_{\alpha=1}^{n} \left\{ \left[\sum_{i=1}^{N} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha} \right] \delta q_\alpha \right\}$$

We will know the expression $\sum_{i=1}^{N} \vec{f}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha}$ by generalized Forces and we will symbolize (denote) it with a symbol (we will denote it by a symbol) Q_α, that is $Q_\alpha = \sum_{i=1}^{N} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha}$.

So the work can be written in the form $\delta W = \sum_{\alpha=1}^{n} Q_\alpha \delta q_\alpha$.

The Momentum

The momentum defends as $p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha}$. It depends on the generalized coordinates of the system and the generalized velocities, and sometime maybe it depends explicitly on time t, i. e. $p_\alpha = p_\alpha(q_\alpha, \dot{q}_\alpha, t)$

Total Kinetic Energy

The total kinetic energy T of a system of N particles is given by

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i (\dot{r}_i)^2 = \frac{1}{2} \frac{d}{dt} \sum_{i=1}^{N} m_i (\dot{r}_i)^2$$

Degrees of freedom

Number of independent coordinates required to completely specify the dynamics of particles (system of particles) is known as degree's of freedom.

Important relations

(1) Proof that $\frac{\partial \vec{r}_i}{\partial q_\alpha} = \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_\alpha}$?

Proof

We know that $\vec{r}_i = \vec{r}_i(q_\alpha, t)$ and derive with respect to time we have

$$\frac{d\vec{r}_i}{dt} = \sum_{\alpha=1}^{n} \left\{ \frac{\partial \vec{r}_i}{\partial q_\alpha} \frac{dq_\alpha}{dt} + \frac{\partial \vec{r}_i}{\partial t} \frac{dt}{dt} \right\}. \text{ Then } \qquad \frac{d\vec{r}_i}{dt} = \dot{\vec{r}}_i = \frac{\partial \vec{r}_i}{\partial q_\alpha} \dot{q}_\alpha + \frac{\partial \vec{r}_i}{\partial t} \qquad (1)$$

Deriving Eq. (1) with respect to $\dot{q}_\alpha$ (where $\vec{r}_i \neq \vec{r}_i(\dot{q}_\alpha)$), we have

$$\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_\alpha} = \frac{\partial}{\partial \dot{q}_\alpha}\left\{\frac{\partial \vec{r}_i}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial \vec{r}_i}{\partial t}\right\} = \frac{\partial}{\partial \dot{q}_\alpha}\left\{\frac{\partial \vec{r}_i}{\partial q_\alpha}\dot{q}_\alpha\right\} + \frac{\partial}{\partial \dot{q}_\alpha}\left\{\frac{\partial \vec{r}_i}{\partial t}\right\} =$$

$$\underbrace{\frac{\partial}{\partial \dot{q}_\alpha}\left\{\frac{\partial \vec{r}_i}{\partial q_\alpha}\right\}}_{=0}\dot{q}_\alpha + \frac{\partial \vec{r}_i}{\partial q_\alpha}\underbrace{\frac{\partial \dot{q}_\alpha}{\partial \dot{q}_\alpha}}_{=1} + \underbrace{\frac{\partial}{\partial \dot{q}_\alpha}\left\{\frac{\partial \vec{r}_i}{\partial t}\right\}}_{=0} = \frac{\partial \vec{r}_i}{\partial q_\alpha}$$

Then $\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_\alpha} = \frac{\partial \vec{r}_i}{\partial q_\alpha}$

(2) Proof that $\ddot{\vec{r}} \cdot \frac{\partial \vec{r}}{\partial q_\alpha} = \frac{d}{dt}\left(\dot{\vec{r}} \cdot \frac{\partial \dot{\vec{r}}}{\partial \dot{q}_\alpha}\right) - \dot{\vec{r}} \cdot \left(\frac{\partial \dot{\vec{r}}}{\partial q_\alpha}\right)$?

Proof

We known that

$$\vec{r}_i = \vec{r}_i(q_\alpha, t) = \vec{r}_i(q_1, q_2, q_3, q_4, q_5, \ldots\ldots\ldots\ldots q_n, t) \tag{1}$$

If we derive Eq. (1) with respect to time, we get

$$\frac{d\vec{r}_i}{dt} = \frac{\partial \vec{r}_i}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial \vec{r}_i}{\partial q_2}\frac{dq_2}{dt} + \frac{\partial \vec{r}_i}{\partial q_3}\frac{dq_3}{dt} + \ldots\ldots\ldots\ldots\ldots + \frac{\partial \vec{r}_i}{\partial q_n}\frac{dq_n}{dt} + \frac{\partial \vec{r}_i}{\partial t}\frac{dt}{dt}$$

$$= \frac{\partial \vec{r}_i}{\partial q_1}\dot{q}_1 + \frac{\partial \vec{r}_i}{\partial q_2}\dot{q}_2 + \frac{\partial \vec{r}_i}{\partial q_3}\dot{q}_3 + \ldots\ldots\ldots\ldots\ldots + \frac{\partial \vec{r}_i}{\partial q_n}\dot{q}_n + \frac{\partial \vec{r}_i}{\partial t}$$

Again, we derive the above relation with respect to q_α.

$$\frac{\partial}{\partial q_\alpha}\left(\frac{d\vec{r}_i}{dt}\right) = \frac{\partial}{\partial q_\alpha}\left\{\frac{\partial \vec{r}_i}{\partial q_1}\dot{q}_1 + \frac{\partial \vec{r}_i}{\partial q_2}\dot{q}_2 + \frac{\partial \vec{r}_i}{\partial q_3}\dot{q}_3 + \ldots\ldots\ldots\ldots\ldots + \frac{\partial \vec{r}_i}{\partial q_n}\dot{q}_n + \frac{\partial \vec{r}_i}{\partial t}\right\}$$

$$= \left\{\frac{\partial^2 \vec{r}_i}{\partial q_\alpha \partial q_1}\dot{q}_1 + \frac{\partial^2 \vec{r}_i}{\partial q_\alpha \partial q_2}\dot{q}_2 + \frac{\partial^2 \vec{r}_i}{\partial q_\alpha \partial q_3}\dot{q}_3 + \ldots\ldots\ldots\ldots\ldots + \frac{\partial^2 \vec{r}_i}{\partial q_\alpha \partial q_n}\dot{q}_n + \frac{\partial^2 \vec{r}_i}{\partial q_\alpha \partial t}\right\}$$

$$\frac{\partial}{\partial q_\alpha}\left(\frac{d\,()}{dt}\right) = \frac{\partial^2\,()}{\partial q_\alpha \partial q_1}\dot{q}_1 + \frac{\partial^2\,()}{\partial q_\alpha \partial q_2}\dot{q}_2 + \frac{\partial^2\,()}{\partial q_\alpha \partial q_3}\dot{q}_3 + \ldots\ldots\ldots\ldots\ldots + \frac{\partial^2\,()}{\partial q_\alpha \partial q_n}\dot{q}_n + \frac{\partial^2\,()}{\partial q_\alpha \partial t} \tag{2}$$

While, if we derive Eq. (1) with respect to q_α, we have

$$\frac{\partial \vec{r}_i}{\partial q_\alpha} = \frac{\partial \vec{r}_i(q_1, q_2, q_3, q_4, q_5, \ldots\ldots\ldots\ldots q_n, t)}{\partial q_\alpha}$$

Again, we derive the above relation with respect to t, we get

$$\frac{d}{dt}\left(\frac{\partial\vec{r}_i}{\partial q_\alpha}\right)=\frac{d}{dt}\left(\frac{\partial\vec{r}_i(q_1,q_2,q_3,q_4,q_5,..........q_n,t)}{\partial q_\alpha}\right)$$

$$=\frac{\partial^2\vec{r}_i(q_1,q_2,q_3,q_4,q_5,..........q_n,t)}{\partial q_1\,\partial q_\alpha}\;\frac{dq_1}{dt}+\frac{\partial^2\vec{r}_i(q_1,q_2,q_3,q_4,q_5,..........q_n,t)}{\partial q_2\,\partial q_\alpha}\;\frac{dq_2}{dt}+$$

$$\frac{\partial^2\vec{r}_i(q_1,q_2,q_3,q_4,q_5,..........q_n,t)}{\partial q_3\,\partial q_\alpha}\;\frac{dq_3}{dt}+...................+\frac{\partial^2\vec{r}_i(q_1,q_2,q_3,q_4,q_5,..........q_n,t)}{\partial q_n\,\partial q_\alpha}\;\frac{dq_n}{dt}+$$

$$\frac{\partial^2\vec{r}_i(q_1,q_2,q_3,q_4,q_5,..........q_n,t)}{\partial t\,\partial q_\alpha}\;\frac{dt}{dt}$$

$$==\frac{\partial^2\vec{r}_i}{\partial q_1\,\partial q_\alpha}\;\frac{dq_1}{dt}+\frac{\partial^2\vec{r}_i}{\partial q_2\,\partial q_\alpha}\;\frac{dq_2}{dt}+\frac{\partial^2\vec{r}_i}{\partial q_3\,\partial q_\alpha}\;\frac{dq_3}{dt}+...................+\frac{\partial^2\vec{r}_i}{\partial q_n\,\partial q_\alpha}\;\frac{dq_n}{dt}+\frac{\partial^2\vec{r}_i}{\partial t\,\partial q_\alpha}$$

$$\frac{d}{dt}\left(\frac{\partial\vec{r}_i}{\partial q_\alpha}\right)=\frac{\partial^2()}{\partial q_1\,\partial q_\alpha}\dot{q}_1+\frac{\partial^2()}{\partial q_2\,\partial q_\alpha}\dot{q}_2+\frac{\partial^2()}{\partial q_3\,\partial q_\alpha}\dot{q}_3+.................+\frac{\partial^2()}{\partial q_n\partial q_\alpha}\dot{q}_n+\frac{\partial^2()}{\partial t\,\partial q_\alpha}$$

That we can be written as

$$\frac{d}{dt}\left(\frac{\partial\vec{r}_i}{\partial q_\alpha}\right)=\frac{\partial^2()}{\partial q_\alpha\partial q_1}\dot{q}_1+\frac{\partial^2()}{\partial q_\alpha\partial q_2}\dot{q}_2+\frac{\partial^2()}{\partial q_\alpha\partial q_3}\dot{q}_3+.................+\frac{\partial^2()}{\partial q_\alpha\partial q_n}\dot{q}_n+\frac{\partial^2()}{\partial q_\alpha\partial t}\qquad(3)$$

Now, from Eq. (1) and Eq. (2) it can be said that $\frac{\partial}{\partial q_\alpha}\left(\frac{d()}{dt}\right)=\frac{d}{dt}\left(\frac{\partial()}{\partial q_\alpha}\right)$

(3) Proof that $\frac{\partial T}{\partial q_\alpha}=m\dot{\vec{r}}_i\cdot\frac{\partial\dot{\vec{r}}_i}{\partial q_\alpha}$?

Proof

The kinetic energy is given as $T=\frac{1}{2}m\dot{r}^2$, that can be written as $T=\frac{1}{2}m\dot{\vec{r}}_i\cdot\dot{\vec{r}}_i$ and partial derivative it with respect to q_α, we have

$$\frac{\partial T}{\partial q_\alpha}=\frac{\partial}{\partial q_\alpha}\left(\frac{1}{2}m\dot{\vec{r}}_i\cdot\dot{\vec{r}}_i\right)=\frac{1}{2}m\left(\frac{\partial\dot{\vec{r}}_i}{\partial q_\alpha}\cdot\dot{\vec{r}}_i+\dot{\vec{r}}_i\cdot\frac{\partial\dot{\vec{r}}_i}{\partial q_\alpha}\right)=\frac{1}{2}m\left(\dot{\vec{r}}_i\cdot\frac{\partial\dot{\vec{r}}_i}{\partial q_\alpha}+\dot{\vec{r}}_i\cdot\frac{\partial\dot{\vec{r}}_i}{\partial q_\alpha}\right)=\frac{1}{2}m\left(2\dot{\vec{r}}_i\cdot\frac{\partial\dot{\vec{r}}_i}{\partial q_\alpha}\right)$$

So $\frac{\partial T}{\partial q_\alpha}=m\dot{\vec{r}}_i\cdot\frac{\partial\dot{\vec{r}}_i}{\partial q_\alpha}$

(4) Proof that $\frac{\partial T}{\partial\dot{q}_\alpha}=m\dot{\vec{r}}_i\cdot\frac{\partial\dot{\vec{r}}_i}{\partial\dot{q}_\alpha}$?

Proof

The kinetic energy is given as $T=\frac{1}{2}m\dot{r}^{2}$, that can be written as $T=\frac{1}{2}m\dot{\vec{r}}_i.\dot{\vec{r}}_i$ and partial derivative it with respect to $\dot{q}_\alpha$, we have

$$\frac{\partial T}{\partial \dot{q}_\alpha}=\frac{\partial}{\partial \dot{q}_\alpha}\left(\frac{1}{2}m\dot{\vec{r}}_i.\dot{\vec{r}}_i\right)=\frac{1}{2}m\left(\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_\alpha}.\dot{\vec{r}}_i+\dot{\vec{r}}_i.\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_\alpha}\right)=\frac{1}{2}m\left(\dot{\vec{r}}_i.\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_\alpha}+\dot{\vec{r}}_i.\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_\alpha}\right)=\frac{1}{2}m\left(2\dot{\vec{r}}_i.\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_\alpha}\right)$$

So $\frac{\partial T}{\partial \dot{q}_\alpha}=m\dot{\vec{r}}_i.\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_\alpha}$

Conservative Force Fields (conservative dynamical system)

For a conservative dynamical system, the force can be written as the negative gradient of a potential energy (V) in the form $F=-\nabla V$.

The generalized force can be written as

$$Q_\alpha=\sum_{i=1}^{N}\vec{F}_i.\frac{\partial \vec{r}_i}{\partial q_\alpha}=\sum_{i=1}^{N}(-\nabla V).\frac{\partial \vec{r}_i}{\partial q_\alpha}=-\sum_{i=1}^{N}\frac{\partial V}{\partial \vec{r}_i}.\frac{\partial \vec{r}_i}{\partial q_\alpha}=-\frac{\partial V}{\partial q_\alpha},$$

i, e. the generalized force for conservative dynamical system can be in the form

$$Q_\alpha=-\frac{\partial V}{\partial q_\alpha}.$$

Chapter 2

Lagrange's Equations of Motion

The usual way of using newtonian mechanics to solve a problem in dynamics is first of all to draw a large, clear diagram of the system, using a ruler and a compass. Then mark in the forces on the various parts of the system with red arrows and the accelerations of the various parts with green arrows. Then apply the equation $F = ma$ in two different directions if it is a two-dimensional problem or in three directions if it is a three-dimensional problem, or $\tau = I\theta$ if torques are involved. More correctly, if a mass or a moment of inertia is not constant, the equations are $F = p$ and $\tau = L$. In any case, we arrive at one or more equations of motion, which are differential equations which we integrate with respect to space or time to find the desired solution. Most of us will have done many, many problems of that sort.

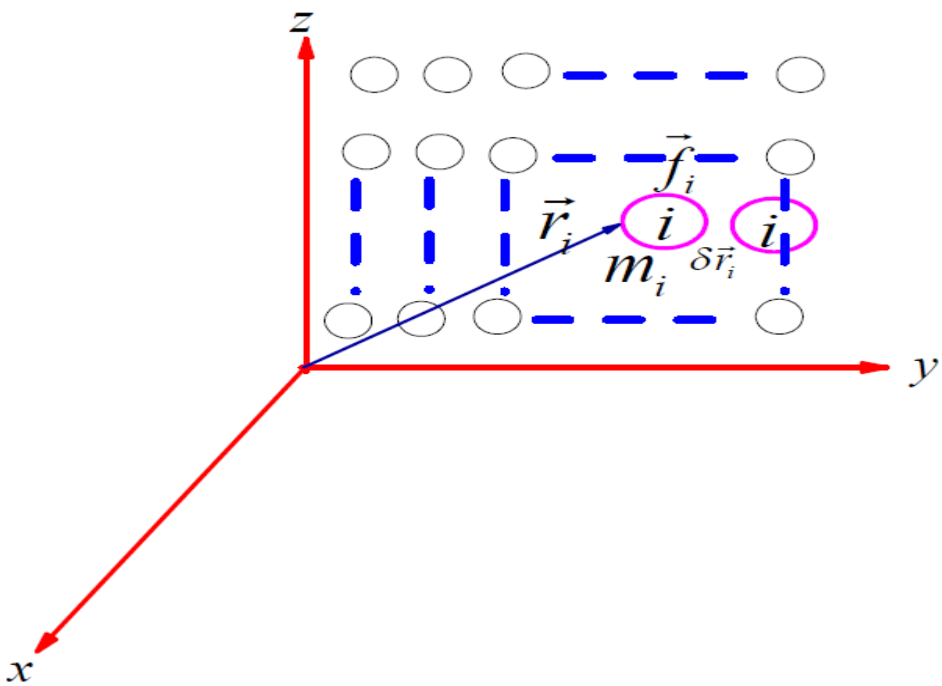
Sometimes it is not all that easy to find the equations of motion as described above. There is an alternative approach known as lagrangian mechanics which enables us to find the equations of motion when the newtonian method is proving difficult. In lagrangian mechanics we start, as usual, by drawing a large, clear diagram of the system, using a ruler and a compass. But, rather than drawing the forces and accelerations with red and green arrows, we draw the velocity vectors (including angular velocities) with blue arrows, and, from these we write down the kinetic energy of the system. If the forces are conservative forces (gravity, springs and stretched strings), we write down also the potential energy. That done, the next step is to write down the lagrangian equations of motion for each coordinate. These equations involve the kinetic and potential energies, and are a little bit more involved than $F = ma$, though they do arrive at the same results.

I shall derive the lagrangian equations of motion, and while I am doing so, you will think that the going is very heavy, and you will be discouraged. At the end of the derivation you will see that the lagrangian equations of motion are indeed rather more involved than $F = ma$, and you will begin to despair – but do not do so! In a very short time after that you will be able to solve difficult problems in mechanics that you would not be able to start using the familiar newtonian methods, and the speed at which you do so will be limited solely by the speed at which you can write. Indeed, you scarcely have to stop and think. You know straight away what you have to do. Draw the diagram. Mark the velocity vectors. Write down expressions for the kinetic and potential energies, and apply the lagrangian equations. It is automatic, fast, and enjoyable.

Incidentally, when Lagrange first published his great work La méchanique analytique (the modern French spelling would be mécanique), he pointed out with some pride in his introduction that there were no drawings or diagrams in the book – because all of mechanics could be done analytically – i.e. with algebra and calculus. Not all of us, however, are as gifted as Lagrange, and we cannot omit the first and very important step of drawing a large and clear diagram with ruler and compass and marking all the velocity vectors.

The lagrange's equations

Suppose that we have a dynamical system of N particles, and select one of these particles, let it be a number i, which has a mass m_i, its position is $\vec{r}_i$ and it is affected by the force $\vec{F}_i$, then it moves a displacement $\delta \vec{r}_i$.



Writing the equation of motion for this particle, it will be according to Newton's second law on the form $\vec{F}_i = m_i \ddot{\vec{r}}$ and for the all dynamical system it can be written

$$\sum_{i=1}^{N} \vec{F}_i = \sum_{i=1}^{N} m_i \ddot{\vec{r}} \tag{1}$$

Multiplying Eq. (1) dot product by $\dfrac{\partial \vec{r}}{\partial q_\alpha}$, we have

$$\sum_{i=1}^{N}\vec{F}_i \cdot \frac{\partial \vec{r}}{\partial q_\alpha} = \sum_{i=1}^{N} m_i \ddot{\vec{r}} \cdot \frac{\partial \vec{r}}{\partial q_\alpha} \tag{2}$$

But

$$\frac{d}{dt}\left(\dot{\vec{r}} \cdot \frac{\partial \vec{r}}{\partial q_\alpha}\right) = \ddot{\vec{r}} \cdot \frac{\partial \vec{r}}{\partial q_\alpha} + \dot{\vec{r}} \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}}{\partial q_\alpha}\right) = \ddot{\vec{r}} \cdot \frac{\partial \vec{r}}{\partial q_\alpha} + \dot{\vec{r}} \cdot \left(\frac{\partial}{\partial q_\alpha}(\frac{d\vec{r}}{dt})\right) = \ddot{\vec{r}} \cdot \frac{\partial \vec{r}}{\partial q_\alpha} + \dot{\vec{r}} \cdot \left(\frac{\partial \dot{\vec{r}}}{\partial q_\alpha}\right)$$

Then $\ddot{\vec{r}} \cdot \frac{\partial \vec{r}}{\partial q_\alpha} = \frac{d}{dt}\left(\dot{\vec{r}} \cdot \frac{\partial \vec{r}}{\partial q_\alpha}\right) - \dot{\vec{r}} \cdot \left(\frac{\partial \dot{\vec{r}}}{\partial q_\alpha}\right)$

Also, we know that $\frac{\partial \vec{r}}{\partial q_\alpha} = \frac{\partial \dot{\vec{r}}}{\partial \dot{q}_\alpha}$

$$\ddot{\vec{r}} \cdot \frac{\partial \vec{r}}{\partial q_\alpha} = \frac{d}{dt}\left(\dot{\vec{r}} \cdot \frac{\partial \dot{\vec{r}}}{\partial \dot{q}_\alpha}\right) - \dot{\vec{r}} \cdot \left(\frac{\partial \dot{\vec{r}}}{\partial q_\alpha}\right) \tag{3}$$

Substituting from Eq. (3) into Eq. (2) we get

$$\sum_{i=1}^{N}\vec{F}_i \cdot \frac{\partial \vec{r}}{\partial q_\alpha} = \sum_{i=1}^{N} m_i \ddot{\vec{r}} \cdot \frac{\partial \vec{r}}{\partial q_\alpha} = \sum_{i=1}^{N} m_i \left\{\frac{d}{dt}\left(\dot{\vec{r}} \cdot \frac{\partial \dot{\vec{r}}}{\partial \dot{q}_\alpha}\right) - \dot{\vec{r}} \cdot \left(\frac{\partial \dot{\vec{r}}}{\partial q_\alpha}\right)\right\} = \sum_{i=1}^{N} \left\{\frac{d}{dt}\left(m_i \dot{\vec{r}} \cdot \frac{\partial \dot{\vec{r}}}{\partial \dot{q}_\alpha}\right) - m_i \dot{\vec{r}} \cdot \frac{\partial \dot{\vec{r}}}{\partial q_\alpha}\right\} \tag{4}$$

$$Q_\alpha = \vec{F}_i \cdot \frac{\partial \vec{r}}{\partial q_\alpha}, \qquad \frac{\partial T}{\partial \dot{q}_\alpha} = m_i \dot{\vec{r}} \cdot \frac{\partial \dot{\vec{r}}}{\partial \dot{q}_\alpha}, \qquad \frac{\partial T}{\partial q_\alpha} = m_i \dot{\vec{r}} \cdot \frac{\partial \dot{\vec{r}}}{\partial q_\alpha}$$

Then $\sum_{i=1}^{N} Q_\alpha = \sum_{i=1}^{N} \left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha}\right\}$

Now substituting into Eq. (4) we have

$$Q_\alpha = \frac{d}{dt}\left\{\frac{\partial T}{\partial \dot{q}_\alpha}\right\} - \frac{\partial T}{\partial q_\alpha} \tag{5}$$

This the Lagrangian equations is in terms of generalized forces, where $\alpha = 1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots, n$

lagrangian equation for conservative system

If the system is conservative so that particles move under the influence of a potential which is dependent on coordinates only, then the forces are derived from the potential (V) given by $Q_\alpha = -\dfrac{\partial V}{\partial q_\alpha}$.

Since the potential energy (V) is dependent on coordinates only (q_α), and not on the velocities, $(\dot{q}_\alpha)$, i. e. $V = V(q_\alpha, t)$.

Then the Lagrange's equations (Eq. (5)) may be written

$$Q_\alpha = -\frac{\partial V}{\partial q_\alpha} = \frac{d}{dt}\left\{\frac{\partial T}{\partial \dot{q}_\alpha}\right\} - \frac{\partial T}{\partial q_\alpha}$$

$$-\frac{\partial V}{\partial q_\alpha} = \frac{d}{dt}\left\{\frac{\partial T}{\partial \dot{q}_\alpha}\right\} - \frac{\partial T}{\partial q_\alpha} \rightarrow \frac{\partial V}{\partial \dot{q}_\alpha} - \frac{\partial V}{\partial q_\alpha} = \frac{d}{dt}\left\{\frac{\partial T}{\partial \dot{q}_\alpha}\right\} - \frac{\partial T}{\partial q_\alpha}$$

$$0 = \frac{d}{dt}\left\{\frac{\partial (T-V)}{\partial \dot{q}_\alpha}\right\} - \frac{\partial (T-V)}{\partial q_\alpha}$$

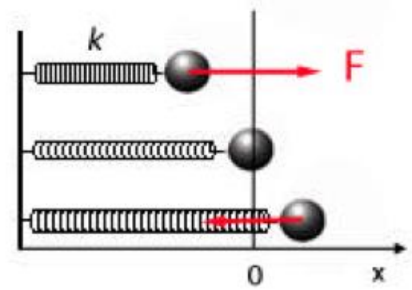
We put $L = T - V$ that is called Lagrange function. So

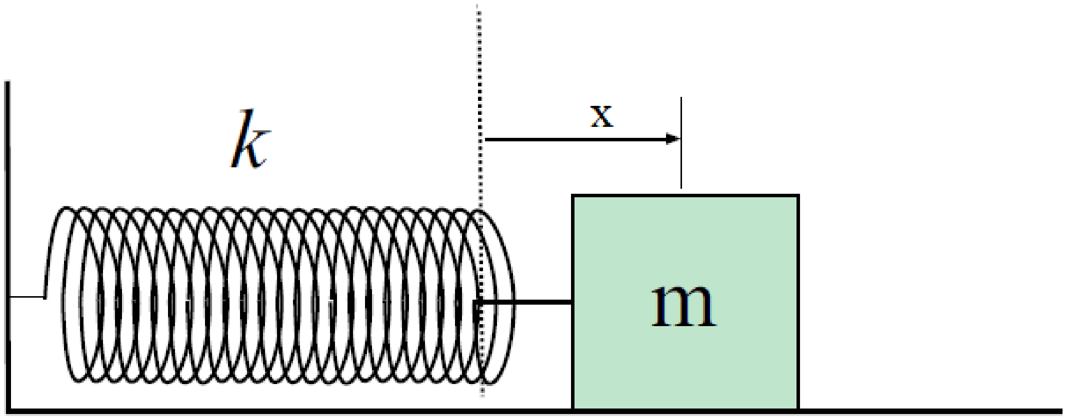
$$\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{q}_\alpha}\right\} - \frac{\partial L}{\partial q_\alpha} = 0$$

This is called the Lagrange equation or Euler-Lagrange equation.

Example 1: Determine the motion of harmonic oscillator in one dimension by Lagrangian Equations?

Solution





We first consider a simple mass spring system. This is a one degree of freedom system, with one x.

<u>The Potential Energy</u>

The force proportional (Inversely proportional) to displacement (x) (directly proportional)

$F = -k\,x$, but $\quad F = -\nabla V = -k\,x$

$$-\frac{dV}{dx} = -k\,x \;\rightarrow\; dV = k\,x\,dx \;\rightarrow\; \int dV = \int k\,x\,dx$$

Then $\quad V = \frac{1}{2} k\,x^2$ (does not dependent on the generalized velocity)

<u>Kinetic Energy</u>

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \dot{x}^2$$

<u>Lagrange Function</u>

The Lagrange Function of simple harmonic oscillator in one dimension can be written as

$$L = T - V = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k\,x^2$$

So the Lagrange Equation $\frac{d}{dt}\left\{ \frac{\partial L}{\partial \dot{q}_\alpha} \right\} - \frac{\partial L}{\partial q_\alpha} = 0$ becomes

$$\frac{d}{dt}\left\{ \frac{\partial L}{\partial \dot{x}} \right\} - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}\left\{\frac{\partial}{\partial x^{\bullet}}\left[\frac{1}{2}m x^{\bullet 2}-\frac{1}{2}k x^{2}\right]\right\}-\frac{\partial}{\partial x}\left[\frac{1}{2}m x^{\bullet 2}-\frac{1}{2}k x^{2}\right]=0$$

$$\frac{d}{dt}\left[\frac{2}{2}m x^{\bullet}\right]-\left[-\frac{2}{2}k x\right]=0$$

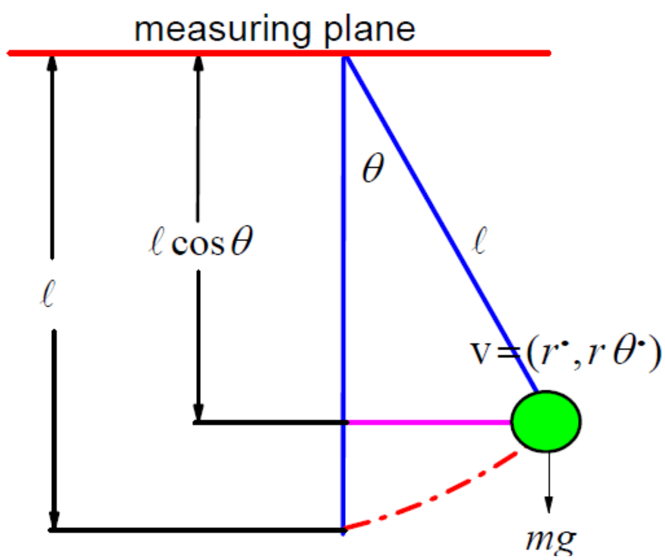
$m x^{\bullet\bullet}+k x=0$. So $x^{\bullet\bullet}=-\frac{k}{m}x$

This is the same as the equation of motion of the simple harmonic oscillator resulted from application of Newton's second law to a mass attached to spring of spring constant k (Hook constant) and displaced to a position x from equilibrium position.

Example: 2 Determine the motion of plane pendulum (simple pendulum) by Lagrangian Equation ?

Solution

A plane pendulum consists of a bob of mass m suspended from a light inextensible cord of length L and swinging in a plane.



For the motion of plane pendulum, we have

$$v=(r^{\bullet}, r\theta^{\bullet}),\ \ \mathrm{r}=L=\text{constant}\ \rightarrow\ v=(0, L\theta^{\bullet})\ \rightarrow\ v=L\theta^{\bullet}$$

Kinetic Energy

$$T = \frac{1}{2}v^2 = \frac{1}{2}L^2\,\dot{\theta}^2$$

The Potential Energy

$$V = -\,mgL\cos\theta$$

Lagrange Function

The Lagrange Function of simple harmonic oscillator in one dimension can be written as

$$L = T - V = \frac{1}{2}L^2\,\dot{\theta}^2 + mgL\cos\theta$$

So the Lagrange Equation $\frac{d}{dt}\left\{ \frac{\partial L}{\partial \dot{q}_\alpha} \right\} - \frac{\partial L}{\partial q_\alpha} = 0$ becomes

$$\frac{d}{dt}\left\{ \frac{\partial L}{\partial \dot{\theta}} \right\} - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\left\{ \frac{\partial}{\partial \dot{\theta}}\left[\frac{1}{2}L^2\,\dot{\theta}^2 + mgL\cos\theta \right] \right\} - \frac{\partial}{\partial \theta}\left[\frac{1}{2}L^2\,\dot{\theta}^2 + mgL\cos\theta \right] = 0$$

$$\frac{d}{dt}\left\{ L^2\,\dot{\theta} \right\} + \left[mgL\sin\theta \right] = 0$$

$$L^2\,\ddot{\theta} + mg\,L\sin\theta = 0 \;\rightarrow\; L\,\ddot{\theta} + mg\sin\theta = 0 \;\rightarrow\; \ddot{\theta} = -\frac{mg}{L}\sin\theta$$

If θ is very small, then $\sin\theta = \theta$

Then

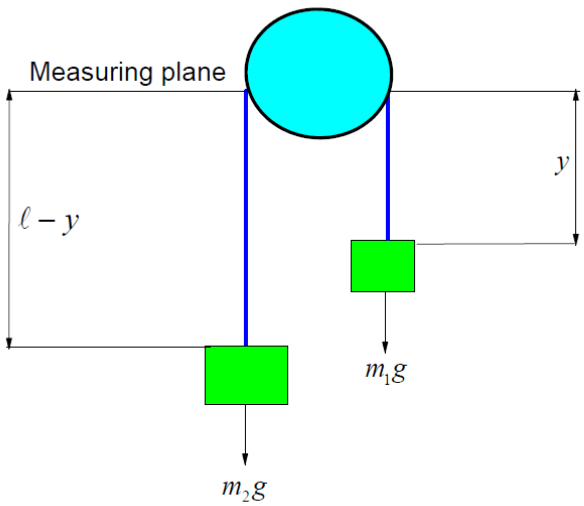
$$\ddot{\theta} = -\frac{mg}{L}\theta.$$

This the equation for simple pendulum with angular frequency

Example: 3 Two masses m_1 and m_2, are hanging by a massless string from a frictionless pulley. If m_2 is greater than m_1, determine the acceleration of the two masses when released from rest by the Lagrangian method.

Solution

First, identify a direction as positive. Since you can easily observe that m_2 will accelerate downward and m_1 will accelerate upward, since $m_2 > m_1$, call the direction of motion around the pulley and down toward m_2 the positive $y-$ direction. Then, you can create free body diagrams for both object m_2 and m_1, as shown below.



Solution

Mass	The vertical distance from the measuring plane	Velocity	Kinetic Energy	Potential Energy = - Work done
m_1	y	$y^{\bullet}$	$\frac{1}{2} m_1 y^{\bullet 2}$	$-m_1 g\ y$
m_2	$l - y$	$-y^{\bullet}$	$\frac{1}{2} m_2 y^{\bullet 2}$	$-m_2 g\ (l - y)$

The total of Kinetic energy is given as

$$T = \frac{1}{2}(m_1 + m_2)\dot{y}^2 .$$

While the total of potential energy is given as

$$V = -m_1 g\, y - m_2 g\, (l - y) = g\,(m_2 - m_1)y - m_2 g\, l \ .$$

So, the lagrangian function is

$$L = T - V = \frac{1}{2}(m_1 + m_2)\,\dot{y}^2 - g\,(m_2 - m_1)y + m_2 g\, l$$

From lagrangian Equation $\frac{d}{dt}\left\{ \frac{\partial L}{\partial \dot{q}_\alpha} \right\} - \frac{\partial L}{\partial q_\alpha} = 0$, where we have one coordinate $(q_1 = y)$. So the Lagrangian Equation of our problem given as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$

$$\frac{d}{dt}\Big((m_1 + m_2)\,\dot{y} \Big) + (m_2 - m_1)g = 0$$

$$(m_1 + m_2)\,\ddot{y} + (m_2 - m_1)g = 0$$

Then: $\ddot{y} = \frac{m_1 - m_2}{m_1 + m_2} g$

Example 4: A particle of mass m moving in a plane under the attractive force $\frac{\mu m}{r^2}$ directed to the origin of polar coordinates (r, θ). Determine the equation of motion by Lagrange equations

Solution

We know that in polar coordinates $\vec{v} = (\dot{r},\ r\dot{\theta})$, so we have

<u>Kinetic Energy</u>

$$T = \frac{1}{2}v^2 = \frac{m}{2}\left(\dot{r}^2 + r^2\,\dot{\theta}^2 \right)$$

The Potential Energy

$$F = -\frac{\mu m}{r^2}, \text{ but} \qquad F = -\nabla V = -\frac{\mu m}{r^2}$$

$$\frac{dV}{dr} = \frac{\mu m}{r^2} \rightarrow dV = \frac{\mu m}{r^2} dr \rightarrow \int dV = \int \frac{\mu m}{r^2} dr \text{ . } \qquad \text{Then} \qquad V = -\frac{\mu m}{r}$$

Lagrange Function

The Lagrange Function in polar coordinates (r, θ) can be written as

$$L = T - V = \frac{m}{2}\left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) + \frac{\mu m}{r}$$

So, the Lagrange Equation $\frac{d}{dt}\left\{ \frac{\partial L}{\partial \dot{q}_\alpha} \right\} - \frac{\partial L}{\partial q_\alpha} = 0$ in polar coordinates (r, θ) becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = 0 \qquad (1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0 \qquad (2)$$

From Eq. (1), we get

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{r}} \left[\frac{m}{2}\left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) + \frac{\mu m}{r} \right] \right) - \frac{\partial}{\partial r}\left[\frac{m}{2}\left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) + \frac{\mu m}{r} \right] = 0$$

$$\frac{d}{dt}(m \dot{r}) - \left[\frac{m}{2}\left(2r \dot{\theta}^2 \right) - \frac{\mu m}{r^2} \right] = 0$$

$$\ddot{r} - r \dot{\theta}^2 + \frac{\mu}{r^2} = 0$$

Then $\qquad \ddot{r} - r \dot{\theta}^2 = -\frac{\mu}{r^2}$

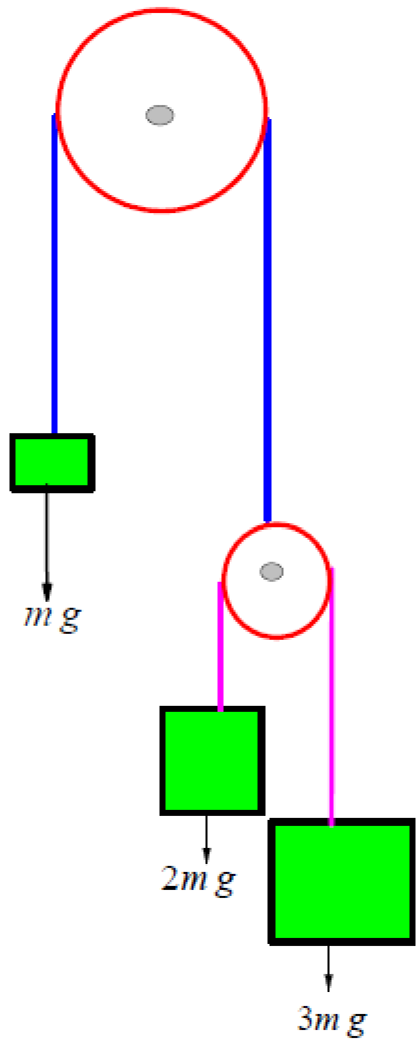
From Eq. (2), we get

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{\theta}} \left[\frac{m}{2}\left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) + \frac{\mu m}{r} \right] \right) - \frac{\partial}{\partial \theta}\left[\frac{m}{2}\left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) + \frac{\mu m}{r} \right] = 0$$

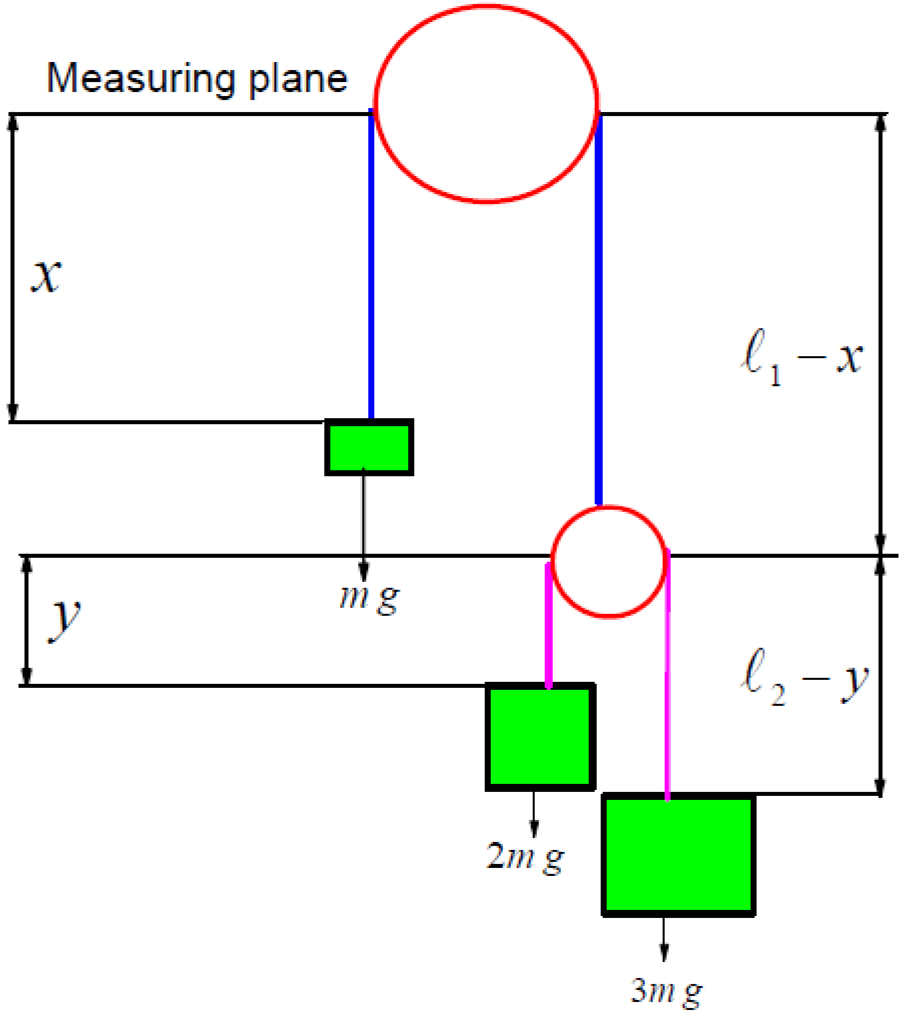
$$\frac{d}{dt}\left(m r^2 \dot{\theta} \right) - 0 = 0$$

$m r^2 \dot{\theta} = \text{constant}$. $\quad$ Then $\qquad r^2 \dot{\theta} = \text{constant}$

Example: 5: Determine the equations of motion of Double Atwood machine which consists of one of the pulleys replaced by an Atwood machine as shown below Figure. Neglect the masses of pulleys.



Solution



Mass	The vertical distance from the measuring plane	Velocity	Kinetic Energy	Potential Energy = - Work done
m	x	$\dot{x}$	$m\dot{x}^2$	$-mgx$
$2m$	$y+l_1-x$	$\dot{y}-\dot{x}$	$m(\dot{y}-\dot{x})^2$	$-2mg\,(y+l_1-x)$
$3m$	l_1+l_2-x-y	$-\dot{x}-\dot{y}$	$3m(-\dot{x}-\dot{y})^2$	$-3mg\,(l_1+l_2-x-y)$

Kinetic Energy

The total of Kinetic energy is given as

$$T=\frac{m}{2}\left\{6\dot{x}^2+2\dot{x}\dot{y}+5\dot{y}^2\right\}$$

The Potential Energy

The total of potential energy is given as

$$V = -mgx - 2mg\,(y + l_1 - x) - 3mg\,(l_1 + l_2 - x - y) = -mg\,(x + 2y + 2l_1 - 2x + 3l_1 + 3l_2 - 3x - 3y)$$
$$= -mg\,(-4x - y + 5l_1 + 3l_1)$$

$$V = mg\,(4x + y) + C \quad , C = -mg\,(5l_1 + 3l_1)$$

Lagrange Function

The Lagrange Function can be written as

$$L = T - V = \frac{m}{2}\left\{6\dot{x}^2 + 2\dot{x}\dot{y} + 5\dot{y}^2\right\} - mg\,(4x + y) - C$$

So, the Lagrange Equation $\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{q}_\alpha}\right\} - \frac{\partial L}{\partial q_\alpha} = 0$ for the plane coordinates (x, y) can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0\,. \qquad (1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \qquad (2)$$

From Eq. (1), we get

$$\frac{d}{dt}\left\{\frac{m}{2}\left(12\dot{x} + 2\dot{y}\right)\right\} + mg\,(4) = 0 \rightarrow 6\ddot{x} + \ddot{y} = -4g \qquad (3)$$

From Eq. (2), we get

$$\frac{d}{dt}\left\{\frac{m}{2}\left(2\dot{x} + 10\dot{y}\right)\right\} + mg = 0 \rightarrow \ddot{x} + 5\ddot{y} = -g \qquad (4)$$

From Eq. (3) and Eq. (4), we get

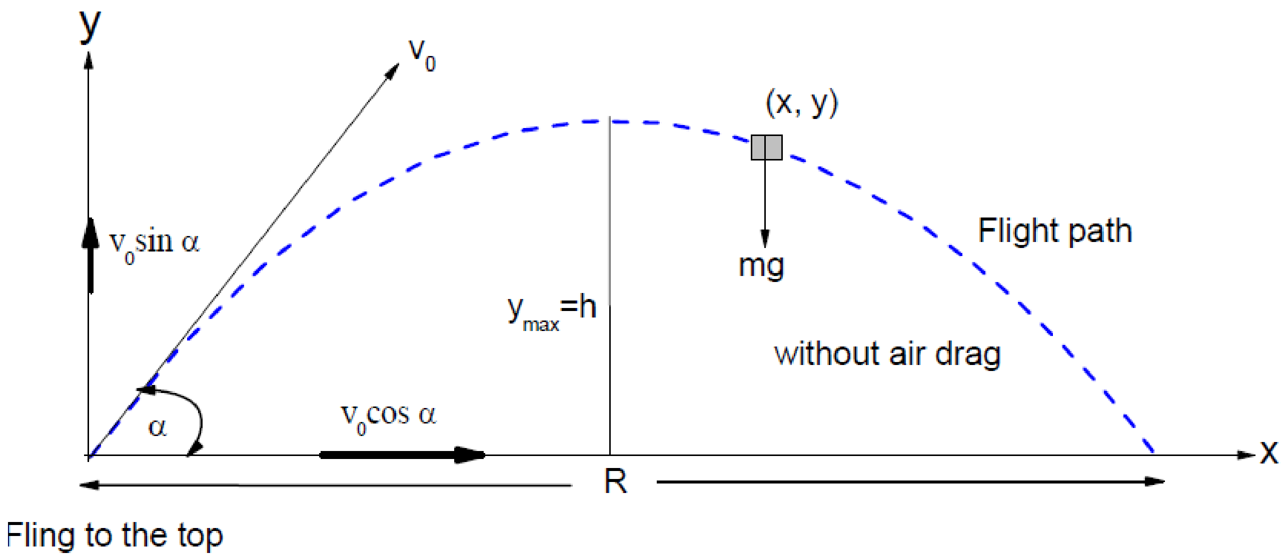
$$\ddot{x} + 5\left(-4g - 6\ddot{x}\right) = -g \;\rightarrow\; -5\ddot{x} = 19g \rightarrow \ddot{x} = -\frac{19}{5}g \qquad (5)$$

From Eq. (5) into Eq. (4), we get

$$\ddot{x} + 5\ddot{y} = -g \rightarrow -\frac{19}{5}g + 5\ddot{y} = -g \rightarrow 5\ddot{y} = -g + \frac{19}{5}g \rightarrow 5\ddot{y} = \frac{14}{5}g \rightarrow \ddot{y} = \frac{14}{25}g \qquad (6)$$

Example 6: A particle of mass m is projected with initial velocity v_0 at an angle α to the horizontal in the uniform gravitational field of the earth. Use Lagrange's equation to describe the motion of the projectile. Ignore the air resistance?

Solution



Let a particle of mass m be projected from the origin point with an initial velocity v_0 making an angle α with the horizontal line referred as x-axis. Let (x, y) be the position of the particle at any instant t. Since x and y are independent and hence the generalized coordinates are $(q_1, q_2) = (x , y)$ and the generalized velocities are $(\dot{q}_1, \dot{q}_2) = (\dot{x} , \dot{y})$.

Kinetic Energy

The kinetic of the projectile is given by $T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$,

The Potential Energy

The total of potential energy is given as

$$F = -mg \;\rightarrow\; F = -mg = -\nabla V \;\rightarrow\; mg = \frac{dV}{dy} \;\rightarrow\; V = mg\, y$$

Lagrange Function

The Lagrange Function can be written as

$$L = T - V = \frac{1}{2} m(x^{\bullet 2} + y^{\bullet 2}) - mg\, y$$

So, the Lagrange Equation $\frac{d}{dt}\left\{ \frac{\partial L}{\partial \dot{q}_\alpha} \right\} - \frac{\partial L}{\partial q_\alpha} = 0$ for the plane coordinates (x, y) can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial x^\bullet}\right) - \frac{\partial L}{\partial x} = 0 . \qquad (1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^\bullet}\right) - \frac{\partial L}{\partial y} = 0 \qquad (2)$$

From Eq. (1), we get

$$\frac{d}{dt}\left(\frac{\partial}{\partial x^\bullet}\left[\frac{1}{2} m(x^{\bullet 2} + y^{\bullet 2}) - mg\, y\right]\right) - \frac{\partial}{\partial x}\left[\frac{1}{2} m(x^{\bullet 2} + y^{\bullet 2}) - mg\, y\right] = 0$$

$$\frac{d}{dt}(mx^\bullet) - (0) = 0 \rightarrow (m\, x^{\bullet\bullet}) = 0$$

Then

$$x^{\bullet\bullet} = 0 \quad or \quad x^\bullet = \text{Constant} = c_1 \qquad (3)$$

From the initial condition at $t = 0 \;\rightarrow\; x^\bullet\big|_{t=0} = v_0 \cos\alpha \rightarrow c_1 = v_0 \cos\alpha$. Then

$$x^\bullet = v_0 \cos\alpha \qquad (4)$$

Integration Eq. (4), we have $x = v_0\, t \cos\alpha + c_2$

From the initial condition at $t = 0$, $x = 0 \;\rightarrow$, then $c_2 = 0$

$$x = v_0\, t \cos\alpha \qquad (5)$$

While, from Eq. (2), we get

$$\frac{d}{dt}\left(\frac{\partial}{\partial y^\bullet}\left[\frac{1}{2} m(x^{\bullet 2} + y^{\bullet 2}) - mg\, y\right]\right) - \frac{\partial}{\partial y}\left[\frac{1}{2} m(x^{\bullet 2} + y^{\bullet 2}) - mg\, y\right] = 0$$

$\frac{d}{dt}(m\, y^\bullet) + (mg) = 0$. Then

$$y^{\bullet\bullet} = -g \quad or \;\; y^\bullet = -gt + \text{Constant} = c_3 - gt \qquad (6)$$

From the initial condition at $t = 0$, $y^\bullet\big|_{t=0} = v_0 \sin\alpha \;\rightarrow\; c_2 = v_0 \sin\alpha$. Then

$$y^{\bullet} = \mathrm{v}_0 \sin\alpha - gt \tag{7}$$

Integration Eq. (7), we have $y = \mathrm{v}_0\, t\, \sin\alpha - \frac{1}{2} g\, t^2 + c_4$

From the initial condition at $t = 0$, $y = 0$, $\rightarrow$ $c_4 = 0$. Then

$$y = \mathrm{v}_0\, t\, \sin\alpha - \frac{1}{2} g\, t^2 \tag{8}$$

Exercises

(i) Both the kinetic energy T and potential energy V for Mechanical system are given, respectively, in the form $T = \frac{1}{2}(x^{\bullet 2} + x^2 y^{\bullet 2})$, $V = \frac{\omega^2}{2} x^2$. Find the equations of motion for the system using lagrangian equations where ω is constant?

(ii) Both the kinetic energy T and potential energy V for Mechanical system are given, respectively, in the form $T = \frac{1}{2} m (r^{\bullet 2} + r^2 \theta^{\bullet 2})$, $V = -m\, g\, r \cos\theta$. Find the equations of motion for the system using lagrangian equations?

(iii) Both the kinetic energy T and potential energy V for Mechanical system are given, respectively, in the form $T = \frac{1}{2}(\theta^{\bullet 2} + \phi^{\bullet 2} \sin^2\theta)$, $V = -2\sqrt{2}\cos\theta$. Find the equations of motion for the system using lagrangian equations?

(iv) Both the kinetic energy T and potential energy V for Mechanical system are given, respectively, in the form $T = \frac{1}{2}(x^{\bullet 2} + y^{\bullet 2} + z^{\bullet 2})$, $V = V(x, y, z)$. Find the equations of motion for the system using lagrangian equations?

Chapter 3

Hamiltonian function and Hamilton's equations of motion

In this chapter we consider a radically different formulation of the dynamical problem. We define the Hamiltonian and derive Hamilton’s “canonical” equations.

These are derived in two different ways, first by using a Legendre transformation

on the Lagrangian and secondly by using the stationary property of the action integral. Hamilton’s approach gives us a whole new way of looking at mechanics problems. Although Hamilton’s approach is often not as convenient as Lagrange’s method for solving practical problems, it is, nevertheless, a far superior tool for theoretical studies. Some of the methods developed in Hamiltonian mechanics carry over directly into quantum mechanics, statistical mechanics, and other fields of physics.

Remarks

It is well-known that, the Generalized momentum is

$$p_\alpha = \sum_\alpha m \dot{r}_i \qquad (1)$$

The Generalized kinetic Generalized is

$$T = \sum_\alpha \frac{1}{2} m \dot{r}_i^2 \qquad (2)$$

$$\frac{\partial T}{\partial \dot{r}_i} = \sum_\alpha m \dot{r}_i \qquad (3)$$

From Eq. (1) and Eq. (3) generally speaking we find that

$$\frac{\partial T}{\partial \dot{q}_\alpha} = p_\alpha \qquad (4)$$

We know that the Lagrange function is $L = T - V$ and if $V = V(q)$

$$L = T(q_\alpha, \dot{q}_\alpha) - V(q_\alpha) \qquad (5)$$

From Eq. (4) and Eq. (5), we find that

$$\frac{\partial L}{\partial \dot{q}_\alpha} = \frac{\partial T}{\partial \dot{q}_\alpha} = p_\alpha . \qquad (6)$$

The Lagrange equation

$$\frac{d}{dt}\left\{\frac{\partial L}{\partial q^{\bullet}_{\alpha}}\right\}-\frac{\partial L}{\partial q_{\alpha}}=0 \qquad (7)$$

Substituting from Eq. (6) and Eq. (7), we get

$$\frac{d}{dt}\left\{\frac{\partial L}{\partial q^{\bullet}_{\alpha}}\right\}-\frac{\partial L}{\partial q_{\alpha}}=0 \rightarrow \frac{d}{dt}\left\{ p_{\alpha} \right\}-\frac{\partial L}{\partial q_{\alpha}}=0 \rightarrow \frac{dp_{\alpha}}{dt}-\frac{\partial L}{\partial q_{\alpha}}=0 \quad \rightarrow p^{\bullet}_{\alpha}=\frac{\partial L}{\partial q_{\alpha}}$$

So, for conservative dynamical system, we have

$$p_{\alpha}=\frac{\partial T}{\partial q^{\bullet}_{\alpha}}, \qquad p_{\alpha}=\frac{\partial L}{\partial q^{\bullet}_{\alpha}}, \qquad p^{\bullet}_{\alpha}=\frac{\partial L}{\partial q_{\alpha}} \qquad (8)$$

Hamilton function (Hamiltonian)

Derive the Hamilton function of a mechanical system. (Dynamical system)?

We want to find a function $H=H(q_{\alpha}, p_{\alpha}, t)$, where $p_{\alpha}=\frac{\partial L}{\partial q^{\bullet}_{\alpha}}$ which we want to think of as depending only on q_{α} and p_{α} but not on $q^{\bullet}_{\alpha}$. This would mean

We apply the Legendre transformation to the Lagrangian, where the Lagrangian function is given as $L=T-V=L(q_{\alpha}, q^{\bullet}_{\alpha}, t)$

Then

$$dL=\sum_{\alpha=1}^{n}\left\{\frac{\partial L}{\partial q_{\alpha}}dq_{\alpha}+\frac{\partial L}{\partial q^{\bullet}_{\alpha}}dq^{\bullet}_{\alpha}\right\}+\frac{\partial L}{\partial t}dt \qquad (1)$$

But

$$p^{\bullet}_{\alpha}=\frac{\partial L}{\partial q_{\alpha}}, \quad p_{\alpha}=\frac{\partial L}{\partial q^{\bullet}_{\alpha}}. \qquad (2)$$

From Eq. (1) and Eq. (2), we have

$$dL=\sum_{\alpha=1}^{n}\left\{p^{\bullet}_{\alpha}\,dq_{\alpha}+p_{\alpha}\,dq^{\bullet}_{\alpha}\right\}+\frac{\partial L}{\partial t}dt \qquad (3)$$

Add and delete the term $q^{\bullet}_{\alpha}dp_{\alpha}$ to Eq. (3), we have

$$dL=\sum_{\alpha=1}^{n}\left\{p^{\bullet}_{\alpha}\,dq_{\alpha}+\underbrace{p_{\alpha}\,dq^{\bullet}_{\alpha}}_{2+3}+\underbrace{q^{\bullet}_{\alpha}dp_{\alpha}}_{2+3}-q^{\bullet}_{\alpha}dp_{\alpha}\right\}+\frac{\partial L}{\partial t}dt=\sum_{\alpha=1}^{n}\left\{p^{\bullet}_{\alpha}\,dq_{\alpha}+d(p_{\alpha}\,q^{\bullet}_{\alpha})-q^{\bullet}_{\alpha}dp_{\alpha}\right\}+\frac{\partial L}{\partial t}dt$$

Then

$$\sum_{\alpha=1}^{n} d(p_\alpha \dot{q}_\alpha) - dL = -\sum_{\alpha=1}^{n}\left\{ \dot{p}_\alpha \, dq_\alpha - \dot{q}_\alpha dp_\alpha \right\} - \frac{\partial L}{\partial t} dt$$

$$d\left\{ \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha - L \right\} = \sum_{\alpha=1}^{n}\left\{ \dot{q}_\alpha dp_\alpha - \dot{p}_\alpha \, dq_\alpha \right\} - \frac{\partial L}{\partial t} dt$$

$$dH = \sum_{\alpha=1}^{n}\left\{ \dot{q}_\alpha dp_\alpha - \dot{p}_\alpha \, dq_\alpha \right\} - \frac{\partial L}{\partial t} dt$$

Now we define the function

$$H = \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha - L \qquad (4)$$

That is called the Hamiltonian function

Where $p_\alpha = \dfrac{\partial L}{\partial \dot{q}_\alpha}$, so we can find $\dot{q}_\alpha = \dot{q}_\alpha(p_\alpha)$.

Then we can write the Hamiltonian function as

$$H = H(q_\alpha, p_\alpha, t) \qquad (5)$$

Hamilton’s equations

It is well-known that, the Lagrange function is

$$L = T - V = L(q_\alpha, \dot{q}_\alpha, t) \qquad (1)$$

While Hamiltonian function is

$$H = \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha - L \qquad (2)$$

Or

$$H = H(q_\alpha, p_\alpha, t) \qquad (3)$$

$$dL = \sum_{\alpha=1}^{n}\left\{ \frac{\partial L}{\partial q_\alpha} dq_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha} d\dot{q}_\alpha \right\} + \frac{\partial L}{\partial t} dt \qquad (4)$$

$$dH = \sum_{\alpha=1}^{n}\left\{ \frac{\partial H}{\partial q_\alpha} dq_\alpha + \frac{\partial H}{\partial p_\alpha} dp_\alpha \right\} + \frac{\partial H}{\partial t} dt \qquad (5)$$

$$dH = \sum_{\alpha=1}^{n} \left\{ \overset{\bullet}{q}_{\alpha}\, dp_{\alpha} + p_{\alpha}\, d\overset{\bullet}{q}_{\alpha} \right\} - dL \qquad (6)$$

Also, we know that

$$\overset{\bullet}{p}_{\alpha} = \frac{\partial L}{\partial q_{\alpha}}, \quad p_{\alpha} = \frac{\partial L}{\partial \overset{\bullet}{q}_{\alpha}}. \qquad (7)$$

Substituting from Eq. (7) into Eq. (4), we have

$$dL = \sum_{\alpha=1}^{n} \left\{ \overset{\bullet}{p}_{\alpha}\, dq_{\alpha} + q_{\alpha}\, d\overset{\bullet}{q}_{\alpha} \right\} + \frac{\partial L}{\partial t} dt \qquad (8)$$

Substituting from Eq. (8) into Eq. (6), we have

$$dH = \sum_{\alpha=1}^{n} \left\{ \overset{\bullet}{q}_{\alpha}\, dp_{\alpha} + \underbrace{p_{\alpha}\, d\overset{\bullet}{q}_{\alpha}}_{2+4=0} \right\} - \sum_{\alpha=1}^{n} \left\{ \overset{\bullet}{p}_{\alpha}\, dq_{\alpha} + \underbrace{p_{\alpha}\, d\overset{\bullet}{q}_{\alpha}}_{2+4=0} \right\} - \frac{\partial L}{\partial t} dt = \sum_{\alpha=1}^{n} \left\{ \overset{\bullet}{q}_{\alpha}\, dp_{\alpha} - \overset{\bullet}{p}_{\alpha}\, dq_{\alpha} \right\} - \frac{\partial L}{\partial t} dt$$

$$dH = \sum_{\alpha=1}^{n} \left\{ \overset{\bullet}{q}_{\alpha}\, dp_{\alpha} - \overset{\bullet}{p}_{\alpha}\, dq_{\alpha} \right\} - \frac{\partial L}{\partial t} dt \qquad (9)$$

So, comparing equation (9) with equation (5), we get

$$\overset{\bullet}{q}_{\alpha} = \frac{\partial H}{\partial p_{\alpha}}, \quad \overset{\bullet}{p}_{\alpha} = -\frac{\partial H}{\partial q_{\alpha}}, \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t} \qquad (10)$$

These equations are called Hamilton's equations

These are Hamilton's equations.

Note that in Lagrange's technique, we have N second order differential equations, while in Hamilton's we have $2N$ first order differential equations. In principle, both are equally easy to solve, and both need $2N$ boundary conditions. In practice, Hamilton's equations are sometimes easier to solve, partly because they naturally embody conservation laws.

Note: If Hamiltonian does not depend on time explicitly, show that Hamiltonian must be constant of motion?

$$H = \sum_{\alpha} H(q_{\alpha}, p_{\alpha})$$

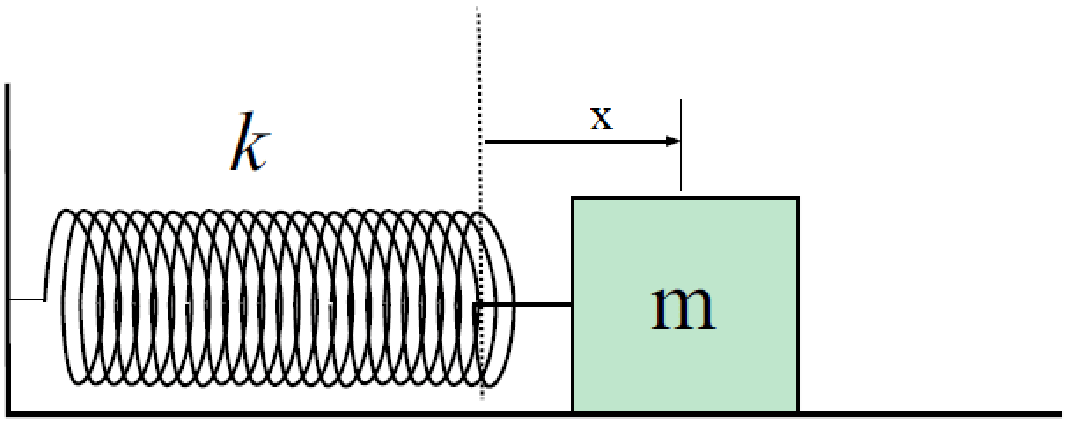
$$\frac{dH}{dt} = \sum_{\alpha} \left(\frac{\partial H}{\partial q_{\alpha}} \frac{dq_{\alpha}}{dt} + \frac{\partial H}{\partial p_{\alpha}} \frac{dp_{\alpha}}{dt} \right) = \sum_{\alpha} \left(\frac{\partial H}{\partial q_{\alpha}} \overset{\bullet}{q}_{\alpha} + \frac{\partial H}{\partial p_{\alpha}} \overset{\bullet}{p}_{\alpha} \right)$$

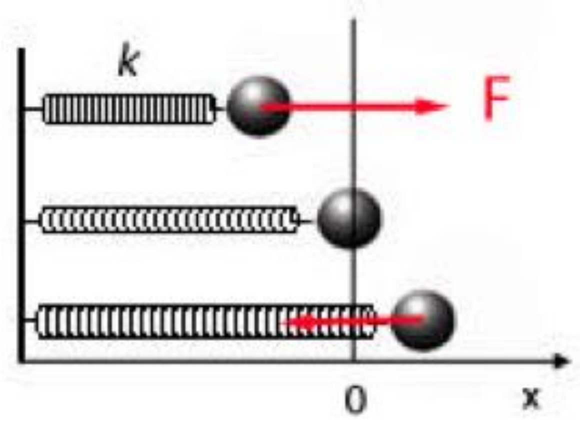
But $\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}, \quad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$

$$\frac{dH}{dt} = \sum_\alpha \left(-\dot{p}_\alpha \dot{q}_\alpha + \dot{q}_\alpha \dot{p}_\alpha\right) = 0$$

$\frac{dH}{dt} = 0$. Then $H = \text{constant}$

Example 1: Determine the motion of harmonic oscillator in one dimension by Hamilton's equations?





We first consider a simple mass spring system. This is a one degree of freedom system, with one x.

The Potential Energy

The force proportional (Inversely proportional) to displacement (x)

$F \,\alpha\, x \;\rightarrow\; F = -k\,x$, but $\quad F = -\nabla V = -k\,x$

$$-\frac{dV}{dx} = -k\,x \;\rightarrow\; dV = k\,x\,dx \;\rightarrow\; \int dV = \int k\,x\,dx$$

Then

$$V = \frac{1}{2}k\,x^2 \qquad (1)$$

<u>Kinetic Energy</u>

$$T = \frac{1}{2}m v^2 = \frac{1}{2}m\,\dot{x}^2 \qquad (2)$$

<u>Lagrange Function</u>

The Lagrange Function of simple harmonic oscillator in one dimension can be written as

$$L = T - V = \frac{1}{2}m\,\dot{x}^2 - \frac{1}{2}k\,x^2 \qquad (3)$$

Using the Hamilton's equations

$$\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}, \qquad \dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}, \qquad (4)$$

where

$$H = \Sigma p_\alpha \dot{q}_\alpha - L, \qquad p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} \qquad (5)$$

From lagrangian function, it clear that the motion in one dimension (i. e. y), so we can write the Hamilton's function

$$H = p_x \dot{q}_x - L = p_x\,\dot{x} - L = p_x\,\dot{x} - \frac{1}{2}m\,\dot{x}^2 + \frac{1}{2}k\,x^2 \qquad (6)$$

From the relation $p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha}$. That can be written for our problem as $p_x = \frac{\partial L}{\partial \dot{x}}$, that becomes as

$$p_x = m\,\dot{x} \rightarrow \dot{x} = \frac{p_x}{m} \qquad (7)$$

From Eq. (7) into Eq. (6), we have

$$H = p_x \dot{x} - \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 = \frac{p_x^2}{m} - \frac{1}{2} \frac{p_x^2}{m} + \frac{1}{2} k x^2 = \frac{1}{2} \frac{p_x^2}{m} + \frac{1}{2} k x^2$$

$$H = \frac{1}{2} \frac{p_x^2}{m} + \frac{1}{2} k x^2 \qquad (8)$$

From Hamilton equation $\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$, and Eq. (8), we have

$$\dot{p}_x = -\frac{\partial H}{\partial q_x} \rightarrow \dot{p}_x = -\frac{\partial H}{\partial x} = -k\,x \rightarrow \dot{p}_x = -\,k\,x \qquad (9)$$

While from Hamilton equation $\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}$ and Eq. (8), we have

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \rightarrow p_x = m\,\dot{x} \qquad (10)$$

From Eq. (10) into Eq. (9), we have

$$\dot{p}_x = m\ddot{x} = -k\,x \rightarrow m\ddot{x} = -k\,x$$

Then $\ddot{x} = -\frac{k}{m}\,x$

This is the same as the equation of motion of the simple harmonic oscillator resulted from application of Newton's second law to a mass attached to spring of spring constant k (Hook constant) and displaced to a position x from equilibrium position.

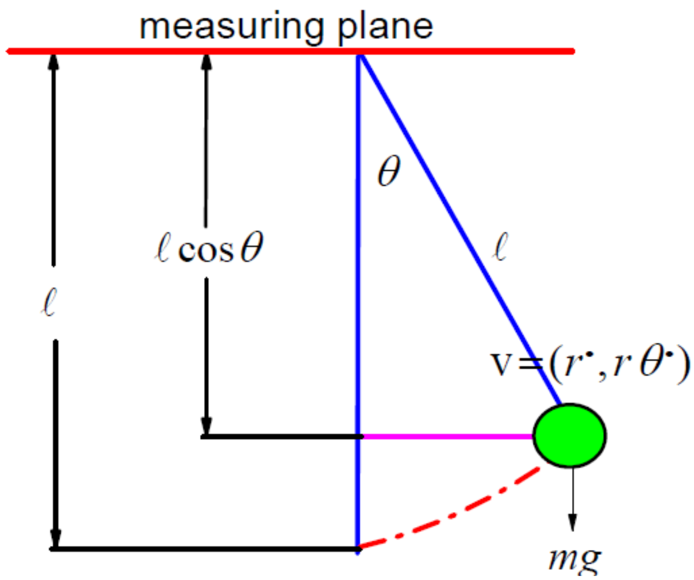
Example: 2 Determine the motion of plane pendulum (simple pendulum) by Hamilton's equations?

Solution

A plane pendulum consists of a bob of mass m suspended from a light inextensible cord of length L and swinging in a plane.

For the motion of plane pendulum, we have

$$v = (\dot{r},\, r\dot{\theta}), \quad \mathrm{r} = L = \text{constant} \rightarrow v = (0,\, L\dot{\theta}) \rightarrow v = L\dot{\theta}$$



Kinetic Energy

$$T = \frac{1}{2}v^2 = \frac{1}{2}\ell^2\,\theta^{\bullet 2} \quad (1)$$

The Potential Energy

$$V = -mg\ell\cos\theta \quad (2)$$

Lagrange Function

The Lagrange Function of simple harmonic oscillator in one dimension can be written as

$$L = T - V = \frac{1}{2}\ell^2\,\theta^{\bullet 2} + mg\ell\cos\theta \quad (3)$$

Using the Hamilton's equations

$$p^{\bullet}_{\alpha} = -\frac{\partial H}{\partial q_{\alpha}}, \qquad q^{\bullet}_{\alpha} = \frac{\partial H}{\partial p_{\alpha}}, \quad (4)$$

where

$$H = \Sigma p_{\alpha} q^{\bullet}_{\alpha} - L, \qquad p_{\alpha} = \frac{\partial L}{\partial q^{\bullet}_{\alpha}} \quad (5)$$

From lagrangian function, it clear that the motion in one dimension (i. e. y), so we can write the Hamilton's function

$$H = p_\theta q_\theta^\bullet - L = p_\theta\,\theta^\bullet - L = p_\theta\,\theta^\bullet - \frac{1}{2}\ell^2\,\theta^{\bullet 2} - mg\ell\,\cos\theta \qquad (6)$$

From the relation $p_\alpha = \frac{\partial L}{\partial q_\alpha^\bullet}$. That can be written for our problem as $p_\theta = \frac{\partial L}{\partial \theta^\bullet}$, that becomes as

$$p_\theta = \ell^2\,\theta^\bullet \rightarrow \theta^\bullet = \frac{p_\theta}{\ell^2} \qquad (7)$$

From Eq. (7) into Eq. (6), we have

$$H = p_\theta\,\theta^\bullet - \frac{1}{2}\ell^2\,\theta^{\bullet 2} - mg\ell\,\cos\theta = p_\theta\frac{p_\theta}{\ell^2} - \frac{1}{2}\ell^2\left(\frac{p_\theta}{\ell^2}\right)^2 - mg\ell\,\cos\theta$$

$$H = \frac{1}{2}\frac{p_\theta^2}{\ell^2} - mg\ell\,\cos\theta \qquad (8)$$

From Hamilton equation $p_\alpha^\bullet = -\frac{\partial H}{\partial q_\alpha}$, and Eq. (8), we have

$$p_\theta^\bullet = -\frac{\partial H}{\partial q_\theta} \rightarrow p_\theta^\bullet = -\frac{\partial H}{\partial \theta} = -mg\ell\,\sin\theta \rightarrow p_\theta^\bullet = -mg\ell\,\sin\theta \qquad (9)$$

While from Hamilton equation $q_\alpha^\bullet = \frac{\partial H}{\partial p_\alpha}$ and Eq. (8), we have

$$\theta^\bullet = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{\ell^2} \rightarrow p_\theta = \ell^2\,\theta^\bullet \qquad (10)$$

From Eq. (10) into Eq. (9), we have

$$p_\theta^\bullet = \ell^2\,\theta^{\bullet\bullet} = -mg\ell\,\sin\theta$$

$$\theta^{\bullet\bullet} = -\frac{mg}{\ell}\sin\theta$$

If θ is very small, then $\sin\theta = \theta$

Then

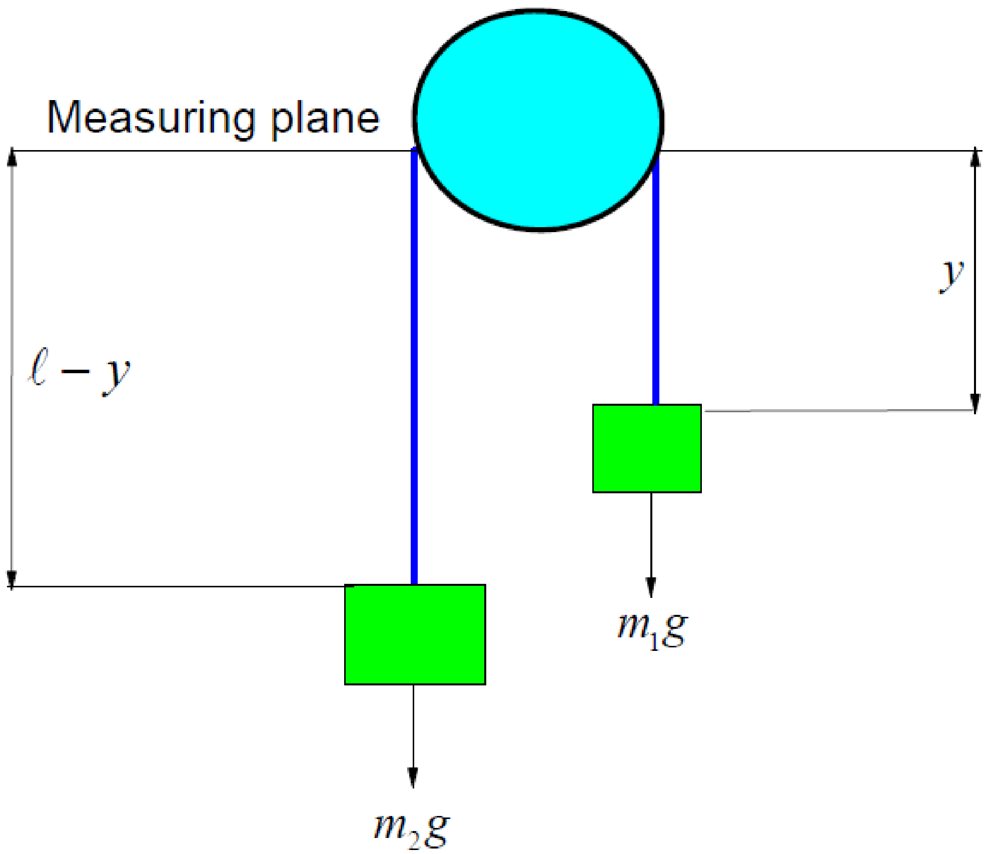
$$\theta^{\bullet\bullet} = -\frac{mg}{\ell}\theta.$$

This the equation for simple pendulum with angular frequency

Example: 3 Two masses m_1 and m_2, are hanging by a massless string from a frictionless pulley. If m_2 is greater than m_1, determine the acceleration of the two masses when released from rest by the Hamilton's equations?

Solution

First, identify a direction as positive. Since you can easily observe that m_2 will accelerate downward and m_1 will accelerate upward, since $m_2 > m_1$, call the direction of motion around the pulley and down toward m_2 the positive $y-$ direction. Then, you can create free body diagrams for both object m_2 and m_1, as shown below.



Mass	The vertical distance from the measuring plane	Velocity	Kinetic Energy	Potential Energy = - Work done
m_1	y	$\dot{y}$	$\frac{1}{2} m_1 \dot{y}^2$	$-m_1 g y$
m_2	$\ell - y$	$-\dot{y}$	$\frac{1}{2} m_2 \dot{y}^2$	$-m_2 g (\ell - y)$

The total of Kinetic energy is given as

$$T = \frac{1}{2}(m_1 + m_2)\,y^{\bullet 2}\,. \tag{1}$$

While the total of potential energy is given as

$$V = -m_1 g\,y - m_2 g\,(\ell - y) = g\,(m_2 - m_1)y - m_2 g\,\ell\ . \tag{2}$$

So, the lagrangian function is

$$L = T - V = \frac{1}{2}(m_1 + m_2)\,y^{\bullet 2} - g\,(m_2 - m_1)\,y + m_2 g\,\ell \tag{3}$$

Using the Hamilton's equations

$$p_\alpha^\bullet = -\frac{\partial H}{\partial q_\alpha}, \qquad q_\alpha^\bullet = \frac{\partial H}{\partial p_\alpha}, \tag{4}$$

where

$$H = \Sigma p_\alpha q_\alpha^\bullet - L, \qquad\qquad p_\alpha = \frac{\partial L}{\partial q_\alpha^\bullet} \tag{5}$$

From lagrangian function, it clear that the motion in one dimension (i. e. y), so we can write the Hamilton's function

$$H = p_y q_y^\bullet - L = p_y\,y^\bullet - L = p_y\,y^\bullet - \frac{1}{2}(m_1 + m_2)\,y^{\bullet 2} + g\,(m_2 - m_1)y - m_2 g\,\ell \tag{6}$$

From the relation $p_\alpha = \frac{\partial L}{\partial q_\alpha^\bullet}$. That can be written for our problem as $p_y = \frac{\partial L}{\partial y^\bullet}$, that becomes as

$$p_y = (m_1 + m_2)\,y^\bullet \rightarrow y^\bullet = \frac{p_y}{m_1 + m_2} \tag{7}$$

From Eq. (7) into Eq. (6), we have

$$H = p_y\,y^\bullet - \frac{1}{2}(m_1 + m_2)\,y^{\bullet 2} + g\,(m_2 - m_1)y - m_2 g\,\ell = \frac{p_y^2}{m_1 + m_2} - \frac{1}{2}(m_1 + m_2)\frac{p_y^2}{(m_1 + m_2)^2} + g\,(m_2 - m_1)y - m_2 g\,\ell$$

$$H = \frac{1}{2}\frac{p_y^2}{m_1 + m_2} + g\,(m_2 - m_1)y - m_2 g\,\ell \tag{8}$$

From Hamilton equation $\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$, and Eq. (8), we have

$$\dot{p}_y = -\frac{\partial H}{\partial q_y} \rightarrow \dot{p}_y = -\frac{\partial H}{\partial y} = -g(m_2 - m_1) \rightarrow \dot{p}_y = -g(m_2 - m_1) \qquad (9)$$

While from Hamilton equation $\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}$ and Eq. (8), we have

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m_1 + m_2} \rightarrow p_y = (m_1 + m_2)\,\dot{y} \qquad (10)$$

From Eq. (10) into Eq. (9), we have

$$\dot{p}_y = (m_1 + m_2)\,\ddot{y} = -(m_2 - m_1)\,g \rightarrow (m_1 + m_2)\,\ddot{y} = (m_1 - m_2)\,g$$

Then $\qquad \ddot{y} = \frac{m_1 - m_2}{m_1 + m_2}\,g$

Example 4: A particle of mass m moving in a plane under the attractive force $\frac{\mu m}{r^2}$ directed to the origin of polar coordinates (r, θ). Using the Hamilton's equations, determine the equation of motion.

Solution

We know that in polar coordinates $\vec{v} = (\dot{r},\ r\dot{\theta})$, so we have

<u>Kinetic Energy</u>

$$T = \frac{1}{2}v^2 = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \qquad (1)$$

<u>The Potential Energy</u>

$$F = -\frac{\mu m}{r^2}, \text{ but } \quad F = -\nabla V = -\frac{\mu m}{r^2}$$

$$\frac{dV}{dr} = \frac{\mu m}{r^2} \rightarrow dV = \frac{\mu m}{r^2}dr \rightarrow \int dV = \int \frac{\mu m}{r^2}dr$$

Then

$$V = -\frac{\mu m}{r} \qquad (2)$$

The Lagrange Function in polar coordinates (r,θ) can be written as

$$L = T - V = \frac{1}{2}m\left(r^{\bullet 2} + r^2\,\theta^{\bullet 2}\right) + \frac{\mu m}{r} \tag{3}$$

Using the Hamilton's equations

$$p_\alpha^\bullet = -\frac{\partial H}{\partial q_\alpha}, \qquad q_\alpha^\bullet = \frac{\partial H}{\partial p_\alpha}, \tag{4}$$

where

$$H = \Sigma p_\alpha q_\alpha^\bullet - L, \qquad p_\alpha = \frac{\partial L}{\partial q_\alpha^\bullet} \tag{5}$$

From lagrangian function, it clear that the motion in one dimension (i. e. (r,θ)), so we can write the Hamilton's function

$$H = p_r q_r^\bullet + p_\theta q_\theta^\bullet - L = p_r\, r^\bullet + p_\theta\, \theta^\bullet - L = p_r\, r^\bullet + p_\theta\, \theta^\bullet - \left(\frac{1}{2}m\left(r^{\bullet 2} + r^2\,\theta^{\bullet 2}\right) + \frac{\mu m}{r}\right) \tag{6}$$

But, $p_\alpha = \frac{\partial L}{\partial q_\alpha^\bullet}$. that, for our problem they can be written as

$p_r = \frac{\partial L}{\partial r^\bullet}$, $p_\theta = \frac{\partial L}{\partial \theta^\bullet}$ and from lagrangian function they can be written as

$$p_r = m r^\bullet \rightarrow r^\bullet = \frac{p_r}{m} \tag{7}$$

$$p_\theta = m r^2 \theta^\bullet \rightarrow \theta^\bullet = \frac{p_\theta}{m r^2} \tag{8}$$

From Eq. (7) and Eq. (8) into Eq. (6), Hamilton's function becomes

$$H = p_r\left(\frac{p_r}{m}\right) + p_\theta\left(\frac{p_\theta}{m r^2}\right) - \frac{1}{2}m\left\{\left(\frac{p_r}{m}\right)^2 + r^2\left(\frac{p_\theta}{m r^2}\right)^2\right\} - \frac{g}{r}$$

$$H = \frac{p_r^2}{m} + \frac{p_\theta^2}{m r^2} - \frac{1}{2}m\left\{\left(\frac{p_r}{m}\right)^2 + r^2\left(\frac{p_\theta}{m r^2}\right)^2\right\} - \frac{g}{r}$$

$$H = \frac{1}{2m}\left\{p_r^2 + \frac{p_\theta^2}{r^2}\right\} - \frac{g}{r} \tag{9}$$

From Hamilton's function (Eq. (9)), Hamilton's equations become

From $p_{\alpha}^{\bullet} = -\frac{\partial H}{\partial q_{\alpha}} \rightarrow \quad p_{r}^{\bullet} = -\frac{\partial H}{\partial r}, \quad p_{\theta}^{\bullet} = -\frac{\partial H}{\partial \theta}$

From $p_{r}^{\bullet} = -\frac{\partial H}{\partial r}$, we get $\quad p_{r}^{\bullet} = -\left[\frac{1}{2m}\left\{-\frac{2p_{\theta}^{2}}{r^{3}}\right\} - \frac{g}{r^{2}}\right]$ (10)

From $p_{\theta}^{\bullet} = -\frac{\partial H}{\partial \theta}$, we get $\quad p_{\theta}^{\bullet} = 0$ (11)

From $q_{\alpha}^{\bullet} = \frac{\partial H}{\partial p_{\alpha}} \rightarrow \quad r^{\bullet} = \frac{\partial H}{\partial p_{r}}, \quad \theta^{\bullet} = \frac{\partial H}{\partial p_{\theta}}.$

From $r^{\bullet} = \frac{\partial H}{\partial p_{r}}$, we get $\quad r^{\bullet} = \frac{\partial H}{\partial p_{r}} = \frac{p_{r}}{m} \rightarrow \quad p_{r} = m r^{\bullet}$ (12)

From $\theta^{\bullet} = \frac{\partial H}{\partial p_{\theta}}$, we get $\quad \theta^{\bullet} = \frac{p_{\theta}}{m r^{2}} \rightarrow \quad p_{\theta} = m r^{2} \theta^{\bullet}$ (13)

Substituting from Eq. (12) inti Eq. (10), we get

$$m r^{\bullet\bullet} = -\frac{1}{2m}\left\{-\frac{2p_{\theta}^{2}}{r^{3}}\right\} - \frac{g}{r^{2}} = -\frac{1}{2m}\left\{-\frac{2\left(\theta^{\bullet} m r^{2}\right)^{2}}{r^{3}}\right\} - \frac{g}{r^{2}} = m r \theta^{\bullet 2} - \frac{g}{r^{2}}$$

Therefore

$$m\left\{r^{\bullet\bullet} - r\theta^{\bullet 2}\right\} = -\frac{g}{r^{2}} \qquad (14)$$

While, if substituting from Eq. (11) intio Eq. (13), we get

$$p_{\theta}^{\bullet} = -\frac{\partial H}{\partial \theta} \rightarrow \frac{d}{dt}(\theta^{\bullet} m r^{2}) = 0$$

Therefore

$$\theta^{\bullet} r^{2} = \text{Constant} \qquad (15)$$

Exercises

(i) Both the kinetic energy T and potential energy V for particle are given as $T=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)$, $V=-mgr\cos\theta$, respectively, where m is the mass particle and g is gravitation. Find the equations of motion for the system using Hamilton's equations?

(ii) Both the kinetic energy T and potential energy V for particle are given as $T=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)$, $V=-\frac{g}{r}$, respectively, where m is the mass particle and g is gravitation . Find the equations of motion for the system using Hamilton's equations?

Routh's Equation

Cyclic Coordinates and the Routhian procedure

Routh's procedure or Routhian mechanics is a hybrid formulation of Lagrangian mechanics and Hamiltonian mechanics developed by Edward John Routh. Correspondingly, the Routhian is the function which replaces both the Lagrangian and Hamiltonian functions. As with the rest of analytical mechanics, Routhian mechanics is completely equivalent to Newtonian mechanics, all other formulations of classical mechanics, and introduces no new physics. It offers an alternative way to solve mechanical problems.

Definitions

The Routhian, like the Hamiltonian, can be obtained from a Legendre transform of the Lagrangian, and has a similar mathematical form to the Hamiltonian, but is not exactly the same. The difference between the Lagrangian, Hamiltonian, and Routhian functions are their variables. For a given set of generalized coordinates representing the degrees of freedom in the system, the Lagrangian is a function of the coordinates and velocities, while the Hamiltonian is a function of the coordinates and momenta.

The Routhian differs from these functions in that some coordinates are chosen to have corresponding generalized velocities, the rest to have corresponding generalized momenta. This choice is arbitrary, and can be done to simplify the problem. It also has the consequence that the Routhian equations are exactly the Hamiltonian equations for some coordinates and corresponding momenta, and the Lagrangian equations for the rest of the coordinates and their velocities. In each case the Lagrangian and Hamiltonian functions are replaced by a single function, the Routhian. The full set thus has the advantages of both sets of equations, with the convenience of splitting one set of coordinates to the Hamilton equations, and the rest to the Lagrangian equations.

Routh's procedure does not guarantee the equations of motion will be simple, however it will lead to fewer equations.

Cyclic coordinates

Often the Routhian approach may offer no advantage, but one notable case where this is useful is when a system has cyclic coordinates (also called "ignorable coordinates"), by definition those coordinates which do not appear in the original Lagrangian. The

Lagrangian equations are powerful results, used frequently in theory and practice, since the equations of motion in the coordinates are easy to set up. However, if cyclic coordinates occur there will still be equations to solve for all the coordinates, including the cyclic coordinates despite their absence in the Lagrangian. The Hamiltonian equations are useful theoretical results, but less useful in practice because coordinates and momenta are related together in the solutions - after solving the equations the coordinates and momenta must be eliminated from each other. Nevertheless, the Hamiltonian equations are perfectly suited to cyclic coordinates because the equations in the cyclic coordinates trivially vanish, leaving only the equations in the non-cyclic coordinates.

The Routhian approach has the best of both approaches, because cyclic coordinates can be split off to the Hamiltonian equations and eliminated, leaving behind the non-cyclic coordinates to be solved from the Lagrangian equations. Overall fewer equations need to be solved compared to the Lagrangian approach.

Routhian Function of mechanical system

Consider a mechanical system with n generalized coordinates, some of these coordinates are cyclic (ignorable) in Lagrange function and the others are non-cyclic.

If we consider the cyclic coordinates are $s_1, s_2, s_3 s_4 ....s_m = s_i\,(i = 1,\ 2,\ 3, ..., \mathrm{m})$, while the non-cyclic coordinates are $q_{m+1}, q_{m+2}, q_{m+3},\ q_{m+4}, ...q_n = q_\alpha\ (\alpha = m+1,\ m+2, ..., n)$. In this case the new Lagrange function will be written in the form $L = L(q_\alpha,\ \dot{q}_\alpha,\ \dot{s}_i)$, and the new Lagrange equations will be written as

Lagrange equations for the noncyclic coordinates:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0$$

Lagrange equations for the cyclic coordinates:

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}_i}\right) - \frac{\partial L}{\partial s_i} = 0$, that can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}_i}\right) - \frac{\partial L}{\partial s_i} = 0 \to \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}_i}\right) = 0 \to \left(\frac{\partial L}{\partial \dot{s}_i}\right) = \text{constant} \to \frac{\partial L}{\partial \dot{s}_i} = \beta_i \qquad (1)$$

From the new $L = L(q_\alpha,\ \dot{q}_\alpha,\ \dot{s}_i)$, we have

$$dL = \sum_{i=1}^{m} \frac{\partial L}{\partial \dot{s}_i} d\dot{s}_i + \sum_{\alpha=m+1}^{n} \left\{ \frac{\partial L}{\partial q_\alpha} dq_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha} d\dot{q}_\alpha \right\} \qquad (2)$$

Subsisting from Eq. (1) into Eq. (2)

$$dL = \sum_{i=1}^{m} \beta_i \, d\dot{s}_i + \sum_{\alpha=m+1}^{n} \left\{ \frac{\partial L}{\partial q_\alpha} dq_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha} d\dot{q}_\alpha \right\}$$

Adding and subtracting the term $\dot{s}_i d\beta_i$, then the previous equation becomes in the form

$$dL = \sum_{i=1}^{m} \left\{ \beta_i \, d\dot{s}_i + \dot{s}_i d\beta_i - \dot{s}_i d\beta_i \right\} + \sum_{\alpha=m+1}^{n} \left\{ \frac{\partial L}{\partial q_\alpha} dq_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha} d\dot{q}_\alpha \right\}$$

$$dL = \sum_{i=1}^{m} \left\{ d\left(\beta_i \, \dot{s}_i\right) - \dot{s}_i d\beta_i \right\} + \sum_{\alpha=m+1}^{n} \left\{ \frac{\partial L}{\partial q_\alpha} dq_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha} d\dot{q}_\alpha \right\}$$

$$dL - \sum_{i=1}^{m} d\left(\beta_i \, \dot{s}_i\right) = -\sum_{i=1}^{m} \dot{s}_i d\beta_i + \sum_{\alpha=m+1}^{n} \left\{ \frac{\partial L}{\partial q_\alpha} dq_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha} d\dot{q}_\alpha \right\}$$

$$d\left\{ L - \sum_{i=1}^{m} \left(\beta_i \, \dot{s}_i\right) \right\} = -\sum_{i=1}^{m} \dot{s}_i d\beta_i + \sum_{\alpha=m+1}^{n} \left\{ \frac{\partial L}{\partial q_\alpha} dq_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha} d\dot{q}_\alpha \right\}$$

Let $R = L - \sum_{i=1}^{m} \beta_i \, \dot{s}_i$, that is called the Routhian Function

Then

$$d\,R = -\sum_{i=1}^{m} \dot{s}_i d\beta_i + \sum_{\alpha=m+1}^{n} \left\{ \frac{\partial L}{\partial q_\alpha} dq_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha} d\dot{q}_\alpha \right\}$$

Routh's equations of motion for a mechanical system

It well-known that Routhian Function is given from

$$dR = -\sum_{i=1}^{m} \dot{s}_i d\beta_i + \sum_{\alpha=m+1}^{n} \left\{ \frac{\partial L}{\partial q_\alpha} dq_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha} d\dot{q}_\alpha \right\} \qquad (1)$$

Where $R = L - \sum_{i=1}^{m} \beta_i \, \dot{s}_i$, $R = R(q_\alpha, \, \dot{q}_\alpha, \, \dot{s}_i, \beta_i)$, and $\frac{\partial L}{\partial \dot{s}_i} = \beta_i$. Then $R = R(q_\alpha, \, \dot{q}_\alpha, \beta_i)$

$$dR = \sum_{i=1}^{m} \frac{\partial R}{\partial \beta_i} d\beta_i + \sum_{\alpha=m+1}^{n} \left\{ \frac{\partial R}{\partial q_\alpha} dq_\alpha + \frac{\partial R}{\partial \dot{q}_\alpha} d\dot{q}_\alpha \right\} \qquad (2)$$

By Comparing the coefficients between Eq. (1) and Eq. (2), we get

$$\frac{\partial R}{\partial \beta_i} = -s_i^\bullet, \qquad \frac{\partial R}{\partial q_\alpha} = \frac{\partial L}{\partial q_\alpha}, \qquad \frac{\partial R}{\partial q_\alpha^\bullet} = \frac{\partial L}{\partial q_\alpha^\bullet} \tag{3}$$

Taking into account, $\frac{\partial L}{\partial s_i^\bullet} = \beta_i$

If the time appears explicitly as a Routhian Function, which becomes in the form, $R = R(q_\alpha,\, q_\alpha^\bullet, \beta_i, t)$, then Routh's equations (3) take the form

$$\frac{\partial R}{\partial \beta_i} = -s_i^\bullet, \qquad \frac{\partial R}{\partial q_\alpha} = \frac{\partial L}{\partial q_\alpha}, \qquad \frac{\partial R}{\partial q_\alpha^\bullet} = \frac{\partial L}{\partial q_\alpha^\bullet}, \qquad \frac{\partial R}{\partial t} = \frac{\partial L}{\partial t} \tag{4}$$

Taking into account, $\frac{\partial L}{\partial s_i^\bullet} = \beta_i$

Example 1: Both the kinetic energy T and potential energy V for mechanical system are given, respectively, in the form $T = \frac{1}{2} m (r^{\bullet 2} + r^2 \theta^{\bullet 2}), \qquad V = -\frac{g}{r}$. Find the equations of motion for the system using Routh's equations?

Solution

Where $T = \frac{1}{2} m (r^{\bullet 2} + r^2 \theta^{\bullet 2}), \qquad V = -\frac{g}{r}$. Then the Lagrange function becomes

$$L = T - V = \frac{1}{2} m (r^{\bullet 2} + r^2 \theta^{\bullet 2}) + \frac{g}{r} \tag{1}$$

From Lagrange Function, it clears that θ the is cyclic coordinate (coordinate number 2), while r is the non-cyclic coordinate

Now, from Routh's equations

$$\frac{d}{dt}\left\{\frac{\partial R}{\partial q_\alpha^\bullet}\right\} - \frac{\partial R}{\partial q_\alpha} = 0, \qquad \frac{\partial R}{\partial \beta_i} = -s_i^\bullet, \qquad R = L - \Sigma \beta_i\, s_i^\bullet, \text{ with } \frac{\partial L}{\partial s_i^\bullet} = \beta_i .$$

We can write Routhian's Function $R = L - \Sigma \beta_i\, s_i^\bullet$ as

$$R = L - \Sigma \beta_i\, s_i^\bullet = L - \beta_2\, s_2^\bullet = L - \beta_2\, \theta^\bullet = \frac{1}{2} m (\dot{r}^2 + r^2\, \dot{\theta}^2) + \frac{g}{r} - \beta_2\, \theta^\bullet \tag{2}$$

From $\frac{\partial L}{\partial \theta^\bullet} = \beta_2$, we find that $\frac{\partial}{\partial \theta^\bullet}\left\{\frac{1}{2} m (r^{\bullet 2} + r^2 \theta^{\bullet 2}) + \frac{g}{r}\right\} = \beta_2$. Then

$$m\,r^2\,\dot{\theta} = \beta_2 \quad Or \quad \dot{\theta} = \frac{\beta_2}{m\,r^2} \qquad (3)$$

From Eq. (3) into Eq. (2), we have

$$R = \frac{1}{2}m\left(\dot{r}^2 + r^2\left(\frac{\beta_2}{m\,r^2}\right)^2\right) + \frac{g}{r} - \beta_2\frac{\beta_2}{m\,r^2} = \frac{1}{2}m\,\dot{r}^2 - \frac{\beta_2^2}{2m\,r^2} + \frac{g}{r} \qquad (4)$$

Now, Routh's equation $\frac{d}{dt}\left\{\frac{\partial R}{\partial \dot{q}_\alpha}\right\} - \frac{\partial R}{\partial q_\alpha} = 0, \qquad \frac{\partial R}{\partial \beta_i} = -\dot{s}_i.$ maybe written as

$$\frac{\partial R}{\partial \beta_i} = -\dot{s}_i \;\rightarrow\; \frac{\partial R}{\partial \beta_2} = -\dot{\theta} \;\rightarrow\; \frac{\partial}{\partial \beta_2}\left\{\frac{1}{2}m\,\dot{r}^2 - \frac{\beta_2^2}{2m\,r^2} + \frac{g}{r}\right\} = -\dot{\theta}$$. Then $-\frac{\beta_2}{m\,r^2} = -\dot{\theta}$

$$\dot{\theta}\,r^2 = \frac{\beta_2}{m} = \text{Constant} \qquad (5)$$

For the non-cyclic coordinate (r), Routh's equation takes the form

$$\frac{d}{dt}\left\{\frac{\partial R}{\partial \dot{q}_\alpha}\right\} - \frac{\partial R}{\partial q_\alpha} = 0 \;\rightarrow\; \frac{d}{dt}\left\{\frac{\partial R}{\partial \dot{r}}\right\} - \frac{\partial R}{\partial r} = 0$$

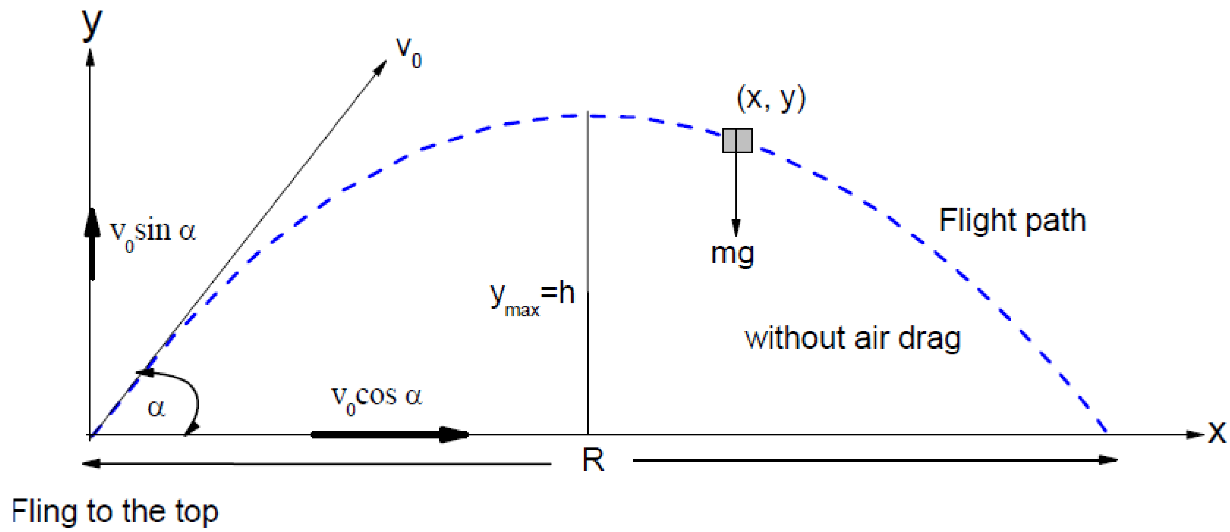
$$\frac{d}{dt}\left\{m\,\dot{r}\right\} - \frac{\partial}{\partial r}\left\{\frac{\beta_2^2}{m\,r^3} - \frac{g}{r^2}\right\} = 0 \;\rightarrow\; m\,\ddot{r} - \left\{\frac{\left(m\,r^2\,\dot{\theta}\right)^2}{m\,r^3} - \frac{g}{r^2}\right\} = 0$$, then

$$m\left\{\ddot{r} - r\,\dot{\theta}^2\right\} = -\frac{g}{r^2} \qquad (6)$$

Example 2: A particle of mass m is projected with initial velocity v_0 at an angle α to the horizontal in the uniform gravitational field of the earth. Use Routh’s equation to describe the motion of the projectile. Ignore the air resistance?

Solution

Let a particle of mass m be projected from the origin point with an initial velocity v_0 making an angle α with the horizontal line referred as x-axis. Let (x, y) be the position of the particle at any instant t. Since x and y are independent and hence the generalized coordinates are $(q_1, q_2) = (x, y)$ and the generalized velocities are $(\dot{q}_1, \dot{q}_2) = (\dot{x}, \dot{y})$.



Kinetic Energy

The kinetic of the projectile is given by $T = \frac{1}{2} m(x^{\bullet 2} + y^{\bullet 2})$,

The Potential Energy

The total of potential energy is given as

$$F = -mg \;\rightarrow\; F = -mg = -\nabla V \;\rightarrow\; mg = \frac{dV}{dy} \;\rightarrow\; V = mg\, y$$

Lagrange Function

The Lagrange Function can be written as

$$L = T - V = \frac{1}{2} m(x^{\bullet 2} + y^{\bullet 2}) - mg\, y \qquad (1)$$

From Lagrange Function, it clear that, x is cyclic coordinate (the coordinate number 1), while y is non-cyclic coordinate.

Now, from Routh's equations

$$\frac{d}{dt}\left\{\frac{\partial R}{\partial q_\alpha^\bullet}\right\} - \frac{\partial R}{\partial q_\alpha} = 0, \qquad \frac{\partial R}{\partial \beta_i} = -s_i^\bullet, \qquad R = L - \Sigma \beta_i\, s_i^\bullet, \qquad \frac{\partial L}{\partial s_i^\bullet} = \beta_i. \quad (2)$$

We can write $R = L - \Sigma \beta_i\, s_i^\bullet$ as

$$R = L - \Sigma \beta_i\, s_i^\bullet = L - \beta_1\, s_1^\bullet = L - \beta_1\, x^\cdot = \frac{1}{2} m(x^{\cdot 2} + y^{\cdot 2}) - mg\, y - \beta_1\, x^\cdot \qquad (3)$$

But from $\frac{\partial L}{\partial s_i^{\bullet}} = \beta_i$, we find that $\frac{\partial L}{\partial x^{\bullet}} = \beta_1$, that can be written as

$$\frac{\partial}{\partial x^{\bullet}}\left\{\frac{1}{2}m(x^{\bullet 2}+y^{\bullet 2})-mg\,y\right\}=\beta_1$$

$$m\,x^{\bullet}=\beta_1 \quad Or \quad x^{\bullet}=\frac{\beta_1}{m} \tag{4}$$

From Eq. (4) into Eq. (3), we can write Routhian's Function as

$$R=\frac{1}{2}m\left((\frac{\beta_1}{m})^2+y^{\bullet 2}\right)-mg\,y-\beta_1\frac{\beta_1}{m} \;\rightarrow\; R=\frac{1}{2}m\,y^{\bullet 2}-\frac{\beta_1^2}{2m}-mg\,y \tag{5}$$

For the non-cyclic coordinate (r), Routh's equation is

$$\frac{d}{dt}\left\{\frac{\partial R}{\partial q_\alpha^{\bullet}}\right\}-\frac{\partial R}{\partial q_\alpha}=0, \qquad \frac{\partial R}{\partial \beta_i}=-s_i^{\bullet}.$$

Where

$$\frac{\partial R}{\partial \beta_1}=-x^{\bullet} \;\rightarrow\; \frac{\partial}{\partial \beta_1}\left\{\frac{1}{2}m\,y^{\bullet 2}-\frac{\beta_1^2}{2m}-mg\,y\right\}=-x^{\bullet}$$

$$x^{\bullet}=\frac{\beta_1}{m}==\text{Constant}=c_1 \tag{6}$$

For the non-cyclic coordinate (r), Routh's equation $\frac{d}{dt}\left\{\frac{\partial R}{\partial q_\alpha^{\bullet}}\right\}-\frac{\partial R}{\partial q_\alpha}=0$ takes the form

$$\frac{d}{dt}\left\{\frac{\partial R}{\partial y^{\bullet}}\right\}-\frac{\partial R}{\partial y}=0$$

$$\frac{d}{dt}\left\{\frac{\partial}{\partial y^{\bullet}}\left(\frac{1}{2}m\,y^{\bullet 2}-\frac{\beta_1^2}{2m}-mg\,y\right)\right\}-\frac{\partial}{\partial y}\left\{\left(\frac{1}{2}m\,y^{\bullet 2}-\frac{\beta_1^2}{2m}-mg\,y\right)\right\}=0$$

$$\frac{d}{dt}\left\{m\,y^{\bullet}\right\}+mg=0$$

$$y^{\bullet\bullet}=-g$$

Exercises

(i) Both the kinetic energy T and potential energy V for Mechanical system are given, respectively, in the form $T = \frac{1}{2} m\left(\dot{r}^2 + r^2 \dot{\theta}^2 \right), \qquad V = -m g r \cos\theta$. Find the equations of motion for the system using Routh's equations?

(ii) Both the kinetic energy T and potential energy V for Mechanical system are given, respectively, in the form $T = \frac{1}{2}(\dot{x}^2 + x^2 \dot{y}^2), \qquad V = \frac{\omega^2}{2} x^2$. Find the equations of motion for the system using Routh's equations?

(iii) A particle of mass m moving in a plane under the attractive force $\frac{\mu m}{r^2}$ directed to the origin of polar coordinates (r, θ). Using the Routh's equations, determine the equation of motion?

Chapter: 5

Poisson Brackets (Poisson's Equation)

In this chapter we discuss an important relation usually called "Poisson's equation." Poisson's name has been given to several equations in mechanics and the study of electricity and magnetism, so we sometimes call the resulting equation "Poisson's equation of motion," to distinguish it from other equations with the name Poisson. We consider a function $f = f(q_\alpha, p_\alpha, t)$ and a Hamiltonian $H(q_\alpha, p_\alpha)$, where p_α, q_α coordinates and momentum. Using the chain rule, we now give an expression for the implicit time derivative of $f = f(q_\alpha, p_\alpha, t)$.

$$\frac{df}{dt} = \sum_{\alpha=1}^{n}\left\{\frac{\partial f}{\partial q_\alpha}\frac{dq_\alpha}{dt} + \frac{\partial f}{\partial p_\alpha}\frac{dp_\alpha}{dt}\right\} + \frac{\partial f}{\partial t}\frac{dt}{dt}$$

$$\frac{df}{dt} = \sum_{\alpha=1}^{n}\left\{\frac{\partial f}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial f}{\partial p_\alpha}\dot{p}_\alpha\right\} + \frac{\partial f}{\partial t}\frac{dt}{dt} \qquad (1)$$

Using Hamilton's equations

$$\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}, \qquad \dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \qquad (2)$$

We can be written Eq. (1) as (substituting from Eq. (2) into Eq. (1))

$$\frac{df}{dt} = \sum_{\alpha=1}^{n}\left\{\frac{\partial f}{\partial q_\alpha}\frac{\partial H}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial H}{\partial q_\alpha}\right\} + \frac{\partial f}{\partial t} \qquad (3)$$

This equation is called the Poisson equation, while the expression

$\left\{\frac{\partial f}{\partial q_\alpha}\frac{\partial H}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial H}{\partial q_\alpha}\right\}$ is called Poisson bracket, that defines as

$$\left\{f, H\right\} = \left\{\frac{\partial f}{\partial q_\alpha}\frac{\partial H}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial H}{\partial q_\alpha}\right\}. \qquad (4)$$

In fact, the Poisson bracket can be defined for any two functions (f, g) defined in phase

$$\left\{f, g\right\} = \sum_{\alpha=1}^{n}\left\{\frac{\partial f}{\partial q_\alpha}\frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial g}{\partial q_\alpha}\right\} \qquad (5)$$

$\alpha = 1,2,3,4,\ldots\ldots\ldots\ldots\ldots\ldots\ldots, n$

Properties of Poisson's brackets

For any three physical quantities f, g, h, which are function of generalized coordinates (q_α) and generalized momentum (p_α), and assuming c is a constant magnitude. Poisson's brackets have the following properties:

(1) $\{f, f\} = 0,$

(2) $\{f, c\} = 0,$

(3) $\{f, g\} = -\{g, f\},$

(4) $\{f + g, h\} = \{f, h\} + \{g, h\},$

(5) $\{f\, g, h\} = f\{g, h\} + g\{f, h\},$

(6) $\frac{\partial}{\partial t}\{f, g\} = \left\{\frac{\partial f}{\partial t}, g\right\} + \left\{f, \frac{\partial g}{\partial t}\right\},$

(7) $\{q_\alpha, f\} = \frac{\partial f}{\partial p_\alpha}, \left[\{f, q_\alpha\} = -\frac{\partial f}{\partial p_\alpha}\right],$

(8) $\{p_\alpha, f\} = -\frac{\partial f}{\partial q_\alpha}\left[\{f, p_\alpha\} = \frac{\partial f}{\partial q_\alpha}\right],$

(9) $\{q_\ell, q_k\} = 0,$

(10) $\{p_\ell, p_k\} = 0,$

(11) $\{q_\ell, p_k\} = \begin{cases} 1, & \ell = k \\ 0, & \ell \neq k \end{cases}$

Proof

From the definition of Poisson brackets

$$\{f(q_\alpha, p_\alpha, t), g(q_\alpha, p_\alpha, t)\} = \left\{\frac{\partial f}{\partial q_\alpha}\frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial g}{\partial q_\alpha}\right\}.$$

We will try to prove the previous properties

(1) $\{f, f\} = \left\{\frac{\partial f}{\partial q_\alpha}\frac{\partial f}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial f}{\partial q_\alpha}\right\} = \left\{\frac{\partial f}{\partial q_\alpha}\frac{\partial f}{\partial p_\alpha} - \frac{\partial f}{\partial q_\alpha}\frac{\partial f}{\partial p_\alpha}\right\} = 0$

(2) $\{f, c\} = \left\{\frac{\partial f}{\partial q_\alpha}\frac{\partial c}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial c}{\partial q_\alpha}\right\} = \left\{\frac{\partial f}{\partial q_\alpha}(0) - \frac{\partial f}{\partial p_\alpha}(0)\right\} = 0$

(3) $\{f, g\} = \left\{\frac{\partial f}{\partial q_\alpha}\frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial g}{\partial q_\alpha}\right\} = -\left\{\frac{\partial f}{\partial p_\alpha}\frac{\partial g}{\partial q_\alpha} - \frac{\partial f}{\partial q_\alpha}\frac{\partial g}{\partial p_\alpha}\right\} = -\left\{\frac{\partial g}{\partial q_\alpha}\frac{\partial f}{\partial p_\alpha} - \frac{\partial g}{\partial p_\alpha}\frac{\partial f}{\partial q_\alpha}\right\} = -\{g, f\}$

$\{f, g\}$ and $\{g, f\}$ are antisymmetric

$$(4)\ \left\{ f+g,h \right\} = \left\{ \frac{\partial\left(f+g\right)}{\partial q_\alpha}\frac{\partial h}{\partial p_\alpha} - \frac{\partial\left(f+g\right)}{\partial p_\alpha}\frac{\partial h}{\partial q_\alpha} \right\} = \left\{ \left[\frac{\partial\left(f\right)}{\partial q_\alpha} + \frac{\partial\left(g\right)}{\partial q_\alpha} \right]\frac{\partial h}{\partial p_\alpha} - \left[\frac{\partial\left(f\right)}{\partial p_\alpha} + \frac{\partial\left(g\right)}{\partial p_\alpha} \right]\frac{\partial h}{\partial q_\alpha} \right\}$$

$$= \left\{ \left[\frac{\partial f}{\partial q_\alpha}\frac{\partial h}{\partial p_\alpha} + \frac{\partial g}{\partial q_\alpha}\frac{\partial h}{\partial p_\alpha} \right] - \left[\frac{\partial f}{\partial p_\alpha}\frac{\partial h}{\partial q_\alpha} + \frac{\partial g}{\partial p_\alpha}\frac{\partial h}{\partial q_\alpha} \right] \right\} = \left\{ \left[\frac{\partial f}{\partial q_\alpha}\frac{\partial h}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial h}{\partial q_\alpha} \right] + \left[\frac{\partial g}{\partial q_\alpha}\frac{\partial h}{\partial p_\alpha} - \frac{\partial g}{\partial p_\alpha}\frac{\partial h}{\partial q_\alpha} \right] \right\}$$

$$= \left\{ \frac{\partial f}{\partial q_\alpha}\frac{\partial h}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial h}{\partial q_\alpha} \right\} + \left\{ \frac{\partial g}{\partial q_\alpha}\frac{\partial h}{\partial p_\alpha} - \frac{\partial g}{\partial p_\alpha}\frac{\partial h}{\partial q_\alpha} \right\} = \left\{ f,h \right\} + \left\{ g,h \right\}$$

$$(5)\ \left\{ f\,g,h \right\} = \left\{ \frac{\partial\left(f\,g\right)}{\partial q_\alpha}\frac{\partial h}{\partial p_\alpha} - \frac{\partial\left(f\,g\right)}{\partial p_\alpha}\frac{\partial h}{\partial q_\alpha} \right\} = \left\{ \left[\frac{\partial f}{\partial q_\alpha}g + f\frac{\partial g}{\partial q_\alpha} \right]\frac{\partial h}{\partial p_\alpha} - \left[\frac{\partial f}{\partial p_\alpha}g + f\frac{\partial g}{\partial p_\alpha} \right]\frac{\partial h}{\partial q_\alpha} \right\}$$

$$= \left\{ f\left[\frac{\partial g}{\partial q_\alpha}\frac{\partial h}{\partial p_\alpha} - \frac{\partial g}{\partial p_\alpha}\frac{\partial h}{\partial q_\alpha} \right] + g\left[\frac{\partial f}{\partial q_\alpha}\frac{\partial h}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial h}{\partial q_\alpha} \right] \right\}$$

$$\left\{ f\,g,h \right\} = f\left\{ \frac{\partial g}{\partial q_\alpha}\frac{\partial h}{\partial p_\alpha} - \frac{\partial g}{\partial p_\alpha}\frac{\partial h}{\partial q_\alpha} \right\} + g\left\{ \frac{\partial f}{\partial q_\alpha}\frac{\partial h}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial h}{\partial q_\alpha} \right\} = f\left\{ g,h \right\} + g\left\{ f,h \right\}$$ Product rule.

$$(6)\ \frac{\partial}{\partial t}\left\{ f,g \right\} = \frac{\partial}{\partial t}\left\{ \frac{\partial f}{\partial q_\alpha}\frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial g}{\partial q_\alpha} \right\} = \left\{ \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial q_\alpha} \right)\frac{\partial g}{\partial p_\alpha} + \frac{\partial f}{\partial q_\alpha}\frac{\partial}{\partial t}\left(\frac{\partial g}{\partial p_\alpha} \right) - \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial p_\alpha} \right)\frac{\partial g}{\partial q_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial}{\partial t}\left(\frac{\partial g}{\partial q_\alpha} \right) \right\}$$

$$= \left\{ \frac{\partial(\frac{\partial f}{\partial t})}{\partial q_\alpha}\frac{\partial g}{\partial p_\alpha} + \frac{\partial f}{\partial q_\alpha}\frac{\partial(\frac{\partial g}{\partial t})}{\partial p_\alpha} - \frac{\partial(\frac{\partial f}{\partial t})}{\partial p_\alpha}\frac{\partial g}{\partial q_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial(\frac{\partial g}{\partial t})}{\partial q_\alpha} \right\} = \left\{ \frac{\partial(\frac{\partial f}{\partial t})}{\partial q_\alpha}\frac{\partial g}{\partial p_\alpha} - \frac{\partial(\frac{\partial f}{\partial t})}{\partial p_\alpha}\frac{\partial g}{\partial q_\alpha} + \frac{\partial f}{\partial q_\alpha}\frac{\partial(\frac{\partial g}{\partial t})}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial(\frac{\partial g}{\partial t})}{\partial q_\alpha} \right\}$$

$$\frac{\partial}{\partial t}\left\{ f,g \right\} = \left\{ \frac{\partial(\frac{\partial f}{\partial t})}{\partial q_\alpha}\frac{\partial g}{\partial p_\alpha} - \frac{\partial(\frac{\partial f}{\partial t})}{\partial p_\alpha}\frac{\partial g}{\partial q_\alpha} \right\} + \left\{ \frac{\partial f}{\partial q_\alpha}\frac{\partial(\frac{\partial g}{\partial t})}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial(\frac{\partial g}{\partial t})}{\partial q_\alpha} \right\} = \left\{ \frac{\partial f}{\partial t},g \right\} + \left\{ f,\frac{\partial g}{\partial t} \right\}$$

$$\frac{\partial}{\partial t}\left\{ f,g \right\} = \left\{ \frac{\partial f}{\partial t},g \right\} + \left\{ f,\frac{\partial g}{\partial t} \right\}$$

$$(7)\ \left\{ q_\alpha, f \right\} = \left\{ \frac{\partial q_\alpha}{\partial q_\alpha}\frac{\partial f}{\partial p_\alpha} - \frac{\partial q_\alpha}{\partial p_\alpha}\frac{\partial f}{\partial q_\alpha} \right\} = \left\{ (1)\frac{\partial f}{\partial p_\alpha} - (0)\frac{\partial f}{\partial q_\alpha} \right\} = \frac{\partial f}{\partial p_\alpha}$$

$$(8)\ \left\{ p_\alpha, f \right\} = \left\{ \frac{\partial p_\alpha}{\partial q_\alpha}\frac{\partial f}{\partial p_\alpha} - \frac{\partial p_\alpha}{\partial p_\alpha}\frac{\partial f}{\partial q_\alpha} \right\} = \left\{ (0)\frac{\partial f}{\partial p_\alpha} - (1)\frac{\partial f}{\partial q_\alpha} \right\} = -\frac{\partial f}{\partial q_\alpha}$$

(9) $\left\{q_\ell,\, q_k\right\} = \sum_{\alpha=1}^{n}\left\{\frac{\partial q_\ell}{\partial q_\alpha}\frac{\partial q_k}{\partial p_\alpha} - \frac{\partial q_\ell}{\partial p_\alpha}\frac{\partial q_k}{\partial q_\alpha}\right\} = \left\{(\frac{\partial q_\ell}{\partial q_\alpha})(0) - (0)(\frac{\partial q_k}{\partial q_\alpha})\right\} = 0$

(10) $\left\{p_\ell,\, p_k\right\} = \sum_{\alpha=1}^{n}\left\{\frac{\partial p_\ell}{\partial q_\alpha}\frac{\partial p_k}{\partial p_\alpha} - \frac{\partial p_\ell}{\partial p_\alpha}\frac{\partial p_k}{\partial q_\alpha}\right\} = \left\{(0)(\frac{\partial p_k}{\partial p_\alpha}) - (\frac{\partial p_\ell}{\partial p_\alpha})(0)\right\} = 0,$

(11) $\left\{q_\ell,\, p_k\right\} = \sum_{\alpha=1}^{n}\left\{\frac{\partial q_\ell}{\partial q_\alpha}\frac{\partial p_k}{\partial p_\alpha} - \frac{\partial q_\ell}{\partial p_\alpha}\frac{\partial p_k}{\partial q_\alpha}\right\} = \sum_{\alpha=1}^{n}\left\{\frac{\partial q_\ell}{\partial q_\alpha}\frac{\partial p_k}{\partial p_\alpha} - (\frac{\partial q_\ell}{\partial p_\alpha})\,0\right\} = \sum_{\alpha=1}^{n}\frac{\partial q_\ell}{\partial q_\alpha}\frac{\partial p_k}{\partial p_\alpha} = \begin{cases}1, & \ell = k\\ 0, & \ell \neq k\end{cases}$

Then $\left\{q_\ell,\, p_k\right\} = \begin{cases}1, & \ell = k\\ 0, & \ell \neq k\end{cases}$

Remarks

$$\sum_{\alpha=1}^{n}\frac{\partial q_\ell}{\partial q_\alpha}\frac{\partial p_k}{\partial p_\alpha} = \frac{\partial q_\ell}{\partial q_1}\frac{\partial p_k}{\partial p_1} + \frac{\partial q_\ell}{\partial q_2}\frac{\partial p_k}{\partial p_2} + \frac{\partial q_\ell}{\partial q_3}\frac{\partial p_k}{\partial p_3} + \frac{\partial q_\ell}{\partial q_4}\frac{\partial p_k}{\partial p_4} + \frac{\partial q_\ell}{\partial q_4}\frac{\partial p_k}{\partial p_4} + \ldots\ldots\ldots\ldots$$

If $\ell = k$. Then

$\ell = k = 1$ or $\ell = k = 2$ or $\ell = k = 3$ or $\ell = k = 4$ or...........

Therefore

$\frac{\partial q_1}{\partial q_1}\frac{\partial p_1}{\partial p_1} = 1$ or $\frac{\partial q_2}{\partial q_2}\frac{\partial p_2}{\partial p_2} = 1$ or $\frac{\partial q_3}{\partial q_3}\frac{\partial p_3}{\partial p_3} = 1$ or $\frac{\partial q_4}{\partial q_4}\frac{\partial p_4}{\partial p_4} = 1$ or $\frac{\partial q_5}{\partial q_5}\frac{\partial p_5}{\partial p_5} = 1$.....

While $\frac{\partial q_1}{\partial q_1}\frac{\partial p_2}{\partial p_1} = 0$.

Generally, we can put $\left\{q_\ell,\, p_k\right\} = \begin{cases}1, & l = k\\ 0, & l \neq k\end{cases}$

Example: Consider a dynamical system (point particle) with position vector $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, momentum vector is $\vec{p} = p_x\vec{i} + p_y\vec{j} + p_z\vec{k}$, and $\vec{M} = \vec{r}\times\vec{p}$ (angular momentum).

Evaluate the Poisson brackets:

(i) $\left\{\vec{r},\, \vec{r}\right\}$, (ii) $\left\{\vec{p},\, \vec{p}\right\}$, (iii) $\left\{\vec{r},\, \vec{p}\right\}$.

(iv) $\left\{\vec{r},\, \vec{M}\right\}$, (v) $\left\{\vec{p},\, \vec{M}\right\}$, (vi) $\left\{\vec{M}, \vec{M}\right\}$.

Answer

The position vector is $\vec{r} = (x, y, z)$, the momentum vector is $\vec{p} = \left(p_x, p_y, p_z\right)$

From the definition of Poisson brackets

$$\left\{f_\alpha, g_\alpha\right\} = \frac{\partial f}{\partial q_\alpha}\frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial g}{\partial q_\alpha}$$

(i) From $\left\{q_l, q_k\right\} = 0$, we get $\left\{\vec{r}, \vec{r}\right\} = 0$

(ii) From $\left\{\vec{p}_l, \vec{p}_k\right\} = 0$, we get $\left\{\vec{p}, \vec{p}\right\} = 0$.

(iii) To find $\left\{\vec{r}, \vec{p}\right\}$, we use the formula $\left\{q_l, p_k\right\} = \begin{cases} 1, & l = k \\ 0, & l \neq k \end{cases}$

We get

$$\left\{x, p_x\right\} = 1, \quad \left\{x, p_y\right\} = 0, \quad \left\{x, p_z\right\} = 0,$$

$$\left\{y, p_x\right\} = 0, \quad \left\{y, p_y\right\} = 1, \quad \left\{y, p_z\right\} = 0,$$

$$\left\{z, p_x\right\} = 0, \quad \left\{z, p_y\right\} = 0, \quad \left\{z, p_z\right\} = 1.$$

Therefore, we can configure the following table

Function	p_x	p_y	p_z
x	1	0	0
y	0	1	0
z	0	0	1

(iv) $\vec{M} = \vec{r} \times \vec{p}$, we find

$$\vec{M} = \vec{r} \times \vec{p} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = \left\{ y\, p_z - z\, p_y \right\} \vec{i} - \left\{ x\, p_z - z\, p_x \right\} \vec{j} + \left\{ x\, p_y - y\, p_x \right\} \vec{k}$$

$$\vec{M} = \vec{r} \times \vec{p} = \left\{ y\, p_z - z\, p_y \right\} \vec{i} + \left\{ z\, p_x - x\, p_z \right\} \vec{j} + \left\{ x\, p_y - y\, p_x \right\} \vec{k} = M_x \vec{i} + M_y \vec{j} + M_z \vec{k}$$

To find the relation $\left\{ \vec{r},\, \vec{M} \right\}$, we use the relation $\left\{ q_\alpha,\, f \right\} = \frac{\partial f}{\partial p_\alpha}$. Therefore, we get

$$\left\{ x, M_x \right\} = \frac{\partial M_x}{\partial p_x} = 0, \qquad \left\{ y, M_x \right\} = \frac{\partial M_x}{\partial p_y} = -z, \qquad \left\{ z, M_x \right\} = \frac{\partial M_x}{\partial p_z} = y,$$

$$\left\{ x, M_y \right\} = \frac{\partial M_y}{\partial p_x} = z, \qquad \left\{ y, M_y \right\} = \frac{\partial M_y}{\partial p_y} = 0, \qquad \left\{ z, M_y \right\} = \frac{\partial M_y}{\partial p_z} = -x,$$

$$\left\{ x, M_z \right\} = \frac{\partial M_z}{\partial p_x} = -y, \qquad \left\{ y, M_z \right\} = \frac{\partial M_z}{\partial p_y} = x, \qquad \left\{ z, M_z \right\} = \frac{\partial M_z}{\partial p_z} = -y.$$

Therefore, we can configure the following table

Function	M_x	M_y	M_z
x	0	z	$-y$
y	$-z$	0	x
z	y	$-x$	0

(v) To find $\left\{ \vec{p},\, \vec{M} \right\}$, we use the relation $\left\{ p_\alpha,\, f \right\} = -\frac{\partial f}{\partial q_\alpha}$. Therefore

$$\left\{ p_x M_x \right\} = -\frac{\partial M_x}{\partial q_x} = -\frac{\partial M_x}{\partial x} = -(0) = 0,$$

$$\left\{ p_x, M_y \right\} = -\frac{\partial M_y}{\partial q_x} = -\frac{\partial M_y}{\partial x} = -(-p_z) = p_z$$

$$\left\{ p_x, M_z \right\} = -\frac{\partial M_z}{\partial q_x} = \frac{\partial M_z}{\partial x} = -(p_y) = -p_y$$

$$\left\{p_y, M_x\right\} = -\frac{\partial M_x}{\partial q_y} = -\frac{\partial M_x}{\partial y} = -(p_z) = -p_z,$$

$$\left\{p_y, M_y\right\} = -\frac{\partial M_y}{\partial q_y} = -\frac{\partial M_y}{\partial y} = -(0) = 0,$$

$$\left\{p_y, M_z\right\} = -\frac{\partial M_z}{\partial q_y} = -\frac{\partial M_z}{\partial y} = -(-p_z) = p_x.$$

$$\left\{p_z, M_x\right\} = -\frac{\partial M_x}{\partial q_z} = -\frac{\partial M_x}{\partial z} = -(-p_y) = p_y,$$

$$\left\{p_z, M_y\right\} = -\frac{\partial M_y}{\partial q_z} = -\frac{\partial M_y}{\partial z} = -(p_x) = -p_x,$$

$$\left\{p_z, M_z\right\} = -\frac{\partial M_z}{\partial q_z} = -\frac{\partial M_z}{\partial z} = -(0) = 0.$$

Therefore, we can configure the following table

Function	M_x	M_y	M_z
p_x	0	p_z	$-p_y$
p_y	$-p_z$	0	p_x
p_z	p_y	$-p_x$	0

(v) To find $\left\{\vec{M}, \vec{M}\right\}$, we use the definition $\left\{f, g\right\} = \left\{\frac{\partial f}{\partial q_\alpha}\frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial g}{\partial q_\alpha}\right\}$.

Where we will try to find the brackets

$\left\{M_x, M_x\right\}, \left\{M_x, M_y\right\}, \left\{M_x, M_z\right\}, \left\{M_y, M_y\right\}, \left\{M_y, M_y\right\}, \left\{M_y, M_z\right\},$

$\left\{M_z, M_x\right\}, \left\{M_z, M_y\right\}, \left\{M_z, M_z\right\}.$

In our problem $\alpha=1,2,3$, the Poisson brackets $\left\{f\,,\,g\right\}=\left\{\frac{\partial f}{\partial q_\alpha}\frac{\partial g}{\partial p_\alpha}-\frac{\partial f}{\partial p_\alpha}\frac{\partial g}{\partial q_\alpha}\right\}$ be written as

$$\left\{f\,,\,g\right\}=\sum_{\alpha=1}^{3}\left\{\frac{\partial f}{\partial q_\alpha}\frac{\partial g}{\partial p_\alpha}-\frac{\partial f}{\partial p_\alpha}\frac{\partial g}{\partial q_\alpha}\right\}=\left\{\frac{\partial f}{\partial q_1}\frac{\partial g}{\partial p_1}-\frac{\partial f}{\partial p_1}\frac{\partial g}{\partial q_1}\right\}+\left\{\frac{\partial f}{\partial q_2}\frac{\partial g}{\partial p_2}-\frac{\partial f}{\partial p_2}\frac{\partial g}{\partial q_2}\right\}+\left\{\frac{\partial f}{\partial q_3}\frac{\partial g}{\partial p_3}-\frac{\partial f}{\partial p_3}\frac{\partial g}{\partial q_3}\right\}$$

$$\left\{f\,,\,g\right\}=\left\{\frac{\partial f}{\partial q_x}\frac{\partial g}{\partial p_x}-\frac{\partial f}{\partial p_x}\frac{\partial g}{\partial q_x}\right\}+\left\{\frac{\partial f}{\partial q_y}\frac{\partial g}{\partial p_y}-\frac{\partial f}{\partial p_y}\frac{\partial g}{\partial q_y}\right\}+\left\{\frac{\partial f}{\partial q_z}\frac{\partial g}{\partial p_z}-\frac{\partial f}{\partial p_z}\frac{\partial g}{\partial q_z}\right\}$$

$$\left\{f\,,\,g\right\}=\left\{\frac{\partial f}{\partial x}\frac{\partial g}{\partial p_x}-\frac{\partial f}{\partial p_x}\frac{\partial g}{\partial x}\right\}+\left\{\frac{\partial f}{\partial y}\frac{\partial g}{\partial p_y}-\frac{\partial f}{\partial p_y}\frac{\partial g}{\partial y}\right\}+\left\{\frac{\partial f}{\partial z}\frac{\partial g}{\partial p_z}-\frac{\partial f}{\partial p_z}\frac{\partial g}{\partial z}\right\}$$

(a) from the Property $\left\{f\,,\,f\right\}=\left\{\frac{\partial f}{\partial q_\alpha}\frac{\partial f}{\partial p_\alpha}-\frac{\partial f}{\partial p_\alpha}\frac{\partial f}{\partial q_\alpha}\right\}=\left\{\frac{\partial f}{\partial q_\alpha}\frac{\partial f}{\partial p_\alpha}-\frac{\partial f}{\partial q_\alpha}\frac{\partial f}{\partial p_\alpha}\right\}=0$, we find that

$$\left\{M_x,M_x\right\}=\left\{M_y,M_y\right\}=\left\{M_z,M_z\right\}=0$$

(b) $$\left\{M_x,\,M_y\right\}=\left\{\frac{\partial M_x}{\partial x}\frac{\partial M_y}{\partial p_x}-\frac{\partial M_x}{\partial p_x}\frac{\partial M_y}{\partial x}\right\}+\left\{\frac{\partial M_x}{\partial y}\frac{\partial M_y}{\partial p_y}-\frac{\partial M_x}{\partial p_y}\frac{\partial M_y}{\partial y}\right\}+\left\{\frac{\partial M_x}{\partial z}\frac{\partial M_y}{\partial p_z}-\frac{\partial M_x}{\partial p_z}\frac{\partial M_y}{\partial z}\right\}$$

$$\left\{M_x,\,M_y\right\}=\quad (0)(z)-(0)\,(-p_z)\quad+\quad (p_z)(0)-(-z)\,(0)\quad+\quad(-p_y)(-x)-(y)\,(p_x)$$

$$\therefore\ \left\{M_x,\,M_y\right\}=xp_y-y\,p_x=M_z$$

$$\left\{M_x,\,M_z\right\}=\left\{\frac{\partial M_x}{\partial x}\frac{\partial M_z}{\partial p_x}-\frac{\partial M_x}{\partial p_x}\frac{\partial M_z}{\partial x}\right\}+\left\{\frac{\partial M_x}{\partial y}\frac{\partial M_z}{\partial p_y}-\frac{\partial M_x}{\partial p_y}\frac{\partial M_z}{\partial y}\right\}+\left\{\frac{\partial M_x}{\partial z}\frac{\partial M_z}{\partial p_z}-\frac{\partial M_x}{\partial p_z}\frac{\partial M_z}{\partial z}\right\}$$

$$\left\{M_x,\,M_z\right\}=\quad (0)(-y)-(0)\,(p_y)\quad+\quad (p_z)(x)-(-z)\,(-p_x)\quad+\quad(-p_y)(0)-(y)\,(0)$$

$$\therefore\ \left\{M_x,\,M_z\right\}=xp_z-z\,p_x=-M_y$$

$$\left\{M_y, M_z\right\} = \left\{\frac{\partial M_y}{\partial x}\frac{\partial M_z}{\partial p_x} - \frac{\partial M_y}{\partial p_x}\frac{\partial M_z}{\partial x}\right\} + \left\{\frac{\partial M_y}{\partial y}\frac{\partial M_z}{\partial p_y} - \frac{\partial M_y}{\partial p_y}\frac{\partial M_z}{\partial y}\right\} + \left\{\frac{\partial M_y}{\partial z}\frac{\partial M_z}{\partial p_z} - \frac{\partial M_y}{\partial p_z}\frac{\partial M_z}{\partial z}\right\}$$

$$\left\{M_y, M_z\right\} = \quad (-p_z)(-y) - (z)\,(p_y) \quad + \quad (0)(x) - (0)\,(-p_x) \quad + \quad (p_x)(0) - (x)\,(0)$$

$$\therefore \left\{M_y, M_z\right\} = y p_z - z\, p_y = M_x$$

Therefore, we can configure the following table

Function	M_x	M_y	M_z
M_x	0	M_z	$-M_y$
M_y	$-M_z$	0	M_x
M_z	M_y	$-M_x$	0

Example: Calculate the formula of Poisson's brackets for $\left\{f_\ell, f_k\right\}$, where $\ell, k = 1, 2, 3, 4$ and

$f_1 = \frac{1}{4}\left(x^2 + p_x^2 - y^2 - p_y^2\right)$, $\qquad f_2 = \frac{1}{2}\left(x\,y + p_x p_y\right)$,

$f_3 = \frac{1}{2}\left(x\,p_y - y p_x\right)$, $\qquad f_4 = \left(x^2 + y^2 + p_x^2 + p_y^2\right)$.

Answer

From the Poisson's brackets

$$\sum_\alpha \left\{f, g\right\} = \sum_\alpha \left\{\frac{\partial f}{\partial q_\alpha}\frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial g}{\partial q_\alpha}\right\},$$

and where α is the number of generalized coordinates ($\alpha = x, y$)

So, the Poisson's brackets will be written as

$$\left\{ f\,,\,g \right\} = \left\{ \frac{\partial f}{\partial q_x}\frac{\partial g}{\partial p_x} - \frac{\partial f}{\partial p_x}\frac{\partial g}{\partial q_x} \right\} + \left\{ \frac{\partial f}{\partial q_y}\frac{\partial g}{\partial p_y} - \frac{\partial f}{\partial p_y}\frac{\partial g}{\partial q_y} \right\}$$

$$\left\{ f\,,\,g \right\} = \left\{ \frac{\partial f}{\partial x}\frac{\partial g}{\partial p_x} - \frac{\partial f}{\partial p_x}\frac{\partial g}{\partial x} \right\} + \left\{ \frac{\partial f}{\partial y}\frac{\partial g}{\partial p_y} - \frac{\partial f}{\partial p_y}\frac{\partial g}{\partial y} \right\}$$

At $\ell,\, k = 1,\, 2, 3,\, 4$, the Poisson's brackets will be written as

$$\sum_{\ell, k=1}^{4} \left\{ f_\ell,\, f_k \right\} = \left\{ \frac{\partial f_\ell}{\partial x}\frac{\partial f_k}{\partial p_x} - \frac{\partial f_\ell}{\partial p_x}\frac{\partial f_k}{\partial x} \right\} + \left\{ \frac{\partial f_\ell}{\partial y}\frac{\partial f_k}{\partial p_y} - \frac{\partial f_\ell}{\partial p_y}\frac{\partial f_k}{\partial y} \right\}$$

$$\ell = 1 \rightarrow \quad \left\{ f_1,\, f_1 \right\}, \quad \left\{ f_1,\, f_2 \right\}, \quad \left\{ f_1,\, f_3 \right\}, \quad \left\{ f_1,\, f_4 \right\}$$

$$\ell = 2 \rightarrow \quad \left\{ f_2,\, f_1 \right\}, \quad \left\{ f_2,\, f_2 \right\}, \quad \left\{ f_2,\, f_3 \right\}, \quad \left\{ f_2,\, f_4 \right\}$$

$$\ell = 3 \rightarrow \quad \left\{ f_3,\, f_1 \right\}, \quad \left\{ f_3,\, f_2 \right\}, \quad \left\{ f_3,\, f_3 \right\}, \quad \left\{ f_3,\, f_4 \right\}$$

$$\ell = 4 \rightarrow \quad \left\{ f_4,\, f_1 \right\}, \quad \left\{ f_4,\, f_2 \right\}, \quad \left\{ f_4,\, f_3 \right\}, \quad \left\{ f_4,\, f_4 \right\}$$

$$\ell = 1: \quad \sum_{\ell=1} \left\{ f_1,\, f_k \right\} = \left\{ \frac{\partial f_1}{\partial x}\frac{\partial f_k}{\partial p_x} - \frac{\partial f_1}{\partial p_x}\frac{\partial f_k}{\partial x} \right\} + \left\{ \frac{\partial f_1}{\partial y}\frac{\partial f_k}{\partial p_y} - \frac{\partial f_1}{\partial p_y}\frac{\partial f_k}{\partial y} \right\}$$

$$\left\{ f_1,\, f_1 \right\} = \left\{ \frac{\partial f_1}{\partial x}\frac{\partial f_1}{\partial p_x} - \frac{\partial f_1}{\partial p_x}\frac{\partial f_1}{\partial x} \right\} + \left\{ \frac{\partial f_1}{\partial y}\frac{\partial f_1}{\partial p_y} - \frac{\partial f_1}{\partial p_y}\frac{\partial f_1}{\partial y} \right\} = 0$$

Likewise, be

$$\left\{ f_2,\, f_2 \right\} = \left\{ f_3,\, f_3 \right\} = \left\{ f_4,\, f_4 \right\} = 0$$

Also, according to the rule $\left\{ f\,,\, f \right\} = 0$, we can calculate these bracket

Therefore

$$\left\{ f_1,\, f_1 \right\} = \left\{ f_2,\, f_2 \right\} = \left\{ f_3,\, f_3 \right\} = \left\{ f_4,\, f_4 \right\} = 0$$

$$\sum_{\ell=1}\left\{ f_1,\ f_2 \right\} = \left\{ \frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial p_x} - \frac{\partial f_1}{\partial p_x}\frac{\partial f_2}{\partial x} \right\} + \left\{ \frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial p_y} - \frac{\partial f_1}{\partial p_y}\frac{\partial f_2}{\partial y} \right\}$$

$$\sum_{\ell=1}\left\{ f_1,\ f_2 \right\} = \left\{ (\frac{1}{2}x)(\frac{1}{2}p_y) - (\frac{1}{2}p_x)(\frac{1}{2}y) \right\} + \left\{ (-\frac{1}{2}y)(\frac{1}{2}p_x) - (-\frac{1}{2}p_y)(\frac{1}{2}x) \right\}$$

$$\sum_{\ell=1}\left\{ f_1,\ f_2 \right\} = \frac{1}{2}\left\{ x\,p_y - y\,p_x \right\}$$

$$\sum_{\ell=1}\left\{ f_1,\ f_3 \right\} = \left\{ \frac{\partial f_1}{\partial x}\frac{\partial f_3}{\partial p_x} - \frac{\partial f_1}{\partial p_x}\frac{\partial f_3}{\partial x} \right\} + \left\{ \frac{\partial f_1}{\partial y}\frac{\partial f_3}{\partial p_y} - \frac{\partial f_1}{\partial p_y}\frac{\partial f_3}{\partial y} \right\}$$

$$\sum_{\ell=1}\left\{ f_1,\ f_3 \right\} = \left\{ (\frac{1}{2}x)(-\frac{1}{2}y) - (\frac{1}{2}p_x)(\frac{1}{2}p_y) \right\} + \left\{ (-\frac{1}{2}y)(\frac{1}{2}x) - (-\frac{1}{2}p_y)(-\frac{1}{2}p_x) \right\}$$

$$\sum_{\ell=1}\left\{ f_1,\ f_3 \right\} = -\frac{1}{2}\left\{ xy + p_x p_y \right\}$$

$$\sum_{\ell=1}\left\{ f_1,\ f_4 \right\} = \left\{ \frac{\partial f_1}{\partial x}\frac{\partial f_4}{\partial p_x} - \frac{\partial f_1}{\partial p_x}\frac{\partial f_4}{\partial x} \right\} + \left\{ \frac{\partial f_1}{\partial y}\frac{\partial f_4}{\partial p_y} - \frac{\partial f_1}{\partial p_y}\frac{\partial f_4}{\partial y} \right\}$$

$$\sum_{\ell=1}\left\{ f_1,\ f_4 \right\} = \left\{ (\frac{1}{2}x)(2p_x) - (\frac{1}{2}p_x)(2x) \right\} + \left\{ (-\frac{1}{2}y)(2p_y) - (-\frac{1}{2}p_y)(2y) \right\}$$

$$\sum_{\ell=1}\left\{ f_1,\ f_4 \right\} = 0$$

$\ell = 2:$ $$\sum_{\ell=2}\left\{ f_2,\ f_k \right\} = \left\{ \frac{\partial f_2}{\partial x}\frac{\partial f_k}{\partial p_x} - \frac{\partial f_2}{\partial p_x}\frac{\partial f_k}{\partial x} \right\} + \left\{ \frac{\partial f_2}{\partial y}\frac{\partial f_k}{\partial p_y} - \frac{\partial f_2}{\partial p_y}\frac{\partial f_k}{\partial y} \right\}$$

$$\sum_{\ell=2}\left\{ f_2,\ f_1 \right\} = \left\{ \frac{\partial f_2}{\partial x}\frac{\partial f_1}{\partial p_x} - \frac{\partial f_2}{\partial p_x}\frac{\partial f_1}{\partial x} \right\} + \left\{ \frac{\partial f_2}{\partial y}\frac{\partial f_1}{\partial p_y} - \frac{\partial f_2}{\partial p_y}\frac{\partial f_1}{\partial y} \right\} = -\left\{ f_1,\ f_2 \right\}$$

$$\sum_{\ell=2}\left\{ f_2,\ f_1 \right\} = \frac{1}{2}\left\{ y\,p_x - x\,p_y \right\}$$

$$\sum_{\ell=2}\left\{ f_2,\ f_3 \right\} = \left\{ \frac{\partial f_2}{\partial x}\frac{\partial f_3}{\partial p_x} - \frac{\partial f_2}{\partial p_x}\frac{\partial f_3}{\partial x} \right\} + \left\{ \frac{\partial f_2}{\partial y}\frac{\partial f_3}{\partial p_y} - \frac{\partial f_2}{\partial p_y}\frac{\partial f_3}{\partial y} \right\}$$

$$\sum_{\ell=2}\left\{f_2,\ f_3\right\}=\left\{(\frac{1}{2}y)(-\frac{1}{2}y)-(\frac{1}{2}p_y)(\frac{1}{2}p_y)\right\}+\left\{(\frac{1}{2}x)(\frac{1}{2}x)-(\frac{1}{2}p_x)(-\frac{1}{2}p_x)\right\}$$

$$\sum_{\ell=2}\left\{f_2,\ f_3\right\}=\frac{1}{4}\left\{x^2-y^2+p_x^2-p_y^2\right\}$$

$$\sum_{\ell=2}\left\{f_2,\ f_4\right\}=\left\{\frac{\partial f_2}{\partial x}\frac{\partial f_4}{\partial p_x}-\frac{\partial f_2}{\partial p_x}\frac{\partial f_4}{\partial x}\right\}+\left\{\frac{\partial f_2}{\partial y}\frac{\partial f_4}{\partial p_y}-\frac{\partial f_2}{\partial p_y}\frac{\partial f_4}{\partial y}\right\}$$

$$\sum_{\ell=2}\left\{f_2,\ f_4\right\}=\left\{(\frac{1}{2}y)(2p_x)-(\frac{1}{2}p_y)(2x)\right\}+\left\{(\frac{1}{2}x)(2p_y)-(\frac{1}{2}p_x)(2y)\right\}$$

$$\sum_{\ell=2}\left\{f_2,\ f_4\right\}=y\,p_x-x\,p_y+x\,p_y-y\,p_x=0$$

$$\sum_{\ell=2}\left\{f_2,\ f_4\right\}=0$$

$\ell=3$: $$\sum_{\ell=3}\left\{f_3,\ f_k\right\}=\left\{\frac{\partial f_3}{\partial x}\frac{\partial f_k}{\partial p_x}-\frac{\partial f_3}{\partial p_x}\frac{\partial f_k}{\partial x}\right\}+\left\{\frac{\partial f_3}{\partial y}\frac{\partial f_k}{\partial p_y}-\frac{\partial f_3}{\partial p_y}\frac{\partial f_k}{\partial y}\right\}$$

$$\sum_{\ell=3}\left\{f_3,\ f_1\right\}=\left\{\frac{\partial f_3}{\partial x}\frac{\partial f_1}{\partial p_x}-\frac{\partial f_3}{\partial p_x}\frac{\partial f_1}{\partial x}\right\}+\left\{\frac{\partial f_3}{\partial y}\frac{\partial f_1}{\partial p_y}-\frac{\partial f_3}{\partial p_y}\frac{\partial f_1}{\partial y}\right\}=-\sum_{\ell=3}\left\{f_1,\ f_3\right\}$$

$$\sum_{\ell=3}\left\{f_3,\ f_1\right\}=\frac{1}{2}\left\{xy+p_xp_y\right\}$$

$$\sum_{\ell=3}\left\{f_3,\ f_2\right\}=\left\{\frac{\partial f_3}{\partial x}\frac{\partial f_2}{\partial p_x}-\frac{\partial f_3}{\partial p_x}\frac{\partial f_2}{\partial x}\right\}+\left\{\frac{\partial f_3}{\partial y}\frac{\partial f_2}{\partial p_y}-\frac{\partial f_3}{\partial p_y}\frac{\partial f_2}{\partial y}\right\}=-\sum_{\ell=3}\left\{f_2,\ f_3\right\}$$

$$\sum_{\ell=3}\left\{f_3,\ f_2\right\}=-\frac{1}{4}\left\{x^2-y^2+p_x^2-p_y^2\right\}$$

$$\sum_{\ell=3}\left\{f_3,\ f_4\right\}=\left\{\frac{\partial f_3}{\partial x}\frac{\partial f_4}{\partial p_x}-\frac{\partial f_3}{\partial p_x}\frac{\partial f_4}{\partial x}\right\}+\left\{\frac{\partial f_3}{\partial y}\frac{\partial f_4}{\partial p_y}-\frac{\partial f_3}{\partial p_y}\frac{\partial f_4}{\partial y}\right\}$$

$$\sum_{\ell=3}\left\{f_3,\ f_4\right\}=\left\{(\frac{1}{2}p_y)(2p_x)-(-\frac{1}{2}y)(2x)\right\}+\left\{(-\frac{1}{2}p_x)(2p_y)-(\frac{1}{2}x)(2y)\right\}$$

$$\sum_{\ell=3}\left\{ f_3,\ f_4 \right\} = p_y p_x + xy - p_y p_x - xy = 0$$

$$\sum_{\ell=3}\left\{ f_3,\ f_4 \right\} = 0$$

$\ell = 4$: $$\sum_{\ell=4}\left\{ f_4,\ f_k \right\} = \left\{ \frac{\partial f_4}{\partial x}\frac{\partial f_k}{\partial p_x} - \frac{\partial f_4}{\partial p_x}\frac{\partial f_k}{\partial x} \right\} + \left\{ \frac{\partial f_4}{\partial y}\frac{\partial f_k}{\partial p_y} - \frac{\partial f_4}{\partial p_y}\frac{\partial f_k}{\partial y} \right\}$$

$$\sum_{\ell=4}\left\{ f_4,\ f_1 \right\} = \left\{ \frac{\partial f_4}{\partial x}\frac{\partial f_1}{\partial p_x} - \frac{\partial f_4}{\partial p_x}\frac{\partial f_1}{\partial x} \right\} + \left\{ \frac{\partial f_4}{\partial y}\frac{\partial f_1}{\partial p_y} - \frac{\partial f_4}{\partial p_y}\frac{\partial f_1}{\partial y} \right\} = -\sum_{\ell=4}\left\{ f_1,\ f_4 \right\}$$

$$\sum_{\ell=4}\left\{ f_4,\ f_1 \right\} = 0$$

$$\sum_{\ell=4}\left\{ f_4,\ f_2 \right\} = \left\{ \frac{\partial f_4}{\partial x}\frac{\partial f_2}{\partial p_x} - \frac{\partial f_4}{\partial p_x}\frac{\partial f_2}{\partial x} \right\} + \left\{ \frac{\partial f_4}{\partial y}\frac{\partial f_2}{\partial p_y} - \frac{\partial f_4}{\partial p_y}\frac{\partial f_2}{\partial y} \right\} = -\sum_{\ell=4}\left\{ f_2,\ f_4 \right\}$$

$$\sum_{\ell=4}\left\{ f_4,\ f_2 \right\} = 0$$

$$\sum_{\ell=4}\left\{ f_4,\ f_3 \right\} = \left\{ \frac{\partial f_4}{\partial x}\frac{\partial f_3}{\partial p_x} - \frac{\partial f_4}{\partial p_x}\frac{\partial f_3}{\partial x} \right\} + \left\{ \frac{\partial f_4}{\partial y}\frac{\partial f_3}{\partial p_y} - \frac{\partial f_4}{\partial p_y}\frac{\partial f_3}{\partial y} \right\} = -\sum_{\ell=4}\left\{ f_3,\ f_4 \right\}$$

$$\sum_{\ell=4}\left\{ f_4,\ f_3 \right\} = 0$$

Chapter 6

Canonical transformations and Generating function for canonical transformation

It is straightforward to transfer coordinate systems using the Lagrangian formulation as minimization of the action can be done in any coordinate system. However, in the Hamiltonian formulation, only some coordinate transformations preserve Hamilton's equations. Canonical transformations, defined here as those that preserve the Poisson brackets or equivalently the symplectic 2-form, also preserve Hamilton's equations. A search for conserved quantities and symmetries is equivalent to a search for a nice coordinate system that preserves Hamilton's equations

In classical mechanics, there is no unique prescription for one to choose the generalized coordinates for a problem. As long as the coordinates and the corresponding momenta span the entire phase space, it becomes an acceptable set. However, it turns out in practice that some choices are better than some others as they make a given problem simpler while still preserving the form of Hamilton's equations. Going over from one set of chosen coordinates and momenta to another set which satisfy Hamilton's equations is done by canonical transformation.

Point Transformations

It's clear that Lagrange's equations are correct for any reasonable choice of parameters labeling the system configuration. Let's call our first choice $q = (q_1, q_2, q_3 ...., q_n)$. Now transform to a new set, maybe even time dependent, $Q_\alpha = Q_\alpha(q, t)$. The derivation of Lagrange's equations by minimizing the action still works, so Hamilton's equations must still also be OK too. This is called a point transformation: we've just moved to a different coordinate system; we're relabeling the points in configuration space (but possibly in a time-dependent way).

General and Canonical Transformations

The ease with which mechanical problems can be solved depends on the choice of the generalized coordinates used. Therefore, it is interesting to examine the transformations of a system of coordinates and moments to another system.

If we call p_α and q_α on one hand and P_α and Q_α on the other hand respectively old and new moments and coordinates, the transformation is $P_\alpha = P_\alpha(p_\alpha, q_\alpha, t)$ and $Q_\alpha = Q_\alpha(p_\alpha, q_\alpha, t)$. One considers only the transformations, called canonical transformations or contact transformations, for which there is a function H, called Hamiltonian in the new coordinates such as $\dot{P}_\alpha = -\dfrac{\partial H}{\partial Q_\alpha}$, $\quad \dot{Q}_\alpha = \dfrac{\partial H}{\partial P_\alpha}$,

where P_α and Q_α are the canonical moments and coordinates.

Condition for a Transformation to be Canonical

We can be proving the transformation $Q_\alpha = Q_\alpha(q_\alpha, p_\alpha, t),\ P_\alpha = P_\alpha(q_\alpha, p_\alpha, t)$ is canonical by in three ways:

(1) The method of the Liouville differential form

This is somewhat less practical, but I include it for completeness. The transformation is canonical if and only if the differential form $\sum_\alpha \left\{ P_\alpha\, dQ_\alpha - p_\alpha\, dq_\alpha \right\}$ is closed (is an exact differential).

(2) The method of Poisson brackets

The transformation is canonical if and only if the fundamental Poisson brackets are preserved Poisson's brackets $\left\{ P_\ell, P_k \right\} = 0, \quad \left\{ Q_\ell, Q_k \right\} = 0, \quad \left\{ Q_\ell, P_k \right\} = \delta_{\ell k}$

Where $\delta_{\ell k}$ kronecker delta,

$$\delta_{\ell k} = \begin{cases} 1, & \ell = k \\ 0, & \ell \neq k \end{cases}$$

(3) $\sum_\alpha \left\{ \delta P_\alpha\, dQ_\alpha - dP_\alpha\, \delta Q_\alpha \right\} = \sum_\alpha \left\{ \delta p_\alpha\, dq_\alpha - dp_\alpha\, \delta q_\alpha \right\}$

Example: 1 Determine whether the next transformations are canonical or non- canonical?

(1) $P = \text{q cotp}, \quad Q = \text{Ln}(\frac{\text{sinp}}{\text{q}})$,

(2) $P = \sqrt{2q}\, k^{\frac{1}{2}} \sin p, \quad Q = \sqrt{2q}\, k^{-\frac{1}{2}} \cos p$, where k is constant.

(3) $P = \sqrt{q} \sin(2p), \quad Q = \sqrt{q} \cos(2p)$,

(4) $P = \frac{1}{2\omega}(p^2 + \omega^2 q^2), \quad Q = \cot^{-1}(\frac{p}{\omega\, q})$, where ω is constant.

(5) $P = -q, \quad Q = p.$

Answer

(1) $P = \text{q cotp}, \quad Q = \text{Ln}(\frac{\text{sinp}}{\text{q}})$

Using Poisson's brackets $\left\{Q_\alpha ,\ P_\alpha \right\} = \frac{\partial Q}{\partial q_\alpha}\frac{\partial P}{\partial p_\alpha} - \frac{\partial Q}{\partial p_\alpha}\frac{\partial P}{\partial q_\alpha}$, we find that

$$\left\{Q_\alpha ,\ P_\alpha \right\} = \left\{ \text{Ln sinp-lnq},\ \text{qcotp} \right\} = \frac{\partial(\text{Ln sinp-lnq})}{\partial q}\frac{\partial(\text{qcotp})}{\partial p} - \frac{\partial(\text{Ln sinp-lnq})}{\partial p}\frac{\partial(\text{qcotp})}{\partial q}$$

$$\left\{Q_\alpha ,\ P_\alpha \right\} = \frac{-1}{q}(-\text{q cosec}^2\text{p}) - \frac{\cos p}{\text{sinp}}\text{cotp} = \text{cosec}^2\text{p - cot}^2\text{p} = 1$$

Therefore, the transformation is canonical.

(2) $P = \sqrt{2q}\, k^{\frac{1}{2}} \sin p, \quad Q = \sqrt{2q}\, k^{-\frac{1}{2}} \cos p$

We will try to calculate the expression $\sum_\alpha \left\{ \delta P_\alpha\, dQ_\alpha - dP_\alpha\, \delta Q_\alpha \right\}$

$$\delta P_\alpha\, dQ_\alpha - dP_\alpha\, \delta Q_\alpha = \delta\left(\sqrt{2q}\, k^{\frac{1}{2}} \sin p\right) d\left(\sqrt{2q}\, k^{-\frac{1}{2}} \cos p\right) - d\left(\sqrt{2q}\, k^{\frac{1}{2}} \sin p\right) \delta\left(\sqrt{2q}\, k^{-\frac{1}{2}} \cos p\right)$$

$$= \left(\frac{2}{2\sqrt{2q}} k^{\frac{1}{2}} \sin p\,(\delta q) + \sqrt{2q}\, k^{\frac{1}{2}} \cos p(\delta p)\right)\left(\frac{2}{2\sqrt{2q}} k^{-\frac{1}{2}} \cos p\,(dq) - \sqrt{2q}\, k^{-\frac{1}{2}} \sin p(dp)\right) -$$

$$\left(\frac{2}{2\sqrt{2q}} k^{\frac{1}{2}} \sin p\,(dq) + \sqrt{2q}\, k^{\frac{1}{2}} \cos p(dp)\right)\left(\frac{2}{2\sqrt{2q}} k^{-\frac{1}{2}} \cos p\,(\delta q) - \sqrt{2q}\, k^{-\frac{1}{2}} \sin p(\delta p)\right)$$

$$= k^{\frac{1}{2}} k^{-\frac{1}{2}} \left\{ \underbrace{\frac{1}{2q} \sin p \cos p\,(\delta q\, dq)}_{1+5=0} - \sin^2 p(\delta q\, dp) + \cos^2 p(\delta p\, dq) - \underbrace{2q \sin p \cos p(\delta p\, dp)}_{4+8=0} \right\} -$$

$$\left\{ \underbrace{\frac{1}{2q} \sin p \cos p\,(dq\, \delta q)}_{1+5=0} - \sin^2 p(dq\, \delta p) + \cos^2 p(dp\, \delta q) - \underbrace{2q \sin p \cos p(dp\, \delta p)}_{4+8=0} \right\}$$

$$= \left\{ \underbrace{-\sin^2 p(\delta q\, dp)}_{1+4} + \underbrace{\cos^2 p(\delta p\, dq) + \sin^2 p(dq\, \delta p)}_{2+3} - \underbrace{\cos^2 p(dp\, \delta q)}_{1+4} \right\}$$

$$= \left\{ -\underbrace{\left(\sin^2 p + \cos^2 p\right)}_{=1}(\delta q\, dp) + \underbrace{\left(\sin^2 p + \cos^2 p\right)}_{=1}(\delta p\, dq) \right\} = \delta p\, dq - \delta q\, dp$$

Then $\delta P_\alpha\, dQ_\alpha - dP_\alpha\, \delta Q_\alpha = \delta p_\alpha\, dq_\alpha - dp_\alpha\, \delta q_\alpha$

This proves that the transformation is canonical.

(3) $P = \sqrt{q} \sin(2p), \quad Q = \sqrt{q} \cos(2p)$

We will try to calculate the expression $\sum_\alpha \left\{ \delta P_\alpha\, dQ_\alpha - dP_\alpha\, \delta Q_\alpha \right\}$

$$\delta P_\alpha\, dQ_\alpha - dP_\alpha\, \delta Q_\alpha = \delta\left(\sqrt{q} \sin 2p\right) d\left(\sqrt{q} \cos 2p\right) - d\left(\sqrt{q} \sin 2p\right) \delta\left(\sqrt{q} \cos 2p\right)$$

$$= \left(\frac{1}{2\sqrt{q}} \sin 2p\,(\delta q) + 2\sqrt{q} \cos p(\delta p)\right)\left(\frac{1}{2\sqrt{q}} \cos 2p\,(dq) - 2\sqrt{q} \sin 2p(dp)\right) -$$

$$\left(\frac{1}{2\sqrt{q}} \sin 2p\,(dq) + 2\sqrt{q} \cos p(dp)\right)\left(\frac{1}{2\sqrt{q}} \cos 2p\,(\delta q) - 2\sqrt{q} \sin 2p(\delta p)\right)$$

$$=\left\{\underbrace{\frac{1}{4p}\sin 2p\cos 2p\,(\delta q\,dq)}_{1+5=0}-\sin^2(2p)(\delta q\,dp)+\cos^2(2p)(\delta p\,dq)-\underbrace{\sin 2p\ \cos 2p\,(\delta p\,dp)}_{4+8=0}\right\}-$$

$$\left\{\underbrace{\frac{1}{4p}\sin 2p\ \cos 2p\ (dq\,\delta q)}_{1+5=0}-\sin^2(2p)\,(dq\,\delta p)+\cos^2(2p)\,(dp\,\delta q)-\underbrace{2q\sin p\cos p(dp\,\delta p)}_{4+8=0}\right\}$$

$$=\left\{\underbrace{-\sin^2(2p)\,(\delta q\,dp)}_{1+4}+\underbrace{\cos^2(2p)\,(\delta p\,dq)+\sin^2(2p)\,(dq\,\delta p)}_{2+3}-\underbrace{\cos^2(2p)\,(dp\,\delta q)}_{1+4}\right\}$$

$$=\left\{-\underbrace{\left(\sin^2(2p)+\cos^2(2p)\right)}_{=1}(\delta q\,dp)+\underbrace{\left(\sin^2(2p)+\cos^2(2p)\right)}_{=1}(\delta p\,dq)\right\}=\delta p\,dq-\delta q\,dp$$

Then $\delta P_\alpha\,dQ_\alpha-dP_\alpha\,\delta Q_\alpha=\delta p_\alpha\,dq_\alpha-dp_\alpha\,\delta q_\alpha$. Hence, the transformation is canonical.

(4) $P=\frac{1}{2\omega}(p^2+\omega^2q^2),\qquad Q=\cot^{-1}(\frac{p}{\omega\,q})$

Using Poisson's brackets $\left\{Q_\alpha,\ P_\alpha\right\}=\frac{\partial Q}{\partial q_\alpha}\frac{\partial P}{\partial p_\alpha}-\frac{\partial Q}{\partial p_\alpha}\frac{\partial P}{\partial q_\alpha}$

$$\left\{Q_\alpha,P_\alpha\right\}=\left\{\cot^{-1}(\frac{p}{\omega\,q}),\ \frac{1}{2\omega}(p^2+\omega^2q^2)\right\}=\frac{\partial(\cot^{-1}(\frac{p}{\omega\,q}))}{\partial q}\frac{\partial(\frac{1}{2\omega}(p^2+\omega^2q^2))}{\partial p}-\frac{\partial(\cot^{-1}(\frac{p}{\omega\,q}))}{\partial p}\frac{\partial(\frac{1}{2\omega}(p^2+\omega^2q^2))}{\partial q}$$

$$\left\{Q_\alpha,P_\alpha\right\}=\frac{-\left(\frac{-p}{\omega\,q^2}\right)}{1+\left(\frac{p}{\omega\,q}\right)^2}\left(\frac{p}{\omega}\right)-\frac{-\left(\frac{1}{\omega\,q}\right)}{1+\left(\frac{p}{\omega\,q}\right)^2}\left(\frac{\omega^2q}{\omega}\right)=\frac{\left(\frac{p^2}{\omega^2\,q^2}\right)}{1+\left(\frac{p}{\omega\,q}\right)^2}+\frac{1}{1+\left(\frac{p}{\omega\,q}\right)^2}=\frac{p^2}{\omega^2\,q^2+p^2}+\frac{\omega^2q^2}{\omega^2q^2+p^2}=\frac{\omega^2\,q^2+p^2}{\omega^2\,q^2+p^2}=1$$

Therefore, the transformation is canonical.

(5) $P=-q, \quad Q=p$

(i) Using Poisson's brackets $\left\{Q_\alpha, P_\alpha\right\}=\frac{\partial Q}{\partial q_\alpha}\frac{\partial P}{\partial p_\alpha}-\frac{\partial Q}{\partial p_\alpha}\frac{\partial P}{\partial q_\alpha}$.

We have $\left\{p,-q\right\}=\frac{\partial p}{\partial q}\frac{\partial(-q)}{\partial p}-\frac{\partial p}{\partial p}\frac{\partial(-q)}{\partial q}=(0)-(1)(-1)=1$. Then $\left\{p,-q\right\}=1$

Therefore, the transformation is canonical.

(ii) We calculate the differential form $\sum_\alpha\left\{P_\alpha\, dQ_\alpha - p_\alpha\, dq_\alpha\right\}$

$$P\,dQ-p\,dq=(-q)(dp)-pdq=\underbrace{-qdp}_{M}-\underbrace{p\,dq}_{N}$$

Then $\frac{\partial M}{\partial q}=-1, \quad \frac{\partial N}{\partial p}=-1 \;\rightarrow\; \frac{\partial M}{\partial q}=\frac{\partial N}{\partial p}=-1$

It means that, the differential form $\sum_\alpha\left\{P_\alpha\, dQ_\alpha - p_\alpha\, dq_\alpha\right\}$ is an exact differential

Therefore, the transformation is canonical.

(iii) We will try to calculate expression $\sum_\alpha\left\{\delta P_\alpha\, dQ_\alpha - dP_\alpha\, \delta Q_\alpha\right\}$

$$\delta P_\alpha\, dQ_\alpha - dP_\alpha\, \delta Q_\alpha = \delta(-q)\,dp-d(-q)\,\delta p = dq\,\delta p-\delta q\,dp == \delta p\,dq-dp\,\delta q=\delta p_\alpha\, dq_\alpha - dp_\alpha\, \delta q_\alpha$$

Then $\delta P_\alpha\, dQ_\alpha - dP_\alpha\, \delta Q_\alpha=\delta p_\alpha\, dq_\alpha - dp_\alpha\, \delta q_\alpha$. This proves that the transformation is canonical (which is indeed an exact differential and the transformation is canonical)

Example 2: Is the transformation $P=Exp(q), \quad Q=Exp(p)$ canonical or non-canonical?

Solution

Using $\sum_\alpha\left\{P_\alpha\, dQ_\alpha - p_\alpha\, dq_\alpha\right\}$:

$$PdQ - pdq = Exp(q)Exp(p) - pdq = Exp(p+q) - pdq$$

which is not an exact differential and the transformation is not canonical.

Exercises:

(i) If $p_1 = \frac{1}{\sqrt{2}} p^2, \quad p_2 = \frac{1}{\sqrt{2}}(p-q), \quad Q_1 = \frac{1}{\sqrt{2}} q^2, \quad Q_2 = \frac{1}{\sqrt{2}}(p+q).$

- Check whether the next transformations are canonical or non- canonical?

Q_1 and Q_2, p_1 and p_2, Q_1 and p_1, Q_1 and p_2, Q_2 and p_1, Q_2 and p_2.

(ii) Using Poisson bracket show that the transformation defined by

$P = \frac{1}{2}(q^2 + p^2), \quad Q = \tan^{-1}(\frac{q}{p}) \quad (q = \sqrt{2P}\sin Q, \quad p = \sqrt{2P}\cos Q)$ is canonical.?

Generating function for canonical transformation

By Hamilton's Variational Principle, the canonical transformations $P_\alpha = P_\alpha\left(p_\alpha, q_\alpha, t\right)$ and $Q_\alpha = Q_\alpha\left(p_\alpha, q_\alpha, t\right)$ must be such that the integrals $\int_{t_1}^{t_2} L\,dt$ and $\int_{t_1}^{t_2} \bar{L}\,dt$ are both extremal, i.e. that one needs to have simultaneously $\delta\int_{t_1}^{t_2} L\,dt = 0$ and $\delta\int_{t_1}^{t_2} \bar{L}\,dt = 0$, which is satisfied if there is a generating function F such that

$$\frac{dF}{dt} = L - \bar{L}\,. \tag{1}$$

Then

$$\delta\int_{t_1}^{t_2}\left(L-\bar{L}\right)dt = \delta\int_{t_1}^{t_2}\frac{dF}{dt}\,dt = 0$$

$$\delta\int_{t_1}^{t_2} dF = \delta\left\{F(t_1) - F(t_1)\right\} = 0 \tag{2}$$

Where F is called generating function for the canonical transformation, that will be as $F\left(p_\alpha, q_\alpha, P_\alpha, Q_\alpha, t\right)$. Depending on the form of the generating functions (which pair of canonical variables being considered as the independent variables for the Generating Function), we can classify canonical transformations into four basic types. The four possible types of generating functions of the first kind, are $F_1\left(q_\alpha, Q_\alpha, t\right)$, $F_2\left(q_\alpha, P_\alpha, t\right)$, $F_3\left(p_\alpha, Q_\alpha, t\right)$ and $F_4\left(p_\alpha, P_\alpha, t\right)$. These four generating functions lead to relatively simple canonical transformations, can be found as shown below.

$$\frac{dF}{dt} = L - \bar{L} \tag{3}$$

But

$$H = \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha - L \quad \rightarrow \quad L = \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha - H \tag{4}$$

$$\bar{H} = \sum_{\alpha=1}^{n} P_\alpha \dot{Q}_\alpha - \bar{L} \quad \rightarrow \quad \bar{L} = \sum_{\alpha=1}^{n} P_\alpha \dot{Q}_\alpha - \bar{H} \tag{5}$$

From Eqs. (4) and (5) into Eq. (3), we get

$$\frac{dF}{dt} = \left\{\sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha - H\right\} - \left\{\sum_{\alpha=1}^{n} P_\alpha \dot{Q}_\alpha - \bar{H}\right\}$$

$$\frac{dF}{dt}=\left\{\sum_{\alpha=1}^{n} p_{\alpha}\, \dot{q}_{\alpha} - P_{\alpha}\, \dot{Q}_{\alpha}\right\}+\left\{\bar{H}-H\right\}$$

$$\frac{dF}{dt}=\left\{\sum_{\alpha=1}^{n} p_{\alpha}\frac{dq_{\alpha}}{dt} - P_{\alpha}\frac{dQ_{\alpha}}{dt}\right\}+\left\{\bar{H}-H\right\} \qquad (6)$$

Therefore Eq. (6) can be written as

$$dF=\left\{\sum_{\alpha=1}^{n} p_{\alpha}\, dq_{\alpha} - P_{\alpha}\, dQ_{\alpha}\right\}+\left\{\bar{H}-H\right\}dt \qquad (7)$$

$$F_1(q_{\alpha}, Q_{\alpha}, t), \quad F_2(q_{\alpha}, P_{\alpha}, t), \quad F_3(p_{\alpha}, Q_{\alpha}, t), \quad F_4(p_{\alpha}, P_{\alpha}, t), \qquad (8)$$

<u>Type 1:</u> $F_1(q_{\alpha}, Q_{\alpha}, t)$:

We consider the exact differential of $F=F_1\left(q_{\alpha}, Q_{\alpha}, t\right)$, we get

$$dF=dF_1\left(q_{\alpha}, Q_{\alpha}, t\right)=\sum_{\alpha}\left\{\frac{\partial F_1}{\partial q_{\alpha}}dq_{\alpha}+\frac{\partial F_1}{\partial Q_{\alpha}}dQ_{\alpha}\right\}+\frac{\partial F_1}{\partial t}dt \qquad (9)$$

If we compare Eq. (8) with Eq. (9), we get

$$p_{\alpha}=\frac{\partial F_1}{\partial q_{\alpha}}, \qquad P_{\alpha}=-\frac{\partial F_1}{\partial Q_{\alpha}}, \qquad \frac{\partial F_1}{\partial t}=\bar{H}-H \qquad (10)$$

<u>Type 2:</u> $F_2(q_{\alpha}, P_{\alpha}, t)$:

$$dF=dF_1\left(q_{\alpha}, Q_{\alpha}, t\right)=\left\{\sum_{\alpha=1}^{n} p_{\alpha}\, dq_{\alpha} - P_{\alpha}\, dQ_{\alpha}\right\}+\left\{\bar{H}-H\right\}dt$$

$$dF_1=\left\{\sum_{\alpha=1}^{n} p_{\alpha}\, dq_{\alpha} - P_{\alpha}\, dQ_{\alpha}+\underbrace{Q_{\alpha}\, dP_{\alpha}-Q_{\alpha}\, dP_{\alpha}}\right\}+\left\{\bar{H}-H\right\}dt$$

$$dF_1=\left\{\sum_{\alpha=1}^{n} p_{\alpha}\, dq_{\alpha} + Q_{\alpha}\, dP_{\alpha}+\underbrace{\left(-P_{\alpha}\, dQ_{\alpha}-Q_{\alpha}\, dP_{\alpha}\right)}\right\}+\left\{\bar{H}-H\right\}dt$$

$$dF_1=\left\{\sum_{\alpha=1}^{n} p_{\alpha}\, dq_{\alpha} + Q_{\alpha}\, dP_{\alpha}-\underbrace{\left(d\left(Q_{\alpha}P_{\alpha}\right)\right)}\right\}+\left\{\bar{H}-H\right\}dt$$

$$dF_1+d\left(Q_{\alpha}, P_{\alpha}\right)=\left\{\sum_{\alpha=1}^{n} p_{\alpha}\, dq_{\alpha} + Q_{\alpha}\, dP_{\alpha}\right\}+\left\{\bar{H}-H\right\}dt$$

$$d\left(F_1+\sum_{\alpha=1}^{n} P_{\alpha}\, Q_{\alpha}\right)=dF_2=\left\{\sum_{\alpha=1}^{n} p_{\alpha}\, dq_{\alpha} + Q_{\alpha}\, dP_{\alpha}\right\}+\left\{\bar{H}-H\right\}dt \qquad (11)$$

We consider the exact differential of $F = F_2(q_\alpha, P_\alpha, t)$, we get

$$dF = dF_2(q_\alpha, P_\alpha, t) = \sum_\alpha \left\{ \frac{\partial F_2}{\partial q_\alpha} dq_\alpha + \frac{\partial F_2}{\partial P_\alpha} dP_\alpha \right\} + \frac{\partial F_2}{\partial t} dt \qquad (12)$$

If we compare Eq. (11) with Eq. (12), we get

$$p_\alpha = \frac{\partial F_2}{\partial q_\alpha}, \qquad Q_\alpha = \frac{\partial F_2}{\partial P_\alpha}, \qquad \frac{\partial F_2}{\partial t} = \bar{H} - H \qquad (13)$$

<u>Type 3:</u> $F_3(p_\alpha, Q_\alpha, t)$

$$dF = dF_1(q_\alpha, Q_\alpha, t) = \left\{ \sum_{\alpha=1}^{n} p_\alpha\, dq_\alpha - P_\alpha\, dQ_\alpha \right\} + \left\{ \bar{H} - H \right\} dt$$

$$dF_1 = \left\{ \sum_{\alpha=1}^{n} p_\alpha\, dq_\alpha - P_\alpha\, dQ_\alpha + \underbrace{q_\alpha\, dp_\alpha - q_\alpha\, dp_\alpha} \right\} + \left\{ \bar{H} - H \right\} dt$$

$$dF_1 = \left\{ \sum_{\alpha=1}^{n} \underbrace{p_\alpha\, dq_\alpha + q_\alpha\, dp_\alpha} - P_\alpha\, dQ_\alpha - q_\alpha\, dp_\alpha \right\} + \left\{ \bar{H} - H \right\} dt$$

$$dF_1 = \left\{ \sum_{\alpha=1}^{n} \underbrace{d(p_\alpha q_\alpha)} - P_\alpha\, dQ_\alpha - q_\alpha\, dp_\alpha \right\} + \left\{ \bar{H} - H \right\} dt$$

$$dF_1 - \sum_{\alpha=1}^{n} d(p_\alpha q_\alpha) = \left\{ \sum_{\alpha=1}^{n} -P_\alpha\, dQ_\alpha - q_\alpha\, dp_\alpha \right\} + \left\{ \bar{H} - H \right\} dt$$

$$d\left(F_1 - \sum_{\alpha=1}^{n} p_\alpha q_\alpha \right) = dF_3 = \left\{ \sum_{\alpha=1}^{n} -P_\alpha\, dQ_\alpha - q_\alpha\, dp_\alpha \right\} + \left\{ \bar{H} - H \right\} dt \qquad (14)$$

We consider the exact differential of $F = F_3(p_\alpha, Q_\alpha, t)$, we get

$$dF = dF_3(p_\alpha, Q_\alpha, t) = \sum_\alpha \left\{ \frac{\partial F_3}{\partial p_\alpha} dp_\alpha + \frac{\partial F_3}{\partial Q_\alpha} dQ_\alpha \right\} + \frac{\partial F_3}{\partial t} dt \qquad (15)$$

If we compare Eq. (14) with Eq. (15), we get

$$q_\alpha = -\frac{\partial F_3}{\partial p_\alpha}, \qquad P_\alpha = -\frac{\partial F_3}{\partial Q_\alpha}, \qquad \frac{\partial F_3}{\partial t} = \bar{H} - H \qquad (16)$$

<u>Type 4:</u> $F_4(p_\alpha, P_\alpha, t)$

$$dF = dF_1(q_\alpha, Q_\alpha, t) = \left\{ \sum_{\alpha=1}^{n} p_\alpha\, dq_\alpha - P_\alpha\, dQ_\alpha \right\} + \left\{ \bar{H} - H \right\} dt$$

$$dF_1 = \left\{ \sum_{\alpha=1}^{n} p_\alpha\, dq_\alpha - P_\alpha\, dQ_\alpha + \underbrace{Q_\alpha\, dP_\alpha - Q_\alpha\, dP_\alpha} + \underbrace{q_\alpha\, dp_\alpha - q_\alpha\, dp_\alpha} \right\} + \left\{ \bar{H} - H \right\} dt$$

$$dF_1 = \left\{ \sum_{\alpha=1}^{n} \underbrace{p_\alpha\, dq_\alpha + q_\alpha\, dp_\alpha} + Q_\alpha\, dP_\alpha - q_\alpha\, dp_\alpha \underbrace{- P_\alpha\, dQ_\alpha - Q_\alpha\, dP_\alpha} \right\} + \left\{ \bar{H} - H \right\} dt$$

$$dF_1 = \left\{ \sum_{\alpha=1}^{n} \underbrace{d(p_\alpha q_\alpha)} + Q_\alpha\, dP_\alpha - q_\alpha\, dp_\alpha \underbrace{- d(P_\alpha Q_\alpha)} \right\} + \left\{ \bar{H} - H \right\} dt$$

$$dF_1 + \sum_{\alpha=1}^{n} d(P_\alpha Q_\alpha) - \sum_{\alpha=1}^{n} \underbrace{d(p_\alpha q_\alpha)} = \left\{ Q_\alpha\, dP_\alpha - q_\alpha\, dp_\alpha \right\} + \left\{ \bar{H} - H \right\} dt$$

$$d\left(F_1 + \sum_{\alpha=1}^{n} \left\{ P_\alpha Q_\alpha - p_\alpha q_\alpha \right\} \right) = \left\{ Q_\alpha\, dP_\alpha - q_\alpha\, dp_\alpha \right\} + \left\{ \bar{H} - H \right\} dt$$

$$d\left(F_1 + \sum_{\alpha=1}^{n} \left\{ P_\alpha Q_\alpha - p_\alpha q_\alpha \right\} \right) = dF_4 = \left\{ Q_\alpha\, dP_\alpha - q_\alpha\, dp_\alpha \right\} + \left\{ \bar{H} - H \right\} dt \qquad (17)$$

We consider the exact differential of $F = F_4(p_\alpha, P_\alpha, t)$, we get

$$dF = dF_4(p_\alpha, P_\alpha, t) = \sum_\alpha \left\{ \frac{\partial F_4}{\partial p_\alpha} dp_\alpha + \frac{\partial F_4}{\partial P_\alpha} dP_\alpha \right\} + \frac{\partial F_4}{\partial t} dt \qquad (18)$$

If we compare Eq. (17) with Eq. (18), we get

$$q_\alpha = -\frac{\partial F_4}{\partial p_\alpha}, \qquad Q_\alpha = \frac{\partial F_4}{\partial P_\alpha}, \qquad \frac{\partial F_4}{\partial t} = \bar{H} - H \qquad (19)$$

Properties of the Four basic canonical transformations

Generating function	**Derivatives of generating function**	**Trivial special cases**	**Transformation**
$F_1(q_i, Q_i, t)$	$p_i = \frac{\partial F_1}{\partial q_i}, P_i = -\frac{\partial F_1}{\partial Q_i}$	$F_1 = q_i Q_i$	$p_i = Q_i,$ $P_i = -q_i$
$F_2(q_i, P_i, t)$	$p_i = \frac{\partial F_2}{\partial q_i}, Q_i = \frac{\partial F_2}{\partial P_i}$	$F_2 = q_i P_i$	$p_i = P_i$ $Q_i = q_i$
$F_3(p_i, Q_i, t)$	$q_i = -\frac{\partial F_3}{\partial p_i}, P_i = -\frac{\partial F_3}{\partial Q_i}$	$F_3 = p_i Q_i$	$q_i = -Q_i$ $P_i = -p_i$
$F_4(p_i, P_i, t)$	$q_i = -\frac{\partial F_4}{\partial p_i}, Q_i = \frac{\partial F_4}{\partial P_i}$	$F_4 = p_i P_i$	$q_i = -P_i$ $Q_i = p_i$

Example 1: Show that the transformation $P=-q, \quad Q=p$ is canonical, and determine the generating functions F_1?

Solution

Using Poisson's brackets $\left\{Q_\alpha, P_\alpha\right\}=\frac{\partial Q}{\partial q_\alpha}\frac{\partial P}{\partial p_\alpha}-\frac{\partial Q}{\partial p_\alpha}\frac{\partial P}{\partial q_\alpha}$.

We have $\left\{Q_\alpha, P_\alpha\right\}=\left\{p,-q\right\}=\frac{\partial p}{\partial q}\frac{\partial(-q)}{\partial p}-\frac{\partial p}{\partial p}\frac{\partial(-q)}{\partial q}=(0)-(1)(-1)=1$

$\left\{Q_\alpha, P_\alpha\right\}=\left\{p,-q\right\}=1$, hence the transformation is canonical.

For the generating functions $F_1=F_1(q_\alpha, Q_\alpha, t)$

$$p_\alpha=\frac{\partial F_1}{\partial q_\alpha} \qquad P_\alpha=-\frac{\partial F_1}{\partial Q_\alpha}.$$

$$p_\alpha=\frac{\partial F_1}{\partial q_\alpha} \rightarrow Q=\frac{\partial F_1}{\partial q} \rightarrow F_1=\int Q\,dq \tag{1}$$

$$P_\alpha=-\frac{\partial F_1}{\partial Q_\alpha} \rightarrow -q=-\frac{\partial F_1}{\partial Q} \rightarrow q=\frac{\partial F_1}{\partial Q}\rightarrow F_1=\int q\,dQ \tag{2}$$

From the transformation $P=-q, \quad Q=p$, we have

From Eq. (1), we get $F_1=\int Q\,dq= qQ$

Also, from Eq. (2), we get $F_1=\int q\,dQ=qQ$

Hence, the generating functions F_1 given by $F_1(q_\alpha, Q_\alpha, t)=qQ$.

Example 2: Show that the following transformation is canonical,

$$P=\mathrm{q}\cot\mathrm{p}, \quad \mathrm{Q}=\mathrm{Log}\left(\frac{\sin\mathrm{p}}{\mathrm{q}}\right)?$$

Find the generating functions $F_4=F_4(p_\alpha, Q_\alpha, t)$?

Solution

(i) Using Poisson's brackets $\left\{Q_\alpha, P_\alpha\right\} = \frac{\partial Q}{\partial q_\alpha}\frac{\partial P}{\partial p_\alpha} - \frac{\partial Q}{\partial p_\alpha}\frac{\partial P}{\partial q_\alpha}$.

We have

$$\left\{Q_\alpha, P_\alpha\right\} = \left\{\sqrt{2q}\,k^{\frac{1}{2}}\cos p,\ \sqrt{2q}\,k^{\frac{1}{2}}\sin p\right\} = \frac{\partial(\sqrt{2q}\,k^{\frac{1}{2}}\cos p)}{\partial q}\frac{\partial(\sqrt{2q}\,k^{\frac{1}{2}}\sin p)}{\partial p} - \frac{\partial(\sqrt{2q}\,k^{\frac{1}{2}}\cos p)}{\partial p}\frac{\partial(\sqrt{2q}\,k^{\frac{1}{2}}\sin p)}{\partial q}$$

$$\left\{Q_\alpha, P_\alpha\right\} = \frac{2}{2\sqrt{2q}}k^{-\frac{1}{2}}\cos p\sqrt{2q}\,k^{\frac{1}{2}}\cos p - \sqrt{2q}\,k^{-\frac{1}{2}}(-\sin p).\ \frac{2}{\sqrt{2q}}k^{\frac{1}{2}}\sin p$$

$\left\{Q_\alpha, P_\alpha\right\} = (\cos p)^2 + (\sin p)^2 = 1.$ Therefore, the transformation is canonical.

(ii) For the generating functions $F_4 = F_4(p_\alpha, P_\alpha, t)$

$$q_\alpha = -\frac{\partial F_4}{\partial p_\alpha} \qquad Q_\alpha = \frac{\partial F_4}{\partial P_\alpha}.$$

Therefore

$$q_\alpha = -\frac{\partial F_4}{\partial p_\alpha} \rightarrow -\frac{P}{\cot p} = \frac{\partial F_4}{\partial p} \rightarrow F_4 = -\int \frac{P}{\cot p}dp \qquad (1)$$

$$Q_\alpha = \frac{\partial F_2}{\partial P_\alpha} \rightarrow \mathrm{Log}\left(\frac{\sin p}{q}\right) = \frac{\partial F_4}{\partial P} \rightarrow F_4 = \int \mathrm{Log}\left(\frac{\sin p}{q}\right)dP \qquad (2)$$

From the transformation $P = q\cot p$, $\quad Q = \mathrm{Log}\left(\frac{\sin p}{q}\right)$, we have

From Eq. (1),

$$F_4 = P\int \frac{-\sin p}{\cos p}dp = P\log(\cos p) + C(P)$$

$$F_4 = P\log(\cos p) + C(P)$$

$C(P) = P\,\text{Log}(P) - P$. Then

$$F_4 = P\left\{\text{Log}\left(\frac{\cos p}{P}\right) - 1\right\}$$

Also, from Eq. (2), we get

$$F_4 = \int \text{Log}\left(\frac{\frac{\sin p}{P}}{\cot p}\right) dP \;\rightarrow\; F_4 = \int \text{Log}\left(\frac{\sin p}{P}\cot p\right) dP \;\rightarrow\; F_4 = \int \text{Log}\left(\frac{\sin p}{P}\frac{\cos p}{\sin p}\right) dP$$

$$F_4 = \int \text{Log}\left(\frac{\cos p}{P}\right) dP \;\rightarrow\; F_4 = \int \left(\text{Log}\cos p - \text{Log}(P)\right) dP$$

$$F_4 = P\,\text{Log}(\cos p) - \left(P\,\text{Log}(P) - P\right)$$

$$F_4 = P\left\{\text{Log}(\cos p) - \text{Log}(P) - 1\right\}$$

$$F_4 = P\left\{\text{Log}\left(\frac{\cos p}{P}\right) - 1\right\}$$

$$\int Log(P)\,dP = P\,Log(P) - \int \frac{1}{P}(P)\,dP = P\,Log(P) - \int dP = P\,Log(P) - P$$

Example 3: Show that the following transformation is canonical,

$P = 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p, \quad Q = \log(1+\sqrt{q}\cos p)$?

Find the generating functions $F_3 = F_3(p_\alpha, Q_\alpha, t)$?

Solution

Using Poisson's brackets $\left\{Q_\alpha, P_\alpha\right\} = \frac{\partial Q}{\partial q_\alpha}\frac{\partial P}{\partial p_\alpha} - \frac{\partial Q}{\partial p_\alpha}\frac{\partial P}{\partial q_\alpha}$. We have

$$\left\{\log(1+\sqrt{q}\cos p),\; 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p\right\} = \frac{\partial(\log(1+\sqrt{q}\cos p))}{\partial q}\frac{\partial\left(2(1+\sqrt{q}\cos p)\sqrt{q}\sin p\right)}{\partial p} - \frac{\partial(\log(1+\sqrt{q}\cos p))}{\partial p}\frac{\partial\left(2(1+\sqrt{q}\cos p)\sqrt{q}\sin p\right)}{\partial q}$$

$$\left\{\log(1+\sqrt{q}\cos p),\ 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p\right\} = \frac{\partial(\log(1+\sqrt{q}\cos p))}{\partial q}\frac{\partial\left(2(\sin p+\sqrt{q}\cos p\sin p)\sqrt{q}\right)}{\partial p} - \frac{\partial(\log(1+\sqrt{q}\cos p))}{\partial p}\frac{\partial\left(2(\sqrt{q}+q\cos p)\sin p\right)}{\partial q}$$

$$\left\{\log(1+\sqrt{q}\cos p),\ 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p\right\} = \frac{\partial(\log(1+\sqrt{q}\cos p))}{\partial q}\frac{\partial\left(2(\sin p+\sqrt{q}\frac{1}{2}\sin 2p)\sqrt{q}\right)}{\partial p} - \frac{\partial(\log(1+\sqrt{q}\cos p))}{\partial p}\frac{\partial\left(2(\sqrt{q}+q\cos p)\sin p\right)}{\partial q}$$

$$\left\{\log(1+\sqrt{q}\cos p),\ 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p\right\} = \frac{\cos p}{2\sqrt{q}(1+\sqrt{q}\cos p)}\left(2(\cos p+\sqrt{q}\cos 2p)\sqrt{q}\right) - \frac{-\sqrt{q}\sin p}{(1+\sqrt{q}\cos p)}\left(2(\frac{1}{2\sqrt{q}}+\cos p)\sin p\right)$$

$$\left\{\log(1+\sqrt{q}\cos p),\ 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p\right\} = \frac{1}{(1+\sqrt{q}\cos p)}\left\{\begin{matrix}\left(\cos p+\sqrt{q}\cos 2p\right)\cos p + \\ \sqrt{q}\left(2(\frac{1}{2\sqrt{q}}+\cos p)\sin p\right)\sin p\end{matrix}\right\}$$

$$\left\{\log(1+\sqrt{q}\cos p),\ 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p\right\} = \frac{1}{(1+\sqrt{q}\cos p)}\left\{\begin{matrix}\cos^2 p+\sqrt{q}\cos p\cos 2p + \\ \sin^2 p+2\sqrt{q}\cos p\sin^2 p\end{matrix}\right\}$$

$$\left\{\log(1+\sqrt{q}\cos p),\ 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p\right\} = \frac{1}{(1+\sqrt{q}\cos p)}\left\{\begin{matrix}\cos^2 p+\sqrt{q}\cos p\left(\cos^2 p-\sin^2 p\right) \\ +\sin^2 p+2\sqrt{q}\cos p\sin^2 p\end{matrix}\right\}$$

$$\left\{\log(1+\sqrt{q}\cos p),\ 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p\right\} = \frac{1}{(1+\sqrt{q}\cos p)}\left\{\begin{matrix}\cos^2 p+\sqrt{q}\cos p\cos^2 p-\sqrt{q}\cos p\sin^2 p + \\ \sin^2 p+\sqrt{q}\cos p\sin^2 p+\sqrt{q}\cos p\sin^2 p\end{matrix}\right\}$$

$$\left\{\log(1+\sqrt{q}\cos p),\ 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p\right\} = \frac{1}{(1+\sqrt{q}\cos p)}\left\{\begin{matrix}\cos^2 p+\sqrt{q}\cos p\cos^2 p + \\ \sin^2 p+\sqrt{q}\cos p\sin^2 p\end{matrix}\right\}$$

$$\left\{\log(1+\sqrt{q}\cos p),\ 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p\right\} = \frac{1}{(1+\sqrt{q}\cos p)}\left\{\begin{matrix}(\cos^2 p+\sin^2 p) + \\ \sqrt{q}\cos p(\cos^2 p+\sin^2 p)\end{matrix}\right\}$$

$$\left\{\log(1+\sqrt{q}\cos p),\ 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p\right\} = \frac{1}{(1+\sqrt{q}\cos p)}\left\{1+\sqrt{q}\cos p\right\} = 1$$

$\left\{\log(1+\sqrt{q}\cos p),\ 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p\right\} = 1$, hence the transformation is canonical.

(ii) For the generating functions $F_3 = F_3(p_\alpha, Q_\alpha, t)$

$$q_\alpha = -\frac{\partial F_3}{\partial p_\alpha}, \qquad P_\alpha = -\frac{\partial F_3}{\partial Q_\alpha}$$

$$Q = \log(1+\sqrt{q}\cos p) \rightarrow e^Q = 1+\sqrt{q}\cos p \rightarrow e^Q - 1 = \sqrt{q}\cos p \rightarrow \frac{e^Q-1}{\cos p} = \sqrt{q}$$

$q = \dfrac{\left(e^Q-1\right)^2}{\cos^2 p}$. Therefore

$$q_\alpha = -\frac{\partial F_3}{\partial p_\alpha} \rightarrow -\frac{\left(e^Q-1\right)^2}{\cos^2 p} = \frac{\partial F_3}{\partial p} \rightarrow F_3 = -\int \frac{\left(e^Q-1\right)^2}{\cos^2 p} dp \tag{1}$$

$$P_\alpha = -\frac{\partial F_3}{\partial Q_\alpha} \rightarrow 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p = -\frac{\partial F_3}{\partial Q} \rightarrow F_3 = -\int 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p\, dQ \tag{2}$$

From the transformation $P = 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p, \quad Q = \log(1+\sqrt{q}\cos p)$, we have

From Eq. (1),

$$F_3 = -\left(e^Q-1\right)^2 \int \sec^2 p\, dp \rightarrow F_3 = -\left(e^Q-1\right)^2 \tan p$$

$$F_3 = -\left(e^Q-1\right)^2 \tan p$$

Also, from Eq. (2), we get

$$F_3 = -\int 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p\, dQ = -\int 2\left(1+\frac{e^Q-1}{\cos p}\cos p\right)\frac{e^Q-1}{\cos p}\sin p\, dQ$$

$$F_3 = -2\tan p\int\left(e^Q-1+\left(e^Q-1\right)^2\right)dQ = -2\tan p\int\left(e^Q-1+\left(e^{2Q}-2e^Q+1\right)\right)dQ$$

$$F_3 = -2\tan p\int\left(e^{2Q}-e^Q\right)dQ = -2\tan p\int\left(e^Q-1\right)e^Q dQ = -2\tan p\int\left(e^Q-1\right)d(e^Q)$$

$$F_3 = -2\,\frac{1}{2}\left(e^Q-1\right)^2\tan p \rightarrow F_3 = -\left(e^Q-1\right)^2\tan p$$

Chapter 7

Hamiltonian-Jacobi mechanics

Hamilton-Jacobi Equations

Introduction

Hamiltonian mechanics is an especially elegant and powerful way to derive the equations of motion for complicated systems. Unfortunately, integrating the equations of motion to derive a solution can be a challenge. Hamilton recognized this difficulty, so he proposed using generating functions to make canonical transformations which transform the equations into a known soluble form. Jacobi, a contemporary mathematician, recognized the importance of Hamilton's pioneering developments in Hamiltonian mechanics, and therefore he developed a sophisticated mathematical framework for exploiting the generating function formalism in order to make the canonical transformations required to solve Hamilton's equations of motion.

In the Lagrange formulation, transforming coordinates $(q_\alpha, \dot{q}_\alpha)$ to cyclic generalized coordinates $(Q_\alpha, \dot{Q}_\alpha)$, simplifies finding the Euler-Lagrange equations of motion. For the Hamiltonian formulation, the concept of coordinate transformations is extended to include simultaneous canonical transformation of both the spatial coordinates q_α and the conjugate momenta p_α from (q_α, p_α) to (Q_α, P_α), where both of the canonical variables are treated equally in the transformation. Compared to Lagrangian mechanics, Hamiltonian mechanics has twice as many variables which is an asset, rather than a liability, since it widens the realm of possible canonical transformations.

Hamiltonian mechanics has the advantage that generating functions can be exploited to make canonical transformations to find solutions, which avoids having to use direct integration. Canonical transformations are the foundation of Hamiltonian mechanics; they underlie Hamilton-Jacobi theory and action-angle variable theory, both of which are powerful means for exploiting Hamiltonian mechanics to solve problems in physics and engineering. The concept underlying canonical transformations is that, if the equations of motion are simplified by using a new set of generalized variables (Q, P), compared to using the original set of variables (q, p), then an advantage has been gained. The solution, expressed in terms of the generalized variables (Q, P) , can be transformed back to express the solution in terms of the original coordinates, (q, p) .

Only a specialized subset of transformations will be considered, namely canonical transformations that preserve the canonical form of Hamilton's equations of motion. That is, given that the original set of variables (q_α, p_α) satisfy Hamilton's equations.

If we consider a canonical transformation. where p_α and q_α is the old moments, while P_α and Q_α are new moments and coordinates, such that $P_\alpha = P_\alpha(p_\alpha, q_\alpha, t)$ and $Q_\alpha = Q_\alpha(p_\alpha, q_\alpha, t)$.

It is note that, the old system (Hamiltonian system) is given as

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}, \quad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha} \tag{1}$$

While, new system (Hamiltonian system), is given by

$$\dot{Q}_\alpha = \frac{\partial \bar{H}}{\partial P_\alpha}, \quad \dot{P}_\alpha = -\frac{\partial \bar{H}}{\partial Q_\alpha}, \tag{2}$$

Now, If we can be found a canonical transformation in which the new Hamiltonian function is equal to zero$\left(\bar{H} = 0\right)$,

Then, we can say that

$$\dot{Q}_\alpha = \dot{P}_\alpha = 0 \tag{3}$$

Therefore

$$Q_\alpha = P_\alpha = \text{constant} = \beta. \tag{4}$$

Let us, we consider the generating function $F = F_2(q_\alpha, P_\alpha, t)$ which produces the canonical transformation according to the formulas, in this case

$$p_\alpha = \frac{\partial F_2}{\partial q_\alpha}, \quad Q_\alpha = \frac{\partial F_2}{\partial P_\alpha}, \qquad \frac{\partial F_2}{\partial t} = \bar{H} - H. \tag{5}$$

Now, if we put $\bar{H} = 0$

$$\frac{\partial F_2}{\partial t} + H(p_\alpha, q_\alpha, t) = 0.$$

$$\frac{\partial F_2}{\partial t} + H\left(\frac{\partial F_2}{\partial q_\alpha}, q_\alpha, t\right) = 0. \tag{6}$$

This equation is called Hamilton-Jacobi equation (Hamilton-Jacobi equations), which can be written on the form

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial q_\alpha}, q_\alpha, t\right) = 0, \tag{7}$$

$$\frac{\partial S}{\partial q_\alpha} = p_\alpha \tag{8}$$

While,

$$\frac{\partial S}{\partial P_\alpha} = \frac{\partial S}{\partial \beta_\alpha} = \gamma_\alpha \tag{9}$$

Where

$P_\alpha = \beta_\alpha$ and $\alpha = 1, 2, 3, \ldots\ldots\ldots\ldots\ldots, n$.

Equation (7) (Hamilton-Jacobi equation) is a partial differential equation of the first order and is in the variables $(q_1, q_2, q_3, \ldots\ldots\ldots\ldots\ldots\ldots, \mathrm{t})$. Which their number is $(n+1)$ and therefore the general solution to this equation will contain $(n+1)$ the constants. By deleting one of these optional constants, we will have (n) of the constants, that are $(\beta_1, \beta_2, \beta_3, ..., \beta_n)$. In this case the solution will be in the form

$$S = S(q_1, q_2, q_3, ..., q_n,\ \ \beta_1, \beta_2, \beta_3, ..., \beta_n, t)$$

Using relations (8), (9) we can define (q_α) as a function of $(\beta_\alpha, \gamma_\alpha, t)$ and thus we have fully defined the mechanical system.

Special case of Hamilton-Jacobi equation

When the old Hamiltonian function does not depend on time, that is $H = H(p_\alpha, q_\alpha)$. Therefore, the Hamilton-Jacobi equation can be written in the form

$$\frac{\partial S(p_\alpha, P_\alpha, t)}{\partial t} + H\left(\frac{\partial S}{\partial q_\alpha}, q_\alpha\right) = 0. \qquad (1)$$

Then we assume that the solution to the Hamilton-Jacobi equation is in the form

$$S = s_1(q_1), + s_2(q_2) + s_3(q_3), ... + s_n(q_n), + s_{n+1}(t) \qquad (2)$$

Thus, Eq (1) becomes in the form

$$H\left(\frac{\partial S}{\partial q_\alpha}, q_\alpha\right) = -\frac{\partial S(t)}{\partial t} \qquad (3)$$

The right side of Eq. (3) depends on time only, while the left side depends on $q_1, q_2, q_3, ..., q_n$ only. Therefore, each of the two sides is equal to a constant (E) where (E) represents the total energy of the mechanical group, i.e.

$$S_{n+1}(t) = -E\,t \ ,$$

$$H\left(\frac{\partial S}{\partial q_\alpha}, q_\alpha\right) = E$$

Example1: Using Hamiltonian-Jacobi system study the motion of harmonic oscillator in one dimension $(1-D)$ and prove that the particle's distance from the equilibrium position is given by $c_1 \sin c_2(t + c_3)$, where c_1, c_2, c_3 are constants?

Answer

It is well-known that the Hamilton-Jacobi system for solving a mechanical system can be written in the form

$$\frac{\partial S(q,\beta,t)}{\partial t}+H(q_\alpha,p_\alpha,t)=0, \qquad \frac{\partial S}{\partial q_\alpha}=p_\alpha, \qquad \frac{\partial S}{\partial \beta_\alpha}=\gamma_\alpha, \tag{1}$$

For the harmonic oscillator in one dimension, the kinetic energy and potential energy are given, respectively, as

$T=\frac{1}{2}m\,x^{\bullet 2}$, $V=\frac{1}{2}k\,x^2$, where m is mass of particle.

Therefore, the total energy is given as $E=T+V=\frac{1}{2}m\,x^{\bullet 2}+\frac{1}{2}k\,x^2$

The Lagrange's function is given as $L=T-V=\frac{1}{2}m\,x^{\bullet 2}-\frac{1}{2}k\,x^2$ (2)

Since time does not appear explicitly in the Lagrange's function, the Hamiltonian function is given in the form

$$H=T+V=\frac{1}{2}m\,x^{\bullet 2}+\frac{1}{2}k\,x^2+E \tag{3}$$

From the relation $p_\alpha=\frac{\partial L}{\partial q_\alpha^\bullet}$, we find that $p_x=\frac{\partial L}{\partial x^\bullet}$

$$p_x=m\,x^\bullet \rightarrow x^\bullet=\frac{p_x}{m} \tag{4}$$

From Eq. (4) into Eq. (3), we get

$$H=T+V=\frac{1}{2}m\left(\frac{p_x}{m}\right)^2+\frac{1}{2}k\,x^2=\frac{1}{2}m\frac{p_x^2}{m}+\frac{1}{2}k\,x^2 \tag{5}$$

Therefore, the Hamilton-Jacobi equation $\frac{\partial S(q,\beta,t)}{\partial t}+H(q_\alpha,p_\alpha,t)=0$ is taken the form

$$\frac{\partial S(q,\beta,t)}{\partial t}+\frac{1}{2}\frac{p_x^2}{m}+\frac{1}{2}k\,x^2=0 \tag{6}$$

Then, the solution of Eq. (6), maybe written in the form (We have one coordinate)

$$s=s_1(x)+s_2(t) \tag{7}$$

From the Hamilton-Jacobi system $\frac{\partial S}{\partial q_\alpha}=p_\alpha$ and where our problem in $x-$coordinate, we have $\frac{\partial S}{\partial q_1}=p_1 \;\rightarrow\; \frac{\partial s}{\partial x}=p_x$, that can be written from Eq. (7) in the form

$$\frac{\partial s_1}{\partial x}=p_x \tag{8}$$

From Eq. (7) and (8), we can write Eq. (6) as

$$\frac{\partial s_2}{\partial t}+\frac{1}{2m}\left(\frac{\partial s_1}{\partial x}\right)^2+\frac{1}{2}k\,x^2=0$$

But every term in this equation is independent each other. Then

$$\frac{1}{2m}\left(\frac{\partial s_1}{\partial x}\right)^2+\frac{1}{2}k\,x^2=-\frac{\partial s_2}{\partial t}=\beta_2 \tag{9}$$

Therefore,

$$-\frac{\partial s_2}{\partial t}=\beta_2 \rightarrow \quad s_2=-\beta_2\,t+c_1(=0) \;\rightarrow\; \quad s_2=-\beta_2\,t \tag{10}$$

$$\frac{1}{2m}\left(\frac{\partial s_1}{\partial x}\right)+\frac{1}{2}k\,x^2=\beta_2$$

$$\frac{1}{2m}\left(\frac{\partial s_1}{\partial x}\right)=\beta_2-\frac{1}{2}k\,x^2$$

$$\frac{\partial s_1}{\partial x}=\sqrt{2m}\;\sqrt{\beta_2-\frac{1}{2}k\,x^2}$$

$$s_1=\sqrt{2m}\int\sqrt{\beta_2-\frac{1}{2}k\,x^2}\;dx+c_2(=0) \tag{11}$$

Now, from Eq. (10) and Eq. (11), the solution of Eq. (5) becomes in the form

$$s = s_1 + s_2 = \sqrt{2m}\int\sqrt{\beta_2 - \frac{1}{2}k\,x^2}\,dx - \beta_2\,t \qquad (12)$$

Again, from the Hamilton-Jacobi system $\frac{\partial s}{\partial \beta_\alpha} = \gamma_\alpha$. Then $\frac{\partial s}{\partial \beta_2} = \gamma_2$ and from Eq. (12), we get

$$\frac{\partial}{\partial \beta_2}\left\{\sqrt{2m}\int\sqrt{\beta_2 - \frac{1}{2}k\,x^2}\,dx - \beta_2\,t\right\} = \gamma_2$$

$$\sqrt{2m}\int\frac{1}{2\sqrt{\beta_2 - \frac{1}{2}k\,x^2}}\,dx - t = \gamma_2$$

$$\sqrt{2m}\int\frac{1}{2\sqrt{\beta_2}\sqrt{1 - \frac{1}{2\beta_2}k\,x^2}}\,dx - t = \gamma_2$$

$$\sqrt{2m}\int\frac{1}{2\sqrt{\beta_2}\sqrt{1 - \left(\sqrt{\frac{1}{2\beta_2}}k\,x\right)^2}}\,dx - t = \gamma_2$$

$$\frac{\frac{\sqrt{2m}}{2\sqrt{\beta_2}}\sin^{-1}\left(\sqrt{\frac{k}{2\beta_2}}\right)x}{\sqrt{\frac{k}{2\beta_2}}} - t = \gamma_2 \quad \to \sqrt{\frac{m}{k}}\sin^{-1}\left(\sqrt{\frac{k}{2\beta_2}}\right)x - t = \gamma_2 \quad \to \sin^{-1}\left(\sqrt{\frac{k}{2\beta_2}}\right)x = \sqrt{\frac{k}{m}}\left(\gamma_2 + t\right)$$

$$\left(\sqrt{\frac{k}{2\beta_2}}\right)x = \sin\sqrt{\frac{k}{m}}\left(\gamma_2 + t\right) \to \quad x = \sqrt{\frac{2\beta_2}{k}}\sin\sqrt{\frac{k}{m}}\left(\gamma_2 + t\right)$$

Therefore

$$x = c_1 \sin c_2\left(c_3 + t\right)$$

Where,

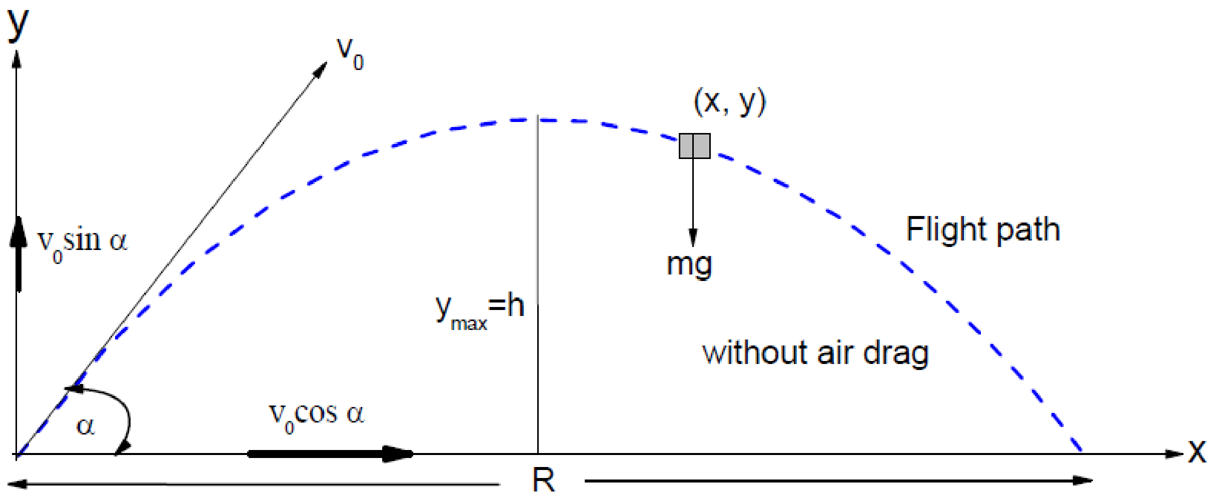
$$c_1 = \sqrt{\frac{2\beta_2}{k}}, \quad c_2 = \sqrt{\frac{k}{m}}, \quad c_3 = \gamma_2$$

Example 2: A particle of mass m is projected with initial velocity v_0 at an angle α to the horizontal in the uniform gravitational field of the earth. Use Hamiltonian-Jacobi system to describe the motion of the projectile. Ignore the air resistance?

Solution

It is well-known that the Hamilton-Jacobi system for solving a mechanical system can be written in the form

$$\frac{\partial S(q,\beta,t)}{\partial t} + H(q_\alpha, p_\alpha, t) = 0, \qquad \frac{\partial S}{\partial q_\alpha} = p_\alpha, \qquad \frac{\partial S}{\partial \beta_\alpha} = \gamma_\alpha, \tag{1}$$



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Let a particle of mass m be projected from the origin point with an initial velocity v_0

making an angle α with the horizontal line referred as x-axis. Let (x, y) be the position of the particle at any instant t. Since x and y are independent and hence the generalized coordinates are $(q_1, q_2) = (x\ ,\ y)$ and the generalized velocities are

$(\dot{q}_1, \dot{q}_2) = (\dot{x}, \dot{y})$.

Kinetic Energy

The kinetic of the projectile is given by $T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$,

The Potential Energy

The total of potential energy is given as

$$F = -mg \;\rightarrow\; F = -mg = -\nabla V \;\rightarrow\; mg = \frac{dV}{dy} \;\rightarrow\; V = mg\,y$$

Lagrange Function

The Lagrange Function can be written as

$$L = T - V = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - mg\,y \tag{2}$$

$$H = p_x \dot{q}_x + p_y \dot{q}_y - L = p_x \dot{x} + p_y \dot{y} - L = p_x \dot{x} + p_y \dot{y} - \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + mg\,y \tag{3}$$

$p_\alpha = \dfrac{\partial L}{\partial \dot{q}_\alpha}$. Then

$$p_x = \frac{\partial L}{\partial \dot{x}}, \quad p_y = \frac{\partial L}{\partial \dot{y}} \tag{4}$$

From Eq. (4) into Eq. (1), we have

$$p_x = m\dot{x} \rightarrow \dot{x} = \frac{p_x}{m} \tag{5}$$

$$p_y = m\dot{y} \rightarrow \dot{y} = \frac{p_y}{m} \tag{6}$$

From Eq. (5) and Eq. (6), then Hamilton Function (Eq. 3) becomes

$$H = p_x\left(\frac{p_x}{m}\right) + p_y\left(\frac{p_y}{m}\right) - \frac{1}{2} m\left\{\left(\frac{p_x}{m}\right)^2 + \left(\frac{p_y}{m}\right)^2\right\} + mg\,y$$

$$H = \frac{p_x^2}{m} + \frac{p_x^2}{m r^2} - \frac{1}{2} m\left\{\left(\frac{p_x}{m}\right)^2 + \left(\frac{p_y}{m}\right)^2\right\} + mg\,y$$

$$H = \frac{1}{2m}\left\{p_x^2 + p_y^2\right\} + mg\,y \tag{7}$$

From Hamiltonian-Jacobi system (1) and Eq. (7), then Hamilton-Jacobi equation

$\frac{\partial S(q,\beta,t)}{\partial t} + H(q_\alpha, p_\alpha, t) = 0$ becomes

$$\frac{\partial S(q,\beta,t)}{\partial t} + \frac{1}{2m}\left\{ p_x^2 + p_y^2 \right\} + mg\, y = 0 \tag{8}$$

Then, the solution of Eq. (8), maybe written in the form (We have two coordinates)

$$s = s_1(x) + s_2(y) + s_3(t) \tag{9}$$

From the Hamilton-Jacobi system $\frac{\partial S}{\partial q_\alpha} = p_\alpha$ and where our problem has two

coordinates x and y, then

$$\frac{\partial S}{\partial q_1} = p_1 \;\rightarrow\; \frac{\partial s_1}{\partial x} = p_x \tag{10}$$

$$\frac{\partial S}{\partial q_2} = p_2 \;\rightarrow\; \frac{\partial s_2}{\partial y} = p_y \tag{11}$$

Now, from Eq. (9), (10) and (11), we can write Eq. (8) in the from

$$\frac{\partial s_3}{\partial t} + \frac{1}{2m}\left\{ \left(\frac{\partial s_1}{\partial x}\right)^2 + \left(\frac{\partial s_2}{\partial y}\right)^2 \right\} + mg\, y = 0 \tag{12}$$

This equation has two parts, one of these parts depends on time and the other on the coordinates, so each of them can be placed on a fixed form in the form

$$\frac{1}{2m}\left\{ \left(\frac{\partial s_1}{\partial x}\right)^2 + \left(\frac{\partial s_2}{\partial y}\right)^2 \right\} + mg\, y = -\frac{\partial s_3}{\partial t} = \beta_3 \tag{13}$$

That can be written as

$$-\frac{\partial s_3}{\partial t} = \beta_3 \rightarrow \quad s_3 = -\beta_3 t + c_1 (=0) \;\rightarrow\; \quad s_3 = -\beta_3 t \tag{14}$$

Also, from Eq. (13), we have

$$\frac{1}{2m}\left\{\left(\frac{\partial s_1}{\partial x}\right)^2+\left(\frac{\partial s_2}{\partial y}\right)^2\right\}+mg\,y=\beta_3$$

$$\left(\frac{\partial s_2}{\partial y}\right)^2+2m\left\{mg\,y-\beta_3\right\}=-\left(\frac{\partial s_1}{\partial x}\right)^2 \quad (15)$$

From Eq. (15), we have

$$\left(\frac{\partial s_1}{\partial x}\right)=\beta_1 \quad (16)$$

From (16), we get

$$ds_1=\beta_1\,dx \;\;\rightarrow\;\; s_1=\beta_1\,x+c_2(=0) \;\;\rightarrow\;\; s_1=\beta_1\,x$$

Then

$$s_1=\beta_1\,x \quad (17)$$

From Eq. (16) into Eq. (15), we have

$$\left(\frac{\partial s_2}{\partial y}\right)^2+2m\left\{mg\,y-\beta_3\right\}=-\beta_1^2 \quad (18)$$

$$s_2=\int\left\{\sqrt{-2m\left\{mg\,y-\beta_3\right\}-\beta_1^2}\right\}dy+c_3(=0)$$

$$s_2=\int\left\{\sqrt{-2m\left\{mg\,y-\beta_3\right\}-\beta_1^2}\right\}dy \quad (19)$$

Now, from Eq. (14), Eq. (17) and Eq. (19), we can write Eq. (9) in the form

$$s=s_1+s_2+s_3=-\beta_3\,t+\beta_1\,x+\int\left\{\sqrt{2m\left\{\beta_3-mg\,y\right\}-\beta_1^2}\right\}dy \quad (20)$$

Again, from the Hamilton-Jacobi system $\frac{\partial s}{\partial \beta_\alpha} = \gamma_\alpha$, we get

$$\frac{\partial s}{\partial \beta_1} = \gamma_1, \qquad \frac{\partial s}{\partial \beta_2} = \gamma_2, \qquad \frac{\partial s}{\partial \beta_3} = \gamma_3 \tag{21}$$

From Eq. (20) and $\frac{\partial s}{\partial \beta_1} = \gamma_1$, we get

$$\frac{\partial}{\partial \beta_1}\left\{-\beta_3 t + \beta_1 x + \int \left\{\sqrt{2m\left\{\beta_3 - mg\, y\right\} - \beta_1^2}\right\} dy\right\} = \gamma_1 \rightarrow \qquad x + \int \frac{-\beta_1}{\sqrt{2m\left\{\beta_3 - mg\, y\right\} - \beta_1^2}}\, dy = \gamma_1$$

$$x + \int (-\beta_1)\left[2m\left\{\beta_3 - mg\, y\right\} - \beta_1^2\right]^{-\frac{1}{2}} dy = \gamma_1 \rightarrow \qquad x + \frac{(-\beta_1)\left[2m\left\{\beta_3 - mg\, y\right\} - \beta_1^2\right]^{\frac{1}{2}}}{\frac{1}{2}(-2m^2 g)} = \gamma_1$$

$$x + \frac{\beta_1\left[2m\left\{\beta_3 - mg\, y\right\} - \beta_1^2\right]^{\frac{1}{2}}}{m^2 g} = \gamma_1$$

$$\beta_1\left[2m\left\{\beta_3 - mg\, y\right\} - \beta_1^2\right]^{\frac{1}{2}} = m^2 g\,(\gamma_1 - x) \tag{22}$$

Second time from Eq. (20) and $\frac{\partial s}{\partial \beta_3} = \gamma_3$, we get

$$\frac{\partial}{\partial \beta_3}\left\{-\beta_3 t + \beta_1 x + \int \left\{\sqrt{2m\left\{\beta_3 - mg\, y\right\} - \beta_1^2}\right\} dy\right\} = \gamma_3$$

$$-t + \int \frac{2m}{2\sqrt{2m\left\{\beta_3 - mg\, y\right\} - \beta_1^2}}\, dy = \gamma_3$$

$$-t+m\int\left[2m\left\{\beta_3-mg\,y\right\}-\beta_1^2\right]^{-\frac{1}{2}}dy=\gamma_3$$

$$-t+\frac{m\left[2m\left\{\beta_3-mg\,y\right\}-\beta_1^2\right]^{\frac{1}{2}}}{\frac{1}{2}(-2m^2g)}=\gamma_3$$

$$-t-\frac{\left[2m\left\{\beta_3-mg\,y\right\}-\beta_1^2\right]^{\frac{1}{2}}}{mg}=\gamma_3$$

$$-\left[2m\left\{\beta_3-mg\,y\right\}-\beta_1^2\right]^{\frac{1}{2}}=m^2g^2\left\{\gamma_3+t\right\} \tag{23}$$

Now from Eqs. (22) and (23), we have for constants $\beta_1,\ \beta_3,\ \gamma_1,\ \gamma_3$, therefore we need for four condition to find these constants.

Squared both two Eqs. (22) and (23), we get

$$\beta_1^2\left[2m\left\{\beta_3-mg\,y\right\}-\beta_1^2\right]=m^4g^2\,(\gamma_1-x)^2 \tag{24}$$

$$2m\left\{\beta_3-mg\,y\right\}-\beta_1^2=m^2g^2\left\{\gamma_3+t\right\}^2 \tag{25}$$

Derivative with respect to both two Eqs. (24) and (25), we get

$$-2m^2g\beta_1^2\,y^{\bullet}=-2m^4g^2\,(\gamma_1-x)\,x^{\bullet} \tag{26}$$

$$-2m^2g\,y^{\bullet}=2m^2g^2\left\{\gamma_3+t\right\} \tag{27}$$

From the initial condition and at $t = 0$, we have $x = y = 0$ and substituting in Eqs. (24) and (25), we get

$$\beta_1^2 \left[2m\beta_3 - \beta_1^2 \right] = m^4 g^2 \gamma_1^2 \tag{28}$$

$$2m\beta_3 - \beta_1^2 = m^2 g^2 \gamma_3^2 \tag{29}$$

Again, From the initial condition and at $t = 0$, we have $x^{\bullet} = v_0 \cos\alpha,\ y^{\bullet} = v_0 \sin\alpha$ and substituting in Eqs. (26) and (27), we get

$$-2m^2 g \beta_1^2 v_0 \sin\alpha = -2m^4 g^2 (\gamma_1 - 0) v_0 \cos\alpha$$

$$\beta_1^2 \sin\alpha = m^2 g \gamma_1 \cos\alpha \ \rightarrow \ \gamma_1 = \frac{\beta_1^2}{m^2 g} \frac{\sin\alpha}{\cos\alpha} = \frac{\beta_1^2}{m^2 g} \tan\alpha$$

$$\gamma_1 = \frac{\beta_1^2}{m^2 g} \tan\alpha \tag{30}$$

$$-2m^2 g v_0 \sin\alpha = 2m^2 g^2 \left\{ \gamma_3 + 0 \right\}$$

$$\gamma_3 = -\frac{v_0}{g} \sin\alpha \tag{31}$$

Substituting (30) into (28), we get

$$\beta_1^2 \left[2m\beta_3 - \beta_1^2 \right] = m^4 g^2 \left(\frac{\beta_1^2}{m^2 g} \tan\alpha \right)^2$$

$$\left[2m\beta_3 - \beta_1^2 \right] = \beta_1^2 \left(\tan\alpha \right)^2 \rightarrow 2m\beta_3 = \beta_1^2 \left(1 + \left(\tan\alpha \right)^2 \right)$$

$$2m\beta_3 - \beta_1^2 = \beta_1^2 \left(\tan\alpha \right)^2$$

$$2m\beta_3 = \beta_1^2\left(1+\left(\tan\alpha\right)^2\right) \tag{32}$$

Substituting from (31) into (29), we get

$$2m\beta_3 - \beta_1^2 = m^2 g^2\left(-\frac{v_0}{g}\sin\alpha\right)^2$$

$$2m\beta_3 - \beta_1^2 = m^2\left(v_0\sin\alpha\right)^2 \tag{33}$$

From Eq. (33) into Eq. (32), we get

$$m^2\left(v_0\sin\alpha\right)^2 = \beta_1^2\left(\tan\alpha\right)^2 \rightarrow \beta_1^2 = \frac{m^2\left(v_0\sin\alpha\right)^2}{\left(\tan\alpha\right)^2} \rightarrow \beta_1^2 = \frac{m^2\left(v_0\sin\alpha\right)^2}{\left(\frac{\sin\alpha}{\cos\alpha}\right)^2} = m^2\left(v_0\cos\alpha\right)^2$$

$$\beta_1 = \pm m v_0\cos\alpha \tag{34}$$

Substituting (34) into (33), we get

$$\gamma_1 = \frac{\left(m v_0\cos\alpha\right)^2}{m^2 g}\tan\alpha$$

$$\gamma_1 = \frac{v_0^2}{g}\sin\alpha\cos\alpha \tag{35}$$

Substituting (34) into (33), we get

$$2m\beta_3 - \beta_1^2 = m^2\left(v_0\sin\alpha\right)^2 \rightarrow 2m\beta_3 - \left(m v_0\cos\alpha\right)^2 = m^2\left(v_0\sin\alpha\right)^2 \rightarrow 2m\beta_3 = m^2 v_0^2\left(\sin\alpha+\cos\alpha\right)^2$$

$$\beta_3 = \frac{m v_0^2}{2} \tag{36}$$

Subtracting Eq. (25) from Eq. (24), we get

$$\frac{m^4 g^2 (\gamma_1 - x)^2}{\beta_1^2} = m^2 g^2 \left\{ \gamma_3 + t \right\}^2$$

$$m^2 (\gamma_1 - x)^2 = \beta_1^2 \left\{ \gamma_3 \ + t \right\}^2$$

$$m(\gamma_1 - x) = \beta_1 \left\{ \gamma_3 \ + t \right\} \tag{37}$$

Substituting by $\gamma_1, \gamma_3, \beta_1$ in Eq. (36), we get

$$m(\gamma_1 - x) = \beta_1 \left\{ \gamma_3 + t \right\} \rightarrow m(\frac{v_0^2}{g} \sin\alpha \cos\alpha - x) = (\pm m v_0 \cos\alpha) \left\{ -\frac{v_0}{g} \sin\alpha + t \right\}$$

$$\underbrace{\frac{v_0^2}{g} \sin\alpha \cos\alpha}_{1+3=0} - x = \underbrace{\frac{v_0^2}{g} \sin\alpha \cos\alpha}_{1+3=0} - v_0 t \cos\alpha$$

$$x = v_0 t \cos\alpha \tag{38}$$

Substituting by $\gamma_3, \beta_1, \beta_3$ in Eq. (25), we get

$$2m \left\{ \beta_3 - mg\, y \right\} - \beta_1^2 = m^2 g^2 \left\{ \gamma_3 + t \right\}^2 \rightarrow 2m \left\{ \beta_3 - mg\, y \right\} - \beta_1^2 = m^2 g^2 \left\{ \gamma_3^2 + 2\gamma_3 \ t + t^2 \right\}$$

$$2m \left\{ \frac{m v_0^2}{2} - mg\, y \right\} - \left(m v_0 \cos\alpha \right)^2 = m^2 g^2 \left\{ \left(-\frac{v_0}{g} \sin\alpha \right)^2 + 2 \left(-\frac{v_0}{g} \sin\alpha \right) t + t^2 \right\}$$

$$v_0^2 - 2g\, y - \left(v_0 \cos\alpha \right)^2 = \left(-v_0 \sin\alpha \right)^2 - 2\, g\, v_0\, t \sin\alpha + g^2 t^2$$

$$v_0^2 (1 - \cos^2\alpha) - 2g\, y = \left(-v_0 \sin\alpha \right)^2 - 2\, g\, v_0\, t \sin\alpha + g^2 t^2$$

$$\underbrace{v_0^2 \sin^2\alpha}_{1+3=0} - 2g\, y = \underbrace{v_0^2 \sin^2\alpha}_{1+3=0} - 2\, g\, v_0\, t \sin\alpha + g^2 t^2$$

$$2g\,y = -g^2 t^2 + 2\,g\,v_0\,t\sin\alpha$$

$$y = -\frac{1}{2}gt^2 + v_0\,t\sin\alpha$$

Then

$$y = v_0\,t\sin\alpha - \frac{1}{2}gt^2 \qquad (39)$$

The two equations (38), (39) represent the required dimensions