CHAPTER 13

Partial Derivatives

This chapter is at the center of multidimensional calculus. Other chapters and other topics may be optional; this chapter and these topics are required. We are back to the basic idea of calculus—***the derivative***. There is a function f, the variables move a little bit, and f moves. The question is how much f moves and how fast. Chapters 1–4 answered this question for $f(x)$, a function of one variable. Now we have $f(x, y)$ or $f(x, y, z)$—with two or three or more variables that move independently. As x and y change, f changes. The fundamental problem of differential calculus is to connect Δx and Δy to Δf.

Calculus solves that problem in the limit. ***It connects*** dx ***and*** dy ***to*** df. In using this language I am building on the work already done. You know that df/dx is the limit of $\Delta f/\Delta x$. Calculus computes the rate of change—which is the slope of the tangent line. The goal is to extend those ideas to

$$f(x, y) = x^2 - y^2 \qquad \text{or} \qquad f(x, y) = \sqrt{x^2 + y^2} \qquad \text{or} \qquad f(x, y, z) = 2x + 3y + 4z.$$

These functions have graphs, they have derivatives, and they must have tangents.

The heart of this chapter is summarized in six lines. The subject is *differential* calculus—small changes in a short time. Still to come is *integral* calculus—adding up those small changes. We give the words and symbols for $f(x, y)$, matched with the words and symbols for $f(x)$. Please use this summary as a guide, to know where calculus is going.

Curve $y = f(x)$ *vs.* ***Surface*** $z = f(x, y)$

$\dfrac{df}{dx}$	becomes two partial derivatives	$\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$
$\dfrac{d^2f}{dx^2}$	becomes four second derivatives	$\dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial x \partial y}, \dfrac{\partial^2 f}{\partial y \partial x}, \dfrac{\partial^2 f}{\partial y^2}$
$\Delta f \approx \dfrac{df}{dx}\Delta x$	becomes the linear approximation	$\Delta f \approx \dfrac{\partial f}{\partial x}\Delta x + \dfrac{\partial f}{\partial y}\Delta y$
tangent line	becomes the tangent plane	$z - z_0 = \dfrac{\partial f}{\partial x}(x - x_0) + \dfrac{\partial f}{\partial y}(y - y_0)$
$\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt}$	becomes the chain rule	$\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$
$\dfrac{df}{dx} = 0$	becomes two maximum-minimum equations	$\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0.$

13.1 Surfaces and Level Curves

The graph of $y=f(x)$ is a curve in the xy plane. There are two variables—x is independent and free, y is dependent on x. Above x on the base line is the point (x, y) on the curve. The curve can be displayed on a two-dimensional printed page.

The graph of $z=f(x, y)$ is a ***surface in** xyz **space***. There are *three variables*—x and y are independent, z is dependent. Above (x, y) in the base plane is the point (x, y, z) on the surface (Figure 13.1). Since the printed page remains two-dimensional, we shade or color or project the surface. The eyes are extremely good at converting two-dimensional images into three-dimensional understanding—they get a lot of practice. The mathematical part of our brain also has something new to work on—***two partial derivatives***.

This section uses examples and figures to illustrate surfaces and their level curves. The next section is also short. Then the work begins.

EXAMPLE 1 Describe the ***surface*** and the ***level curves*** for $z=f(x, y)=\sqrt{x^2+y^2}$.

The surface is a cone. Reason: $\sqrt{x^2+y^2}$ is the distance in the base plane from $(0, 0)$ to (x, y). When we go out a distance 5 in the base plane, we go up the same distance 5 to the surface. The cone climbs with slope 1. The distance out to (x, y) equals the distance up to z (this is a 45° cone).

The level curves are circles. At height 5, the cone contains a circle of points—all at the same "level" on the surface. The plane $z=5$ meets the surface $z=\sqrt{x^2+y^2}$ at those points (Figure 13.1b). The circle below them (in the base plane) is the level curve.

DEFINITION A ***level curve*** or ***contour line*** of $z=f(x, y)$ contains all points (x, y) that share the *same value* $f(x, y)=c$. Above those points, the surface is at the height $z=c$.

There are different level curves for different c. To see the curve for $c=2$, cut through the surface with the horizontal plane $z=2$. The plane meets the surface above the points where $f(x, y)=2$. ***The level curve in the base plane has the equation*** $f(x, y)=2$. Above it are all the points at "level 2" or "level c" on the surface.

Every curve $f(x, y)=c$ is labeled by its constant c. This produces a ***contour map*** (the base plane is full of curves). For the cone, the level curves are given by $\sqrt{x^2+y^2}=c$, and the contour map consists of circles of radius c.

Question What are the level curves of $z=f(x, y)=x^2+y^2$?
Answer *Still circles.* But the surface is not a cone (it bends up like a parabola). The circle of radius 3 is the level curve $x^2+y^2=9$. On the surface above, the height is 9.

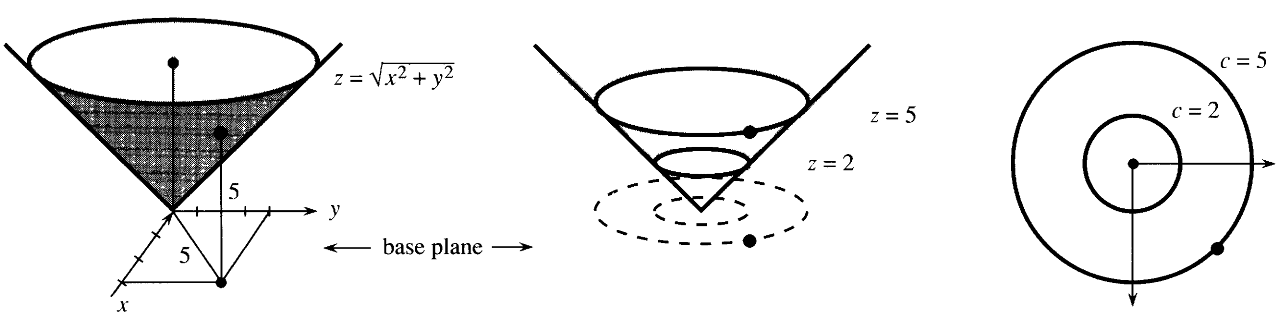


Fig. 13.1 The surface for $z=f(x, y)=\sqrt{x^2+y^2}$ is a cone. The level curves are circles.

EXAMPLE 2 For the *linear function* $f(x, y) = 2x + y$, the surface is a plane. Its level curves are straight lines. ***The surface*** $z = 2x + y$ ***meets the plane*** $z = c$ ***in the line*** $2x + y = c$. That line is above the base plane when c is positive, and below when c is negative. The contour lines are *in* the base plane. Figure 13.2b labels these parallel lines according to their height in the surface.

Question If the level curves are all straight lines, must they be parallel?
Answer No. The surface $z = y/x$ has level curves $y/x = c$. Those lines $y = cx$ swing around the origin, as the surface climbs like a spiral playground slide.

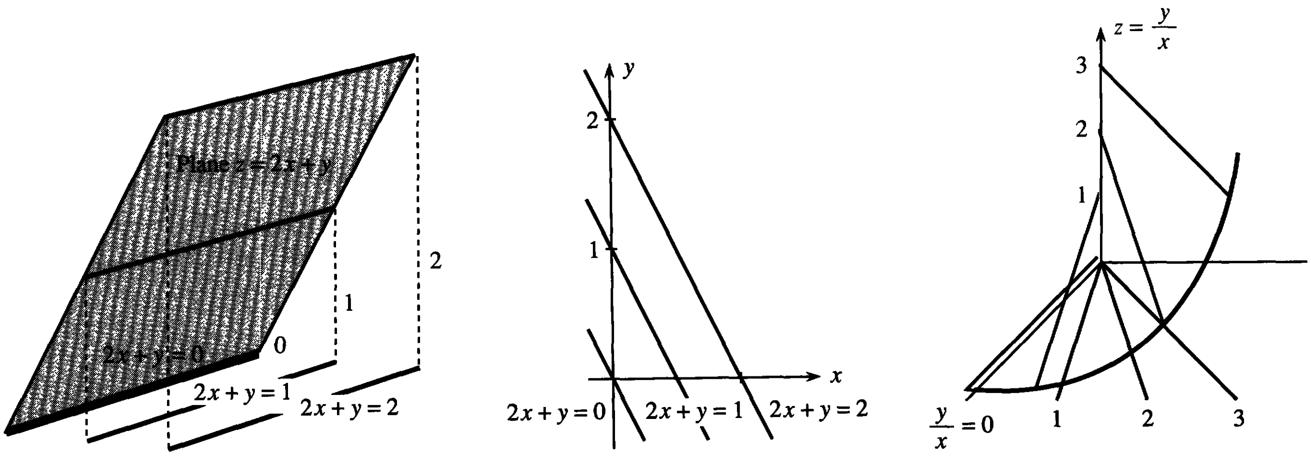


Fig. 13.2 A plane has parallel level lines. The spiral slide $z = y/x$ has lines $y/x = c$.

EXAMPLE 3 The weather map shows contour lines of the *temperature function*. Each level curve connects points at a constant temperature. One line runs from Seattle to Omaha to Cincinnati to Washington. In winter it is painful even to think about the line through L.A. and Texas and Florida. *USA Today* separates the contours by color, which is better. We had never seen a map of universities.

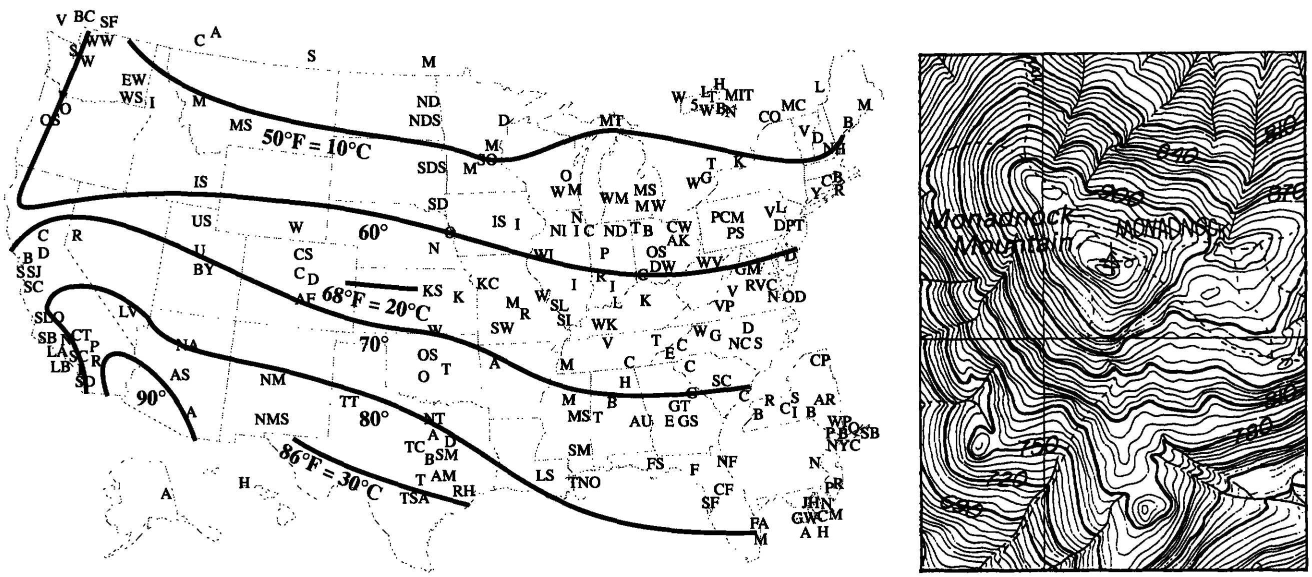


Fig. 13.3 The temperature at many U.S. and Canadian universities. Mt. Monadnock in New Hampshire is said to be the most climbed mountain (except Fuji?) at 125,000/year. Contour lines every 6 meters.

Question From a contour map, how do you find the highest point?
Answer The level curves form *loops* around the maximum point. As c increases the loops become tighter. Similarly the curves squeeze to the lowest point as c decreases.

EXAMPLE 4 A contour map of a mountain may be the best example of all. Normally the level curves are separated by 100 feet in height. On a steep trail those curves are bunched together—the trail climbs quickly. In a flat region the contour lines are far apart. Water runs perpendicular to the level curves. On my map of New Hampshire that is true of creeks but looks doubtful for rivers.

Question Which direction in the base plane is uphill on the surface?
Answer The steepest direction is perpendicular to the level curves. This is important. Proof to come.

EXAMPLE 5 In economics x^2y is a *utility function* and $x^2y = c$ is an *indifference curve.*

The utility function x^2y gives the value of x hours awake and y hours asleep. Two hours awake and fifteen minutes asleep have the value $f = (2^2)(\frac{1}{4})$. This is the same as one hour of each: $f = (1^2)(1)$. Those lie on the same level curve in Figure 13.4a. We are indifferent, and willing to exchange any two points on a level curve.

The indifference curve is "*convex.*" ***We prefer the average of any two points.*** The line between two points is up on higher level curves.

Figure 13.4b shows an extreme case. The level curves are straight lines $4x + y = c$. Four quarters are freely substituted for one dollar. The value is $f = 4x + y$ dollars.

Figure 13.4c shows the other extreme. Extra left shoes or extra right shoes are useless. The value (or utility) is the *smaller* of x and y. That counts *pairs* of shoes.

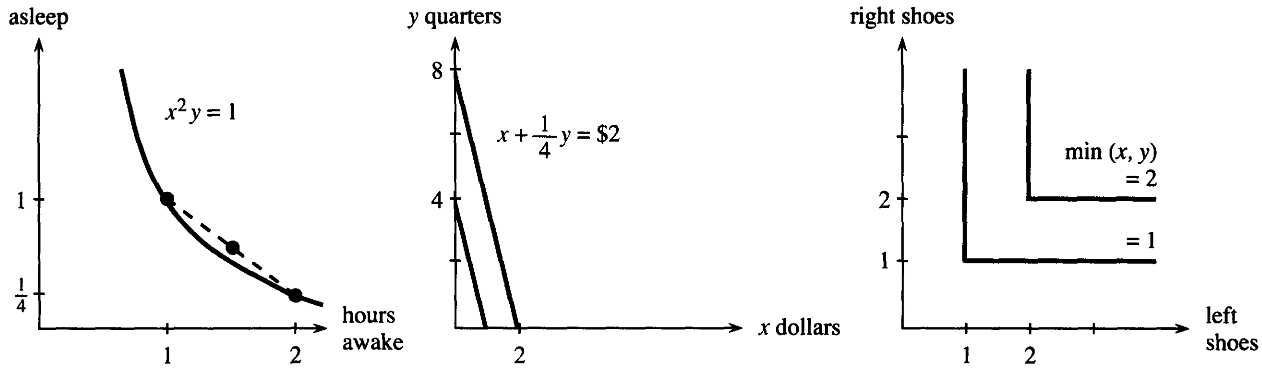


Fig. 13.4 Utility functions x^2y, $4x + y$, $\min(x, y)$. Convex, straight substitution, complements.

13.1 EXERCISES

Read-through questions

The graph of $z = f(x, y)$ is a __a__ in __b__-dimensional space. The __c__ curve $f(x, y) = 7$ lies down in the base plane. Above this level curve are all points at height __d__ in the surface. The __e__ $z = 7$ cuts through the surface at those points. The level curves $f(x, y) =$ __f__ are drawn in the xy plane and labeled by __g__. The family of labeled curves is a __h__ map.

For $z = f(x, y) = x^2 - y^2$, the equation for a level curve is __i__. This curve is a __j__. For $z = x - y$ the curves are __k__. *Level curves never cross because* __l__. They crowd together when the surface is __m__. The curves tighten to a point when __n__. The steepest direction on a mountain is __o__ to the __p__.

1 Draw the surface $z = f(x, y)$ for these four functions:

$$f_1 = \sqrt{4 - x^2 - y^2} \qquad f_2 = 2 - \sqrt{x^2 + y^2}$$
$$f_3 = 2 - \tfrac{1}{2}(x^2 + y^2) \qquad f_4 = 1 + e^{-x^2 - y^2}$$

2 The level curves of all four functions are ______. They enclose the maximum at ______. Draw the four curves $f(x, y) = 1$ and rank them by increasing radius.

3 Set $y = 0$ and compute the x derivative of each function at $x = 2$. Which mountain is flattest and which is steepest at that point?

4 Set $y = 1$ and compute the x derivative of each function at $x = 1$.

For f_5 to f_{10} draw the level curves $f = 0, 1, 2$. Also $f = -4$.

5 $f_5 = x - y$ **6** $f_6 = (x + y)^2$

7 $f_7 = xe^{-y}$ **8** $f_8 = \sin(x - y)$

9 $f_9 = y - x^2$ **10** $f_{10} = y/x^2$

11 Suppose the level curves are parallel straight lines. Does the surface have to be a plane?

12 Construct a function whose level curve $f = 0$ is in two separate pieces.

13 Construct a function for which $f = 0$ is a circle and $f = 1$ is not.

14 Find a function for which $f = 0$ has infinitely many pieces.

15 Draw the contour map for $f = xy$ with level curves $f = -2, -1, 0, 1, 2$. Describe the surface.

16 Find a function $f(x, y)$ whose level curve $f = 0$ consists of a circle and *all points inside it*.

Draw two level curves in 17–20. Are they ellipses, parabolas, or hyperbolas? Write $\sqrt{\quad} - 2x = c$ as $\sqrt{\quad} = c + 2x$ before squaring both sides.

17 $f = \sqrt{4x^2 + y^2}$ **18** $f = \sqrt{4x^2 + y^2} - 2x$

19 $f = \sqrt{5x^2 + y^2} - 2x$ **20** $f = \sqrt{3x^2 + y^2} - 2x$

21 The level curves of $f = (y - 2)/(x - 1)$ are ______ through the point $(1, 2)$ except that this point is not ______.

22 Sketch a map of the US with lines of constant temperature (isotherms) based on today's paper.

23 (a) The contour lines of $z = x^2 + y^2 - 2x - 2y$ are circles around the point ______, where z is a minimum.
(b) The contour lines of $f =$ ______ are the circles $x^2 + y^2 = c + 1$ on which $f = c$.

24 Draw a contour map of any state or country (lines of constant height above sea level). Florida may be too flat.

25 The graph of $w = F(x, y, z)$ is a ______-dimensional surface in $xyzw$ space. Its level sets $F(x, y, z) = c$ are ______-dimensional surfaces in xyz space. For $w = x - 2y + z$ those level sets are ______. For $w = x^2 + y^2 + z^2$ those level sets are ______.

26 The surface $x^2 + y^2 - z^2 = -1$ is in Figure 13.8. There is empty space when z^2 is smaller than 1 because ______.

27 The level sets of $F = x^2 + y^2 + qz^2$ look like footballs when q is ______, like basketballs when q is ______, and like frisbees when q is ______.

28 Let $T(x, y)$ be the driving time from your home at $(0, 0)$ to nearby towns at (x, y). Draw the level curves.

29 (a) The level curves of $f(x, y) = \sin(x - y)$ are ______.
(b) The level curves of $g(x, y) = \sin(x^2 - y^2)$ are ______.
(c) The level curves of $h(x, y) = \sin(x - y^2)$ are ______.

30 Prove that if $x_1y_1 = 1$ and $x_2y_2 = 1$ then their average $x = \tfrac{1}{2}(x_1 + x_2)$, $y = \tfrac{1}{2}(y_1 + y_2)$ has $xy \geqslant 1$. The function $f = xy$ has convex level curves (hyperbolas).

31 The hours in a day are limited by $x + y = 24$. Write x^2y as $x^2(24 - x)$ and maximize to find the optimal number of hours to stay awake.

32 Near $x = 16$ draw the level curve $x^2y = 2048$ and the line $x + y = 24$. Show that the curve is convex and the line is tangent.

33 The surface $z = 4x + y$ is a ______. The surface $z = \min(x, y)$ is formed from two ______. We are willing to exchange 6 left and 2 right shoes for 2 left and 4 right shoes but better is the average ______.

34 Draw a contour map of the top of your shoe.

13.2 Partial Derivatives

The central idea of differential calculus is the derivative. A change in x produces a change in f. The ratio $\Delta f/\Delta x$ approaches the derivative, or slope, or rate of change. What to do if f depends on both x and y?

The new idea is to vary x and y one at a time. First, only x moves. If the function is $x + xy$, then Δf is $\Delta x + y\Delta x$. The ratio $\Delta f/\Delta x$ is $1 + y$. The "x derivative" of $x + xy$

is $1+y$. For all functions the method is the same: ***Keep y constant, change x, take the limit of*** $\Delta f/\Delta x$:

DEFINITION $$\frac{\partial f}{\partial x}(x, y)=\lim_{\Delta x \to 0}\frac{\Delta f}{\Delta x}=\lim_{\Delta x \to 0}\frac{f(x+\Delta x, y)-f(x, y)}{\Delta x}. \tag{1}$$

On the left is a new symbol $\partial f/\partial x$. It signals that only x is allowed to vary—$\partial f/\partial x$ is a ***partial derivative***. The different form ∂ of the same letter (still say "d") is a reminder that x is not the only variable. Another variable y is present but not moving.

EXAMPLE 1 $f(x, y)=x^2y^2+xy+y \qquad \dfrac{\partial f}{\partial x}(x, y)=2xy^2+y+0.$

Do not treat y as zero! Treat it as a constant, like 6. Its x *derivative* is zero. If $f(x)=\sin 6x$ then $df/dx=6\cos 6x$. If $f(x, y)=\sin xy$ then $\partial f/\partial x=y\cos xy$.

Spoken aloud, $\partial f/\partial x$ is still "$d f d\, x$." It is a function of x and y. When more is needed, call it "the partial of f with respect to x." The symbol f' is no longer available, since it gives no special indication about x. Its replacement f_x is pronounced "$f\,x$" or "f sub x," which is shorter than $\partial f/\partial x$ and means the same thing.

We may also want to indicate the point (x_0, y_0) where the derivative is computed:

$$\frac{\partial f}{\partial x}(x_0, y_0) \quad \text{or} \quad f_x(x_0, y_0) \quad \text{or} \quad \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} \quad \text{or just} \quad \left(\frac{\partial f}{\partial x}\right)_0.$$

EXAMPLE 2 $f(x, y)=\sin 2x\cos y \qquad f_x=2\cos 2x\cos y \qquad (\cos y \text{ is constant for } \partial/\partial x)$

The particular point (x_0, y_0) is $(0, 0)$. The height of the surface is $f(0, 0)=0$. The slope in the x direction is $f_x=2$. At a different point $x_0=\pi$, $y_0=\pi$ we find $f_x(\pi, \pi)=-2$.

Now keep x constant and vary y. The ratio $\Delta f/\Delta y$ approaches $\partial f/\partial y$:

$$f_y(x, y)=\lim_{\Delta y \to 0}\frac{\Delta f}{\Delta y}=\lim_{\Delta y \to 0}\frac{f(x, y+\Delta y)-f(x, y)}{\Delta y}. \tag{2}$$

This is the slope in the y direction. Please realize that a surface can go up in the x direction and down in the y direction. The plane $f(x, y)=3x-4y$ has $f_x=3$ (up) and $f_y=-4$ (down). We will soon ask what happens in the 45° direction.

EXAMPLE 3 $f(x, y)=\sqrt{x^2+y^2} \qquad \dfrac{\partial f}{\partial x}=\dfrac{x}{\sqrt{x^2+y^2}}=\dfrac{x}{f} \qquad \dfrac{\partial f}{\partial y}=\dfrac{y}{\sqrt{x^2+y^2}}=\dfrac{y}{f}.$

The x derivative of $\sqrt{x^2+y^2}$ is really one-variable calculus, because y is constant. The exponent drops from $\frac{1}{2}$ to $-\frac{1}{2}$, and there is $2x$ from the chain rule. ***This distance function has the curious derivative*** $\partial f/\partial x=x/f$.

The graph is a cone. Above the point $(0, 2)$ the height is $\sqrt{0^2+2^2}=2$. The partial derivatives are $f_x=0/2$ and $f_y=2/2$. At that point, Figure 13.5 climbs in the y direction. It is level in the x direction. An actual step Δx will increase 0^2+2^2 to $(\Delta x)^2+2^2$. But this change is of order $(\Delta x)^2$ and the x derivative is zero.

Figure 13.5 is rather important. It shows how $\partial f/\partial x$ and $\partial f/\partial y$ are the ordinary derivatives of $f(x, y_0)$ and $f(x_0, y)$. It is natural to call these ***partial functions***. The first has y fixed at y_0 while x varies. The second has x fixed at x_0 while y varies. Their graphs are ***cross sections down the surface***—cut out by the vertical planes $y=y_0$ and $x=x_0$. Remember that the level curve is cut out by the horizontal plane $z=c$.

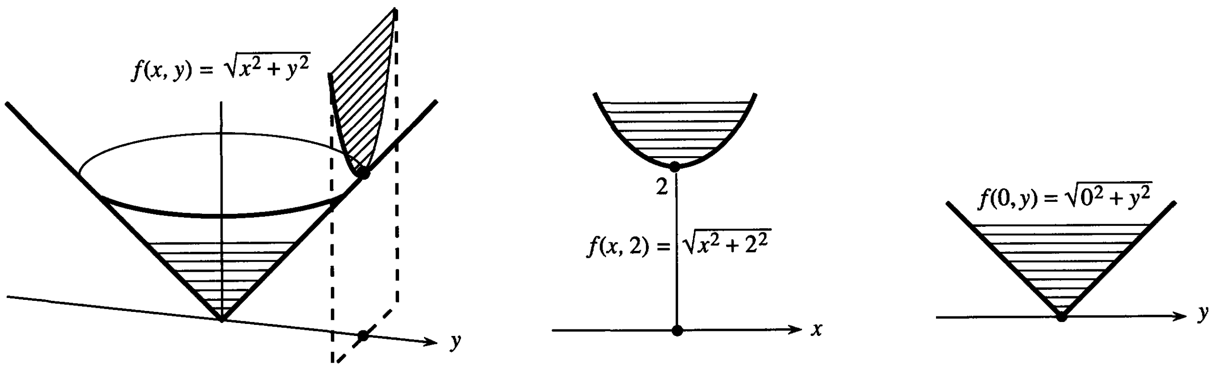


Fig. 13.5 Partial functions $\sqrt{x^2+2^2}$ and $\sqrt{0^2+y^2}$ of the distance function $f = \sqrt{x^2+y^2}$.

The limits of $\Delta f/\Delta x$ and $\Delta f/\Delta y$ are computed as always. With partial functions we are back to a single variable. ***The partial derivative is the ordinary derivative of a partial function*** (constant y or constant x). For the cone, $\partial f/\partial y$ exists at all points except (0, 0). The figure shows how the cross section down the middle of the cone produces the absolute value function: $f(0, y) = |y|$. It has one-sided derivatives but not a two-sided derivative.

Similarly $\partial f/\partial x$ will not exist at the sharp point of the cone. We develop the idea of a *continuous function* $f(x, y)$ as needed (the definition is in the exercises). Each partial derivative involves one direction, but limits and continuity involve all directions. The distance function is continuous at (0, 0), where it is not differentiable.

EXAMPLE 4 $f(x, y) = y^2 - x^2 \qquad \partial f/\partial x = -2x \qquad \partial f/\partial y = 2y$

Move in the x direction from (1, 3). Then $y^2 - x^2$ has the partial function $9 - x^2$. With y fixed at 3, a parabola opens downward. In the y direction (along $x = 1$) the partial function $y^2 - 1$ opens upward. The surface in Figure 13.6 is called a *hyperbolic paraboloid*, because the level curves $y^2 - x^2 = c$ are hyperbolas. Most people call it a saddle, and the special point at the origin is a ***saddle point***.

The origin is special for $y^2 - x^2$ because both derivatives are zero. ***The bottom of the y parabola at*** (0, 0) ***is the top of the x parabola.*** The surface is momentarily flat in all directions. It is the top of a hill and the bottom of a mountain range at the same

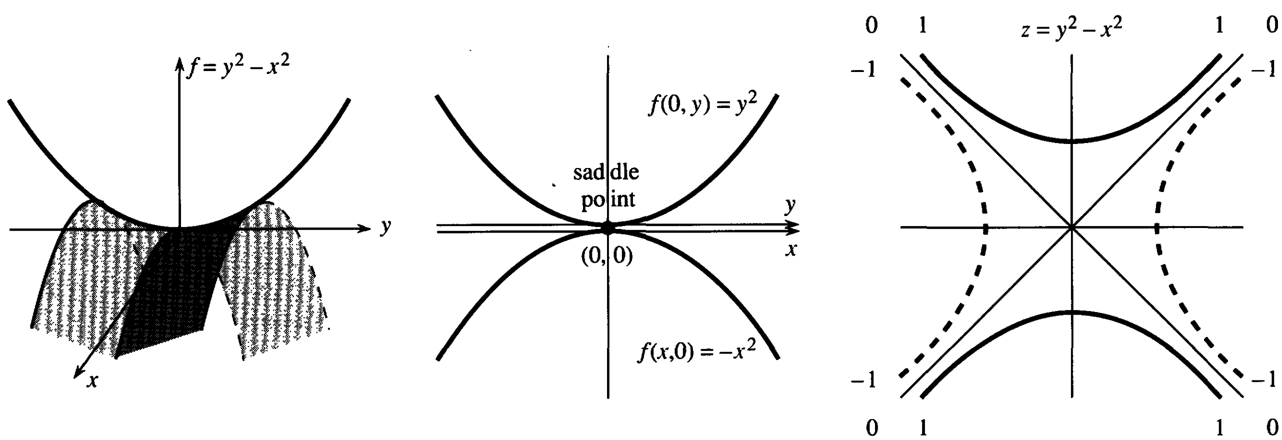


Fig. 13.6 A saddle function, its partial functions, and its level curves.

time. A saddle point is neither a maximum nor a minimum, although both derivatives are zero.

Note Do not think that $f(x, y)$ must contain y^2 and x^2 to have a saddle point. The function $2xy$ does just as well. The level curves $2xy = c$ are still hyperbolas. The partial functions $2xy_0$ and $2x_0y$ now give straight lines—which is remarkable. Along the 45° line $x = y$, the function is $2x^2$ and climbing. Along the $-45°$ line $x = -y$, the function is $-2x^2$ and falling. The graph of $2xy$ is Figure 13.6 rotated by 45°.

EXAMPLES 5–6 $f(x, y, z) = x^2 + y^2 + z^2 \qquad P(T, V) = nRT/V$

Example 5 shows more variables. Example 6 shows that the variables may not be named x and y. Also, the function may not be named f! Pressure and temperature and volume are P and T and V. The letters change but nothing else:

$$\partial P/\partial T = nR/V \qquad \partial P/\partial V = -nRT/V^2 \qquad \text{(note the derivative of } 1/V).$$

There is no $\partial P/\partial R$ because R is a constant from chemistry—not a variable.

Physics produces six variables for a moving body—the coordinates x, y, z and the momenta p_x, p_y, p_z. Economics and the social sciences do better than that. If there are 26 products there are 26 variables—sometimes 52, to show prices as well as amounts. The profit can be a complicated function of these variables. ***The partial derivatives are the marginal profits***, as one of the 52 variables is changed. A spreadsheet shows the 52 values and the effect of a change. An infinitesimal spreadsheet shows the derivative.

SECOND DERIVATIVE

Genius is not essential, to move to second derivatives. The only difficulty is that *two* first derivatives f_x and f_y lead to *four* second derivatives f_{xx} and f_{xy} and f_{yx} and f_{yy}. (Two subscripts: f_{xx} is the x derivative of the x derivative. Other notations are $\partial^2 f/\partial x^2$ and $\partial^2 f/\partial x \partial y$ and $\partial^2 f/\partial y \partial x$ and $\partial^2 f/\partial y^2$.) Fortunately f_{xy} equals f_{yx}, as we see first by example.

EXAMPLE 7 $f = x/y$ has $f_x = 1/y$, which has $f_{xx} = 0$ and $f_{xy} = -1/y^2$.

The function x/y is linear in x (which explains $f_{xx} = 0$). Its y derivative is $f_y = -x/y^2$. This has the x derivative $f_{yx} = -1/y^2$. ***The mixed derivatives f_{xy} and f_{yx} are equal.***

In the pure y direction, the second derivative is $f_{yy} = 2x/y^3$. One-variable calculus is sufficient for all these derivatives, because only one variable is moving.

EXAMPLE 8 $f = 4x^2 + 3xy + y^2$ has $f_x = 8x + 3y$ and $f_y = 3x + 2y$.

Both "*cross derivatives*" f_{xy} and f_{yx} equal 3. The second derivative in the x direction is $\partial^2 f/\partial x^2 = 8$ or $f_{xx} = 8$. Thus "$f\,x\,x$" is "d second $f\,d\,x$ squared." Similarly $\partial^2 f/\partial y^2 = 2$. The only change is from d to ∂.

If $f(x, y)$ has continuous second derivatives then $f_{xy} = f_{yx}$. Problem 43 sketches a proof based on the Mean Value Theorem. For third derivatives almost any example shows that $f_{xxy} = f_{xyx} = f_{yxx}$ is different from $f_{yyx} = f_{yxy} = f_{xyy}$.

Question ***How do you plot a space curve*** $x(t), y(t), z(t)$ ***in a plane***? One way is to look parallel to the direction $(1, 1, 1)$. On your XY screen, plot $X = (y - x)/\sqrt{2}$ and $Y = (2z - x - y)/\sqrt{6}$. The line $x = y = z$ goes to the point $(0, 0)$!

How do you graph a surface $z=f(x, y)$? Use the same X and Y. Fix x and let y vary, for curves one way in the surface. Then fix y and vary x, for the other partial function. For a parametric surface like $x \doteq (2+v \sin \frac{1}{2}u) \cos u$, $y=(2+v \sin \frac{1}{2}u) \sin u$, $z=v \cos \frac{1}{2}u$, vary u and then v. Dick Williamson showed how this draws a one-sided "Möbius strip."

13.2 EXERCISES

Read-through questions

The __a__ derivative $\partial f/\partial y$ comes from fixing __b__ and moving __c__. It is the limit of __d__. If $f=e^{2x} \sin y$ then $\partial f/\partial x=$ __e__ and $\partial f/\partial y=$ __f__. If $f=(x^2+y^2)^{1/2}$ then $f_x=$ __g__ and $f_y=$ __h__. At (x_0, y_0) the partial derivative f_x is the ordinary derivative of the __i__ function $f(x, y_0)$. Similarly f_y comes from $f($__j__$)$. Those functions are cut out by vertical planes $x=x_0$ and __k__, while the level curves are cut out by __l__ planes.

The four second derivatives are f_{xx}, __m__, __n__, __o__. For $f=xy$ they are __p__. For $f=\cos 2x \cos 3y$ they are __q__. In those examples the derivatives __r__ and __s__ are the same. That is always true when the second derivatives are __t__. At the origin, $\cos 2x \cos 3y$ is curving __u__ in the x and y directions, while xy goes __v__ in the 45° direction and __w__ in the $-45°$ direction.

Find $\partial f/\partial x$ and $\partial f/\partial y$ for the functions in 1–12.

1 $3x-y+x^2y^2$

2 $\sin(3x-y)+y$

3 $x^3y^2-x^2-e^y$

4 xe^{x+4}

5 $(x+y)/(x-y)$

6 $1/\sqrt{x^2+y^2}$

7 $(x^2+y^2)^{-1}$

8 $\ln(x+2y)$

9 $\ln\sqrt{x^2+y^2}$

10 y^x

11 $\tan^{-1}(y/x)$

12 $\ln(xy)$

Compute f_{xx}, $f_{xy}=f_{yx}$, and f_{yy} for the functions in 13–20.

13 $x^2+3xy+2y^2$

14 $(x+3y)^2$

15 $(x+iy)^3$

16 e^{ax+by}

17 $1/\sqrt{x^2+y^2}$

18 $(x+y)^n$

19 $\cos ax \cos by$

20 $1/(x+iy)$

Find the domain and range (all inputs and outputs) for the functions 21–26. Then compute f_x, f_y, f_z, f_t.

21 $1/(x-y)^2$

22 $\sqrt{x^2+y^2-t^2}$

23 $(y-x)/(z-t)$

24 $\ln(x+t)$

25 $x^{\ln t}$ *Why does this equal* $t^{\ln x}$?

26 $\cos x \cos^{-1} y$

27 Verify $f_{xy}=f_{yx}$ for $f=x^my^n$. If $f_{xy}=0$ then f_x does not depend on ______ and f_y is independent of ______. The function must have the form $f(x, y)=G(x)+$ ______.

28 In terms of v, compute f_x and f_y for $f(x, y)=\int_x^y v(t)\,dt$. First vary x. Then vary y.

29 Compute $\partial f/\partial x$ for $f=\int_0^{xy} v(t)dt$. Keep y constant.

30 What is $f(x, y)=\int_x^y dt/t$ and what are f_x and f_y?

31 Calculate all eight third derivatives $f_{xxx}, f_{xxy}, \ldots$ of $f=x^3y^3$. How many are different?

In 32–35, choose $g(y)$ so that $f(x, y)=e^{cx}g(y)$ satisfies the equation.

32 $f_x+f_y=0$

33 $f_x=7f_y$

34 $f_y=f_{xx}$

35 $f_{xx}=4f_{yy}$

36 Show that $t^{-1/2}e^{-x^2/4t}$ satisfies the ***heat equation*** $f_t=f_{xx}$. This $f(x, t)$ is the temperature at position x and time t due to a point source of heat at $x=0, t=0$.

37 The equation for heat flow in the xy plane is $f_t=f_{xx}+f_{yy}$. Show that $f(x, y, t)=e^{-2t} \sin x \sin y$ is a solution. What exponent in $f=e^{\text{———}} \sin 2x \sin 3y$ gives a solution?

38 Find solutions $f(x, y)=e^{\text{———}} \sin mx \cos ny$ of the heat equation $f_t=f_{xx}+f_{yy}$. Show that $t^{-1}e^{-x^2/4t}e^{-y^2/4t}$ is also a solution.

39 The basic ***wave equation*** is $f_{tt}=f_{xx}$. Verify that $f(x, t)=\sin(x+t)$ and $f(x, t)=\sin(x-t)$ are solutions. Draw both graphs at $t=\pi/4$. Which wave moved to the left and which moved to the right?

40 Continuing 39, the peaks of the waves moved a distance $\Delta x=$ ______ in the time step $\Delta t=\pi/4$. The wave velocity is $\Delta x/\Delta t=$ ______.

41 Which of these satisfy the wave equation $f_{tt}=c^2f_{xx}$?

$$\sin(x-ct), \quad \cos(x+ct), \quad e^{x-ct}, \quad e^x-e^{ct}, \quad e^x \cos ct.$$

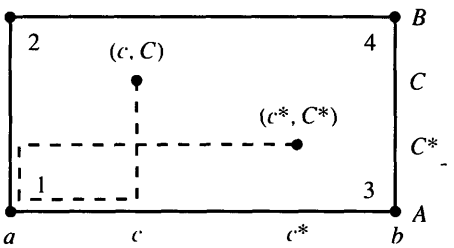
42 Suppose $\partial f/\partial t=\partial f/\partial x$. Show that $\partial^2 f/\partial t^2=\partial^2 f/\partial x^2$.

43 The proof of $f_{xy}=f_{yx}$ studies $f(x, y)$ in a small rectangle. The top-bottom difference is $g(x)=f(x, B)-f(x, A)$. The difference at the corners 1, 2, 3, 4 is:

$$\begin{aligned} Q &= [f_4-f_3]-[f_2-f_1] \\ &= g(b)-g(a) \quad \text{(definition of } g) \\ &= (b-a)g_x(c) \quad \text{(Mean Value Theorem)} \\ &= (b-a)[f_x(c, B)-f_x(c, A)] \quad \text{(compute } g_x) \\ &= (b-a)(B-A)f_{xy}(c, C) \quad \text{(MVT again)} \end{aligned}$$

(a) The right-left difference is $h(y)=f(b, y)-f(a, y)$. The same Q is $h(B)-h(A)$. Change the steps to reach $Q=(B-A)(b-a)f_{yx}(c^*, C^*)$.

(b) The two forms of Q make f_{xy} at (c, C) equal to f_{yx} at (c^*, C^*). Shrink the rectangle toward (a, A). What assumption yields $f_{xy}=f_{yx}$ at that typical point?



44 Find $\partial f/\partial x$ and $\partial f/\partial y$ where they exist, based on equations (1) and (2).

(a) $f=|xy|$ (b) $f=x^2+y^2$ if $x \neq 0$, $f=0$ if $x=0$

Questions 45–52 are about limits in two dimensions.

45 Complete these *four correct definitions of limit*: **1** The points (x_n, y_n) approach the point (a, b) if x_n converges to a and _______. **2** For any circle around (a, b), the points (x_n, y_n) eventually go _______ the circle and stay _______. **3** The distance from (x_n, y_n) to (a, b) is _______ and it approaches _______. **4** For any $\varepsilon > 0$ there is an N such that the distance _______ $< \varepsilon$ for all $n >$ _______.

46 Find (x_2, y_2) and (x_4, y_4) and the limit (a, b) if it exists. Start from $(x_0, y_0)=(1, 0)$.

(a) $(x_n, y_n)=(1/(n+1), n/(n+1))$
(b) $(x_n, y_n)=(x_{n-1}, y_{n-1})$
(c) $(x_n, y_n)=(y_{n-1}, x_{n-1})$
(d) $(x_n, y_n)=(x_{n-1}+y_{n-1}, x_{n-1}-y_{n-1})$

47 (*Limit of* $f(x, y)$) **1** Informal definition: the numbers $f(x_n, y_n)$ approach L when the points (x_n, y_n) approach (a, b). **2** *Epsilon-delta definition*: For each $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x, y)-L|$ is less than _______ when the distance from (x, y) to (a, b) is _______. The value of f *at* (a, b) is not involved.

48 Write down the limit L as $(x, y) \to (a, b)$. At which points (a, b) does $f(x, y)$ have no limit?

(a) $f(x, y)=\sqrt{x^2+y^2}$ (b) $f(x, y)=x/y$
(c) $f(x, y)=1/(x+y)$ (d) $f(x, y)=xy/(x^2+y^2)$

In (d) find the limit at (0, 0) along the line $y=mx$. The limit changes with m, so L does not exist at (0, 0). Same for x/y.

49 *Definition of continuity*: $f(x, y)$ is continuous at (a, b) if $f(a, b)$ is defined and $f(x, y)$ approaches the limit _______ as (x, y) approaches (a, b). Construct a function that is *not* continuous at (1, 2).

50 Show that $x^2y/(x^4+y^2) \to 0$ along every straight line $y=mx$ to the origin. But traveling down the parabola $y=x^2$, the ratio equals _______.

51 Can you define $f(0, 0)$ so that $f(x, y)$ is continuous at (0, 0)?

(a) $f=|x|+|y-1|$ (b) $f=(1+x)^y$ (c) $f=x^{1+y}$.

52 Which functions approach zero as $(x, y) \to (0, 0)$ and why?

(a) $\dfrac{xy^2}{x^2+y^2}$ (b) $\dfrac{x^2y^2}{x^4+y^4}$ (c) $\dfrac{x^my^n}{x^m+y^n}$.

13.3 Tangent Planes and Linear Approximations

Over a short range, a smooth curve $y=f(x)$ is almost straight. The curve changes direction, but the tangent line $y-y_0=f'(x_0)(x-x_0)$ keeps the same slope forever. The tangent line immediately gives the linear approximation to $y=f(x)$: $y \approx y_0+f'(x_0)(x-x_0)$.

What happens with two variables? The function is $z=f(x, y)$, and its graph is a ***surface***. We are at a point on that surface, and we are near-sighted. We don't see far away. The surface may curve out of sight at the horizon, or it may be a bowl or a saddle. To our myopic vision, the surface looks flat. We believe we are on a plane (not necessarily horizontal), and we want the equation of this ***tangent plane***.

Notation The basepoint has coordinates x_0 and y_0. The height on the surface is $z_0 = f(x_0, y_0)$. Other letters are possible: the point can be (a, b) with height w. The subscript $_0$ indicates the value of x or y or z or $\partial f/\partial x$ or $\partial f/\partial y$ *at the point*.

With one variable the tangent line has slope df/dx. With two variables there are two derivatives $\partial f/\partial x$ and $\partial f/\partial y$. At the particular point, they are $(\partial f/\partial x)_0$ and $(\partial f/\partial y)_0$. ***Those are the slopes of the tangent plane.*** Its equation is the key to this chapter:

> **13A** The tangent plane at (x_0, y_0, z_0) has the same slopes as the surface $z = f(x, y)$. The equation of the tangent plane (a linear equation) is
>
> $$z - z_0 = \left(\frac{\partial f}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y - y_0). \tag{1}$$
>
> The normal vector **N** to that plane has components $(\partial f/\partial x)_0, (\partial f/\partial y)_0, -1$.

EXAMPLE 1 Find the tangent plane to $z = 14 - x^2 - y^2$ at $(x_0, y_0, z_0) = (1, 2, 9)$.

Solution The derivatives are $\partial f/\partial x = -2x$ and $\partial f/\partial y = -2y$. When $x = 1$ and $y = 2$ those are $(\partial f/\partial x)_0 = -2$ and $(\partial f/\partial y)_0 = -4$. The equation of the tangent plane is

$$z - 9 = -2(x - 1) - 4(y - 2) \quad \text{or} \quad z + 2x + 4y = 19. \tag{2}$$

This $z(x, y)$ has derivatives -2 and -4, just like the surface. So the plane is tangent.

The normal vector **N** has components $-2, -4, -1$. ***The equation of the normal line is*** $(x, y, z) = (1, 2, 9) + t(-2, -4, -1)$. Starting from $(1, 2, 9)$ the line goes out along **N**—perpendicular to the plane and the surface.

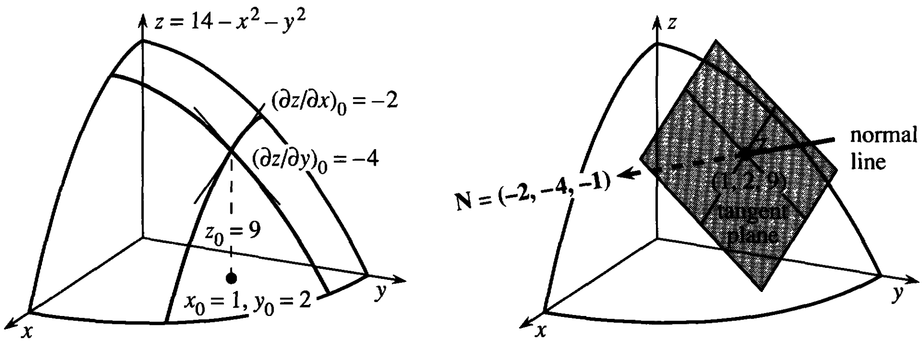


Fig. 13.7 The tangent plane contains the x and y tangent lines, perpendicular to **N**.

Figure 13.7 shows more detail about the tangent plane. The dotted lines are the x and y tangent lines. They lie in the plane. All tangent lines lie in the tangent plane! These particular lines are tangent to the "partial functions"—where y is fixed at $y_0 = 2$ or x is fixed at $x_0 = 1$. The plane is balancing on the surface and touching at the tangent point.

More is true. In the surface, *every curve through the point is tangent to the plane.* Geometrically, the curve goes up to the point and "kisses" the plane.† The tangent **T** to the curve and the normal **N** to the surface are perpendicular: $\mathbf{T} \cdot \mathbf{N} = 0$.

†A safer word is "osculate." At saddle points the plane is kissed from both sides.

EXAMPLE 2 Find the tangent plane to the sphere $z^2 = 14 - x^2 - y^2$ at (1, 2, 3).

Solution Instead of $z = 14 - x^2 - y^2$ we have $z = \sqrt{14 - x^2 - y^2}$. At $x_0 = 1$, $y_0 = 2$ the height is now $z_0 = 3$. The surface is a sphere with radius $\sqrt{14}$. The only trouble from the square root is its derivatives:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\sqrt{14 - x^2 - y^2} = \frac{\frac{1}{2}(-2x)}{\sqrt{14 - x^2 - y^2}} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{\frac{1}{2}(-2y)}{\sqrt{14 - x^2 - y^2}} \tag{3}$$

At (1, 2) those slopes are $-\frac{1}{3}$ and $-\frac{2}{3}$. The equation of the tangent plane is linear: $z - 3 = -\frac{1}{3}(x - 1) - \frac{2}{3}(y - 2)$. I cannot resist improving the equation, by multiplying through by 3 and moving all terms to the left side:

$$\textit{tangent plane to sphere}: \quad 1(x - 1) + 2(y - 2) + 3(z - 3) = 0. \tag{4}$$

If mathematics is the "science of patterns," equation (4) is a prime candidate for study. The numbers 1, 2, 3 appear twice. The coordinates are $(x_0, y_0, z_0) = (1, 2, 3)$. The normal vector is $1\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. The tangent equation is $1x + 2y + 3z = 14$. None of this can be an accident, but the square root of $14 - x^2 - y^2$ made a simple pattern look complicated.

This square root is not necessary. Calculus offers a direct way to find dz/dx—*implicit differentiation*. Just differentiate every term as it stands:

$$x^2 + y^2 + z^2 = 14 \quad \text{leads to} \quad 2x + 2z\,\partial z/\partial x = 0 \quad \text{and} \quad 2y + 2z\,\partial z/\partial y = 0. \tag{5}$$

Canceling the 2's, the derivatives on a sphere are $-x/z$ and $-y/z$. Those are the same as in (3). The equation for the tangent plane has an extremely symmetric form:

$$z - z_0 = -\frac{x_0}{z_0}(x - x_0) - \frac{y_0}{z_0}(y - y_0) \quad \text{or} \quad x_0(x - x_0) + y_0(y - y_0) + z_0(z - z_0) = 0. \tag{6}$$

Reading off $\mathbf{N} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$ from the last equation, calculus proves something we already knew: **The normal vector to a sphere points outward along the radius.**

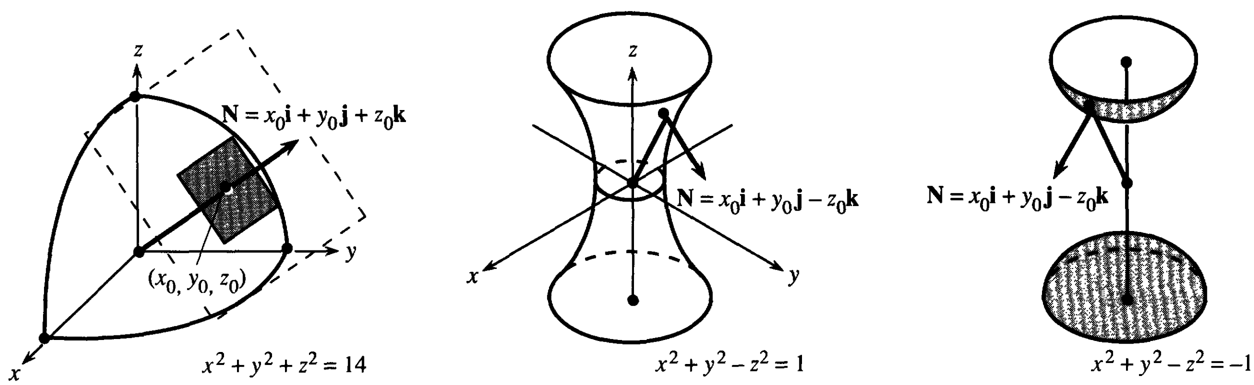


Fig. 13.8 Tangent plane and normal N for a sphere. Hyperboloids of 1 and 2 sheets.

THE TANGENT PLANE TO $F(x, y, z) = c$

The sphere suggests a question that is important for other surfaces. Suppose the equation is $F(x, y, z) = c$ instead of $z = f(x, y)$. Can the partial derivatives and tangent plane be found directly from F?

The answer is *yes*. It is not necessary to solve first for z. The derivatives of F, computed at (x_0, y_0, z_0), give a second formula for the tangent plane and normal vector.

13B The tangent plane to the surface $F(x, y, z) = c$ has the linear equation

$$\left(\frac{\partial F}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial F}{\partial y}\right)_0 (y - y_0) + \left(\frac{\partial F}{\partial z}\right)_0 (z - z_0) = 0. \tag{7}$$

The normal vector is $\mathbf{N} = \left(\frac{\partial F}{\partial x}\right)_0 \mathbf{i} + \left(\frac{\partial F}{\partial y}\right)_0 \mathbf{j} + \left(\frac{\partial F}{\partial z}\right)_0 \mathbf{k}$.

Notice how this includes the original case $z = f(x, y)$. ***The function F becomes*** $f(x, y) - z$. Its partial derivatives are $\partial f/\partial x$ and $\partial f/\partial y$ and -1. (The -1 is from the derivative of $-z$.) Then equation (7) is the same as our original tangent equation (1).

EXAMPLE 3 The surface $F = x^2 + y^2 - z^2 = c$ is a ***hyperboloid***. Find its tangent plane.

Solution The partial derivatives are $F_x = 2x$, $F_y = 2y$, $F_z = -2z$. Equation (7) is

$$\textbf{\textit{tangent plane}}: \quad 2x_0(x - x_0) + 2y_0(y - y_0) - 2z_0(z - z_0) = 0. \tag{8}$$

We can cancel the 2's. The normal vector is $\mathbf{N} = x_0\mathbf{i} + y_0\mathbf{j} - z_0\mathbf{k}$. For $c > 0$ *this hyperboloid has one sheet* (Figure 13.8). For $c = 0$ it is a cone and for $c < 0$ it breaks into two sheets (Problem 13.1.26).

DIFFERENTIALS

Come back to the linear equation $z - z_0 = (\partial z/\partial x)_0(x - x_0) + (\partial z/\partial y)_0(y - y_0)$ for the tangent plane. That may be the most important formula in this chapter. Move along the tangent plane instead of the curved surface. Movements in the plane are dx and dy and dz—while Δx and Δy and Δz are movements in the surface. The d's are governed by the tangent equation—the Δ's are governed by $z = f(x, y)$. In Chapter 2 the d's were ***differentials*** along the tangent line:

$$dy = (dy/dx)dx \quad \text{(straight line)} \quad \text{and} \quad \Delta y \approx (dy/dx)\Delta x \quad \text{(on the curve)}. \tag{9}$$

Now y is independent like x. The dependent variable is z. The idea is the same. The distances $x - x_0$ and $y - y_0$ and $z - z_0$ (on the tangent plane) are dx and dy and dz. *The equation of the plane is*

$$dz = (\partial z/\partial x)_0 dx + (\partial z/\partial y)_0 dy \quad \text{or} \quad df = f_x dx + f_y dy. \tag{10}$$

This is the ***total differential***. All letters dz and df and dw can be used, but ∂z and ∂f are not used. Differentials suggest small movements in x and y; then dz is the resulting movement in z. On the tangent plane, equation (10) holds exactly.

A "centering transform" has put x_0, y_0, z_0 at the center of coordinates. Then the "zoom transform" stretches the surface into its tangent plane.

EXAMPLE 4 The area of a triangle is $A = \frac{1}{2}ab \sin \theta$. Find the total differential dA.

Solution The base has length b and the sloping side has length a. The angle between them is θ. You may prefer $A = \frac{1}{2}bh$, where h is the perpendicular height $a \sin \theta$. Either way we need the partial derivatives. If $A = \frac{1}{2}ab \sin \theta$, then

$$\frac{\partial A}{\partial a} = \frac{1}{2} b \sin \theta \qquad \frac{\partial A}{\partial b} = \frac{1}{2} a \sin \theta \qquad \frac{\partial A}{\partial \theta} = \frac{1}{2} ab \cos \theta. \tag{11}$$

These lead immediately to the total differential dA (like a product rule):

$$dA = \left(\frac{\partial A}{\partial a}\right)da + \left(\frac{\partial A}{\partial b}\right)db + \left(\frac{\partial A}{\partial \theta}\right)d\theta = \frac{1}{2}b \sin\theta \, da + \frac{1}{2}a \sin\theta \, db + \frac{1}{2}ab \cos\theta \, d\theta.$$

EXAMPLE 5 The volume of a cylinder is $V = \pi r^2 h$. Decide whether V is more sensitive to a change from $r = 1.0$ to $r = 1.1$ or from $h = 1.0$ to $h = 1.1$.

Solution The partial derivatives are $\partial V/\partial r = 2\pi rh$ and $\partial V/\partial h = \pi r^2$. ***They measure the sensitivity to change***. Physically, they are the side area and base area of the cylinder. The volume differential dV comes from a shell around the side plus a layer on top:

$$dV = \text{shell} + \text{layer} = 2\pi rh \, dr + \pi r^2 dh. \tag{12}$$

Starting from $r = h = 1$, that differential is $dV = 2\pi dr + \pi dh$. With $dr = dh = .1$, the shell volume is $.2\pi$ and the layer volume is only $.1\pi$. So V is sensitive to dr.

For a short cylinder like a penny, the layer has greater volume. V is more sensitive to dh. In our case $V = \pi r^2 h$ increases from $\pi(1)^3$ to $\pi(1.1)^3$. ***Compare*** ΔV ***to*** dV:

$$\Delta V = \pi(1.1)^3 - \pi(1)^3 = .331\pi \quad \text{and} \quad dV = 2\pi(.1) + \pi(.1) = .300\pi.$$

The difference is $\Delta V - dV = .031\pi$. The shell and layer missed a small volume in Figure 13.9, just above the shell and around the layer. The mistake is of order $(dr)^2 + (dh)^2$. For $V = \pi r^2 h$, the differential $dV = 2\pi rh \, dr + \pi r^2 dh$ is a ***linear approximation*** to the true change ΔV. We now explain that properly.

LINEAR APPROXIMATION

Tangents lead immediately to linear approximations. That is true of tangent planes as it was of tangent lines. The plane stays close to the surface, as the line stayed close to the curve. Linear functions are simpler than $f(x)$ or $f(x, y)$ or $F(x, y, z)$. All we need are first derivatives *at the point*. Then the approximation is good *near the point*.

This key idea of calculus is already present in differentials. On the plane, df equals $f_x dx + f_y dy$. On the curved surface that is a linear approximation to Δf:

13C The linear approximation to $f(x, y)$ near the point (x_0, y_0) is

$$f(x, y) \approx f(x_0, y_0) + \left(\frac{\partial f}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y - y_0). \tag{13}$$

In other words $\Delta f \approx f_x \Delta x + f_y \Delta y$, as proved in Problem 24. The right side of (13) is a linear function $f_L(x, y)$. At (x_0, y_0), the functions f and f_L have the same slopes. Then $f(x, y)$ curves away from f_L with an error of "second order:"

$$|f(x, y) - f_L(x, y)| \leqslant M[(x - x_0)^2 + (y - y_0)^2]. \tag{14}$$

This assumes that f_{xx}, f_{xy}, and f_{yy} are continuous and bounded by M along the line from (x_0, y_0) to (x, y). Example 3 of Section 13.5 shows that $|f_{tt}| \leqslant 2M$ along that line. A factor $\frac{1}{2}$ comes from equation 3.8.12, for the error $f - f_L$ with one variable.

For the volume of a cylinder, r and h went from 1.0 to 1.1. The second derivatives of $V = \pi r^2 h$ are $V_{rr} = 2\pi h$ and $V_{rh} = 2\pi r$ and $V_{hh} = 0$. They are below $M = 2.2\pi$. Then (14) gives the error bound $2.2\pi(.1^2 + .1^2) = .044\pi$, not far above the actual error $.031\pi$. The main point is that ***the error in linear approximation comes from the quadratic terms***—those are the first terms to be ignored by f_L.

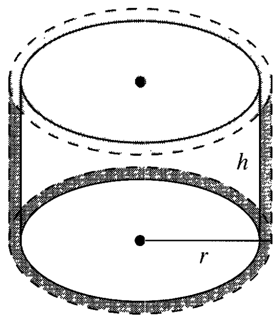


Fig. 13.9 Shell plus layer gives $dV = .300\pi$. Including top ring gives $\Delta V = .331\pi$.

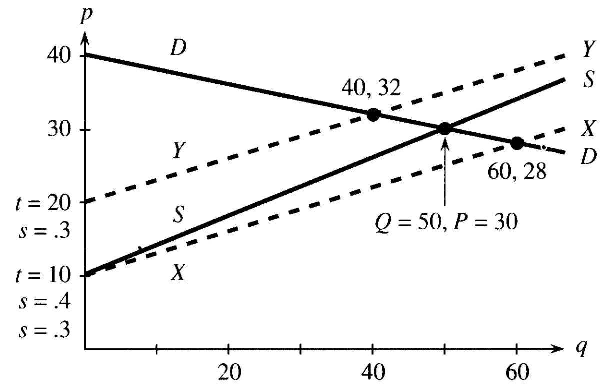


Fig. 13.10 Quantity Q and price P move with the lines.

EXAMPLE 6 Find a linear approximation to the distance function $r = \sqrt{x^2 + y^2}$.

Solution The partial derivatives are x/r and y/r. Then $\Delta r \approx (x/r)\Delta x + (y/r)\Delta y$.

For (x, y, r) near $(1, 2, \sqrt{5})$: $\sqrt{x^2 + y^2} \approx \sqrt{1^2 + 2^2} + (x - 1)/\sqrt{5} + 2(y - 2)/\sqrt{5}$.

If y is fixed at 2, this is a one-variable approximation to $\sqrt{x^2 + 2^2}$. If x is fixed at 1, it is a linear approximation in y. Moving both variables, you might think dr would involve dx and dy in a square root. It doesn't. Distance involves x and y in a square root, but *change of distance* is linear in Δx and Δy—to a first approximation.

There is a rough point at $x = 0$, $y = 0$. Any movement from $(0, 0)$ gives $\Delta r = \sqrt{(\Delta x)^2 + (\Delta y)^2}$. The square root has returned. The reason is that ***the partial derivatives*** x/r ***and*** y/r ***are not continuous at*** $(0, 0)$. The cone has a sharp point with no tangent plane. *Linear approximation breaks down.*

The next example shows how to approximate Δz from Δx and Δy, when the equation is $F(x, y, z) = c$. We use the implicit derivatives in (7) instead of the explicit derivatives in (1). The idea is the same: Look at the tangent equation as a way to find Δz, instead of an equation for z. Here is Example 6 with new letters.

EXAMPLE 7 From $F = -x^2 - y^2 + z^2 = 0$ find a linear approximation to Δz.

Solution (implicit derivatives) Use the derivatives of F: $-2x\Delta x - 2y\Delta y + 2z\Delta z \approx 0$. Then solve for Δz, which gives $\Delta z \approx (x/z)\Delta x + (y/z)\Delta y$—the same as Example 6.

EXAMPLE 8 How does the equilibrium price change when the supply curve changes?

The equilibrium price is at the intersection of the supply and demand curves (***supply = demand***). As the price p rises, the demand q drops (the slope is $-.2$):

$$\text{demand line } DD\text{: } p = -.2q + 40. \tag{15}$$

The supply (also q) goes *up* with the price. The slope s is positive (here $s = .4$):

$$\text{supply line } SS\text{: } p = sq + t = .4q + 10.$$

Those lines are in Figure 13.10. They meet at the ***equilibrium price*** $P = \$30$. The quantity $Q = 50$ is available at P (on SS) and demanded at P (on DD). So it is sold.

Where do partial derivatives come in? The reality is that those lines DD and SS are not fixed for all time. Technology changes, and competition changes, and the value of money changes. Therefore the lines move. Therefore the crossing point (Q, P) also moves. Please recognize that derivatives are hiding in those sentences.

Main point: ***The equilibrium price P is a function of*** s ***and*** t. Reducing s by better technology lowers the supply line to $p = .3q + 10$. The demand line has not changed. The customer is as eager or stingy as ever. But the price P and quantity Q are different. The new equilibrium is at $Q = 60$ and $P = \$28$, where the new line XX crosses DD.

If the technology is expensive, the supplier will raise t when reducing s. Line YY is $p = .3q + 20$. That gives a higher equilibrium $P = \$32$ at a lower quantity $Q = 40$—the demand was too weak for the technology.

Calculus question Find $\partial P/\partial s$ and $\partial P/\partial t$. The difficulty is that P is not given as a function of s and t. So take implicit derivatives of the supply = demand equations:

$$\begin{aligned} \text{supply = demand:} \quad & P = -.2Q + 40 = sQ + t && (16)\\ s \text{ derivative:} \quad & P_s = -.2Q_s = sQ_s + Q \quad (\text{note } t_s = 0) \\ t \text{ derivative:} \quad & P_t = -.2Q_t = sQ_t + 1 \quad (\text{note } t_t = 1) \end{aligned}$$

Now substitute $s = .4$, $t = 10$, $P = 30$, $Q = 50$. That is the starting point, around which we are finding a linear approximation. The last two equations give $P_s = 50/3$ and $P_t = 1/3$ (Problem 25). The linear approximation is

$$P = 30 + 50(s - .4)/3 + (t - 10)/3. \tag{17}$$

Comment This example turned out to be subtle (so is economics). I hesitated before including it. The equations are linear and their derivatives are easy, but something in the problem is hard—there is no explicit formula for P. The function $P(s, t)$ is not known. Instead of a point on a surface, we are following the intersection of two lines. *The solution changes as the equation changes.* ***The derivative of the solution comes from the derivative of the equation.***

Summary The foundation of this section is equation (1) for the tangent plane. Everything builds on that—total differential, linear approximation, sensitivity to small change. Later sections go on to the chain rule and "directional derivatives" and "gradients." The central idea of differential calculus is $\Delta f \approx f_x \Delta x + f_y \Delta y$.

NEWTON'S METHOD FOR TWO EQUATIONS

Linear approximation is used ***to solve equations***. To find out where a function is zero, look first to see where its approximation is zero. To find out where a graph crosses the xy plane, look to see where its tangent plane crosses.

Remember Newton's method for $f(x) = 0$. The current guess is x_n. Around that point, $f(x)$ is close to $f(x_n) + (x - x_n)f'(x_n)$. This is zero at the next guess $x_{n+1} = x_n - f(x_n)/f'(x_n)$. That is where the tangent line crosses the x axis.

With two variables the idea is the same— but two unknowns x and y require *two equations*. We solve $g(x, y) = 0$ and $h(x, y) = 0$. Both functions have linear approximations that start from the current point (x_n, y_n)—where derivatives are computed:

$$\begin{aligned} g(x, y) &\approx g(x_n, y_n) + (\partial g/\partial x)(x - x_n) + (\partial g/\partial y)(y - y_n) \\ h(x, y) &\approx h(x_n, y_n) + (\partial h/\partial x)(x - x_n) + (\partial h/\partial y)(y - y_n). \end{aligned} \tag{18}$$

The natural idea is to ***set these approximations to zero***. That gives linear equations for $x - x_n$ and $y - y_n$. Those are the steps Δx and Δy that take us to the next guess in Newton's method:

> **13D** Newton's method to solve $g(x, y) = 0$ and $h(x, y) = 0$ has linear equations for the steps Δx and Δy that go from (x_n, y_n) to (x_{n+1}, y_{n+1}):
>
> $$\left(\frac{\partial g}{\partial x}\right)\Delta x + \left(\frac{\partial g}{\partial y}\right)\Delta y = -g(x_n, y_n) \quad \text{and} \quad \left(\frac{\partial h}{\partial x}\right)\Delta x + \left(\frac{\partial h}{\partial y}\right)\Delta y = -h(x_n, y_n). \qquad (19)$$

EXAMPLE 9 $g = x^3 - y = 0$ and $h = y^3 - x = 0$ have 3 solutions $(1, 1), (0, 0), (-1, -1)$.

I will start at different points (x_0, y_0). The next guess is $x_1 = x_0 + \Delta x$, $y_1 = y_0 + \Delta y$. It is of extreme interest to know which solution Newton's method will choose—if it converges at all. I made three small experiments.

1. Suppose $(x_0, y_0) = (2, 1)$. At that point $g = 2^3 - 1 = 7$ and $h = 1^3 - 2 = -1$. The derivatives are $g_x = 3x^2 = 12$, $g_y = -1$, $h_x = -1$, $h_y = 3y^2 = 3$. The steps Δx and Δy come from solving (19):

$$\begin{array}{l} 12\Delta x - \Delta y = -7 \\ -\Delta x + 3\Delta y = +1 \end{array} \Rightarrow \begin{array}{l} \Delta x = -4/7 \\ \Delta y = +1/7 \end{array} \Rightarrow \begin{array}{l} x_1 = x_0 + \Delta x = 10/7 \\ y_1 = y_0 + \Delta y = 8/7. \end{array}$$

This new point (10/7, 8/7) is closer to the solution at (1, 1). The next point is (1.1, 1.05) and convergence is clear. Soon convergence is fast.

2. Start at $(x_0, y_0) = (\frac{1}{2}, 0)$. There we find $g = 1/8$ and $h = -1/2$:

$$\begin{array}{l} (3/4)\Delta x - \Delta y = -1/8 \\ -\Delta x + 0\Delta y = +1/2 \end{array} \Rightarrow \begin{array}{l} \Delta x = -1/2 \\ \Delta y = +1/4 \end{array} \Rightarrow \begin{array}{l} x_1 = x_0 + \Delta x = 0 \\ y_1 = y_0 + \Delta y = -1/4. \end{array}$$

Newton has jumped from $(\frac{1}{2}, 0)$ on the x axis to $(0, -\frac{1}{4})$ on the y axis. The next step goes to (1/32, 0), back on the x axis. We are in the "basin of attraction" of (0, 0).

3. Now start further out the axis at (1, 0), where $g = 1$ and $h = -1$:

$$\begin{array}{l} 3\Delta x - \Delta y = -1 \\ -\Delta x + 0\Delta y = +1 \end{array} \Rightarrow \begin{array}{l} \Delta x = -1 \\ \Delta y = -2 \end{array} \Rightarrow \begin{array}{l} x_1 = x_0 + \Delta x = 0 \\ y_1 = y_0 + \Delta y = -2. \end{array}$$

Newton moves from (1, 0) to (0, −2) to (16, 0). Convergence breaks down—the method blows up. This danger is ever-present, when we start far from a solution.

Please recognize that even a small computer will uncover amazing patterns. It can start from hundreds of points (x_0, y_0), and follow Newton's method. Each solution has a ***basin of attraction***, containing all (x_0, y_0) leading to that solution. There is also a basin leading to infinity. The basins in Figure 13.11 are completely mixed together—a color figure shows them as ***fractals***. The most extreme behavior is on the borderline between basins, when Newton can't decide which way to go. Frequently we see chaos.

Chaos is irregular movement that follows a definite rule. Newton's method determines an *iteration* from each point (x_n, y_n) to the next. In scientific problems it normally converges to the solution we want. (We start close enough.) But the computer makes it posible to study iterations from faraway points. This has created a new part of mathematics—so new that any experiments you do are likely to be original.

Section 3.7 found chaos when trying to solve $x^2+1=0$. But don't think Newton's method is a failure. On the contrary, it is the best method to solve nonlinear equations. The error is squared as the algorithm converges, because linear approximations have errors of order $(\Delta x)^2+(\Delta y)^2$. Each step doubles the number of correct digits, *near the solution.* The example shows why it is important to be near.

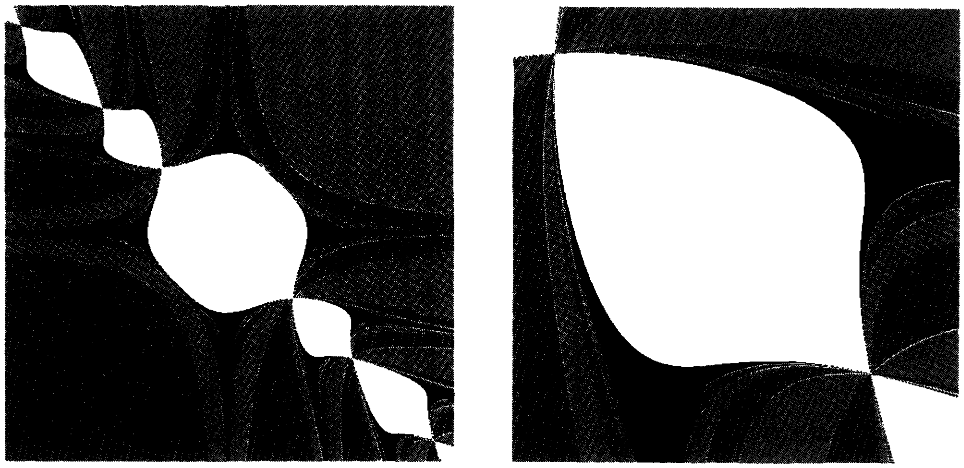


Fig. 13.11 The basins of attraction to (1, 1), (0, 0), (−1, −1), and infinity.

13.3 EXERCISES

Read-through questions

The tangent line to $y=f(x)$ is $y-y_0=$ ____a____. The tangent plane to $w=f(x,y)$ is $w-w_0=$ ____b____. The normal vector is $\mathbf{N}=$ ____c____. For $w=x^3+y^3$ the tangent equation at (1, 1, 2) is ____d____. The normal vector is $\mathbf{N}=$ ____e____. For a sphere, the direction of $\mathbf{N}$ is ____f____.

The surface given implicitly by $F(x,y,z)=c$ has tangent equation $(\partial F/\partial x)_0(x-x_0)+$ ____g____. For $xyz=6$ at (1, 2, 3) the tangent plane is ____h____. On that plane the differentials satisfy ____i____ $dx+$ ____j____ $dy+$ ____k____ $dz=0$. The differential of $z=f(x,y)$ is $dz=$ ____l____. This holds exactly on the tangent plane, while $\Delta z\approx$ ____m____ holds approximately on the ____n____. The height $z=3x+7y$ is more sensitive to a change in ____o____ than in x, because the partial derivative ____p____ is larger than ____q____.

The linear approximation to $f(x,y)$ is $f(x_0,y_0)+$ ____r____. This is the same as $\Delta f\approx$ ____s____ $\Delta x+$ ____t____ Δy. The error is of order ____u____. For $f=\sin xy$ the linear approximation around (0, 0) is $f_L=$ ____v____. We are moving along the ____w____ instead of the ____x____. When the equation is given as $F(x,y,z)=c$, the linear approximation is ____y____ $\Delta x+$ ____z____ $\Delta y+$ ____A____ $\Delta z=0$.

Newton's method solves $g(x,y)=0$ and $h(x,y)=0$ by a ____B____ approximation. Starting from x_n, y_n the equations are replaced by ____C____ and ____D____. The steps Δx and Δy go to the next point ____E____. Each solution has a basin of ____F____. Those basins are likely to be ____G____.

In 1–8 find the tangent plane and the normal vector at P.

1 $z=\sqrt{x^2+y^2}$, $P=(0,1,1)$

2 $x+y+z=17$, $P=(3,4,10)$

3 $z=x/y$, $P=(6,3,2)$

4 $z=e^{x+2y}$, $P=(0,0,1)$

5 $x^2+y^2+z^2=6$, $P=(1,2,1)$

6 $x^2+y^2+2z^2=7$, $P=(1,2,1)$

7 $z=x^y$, $P=(1,1,1)$

8 $V=\pi r^2h$, $P=(2,2,8\pi)$.

9 Show that the tangent plane to $z^2-x^2-y^2=0$ goes through the origin and makes a 45° angle with the z axis.

10 The planes $z=x+4y$ and $z=2x+3y$ meet at (1, 1, 5). The whole line of intersection is $(x,y,z)=(1,1,5)+\mathbf{v}t$. Find $\mathbf{v}=\mathbf{N}_1\times\mathbf{N}_2$.

11 If $z=3x-2y$ find dz from dx and dy. If $z=x^3/y^2$ find dz from dx and dy at $x_0=1$, $y_0=1$. If x moves to 1.02 and y moves to 1.03, find the approximate dz and exact Δz for both functions. The first surface is the ________ to the second surface.

12 The surfaces $z = x^2 + 4y$ and $z = 2x + 3y^2$ meet at $(1, 1, 5)$. Find the normals $\mathbf{N}_1$ and $\mathbf{N}_2$ and also $\mathbf{v} = \mathbf{N}_1 \times \mathbf{N}_2$. The line in this direction $\mathbf{v}$ is tangent to what curve?

13 The normal $\mathbf{N}$ to the surface $F(x, y, z) = 0$ has components F_x, F_y, F_z. The *normal line* has $x = x_0 + F_x t$, $y = y_0 + F_y t$, $z =$ ________. For the surface $xyz - 24 = 0$, find the tangent plane and normal line at $(4, 2, 3)$.

14 For the surface $x^2y^2 - z = 0$, the normal line at $(1, 2, 4)$ has $x =$ ________, $y =$ ________, $z =$ ________.

15 For the sphere $x^2 + y^2 + z^2 = 9$, find the equation of the tangent plane through $(2, 1, 2)$. Also find the equation of the normal line and show that it goes through $(0, 0, 0)$.

16 If the normal line at every point on $F(x, y, z) = 0$ goes through $(0, 0, 0)$, show that $F_x = cx$, $F_y = cy$, $F_z = cz$. The surface must be a sphere.

17 For $w = xy$ near (x_0, y_0), the linear approximation is $dw =$ ________. This looks like the ________ rule for derivatives. The difference between $\Delta w = xy - x_0y_0$ and this approximation is ________.

18 If $f = xyz$ (3 independent variables) what is df?

19 You invest $P = \$4000$ at $R = 8\%$ to make $I = \$320$ per year. If the numbers change by dP and dR what is dI? If the rate drops by $dR = .002$ (to 7.8%) what change dP keeps $dI = 0$? Find the exact interest I after those changes in R and P.

20 Resistances R_1 and R_2 have parallel resistance R, where $1/R = 1/R_1 + 1/R_2$. Is R more sensitive to ΔR_1 or ΔR_2 if $R_1 = 1$ and $R_2 = 2$?

21 (a) If your batting average is $A = (25 \text{ hits})/(100 \text{ at bats}) = .250$, compute the increase (to 26/101) with a hit and the decrease (to 25/101) with an out.

(b) If $A = x/y$ then $dA =$ ________ $dx +$ ________ dy. A hit $(dx = dy = 1)$ gives $dA = (1 - A)/y$. An out $(dy = 1)$ gives $dA = -A/y$. So at $A = .250$ a hit has ________ times the effect of an out.

22 (a) 2 hits and 3 outs $(dx = 2, dy = 5)$ will raise your average $(dA > 0)$ provided A is less than ________.

(b) A player batting $A = .500$ with $y = 400$ at bats needs $dx =$ ________ hits to raise his average to .505.

23 If x and y change by Δx and Δy, find the approximate change $\Delta\theta$ in the angle $\theta = \tan^{-1}(y/x)$.

24 The *Fundamental Lemma* behind equation (13) writes $\Delta f = a\Delta x + b\Delta y$. The Lemma says that $a \to f_x(x_0, y_0)$ and $b \to f_y(x_0, y_0)$ when $\Delta x \to 0$ and $\Delta y \to 0$. The proof takes Δx first and then Δy:

(1) $f(x_0 + \Delta x, y_0) - f(x_0, y_0) = \Delta x f_x(c, y_0)$ where c is between ________ and ________ (by which theorem?)

(2) $f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 + \Delta x, y_0) = \Delta y f_y(x_0 + \Delta x, C)$ where C is between ________ and ________.

(3) $a = f_x(c, y_0) \to f_x(x_0, y_0)$ provided f_x is ________.

(4) $b = f_y(x_0 + \Delta x, C) \to f_y(x_0, y_0)$ provided f_y is ________.

25 If the supplier reduces s, Figure 13.10 shows that P decreases and Q ________.

(a) Find $P_s = 50/3$ and $P_t = 1/3$ in the economics equation (17) by solving the equations above it for Q_s and Q_t.

(b) What is the linear approximation to Q around $s = .4$, $t = 10$, $P = 30$, $Q = 50$?

26 Solve the equations $P = -.2Q + 40$ and $P = sQ + t$ for P and Q. Then find $\partial P/\partial s$ and $\partial P/\partial t$ explicitly. At the same s, t, P, Q check 50/3 and 1/3.

27 If the supply = demand equation (16) changes to $P = sQ + t = -Q + 50$, find P_s and P_t at $s = 1$, $t = 10$.

28 To find out how the roots of $x^2 + bx + c = 0$ vary with b, take partial derivatives of the equation with respect to ________. Compare $\partial x/\partial b$ with $\partial x/\partial c$ to show that a root at $x = 2$ is more sensitive to b.

29 Find the tangent planes to $z = xy$ and $z = x^2 - y^2$ at $x = 2$, $y = 1$. Find the Newton point where those planes meet the xy plane (set $z = 0$ in the tangent equations).

30 (a) To solve $g(x, y) = 0$ and $h(x, y) = 0$ is to find the meeting point of three surfaces: $z = g(x, y)$ and $z = h(x, y)$ and ________.

(b) Newton finds the meeting point of three planes: the tangent plane to the graph of g, ________, and ________.

Problems 31–36 go further with Newton's method for $g = x^3 - y$ and $h = y^3 - x$. This is Example 9 with solutions $(1, 1)$, $(0, 0)$, $(-1, -1)$.

31 Start from $x_0 = 1$, $y_0 = 1$ and find Δx and Δy. Where are x_1 and y_1, and what line is Newton's method moving on?

32 Start from $(\frac{1}{2}, \frac{1}{2})$ and find the next point. This is in the basin of attraction of which solution?

33 Starting from $(a, -a)$ find Δy which is also $-\Delta x$. Newton goes toward $(0, 0)$. But can you find the sharp point in Figure 13.11 where the lemon meets the spade?

34 Starting from $(a, 0)$ show that Newton's method goes to $(0, -2a^3)$ and find the next point (x_2, y_2). Which numbers a lead to convergence? Which special number a leads to a cycle, in which (x_2, y_2) is the same as the starting point $(a, 0)$?

35 Show that $x^3 = y$, $y^3 = x$ has exactly three solutions.

36 Locate a point from which Newton's method diverges.

37 Apply Newton's method to a linear problem: $g = x + 2y - 5 = 0$, $h = 3x - 3 = 0$. From any starting point show that (x_1, y_1) is the exact solution (convergence in *one step*).

38 The complex equation $(x+iy)^3=1$ contains two real equations, $x^3-3xy^2=1$ from the real part and $3x^2y-y^3=0$ from the imaginary part. Search by computer for the basins of attraction of the three solutions $(1, 0)$, $(-1/2, \sqrt{3}/2)$, and $(-1/2, -\sqrt{3}/2)$—which give the cube roots of 1.

39 In Newton's method the new guess comes from (x_n, y_n) by an *iteration*: $x_{n+1}=G(x_n, y_n)$ and $y_{n+1}=H(x_n, y_n)$. What are G and H for $g=x^2-y=0$, $h=x-y=0$? First find Δx and Δy; then $x_n+\Delta x$ gives G and $y_n+\Delta y$ gives H.

40 In Problem 39 find the basins of attraction of the solution $(0, 0)$ and $(1, 1)$.

41 The matrix in Newton's method is the ***Jacobian***:

$$J=\begin{bmatrix}\partial g/\partial x & \partial g/\partial y\\ \partial h/\partial x & \partial h/\partial y\end{bmatrix} \quad\text{and}\quad J\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix}=\begin{bmatrix}-g_n\\ -h_n\end{bmatrix}.$$

Find J and Δx and Δy for $g=e^x-1$, $h=e^y+x$.

42 Find the Jacobian matrix at $(1, 1)$ when $g=x^2+y^2$ and $h=xy$. This matrix is ________ and Newton's method fails. The graphs of g and h have ________ tangent planes.

43 Solve $g=x^2-y^2+1=0$ and $h=2xy=0$ by Newton's method from three starting points: $(0, 2)$ and $(-1, 1)$ and $(2, 0)$. Take ten steps by computer or one by hand. The solution $(0, 1)$ attracts when $y_0>0$. If $y_0=0$ you should find the chaos iteration $x_{n+1}=\frac{1}{2}(x_n-x_n^{-1})$.

13.4 Directional Derivatives and Gradients

As x changes, we know how $f(x, y)$ changes. The partial derivative $\partial f/\partial x$ treats y as constant. Similarly $\partial f/\partial y$ keeps x constant, and gives the slope in the y direction. But east-west and north-south are not the only directions to move. We could go along a 45° line, where $\Delta x=\Delta y$. In principle, before we draw axes, no direction is preferred. The graph is a surface with slopes in *all* directions.

On that surface, calculus looks for the rate of change (or the slope). There is a ***directional derivative***, whatever the direction. In the 45° case we are inclined to divide Δf by Δx, but we would be wrong.

Let me state the problem. We are given $f(x, y)$ around a point $P=(x_0, y_0)$. We are also given a direction $\mathbf{u}$ (a unit vector). There must be a natural definition of $D_{\mathbf{u}}f$—***the derivative of f in the direction*** $\mathbf{u}$. To compute this slope at P, we need a formula. Preferably the formula is based on $\partial f/\partial x$ and $\partial f/\partial y$, which we already know.

Note that the 45° direction has $\mathbf{u}=\mathbf{i}/\sqrt{2}+\mathbf{j}/\sqrt{2}$. The square root of 2 is going to enter the derivative. This shows that dividing Δf by Δx is wrong. We should divide by the step length Δs.

EXAMPLE 1 Stay on the surface $z=xy$. When (x, y) moves a distance Δs in the 45° direction from $(1, 1)$, what is $\Delta z/\Delta s$?

Solution The step is Δs times the unit vector $\mathbf{u}$. Starting from $x=y=1$ the step ends at $x=y=1+\Delta s/\sqrt{2}$. (The components of $\mathbf{u}\Delta s$ are $\Delta s/\sqrt{2}$.) Then $z=xy$ is

$$z=(1+\Delta s/\sqrt{2})^2=1+\sqrt{2}\Delta s+\tfrac{1}{2}(\Delta s)^2, \text{ which means } \Delta z=\sqrt{2}\Delta s+\tfrac{1}{2}(\Delta s)^2.$$

The ratio $\Delta z/\Delta s$ approaches $\sqrt{2}$ as $\Delta s\to 0$. That is the slope in the 45° direction.

DEFINITION *The derivative of f in the direction* $\mathbf{u}$ *at the point P is* $D_{\mathbf{u}}f(P)$:

$$D_{\mathbf{u}}f(P)=\lim_{\Delta s\to 0}\frac{\Delta f}{\Delta s}=\lim_{\Delta s\to 0}\frac{f(P+\mathbf{u}\Delta s)-f(P)}{\Delta s}. \tag{1}$$

The step from $P=(x_0, y_0)$ has length Δs. It takes us to $(x_0+u_1\Delta s, y_0+u_2\Delta s)$. We compute the change Δf and divide by Δs. But formula (2) below saves time.

The x direction is $\mathbf{u}=(1,0)$. Then $\mathbf{u}\Delta s$ is $(\Delta s, 0)$ and we recover $\partial f/\partial x$:

$$\frac{\Delta f}{\Delta s}=\frac{f(x_0+\Delta s, y_0)-f(x_0,y_0)}{\Delta s} \text{ approaches } D_{(1,0)}f=\frac{\partial f}{\partial x}.$$

Similarly $D_{\mathbf{u}}f=\partial f/\partial y$, when $\mathbf{u}=(0,1)$ is in the y direction. *What is $D_{\mathbf{u}}f$ when $\mathbf{u}=(0,-1)$?* That is the negative y direction, so $D_{\mathbf{u}}f=-\partial f/\partial y$.

CALCULATING THE DIRECTIONAL DERIVATIVE

$D_{\mathbf{u}}f$ is the slope of the surface $z=f(x,y)$ in the direction $\mathbf{u}$. How do you compute it? From $\partial f/\partial x$ and $\partial f/\partial y$, in two special directions, there is a quick way to find $D_{\mathbf{u}}f$ in all directions. **Remember that u is a unit vector.**

13E The ***directional derivative*** $D_{\mathbf{u}}f$ in the direction $\mathbf{u}=(u_1,u_2)$ equals

$$D_{\mathbf{u}}f=\frac{\partial f}{\partial x}u_1+\frac{\partial f}{\partial y}u_2. \tag{2}$$

The reasoning goes back to the linear approximation of Δf:

$$\Delta f\approx\frac{\partial f}{\partial x}\Delta x+\frac{\partial f}{\partial y}\Delta y=\frac{\partial f}{\partial x}u_1\Delta s+\frac{\partial f}{\partial y}u_2\Delta s.$$

Divide by Δs and let Δs approach zero. Formula (2) is the limit of $\Delta f/\Delta s$, as the approximation becomes exact. A more careful argument guarantees this limit provided f_x and f_y are continuous at the basepoint (x_0, y_0).

Main point: ***Slopes in all directions are known from slopes in two directions.***

EXAMPLE 1 (repeated) $f=xy$ and $P=(1,1)$ and $\mathbf{u}=(1/\sqrt{2}, 1/\sqrt{2})$. Find $D_{\mathbf{u}}f(P)$.

The derivatives $f_x=y$ and $f_y=x$ equal 1 at P. The 45° derivative is

$$D_{\mathbf{u}}f(P)=f_xu_1+f_yu_2=1(1/\sqrt{2})+1(1/\sqrt{2})=\sqrt{2} \text{ as before.}$$

EXAMPLE 2 The linear function $f=3x+y+1$ has slope $D_{\mathbf{u}}f=3u_1+u_2$.

The x direction is $u=(1,0)$, and $D_{\mathbf{u}}f=3$. That is $\partial f/\partial x$. In the y direction $D_{\mathbf{u}}f=1$. ***Two other directions are special—along the level lines of*** $f(x,y)$ ***and perpendicular***:

Level direction: $D_{\mathbf{u}}f$ is zero because f is constant

Steepest direction: $D_{\mathbf{u}}f$ is as large as possible (with $u_1^2+u_2^2=1$).

To find those directions, look at $D_{\mathbf{u}}f=3u_1+u_2$. The level direction has $3u_1+u_2=0$. Then u is proportional to $(1,-3)$. Changing x by 1 and y by -3 produces no change in $f=3x+y+1$.

In the steepest direction $\mathbf{u}$ is proportional to (3, 1). Note the partial derivatives $f_x=3$ and $f_y=1$. The dot product of (3, 1) and $(1,-3)$ is zero—*steepest direction is perpendicular to level direction.* To make (3, 1) a unit vector, divide by $\sqrt{10}$.

Steepest climb: $D_{\mathbf{u}}f=3(3/\sqrt{10})+1(1/\sqrt{10})=10/\sqrt{10}=\sqrt{10}$

Steepest descent: Reverse to $\mathbf{u}=(-3/\sqrt{10}, -1/\sqrt{10})$ and $D_{\mathbf{u}}f=-\sqrt{10}$.

The contour lines around a mountain follow $D_{\mathbf{u}}f=0$. The creeks are perpendicular. On a plane like $f=3x+y+1$, those directions stay the same at all points (Figure 13.12). On a mountain the steepest direction changes as the slopes change.

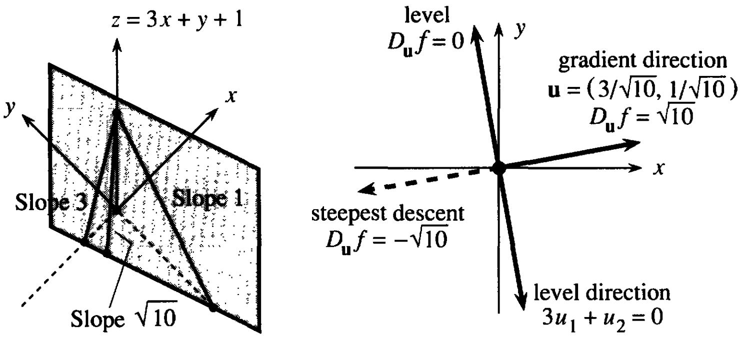


Fig. 13.12 Steepest direction is along the gradient. Level direction is perpendicular.

THE GRADIENT VECTOR

Look again at $f_x u_1 + f_y u_2$, which is the directional derivative $D_u f$. *This is the dot product of two vectors.* One vector is $\mathbf{u} = (u_1, u_2)$, which sets the direction. The other vector is (f_x, f_y), which comes from the function. This second vector is the ***gradient.***

DEFINITION The ***gradient*** of $f(x, y)$ is the vector whose components are $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$:

$$\text{grad}\, f = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} \qquad \left(\text{add } \frac{\partial f}{\partial z}\mathbf{k} \text{ in three dimensions}\right).$$

The space-saving symbol ∇ is read as "grad." In Chapter 15 it becomes "del."

For the linear function $3x + y + 1$, the gradient is the constant vector (3, 1). It is the way to climb the plane. For the nonlinear function $x^2 + xy$, the gradient is the non-constant vector $(2x + y, x)$. Notice that grad f shares the two derivatives in $\mathbf{N} = (f_x, f_y, -1)$. But the gradient is not the normal vector. $\mathbf{N}$ is in three dimensions, pointing away from the surface $z = f(x, y)$. ***The gradient vector is in the** xy **plane**!* The gradient tells which way on the surface is up, but it does that from down in the base.

The level curve is also in the xy plane, perpendicular to the gradient. The contour map is a projection on the base plane of what the hiker sees on the mountain. The vector grad f tells the ***direction*** of climb, and its length $|\text{grad}\, f|$ gives the ***steepness.***

> **13F** The directional derivative is $D_{\mathbf{u}}f = (\text{grad}\, f) \cdot \mathbf{u}$. The level direction is perpendicular to grad f, since $D_{\mathbf{u}}f = 0$. ***The slope** $D_{\mathbf{u}}f$ **is largest when** $\mathbf{u}$ **is parallel to*** grad f. That maximum slope is the length $|\text{grad}\, f| = \sqrt{f_x^2 + f_y^2}$:
>
> $$\textit{for} \quad \mathbf{u} = \frac{\text{grad}\, f}{|\text{grad}\, f|} \quad \textit{the slope is} \quad (\text{grad}\, f) \cdot \mathbf{u} = \frac{|\text{grad}\, f|^2}{|\text{grad}\, f|} = |\text{grad}\, f|.$$

The example $f = 3x + y + 1$ had grad $f = (3, 1)$. Its steepest slope was in the direction $\mathbf{u} = (3, 1)/\sqrt{10}$. The maximum slope was $\sqrt{10}$. That is $|\text{grad}\, f| = \sqrt{9 + 1}$.

Important point: ***The maximum of*** $(\text{grad}\, f) \cdot \mathbf{u}$ ***is the length*** $|\text{grad}\, f|$. In nonlinear examples, the gradient and steepest direction and slope will vary. But look at one particular point in Figure 13.13. Near that point, and near any point, the linear picture takes over.

On the graph of f, the special vectors are the level direction $\mathbf{L} = (f_y, -f_x, 0)$ and the uphill direction $\mathbf{U} = (f_x, f_y, f_x^2 + f_y^2)$ and the normal $\mathbf{N} = (f_x, f_y, -1)$. Problem 18 checks that those are perpendicular.

EXAMPLE 3 The gradient of $f(x, y) = (14 - x^2 - y^2)/3$ is $\nabla f = (-2x/3, -2y/3)$.

On the surface, the normal vector is $\mathbf{N} = (-2x/3, -2y/3, -1)$. At the point $(1, 2, 3)$, this perpendicular is $\mathbf{N} = (-2/3, -4/3, -1)$. At the point $(1, 2)$ down in the base, the gradient is $(-2/3, -4/3)$. The length of grad f is the slope $\sqrt{20}/3$.

Probably a hiker does not go straight up. A "grade" of $\sqrt{20}/3$ is fairly steep (almost 150%). To estimate the slope in other directions, measure the distance along the path between two contour lines. If $\Delta f = 1$ in a distance $\Delta s = 3$ the slope is about $1/3$. This calculation is not exact until the limit of $\Delta f/\Delta s$, which is $D_{\mathbf{u}}f$.

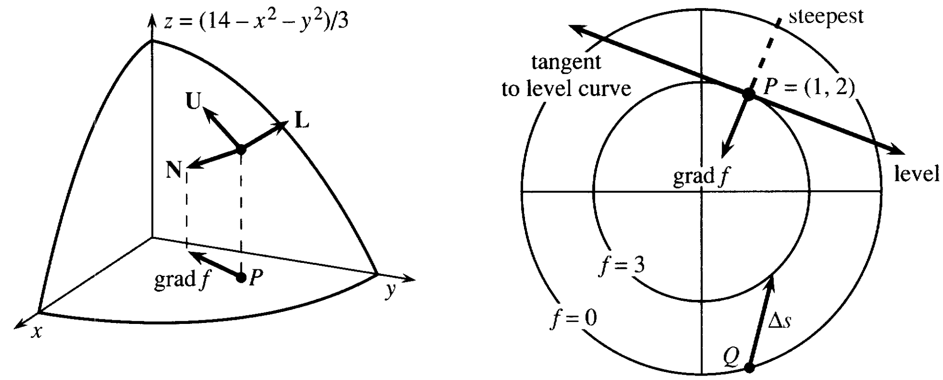


Fig. 13.13 N perpendicular to surface and grad f perpendicular to level line (in the base).

EXAMPLE 4 The gradient of $f(x, y, z) = xy + yz + xz$ has three components.

The pattern extends from $f(x, y)$ to $f(x, y, z)$. The gradient is now the three-dimensional vector (f_x, f_y, f_z). For this function grad f is $(y + z, x + z, x + y)$. To draw the graph of $w = f(x, y, z)$ would require a four-dimensional picture, with axes in the $xyzw$ directions.

Notice the dimensions. The graph is a 3-dimensional "surface" in 4-dimensional space. The gradient is down below in the 3-dimensional base. The level sets of f come from $xy + yz + zx = c$—they are 2-dimensional. The gradient is perpendicular to that level set (still down in 3 dimensions). The gradient is not **N**! The normal vector is $(f_x, f_y, f_z, -1)$, perpendicular to the surface up in 4-dimensional space.

EXAMPLE 5 Find grad z when $z(x, y)$ is given implicitly: $F(x, y, z) = x^2 + y^2 - z^2 = 0$.

In this case we find $z = \pm\sqrt{x^2 + y^2}$. The derivatives are $\pm x/\sqrt{x^2 + y^2}$ and $\pm y/\sqrt{x^2 + y^2}$, which go into grad z. But the point is this: To find that gradient faster, differentiate $F(x, y, z)$ as it stands. Then divide by F_z:

$$F_x dx + F_y dy + F_z dz = 0 \quad \text{or} \quad dz = (-F_x dx - F_y dy)/F_z. \tag{3}$$

The gradient is $(-F_x/F_z, -F_y/F_z)$. Those derivatives are evaluated at (x_0, y_0). The computation does not need the explicit function $z = f(x, y)$:

$$F = x^2 + y^2 - z^2 \Rightarrow F_x = 2x,\ F_y = 2y,\ F_z = -2z \Rightarrow \text{grad } z = (x/z, y/z).$$

To go uphill on the cone, move in the direction $(x/z, y/z)$. That gradient direction goes radially outward. The steepness of the cone is the length of the gradient vector:

$$|\text{grad } z| = \sqrt{(x/z)^2 + (y/z)^2} = 1 \text{ because } z^2 = x^2 + y^2 \text{ on the cone.}$$

DERIVATIVES ALONG CURVED PATHS

On a straight path the derivative of f is $D_{\mathbf{u}}f = (\text{grad} f) \cdot \mathbf{u}$. What is the derivative on a curved path? ***The path direction* u *is the tangent vector* T**. So replace **u** by **T**, which gives the "direction" of the curve.

The path is given by the position vector $\mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. The velocity is $\mathbf{v} = (dx/dt)\mathbf{i} + (dy/dt)\mathbf{j}$. The tangent vector is $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$. Notice the choice—to move at any speed (with $\mathbf{v}$) or to go at unit speed (with $\mathbf{T}$). There is the same choice for the derivative of $f(x, y)$ along this curve:

$$\textbf{\textit{rate of change}} \quad \frac{df}{dt} = (\text{grad } f) \cdot \mathbf{v} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \tag{4}$$

$$\textbf{\textit{slope}} \quad \frac{df}{ds} = (\text{grad } f) \cdot \mathbf{T} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} \tag{5}$$

The first involves *time*. If we move faster, df/dt increases. The second involves *distance*. If we move a distance ds, at any speed, the function changes by df. So the slope in that direction is df/ds. Chapter 1 introduced velocity as df/dt and slope as dy/dx and mixed them up. Finally we see the difference.

Uniform motion on a straight line has $\mathbf{R} = \mathbf{R}_0 + \mathbf{v}t$. The velocity $\mathbf{v}$ is constant. The direction $\mathbf{T} = \mathbf{u} = \mathbf{v}/|\mathbf{v}|$ is also constant. The directional derivative is $(\text{grad} f) \cdot \mathbf{u}$, but the rate of change is $(\text{grad} f) \cdot \mathbf{v}$.

Equations (4) and (5) look like chain rules. They are chain rules. The next section extends $df/dt = (df/dx)(dx/dt)$ to more variables, proving (4) and (5). Here we focus on the meaning: ***df/ds is the derivative of f in the direction* $\mathbf{u} = \mathbf{T}$ *along the curve.***

EXAMPLE 7 Find df/dt and df/ds for $f = r$. The curve is $x = t^2$, $y = t$ in Figure 13.14a.

Solution The velocity along the curve is $\mathbf{v} = 2t\mathbf{i} + \mathbf{j}$. At the typical point $t = 1$ it is $\mathbf{v} = 2\mathbf{i} + \mathbf{j}$. The unit tangent is $\mathbf{T} = \mathbf{v}/\sqrt{5}$. The gradient is a unit vector $\mathbf{i}/\sqrt{2} + \mathbf{j}/\sqrt{2}$ pointing outward, when $f(x, y)$ is the distance r from the center. The dot product with $\mathbf{v}$ is $df/dt = 3/\sqrt{2}$. The dot product with $\mathbf{T}$ is $df/ds = 3/\sqrt{10}$.

When we slow down to speed 1 (with $\mathbf{T}$), the changes in $f(x, y)$ slow down too.

EXAMPLE 8 Find df/ds for $f = xy$ along the circular path $x = \cos t$, $y = \sin t$.

First take a direct approach. On the circle, xy equals $(\cos t)(\sin t)$. Its derivative comes from the product rule: $df/dt = \cos^2 t - \sin^2 t$. Normally this is different from df/ds, because the time t need not equal the arc length s. There is a speed factor ds/dt to divide by—but here the speed is 1. (A circle of length $s = 2\pi$ is completed at $t = 2\pi$.) Thus the slope df/ds along the roller-coaster in Figure 13.14 is $\cos^2 t - \sin^2 t$.

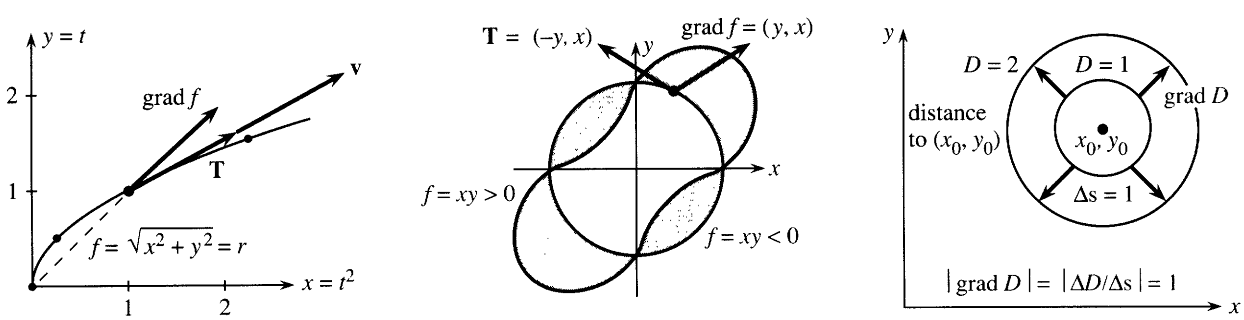


Fig. 13.14 The distance $f = r$ changes along the curve. The slope of the roller-coaster is $(\text{grad} f) \cdot \mathbf{T}$. The distance D from (x_0, y_0) has grad D = unit vector.

The second approach uses the vectors grad f and $\mathbf{T}$. The gradient of $f = xy$ is $(y, x) = (\sin t, \cos t)$. The unit tangent vector to the path is $\mathbf{T} = (-\sin t, \cos t)$. Their dot product is the same df/ds:

$$\textit{slope along path} = (\text{grad } f) \cdot \mathbf{T} = -\sin^2 t + \cos^2 t.$$

GRADIENTS WITHOUT COORDINATES

This section ends with a little "philosophy." What is the *coordinate-free definition* of the gradient? Up to now, grad $f = (f_x, f_y)$ depended totally on the choice of x and y axes. But the steepness of a surface is independent of the axes. Those are added later, to help us compute.

The steepness df/ds involves only f and the direction, nothing else. The gradient should be a "tensor"—its meaning does not depend on the coordinate system. The gradient has different formulas in different systems (xy or $r\theta$ or ...), but the direction and length of grad f are determined by df/ds—without any axes:

The ***direction*** of grad f is the one in which df/ds is largest.
The ***length*** $|\text{grad } f|$ is that largest slope.

The key equation is (***change in*** f) $\approx$ (***gradient of*** f) $\cdot$ (***change in position***). That is another way to write $\Delta f \approx f_x \Delta x + f_y \Delta y$. It is the multivariable form—we used two variables—of the basic linear approximation $\Delta y \approx (dy/dx)\Delta x$.

EXAMPLE 9 $D(x, y) =$ distance from (x, y) to (x_0, y_0). Without derivatives prove $|\text{grad } D| = 1$. The graph of $D(x, y)$ is a cone with slope 1 and sharp point (x_0, y_0).

First question In which direction does the distance $D(x, y)$ increase fastest?
Answer Going directly away from (x_0, y_0). Therefore this is the direction of grad D.

Second question How quickly does D increase in that steepest direction?
Answer A step of length Δs increases D by Δs. Therefore $|\text{grad } D| = \Delta s/\Delta s = 1$.

Conclusion grad D ***is a unit vector***. The derivatives of D in Problem 48 are $(x - x_0)/D$ and $(y - y_0)/D$. The sum of their squares is 1, because $(x - x_0)^2 + (y - y_0)^2$ equals D^2.

13.4 EXERCISES

Read-through questions

$D_\mathbf{u} f$ gives the rate of change of __a__ in the direction __b__. It can be computed from the two derivatives __c__ in the special directions __d__. In terms of u_1, u_2 the formula is $D_\mathbf{u} f =$ __e__. This is a __f__ product of $\mathbf{u}$ with the vector __g__, which is called the __h__. For the linear function $f = ax + by$, the gradient is grad $f =$ __i__ and the directional derivative is $D_\mathbf{u} f =$ __j__ $\cdot$ __k__.

The gradient $\nabla f = (f_x, f_y)$ is not a vector in __l__ dimensions, it is a vector in the __m__. It is perpendicular to the __n__ lines. It points in the direction of __o__ climb. Its magnitude $|\text{grad } f|$ is __p__. For $f = x^2 + y^2$ the gradient points __q__ and the slope in that steepest direction is __r__.

The gradient of $f(x, y, z)$ is __s__. This is different from the gradient on the surface $F(x, y, z) = 0$, which is $-(F_x/F_z)\mathbf{i} +$ __t__. Traveling with velocity $\mathbf{v}$ on a curved path, the rate of change of f is $df/dt =$ __u__. When the tangent direction is $\mathbf{T}$, the slope of f is $df/ds =$ __v__. In a straight direction $\mathbf{u}$, df/ds is the same as __w__.

Compute grad f, then $D_\mathbf{u} f = (\text{grad } f) \cdot \mathbf{u}$, then $D_\mathbf{u} f$ at P.

1 $f(x, y) = x^2 - y^2$	$\mathbf{u} = (\sqrt{3}/2, 1/2)$	$P = (1, 0)$
2 $f(x, y) = 3x + 4y + 7$	$\mathbf{u} = (3/5, 4/5)$	$P = (0, \pi/2)$
3 $f(x, y) = e^x \cos y$	$\mathbf{u} = (0, 1)$	$P = (0, \pi/2)$
4 $f(x, y) = y^{10}$	$\mathbf{u} = (0, -1)$	$P = (1, -1)$

5 $f(x, y)$ = distance to (0, 3) $\mathbf{u}=(1, 0)$ $P=(1, 1)$

Find grad $f=(f_x,f_y,f_z)$ for the functions 6–8 from physics.

6 $1/\sqrt{x^2+y^2+z^2}$ (point source at the origin)

7 $\ln(x^2+y^2)$ (line source along z axis)

8 $1/\sqrt{(x-1)^2+y^2+z^2}-1/\sqrt{(x+1)^2+y^2+z^2}$ (dipole)

9 For $f=3x^2+2y^2$ find the steepest direction and the level direction at (1, 2). Compute $D_{\mathbf{u}}f$ in those directions.

10 Example 2 claimed that $f=3x+y+1$ has steepest slope $\sqrt{10}$. Maximize $D_{\mathbf{u}}f=3u_1+u_2=3u_1+\sqrt{1-u_1^2}$.

11 **True or false**, when $f(x, y)$ is any smooth function:

(a) There is a direction $\mathbf{u}$ at P in which $D_{\mathbf{u}}f=0$.

(b) There is a direction $\mathbf{u}$ in which $D_{\mathbf{u}}f=\text{grad} f$.

(c) There is a direction $\mathbf{u}$ in which $D_{\mathbf{u}}f=1$.

(d) The gradient of $f(x)g(x)$ equals g grad $f+f$ grad g.

12 What is the gradient of $f(x)$? (One component only.) What are the two possible directions $\mathbf{u}$ and the derivatives $D_{\mathbf{u}}f$? What is the normal vector $\mathbf{N}$ to the curve $y=f(x)$? (Two components.)

In 13–16 find the direction u in which f increases fastest at $P=(1, 2)$. How fast?

13 $f(x, y)=ax+by$ **14** $f(x, y)$ = smaller of $2x$ and y

15 $f(x, y)=e^{x-y}$ **16** $f(x, y)=\sqrt{5-x^2-y^2}$ (careful)

17 (Looking ahead) At a point where $f(x, y)$ is a maximum, what is grad f? Describe the level curve containing the maximum point (x, y).

18 (a) Check by dot products that the normal and uphill and level directions on the graph are perpendicular: $\mathbf{N}=(f_x,f_y,-1), \mathbf{U}=(f_x,f_y,f_x^2+f_y^2), \mathbf{L}=(f_y,-f_x,0)$.

(b) $\mathbf{N}$ is _______ to the tangent plane, $\mathbf{U}$ and $\mathbf{L}$ are _______ to the tangent plane.

(c) The gradient is the xy projection of _______ and also of _______. The projection of $\mathbf{L}$ points along the _______.

19 Compute the $\mathbf{N}$, $\mathbf{U}$, $\mathbf{L}$ vectors for $f=1-x+y$ and draw them at a point on the flat surface.

20 Compute $\mathbf{N}$, $\mathbf{U}$, $\mathbf{L}$ for $x^2+y^2-z^2=0$ and draw them at a typical point on the cone.

With gravity in the negative z direction, in what direction $-\mathbf{U}$ will water flow down the roofs 21–24?

21 $z=2x$ (flat roof) **22** $z=4x-3y$ (flat roof)

23 $z=\sqrt{1-x^2-y^2}$ (sphere) **24** $z=-\sqrt{x^2+y^2}$ (cone)

25 Choose two functions $f(x, y)$ that depend only on $x+2y$. Their gradients at (1, 1) are in the direction _______. Their level curves are _______.

26 The level curve of $f=y/x$ through (1, 1) is _______. The direction of the gradient must be _______. Check grad f.

27 Grad f is perpendicular to $2\mathbf{i}+\mathbf{j}$ with length 1, and grad g is parallel to $2\mathbf{i}+\mathbf{j}$ with length 5. Find grad f, grad g, f, and g.

28 **True or false**:

(a) If we know grad f, we know f.

(b) The line $x=y=-z$ is perpendicular to the plane $z=x+y$.

(c) The gradient of $z=x+y$ lies along that line.

29 Write down the level direction $\mathbf{u}$ for $\theta=\tan^{-1}(y/x)$ at the point (3, 4). Then compute grad θ and check $D_{\mathbf{u}}\theta=0$.

30 On a circle around the origin, distance is $\Delta s=r\Delta\theta$. Then $d\theta/ds=1/r$. Verify by computing grad θ and $\mathbf{T}$ and (grad θ)$\cdot\mathbf{T}$.

31 At the point (2, 1, 6) on the mountain $z=9-x-y^2$, which way is up? On the roof $z=x+2y+2$, which way is down? The roof is _______ to the mountain.

32 Around the point $(1, -2)$ the temperature $T=e^{-x^2-y^2}$ has $\Delta T\approx$ _______ $\Delta x+$ _______ Δy. In what direction $\mathbf{u}$ does it get hot fastest?

33 Figure A shows level curves of $z=f(x, y)$.

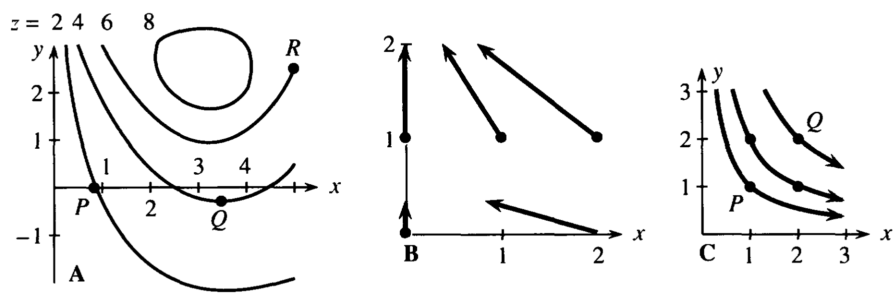
(a) Estimate the direction and length of grad f at P, Q, R.

(b) Locate two points where grad f is parallel to $\mathbf{i}+\mathbf{j}$.

(c) Where is |grad f| largest? Where is it smallest?

(d) What is your estimate of z_{max} on this figure?

(e) On the straight line from P to R, describe z and estimate its maximum.



34 A quadratic function $ax^2 + by^2 + cx + dy$ has the gradients shown in Figure B. Estimate a, b, c, d and sketch two level curves.

35 The level curves of $f(x, y)$ are circles around (1, 1). The curve $f = c$ has radius $2c$. What is f? What is grad f at (0, 0)?

36 Suppose grad f is tangent to the hyperbolas $xy =$ constant in Figure C. Draw three level curves of $f(x, y)$. Is $|\text{grad } f|$ larger at P or Q? Is $|\text{grad } f|$ constant along the hyperbolas? Choose a function that could be f: $x^2 + y^2$, $x^2 - y^2$, xy, x^2y^2.

37 Repeat Problem 36, if grad f is *perpendicular* to the hyperbolas in Figure C.

38 If $f = 0, 1, 2$ at the points (0, 1), (1, 0), (2, 1), estimate grad f by assuming $f = Ax + By + C$.

39 What functions have the following gradients?

(a) $(2x + y, x)$ (b) $(e^{x-y}, -e^{x-y})$ (c) $(y, -x)$ (careful)

40 Draw level curves of $f(x, y)$ if grad $f = (y, x)$.

In 41–46 find the velocity v and the tangent vector T. Then compute the rate of change $df/dt = \text{grad } f \cdot \mathbf{v}$ and the slope $df/ds = \text{grad } f \cdot \mathbf{T}$.

41 $f = x^2 + y^2$ $\quad x = t \quad y = t^2$

42 $f = x$ $\quad x = \cos 2t \quad y = \sin 2t$

43 $f = x^2 - y^2$ $\quad x = x_0 + 2t \quad y = y_0 + 3t$

44 $f = xy$ $\quad x = t^2 + 1 \quad y = 3$

45 $f = \ln xyz$ $\quad x = e^t \quad y = e^{2t} \quad z = e^{-t}$

46 $f = 2x^2 + 3y^2 + z^2$ $\quad x = t \quad y = t^2 \quad z = t^3$

47 (a) Find df/ds and df/dt for the roller-coaster $f = xy$ along the path $x = \cos 2t$, $y = \sin 2t$. (b) Change to $f = x^2 + y^2$ and explain why the slope is zero.

48 The distance D from (x, y) to (1, 2) has $D^2 = (x - 1)^2 + (y - 2)^2$. Show that $\partial D/\partial x = (x - 1)/D$ and $\partial D/\partial y = (y - 2)/D$ and $|\text{grad } D| = 1$. The graph of $D(x, y)$ is a ________ with its vertex at ________.

49 If $f = 1$ and grad $f = (2, 3)$ at the point (4, 5), find the tangent plane at (4, 5). If f is a linear function, find $f(x, y)$.

50 Define the derivative of $f(x, y)$ in the direction $\mathbf{u} = (u_1, u_2)$ at the point $P = (x_0, y_0)$. What is Δf (approximately)? What is $D_{\mathbf{u}} f$ (exactly)?

51 The slope of f along a level curve is $df/ds =$ ________ $= 0$. This says that grad f is perpendicular to the vector ________ in the level direction.

13.5 The Chain Rule

Calculus goes back and forth between solving problems and getting ready for harder problems. The first is "*application*," the second looks like "*theory*." If we minimize f to save time or money or energy, that is an application. If we don't take derivatives to find the minimum—maybe because f is a function of other functions, and we don't have a chain rule—then it is time for more theory. The chain rule is a fundamental working tool, because $f(g(x))$ appears all the time in applications. So do $f(g(x, y))$ and $f(x(t), y(t))$ and worse. We have to know their derivatives. Otherwise calculus can't continue with the applications.

You may instinctively say: Don't bother with the theory, just teach me the formulas. That is not possible. You now regard the derivative of sin $2x$ as a trivial problem, unworthy of an answer. That was not always so. Before the chain rule, the slopes of $\sin 2x$ and $\sin x^2$ and $\sin^2 x^2$ were hard to compute from $\Delta f/\Delta x$. We are now at the same point for $f(x, y)$. We know the *meaning* of $\partial f/\partial x$, but if $f = r \tan \theta$ and $x = r \cos \theta$ and $y = r \sin \theta$, we need a way to *compute* $\partial f/\partial x$. A little theory is unavoidable, if the problem-solving part of calculus is to keep going.

To repeat: ***The chain rule applies to a function of a function.*** In one variable that was $f(g(x))$. With two variables there are more possibilities:

1. $f(z)$ with $z = g(x, y)$ Find $\partial f/\partial x$ and $\partial f/\partial y$
2. $f(x, y)$ with $x = x(t)$, $y = y(t)$ Find df/dt
3. $f(x, y)$ with $x = x(t, u)$, $y = y(t, u)$ Find $\partial f/\partial t$ and $\partial f/\partial u$