South valley University

Faculty of science-Qena
Mathematics Department

Statics: Lecture Notes

Part I

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Contents

Chapter: 1
Introduction

General principles

Mechanics: Mechanics is a branch of physical sciences which describes or predicts the conditions of rest or motion of bodies under the action of forces.

Mechanics: Mechanics is a branch of physical sciences that is concerned with the state of rest or motion of bodies that are subjected to the action of forces.

Mechanics can be subdivided into three branches: rigid body mechanics- deformable body mechanics- fluid mechanics

1- Rigid bodies: (i) Statics (ii) Dynamics 2- Deformable bodies

3- Fluid Mechanics: (i) Compressible – gas (ii) incompressible - liquids

Here we will study only the rigid body Mechanics. In the first term we will study some subjects in Statics

In Statics we will assume the bodies to be perfectly rigid, no deformation.

This is never true in the real world, everything deforms a little when a load is applied.

These deformations are small and will not significantly affect the conditions of equilibrium or motion, so we will neglect the deformations.

Basic Quantities

Basic Concepts: There are four basic quantities in Mechanics space, time, mass, force:

(1) - Length: Length is used to locate the position of a point and describe the size of physical systems.

(2) - Time: Time is the measure of the succession of events and it is important in Dynamics

(3) - Mass: Mass is a measure of a quantity of matter that is used to compare the action of one body with that of another. This property manifests itself as a gravitational attraction between two bodies and provides a measure of the resistance of matter to a change in velocity.

(4) - Force: Force is push or pull exerted by one body on anther. This interaction can occur when there is direct contact between the bodies, such as a person pushing on a wall, or it can occur through a distance when the bodies are physically separated.

Idealizations: Models or idealizations are used in mechanics in order to simplify application of the theory. Here we will consider three important idealizations.

(i) - Particle: A particle has a mass, but a size that can be neglected. For example, the size of the earth is insignificant compared to the size of its orbit, and therefore the earth can be modeled as a particle when studying its orbital motion.

(ii) - Rigid Body: A rigid body is a combination of a large number of particles in which all the particles remain at a fixed distance from one another, both before and after applying a load

A rigid body is considered rigid when the relative movement between its parts is negligible.

(iii) - Concentrated Force. A concentrated force represents the effect of a loading which is assumed to act at a point on a body when the contact area is small compared with the overall size. We can represent a load by a concentrated force, provided the area over which the load is applied is very small compared to the overall size of the body. An example would be the contact force between a wheel and the ground

Weight and Mass

Weight is the measure of how heavy an object

The unit of measurement for weight is that of force, which in the International System of Units (SI) is the newton.

Mass is both a property of a physical body and a measure of its resistance to acceleration (a change in its state of motion) when a net force is applied (The mass of an object is the amount of material it contains.). An object's mass also determines the strength of its gravitational attraction to other bodies.

The unit of measurement for mass in the International System of Units (SI) is the kilogram (kg).

Weight is not the same thing as mass. Mass is a literal representation of the amount of matter in a particle or object, and is independent of external factors such as speed, acceleration, or applied force (as long as relativistic effects are small enough to be neglected). Weight has meaning only when an object having a specific mass is placed in an acceleration field. At the Earth's surface, a kilogram mass weighs about 2.2 pounds, for example. But on Mars, the same kilogram mass would weigh only about 0.8 pounds

Newton's Three Laws of motion:

Engineering Mechanics is formulated on the basis of Newton's three Laws of motion:

First Law (1st Law):

A particle originally at rest, or moving in a straight line with constant velocity, tends to remain in this state provided the particle is not subjected to an unbalanced force.
A particle remains at rest or continues to move in a straight line with a constant speed if there is no unbalanced force acting on it (resultant force = 0).

Second Law (2nd Law):

A particle acted upon by an unbalanced force (F) experiences an acceleration (a) that has the same direction as the force and a magnitude that is directly proportional to the force.
If (F) is applied to a particle of mass (M), this law may be expressed mathematically as

$$F = M\,a$$

Third Law (3rd Law):

For every action there is an equal and opposite reaction.
The mutual forces of action and reaction between two particles are equal, opposite, and collinear
The forces of action and reaction between interacting bodies are equal in magnitude, opposite in direction, and act along the same line of action (Collinear).

Newton's Law of Gravitation Attraction

The gravitational attraction force between any two particles is

$$F = G\frac{M\,m}{r^2},$$

F = mutual force of attraction between two particles
G = universal constant known as the constant of gravitation
M, m = masses of each of the two particles
r = distance between the two particles

What am I talking about? Weight.

The weight of a particle is the gravitational force between a particle and earth;

By using the equation $g = \frac{G\,M}{r^2}$, where

M = mass of earth, m = mass of a particle

r = radius of earth, g = acceleration of gravity at earth's surface

Using $g = \frac{GM}{r^2} \rightarrow G = \frac{gr^2}{M}$

Substituting into $F = G\frac{Mm}{r^2}$, we have $F = \frac{gr^2}{M}\left(\frac{Mm}{r^2}\right)$

$$F = mg \Rightarrow \text{ weight is } \quad W = mg$$

Or using $F = ma$ and at the surface of the Earth $a = g$

$$\therefore \quad F = mg\,,\ then \quad W = mg$$

g is dependent upon r. Most cases use g = 9.81 m/s² = 32.2 ft/ s²

Units of Measurement

A unit of measurement is a definite magnitude of a physical quantity.

There are two main measurement systems:

(1)- Metric system (International system SI):

This system is based on three main units:

Meter – Kilogram – Second (It is called mks System).

SI is an abbreviation of French expression (**S**ysteme **I**nternational d'Unités) in English (International system)

(2) – English system (British System or Imperial System or US Customary System)

This system is based on three main units:

Foot - Pond - Second (It is also called FPS system). See below Table

Name	Length	Time	Mass	Force
International system of Units (SI)	Meter	Second	Kilogram	Newton
	m	s	kg	$N = \frac{kg.m}{s^2}$
U. S. Customary FPS	Foot	Second	Slug	Pound
	ft	s	$N = \frac{Ib.s^2}{ft}$	Ib

Pound (Ib) , unit of avoirdupois weight, equal to 16 ounces, 7,000 grains, or 0.45359237 kg,

Newton

The newton is a unit to for measuring force equal to the force needed to move one kilogram of mass at a rate of one meter per second squared.
The newton is the SI derived unit for force in the metric system. Newtons can be abbreviated as N, for example 1 newton can be written as $1N$.
Newtons can be expressed using the formula: (1) $N = (1\,\text{kg})\ (1\,\text{m/ s}^2)$.

Pound-Force

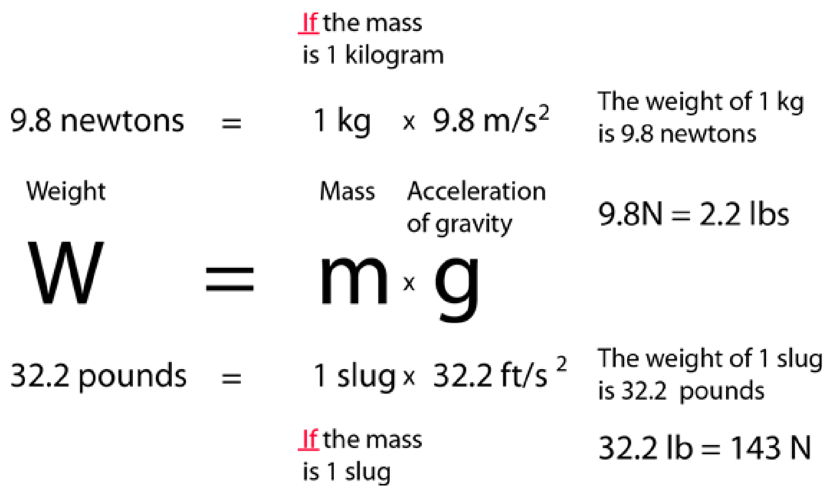
Pound-force is a unit of force equal to the force needed to move one pound of mass at a rate of 32.174049. 32 foot per second squared.

The pound-force is a US customary and imperial unit of force. A pound-force is sometimes also referred to as a pound of force. Pound-force can be abbreviated as IbF or Ib_F. For example, 1 pound-force can be written as $1IbF$ or $1Ib_F$.

Pound-force can be expressed using the formula: $1Ib = 32.174049\text{lb}\ \frac{\text{ft}}{\text{s}^2}$.

How to convert kilograms to pounds

1 kilogram = 2.2046226218488 pounds
1 pound = 0.45359237 kilograms



Force: Newton (N)

(1) $N = (1\,\text{kg})\ (1\,\text{m/ s}^2)$

1 Newton is the force required to give a mass of 1 kg an acceleration of 1 m/ s².

Weight is a force. The weight of 1 kg Mass is:

$$W = \text{m g} \Rightarrow \text{W} = (1\,\text{kg})\ (9.81\,\text{m/ s}^2) = \ 9.81\ \ \text{N}$$

Chapter: 2
Vectors Forces
Part 1: Vectors in 2D and 3D

Introduction

Statics: The study of bodies when they are at rest and all forces are in equilibrium. Static equations are often used in truss problems. To solve a static equation, engineers use a free body diagram. If an object is at rest, as it is in statics, the sum of the forces acting upon the object will equal zero. The sum of the moments will also equal zero.

Scalars and Vectors

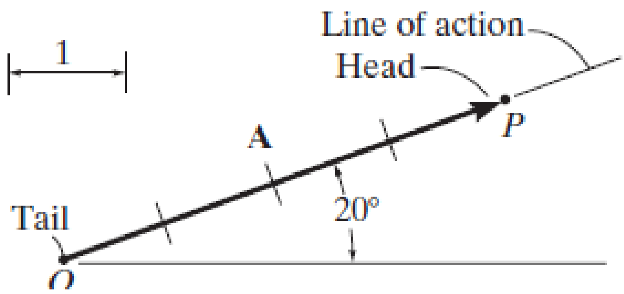
Scalar: A scalar is any positive or negative physical quantity that can be completely specified by its magnitude

Examples: Examples of scalar quantities include length, mass, and time.

Vector: A vector is any physical quantity that requires both a magnitude and a direction for its complete description.

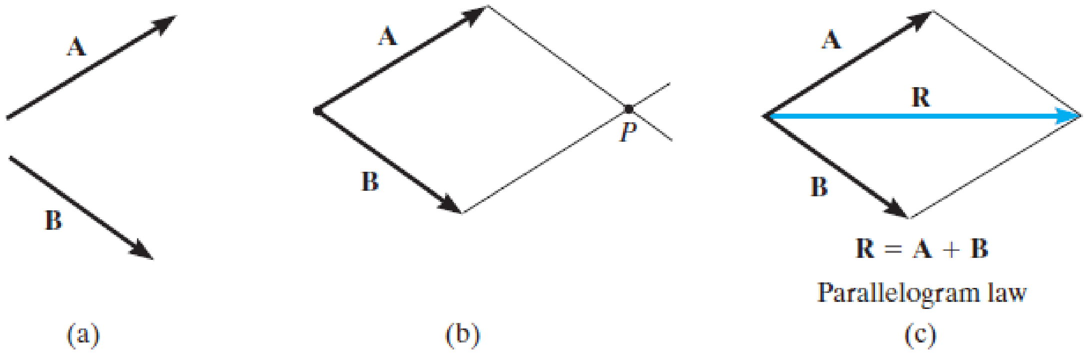
Examples(For instance): Examples of vectors encountered in statics are force, position, and moment.

Vector A vector is shown graphically by an arrow. The length of the arrow represents the magnitude of the vector, and the angle between the vector and a fixed axis defines the direction of its line of action . The head or tip of the arrow indicates the sense of direction of the vector (see below Figure)



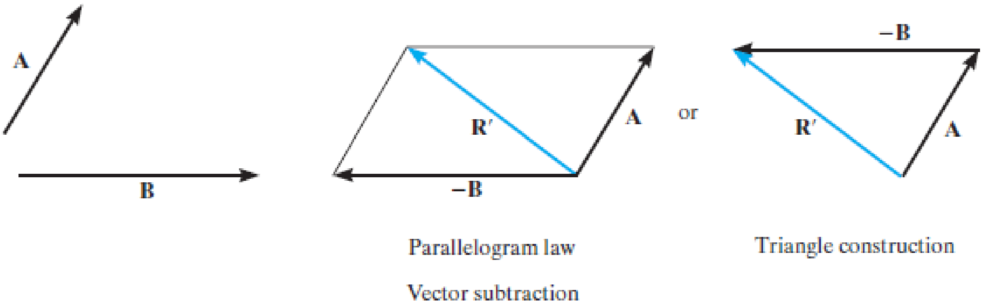
Vector Addition:

All vector quantities obey the parallelogram law of Addition



$$\vec{R}=\vec{A}+\vec{B}$$

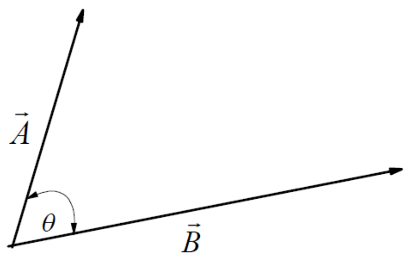
Vector Subtraction. The resultant of the difference between two vectors A and B of the same type may be expressed as



$$\vec{R}=\vec{A}-\vec{B}=+(-\vec{B})$$

Dot Product

The Dot Product gives a scalar (ordinary number) answer, and is sometimes called the scalar product.



The Dot product define as $\vec{A}\bullet\vec{B}=AB\cos\theta$

Laws of Operation

1. Commutative law : $\vec{A}\bullet\vec{B}=\vec{B}\bullet\vec{A}$

2. Multiplication by a scalar $\lambda\left(\vec{A}\bullet\vec{B}\right)=\left(\lambda\vec{A}\right)\bullet\vec{B}=\vec{A}\bullet\left(\lambda\vec{B}\right)=\left(\vec{A}\bullet\vec{B}\right)\lambda$

3. Distribution law : $\vec{A}\bullet(\vec{B}+\vec{D})=(\vec{A}\bullet\vec{B})+(\vec{A}\bullet\vec{D})$

4. Cartesian Vector Formulation

Dot product of two vectors $\vec{A}=a_X\vec{I}+a_Y\vec{J}+a_Z\vec{K}$ and $\vec{B}=b_X\vec{I}+b_Y\vec{J}+b_Z\vec{K}$,

then . $\vec{A}\cdot\vec{B}=a_Xb_X+a_Yb_Y+a_Zb_Z$

5. Applications: The angle formed between two vectors given by

$$\cos\theta=\frac{\vec{A}\bullet\vec{B}}{|\vec{A}|\,|\vec{B}|}=\frac{a_Xb_X+a_Yb_Y+a_Zb_Z}{\sqrt{a_X^{\ 2}+a_Y^{\ 2}+a_Z^{\ 2}}\sqrt{b_X^{\ 2}+b_Y^{\ 2}+b_Z^{\ 2}}}$$

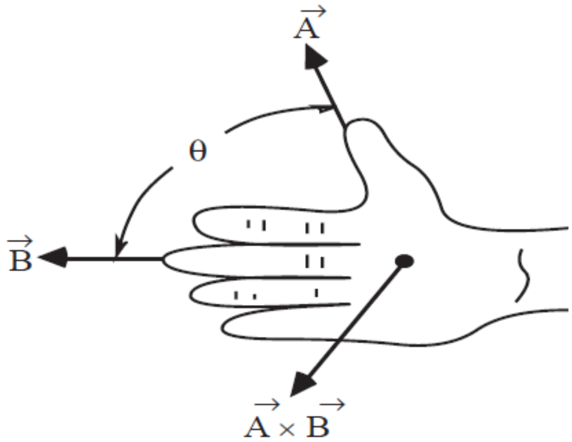
If $\vec{A}\cdot\vec{B}=0\quad\rightarrow\quad\vec{A}$ perpendicular $\vec{B}$

6- Dot product of Cartesian unit vectors $\vec{i}\cdot\vec{j}=\vec{j}\cdot\vec{k}=\vec{k}\cdot\vec{i}=0$, while

$$\vec{i}\cdot\vec{i}=i^2=1,\quad \vec{j}\cdot\vec{j}=j^2=1,\quad \vec{k}\cdot\vec{k}=k^2=1.$$

Cross Product

The Cross Product which gives a vector as an answer, and is sometimes called the vector product. For two vectors define as $\vec{A}\wedge\vec{B}=AB\sin\theta\,\vec{e}$, where $\vec{e}$ is unite vector in the direction of $\vec{A}\wedge\vec{B}$



Laws of Operation

1. Commutative law $\vec{A}\wedge\vec{B}\neq\vec{B}\wedge\vec{A}$, but $\vec{A}\wedge\vec{B}=-\vec{A}\wedge\vec{B}$

2. Multiplication by a scalar $\lambda(\vec{A}\wedge\vec{B})=(\lambda\vec{A})\wedge\vec{B}=\vec{A}\wedge(\lambda\vec{B})=(\vec{A}\wedge\vec{B})\lambda$

3. Distribution law $\vec{A}\wedge(\vec{B}+\vec{D})=(\vec{A}\wedge\vec{B})+(\vec{A}\wedge\vec{D})$

4. Cartesian Vector Formulation

Cross product of two vectors $\vec{A}=a_X\vec{I}+a_Y\vec{J}+a_Z\vec{K}$ and $\vec{B}=b_X\vec{I}+b_Y\vec{J}+b_Z\vec{K}$

Then $\vec{A}\wedge\vec{B}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_X & a_Y & a_Z\\ b_X & b_Y & b_Z\end{vmatrix}$.

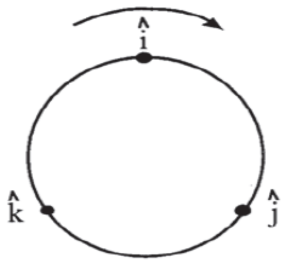
5. Applications :

The angle formed between two vectors given by $\sin\theta=\frac{\lfloor\vec{A}\wedge\vec{B}\rfloor}{|\vec{A}||\vec{B}|}$

6- Cross product of Cartesian unit vectors

$\vec{i}\wedge\vec{j}=\vec{k}$, $\vec{j}\wedge\vec{k}=\vec{i}$, $\vec{k}\wedge\vec{i}=\vec{j}$, while

$\vec{i}\wedge\vec{i}=0$, $\vec{j}\wedge\vec{j}=0$, $\vec{k}\wedge\vec{k}=0$., see below figure



Triple Scalar Product

For three vectors $\vec{A}, \vec{B}, \vec{C}$ the Triple Scalar Product define as

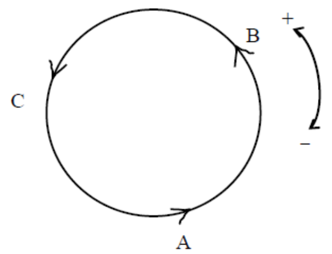
$$\vec{A}\cdot(\vec{B}\wedge\vec{C})=\begin{vmatrix}a_X & a_Y & a_Z\\ b_X & b_Y & b_Z\\ c_X & c_Y & c_Z\end{vmatrix}$$

Note that

(1) $\vec{A}\cdot(\vec{B}\wedge\vec{C})=(\vec{B}\wedge\vec{C})\cdot A$,

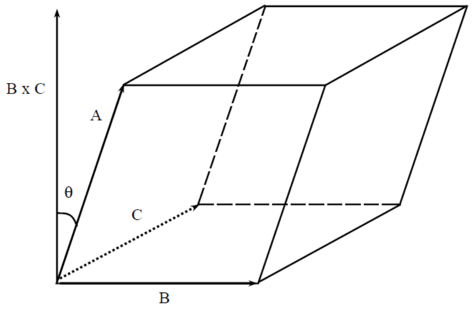
(2) $\vec{A}\cdot(\vec{B}\wedge\vec{C})=\vec{C}\cdot(\vec{A}\wedge\vec{B})=\vec{B}\cdot(\vec{C}\wedge\vec{A})$

(3) $\vec{A}\cdot(\vec{B}\wedge\vec{C})=-\vec{A}\cdot(\vec{C}\wedge\vec{B})=-\vec{B}\cdot(\vec{A}\wedge\vec{C})$, see below figure



(4)- Triple Scalar Product represents the volume of parallelepiped

$\vec{A}\cdot\left(\vec{B}\wedge\vec{C}\right)=$ the volume of parallelepiped



Triple Vector Product

For three vectors $\vec{A}, \vec{B}, \vec{C}$ the Triple Vector Product define as $\vec{A}\wedge\left(\vec{B}\wedge\vec{C}\right)$

Note that

(1)- $\vec{A}\wedge\left(\vec{B}\wedge\vec{C}\right)\neq\vec{A}\wedge\left(\vec{C}\wedge\vec{B}\right)$,

(2)- $\vec{A}\wedge\left(\vec{B}\wedge\vec{C}\right)=\left(\vec{A}\cdot\vec{C}\right)\vec{B}-\left(\vec{A}\cdot\vec{B}\right)\vec{C}$,(prove that this property?)

Unit vector

A unit vector is a vector that has a magnitude of 1

$$\text{Unit vector}=\frac{\text{Vector}}{\text{Magnitude of the vector}}$$

Example **1:** Given a vector $\vec{r}=4\vec{i}-3\vec{j}$, find the unit vector?

Solution

$$r=|\vec{r}|=\sqrt{(4)^2+(-3)^2}=\sqrt{16+9}=\sqrt{25}=5$$

$$\vec{u}=\frac{\vec{r}}{|\vec{r}|}=\frac{4}{5}\vec{i}-\frac{3}{5}\vec{j}=\frac{1}{5}\left(4,-3\right)$$

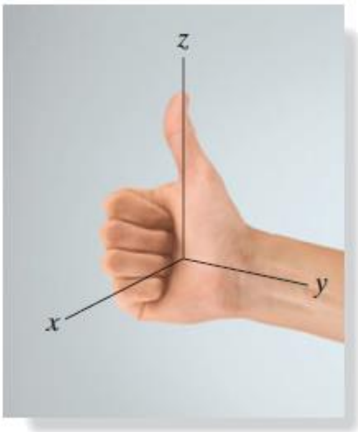
Cartesian vector

Introduction

The operations of vector algebra, when applied to solving problems in three dimensions , are greatly simplified if the vectors are first represented in Cartesian vector form. In this section we will present a general method for doing this; then in the next section we will use this method for finding the resultant force of a system of concurrent forces.

Right-Handed Coordinate System.

We will use a right-handed coordinate system to develop the theory of vector algebra that follows. A rectangular coordinate system is said to be right-handed if the thumb of the right hand points in the direction of the positive z axis when the right-hand fingers are curled about this axis and directed from the positive x towards the positive y axis, as in Figure.



Position Vectors

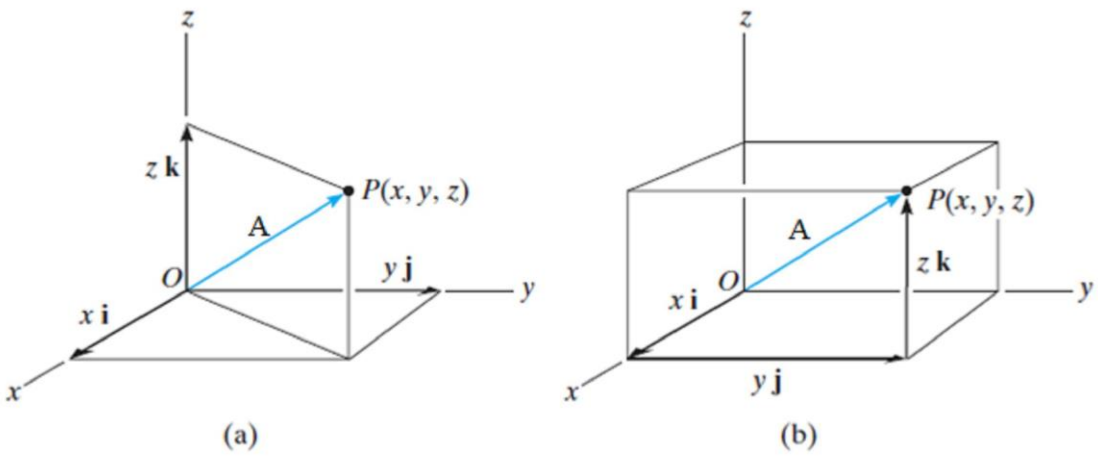
Here we will introduce the concept of a position vector. It will be shown that this vector is of importance in formulating a Cartesian force vector directed between two points in space.

A position vector $\vec{A}$ is defined as a fixed vector which locates a point in space relative to another point. For example, if $\vec{A}$ extends from the origin of coordinates, O, to point $P(x, y, z)$ (Fig. a), then $\vec{A}$ can be expressed in Cartesian vector form as

$$\vec{A} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$$

Note how the head-to-tail vector addition of the three components yields vector r (Fig. b) . Starting at the origin O , one "travels" x in the $+i$

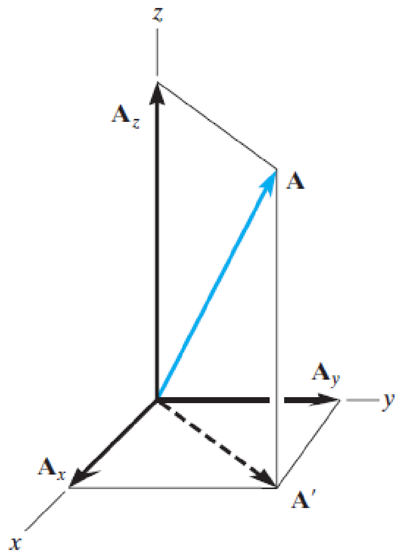
direction, then y in the $+j$ direction, and finally z in the $+k$ direction to arrive at point $P(x, y, z)$



<u>**Rectangular Components of a Vector.**</u>

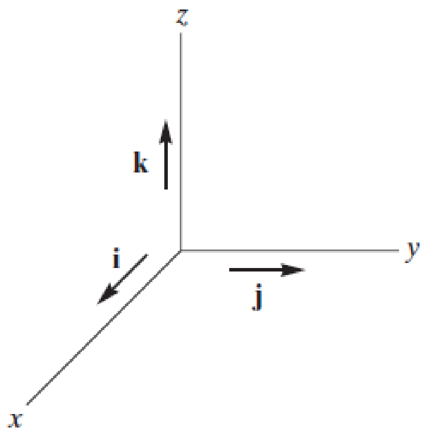
A vector A may have one, two, or three rectangular components along the x, y, z coordinate axes, depending on how the vector is oriented relative to the axes. In general, though, when A is directed within an octant of the x, y, z frame (see front Figure), then by two successive applications of the parallelogram law, we may resolve the vector into components as $A = A' + A_z$ and then $A' = A_x + A_y$ Combining these equations, to eliminate A', A is represented by the vector sum of its three rectangular components,

$A = A_x + A_y + A_z$



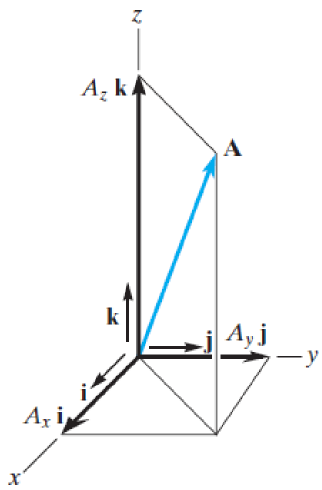
Cartesian Unit Vectors.

In three dimensions, the set of Cartesian unit vectors, i, j, k is used to designate the directions of the x, y, z axes, respectively. The sense (or arrowhead) of these vectors will be represented analytically by a plus or minus sign, depending on whether they are directed along the positive or negative x, y or z axes. The positive Cartesian unit vectors are shown in Figure.



Cartesian Vector Representation.

Since the three components of $\vec{A}$ in above equation act in the positive $\vec{i}$, $\vec{j}$ and $\vec{k}$ directions (see Figure) , we can write $\vec{A}$ in Cartesian vector form as $\vec{A} = a_x\vec{i} + a_y\ \vec{j} + a_z\vec{k}$



Magnitude of a Cartesian Vector.

The magnitude of a Cartesian vector is $\vec{A} = \sqrt{(a_x)^2 + (a_y)^2 + (a_z)^2}$. The

magnitude of $\vec{A}$ is equal to the positive square root of the sum of the squares of its components.

Direction of a Cartesian Vector.

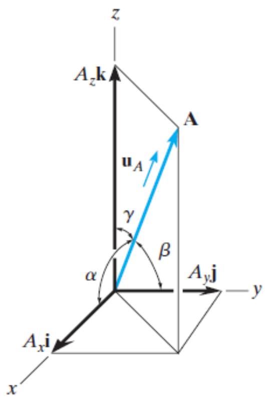
We will define the direction of $\vec{A}$ by the coordinate direction angles a α, β, γ , measured between the tail of $\vec{A}$ and the positive x, y, z axes provided they are located at the tail of $\vec{A}$ (see Figure). Note that regardless of where $\vec{A}$ is directed, each of these angles will be between 0^O and 180^O.

The angles α, β, γ given by $\cos\alpha = \frac{a_x}{A}, \quad \cos\beta = \frac{a_y}{A}, \quad \cos\gamma = \frac{a_z}{A}$

From this Eq. we have $A^2\cos\alpha = a_x^2, \quad A^2\cos\beta = a_y^2, \quad A^2\cos\gamma = a_z^2$

$A^2\left(\cos^2\alpha + \cos^2\beta + \cos^2\gamma\right) = a_x^2 + a_y^2 + a_z^2 = A^2$. Then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$



Note that, the unit vector given by

$\vec{A}_u = \frac{a_x}{A}\vec{i} + \frac{a_y}{A}\vec{j} + \frac{a_z}{A}\vec{k}$. Also given by $\vec{A}_u = \cos\alpha\,\vec{i} + \cos\beta\,\vec{j} + \cos\gamma\,\vec{k}$.

Then the direction of vector $\vec{A}$ given by

$$\cos\alpha = \frac{a_x}{A},\quad \cos\beta = \frac{a_y}{A},\quad \cos\gamma = \frac{a_z}{A}$$

In the space represent the force as:

(1) If we know two angles with two axes

In this case $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ and the force given by

$\vec{F} = F\cos\alpha\,\vec{i} + F\cos\beta\,\vec{j} + F\cos\gamma\,\vec{k}$

(2) If we know an angles with two axes in plane

In this case we resolve the force in the vertical axis and in the other plane

(3) If we know the unite vector in space

In this case $\vec{A}_u = \frac{a_x}{A}\vec{i} + \frac{a_y}{A}\vec{j} + \frac{a_z}{A}\vec{k}$, $\vec{A}_u = \cos\alpha\,\vec{i} + \cos\beta\,\vec{j} + \cos\gamma\,\vec{k}$, and

$\cos\alpha = \frac{a_x}{A},\quad \cos\beta = \frac{a_y}{A},\quad \cos\gamma = \frac{a_z}{A}$.

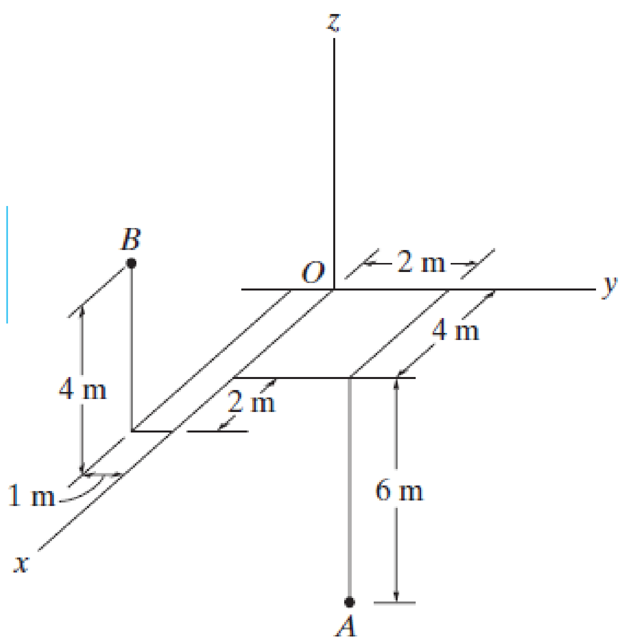
Example 2: Given a vector $\vec{r} = 12\vec{i} - 3\vec{j} - 4\vec{k}$, find the unit vector?

Solution

$$r = |\vec{r}| = \sqrt{(12)^2 + (-3)^2 + (-4)^2} = \sqrt{144 + 9 + 16} = \sqrt{169} = 13$$

$$\vec{u} = \frac{\vec{r}}{|\vec{r}|} = \frac{12}{13}\vec{i} - \frac{3}{13}\vec{j} - \frac{4}{13}\vec{k} = \frac{1}{13}\left(12, -3, 4\right)$$

Example 3: Determine the length and its direction measured from B toward A as shown in Figure?



Solution

The coordinates of A and B are $B(6,-1,4)$, $A(4,2,-6)$

$$\vec{AB} = \vec{r} = (6-4)\vec{i} + (-1-2)\vec{j} + (4-(-6))\vec{k}$$

$$\vec{r} = 2\vec{i} - 3\vec{j} + 10\vec{k}$$

$$r = |\vec{r}| = \sqrt{(2)^2 + (-3)^2 + (10)^2} = \sqrt{4+9+100} = \sqrt{113}$$

$$\vec{u} = \frac{\vec{AB}}{|\vec{AB}|} = \frac{\vec{r}}{|\vec{r}|} = \frac{2}{\sqrt{113}}\vec{i} - \frac{3}{\sqrt{113}}\vec{j} + \frac{10}{\sqrt{113}}\vec{k}$$

The direction of $\vec{AB} = \vec{r}$ given by

$$\cos\alpha = \frac{\vec{r}_x}{r}, \quad \cos\beta = \frac{\vec{r}_y}{r}, \quad \cos\gamma = \frac{\vec{r}_z}{r}$$

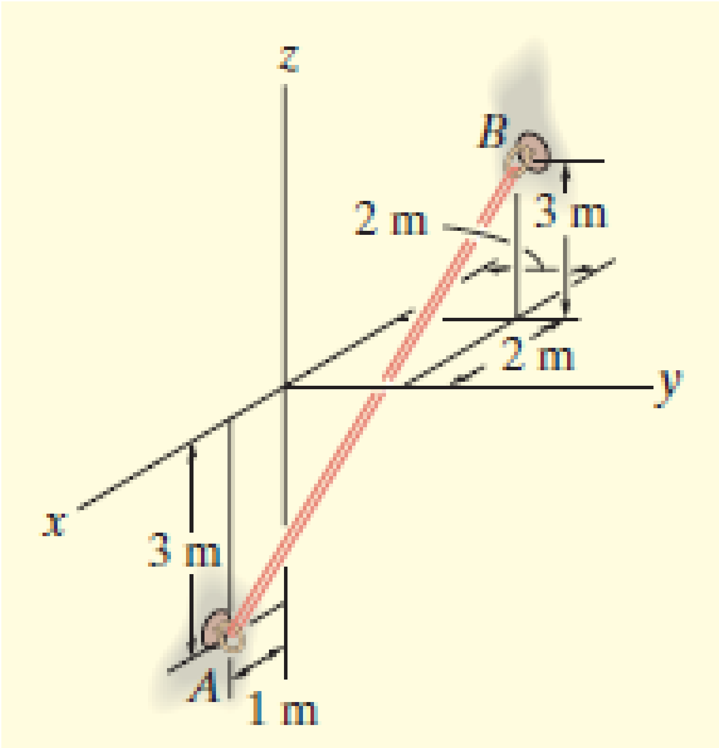
$$\cos\alpha = \frac{2}{\sqrt{113}}, \quad \cos\beta = -\frac{3}{\sqrt{113}}, \quad \cos\gamma = \frac{10}{\sqrt{113}}$$

$$\alpha = \cos^{-1}\left(\frac{2}{\sqrt{113}}\right) = 79.15545231^O,$$

$$\beta = \cos^{-1}\left(-\frac{3}{\sqrt{113}}\right) = 106.3927403^O,$$

$$\gamma = \cos^{-1}\left(\frac{10}{\sqrt{113}}\right) = 19.8136571^O$$

Example 4: An elastic rubber band is attached to points A and B as shown in Figure . Determine its length and its direction measured from A toward B ?



Solution

The coordinates A and B are $A(1,0,-3)$, $B(-2,2,3)$

$$\overrightarrow{AB} = \vec{r} = (-2-(-1))\vec{i} + (2-0)\vec{j} + (3-(-3))\vec{k}$$

$$\vec{r} = -3\vec{i} + 2\vec{j} + 6\vec{k}$$

$$r = |\vec{r}| = \sqrt{(-3)^2 + (2)^2 + (6)^2} = \sqrt{9+4+36} = \sqrt{49} = 7$$

he T $\vec{u} = \dfrac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|} = \dfrac{\vec{r}}{|\vec{r}|} = -\dfrac{3}{7}\vec{i} + \dfrac{2}{7}\vec{j} + \dfrac{6}{7}\vec{k}$

direction of $\overrightarrow{AB} = \vec{r}$ given by. $\cos\alpha = \dfrac{\vec{r}_x}{r}, \quad \cos\beta = \dfrac{\vec{r}_y}{r}, \quad \cos\gamma = \dfrac{\vec{r}_z}{r}$

Then $\cos\alpha = -\dfrac{3}{7}, \quad \cos\beta = \dfrac{2}{7}, \quad \cos\gamma = \dfrac{6}{7}$. This tends to

$$\alpha = \cos^{-1}\left(-\frac{3}{7}\right) = 115.376^O, \ \beta = \cos^{-1}\left(\frac{2}{7}\right) = 73.398^O,$$

$$\gamma = \cos^{-1}\left(\frac{6}{7}\right) = 31^O$$

Part 2: Vectors Forces in 2D and 3D

Resultant of Concurrent Coplanar Forces

How to calculate the resultant force acting on an object? Several forces can act on a body or point, each force having different direction and magnitude. In engineering the focus is on the resultant force acting on the body. The resultant of concurrent forces (acting in the same plane) can be found using the parallelogram law, the triangle rule or the polygon rule.. Two or more forces are concurrent is their direction crosses through a common point. For example, two concurrent forces F_1 and F_2 are acting on the same point P. In order to find their resultant F, we can apply either the parallelogram law, triangle rule.

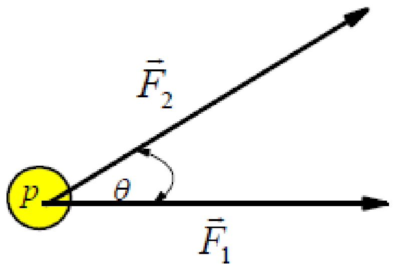


Fig. 1

A force is a vector quantity since it has magnitude and direction. There force, the force addition will be according to the Parallelogram law

Parallelogram in 2D:

Redraw a half portion of the parallelogram to illustrate the triangular head-to-tail addition of the components. From this triangle, the magnitude of the resultant force can be determined using the law of *Cosine*, and its direction is determined from the law of *Sine*. The magnitudes of two force components are determined from the law of *Sine*. The formulas are given in Figure.

Cosine law

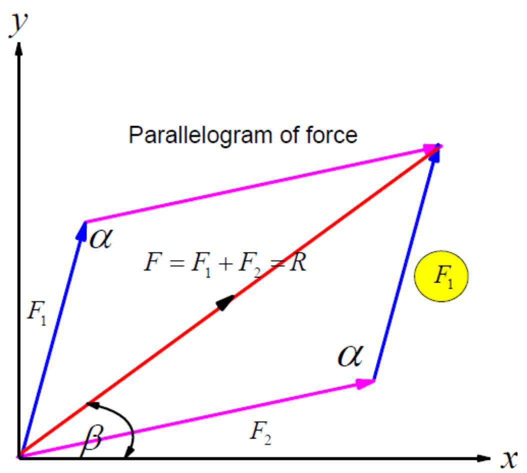


Fig. 2 (a)

From the triangle force, the resultant force is the vector sum between the components:

$$\vec{F} = \vec{F}_1 + \vec{F}_2$$

Cosine law is $F = \sqrt{F_1^2 + F_2^2 - 2F_1F_2\cos\alpha}$

Sine law

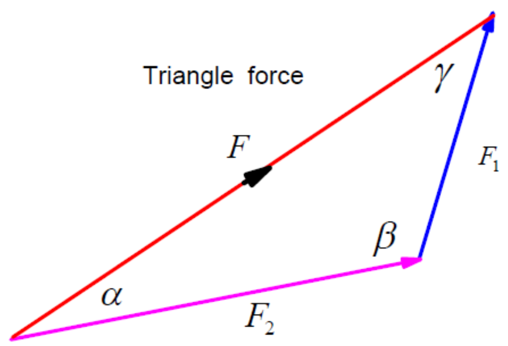


Fig. 2 (b)

Sine law is $\dfrac{F_1}{\sin\alpha} = \dfrac{F_2}{\sin\gamma} = \dfrac{F}{\sin\beta}$

Components

Also, we can find the resultant of Concurrent Coplanar Forces as

Step 1: to resolve each force into its $x-y$ components.

Step 2: to add all the x components together and add all the y components together. These two totals become the resultant vector.

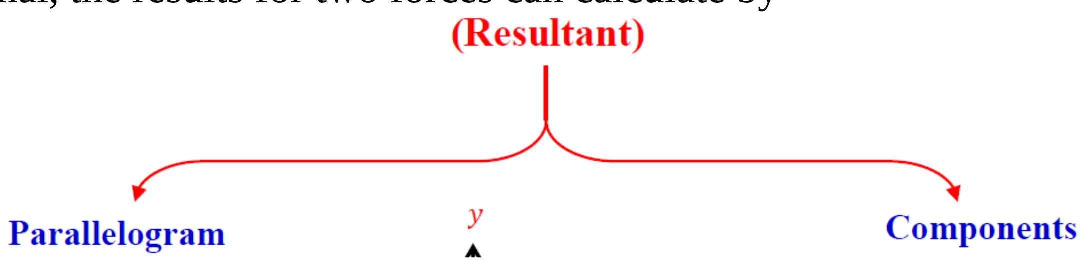
Step 3: find the magnitude and the angle of the resultant vector.

I. e, we calculate both $\sum F_x$ and $\sum F_y$, then the resultant is

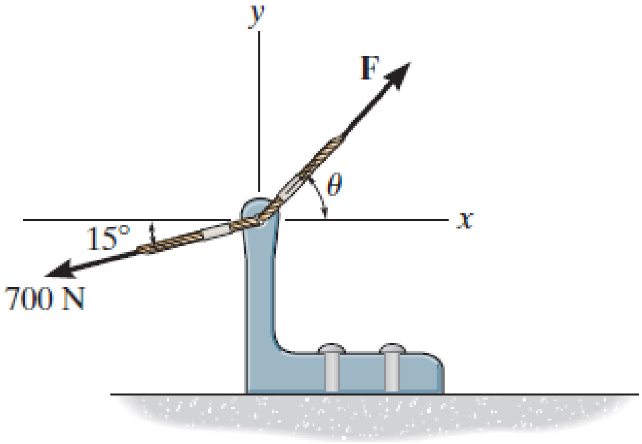
$$F = \sqrt{\left(\sum F_x\right)^2 + \left(\sum F_y\right)^2}$$

The direction is $\theta = \tan^{-1}\left(\dfrac{\sum F_y}{\sum F_x}\right)$.

Final, the results for two forces can calculate by

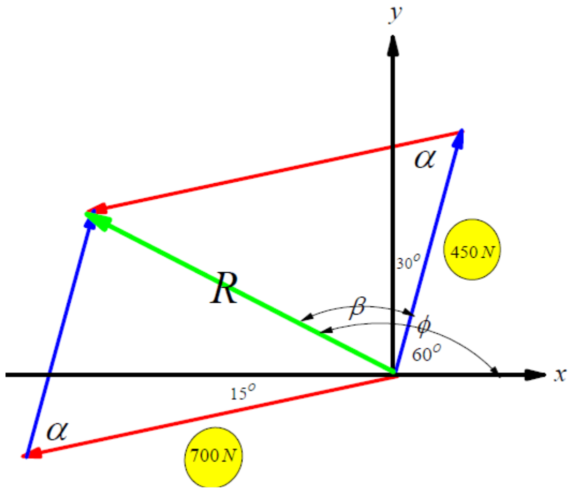


Example 1: If $\theta = 60^{o}$ and $F = 450\ N$, determine the magnitude of the resultant force and its direction, measured counterclockwise from the positive $x-$ axis.



Solution

We can draw the forces as in below Figure (Free-Body Diagrams)



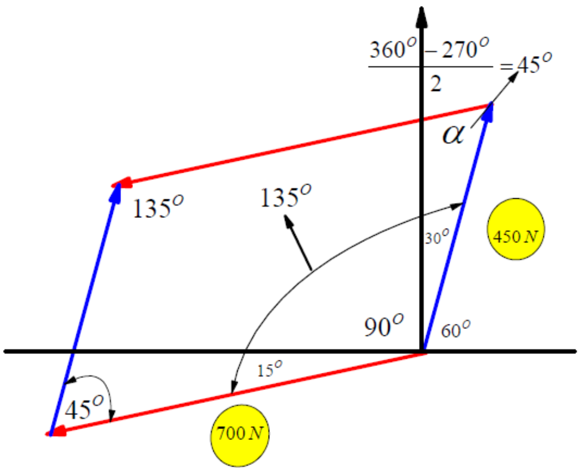


Fig. (1)

Applying the law of Cosine to Fig. (1),This yields

$F_R = \sqrt{(700)^2 + (450)^2 - 2(700)(450)\cos 45^O} = 497\,N$ Again,

applying the law of Sine to Fig. (1), and using this result, yields

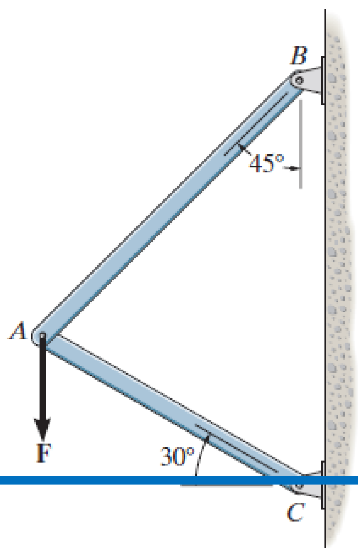
$$\frac{\sin(\beta + 30^o)}{700} = \frac{\sin 45^o}{497.01} \rightarrow \beta = 95.19^O$$

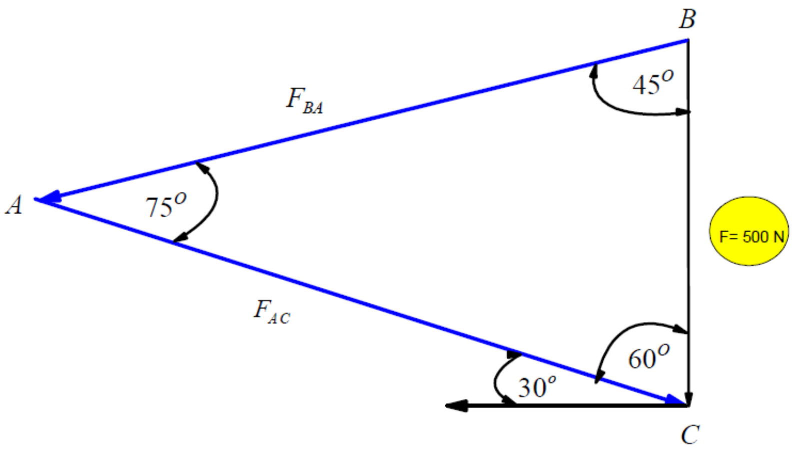
Thus, the direction of angle of measured counterclockwise from the positive $x-$ axis is

$\phi = \alpha + 60^O = 95.19^O + 60^O = 155.19^O$

Solve this problem using the components method?

Example 2: The vertical force F acts downward at A on the two membered frame. Determine the magnitudes of the two components of F directed along the axes of BA and AC. Set $F = 500$?





From the Cosine low $F=\sqrt{F_1^2+F_2^2-2F_1F_2\cos\alpha}$

$F=\sqrt{(6)^2+(4)^2-2(6)(4)\cos105^O}=\sqrt{36+16-(48)(-0.2588)}$

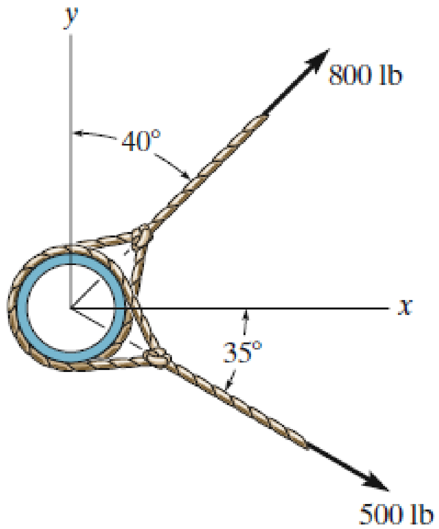
$=\sqrt{52+12.423}=\sqrt{64.423}=8.03\ N$

From the Sine law $\frac{F}{\sin105^o}=\frac{F_1}{\sin\alpha}\rightarrow\sin\alpha=\frac{F_1}{F}\sin105^o=$

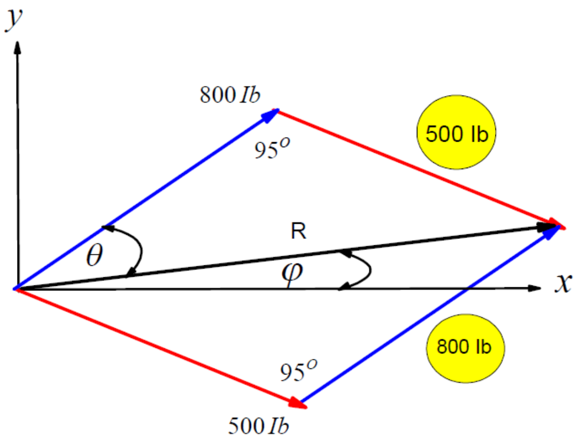
$\frac{4}{8.03}(0.9659)=\frac{3.8637}{8.03}=0.4811\rightarrow\alpha=28.761$

Then $\theta=28.761-25=3.761^O$

Example 3: Determine the magnitude of the resultant force (see below Figure) and its direction, measured counterclockwise from the positive x axis ?



Solution



From the Cosine low $F = \sqrt{F_1^2 + F_2^2 - 2F_1F_2\cos\alpha}$

$R = \sqrt{(500)^2 + (800)^2 - 2(500)(800)\cos 95^O} = \sqrt{(500)^2 + (800)^2 - 2(500)(800)(-0.08715)}$

$= \sqrt{250000 + 640000 + 69724.594} = \sqrt{959724.594} = 979.65539 = 980 Ib$

Using this result to apply the sine law

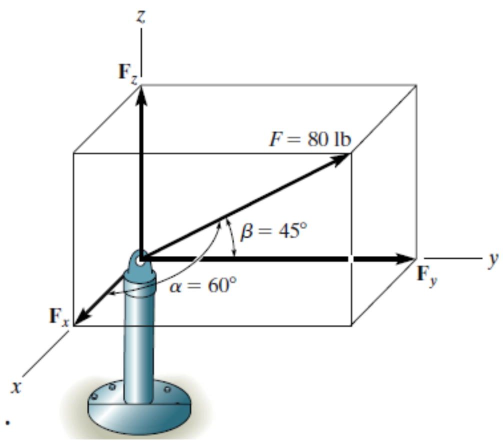
$\frac{R}{\sin 95^o} = \frac{500}{\sin\theta} \rightarrow \frac{980}{\sin 95^o} = \frac{500}{\sin\theta} \rightarrow \sin\theta = 500\frac{980}{\sin 95^o} = 500\left(\frac{0.9961}{980}\right) = 0.5082$

$\theta = \sin^{-1}(0.5082) \rightarrow \theta = 30.54495^O$

Thus, the direction f of R measured counterclockwise from the positive x axis is

$\theta = 50 - 30.54495^O = 19.54^O$

Example 4: The force F has a magnitude of $80\ Ib$ and acts within the octant shown. Determine the magnitudes of the x, y, z components of F?



Solution

From the Figure clears that $\alpha = 60^O$ and $\beta = 45^O$ and using the relation

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, we find that $(\cos(60^O))^2 + (\cos(45^O))^2 + \cos^2\gamma = 1$

$(0.5)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2\gamma = 1 \rightarrow 0.25 + 0.5 + \cos^2\gamma = 1 \rightarrow +\cos^2\gamma = 0.25\ \cos^2\gamma = \pm 0.5 \rightarrow \gamma = 60^o$

Or $\gamma = 120^o$

By inspection it is necessary that $\gamma = 60^o$, since F_x must be in the $+x$

Now using the relation $\vec{F} = F\cos\alpha\,\vec{i} + F\cos\beta\,\vec{j} + F\cos\gamma\,\vec{k}$, we have

$\vec{F} = 80\left(\cos 60^o\ \vec{i} + \cos 45^o\ \vec{j} + \cos 60^o\ \vec{k}\right)$

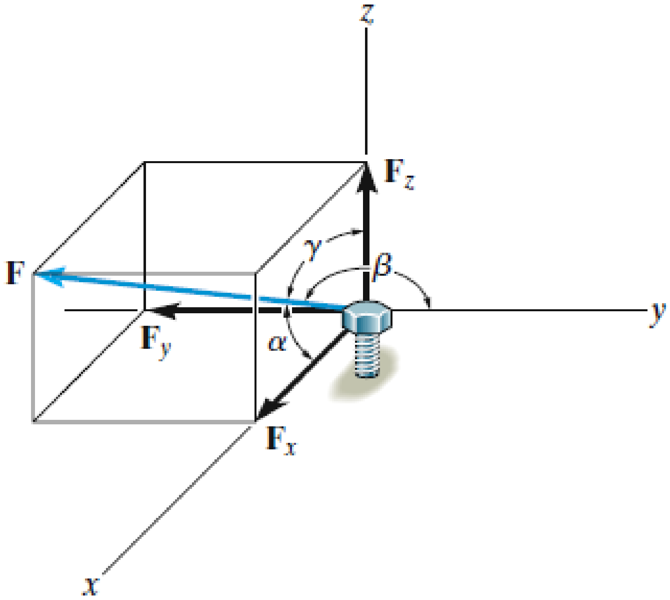
$\vec{F} = 80\left(0.5\vec{i} + \frac{1}{\sqrt{2}}\ \vec{j} + 0.5\vec{k}\right) = 80\left(0.5\vec{i} + \frac{\sqrt{2}}{(\sqrt{2})(\sqrt{2})}\ \vec{j} + 0.5\vec{k}\right)$

$\vec{F} = 80\left(0.5\vec{i} + (0.5)(\sqrt{2})\ \vec{j} + 0.5\vec{k}\right) = 40\left(\vec{i} + \sqrt{2}\ \ \vec{j} + \vec{k}\right)$

So, $\vec{F}_X = 40\,Ib,\quad \vec{F}_y = 40\sqrt{2}\ Ib,\quad \vec{F}_Z = 40\,Ib$

Not that $F = 40\sqrt{1+2++1} = 40\sqrt{4} = 40(2) = 80\ Ib$

Example 5: The bolt is subjected to the force F, which has components acting along the x, y, z axes as shown. If the magnitude of F is $80\ N$, $\alpha = 60^O$ and $\gamma = 45^O$ determine the magnitudes of its components (see below Figure).



Solution

From the Figure clears that $\alpha = 60^O$ and $\gamma = 45^O$ and using the relation

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, we find that $\left(\cos(60^O)\right)^2 + \cos^2\beta + \left(\cos(45^O)\right)^2 = 1$

$(0.5)^2 + \cos^2\beta + \left(\frac{1}{\sqrt{2}}\right)^2 = 1 \rightarrow 0.25 + \cos^2\beta + 0.5 = 1 \rightarrow \cos^2\beta = 0.25$

$\cos^2\beta = \pm 0.5 \quad \rightarrow \quad \beta = 60^o$ Or $\beta = 120^o$

By inspection it is necessary that $\beta = 120^o$, since F_x must be in the $+x$

Now using the relation $\vec{F} = F\cos\alpha\,\vec{i} + F\cos\beta\ \vec{j} + F\cos\gamma\,\vec{k}$, we have

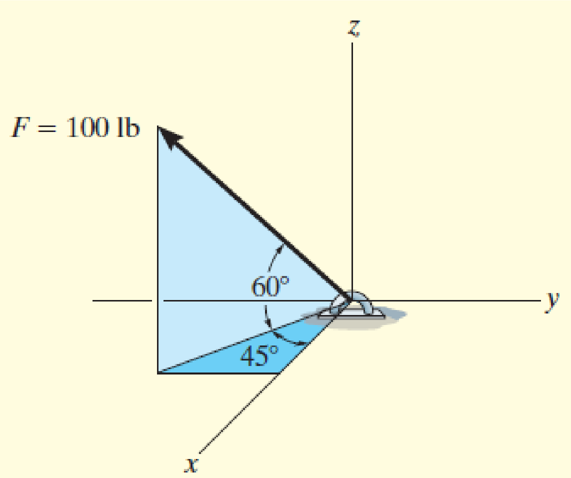
$\vec{F} = 80\left(\cos 60^o\ \vec{i} + \cos 120^o\ \vec{j} + \cos 45^o\ \vec{k}\right)$

$\vec{F} = 80\left(0.5\vec{i} - 0.5\ \vec{j} + \frac{1}{\sqrt{2}}\vec{k}\right) = 80\left(0.5\vec{i} - 0.5\ \vec{j} + \frac{\sqrt{2}}{(\sqrt{2})(\sqrt{2})}\vec{k}\right)$

$\vec{F} = 80\left(0.5\vec{i} - (0.5)\ \vec{j} + 0.5\left(\sqrt{2}\right)\vec{k}\right) = 40\left(\vec{i} - \ \vec{j} + \sqrt{2}\ \vec{k}\right)$

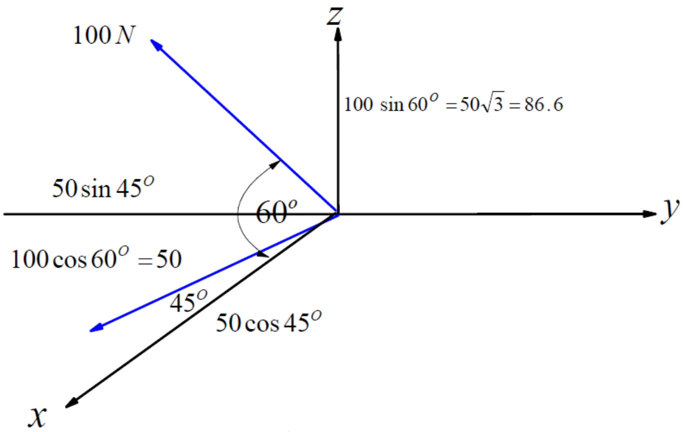
So, $\vec{F}_X = 40\ N,\quad \vec{F}_y = 40\ N,\quad \vec{F}_Z = 40\sqrt{2}\ N$

Example 6: Express the force F shown in Figure as a Cartesian vector and its direction?



Solution

If we resolve the force as in Figure



Then $\vec{F} = \left(50\cos 45^{O}\ \vec{i} - 50\sin 45^{O}\ \vec{j} + 86.6\vec{k}\right)$

$$\vec{F} = \left(\frac{50}{\sqrt{2}}\ \vec{i} - \frac{50}{\sqrt{2}}\ \vec{j} + 86.6\vec{k}\right)$$

$$\vec{F} = \left(35.4\ \vec{i} - 35.4\ \vec{j} + 86.6\vec{k}\right)$$

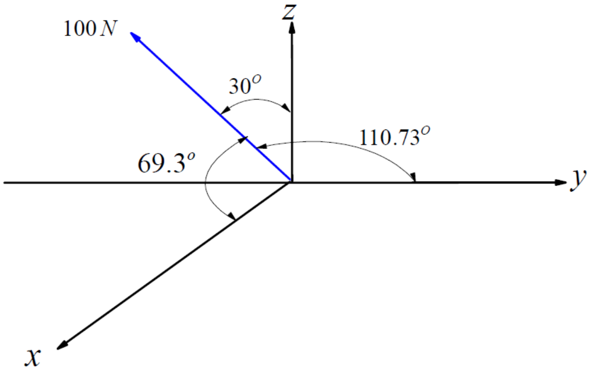
Not that $F = \sqrt{(35.4)^2 + (35.4)^2 + (86.6)^2} = \sqrt{10005} \cong 100\ N$

The direction of the force given by

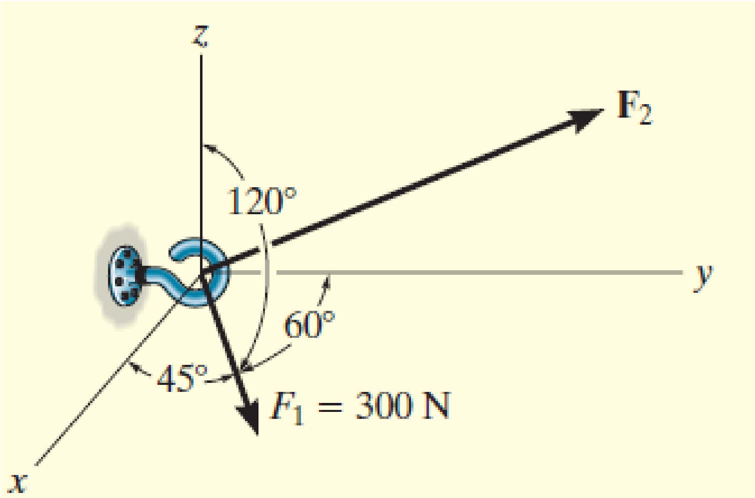
$$\cos\alpha = \frac{F_x}{F},\quad \cos\beta = \frac{F_y}{F},\quad \cos\gamma = \frac{F_z}{F}$$

$$\cos\alpha = \frac{35.4}{100},\quad \cos\beta = -\frac{35.4}{100},\quad \cos\gamma = \frac{86.6}{100}$$

$$\alpha = \cos^{-1}\left(\frac{35.4}{100}\right) = 69.3^{O},\quad \beta = \cos^{-1}\left(-\frac{35.4}{100}\right) = 110.73^{O},\quad \gamma = \cos^{-1}\left(\frac{86.6}{100}\right) = 30^{O}$$

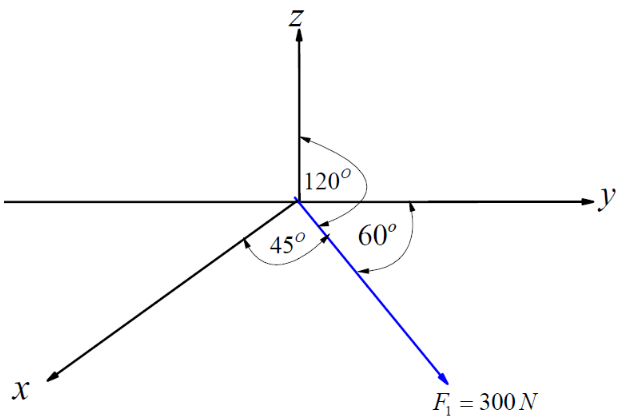


Example 7: Two forces act on the hook shown in below Figure. Specify the magnitude of F_2 and its coordinate direction angles so that the resultant force F_R acts along the positive y axis and has a magnitude of $800N$.



Solution

Free-Body Diagram. we can plot the Free-Body Diagram as in figure

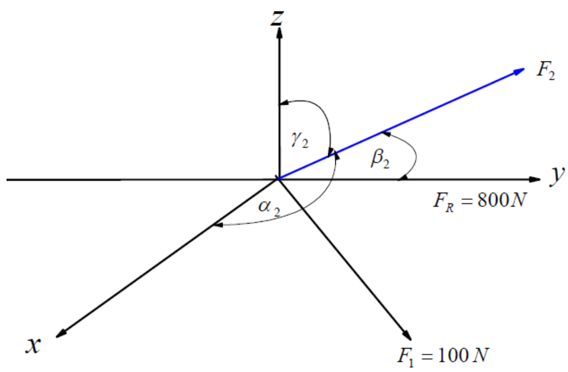


So, we can expresses $\vec{F}_1$ as follows:

$$\vec{F}_1 = 300\left(\cos 45^O\,\vec{i} + \cos 60^O\,\vec{j} + \cos 120^O\,\vec{k}\right)\vec{F}_1 = 300\left(\frac{1}{\sqrt{2}}\,\vec{i} + 0.5\,\vec{j} - 0.5\,\vec{k}\right) = 150\left(\sqrt{2}\,\vec{i} + \vec{j} - \vec{k}\right).$$

Also, $\vec{F}_R = 800\,\vec{j}$

We require $\vec{F}_R = \vec{F}_1 + \vec{F}_2$ (see below Figure)



Then

$$800\vec{j} = 150\left(\sqrt{2}\,\vec{i} + \vec{j} - \vec{k}\right) + F_{2x}\,\vec{i} + F_{2y}\,\vec{j} + F_{2z}\,\vec{k}$$

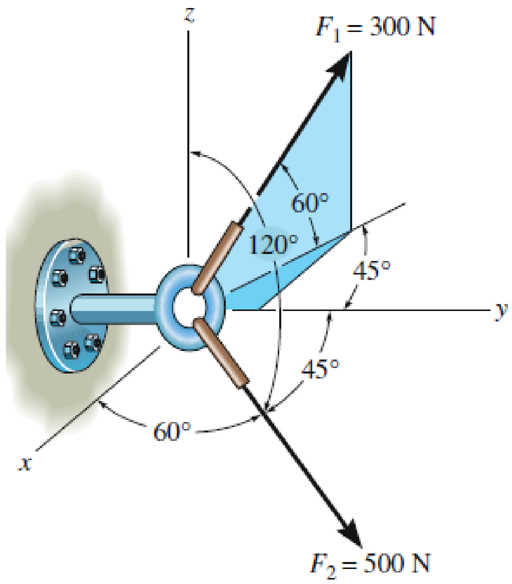
$$F_{2x} = -150\sqrt{2}, \quad F_{2y} = 650, \quad F_{2z} = 150$$

$$F_2 = \sqrt{(F_{2x})^2 + (F_{2y})^2 + (F_{2z})^2} = \sqrt{45000 + 422500 + 22500} = \sqrt{490000} = 700$$

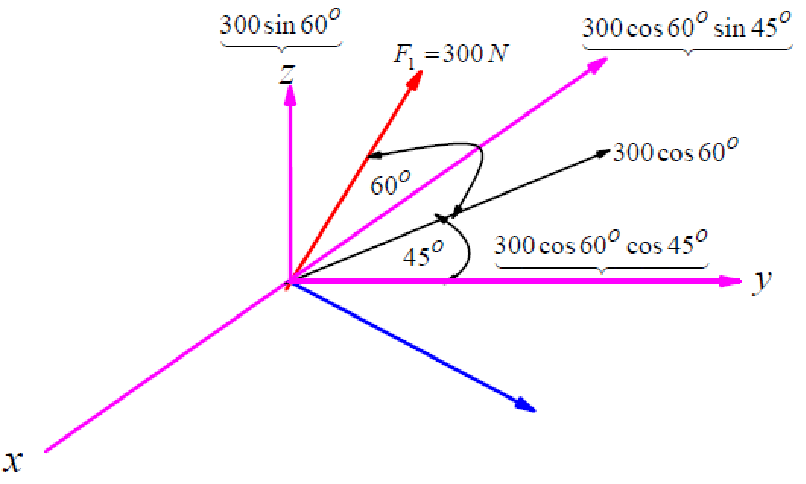
The direction of F_2 given from $\cos\alpha_2 = \frac{F_{x2}}{F_2}$, $\cos\beta_2 = \frac{F_{y2}}{F_2}$, $\cos\gamma_2 = \frac{F_{z2}}{F_2}$

$$\alpha_2 = \cos^{-1}\left(-\frac{150\sqrt{2}}{700}\right) = 107.64^O, \quad \beta_2 = \cos^{-1}\left(\frac{650}{700}\right) = 21.8^O, \quad \gamma_2 = \cos^{-1}\left(\frac{150}{700}\right) = 77.6^O$$

Example 8: The screw eye is subjected to the two forces as shown below Figure. Express each force in Cartesian vector form and then determine the resultant force. Find the magnitude and coordinate direction angles of the resultant force.



Solution

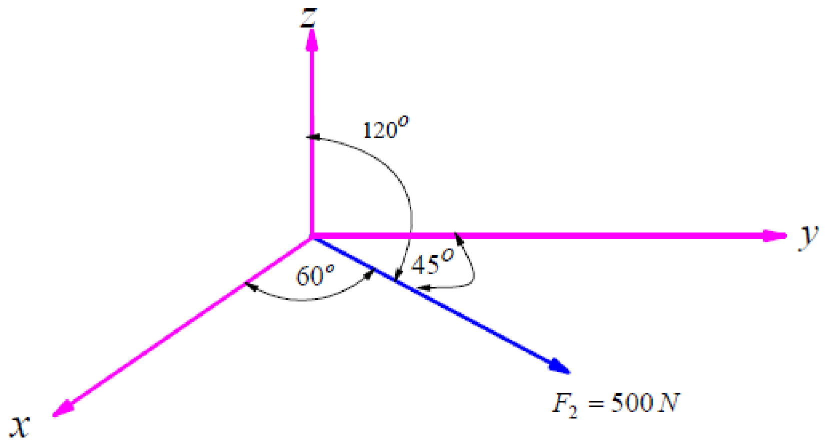


$$\vec{F}_1 = 300\left(-\cos 60^o \sin 45^O \vec{i} + \cos 60^o \cos 45^O \vec{j} + \sin 60^O \vec{k}\right)$$

$$\vec{F}_1 = 300\left(-(0.5)(\frac{1}{\sqrt{2}})\ \vec{i} + (0.5)\ (\frac{1}{\sqrt{2}})\ \vec{j} + \frac{\sqrt{3}}{2}\ \vec{k}\right)$$

$$\vec{F}_1 = 300\left(-(0.3535)\ \vec{i} + (0.3535)\ \vec{j} + 1.732\vec{k}\right)$$

$$\vec{F}_1 = \left(-(106.07)\ \vec{i} + 106.07\ \vec{j} + 259.81\vec{k}\right)$$



Using the relation $\vec{F} = F\cos\alpha\vec{i} + F\cos\beta\ \vec{j} + F\cos\gamma\vec{k}$, we have

$$\vec{F}_2 = 500\left(\cos 60^O \vec{i} + \cos 45^O\ \ \vec{j} + \cos 120^O \vec{k}\right)$$

$$\vec{F}_2 = 500\left(\ (0.5)\ \vec{i} + \frac{1}{\sqrt{2}}\ \vec{j} - (0.5)\ \vec{k}\right)$$

$$\vec{F}_2 = 500\left((0.5)\ \vec{i} + (0.7071)\ \vec{j} - (0.5)\ \vec{k}\right)$$

$$\vec{F}_2 = \left(250\ \vec{i} + (353.5542)\ \vec{j} - 250\ \vec{k}\right)$$

So, the resultant is given by

$$\vec{F}_R = \vec{F}_1 + \vec{F}_2 = \left(-(106.07)\ \vec{i} + 106.07\ \vec{j} + 259.81\vec{k}\right) + \left(250\ \vec{i} + (353.5542)\ \vec{j} - 250\ \vec{k}\right)$$

$$\vec{F}_R = \left(143.93\vec{i} + 459.62\ \vec{j} + 9.81\vec{k}\right)$$

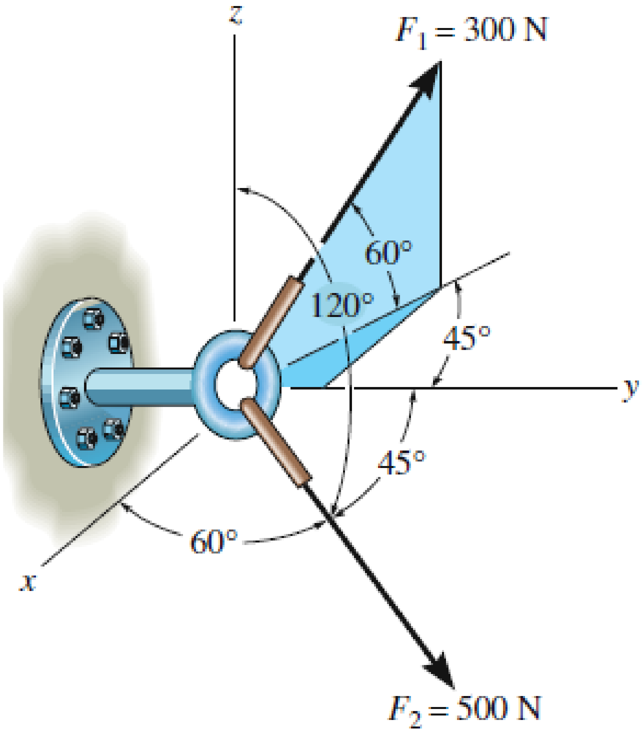
$$F_R = \sqrt{(143.93)^2 + (459.62)^2 + (9.81)^2} = 481.73\ N$$

$$\cos\alpha = \frac{143.93}{481.73} = 0.298777 \quad \rightarrow \quad \alpha = \cos^{-1}(0.298777) = 72.6158^{O}$$

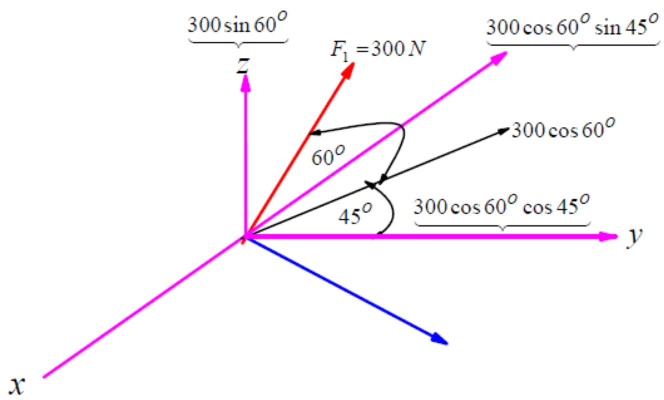
$$\cos\beta = \frac{459.62}{481.73} = 0.9541 \quad \rightarrow \quad \beta = \cos^{-1}(0.9451) = 17.42^{O}$$

$$\cos\gamma = \frac{9.81}{481.73} = 0.0203 \quad \rightarrow \quad \gamma = \cos^{-1}(0.0203) = 88.833^{O}$$

Example 9: Determine the coordinate direction angles of F_1.



Solution



$$\vec{F}_1 = 300\left(-\cos 60^o \sin 45^O \vec{i} + \cos 60^o \cos 45^O \vec{j} + \sin 60^O \vec{k}\right)$$

$$\vec{F}_1 = 300\left(-(0.5)(\frac{1}{\sqrt{2}})\ \vec{i} + (0.5)\ (\frac{1}{\sqrt{2}})\ \vec{j} + \frac{\sqrt{3}}{2}\ \vec{k}\right) \quad \vec{F}_1 = 300\left(-(0.3535)\ \vec{i} + (0.3535)\ \vec{j} + 1.732\vec{k}\right)$$

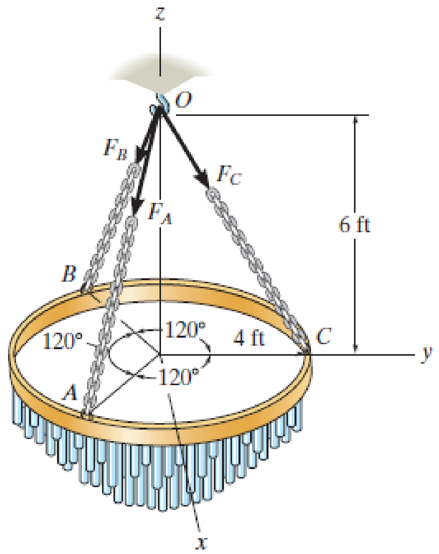
$\vec{F}_1 = \left(-(106.07)\,\vec{i} + 106.07\,\vec{j} + 259.81\vec{k}\right)$

$\cos\alpha = -\dfrac{106.07}{300} = -0.3569 \quad \rightarrow \quad \alpha = \cos^{-1}(-0.3569) = 110.7056^O$

$\cos\beta = \dfrac{106.07}{300} = 0.3569 \quad \rightarrow \quad \beta = \cos^{-1}(0.3569) = 69.09^O$

$\cos\gamma = \dfrac{259.81}{300} = 0.688033 \quad \rightarrow \quad \gamma = \cos^{-1}(0.688033) = 29.99909^O = 30^O$

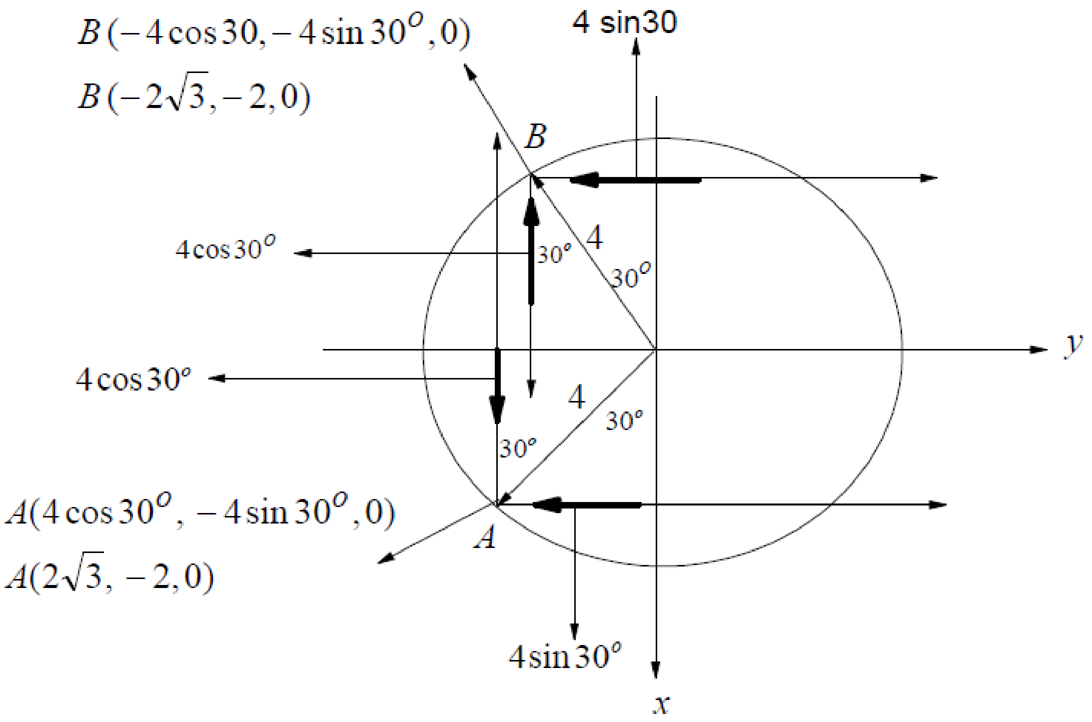
Example 10: A chandelier is supported by three chains which are concurrent at point O. If the resultant force at O has a magnitude of $130\ Ib$ and is directed along the negative $Z-$ axis, determine the force in each chain.



Solution

Equations of Equilibrium. First we will express each force in Cartesian vector form. Since the coordinates of points O, A, B and C are $O(0, 0, 6)$,

Figure) we havebelow (see $A(2\sqrt{3}, -2, 0)$, $B(-2\sqrt{3}, -2, 0)$, $C(0, 4, 0)$



$$\vec{F}_{OA} = \frac{F_{OA}}{\sqrt{52}}(2\sqrt{3}\,\vec{i} - 2\,\vec{j} - 6\,\vec{k})$$

$$\vec{F}_{OB} = \frac{F_{OB}}{\sqrt{52}}(-2\sqrt{3}\,\vec{i} - 2\,\vec{j} - 6\,\vec{k})$$

$$\vec{F}_{OC} = \frac{F_{OC}}{\sqrt{52}}(4\,\vec{j} - 6\,\vec{k})$$

The magnitude of resultant force is, i. e. $\vec{F}_R = \vec{F}_{OA} + \vec{F}_{OB} + \vec{F}_{OC}$

$$\vec{F}_R = 130\vec{k} = \frac{F_{OA}}{\sqrt{52}}(2\sqrt{3}\,\vec{i} - 2\,\vec{j} - 6\,\vec{k}) + \frac{F_{OB}}{\sqrt{52}}(-2\sqrt{3}\,\vec{i} - 2\,\vec{j} - 6\,\vec{k}) + \frac{F_{OC}}{\sqrt{52}}(4\,\vec{j} - 6\,\vec{k}) \qquad (1)$$

From Eq. (1), we have

$$0 = \frac{F_{OA}}{\sqrt{52}}\left(2\sqrt{3}\right) + \frac{F_{OB}}{\sqrt{52}}(-2\sqrt{3}) \quad \rightarrow \quad F_{OA} = F_{OB} \qquad (2)$$

Also, from Eq. (1), we have

$$0 = \frac{F_{OA}}{\sqrt{52}}(-2) + \frac{F_{OB}}{\sqrt{52}}(-2) + \frac{F_{OC}}{\sqrt{52}}(4) \;\rightarrow\; 0 = -F_{OA} - F_{OB} + 2F_{OC}$$

From Eq. (2) , we have $0 = -F_{OA} - F_{OA} + 2F_{OC} \;\rightarrow\; -2F_{OA} + 2F_{OC} = 0$

$$F_{OA} = F_{OC} \qquad (3)$$

From Eqs. (2) and (3) , we have

$$F_{OA} = F_{OB} = F_{OC} \qquad (4)$$

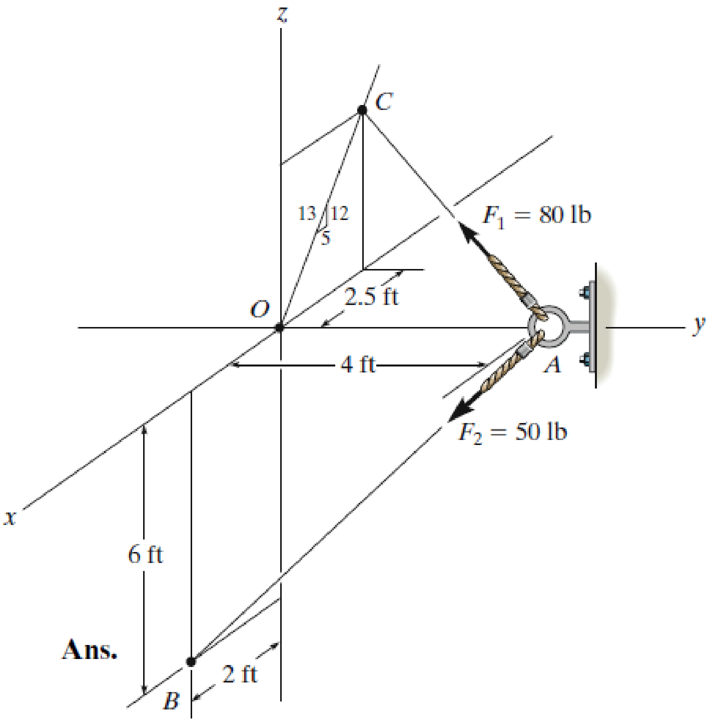
Again, from Eq. (1), we have

$130 = \frac{F_{OA}}{\sqrt{52}}(-6) + \frac{F_{OB}}{\sqrt{52}}(-6) + \frac{F_{OC}}{\sqrt{52}}(-6)$, and from Eq. (4), we have

$$130 = 3\left(\frac{F_{OA}}{\sqrt{52}}(-6)\right) \rightarrow \qquad F_{OA} = -\frac{130\sqrt{52}}{18} = -52.1\ Ib$$

Then the force in each chain are $F_{OA} = F_{OB} = F_{OC} = 52.1\ Ib$

Example: 11 Express each of the forces in Cartesian vector form and determine the magnitude and coordinate direction angles of the resultant force for the two forces act in below figure ?



Solution

Equations of Equilibrium. First we will express each force in Cartesian vector form. Since the coordinates of points A, B are given $A(0, 4, 0)$, $B(2, 0, -6)$

While, C is given as (see blow Figure) $C(-2.5, 0, \frac{12}{5}(2.5)) \to C(-2.5, 0, 6)$.

Since the force F_{AB} acts across the two points $A(0, 4, 0)$ and $B(2, 0, -6)$. So the unit vector in this direction is $\vec{e}_{AB} = \frac{1}{\sqrt{56}}(2, -4, -6)$, i. e. Then

$$\vec{F}_{AB} = F_{AB}\,\vec{e}_{AB} = \frac{F_{AB}}{\sqrt{56}}(2, -4, -6) = \frac{50}{\sqrt{56}}(2, -4, -6).$$

Also, force F_{AC} acts across the two points $A(0, 4, 0)$ and rSo the unit vecto .$C(-2.5, 0, 6)$ in this direction is $\vec{e}_{AC} = \frac{1}{\sqrt{58.25}}(-2.5, -4, 6)$, i. e. Then

$$.\vec{F}_{AC} = F_{AC}\,\vec{e}_{AC} = \frac{F_{AC}}{\sqrt{58.25}}(2.5, -4, 6) = \frac{80}{\sqrt{58.25}}(-2.5, -4, 6)$$

$$\vec{F} = \vec{F}_{AB} + \vec{F}_{AC}$$

$$\vec{F} = \frac{50}{\sqrt{56}}(2, -4, -6) + \frac{80}{\sqrt{58.25}}(-2.5, -4, 6)$$

$$\vec{F} = \left(\frac{100}{\sqrt{56}} - \frac{200}{\sqrt{58.25}}\right)\vec{i} - \left(\frac{200}{\sqrt{56}} + \frac{320}{\sqrt{58.25}}\right)\vec{j} + \left(-\frac{300}{\sqrt{56}} + \frac{480}{\sqrt{58.25}}\right)\vec{k}$$

$$\vec{F} = \left(\frac{100}{7.4833} - \frac{200}{7.632}\right)\vec{i} - \left(\frac{200}{7.4833} + \frac{320}{7.632}\right)\vec{j} + \left(-\frac{300}{7.4833} + \frac{480}{7.632}\right)\vec{k}$$

$$\vec{F} = (13.3511 - 26.2054)\vec{i} - (26.7261 + 41.9287)\vec{j} + (-40.0892 + 62.893)\vec{k}$$

$$\vec{F} = (12.8543)\,\vec{i} - (68.6548)\vec{j} + (22.807)\vec{k}$$

Then $F = \sqrt{(-12.8543)^2 + (-68.6548)^2 + (22.807)^2} = 73.5 \ Ib$ The direction of F given from $\cos\alpha = \frac{F_x}{F},\ \cos\beta = \frac{F_y}{F},\ \cos\gamma = \frac{F_z}{F}$

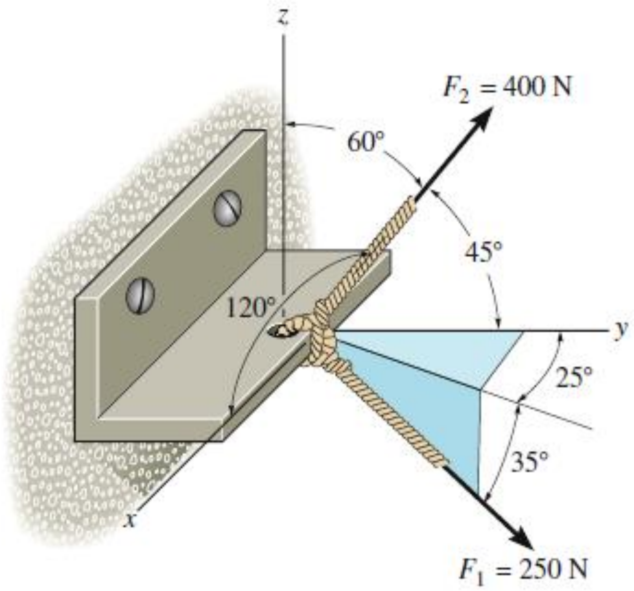
Then

$$\cos\alpha = -\frac{12.8543}{73.5} = -0.174049 \quad \rightarrow \quad \alpha = \cos^{-1}(0.8) = 100.0233^O$$

$$\cos\beta = -\frac{68.6548}{73.5} = -0.934078 \quad \rightarrow \quad \beta = \cos^{-1}(-0.934078) = 159.0798^O$$

$$\cos\gamma = \frac{22.807}{73.5} = 0.310299 \quad \rightarrow \quad \gamma = \cos^{-1}(0.310299) = 71.9227^O$$

Example: 12 The bracket is subjected to the two forces shown in below Figure. Express each force in Cartesian vector form and then determine the resultant force Find the magnitude and coordinate direction angles of the resultant force



Solution

$$\vec{F}_1 = 250\left(\cos35^o \sin25^O \vec{i} + \cos35^o \cos25^O \vec{j} - \sin35^O \vec{k}\right)$$

$$\vec{F}_1 = 250\left((0.81915)(0.42261)\,\vec{i} + (0.81915)(0.9063)\,\vec{j} - (0.57357)\,\vec{k}\right)$$

$$\vec{F}_1 = \left(86.545)\,\vec{i} + 185.5989)\,\vec{j} - 143.394\vec{k}\right)$$

Using the relation $\vec{F} = F\cos\alpha\vec{i} + F\cos\beta\,\vec{j} + F\cos\gamma\vec{k}$, we have

$$\vec{F}_2 = 400\left(\cos120^O \vec{i} + \cos45^O \ \vec{j} + \cos60^O \vec{k}\right)$$

$$\vec{F}_2 = 400\left((-0.5)\,\vec{i} + \frac{1}{\sqrt{2}}\,\vec{j} + (0.5)\,\vec{k}\right)$$

$$\vec{F}_2 = \left(-200\,\vec{i} + (282.8427)\,\vec{j} + 200\,\vec{k}\right)$$

So, the resultant given by

$$\vec{F}_R = \vec{F}_1 + \vec{F}_2 = \left(86.545\vec{i} + 185.5989\,\vec{j} - 143.394\vec{k}\right) + \left(-200\,\vec{i} + 282.8427\,\vec{j} + 200\,\vec{k}\right)$$

$$\vec{F}_R = \left(-113.455\vec{i} + 468.44161\vec{j} + 56.606\vec{k}\right)$$

$$F_R = \sqrt{(-113.455)^2 + (468.44161)^2 + (56.606)^2} = 485.3\ N$$

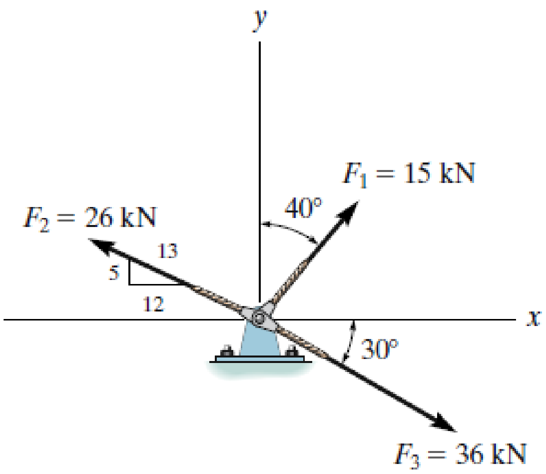
$$\cos\alpha = -\frac{113.455}{485.3} = -0.23378 \quad \rightarrow \quad \alpha = \cos^{-1}(-0.23378) = 103.5197^{O}$$

$$\cos\beta = \frac{468.44161}{485.3} = 0.96526 \quad \rightarrow \quad \beta = \cos^{-1}(0.96526) = 15.1462^{O}$$

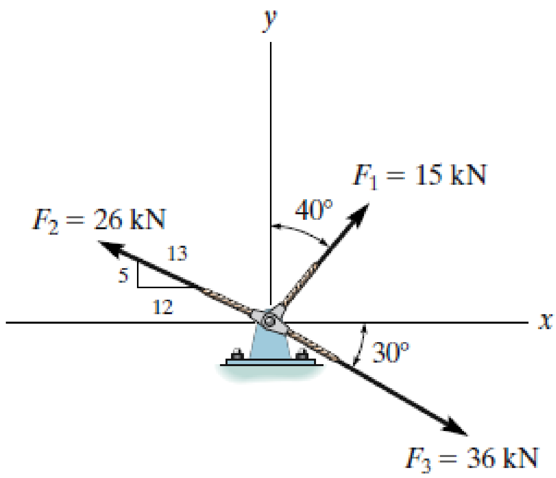
$$\cos\gamma = \frac{56.606}{485.3} = 0.11664 \quad \rightarrow \quad \gamma = \cos^{-1}(0.11664) = 83.3017^{O}$$

Problems

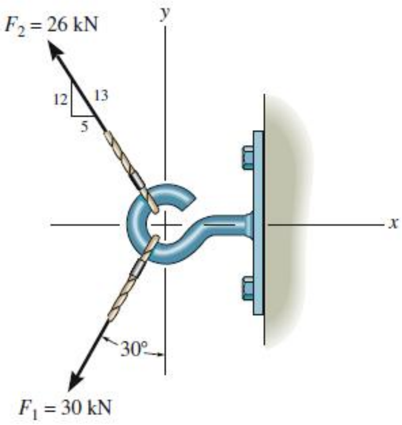
(1) Express F1, F2, and F3 as Cartesian vectors.



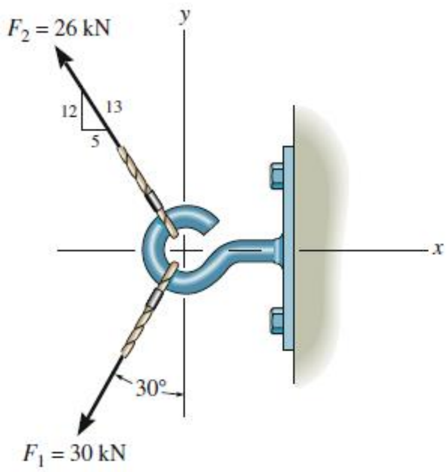
(2) Determine the magnitude of the resultant force and its orientation measured counterclockwise from the positive x axis



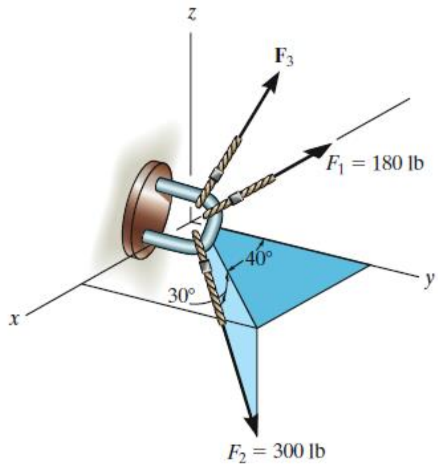
(3) Express F1 and F2 as Cartesian vectors.



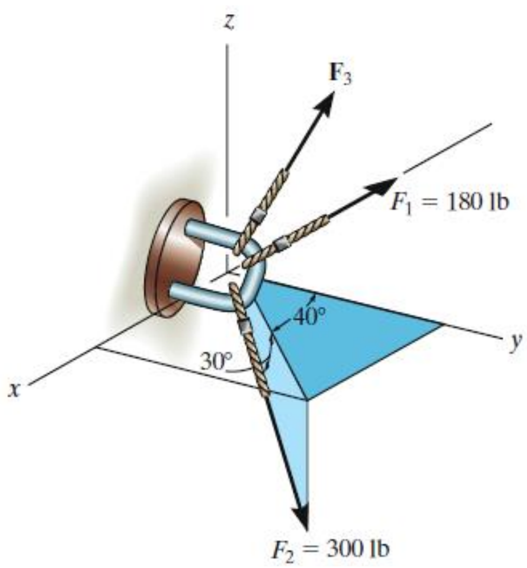
(4) Determine the magnitude of the resultant force and its direction measured counterclockwise from the positive x axis.



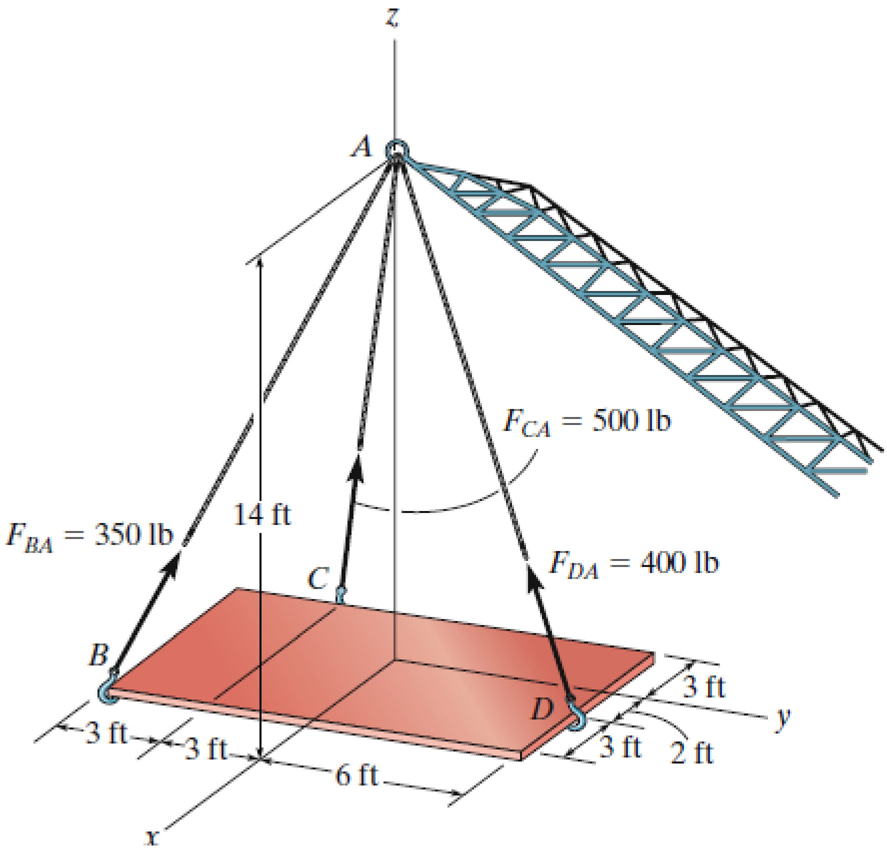
(5) Determine the magnitude and coordinate direction angles of f3 so that the resultant of the three forces acts along the positive y axis and has a magnitude of 600 lb.



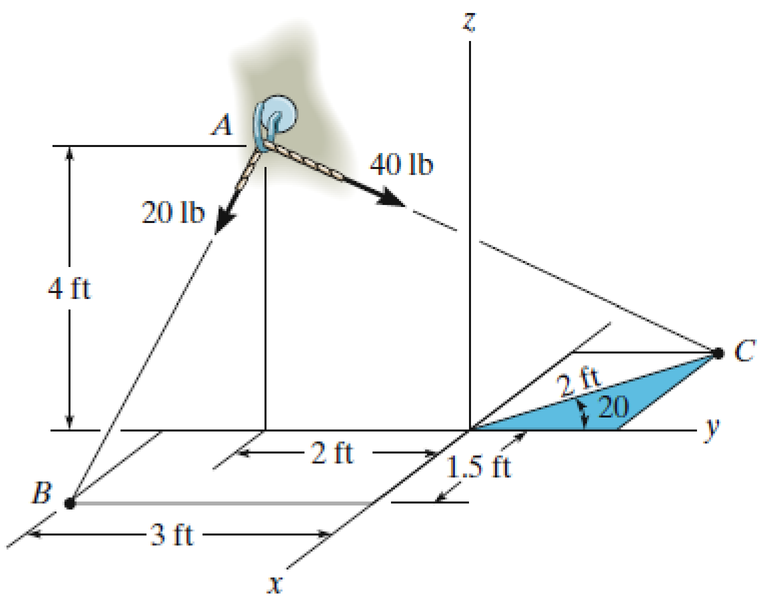
(6) Determine the magnitude and coordinate direction angles of F3 so that the resultant of the three forces is zero



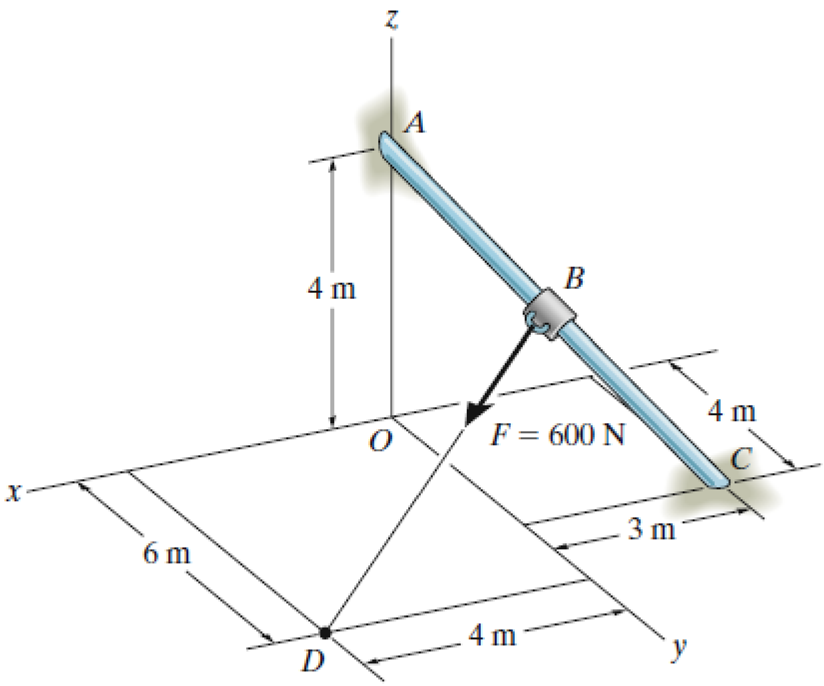
(7) The plate is suspended using the three cables which exert the forces shown. Express each force as a Cartesian vector.



(8) Determine the magnitude and coordinates on angles of the resultant force.



(9) Express force F in Cartesian vector form if point B is located 3 m along the rod end C.



Chapter: 3

Condition for the Equilibrium of a particle

A particle is said to be in equilibrium if it remains at rest if originally at rest, or has a constant velocity if originally in motion. Most often, however, the term "equilibrium" or, more specifically, "static equilibrium" is used to describe an object at rest. To maintain equilibrium, it is necessary to satisfy Newton's first law of motion, which requires the resultant force acting on a particle to be equal to zero . This condition may be stated mathematically as

$$F = 0 \qquad (1)$$

where F is the vector sum of all the forces acting on the particle.

Not only is $F = 0$ a necessary condition for equilibrium, it is also a sufficient condition. This follows from Newton's second law of motion, which can be written as $F = ma$. Since the force system satisfies Eq. (1) , then $ma = 0$, and therefore the particle's acceleration $a = 0$ Consequently, the particle indeed moves with constant velocity or remains at rest.

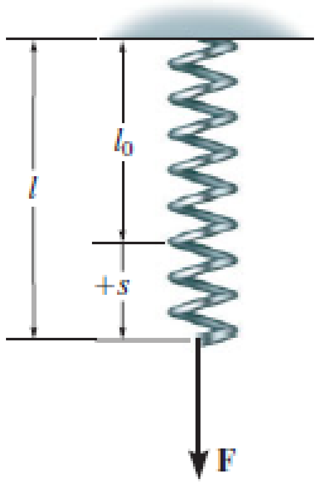


Fig. 1

1. Coplanar Force Systems

plane, as in Fig. (2) , then each force can be resolved into its $\vec{i}$ and $\vec{j}$ components. For equilibrium, these forces must sum to produce a zero

force resultant, i.e., $\sum F = 0 \quad \rightarrow \sum F_x \vec{i} + \sum F_y \vec{j} = 0$

For this vector equation to be satisfied, the resultant force's x and y components must both be equal to zero. Hence,. $\sum F_x = 0, \quad \sum F_y = 0$

These two equations can be solved for at most two unknowns, generally represented as angles and magnitudes of forces shown on the particle's free-body diagram.

When applying each of the two equations of equilibrium, we must account for the sense of direction of any component by using an algebraic sign which corresponds to the arrowhead direction of the component along the x or y axis. It is important to note that if a force has an unknown magnitude , then the arrowhead sense of the force on the free-body diagram can be assumed . Then if the solution yields a negative scalar , this indicates that the sense of the force is opposite to that which was assumed.

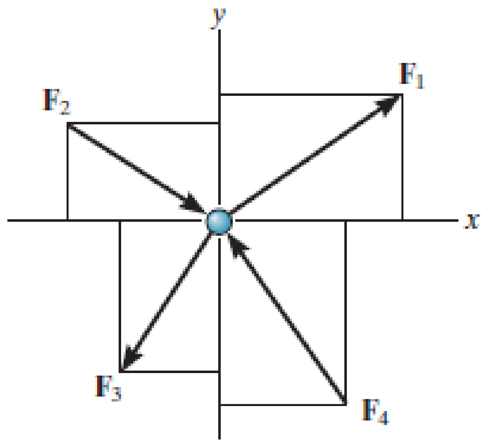
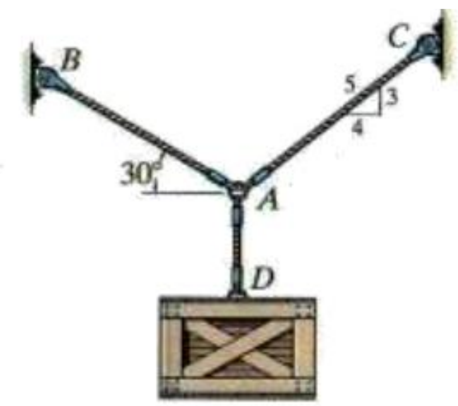


Fig. 2

Example 1: The Crate has a weight of $550N = (\approx 55\,kg)$. Determine the tension in each supporting cable in Figure



Solution

Applying the equations of equilibrium along the x and y axes, we have

$\sum F_x = 0 \quad \rightarrow \quad T_{AC}\left(\frac{4}{5}\right) - T_{AB}\cos 30^O = 0 \quad \rightarrow \quad T_{AC}\left(\frac{4}{5}\right) - \frac{\sqrt{3}}{2}T_{AB} = 0$

$\sum F_y = 0 \quad \rightarrow \quad T_{AC}\left(\frac{3}{5}\right) + T_{AB}\sin 30^O - 550 = 0 \quad \rightarrow \quad T_{AC}\left(\frac{3}{5}\right) + \frac{1}{2}T_{AB} = 550$

$8\,T_{AC} - 5\sqrt{3}T_{AB} = 0,$ (1), $6T_{AC} + 5T_{AB} = 5500$ (2)

$T_{AC} = \frac{5500\sqrt{3}}{\left(8+6\sqrt{3}\right)} = 518\ N$ $\left(8+6\sqrt{3}\right)T_{AC} = 5500\sqrt{3} \rightarrow$, Then $6\sqrt{3}T_{AC} + 5\sqrt{3}T_{AB} = 5500\sqrt{3}$

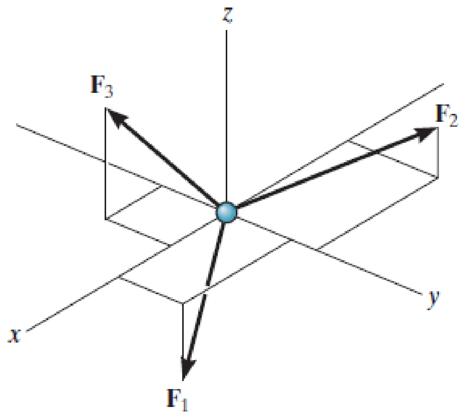
In Eq. (1), we have $8\,\frac{5500\sqrt{3}}{\left(8+6\sqrt{3}\right)} - 5\sqrt{3}T_{AB} = 0 \quad \rightarrow \quad 8\,\frac{1100}{\left(8+6\sqrt{3}\right)} - T_{AB} = 0,$

$T_{AB} = \frac{8800}{\left(8+6\sqrt{3}\right)} = 478.5\ N$

2. Three-dimensional force system

The necessary and sufficient condition for particle equilibrium is $\sum F = 0$

In the case of a three-dimensional force system, as in Figure .



We can resolve the forces into their respective $\vec{i}, \vec{j}, \vec{k}$ components, so that

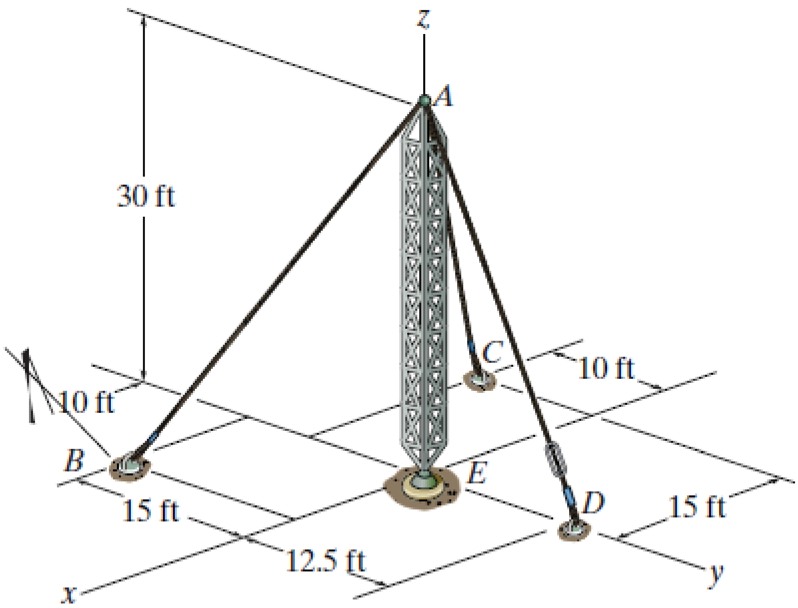
$\sum F_x\vec{i} + F_y\,\vec{j} + F_z\,\vec{k} = 0$.

To satisfy this equation we require

$$F_x = 0, \quad F_y = 0, \quad F_z = 0$$

These three equations state that the algebraic sum of the components of all the forces acting on the particle along each of the coordinate axes must be zero. Using them we can solve for at most three unknowns, generally represented as coordinate direction angles or magnitudes of forces shown on the particle's free-body diagram.

Example 2: If cable AD is tightened by a turnbuckle and develops a tension of $1300\ lb$. Determine the tension developed in cables AB and AC and the force developed along the antenna tower AE at point A.



Free-Body Diagram. First we will express each force in Cartesian vector form. Since the coordinates of points O, A, B and C are (see below Figure)

$A(0, 0, 30)$, $B(10, -15, 0)$, $C(-15, -10, 0)$, $D(0, 12.5, 0)$, $E(0, 0, 0)$,

Form the Figure we can express T_{AB}, T_{AC}, T_{AD} and F as follows

Since the tension $\vec{T}_{AB}$ acts across the two points A and B. So the unit vector in this direction is $\vec{e}_{AB} = B - A = \frac{1}{\sqrt{1225}}(10, -15, -30) = \frac{1}{35}(10, -15, -30) = \frac{1}{7}(2, -3, -6)$, i. e

$\vec{e}_{AB} = \frac{1}{7}(2, -3, -6)$. Then

$$\vec{T}_{AB} = T_{AB}\,\vec{e}_{AB} = \vec{e}_{AB} = \frac{T_{AB}}{7}(2, -3, -6).$$

Also, the tension $\vec{T}_{AC}$ acts across the two points A and C. So the unit vector in this direction is

$\vec{e}_{AC} = C - A = \frac{1}{\sqrt{1225}}(-15, -10, -30) = \frac{1}{35}(-15, -10, -30) = \frac{1}{7}(-3, -2, -6)$, i. e

$\vec{e}_{AC} = \frac{1}{7}(-3, -2, -6)$. Then

$$\vec{T}_{AC} = T_{AC}\,\vec{e}_{AC} = \frac{T_{AC}}{7}(-3, -2, -6)$$

A third time, the tension T_{AD} acts across the two points A and D. So the unit vector in this direction is

$$\vec{e}_{AD} = D - A = \frac{1}{\sqrt{1056.5}}(0, 12.5, -30) = \frac{1}{32.5}(0, 12.5, -30) = \frac{1}{325}(0, 125, -300),$$

i. e $\vec{e}_{AD} = \frac{1}{13}(0, 5, -12)$. Then

$$\vec{T}_{AD} = T_{AD}\,\vec{e}_{AD} = \frac{T_{AD}}{13}(0, 5, -12) = \frac{1300}{13}(0, 5, -12) = 100(0, 5, -12)$$

$$\vec{F} = F\,\vec{k}$$

Equations of Equilibrium: Equilibrium requires

$\sum F = 0 \quad \rightarrow \vec{T}_{AB} + \vec{T}_{AC} + \vec{T}_{AD} + F = 0$, i. e.

$$\frac{T_{AB}}{7}(2) + \frac{T_{AC}}{7}(-3) + 100(0) + F(0) = 0 \quad \rightarrow \quad 2T_{AB} - 3T_{AC} = 0 \qquad (1)$$

$$\frac{T_{AB}}{7}(-3) + \frac{T_{AC}}{7}(-2) + 100(5) + F(0) = 0 \quad \rightarrow \quad -3T_{AB} - 2T_{AC} + 3500 = 0 \qquad (2)$$

$$\frac{T_{AB}}{7}(-6) + \frac{T_{AC}}{7}(-6) + 100(-12) + F(1) = 0 \quad \rightarrow -6T_{AB} - 6T_{AC} - 8400 + 7F = 0 \qquad (3)$$

From Eq. (1) and Eq. (2), we have

$$3\left\{2T_{AB} - 3T_{AC}\right\} + 2\left\{-3T_{AB} - 2T_{AC} + 3500\right\} = 0$$

$$3\left\{-3T_{AC}\right\} + 2\left\{-2T_{AC} + 3500\right\} = 0 \;\rightarrow\; 13T_{AC} = 7000 \;\rightarrow\; T_{AC} = 538.461\,N$$

In Eq. (1), we have

$$T_{AB} = \frac{3}{2}T_{AC} = \left(\frac{3}{2}\right)538.461 = 807.692\ N \;\rightarrow\; T_{AB} = 807.692\ N$$

While in Eq. (3), we have

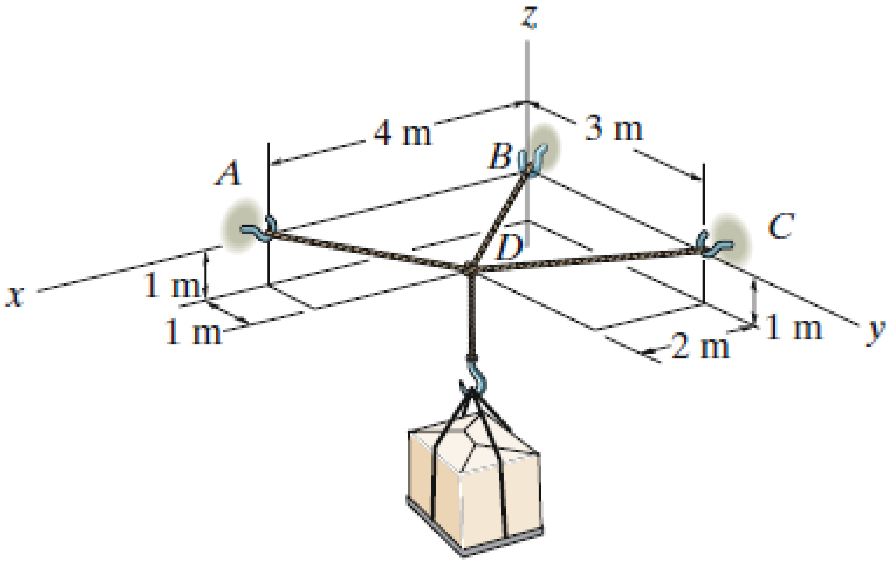
$$-6(807.692) - 6(538.461) - 8400 + 7F = 0 \;\rightarrow\; 7F = 6(807.692 + 538.461) + 8400$$

$$F = 2353.845\,N$$

Final

$$T_{AB} = 807.692\ N, \quad T_{AC} = 538.461\,N, \quad F = 2353.845\,N.$$

Example 3: The crate has a mass of $130\,kg$. Determine the tension developed in each cable for equilibrium.



Solution

Free-Body Diagram. First we will express each force in Cartesian vector form. Since

$A(4,0,1),\ \ B(0,0,1),\ \ C(0,3,1),\ \ \ D(2,1,0)$

Form the Figure we can express T_{AD}, T_{BD}, T_{CD} and F as follows

Since the tension $\vec{T}_{AD}$ acts across the two points A and D. So the unit vector in this direction is $\vec{e}_{AD} = D - A = \frac{1}{\sqrt{6}}(-2,1,-1)$. Then

$$\vec{T}_{AD} = T_{AD}\,\vec{e}_{AD} = \vec{e}_{AD} = \frac{T_{AD}}{\sqrt{6}}(-2,1,-1).$$

Since the tension $\vec{T}_{BD}$ acts across the two points B and D. So the unit vector in this direction is $\vec{e}_{BD} = D - B = \frac{1}{\sqrt{6}}(2,1,-1)$. Then

$$\vec{T}_{BD} = T_{AD}\,\vec{e}_{AD} = \vec{e}_{AD} = \frac{T_{AD}}{\sqrt{6}}(2,1,-1).$$

Since the tension $\vec{T}_{CD}$ acts across the two points B and D. So the unit vector in this direction is $\vec{e}_{CD} = D - C = \frac{1}{\sqrt{5}}(2,0,-1)$. Then

$$\vec{T}_{CD} = T_{CD}\,\vec{e}_{CD} = \vec{e}_{CD} = \frac{T_{CD}}{\sqrt{5}}(2,0,-1).$$

Equations of Equilibrium: Equilibrium requires

$\sum F = 0 \ \ \rightarrow \vec{T}_{AB} + \vec{T}_{AC} + \vec{T}_{AD} + F = 0$, i. e.

$$\frac{T_{AB}}{7}(2,-3,-6)+\frac{T_{AC}}{7}(-3,-2,-6)+100(0,5,-12)+F(0,0,1)=0$$

$$\frac{T_{AB}}{7}(2)+\frac{T_{AC}}{7}(-3)+100(0)+F(0)=0 \quad \rightarrow \quad 2T_{AB}-3T_{AC}=0 \qquad (1)$$

$$\frac{T_{AB}}{7}(-3)+\frac{T_{AC}}{7}(-2)+100(5)+F(0)=0 \quad \rightarrow \quad -3T_{AB}-2T_{AC}+3500=0 \qquad (2)$$

$$\frac{T_{AB}}{7}(-6)+\frac{T_{AC}}{7}(-6)+100(-12)+F(1)=0 \quad \rightarrow \quad -6T_{AB}-6T_{AC}-8400+7F=0 \qquad (3)$$

From Eq. (1) and Eq. (2), we have

$$3\left\{2T_{AB}-3T_{AC}\right\}+2\left\{-3T_{AB}-2T_{AC}+3500\right\}=0$$

$$3\left\{-3T_{AC}\right\}+2\left\{-2T_{AC}+3500\right\}=0 \quad \rightarrow \quad 13T_{AC}=7000 \quad \rightarrow \quad T_{AC}=538.461\,N$$

In Eq. (1), we have

$$T_{AB}=\frac{3}{2}T_{AC}=\left(\frac{3}{2}\right)538.461=807.692\ N \quad \rightarrow \quad T_{AB}=807.692\ N$$

While in Eq. (3), we have

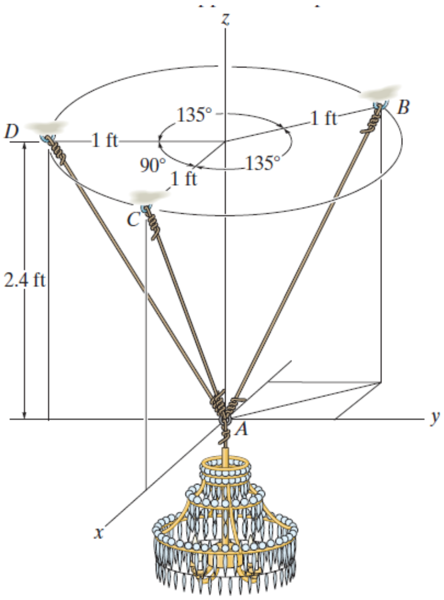
$$-6(807.692)-6(538.461)-8400+7F=0 \quad \rightarrow \quad 7F=6(807.692+538.461)+8400$$

$$F=2353.845\,N$$

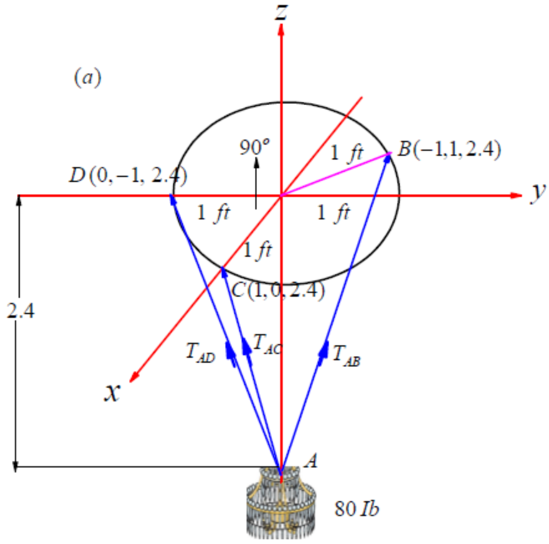
Final

$$T_{AB}=807.692\ N, \quad T_{AC}=538.461\,N, \quad F=2353.845\,N.$$

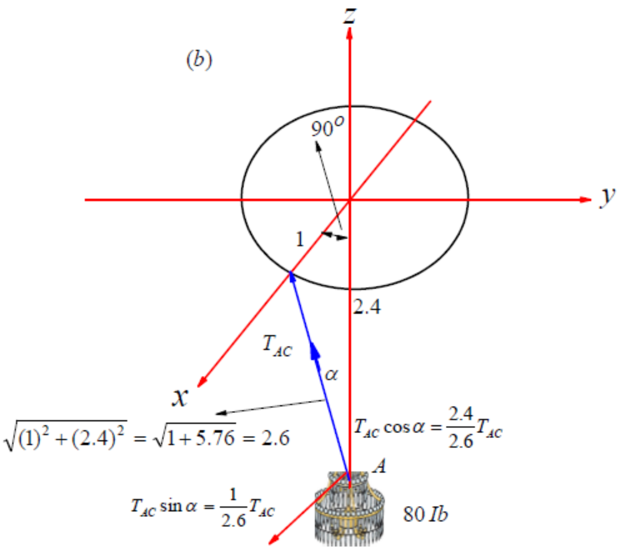
Example 4: The $80\,lb$ chandelier is supported by three wires as shown. Determine the force in each wire for equilibrium.

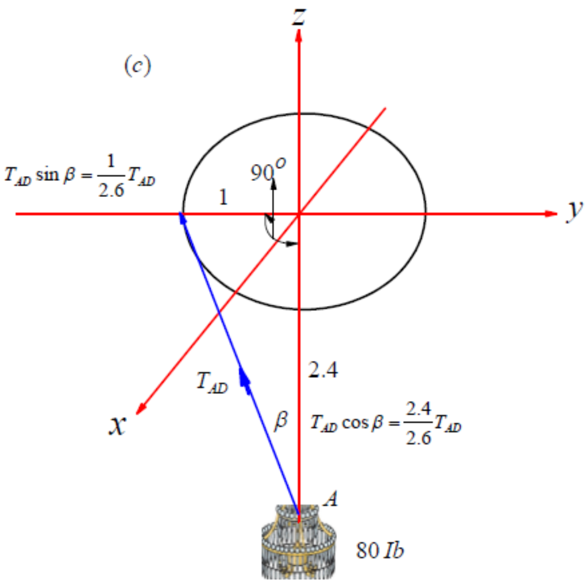


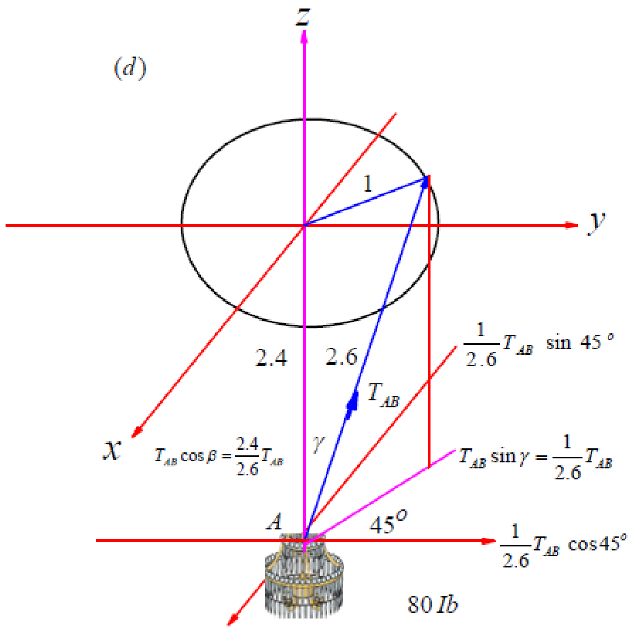
Solution



From this Figure we can organize the next three Figures (b-d)







From Figure (b-d) we can find that

$$\frac{1}{2.6}T_{AC} - \frac{1}{2.6}T_{AB}\sin 45^o = 0 \quad \rightarrow \quad T_{AC} - 0.7071\,T_{AB} = 0 \qquad (1)$$

$$-\frac{1}{2.6}T_{AD} + \frac{1}{2.6}T_{AB}\sin 45^o = 0 \quad \rightarrow \quad T_{AD} - 0.7071 T_{AB} = 0 \qquad (2)$$

$$\frac{2.4}{2.6}T_{AC} + \frac{2.4}{2.6}T_{AD} + \frac{2.4}{2.6}T_{AB} - 80 = 0 \quad \rightarrow \quad T_{AC} + T_{AD} + T_{AB} = 86.666 \qquad (3)$$

Solving Eqs. (1)- (3)

Subtracting Eq. (1) and (2), we have $T_{AC} = T_{AD}$

Then substituting into Eq. (3)

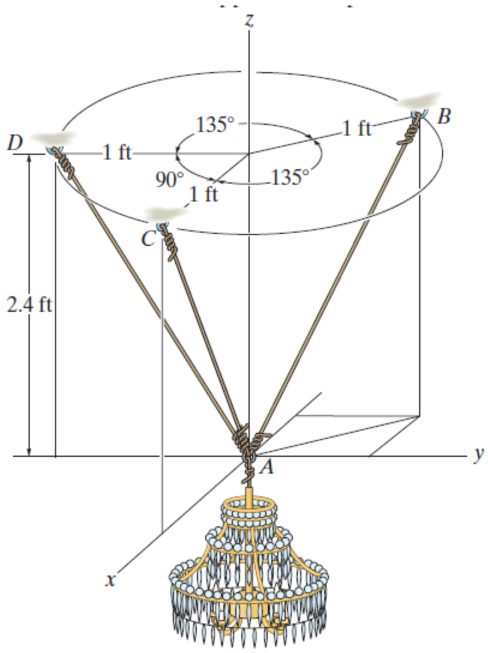
$$T_{AC} + T_{AC} + T_{AB} = 86.666 \rightarrow T_{AC} + \frac{1}{2}T_{AB} = 43.333 \qquad (4)$$

Again substituting into Eq. (3) $\quad 0.7071\,T_{AB} + 0.5T_{AB} = 43.333 \rightarrow T_{AB} = 35.9\ Ib$

Then in Eq. (4), we have $T_{AC} + \frac{1}{2}(35.9) = 43.333 \rightarrow T_{AC} = 25.4\ Ib$

Final $T_{AB} = 35.9, \quad T_{AC} = T_{AD} = 25.4\ Ib$ (Ans.)

Example 5: If each wire can sustain a maximum tension of 120 *Ib* before it fails, determine the greatest weight of the chandelier the wires will support in the position shown.



Solution

From the above example

$$\frac{1}{2.6}T_{AC}-\frac{1}{2.6}T_{AB}\sin 45^{o}=0 \quad\rightarrow\quad T_{AC}-0.7071\,T_{AB}=0 \qquad (1)$$

$$-\frac{1}{2.6}T_{AD}+\frac{1}{2.6}T_{AB}\sin 45^{o}=0 \quad\rightarrow\quad T_{AD}-0.7071T_{AB}=0 \qquad (2)$$

$$\frac{2.4}{2.6}T_{AC}+\frac{2.4}{2.6}T_{AD}+\frac{2.4}{2.6}T_{AB}-W=0 \quad\rightarrow\quad T_{AC}+T_{AD}+T_{AB}=1.08\,W \qquad (3)$$

Now if we put $T_{AC}=120$ in Eq. (1), we get

$$120-0.7071\,T_{AB}=0 \quad\rightarrow\quad T_{AB}=\frac{120}{0.7071}=169.7072>120$$ rejected solution

Again, if we put $T_{AB}=120$ in Eq. (1) , we get

$$T_{AC}-0.7071\,T_{AB}=0 \quad\rightarrow\quad T_{AC}=120(0.7071)=84.85<120$$ A reasonable solution

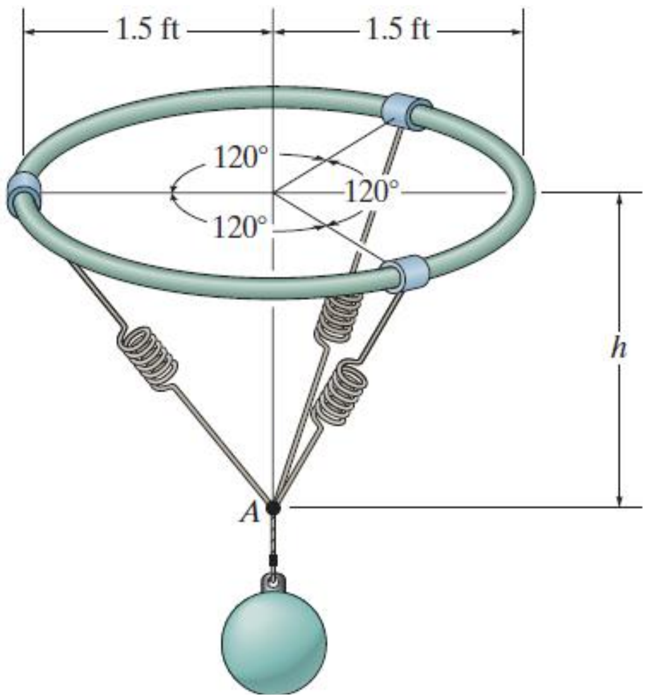
Then in Eq. (2), we have

$$T_{AD}-0.7071\,T_{AB}=0 \quad\rightarrow\quad T_{AB}=(120)(0.7071)=84.85<120$$ A reasonable solution

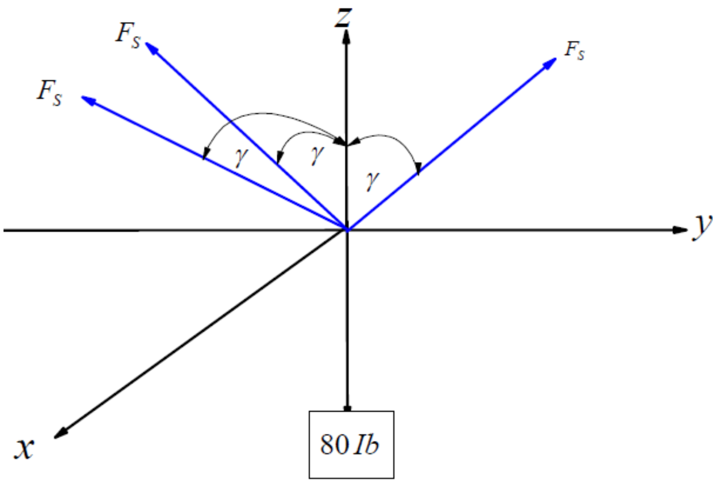
In Eq. (3)

$$84.85+84.85+120=1.08\,W \quad\rightarrow\quad 1.08\,W=289.7 \quad\rightarrow\quad W=268.24$$

Example 6: The $80\,Ib$ ball is suspended from the horizontal ring using three springs each having an unstretched length of $1.5\,ft$ and stiffness of $50\,Ib/ft$. Determine the vertical distance h from the ring to point A for equilibrium.



Solution



It clears that, the three springs are symmetric and subjected to a same tensile force. If one realizes this forces to the $z-$axis, we have

$$F_s\cos\gamma + F_s\cos\gamma + F_s\cos\gamma - 80 = 0 \quad \rightarrow \quad 3F_s\cos\gamma = 80 \qquad (1)$$

But we know that , the relation between spring force and stiffness (k) given by

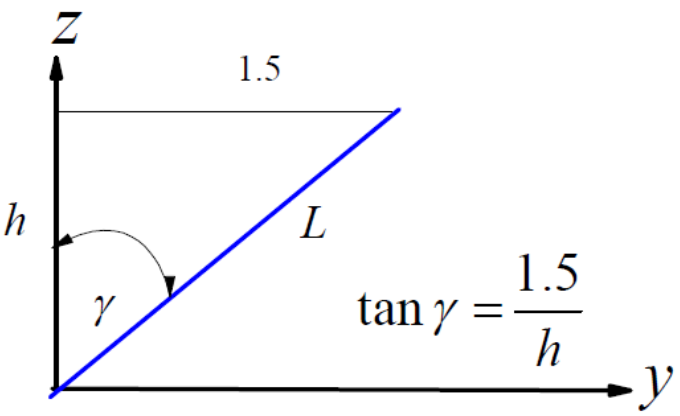
$$F_s = s\,k = k\,(L - L_0) = 50\left(\frac{1.5}{\sin\gamma} - 1.5\right) = 75\left(\frac{1}{\sin\gamma} - 1\right) \qquad (2)$$

Substituting from Eq. (1) into Eq. (2)

$$\frac{80}{3\cos\gamma} = 75\left(\frac{1}{\sin\gamma} - 1\right) \rightarrow \frac{\sin\gamma}{\cos\gamma}\,\frac{80}{225} = 1 - \sin\gamma \rightarrow \quad 0.3555\tan\gamma + \sin\gamma = 1 \qquad (3)$$

γ	$0.3555\tan\gamma$	$\sin\gamma$	$0.3555\tan\gamma+\sin\gamma=1$
30	$(0.3555)(0.5773)=0.20524$	0.5	0.70524
40	$(0.3555)(0.839)=0.29829$	0.6427	0.9299
42	$(0.3555)(0.9)=0.32$	0.6691	0.9891
42.5	$(0.3555)(0.916)=0.32575$	0.67559	1.0013

Then $\gamma=42.5^{o}$



(b)

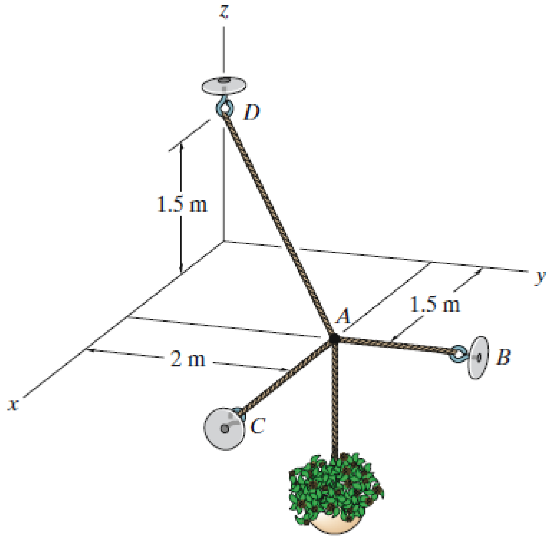
From Fig. (b) we note that

$$\tan\gamma=\frac{1.5}{h}\rightarrow h=\frac{1.5}{\tan(42.5^{o})}=\frac{1.5}{0.916}=1.64\ ft$$

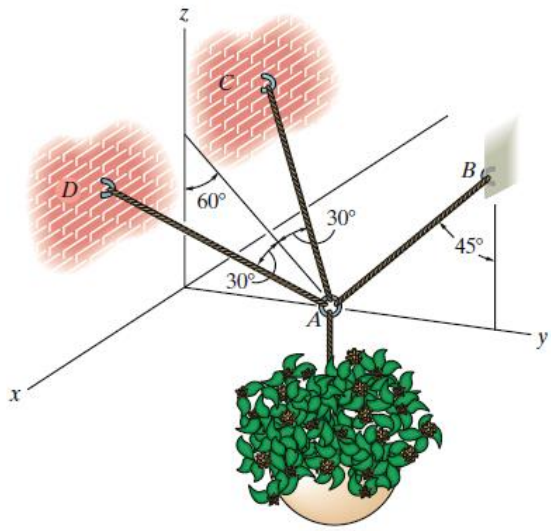
Then, the vertical distance h from the ring to point A for equilibrium is $1.64\ ft$

Problems

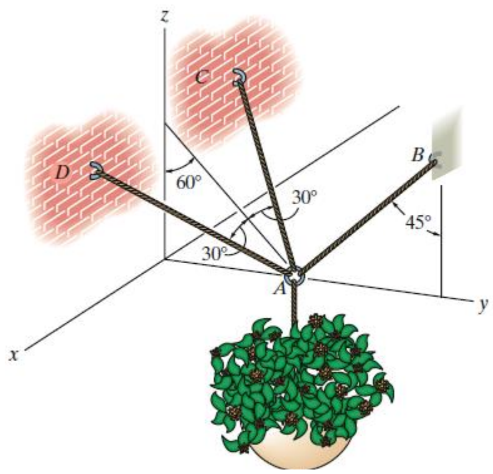
(1) The three cables are used to support the 40-kg flowerpot. Determine the force developed in each cable for equilibrium ?



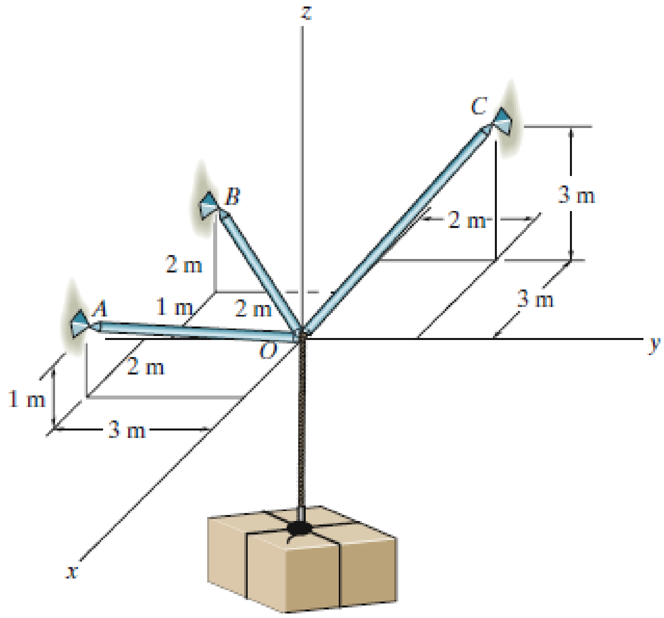
(2) The 25-kg flowerpot is supported at A by the three cords. Determine the force acting in each cord for equilibrium ?



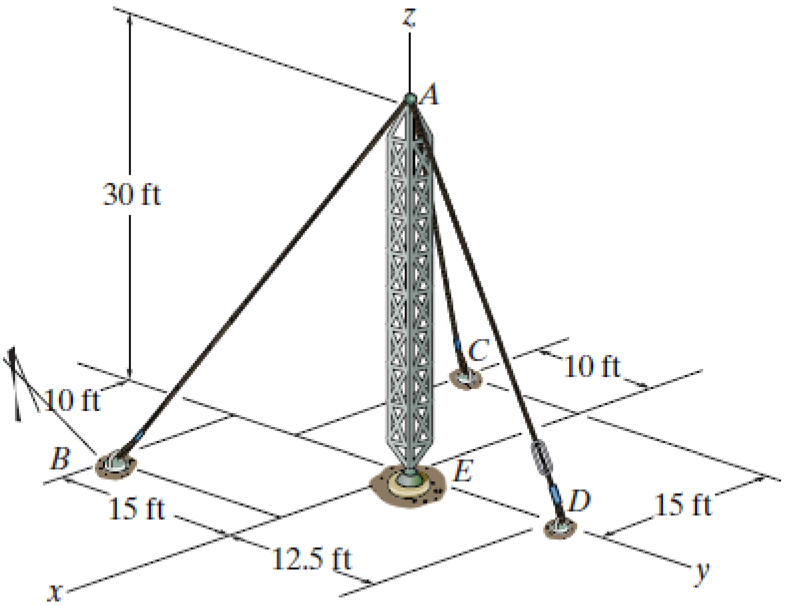
(3) If each cord can sustain a maximum tension of 50 N before it fails, determine the greatest weight of the flowerpot the cords can support ?



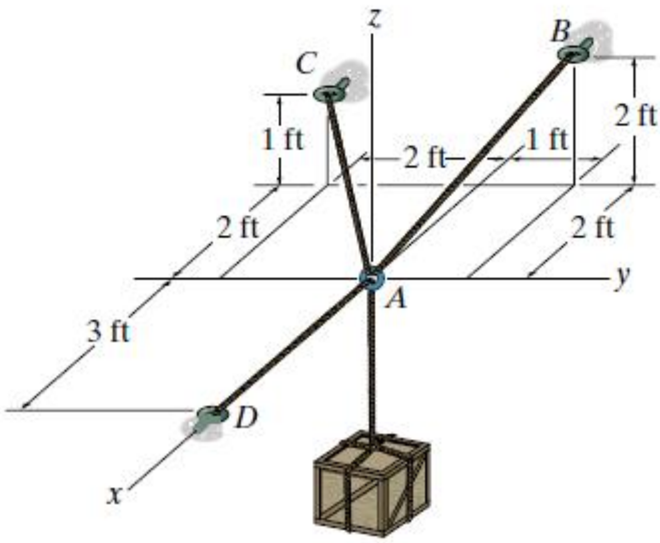
(4) If the maximum force in each rod can not exceed 1500 N, determine the greatest mass of the crate that can be supported.



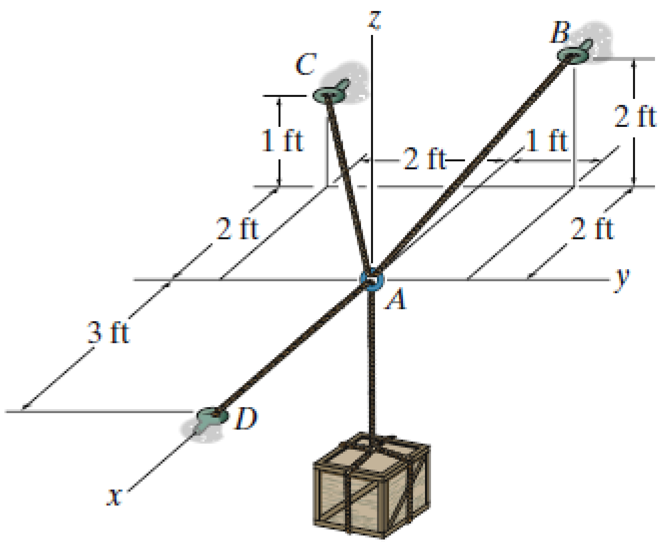
(5) If the tension developed in either cable AB or AC cannot exceed 1000 lb, determine the maximum tension that can be developed in cable AD when it is tightened by the turnbuckle. Also, what is the force developed along the antenna tower at point A ?



(6) Determine the tension developed in cables AB, AC and AD required for equilibrium of the 300-lb crate.



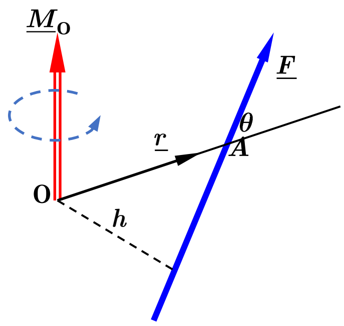
(7) Determine the maximum weight of the crate so that the tension developed in any cable does not exceed 450 lb.



Moment and Couples Forces

In this chapter we will obtain the moment of a force about a point or about an axis, reduction the forces at a point.

♦ The Moment



The **moment of a force** is the tendency of some forces to cause rotation. The moment of a force about a point is defined to be the product of the force and the perpendicular distance of its line of action from the point. On the other hand The moment of a force $\underline{F}$ about point O, or actually about the moment axis passing through O and perpendicular to the plane containing O and $\underline{F}$, as shown, can be expressed using the vector cross product, namely,

$$\underline{M}_O = \underline{r} \wedge \underline{F}$$

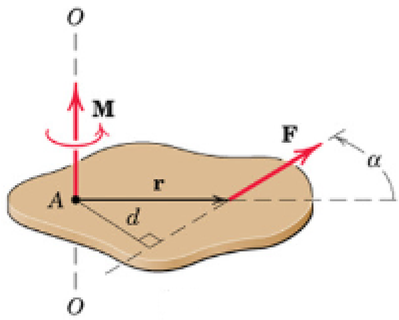
Here $\underline{r}$ represents a position vector directed from O to any point on the line of action of $\underline{F}$. Note that

$$|\underline{M}_O| = |\underline{r} \wedge \underline{F}| = rF\sin\theta = h$$

So if the force $\underline{F}$ in Cartesian coordinates is $\underline{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$ and the vector $\underline{r}$ is given by $\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then

$$\underline{M}_O = \underline{r} \wedge \underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= (yF_z - zF_y)\hat{i} - (xF_z - zF_x)\hat{j} + (xF_y - yF_x)\hat{k}$$



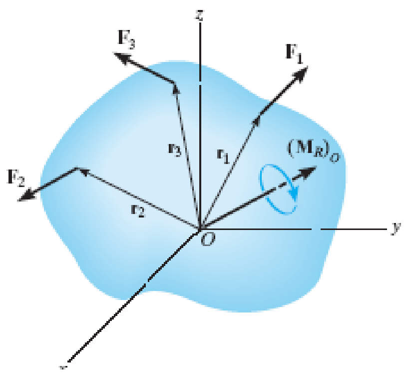
◈ Varginon's Theorem If a number of coplanar forces acting at a point of a rigid body have a resultant, then the vector sum of the moments of the all forces about any arbitrary point is equal to the moments of the resultant about the same point.

Proof.

Let the coplanar forces $\underline{F}_1, \underline{F}_2, ..., \underline{F}_n$ acting at a

a rigid body have the resultant $\underline{F}$.

$$\underline{F} = \underline{F}_1 + \underline{F}_2 + ... + \underline{F}_n = \sum \underline{F}_i$$



Let O be an arbitrary point and $\underline{r}_i$ be the position vector directed from O to any point on the line of action of $\underline{F}$. The sum of the moment of the forces $\underline{F}_1, \underline{F}_2, ..., \underline{F}_n$ about O is

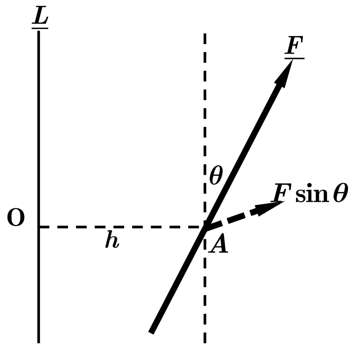
$$\sum \underline{r} \wedge \underline{F}_i = \underline{r} \wedge \underline{F}_1 + \underline{r} \wedge \underline{F}_2 + ... + \underline{r} \wedge \underline{F}_n$$
$$= \underline{r} \wedge \left(\underline{F}_1 + \underline{F}_2 + ... + \underline{F}_n \right)$$
$$= \underline{r} \wedge \underline{F}$$

which is equal to the moment of the resultant about O.

Any system of forces, acting in one plane upon a rigid body, can be reduced to either a single force or a single couple.

◈Three forces represented in magnitude, direction and position by the sides of a triangle taken the same way round are equivalent to a couple.

♦ <u>Moment of a force about an axis</u>



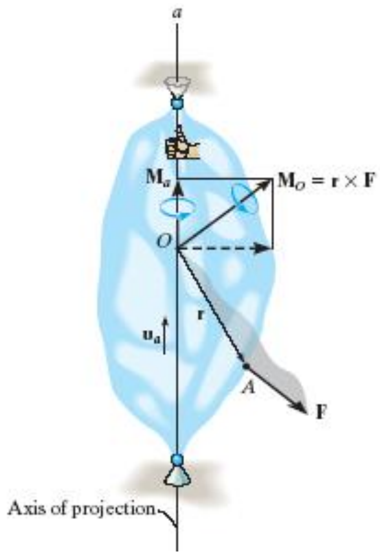
Thus if $\underline{F}$ be a force and $\underline{L}$ be a line which does not intersect $\underline{F}$, $\mathbf{OA} = h$ the shortest distance between $\underline{F}$ and $\underline{L}$, and θ the angle between $\underline{F}$ and a line through A parallel to $\underline{L}$, then $F\sin\theta$ is the resolved part of $\underline{F}$ at right angles to $\underline{L}$ and $Fh\sin\theta$ is the moment of $\underline{F}$ about $\underline{L}$ notation by $\underline{M}_L$. If $\underline{F}$ intersects the line $\underline{L}$ or is parallel to $\underline{L}$, then the moment of $\underline{F}$ about $\underline{L}$ is zero, because in the one case $h = 0$ and in the other $\sin\theta = 0$.

Or on the other hand $\underline{M}_L = \left(\underline{M}_o \bullet \hat{n} \right) \hat{n}$ where $\hat{n}$ is a unit vector of axis $\underline{L}$ and $\underline{M}_o$ represents the moment of the force $\underline{F}$ about a point $\mathbf{O}$ (say) lies on the axis $\underline{L}$, here

$$\left| \underline{M}_L \right| = \hat{n} \bullet \left(\underline{r} \wedge \underline{F} \right) = \begin{vmatrix} \ell & m & n \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \ell(yF_z - zF_y) - m(xF_z - zF_x) + n(xF_y - yF_x)$$

◈ When two forces act at a point the algebraical sum of their moments about any line is equal to the moment of their resultant about this line.



♦ In brief to calculate the moment of a force about an axis, one does the following three steps

(i) Obtain a unit vector of the axis (say $\hat{n}$)

(ii) Determine the moment $\underline{M}_o$ of the force $\underline{F}$

(iii) about a point lies on the axis, say $\mathbf{O}$.

(iv) The moment of a force about an axis is

$$\underline{M}_L = \underline{M}_o \bullet \hat{n}\ \hat{n}$$

◈ <u>Particular cases</u>

The moment of a force $\underline{F}$ about $\mathbf{X}$ axis is $\underline{M}_{OX} = \underline{M}_o \bullet \hat{i}\ \hat{i}$

The moment of a force $\underline{F}$ about $\mathbf{Y}$ axis is $\underline{M}_{OX} = \underline{M}_o \bullet \hat{j}\ \hat{j}$

The moment of a force $\underline{F}$ about $\mathbf{Z}$ axis is $\underline{M}_{OX} = \underline{M}_o \bullet \hat{k}\ \hat{k}$

♦ <u>Couples</u>

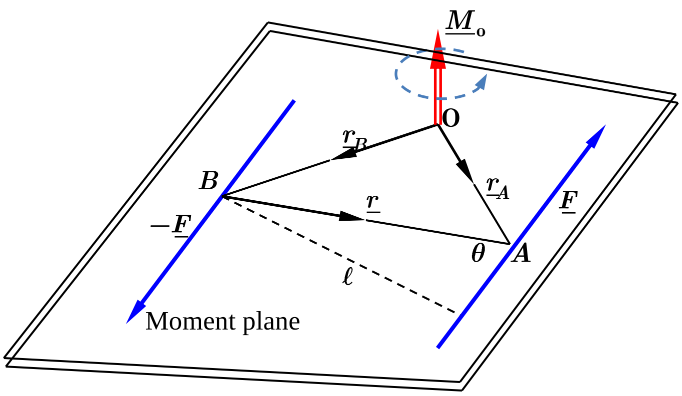
Couples play an important part in the general theory of systems of forces and we shall now establish some of their principal properties. Since a couple consists of two equal and opposite parallel forces (unlike forces), the algebraical sum of the resolved parts of the forces in every direction is zero, so that there is no tendency for the couple to produce in any direction a displacement of translation of the body upon which it acts; and the couple cannot be replaced by a single force. The effect of a couple must therefore be measured in some other way, and, since it has no tendency to produce translation, we next consider what tendency it has to produce rotation.

Let the couple consist of two forces of magnitude F . It is of course assumed that they are both acting upon the same rigid body. Let us take the algebraical sum of the moments of the forces about any point $\mathbf{O}$ in their plane as the measure of their tendency to turn the body upon which they act about the point $\mathbf{O}$.

$$\underline{M}_o = \underline{r}_A \wedge \underline{F} + \underline{r}_B \wedge (-\underline{F})$$

$$\underline{M}_o = \underline{r}_A - \underline{r}_B \wedge \underline{F} = \underline{r} \wedge \underline{F}$$

Where its magnitude is $\left|\underline{M}_o\right| = \left|\underline{r} \wedge \underline{F}\right| = rF\sin\theta = F\ell$



◈ Forces completely represented by the sides of a plane polygon taken the same way round are equivalent to a couple whose moment is represented by twice the area of the polygon.

♦ <u>Reduction a system of forces</u>

When a number of forces and couple moments are acting on a body, it is easier to understand their overall effect on the body if they are combined into a single force and couple moment having the same external effect. The two force and couple systems are called equivalent systems since they have the same external effect on the body. Suppose a system of forces $\underline{F}_1, \underline{F}_2, \ldots\ldots, \underline{F}_i, \ldots\ldots, \underline{F}_n$ is reduced at a chosen point O to a single force $\underline{F}$ and a single couple $\underline{M}$ viz. the obtaining result is $\underline{M}_{\mathrm{o}}, \underline{F}$ where

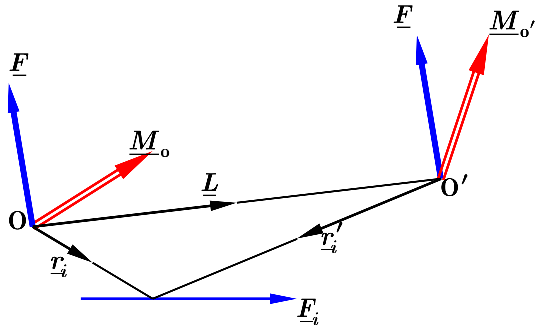
$$\underline{M}_{\mathrm{o}} = \sum_{i=1}^{n} \underline{r}_i \wedge \underline{F}_i, \qquad \underline{F} = \sum_{i=1}^{n} \underline{F}_i$$

Once again if the system of these forces reduced at another point O' where the obtaining results is

$$\underline{M}_{\mathrm{o}'} = \sum_{i=1}^{n} \underline{r}_i' \wedge \underline{F}_i, \qquad \underline{F} = \sum_{i=1}^{n} \underline{F}_i$$

That is when the point of reduction changed from O to O', the resultant of the forces does not change while the moment altered, such that

$$\begin{aligned} \therefore \underline{M}_{\mathrm{o}'} &= \sum_{i=1}^{n} \underline{r}_i' \wedge \underline{F}_i \\ &= \sum_{i=1}^{n} \; \underline{r}_i - \underline{L} \; \wedge \underline{F}_i \\ &= \sum_{i=1}^{n} \underline{r}_i \wedge \underline{F}_i - \sum_{i=1}^{n} \underline{L} \wedge \underline{F}_i \\ &= \underline{M}_{\mathrm{o}} - \sum_{i=1}^{n} \underline{L} \wedge \underline{F}_i \\ &= \underline{M}_{\mathrm{o}} - \underline{L} \wedge \sum_{i=1}^{n} \underline{F}_i \end{aligned}$$



$$\therefore \underline{M}_{\mathrm{o}'} = \underline{M}_{\mathrm{o}} - \underline{L} \wedge \underline{F}$$

Also it is obvious

$$\therefore \underline{F}.\underline{M}_{\mathrm{o}'} = \underline{F}. \; \underline{M}_{\mathrm{o}} - \underline{L} \wedge \underline{F} \; = \underline{F}.\underline{M}_{\mathrm{o}} - \overbrace{\underline{F}. \; \underline{L} \wedge \underline{F}}^{0} = \underline{F}.\underline{M}_{\mathrm{o}} = \mathbf{const.}$$

The quantity $\underline{F}.\underline{M}_{\mathrm{o}}$ is called invariant quantity

◈Wrench

Suppose a system of forces is reduced to a single force $\underline{F}$ and a single couple $\underline{M}$ such that the axis of the couple is coincides with the line of action of the force $\underline{F}$, then that line is called central axis. In addition, $\underline{F}$ and $\underline{M}$ taken together are called wrench of the system and are written as $(\underline{F}, \underline{M})$. The single force $\underline{F}$ is called the intensity of the wrench and the ratio $\underline{M} / \underline{F}$ is called the pitch of the system and is denoted by λ. Since $\underline{F}$ and $\underline{M}_{\mathrm{o}'}$ have the same direction so

$$\underline{M}_{\mathrm{o}'} = \underline{M}_{\mathrm{o}} - \underline{r} \wedge \underline{F} = \lambda \underline{F} \qquad \text{multiply by } \underline{F} \text{ using scalar product}$$

$$\Rightarrow \underline{F} \bullet \; \underline{M}_{\mathrm{o}} - \underline{r} \wedge \underline{F} \; = \lambda F^2 \qquad \therefore \lambda = \frac{\underline{F} \bullet \underline{M}_{\mathrm{o}}}{F^2} = \frac{F M_{\mathrm{O}}}{F^2} = \frac{M_{\mathrm{O}}}{F}$$

Where λ is known as the pitch of equivalent wrench

Also since $\underline{F} \wedge \underline{M}_{\mathrm{o}'} = \underline{0}$ multiply by $\underline{F}$ using cross product we have,

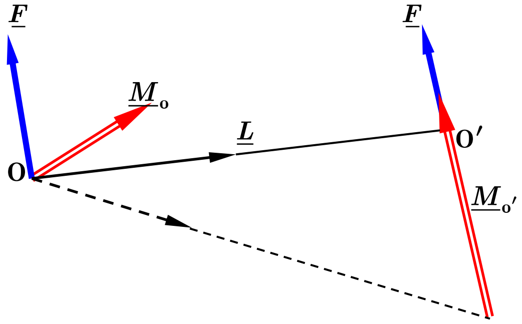
$$\therefore \underline{F} \wedge \underbrace{\underline{M}_{\mathrm{o}} - \underline{r} \wedge \underline{F}}_{\underline{M}_{\mathrm{o}'}} = \underline{F} \wedge \underline{M}_{\mathrm{o}} - \underline{F} \wedge \; \underline{r} \wedge \underline{F} \; = \underline{0}$$

According to the properties of triple vector product

$$\underline{F}\wedge(\underline{r}\wedge\underline{F}) = (\underline{F}.\underline{F})\underline{r}-(\underline{F}.\underline{r})\underline{F} = F^2\underline{r}-(\underline{F}.\underline{r})\underline{F}$$

$$\therefore \underline{F}\wedge\underline{M}_o - \left(F^2\underline{r}-(\underline{F}.\underline{r})\underline{F}\right) = \underline{0}$$

$$\therefore \underline{r} = \underbrace{\frac{\underline{F}\wedge\underline{M}_o}{F^2}}_{\underline{r}_1} + \underbrace{\frac{\underline{r}.\underline{F}}{F^2}}_{\mu}\underline{F} \qquad \textbf{Or} \qquad \underline{r} = \underline{r}_1 + \mu\underline{F}$$



The previous equation represents the equation of the central axis or axis of equivalent wrench in vector form and to get the Cartesian form let

$$\underline{r} = (x, y, z), \quad \underline{r}_1 = (a, b, c), \quad \underline{F} = (F_x, F_y, F_z)$$

Therefore, the Cartesian form of central axis is

$$\frac{x-a}{F_x} = \frac{y-b}{F_y} = \frac{z-c}{F_z}$$

◈ Special Cases

(i) $\underline{F}.\underline{M}_o = 0$ **and** $\underline{F} \neq 0,\ \underline{M}_o = 0$

The system reduced to a single force that acts along the line $\underline{r} = \lambda\underline{F}$

(ii) $\underline{F}.\underline{M}_o = 0$ **and** $\underline{F} = 0,\ \underline{M}_o \neq 0$

The system reduced to a single moment

(iii) $\underline{F}.\underline{M}_o = 0$ **and** $\underline{F} \neq 0,\ \underline{M}_o \neq 0$

In this case $\underline{M}_o$ will be perpendicular to $\underline{F}$ and the system can be reduced to wrench in which the central axis is

$$\therefore \underline{r} = \frac{\underline{F}\wedge\underline{M}_o}{F^2} + \mu\underline{F}$$

(iv) $\underline{F} = 0$ **and** $\underline{M}_o = 0$

The system of forces will be in equilibrium or it is a balanced system of forces.

■ Illustrative Examples ■

❒ EXAMPLE 1

Determine the moment of the force $\underline{F} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ acting at the point $A(3,2,0)$ about the origin and the point $B(2,1,-1)$.

❒ SOLUTION

Since the moment is given by $\underline{M}_o = \underline{r} \wedge \underline{F}$ where

$\underline{r} = \underline{OA} = \underline{A} - \underline{O} = (3,2,0) - (0,0,0) = 3\hat{i} + 2\hat{j}$

Therefore the moment of the given force about the origin is

$$\underline{M}_o = \underline{r} \wedge \underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 0 \\ 2 & 3 & 4 \end{vmatrix} = 8\hat{i} - 12\hat{j} + 5\hat{k}$$

Again, $\underline{r}' = \underline{BA} = \underline{A} - \underline{B} = (3,2,0) - (2,1,-1) = \hat{i} + \hat{j} + \hat{k}$

Hence, the moment f the given force about the point $B(2,1,-1)$ is

$$\underline{M}_B = \underline{r}' \wedge \underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3-2 & 2-1 & 0+1 \\ 2 & 3 & 4 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{vmatrix} = \hat{i} - 2\hat{j} + \hat{k}$$

❒ EXAMPLE 2

Calculate the moment of the force of magnitude $10\sqrt{3}$ and passing through the point $A(5,3,-3)$ to $B(4,4,-4)$ about the origin.

❒ SOLUTION

We have to write the force in vector form, to do this the unit vector in the direction of the force $\hat{F}$, viz. from point $A(5,3,-3)$ to $B(4,4,-4)$ so

$\because \underline{AB} = \underline{B} - \underline{A} = (4,4,-4) - (5,3,-3) = -\hat{i} + \hat{j} - \hat{k}$

$$\Rightarrow \hat{F} = \frac{\underline{AB}}{AB} = \frac{-\hat{i} + \hat{j} - \hat{k}}{\sqrt{3}} \equiv \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$$

Therefore the force be

$$\therefore \underline{F} = F\hat{F} = 10\sqrt{3}\left\{\frac{-\hat{i} + \hat{j} - \hat{k}}{\sqrt{3}}\right\} \equiv -10\hat{i} + 10\hat{j} - 10\hat{k}$$

Choosing any point as an acting point of the force, then the moment of the force about the origin O (consider $A(5,3,-3)$ as an acting point)

$$\because \underline{r} = (5,3,-3) - (0,0,0) = (5,3,-3)$$

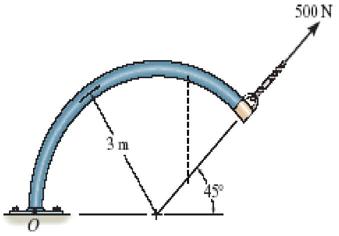
$$\Rightarrow \underline{M}_o = \underline{r} \wedge \underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 3 & -3 \\ -10 & 10 & -10 \end{vmatrix} = 80\hat{j} + 80\hat{k}$$

Also if we choose the point $B(4,4,-4)$ as an acting point

$$\Rightarrow \underline{M}_o = \underline{r}' \wedge \underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & -4 \\ -10 & 10 & -10 \end{vmatrix} = 40 \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ -1 & 1 & -1 \end{vmatrix} = 80\hat{j} + 80\hat{k}$$

❒ EXAMPLE 3

Determine the moment of the force as shown about point O.



❒ SOLUTION

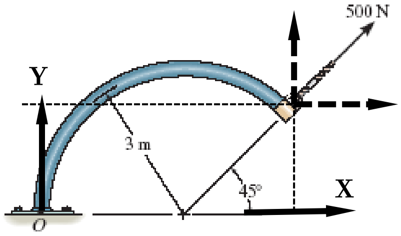
Taking horizontal axis X as shown, the force 500 can be reso

$$500\cos 45^0\,\hat{i} + 500\sin 45^0\,\hat{j} = 250\sqrt{2}(\hat{i} + \hat{j})$$

Therefore, the moment is given by,

$$M_o = 250\sqrt{2}\left(3 + \frac{3}{\sqrt{2}}\right) - 250\sqrt{2}\left(\frac{3}{\sqrt{2}}\right) = 750\sqrt{2}$$

Or by cross product where



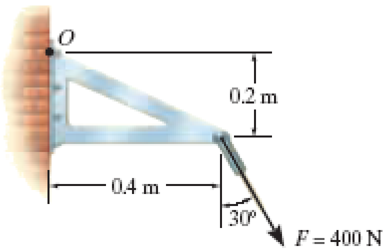
$$\underline{r} = \left(3 + \frac{3}{\sqrt{2}}\right)\hat{i} + \frac{3}{\sqrt{2}}\hat{j}$$

$$\underline{F} = 500\cos 45^0\,\hat{i} + 500\sin 45^0\,\hat{j} = 250\sqrt{2}(\hat{i} + \hat{j})$$

$$\Rightarrow \underline{M}_o = \underline{r} \wedge \underline{F} = 250\sqrt{2} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 + \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} & 0 \\ 1 & 1 & 0 \end{vmatrix} = 750\sqrt{2}\,\hat{k}$$

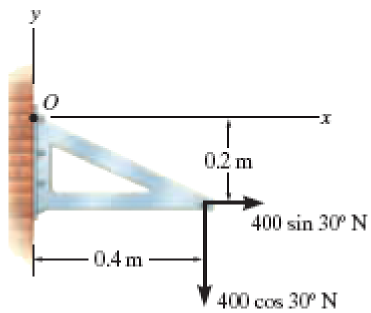
❒ EXAMPLE 4

Force F acts at the end of the angle bracket as shown. Determine the moment of the force about point O.



❒ SOLUTION

Using a Cartesian vector approach, the force and position vectors are



$$\underline{r} = 0.4\hat{i} - 0.2\hat{j}$$

$$\underline{F} = 400\sin 30^0\,\hat{i} - 400\cos 30^0\,\hat{j} = 200\hat{i} - 346.4\hat{j}$$

The moment is therefore,

$$\Rightarrow \underline{M}_o = \underline{r} \wedge \underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0.4 & -0.2 & 0 \\ 200 & -346.4 & 0 \end{vmatrix} = -98.6\hat{k}$$

❒ EXAMPLE 5

Find the sum of moment of the forces, $\underline{F} = 2\hat{i}$ acts at the origin, the force $-\frac{1}{2}\underline{F}$ acts at $\underline{r}_2 = 3\hat{j}$ and the force $-\frac{1}{2}\underline{F}$ acts at $\underline{r}_3 = 5\hat{k}$ about the origin.

❒ SOLUTION

As clear the resultant of these three forces is zero but the moment about the origin is given by

$$\Rightarrow \underline{M}_o = \sum_{i=1}^{3} \underline{r}_i \wedge \underline{F}_i = \underline{r}_1 \wedge \underline{F}_1 + \underline{r}_2 \wedge \underline{F}_2 + \underline{r}_3 \wedge \underline{F}_3$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 3 & 0 \\ -1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 5 \\ -1 & 0 & 0 \end{vmatrix}$$

$$\therefore \underline{M}_o = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 3 & 5 \\ -1 & 0 & 0 \end{vmatrix} = -5\hat{j} + 3\hat{k} \qquad \text{and} \qquad \left|\underline{M}_o\right| = \sqrt{34}$$

❒ EXAMPLE 6

The force $2\hat{i} - \hat{j}$ acts along the line that passing through the point $(4,4,5)$ and the force $3\hat{k}$ acting at the origin. Find the pitch and axis of equivalent wrench.

❒ SOLUTION

The two forces reduced at the origin to a resultant force $\underline{F}$ and a moment $\underline{M}_O$ so that

$$\underline{F} = \underline{F}_1 + \underline{F}_2 = 2\hat{i} - \hat{j} + 3\hat{k}, \qquad \therefore F^2 = 14$$

$$\underline{M}_O = \underline{r}_1 \wedge \underline{F}_1 + \underline{r}_2 \wedge \underline{F}_2 = \underline{0} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 5 \\ 2 & -1 & 0 \end{vmatrix} = 5\hat{i} + 10\hat{j} - 12\hat{k}$$

Thus the pitch of equivalent wrench is given by $\lambda = \dfrac{\underline{F} \bullet \underline{M}}{F^2}$ that is

$$\lambda = \frac{\underline{F} \bullet \underline{M}}{F^2} = \frac{2\hat{i} - \hat{j} + 3\hat{k} \bullet 5\hat{i} + 10\hat{j} - 12\hat{k}}{14} = -\frac{36}{14} = -\frac{18}{7}$$

In addition the equation of axis of wrench $\underline{r} = \underline{r}_1 + \mu \underline{F}$

$$\underline{r}_1 = \frac{\underline{F} \wedge \underline{M}}{F^2} = \frac{1}{14} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 3 \\ 5 & 10 & -12 \end{vmatrix} = \frac{1}{14} \; -18\hat{i} + 39\hat{j} + 25\hat{k}$$

Then the vector form of the axis becomes

$$\underline{r} = \frac{1}{14} \; -18\hat{i} + 39\hat{j} + 25\hat{k} \; + \mu(2\hat{i} - \hat{j} + 3\hat{k})$$

And Cartesian form is

$$\frac{x + \frac{18}{14}}{2} = \frac{y - \frac{39}{14}}{-1} = \frac{z - \frac{25}{14}}{3} \qquad \text{Or} \qquad \frac{14x + 18}{2} = \frac{14y - 39}{-1} = \frac{14z - 25}{3}$$

❒ EXAMPLE 7

A force P acts along the axis of $\mathbf{OX}$ and another force nP acts along a generator of the cylinder $x^2+y^2=a^2$ at the point $(a\cos\theta, a\sin\theta, 0)$; show that the central axis lies on the cylinder $n^2(nx-z)^2+(1+n^2)^2y^2=n^4a^2$

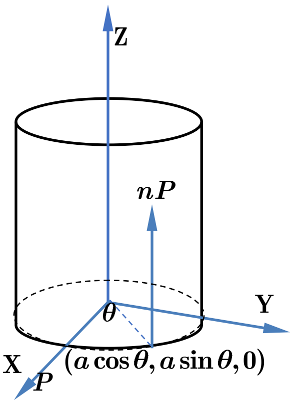
❒ SOLUTION

Generators of the cylinder are parallel to the axis of . Let one generator of it pass through the point and its unit vector is and the force acts along this line. Also the force acts along axis then

$\underline{F}_1 = P\hat{i}, \qquad \text{acts at } (0,0,0)$

$\underline{F}_2 = nP\hat{k}, \qquad \text{acts at } (a\cos\theta, a\sin\theta, 0)$

$\underline{F} = P(\hat{i}+n\hat{k}), \qquad (F^2 = (1+n^2)P^2)$



The system reduces to a single force and a moment so that

$\because \underline{M}_o = \underline{r}_1 \wedge \underline{F}_1 + \underline{r}_2 \wedge \underline{F}_2$

$$\therefore \underline{M}_o = P\left\{\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 0 & 0\\ 1 & 0 & 0\end{vmatrix} + \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a\cos\theta & a\sin\theta & 0\\ 0 & 0 & n\end{vmatrix}\right\}$$

$$= anP(\sin\theta\hat{i} - \cos\theta\hat{j})$$

The pitch of equivalent wrench is given by $\lambda = \dfrac{\underline{F}\bullet\underline{M}}{F^2}$ that is

$$\lambda = \frac{\underline{F}\bullet\underline{M}}{F^2} = \frac{P\ \hat{i}+n\hat{k}\ \bullet anP(\sin\theta\hat{i}-\cos\theta\hat{j})}{(1+n^2)P^2} = \frac{an\sin\theta}{1+n^2}$$

In addition, the equation of axis of wrench $\underline{r} = \underline{r}_1 + \mu\underline{F}$

$$\underline{r}_1 = \frac{\underline{F}\wedge\underline{M}}{F^2} = \frac{anP^2}{(1+n^2)P^2}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 0 & n\\ \sin\theta & -\cos\theta & 0\end{vmatrix}$$

$$= \frac{an}{1+n^2}\ \ n\cos\theta\hat{i} + n\sin\theta\hat{j} - \cos\theta\hat{k}$$

Then the vector form of the axis becomes

$$\underline{r} = \frac{an}{1+n^2}\ \ n\cos\theta\hat{i} + n\sin\theta\hat{j} - \cos\theta\hat{k}\ \ + \mu(\hat{i}+\hat{k})$$

And Cartesian form is given by

$$\frac{x - \dfrac{an^2\cos\theta}{1+n^2}}{1} = \frac{y - \dfrac{an^2\sin\theta}{1+n^2}}{0} = \frac{z + \dfrac{an\cos\theta}{1+n^2}}{n} \qquad \text{Or}$$

$$y - \frac{an^2\sin\theta}{1+n^2} = 0, \qquad n\left(x - \frac{an^2\cos\theta}{1+n^2}\right) = z + \frac{an\cos\theta}{1+n^2}$$

$$y = \frac{an^2}{1+n^2}\sin\theta, \qquad nx - z = \frac{an(1+n^2)\cos\theta}{1+n^2}$$

Squaring these equations

$$y^2 = n^2\left(\frac{an}{1+n^2}\right)^2 \sin^2\theta, \qquad (nx-z)^2 = (1+n^2)^2\left(\frac{an}{1+n^2}\right)^2 \cos^2\theta$$

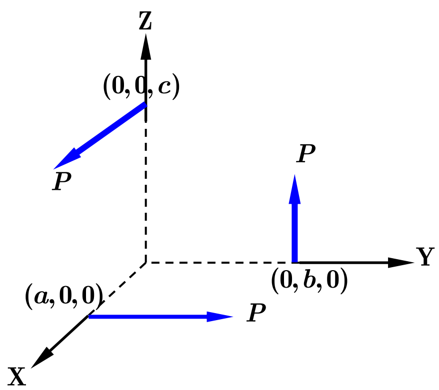
then multiply first equation by $(1+n^2)^2$ and the second by n^2 then adding the result we get

$$(1+n^2)^2y^2 + n^2(nx-z)^2 = n^2(1+n^2)^2\left(\frac{an}{1+n^2}\right)^2 = a^2n^4$$

❒ EXAMPLE 8

Three forces each equal to P act on a body, one at point $(a,0,0)$ parallel to $\mathbf{OY}$, the second at the point $(0,b,0)$ parallel to $\mathbf{OZ}$ and the third at the point $(0,0,c)$ parallel to $\mathbf{OX}$, the axes being rectangular. Find the resultant wrench.

❒ SOLUTION



As given we see

$$\begin{aligned}
\underline{F}_1 &= P\hat{i}, && \text{acts at } (0,0,c)\\
\underline{F}_2 &= P\hat{j}, && \text{acts at } (a,0,0)\\
\underline{F}_3 &= P\hat{k}, && \text{acts at } (0,b,0)\\
\underline{F} &= P(\hat{i}+\hat{j}+\hat{k}), && (F^2 = 3P^2)
\end{aligned}$$

The system reduces to a single force and a moment so that

$$\because\ \underline{M}_o = \underline{r}_1 \wedge \underline{F}_1 + \underline{r}_2 \wedge \underline{F}_2 + \underline{r}_3 \wedge \underline{F}_3$$

$$\therefore\ \underline{M}_o = P\left\{\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 0 & c\\ 1 & 0 & 0\end{vmatrix} + \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a & 0 & 0\\ 0 & 1 & 0\end{vmatrix} + \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & b & 0\\ 0 & 0 & 1\end{vmatrix}\right\}$$

$$= P(b\hat{i} + c\hat{j} + a\hat{k})$$

The pitch of equivalent wrench is given by $\lambda = \dfrac{\underline{F}\bullet\underline{M}}{F^2}$ that is

$$\lambda = \frac{\underline{F}\bullet\underline{M}}{F^2} = \frac{P\ \hat{i}+\hat{j}+\hat{k}\ \bullet P(b\hat{i}+c\hat{j}+a\hat{k})}{3P^2} = \frac{a+b+c}{3}$$

In addition the equation of axis of wrench $\underline{r} = \underline{r}_1 + \mu\underline{F}$

$$\underline{r}_1 = \frac{\underline{F}\wedge\underline{M}}{F^2} = \frac{P^2}{3P^2}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 1\\ b & c & a\end{vmatrix}$$

$$= \frac{1}{3}\ (a-c)\hat{i} + (b-a)\hat{j} + (c-b)\hat{k}$$

Then the vector form of the axis becomes

$$\underline{r} = \frac{1}{3}\ (a-c)\hat{i} + (b-a)\hat{j} + (c-b)\hat{k}\ + \mu(\hat{i}+\hat{j}+\hat{k})$$

and Cartesian form is

$$\frac{x-\frac{1}{3}(a-c)}{1}=\frac{y-\frac{1}{3}(b-a)}{1}=\frac{z-\frac{1}{3}(c-b)}{1} \qquad \text{Or}$$

$$3\ \ y-x\ \ = b+c-2a \qquad \text{and} \qquad 3\ \ z-y\ \ = a+c-2b$$

❒ EXAMPLE 9

Forces X, Y, Z act along three lines given by the equations

$$y=0, z=c; \qquad z=0, x=a; \qquad x=0, y=b$$

Prove that the pitch of the equivalent wrench is

$$(aYZ+bZX+cXY)\ /\ (X^2+Y^2+Z^2)$$

If the wrench reduces to a single force, show that the line of the action of the force lies on the hyperboloid $(x-a)(y-b)(z-c)=xyz$

❒ SOLUTION

As given
$$\begin{aligned} \underline{F}_1 &= X\hat{i}, && \text{acts at } (0,0,c) \\ \underline{F}_2 &= Y\hat{j}, && \text{acts at } (a,0,0) \\ \underline{F}_3 &= Z\hat{k} && \text{acts at } (0,b,0) \\ \underline{F} &= X\hat{i}+Y\hat{j}+Z\hat{k}, && F^2 = X^2+Y^2+Z^2 \end{aligned}$$

The system reduces to a single force and a moment so that

$$\because\ \underline{M}_o = \underline{r}_1 \wedge \underline{F}_1 + \underline{r}_2 \wedge \underline{F}_2 + \underline{r}_3 \wedge \underline{F}_3$$

$$\therefore\ \underline{M}_o = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & c \\ X & 0 & 0 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & 0 & 0 \\ 0 & Y & 0 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & b & 0 \\ 0 & 0 & Z \end{vmatrix}$$

$$= bZ\hat{i} + cX\hat{j} + aY\hat{k}$$

The pitch of equivalent wrench is given by $\lambda = \frac{\underline{F} \bullet \underline{M}}{F^2}$ that is

$$\lambda = \frac{\underline{F} \bullet \underline{M}}{F^2} = \frac{(X\hat{i}+Y\hat{j}+Z\hat{k}) \bullet\ bZ\hat{i}+cX\hat{j}+aY\hat{k}}{X^2+Y^2+Z^2}$$

$$= \frac{bXZ+cXY+aYZ}{X^2+Y^2+Z^2}$$

Besides, the equation of axis of wrench $\underline{r} = \underline{r}_1 + \mu \underline{F}$

$$\underline{r}_1 = \frac{\underline{F} \wedge \underline{M}}{F^2} = \frac{1}{(X^2+Y^2+Z^2)} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ X & Y & Z \\ bZ & cX & aY \end{vmatrix}$$

$$= \frac{1}{(X^2+Y^2+Z^2)}\ (aY^2-cXZ)\hat{i} + (bZ^2-aXY)\hat{j} + (cX^2-bYZ)\hat{k}$$

Then the vector form of the axis becomes

$$\underline{r} = \frac{(aY^2 - cXZ)\hat{i} + (bZ^2 - aXY)\hat{j} + (cX^2 - bYZ)\hat{k}}{(X^2 + Y^2 + Z^2)} + \mu(X\hat{i} + Y\hat{j} + Z\hat{k})$$

And Cartesian form is

$$\frac{x - \dfrac{aY^2 - cXZ}{X^2 + Y^2 + Z^2}}{X} = \frac{y - \dfrac{bZ^2 - aXY}{X^2 + Y^2 + Z^2}}{Y} = \frac{z - \dfrac{cX^2 - bYZ}{X^2 + Y^2 + Z^2}}{Z}$$

Or

$$Y\left(x - \frac{aY^2 - cXZ}{X^2 + Y^2 + Z^2}\right) = X\left(y - \frac{bZ^2 - aXY}{X^2 + Y^2 + Z^2}\right)$$

$$Y\left(z - \frac{cX^2 - bYZ}{X^2 + Y^2 + Z^2}\right) = Z\left(y - \frac{bZ^2 - aXY}{X^2 + Y^2 + Z^2}\right)$$ ……Complete

❒ EXAMPLE 10

Two forces each equal to P act along the lines $\frac{x \mp a\cos\theta}{a\sin\theta} = \frac{y - b\sin\theta}{\mp b\cos\theta} = \frac{z}{c}$ show that the axis of equivalent wrench lays on the surface $y\left(\frac{x}{z} + \frac{z}{x}\right) = b\left(\frac{a}{c} + \frac{c}{a}\right)$

❒ SOLUTION

First line is $\frac{x - a\cos\theta}{a\sin\theta} = \frac{y - b\sin\theta}{-b\cos\theta} = \frac{z}{c}$ passing through $(a\cos\theta, b\sin\theta, 0)$

the second line is $\frac{x + a\cos\theta}{a\sin\theta} = \frac{y - b\sin\theta}{b\cos\theta} = \frac{z}{c}$ passing $(-a\cos\theta, b\sin\theta, 0)$

The unit vector of first line is

$$\hat{n}_1 = \frac{1}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta + c^2}}(a\sin\theta, -b\cos\theta, c)$$
$$= \frac{1}{\mu}\ a\sin\theta\hat{i} - b\cos\theta\hat{j} + c\hat{k}$$

The unit vector of second line is

$$\hat{n}_2 = \frac{1}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta + c^2}}(a\sin\theta, b\cos\theta, c)$$
$$= \frac{1}{\mu}\ a\sin\theta\hat{i} + b\cos\theta\hat{j} + c\hat{k} \qquad \left(\mu = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta + c^2}\right)$$

Therefore,

$$\underline{F}_1 = P\hat{n}_1 = \frac{P}{\mu}\ a\sin\theta\hat{i} - b\cos\theta\hat{j} + c\hat{k}$$

$$\underline{F}_2 = P\hat{n}_2 = \frac{P}{\mu}\ a\sin\theta\hat{i} + b\cos\theta\hat{j} + c\hat{k}$$

The system reduces to a single force and a moment so that

$$\underline{F} = \underline{F}_1 + \underline{F}_2$$
$$= \frac{P}{\mu} \; a\sin\theta\hat{i} - b\cos\theta\hat{j} + c\hat{k} \;\; + \frac{P}{\mu} \; a\sin\theta\hat{i} + b\cos\theta\hat{j} + c\hat{k}$$
$$= \frac{2P}{\mu} \; a\sin\theta\hat{i} + c\hat{k} \qquad \text{and} \quad F^2 = \frac{4P^2}{\mu^2} \; a^2\sin^2\theta + c^2$$

$$\underline{M} = \underline{r}_1 \wedge \underline{F}_1 + \underline{r}_2 \wedge \underline{F}_2$$
$$= \frac{P}{\mu}\left\{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\theta & b\sin\theta & 0 \\ a\sin\theta & -b\cos\theta & c \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\cos\theta & b\sin\theta & 0 \\ a\sin\theta & b\cos\theta & c \end{vmatrix}\right\}$$

$$\therefore \underline{M} = \frac{2P}{\mu} \;\; cb\sin\theta\hat{i} - ab\hat{k}$$

since the equation of axis of equivalent wrench is $\underline{r} = \underline{r}_1 + \mu\underline{F}$

$$\underline{r}_1 = \frac{\underline{F} \wedge \underline{M}}{F^2} = \frac{1}{a^2\sin^2\theta + c^2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\sin\theta & 0 & c \\ cb\sin\theta & 0 & -ab \end{vmatrix} = \frac{c^2 + a^2 \;\; b\sin\theta}{a^2\sin^2\theta + c^2}\hat{j}$$

Then the vector form of the axis becomes

$$\underline{r} = \frac{(c^2 + a^2)b\sin\theta}{a^2\sin^2\theta + c^2}\hat{j} + \mu \;\; a\sin\theta\hat{i} + c\hat{k}$$

While the Cartesian form is

$$\frac{x - 0}{a\sin\theta} = \frac{y - \dfrac{(c^2 + a^2)b\sin\theta}{a^2\sin^2\theta + c^2}}{0} = \frac{z - 0}{c}$$

Thus we can deduce from these equation

$$y = \frac{(c^2 + a^2)b\sin\theta}{a^2\sin^2\theta + c^2} \qquad \text{and} \qquad \frac{x}{z} = \frac{a\sin\theta}{c}$$

$$y \;\; a^2\sin^2\theta + c^2 \;\; = \;\; c^2 + a^2 \;\; b\sin\theta$$
$$\Rightarrow y\left(a^2\sin\theta + \frac{c^2}{\sin\theta}\right) = b \;\; c^2 + a^2$$

Dividing by ac and substituting $\frac{x}{z} = \frac{a\sin\theta}{c}$ we get

$$y\left(\frac{x}{z} + \frac{z}{x}\right) = b\left(\frac{c}{a} + \frac{a}{c}\right)$$

❒ EXAMPLE 11

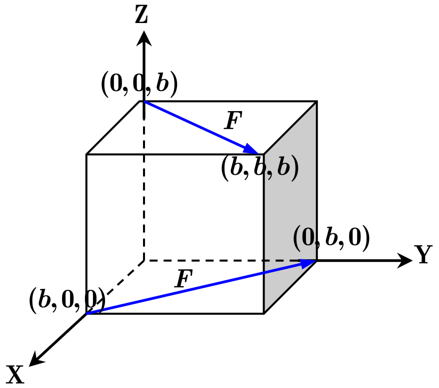
Two forces each equal to F act along the sides of a cube of length b as shown, Fin the axis of equivalent wrench.

❒ SOLUTION

By calculating the unit vectors of the forces, we get,

$$\hat{n}_1 = (b,b,b) - (0,0,b) = \frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$$

$$\therefore \underline{F}_1 = F\hat{n}_1 = \frac{F}{\sqrt{2}}(\hat{i} + \hat{j})$$



And for the second force

$$\hat{n}_2 = (0,b,0) - (b,0,0) = \frac{1}{\sqrt{2}}(-\hat{i} + \hat{j})$$

$$\therefore \underline{F}_2 = F\hat{n}_2 = \frac{F}{\sqrt{2}}(-\hat{i} + \hat{j})$$

The system reduces to a single force and a moment at the origin so that

$$\underline{R} = \underline{F}_1 + \underline{F}_2 = \sqrt{2}F\hat{j} \qquad \therefore R^2 = 2F^2$$

$$\underline{M} = \underline{r}_1 \wedge \underline{F}_1 + \underline{r}_2 \wedge \underline{F}_2 = \frac{Fb}{\sqrt{2}}\left\{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}\right\} = \frac{Fb}{\sqrt{2}}\ \hat{i} - \hat{j} - \hat{k}$$

Here we choose the point $(0,0,b)$ as an acting point of first force and the point $(b,0,0)$ of the second force. The pitch of equivalent wrench is given by $\lambda = \frac{\underline{F} \bullet \underline{M}}{F^2}$ that is

$$\lambda = \frac{\underline{R} \bullet \underline{M}}{R^2} = \frac{\sqrt{2}F\hat{j} \bullet \frac{Fb}{\sqrt{2}}\ \hat{i} - \hat{j} - \hat{k}}{2F^2} = -\frac{b}{2}$$

since the equation of axis of equivalent wrench is $\underline{r} = \underline{r}_1 + \mu\underline{F}$ so

$$\underline{r}_1 = \frac{\underline{R} \wedge \underline{M}}{R^2} = \frac{F^2 b}{2F^2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{vmatrix} = -\frac{b}{2}(\hat{i} + \hat{k})$$

Then the vector form of the axis becomes

$$\underline{r} = -\frac{b}{2}\ \hat{i} + \hat{k}\ + \mu\hat{j}$$

While the Cartesian form is given by

$$\frac{x + \frac{b}{2}}{0} = \frac{y - 0}{1} = \frac{z + \frac{b}{2}}{0} \qquad \text{Or} \qquad z = -\frac{b}{2} \text{ and } x = -\frac{b}{2}$$

PROBLEMS

❒ If the force $\underline{F} = 3\hat{i} - \hat{j} + 7\hat{k}$ acts at the origin, determine its moment about the point $(4, 4, 6)$.

❒ A force of magnitude 100 acts along the line passing through the point $(0,1,0)$ to $(1,0,0)$. Obtain its moment about the origin point and about the axes.

❒ The three forces $2\hat{i} + 2\hat{j}$, $\hat{j} - 2\hat{k}$, $-\hat{i} + 2\hat{j} + \hat{k}$ act at the points $(0,1,0)$, $(1,0,0)$, $(0,0,1)$ respectively, Find the pitch of the equivalent wrench.

❒ Two forces each equal to $3F$ act along the lines $\frac{x-1}{2} = \frac{y+1}{2} = \frac{z-2}{1}$ and $\frac{x-2}{1} = \frac{y+1}{-2} = \frac{z-1}{2}$. Find the equivalent wrench.

❒ The magnitude of two forces is F_2, F_1 act along the lines $(z = -c,\ y = -x\tan\alpha)$ and $(z = c,\ y = x\tan\alpha)$. Determine the central axis of equivalent wrench.

APPLIED MATHEMATICS 1

STATICS AND DYNAMICS

First Year
Mathematics Dept.
Faculty of Education

Dynamics

INTRODUCTION

The subject of Dynamics is generally divided into two branches: the first one, is called *Kinematics*, is concerned with the geometry of motion apart from all considerations of force, mass or energy; the second, is called *Kinetics*, is concerned with the effects of forces on the motion of bodies.

In order to describe the motion of a particle (or point) two things are needed,

(i) a frame of reference,
(ii) a time-keeper.

It is not possible to describe absolute motion, but only motion relative to surrounding objects; and a suitable frame of reference depends on the kind of motion that it is desired to describe. Thus if the motion is rectilinear the distance from a fixed point on the line is a sufficient description of the position of the moving point; and in more general cases systems of two or of three rectangular axes may be chosen as a frame of reference. For example, in the case of a body projected from the surface of the Earth a set of axes with the origin at the point of projection would be suitable for the description of motion relative to the Earth. But, for the description of the motion of the planets, it would be more convenient to take a frame of axes with an origin at the Sun's center (Polar co-ordinates).

■ Definitions

1. Mass: The mass of a body is the quantity of matter in the body. The unit of mass used in England is a pound and is defined to be the mass of a certain piece of platinum kept in the Exchequer Office.

2. A Particle (point): is a portion of matter which is indefinitely small in size, or which, for the purpose of our investigations, is so small that the distances between its different parts may be neglected.

3. A Body: may be regarded as an indefinitely large number of indefinitely small portions, or as a conglomeration of particles.

4. A Rigid Body is a body whose parts always preserve an invariable position with respect to one another.

5. Space is the boundless, three-dimensional extent in which objects and events occur and have relative position and direction. Two-dimensional space is described with two coordinates (x, y), while three-dimensional space (physical reality) is described in three coordinates (x, y, z).

6. Time is a part of the measuring system used to sequence events, to compare the durations of events and the intervals between them, and to quantify rates of change such as the motions of object (not related to analysis of statics problems).

7. Force is any influence that causes an object to undergo a change in speed, a change in direction, or in a change in shape. Force can also be described by intuitive concepts such as a push or pull that can cause an object with mass to change its velocity, i.e. accelerate. A force has both magnitude and direction, which is a vector quantity.

KINEMATICS IN ONE DIMENSION

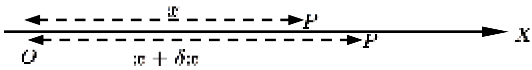
RECTILINEAR MOTION

Although motion in a straight line or rectilinear motion constitute the simplest of dynamical problems, yet it is very important because many physical problems reduce to this category, e.g., simple harmonic motion, motion under inverse square law, motion in a resisting medium and motion of a rocket. Therefore, in this chapter, we first proceed to determine the solution of the one dimensional equation of motion with subject to initial conditions. When a point (or particle) moves along a straight line, its motion is said to be a rectilinear motion. Here in this chapter we shall discuss the motion of a point (or particle) along a straight line which may be either horizontal or vertical. When a point (or particle) moves along a straight line, its motion is said to be a rectilinear motion. Here in this chapter we shall discuss the motion of a point (or particle) along a straight line which may be either horizontal or vertical.

■ Velocity and Acceleration

Suppose a particle moves along a straight line $\mathbf{OX}$ where $\mathbf{O}$ represents a fixed point on the line. Let $\mathbf{P}$ be the position of the particle at time t, where $\mathbf{OP} = x$ and $\mathbf{P'}$ be the position of the particle at time $t + \delta t$, with $\mathbf{OP'} = x + \delta x$. Therefore $\delta x/\delta t$ represents the average rate of displacement or the average velocity during the interval δt. If this ratio be independent of the interval δt, i.e. if it has the same value for all intervals of time, then the velocity is constant or uniform, and equal distances will be traversed in equal times. Whether the ratio $\delta x / \delta t$ be constant or not, its limiting value as δt tends to zero is defined to be the measure of the *velocity* (also known as instantaneous velocity) of the moving point at time t. But this limiting value is the differential coefficient of x with regard to t, so that if we denote the velocity by v, we have

$$v = \lim_{\delta t \to 0} \frac{\delta x}{\delta t} = \frac{dx}{dt} = \dot{x}$$



Again, *Acceleration* is similarly defined as the rate of change of velocity. Thus, if $v, v + \delta v$ denote the velocities of the moving point at times $t, t + \delta t$, then δv is the change of velocity in time δt and $\delta v / \delta t$ is the average rate of change of velocity during the interval δt. If this ratio is independent of the interval δt, then the acceleration is constant or uniform, or equal increments of velocity take place in equal intervals. Whether the ratio $\delta v / \delta t$ be constant or not, its limiting value as δt tends to zero is defined to be the measure of the acceleration of the moving point at time t. But this limiting value is the differential coefficient of v with regard to t, so that if we denote the acceleration by a, we have

$$\begin{aligned} a = \lim_{\delta t \to 0} \frac{\delta v}{\delta t} &= \frac{dv}{dt} \\ &= \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} = \ddot{x} \end{aligned}$$

■ Other Expression for Acceleration

Let $v = \dfrac{dx}{dt}$. We can write (using chain rule in Differentiation)

$$\begin{aligned} a = \ddot{x} = \frac{d^2x}{dt^2} &= \frac{d}{dt}\left(\frac{dx}{dt}\right) \\ &= \frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt} = v\frac{dv}{dx} \end{aligned}$$

v

Therefore, $\frac{d^2x}{dt^2}$, $\frac{dv}{dt}$ and $v\frac{dv}{dx}$ are three expressions for representing the acceleration and any one of them can be used to suit the convenience in working out the problems.

■ Remember

The law of acceleration in a particular problem may be given by expressing the acceleration as a function of the time t, or the distance x, or the velocity v. The problem of further investigating the motion can then be solved as follows:

➤If acceleration is given as a function of the time t say $\varphi(t)$ so

$$a = \varphi(t) \qquad \Rightarrow \frac{dv}{dt} = \varphi(t)$$
$$\Rightarrow dv = \varphi(t)dt$$
$$\Rightarrow v = \int \varphi(t)dt + c_1$$

And then $\because \quad v = \int \varphi(t)dt + c_1 \qquad \Rightarrow \frac{dx}{dt} = \int \varphi(t)dt + c_1$

$$\Rightarrow dx = \left(\int \varphi(t)dt + c_1\right) dt$$

$$\therefore x = \int \left(\int \varphi(t)dt + c_1\right) dt + c_2$$

➤If acceleration is given as a function of the distance x say $f(x)$ so

$$a = f(x) \qquad \Rightarrow v\frac{dv}{dx} = f(x) \qquad \Rightarrow vdv = f(x)dx$$
$$\Rightarrow v^2 = 2\int f(x)dx + c_3$$

Further, $\because v^2 = 2\int f(x)dx + c_3$

$$\Rightarrow \frac{dx}{dt} = \mp\sqrt{2\int f(x)dx + c_3}$$

$$\Rightarrow \mp\frac{dx}{\sqrt{2\int f(x)dx + c_3}} = dt$$

$$\Rightarrow t + c_4 = \mp\int \frac{dx}{\sqrt{2\int f(x)dx + c_3}}$$

➤Again, Acceleration is given as a function of velocity v say $\varphi(v)$

$$a = \varphi(v) \qquad \Rightarrow \frac{dv}{dt} = \varphi(v)$$

$$\Rightarrow \frac{dv}{\varphi(v)} = dt \qquad \text{by integrating}$$

$$\Rightarrow t = \int \frac{dv}{\varphi(v)} + c_5$$

or we may connect velocity with distance by writing

$$v\frac{dv}{dx} = \varphi(v) \quad \Rightarrow \frac{vdv}{\varphi(v)} = dx \qquad \therefore x = \int \frac{vdv}{\varphi(v)} + c_6$$

where, $c_1 - c_6$ are constants of integration.

■ Illustrative Examples ■

Example

A car moves along a straight line such that its displacement x from a fixed point on the line (origin) at time t is given by $x = t^3 - 9t^2 + 24t + 6$. Determine the instant when the acceleration becomes zero, the position of the car at this instant and the velocity of the particle then.

Solution

Since, $x = t^3 - 9t^2 + 24t + 6$. Differentiating with respect to time (w.r.t),

the velocity $$v = \frac{dx}{dt} = 3t^2 - 18t + 24,$$

and the acceleration is $$a = \frac{dv}{dt} = 6t - 18$$

Now the acceleration vanishes i.e. $a = 0$ when $6t - 18 = 0 \Rightarrow t = 3$

When $t = 3$, the position is given by $x = 3^3 - 9(3^2) + 24(3) + 6 = 24$ units.

Again when $t = 3$ the velocity is given by $v = 3(3^2) - 18(3) + 24 = -3$, this means that at $t = 3$ the velocity of the particle equals 3 units and in the opposite direction of x.

Example

If at time t the displacement x of a particle moving away from the origin is given by $x = A\cos t + B\sin t$, where A, B are constants. Find the velocity and acceleration of the particle at in terms of time.

Solution

Given that $x = A\cos t + B\sin t$

Differentiating with respect to time (w.r.t), we obtain the velocity of the particle

$$v=\frac{dx}{dt}=B\cos t-A\sin t$$

Differentiating again, one get the acceleration at any time,

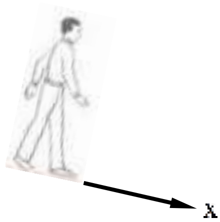
$$\begin{aligned}a=\frac{dv}{dt}&=-A\cos t-B\sin t\\&=-(\underbrace{A\cos t+B\sin t}_{x})\\&=-x\end{aligned}$$

Note that the acceleration proportional to the displacement.

Example

A man moves along a straight line where its distance x from a fixed point on the line is given by $x=A\cos(\mu t+\epsilon)$. Prove that its acceleration varies as the distance measured from the origin and is directed towards the origin.

Solution



Since we have $x=A\cos(\mu t+\epsilon)$

Differentiating w.r.t $x=A\cos(\mu t+\epsilon)$, we get

$$\frac{dx}{dt}=-\mu A\sin(\mu t+\epsilon)$$

Differentiation again $\dfrac{d^2x}{dt^2}=-\mu^2\underbrace{A\cos(\mu t+\epsilon)}_{x}=-\mu^2x$

That is the acceleration varies as the distance x from the origin. The negative sign “-“ indicates that it is in the negative sense of the x-axis, i.e., towards the origin.

Example

A truck moves along a straight line such that its distance x from a fixed point on it and the velocity v are related by $v^2=\mu(b^2-x^2)$. Prove that the acceleration varies as the distance from the origin and is directed towards the origin.

Solution

Since we have $v^2 = \mu(b^2 - x^2)$

Differentiating w.r.t x, we obtain

$$2v\frac{dv}{dx} = \mu(-2x) \qquad \therefore v\frac{dv}{dx} = a = -\mu x$$

Hence the acceleration varies as the distance x from the origin. The negative sign "-" indicates that it is in the direction of x decreasing, i.e., towards the origin.

Example

A particle moves along a straight line such that its distance x from a fixed point on it and the time at any time t are related by $x = 2(1 - e^{-t})$. Find the velocity in terms of distance and the acceleration in terms of velocity.

Solution

In order to obtain the velocity with differentiating the function of position x with respect to time, we get

$$x = 2\ 1 - e^{-t} \qquad \Rightarrow v = \frac{dx}{dt} = 2e^{-t} \qquad \textbf{Note} \quad \frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$$

$$\because x - 2 = -2e^{-t} \qquad \Rightarrow v = 2 - x$$

This equation illustrates the relation between velocity and distance.

Now to get the relation between acceleration and velocity

$$\because a = v\frac{dv}{dx} = v(-1) = -v \qquad \textbf{Note} \quad \frac{dv}{dx} = -1 \qquad \therefore a = -v$$

Example

A car moves along a straight line such that its acceleration at any time t is given by $6t + 2$. Initially the mass at rest placed at the origin point. Determine the velocity and distance as a function of time. Determine the position of the car after 5 sec.

Solution

Since we have $a = 6t + 2,$ $\quad a = \dfrac{dv}{dt} \quad \Rightarrow \dfrac{dv}{dt} = 6t + 2$

Thus, by separation of variables we get

$$dv = \ 6t + 2 \ \ dt \qquad \Rightarrow \int dv = \int \ 6t + 2 \ \ dt$$

$$\therefore \ v = 3t^2 + 2t + c_1$$

From initial conditions at $t = 0$, $v = 1$ then $c_1 = 0$

Again, $\therefore \ v = 3t^2 + 2t$ this equation gives the relation between velocity and time. Since $v = \dfrac{dx}{dt}$ that is

$\dfrac{dx}{dt} = 3t^2 + 2t \qquad \Rightarrow dx = \ 3t^2 + 2t \ \ dt$ (Separation variables)

$\int dx = \int \ 3t^2 + 2t \ \ dt$ or $x = t^3 + t^2 + c_2$

From initial conditions at $t = 0$, $x = 0$ then $c_2 = 0$, i.e.

$x = t^3 + t^2$

this equation gives the relation between distance and time.

The position at $t = 5$ is $x\big|_{t=5} = 5^3 + 5^2 = 150$

Example

A point moves along a straight line according to $v = u + bx$, where u, b are constants. Find the velocity and acceleration in terms of time and the acceleration in terms of distance and also as a function of velocity.

Solution

Velocity and acceleration can be obtained by differentiation the function of position and then velocity with respect to time, therefore

$$v = u + bx \quad \Rightarrow a = \frac{dv}{dt} = b\frac{dx}{dt} = bv = b(u+bx) \qquad \Rightarrow a = b(u+bx)$$

This equation gives the acceleration as a function of velocity $a = bv$ and as a function of distance $a = b(u+bx)$

Again to get the velocity and acceleration as functions of time

$$\because v = u + bx \quad \Rightarrow \frac{dx}{dt} = b(u+bx) \qquad \Rightarrow \frac{dx}{u+bx} = bdt$$

Multiply the previous relation by b and then integrate

$$\int \frac{bdx}{u+bx} = \int b^2 dt \quad \Rightarrow \ln(u+bx) = b^2\, t + C$$

Where C is integration constant , the last relation can be rewritten as

$$\because \ln(u+bx) = b^2\, t + C \qquad \Rightarrow \ln v = b^2\, t + C \quad \text{Or}$$
$$\Rightarrow v = Ae^{b^2 t}, \quad A = e^C$$

This is the relation between velocity and time, also the acceleration given by

$$a = bv = bAe^{b^2 t}$$

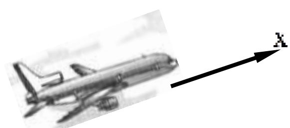
Example

A plane flies along a straight line with retardation $a = -2v^2$. Find the position at any instance if the point starts from origin with initial velocity equals unity.

Solution

The motion under retardation where $a = -2v^2$ but we know $a = \dfrac{dv}{dt}$, so

$$\because a = -2v^2 \qquad \Rightarrow \frac{dv}{dt} = -2v^2$$



By separation of variables and integrate, we obtain

$$-\int \frac{dv}{v^2} = \int 2dt + c_1 \qquad \Rightarrow \frac{1}{v} = 2t + c_1$$

The integration constant c_1 can be evaluated as $v = 1$ when $t = 0$, hence $1 = 2(0) + c_1 \quad \therefore\ c_1 = 1$ then the velocity can be obtained by

$$\frac{1}{v} = 2t + 1 \quad \text{but} \quad v = \frac{dx}{dt} \quad \therefore \frac{dt}{dx} = 2t + 1 \quad \text{Or} \quad \frac{dt}{2t+1} = dx$$

Again by integrating we get

$$\frac{2dt}{2t+1} = 2dx \quad \Rightarrow \ln(2t+1) = 2x + c_2$$

From initial condition $x = 0$ when $t = 0$ then $c_2 = 0$ and the relation between distance and time becomes

$$x = \frac{1}{2}\ln(2t+1)$$

Example

A particle starts from rest at a distance h from the origin $\mathbf{O}$ with retardation $-4x^{-3}$. Prove that the particle reach to distance ℓ from $\mathbf{O}$ in time $\frac{h}{2}\sqrt{h^2 - \ell^2}$ and then find its velocity at this position.

Solution

Since we have been given the retardation as $a = -16x^{-3}$ and $a = v\frac{dv}{dx}$ therefore,

$$\therefore\ v\frac{dv}{dx} = -4x^{-3} \qquad \Rightarrow vdv = -4x^{-3}\ dx$$

By integrating, we obtain

$$\therefore \int vdv = -\int 4x^{-3}\ dx + c_1 \qquad \text{Or} \qquad \frac{1}{2}v^2 = \frac{2}{x^2} + c_1 \quad \text{Or} \quad v^2 = \frac{4}{x^2} + c$$

The integration constant c can be evaluated as $v = 0$ when $x = h$, hence $0 = \frac{4}{h^2} + c$ i.e. $c_1 = -\frac{4}{h^2}$ and then we get

$$v^2 = \frac{4}{x^2} - \frac{4}{h^2} = \frac{4(h^2 - x^2)}{x^2 h^2} \qquad \therefore v = \pm \frac{2}{h} \frac{\sqrt{h^2 - x^2}}{x}$$

We will consider the minus sign since the motion of the particle towards the origin –in decreasing x - and use $v = \frac{dx}{dt}$

$$\therefore \frac{dx}{dt} = -\frac{2}{h} \frac{\sqrt{h^2 - x^2}}{x} \qquad \Rightarrow -\frac{xdx}{\sqrt{h^2 - x^2}} = \frac{2}{h} dt \quad \text{Or}$$

$$\Rightarrow -\int \frac{xdx}{\sqrt{h^2 - x^2}} = \int \frac{2}{h} dt + c_2$$

$$\Rightarrow \sqrt{h^2 - x^2} = \frac{2}{h} t + c_2$$

To obtain the constant c_2 when $x = h$ as $t = 0$ and then $c_2 = 0$ so

$$\therefore \sqrt{h^2 - x^2} = \frac{2}{h} t \qquad \text{Or} \qquad t = \frac{h}{2} \sqrt{h^2 - x^2}$$

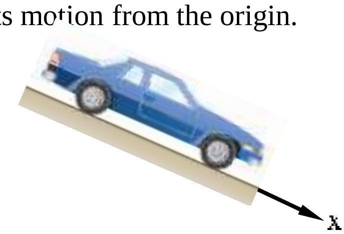
The spent time to reach to a distance ℓ from origin point is $t = \frac{h}{2} \sqrt{h^2 - \ell^2}$, to determine the velocity at this position, we put $x = \ell$ in velocity relation, that is

$$v\Big|_{x=\ell} = \frac{2\sqrt{h^2 - \ell^2}}{h\ell}$$

Example

A car moves along a straight line according to the relation $v = (1 + x^2)t$. Find the distance as a function of time if the point starts its motion from the origin.

Solution



Since $v = (1 + x^2)t$ thus

$$\frac{dx}{dt} = (1 + x^2)t \qquad \Rightarrow \frac{dx}{1 + x^2} = tdt$$

$$\therefore \int \frac{dx}{1+x^2} = \int t\,dt + c_1 \qquad \Rightarrow \tan^{-1} x = \frac{1}{2}t^2 + c_1$$

From initial condition where the point starts its motion at origin

$$\therefore \tan^{-1} 0 = \frac{1}{2}0^2 + c_1 \quad \Rightarrow 0 = 0 + c_1 \therefore c_1 = 0 \qquad \therefore x = \tan\left(\frac{1}{2}t^2\right)$$

Note that

$$\int \frac{f' dx}{1+f^2} = \tan^{-1} f$$

Example

If t be regarded as a function of velocity v, prove that the rate of decrease of acceleration is given by $a^3 \dfrac{d^2t}{dv^2}$, a being the acceleration.

Solution

Let a be the acceleration at time t. Then $a = \dfrac{dv}{dt}$. Now the rate of decrease of acceleration = $-\dfrac{da}{at}$

$$= -\frac{d}{at}\left(\frac{dv}{dt}\right) = -\frac{d}{at}\left(\frac{dt}{dv}\right)^{-1} \text{ regarded } t \text{ as a function of } v$$

$$= -\left(\frac{d}{av}\left(\frac{dt}{dv}\right)^{-1}\right)\frac{dv}{dt} = \left(\left(\frac{dt}{dv}\right)^{-2}\frac{d^2t}{dv^2}\right)\frac{dv}{dt}$$

$$= \left(\left(\frac{dv}{dt}\right)^{2}\frac{d^2t}{dv^2}\right)\frac{dv}{dt} = \underbrace{\left(\frac{dv}{dt}\right)^{3}}_{a}\frac{d^2t}{dv^2} = a^3\frac{d^2t}{dv^2}$$

Example

Prove that if a point moves with a velocity varying as any power (not less than unity) of its distance from a fixed point which it is approaching, it will never reach that point.

Solution

If x is the distance of the particle from the fixed point $\mathbf{O}$ at any time t, then its speed v at this time is given by $v = kx^n$, where k is a constant and n is not less than 1. Since the particle is moving towards the fixed point i.e., in the direction decreasing, therefore

$$\frac{dx}{dt} = -v \qquad \text{or} \qquad \frac{dx}{dt} = -kx^n \qquad \text{....(1)}$$

Case 1. If $n = 1$, then from (1), we have

$$\frac{dx}{dt} = -kx \qquad \text{or} \qquad dt = -\frac{1}{k}\frac{dx}{x}$$

Integrating, $t = -\frac{1}{k}\ln x + A$ where A is a constant.

Putting $x = 0$ then the time t to reach the fixed point $\mathbf{O}$ is given by

$$t = -\frac{1}{k}\ln 0 + A = \infty$$

i.e., the particle will never reach the fixed point $\mathbf{O}$

Case 2. If $n > 1$, then from (1), we have

$$dt = -\frac{1}{k}x^{-n}dx$$

Integrating, $t = -\frac{1}{k}\frac{x^{1-n}}{1-n} + B$ where B is a constant.

Or
$$t = \frac{1}{k(n-1)x^{n-1}} + B$$

Putting $x = 0$ then the time t to reach the fixed point $\mathbf{O}$ is given by

$$t = \infty + B = \infty$$

i.e., the particle will never reach the fixed point $\mathbf{O}$

Hence if $n \geq 1$, the particle will never reach the fixed point, it is approaching.

PROBLEMS

❒ A particle moving in a straight line is subject to a resistance which produces the retardation kv^3, where v is the velocity and k is a constant. Show that v and t (the time) are given in terms of x (the distance) by the equations $v = \dfrac{u}{kux+1}$, $t = \dfrac{1}{2}kx^2 + \dfrac{x}{u}$, where u is the initial velocity.

❒ If the relation between x and t is of the form $t = bx^2 + kx$, find the velocity v as a function of x, and prove that the retardation of the particle is $2bv^3$.

❒ A particle is projected vertically upwards with speed u and moves in a vertical straight line under uniform gravity with no air resistance. Find the maximum height achieved by the particle and the time taken for it to return to its starting point.

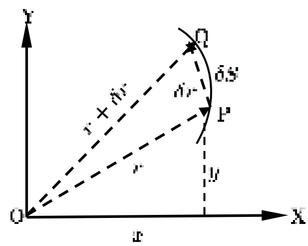
Kinematics in Two Dimensions

■ Velocity in Cartesian Coordinates

The velocity vector of a particle (or point) moving along a curve is the rate of change of its displacement with respect to time.

Let P and Q be the positions of a particle moving along a curve at times t and $t+\delta t$ respectively. With respect to $\mathbf{O}$ as the origin of vectors, let $\underline{\mathbf{O}P} = \underline{r}$ and $\underline{\mathbf{O}Q} = \underline{r} + \delta\underline{r}$ Then $\underline{\mathbf{P}Q} = \underline{\mathbf{O}Q} - \underline{\mathbf{OP}} = \delta\underline{r}$ represents the displacement of the particle in time δt and $\frac{\delta \underline{r}}{\delta t}$ indicates the average rate of displacement (or average velocity) during the interval δt. The limiting value of the average velocity $\frac{\delta \underline{r}}{\delta t}$ as δt tends to zero ($\delta t \to 0$) is the velocity. Therefore if the vector $\underline{v}$ represents the velocity of the particle at time t then

$$\underline{v} = \lim_{\delta t \to 0} \frac{\delta \underline{r}}{\delta t} = \frac{d\underline{r}}{dt} = \dot{\underline{r}}$$



Where $\underline{r}$ is the position vector of the particle.

Now, if $\quad \underline{r} = x\hat{i} + y\hat{j}$

Then $\quad \underline{v} = \frac{d\underline{r}}{dt} = \underbrace{\frac{dx}{dt}}_{v_x}\hat{i} + \underbrace{\frac{dy}{dt}}_{v_y}\hat{j} = \dot{x}\hat{i} + \dot{y}\hat{j}$

Note that $(\dot{x}, \dot{y})$ are called the components or resolved parts of the velocity $\underline{v}$ along the axes x and y respectively. The speed of the particle at P is given by

$$|\underline{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \frac{ds}{dt}$$

Also the angle θ which the direction of $\underline{v}$ makes with $\mathbf{OX}$ is

$$\tan\theta = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{dy}{dx}$$

■ Acceleration in Cartesian Coordinates

The acceleration vector of a particle moving along a curve is defined as the rate of change of its velocity vector.

if $\underline{v}$ and $\underline{v} + \delta\underline{v}$ are the velocities of a particle moving along a curve at times t and $t + \delta t$ respectively, then $\delta\underline{v}$ is the change in velocity vector in time δt and $\dfrac{\delta\underline{v}}{\delta t}$ is the average and then

$$\underline{a} = \lim_{\delta t \to 0} \frac{\delta\underline{v}}{\delta t} = \frac{d\underline{v}}{dt} = \frac{d}{dt}\left(\frac{d\underline{r}}{dt}\right) = \frac{d^2\underline{r}}{dt^2}$$

Substituting for $\underline{v} = \dfrac{dx}{dt}\hat{i} + \dfrac{dy}{dt}\hat{j}$ we have,

$$\underline{a} = \frac{d}{dt}\left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}\right) = \underbrace{\frac{d^2x}{dt^2}}_{a_x}\hat{i} + \underbrace{\frac{d^2y}{dt^2}}_{a_y}\hat{j} = \ddot{x}\,\hat{i} + \ddot{y}\,\hat{j}$$

Here, $(\ddot{x}, \ddot{y})$ are called the components of the acceleration $\underline{a}$ along the axes x and y respectively. The magnitude of the acceleration is given by

$$|\underline{a}| = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}$$

Again, the angle φ which the direction of $\underline{a}$ makes with OX is

$$\tan\varphi = \frac{d^2y}{dt^2} \Big/ \frac{d^2x}{dt^2}$$

■ Illustrative Examples ■

Example

A point moves along the curve $x = t^3 + 1, \quad y = t^2$ where, t is the time. Determine the components of velocity and acceleration at $t = 1$

Solution

Let $\underline{r}$ be the position vector of the particle at time t, therefore

$$\underline{r} = x\,\hat{i} + y\,\hat{j} = (t^3 + 1)\,\hat{i} + t^2\,\hat{j}$$

Then the velocity vector is

$$\underline{v} = \frac{d\underline{r}}{dt} = 3t^2\,\hat{i} + 2t\,\hat{j} \quad \text{and} \quad \underline{v}\Big|_{t=1} = 3(1)^2\,\hat{i} + 2(1)\,\hat{j} = 3\,\hat{i} + 2\,\hat{j}$$

Again the vector of acceleration is

$$\underline{a} = \frac{d\underline{v}}{dt} = 6t\,\hat{i} + 2\,\hat{j} \quad \text{and} \quad \underline{a}\Big|_{t=1} = 6(1)\,\hat{i} + 2\,\hat{j} = 6\,\hat{i} + 2\,\hat{j}$$

Example

The position of a moving point at time t is given by $x = 3\cos t, \quad y = 2\sin t$

Find its path velocity and acceleration vectors.

Solution

Since the parametric equations are $x = 3\cos t, \quad y = 2\sin t$ then

$$\left(\frac{x}{3}\right)^2 = \cos^2 t, \quad \left(\frac{y}{2}\right)^2 = \sin^2 t \quad \Rightarrow \left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \quad \text{or } 4x^2 + 9y^2 = 36$$

This is a the path equation which represents an Ellipse

Velocity vector is $\underline{v} = -3\sin t\;\hat{i} + 2\cos t\;\hat{j}$

While the acceleration vector is

$$\underline{a} = -3\cos t\ \hat{i} - 2\sin t\ \hat{j} = -\underbrace{(3\cos t\ \hat{i} + 2\sin t\ \hat{j})}_{\underline{r}} = -\underline{r}$$

Example

A particle moves along the curve $y = 2x^2$ such that its horizontal component of velocity is constant and equals 2. Calculate the components of acceleration and velocity when $y = 8$.

Solution

Since the horizontal component of velocity equals 2, i.e. $\dot{x} = 2$, therefore by differentiating w.r.t t we get

$\ddot{x} = 0$ and $y = 2x^2 \quad \Rightarrow \dot{y} = 4x\dot{x} = 8x \quad \therefore \ddot{y} = 8\dot{x} = 16$

That is the acceleration vector is given by

$\underline{a} = 16\,\hat{j}$

and the velocity components are $\dot{x} = 2$ and $\dot{y} = 8x$

Since as $y = 8$ gives $x = \pm 2$ thus, $\quad \underline{v} = 2\,\hat{i} + 8(\pm 2)\,\hat{j}, \quad |\underline{v}| = \sqrt{260}$

Example

A particle describes a plane curve such that its components of acceleration equal $(0, -\mu / y^2)$ with initial velocity $\sqrt{2\mu / b}$ parallel to **X-** axis and the initial position $(0, b)$. Find the path equation.

Solution

Here we are given that

$$\frac{d^2x}{dt^2} = 0, \qquad \frac{d^2y}{dt^2} = -\frac{\mu}{y^2}$$

Note that $\dfrac{d^2y}{dt^2} = \dfrac{d}{dt}\left(\dfrac{dy}{dt}\right) = \dfrac{d}{dy}\left(\dfrac{dy}{dt}\right) \times \dfrac{dy}{dt} = \dot{y}\dfrac{d\dot{y}}{dy}$ chain rule

Then $\dot{y}\dfrac{d\dot{y}}{dy} = -\dfrac{\mu}{y^2} \quad \Rightarrow \dot{y}d\dot{y} = -\dfrac{\mu}{y^2}dy \quad \Rightarrow \int \dot{y}d\dot{y} = -\int \dfrac{\mu}{y^2}dy$

$\dot{y}^2 = \dfrac{2\mu}{y} + c_1 \quad \left(\dot{y} = \dfrac{dy}{dt}\right)$

Initially $\dfrac{dy}{dt} = 0$ when $y = b$, thus $\quad c_1 = -\dfrac{2\mu}{b}$

$\dot{y}^2 = \dfrac{2\mu}{y} - \dfrac{2\mu}{b} \quad \Rightarrow \dot{y}^2 = \dfrac{2\mu}{y} - \dfrac{2\mu}{b} = 2\mu\left(\dfrac{1}{y} - \dfrac{1}{b}\right) = \dfrac{2\mu}{b}\dfrac{b-y}{y}$

Hence

$$\frac{dy}{dt} = -\sqrt{\frac{2\mu}{b}}\sqrt{\frac{b-y}{y}} \qquad (1)$$

(Negative sign has been taken because the particle is moving in the direction of y decreasing)

Again from $\dfrac{d^2x}{dt^2} = 0 \quad \Rightarrow \dfrac{dx}{dt} = c_3$

Initially when $t = 0$, $\dfrac{dx}{dt} = \sqrt{\dfrac{2\mu}{b}}$ thus $\quad c_3 = \sqrt{\dfrac{2\mu}{b}}$

$$\therefore \frac{dx}{dt} = \sqrt{\frac{2\mu}{b}} \qquad (2)$$

By dividing the two equations (1) and (2) we get

$\dfrac{dy}{dx} = -\sqrt{\dfrac{b-y}{y}} \quad \Rightarrow \sqrt{\dfrac{y}{b-y}}\,dy = -dx$, then by integrating

$$b\left(\sin^{-1}\sqrt{\frac{y}{b}} - \sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}}\right) = -x + c_2$$

Hint to get the integration $\int \sqrt{\dfrac{y}{b-y}}\,dy$ let us use the transformation

$$y = b\sin^2\theta \qquad \Rightarrow dy = 2b\sin\theta\cos\theta\, d\theta$$

$$\therefore \int\sqrt{\frac{y}{b-y}}\,dy = \int\sqrt{\frac{b\sin^2\theta}{b-b\sin^2\theta}}\,2b\sin\theta\cos\theta\,d\theta$$

$$=\int\sqrt{\frac{b\sin^2\theta}{b\cos^2\theta}}\,2b\sin\theta\cos\theta\,d\theta$$

$$=\int\frac{\sin\theta}{\cancel{\cos\theta}}\,2b\sin\theta\,\cancel{\cos\theta}\,d\theta = 2b\int\sin^2\theta\,d\theta$$

$$\because \sin^2\theta = \frac{1}{2}\ \ 1-\cos 2\theta$$

$$\Rightarrow 2b\int\sin^2\theta\,d\theta = 2b\int\frac{1}{2}\ \ 1-\cos 2\theta\ \ d\theta$$

$$= b\int\ \ 1-\cos 2\theta\ \ d\theta = b\left(\theta-\frac{\sin 2\theta}{2}\right)$$

$$\therefore \int\sqrt{\frac{y}{b-y}}\,dy = b\left(\sin^{-1}\left(\sqrt{\frac{y}{b}}\right)-\sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}}\right)$$

The initial condition is $t=0\ \ x=0,\ \ y=b$ then from the equation

$$b\left(\sin^{-1}\sqrt{\frac{y}{b}}-\sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}}\right) = -x+c_2 \qquad \Rightarrow c_2 = b\frac{\pi}{2}$$

$$\therefore b\left(\sin^{-1}\left(\sqrt{\frac{y}{b}}\right)-\sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}}\right) = b\frac{\pi}{2}-x$$

$$\Rightarrow \sin^{-1}\sqrt{\frac{y}{b}} = \sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}}+\frac{\pi}{2}-\frac{x}{b}$$

$$\sqrt{\frac{y}{b}} = \sin\left(\sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}}+\frac{\pi}{2}-\sqrt{2\mu b}\,t\ \right)$$

$$= \cos\left(x-\sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}}\ \right)$$

$$y = b\cos^2\left(x-\sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}}\ \right)$$

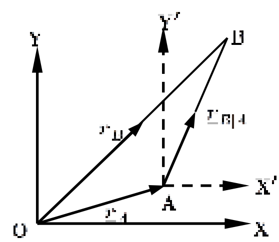
■ Relative motion of two particles

Motion does not happen in isolation. If you're riding in a train moving at 10 ms^{-1} east, this velocity is measured relative to the ground on which you're traveling. However, if another train passes you at 15 ms^{-1} east, your velocity relative to this other train is different from your velocity relative to the ground. Your velocity relative to the other train is 5 ms^{-1} west. To explore this idea further, we first need to establish some terminology.

♦ Reference Frames

To discuss relative motion in one or more dimensions, we first introduce the concept of reference frames. When we say an object has a certain velocity, we must state it has a velocity with respect to a given reference frame. In most examples we have examined so far, this reference frame has been Earth. If you say a person is sitting in a train moving at 10 m/s east, then you imply the person on the train is moving relative to the surface of Earth at this velocity, and Earth is the reference frame. We can expand our view of the motion of the person on the train and say Earth is spinning in its orbit around the Sun, in which case the motion becomes more complicated. In this case, the solar system is the reference frame. In summary, all discussion of relative motion must define the reference frames involved. We now develop a method to refer to reference frames in relative motion.

For two particles A and B moving in plane as shown, we considered the relative motion of B with respect to A, or more precisely, with respect to a moving frame attached to A and in translation with A. Denoting by $\underline{r}_{B|A}$ the relative position vector of B with respect to A, we had



$$\underline{r}_B = \underline{r}_A + \underline{r}_{B|A} \qquad \text{or} \qquad \underline{r}_{B|A} = \underline{r}_B - \underline{r}_A$$

Denoting by $\underline{v}_{B|A}$ and $\underline{a}_{B|A}$, respectively, the relative velocity and the relative acceleration of B with respect to A, we also showed that

Differentiating previous equation with respect to time

$$\frac{d\underline{r}_{B|A}}{dt} = \frac{d\underline{r}_B}{dt} - \frac{d\underline{r}_A}{dt} \quad \textbf{or} \quad \underline{v}_{B|A} = \underline{v}_B - \underline{v}_A$$

Differentiating previous equation with respect to time

$$\frac{d\underline{v}_{B|A}}{dt} = \frac{d\underline{v}_B}{dt} - \frac{d\underline{v}_A}{dt} \quad \textbf{or} \quad \underline{a}_{B|A} = \underline{a}_B - \underline{a}_A$$

■ Illustrative Examples ■

Example

Two points A and B are moving along a straight line such that $\underline{x}_A = t^3 - 2t$ and $\underline{x}_B = 2t^3 + t^2 - 5$. Find the relative velocity $\underline{v}_{BA}$ and acceleration $\underline{a}_{BA}$.

Solution

Since the relative position of point B with respect to point A, $\underline{x}_{B|A}$, is given by

$$\underline{x}_{B|A} = \underline{x}_B - \underline{x}_A$$

$$\Rightarrow \underline{x}_{B|A} = (2t^3 + t^2 - 5) - (t^3 - 2t) = t^3 + t^2 + 2t - 5$$

Hence the relative velocity $\underline{v}_{B|A}$ is obtained by

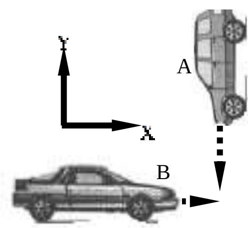
$$\underline{v}_{B|A} = \frac{d\underline{x}_{B|A}}{dt} = 3t^2 + 2t + 2$$

Again the relative acceleration $\underline{a}_{BA}$ is given by

$$\underline{a}_{B|A} = \frac{d\underline{v}_{B|A}}{dt} = 6t + 2$$

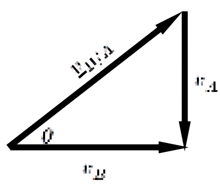
Example

A car A is traveling south at a speed of 70 km/h toward an intersection. A car B is traveling east toward the intersection at a speed of 80 km/h, as shown. Determine the velocity of the car B relative to the car A.



Solution

According to the given data the velocity of car A is $\underline{v}_A = -70\,\hat{j}$ and velocity of car B is $\underline{v}_B = 80\,\hat{i}$ then



$$\begin{aligned}\underline{v}_{B|A} &= \underline{v}_B - \underline{v}_A \\ &= 80\hat{i} - (-70\hat{j}) \\ &= 80\hat{i} + 70\hat{j}\end{aligned}$$

$$\Rightarrow \left|\underline{v}_{B|A}\right| = \sqrt{(80)^2 + (70)^2} = \sqrt{11300} \simeq 106.3 \ \text{km h}^{-1}$$

And make an angle θ with the velocity direction of car B obtained by

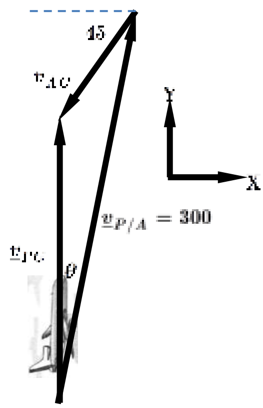
$$\tan\theta = \frac{70}{80} = \frac{7}{8} \qquad \Rightarrow \theta = \tan^{-1}\left(\frac{7}{8}\right)$$

Example

A pilot must fly his plane due north to reach his destination. The plane can fly at 300 km/h in still air. A wind is blowing out of the northeast at 90 km/h. Calculate the speed of the plane relative to the ground and in what direction must the pilot head her plane to fly due north.

Solution

The pilot must point her plane somewhat east of north to compensate for the wind velocity. We need to construct a vector equation that contains the velocity of the plane with respect to the ground, the velocity of the plane with respect to the air, and the velocity of the air with respect to the ground. Since these last two quantities are known, we can solve for the velocity of the plane with respect to the ground. We can graph the vectors and use this diagram to evaluate the magnitude of the plane's velocity with respect to the ground. The diagram will also tell us the angle the plane's velocity makes with north with respect to the air, which is the direction the pilot must head her plane.



From the given data the velocity of plane P is $\underline{v}_{P|A} = 300(\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$ and

velocity of air A is $\underline{v}_{A|G} = -90\,(\cos 45^0\,\hat{i} + \sin 45^0\,\hat{j})$ and $\underline{v}_{P|G} = v_{P|G}\,\hat{j}$ then

$$\begin{aligned} \underline{v}_{P|G} &= \underline{v}_{P|A} + \underline{v}_{A|G} \\ &= 300(\sin\theta\,\hat{i} + \cos\theta\,\hat{j}) + -90\,(\cos 45^0\,\hat{i} + \sin 45^0\,\hat{j}) \\ &= (300\sin\theta - 90\cos 45^0)\hat{i} + (300\cos\theta - 90\sin 45^0)\hat{j} \end{aligned}$$

$$\Rightarrow 300\sin\theta - 90\cos 45^0 = 0$$

$$\sin\theta = \frac{45\sqrt{2}}{300} \qquad \text{And } v_{P|G} = 300\cos\theta - 90\sin 45^0 \simeq 230 \text{ km h}^{-1}$$

PROBLEMS

❒ The position of a moving point at time t is given by $x = at^2,\ y = 2at$

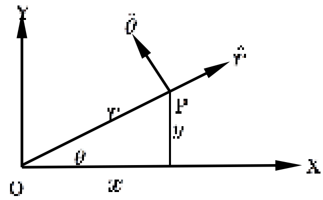
Find its velocity and acceleration

❒ A particle moves with constant velocity parallel to the axis of $\mathbf{Y}$ and a velocity proportional to y parallel to the axis of $\mathbf{X}$. Prove that it will describe a parabola

❒ A particle is acted on by a force parallel to the axis of $\mathbf{Y}$ whose acceleration is λy and is initially projected with a velocity $a\sqrt{\lambda}$ parallel to the axis of $\mathbf{X}$ at a point where $y = a$. Prove that it will describe the catenary $y = a\cosh(x / a)$

❒ A boat heads north in still water at 4.5 ms^{-1} directly across a river that is running east at 3.0 ms^{-1}. Find the velocity of the boat with respect to Earth.

POLAR COORDINATES



In some problems it is convenient to employ another coordinates not Cartesian coordinates as polar coordinates. Let the position of a point $\mathbf{P}$ be defined by its distance r from a fixed origin $\mathbf{O}$ and the angle θ that OP makes with a fixed axis $\mathbf{OX}$.

The Cartesian coordinates (x, y) of P are connected with the polar coordinates (r, θ) by the relations $x = r\cos\theta, \quad y = r\sin\theta$.

Note that $\hat{r}$ and $\hat{\theta}$ represent unit vectors in direction of increasing r and normal to r in the direction of increasing θ as illustrated in the figure.

■ Angular Velocity and Acceleration

Let $\mathbf{P}$ be a moving point in a plane. If $\mathbf{O}$ be a fixed point (pole) and $\mathbf{O}X$ is a fixed line through $\mathbf{O}$ in the plane of motion, then the angular velocity of the moving point $\mathbf{P}$ about $\mathbf{O}$ (or the line $\mathbf{OP}$ in the plane $\mathbf{XOP}$) is the rate of change of the angle $\mathbf{XOP}$ Figure.

Let $\mathbf{P}$ and $\mathbf{Q}$ be the positions of a moving particle at times t and $t + \delta t$ respectively such that $\angle \mathbf{POX} = \theta$ and $\angle \mathbf{QOX} = \theta + \delta\theta$. Therefore, the angle turned by the particle in time δt is $\delta\theta$. That is the average rate of change of the angle of $\mathbf{P}$ about $\mathbf{O}$ is $\dfrac{\delta\theta}{\delta t}$

Then the angular velocity of the point $\mathbf{P}$ about $\mathbf{O}$ is

$$\lim_{\delta t \to 0} \frac{\delta\theta}{\delta t} = \frac{d\theta}{dt} = \dot{\theta}$$

Where the dot placed over θ denotes differentiation with respect to time, and the units of angular velocity is radian/sec.

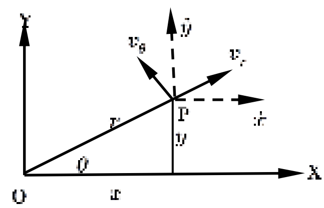
Now the rate of change of angular velocity is called angular acceleration

That is the angular acceleration

$$\lim_{\delta t \to 0} \frac{\delta \dot{\theta}}{\delta t} = \frac{d\dot{\theta}}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2} = \ddot{\theta}$$

The units of angular acceleration is radian/sec^2

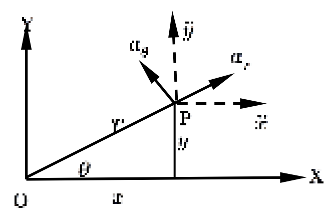
■ Velocity and Acceleration in Polar Coordinates



Let the position of a point **P** be defined by its distance r from a fixed origin **O** and the angle θ that OP makes with a fixed axis **OX**.

The Cartesian coordinates (x, y) of P are connected with the polar coordinates (r, θ) by the relations $x = r\cos\theta, \;\; y = r\sin\theta$.

Let v_r, v_θ denote the components of velocity of **P** in the direction **OP** and at right angles to **OP** in the sense in which θ increases. The resultant of the components v_r, v_θ is also the resultant of the components $\dot{x}$, $\dot{y}$. Therefore by resolving parallel to **OX** and **OY** we get



$$\begin{aligned} v_r \cos\theta - v_\theta \sin\theta = \dot{x} &= \frac{d}{dt}(r\cos\theta) \\ &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta \end{aligned}$$

And

$$\begin{aligned} v_r \sin\theta + v_\theta \cos\theta = \dot{y} &= \frac{d}{dt}(r\sin\theta) \\ &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{aligned}$$

Solving these equations for u and v clearly gives (comparing)

$$v_r = \dot{r}, \qquad v_\theta = r\dot{\theta},$$

and these are the polar components of velocity.

In like manner if a_r, a_θ denote the components of acceleration along and at right angles to **OP** , since these have the same resultant as $\ddot{x}$ and $\ddot{y}$, we get

$$a_r \cos\theta - a_\theta \sin\theta = \ddot{x} = \frac{d^2}{dt^2}(r\cos\theta)$$
$$= \ \ddot{r} - r\dot{\theta}^2 \ \ \cos\theta - \ \ r\ddot{\theta} + 2\dot{r}\dot{\theta} \ \ \sin\theta$$

Again for $\ddot{y}$ we have

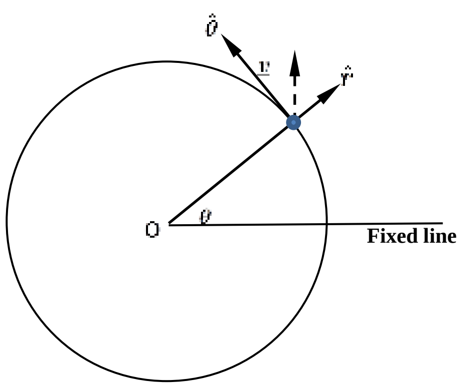
$$a_r \sin\theta + a_\theta \cos\theta = \ddot{y} = \frac{d^2}{dt^2}(r\sin\theta)$$
$$= \ \ddot{r} - r\dot{\theta}^2 \ \ \sin\theta + \ \ r\ddot{\theta} + 2\dot{r}\dot{\theta} \ \ \cos\theta$$

giving on solution $a_r = \ddot{r} - r\dot{\theta}^2$ and $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$

These components constitute a third representation of the velocity and acceleration of a point moving in a plane; they are sometimes called radial and transverse components**,** and we note that the transverse component of acceleration may also be written

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}\ \ r^2\dot{\theta}$$

■ Special Case: If the particle moves in a circle with radius ℓ**,** i.e. $r = \ell$ then $\dot{r} = \ddot{r} = 0$ and hence the velocity of the particle is given by $\underline{v} = \ell\dot{\theta}\,\hat{\theta}$ and its direction will be along the normal to tangent to the circle and also the acceleration will be $\underline{a} = -\ell\dot{\theta}^2\,\hat{r} + \ell\ddot{\theta}\,\hat{\theta}$.



■ Another method for Velocity and Acceleration in Polar Coordinates

We will now evaluate the two derivatives $\dfrac{d\hat{r}}{d\theta}$ and $\dfrac{d\hat{\theta}}{d\theta}$. These will be needed when we derive the formulae for the velocity and acceleration of *P* in polar co-ordinates. First we expand $(\hat{r}, \hat{\theta})$ in terms of the Cartesian basis vectors $(\hat{i}, \hat{j})$. This gives

$$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}, \qquad \hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

Since $\hat{r}, \hat{\theta}$ are now expressed in terms of the constant vectors $(\hat{i}, \hat{j})$, the differentiations with respect to θ are simple and give

$$\frac{d\hat{r}}{d\theta} = \hat{\theta}, \qquad \frac{d\hat{\theta}}{d\theta} = -\hat{r} \qquad (1)$$

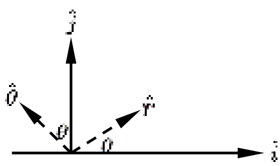
Suppose now that *P* is a moving particle with polar co-ordinates r, θ that are functions of the time t. The position vector of **P** relative to **O** has magnitude $\mathbf{OP} = r$ and direction $\hat{r}$ and can therefore be written

$$\underline{r} = r\,\hat{r} \qquad (2)$$

In what follows, one must distinguish carefully between the position vector $\underline{r}$, which is the vector $\underline{\mathbf{OP}}$, the co-ordinate r, which is the *distance* **OP**, and the polar unit vector $\hat{r}$.

To obtain the polar formula for the velocity of **P**, we differentiate formula (2) with respect to time t. This gives

$$\begin{aligned} \underline{v} = \frac{d\underline{r}}{dt} = \frac{d}{dt}(r\hat{r}) &= \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} \\ &= \dot{r}\hat{r} + r\frac{d\hat{r}}{d\theta}\frac{d\theta}{dt} \\ &= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \end{aligned}$$



We used the chain rule and formula (1), which is the polar formula for the velocity of **P**.

In order to obtain the polar formula for acceleration, we differentiate the velocity formula $\underline{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$ with respect to t again. This gives

$$
\begin{aligned}
\underline{a} = \frac{d\underline{v}}{dt} &= \frac{d}{dt}\ \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \\
&= \ddot{r}\,\hat{r} + \dot{r}\underbrace{\frac{d\hat{r}}{d\theta}}_{\hat{\theta}}\underbrace{\frac{d\theta}{dt}}_{\dot{\theta}} + (r\ddot{\theta} + \dot{r}\dot{\theta})\hat{\theta} + r\dot{\theta}\underbrace{\frac{d\hat{\theta}}{d\theta}}_{-\hat{r}}\underbrace{\frac{d\theta}{dt}}_{\dot{\theta}} \\
&= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}
\end{aligned}
$$

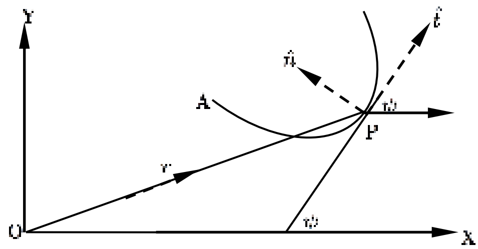
which is the polar formula for the acceleration of **P**.

The formula $\underline{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$ shows that the velocity of **P** is the vector sum of an outward radial velocity $\dot{r}$ and a transverse velocity $r\dot{\theta}$; in other words $\boldsymbol{v}$ is just the sum of the velocities that **P** would have if r and θ varied separately. This is *not* true for the acceleration as it will be observed that adding together the separate accelerations would not yield the term $2\dot{r}\dot{\theta}\,\hat{\theta}$. This 'Coriolis term' is certainly present however, but is difficult to interpret intuitively.

INTRINSIC COORDINATES

Let **P** be the position of a moving particle at time t and $\underline{r}$ its position vector with respect to the origin **O**. Let $\mathbf{AP} = S$ and let ψ be the angle which the tangent at **P** to the path of the particle makes with **OX**. Then $(\mathbf{S},\psi)$ are the intrinsic coordinates of **P**.

Let $\hat{t}$ denote the unit vector along the tangent at in the direction of S increasing and $\hat{n}$ be the unit vector normal at **P** in the direction of ψ increasing i.e., in the direction of inwards drawn normal.



In the same manner we will now evaluate the two derivatives $\frac{d\hat{t}}{d\psi}$ and $\frac{d\hat{n}}{d\psi}$. These will be needed when we derive the formulae for the velocity and acceleration of **P** in intrinsic co-ordinates. First we expand $(\hat{t},\, \hat{n})$ in terms of the Cartesian basis vectors $(\hat{i},\, \hat{j})$. This gives

$$\hat{t} = \cos\psi\,\hat{i} + \sin\psi\,\hat{j}, \qquad \hat{n} = -\sin\psi\,\hat{i} + \cos\psi\,\hat{j}$$

Since $(\hat{t},\, \hat{n})$ are now expressed in terms of the constant vectors $(\hat{i},\, \hat{j})$, the differentiations with respect to ψ are simple and give

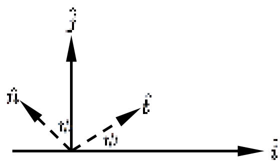
$$\frac{d\hat{t}}{d\psi} = \hat{n}, \qquad \frac{d\hat{n}}{d\psi} = -\hat{t} \qquad (1)$$

Since the velocity be in tangent so

$$\underline{v} = \frac{dS}{dt}\hat{t} \qquad (2)$$

Now, in order to obtain the polar formula for acceleration, we differentiate the velocity formula $\underline{v} = \frac{dS}{dt}\hat{t}$ with respect to time t . This leads to

$$\begin{aligned}
\underline{a} = \frac{d\underline{v}}{dt} &= \frac{d}{dt}\left(\frac{dS}{dt}\hat{t}\right) \\
&= \frac{d^2S}{dt^2}\hat{t} + \frac{dS}{dt}\frac{d\hat{t}}{dt} \\
&= \frac{d^2S}{dt^2}\hat{t} + \frac{dS}{dt}\frac{d\hat{t}}{d\psi}\frac{d\psi}{dS}\frac{dS}{dt} \\
&= \frac{d^2S}{dt^2}\hat{t} + \left(\frac{dS}{dt}\right)^2 \underbrace{\frac{d\hat{t}}{d\psi}}_{\hat{n}}\ \underbrace{\frac{d\psi}{dS}}_{1/\rho} \\
&= \frac{d^2S}{dt^2}\hat{t} + \frac{v^2}{\rho}\hat{n}
\end{aligned}$$



Where $\rho\left(= \frac{dS}{d\psi}\right)$ is the radius of curvature at the point $\mathbf{P}$ which is the tangential $a_t = \frac{d^2S}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dS}$ and normal $a_n = \frac{v^2}{\rho}$ acceleration at $\mathbf{P}$.

The formula $\underline{v} = \frac{dS}{dt}\hat{t}$ illustrates that the velocity of $\mathbf{P}$ is in the tangent at $\mathbf{P}$, while the acceleration has two components $(\mathbf{a}_t, a_n)$ and the resultant of acceleration is $a = \sqrt{a_t^2 + a_n^2} = \sqrt{\left(\frac{dv}{dt}\right)^2 + \left(\frac{v^2}{\rho}\right)^2}$

■ ILLUSTRATIVE EXAMPLES ■

Example

A point **P** describes, with a constant angular velocity about the origin and $r = ae^{\theta}$ Obtain the radial and transverse acceleration of **P**.

Solution

Since $\because \underline{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$ and given $\frac{d\theta}{dt} = \omega$ **(constant)**

then by differentiating $r = ae^{\theta}$ with respect to time we have

$$\dot{r} = ae^{\theta}\dot{\theta} = a\omega e^{\theta} = \omega r \quad \textbf{and} \quad \Rightarrow \ddot{r} = a\omega\dot{\theta}e^{\theta} = a\omega^2 e^{\theta} = \omega^2 r$$

Also $\frac{d\theta}{dt} = \omega \Rightarrow \frac{d^2\theta}{dt^2} = 0$ then the radial a_r and transverse a_θ acceleration are

$$\because a_r = \omega^2 r - r\omega^2 = 0, \qquad a_\theta = 0 + 2\omega^2 r$$

That is $\underline{a} = 2\omega^2 r\hat{\theta}$

Example

The velocities of a particle along and perpendicular to the radius vector are constants. Prove that the acceleration inversely varies as the radius r.

Solution

Since $\dot{r} = A$ and $r\dot{\theta} = B$ where A, B are constants then by differentiating with respect to time we have

$$\dot{r} = A \quad \Rightarrow \ddot{r} = 0 \qquad \text{and} \qquad r\dot{\theta} = B \quad \Rightarrow r\ddot{\theta} + \dot{r}\dot{\theta} = 0$$

then, the radial acceleration is $\because a_r = \ddot{r} - r\dot{\theta}^2$ thus $a_r = 0 - \frac{B^2}{r} = -\frac{B^2}{r}$

And for transverse acceleration a_θ

$$\because\ a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} \qquad \text{thus} \qquad a_\theta = \dot{r}\dot{\theta} = \frac{AB}{r}$$

The magnitude of acceleration is given by

$$\begin{aligned}
\left|\underline{a}\right| = \sqrt{a_r^2 + a_\theta^2} &= \sqrt{\frac{B^4}{r^2} + \frac{A^2B^2}{r^2}} \\
&= \sqrt{\frac{B^4 + A^2B^2}{r^2}} \\
&= \frac{C}{r}, \qquad\qquad C = \sqrt{B^4 + A^2B^2}
\end{aligned}$$

that is the acceleration inversely varies as the radius vector r.

Example

A particle moves along the circle $r = 2\cos\theta$ in such a way that its acceleration perpendicular to the radius vector is always zero. Find the velocity of the moving particle in terms of r.

Solution

Since $r = 2\cos\theta$ then by differentiating with respect to time we have

$$r = 2\cos\theta \qquad \Rightarrow \dot{r} = -2\sin\theta\ \dot{\theta}$$

Also we have that the acceleration towards the origin is always zero i.e.,

$$\begin{aligned}
a_\theta = 0 \qquad &\Rightarrow \frac{1}{r}\frac{d}{dt}\ r^2\dot{\theta}\ = 0 \\
&\Rightarrow d\ \ r^2\dot{\theta}\ = 0 \\
&\Rightarrow r^2\dot{\theta} = h\ \text{(const.)}
\end{aligned}$$

Therefore the velocity magnitude is

$$\begin{aligned}
\left|\underline{v}\right| = \sqrt{v_r^2 + v_\theta^2} &= \sqrt{\dot{r}^2 + (r\dot{\theta})^2} \\
&= \sqrt{(-2\sin\theta\,\dot{\theta})^2 + (r\dot{\theta})^2} \\
&= \sqrt{4\sin^2\theta + r^2}\ \dot{\theta} \\
&= \sqrt{4(1-\cos^2\theta) + r^2}\ \dot{\theta} \\
&= \sqrt{4 - 4\cos^2\theta + r^2}\ \dot{\theta} \\
&= \sqrt{4 - r^2 + r^2}\ \dot{\theta} = 2\ \underbrace{\dot{\theta}}_{h/r^2} = \frac{2h}{r^2}
\end{aligned}$$

Example

A particle describes a curve with constant velocity and its angular velocity about a given fixed point **O** varies inversely as its distance from **O**. Find the path equation.

Solution

Let the velocity of the particle be equal to v (constant). Given that the angular velocity $d\theta / dt$ of the particle about a fixed point **O** varies inversely as its distance r from **O**, we have

$$\frac{d\theta}{dt} \propto \frac{1}{r} \qquad \Rightarrow \frac{d\theta}{dt} = \frac{k}{r} \quad (k \text{ is constant})$$

Since $v = \sqrt{\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2} = \lambda$ (λ is constant)

$$\Rightarrow v^2 = \left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2 = \lambda^2$$

$$\Rightarrow v^2 = \left(\frac{dr}{dt}\right)^2 + \left(r\frac{k}{r}\right)^2 = \lambda^2$$

$$\Rightarrow \left(\frac{dr}{dt}\right)^2 = \lambda^2 - k^2$$

$$\Rightarrow \frac{dr}{dt} = \sqrt{\lambda^2 - k^2} = \mu$$

$$\Rightarrow \frac{dr}{d\theta}\frac{d\theta}{dt} = \mu \qquad \Rightarrow \frac{dr}{d\theta}\frac{k}{r} = \mu \qquad \Rightarrow \frac{dr}{r} = \frac{\mu}{k}d\theta$$

By integrating

$$\therefore \ln r = \frac{\mu}{k}\theta + \ln c \quad (\ln c \text{ is integration constant})$$

$$\therefore \ln\frac{r}{c} = \frac{\mu}{k}\theta \qquad \Rightarrow r = ce^{\mu/k\,\theta}$$

This is the equation of equiangular spiral

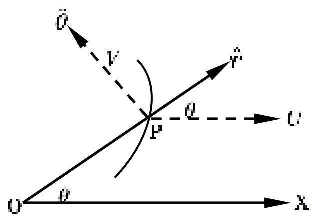
Example

Find the path equation of a point P which possesses two constant velocities U and $\mathbf{V}$, the first of which is in $\mathbf{OX}$ direction and the other is perpendicular to the radius $\mathbf{O}P$ drown from a fixed point $\mathbf{O}$.

Solution

Take the fixed point $\mathbf{O}$ as the pole and the fixed direction as the initial line $\mathbf{OX}$. Let $P(r,\theta)$ be the position of the particle at any time. Resolve the velocities in the direction of and perpendicular to the radius $\mathbf{O}P$ we have

$$\frac{dr}{dt} = U\cos\theta \qquad \text{and} \qquad r\frac{d\theta}{dt} = V - U\sin\theta$$



Dividing these two equations we have

$$\frac{dr}{rd\theta} = \frac{U\cos\theta}{V - U\sin\theta} \qquad \Rightarrow \frac{dr}{r} = \frac{U\cos\theta}{V - U\sin\theta}d\theta$$

By integrating we get

$$\ln r = -\ln(V - U\sin\theta) + \ln c$$

or $\qquad \ln\frac{c}{r} = \ln(V - U\sin\theta) \qquad$ ($\ln c$ is integration constant)

$$\therefore \frac{c}{r} = V - U\sin\theta$$

Example

A particle moves in a plane curve so that its tangential acceleration is constant, and the magnitude of tangential velocity and normal acceleration are in a constant ratio. Find the intrinsic equation of the curve.

Solution

In our problem it is given that

tangential acceleration $\dfrac{dv}{dt} = \lambda$ (a constant) and (1)

$$\frac{\textbf{tangential velocity}}{\textbf{normal acceleration}} = \frac{v}{v^2 / \rho} = \frac{\rho}{v} = \mu \text{ (a constant)} \qquad (2)$$

Since $\quad v = \dfrac{dS}{dt}, \quad \rho = \dfrac{dS}{d\psi}$

Then from formula (2)

$$\frac{\rho}{v} = \frac{dS / d\psi}{dS / dt} = \mu \quad \Rightarrow \frac{dt}{d\psi} = \mu \quad \textbf{Or} \quad \frac{d\psi}{dt} = \frac{1}{\mu}$$

From formula (1)

$$\frac{dv}{dt} = \frac{dv}{d\psi} \times \underbrace{\frac{d\psi}{dt}}_{1/\mu} = \lambda \quad \textbf{Or} \quad \frac{dv}{d\psi} = \lambda\mu \quad \Rightarrow dv = \mu\lambda d\psi$$

By integrating $\quad v = \mu\lambda\psi + c_1$

Where c_1 is a constant, again from equation (2)

$$\rho = \mu v \quad \Rightarrow \frac{dS}{d\psi} = \mu \ \ \mu\lambda\psi + c_1 \quad \textbf{Or} \quad dS = \mu \ \ \mu\lambda\psi + c_1 \ \ d\psi$$

Integrating

$$S = \frac{1}{2}\mu^2\lambda\psi^2 + \mu c_1\psi + C \quad \textbf{Or}$$

$$S = A\psi^2 + B\psi + C, \qquad A = \frac{1}{2}\lambda\mu^2, \ B = \mu c_1$$

Example

A particle moves in a catenary $s = c\tan\psi$, the direction of its acceleration at any point makes equal angles with the tangent and normal to the path at the point. If the speed at the vertex ($\psi = 0$) be u, show that the velocity and acceleration at any other point ψ are ue^{ψ} and $(\sqrt{2} / c)u^2e^{2\psi}\cos^2\psi$.

Solution

It is given that the direction of acceleration at any point makes equal angles with the tangent and normal to the path at the point. Therefore the tangential and normal accelerations will be equal at any time i.e. $v\dfrac{dv}{ds}=\dfrac{v^2}{\rho}$

$$v\frac{dv}{ds}=\frac{v^2}{\rho} \qquad \Rightarrow \frac{dv}{ds}\rho=v \quad \text{or}$$

$$\frac{dv}{ds}\frac{ds}{d\psi}=v \qquad \Rightarrow \frac{dv}{d\psi}=v \qquad \Rightarrow \frac{dv}{v}=d\psi$$

$\ln v=\psi+c$ from the initial conditions ($v=u, \quad \psi=0 \qquad \therefore c=\ln u$)

$$\therefore \ln v=\psi+\ln u \qquad \Rightarrow \ln\frac{v}{u}=\psi \qquad \Rightarrow v=ue^{\psi}$$

which gives the velocity of the particle at any point ψ .

Further it is given that the path of the particle is the catenary $s=c\tan\psi$

$$\therefore \rho=\frac{ds}{d\psi}=c\sec^2\psi$$

And since the acceleration magnitude is given by

$$=\sqrt{\left(v\frac{dv}{ds}\right)^2+\left(\frac{v^2}{\rho}\right)^2}=\sqrt{\left(\frac{v^2}{\rho}\right)^2+\left(\frac{v^2}{\rho}\right)^2}$$

$$=\sqrt{2}\left(\frac{v^2}{\rho}\right)=\sqrt{2}\,u^2e^{2\psi}\frac{1}{c\sec^2\psi}=\frac{\sqrt{2}}{c}u^2e^{2\psi}\cos^2\psi$$

Example

The relation between the velocity of a particle moving in a plane v and its tangent acceleration a_t is $a_t=\dfrac{1}{1+v}$. Find the relation between v,S and v,t if the particle start from rest at the position $S=0$.

Solution

To obtain the relation between v,t where $a_t=\dfrac{dv}{dt}$ then

$$\frac{dv}{dt}=\frac{1}{1+v} \qquad \Rightarrow (1+v)dv=dt$$

By integrating $$v + \frac{1}{2}v^2 = t + c_1$$

The integration constant c_1 can be evaluated from initial conditions, i.e. $v = 0$ at $t = 0$ and hence $c_1 = 0$ then the last formula becomes

$$v + \frac{1}{2}v^2 = t$$

Again since, $a_t = v\frac{dv}{dS}$ and therefore,

$$v\frac{dv}{dS} = \frac{1}{1+v} \qquad \Rightarrow (v + v^2)dv = dS$$

Integration again $$\frac{1}{2}v^2 + \frac{1}{3}v^3 = S + c_2$$

Where c_2 is a constant where $v = 0$ at $S = 0$ and hence $c_2 = 0$ then the relation between v, S is $\frac{1}{2}v^2 + \frac{1}{3}v^3 = S$

Example

A particle describes a curve (for which and vanish simultaneously) with uniform v, if acceleration at any point S be $\mu v^2 / (S^2 + \mu^2)$. Find the intrinsic equation of the curve.

Solution

It is given that the velocity is constant i.e.,

$$v = \frac{dS}{dt} = c \qquad \Rightarrow \frac{d^2S}{dt^2} = 0$$

And since the resultant of acceleration is

$$a = \sqrt{\left(\frac{d^2S}{dt^2}\right)^2 + \left(\frac{v^2}{\rho}\right)^2} = \frac{v^2}{\rho}$$

But it is given that

$$a = \frac{\mu v^2}{S^2 + \mu^2} \qquad \Rightarrow \frac{v^2}{\rho} = \frac{\mu v^2}{S^2 + \mu^2}$$

$$\Rightarrow \frac{v^2 d\psi}{dS} = \frac{\mu v^2}{S^2 + \mu^2}$$

$$\Rightarrow \frac{\mu dS}{S^2 + \mu^2} = d\psi$$

$$\Rightarrow \frac{1}{\mu}\frac{dS}{1 + \left(\frac{S}{\mu}\right)^2} = d\psi$$

By integrating

$$\tan^{-1}\left(\frac{S}{\mu}\right) = \psi + C \quad \text{C is a constant}$$

Given that $\psi = 0$ when $S = 0$, gives $C = 0$

Therefore, $\tan^{-1}\left(\frac{S}{\mu}\right) = \psi$ **Or** $S = \mu \tan(\psi)$

Example

A particle moves over a circle with radius 2 ft according to a constant tangent acceleration 4 ft sec^{-2}. If initially, the particle at the point A on a circumference and have zero velocity. Find the velocity of the particle after it returns to the point A and time spent to reach. Find its acceleration after return to point a.

Solution

It's given that $a_t = 4$ thus

$$\frac{dv}{dt} = 4 \quad \Rightarrow dv = 4dt \quad \Rightarrow v = 4t + c_1$$

To get the constant c_1 we apply the initial conditions i.e. $v = 0$ when $t = 0$ so $c_1 = 0$

Then the last equation turns into $v = 4t$

Again since $v = \frac{dS}{dt}$ hence

$\frac{dS}{dt} = 4t \quad \Rightarrow dS = 4tdt \quad \Rightarrow S = 2t^2 + c_2$

Again To get the constant c_2 we apply the initial conditions i.e. $S = 0$ when $t = 0$ so $c_1 = 0$ (consider A be the fixed point) then the relation between S, t is $\quad S = 2t^2$

From this equation we can obtain the tome spent to reach to the point A again – note $S = 4\pi$ - thus the time $\quad 4\pi = 2t^2 \quad \therefore t = \sqrt{2\pi}$

And its velocity is $v = 4\sqrt{2\pi}$

Moreover the acceleration has two components namely tangential component $a_t = 4\,\text{ft}\sec^{-2}$ and normal component a_n, where

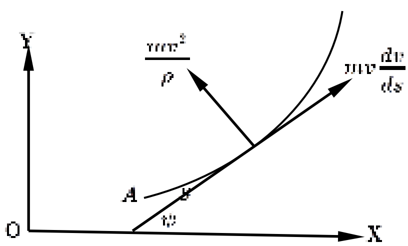
$$a_n = \frac{v^2}{\rho} = \frac{16(2\pi)}{2} = 16\pi\ \text{ft}\sec^{-2} \quad (\text{note } \rho = 2\ \text{ft})$$

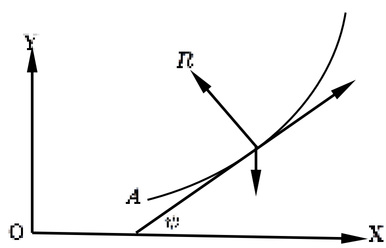
MOTION UNDER CONSTRAINT

A particle may be constrained to move along a given curve or surface, and the constraint may be one-sided, as for example when a heavy particle slides on the inside of a spherical surface and is free to break contact with the surface on the inside of the sphere but cannot get outside. There will then be a normal pressure inwards exerted by the sphere on the particle so long as contact persists, and the pressure will vanish at the point where the particle leaves the surface. On the other hand if the constraint is two-sided as when a particle moves in a fine tube, or a bead moves along a wire, then the normal reaction may vanish and change sign but the particle persists in the prescribed path.

■ Motion of a Heavy Particle on a Smooth Curve in a Vertical Plane

The motion is determined by the tangential and normal components of acceleration. The beginner may find it useful in such problems as this to make two diagrams, one showing the components of acceleration multiplied by the mass and the other showing the forces. It is then only necessary to realize that the two diagrams are equivalent representations of the same vector, so that the resolved parts in any assigned direction in the two diagrams are equal.





If m is the mass of the particle, the forces acting on it are the weight mg and the reaction R along the normal. The components of acceleration are $v\dfrac{dv}{ds}$ along the tangent and $\dfrac{v^2}{\rho}$ along the inward normal. Hence, by resolving along the tangent, we get

$$mv\frac{dv}{dS} = -mg\sin\psi = -mg\frac{dy}{ds},$$

therefore, by integration,

$$\frac{1}{2}mv^2 = c - mgy$$

or, if u is the velocity when the ordinate is y_0, we have

$$\frac{1}{2}m \;\; v^2 - u^2 \;\; = mg \;\; y_0 - y \qquad (1)$$

This is the equation of energy and might have been written down at once; for since the curve is smooth no work is done by the reaction R in any displacement, so the increase in kinetic energy is equal to the work done by the weight.

Again, resolving along the normal, we get

$$m\frac{v^2}{\rho} = R - mg\cos\psi \qquad (2)$$

Substituting for v from equation (1), we have

$$R = mg\cos\psi + m \;\; u^2 + 2g(y_0 - y) \;\; / \rho \qquad (3)$$

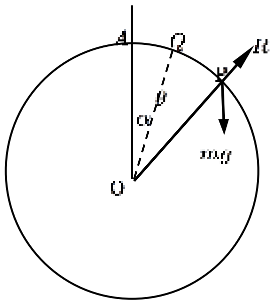
Assuming that the form of the curve is given, the values of ρ and ψ at any point can be determined, and thus R is known; and if we equate to zero the value of R we shall have an equation to determine the point, if any, at which the particle leaves the curve.

■ Motion of a Heavy Particle, placed on the outside of a Smooth Circle in a Vertical Plane and allowed to slide down

If the particle starts with zero initial velocity from position *Q* at an angular distance $\alpha = \angle \mathbf{AOQ}$ from the highest point *A*, and a is the radius of the circle and v the velocity at *P* where the angular distance from *A* is $\theta = \angle \mathbf{AOP}$, then,

$$v^2 = 2ga(\cos\alpha - \cos\theta)$$

Also by resolving along the inward normal



$$m\frac{v^2}{a} = R - mg\cos\theta$$

where R is the outward reaction of the curve.

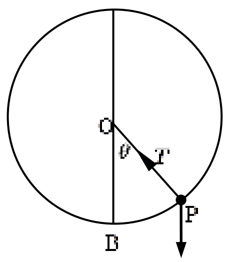
Therefore $\quad R = mg(3\cos\theta - 2\cos\alpha)$

showing that the pressure vanishes, and that the particle flies off the curve, when $\cos\theta = \frac{2}{3}\cos\alpha$.

■ Motion in a Vertical Plane of a Heavy Particle attached by a Pine String to a Fixed Point

Suppose that the particle starts with velocity u from its lowest position *B*. If v is the velocity at $\mathbf{P}$ and θ is the angle that the string makes with the vertical, the equation of energy is

$$\frac{1}{2}m\ \ v^2 - u^2\ \ = -mga\ \ 1 - \cos\theta \qquad (1)$$



and by resolving along the inward normal

$$m\frac{v^2}{a} = T - mg\cos\theta$$

where T is the tension of the string.

Therefore

$$T = m(3g\cos\theta - 2g + u^2\ /\ a) \qquad (2)$$

In order to find the height of ascent we put $v = 0$ in (1), and get

$$2ga\cos\theta = 2ga - u^2 \qquad (3)$$

and by putting $T = 0$ in equation (2), we find that the tension vanishes when

$$3ga\cos\theta = 2ga - u^2 \qquad (4)$$

Now we have the following cases:

(i) If $u^2 < 2ga$, the string does not reach the horizontal position and the tension does not vanish.

(ii) If $u^2 = 2ga$, the string just reaches the horizontal position, the tension vanishes for $\theta = \frac{\pi}{2}$, and the particle swings through a quadrant on each side of the vertical.

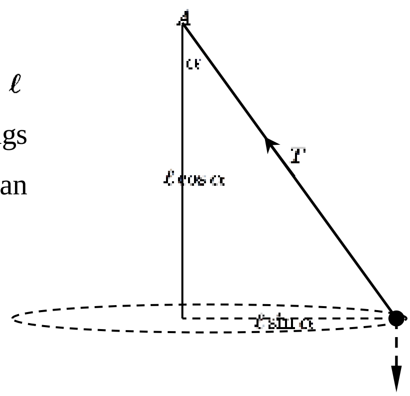
(iii) If $2ga < u^2 < 5ga$, we find that there is a value of θ, an obtuse angle, given by (4) smaller than that given by (3), so that the string becomes slack before the velocity vanishes and the particle will fall away from the circular path and move in a parabola till the string again becomes taut.

(iv) If $u^2 = 5ga$, the tension just vanishes in the highest position, but v does not vanish, so that circular motion persists.

(v) If $u^2 > 5ga$, neither v nor T vanish. This is an example of a one-sided constraint; if instead of the problem of a particle attached to a string, we consider that of a bead sliding on a wire, we find that if $u^2 = 4ga$ the bead will reach the highest point of the wire and for any greater value of u it will describe the complete circle.

■ **Circular pendulum:**

A mass hangs from a massless string of length ℓ Conditions have been set up so that the mass swings around in a horizontal circle, with the string making an angle α with the vertical (see Figure).



The mass travels in a circle, so the horizontal radial
force must be

$$F_n = m\left(\frac{v^2}{\ell \sin\alpha}\right) = m\left(\frac{\omega^2 \ell^2 \sin^2\alpha}{\ell \sin\alpha}\right) = \omega^2 \ell \sin\alpha$$

directed radially inward. The forces on the mass are the tension in the string, T, and gravity, mg as illustrated. There is no acceleration in the vertical direction, so $F = ma$ in the vertical and radial directions give, respectively,

$T\cos\alpha = mg$ and $T\sin\alpha = m\omega^2 \ell \sin\alpha$

Solving for ω gives

$$\omega = \sqrt{\frac{g}{\ell\cos\alpha}}$$

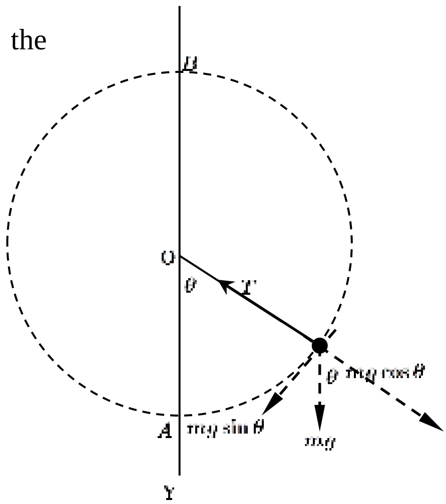
Note that if $\alpha \approx 0$, then $\omega = \sqrt{\frac{g}{\ell}}$, which equals the frequency of a plane pendulum of length ℓ. And if $\alpha \approx 90$, then $\omega \to \infty$, which makes sense.

■ Illustrative Examples ■

Example

A heavy particle of weight mg, attached to a fixed point by a light inextensible string with length ℓ, describes a circle in a vertical plane. The tension in the string has the values $n\,mg$ and $n'\,mg$, respectively, when the particle is at the highest and lowest points in its path. Show that $n' = n + 6$.

Solution



The equation of motion along and perpendicular to the radius r

$$m\ell\dot{\theta}^2 = T - mg\cos\theta \qquad \textbf{(1)}$$

$$m\ell\ddot{\theta} = -mg\sin\theta \quad \Rightarrow \ell\ddot{\theta} = -g\sin\theta \qquad \textbf{(2)}$$

Since we have $\ddot{\theta} = \dfrac{d\dot{\theta}}{dt} = \dfrac{d\dot{\theta}}{d\theta}\dfrac{d\theta}{dt} = \dot{\theta}\dfrac{d\dot{\theta}}{d\theta}$ and substituting in formula (1) and integrate

$$\Rightarrow \ell\dot{\theta}\frac{d\dot{\theta}}{d\theta} = -g\sin\theta$$

$$\Rightarrow \ell\dot{\theta}d\dot{\theta} = -g\sin\theta d\theta$$

$$\Rightarrow \ell\dot{\theta}^2 = 2g\cos\theta + c \qquad \textbf{(3)}$$

Where c the integration constant and substituting from equation (3) in equation (1) we get

$$T = m(2g\cos\theta + c) + mg\cos\theta = 3mg\cos\theta + C \qquad \textbf{(4)}$$

It's given that the tension at highest position is $n\,mg$ i.e. at point B and at the lowest position $n'\,mg$ i.e. at point A then

$$n\ mg = 3mg\cos\pi + C \quad \Rightarrow C = (n+3)mg \qquad \text{and}$$

$$n'\ mg = 3mg\cos 0 + C \quad \Rightarrow C = (n'-3)mg$$

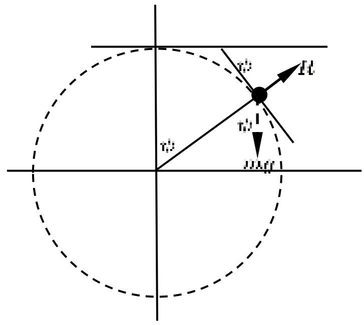
Therefore, by subtracting $n - n' + 6 = 0$ **Or** $n' = n + 6$

<u>The reader can resolve this problem by using intrinsic coordinates.</u>

Example

A particle slides outside a smooth vertical circle with radius. At time $t = 0$ the particle was at the top of the circle and has zero initial velocity. Determine the velocity at any position and the reaction of the circle the find the position that the particle will leave the circle.

Solution



The forces acting on the particle are mg the weight and reaction R as illustrated in the figure

The equation of motion along perpendicular to the radius is

$$mv\frac{dv}{dS} = mg\sin\psi \qquad \text{Or} \quad v\frac{dv}{d\psi}\frac{d\psi}{dS} = g\sin\psi \quad \left(\frac{dS}{d\psi} = b\right)$$

$$\Rightarrow vdv = bg\sin\psi\ d\psi$$

By integration

$$v^2 = C - 2bg\cos\psi$$

Since $v = 0$ at $t = 0$ thus $C = 2bg$ then $\qquad v^2 = 2bg(1 - \cos\psi)$

The equation of motion along the radius we get

$$m\frac{v^2}{b} = mg\cos\psi - R \qquad \Rightarrow R = mg\cos\psi - m\frac{v^2}{b}$$

$$\begin{aligned} \Rightarrow \quad R &= mg\cos\psi - 2mg(1 - \cos\psi) \\ &= mg(3\cos\psi - 2) \end{aligned}$$

The last equation gives the reaction of the circle at any position and the particle will leave the circle when the reaction vanishes, i.e.

$$mg(3\cos\psi - 2) = 0 \qquad \Rightarrow \cos\psi = \frac{2}{3}$$

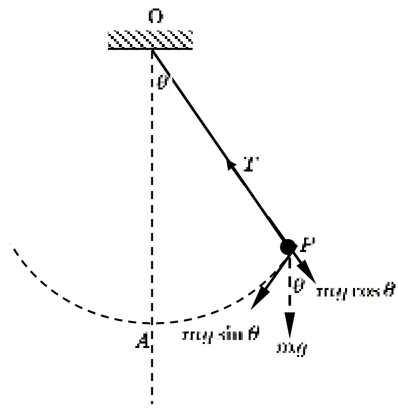
That is the particle will leave the circle after sliding a vertical distance equals

Example

Prove that the simple pendulum executes simple harmonic motion and determine the periodic time.

Solution

If a heavy particle is tied to one end of a light inextensible string the other end of which is fixed, and oscillates in a vertical circle, we have what is called a Simple Pendulum



We now obtain the time of oscillation of such a pendulum when it is allowed to oscillate through a small angle only. Let O be the fixed point, A the lowest position of the particle, and P any position such that $\angle \mathbf{AOP}=\theta$

Since in polar co-ordinate

$$(a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = L\ddot{\theta} \quad (r = L \therefore \dot{r} = 0))$$

The equations of motion in θ direction is

$$mL\ddot{\theta} = -mg\sin\theta \quad \Rightarrow \ddot{\theta} = -\frac{g}{L}\sin\theta$$

When the angle θ is small enough so the approximation $\sin\theta \cong \theta$ can be applied and the equation of motion, $\ddot{\theta} = -\frac{g}{L}\sin\theta$ becomes $\ddot{\theta} = -\frac{g}{L}\theta$

Which is similar to $\ddot{\theta} = -w^2\theta$, with $w^2 = \frac{g}{L}$ **Or** $w=\sqrt{\frac{g}{L}}$

So, a simple pendulum moves like a SHM with periodic time of motion equals

$$2\pi\sqrt{\frac{L}{g}}$$

PROBLEMS

❒ A point starts from the origin in the direction of the initial line with velocity $\frac{u}{\omega}$ and moves with angular velocity ω about the origin and with constant negative radial acceleration u. Find the equation of path

❒ A point describes the cycloid $S = 4a\sin\psi$ with uniform speed v. Find its acceleration at any point.

❒ If the tangential and normal acceleration of a particle describing a plane curve be constant throughout, prove that the radius of curvature ρ at any point is given by $\rho = (at + b)^2$, where a, b are constants

❒ The velocities of a particle along and perpendicular to the radius vector are λr and $\mu\theta$ respectively. Find the path equation and obtain the accelerations along and perpendicular to the radius vector.

❒ The velocities of a particle along and perpendicular to the radius vector are λr^2 and $\mu\theta^2$ respectively. Find the path equation and obtain the accelerations along and perpendicular to the radius vector.

KINETICS OF A PARTICLE

This chapter is concerned with the foundations of dynamics and gravitation. Kinematics is concerned purely with geometry of motion, but dynamics seeks to answer the question as to what motion will actually occur when specified forces act on a body. The rules that allow one to make this connection are Newton's laws of motion. These are laws of physics that are founded upon experimental evidence and stand or fall according to the accuracy of their predictions. In fact, Newton's formulation of mechanics has been astonishingly successful in its accuracy and breadth of application, and has survived, essentially intact, for more than three centuries. The same is true for Newton's universal law of gravitation which specifies the forces that all masses exert upon each other.

Taken together, these laws represent virtually the entire foundation of classical mechanics and provide an accurate explanation for a vast range of motions from large molecules to entire galaxies.

■ Newton's Laws

Isaac Newton's∗ three famous laws of motion were laid down in Principia, written in Latin and published in 1687. These laws set out the founding principles of mechanics and have survived, essentially unchanged, to the present day. Even when translated into English, Newton's original words are hard to understand, mainly because the terminology of the seventeenth century is now archaic. Also, the laws are now formulated as applying to particles, a concept never used by Newton. A particle is an idealized body that occupies only a single point of space and has no internal structure. True particles do not exist in nature, but it is convenient to regard realistic bodies as being made up of particles. Using modern terminology, Newton's laws may be stated as follows:

➢**First Law:** When all external influences on a particle are removed, the particle moves with constant velocity. {This velocity may be zero in which case the particle remains at rest.}

➢**Second Law:** When a force $\boldsymbol{F}$ acts on a particle of mass m, the particle moves with instantaneous acceleration $\underline{\boldsymbol{a}}$ given by the formula

$$\boldsymbol{F} = m\underline{\boldsymbol{a}}$$

where the unit of force is implied by the units of mass and acceleration.

➢**Third Law:** When two particles exert forces upon each other, these forces are (i) equal in magnitude, (ii) opposite in direction, and (iii) parallel to the straight line joining the two particles.

■ The Law of Gravitation

Physicists recognize only four distinct kinds of interaction forces that exist in nature. These are gravitational forces, electromagnetic forces and weak/strong nuclear forces. The nuclear forces are important only within the atomic nucleus and will not concern us at all. The electromagnetic forces include electrostatic attraction and repulsion, but we will encounter them mainly as 'forces of contact' between material bodies. Since such forces are intermolecular, they are ultimately electromagnetic although we will make no use of this fact! The present section however is concerned with gravitation.

It is an observed fact that any object with mass attracts any other object with mass with a force called gravitation. When gravitational interaction occurs between particles, the Third Law implies that the interaction forces must be equal in magnitude, opposite in direction and parallel to the straight line joining the particles. The magnitude of the gravitational interaction forces is given by:

The gravitational forces that two particles exert upon each other each have magnitude

$$\underline{\boldsymbol{F}} = \gamma \frac{\boldsymbol{Mm}}{\boldsymbol{R}^2} \hat{\boldsymbol{F}} \tag{1}$$

where $\boldsymbol{M}, \boldsymbol{m}$ are the particle masses, $\boldsymbol{R}$ is the distance between the particles, and γ, the constant of gravitation, is a universal constant. Since γ is not dimensionless, its numerical value depends on the units of mass, length and force.

This is the famous inverse square law of gravitation originally suggested by Robert Hooke, a scientific contemporary (and adversary) of Newton. In SI units, the constant of gravitation is given approximately by

$$\gamma = \mathbf{6.67 \times 10^{-11}\ Nm^2kg^{-2}}$$

this value being determined by observation and experiment. There is presently no theory (general relativity included) that is able to predict the value of γ. Indeed, the theory of general relativity does not exclude *repulsion* between masses!

To give some idea of the magnitudes of the forces involved, suppose we have two uniform spheres of lead, each with mass 5000 kg (five metric tons). Their common radius is about 47 cm which means that they can be placed with their centers 1 m apart. What gravitational force do they exert upon each other when they are in this position? We will show later that the gravitational force between uniform spheres of matter is exactly the same *as if* all the mass of each sphere were concentrated at its center. Given that this result is true, we can find the force that each sphere exerts on the other simply by substituting $\boldsymbol{M} = \boldsymbol{m} = \mathbf{5000}$ and $\boldsymbol{R} = \mathbf{1}$ into equation (1). This gives $\boldsymbol{F} = \mathbf{0.00167\,N}$ approximately, the weight of a few grains of salt! Such forces seem insignificant, but gravitation is the force that keeps the Moon in orbit around the Earth, and the Earth in orbit around the Sun. The reason for this disparity is that the masses involved are so much larger than those of the lead spheres in our example. For instance, the mass of the Sun is about 2×10^{30} kg.

■ Motion Through a Resisting Medium

When a body moves in a medium like air or any other fluid, it experiences a resistance to its motion. The resistance which we have been neglecting so far, generally varies with the velocity. For small velocities the resistance is approximately proportional to the velocity, for greater velocities it varies as the square of the velocity and for still greater velocities, the resistance varies as the cube or even a higher power of the velocity. The forces of resistance being non-conservative, the principle of Conservation of Energy is not applicable to such cases.

■ Bodies Falling Vertically in a Resisting Medium

Suppose a particle with mass m is allowed to fall vertically subject to a resistance proportional to some power of the velocity v, e.g. a resistance force $\mu m v$, then we have the equation of motion

$$m\frac{dv}{dt} = mg - \mu m v \qquad \text{Or} \qquad \frac{dv}{dt} = g - \mu v$$

where μm is the constant of proportionality and g, the acceleration due to gravity, is supposed to remain constant. The equation shows that the acceleration of the particle decreases as its velocity increases and that it vanishes when g / μ . Separation of variables for the previous equation we get

$$\frac{dv}{g - \mu v} = dt \qquad \Rightarrow \frac{-\mu dv}{g - \mu v} = -\mu dt$$

Integration we have

$$\ln(g - \mu v) = -\mu t + c_1$$

If the initial velocity is u therefore, the constant c_1 may be obtained as $\ln(g - \mu u) = c_1$ then

$$\ln(g - \mu v) = -\mu t + \ln(g - \mu u) \qquad \Rightarrow \ln\frac{g - \mu v}{g - \mu u} = -\mu t$$

$$\Rightarrow g - \mu v = (g - \mu u)e^{-\mu t} \qquad Or \qquad v = \frac{g}{\mu} - \frac{1}{\mu}(g - \mu u)e^{-\mu t}$$

The value $\frac{g}{\mu}$ is the greatest velocity attainable by the particle and is called the limiting or terminal velocity.

To get the height since $v = \frac{dy}{dt}$ then

$$\frac{dy}{dt} = \frac{g}{\mu} - \frac{1}{\mu}(g - \mu u)e^{-\mu t} \qquad \Rightarrow dy = \left(\frac{g}{\mu} - \frac{1}{\mu}(g - \mu u)e^{-\mu t}\right)dt$$

And integrate we get

$$y = \frac{g}{\mu}t + \frac{1}{\mu^2}(g - \mu u)e^{-\mu t} + c_2$$

Where $c_2 = -\frac{1}{\mu^2}(g - \mu u)$ sine $y = 0$ when $t = 0$ that is

$$y = \frac{g}{\mu}t + \frac{1}{\mu^2}(g - \mu u) \;\; e^{-\mu t} - 1$$

Subsequently the particle moves uniformly with this limiting velocity. The velocity for the rain drops at the surface of the earth cannot, therefore, give us any idea of the height from which they might have fallen, for after moving for some time they acquire the terminal velocity and continue to move uniformly with that velocity.

■ Illustrative Examples ■

‖ Example ▸

A particle with mass m moves horizontally through a resisting medium where its resistance proportional to v and the proportional constant is αm. If the particle starts its motion from the origin point with initial velocity u. Find the distance after time t.

‖ Solution ▸

The equation of motion of the particle is (horizontally)

$$m\frac{dv}{dt} = -\alpha m v \quad \Rightarrow \frac{dv}{v} = -\alpha\, dt$$

By integrating we have $\ln(v) = c_1 - \alpha t$ **(1)**

The constant c_1 can be determined from the initial conditions, $v = u$ at $t = 0$, therefore $c_1 = \ln u$ and equation (1) becomes

$$\ln(v) = \ln u - \alpha t \qquad \textbf{Or} \qquad v = ue^{-\alpha t} \qquad \textbf{(2)}$$

Equation (2) gives the velocity of the particle at any instance, and the position of the particle x can be obtained as follows

$$\frac{dx}{dt} = ue^{-\alpha t} \quad \Rightarrow \quad dx = ue^{-\alpha t} dt$$

v

$$\Rightarrow \int dx = \int ue^{-\alpha t} dt + c_2 \qquad \textbf{Or}$$

$$x = -\frac{u}{\alpha} e^{-\alpha t} + c_2 \qquad \textbf{(3)}$$

Where c_2 is integration constant that can be calculated from the initial conditions, $x = 0$ at $t = 0$, therefore $c_2 = \frac{u}{\alpha}$ and equation (3) turns into

$$x = \frac{u}{\alpha}\ 1 - e^{-\alpha t}$$

Example

A moving point with mass equals unity subject to a resistance $\lambda v + \mu v^2$ If the resisting force is the only force acting on the point. Find the distance where u is the initial velocity of the point.

Solution

Equation of motion is ($m = 1$) – Note resisting force is the only acting force-

$$v\frac{dv}{dx} = -(\lambda v + \mu v^2) \qquad \Rightarrow \frac{\mu dv}{\lambda + \mu v} = -\mu dx$$

By integration we get

$$\ln(\lambda + \mu v) = c_1 - \mu x \tag{1}$$

Where c_1 represents integration constant and can be obtained from the initial conditions, $v = u$ at $x = 0$, therefore $c_1 = \ln(\lambda + \mu u)$ and equation (1) turns into

$$\ln(\lambda + \mu v) = \ln(\lambda + \mu u) - \mu x \qquad \text{Or} \qquad \ln\left(\frac{\lambda + \mu u}{\lambda + \mu v}\right) = \mu x \tag{2}$$

Again from the last equation we can obtain the position of the point as the velocity vanishes

$$x\Big|_{v=0} = \frac{1}{\mu}\ln\left(\frac{\lambda + \mu u}{\lambda}\right) = \frac{1}{\mu}\ln\left(1 + \frac{\mu}{\lambda}u\right)$$

Example

Two equal particles with mass m projected downwards from the same point and at the same instance with initial velocities u_1, u_2 subject to a resistance $\mu m v$ If u_1', u_2' are the velocities of the particles after time T. Prove that $u_1' - u_2' = (u_1 - u_2)e^{-\mu T}$.

Solution

With respect the first particle we suppose that its velocity at any time is v therefore, the equation of motion is

$$m\frac{dv}{dt} = mg - \mu mv \quad \Rightarrow \frac{dv}{g-\mu v} = dt \quad \textbf{Or} \quad \frac{-\mu dv}{g-\mu v} = -\mu dt$$

By integration we have

$$\ln(g-\mu v) = c - \mu t$$

look c indicates the integration constant which can be calculated from the initial conditions, $v = u_1$ when $t = 0$, therefore $c = \ln(g-\mu u_1)$ and the last equation become

$$\ln(g-\mu v) = \ln(g-\mu u_1) - \mu t \quad \textbf{Or} \quad g-\mu v = (g-\mu u_1)e^{-\mu t}$$

Now after time T, the velocity become u_1' that is

$$g-\mu u_1' = (g-\mu u_1)e^{-\mu T} \quad \textbf{(1)}$$

Now with respect the second particle we suppose that its velocity at any time is v' therefore, the equation of motion is

$$m\frac{dv'}{dt} = mg - \mu mv' \quad \Rightarrow \frac{dv'}{g-\mu v} = dt \quad \textbf{Or} \quad \frac{-\mu dv'}{g-\mu v'} = -\mu dt$$

By integration we have

$$\ln(g-\mu v') = c' - \mu t$$

where c refers to the integration constant which can be obtained from the initial conditions, $v' = u_2$ when $t = 0$, therefore $c' = \ln(g-\mu u_2)$ and the previous equation converted to

$$\ln(g-\mu v') = \ln(g-\mu u_2) - \mu t \quad \textbf{Or} \quad g-\mu v' = (g-\mu u_2)e^{-\mu t}$$

Again, Now after time T, the velocity become u_2' that is

$$g-\mu u_2' = (g-\mu u_2)e^{-\mu T} \quad \textbf{(2)}$$

By subtracting Equations (1) and (2) we obtain

$$\mu \;\; u_1' - u_2' \;\; = \mu(u_1 - u_2)e^{-\mu T} \qquad \textbf{Or} \qquad u_1' - u_2' = (u_1 - u_2)e^{-\mu T}$$

Example

A point with mass m is projected vertically upwards with initial velocity $\sqrt{g\mu^{-1}}$ and the resistance of air produces retardation per unit mass μv^2 where v is the velocity and μ is constant. Find the highest position and the time spent to reach is $\frac{\pi}{4\sqrt{g\mu}}$.

Solution

The equation of motion – let the projection point be the origin-then

$$mv\frac{dv}{dy} = -mg - \mu m v^2 \quad \Rightarrow \frac{vdv}{g + \mu v^2} = -dy \quad \textbf{Or} \quad \frac{2\mu v dv}{g + \mu v^2} = -2\mu dy$$

By integration we get

$$\ln(g + \mu v^2) = c_1 - 2\mu y \qquad \textbf{(1)}$$

Note c_1 indicates the integration constant which can be obtained from the initial conditions, $v = \sqrt{g\mu^{-1}}$ when $y = 0$, therefore $c_1 = \ln 2g$ and equation (1) be

$$\ln(g + \mu v) = \ln 2g - 2\mu y \qquad \textbf{Or} \quad y = \frac{1}{2\mu}\ln\frac{2g}{g + \mu v} \qquad \textbf{(2)}$$

Equation (2) gives the position of the point at any instance t and at highest position the velocity is zero $v = 0$ and then

$$y = \frac{1}{2\mu}\ln\frac{2g}{g} \qquad \Rightarrow Y = \frac{1}{2\mu}\ln 2$$

And this is the highest position and to evaluate the spent time to reach since

$$m\frac{dv}{dt} = -mg - \mu m v^2 \quad \Rightarrow \frac{dv}{g+\mu v^2} = -dt \quad \text{Or} \quad \frac{\sqrt{\frac{\mu}{g}}dv}{1+\left(\sqrt{\frac{\mu}{g}}\,v\right)^2} = -\sqrt{g\mu}\,dt$$

By integration we obtain

$$\tan^{-1}\left(\sqrt{\frac{\mu}{g}}\,v\right) = c_2 - \sqrt{g\mu}\,t \qquad \textbf{(3)}$$

Note c_2 is the integration constant which its value can be evaluated by the initial conditions, $v = \sqrt{g\mu^{-1}}$ when $t = 0$, therefore $c_2 = \frac{\pi}{4}$ and equation (3) turn into

$$\tan^{-1}\left(\sqrt{\frac{\mu}{g}}\,v\right) = \frac{\pi}{4} - \sqrt{g\mu}\,t \qquad \text{Or} \qquad v = \sqrt{\frac{g}{\mu}}\tan\left(\frac{\pi}{4} - \sqrt{g\mu}\,t\right)$$

This equation gives the velocity at any time t, and when $v = 0$ then t

$$\Rightarrow 0 = \sqrt{\frac{g}{\mu}}\tan\left(\frac{\pi}{4} - \sqrt{g\mu}\,t\right)$$

$$\Rightarrow \quad \left(\frac{\pi}{4} - \sqrt{g\mu}\,t\right) = 0 \qquad \text{Or} \quad t = \frac{\pi}{4\sqrt{g\mu}}$$

Example

A point with mass m is projected vertically upwards where the resistance of air produces a retardation $m\mu v$ where v is the velocity and μ is constant. If the velocity vanish at time T with a height ℓ from the point of projection Show that the initial velocity of the point is $\mu\ell + gT$.

Solution

The equation of motion –the point of projection is chosen to be the origin point-

$$m\frac{dv}{dt} = -mg - \mu mv \quad \Rightarrow \frac{\mu dv}{g + \mu v} = -\mu dt$$

By integration we get

$$\ln(g + \gamma v) = c_1 - \mu t \qquad (1)$$

here c_1 gives the integration constant which can be obtained from the initial conditions, $v = u$ when $t = 0$, -we suppose that the initial velocity is u which we need to obtain- therefore $c_1 = \ln(g + \mu u)$ and equation (1) takes the following formula

$$\ln(g + \mu v) = \ln(g + \mu u) - \mu t \quad \text{Or} \quad g + \mu v = \ g + \mu u \ \ e^{-\mu t}$$

$$\text{at } \ t = T, \ v = 0 \quad \Rightarrow \ g = \ g + \mu u \ \ e^{-\mu T} \qquad (2)$$

In order to determine the height of the point we have

$$\because \ g + \mu v = (g + \mu u)e^{-\mu t} \qquad \text{Or} \qquad v = \frac{1}{\mu} \ (g + \mu u)e^{-\mu t} \ - g$$

But $v = \dfrac{dy}{dt}$ then

$$\frac{dy}{dt} = \frac{1}{\mu} \ (g + \mu u)e^{-\mu t} \ - g \qquad \Rightarrow dy = \frac{1}{\mu} \ (g + \mu u)e^{-\mu t} \ - g \ \ dt$$

By integration we get

$$y = -\frac{1}{\mu}\left(\frac{(g + \mu u)}{\mu}e^{-\mu t} \ + gt\right) + c_2 \qquad (3)$$

here c_2 gives the integration constant which can be obtained from the initial conditions, $y = 0$ when $t = 0$, therefore $c_2 = \dfrac{(g + \mu u)}{\mu^2}$ and equation (2) become

$$y = \frac{g + \mu u}{\mu^2} - \frac{1}{\mu}\left(\frac{(g + \mu u)}{\mu}e^{-\mu t} \ + gt\right)$$

Now let $y = \ell$ when $t = T$

$$\ell = \frac{g+\mu u}{\mu^2} - \frac{(g+\mu u)}{\mu^2}e^{-\mu T} - \frac{gT}{\mu}$$

$$\Rightarrow \frac{(g+\mu u)}{\mu^2}e^{-\mu T} = \frac{g+\mu u}{\mu^2} - \frac{gT}{\mu} - \ell \quad \textbf{Or} \quad \frac{g}{\mu^2} = \frac{g+\mu u}{\mu^2} - \frac{gT}{\mu} - \ell$$

We use equation (2)

$$\frac{g}{\mu^2} = \frac{g+\mu u}{\mu^2} - \frac{gT}{\mu} - \ell \qquad \Rightarrow u = gT + \mu\ell$$

Example

A point with mass m is projected vertically upwards with initial velocity u and the resistance of air produces a retardation $m\gamma v^2$ where v is the velocity and γ is constant. Show that the velocity with which the point will return to the point of projection is $\dfrac{uu'}{\sqrt{u^2+u'^2}}$ where $u' = \sqrt{g\gamma^{-1}}$.

Solution

To determine the velocity with which the point will return to the point of projection, we will consider the motion of the point upwards until it stop then it return.

The equation of motion of the point – consider $\mathbf{Y}$ axis to be vertically and the point of projection is chosen to be the origin point-

$$mv\frac{dv}{dy} = -mg - \gamma m v^2 \quad \Rightarrow \frac{2\gamma v dv}{g+\gamma v^2} = -2\gamma dy$$

By integration we get

$$\ln(g+\gamma v^2) = c_1 - 2\gamma y \qquad \textbf{(1)}$$

Where c_1 points out integration constant which can be obtained from the initial conditions, $v = u$ at $y = 0$, therefore $c_1 = \ln(g+\gamma u^2)$ and equation (1) takes the following formula

$$\ln(g+\gamma v^2) = \ln(g+\gamma u^2) - 2\gamma y \quad Or \quad y = \frac{1}{2\gamma}\ln\left(\frac{g+\gamma u^2}{g+\gamma v^2}\right)$$

The point will stop as $v = 0$, therefore

$$y\big|_{v=0} = Y = \frac{1}{2\gamma}\ln\left(\frac{g+\gamma u^2}{g}\right) = \frac{1}{2\gamma}\ln\left(1+\frac{u^2}{u'^2}\right), \quad (u'^2 = \frac{g}{\gamma})$$

Now by taking the motion where the point moves downwards, let the highest position represents the new origin point and the $\mathbf{Y}$ axis is chosen to be vertically downward. Moreover, the initial condition will be $v = 0$ when $y = 0$ where v is the velocity. The equation of motion

$$mv\frac{dv}{dy} = mg - \gamma mv^2 \quad \Rightarrow \frac{2\gamma v dv}{g - \gamma v^2} = -2\gamma dy$$

By integration we get

$$\ln(g - \gamma v^2) = c_2 - 2\gamma y \tag{2}$$

Constant of integration c_2 can be obtained from the initial conditions, $v = 0$ at $y = 0$, therefore $c_2 = \ln g$ and equation (2) becomes

$$\ln(g - \gamma v^2) = \ln g - 2\gamma y \quad \text{Or} \quad y = \frac{1}{2\gamma}\ln\left(\frac{g}{g-\gamma v^2}\right)$$

And the velocity of the point with which the point will return to the point of projection is that is at $y = Y = \frac{1}{2\gamma}\ln\left(1+\frac{u^2}{u'^2}\right)$ hence

$$\frac{1}{2\gamma}\ln\left(1+\frac{u^2}{u'^2}\right) = \frac{1}{2\gamma}\ln\left(\frac{g}{g-\gamma v^2}\right) \quad \text{Or} \quad 1+\frac{u^2}{u'^2} = \frac{g}{g-\gamma v^2}$$

$$\therefore \frac{u'^2 + u^2}{u'^2} = \frac{g}{g-\gamma v^2} \quad \Rightarrow g - \gamma v^2 = \frac{gu'^2}{u'^2+u^2} \quad \Rightarrow \gamma v^2 = g - \frac{gu'^2}{u'^2+u^2}$$

$$v^2 = u'^2 - \frac{u'^4}{u'^2 + u^2}$$

$$= \frac{u'^2(u'^2 + u^2)}{u'^2 + u^2} - \frac{u'^4}{u'^2 + u^2}$$

$$= \frac{u'^2 u^2}{u'^2 + u^2}$$

$$\therefore v = \frac{uu'}{\sqrt{u^2 + u'^2}} \qquad \left(u'^2 = \frac{g}{\gamma}\right)$$

PROBLEMS

❒ A particle is projected with velocity V along a smooth horizontal plane in a medium whose resistance per unit mass is γv, γ is a constant. Obtain the velocity v and the distance after a time t .

❒ A particle is projected vertically upwards with velocity u and the resistance of the air produces a retardation kv where v is the velocity. Determine the velocity with which the particle will return to the point of projection.

❒ A particle $\mathbf{P}$ moving along a horizontal straight line has retardation μv, where v is the velocity at time t. When $t = 0$, the particle is at $\mathbf{O}$ and has velocity u. Show that $u - v$ is proportional to $\mathbf{OP}$.

❒ A particle subject to gravity describes a curved path in a resisting medium which causes retardation hv. Show that the resultant acceleration has a constant direction, and equals $a_0 e^{-ht}$ where a_0 is the acceleration when $t = 0$.

PROJECTILE MOTION

Let us consider that u, v denote the resolved parts of the velocity of the particle parallel to the axes at time t and $u+\delta u$, $v+\delta v$ refer to the resolved parts at time $t+\delta t$ then the resolved parts of the acceleration are given as

$$a_x = \lim_{\delta t \to 0} \frac{\delta u}{\delta t} = \frac{du}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} = \ddot{x}$$

$$a_y = \lim_{\delta t \to 0} \frac{\delta v}{\delta t} = \frac{du}{dt} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2} = \ddot{y}$$

The consideration of component velocities and accelerations is of great importance when we have to deal with cases of motion where the path is not a straight line.

■ Equations of Motion of a Particle Moving in a Plane

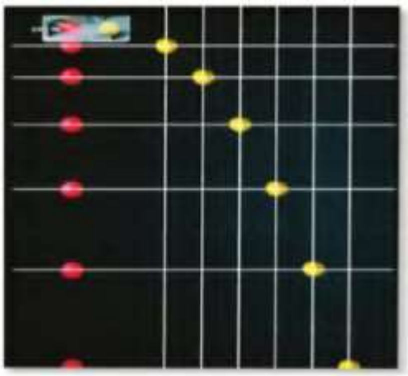
The position of a point in a straight line being determined by one co-ordinate, only one equation of motion is sufficient to determine the motion completely. In the case of a particle moving in a plane, two equations of motion are required in order to obtain the two co-ordinates which define the position of a point in a plane. The two equations of motion are obtained by resolving the forces in any two convenient directions at right angles to one another. If the two directions are taken parallel to the co-ordinate axes the equations of motion, as deduced from the second law of motion, will be of the form

$$m\frac{d^2x}{dt^2} = F_x \quad \text{and} \quad m\frac{d^2y}{dt^2} = F_y$$

where F_x, F_y are the sums of the resolved parts of the forces parallel to the axes of x and y:

■ Projectiles

As an example of motion in two dimension is the projectile motion. Recall that a particle has a mass but negligible size and shape. Therefore, we must limit application to those objects that have dimensions that are of no consequence in the analysis of the motion. In most problems, we will be focused in bodies of finite size, such as rockets, projectiles, or vehicles. Each of these objects can be considered as a particle, as long as the motion is characterized by the motion of its mass center and any rotation of the body is neglected. The free-flight motion of a projectile is often studied in terms of its rectangular components. The acceleration is of approximately 9.81 ms^{-2} or 32.2 ft s^{-2}.

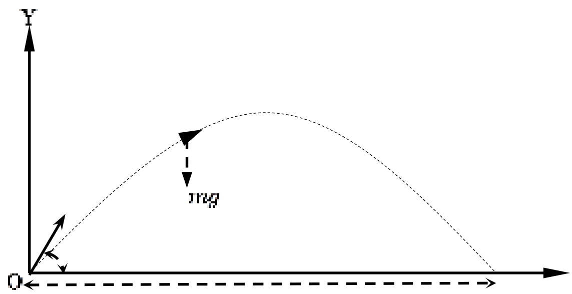


Each picture in this sequence is taken after the same time interval. The red ball falls from rest, whereas the yellow ball is given a horizontal velocity when released. Both balls accelerate downward at the same rate, and so they remain at the same elevation at any instant. This acceleration causes the difference in elevation between the balls to increase between successive photos. Also, note the horizontal distance between successive photos of the yellow ball is constant since the velocity in the horizontal direction remains constant.

We will discuss the motion of a particle projected in the field of gravity. We now consider the motion of a ***projectile***, that is, the motion of a body which is small enough to be regarded as a particle and which is projected in a direction oblique to the direction of gravity. A body that moves freely under uniform gravity, and possibly air resistance, is called a projectile. Projectile motion is very common. In ball games, the ball is a projectile, and controlling its trajectory is a large part of the skill of the game. On a larger scale, artillery shells are projectiles, but guided missiles, which have rocket propulsion, are not.

Note: Near the Earth's surface, we assume that the downward acceleration due to gravity is constant and the effect of air resistance is negligible.
We shall suppose the body to be projected in *vacuum* near the surface of the earth or, in other words, we shall suppose the resistance due to air and the slight variation in the force of gravity to be negligible. A particle of mass m is projected into the air with velocity u, in a direction making an angle α with the horizontal, to find its motion and the path described.

Let O, the point of projection, be taken as the origin and let the horizontal and the vertical lines through be taken as the axes of $\mathbf{X}$ and $\mathbf{Y}$. Again, let P be the position of the moving point, after time t. During the motion of the projectile, the only force acting on it is its weight acting downwards. The equations of motion, therefore, are



$$m\frac{d^2x}{dt^2} = 0 \qquad \text{and} \qquad m\frac{d^2y}{dt^2} = -mg$$

Or in other formula

$$\frac{d^2x}{dt^2} = 0 \qquad \text{and} \qquad \frac{d^2y}{dt^2} = -g$$

Integrating these equations, we get

$$\frac{dx}{dt} = C_1 \qquad \text{and} \qquad \frac{d^2y}{dt^2} = C_2 - gt \tag{1}$$

where C_1, C_2 are integration constants

Initially at O when $t = 0$, $\frac{dx}{dt} = u\cos\alpha$ and $\frac{dy}{dt} = u\sin\alpha$ then Equation (1) becomes

$$\frac{dx}{dt} = u\cos\alpha \qquad \text{and} \qquad \frac{dy}{dt} = u\sin\alpha - gt \qquad (2)$$

Integrating these equations again and applying initial conditions, *viz.*, when $t = 0,\ x = y = 0$, we obtain

$$x = u\cos\alpha\ t \qquad \text{and} \qquad y = u\sin\alpha\ t - \frac{1}{2}gt^2 \qquad (3)$$

Equation (2) gives the components of the velocity and (3) the displacements of the particle in the horizontal and vertical directions at any time t. These equations could also be written down at once by regarding the particle to be projected with a constant velocity $u\cos\alpha$ in the horizontal direction and with an initial velocity $u\sin\alpha$ under a retardation g in the vertical direction. Eliminating the time t the two parts of Equation (3) we have,

$$y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{u^2\cos^2\alpha} \qquad (4)$$

We now deduce the following facts from the five equations just obtained:

■ The Path Equation of Projectile

Equation (4) is of the second degree and the second degree term x^2 is a perfect square. It follow, therefore, that the path of the particle is a parabola. Equation (4) can be re-written in the form

$$y - \frac{u^2\sin^2\alpha}{2g} = \frac{-g}{2u^2\cos^2\alpha}\left(x - \frac{u^2\sin 2\alpha}{2g}\right)^2$$

It shows that the latus-rectum of the parabolic path $= 2u^2\cos^2\alpha\ /\ g$.

In the particular case when the particle is projected horizontally, $\alpha = 0$, and the Equation (4) of the path reduces to

$$y = -\frac{g}{2u^2}x^2$$

which is obviously a parabola the length of whose latus-rectum is $2u^2 / g$. The path of a projectile is called its ***trajectory***.

■ The Time of Flight

Let T, represents the time which the particle takes in reaching the horizontal plane through the point of projection.

Putting $y = 0$, in the second part of Equation (3) we get either $t = 0$ (at **O**)

And $t = \frac{2u\sin\alpha}{g} \quad \Rightarrow T = \frac{2u\sin\alpha}{g}$

■ Greatest Height

This is also obtained either by finding by differentiation, the, maximum value of y from the second part of Equation (3) or by the fact that at the greatest height the vertical component of the velocity must vanish, i.e. from the second part of Equation (2)

$$\frac{dy}{dt} = u\sin\alpha - gt = 0 \qquad \Rightarrow t = \frac{u\sin\alpha}{g}$$

Substituting this in Equation (3) and simplifying we get

$$Y = \frac{u^2\sin^2\alpha}{2g}$$

■ Horizontal Range

The range $R = \text{OB}$, on the horizontal plane through the point of projection the horizontal distance described by the particle in the time of flight T.

$$R = u\cos\alpha.T = u\cos\alpha\frac{2u\sin\alpha}{g} = \frac{u^2\sin 2\alpha}{g}$$

R can also be obtained by putting $y = 0$ in Equation (4).

Since, $R = \frac{u^2 \sin 2\alpha}{g}$ so R can be obtained by two values of projected angles

because $\sin 2\alpha = \sin(\pi - 2\alpha) = \sin 2\left(\frac{\pi}{2} - \alpha\right) \quad (\alpha, \frac{\pi}{2} - \alpha)$

■ Maximum Horizontal Range

The range R is maximum when $\sin 2\alpha = 1$, i.e., when $\alpha = \frac{\pi}{4}$ Or $\alpha = 45^O$

therefore, the maximum range $R_{\max} = \frac{u^2}{g}$.

For a given velocity of projection, the horizontal range is the greatest when the angle of projection is 45°.

■ Range on an Inclined Plane

Let a particle be projected from a point O on a plane of inclination β, in the vertical plane through OP, the line of greatest slope of the inclined plane.

Let the velocity of projection be u at an elevation α to the horizontal. The equation to the path of the particle is

$$y = x \tan \alpha - \frac{1}{2} \frac{gx^2}{u^2 \cos^2 \alpha} \qquad (10)$$

If the particle strikes the inclined Q plane at the point P, the distance, OP is called the range on the inclined plane. If OP$= R$ then the co-ordinates of P $(R\cos\beta, R\sin\beta)$ must satisfy Equation (10).

$$R \sin \beta = R \cos \beta \tan \alpha - \frac{1}{2} \frac{gr^2 \cos^2 \beta}{u^2 \cos^2 \alpha}$$

Then the range r

$$R = \frac{2u^2}{g} \cdot \frac{(\cos \beta \tan \alpha - \sin \beta) \cos^2 \alpha}{\cos^2 \beta} = \frac{2u^2}{g} \cdot \frac{\sin(\alpha - \beta) \cos \alpha}{\cos^2 \beta}$$

$$= \frac{u^2}{g \cos^2 \beta} \ \sin(2\alpha - \beta) - \sin \beta$$

The range down the plane may be obtained by putting $-\beta$ for β in this case the slope of the inclined plane is downwards.

■ Maximum Range on an Inclined Plane

u and β being known, the range varies with α, and it will be maximum when $\sin(2\alpha - \beta)$ is maximum. When $2\alpha - \beta = \frac{\pi}{2}$ **Or** $\alpha - \beta = \frac{\pi}{2} - \alpha$

Hence for maximum range, the direction of projection must bisect the angle between the vertical and the inclined plane. If OT be the direction of projection, then OT is tangent to the path at O, and the vertical through is perpendicular to the directrix. OT being equally inclined to OP and the vertical, the focus to the path must, therefore, lie on the line OP of the inclined plane, i.e., in the case of maximum range the focus lies in the range. The value of the maximum range is

$$R_{\max} = \frac{u^2(1 - \sin\beta)}{g\cos^2\beta}$$

$$= \frac{u^2(1 - \sin\beta)}{g(1 - \sin^2\beta)}$$

$$= \frac{u^2 \cancel{(1 - \sin\beta)}}{g(1 + \sin\beta)\cancel{(1 - \sin\beta)}}$$

$$= \frac{u^2}{g(1 + \sin\beta)}$$

■ Illustrative Examples ■

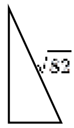
Example

If the maximum height for a projectile is $900\ \text{ft}$ and the horizontal range is $400\ \text{ft}$. Find the velocity and its direction.

Solution

Since the maximum height and horizontal range are given by formulas

$$Y = \frac{u^2 \sin^2 \alpha}{2g}, \qquad R = \frac{u^2 \sin 2\alpha}{g}$$



Then using given values we get

$$900 = \frac{u^2 \sin^2 \alpha}{2g}, \qquad 400 = \frac{2u^2 \sin\alpha \cos\alpha}{g}$$

Then by dividing these two equations

$$\frac{9}{4} = \frac{u^2 \sin^2 \alpha}{2g} \Big/ \frac{2u^2 \sin\alpha\cos\alpha}{g} \Rightarrow \frac{9}{4} = \frac{\tan\alpha}{4} \quad \therefore \alpha = \tan^{-1}\ 9$$

which gives the angle of projection and the magnitude of the velocity of projection by using first equation

$$900 = \frac{u^2}{2g} \times \frac{81}{82} \Rightarrow u^2 = \frac{1800 \times 82 \times 32.2}{81} \quad \text{Or} \quad u \cong 242.23 \quad (g = 32.2\ \text{ft}\sec^{-2})$$

Example

If the ratio between the magnitude of the velocity at maximum height and a height equals half of maximum height is $\sqrt{\frac{6}{7}}$. Show that the angle of projection is 30^0.

Example

As it is obtained that $y = (u\sin\alpha)\ t - \frac{1}{2}gt^2$

Let the point A be the maximum height and hence $Y_A = \frac{u^2\sin^2\alpha}{2g}$

And **B** be the point where its height equals half of maximum height i.e.,

$$Y_B = \frac{1}{2}Y_A = \frac{u^2\sin^2\alpha}{4g}$$

The time spent from the projection of the particle reach point **B** is given by

$$\frac{u^2\sin^2\alpha}{4g} = (u\sin\alpha)\ t - \frac{1}{2}gt^2$$

Rewrite this equation again as (multiply by $4g$)

$$2(gt)^2 - 4(gt)u\sin\alpha + u^2\sin^2\alpha = 0 \qquad \Rightarrow gt = \left(1 - \frac{1}{\sqrt{2}}\right)u\sin\alpha$$

The components of velocity at point B are

$$\dot{x}_B = u\cos\alpha, \quad \dot{y}_B = u\sin\alpha - gt = u\sin\alpha - \left(1 - \frac{1}{\sqrt{2}}\right)u\sin\alpha = \frac{1}{\sqrt{2}}u\sin\alpha$$

The resultant of the velocity at point B

$$v_B = \sqrt{\dot{x}_B^2 + \dot{y}_B^2} = \sqrt{(u\cos\alpha)^2 + \left(\frac{1}{\sqrt{2}}u\sin\alpha\right)^2} = \frac{u}{\sqrt{2}}\sqrt{1 + \cos^2\alpha}$$

Since at the maximum height $\dot{x}_A = u\cos\alpha, \quad \dot{y}_A = 0$ then

$$v_A = \sqrt{\dot{x}_A^2 + \dot{y}_A^2} = u\cos\alpha$$

But as given $\frac{v_A}{v_B} = \sqrt{\frac{6}{7}}$ therefore,

$$\Rightarrow \frac{\sqrt{2}u\cos\alpha}{u\sqrt{1 + \cos^2\alpha}} = \sqrt{\frac{6}{7}}$$
$$\Rightarrow \frac{\cos\alpha}{\sqrt{1 + \cos^2\alpha}} = \sqrt{\frac{3}{7}}$$
$$\Rightarrow \frac{\cos^2\alpha}{1 + \cos^2\alpha} = \frac{3}{7}$$

$7\cos^2\alpha = 3 + 3\cos^2\alpha \qquad \Rightarrow 4\cos^2\alpha = 3$

$$\Rightarrow \cos\alpha = \frac{\sqrt{3}}{2} \quad \text{Or} \quad \alpha = 30^0$$

Example

A particle is projected with a velocity of 24 ft sec^{-1} at an angle of elevation 60. Find (a) the equation to its path, (b) the greatest height attained, (c) the time for the range, (d) the length of the range.,

Solution

Since $u = 24$ and $\alpha = 60^o$, $\quad g \simeq 32.2$ ftsec^{-2}

(a) the equation to the path is

$$y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{u^2\cos^2\alpha}, \qquad \text{therefore} \qquad y = \sqrt{3}\,x - \frac{1}{9}x^2$$

(b) The maximum height $= \dfrac{u^2\sin^2\alpha}{2g} = \dfrac{24\times 24}{2\times 32.2}\times\dfrac{3}{4} \simeq 6.71$ ft

(c) The time for the range $= \dfrac{2u\sin\alpha}{g} = \dfrac{2\times 24}{32.2}\times\dfrac{\sqrt{3}}{2} \simeq 1.29$ sec

(d) the length of the ran $= u\cos\alpha\; T = 24\times\dfrac{1}{2}\times\dfrac{3\sqrt{3}}{4} \simeq 15.49$ ft

Example

Find the maximum horizontal range of cricket ball projected with a velocity of 48 ft. per sec. If the ball is to have a range of $36\sqrt{3}$ ft., find the least angle of projection and the least time taken (let $g \simeq 32$ ftsec^{-2}).

Solution

We have $u = 48$ and $\alpha = 45^o$, $\quad g \simeq 32$ ftsec^{-2}

$$R_{\max} = \frac{u^2}{g} \qquad \Rightarrow R_{\max} = \frac{48\times 48}{32} \simeq 72 \text{ ft}$$

$$\text{If } R = \frac{u^2\sin 2\alpha}{g} = 36\sqrt{3} \qquad \Rightarrow \sin 2\alpha = \frac{36\sqrt{3}\times 32}{48\times 48} = \frac{\sqrt{3}}{2}$$

Then $2\alpha = 60^o$ or 120^o that is $\alpha = 30^o$ or 60^o

Thus, the least angle of projection $\alpha = 30^o$

and the least time taken $= \dfrac{2u\sin\alpha}{g} = \dfrac{2\times 48}{32}\times\dfrac{1}{2} \simeq 1.5$ sec

Example

A ball is projected from a point on the ground distant a from the foot of a vertical wall of height b, the angle of projection being α to the horizontal. If the ball just clears the wall prove that the greatest height reached is

$$\frac{a^2\tan^2\alpha}{4(a\tan\alpha - b)}$$

Solution

Let u be the velocity of projection, then since the ball passes through the top of the wall, a point (a,b), we have

$$b = a\tan\alpha - \frac{ga^2}{2u^2\cos^2\alpha} \qquad \text{Or} \qquad a\tan\alpha - b = \frac{ga^2}{2u^2\cos^2\alpha}$$

$$\therefore\ u^2 = \frac{ga^2}{2(a\tan\alpha - b)\cos^2\alpha}$$

Now the greatest height Y reached by the ball

$$\begin{aligned} Y &= \frac{u^2\sin^2\alpha}{2g} \\ &= \frac{\sin^2\alpha}{2g}\,\frac{ga^2}{2(a\tan\alpha - b)\cos^2\alpha} \\ &= \frac{a^2\tan^2\alpha}{4(a\tan\alpha - b)} \end{aligned}$$

Example

If T be the time taken to reach the other common point A of its path and T' the time to reach the horizontal plane through the point of projection. Find the height of the point A.

Solution

Since $x = u\cos\alpha\ t$ and the time of flight is $T + T'$ also $R = \dfrac{u^2 \sin 2\alpha}{g}$

Hence $u\cos\alpha\ (T + T') = \dfrac{2u^2 \cos\alpha \sin\alpha}{g} \quad \Rightarrow u\sin\alpha = \dfrac{1}{2}g(T + T')$

$\because\ y\big|_A = u\sin\alpha\ T - \dfrac{1}{2}gT^2 \quad \Rightarrow \quad y\big|_A = \dfrac{1}{2}gT(T + T') - \dfrac{1}{2}gT^2 = \dfrac{1}{2}gTT'$

Example

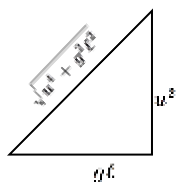
A particle is projected with a velocity u so as just to pass over the highest possible post at a horizontal distance ℓ from the point of projection **O**. Prove that the greatest height above O attained by the particle in its flight is $\dfrac{u^6}{2g(u^4 + g^2\ell^2)}$.

Solution

Taking θ as the angle of projection and substituting ℓ for x the equation to the path, we have

$$y = \ell\tan\theta - \frac{g\ell^2}{2u^2\cos^2\theta} = \ell\tan\theta - \frac{g\ell^2}{2u^2}(1 + \tan^2\theta)$$

$$\therefore\ \frac{dy}{d\theta} = \ell\sec^2\theta - \frac{g\ell^2}{u^2}\tan\theta\sec^2\theta = \ell\sec^2\theta\left(1 - \frac{gd^2}{u^2}\tan\theta\right)$$



$$\frac{dy}{d\theta} = 0 \Rightarrow \tan\theta = \frac{u^2}{g\ell} \qquad \text{or} \qquad \sin^2\theta = \frac{u^4}{u^4 + g^2\ell^2}$$

y being positive and its minimum value being zero, the value of θ given in previous equation gives the maximum value of y. Now the greatest height attained by the particle

$$Y = \frac{u^2\sin^2\theta}{2g} = \frac{u^2}{2g}\left(\frac{u^4}{u^4 + g^2\ell^2}\right) = \frac{u^6}{2g\ \ u^4 + g^2\ell^2}$$

Example

Two particles are projected from the same point in the same vertical plane with equal velocities. If t, t' be the times taken to reach the common point of their paths and T, T' the times to the highest point, show that $tT + t'T'$ is independent of the directions of projection

Solution

Let α, β be the directions of projection

$$T = \frac{u\sin\alpha}{g}, \quad T' = \frac{u\sin\beta}{g}$$

If x is the horizontal distance of the common point, then

$$x = u\cos\alpha\ t, \ x = u\cos\beta\ t'$$

$$\therefore tT + t'T' = \frac{x}{u\cos\alpha}\frac{u\sin\alpha}{g} + \frac{x}{u\cos\beta}\frac{u\sin\beta}{g} = \frac{x}{g}(\tan\alpha + \tan\beta) \qquad (*)$$

Now the equations of the two- paths arc

$$y = x\tan\alpha - \frac{1}{2}\frac{gx^2\sec^2\alpha}{u^2}, \quad y = x\tan\beta - \frac{1}{2}\frac{gx^2\sec^2\beta}{u^2}$$

Subtracting we have,

$$x(\tan\alpha - \tan\beta) = \frac{1}{2}\frac{gx^2}{u^2}\ \sec^2\alpha - \sec^2\beta \ = \frac{1}{2}\frac{gx^2}{u^2}\ \tan^2\alpha - \tan^2\beta$$

$$\frac{x}{g}(\tan\alpha + \tan\beta) = \frac{2u^2}{g^2}$$

Hence from Equation (*)

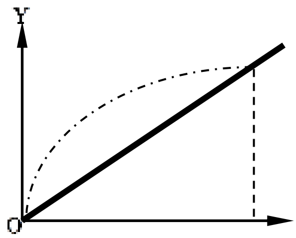
$$\therefore tT + t'T' = \frac{2u^2}{g^2}$$ which is independent of the directions of projection.

Example

A particle is projected with velocity u from a point on an inclined plate. If v_1 be its velocity on striking the plane when the range up the plane is maximum and v_2 the velocity on striking the plane when the range down the plane is maximum, prove that $u^2 = v_1 v_2$

Solution

Let R be the maximum range up the plane and α be the inclination of the plane, then



$$R = \frac{u^2}{g(1+\sin\alpha)}, \text{ and } v_1^2 = u^2 - 2gy = u^2 - 2gR\sin\alpha$$

$$\therefore v_1^2 = u^2 - 2g\sin\alpha \times \frac{u^2}{g(1+\sin\alpha)} = u^2 \times \frac{1-\sin\alpha}{1+\sin\alpha}$$

Similarly, by changing the sign of α, we have

$$\therefore v_2^2 = u^2 \times \frac{1+\sin\alpha}{1-\sin\alpha} \quad \text{Hence} \quad u^4 = v_1^2 v_2^2 \quad \text{Or} \quad u^2 = v_1 v_2$$

Example

A particle is projected and it paths through the two points $(12,12)$ and $(36,12)$ Find its velocity and the direction of projection.

Solution

The trajectory or path equation is $$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$$

The two points $(12,12)$ and $(36,12)$ lies on the path so that

With regard the point $(36,12)$ $$12 = 36\tan\alpha - \frac{g(36)^2}{2u^2\cos^2\alpha}$$

With regard the point $(12,12)$ $$12 = 12\tan\alpha - \frac{g(12)^2}{2u^2\cos^2\alpha}$$

By multiplying the second equation by 9 then subtracting, we have

$$96 = 72\tan\alpha \qquad \Rightarrow \tan\alpha = \frac{96}{72} = \frac{4}{3}$$

which gives the direction of velocity of projection, and to obtain the magnitude of the projection velocity, from first equation

$$\Rightarrow 12 = 36\left(\frac{4}{3}\right) - \frac{g(36)^2}{2u^2\left(\frac{3}{5}\right)^2} \qquad \Rightarrow \frac{g(36)^2}{2u^2\left(\frac{3}{5}\right)^2} = 36$$

$$\Rightarrow u^2 = 50\,g \qquad \text{Or} \quad u = 5\sqrt{2g}$$

Example

A particle is projected and it paths through the two points (a,b) and (b,a) where (a,b) and (b,a) Prove that the range is given by $\frac{a^2+ab+b^2}{a+b}$.

Solution

The trajectory or path equation is $\quad y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$

The two points (a,b) and (b,a) lies on the path so that

With regard the point (a,b) $\quad a = b\tan\alpha - \frac{gb^2}{2u^2\cos^2\alpha}$

With regard the point (b,a) $\quad b = a\tan\alpha - \frac{ga^2}{2u^2\cos^2\alpha}$

By multiplying the first equation by a and the second by b then subtracting, we have

$$\underbrace{a^2 - b^2}_{(a+b)\cancel{(a-b)}} = \frac{gab}{2u^2\cos^2\alpha}\cancel{(a-b)} \qquad \Rightarrow a+b = \frac{gab}{2u^2\cos^2\alpha}$$

$$\text{Or} \quad \frac{ab}{a+b} = \frac{2u^2\cos^2\alpha}{g}$$

Once again by multiplying the first equation by a^2 and the second by b^2 then subtracting, we have

$$\underbrace{a^3 - b^3}_{(a^2+ab+b^2)\cancel{(a-b)}} = ab\cancel{(a-b)}\tan\alpha \qquad \Rightarrow ab\tan\alpha = a^2 + ab + b^2$$

Since the range is given by $R = \dfrac{u^2 \sin 2\alpha}{g}$ therefore,

$$R = \frac{u^2 \sin 2\alpha}{g} = \frac{2u^2 \cos\alpha \sin\alpha}{g}$$

$$= \underbrace{\frac{2u^2 \cos^2\alpha}{g}}_{ab/(a+b)}\tan\alpha = \frac{ab}{a+b}\tan\alpha = \frac{a^2 + ab + b^2}{a+b}$$

$$\therefore R = \frac{a^2 + ab + b^2}{a+b}$$

Example

A particle is projected to reach a certain object located in the same horizontal plate of projection point, when it projected with angle α it falls down before the object by distance ℓ and when it projected with angle β it falls down after the object by distance ℓ. Find the exact angle to reach the object.

Solution

Let u be the velocity of projection and R is the exact range of the object then the range in first case is $R - \ell$ and the range in second case is $R + \ell$ therefore

$$R - \ell = \frac{u^2 \sin 2\alpha}{g} \qquad \text{and} \qquad R + \ell = \frac{u^2 \sin 2\beta}{g}$$

By addition the two equations, we get

$$\therefore 2R = \frac{u^2}{g}\ \sin 2\alpha + \sin 2\beta \qquad \Rightarrow R = \frac{u^2}{2g}\ \sin 2\alpha + \sin 2\beta$$

Now, let θ be the exact angle to reach the object so $R = \dfrac{u^2 \sin 2\theta}{g}$

By comparing (or dividing) the last two equations then

$$\Rightarrow \frac{u^2 \sin 2\theta}{g} = \frac{u^2}{2g}\ \sin 2\alpha + \sin 2\beta$$

$$\Rightarrow \sin 2\theta = \frac{\sin 2\alpha + \sin 2\beta}{2} \qquad \Rightarrow \theta = \frac{1}{2}\sin^{-1}\left(\frac{\sin 2\alpha + \sin 2\beta}{2}\right)$$

■ Projectiles with Resistance

We now proceed to include the effect of air resistance. From our earlier discussion of fluid drag, it is evident that in most practical instances of projectile motion through the Earth's atmosphere, it is the **quadratic law** of resistance that is appropriate. On the other hand, only the **linear law** of resistance gives rise to linear equations of motion and simple analytical solutions. This explains why mechanics textbooks contain extensive coverage of the linear case, even though this case is almost never appropriate in practice; the case that is appropriate cannot be solved! In the following example, we treat the linear resistance case.

Now suppose that the motion is opposed by a force proportional to the velocity. Thus if m denote the mass and v the velocity, let $m\gamma v$ denote the magnitude of the resistance. Therefore the components of the resistance parallel to horizontal and vertical axes **OX** , **OY** are

$$-m\gamma\dot{x}, \; -m\gamma\dot{y}$$

Let u denote the initial velocity in a direction making an angle α with the horizontal. The equations of motion give

$$\ddot{x} = -\gamma\dot{x} \quad \text{and} \quad \ddot{y} = -g - \gamma\dot{y}$$

By integrating we obtain

$$\ln\dot{x} = c_1 - \gamma t \quad \text{and} \quad \ln\left(\dot{y} + \frac{g}{\gamma}\right) = c_2 - \gamma t$$

since initially $x = y = 0$ and $\dot{x} = u\cos\alpha, \quad \dot{y} = u\sin\alpha$,then $c_1 = \ln u\cos\alpha$

and $c_2 = \ln\left(u\sin\alpha + \dfrac{g}{\gamma}\right)$, and hence

$$\dot{x} = u\cos\alpha \; e^{-\gamma t} \quad \text{and} \quad \dot{y} = \left(u\sin\alpha + \frac{g}{\gamma}\right)e^{-\gamma t} - \frac{g}{\gamma}$$

Once again integrate the previous formula

$$x = -\frac{u\cos\alpha}{\gamma}\, e^{-\gamma t} + c_3 \quad \text{and} \quad y = -\frac{1}{\gamma}\left(u\sin\alpha + \frac{g}{\gamma}\right)e^{-\gamma t} - \frac{g}{\gamma}t + c_4$$

Where, c_4, c_3 are constant, and $x = y = 0$ at $t = 0$ so that

$$c_3 = \frac{u\cos\alpha}{\gamma}, \qquad c_4 = \frac{1}{\gamma}\left(u\sin\alpha + \frac{g}{\gamma}\right)$$

So the last equation becomes

$$x = \frac{u\cos\alpha}{\gamma}\ \ 1 - e^{-\gamma t} \quad \text{and} \quad y = \frac{1}{\gamma}\left(u\sin\alpha + \frac{g}{\gamma}\right)\ 1 - e^{-\gamma t} \ \ - \frac{g}{\gamma}t$$

- The time spent to reach the maximum height is

$$T = \frac{1}{\gamma}\ln\left(\frac{\gamma u\sin\alpha}{g} + 1\right)$$

- The maximum height is

$$y = \frac{u\sin\alpha}{\gamma} - \frac{g}{\gamma^2}\ln\left(1 + \frac{\gamma u\sin\alpha}{g}\right)$$

- The time of flight is

$$T' = \frac{1}{\gamma}\left(\frac{\gamma u\sin\alpha}{g} + 1\right)\ 1 - e^{-\gamma T'}$$

- The path equation is

$$y = \frac{g}{\gamma u\cos\alpha}\left(\frac{\gamma u\sin\alpha}{g} + 1\right)x + \frac{g}{\gamma^2}\ln\left(1 - \frac{\gamma x}{u\cos\alpha}\right)$$

For instance to evaluate the spent time to reach the maximum height

Since $$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - .....$$

this is true for $|x| < 1$, and now let $\gamma \to 0$ in formula

$$T = \frac{1}{\gamma}\ln\left(\frac{\gamma u\sin\alpha}{g} + 1\right)$$

We get

$$T = \lim_{\gamma \to 0} \frac{1}{\gamma}\left(\frac{\gamma u \sin\alpha}{g} - \frac{\gamma^2 u^2 \sin^2\alpha}{2g^2} + \frac{\gamma^3 u^3 \sin^3\alpha}{3g} + ...\right)$$
$$= \lim_{\gamma \to 0}\left(\frac{u \sin\alpha}{g} - \frac{\gamma u^2 \sin^2\alpha}{2g^2} + \frac{\gamma u^3 \sin^3\alpha}{3g} + ...\right) = \frac{u\sin\alpha}{g}$$

This result obtained before when we neglected the resistance of air.

Example

A particle of mass m is projected with initial velocity u at an angle of elevation α through a resisting medium where its resistance proportional to v and the proportional constant is μm. Prove that the direction of the velocity makes an angle α with the horizontal $\frac{1}{\mu}\ln\left(1 + \frac{\mu u}{g}(\sin\alpha + \cos\alpha)\right)$

Solution

By writing the equation of motion in **OX, OY** and then integrating and use the initial conditions as illustrated before we obtain the components of velocity of the particle at any instance

$$\dot{x} = u\cos\alpha \;\; e^{-\mu t} \qquad \text{and} \qquad \dot{y} = \left(u\sin\alpha + \frac{g}{\mu}\right)e^{-\mu t} - \frac{g}{\mu}$$

Since the angle of projection is α and the angle that the direction of velocity makes with the horizontal axis decreases until vanish at the highest position then it reverse to be α again downwards after time t which determines from

$$\tan\ -\alpha \ = \frac{\dot{y}}{\dot{x}} = \frac{\left(u\sin\alpha + \frac{g}{\mu}\right)e^{-\mu t} - \frac{g}{\mu}}{u\cos\alpha \;\; e^{-\mu t}} = -\tan\ \alpha$$

That is

$$\left(u\sin\alpha + \frac{g}{\mu}\right)e^{-\mu t} - \frac{g}{\mu} = -u\sin\alpha \;\; e^{-\mu t} \;\; \Rightarrow \left(2u\sin\alpha + \frac{g}{\mu}\right)e^{-\mu t} = \frac{g}{\mu}$$
$$\left(\frac{2\mu u\sin\alpha}{g} + 1\right) = e^{\mu t} \qquad \Rightarrow t = \frac{1}{\mu}\ln\left(\frac{2\mu u \sin\alpha}{g} + 1\right)$$

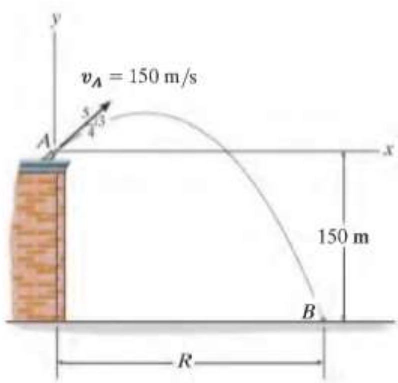
PROBLEMS

❒ A body, projected with a velocity of 120 ft sec^{-1} just clears a vertical wall 72 ft high and 360 ft. distant, find the two possible angles of projection and the corresponding horizontal ranges.

❒ A particle is projected so as just to clear a wall of height b at a horizontal distance a, and to have a range c from the point of projection, show that the velocity of projection V is given by

$$\frac{2V^2}{g} = \frac{a^2(c-a)^2 + b^2c^2}{ab(c-a)}.$$

❒ A projectile is fired with an initial velocity of V_A = 150 m/s off the roof of the building. Determine the range R where it strikes the ground at B.



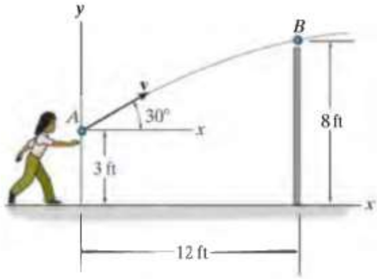
❒ A stone is projected with velocity V and elevation from a point O on level ground so as to hit a mark P on a wall whose distance from O is a, the height of P above the ground being 6. Prove that

$$2V^2(a\sin\theta\cos\theta - b\cos^2\theta) = ga^2 .$$

❒ A particle in projected with a velocity of 120 ft. per sec. at an angle of 60 with the horizontal from the foot of an inclined plane of inclination 30. Find the time of flight and the range on the inclined plane.

❒ A particle is projected from a point on a plane of inclination β with velocity u. Show that the maximum range down the plane is

$$\frac{u^2}{g}\left(\frac{1+\sin\beta}{\cos^2\beta}\right).$$



❒ A ball is thrown from A. If it is required to clear the wall at B, determine the minimum magnitude of its initial velocity V_A.

❒ A boy throws a ball at 0 in the air with a speed Vo at an angle θ_1. If he then throws another ball with the same speed v_0 at an angle $\theta_2 < \theta_1$ determine the time between the throws so that the balls collide in midair at B.