Faculty of Science
Department of Mathematics

Course:

Algebra

For

First year-Edu-Math

Program

Algebra I Content

Mathematical Induction:
is a special method of proof used to prove a Statement, a Theorem, or a Formula, that is asserted about every natural number.
The natural numbers are the counting numbers: $1,2,3,4,\ldots\ etc.$, also called positive integers.

Principle of Mathematical Induction:
Let $P(n)$ be a statement involving the positive integer n .
IF the statement is true when $n=1$, and whenever the statement is true for $n=k$, then it is also true for $n=k+1$, Then the statement is true for all integers $n\geq 1$.
There is nothing special about the integer 1 in the statement above.
It can be replaced (in both places it occurs) by any other positive integer, and the Principle still works.

Steps of Mathematical Induction:
(STEP 1): We show that $P(1)$ is true.
(STEP 2): We assume that $P(k)$ is true.
(STEP 3): We show that $P(k+1)$ is true.
As shown in the following examples:

1- Use mathematical induction to prove that:

$$1+2+3+\ldots.+n=\frac{n(n+1)}{2}.$$

Solution: Let the statement $P(n)$ be $1+2+3+\ldots.+n=\dfrac{n(n+1)}{2}$

(STEP 1): We show that $P(1)$ is true:

$$L.H.S.=1 \quad , \quad R.H.S.=\frac{1(1+1)}{2}=1$$

Both sides of the statement are equal hence $P(1)$ is true.
(STEP 2): We assume that $P(k)$ is true:

$$1+2+3+\ldots.+k=\frac{k(k+1)}{2}.$$

(STEP 3): We show that $P(k+1)$ is true:

$$L.H.S.=1+2+3+\ldots+k+(k+1)$$
$$=\frac{k(k+1)}{2}+(k+1)$$
$$=\frac{(k+1)}{2}[k+2]$$
$$=R.H.S.$$

Which is the statement $P(k+1)$.

Then the statement $P(n)$ is true for all positive integers n.

We can rewrite the solution as follow:

Solution: Let $P(n)$ be $1+2+3+....+n=\frac{n(n+1)}{2}$

(1) at $n=1$:

$L.H.S.=1$, $R.H.S.=\frac{1(1+1)}{2}=1$

$\therefore P(1)$ is true.

(2) let $n=k$:

$1+2+3+....+k=\frac{k(k+1)}{2}$.

(3) at $n=k+1$:

$$L.H.S.=1+2+3+...+k+(k+1)$$
$$=\frac{k(k+1)}{2}+(k+1)$$
$$=\frac{(k+1)}{2}[k+2]$$
$$=R.H.S.$$

$\therefore P(k+1)$ is true.

Then $P(n)$ is true for all positive integers n .

2- Use mathematical induction to prove that:

$$1^2+2^2+3^2+....+n^2=\frac{n(n+1)(2n+1)}{6}.$$

Solution: Let $P(n)$ be $1^2+2^2+3^2+....+n^2=\frac{n(n+1)(2n+1)}{6}$

(1) at $n=1$: $L.H.S.=1^2=1$, $R.H.S.=\frac{1(1+1)(2+1)}{6}=1$

$\therefore P(1)$ is true.

(2) let $n=k$: $1^2+2^2+3^2+....+k^2=\frac{k(k+1)(2k+1)}{6}$.

(3) at $n=k+1$:

$$L.H.S.=1^2+2^2+3^2+....+k^2+(k+1)^2=\frac{k(k+1)(2k+1)}{6}+(k+1)^2$$

$$=\frac{(k+1)}{6}[k(2k+1)+6(k+1)]$$

$$=\frac{(k+1)}{6}[2k^2+k+6k+6]$$

$$=\frac{(k+1)}{6}[2k^2+7k+6]$$

$$=\frac{(k+1)}{6}[(2k+3)(k+2)]$$

$$=\frac{(k+1)(k+2)(2k+3)}{6}$$

$$=R.H.S.$$

$\therefore P(k+1)$ is true.

Then $P(n)$ is true for all positive integers n.

3- Prove that (n^3+2n) is divisible by 3 for all positive integers n.

Solution: Let $P(n)$ be "(n^3+2n) is divisible by 3"

(1) at $n=1$:

$1^3+2(1)=3$ is divisible by 3.

$\therefore P(1)$ is true.

(2) let $n=k$:

"(k^3+2k) is divisible by 3".

(3) at $n=k+1$:

$$(k+1)^3+2(k+1)=(k^3+3k^2+3k+1)+(2k+2)$$
$$=k^3+3k^2+5k+3$$
$$=(k^3+2k)+(3k^2+3k+3)$$
$$=(k^3+2k)+3(k^2+k+1)$$

(k^3+2k) is divisible by 3 from (2), and $3(k^2+k+1)$ is also divisible by 3

$\therefore P(k+1)$ is true.

Then $P(n)$ is true for all positive integers n.

4- Prove that $2^{n-1}\le n!$ for all positive integers n.

Solution: Let $P(n)$ be $2^{n-1}\le n!$

(1) at $n=1$:

$2^{1-1}=2^0=1\le 1!=1$

$\therefore P(1)$ is true.

(2) let $n=k$:

$2^{k-1}\le k!$

(3) at $n=k+1$:

$2^{k-1}\le k!\Rightarrow(2)(2^{k-1})\le(2)(k!)\Rightarrow(2)(2^{k-1})\le(k+1)(k!)\Rightarrow 2^k\le(k+1)!$;

$2\le k+1\ \ \forall\ k\in Z^+$

$\therefore P(k+1)$ is true.

Then $P(n)$ is true for all positive integers n.

H.W:

1- Use mathematical induction to prove that:

$(i)\ 2+4+6+...+2n = n(n+1).$

$(ii)\ 1+\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2^{n-1}} = 2-\frac{1}{2^{n-1}}.$

2- Prove that $(x^n - 1)$ is divisible by $(x-1)$ for all positive integers n.

Exercises:

1- Use mathematical induction to prove that:

$(i)\quad 1+3+5+....+(2n-1)=n^2$

$(ii)\quad 1+4+7+....+(3n-2)=\frac{n(3n-1)}{2}$

$(iii)\quad 2+6+12+....+n(n+1)=\frac{n(n+1)(n+2)}{3}$

2- Prove that $(3n^2-n)$ is divisible by 2 for all positive integers n.

3- Prove that (7^n-2^n) is divisible by 5 for all positive integers n.

4- Prove that (x^n-y^n) is divisible by $(x-y)$ for all positive integers n.

Rational Fraction:

The algebraic formula $p(x)=a_0x^n+a_1x^{n-1}+...+a_n$ is called a polynomial of a variable x of degree n; the coefficients $a_0, a_1,..., a_n$ are real numbers.

If $p(x)$ and $q(x)$ are two polynomials, then the ratio $\frac{p(x)}{q(x)}$

of these two polynomials is called Rational Fraction,
$p(x)$ the numerator, and $q(x)$ the denominator.

We have two types of Rational Fraction:

1. Proper Rational Fraction.
2. Improper Rational Fraction.

Proper Rational Fraction:

If the degree of the numerator of the rational fraction is less than the degree of the denominator of the rational fraction, then that fraction is called the proper rational fraction.

Improper Rational Fraction:

If the degree of the numerator of the rational fraction is equal or greater than the degree of the denominator of the rational fraction, then that fraction is called the improper rational fraction. Suppose, the improper fraction is reducible to an integer added to a proper fraction, then the improper rational fraction can be reduced as a sum of polynomial and a proper rational fraction.

Let us take if $\frac{p(x)}{q(x)}$ is a improper rational fraction, then $\frac{p(x)}{q(x)}=h(x)+\frac{p_1(x)}{q(x)}$

Where, $h(x)$ is a polynomial and $\frac{p_1(x)}{q(x)}$ is a proper rational fraction.

Partial-Fraction Decomposition

You have added and simplified rational expressions, such as:

$$\frac{2}{x}+\frac{1}{x+1}=\frac{2(x+1)+x}{x(x+1)}=\frac{3x+2}{x^2+x}.$$

Partial-fraction decomposition is the process of starting with the simplified answer and taking it back apart, of "decomposing" the final expression into its initial polynomial fractions.

Partial-fraction decomposition rules:
The following tables indicates the simpler partial fractions associated to proper rational fractions.

1- The denominator factor as distinct linear factors:

Form of the rational fraction	Form of the partial fractions
$\dfrac{f(x)}{(a_1x+b_1)(a_2x+b_2)\ldots}$	$\dfrac{A}{a_1x+b_1}+\dfrac{B}{a_2x+b_2}+\ldots$

2- The denominator factor as repeated linear factors:

Form of the rational fraction	Form of the partial fractions
$\dfrac{f(x)}{(ax+b)^k}$	$\dfrac{A_1}{(ax+b)}+\dfrac{A_2}{(ax+b)^2}+\ldots+\dfrac{A_k}{(ax+b)^k}$

3- The denominator factor as distinct quadratic factors can not be factored further:

Form of the rational fraction	Form of the partial fractions
$\dfrac{f(x)}{(a_1x^2+b_1x+c_1)(a_2x^2+b_2x+c_2)\ldots}$	$\dfrac{Ax+B}{a_1x^2+b_1x+c_1}+\dfrac{Cx+D}{a_2x^2+b_2x+c_2}+\ldots$

4- The denominator factor as repeated quadratic factors:

Form of the rational fraction	Form of the partial fractions
$\dfrac{f(x)}{(ax^2+bx+c)^k}$	$\dfrac{A_1x+B_1}{(ax^2+bx+c)}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}+\ldots+\dfrac{A_kx+B_k}{(ax^2+bx+c)^k}$

In the above tables A, B, C and D are real numbers to be determined suitably.

To decompose the improper fraction:
Divide the numerator by the denominator, and then use the above rules to decompose the remainder (be proper fraction).

Examples:

(1) Express the following in partial fractions:

$$\frac{3x+2}{x^2+x}.$$

Solution: To decompose a fraction, you first factor the denominator x^2+x, which factors as $x(x+1)$.

$$\therefore \frac{3x+2}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} \Rightarrow \frac{3x+2}{x(x+1)} = \frac{A(x+1)+Bx}{x(x+1)}$$

$$\Rightarrow 3x+2 = A(x+1)+Bx$$

$$\Rightarrow 3x+2 = (A+B)x+A$$

For the two sides of the equation "$3x+2=(A+B)x+A$" to be equal, the coefficients of the two polynomials must be equal.

So you **"equate the coefficients of x"** to get: $\begin{matrix} 3 = A+B \\ 2 = A \end{matrix} \Rightarrow \begin{matrix} A = 2 \\ B = 1 \end{matrix}$

There is another method for solving for the values of A and B:

The equation "$3x+2=A(x+1)+Bx$" is supposed to be true for any value of x, we can **"pick useful values of x"**, and find the values for A and B.

Looking at the equation "$3x+2=A(x+1)+Bx$", you can see that,

if $x=0$, then we quickly find that $2=A$, and

if $x=-1$, then we easily get $-3+2=-B$, so $B=1$.

$$\therefore \frac{3x+2}{x^2+x} = \frac{2}{x} + \frac{1}{x+1}.$$

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(2) Express the following in partial fractions:

$$\frac{4x^2-3x+5}{(x-1)^2(x+2)}$$

Solution:

$$\frac{4x^2-3x+5}{(x-1)^2(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}$$

$$\Rightarrow \frac{4x^2-3x+5}{(x-1)^2(x+2)} = \frac{A(x-1)(x+2)+B(x+2)+C(x-1)^2}{(x-1)^2(x+2)}$$

$$\Rightarrow 4x^2-3x+5 = A(x-1)(x+2)+B(x+2)+C(x-1)^2.$$

Pick useful values of x:

$x=1 \Rightarrow 6=3B \Rightarrow B=2$, $x=-2 \Rightarrow 27=9C \Rightarrow C=3$,

and equate the coefficients of x^2 to get: $4=A+C \Rightarrow 4=A+3 \Rightarrow A=1$.

$$\therefore \frac{4x^2-3x+5}{(x-1)^2(x+2)}=\frac{1}{(x-1)}+\frac{2}{(x-1)^2}+\frac{3}{(x+2)}.$$

(3) <u>Express the following in partial fractions</u>:

$$\frac{x+1}{x^3+x^2-6x}$$

<u>Solution</u>: $x^3+x^2-6x=x(x^2+x-6)=x(x-2)(x+3)$

$$\therefore \frac{x+1}{x^3+x^2-6x}=\frac{x+1}{x(x-2)(x+3)}=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+3}$$

$$\Rightarrow \frac{x+1}{x(x-2)(x+3)}=\frac{A(x-2)(x+3)+Bx(x+3)+Cx(x-2)}{x(x-2)(x+3)}$$

$$\Rightarrow x+1=A(x-2)(x+3)+Bx(x+3)+Cx(x-2).$$

Pick useful values of x :

$$x=0\Rightarrow 1=-6A\Rightarrow A=-\frac{1}{6} ,$$

$$x=2\Rightarrow 3=10B\Rightarrow B=\frac{3}{10} ,$$

$$x=-3\Rightarrow -2=15C\Rightarrow C=-\frac{2}{15} ,$$

$$\therefore \frac{x+1}{x^3+x^2-6x}=\frac{-1}{6x}+\frac{3}{10(x-2)}+\frac{-2}{15(x+3)}.$$

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(4) <u>Express the following in partial fractions</u>:

$$\frac{1}{x^4-1}$$

<u>Solution</u>: $x^4-1=(x^2-1)(x^2+1)=(x-1)(x+1)(x^2+1)$

$$\therefore \frac{1}{x^4-1}=\frac{1}{(x-1)(x+1)(x^2+1)}=\frac{A}{(x-1)}+\frac{B}{(x+1)}+\frac{Cx+D}{(x^2+1)}$$

$$\Rightarrow \frac{1}{(x-1)(x+1)(x^2+1)}=\frac{A(x+1)(x^2+1)+B(x-1)(x^2+1)+(Cx+D)(x^2-1)}{(x-1)(x+1)(x^2+1)}$$

$$\Rightarrow 1=A(x+1)(x^2+1)+B(x-1)(x^2+1)+(Cx+D)(x^2-1).$$

Pick useful values of x :

$$x=1\Rightarrow 1=4A\Rightarrow A=\frac{1}{4},$$

$$x=-1\Rightarrow 1=-4B\Rightarrow B=-\frac{1}{4},$$

and equate the coefficients of x^3 and x^2 to get:

$0 = A + B + C \Rightarrow 0 = \frac{1}{4} - \frac{1}{4} + C \Rightarrow C = 0,$

$0 = A - B + D \Rightarrow 0 = \frac{1}{4} + \frac{1}{4} + D \Rightarrow D = -\frac{1}{2}.$

$\therefore \frac{1}{x^4 - 1} = \frac{1}{4(x-1)} - \frac{1}{4(x+1)} - \frac{1}{2(x^2+1)}.$

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(5) <u>Express the following in partial fractions</u>:

$\frac{x^2 + x + 1}{x^2 + 2x + 1}$

<u>Solution</u>: the given fraction is improper rational fraction, then we divide the numerator by the denominator:

$$\begin{array}{r|l} x^2 + 2x + 1 & x^2 + x + 1 \\ \hline 1 & x^2 + 2x + 1 \\ \cline{2-2} & \quad -x \end{array}$$

$\therefore \frac{x^2 + x + 1}{x^2 + 2x + 1} = 1 - \frac{x}{x^2 + 2x + 1},$

We decompose the proper fraction $\frac{x}{x^2 + 2x + 1}$ as follow:

$\frac{x}{x^2 + 2x + 1} = \frac{x}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} = \frac{A(x+1) + B}{(x+1)^2}$

$\Rightarrow x = A(x+1) + B.$

Equate the coefficients of x and x^0 (constant terms) to get:

$1 = A$ and $0 = A + B \Rightarrow A = 1,\ B = -1$

$\therefore \frac{x}{x^2 + 2x + 1} = \frac{1}{x+1} - \frac{1}{(x+1)^2},$

$\therefore \frac{x^2 + x + 1}{x^2 + 2x + 1} = 1 - \frac{1}{x+1} + \frac{1}{(x+1)^2}.$

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(6) <u>Express the following in partial fractions</u>:

$$\frac{3x^4 - x^3 + x^2 - x - 1}{x^3 - x^2 + x - 1}$$

<u>Solution</u>: the given fraction is improper rational fraction, then we divide the numerator by the denominator:

$x^3 - x^2 + x - 1$	$3x^4 - x^3 + x^2 - x - 1$
	$3x^4 - 3x^3 + 3x^2 - 3x$
$3x + 2$	$2x^3 - 2x^2 + 2x - 1$
	$2x^3 - 2x^2 + 2x - 2$
	$0 + 0 + 0 + 1$

$$\therefore \frac{3x^4 - x^3 + x^2 - x - 1}{x^3 - x^2 + x - 1} = (3x+2) + \frac{1}{x^3 - x^2 + x - 1} \text{ ,}$$

We decompose the proper fraction $\dfrac{1}{x^3 - x^2 + x - 1}$ as follow:

$$x^3 - x^2 + x - 1 = x^2(x-1) + (x-1) = (x-1)(x^2+1).$$

$$\therefore \frac{1}{x^3 - x^2 + x - 1} = \frac{1}{(x-1)(x^2+1)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+1)}$$

$$\Rightarrow \frac{1}{(x-1)(x^2+1)} = \frac{A(x^2+1) + (Bx+C)(x-1)}{(x-1)(x^2+1)}$$

$$\Rightarrow 1 = A(x^2+1) + (Bx+C)(x-1).$$

Pick useful values of x :

$$x = 1 \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2},$$

and equate the coefficients of x^2 and x to get:

$$0 = A + B \Rightarrow 0 = \frac{1}{2} + B \Rightarrow B = -\frac{1}{2} \text{ ,}$$

$$0 = -B + C \Rightarrow 0 = \frac{1}{2} + C \Rightarrow C = -\frac{1}{2} \text{ .}$$

$$\therefore \frac{1}{x^3 - x^2 + x - 1} = \frac{1}{2(x-1)} - \frac{x+1}{2(x^2+1)} \text{ ,}$$

$$\therefore \frac{3x^4 - x^3 + x^2 - x - 1}{x^3 - x^2 + x - 1} = (3x+2) + \frac{1}{2(x-1)} - \frac{x+1}{2(x^2+1)}\,.$$

Exercises:

Express each of the following in partial fractions:

$(i)\ \dfrac{3x+4}{x^2+x-6}$ $\quad(ii)\ \dfrac{2x+1}{x^3+x^2+x+1}$ $\quad(iii)\ \dfrac{x+1}{x^3+x^2-2x}$

$(iv)\ \dfrac{1}{x^4+x^2-2}$ $\quad(v)\ \dfrac{x^3-2x+2}{x^3-2x+1}$.

Matrices:

A *matrix* is a rectangular array of numbers (elements), the general form of a matrix with m rows and n columns is:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

We denote such a matrix by $(a_{ij})_{m,n}$ or simply (a_{ij}), and the type of the matrix is $m \times n$.

Example1: consider the 2×3 matrix $\begin{pmatrix} 1 & -3 & 4 \\ 0 & 5 & -2 \end{pmatrix}$.

Its rows are $(1,-3,4)$ and $(0,5,-2)$, and its columns are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 5 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \end{pmatrix}$.

Capital letters $A, B, \ldots$ denote matrices, whereas lower case letters $a, b, \ldots$ denote elements.

Example2: build a matrix $A = (a_{ij})_{2\times 3}$; $a_{ij} = \begin{cases} i+j & if \;\; i<j \\ i & if \;\; i=j \\ i-j & if \;\; i>j \end{cases}$

Solution:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}, \quad \begin{array}{l} a_{11} = 1,\ a_{12} = 1+2=3,\ a_{13} = 1+3=4, \\ a_{21} = 2-1=1,\ a_{22} = 2,\ a_{23} = 2+3=5 \end{array}$$

$$\therefore A = \begin{pmatrix} 1 & 3 & 4 \\ 1 & 2 & 5 \end{pmatrix}.$$

Example3: build a matrix $B = (b_{ij})_{3\times 3}$; $b_{ij} = \begin{cases} i+j & if \;\; i<j \\ 0 & if \;\; i=j \\ i^2-j^2 & if \;\; i>j \end{cases}$

Solution:

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \quad \begin{array}{l} b_{11} = 0,\ b_{12} = 1+2=3,\ b_{13} = 1+3=4, \\ b_{21} = 2^2-1^2=3,\ b_{22} = 0,\ b_{23} = 2+3=5 \\ b_{31} = 3^2-1^2=8,\ b_{32} = 3^2-2^2=5,\ b_{33} = 0 \end{array}$$

$$\therefore B = \begin{pmatrix} 0 & 3 & 4 \\ 3 & 0 & 5 \\ 8 & 5 & 0 \end{pmatrix}.$$

✓ Two matrices A and B are equal, if they have the same number of rows and the number of columns.
✓ A matrix whose elements are all zero is called a *zero matrix*, and denoted by 0.

Matrix Addition:
The sum of the two matrices A and B, written $A+B$, is the matrix obtained by adding corresponding element from A and B.
Note that: $A+B$ have the same type as A and B,
The sum of two matrices with different types is not defined.

Example: $\begin{pmatrix} 1 & -2 & 3 \\ 0 & 4 & 5 \end{pmatrix} + \begin{pmatrix} 3 & 0 & -6 \\ 2 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -2 & -3 \\ 2 & 1 & 6 \end{pmatrix}$,

The sum $\begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 5 & -2 \\ 1 & -3 & -1 \end{pmatrix}$ is not defined.

Properties: For matrices A, B and C (with the same type),

(i) $(A+B)+C = A+(B+C)$

(ii) $A+B = B+A$

(iii) $A+0 = 0+A = A$

Scalar Multiplication:
The product of a scalar k and a matrix A, written kA is the matrix obtained by multiplying each element of A by k.

Example: $3\begin{pmatrix} 1 & -2 & 0 \\ 4 & 3 & -3 \end{pmatrix} = \begin{pmatrix} 3 & -6 & 0 \\ 12 & 9 & -15 \end{pmatrix}$.

Matrix Multiplication:
Let A and B be matrices such that the number of columns of A is equal to the number of rows of B. Then the product of A and B, written AB, Is the matrix with the same number of rows as A and of columns as B, and whose element in the i_th row and the j_th column is obtained by multiplying the i_th row of A by the j_th column of B.

Example: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 12 & -1 \\ 23 & -3 \end{pmatrix}$,

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & -2 & 3 \\ 0 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & -6 & 13 \\ 3 & 10 & 29 \end{pmatrix}$,

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & -5 \\ 3 & 6 \end{pmatrix}$ is not defined ,

also $\begin{pmatrix} 1 & -2 & 3 \\ 0 & 4 & 5 \end{pmatrix}\begin{pmatrix} 3 & 0 & -6 \\ 2 & -3 & 1 \end{pmatrix}$ is not defined.

Properties:
Matrix Multiplication does,however, satisfy the following properties:

$(i)\ (AB)C = A(BC)$

$(ii)\ A(B+C) = AB + AC$

$(iii)\ (B+C)A = BA + CA$

$(iv)\ k(AB) = (kA)B = A(kB)$ *where k is a scalar.*

Square Matrix: A matrix with the same number of rows as columns is called a square matrix. A square matrix with $n\_$ rows and $n\_$ columns is called an $n\_$ square matrix. The main diagonal or simply diagonal of a square matrix $A=(a_{ij})$ is the numbers $a_{11}, a_{22}, \ldots, a_{nn}$.
The square matrix with $1's$ along the main diagonal and $0's$ elsewhere is called *the unit matrix* or *the identity matrix* and will be denoted by I.
For any square matrix A, $AI = IA = A$.

Example: The matrix $\begin{pmatrix} 1 & -2 & 0 \\ 0 & -4 & -1 \\ 5 & 3 & 2 \end{pmatrix}$ is $3\_$ square matrix,

the numbers along the main diagonal are $1, -4, 2$.

And the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a unit matrix.

Transpose: The *transpose of a matrix* A, written by A^t is the matrix obtainad by writting the rows of A, in order, as columns.

Example: $\begin{pmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \end{pmatrix}^t = \begin{pmatrix} 1 & 4 \\ 2 & -5 \\ 3 & 6 \end{pmatrix}$.

Properties:
The transpose operation on a matrices satisfies the following properties:

$(i)\ (A+B)^t = A^t + B^t$

$(ii)\ (A^t)^t = A$

$(iii)\ (kA)^t = kA^t \ ,\ for\ k\ a\ scalar$

$(iv)\ (AB)^t = B^t A^t$

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<u>Exercises</u>:

1- Build a matrices $A=(a_{ij})_{3\times 2}$, $B=(b_{ij})_{2\times 3}$;

$$a_{ij}=\begin{cases} i+j & if\ i<j \\ i & if\ i=j \\ i-j & if\ i>j \end{cases} \quad , \quad b_{ij}=\begin{cases} 2i-1 & if\ i=j \\ i+j-2 & if\ i\neq j \end{cases}$$

1- If $A=\begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & -3 \end{pmatrix}$, $B=\begin{pmatrix} 1 & -4 & 0 & 1 \\ 2 & -1 & 3 & -1 \\ 4 & 0 & -2 & 0 \end{pmatrix}$. Compute AB

2- If $A=\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}$, $B=\begin{pmatrix} -11 & -4 & 6 \\ 2 & 0 & -1 \\ 2 & 1 & -1 \end{pmatrix}$. Compute AB^t ,

where B^t the transpose of B.

3- If $A=\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & -2 \end{pmatrix}$, $B=\begin{pmatrix} 2 & -1 & 1 \\ -4 & 3 & -2 \\ 3 & -2 & 1 \end{pmatrix}$. Compute AB^t

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Determinants:

To every square matrix there is assigned a specific number called *determinant of the matrix*.

We write $\det(A)$ or $|A|$ for the determinant of the square matrix A.

Usually a square matrix is said to be singular if its determinant is zero, and nonsingular otherwise.

Determinants of order two:

The determinant of the 2×2 square matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is denoted and defined as follows: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Example:

$$\begin{vmatrix} 5 & 4 \\ 2 & 3 \end{vmatrix} = (5)(3) - (4)(2) = 15 - 8 = 7 \ ,$$

$$\begin{vmatrix} 2 & 1 \\ -4 & 6 \end{vmatrix} = (2)(6) - (1)(-4) = 12 + 4 = 16 \ .$$

Determinants of order three:

The determinant of the 3×3 square matrix is defined as follows:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (-1)^{1+1} a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + (-1)^{1+2} b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + (-1)^{1+3} c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Example:

The determinant of a matrix $A = \begin{pmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{pmatrix}$ is:

$$\begin{vmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{vmatrix} = 2\begin{vmatrix} -4 & 2 \\ -1 & 5 \end{vmatrix} - 3\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} + (-4)\begin{vmatrix} 0 & -4 \\ 1 & -1 \end{vmatrix}$$

$$= 2(-20+2) - 3(0-2) - 4(0+4)$$

$$= -36 + 6 - 16 = -46$$

also,

$$\begin{vmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{vmatrix} = 2\begin{vmatrix} -4 & 2 \\ -1 & 5 \end{vmatrix} - 0\begin{vmatrix} 3 & -4 \\ -1 & 5 \end{vmatrix} + 1\begin{vmatrix} 3 & -4 \\ -4 & 2 \end{vmatrix}$$

$$= 2(-20+2)+(6-16)$$

$$= -36-10 = -46$$

Linear equations in three unknowns and determinants:

Consider three linear equations in the three unknowns $x, y \ and \ z$:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

The above system has a unique solution iff the determinant of the matrix of coefficients is not zero;

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$$

In this case, the unique solution of the system can be expressed as quotients of determinants,

$$x = \frac{N_x}{D} \ , \ y = \frac{N_y}{D} \ , \ z = \frac{N_z}{D}$$

Where the denominator D in each quotient is the determinant of the matrix of coefficients,as avove,and the numerators $N_x, N_y \ and \ N_z$ are obtained by replacing the column of coefficients of the unknown in the matrix of coefficients by the column of constant terms:

$$N_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \ , \ N_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \ , \ N_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

We emphasize that if the determinant D of the matrix of coefficients is zero then the system has either no solution or an infinite number of solutions.

Example: Solve the following system by determinants.

$$2x+y-z=3$$

$$x+y+z=1$$

$$x-2y-3z=4$$

Solution:

$$D=\begin{vmatrix}2 & 1 & -1\\ 1 & 1 & 1\\ 1 & -2 & -3\end{vmatrix}=2(-3+2)-1(-3-1)-1(-2-1)=-2+4+3=5 \quad ,$$

$$N_x=\begin{vmatrix}3 & 1 & -1\\ 1 & 1 & 1\\ 4 & -2 & -3\end{vmatrix}=3(-3+2)-1(-3-4)-1(-2-4)=-3+7+6=10,$$

$$N_y=\begin{vmatrix}2 & 3 & -1\\ 1 & 1 & 1\\ 1 & 4 & -3\end{vmatrix}=2(-3-4)-3(-3-1)-1(4-1)=-14+12-3=-5,$$

$$N_z=\begin{vmatrix}2 & 1 & 3\\ 1 & 1 & 1\\ 1 & -2 & 4\end{vmatrix}=2(4+2)-1(4-1)+3(-2-1)=12-3-9=0$$

$$\therefore x=\frac{N_x}{D}=\frac{10}{5}=2,\quad y=\frac{N_y}{D}=\frac{-5}{5}=-1,\quad z=\frac{N_z}{D}=\frac{0}{5}=0$$

Invertible matrices and determinants:

A square matrix A is said to be *invertible* if there exists a matrix B with the property that $AB = BA = I$, the identity matrix,

we call such a matrix B the inverse *of* A and denote it by A^{-1}.

Observe that the above relation is symmetric; that is, if B is the inverse of A, then A is also the inverse of B.

Example: The matrix $\begin{pmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{pmatrix}$ is the inverse of $\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}$

Such that $\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}\begin{pmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Minors and cofactors: Consider an $n\_$ square matrix $A = (a_{ij})$.

Let M_{ij} denote $(n-1)_$square submatrix of A obtained by deleting its i_th row and j_th column.

The determinant $|M_{ij}|$ is called the *minor* of the element a_{ij} of A, and we define the *cofactor* of a_{ij} to be the "signed" minor $(-1)^{i+j}|M_{ij}| = \Delta_{ij}$.

(Δ_{ij}) is called the *matrix of cofactors* of A, and will be denoted by $\tilde{A}$.

Example: Let $A = \begin{pmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{pmatrix}$. The cofactors of A are:

$$\Delta_{11} = +\begin{vmatrix} -4 & 2 \\ -1 & 5 \end{vmatrix} = -18 \;,\; \Delta_{12} = -\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} = 2 \;,\; \Delta_{13} = +\begin{vmatrix} 0 & -4 \\ 1 & -1 \end{vmatrix} = 4 \;,$$

$$\Delta_{21} = -\begin{vmatrix} 3 & -4 \\ -1 & 5 \end{vmatrix} = -11 \;,\; \Delta_{22} = +\begin{vmatrix} 2 & -4 \\ 1 & 5 \end{vmatrix} = 14 \;,\; \Delta_{23} = -\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = 5 \;,$$

$$\Delta_{31} = +\begin{vmatrix} 3 & -4 \\ -4 & 2 \end{vmatrix} = -10 \;,\; \Delta_{32} = -\begin{vmatrix} 2 & -4 \\ 0 & 2 \end{vmatrix} = -4 \;,\; \Delta_{33} = +\begin{vmatrix} 2 & 3 \\ 0 & -4 \end{vmatrix} = -8 \;.$$

$$\therefore \tilde{A} = \begin{pmatrix} -18 & 2 & 4 \\ -11 & 14 & 5 \\ -10 & -4 & -8 \end{pmatrix}.$$

The transpose of the matrix of cofactors of A is called the *adjoint* of A, denoted $adj\ A = (\tilde{A})^t$. And the *inverse of a nonsingular matrix* A is to be $A^{-1} = \dfrac{adj\, A}{|A|} = \dfrac{(\tilde{A})^t}{|A|}$.

For the matrix $A = \begin{pmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{pmatrix}$ in the above example:

$$|A| = \begin{vmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{vmatrix} = 2(-20+2)+(6-16) = -36-10 = -46 ,$$

$$A^{-1} = \frac{(\tilde{A})^t}{|A|} = -\frac{1}{46}\begin{pmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{pmatrix}.$$

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Exercises:

1- Compute the determinant of each matrix:

$$(i)\begin{pmatrix}1 & 2 & 3\\ 4 & -2 & 3\\ 0 & 5 & -1\end{pmatrix} \qquad (ii)\begin{pmatrix}4 & -1 & -2\\ 0 & 2 & -3\\ 5 & 2 & 1\end{pmatrix} \qquad (iii)\begin{pmatrix}2 & -3 & 4\\ 1 & 2 & -3\\ -1 & -2 & 5\end{pmatrix}$$

2- Solve the following system by determinants:
$$\begin{aligned}2x+3y-z&=1\\ 3x+5y+2z&=8\\ x-2y-3z&=-1\end{aligned}$$

3- Verify that the inverse of $A=\begin{pmatrix}2 & 3 & -1\\ 1 & 2 & 1\\ -1 & -1 & 3\end{pmatrix}$ is $\begin{pmatrix}7 & -8 & 5\\ -4 & 5 & -3\\ 1 & -1 & 1\end{pmatrix}$.

4- Verify that the inverse of a matrix $A=(a_{ij})_{3\times 3}$; $a_{ij}=\begin{cases}2i & if\ i<j\\ i & if\ i=j\\ 2j & if\ i>j\end{cases}$

is $A^{-1}=\frac{1}{2}\begin{pmatrix}-10 & 2 & 4\\ 2 & -1 & 0\\ 4 & 0 & -2\end{pmatrix}$.

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References:

[1] S.Lipschutz:“Theory and Problems of Finite Mathematics”_Schaum’s outline Series_McGraw-Hill Book Company_(1966).

[2] S.Lipschutz:“Theory and Problems of Linear Algebra”_Schaum’s outline Series_McGraw-Hill Book Company_(1974).

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Definition: A *complex number* is a number consisting of a real and imaginary part.

Its standard form is $z = x + iy$; $i = \sqrt{-1}$, $\mathrm{Re}(z) = x$, $\mathrm{Im}(z) = y$.

✓ The complex conjugate of a complex number $z = x + iy$, denoted by $\bar{z}$ is given by $\bar{z} = x - iy$.

✓ The complex number $-z = -x - iy$ is the addition inverse of a complex number $z = x + iy$, and the multiplication inverse of a complex number $0 \neq z = x + iy$ is $z^{-1} = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{x - iy}{(x+iy)(x-iy)} = \frac{x - iy}{x^2 + y^2}$.

Examples: Find $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, $\bar{z}$, $-z$, z^{-1} for each comlex number z of the following:

$1-2i$, $2+i$, i , $2i$, $\frac{1}{1+i}$, -1

Solution: $z = 1 - 2i$

$\mathrm{Re}(z) = 1$, $\mathrm{Im}(z) = -2$, $\bar{z} = 1 + 2i$, $-z = -1 + 2i$,

$$z^{-1} = \frac{1}{1-2i} = \frac{1+2i}{(1-2i)(1+2i)} = \frac{1+2i}{1^2 - (2i)^2} = \frac{1}{5}(1+2i)$$

✓ Two complex numbers are equal if their real parts are equal and their imaginary parts are equal

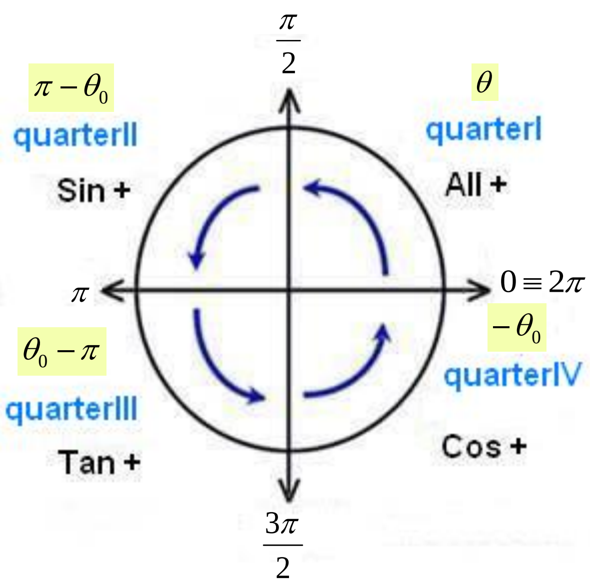
(i.e. If $x_1 + iy_1 = x_2 + iy_2$ Then $x_1 = x_2$ and $y_1 = y_2$).

<u>The polar form of a complex number</u>:

$z = r(\cos\theta + i\sin\theta)$ is called the *polar form* of a comlex number $z = x + iy$ such that:

$$x = r\cos\theta\ ,\ \ y = r\sin\theta\ \ ,\ \ r = |z| = \sqrt{x^2 + y^2}\ ,\ \ \theta = \tan^{-1}\frac{y}{x}$$

θ is called the *argument* of z, denoted by $\arg(z)$.
The *principal argument* of z is $-\pi \le \theta \le \pi$
(determined according to in which quarter lies?)
As shown in the following diagram:



" <u>A</u>ll <u>S</u>tudents <u>T</u>ake <u>C</u>alculus "

(θ_0 will be one of the famous angles $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{4}, \ldots$ rad)

<u>In other words</u>:
- ✓ The comlex number $z = x + iy$ lies in quarterI.
- ✓ The comlex number $z = -x + iy$ lies in quarterII.
- ✓ The comlex number $z = -x - iy$ lies in quarterIII.
- ✓ The comlex number $z = x - iy$ lies in quarterIV.

Examples: Write each of the following comlex number z in polar form:

$1+i$, $-\sqrt{3}+i$, $-1-i\sqrt{3}$, $1-i$

(1) $z=1+i$

$$r=\sqrt{x^2+y^2}=\sqrt{1+1}=\sqrt{2},$$

$$\sin\theta=\frac{y}{r}=\frac{1}{\sqrt{2}},$$

$$\cos\theta=\frac{x}{r}=\frac{1}{\sqrt{2}},$$

$$\tan\theta=\frac{y}{x}=\frac{1}{1}=1.$$

$$\therefore\theta=\frac{\pi}{4},$$

$$\therefore 1+i=\sqrt{2}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}).$$

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(2) $z=-\sqrt{3}+i$

$$r=\sqrt{x^2+y^2}=\sqrt{3+1}=2,$$

$$\sin\theta=\frac{y}{r}=\frac{1}{2},$$

$$\cos\theta=\frac{x}{r}=\frac{-\sqrt{3}}{2},$$

$$\tan\theta=\frac{y}{x}=\frac{1}{-\sqrt{3}}.$$

$$\therefore\theta=\pi-\frac{\pi}{6}=\frac{5\pi}{6},$$

$$\therefore -\sqrt{3}+i=2[\cos(\frac{5\pi}{6})+i\sin(\frac{5\pi}{6})].$$

---

(3) $z=-1-i\sqrt{3}$

$r=\sqrt{x^2+y^2}=\sqrt{1+3}=2,$

$\sin\theta=\frac{y}{r}=\frac{-\sqrt{3}}{2}$,

$\cos\theta=\frac{x}{r}=\frac{-1}{2}$,

$\tan\theta=\frac{y}{x}=\frac{-\sqrt{3}}{-1}=\sqrt{3}.$

$\therefore\theta=\frac{\pi}{3}-\pi=-\frac{2\pi}{3}$,

$\therefore -1-i\sqrt{3}=2[\cos(-\frac{2\pi}{3})+i\sin(-\frac{2\pi}{3})].$

---

(4) $z=1-i$

$r=\sqrt{x^2+y^2}=\sqrt{1+1}=\sqrt{2},$

$\sin\theta=\frac{y}{r}=\frac{-1}{\sqrt{2}}$,

$\cos\theta=\frac{x}{r}=\frac{1}{\sqrt{2}}$,

$\tan\theta=\frac{y}{x}=\frac{-1}{1}=-1.$

$\therefore\theta=-\frac{\pi}{4}$,

$\therefore 1-i=\sqrt{2}[\cos(-\frac{\pi}{4})+i\sin(-\frac{\pi}{4})].$

✓ **H.W:**

1- Write the complex number $z=\frac{2}{1+i}$ in the form $z=x+iy$, and find $\text{Re}(z)$, $\text{Im}(z)$, $\bar{z}$, $|z|$, $\arg(z)$.

2- Write the complex number $z=\frac{4}{-\sqrt{3}+i}$ in the form $z=x+iy$, and find

$\operatorname{Re}(z)$, $\operatorname{Im}(z)$, $\bar{z}$, $|z|$, $\arg(z)$.

De Moivre`s Theorem: Let $z = r(\cos\theta + i\sin\theta)$ be a complex number and n be any real number. Then $z^n = r^n(\cos n\theta + i\sin n\theta)$.

Examples:

(1) Using De Moivre`s Theorem, find the value of $(1+i)^8$

Solution:

we put the complex number $z = 1+i$ in the polar form as follows:

$$r = \sqrt{1+1} = \sqrt{2}, \theta = \tan^{-1}\frac{1}{1} = \tan^{-1}1 = \frac{\pi}{4}$$

$$\therefore 1+i = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) \; ,$$

$$\therefore (1+i)^8 = (\sqrt{2})^8(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})^8 = 16(\cos 2\pi + i\sin 2\pi) = 16.$$

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(2) Using De Moivres` Theorem, reduce the complex number:

$$z = \frac{(\cos 2\theta - i\sin 2\theta)^5(\cos 3\theta + i\sin 3\theta)^7}{(\cos 4\theta + i\sin 4\theta)^{11}(\cos 5\theta - i\sin 5\theta)^9}$$, and find its value at $\theta = \frac{\pi}{6}$.

Solution:

$$z = \frac{[\cos(-2\theta) + i\sin(-2\theta)]^5[\cos 3\theta + i\sin 3\theta]^7}{[\cos 4\theta + i\sin 4\theta]^{11}[\cos(-5\theta) + i\sin(-5\theta)]^9}$$

$$= \frac{[\cos\theta + i\sin\theta]^{-10}[\cos\theta + i\sin\theta]^{21}}{[\cos\theta + i\sin\theta]^{44}[\cos\theta + i\sin\theta)]^{-45}}$$

$$= (\cos\theta + i\sin\theta)^{12} = \cos 12\theta + i\sin 12\theta.$$

and at $\theta = \frac{\pi}{6}$: $z = \cos(12)(\frac{\pi}{6}) + i\sin(12)(\frac{\pi}{6}) = \cos 2\pi + i\sin 2\pi = 1$

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(3) Using De Moivre`s Theorem, reduce the complex number:

$\dfrac{(1+i\tan\theta)^5}{(1-i\tan\theta)^7}$, and find its value at $\theta=\dfrac{\pi}{6}$.

Solution:

$$z=\frac{(1+i\tan\theta)^5}{(1-i\tan\theta)^7}=\frac{(1+i\dfrac{\sin\theta}{\cos\theta})^5}{(1-i\dfrac{\sin\theta}{\cos\theta})^7}$$

$$=\frac{(\cos\theta)^2(\cos\theta+i\sin\theta)^5}{(\cos\theta-i\sin\theta)^7}$$

$$=\frac{(\cos\theta)^2(\cos\theta+i\sin\theta)^5}{(\cos\theta+i\sin\theta)^{-7}}$$

$$=(\cos\theta)^2(\cos\theta+i\sin\theta)^{12}$$

$$=(\cos\theta)^2[\cos(12\theta)+i\sin(12\theta)].$$

and at $\theta=\dfrac{\pi}{6}$:

$$z=(\cos(\frac{\pi}{6}))^2[\cos(12)(\frac{\pi}{6})+i\sin(12)(\frac{\pi}{6})]=(\frac{\sqrt{3}}{2})^2[\cos 2\pi+i\sin 2\pi]=\frac{3}{4}.$$

✓ **H.W:**

Using De Moivre`s Theorem, find the value of $(1+i\sqrt{3})^6$, $(\sqrt{3}+i)^{12}$

The Series

The sum of a finite or an infinite sequence of numbers $a_1, a_2, ..., a_n$ is called *series* :

$$a_1 + a_2 + ... + a_n = \sum_{k=1}^{n} a_k \quad \text{or} \quad a_1 + a_2 + ... + a_n + ... = \sum_{k=1}^{\infty} a_k \ .$$

A series is briefly written as $\sum_{n=1}^{\infty} a_n$ and a_n is called the n*th* term or the general term of the series.

Examples of Series:

(1) $\sum_{k=2}^{5} \frac{1}{k^2 + k} = \frac{1}{2^2 + 2} + \frac{1}{3^2 + 3} + \frac{1}{4^2 + 4} + \frac{1}{5^2 + 5} = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30}$

(2) $\sum_{r=0}^{n} 2^{-r} = 1 + \frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2^n}$

(3) $\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + ...$

Numerical Series:

$a + (a + d) + (a + 2d) + ... + [a + (n-1)d] + ...$;

the n*th* term: $a_n = a + (n-1)d$,

the n*th* partial sum: $S_n = \frac{n}{2}[2a + (n-1)d]$

Geomtric Series:

$a + ar + ar^2 + ... + ar^{n-1} + ...$;

the n*th* term: $a_n = ar^{n-1}$,

$$S_n = \frac{a(1-r^n)}{1-r} \ ; \ r \neq 1$$

Sum of Finite Series:

The Differences Method: we put the r*th* term a_r as difference between two consecutive quantities [$a_r = \alpha(r+1) - \alpha(r)$ *or* $a_r = \alpha(r) - \alpha(r+1)$].

Theorem:

For a series $\sum_{r=1}^{\infty} a_r$ if $a_r = \alpha(r+1) - \alpha(r)$ then $S_n = \alpha(n+1) - \alpha(1)$

Also, if $a_r = \alpha(r) - \alpha(r+1)$ then $S_n = \alpha(1) - \alpha(n+1)$

proof:

$a_r = \alpha(r+1) - \alpha(r)$ put $r = 1,2,3,\ldots,n$:

$a_1 = \alpha(2) - \alpha(1)$

$a_2 = \alpha(3) - \alpha(2)$

$\vdots$

$a_{n-1} = \alpha(n) - \alpha(n-1)$

$a_n = \alpha(n+1) - \alpha(n)$

$\therefore S_n = a_1 + a_2 + \ldots + a_n = \alpha(n+1) - \alpha(1)$

Similarly, we can prove the second part of the theorem.

Examples:

1- Find the sum of a series $(1)(2)+(2)(3)+(3)(4)+...+n(n+1)$

Solution:

$a_r = r(r+1)$,

as a_r parts are consecutive, we suggest the differences as follows:

$$r(r+1)(r+2)-(r-1)r(r+1)=3r(r+1)=3a_r \text{ ,}$$

$$\therefore a_r = \frac{1}{3}[r(r+1)(r+2)]-\frac{1}{3}[(r-1)r(r+1)] = \alpha(r+1)-\alpha(r) \text{ ,}$$

$$\therefore S_n = \alpha(n+1)-\alpha(1)$$

$$=\frac{1}{3}[n(n+1)(n+2)]-\frac{1}{3}[(1-1)1(1+1)]$$

$$=\frac{1}{3}[n(n+1)(n+2)].$$

2- Find the sum S_n of a series $(1)(2)(3)+(2)(3)(4)+(3)(4)(5)+...$

Solution:

$a_r = r(r+1)(r+2)$,

as a_r parts are consecutive, we suggest the differences as follows:

$$r(r+1)(r+2)(r+3)-(r-1)r(r+1)(r+2)=4r(r+1)(r+2)=4a_r \text{ ,}$$

$$\therefore a_r = \frac{1}{4}[r(r+1)(r+2)(r+3)]-\frac{1}{4}[(r-1)r(r+1)(r+2)] = \alpha(r+1)-\alpha(r) \text{ ,}$$

$$\therefore S_n = \alpha(n+1)-\alpha(1)$$

$$=\frac{1}{4}[n(n+1)(n+2)(n+3)]-\frac{1}{4}[(1-1)1(1+1)(1+2)]$$

$$=\frac{1}{4}[n(n+1)(n+2)(n+3)].$$

3- Find the sum S_n of a series $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+....$

Solution:

$$a_r = \frac{r}{(r+1)!} = \frac{r+1-1}{(r+1)!} = \frac{r+1}{(r+1)!}-\frac{1}{(r+1)!} = \frac{1}{r!}-\frac{1}{(r+1)!} = \alpha(r)-\alpha(r+1) \text{ ,}$$

$$\therefore S_n = \alpha(1)-\alpha(n+1) = 1-\frac{1}{(n+1)!}.$$

4- Find the sum S_n of a series $\frac{1}{(1)(4)}+\frac{1}{(4)(7)}+\frac{1}{(7)(10)}+\dots$

Solution:

$$a_r=\frac{1}{(3r-2)(3r+1)} \ ,$$

as a_r is a fraction its denominator consists of two consecutive parts, we use the parital fractions as follows:

$$a_r=\frac{1}{(3r-2)(3r+1)}=\frac{A}{3r-2}+\frac{B}{3r+1}$$

$$\therefore 1=A(3r+1)+B(3r-2)$$

$$put\ r=\frac{2}{3}\Rightarrow A=\frac{1}{3}\ ,\quad put\ r=\frac{-1}{3}\Rightarrow B=\frac{-1}{3}\ ,$$

$$\therefore a_r=\frac{1}{(3r-2)(3r+1)}=(\frac{1}{3})(\frac{1}{3r-2}-\frac{1}{3r+1})=\alpha(r)-\alpha(r+1)\ ,$$

$$\therefore S_n=\alpha(1)-\alpha(n+1)=\frac{1}{3}[1-\frac{1}{3n+1)}].$$

5- Find the sum S_n of a series $\frac{1}{(1)(2)(3)}+\frac{1}{(2)(3)(4)}+\frac{1}{(3)(4)(5)}+\dots$

Solution:

$$a_r=\frac{1}{r(r+1)(r+2)} \ ,$$

as a_r is a fraction its denominator consists of three consecutive parts, we suggest the differences as follows:

$$\frac{1}{r(r+1)}-\frac{1}{(r+1)(r+2)}=\frac{r+2-r}{r(r+1)(r+2)}=\frac{2}{r(r+1)(r+2)}=2a_r$$

$$\therefore a_r=\frac{1}{2}[\frac{1}{r(r+1)}-\frac{1}{(r+1)(r+2)}]=\alpha(r)-\alpha(r+1)$$

$$\therefore S_n=\alpha(1)-\alpha(n+1)=\frac{1}{2}[\frac{1}{2}-\frac{1}{(n+1)(n+2)}]=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}.$$

<u>H.W</u>: (Using The Differences Method)

1- Verify that $1+3+6+...+\frac{n(n+1)}{2}=\frac{n(n+1)(n+2)}{6}$.

2- Verify that $\frac{1}{(1)(3)}+\frac{1}{(3)(5)}+\frac{1}{(5)(7)}+...+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$.

Convergence & divergence of Infinite Series:

Definition: The series $\sum_{k=1}^{\infty} a_k$ is *convergent* if $\lim_{n\to\infty} S_n = \lim_{n\to\infty}(\sum_{k=1}^{n} a_k) = s$;

s defined number , and the series $\sum_{k=1}^{\infty} a_k$ is *divergent* if $\lim_{n\to\infty} S_n = \infty$.

Results:

(**1**) If the series $\sum_{k=1}^{\infty} a_k$ is convergent then $\lim_{n\to\infty} a_n = 0$, but the reverse is not true in general ($\lim_{n\to\infty} \frac{1}{n} = 0$ but $\sum_{k=1}^{\infty} \frac{1}{k}$ divergen*t*).

(**2**) If $\lim_{n\to\infty} a_n \neq 0$ then the series $\sum_{k=1}^{\infty} a_k$ is divergent.

Examples:

1- Using The Differences Method, find S_n of the series

$\frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \frac{1}{(3)(4)} + \ldots$, and then determine whether the series is convergent or divergent.

Solution:

$$a_r = \frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1} = \alpha(r) - \alpha(r+1)$$

$$\therefore S_n = \alpha(1) - \alpha(n+1) = 1 - \frac{1}{n+1} \ ,$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (1 - \frac{1}{n+1}) = 1 \ ,$$

Then the series is convergent.

2- Using The Differences Method, find S_n of the series $\sum_{n=1}^{\infty} \log(1+\frac{1}{n})$, and then determine whether the series is convergent or divergent.

Solution:

$$a_r = \log(1+\frac{1}{r}) = \log(\frac{r+1}{r}) = \log(r+1) - \log(r) = \alpha(r+1) - \alpha(r)$$

$$\therefore S_n = \alpha(n+1) - \alpha(1) = \log(n+1) - \log(1) = \log(n+1) \ ,$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \log(n+1) = \log \lim_{n\to\infty} (n+1) = \infty \ ,$$

Then the series is divergent.

3- Discuss the convergence or divergence of a series $1+\frac{2}{3}+\frac{3}{5}+\frac{4}{7}+\ldots$

Solution:

$$a_n = \frac{n}{2n-1}.$$

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2n-1} = \lim_{n\to\infty} \frac{1}{2-\frac{1}{n}} = \frac{1}{2} \neq 0.$$

Then the series is divergent.

Convergence & Divergence Tests of positive series:

(1) Comparison test(II):

Consider two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$; $a_n \geq 0, b_n \geq 0 \ \forall n$

If $\lim_{n\to\infty} \frac{a_n}{b_n} = k$; k nonzero, positive or negative defined number

then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent or divergent together.

Examples:

1- Discuss the convergence or divergence of a series $\sum_{n=1}^{\infty} \frac{1}{n}$

Solution:

We compare the series $\sum_{n=1}^{\infty} \frac{1}{n}$ with a series $\sum_{n=1}^{\infty} \log(1+\frac{1}{n})$

$$a_n = \frac{1}{n} \ , \ b_n = \log(1+\frac{1}{n}) \ ,$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\log(1+\frac{1}{n})} = \lim_{n\to\infty} \frac{1}{\log(1+\frac{1}{n})^n} = \frac{1}{\log e} = 1 \ ,$$

and the series $\sum_{n=1}^{\infty} \log(1+\frac{1}{n})$ is divergent, then also $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

2- Discuss the convergence or divergence of a series $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Solution:

We compare the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ with a series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$$a_n = \frac{1}{n^2} \ , \ b_n = \frac{1}{n(n+1)} \ ,$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n(n+1)}{n^2} = 1 \ ,$$

and the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, then also $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

(2) D`Alembert`s test (Ratio test(I)):

Consider the series $\sum_{n=1}^{\infty} a_n$; $a_n \geq 0 \ \ \forall n$

(i) If $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$ then $\sum_{n=1}^{\infty} a_n$ is convergent.

(i) If $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$ then $\sum_{n=1}^{\infty} a_n$ is divergent.

(i) If $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ then the test failure.

Examples: Using Ratio Test, discuss the convergence or divergence of each of the following series:

$$(i)\ \sum_{n=1}^{\infty} \frac{n}{3^n} \qquad (ii)\ \sum_{n=1}^{\infty} \frac{n!}{3^n} \qquad (iii)\ \sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

Solution:

$$(i)\ a_n = \frac{n}{3^n}\ ,\ a_{n+1} = \frac{(n+1)}{3^{(n+1)}}\ ,$$

$$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} (\frac{(n+1)}{3^{(n+1)}})(\frac{3^n}{n}) = \lim_{n\to\infty} (\frac{n+1}{n})(\frac{1}{3}) = \frac{1}{3} \lim_{n\to\infty} (1+\frac{1}{n}) = \frac{1}{3} < 1.$$

Then the series $\sum_{n=1}^{\infty} \frac{n}{3^n}$ is convergent.

$$(ii)\ a_n = \frac{n!}{3^n}\ ,\ a_{n+1} = \frac{(n+1)!}{3^{(n+1)}}\ ,$$

$$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} (\frac{(n+1)!}{3^{(n+1)}})(\frac{3^n}{n!}) = \lim_{n\to\infty} \frac{n+1}{3} = \infty > 1.$$

Then the series $\sum_{n=1}^{\infty} \frac{n!}{3^n}$ is divergent.

$$(iii)\ a_n = \frac{n}{n^2+1}\ ,\ a_{n+1} = \frac{n+1}{(n+1)^2+1}\ ,$$

$$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} [\frac{n+1}{(n+1)^2+1}][\frac{n^2+1}{n}] = \lim_{n\to\infty} \frac{n^3+n^2+n+1}{n^3+2n^2+2n} = 1.$$

Then the test failure.

(use the comparison test: compare the series with a series $\sum_{n=1}^{\infty} \frac{1}{n}$).

Exercises:

1- Using The Differences Method, find S_n of each of the following series, and then determine whether the series is convergent or divergent:

$(i)\ \sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}$ $(ii)\ \sum_{n=1}^{\infty}\frac{1}{(n+2)(n+3)}$ $(iii)\ \sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}$.

2- Discuss the convergence or divergence of each of the following series:

$(i)\ \frac{1}{11}+\frac{2}{21}+\frac{3}{31}+....$ $(ii)\ \sum_{n=1}^{\infty}\frac{n^4}{2^n}$ $(iii)\ \sum_{n=1}^{\infty}\frac{2^n}{n}$

$(iv)\ \sum_{n=1}^{\infty}\frac{2^n n!}{n^2}$ $(v)\ \sum_{n=1}^{\infty}\frac{1}{n2^n}$

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