South valley University

Faculty of science-Qena
Mathematics Department

Statics: Lecture Notes

Part I

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Contents

Chapter: 1
Introduction

General principles

Mechanics: Mechanics is a branch of physical sciences which describes or predicts the conditions of rest or motion of bodies under the action of forces.

Mechanics: Mechanics is a branch of physical sciences that is concerned with the state of rest or motion of bodies that are subjected to the action of forces.

Mechanics can be subdivided into three branches: rigid body mechanics- deformable body mechanics- fluid mechanics

1- Rigid bodies: (i) Statics (ii) Dynamics 2- Deformable bodies

3- Fluid Mechanics: (i) Compressible – gas (ii) incompressible - liquids

Here we will study only the rigid body Mechanics. In the first term we will study some subjects in Statics

In Statics we will assume the bodies to be perfectly rigid, no deformation.

This is never true in the real world, everything deforms a little when a load is applied.

These deformations are small and will not significantly affect the conditions of equilibrium or motion, so we will neglect the deformations.

Basic Quantities

Basic Concepts: There are four basic quantities in Mechanics space, time, mass, force:

(1) - Length: Length is used to locate the position of a point and describe the size of physical systems.

(2) - Time: Time is the measure of the succession of events and it is important in Dynamics

(3) - Mass: Mass is a measure of a quantity of matter that is used to compare the action of one body with that of another. This property manifests itself as a gravitational attraction between two bodies and provides a measure of the resistance of matter to a change in velocity.

(4) - Force: Force is push or pull exerted by one body on anther. This interaction can occur when there is direct contact between the bodies, such as a person pushing on a wall, or it can occur through a distance when the bodies are physically separated.

Idealizations: Models or idealizations are used in mechanics in order to simplify application of the theory. Here we will consider three important idealizations.

(i) - Particle: A particle has a mass, but a size that can be neglected. For example, the size of the earth is insignificant compared to the size of its orbit, and therefore the earth can be modeled as a particle when studying its orbital motion.

(ii) - Rigid Body: A rigid body is a combination of a large number of particles in which all the particles remain at a fixed distance from one another, both before and after applying a load

A rigid body is considered rigid when the relative movement between its parts is negligible.

(iii) - Concentrated Force. A concentrated force represents the effect of a loading which is assumed to act at a point on a body when the contact area is small compared with the overall size. We can represent a load by a concentrated force, provided the area over which the load is applied is very small compared to the overall size of the body. An example would be the contact force between a wheel and the ground

Weight and Mass

Weight is the measure of how heavy an object

The unit of measurement for weight is that of force, which in the International System of Units (SI) is the newton.

Mass is both a property of a physical body and a measure of its resistance to acceleration (a change in its state of motion) when a net force is applied (The mass of an object is the amount of material it contains.). An object's mass also determines the strength of its gravitational attraction to other bodies.

The unit of measurement for mass in the International System of Units (SI) is the kilogram (kg).

Weight is not the same thing as mass. Mass is a literal representation of the amount of matter in a particle or object, and is independent of external factors such as speed, acceleration, or applied force (as long as relativistic effects are small enough to be neglected). Weight has meaning only when an object having a specific mass is placed in an acceleration field. At the Earth's surface, a kilogram mass weighs about 2.2 pounds, for example. But on Mars, the same kilogram mass would weigh only about 0.8 pounds

Newton's Three Laws of motion:

Engineering Mechanics is formulated on the basis of Newton's three Laws of motion:

First Law (1st Law):

A particle originally at rest, or moving in a straight line with constant velocity, tends to remain in this state provided the particle is not subjected to an unbalanced force.
A particle remains at rest or continues to move in a straight line with a constant speed if there is no unbalanced force acting on it (resultant force = 0).

Second Law (2nd Law):

A particle acted upon by an unbalanced force (F) experiences an acceleration (a) that has the same direction as the force and a magnitude that is directly proportional to the force.
If (F) is applied to a particle of mass (M), this law may be expressed mathematically as

$$F = M\,a$$

Third Law (3rd Law):

For every action there is an equal and opposite reaction.
The mutual forces of action and reaction between two particles are equal, opposite, and collinear

The forces of action and reaction between interacting bodies are equal in magnitude, opposite in direction, and act along the same line of action (Collinear).

Newton's Law of Gravitation Attraction

The gravitational attraction force between any two particles is

$$F = G\frac{M\,m}{r^2},$$

F = mutual force of attraction between two particles
G = universal constant known as the constant of gravitation
M, m = masses of each of the two particles
r = distance between the two particles

What am I talking about? Weight.

The weight of a particle is the gravitational force between a particle and earth;

By using the equation $g = \dfrac{G\,M}{r^2}$, where

M = mass of earth, m = mass of a particle

r = radius of earth, g = acceleration of gravity at earth's surface

Using $g = \frac{GM}{r^2} \rightarrow G = \frac{g r^2}{M}$

Substituting into $F = G\frac{Mm}{r^2}$, we have $F = \frac{g r^2}{M}\left(\frac{Mm}{r^2}\right)$

$F = mg \Rightarrow$ weight is $W = mg$

Or using $F = ma$ and at the surface of the Earth $a = g$

$\therefore \quad F = mg$, *then* $W = mg$

g is dependent upon r. Most cases use g = 9.81 m/s² = 32.2 ft/ s²

Units of Measurement

A unit of measurement is a definite magnitude of a physical quantity.

There are two main measurement systems:

(1)- Metric system (International system SI):

This system is based on three main units:

Meter – Kilogram – Second (It is called mks System).

SI is an abbreviation of French expression (**S**ysteme **I**nternational d'Unités) in English (International system)

(2) – English system (British System or Imperial System or US Customary System)

This system is based on three main units:

Foot - Pond - Second (It is also called FPS system). See below Table

Name	Length	Time	Mass	Force
International system of Units (SI)	Meter	Second	Kilogram	Newton
	m	s	kg	$N = \frac{kg.m}{s^2}$
U. S. Customary FPS	Foot	Second	Slug	Pound
	ft	s	$N = \frac{Ib.s^2}{ft}$	Ib

Pound (Ib) , unit of avoirdupois weight, equal to 16 ounces, 7,000 grains, or 0.45359237 kg,

Newton

The newton is a unit to for measuring force equal to the force needed to move one kilogram of mass at a rate of one meter per second squared.
The newton is the SI derived unit for force in the metric system. Newtons can be abbreviated as N , for example 1 newton can be written as $1N$.
Newtons can be expressed using the formula: (1) $N = (1\,\text{kg})\ (1\,\text{m/ s}^2)$.

Pound-Force

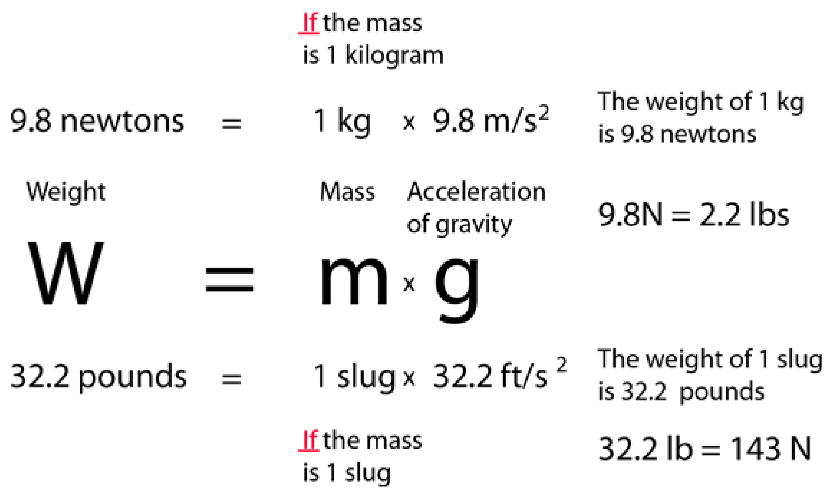
Pound-force is a unit of force equal to the force needed to move one pound of mass at a rate of 32.174049. 32 foot per second squared.

The pound-force is a US customary and imperial unit of force. A pound-force is sometimes also referred to as a pound of force. Pound-force can be abbreviated as IbF or Ib_F . For example, 1 pound-force can be written as $1IbF$ or $1Ib_F$.

Pound-force can be expressed using the formula: $1Ib = 32.174049\text{lb}\ \frac{\text{ft}}{\text{s}^2}$.

How to convert kilograms to pounds

1 kilogram = 2.2046226218488 pounds
1 pound = 0.45359237 kilograms



Force: Newton (N)

(1) $N = (1\,\text{kg})\ (1\,\text{m/ s}^2)$

1 Newton is the force required to give a mass of 1 kg an acceleration of 1 m/ s².

Weight is a force. The weight of 1 kg Mass is:

$$W = \text{m g} \Rightarrow \text{W} = (1\,\text{kg})\ (9.81\,\text{m/ s}^2) = \ 9.81\ \ \text{N}$$

Chapter: 2
Vectors Forces
Part 1: Vectors in 2D and 3D

Introduction

Statics: The study of bodies when they are at rest and all forces are in equilibrium. Static equations are often used in truss problems. To solve a static equation, engineers use a free body diagram. If an object is at rest, as it is in statics, the sum of the forces acting upon the object will equal zero. The sum of the moments will also equal zero.

Scalars and Vectors

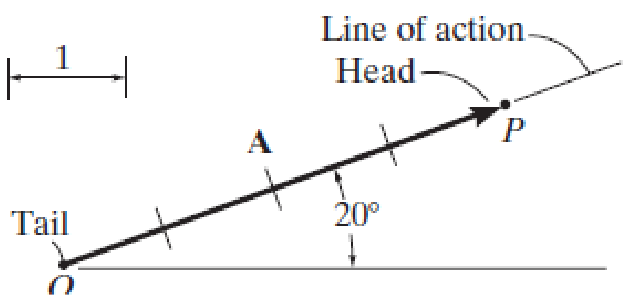
Scalar: A scalar is any positive or negative physical quantity that can be completely specified by its magnitude

Examples: Examples of scalar quantities include length, mass, and time.

Vector: A vector is any physical quantity that requires both a magnitude and a direction for its complete description.

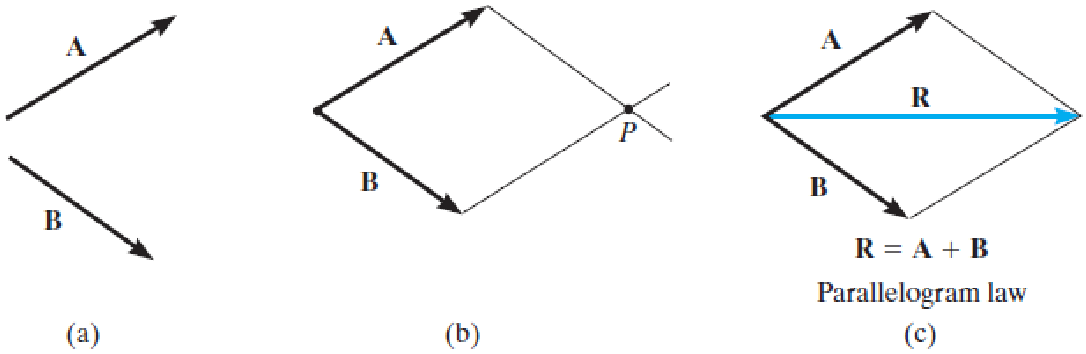
Examples(For instance): Examples of vectors encountered in statics are force, position, and moment.

Vector A vector is shown graphically by an arrow. The length of the arrow represents the magnitude of the vector, and the angle between the vector and a fixed axis defines the direction of its line of action . The head or tip of the arrow indicates the sense of direction of the vector (see below Figure)



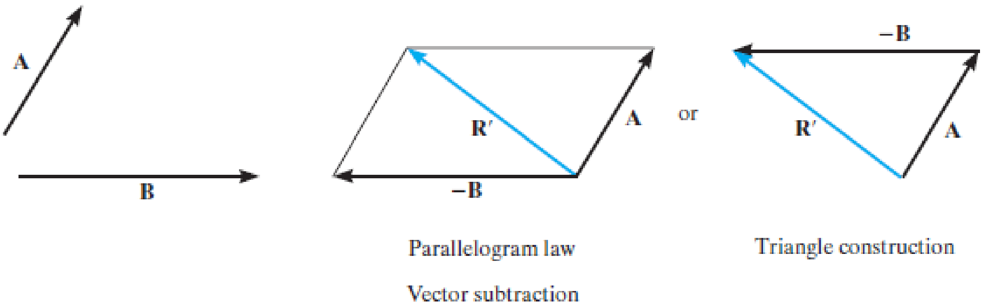
Vector Addition:

All vector quantities obey the parallelogram law of Addition



$$\vec{R} = \vec{A} + \vec{B}$$

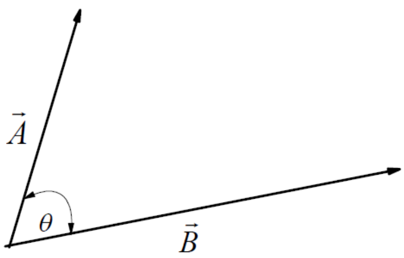
Vector Subtraction. The resultant of the difference between two vectors A and B of the same type may be expressed as



$$\vec{R} = \vec{A} - \vec{B} = +(-\vec{B})$$

Dot Product

The Dot Product gives a scalar (ordinary number) answer, and is sometimes called the scalar product.



The Dot product define as $\vec{A} \bullet \vec{B} = AB\cos\theta$

Laws of Operation

1. Commutative law : $\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A}$

2. Multiplication by a scalar $\lambda\left(\vec{A} \bullet \vec{B}\right) = \left(\lambda \vec{A}\right) \bullet \vec{B} = \vec{A} \bullet \left(\lambda \vec{B}\right) = \left(\vec{A} \bullet \vec{B}\right)\lambda$

3. Distribution law : $\vec{A}\bullet\left(\vec{B}+\vec{D}\right)=\left(\vec{A}\bullet\vec{B}\right)+\left(\vec{A}\bullet\vec{D}\right)$

4. Cartesian Vector Formulation

Dot product of two vectors $\vec{A}=a_X\vec{I}+a_Y\vec{J}+a_Z\vec{K}$ and $\vec{B}=b_X\vec{I}+b_Y\vec{J}+b_Z\vec{K}$,

then . $\vec{A}\cdot\vec{B}=a_Xb_X+a_Yb_Y+a_Zb_Z$

5. Applications: The angle formed between two vectors given by

$$\cos\theta=\frac{\vec{A}\bullet\vec{B}}{\left|\vec{A}\right|\left|\vec{B}\right|}=\frac{a_Xb_X+a_Yb_Y+a_Zb_Z}{\sqrt{a_X{}^2+a_Y{}^2+a_Z{}^2}\sqrt{b_X{}^2+b_Y{}^2+b_Z{}^2}}$$

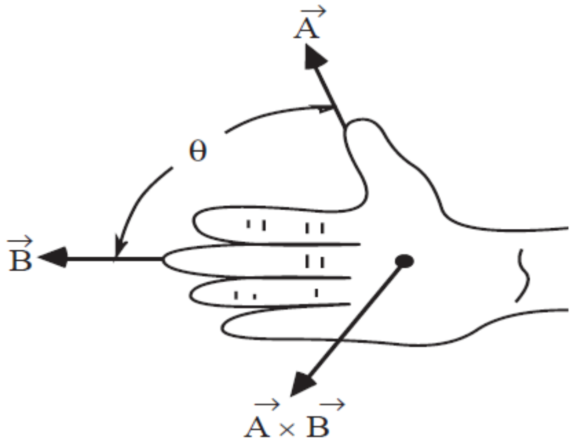
If $\vec{A}\cdot\vec{B}=0$ $\rightarrow$ $\vec{A}$ perpendicular $\vec{B}$

6- Dot product of Cartesian unit vectors $\vec{i}\cdot\vec{j}=\vec{j}\cdot\vec{k}=\vec{k}\cdot\vec{i}=0$, while

$\vec{i}\cdot\vec{i}=i^2=1,\quad \vec{j}\cdot\vec{j}=j^2=1,\quad \vec{k}\cdot\vec{k}=k^2=1.$

Cross Product

The Cross Product which gives a vector as an answer, and is sometimes called the vector product. For two vectors define as $\vec{A}\wedge\vec{B}=AB\sin\theta\,\vec{e}$, where $\vec{e}$ is unite vector in the direction of $\vec{A}\wedge\vec{B}$



Laws of Operation

1. Commutative law $\vec{A}\wedge\vec{B}\neq\vec{B}\wedge\vec{A}$, but $\vec{A}\wedge\vec{B}=-\vec{A}\wedge\vec{B}$

2. Multiplication by a scalar $\lambda\left(\vec{A}\wedge\vec{B}\right)=\left(\lambda\vec{A}\right)\wedge\vec{B}=\vec{A}\wedge\left(\lambda\vec{B}\right)=\left(\vec{A}\wedge\vec{B}\right)\lambda$

3. Distribution law $\vec{A}\wedge\left(\vec{B}+\vec{D}\right)=\left(\vec{A}\wedge\vec{B}\right)+\left(\vec{A}\wedge\vec{D}\right)$

4. Cartesian Vector Formulation

Cross product of two vectors $\vec{A}=a_X\vec{I}+a_Y\vec{J}+a_Z\vec{K}$ and $\vec{B}=b_X\vec{I}+b_Y\vec{J}+b_Z\vec{K}$

Then $\vec{A}\wedge\vec{B}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_X & a_Y & a_Z\\ b_X & b_Y & b_Z\end{vmatrix}$.

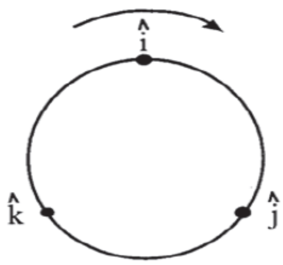
5. Applications :

The angle formed between two vectors given by $\sin\theta=\dfrac{\lfloor\vec{A}\wedge\vec{B}\rfloor}{|\vec{A}|\,|\vec{B}|}$

6- Cross product of Cartesian unit vectors

$\vec{i}\wedge\vec{j}=\vec{k},\quad \vec{j}\wedge\vec{k}=\vec{i},\quad \vec{k}\wedge\vec{i}=\vec{j}$, while

$\vec{i}\wedge\vec{i}=0,\quad \vec{j}\wedge\vec{j}=0,\quad \vec{k}\wedge\vec{k}=0.$, see below figure



Triple Scalar Product

For three vectors $\vec{A},\vec{B},\vec{C}$ the Triple Scalar Product define as

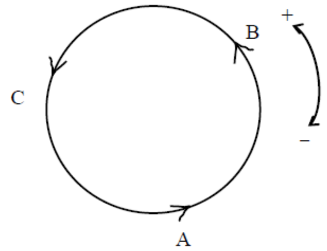
$$\vec{A}\cdot\left(\vec{B}\wedge\vec{C}\right)=\begin{vmatrix}a_X & a_Y & a_Z\\ b_X & b_Y & b_Z\\ c_X & c_Y & c_Z\end{vmatrix}$$

Note that

(1) $\vec{A}\cdot\left(\vec{B}\wedge\vec{C}\right)=\left(\vec{B}\wedge\vec{C}\right)\cdot A$,

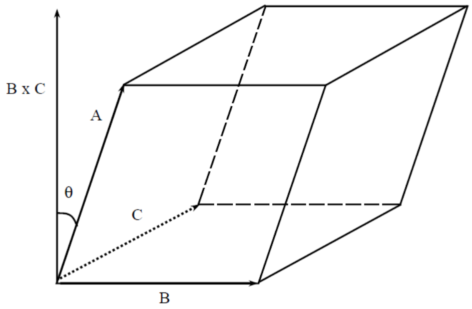
(2) $\vec{A}\cdot\left(\vec{B}\wedge\vec{C}\right)=\vec{C}\cdot\left(\vec{A}\wedge\vec{B}\right)=\vec{B}\cdot\left(\vec{C}\wedge\vec{A}\right)$

(3) $\vec{A}\cdot\left(\vec{B}\wedge\vec{C}\right)=-\vec{A}\cdot\left(\vec{C}\wedge\vec{B}\right)=-\vec{B}\cdot\left(\vec{A}\wedge\vec{C}\right)$, see below figure



(4)- Triple Scalar Product represents the volume of parallelepiped

$\vec{A}\cdot\left(\vec{B}\wedge\vec{C}\right)=$ the volume of parallelepiped



Triple Vector Product

For three vectors $\vec{A}, \vec{B}, \vec{C}$ the Triple Vector Product define as $\vec{A}\wedge\left(\vec{B}\wedge\vec{C}\right)$

Note that

(1)- $\vec{A}\wedge\left(\vec{B}\wedge\vec{C}\right)\neq\vec{A}\wedge\left(\vec{C}\wedge\vec{B}\right)$,

(2)- $\vec{A}\wedge\left(\vec{B}\wedge\vec{C}\right)=\left(\vec{A}\cdot\vec{C}\right)\vec{B}-\left(\vec{A}\cdot\vec{B}\right)\vec{C}$,(prove that this property?)

Unit vector

A unit vector is a vector that has a magnitude of 1

$$\text{Unit vector}=\frac{\text{Vector}}{\text{Magnitude of the vector}}$$

Example **1:** Given a vector $\vec{r}=4\vec{i}-3\vec{j}$, find the unit vector?

Solution

$$r=\left|\vec{r}\right|=\sqrt{(4)^2+(-3)^2}=\sqrt{16+9}=\sqrt{25}=5$$

$$\vec{u}=\frac{\vec{r}}{\left|\vec{r}\right|}=\frac{4}{5}\vec{i}-\frac{3}{5}\vec{j}=\frac{1}{5}\left(4,-3\right)$$

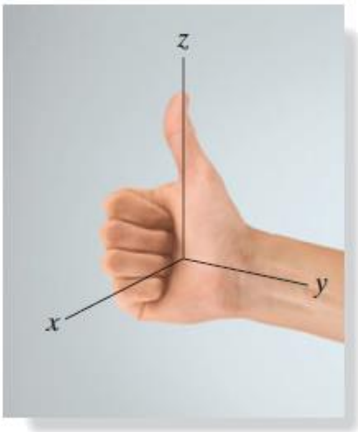
Cartesian vector

Introduction

The operations of vector algebra, when applied to solving problems in three dimensions , are greatly simplified if the vectors are first represented in Cartesian vector form. In this section we will present a general method for doing this; then in the next section we will use this method for finding the resultant force of a system of concurrent forces.

Right-Handed Coordinate System.

We will use a right-handed coordinate system to develop the theory of vector algebra that follows. A rectangular coordinate system is said to be right-handed if the thumb of the right hand points in the direction of the positive z axis when the right-hand fingers are curled about this axis and directed from the positive x towards the positive y axis, as in Figure.



Position Vectors

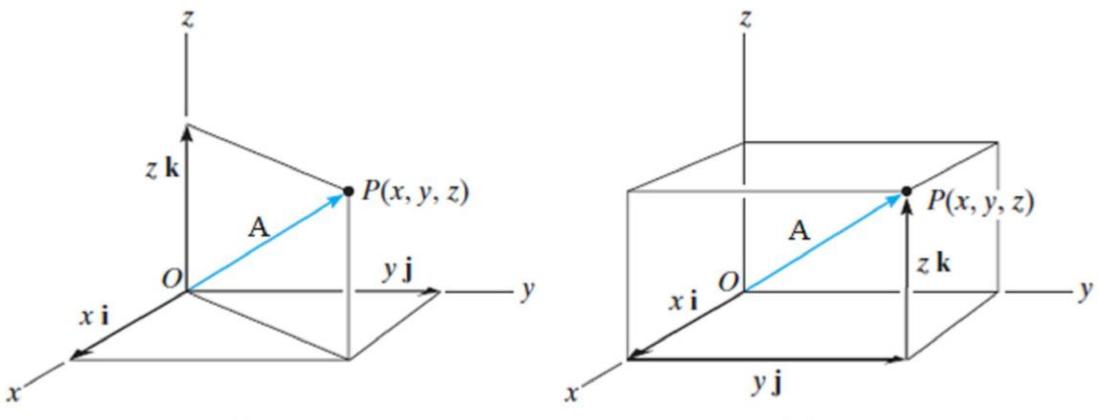
Here we will introduce the concept of a position vector. It will be shown that this vector is of importance in formulating a Cartesian force vector directed between two points in space.

A position vector $\vec{A}$ is defined as a fixed vector which locates a point in space relative to another point. For example, if $\vec{A}$ extends from the origin of coordinates, O , to point $P(x, y, z)$ (Fig. a), then $\vec{A}$ can be expressed in Cartesian vector form as

$$\vec{A} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k}$$

Note how the head-to-tail vector addition of the three components yields vector r (Fig. b) . Starting at the origin O , one "travels" x in the $+i$

direction, then y in the $+j$ direction, and finally z in the $+k$ direction to arrive at point $P(x, y, z)$

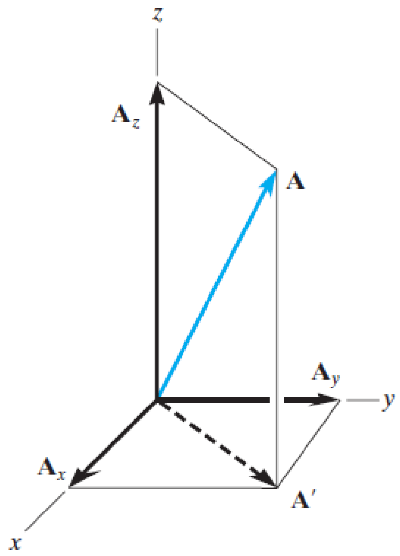


(a) (b)

Rectangular Components of a Vector.

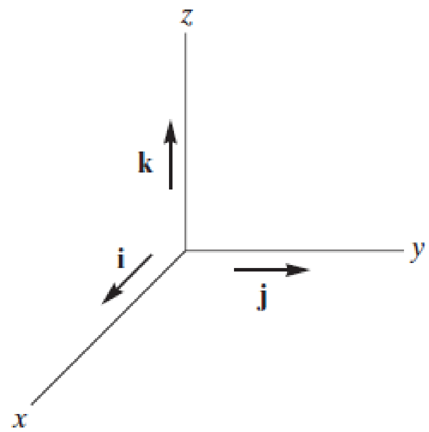
A vector A may have one, two, or three rectangular components along the x, y, z coordinate axes, depending on how the vector is oriented relative to the axes. In general, though, when A is directed within an octant of the x, y, z frame (see front Figure), then by two successive applications of the parallelogram law, we may resolve the vector into components as $A = A' + A_z$ and then $A' = A_x + A_y$ Combining these equations, to eliminate A', A is represented by the vector sum of its three rectangular components,

$A = A_x + A_y + A_z$



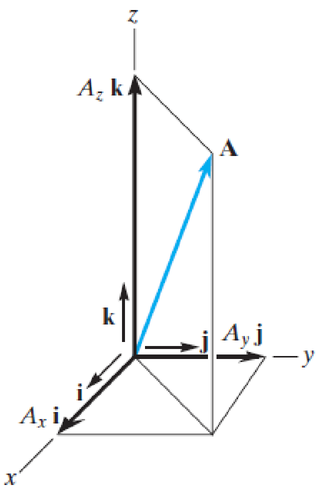
Cartesian Unit Vectors.

In three dimensions, the set of Cartesian unit vectors, i, j, k is used to designate the directions of the x, y, z axes, respectively. The sense (or arrowhead) of these vectors will be represented analytically by a plus or minus sign, depending on whether they are directed along the positive or negative x, y or z axes. The positive Cartesian unit vectors are shown in Figure.



Cartesian Vector Representation.

Since the three components of $\vec{A}$ in above equation act in the positive $\vec{i}, \vec{j}$ and $\vec{k}$ directions (see Figure) , we can write $\vec{A}$ in Cartesian vector form as $\vec{A} = a_x\vec{i} + a_y\ \vec{j} + a_z\vec{k}$



Magnitude of a Cartesian Vector.

The magnitude of a Cartesian vector is $\vec{A}=\sqrt{(a_x)^2+(a_y)^2+(a_z)^2}$. The

magnitude of $\vec{A}$ is equal to the positive square root of the sum of the squares of its components.

Direction of a Cartesian Vector.

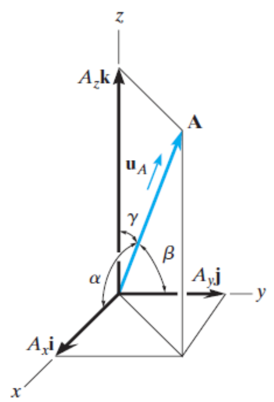
We will define the direction of $\vec{A}$ by the coordinate direction angles a α, β, γ , measured between the tail of $\vec{A}$ and the positive x, y, z axes provided they are located at the tail of $\vec{A}$ (see Figure). Note that regardless of where $\vec{A}$ is directed, each of these angles will be between 0^O and 180^O .

The angles α, β, γ given by $\cos\alpha=\dfrac{a_x}{A}, \quad \cos\beta=\dfrac{a_y}{A}, \quad \cos\gamma=\dfrac{a_z}{A}$

From this Eq. we have $A^2\cos\alpha=a_x^2, \quad A^2\cos\beta=a_y^2, \quad A^2\cos\gamma=a_z^2$

$A^2\left(\cos^2\alpha+\cos^2\beta+\cos^2\gamma\right)=a_x^2+a_y^2+a_z^2=A^2$. Then

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$$



Note that, the unit vector given by

$\vec{A}_u = \frac{a_x}{A}\vec{i} + \frac{a_y}{A}\vec{j} + \frac{a_z}{A}\vec{k}$. Also given by $\vec{A}_u = \cos\alpha\,\vec{i} + \cos\beta\,\vec{j} + \cos\gamma\,\vec{k}$.

Then the direction of vector $\vec{A}$ given by

$$\cos\alpha = \frac{a_x}{A},\quad \cos\beta = \frac{a_y}{A},\quad \cos\gamma = \frac{a_z}{A}$$

In the space represent the force as:

(1) If we know two angles with two axes

In this case $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ and the force given by

$\vec{F} = F\cos\alpha\,\vec{i} + F\cos\beta\,\vec{j} + F\cos\gamma\,\vec{k}$

(2) If we know an angles with two axes in plane

In this case we resolve the force in the vertical axis and in the other plane

(3) If we know the unite vector in space

In this case $\vec{A}_u = \frac{a_x}{A}\vec{i} + \frac{a_y}{A}\vec{j} + \frac{a_z}{A}\vec{k}$, $\vec{A}_u = \cos\alpha\,\vec{i} + \cos\beta\,\vec{j} + \cos\gamma\,\vec{k}$, and

$\cos\alpha = \frac{a_x}{A},\quad \cos\beta = \frac{a_y}{A},\quad \cos\gamma = \frac{a_z}{A}$.

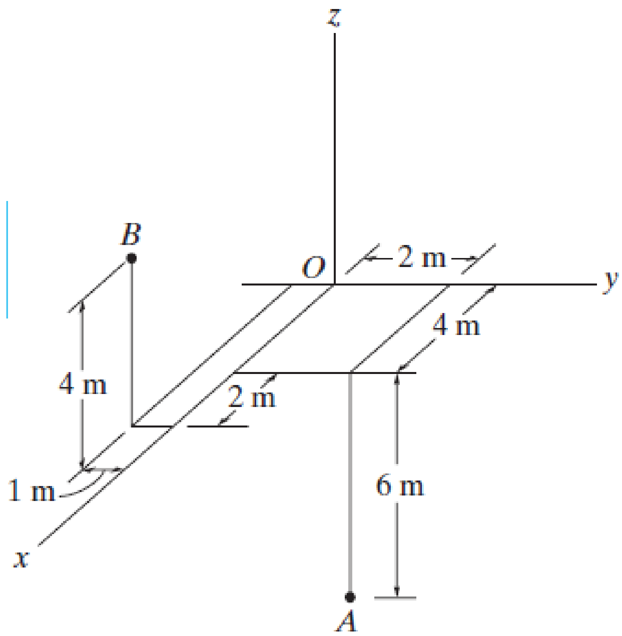
Example 2: Given a vector $\vec{r} = 12\vec{i} - 3\vec{j} - 4\vec{k}$, find the unit vector?

Solution

$$r = |\vec{r}| = \sqrt{(12)^2 + (-3)^2 + (-4)^2} = \sqrt{144 + 9 + 16} = \sqrt{169} = 13$$

$$\vec{u} = \frac{\vec{r}}{|\vec{r}|} = \frac{12}{13}\vec{i} - \frac{3}{13}\vec{j} - \frac{4}{13}\vec{k} = \frac{1}{13}\left(12, -3, 4\right)$$

Example 3: Determine the length and its direction measured from B toward A as shown in Figure?



Solution

The coordinates of A and B are $B(6,-1,\,4)$, $A(4,2,\,-6)$

$$\overrightarrow{AB}=\vec{r}=(6-4)\vec{i}+(-1-2)\,\vec{j}+(4-(-6))\,\vec{k}$$

$$\vec{r}=2\vec{i}-3\vec{j}+10\vec{k}$$

$$r=|\vec{r}|=\sqrt{(2)^2+(-3)^2+(10)^2}=\sqrt{4+9+100}=\sqrt{113}$$

$$\vec{u}=\frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|}=\frac{\vec{r}}{|\vec{r}|}=\frac{2}{\sqrt{113}}\vec{i}-\frac{3}{\sqrt{113}}\vec{j}+\frac{10}{\sqrt{113}}\vec{k}$$

The direction of $\overrightarrow{AB}=\vec{r}$ given by

$$\cos\alpha=\frac{\vec{r}_x}{r},\quad \cos\beta=\frac{\vec{r}_y}{r},\quad \cos\gamma=\frac{\vec{r}_z}{r}$$

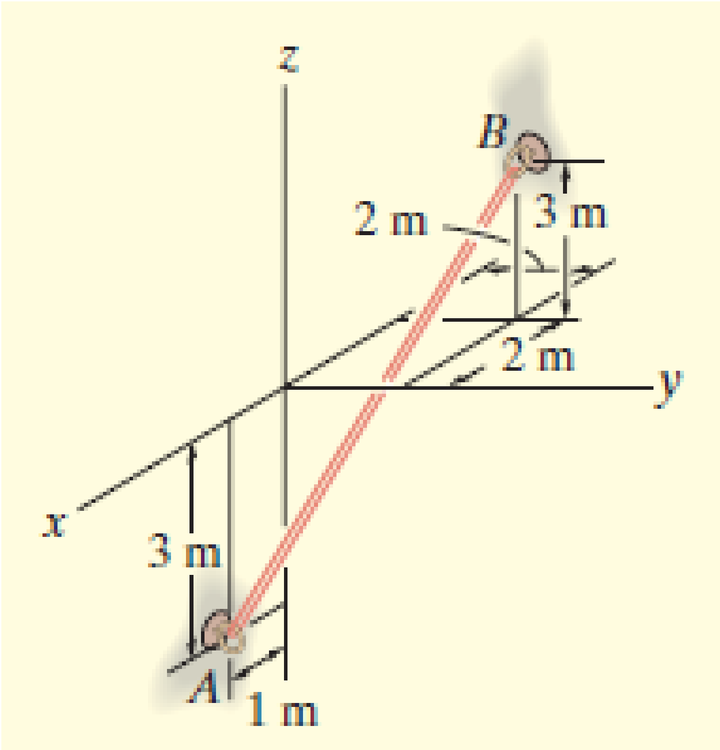
$$\cos\alpha=\frac{2}{\sqrt{113}},\quad \cos\beta=-\frac{3}{\sqrt{113}},\quad \cos\gamma=\frac{10}{\sqrt{113}}$$

$$\alpha=\cos^{-1}\left(\frac{2}{\sqrt{113}}\right)=79.15545231^{O},$$

$$\beta=\cos^{-1}\left(-\frac{3}{\sqrt{113}}\right)=106.39274603^{O},$$

$$\gamma=\cos^{-1}\left(\frac{10}{\sqrt{113}}\right)=19.81365713^{O}$$

Example 4: An elastic rubber band is attached to points A and B as shown in Figure . Determine its length and its direction measured from A toward B ?



Solution

The coordinates A and B are $A(1,0,-3)$, $B(-2,2,3)$

$$\vec{AB}=\vec{r}=(-2-(-1))\vec{i}+(2-0)\vec{j}+(3-(-3))\vec{k}$$

$$\vec{r}=-3\vec{i}+2\vec{j}+6\vec{k}$$

$$r=|\vec{r}|=\sqrt{(-3)^2+(2)^2+(6)^2}=\sqrt{9+4+36}=\sqrt{49}=7$$

he T $$\vec{u}=\frac{\vec{AB}}{\left|\vec{AB}\right|}=\frac{\vec{r}}{|\vec{r}|}=-\frac{3}{7}\vec{i}+\frac{2}{7}\vec{j}+\frac{6}{7}\vec{k}$$

direction of $\vec{AB}=\vec{r}$ given by. $\cos\alpha=\frac{\vec{r}_x}{r}$, $\cos\beta=\frac{\vec{r}_y}{r}$, $\cos\gamma=\frac{\vec{r}_z}{r}$

Then $\cos\alpha=-\frac{3}{7}$, $\cos\beta=\frac{2}{7}$, $\cos\gamma=\frac{6}{7}$. This tends to

$$\alpha = \cos^{-1}\left(-\frac{3}{7}\right) = 115.376^{O},\ \beta = \cos^{-1}\left(\frac{2}{7}\right) = 73.398^{O},$$

$$\gamma = \cos^{-1}\left(\frac{6}{7}\right) = 31^{O}$$

Part 2: Vectors Forces in 2D and 3D

Resultant of Concurrent Coplanar Forces

How to calculate the resultant force acting on an object? Several forces can act on a body or point, each force having different direction and magnitude. In engineering the focus is on the resultant force acting on the body. The resultant of concurrent forces (acting in the same plane) can be found using the parallelogram law, the triangle rule or the polygon rule.. Two or more forces are concurrent is their direction crosses through a common point. For example, two concurrent forces F_1 and F_2 are acting on the same point P. In order to find their resultant F, we can apply either the parallelogram law, triangle rule.

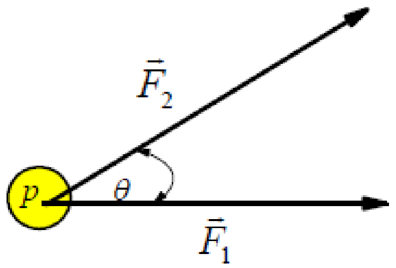


Fig. 1

A force is a vector quantity since it has magnitude and direction. There force, the force addition will be according to the Parallelogram law

Parallelogram in 2D:

Redraw a half portion of the parallelogram to illustrate the triangular head-to-tail addition of the components. From this triangle, the magnitude of the resultant force can be determined using the law of *Cosine*, and its direction is determined from the law of *Sine*. The magnitudes of two force components are determined from the law of *Sine*. The formulas are given in Figure.

Cosine law

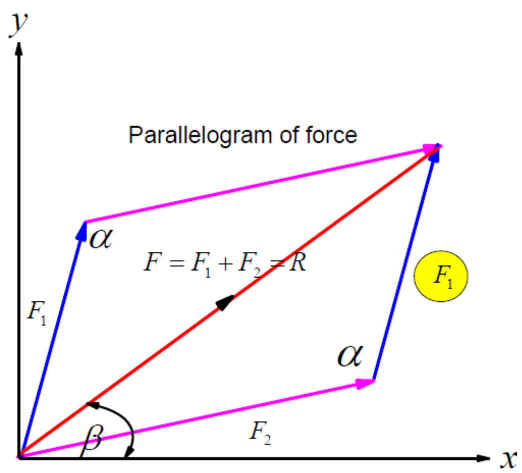


Fig. 2 (a)

From the triangle force, the resultant force is the vector sum between the components:

$$\vec{F} = \vec{F}_1 + \vec{F}_2$$

Cosine law is $\quad F = \sqrt{F_1^2 + F_2^2 - 2F_1F_2\cos\alpha}$

Sine law

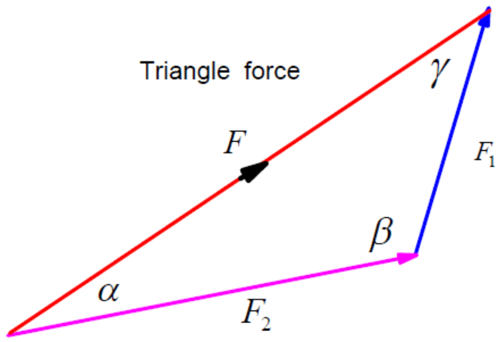


Fig. 2 (b)

Sine law is $\quad \dfrac{F_1}{\sin\alpha} = \dfrac{F_2}{\sin\gamma} = \dfrac{F}{\sin\beta}$

Components

Also, we can find the resultant of Concurrent Coplanar Forces as

Step 1: to resolve each force into its $x-y$ components.

Step 2: to add all the x components together and add all the y components together. These two totals become the resultant vector.

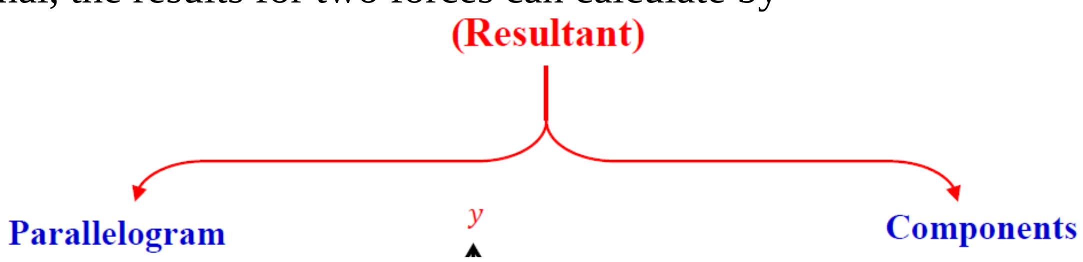
Step 3: find the magnitude and the angle of the resultant vector.

I. e, we calculate both $\sum F_x$ and $\sum F_y$, then the resultant is

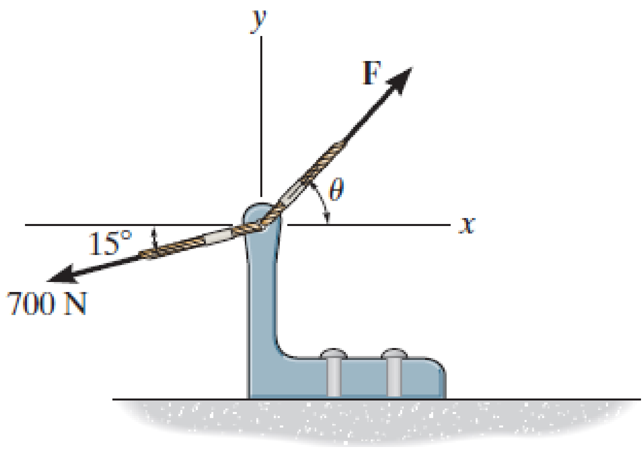
$$F = \sqrt{\left(\sum F_x\right)^2 + \left(\sum F_y\right)^2}$$

The direction is $\quad \theta = \tan^{-1}\left(\dfrac{\sum F_y}{\sum F_x}\right)$.

Final, the results for two forces can calculate by



Example 1: If $\theta = 60^o$ and $F = 450\ N$, determine the magnitude of the resultant force and its direction, measured counterclockwise from the positive $x-$ axis.



Solution

We can draw the forces as in below Figure (Free-Body Diagrams)

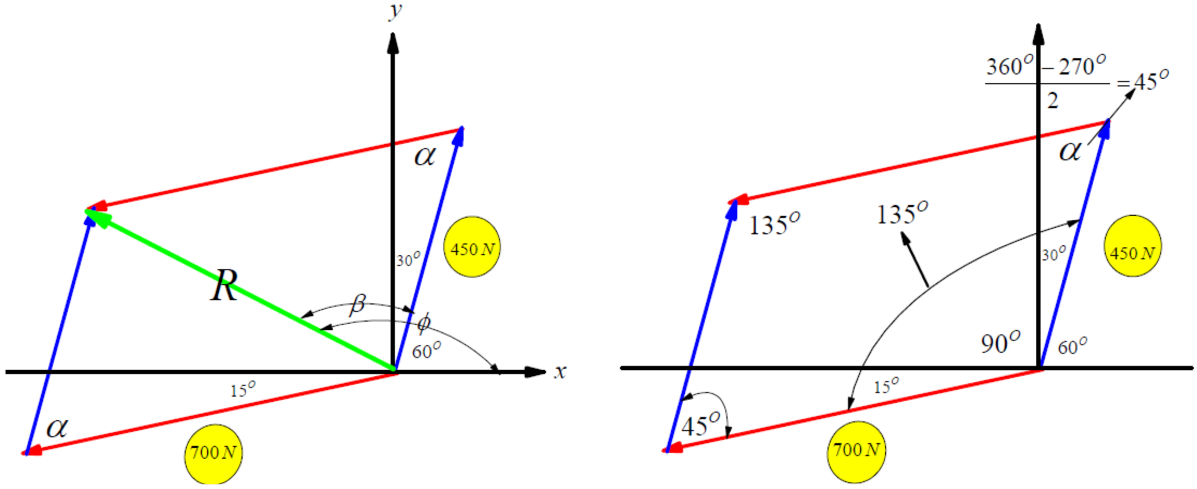


Fig. (1)

Applying the law of Cosine to Fig. (1),This yields

$F_R = \sqrt{(700)^2 + (450)^2 - 2(700)(450)\cos 45^O} = 497\,N$ Again,

applying the law of Sine to Fig. (1), and using this result, yields

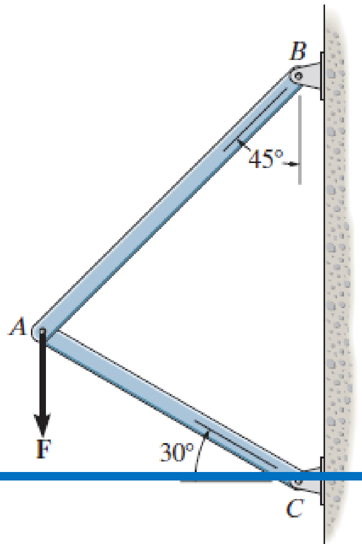
$$\frac{\sin(\beta + 30^o)}{700} = \frac{\sin 45^o}{497.01} \quad \rightarrow \quad \beta = 95.19^O$$

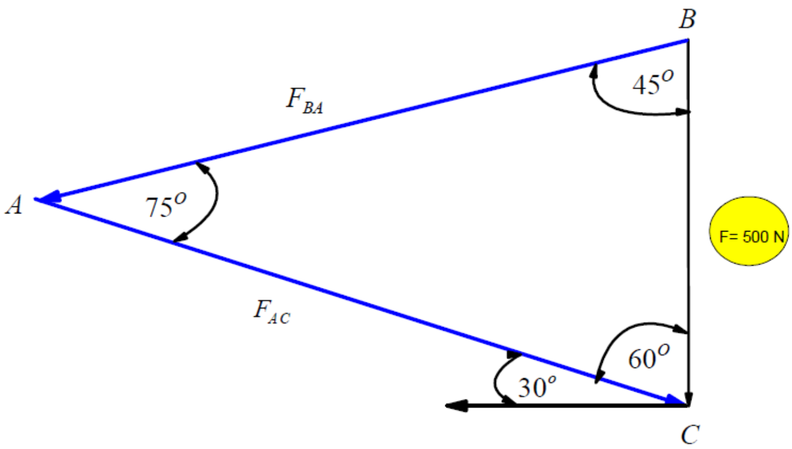
Thus, the direction of angle of measured counterclockwise from the positive $x-$ axis is

$\phi = \alpha + 60^O = 95.19^O + 60^O = 155.19^O$

Solve this problem using the components method?

Example 2: The vertical force F acts downward at A on the two membered frame. Determine the magnitudes of the two components of F directed along the axes of BA and AC. Set $F = 500$?





From the Cosine low $F = \sqrt{F_1^2 + F_2^2 - 2F_1F_2\cos\alpha}$

$$F = \sqrt{(6)^2 + (4)^2 - 2(6)(4)\cos105^O} = \sqrt{36+16-(48)(-0.2588)}$$

From the

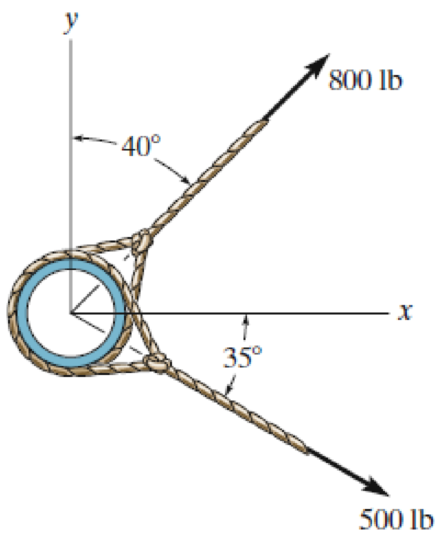
$$= \sqrt{52+12.423} = \sqrt{64.423} = 8.03\ N$$

Sine law $\dfrac{F}{\sin105^o} = \dfrac{F_1}{\sin\alpha} \rightarrow \quad \sin\alpha = \dfrac{F_1}{F}\sin105^o =$

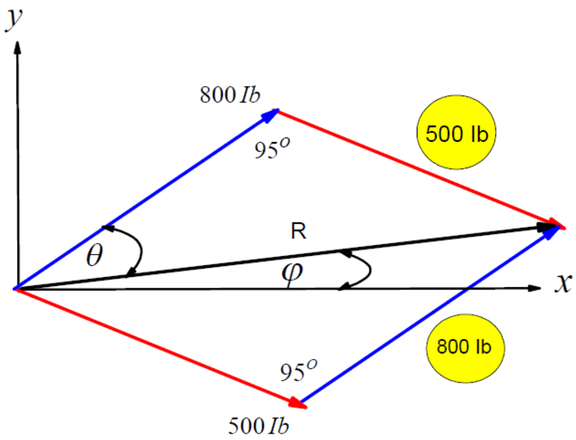
$$\frac{4}{8.03}(0.9659) = \frac{3.8637}{8.03} = 0.4811 \rightarrow \alpha = 28.761$$

Then $\theta = 28.761 - 25 = 3.761^O$

Example 3: Determine the magnitude of the resultant force (see below Figure) and its direction, measured counterclockwise from the positive x axis ?



Solution



From the Cosine low $F=\sqrt{F_1^2+F_2^2-2F_1F_2\cos\alpha}$

$R=\sqrt{(500)^2+(800)^2-2(500)(800)\cos 95^O}=\sqrt{(500)^2+(800)^2-2(500)(800)(-0.08715)}$

$=\sqrt{250000+640000+69724.594}=\sqrt{959724.594}=979.65539=980\,Ib$

Using this result to apply the sine law

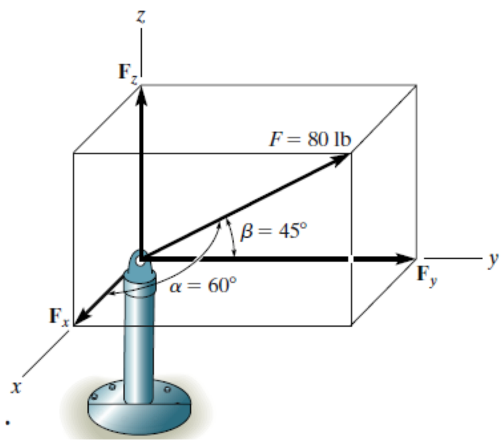
$\frac{R}{\sin 95^o}=\frac{500}{\sin\theta}\rightarrow\frac{980}{\sin 95^o}=\frac{500}{\sin\theta}\rightarrow\ \sin\theta=500\frac{980}{\sin 95^o}=500\left(\frac{0.9961}{980}\right)=0.5082$

$\theta=\sin^{-1}(0.5082)\ \rightarrow\theta=30.54495^O$

Thus, the direction f of R measured counterclockwise from the positive x axis is

$\theta=50-30.54495^O=19.54^O$

Example 4: The force F has a magnitude of 80 Ib and acts within the octant shown. Determine the magnitudes of the x, y, z components of F?



Solution

From the Figure clears that $\alpha=60^O$ and $\beta=45^O$ and using the relation

$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$, we find that $(\cos(60^O))^2+(\cos(45^O))^2+\cos^2\gamma=1$

$(0.5)^2+\left(\frac{1}{\sqrt{2}}\right)^2+\cos^2\gamma=1\rightarrow 0.25+0.5+\cos^2\gamma=1\rightarrow\ +\cos^2\gamma=0.25\ \cos^2\gamma=\pm 0.5\ \rightarrow\ \ \gamma=60^o$

Or $\gamma=120^o$

By inspection it is necessary that $\gamma=60^o$, since F_x must be in the $+x$

Now using the relation $\vec{F}=F\cos\alpha\vec{i}+F\cos\beta\,\vec{j}+F\cos\gamma\vec{k}$, we have

$\vec{F} = 80\left(\cos 60^o \ \vec{i} + \cos 45^o \ \vec{j} + \cos 60^o \ \vec{k}\right)$

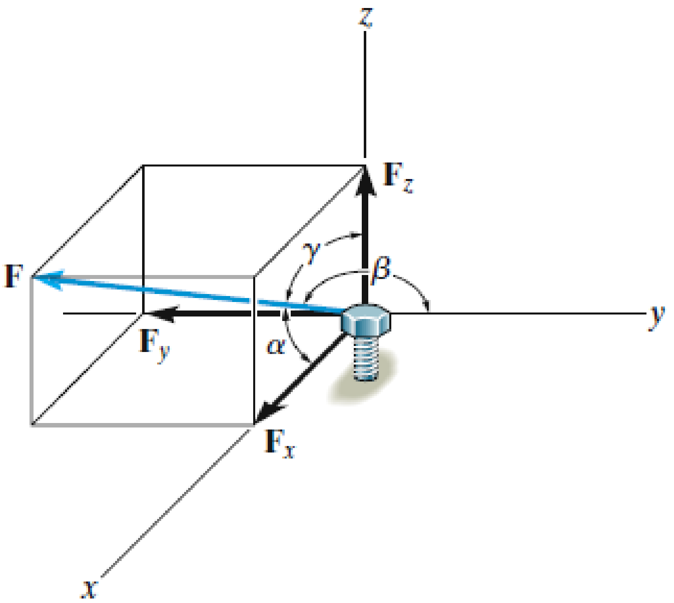
$\vec{F} = 80\left(0.5\vec{i} + \frac{1}{\sqrt{2}} \ \vec{j} + 0.5\vec{k} \right) = 80\left(0.5\vec{i} + \frac{\sqrt{2}}{\left(\sqrt{2}\right)\left(\sqrt{2}\right)} \ \vec{j} + 0.5\vec{k} \right)$

$\vec{F} = 80\left(0.5\vec{i} + (0.5)\left(\sqrt{2}\right) \ \vec{j} + 0.5\vec{k}\right) = 40\left(\vec{i} + \sqrt{2} \ \ \vec{j} + \vec{k}\right)$

So, $\vec{F}_X = 40 Ib, \quad \vec{F}_y = 40\sqrt{2} \ Ib, \quad \vec{F}_Z = 40 Ib$

Not that $F = 40\sqrt{1+2++1} = 40\sqrt{4} = 40(2) = 80 \ Ib$

Example 5: The bolt is subjected to the force F, which has components acting along the x, y, z axes as shown. If the magnitude of F is $80 \ N$, $\alpha = 60^O$ and $\gamma = 45^O$ determine the magnitudes of its components (see below Figure).



Solution

From the Figure clears that $\alpha = 60^O$ and $\gamma = 45^O$ and using the relation

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, we find that $\left(\cos(60^O)\right)^2 + \cos^2\beta + \left(\cos(45^O)\right)^2 = 1$

$(0.5)^2 + \cos^2\beta + \left(\frac{1}{\sqrt{2}}\right)^2 = 1 \rightarrow 0.25 + \cos^2\beta + 0.5 = 1 \rightarrow \cos^2\beta = 0.25$

$\cos^2\beta = \pm 0.5 \quad \rightarrow \quad \beta = 60^o \quad \text{Or} \quad \beta = 120^o$

By inspection it is necessary that $\beta = 120^o$, since F_x must be in the $+x$

Now using the relation $\vec{F} = F\cos\alpha\vec{i} + F\cos\beta \ \vec{j} + F\cos\gamma\vec{k}$, we have

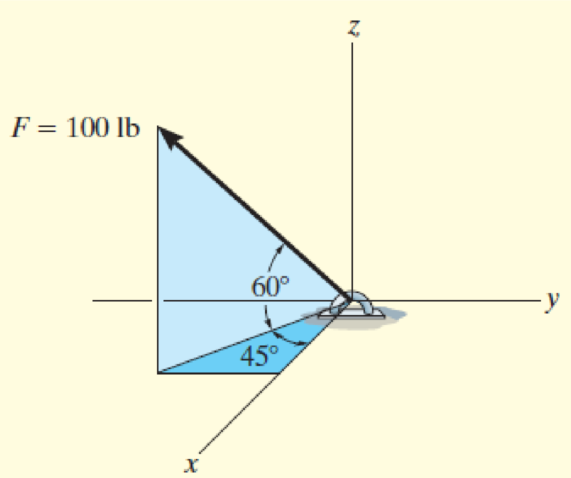
$\vec{F} = 80\left(\cos 60^o \ \vec{i} + \cos 120^o \ \vec{j} + \cos 45^o \ \vec{k}\right)$

$\vec{F} = 80\left(0.5\vec{i} - 0.5 \ \vec{j} + \frac{1}{\sqrt{2}}\vec{k} \right) = 80\left(0.5\vec{i} - 0.5 \ \vec{j} + \frac{\sqrt{2}}{\left(\sqrt{2}\right)\left(\sqrt{2}\right)}\vec{k} \right)$

$\vec{F} = 80\left(0.5\vec{i} - (0.5) \ \vec{j} + 0.5\left(\sqrt{2}\right)\vec{k}\right) = 40\left(\vec{i} - \ \vec{j} + \sqrt{2} \ \vec{k}\right)$

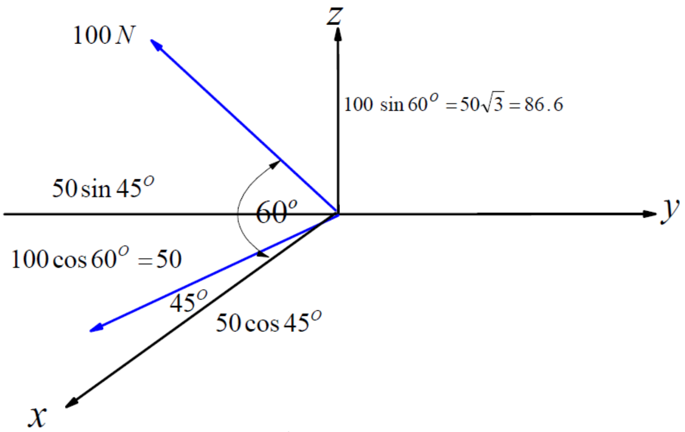
So, $\vec{F}_X = 40 \ N, \quad \vec{F}_y = 40 \ N, \quad \vec{F}_Z = 40\sqrt{2} \ N$

Example 6: Express the force F shown in Figure as a Cartesian vector and its direction?



Solution

If we resolve the force as in Figure



Then $\vec{F} = \left(50\cos 45^{O}\,\vec{i} - 50\sin 45^{O}\,\vec{j} + 86.6\vec{k}\right)$

$$\vec{F} = \left(\frac{50}{\sqrt{2}}\,\vec{i} - \frac{50}{\sqrt{2}}\,\vec{j} + 86.6\vec{k}\right)$$

$$\vec{F} = \left(35.4\,\vec{i} - 35.4\,\vec{j} + 86.6\vec{k}\right)$$

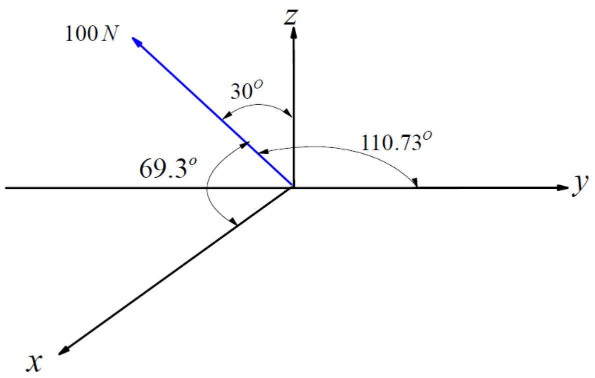
Not that $F = \sqrt{(35.4)^2 + (35.4)^2 + (86.6)^2} = \sqrt{10005} \cong 100\ N$

The direction of the force given by

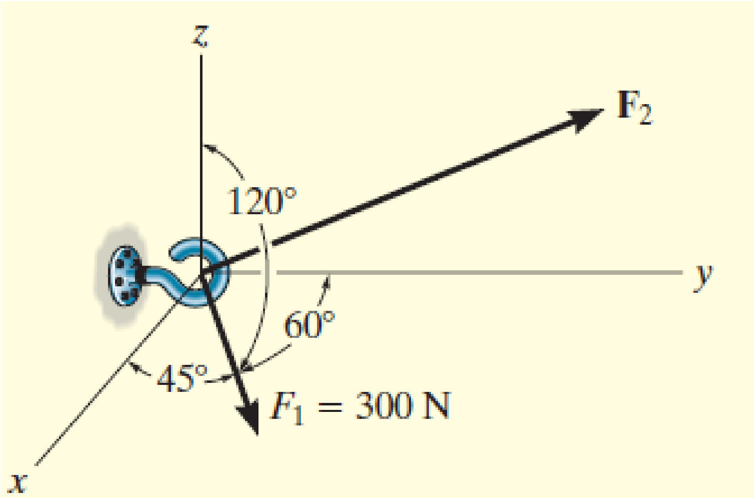
$$\cos\alpha = \frac{F_x}{F},\quad \cos\beta = \frac{F_y}{F},\quad \cos\gamma = \frac{F_z}{F}$$

$$\cos\alpha = \frac{35.4}{100},\quad \cos\beta = -\frac{35.4}{100},\quad \cos\gamma = \frac{86.6}{100}$$

$$\alpha = \cos^{-1}\left(\frac{35.4}{100}\right) = 69.3^{O},\quad \beta = \cos^{-1}\left(-\frac{35.4}{100}\right) = 110.73^{O},\quad \gamma = \cos^{-1}\left(\frac{86.6}{100}\right) = 30^{O}$$

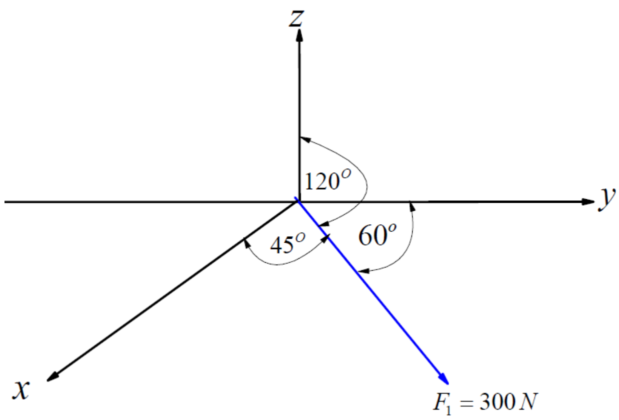


Example 7: Two forces act on the hook shown in below Figure. Specify the magnitude of F_2 and its coordinate direction angles so that the resultant force F_R acts along the positive y axis and has a magnitude of $800N$.



Solution

Free-Body Diagram. we can plot the Free-Body Diagram as in figure

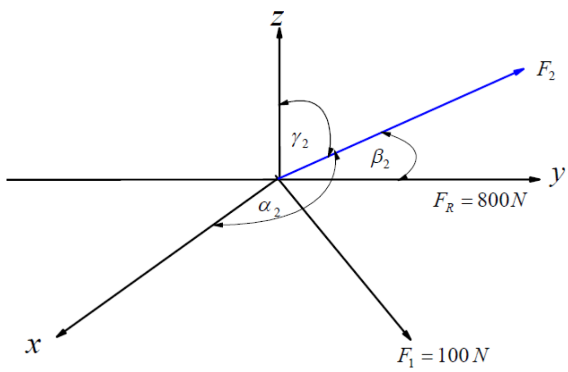


So, we can expresses $\vec{F}_1$ as follows:

$$\vec{F}_1 = 300\left(\cos45^O\ \vec{i} + \cos60^O\ \vec{j} + \cos120^O\ \vec{k}\right)\vec{F}_1 = 300\left(\frac{1}{\sqrt{2}}\ \vec{i} + 0.5\ \vec{j} - 0.5\vec{k}\right) = 150\left(\sqrt{2}\ \vec{i} + \vec{j} - \vec{k}\right).$$

Also, $\vec{F}_R = 800\ \vec{j}$

We require $\vec{F}_R = \vec{F}_1 + \vec{F}_2$ (see below Figure)



Then

$$800\vec{j} = 150\left(\sqrt{2}\,\vec{i} + \vec{j} - \vec{k}\right) + F_{2x}\,\vec{i} + F_{2y}\,\vec{j} + F_{2z}\,\vec{k}$$

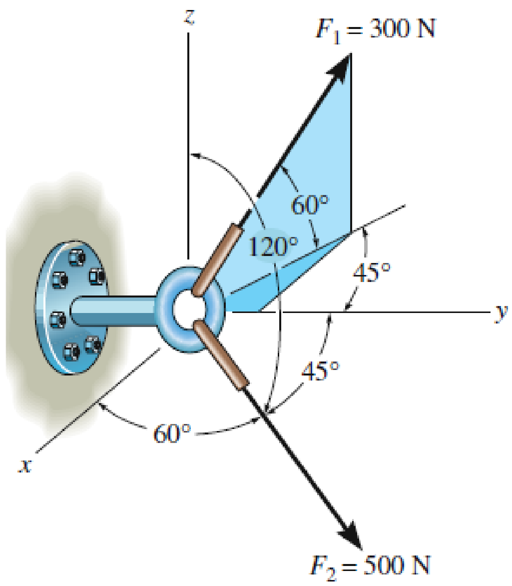
$$F_{2x} = -150\sqrt{2}, \quad F_{2y} = 650, \quad F_{2z} = 150$$

$$F_2 = \sqrt{(F_{2x})^2 + (F_{2y})^2 + (F_{2z})^2} = \sqrt{45000 + 422500 + 22500} = \sqrt{490000} = 700$$

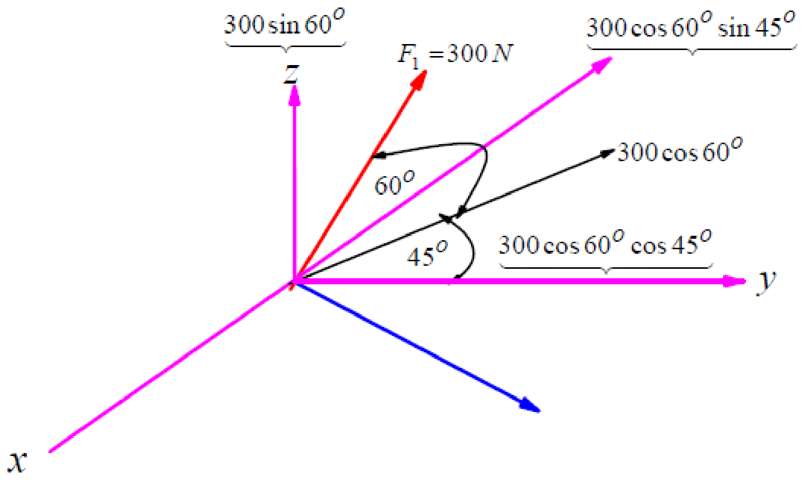
The direction of F_2 given from $\cos\alpha_2 = \dfrac{F_{x2}}{F_2}$, $\cos\beta_2 = \dfrac{F_{y2}}{F_2}$, $\cos\gamma_2 = \dfrac{F_{z2}}{F_2}$

$$\alpha_2 = \cos^{-1}\left(-\frac{150\sqrt{2}}{700}\right) = 107.64^O, \quad \beta_2 = \cos^{-1}\left(\frac{650}{700}\right) = 21.8^O, \quad \gamma_2 = \cos^{-1}\left(\frac{150}{700}\right) = 77.6^O$$

Example 8: The screw eye is subjected to the two forces as shown below Figure. Express each force in Cartesian vector form and then determine the resultant force. Find the magnitude and coordinate direction angles of the resultant force.



Solution

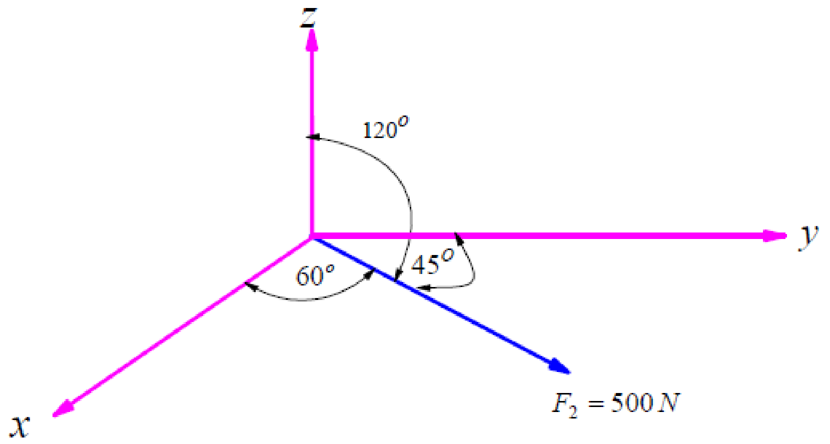


$$\vec{F}_1 = 300\left(-\cos 60^o \sin 45^O \vec{i} + \cos 60^o \cos 45^O \vec{j} + \sin 60^O \vec{k}\right)$$

$$\vec{F}_1 = 300\left(-(0.5)\left(\frac{1}{\sqrt{2}}\right)\vec{i} + (0.5)\left(\frac{1}{\sqrt{2}}\right)\vec{j} + \frac{\sqrt{3}}{2}\vec{k}\right)$$

$$\vec{F}_1 = 300\left(-(0.3535)\vec{i} + (0.3535)\vec{j} + 1.732\vec{k}\right)$$

$$\vec{F}_1 = \left(-(106.07)\vec{i} + 106.07\vec{j} + 259.81\vec{k}\right)$$



Using the relation $\vec{F} = F\cos\alpha\vec{i} + F\cos\beta\,\vec{j} + F\cos\gamma\,\vec{k}$, we have

$$\vec{F}_2 = 500\left(\cos 60^O \vec{i} + \cos 45^O\ \vec{j} + \cos 120^O \vec{k}\right)$$

$$\vec{F}_2 = 500\left((0.5)\,\vec{i} + \frac{1}{\sqrt{2}}\,\vec{j} - (0.5)\,\vec{k}\right)$$

$$\vec{F}_2 = 500\left((0.5)\,\vec{i} + (0.7071)\,\vec{j} - (0.5)\,\vec{k}\right)$$

$$\vec{F}_2 = \left(250\,\vec{i} + (353.5542)\,\vec{j} - 250\,\vec{k}\right)$$

So, the resultant is given by

$$\vec{F}_R = \vec{F}_1 + \vec{F}_2 = \left(-(106.07)\vec{i} + 106.07\vec{j} + 259.81\vec{k}\right) + \left(250\,\vec{i} + (353.5542)\,\vec{j} - 250\,\vec{k}\right)$$

$$\vec{F}_R = \left(143.93\vec{i} + 459.62\,\vec{j} + 9.81\vec{k}\right)$$

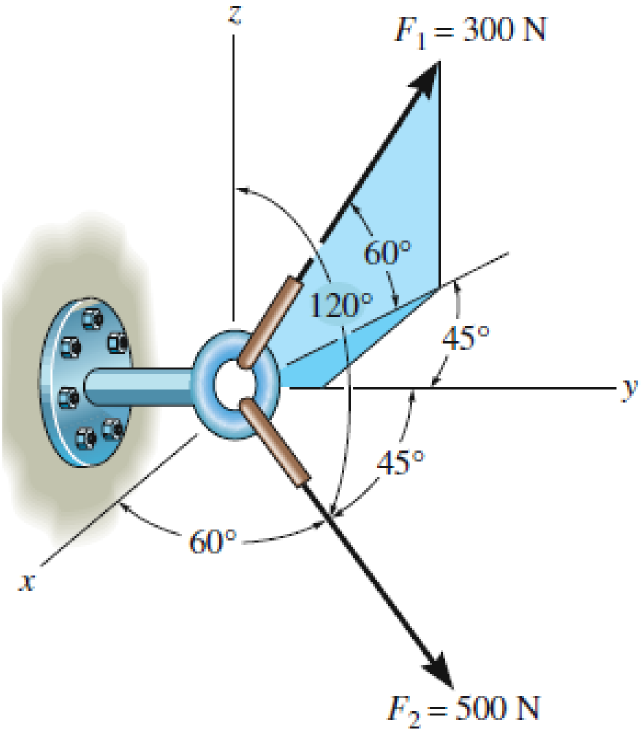
$$F_R = \sqrt{(143.93)^2 + (459.62)^2 + (9.81)^2} = 481.73\,N$$

$$\cos\alpha=\frac{143.93}{481.73}=0.298777 \quad\rightarrow\quad \alpha=\cos^{-1}(0.298777)=72.6158^{O}$$

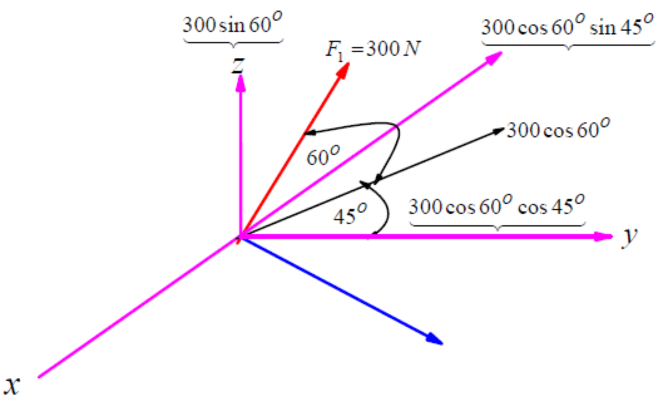
$$\cos\beta=\frac{459.62}{481.73}=0.9541 \quad\rightarrow\quad \beta=\cos^{-1}(0.9451)=17.42^{O}$$

$$\cos\gamma=\frac{9.81}{481.73}=0.0203 \quad\rightarrow\quad \gamma=\cos^{-1}(0.0203)=88.833^{O}$$

Example 9: Determine the coordinate direction angles of F_1.



Solution



$$\vec{F}_1=300\left(-\cos 60^{o}\sin 45^{O}\,\vec{i}+\cos 60^{o}\cos 45^{O}\,\vec{j}+\sin 60^{O}\,\vec{k}\right)$$

$$\vec{F}_1=300\left(-(0.5)(\frac{1}{\sqrt{2}})\,\vec{i}+(0.5)\,(\frac{1}{\sqrt{2}})\,\vec{j}+\frac{\sqrt{3}}{2}\,\vec{k}\right) \quad \vec{F}_1=300\left(-(0.3535)\,\vec{i}+(0.3535)\,\vec{j}+1.732\vec{k}\right)$$

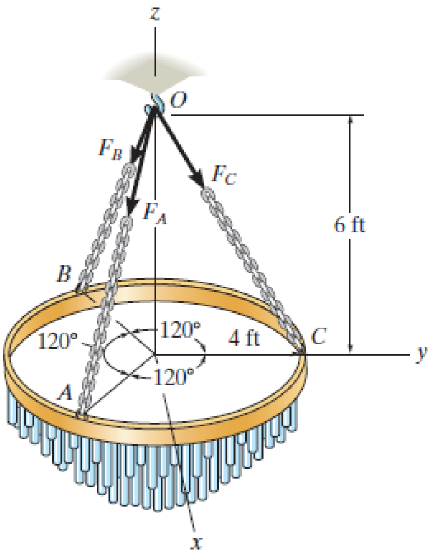
$$\vec{F}_1 = \left(-(106.07)\,\vec{i} + 106.07\,\vec{j} + 259.81\vec{k}\right)$$

$$\cos\alpha = -\frac{106.07}{300} = -0.3569 \quad \rightarrow \quad \alpha = \cos^{-1}(-0.3569) = 110.7056^{O}$$

$$\cos\beta = \frac{106.07}{300} = 0.3569 \quad \rightarrow \quad \beta = \cos^{-1}(0.3569) = 69.09^{O}$$

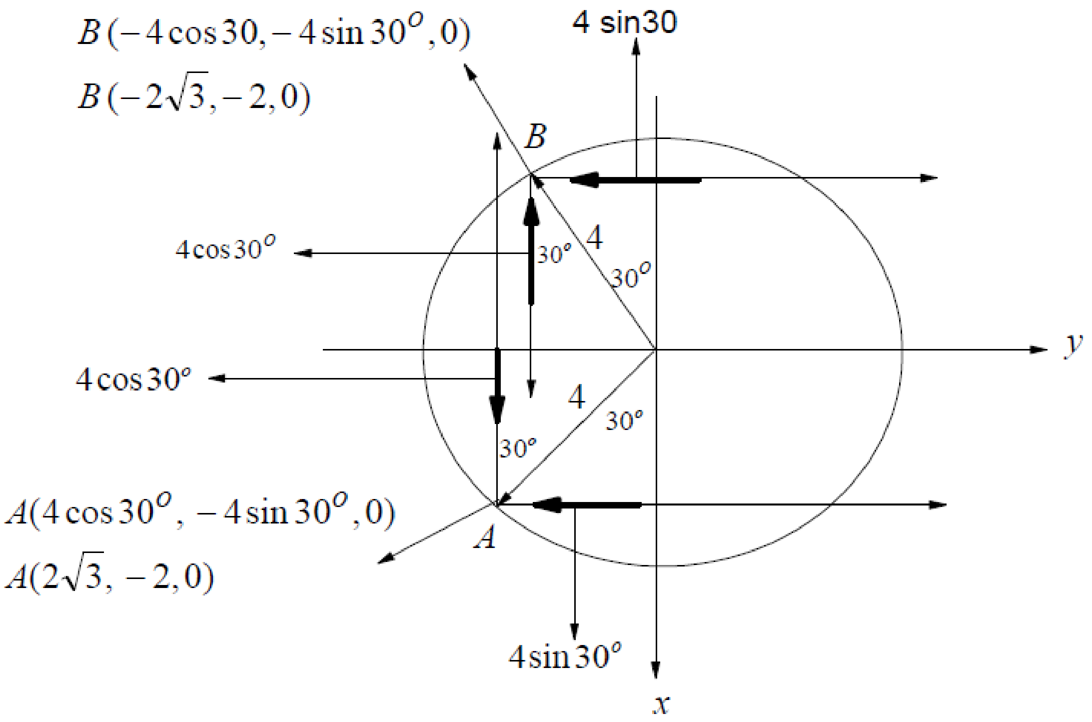
$$\cos\gamma = \frac{259.81}{300} = 0.688033 \quad \rightarrow \quad \gamma = \cos^{-1}(0.688033) = 29.99909^{O} = 30^{O}$$

Example 10: A chandelier is supported by three chains which are concurrent at point O. If the resultant force at O has a magnitude of $130\ Ib$ and is directed along the negative $Z-$ axis, determine the force in each chain.



Solution

Equations of Equilibrium. First we will express each force in Cartesian vector form. Since the coordinates of points O, A, B and C are $O(0,\ 0,\ 6)$,
Figure) we havebelow (see $A(2\sqrt{3},\ -2,\ 0)$, $B(-2\sqrt{3},\ -2,\ 0)$, $C(0,4,\ 0)$



$$\vec{F}_{OA} = \frac{F_{OA}}{\sqrt{52}}(2\sqrt{3}\,\vec{i} - 2\,\vec{j} - 6\,\vec{k})$$

$$\vec{F}_{OB} = \frac{F_{OB}}{\sqrt{52}}(-2\sqrt{3}\,\vec{i} - 2\,\vec{j} - 6\,\vec{k})$$

$$\vec{F}_{OC} = \frac{F_{OC}}{\sqrt{52}}(4\,\vec{j} - 6\,\vec{k})$$

The magnitude of resultant force is, i. e. $\vec{F}_R = \vec{F}_{OA} + \vec{F}_{OB} + \vec{F}_{OC}$

$$\vec{F}_R = 130\vec{k} = \frac{F_{OA}}{\sqrt{52}}(2\sqrt{3}\,\vec{i} - 2\,\vec{j} - 6\,\vec{k}) + \frac{F_{OB}}{\sqrt{52}}(-2\sqrt{3}\,\vec{i} - 2\,\vec{j} - 6\,\vec{k}) + \frac{F_{OC}}{\sqrt{52}}(4\,\vec{j} - 6\,\vec{k}) \qquad (1)$$

From Eq. (1), we have

$$0 = \frac{F_{OA}}{\sqrt{52}}\left(2\sqrt{3}\right) + \frac{F_{OB}}{\sqrt{52}}(-2\sqrt{3}) \quad \rightarrow \quad F_{OA} = F_{OB} \qquad (2)$$

Also, from Eq. (1), we have

$$0 = \frac{F_{OA}}{\sqrt{52}}(-2) + \frac{F_{OB}}{\sqrt{52}}(-2) + \frac{F_{OC}}{\sqrt{52}}(4) \quad \rightarrow \quad 0 = -F_{OA} - F_{OB} + 2F_{OC}$$

From Eq. (2) , we have $0 = -F_{OA} - F_{OA} + 2F_{OC} \rightarrow -2F_{OA} + 2F_{OC} = 0$

$$F_{OA} = F_{OC} \qquad (3)$$

From Eqs. (2) and (3) , we have

$$F_{OA} = F_{OB} = F_{OC} \qquad (4)$$

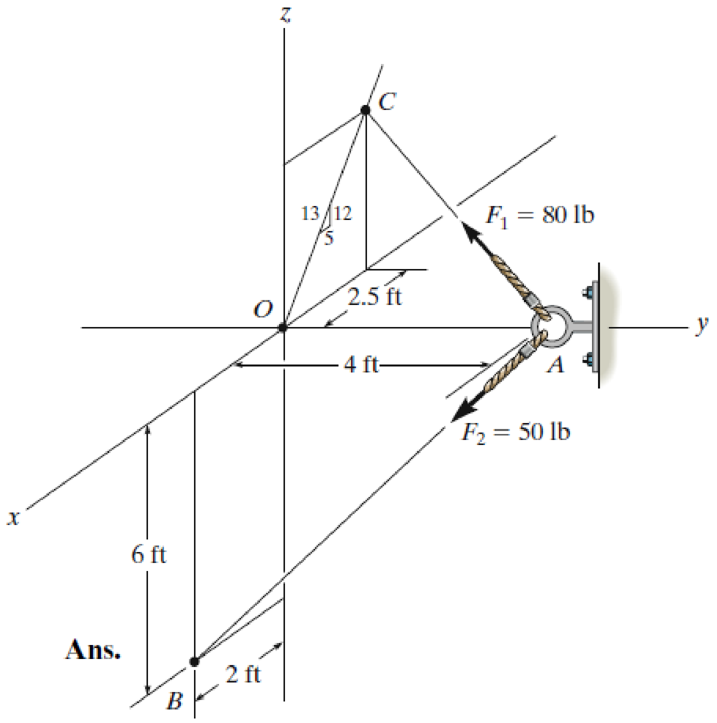
Again, from Eq. (1), we have

$130 = \frac{F_{OA}}{\sqrt{52}}(-6) + \frac{F_{OB}}{\sqrt{52}}(-6) + \frac{F_{OC}}{\sqrt{52}}(-6)$, and from Eq. (4), we have

$$130 = 3\left(\frac{F_{OA}}{\sqrt{52}}(-6)\right) \rightarrow \qquad F_{OA} = -\frac{130\sqrt{52}}{18} = -52.1\ Ib$$

Then the force in each chain are $F_{OA} = F_{OB} = F_{OC} = 52.1\ Ib$

Example: 11 Express each of the forces in Cartesian vector form and determine the magnitude and coordinate direction angles of the resultant force for the two forces act in below figure ?



Solution

Equations of Equilibrium. First we will express each force in Cartesian vector form. Since the coordinates of points A, B are given $A(0, 4, 0)$, $B(2, 0, -6)$

While, C is given as (see blow Figure) $C(-2.5, 0, \frac{12}{5}(2.5)) \to C(-2.5, 0, 6)$.

Since the force F_{AB} acts across the two points $A(0, 4, 0)$ and $B(2, 0, -6)$. So the unit vector in this direction is $\vec{e}_{AB} = \frac{1}{\sqrt{56}}(2, -4, -6)$, i. e. Then

$$\vec{F}_{AB} = F_{AB}\,\vec{e}_{AB} = \frac{F_{AB}}{\sqrt{56}}(2, -4, -6) = \frac{50}{\sqrt{56}}(2, -4, -6).$$

Also, force F_{AC} acts across the two points $A(0, 4, 0)$ and rSo the unit vecto .$C(-2.5, 0, 6)$ in this direction is $\vec{e}_{AC} = \frac{1}{\sqrt{58.25}}(-2.5, -4, 6)$, i. e. Then

$$.\vec{F}_{AC} = F_{AC}\,\vec{e}_{AC} = \frac{F_{AC}}{\sqrt{58.25}}(2.5, -4, 6) = \frac{80}{\sqrt{58.25}}(-2.5, -4, 6)$$

$$\vec{F} = \vec{F}_{AB} + \vec{F}_{AC}$$

$$\vec{F} = \frac{50}{\sqrt{56}}(2, -4, -6) + \frac{80}{\sqrt{58.25}}(-2.5, -4, 6)$$

$$\vec{F} = \left(\frac{100}{\sqrt{56}} - \frac{200}{\sqrt{58.25}}\right)\vec{i} - \left(\frac{200}{\sqrt{56}} + \frac{320}{\sqrt{58.25}}\right)\vec{j} + \left(-\frac{300}{\sqrt{56}} + \frac{480}{\sqrt{58.25}}\right)\vec{k}$$

$$\vec{F} = \left(\frac{100}{7.4833} - \frac{200}{7.632}\right)\vec{i} - \left(\frac{200}{7.4833} + \frac{320}{7.632}\right)\vec{j} + \left(-\frac{300}{7.4833} + \frac{480}{7.632}\right)\vec{k}$$

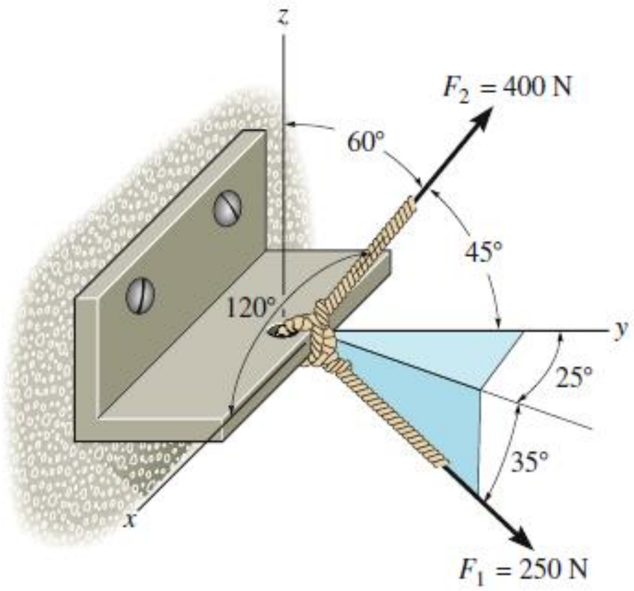
$$\vec{F} = (13.3511 - 26.2054)\vec{i} - (26.7261 + 41.9287)\vec{j} + (-40.0892 + 62.893)\vec{k}$$
$$\vec{F} = (12.8543)\vec{i} - (68.6548)\vec{j} + (22.807)\vec{k}$$

Then $F = \sqrt{(-12.8543)^2 + (-68.6548)^2 + (22.807)^2} = 73.5 \ Ib$ The direction of F given from $\cos\alpha = \frac{F_x}{F}, \ \cos\beta = \frac{F_y}{F}, \ \cos\gamma = \frac{F_z}{F}$

Then

$$\cos\alpha = -\frac{12.8543}{73.5} = -0.174049 \qquad \rightarrow \qquad \alpha = \cos^{-1}(0.8) = 100.0233^O$$
$$\cos\beta = -\frac{68.6548}{73.5} = -0.934078 \qquad \rightarrow \qquad \beta = \cos^{-1}(-0.934078) = 159.0798^O$$
$$\cos\gamma = \frac{22.807}{73.5} = 0.310299 \qquad \rightarrow \qquad \gamma = \cos^{-1}(0.310299) = 71.9227^O$$

Example: 12 The bracket is subjected to the two forces shown in below Figure. Express each force in Cartesian vector form and then determine the resultant force Find the magnitude and coordinate direction angles of the resultant force



Solution

$$\vec{F}_1 = 250\left(\cos35^o \sin25^O \vec{i} + \cos35^o \cos25^O \vec{j} - \sin35^O \vec{k}\right)$$
$$\vec{F}_1 = 250\left((0.81915)(0.42261)\vec{i} + (0.81915)(0.9063)\vec{j} - (0.57357)\vec{k}\right)$$
$$\vec{F}_1 = \left(86.545)\vec{i} + 185.5989)\vec{j} - 143.394\vec{k}\right)$$

Using the relation $\vec{F} = F\cos\alpha\vec{i} + F\cos\beta\vec{j} + F\cos\gamma\vec{k}$, we have

$$\vec{F}_2 = 400\left(\cos120^O \vec{i} + \cos45^O \vec{j} + \cos60^O \vec{k}\right)$$
$$\vec{F}_2 = 400\left((-0.5)\vec{i} + \frac{1}{\sqrt{2}}\vec{j} + (0.5)\vec{k}\right)$$
$$\vec{F}_2 = \left(-200\vec{i} + (282.8427)\vec{j} + 200\vec{k}\right)$$

So, the resultant given by

$$\vec{F}_R = \vec{F}_1 + \vec{F}_2 = \left(86.545\vec{i} + 185.5989\vec{j} - 143.394\vec{k}\right) + \left(-200\vec{i} + 282.8427\vec{j} + 200\vec{k}\right)$$

$$\vec{F}_R = \left(-113.455\vec{i} + 468.44161\vec{j} + 56.606\vec{k}\right)$$

$$F_R = \sqrt{(-113.455)^2 + (468.44161)^2 + (56.606)^2} = 485.3\ N$$

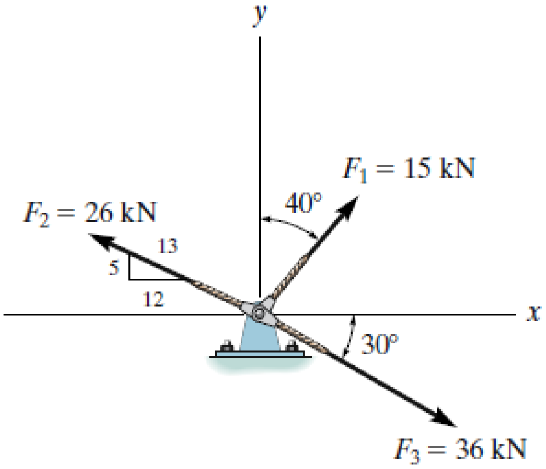
$$\cos\alpha = -\frac{113.455}{485.3} = -0.23378 \quad \rightarrow \quad \alpha = \cos^{-1}(-0.23378) = 103.5197^O$$

$$\cos\beta = \frac{468.44161}{485.3} = 0.96526 \quad \rightarrow \quad \beta = \cos^{-1}(0.96526) = 15.1462^O$$

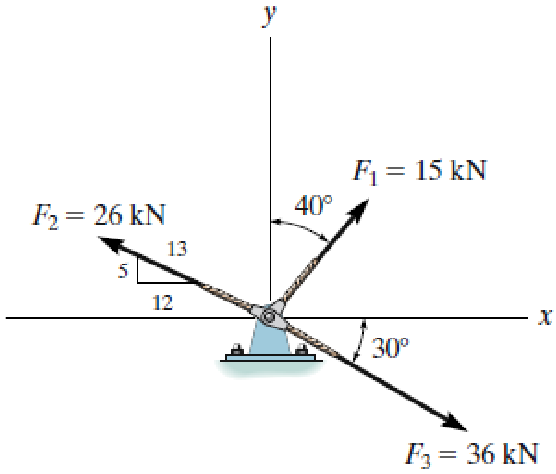
$$\cos\gamma = \frac{56.606}{485.3} = 0.11664 \quad \rightarrow \quad \gamma = \cos^{-1}(0.11664) = 83.3017^O$$

Problems

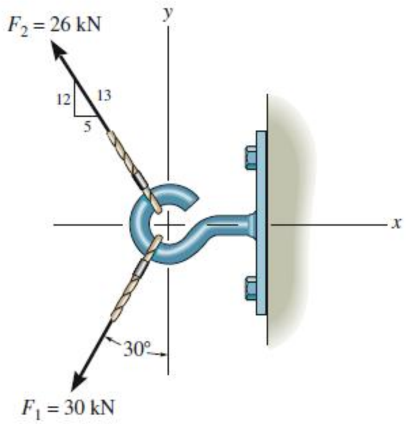
(1) Express F1, F2, and F3 as Cartesian vectors.



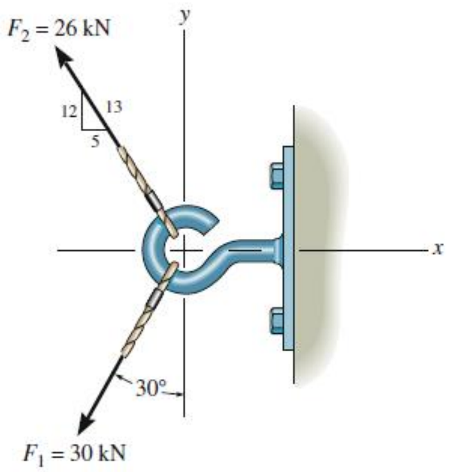
(2) Determine the magnitude of the resultant force and its orientation measured counterclockwise from the positive x axis



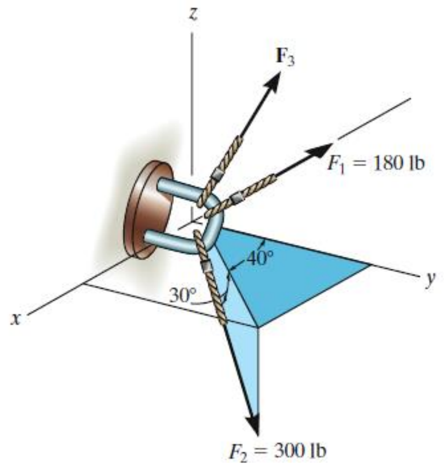
(3) Express F1 and F2 as Cartesian vectors.



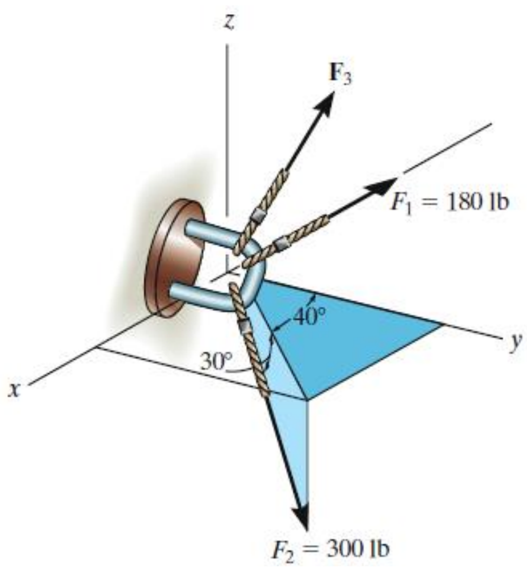
(4) Determine the magnitude of the resultant force and its direction measured counterclockwise from the positive x axis.



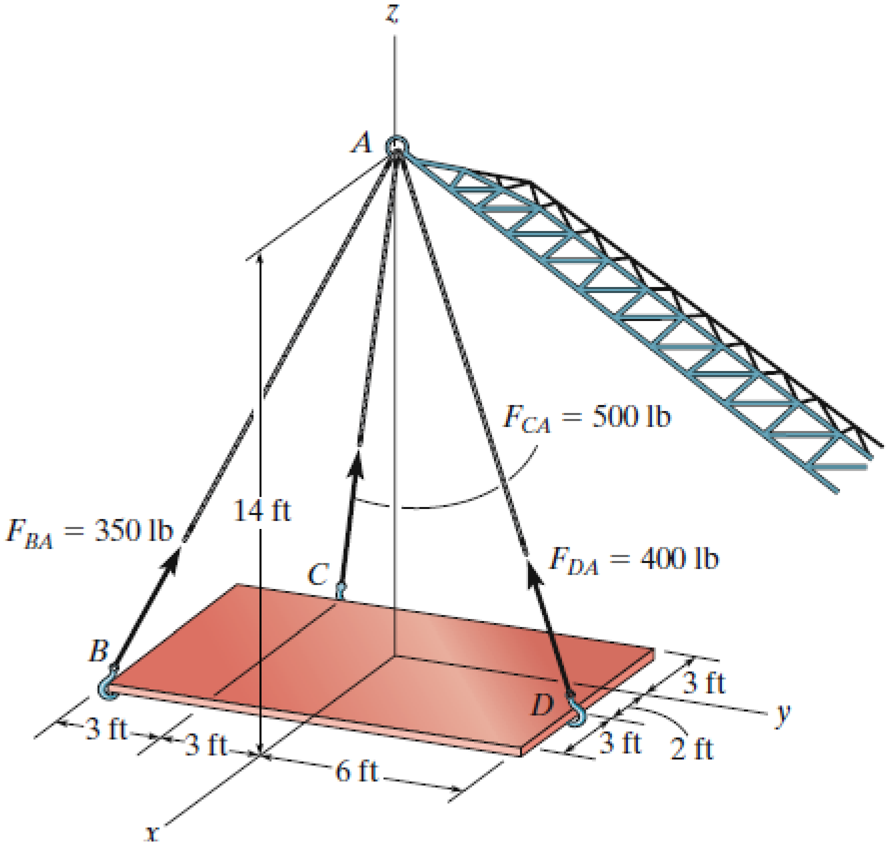
(5) Determine the magnitude and coordinate direction angles of f3 so that the resultant of the three forces acts along the positive y axis and has a magnitude of 600 lb.



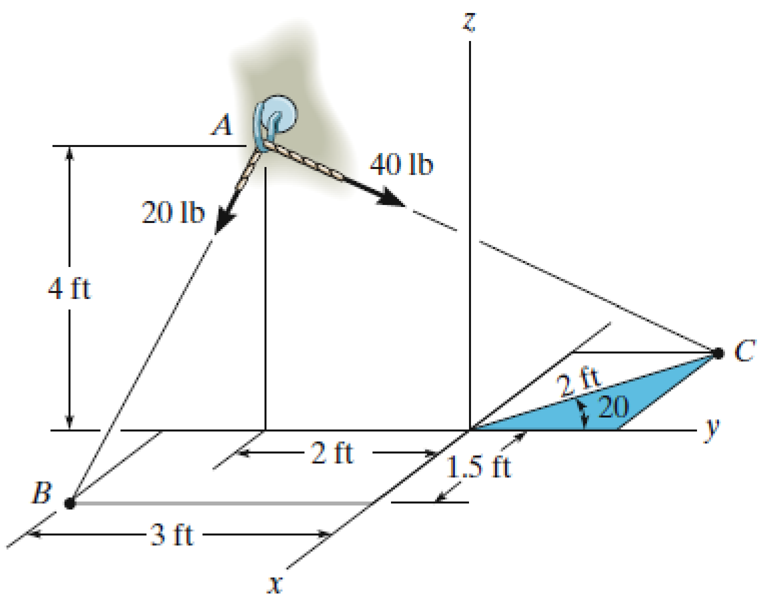
(6) Determine the magnitude and coordinate direction angles of F3 so that the resultant of the three forces is zero



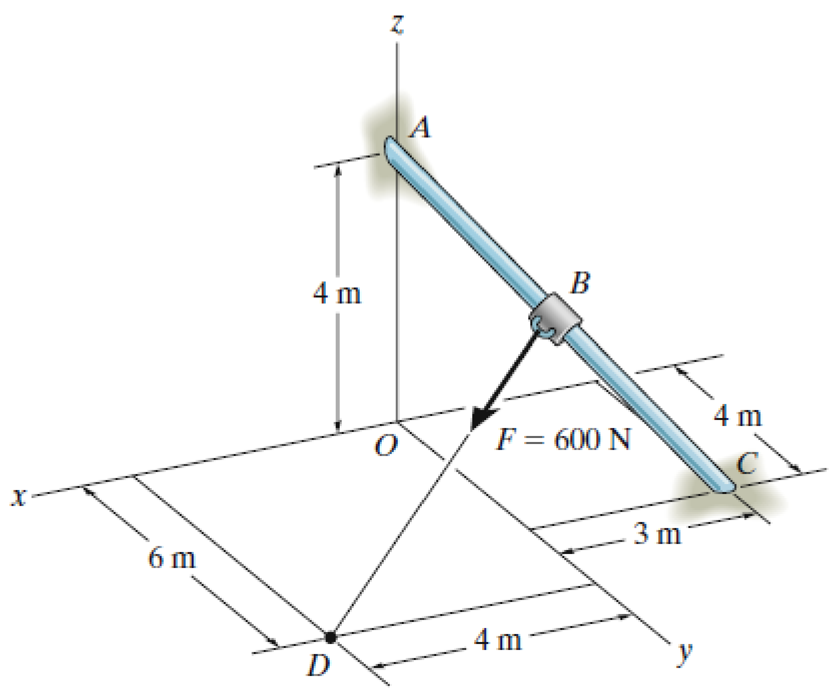
(7) The plate is suspended using the three cables which exert the forces shown. Express each force as a Cartesian vector.



(8) Determine the magnitude and coordinates on angles of the resultant force.



(9) Express force F in Cartesian vector form if point B is located 3 m along the rod end C.



Chapter: 3

Condition for the Equilibrium of a particle

A particle is said to be in equilibrium if it remains at rest if originally at rest, or has a constant velocity if originally in motion. Most often, however, the term "equilibrium" or, more specifically, "static equilibrium" is used to describe an object at rest. To maintain equilibrium, it is necessary to satisfy Newton's first law of motion, which requires the resultant force acting on a particle to be equal to zero . This condition may be stated mathematically as

$$F = 0 \qquad (1)$$

where F is the vector sum of all the forces acting on the particle.

Not only is $F = 0$ a necessary condition for equilibrium, it is also a sufficient condition. This follows from Newton's second law of motion, which can be written as $F = ma$. Since the force system satisfies Eq. (1) , then $ma = 0$, and therefore the particle's acceleration $a = 0$ Consequently, the particle indeed moves with constant velocity or remains at rest.

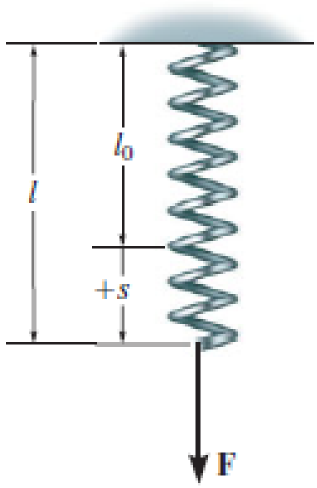


Fig. 1

1. Coplanar Force Systems

plane, as in Fig. (2) , then each force can be resolved into its $\vec{i}$ and $\vec{j}$ components. For equilibrium, these forces must sum to produce a zero

force resultant, i.e., $\sum F = 0 \quad \rightarrow \sum F_x \vec{i} + \sum F_y \vec{j} = 0$

For this vector equation to be satisfied, the resultant force's x and y components must both be equal to zero. Hence,. $\sum F_x = 0, \quad \sum F_y = 0$

These two equations can be solved for at most two unknowns, generally represented as angles and magnitudes of forces shown on the particle's free-body diagram.

When applying each of the two equations of equilibrium, we must account for the sense of direction of any component by using an algebraic sign which corresponds to the arrowhead direction of the component along the x or y axis. It is important to note that if a force has an unknown magnitude , then the arrowhead sense of the force on the free-body diagram can be assumed . Then if the solution yields a negative scalar , this indicates that the sense of the force is opposite to that which was assumed.

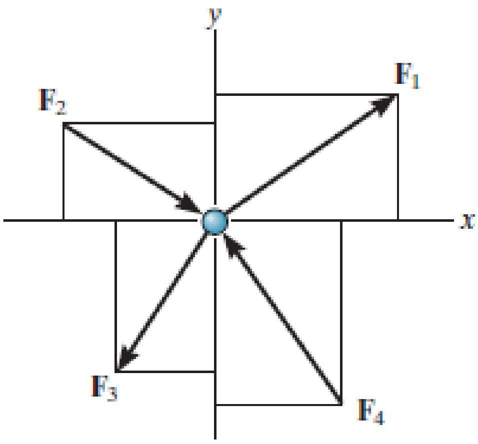
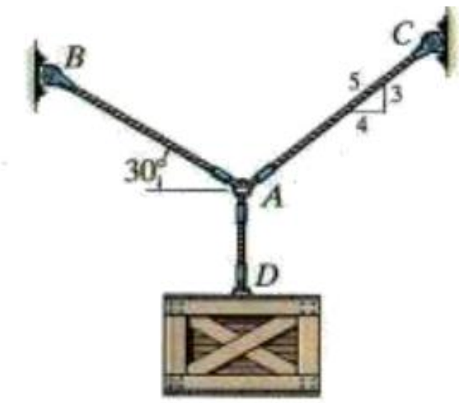


Fig. 2

Example 1: The Crate has a weight of $550N = (\approx 55\,kg)$. Determine the tension in each supporting cable in Figure



Solution

Applying the equations of equilibrium along the x and y axes, we have

$$\sum F_x = 0 \quad \rightarrow \quad T_{AC}\left(\frac{4}{5}\right) - T_{AB}\cos 30^O = 0 \quad \rightarrow \quad T_{AC}\left(\frac{4}{5}\right) - \frac{\sqrt{3}}{2}T_{AB} = 0$$

$$\sum F_y = 0 \quad \rightarrow \quad T_{AC}\left(\frac{3}{5}\right) + T_{AB}\sin 30^O - 550 = 0 \quad \rightarrow \quad T_{AC}\left(\frac{3}{5}\right) + \frac{1}{2}T_{AB} = 550$$

$8\,T_{AC} - 5\sqrt{3}T_{AB} = 0,$ (1), $6T_{AC} + 5T_{AB} = 5500$ (2)

$$T_{AC} = \frac{5500\sqrt{3}}{\left(8+6\sqrt{3}\right)} = 518\ N \qquad \left(8+6\sqrt{3}\right)T_{AC} = 5500\sqrt{3} \rightarrow , \quad \text{Then} \quad 6\sqrt{3}T_{AC} + 5\sqrt{3}\,T_{AB} = 5500\sqrt{3}$$

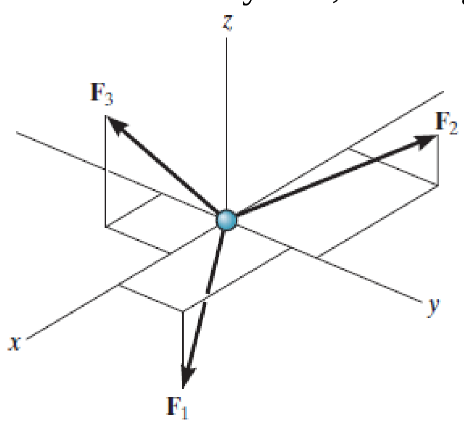
In Eq. (1), we have $8\,\frac{5500\sqrt{3}}{\left(8+6\sqrt{3}\right)} - 5\sqrt{3}\,T_{AB} = 0 \quad \rightarrow \quad 8\,\frac{1100}{\left(8+6\sqrt{3}\right)} - T_{AB} = 0,$

$$T_{AB} = \frac{8800}{\left(8+6\sqrt{3}\right)} = 478.5\ N$$

2. Three-dimensional force system

The necessary and sufficient condition for particle equilibrium is $\sum F = 0$

In the case of a three-dimensional force system, as in Figure .



We can resolve the forces into their respective $\vec{i}, \vec{j}, \vec{k}$ components, so that

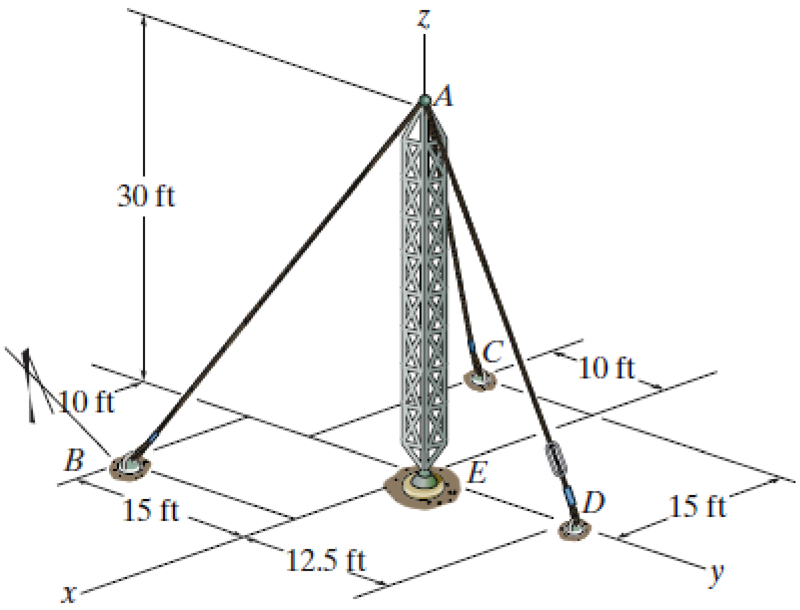
$\sum F_x\vec{i} + F_y\,\vec{j} + F_z\,\vec{k} = 0$.

To satisfy this equation we require

$$F_x = 0, \quad F_y = 0, \quad F_z = 0$$

These three equations state that the algebraic sum of the components of all the forces acting on the particle along each of the coordinate axes must be zero. Using them we can solve for at most three unknowns, generally represented as coordinate direction angles or magnitudes of forces shown on the particle's free-body diagram.

Example 2: If cable AD is tightened by a turnbuckle and develops a tension of $1300\,Ib$. Determine the tension developed in cables AB and AC and the force developed along the antenna tower AE at point A.



Free-Body Diagram. First we will express each force in Cartesian vector form. Since the coordinates of points O, A, B and C are (see below Figure)

$A(0, 0, 30)$, $B(10, -15, 0)$, $C(-15, -10, 0)$, $D(0, 12.5, 0)$, $E(0, 0, 0)$,

Form the Figure we can express T_{AB}, T_{AC}, T_{AD} and F as follows

Since the tension $\vec{T}_{AB}$ acts across the two points A and B. So the unit vector in this direction is $\vec{e}_{AB} = B - A = \frac{1}{\sqrt{1225}}(10, -15, -30) = \frac{1}{35}(10, -15, -30) = \frac{1}{7}(2, -3, -6)$, i. e

$\vec{e}_{AB} = \frac{1}{7}(2, -3, -6)$. Then

$$\vec{T}_{AB} = T_{AB}\vec{e}_{AB} = \vec{e}_{AB} = \frac{T_{AB}}{7}(2, -3, -6).$$

Also, the tension $\vec{T}_{AC}$ acts across the two points A and C. So the unit vector in this direction is

$\vec{e}_{AC} = C - A = \frac{1}{\sqrt{1225}}(-15, -10, -30) = \frac{1}{35}(-15, -10, -30) = \frac{1}{7}(-3, -2, -6)$, i. e

$\vec{e}_{AC} = \frac{1}{7}(-3, -2, -6)$. Then

$$\vec{T}_{AC} = T_{AC}\vec{e}_{AC} = \frac{T_{AC}}{7}(-3, -2, -6)$$

A third time, the tension T_{AD} acts across the two points A and D. So the unit vector in this direction is

$$\vec{e}_{AD} = D - A = \frac{1}{\sqrt{1056.5}}(0,12.5,-30) = \frac{1}{32.5}(0,12.5,-30) = \frac{1}{325}(0,125,-300),$$

i. e $\vec{e}_{AD} = \frac{1}{13}(0,5,-12)$. Then

$$\vec{T}_{AD} = T_{AD}\,\vec{e}_{AD} = \frac{T_{AD}}{13}(0,5,-12) = \frac{1300}{13}(0,5,-12) = 100(0,5,-12)$$

$$\vec{F} = F\,\vec{k}$$

Equations of Equilibrium: Equilibrium requires

$\sum F = 0 \quad \rightarrow \vec{T}_{AB} + \vec{T}_{AC} + \vec{T}_{AD} + F = 0$, i. e.

$$\frac{T_{AB}}{7}(2) + \frac{T_{AC}}{7}(-3) + 100(0) + F(0) = 0 \quad \rightarrow \quad 2T_{AB} - 3T_{AC} = 0 \qquad (1)$$

$$\frac{T_{AB}}{7}(-3) + \frac{T_{AC}}{7}(-2) + 100(5) + F(0) = 0 \quad \rightarrow \quad -3T_{AB} - 2T_{AC} + 3500 = 0 \qquad (2)$$

$$\frac{T_{AB}}{7}(-6) + \frac{T_{AC}}{7}(-6) + 100(-12) + F(1) = 0 \quad \rightarrow -6T_{AB} - 6T_{AC} - 8400 + 7F = 0 \qquad (3)$$

From Eq. (1) and Eq. (2), we have

$$3\left\{2T_{AB} - 3T_{AC}\right\} + 2\left\{-3T_{AB} - 2T_{AC} + 3500\right\} = 0$$

$$3\left\{-3T_{AC}\right\} + 2\left\{-2T_{AC} + 3500\right\} = 0 \;\rightarrow\; 13T_{AC} = 7000 \;\rightarrow\; T_{AC} = 538.461\,N$$

In Eq. (1), we have

$$T_{AB} = \frac{3}{2}T_{AC} = \left(\frac{3}{2}\right)538.461 = 807.692\ N \;\rightarrow\; T_{AB} = 807.692\ N$$

While in Eq. (3), we have

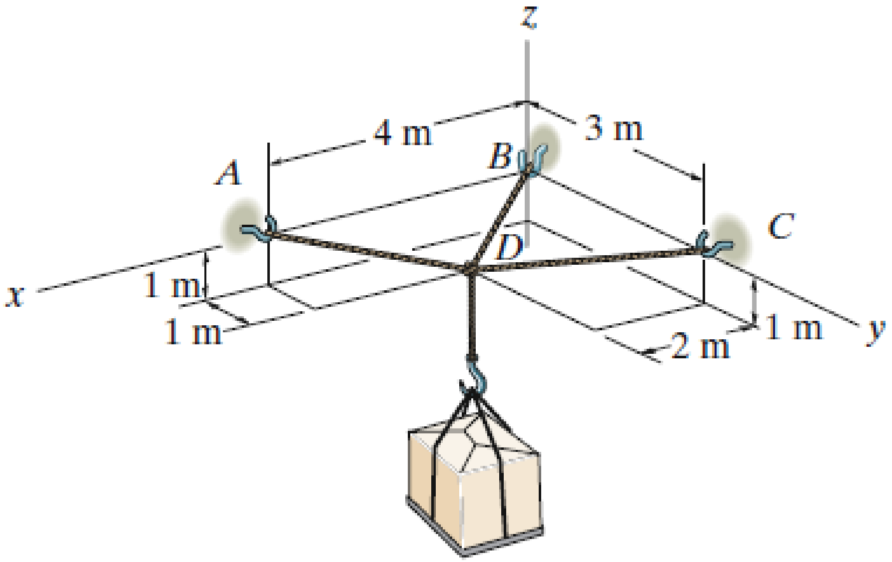
$$-6(807.692) - 6(538.461) - 8400 + 7F = 0 \;\rightarrow\; 7F = 6(807.692 + 538.461) + 8400$$

$$F = 2353.845\,N$$

Final

$$T_{AB} = 807.692\ N, \quad T_{AC} = 538.461\,N, \quad F = 2353.845\,N.$$

Example 3: The crate has a mass of $130\,kg$. Determine the tension developed in each cable for equilibrium.



Solution

Free-Body Diagram. First we will express each force in Cartesian vector form. Since

$A(4, 0, 1),\ \ B(0, 0, 1),\ \ C(0, 3, 1),\ \ \ D(2, 1, 0)$

Form the Figure we can express T_{AD}, T_{BD}, T_{CD} and F as follows

Since the tension $\vec{T}_{AD}$ acts across the two points A and D. So the unit vector in this direction is $\vec{e}_{AD} = D - A = \frac{1}{\sqrt{6}}(-2, 1, -1)$. Then

$$\vec{T}_{AD} = T_{AD}\,\vec{e}_{AD} = \vec{e}_{AD} = \frac{T_{AD}}{\sqrt{6}}(-2, 1, -1).$$

Since the tension $\vec{T}_{BD}$ acts across the two points B and D. So the unit vector in this direction is $\vec{e}_{BD} = D - B = \frac{1}{\sqrt{6}}(2, 1, -1)$. Then

$$\vec{T}_{BD} = T_{AD}\,\vec{e}_{AD} = \vec{e}_{AD} = \frac{T_{AD}}{\sqrt{6}}(2, 1, -1).$$

Since the tension $\vec{T}_{CD}$ acts across the two points B and D. So the unit vector in this direction is $\vec{e}_{CD} = D - C = \frac{1}{\sqrt{5}}(2, 0, -1)$. Then

$$\vec{T}_{CD} = T_{CD}\,\vec{e}_{CD} = \vec{e}_{CD} = \frac{T_{CD}}{\sqrt{5}}(2, 0, -1).$$

Equations of Equilibrium: Equilibrium requires

$\sum F = 0 \ \ \rightarrow \vec{T}_{AB} + \vec{T}_{AC} + \vec{T}_{AD} + F = 0$, i. e.

$$\frac{T_{AB}}{7}(2,-3,-6)+\frac{T_{AC}}{7}(-3,-2,-6)+100(0,5,-12)+F(0,0,1)=0$$

$$\frac{T_{AB}}{7}(2)+\frac{T_{AC}}{7}(-3)+100(0)+F(0)=0 \quad \rightarrow \quad 2T_{AB}-3T_{AC}=0 \qquad (1)$$

$$\frac{T_{AB}}{7}(-3)+\frac{T_{AC}}{7}(-2)+100(5)+F(0)=0 \quad \rightarrow \quad -3T_{AB}-2T_{AC}+3500=0 \qquad (2)$$

$$\frac{T_{AB}}{7}(-6)+\frac{T_{AC}}{7}(-6)+100(-12)+F(1)=0 \quad \rightarrow -6T_{AB}-6T_{AC}-8400+7F=0 \qquad (3)$$

From Eq. (1) and Eq. (2), we have

$$3\left\{2T_{AB}-3T_{AC}\right\}+2\left\{-3T_{AB}-2T_{AC}+3500\right\}=0$$

$$3\left\{-3T_{AC}\right\}+2\left\{-2T_{AC}+3500\right\}=0 \;\rightarrow\; 13T_{AC}=7000 \;\rightarrow\; T_{AC}=538.461\,N$$

In Eq. (1), we have

$$T_{AB}=\frac{3}{2}T_{AC}=\left(\frac{3}{2}\right)538.461=807.692\;\;N \;\rightarrow\; T_{AB}=807.692\;\;N$$

While in Eq. (3), we have

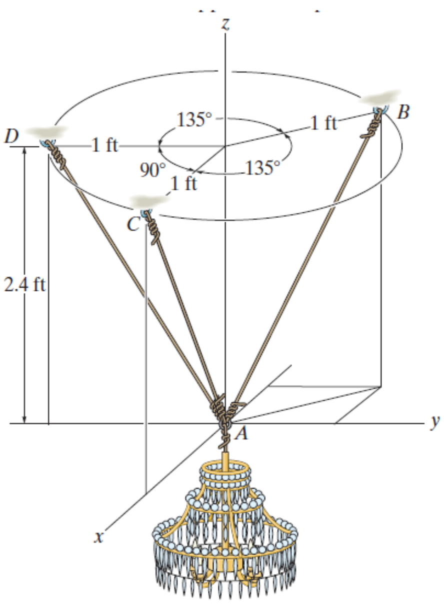
$$-6(807.692)-6(538.461)-8400+7F=0 \;\rightarrow\; 7F=6(807.692+538.461)+8400$$

$$F=2353.845\,N$$

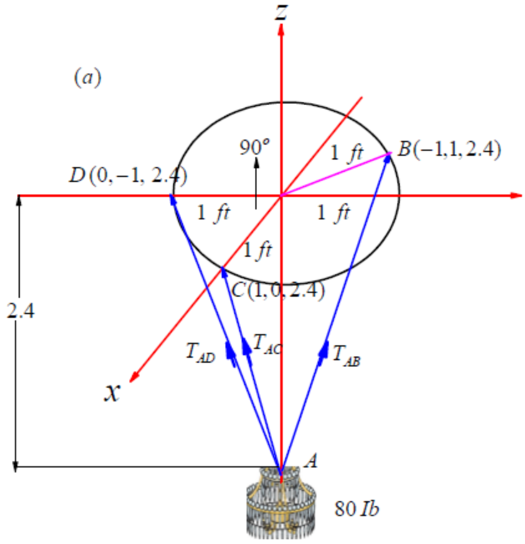
Final

$$T_{AB}=807.692\;\;N, \quad T_{AC}=538.461\,N, \quad F=2353.845\,N.$$

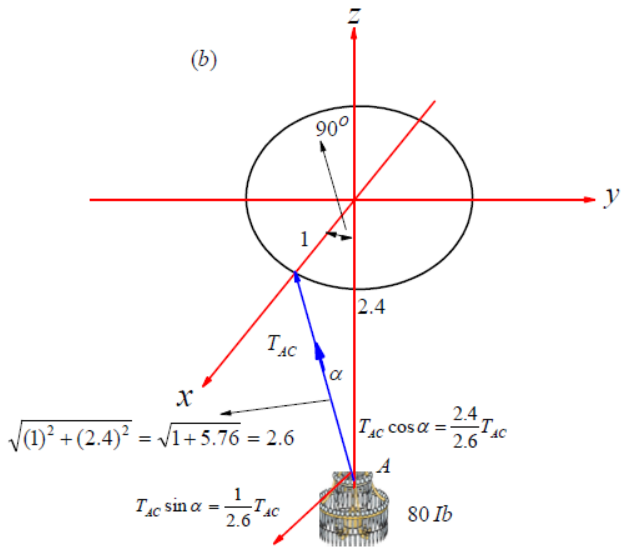
Example 4: The $80\,lb$ chandelier is supported by three wires as shown. Determine the force in each wire for equilibrium.

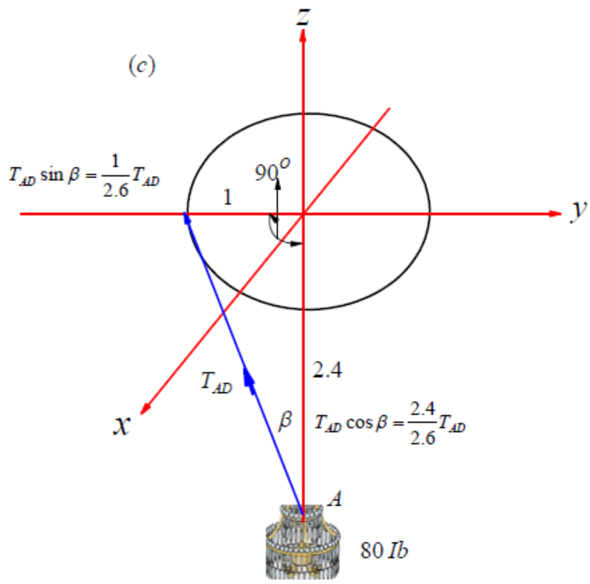


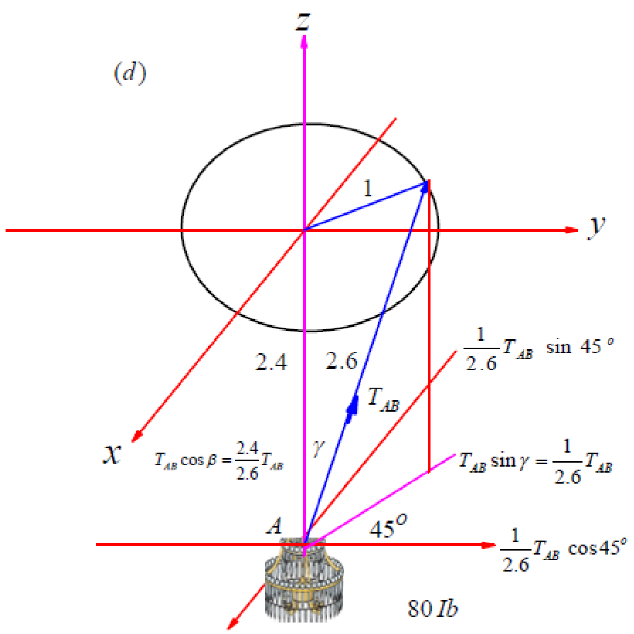
Solution



From this Figure we can organize the next three Figures (b-d)







From Figure (b-d) we can find that

$$\frac{1}{2.6}T_{AC} - \frac{1}{2.6}T_{AB}\sin 45^o = 0 \quad \rightarrow \quad T_{AC} - 0.7071\,T_{AB} = 0 \qquad (1)$$

$$-\frac{1}{2.6}T_{AD} + \frac{1}{2.6}T_{AB}\sin 45^o = 0 \quad \rightarrow \quad T_{AD} - 0.7071T_{AB} = 0 \qquad (2)$$

$$\frac{2.4}{2.6}T_{AC} + \frac{2.4}{2.6}T_{AD} + \frac{2.4}{2.6}T_{AB} - 80 = 0 \quad \rightarrow \quad T_{AC} + T_{AD} + T_{AB} = 86.666 \qquad (3)$$

Solving Eqs. (1)- (3)

Subtracting Eq. (1) and (2), we have $T_{AC} = T_{AD}$

Then substituting into Eq. (3)

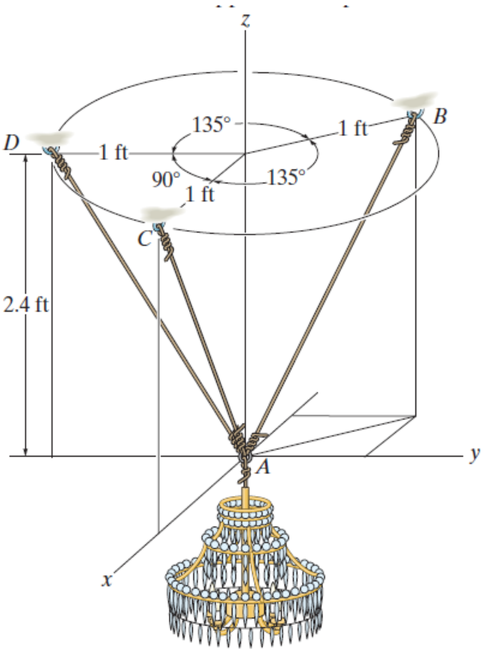
$$T_{AC} + T_{AC} + T_{AB} = 86.666 \rightarrow T_{AC} + \frac{1}{2}T_{AB} = 43.333 \qquad (4)$$

Again substituting into Eq. (3) $\quad 0.7071\,T_{AB} + 0.5T_{AB} = 43.333 \rightarrow T_{AB} = 35.9\ \ Ib$

Then in Eq. (4), we have $T_{AC} + \frac{1}{2}(35.9) = 43.333 \rightarrow T_{AC} = 25.4\ Ib$

Final $T_{AB} = 35.9, \quad T_{AC} = T_{AD} = 25.4\ Ib$ (Ans.)

Example 5: If each wire can sustain a maximum tension of $120\ Ib$ before it fails, determine the greatest weight of the chandelier the wires will support in the position shown.



Solution

From the above example

$$\frac{1}{2.6}T_{AC}-\frac{1}{2.6}T_{AB}\sin 45^{o}=0 \quad \rightarrow \quad T_{AC}-0.7071\,T_{AB}=0 \qquad (1)$$

$$-\frac{1}{2.6}T_{AD}+\frac{1}{2.6}T_{AB}\sin 45^{o}=0 \quad \rightarrow \quad T_{AD}-0.7071T_{AB}=0 \qquad (2)$$

$$\frac{2.4}{2.6}T_{AC}+\frac{2.4}{2.6}T_{AD}+\frac{2.4}{2.6}T_{AB}-W=0 \quad \rightarrow \quad T_{AC}+T_{AD}+T_{AB}=1.08\,W \qquad (3)$$

Now if we put $T_{AC}=120$ in Eq. (1), we get

$$120-0.7071\,T_{AB}=0 \quad \rightarrow \quad T_{AB}=\frac{120}{0.7071}=169.7072>120 \quad \text{rejected solution}$$

Again, if we put $T_{AB}=120$ in Eq. (1) , we get

$$T_{AC}-0.7071\,T_{AB}=0 \quad \rightarrow \quad T_{AC}=120(0.7071)=84.85<120 \quad \text{A reasonable solution}$$

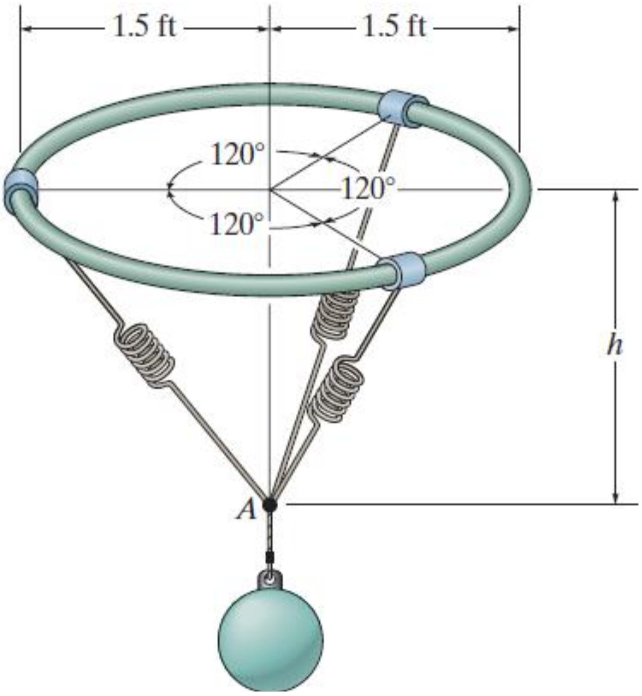
Then in Eq. (2), we have

$$T_{AD}-0.7071\,T_{AB}=0 \quad \rightarrow \quad T_{AB}=(120)(0.7071)=84.85<120 \quad \text{A reasonable solution}$$

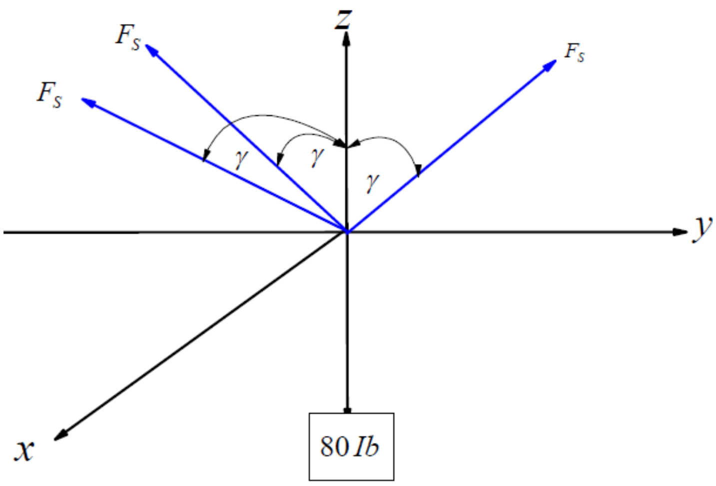
In Eq. (3)

$$84.85+84.85+120=1.08\,W \quad \rightarrow \quad 1.08\,W=289.7 \quad \rightarrow \quad W=268.24$$

Example 6: The $80\,Ib$ ball is suspended from the horizontal ring using three springs each having an unstretched length of $1.5\,ft$ and stiffness of $50\,Ib/ft$. Determine the vertical distance h from the ring to point A for equilibrium.



Solution



It clears that, the three springs are symmetric and subjected to a same tensile force. If one realizes this forces to the $z-$axis, we have

$$F_s\cos\gamma + F_s\cos\gamma + F_s\cos\gamma - 80 = 0 \quad \rightarrow \quad 3F_s\cos\gamma = 80 \qquad (1)$$

But we know that , the relation between spring force and stiffness (k) given by

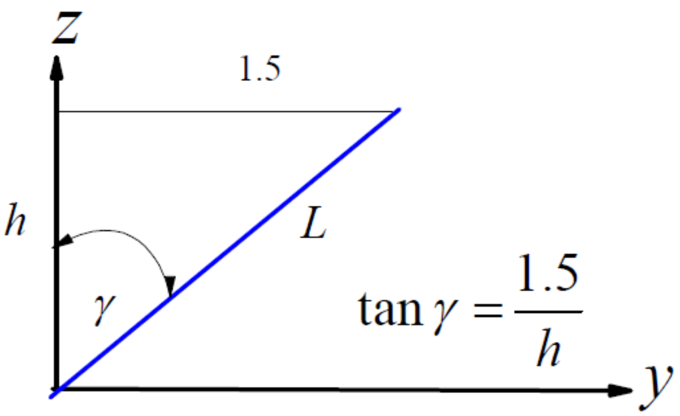
$$F_s = s\,k = k\,(L - L_0) = 50\left(\frac{1.5}{\sin\gamma} - 1.5\right) = 75\left(\frac{1}{\sin\gamma} - 1\right) \qquad (2)$$

Substituting from Eq. (1) into Eq. (2)

$$\frac{80}{3\cos\gamma} = 75\left(\frac{1}{\sin\gamma} - 1\right) \rightarrow \frac{\sin\gamma}{\cos\gamma}\,\frac{80}{225} = 1 - \sin\gamma \rightarrow \quad 0.3555\tan\gamma + \sin\gamma = 1 \qquad (3)$$

γ	$0.3555\tan\gamma$	$\sin\gamma$	$0.3555\tan\gamma+\sin\gamma=1$
30	$(0.3555)(0.5773)=0.20524$	0.5	0.70524
40	$(0.3555)\,(0.839)=0.29829$	0.6427	0.9299
42	$(0.3555)\,(0.9)=0.32$	0.6691	0.9891
42.5	$(0.3555)\,(0.916)=0.32575$	0.67559	1.0013

Then $\gamma=42.5^{o}$



(b)

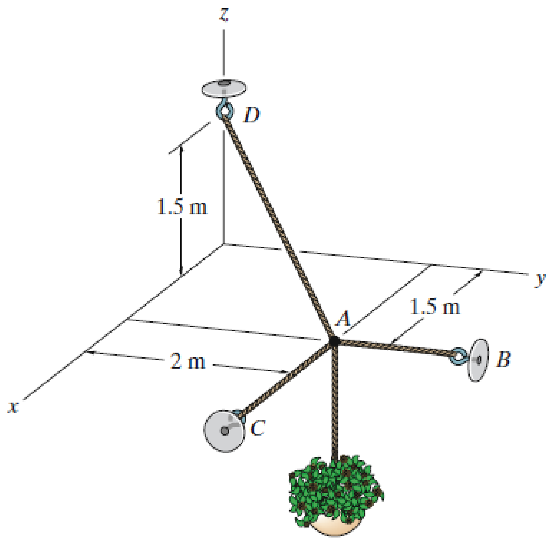
From Fig. (b) we note that

$$\tan\gamma=\frac{1.5}{h}\;\rightarrow\; h=\frac{1.5}{\tan(42.5^{o})}=\frac{1.5}{0.916}=1.64\ ft$$

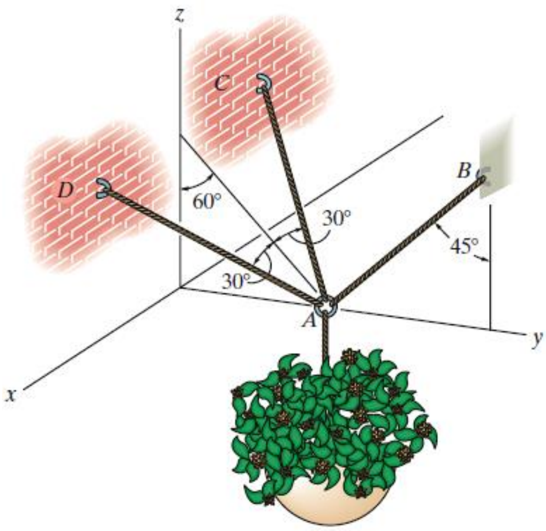
Then, the vertical distance h from the ring to point A for equilibrium is $1.64\ ft$

Problems

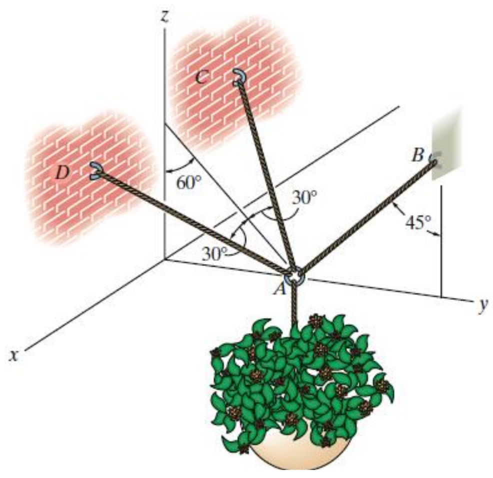
(1) The three cables are used to support the 40-kg flowerpot. Determine the force developed in each cable for equilibrium ?



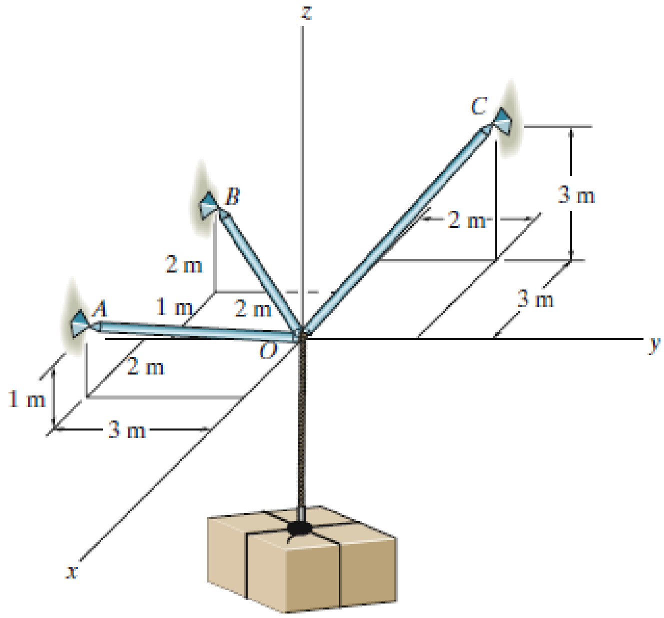
(2) The 25-kg flowerpot is supported at A by the three cords. Determine the force acting in each cord for equilibrium ?



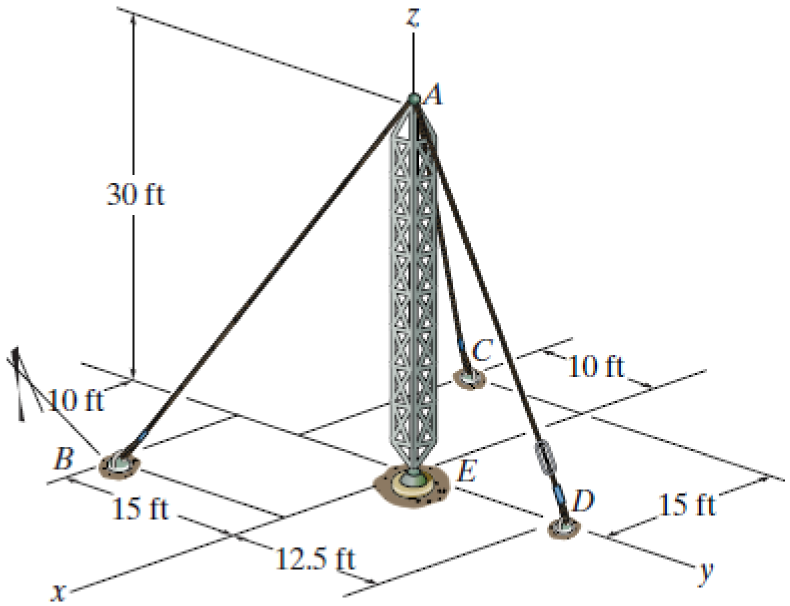
(3) If each cord can sustain a maximum tension of 50 N before it fails, determine the greatest weight of the flowerpot the cords can support ?



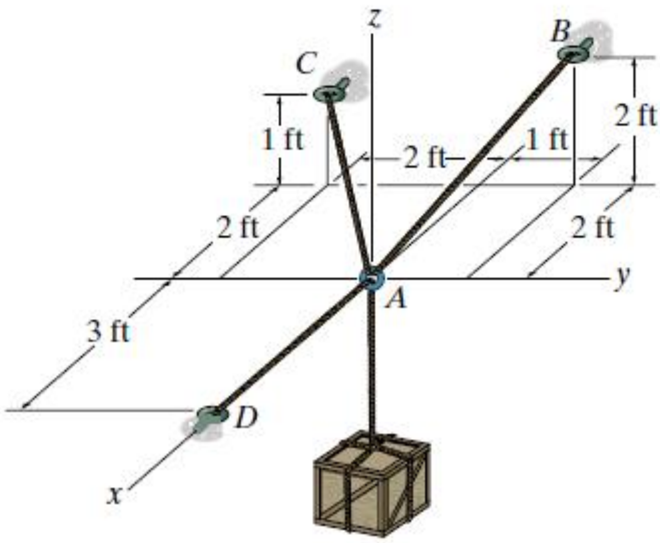
(4) If the maximum force in each rod can not exceed 1500 N, determine the greatest mass of the crate that can be supported.



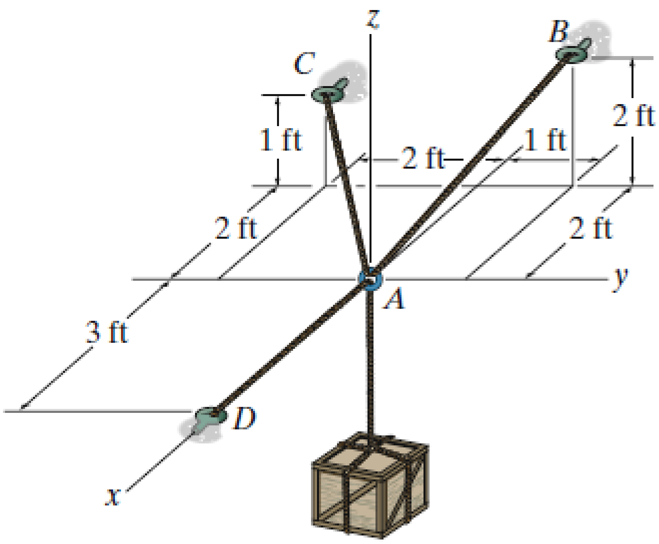
(5) If the tension developed in either cable AB or AC cannot exceed 1000 lb, determine the maximum tension that can be developed in cable AD when it is tightened by the turnbuckle. Also, what is the force developed along the antenna tower at point A ?



(6) Determine the tension developed in cables AB, AC and AD required for equilibrium of the 300-lb crate.



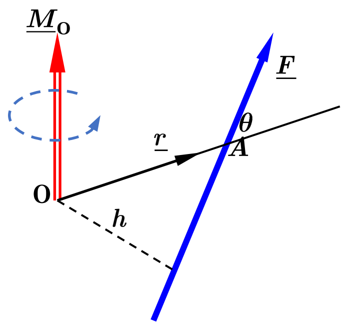
(7) Determine the maximum weight of the crate so that the tension developed in any cable does not exceed 450 lb.



Moment and Couples Forces

In this chapter we will obtain the moment of a force about a point or about an axis, reduction the forces at a point.

♦ <u>The Moment</u>



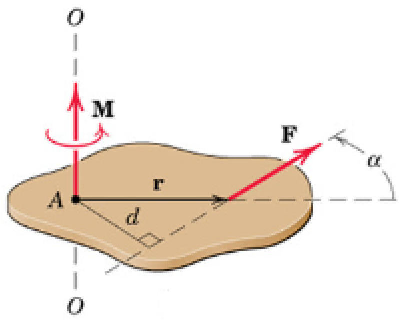
The **moment of a force** is the tendency of some forces to cause rotation. The moment of a force about a point is defined to be the product of the force and the perpendicular distance of its line of action from the point. On the other hand The moment of a force $\underline{F}$ about point O, or actually about the moment axis passing through O and perpendicular to the plane containing O and $\underline{F}$, as shown, can be expressed using the vector cross product, namely,

$$\underline{M}_O = \underline{r} \wedge \underline{F}$$

Here $\underline{r}$ represents a position vector directed from O to any point on the line of action of $\underline{F}$. Note that

$$\left|\underline{M}_O\right| = \left|\underline{r} \wedge \underline{F}\right| = rF\sin\theta = h$$

So if the force $\underline{F}$ in Cartesian coordinates is $\underline{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$ and the vector $\underline{r}$ is given by $\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then



$$\underline{M}_O = \underline{r} \wedge \underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= (yF_z - zF_y)\hat{i} - (xF_z - zF_x)\hat{j} + (xF_y - yF_x)\hat{k}$$

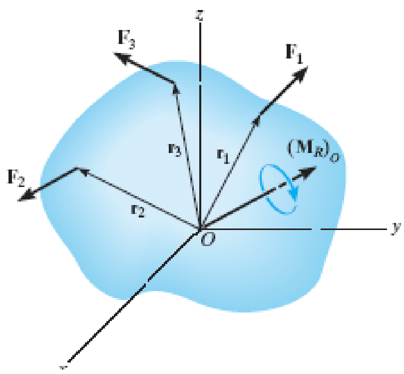
◈ <u>Varginon's Theorem</u> If a number of coplanar forces acting at a point of a rigid body have a resultant, then the vector sum of the moments of the all forces about any arbitrary point is equal to the moments of the resultant about the same point.

Proof.

Let the coplanar forces $\underline{F}_1, \underline{F}_2, \ldots\ldots, \underline{F}_n$ acting at a

a rigid body have the resultant $\underline{F}$.

$$\underline{F} = \underline{F}_1 + \underline{F}_2 + \ldots\ldots + \underline{F}_n = \sum \underline{F}_i$$



Let O be an arbitrary point and $\underline{r}_i$ be the

position vector directed from O to any point on the line of action of $\underline{F}$. The sum of the moment of the forces $\underline{F}_1, \underline{F}_2, \ldots\ldots, \underline{F}_n$ about O is

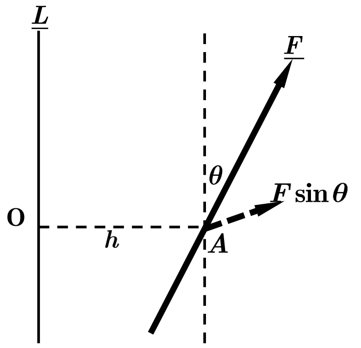
$$\sum \underline{r} \wedge \underline{F}_i = \underline{r} \wedge \underline{F}_1 + \underline{r} \wedge \underline{F}_2 + ... + \underline{r} \wedge \underline{F}_n$$
$$= \underline{r} \wedge \ \underline{F}_1 + \underline{F}_2 + ... + \underline{F}_n$$
$$= \underline{r} \wedge \underline{F}$$

which is equal to the moment of the resultant about O.

Any system of forces, acting in one plane upon a rigid body, can be reduced to either a single force or a single couple.

◈Three forces represented in magnitude, direction and position by the sides of a triangle taken the same way round are equivalent to a couple.

♦ <u>Moment of a force about an axis</u>

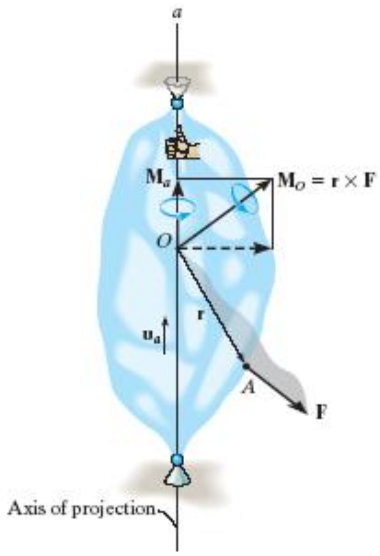


Thus if $\underline{F}$ be a force and $\underline{L}$ be a line which does not intersect $\underline{F}$, $\mathbf{OA} = h$ the shortest distance between $\underline{F}$ and $\underline{L}$, and θ the angle between $\underline{F}$ and a line through A parallel to $\underline{L}$, then $F\sin\theta$ is the resolved part of $\underline{F}$ at right angles to $\underline{L}$ and $Fh\sin\theta$ is the moment of $\underline{F}$ about $\underline{L}$ notation by $\underline{M}_L$. If $\underline{F}$ intersects the line $\underline{L}$ or is parallel to $\underline{L}$, then the moment of $\underline{F}$ about $\underline{L}$ is zero, because in the one case $h = 0$ and in the other $\sin\theta = 0$.

Or on the other hand $\underline{M}_L = \underline{M}_o \bullet \hat{n}\ \hat{n}$ where $\hat{n}$ is a unit vector of axis $\underline{L}$ and $\underline{M}_o$ represents the moment of the force $\underline{F}$ about a point $\mathbf{O}$ (say) lies on the axis $\underline{L}$, here

$$\left|\underline{M}_L\right| = \hat{n} \bullet \underline{r} \wedge \underline{F} = \begin{vmatrix} \ell & m & n \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \ell(yF_z - zF_y) - m(xF_z - zF_x) + n(xF_y - yF_x)$$



◈ When two forces act at a point the algebraical sum of their moments about any line is equal to the moment of their resultant about this line.

♦ In brief to calculate the moment of a force about an axis, one does the following three steps

(i) Obtain a unit vector of the axis (say $\hat{n}$)

(ii) Determine the moment $\underline{M}_o$ of the force $\underline{F}$

(iii) about a point lies on the axis, say $\mathbf{O}$.

(iv) The moment of a force about an axis is

$$\underline{M}_L = \underline{M}_o \bullet \hat{n}\ \hat{n}$$

◈ <u>Particular cases</u>

The moment of a force $\underline{F}$ about $\mathbf{X}$ axis is $\underline{M}_{\mathrm{OX}} = \underline{M}_o \bullet \hat{i}\ \hat{i}$

The moment of a force $\underline{F}$ about $\mathbf{Y}$ axis is $\underline{M}_{\mathrm{OX}} = \underline{M}_o \bullet \hat{j}\ \hat{j}$

The moment of a force $\underline{F}$ about $\mathbf{Z}$ axis is $\underline{M}_{\mathrm{OX}} = \underline{M}_o \bullet \hat{k}\ \hat{k}$

♦ <u>Couples</u>

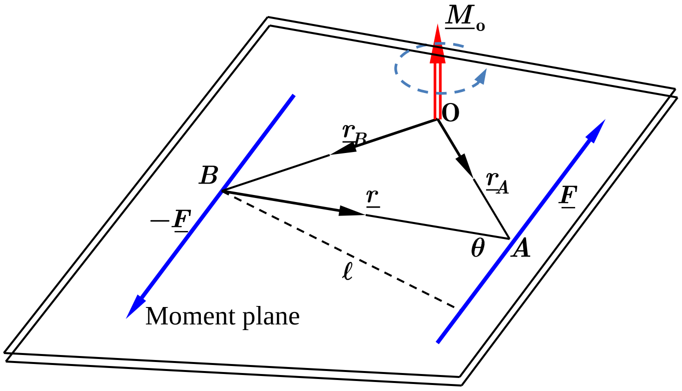
Couples play an important part in the general theory of systems of forces and we shall now establish some of their principal properties. Since a couple consists of two equal and opposite parallel forces (unlike forces), the algebraical sum of the resolved parts of the forces in every direction is zero, so that there is no tendency for the couple to produce in any direction a displacement of translation of the body upon which it acts; and the couple cannot be replaced by a single force. The effect of a couple must therefore be measured in some other way, and, since it has no tendency to produce translation, we next consider what tendency it has to produce rotation.

Let the couple consist of two forces of magnitude $\boldsymbol{F}$. It is of course assumed that they are both acting upon the same rigid body. Let us take the algebraical sum of the moments of the forces about any point $\mathbf{O}$ in their plane as the measure of their tendency to turn the body upon which they act about the point $\mathbf{O}$.

$$\underline{M}_o = \underline{r}_A \wedge \underline{F} + \underline{r}_B \wedge (-\underline{F})$$

$$\underline{M}_o = \left(\underline{r}_A - \underline{r}_B\right) \wedge \underline{F} = \underline{r} \wedge \underline{F}$$

Where its magnitude is $\left|\underline{M}_o\right| = \left|\underline{r} \wedge \underline{F}\right| = rF\sin\theta = F\ell$



◈ Forces completely represented by the sides of a plane polygon taken the same way round are equivalent to a couple whose moment is represented by twice the area of the polygon.

♦ <u>Reduction a system of forces</u>

When a number of forces and couple moments are acting on a body, it is easier to understand their overall effect on the body if they are combined into a single force and couple moment having the same external effect. The two force and couple systems are called equivalent systems since they have the same external effect on the body. Suppose a system of forces $\underline{F}_1, \underline{F}_2, \ldots\ldots, \underline{F}_i, \ldots\ldots, \underline{F}_n$ is reduced at a chosen point O to a single force $\underline{F}$ and a single couple $\underline{M}$ viz. the obtaining result is $(\underline{M}_{\mathrm{o}}, \underline{F})$ where

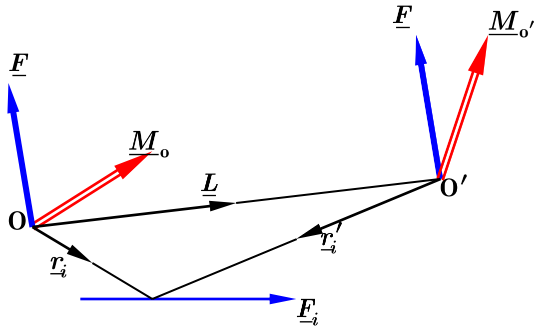
$$\underline{M}_{\mathrm{o}} = \sum_{i=1}^{n} \underline{r}_i \wedge \underline{F}_i, \qquad \underline{F} = \sum_{i=1}^{n} \underline{F}_i$$

Once again if the system of these forces reduced at another point O' where the obtaining results is

$$\underline{M}_{\mathrm{o}'} = \sum_{i=1}^{n} \underline{r}_i' \wedge \underline{F}_i, \qquad \underline{F} = \sum_{i=1}^{n} \underline{F}_i$$

That is when the point of reduction changed from O to O', the resultant of the forces does not change while the moment altered, such that

$$\begin{aligned}
\therefore \underline{M}_{\mathrm{o}'} &= \sum_{i=1}^{n} \underline{r}_i' \wedge \underline{F}_i \\
&= \sum_{i=1}^{n} (\underline{r}_i - \underline{L}) \wedge \underline{F}_i \\
&= \sum_{i=1}^{n} \underline{r}_i \wedge \underline{F}_i - \sum_{i=1}^{n} \underline{L} \wedge \underline{F}_i \\
&= \underline{M}_{\mathrm{o}} - \sum_{i=1}^{n} \underline{L} \wedge \underline{F}_i \\
&= \underline{M}_{\mathrm{o}} - \underline{L} \wedge \sum_{i=1}^{n} \underline{F}_i
\end{aligned}$$



$$\therefore \underline{M}_{\mathrm{o}'} = \underline{M}_{\mathrm{o}} - \underline{L} \wedge \underline{F}$$

Also it is obvious

$$\therefore \underline{F}.\underline{M}_{\mathrm{o}'} = \underline{F}.(\underline{M}_{\mathrm{o}} - \underline{L} \wedge \underline{F}) = \underline{F}.\underline{M}_{\mathrm{o}} - \overbrace{\cancel{\underline{F}.(\underline{L} \wedge \underline{F})}}^{0} = \underline{F}.\underline{M}_{\mathrm{o}} = \mathbf{const.}$$

The quantity $\underline{F}.\underline{M}_{\mathrm{o}}$ is called invariant quantity

◈Wrench

Suppose a system of forces is reduced to a single force $\underline{F}$ and a single couple $\underline{M}$ such that the axis of the couple is coincides with the line of action of the force $\underline{F}$, then that line is called central axis. In addition, $\underline{F}$ and $\underline{M}$ taken together are called wrench of the system and are written as $(\underline{F}, \underline{M})$. The single force $\underline{F}$ is called the intensity of the wrench and the ratio $\underline{M} / \underline{F}$ is called the pitch of the system and is denoted by λ. Since $\underline{F}$ and $\underline{M}_{\mathrm{o}'}$ have the same direction so

$$\underline{M}_{\mathrm{o}'} = \underline{M}_{\mathrm{o}} - \underline{r} \wedge \underline{F} = \lambda \underline{F}$$ multiply by $\underline{F}$ using scalar product

$$\Rightarrow \underline{F} \bullet (\underline{M}_{\mathrm{o}} - \underline{r} \wedge \underline{F}) = \lambda F^2 \qquad \therefore \lambda = \frac{\underline{F} \bullet \underline{M}_{\mathrm{o}}}{F^2} = \frac{F M_{\mathrm{O}}}{F^2} = \frac{M_{\mathrm{O}}}{F}$$

Where λ is known as the pitch of equivalent wrench

Also since $\underline{F} \wedge \underline{M}_{\mathrm{o}'} = \underline{0}$ multiply by $\underline{F}$ using cross product we have,

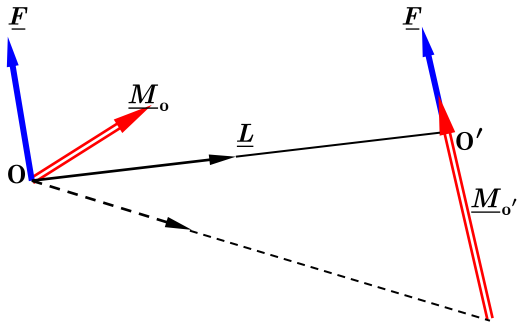
$$\therefore \underline{F} \wedge \underbrace{(\underline{M}_{\mathrm{o}} - \underline{r} \wedge \underline{F})}_{\underline{M}_{\mathrm{o}'}} = \underline{F} \wedge \underline{M}_{\mathrm{o}} - \underline{F} \wedge (\underline{r} \wedge \underline{F}) = \underline{0}$$

According to the properties of triple vector product

$$\underline{F} \wedge (\underline{r} \wedge \underline{F}) = (\underline{F}.\underline{F})\underline{r} - (\underline{F}.\underline{r})\underline{F} = F^2\underline{r} - (\underline{F}.\underline{r})\underline{F}$$

$$\therefore \underline{F} \wedge \underline{M}_o - (F^2\underline{r} - (\underline{F}.\underline{r})\underline{F}) = \underline{0}$$

$$\therefore \underline{r} = \underbrace{\frac{\underline{F} \wedge \underline{M}_o}{F^2}}_{\underline{r}_1} + \underbrace{\frac{\underline{r}.\underline{F}}{F^2}}_{\mu}\underline{F} \qquad \textbf{Or} \qquad \underline{r} = \underline{r}_1 + \mu\underline{F}$$



The previous equation represents the equation of the central axis or axis of equivalent wrench in vector form and to get the Cartesian form let

$$\underline{r} = (x, y, z), \quad \underline{r}_1 = (a, b, c), \quad \underline{F} = (F_x, F_y, F_z)$$

Therefore, the Cartesian form of central axis is

$$\frac{x-a}{F_x} = \frac{y-b}{F_y} = \frac{z-c}{F_z}$$

◈ Special Cases

(i) $\underline{F}.\underline{M}_o = 0$ **and** $\underline{F} \neq 0, \underline{M}_o = 0$

The system reduced to a single force that acts along the line $\underline{r} = \lambda\underline{F}$

(ii) $\underline{F}.\underline{M}_o = 0$ **and** $\underline{F} = 0, \underline{M}_o \neq 0$

The system reduced to a single moment

(iii) $\underline{F}.\underline{M}_o = 0$ **and** $\underline{F} \neq 0, \underline{M}_o \neq 0$

In this case $\underline{M}_o$ will be perpendicular to $\underline{F}$ and the system can be reduced to wrench in which the central axis is

$$\therefore \underline{r} = \frac{\underline{F} \wedge \underline{M}_o}{F^2} + \mu\underline{F}$$

(iv) $\underline{F} = 0$ **and** $\underline{M}_o = 0$

The system of forces will be in equilibrium or it is a balanced system of forces.

■ Illustrative Examples ■

❒ EXAMPLE 1

Determine the moment of the force $\underline{F} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ acting at the point $A(3,2,0)$ about the origin and the point $B(2,1,-1)$.

❒ SOLUTION

Since the moment is given by $\underline{M}_o = \underline{r} \wedge \underline{F}$ where

$\underline{r} = \underline{OA} = \underline{A} - \underline{O} = (3,2,0) - (0,0,0) = 3\hat{i} + 2\hat{j}$

Therefore the moment of the given force about the origin is

$$\underline{M}_o = \underline{r} \wedge \underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 0 \\ 2 & 3 & 4 \end{vmatrix} = 8\hat{i} - 12\hat{j} + 5\hat{k}$$

Again, $\underline{r}' = \underline{BA} = \underline{A} - \underline{B} = (3,2,0) - (2,1,-1) = \hat{i} + \hat{j} + \hat{k}$

Hence, the moment f the given force about the point $B(2,1,-1)$ is

$$\underline{M}_B = \underline{r}' \wedge \underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3-2 & 2-1 & 0+1 \\ 2 & 3 & 4 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{vmatrix} = \hat{i} - 2\hat{j} + \hat{k}$$

❒ EXAMPLE 2

Calculate the moment of the force of magnitude $10\sqrt{3}$ and passing through the point $A(5,3,-3)$ to $B(4,4,-4)$ about the origin.

❒ SOLUTION

We have to write the force in vector form, to do this the unit vector in the direction of the force $\hat{F}$, viz. from point $A(5,3,-3)$ to $B(4,4,-4)$ so

$$\because \underline{AB} = \underline{B} - \underline{A} = (4,4,-4) - (5,3,-3) = -\hat{i} + \hat{j} - \hat{k}$$

$$\Rightarrow \hat{F} = \frac{\underline{AB}}{AB} = \frac{-\hat{i} + \hat{j} - \hat{k}}{\sqrt{3}} \equiv \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$$

Therefore the force be

$$\therefore \underline{F} = F\hat{F} = 10\sqrt{3}\left\{\frac{-\hat{i} + \hat{j} - \hat{k}}{\sqrt{3}}\right\} \equiv -10\hat{i} + 10\hat{j} - 10\hat{k}$$

Choosing any point as an acting point of the force, then the moment of the force about the origin O (consider $A(5,3,-3)$ as an acting point)

$$\because \underline{r} = (5,3,-3) - (0,0,0) = (5,3,-3)$$

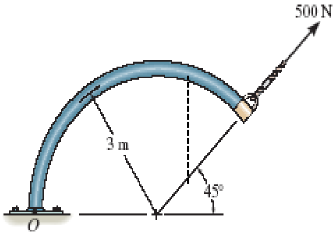
$$\Rightarrow \underline{M}_o = \underline{r} \wedge \underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 3 & -3 \\ -10 & 10 & -10 \end{vmatrix} = 80\hat{j} + 80\hat{k}$$

Also if we choose the point $B(4,4,-4)$ as an acting point

$$\Rightarrow \underline{M}_o = \underline{r}' \wedge \underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & -4 \\ -10 & 10 & -10 \end{vmatrix} = 40 \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ -1 & 1 & -1 \end{vmatrix} = 80\hat{j} + 80\hat{k}$$

❒ EXAMPLE 3

Determine the moment of the force as shown about point O.



❒ SOLUTION

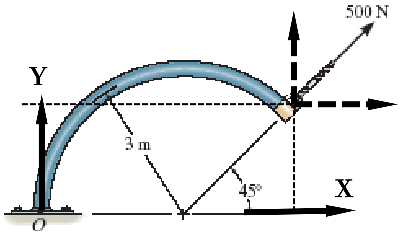
Taking horizontal axis X as shown, the force 500 can be reso

$$500\cos 45^0\,\hat{i} + 500\sin 45^0\,\hat{j} = 250\sqrt{2}(\hat{i} + \hat{j})$$

Therefore, the moment is given by,

$$M_o = 250\sqrt{2}\left(3 + \frac{3}{\sqrt{2}}\right) - 250\sqrt{2}\left(\frac{3}{\sqrt{2}}\right) = 750\sqrt{2}$$

Or by cross product where



$$\underline{r} = \left(3 + \frac{3}{\sqrt{2}}\right)\hat{i} + \frac{3}{\sqrt{2}}\hat{j}$$

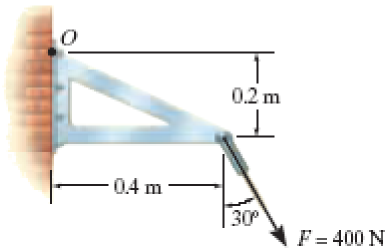
$$\underline{F} = 500\cos 45^0\,\hat{i} + 500\sin 45^0\,\hat{j} = 250\sqrt{2}(\hat{i} + \hat{j})$$

$$\Rightarrow \underline{M}_o = \underline{r} \wedge \underline{F} = 250\sqrt{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 + \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} & 0 \\ 1 & 1 & 0 \end{vmatrix} = 750\sqrt{2}\,\hat{k}$$

❒ EXAMPLE 4

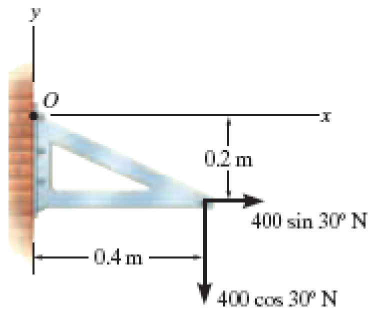
Force F acts at the end of the angle bracket as shown.

Determine the moment of the force about point O.



❒ SOLUTION

Using a Cartesian vector approach, the force and position vectors are



$$\underline{r} = 0.4\hat{i} - 0.2\hat{j}$$

$$\underline{F} = 400\sin 30^0\,\hat{i} - 400\cos 30^0\,\hat{j} = 200\hat{i} - 346.4\hat{j}$$

The moment is therefore,

$$\Rightarrow \underline{M}_o = \underline{r} \wedge \underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0.4 & -0.2 & 0 \\ 200 & -346.4 & 0 \end{vmatrix} = -98.6\hat{k}$$

❒ EXAMPLE 5

Find the sum of moment of the forces, $\underline{F} = 2\hat{i}$ acts at the origin, the force $-\frac{1}{2}\underline{F}$ acts at $\underline{r}_2 = 3\hat{j}$ and the force $-\frac{1}{2}\underline{F}$ acts at $\underline{r}_3 = 5\hat{k}$ about the origin.

❒ SOLUTION

As clear the resultant of these three forces is zero but the moment about the origin is given by

$$\Rightarrow \underline{M}_o = \sum_{i=1}^{3} \underline{r}_i \wedge \underline{F}_i = \underline{r}_1 \wedge \underline{F}_1 + \underline{r}_2 \wedge \underline{F}_2 + \underline{r}_3 \wedge \underline{F}_3$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 3 & 0 \\ -1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 5 \\ -1 & 0 & 0 \end{vmatrix}$$

$$\therefore \underline{M}_o = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 3 & 5 \\ -1 & 0 & 0 \end{vmatrix} = -5\hat{j} + 3\hat{k} \qquad \text{and} \qquad \left|\underline{M}_o\right| = \sqrt{34}$$

❒ EXAMPLE 6

The force $2\hat{i} - \hat{j}$ acts along the line that passing through the point $(4,4,5)$ and the force $3\hat{k}$ acting at the origin. Find the pitch and axis of equivalent wrench.

❒ SOLUTION

The two forces reduced at the origin to a resultant force $\underline{F}$ and a moment $\underline{M}_O$ so that

$$\underline{F} = \underline{F}_1 + \underline{F}_2 = 2\hat{i} - \hat{j} + 3\hat{k}, \qquad \therefore F^2 = 14$$

$$\underline{M}_O = \underline{r}_1 \wedge \underline{F}_1 + \underline{r}_2 \wedge \underline{F}_2 = \underline{0} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 5 \\ 2 & -1 & 0 \end{vmatrix} = 5\hat{i} + 10\hat{j} - 12\hat{k}$$

Thus the pitch of equivalent wrench is given by $\lambda = \dfrac{\underline{F} \bullet \underline{M}}{F^2}$ that is

$$\lambda = \frac{\underline{F} \bullet \underline{M}}{F^2} = \frac{2\hat{i} - \hat{j} + 3\hat{k} \bullet 5\hat{i} + 10\hat{j} - 12\hat{k}}{14} = -\frac{36}{14} = -\frac{18}{7}$$

In addition the equation of axis of wrench $\underline{r} = \underline{r}_1 + \mu \underline{F}$

$$\underline{r}_1 = \frac{\underline{F} \wedge \underline{M}}{F^2} = \frac{1}{14}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 3 \\ 5 & 10 & -12 \end{vmatrix} = \frac{1}{14} \; -18\hat{i} + 39\hat{j} + 25\hat{k}$$

Then the vector form of the axis becomes

$$\underline{r} = \frac{1}{14} \; -18\hat{i} + 39\hat{j} + 25\hat{k} \; + \mu(2\hat{i} - \hat{j} + 3\hat{k})$$

And Cartesian form is

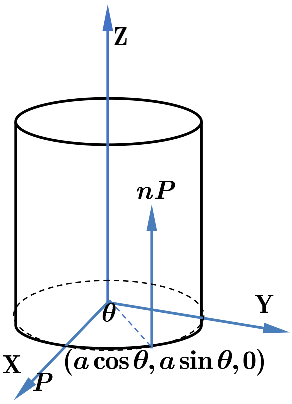
$$\frac{x + \frac{18}{14}}{2} = \frac{y - \frac{39}{14}}{-1} = \frac{z - \frac{25}{14}}{3} \qquad \text{Or} \qquad \frac{14x + 18}{2} = \frac{14y - 39}{-1} = \frac{14z - 25}{3}$$

❐ EXAMPLE 7

A force P acts along the axis of $\mathbf{OX}$ and another force nP acts along a generator of the cylinder $x^2+y^2=a^2$ at the point $(a\cos\theta, a\sin\theta, 0)$; show that the central axis lies on the cylinder $n^2(nx-z)^2+(1+n^2)^2y^2=n^4a^2$

❐ SOLUTION

Generators of the cylinder are parallel to the axis of . Let one generator of it pass through the point and its unit vector is and the force acts along this line. Also the force acts along axis then



$\underline{F}_1 = P\hat{i}, \qquad \text{acts at } (0,0,0)$

$\underline{F}_2 = nP\hat{k}, \qquad \text{acts at } (a\cos\theta, a\sin\theta, 0)$

$\underline{F} = P(\hat{i}+n\hat{k}), \qquad (F^2 = (1+n^2)P^2)$

The system reduces to a single force and a moment so that

$$\because \underline{M}_o = \underline{r}_1 \wedge \underline{F}_1 + \underline{r}_2 \wedge \underline{F}_2$$

$$\therefore \underline{M}_o = P\left\{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\theta & a\sin\theta & 0 \\ 0 & 0 & n \end{vmatrix}\right\}$$

$$= anP(\sin\theta\hat{i} - \cos\theta\hat{j})$$

The pitch of equivalent wrench is given by $\lambda = \dfrac{\underline{F}\bullet\underline{M}}{F^2}$ that is

$$\lambda = \frac{\underline{F}\bullet\underline{M}}{F^2} = \frac{P\ \hat{i}+n\hat{k}\ \bullet anP(\sin\theta\hat{i}-\cos\theta\hat{j})}{(1+n^2)P^2} = \frac{an\sin\theta}{1+n^2}$$

In addition, the equation of axis of wrench $\underline{r} = \underline{r}_1 + \mu\underline{F}$

$$\underline{r}_1 = \frac{\underline{F}\wedge\underline{M}}{F^2} = \frac{anP^2}{(1+n^2)P^2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & n \\ \sin\theta & -\cos\theta & 0 \end{vmatrix}$$

$$= \frac{an}{1+n^2}\ n\cos\theta\hat{i} + n\sin\theta\hat{j} - \cos\theta\hat{k}$$

Then the vector form of the axis becomes

$$\underline{r} = \frac{an}{1+n^2}\ n\cos\theta\hat{i} + n\sin\theta\hat{j} - \cos\theta\hat{k}\ + \mu(\hat{i}+\hat{k})$$

And Cartesian form is given by

$$\frac{x - \dfrac{an^2\cos\theta}{1+n^2}}{1} = \frac{y - \dfrac{an^2\sin\theta}{1+n^2}}{0} = \frac{z + \dfrac{an\cos\theta}{1+n^2}}{n} \qquad \text{Or}$$

$$y - \frac{an^2\sin\theta}{1+n^2} = 0, \qquad n\left(x - \frac{an^2\cos\theta}{1+n^2}\right) = z + \frac{an\cos\theta}{1+n^2}$$

$$y = \frac{an^2}{1+n^2}\sin\theta, \qquad nx - z = \frac{an(1+n^2)\cos\theta}{1+n^2}$$

Squaring these equations

$$y^2 = n^2\left(\frac{an}{1+n^2}\right)^2 \sin^2\theta, \qquad (nx - z)^2 = (1+n^2)^2\left(\frac{an}{1+n^2}\right)^2 \cos^2\theta$$

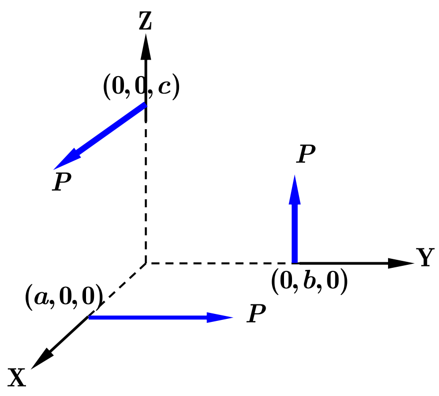
then multiply first equation by $(1+n^2)^2$ and the second by n^2 then adding the result we get

$$(1+n^2)^2 y^2 + n^2(nx-z)^2 = n^2(1+n^2)^2\left(\frac{an}{1+n^2}\right)^2 = a^2n^4$$

❒ EXAMPLE 8

Three forces each equal to P act on a body, one at point $(a,0,0)$ parallel to $\mathbf{OY}$, the second at the point $(0,b,0)$ parallel to $\mathbf{OZ}$ and the third at the point $(0,0,c)$ parallel to $\mathbf{OX}$, the axes being rectangular. Find the resultant wrench.

❒ SOLUTION



As given we see

$$\begin{aligned} \underline{F}_1 &= P\hat{i}, && \text{acts at } (0,0,c) \\ \underline{F}_2 &= P\hat{j}, && \text{acts at } (a,0,0) \\ \underline{F}_3 &= P\hat{k}, && \text{acts at } (0,b,0) \\ \underline{F} &= P(\hat{i}+\hat{j}+\hat{k}), && (F^2 = 3P^2) \end{aligned}$$

The system reduces to a single force and a moment so that

$$\because \underline{M}_o = \underline{r}_1 \wedge \underline{F}_1 + \underline{r}_2 \wedge \underline{F}_2 + \underline{r}_3 \wedge \underline{F}_3$$

$$\therefore \underline{M}_o = P\left\{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & c \\ 1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & b & 0 \\ 0 & 0 & 1 \end{vmatrix}\right\}$$

$$= P(b\hat{i} + c\hat{j} + a\hat{k})$$

The pitch of equivalent wrench is given by $\lambda = \dfrac{\underline{F} \bullet \underline{M}}{F^2}$ that is

$$\lambda = \frac{\underline{F} \bullet \underline{M}}{F^2} = \frac{P\ \hat{i}+\hat{j}+\hat{k}\ \bullet P(b\hat{i}+c\hat{j}+a\hat{k})}{3P^2} = \frac{a+b+c}{3}$$

In addition the equation of axis of wrench $\underline{r} = \underline{r}_1 + \mu \underline{F}$

$$\underline{r}_1 = \frac{\underline{F} \wedge \underline{M}}{F^2} = \frac{P^2}{3P^2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ b & c & a \end{vmatrix}$$

$$= \frac{1}{3}\ (a-c)\hat{i} + (b-a)\hat{j} + (c-b)\hat{k}$$

Then the vector form of the axis becomes

$$\underline{r} = \frac{1}{3}\ (a-c)\hat{i} + (b-a)\hat{j} + (c-b)\hat{k}\ + \mu(\hat{i}+\hat{j}+\hat{k})$$

and Cartesian form is

$$\frac{x-\frac{1}{3}(a-c)}{1}=\frac{y-\frac{1}{3}(b-a)}{1}=\frac{z-\frac{1}{3}(c-b)}{1} \qquad \text{Or}$$

$$3\ \ y-x\ \ =b+c-2a \qquad \text{and} \qquad 3\ \ z-y\ \ =a+c-2b$$

❒ EXAMPLE 9

Forces X, Y, Z act along three lines given by the equations

$$y=0, z=c; \qquad z=0, x=a; \qquad x=0, y=b$$

Prove that the pitch of the equivalent wrench is

$$(aYZ+bZX+cXY)\,/\,(X^2+Y^2+Z^2)$$

If the wrench reduces to a single force, show that the line of the action of the force lies on the hyperboloid $(x-a)(y-b)(z-c)=xyz$

❒ SOLUTION

As given
$$\begin{aligned} \underline{F}_1 &= X\hat{i}, && \text{acts at } (0,0,c) \\ \underline{F}_2 &= Y\hat{j}, && \text{acts at } (a,0,0) \\ \underline{F}_3 &= Z\hat{k} && \text{acts at } (0,b,0) \\ \underline{F} &= X\hat{i}+Y\hat{j}+Z\hat{k}, && F^2 = X^2+Y^2+Z^2 \end{aligned}$$

The system reduces to a single force and a moment so that

$$\because\ \underline{M}_o=\underline{r}_1\wedge\underline{F}_1+\underline{r}_2\wedge\underline{F}_2+\underline{r}_3\wedge\underline{F}_3$$

$$\therefore\ \underline{M}_o=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\0&0&c\\X&0&0\end{vmatrix}+\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\a&0&0\\0&Y&0\end{vmatrix}+\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\0&b&0\\0&0&Z\end{vmatrix}$$

$$=bZ\hat{i}+cX\hat{j}+aY\hat{k}$$

The pitch of equivalent wrench is given by $\lambda=\dfrac{\underline{F}\bullet\underline{M}}{F^2}$ that is

$$\lambda=\frac{\underline{F}\bullet\underline{M}}{F^2}=\frac{(X\hat{i}+Y\hat{j}+Z\hat{k})\bullet\ bZ\hat{i}+cX\hat{j}+aY\hat{k}}{X^2+Y^2+Z^2}$$

$$=\frac{bXZ+cXY+aYZ}{X^2+Y^2+Z^2}$$

Besides, the equation of axis of wrench $\underline{r}=\underline{r}_1+\mu\underline{F}$

$$\underline{r}_1=\frac{\underline{F}\wedge\underline{M}}{F^2}=\frac{1}{(X^2+Y^2+Z^2)}\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\X&Y&Z\\bZ&cX&aY\end{vmatrix}$$

$$=\frac{1}{(X^2+Y^2+Z^2)}\ (aY^2-cXZ)\hat{i}+(bZ^2-aXY)\hat{j}+(cX^2-bYZ)\hat{k}$$

Then the vector form of the axis becomes

$$\underline{r} = \frac{(aY^2 - cXZ)\hat{i} + (bZ^2 - aXY)\hat{j} + (cX^2 - bYZ)\hat{k}}{(X^2 + Y^2 + Z^2)} + \mu(X\hat{i} + Y\hat{j} + Z\hat{k})$$

And Cartesian form is

$$\frac{x - \dfrac{aY^2 - cXZ}{X^2 + Y^2 + Z^2}}{X} = \frac{y - \dfrac{bZ^2 - aXY}{X^2 + Y^2 + Z^2}}{Y} = \frac{z - \dfrac{cX^2 - bYZ}{X^2 + Y^2 + Z^2}}{Z}$$

Or

$$Y\left(x - \frac{aY^2 - cXZ}{X^2 + Y^2 + Z^2}\right) = X\left(y - \frac{bZ^2 - aXY}{X^2 + Y^2 + Z^2}\right)$$

$$Y\left(z - \frac{cX^2 - bYZ}{X^2 + Y^2 + Z^2}\right) = Z\left(y - \frac{bZ^2 - aXY}{X^2 + Y^2 + Z^2}\right)$$ ……Complete

❒ EXAMPLE 10

Two forces each equal to P act along the lines $\frac{x \mp a\cos\theta}{a\sin\theta} = \frac{y - b\sin\theta}{\mp b\cos\theta} = \frac{z}{c}$ show that the axis of equivalent wrench lays on the surface $y\left(\frac{x}{z} + \frac{z}{x}\right) = b\left(\frac{a}{c} + \frac{c}{a}\right)$

❒ SOLUTION

First line is $\frac{x - a\cos\theta}{a\sin\theta} = \frac{y - b\sin\theta}{-b\cos\theta} = \frac{z}{c}$ passing through $(a\cos\theta, b\sin\theta, 0)$

the second line is $\frac{x + a\cos\theta}{a\sin\theta} = \frac{y - b\sin\theta}{b\cos\theta} = \frac{z}{c}$ passing $(-a\cos\theta, b\sin\theta, 0)$

The unit vector of first line is

$$\begin{aligned}\hat{n}_1 &= \frac{1}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta + c^2}}(a\sin\theta, -b\cos\theta, c)\\ &= \frac{1}{\mu}\ a\sin\theta\hat{i} - b\cos\theta\hat{j} + c\hat{k}\end{aligned}$$

The unit vector of second line is

$$\begin{aligned}\hat{n}_2 &= \frac{1}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta + c^2}}(a\sin\theta, b\cos\theta, c)\\ &= \frac{1}{\mu}\ a\sin\theta\hat{i} + b\cos\theta\hat{j} + c\hat{k} \qquad (\mu = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta + c^2})\end{aligned}$$

Therefore,

$$\underline{F}_1 = P\hat{n}_1 = \frac{P}{\mu}\ a\sin\theta\hat{i} - b\cos\theta\hat{j} + c\hat{k}$$

$$\underline{F}_2 = P\hat{n}_2 = \frac{P}{\mu}\ a\sin\theta\hat{i} + b\cos\theta\hat{j} + c\hat{k}$$

The system reduces to a single force and a moment so that

$$\underline{F} = \underline{F}_1 + \underline{F}_2$$
$$= \frac{P}{\mu}\ a\sin\theta\hat{i} - b\cos\theta\hat{j} + c\hat{k}\ \ + \frac{P}{\mu}\ a\sin\theta\hat{i} + b\cos\theta\hat{j} + c\hat{k}$$
$$= \frac{2P}{\mu}\ a\sin\theta\hat{i} + c\hat{k} \qquad\qquad \text{and} \quad F^2 = \frac{4P^2}{\mu^2}\ a^2\sin^2\theta + c^2$$

$$\underline{M} = \underline{r}_1 \wedge \underline{F}_1 + \underline{r}_2 \wedge \underline{F}_2$$
$$= \frac{P}{\mu}\left\{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\theta & b\sin\theta & 0 \\ a\sin\theta & -b\cos\theta & c \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\cos\theta & b\sin\theta & 0 \\ a\sin\theta & b\cos\theta & c \end{vmatrix}\right\}$$

$$\therefore\ \underline{M} = \frac{2P}{\mu}\ cb\sin\theta\hat{i} - ab\hat{k}$$

since the equation of axis of equivalent wrench is $\underline{r} = \underline{r}_1 + \mu\underline{F}$

$$\underline{r}_1 = \frac{\underline{F} \wedge \underline{M}}{F^2} = \frac{1}{a^2\sin^2\theta + c^2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\sin\theta & 0 & c \\ cb\sin\theta & 0 & -ab \end{vmatrix} = \frac{c^2 + a^2\ \ b\sin\theta}{a^2\sin^2\theta + c^2}\hat{j}$$

Then the vector form of the axis becomes

$$\underline{r} = \frac{(c^2 + a^2)b\sin\theta}{a^2\sin^2\theta + c^2}\hat{j} + \mu\ \ a\sin\theta\hat{i} + c\hat{k}$$

While the Cartesian form is

$$\frac{x - 0}{a\sin\theta} = \frac{y - \dfrac{(c^2 + a^2)b\sin\theta}{a^2\sin^2\theta + c^2}}{0} = \frac{z - 0}{c}$$

Thus we can deduce from these equation

$$y = \frac{(c^2 + a^2)b\sin\theta}{a^2\sin^2\theta + c^2} \qquad\qquad \text{and} \qquad \frac{x}{z} = \frac{a\sin\theta}{c}$$

$$y\ \ a^2\sin^2\theta + c^2\ \ = \ c^2 + a^2\ \ b\sin\theta$$
$$\Rightarrow y\left(a^2\sin\theta + \frac{c^2}{\sin\theta}\right) = b\ \ c^2 + a^2$$

Dividing by ac and substituting $\frac{x}{z} = \frac{a\sin\theta}{c}$ we get

$$y\left(\frac{x}{z} + \frac{z}{x}\right) = b\left(\frac{c}{a} + \frac{a}{c}\right)$$

❒ EXAMPLE 11

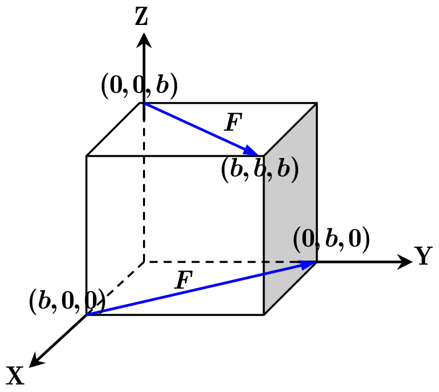
Two forces each equal to F act along the sides of a cube of length b as shown, Fin the axis of equivalent wrench.

❒ SOLUTION

By calculating the unit vectors of the forces, we get,

$$\hat{n}_1 = (b,b,b) - (0,0,b) = \frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$$

$$\therefore \underline{F}_1 = F\hat{n}_1 = \frac{F}{\sqrt{2}}(\hat{i} + \hat{j})$$



And for the second force

$$\hat{n}_2 = (0,b,0) - (b,0,0) = \frac{1}{\sqrt{2}}(-\hat{i} + \hat{j})$$

$$\therefore \underline{F}_2 = F\hat{n}_2 = \frac{F}{\sqrt{2}}(-\hat{i} + \hat{j})$$

The system reduces to a single force and a moment at the origin so that

$$\underline{R} = \underline{F}_1 + \underline{F}_2 = \sqrt{2}F\hat{j} \qquad \therefore R^2 = 2F^2$$

$$\underline{M} = \underline{r}_1 \wedge \underline{F}_1 + \underline{r}_2 \wedge \underline{F}_2 = \frac{Fb}{\sqrt{2}}\left\{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}\right\} = \frac{Fb}{\sqrt{2}}\ \hat{i} - \hat{j} - \hat{k}$$

Here we choose the point $(0,0,b)$ as an acting point of first force and the point $(b,0,0)$ of the second force. The pitch of equivalent wrench is given by $\lambda = \dfrac{\underline{F} \bullet \underline{M}}{F^2}$ that is

$$\lambda = \frac{\underline{R} \bullet \underline{M}}{R^2} = \frac{\sqrt{2}F\hat{j} \bullet \frac{Fb}{\sqrt{2}}\ \hat{i} - \hat{j} - \hat{k}}{2F^2} = -\frac{b}{2}$$

since the equation of axis of equivalent wrench is $\underline{r} = \underline{r}_1 + \mu\underline{F}$ so

$$\underline{r}_1 = \frac{\underline{R} \wedge \underline{M}}{R^2} = \frac{\cancel{F^2}b}{2\cancel{F^2}}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{vmatrix} = -\frac{b}{2}(\hat{i} + \hat{k})$$

Then the vector form of the axis becomes

$$\underline{r} = -\frac{b}{2}\ \hat{i} + \hat{k}\ + \mu\hat{j}$$

While the Cartesian form is given by

$$\frac{x + \frac{b}{2}}{0} = \frac{y - 0}{1} = \frac{z + \frac{b}{2}}{0} \qquad \text{Or} \qquad z = -\frac{b}{2} \text{ and } x = -\frac{b}{2}$$

PROBLEMS

❒ If the force $\underline{F} = 3\hat{i} - \hat{j} + 7\hat{k}$ acts at the origin, determine its moment about the point $(4, 4, 6)$.

❒ A force of magnitude 100 acts along the line passing through the point $(0, 1, 0)$ to $(1, 0, 0)$. Obtain its moment about the origin point and about the axes.

❒ The three forces $2\hat{i} + 2\hat{j}$, $\hat{j} - 2\hat{k}$, $-\hat{i} + 2\hat{j} + \hat{k}$ act at the points $(0, 1, 0)$, $(1, 0, 0)$, $(0, 0, 1)$ respectively, Find the pitch of the equivalent wrench.

❒ Two forces each equal to $3F$ act along the lines $\frac{x-1}{2} = \frac{y+1}{2} = \frac{z-2}{1}$ and $\frac{x-2}{1} = \frac{y+1}{-2} = \frac{z-1}{2}$. Find the equivalent wrench.

❒ The magnitude of two forces is F_2, F_1 act along the lines $(z = -c,\ y = -x\tan\alpha)$ and $(z = c,\ y = x\tan\alpha)$. Determine the central axis of equivalent wrench.