Chapter 11

LAGRANGE'S EQUATIONS

GENERAL METHODS OF MECHANICS

Up to now we have dealt primarily with the formulation of problems in mechanics by Newton's laws of motion. It is possible to give treatments of mechanics from rather general viewpoints, in particular those due to Lagrange and Hamilton.

Although such treatments reduce to Newton's laws, they are characterized not only by the relative ease with which many problems can be formulated and solved but by their relationship in both theory and application to such advanced fields as quantum mechanics, statistical mechanics, celestial mechanics and electrodynamics.

GENERALIZED COORDINATES

Suppose that a particle or a system of N particles moves subject to possible constraints, as for example a particle moving along a circular wire or a rigid body moving along an inclined plane. Then there will be a minimum number of independent coordinates needed to specify the motion. These coordinates denoted by

$$q_1, q_2, \ldots, q_n \tag{1}$$

are called *generalized coordinates* and can be distances, angles or quantities relating to them. The number n of generalized coordinates is the number of degrees of freedom [see page 165].

Many sets of generalized coordinates may be possible in a given problem, but a strategic choice can simplify the analysis considerably.

NOTATION

In the following the subscript α will range from 1 to n, the number of degrees of freedom, while the subscript ν will range from 1 to N, the number of particles in the system.

TRANSFORMATION EQUATIONS

Let $\mathbf{r}_\nu = x_\nu\mathbf{i} + y_\nu\mathbf{j} + z_\nu\mathbf{k}$ be the position vector of the νth particle with respect to an xyz coordinate system. The relationships of the generalized coordinates (*1*) to the position coordinates are given by the *transformation equations*

$$\left.\begin{aligned} x_\nu &= x_\nu(q_1, q_2, \ldots, q_n, t) \\ y_\nu &= y_\nu(q_1, q_2, \ldots, q_n, t) \\ z_\nu &= z_\nu(q_1, q_2, \ldots, q_n, t) \end{aligned}\right\} \tag{2}$$

where t denotes the time. In vector form, (*2*) can be written

$$\mathbf{r}_\nu = \mathbf{r}_\nu(q_1, q_2, \ldots, q_n, t) \tag{3}$$

The functions in (*2*) or (*3*) are supposed to be continuous and to have continuous derivatives.

CLASSIFICATION OF MECHANICAL SYSTEMS

Mechanical systems can be classified according as they are *scleronomic* or *rheonomic*, *holonomic* or *non-holonomic*, and *conservative* or *non-conservative* as defined below.

SCLERONOMIC AND RHEONOMIC SYSTEMS

In many mechanical systems of importance the time t does not enter explicitly in the equations (*2*) or (*3*). Such systems are sometimes called *scleronomic*. In others, as for example those involving moving constraints, the time t does enter explicitly. Such systems are called *rheonomic*.

HOLONOMIC AND NON-HOLONOMIC SYSTEMS

Let $q_1, q_2, \ldots, q_n$ denote the generalized coordinates describing a system and let t denote the time. If all the constraints of the system can be expressed as equations having the form $\phi(q_1, q_2, \ldots, q_n, t) = 0$ or their equivalent, then the system is said to be *holonomic*; otherwise the system is said to be *non-holonomic*. Compare page 170.

CONSERVATIVE AND NON-CONSERVATIVE SYSTEMS

If all forces acting on a system of particles are derivable from a potential function [or potential energy] V, then the system is called *conservative*, otherwise it is *non-conservative*.

KINETIC ENERGY. GENERALIZED VELOCITIES

The total kinetic energy of the system is

$$T = \frac{1}{2}\sum_{\nu=1}^{N} m_\nu \dot{\mathbf{r}}_\nu^2 \tag{4}$$

The kinetic energy can be written as a *quadratic form* in the *generalized velocities* $\dot{q}_\alpha$. If the system is scleronomic [i.e. independent of time t explicitly], then the quadratic form has only terms of the form $a_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta$. If it is rheonomic, linear terms in $\dot{q}_\alpha$ are also present.

GENERALIZED FORCES

If W is the total work done on a system of particles by forces $\mathbf{F}_\nu$ acting on the νth particle, then

$$dW = \sum_{\alpha=1}^{n} \Phi_\alpha \, dq_\alpha \tag{5}$$

where

$$\Phi_\alpha = \sum_{\nu=1}^{N} \mathbf{F}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha} \tag{6}$$

is called the *generalized force* associated with the generalized coordinate q_α. See Problem 11.6.

LAGRANGE'S EQUATIONS

The generalized force can be related to the kinetic energy by the equations [see Problem 11.10]

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = \Phi_\alpha \tag{7}$$

If the system is conservative so that the forces are derivable from a potential or potential energy V, we can write *(7)* as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0 \tag{8}$$

where

$$L = T - V \tag{9}$$

is called the *Lagrangian function* of the system, or simply the *Lagrangian.*

The equations *(7)* or *(8)* are called *Lagrange's equations* and are valid for holonomic systems which may be scleronomic or rheonomic.

If some of the forces in a system are conservative so as to be derivable from a potential V' while other forces such as friction, etc., are non-conservative, we can write Lagrange's equations as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = \Phi'_\alpha \tag{10}$$

where $L = T - V'$ and Φ'_α are the generalized forces associated with the non-conservative forces in the system.

GENERALIZED MOMENTA

We define

$$p_\alpha = \frac{\partial T}{\partial \dot{q}_\alpha} \tag{11}$$

to be the *generalized momentum* associated with the generalized coordinate q_α. We often call p_α the momentum *conjugate* to q_α, or the *conjugate momentum.*

If the system is conservative with potential energy depending only on the generalized coordinates, then *(11)* can be written in terms of the Lagrangian $L = T - V$ as

$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} \tag{12}$$

LAGRANGE'S EQUATIONS FOR NON-HOLONOMIC SYSTEMS

Suppose that there are m equations of constraint having the form

$$\sum_\alpha A_\alpha\, dq_\alpha + A\, dt = 0, \quad \sum_\alpha B_\alpha\, dq_\alpha + B\, dt = 0, \quad \ldots \tag{13}$$

or equivalently

$$\sum_\alpha A_\alpha\, \dot{q}_\alpha + A = 0, \quad \sum_\alpha B_\alpha\, \dot{q}_\alpha + B = 0, \quad \ldots \tag{14}$$

We must of course have $m < n$ where n is the number of coordinates q_α.

The equations *(13)* or *(14)* may or may not be integrable so as to obtain a relationship involving the q_α's. If they are not integrable the constraints are *non-holonomic* or *non-integrable*; otherwise they are *holonomic* or *integrable.*

In either case Lagrange's equations can be replaced by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = \Phi_\alpha + \lambda_1 A_\alpha + \lambda_2 B_\alpha + \cdots \tag{15}$$

where the m parameters $\lambda_1, \lambda_2, \ldots$ are called *Lagrange multipliers* [see Problem 11.18].

If the forces are conservative, *(15)* can be written in terms of the Lagrangian $L = T - V$ as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = \lambda_1 A_\alpha + \lambda_2 B_\alpha + \cdots \tag{16}$$

It should be emphasized that the above results are applicable to holonomic (as well as non-holonomic) systems since a constraint condition of the form

$$\phi(q_1, q_2, \ldots, q_n, t) = 0 \tag{17}$$

can by differentiation be written as

$$\sum_\alpha \frac{\partial\phi}{\partial q_\alpha} dq_\alpha + \frac{\partial\phi}{\partial t} dt = 0 \tag{18}$$

which has the form (*13*).

LAGRANGE'S EQUATIONS WITH IMPULSIVE FORCES

Suppose that the forces $\mathbf{F}_\nu$ acting on a system are such that

$$\lim_{\tau\to 0} \int_0^\tau \mathbf{F}_\nu\, dt = \mathcal{J}_\nu \tag{19}$$

where τ represents a time interval. Then we call $\mathbf{F}_\nu$ *impulsive forces* and $\mathcal{J}_\nu$ are called *impulses*.

If we let the subscripts 1 and 2 denote respectively quantities before and after application of the impulsive forces, Lagrange's equations become [see Problem 11.23]

$$\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_2 - \left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_1 = \mathcal{F}_\alpha \tag{20}$$

where

$$\mathcal{F}_\alpha = \sum_\nu \mathcal{J}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha} \tag{21}$$

If we call $\mathcal{F}_\alpha$ the *generalized impulse*, (*20*) can be written

$$\text{Generalized impulse} = \text{change in generalized momentum} \tag{22}$$

which is a generalization of Theorem 2.6, page 36.

Solved Problems

GENERALIZED COORDINATES AND TRANSFORMATION EQUATIONS

11.1. Give a set of generalized coordinates needed to completely specify the motion of each of the following: (*a*) a particle constrained to move on an ellipse, (*b*) a circular cylinder rolling down an inclined plane, (*c*) the two masses in a double pendulum [Fig. 11-3] constrained to move in a plane.

(*a*) Let the ellipse be chosen in the xy plane of Fig. 11-1. The particle of mass m moving on the ellipse has coordinates (x, y). However, since we have the transformation equations $x = a\cos\theta$, $y = b\sin\theta$, we can specify the motion completely by use of the generalized coordinate θ.

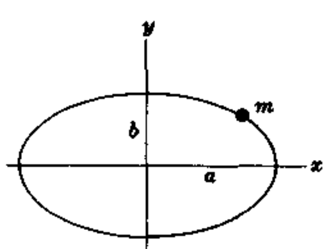

Fig. 11-1

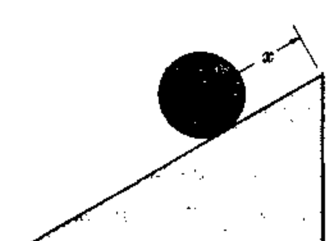

Fig. 11-2

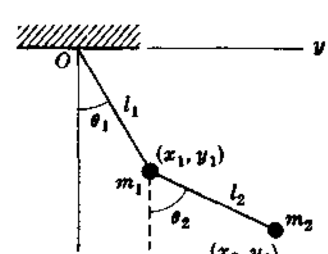

Fig. 11-3

(b) The position of the cylinder [Fig. 11-2 above] on the inclined plane can be completely specified by giving the distance x traveled by the center of mass and the angle θ of rotation turned through by the cylinder about its axis.

If there is no slipping, x is related to θ so that only one generalized coordinate [either x or θ] is needed. If there is slipping, two generalized coordinates x and θ are needed.

(c) Two coordinates θ_1 and θ_2 completely specify the positions of masses m_1 and m_2 [see Fig. 11-3 above] and can be considered as the required generalized coordinates.

11.2. Write the transformation equations for the system in Problem 11.1(c).

Choose an xy coordinate system as shown in Fig. 11-3. Let (x_1, y_1) and (x_2, y_2) be the rectangular coordinates of m_1 and m_2 respectively. Then from Fig. 11-3 we see that

$$x_1 = l_1 \cos\theta_1 \qquad y_1 = l_1 \sin\theta_1$$
$$x_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2 \qquad y_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$$

which are the required transformation equations.

11.3. Prove that $\dfrac{\partial \dot{\mathbf{r}}_\nu}{\partial \dot{q}_\alpha} = \dfrac{\partial \mathbf{r}_\nu}{\partial q_\alpha}$.

We have $\mathbf{r}_\nu = \mathbf{r}_\nu(q_1, q_2, \ldots, q_n, t)$. Then

$$\dot{\mathbf{r}}_\nu = \frac{\partial \mathbf{r}_\nu}{\partial q_1}\dot{q}_1 + \cdots + \frac{\partial \mathbf{r}_\nu}{\partial q_n}\dot{q}_n + \frac{\partial \mathbf{r}_\nu}{\partial t} \tag{1}$$

Thus
$$\frac{\partial \dot{\mathbf{r}}_\nu}{\partial \dot{q}_\alpha} = \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha} \tag{2}$$

We can look upon this result as a "cancellation of the dots".

11.4. Prove that $\dfrac{d}{dt}\left(\dfrac{\partial \mathbf{r}_\nu}{\partial q_\alpha}\right) = \dfrac{\partial \dot{\mathbf{r}}_\nu}{\partial q_\alpha}$.

We have from (1) of Problem 11.3,

$$\dot{\mathbf{r}}_\nu = \frac{\partial \mathbf{r}_\nu}{\partial q_1}\dot{q}_1 + \cdots + \frac{\partial \mathbf{r}_\nu}{\partial q_n}\dot{q}_n + \frac{\partial \mathbf{r}_\nu}{\partial t} \tag{1}$$

Then
$$\frac{\partial \dot{\mathbf{r}}_\nu}{\partial q_\alpha} = \frac{\partial^2 \mathbf{r}_\nu}{\partial q_\alpha\, \partial q_1}\dot{q}_1 + \cdots + \frac{\partial^2 \mathbf{r}_\nu}{\partial q_\alpha\, \partial q_n}\dot{q}_n + \frac{\partial^2 \mathbf{r}_\nu}{\partial q_\alpha\, \partial t} \tag{2}$$

Now
$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}\right) &= \frac{\partial}{\partial q_1}\left(\frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}\right)\frac{dq_1}{dt} + \cdots + \frac{\partial}{\partial q_n}\left(\frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}\right)\frac{dq_n}{dt} + \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}\right) \\ &= \frac{\partial^2 \mathbf{r}_\nu}{\partial q_1\, \partial q_\alpha}\dot{q}_1 + \cdots + \frac{\partial^2 \mathbf{r}_\nu}{\partial q_n\, \partial q_\alpha}\dot{q}_n + \frac{\partial^2 \mathbf{r}_\nu}{\partial t\, \partial q_\alpha}\end{aligned} \tag{3}$$

Since $\mathbf{r}_\nu$ is assumed to have continuous second order partial derivatives, the order of differentiation does not matter. Thus from (2) and (3) the required result follows.

The result can be interpreted as an interchange of order of the operators, i.e.,

$$\frac{d}{dt}\left(\frac{\partial}{\partial q_\alpha}\right) = \frac{\partial}{\partial q_\alpha}\left(\frac{d}{dt}\right)$$

CLASSIFICATION OF MECHANICAL SYSTEMS

11.5. Classify each of the following according as they are (i) scleronomic or rheonomic, (ii) holonomic or non-holonomic and (iii) conservative or non-conservative.

(a) A sphere rolling down from the top of a fixed sphere.

(b) A cylinder rolling without slipping down a rough inclined plane of angle α.

(c) A particle sliding down the inner surface, with coefficient of friction μ, of a paraboloid of revolution having its axis vertical and vertex downward.

(d) A particle moving on a very long frictionless wire which rotates with constant angular speed about a horizontal axis.

(a) scleronomic [equations do not involve time t explicitly]
non-holonomic [since rolling sphere leaves the fixed sphere at some point]
conservative [gravitational force acting is derivable from a potential]

(b) scleronomic
holonomic [equation of constraint is that of a line or plane]
conservative

(c) scleronomic
holonomic
non-conservative [since force due to friction is not derivable from a potential]

(d) rheonomic [constraint involves time t explicitly]
holonomic [equation of constraint is that of a line which involves t explicitly]
conservative

WORK, KINETIC ENERGY AND GENERALIZED FORCES

11.6. Derive equations (5) and (6), page 283, for the work done on a system of particles.

Suppose that a system undergoes increments $dq_1, dq_2, \ldots, dq_n$ of the generalized coordinates. Then the νth particle undergoes a displacement

$$d\mathbf{r}_\nu = \sum_{\alpha=1}^{n} \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha} dq_\alpha$$

Thus the total work done is

$$dW = \sum_{\nu=1}^{N} \mathbf{F}_\nu \cdot d\mathbf{r}_\nu = \sum_{\nu=1}^{N} \left\{ \sum_{\alpha=1}^{n} \mathbf{F}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha} \right\} dq_\alpha = \sum_{\alpha=1}^{n} \Phi_\alpha \, dq_\alpha$$

where

$$\Phi_\alpha = \sum_{\nu=1}^{N} \mathbf{F}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}$$

We call Φ_α the *generalized force* associated with the generalized coordinate q_α.

11.7. Prove that $\Phi_\alpha = \partial W / \partial q_\alpha$.

We have $dW = \sum \frac{\partial W}{\partial q_\alpha} dq_\alpha$. Also, by Problem 11.6, $dW = \sum \Phi_\alpha \, dq_\alpha$. Then

$$\sum \left(\Phi_\alpha - \frac{\partial W}{\partial q_\alpha} \right) dq_\alpha = 0$$

If the dq_α are independent, all coefficients of dq_α must be zero, so that $\Phi_\alpha = \partial W / \partial q_\alpha$.

11.8. Let $\mathbf{F}_\nu$ be the net external force acting on the νth particle of a system. Prove that

$$\frac{d}{dt}\left\{ \sum_\nu m_\nu \dot{\mathbf{r}}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha} \right\} - \sum_\nu m_\nu \dot{\mathbf{r}}_\nu \cdot \frac{\partial \dot{\mathbf{r}}_\nu}{\partial q_\alpha} = \sum_\nu \mathbf{F}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}$$

By Newton's second law applied to the νth particle, we have

$$m_\nu \ddot{\mathbf{r}}_\nu = \mathbf{F}_\nu \qquad (1)$$

Then $$m_\nu \ddot{\mathbf{r}}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha} = \mathbf{F}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha} \tag{2}$$

Now by Problem 11.4, $$\frac{d}{dt}\left(\dot{\mathbf{r}}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}\right) = \ddot{\mathbf{r}}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha} + \dot{\mathbf{r}}_\nu \cdot \frac{d}{dt}\left(\frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}\right)$$
$$= \ddot{\mathbf{r}}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha} + \dot{\mathbf{r}}_\nu \cdot \frac{\partial \dot{\mathbf{r}}_\nu}{\partial q_\alpha} \tag{3}$$

Thus $$\ddot{\mathbf{r}}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha} = \frac{d}{dt}\left(\dot{\mathbf{r}}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}\right) - \dot{\mathbf{r}}_\nu \cdot \frac{\partial \dot{\mathbf{r}}_\nu}{\partial q_\alpha} \tag{4}$$

Hence from (2) we have, since m_ν is constant,

$$\frac{d}{dt}\left(m_\nu \dot{\mathbf{r}}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}\right) - m_\nu \dot{\mathbf{r}}_\nu \cdot \frac{\partial \dot{\mathbf{r}}_\nu}{\partial q_\alpha} = \mathbf{F}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}$$

Summing both sides with respect to ν over all particles, we have

$$\frac{d}{dt}\left\{\sum_\nu m_\nu \dot{\mathbf{r}}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}\right\} - \sum_\nu m_\nu \dot{\mathbf{r}}_\nu \cdot \frac{\partial \dot{\mathbf{r}}_\nu}{\partial q_\alpha} = \sum_\nu \mathbf{F}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}$$

11.9. Let T be the kinetic energy of a system of particles. Prove that

(a) $\dfrac{\partial T}{\partial q_\alpha} = \sum_\nu m_\nu \dot{\mathbf{r}}_\nu \cdot \dfrac{\partial \dot{\mathbf{r}}_\nu}{\partial q_\alpha}$, (b) $\dfrac{\partial T}{\partial \dot{q}_\alpha} = \sum_\nu m_\nu \dot{\mathbf{r}}_\nu \cdot \dfrac{\partial \mathbf{r}_\nu}{\partial q_\alpha}$

(a) The kinetic energy is $T = \frac{1}{2}\sum_\nu m_\nu \dot{r}_\nu^2 = \frac{1}{2}\sum_\nu m_\nu \dot{\mathbf{r}}_\nu \cdot \dot{\mathbf{r}}_\nu$. Thus

$$\frac{\partial T}{\partial q_\alpha} = \sum_\nu m_\nu \dot{\mathbf{r}}_\nu \cdot \frac{\partial \dot{\mathbf{r}}_\nu}{\partial q_\alpha}$$

(b) We have by the "cancellation of the dots" [Problem 11.3, page 286],

$$\frac{\partial T}{\partial \dot{q}_\alpha} = \sum m_\nu \dot{\mathbf{r}}_\nu \cdot \frac{\partial \dot{\mathbf{r}}_\nu}{\partial \dot{q}_\alpha} = \sum m_\nu \dot{\mathbf{r}}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}$$

LAGRANGE'S EQUATIONS

11.10. Prove that $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{q}_\alpha}\right) - \dfrac{\partial T}{\partial q_\alpha} = \Phi_\alpha$, $\alpha = 1, \ldots, n$ where $\Phi_\alpha = \sum_\nu \mathbf{F}_\nu \cdot \dfrac{\partial \mathbf{r}_\nu}{\partial q_\alpha}$.

From Problem 11.8,

$$\frac{d}{dt}\left\{\sum_\nu m_\nu \dot{\mathbf{r}}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}\right\} - \sum_\nu m_\nu \dot{\mathbf{r}}_\nu \cdot \frac{\partial \dot{\mathbf{r}}_\nu}{\partial q_\alpha} = \sum_\nu \mathbf{F}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha} \tag{1}$$

From Problems 11.9(a) and 11.9(b),

$$\sum_\nu m_\nu \dot{\mathbf{r}}_\nu \cdot \frac{\partial \dot{\mathbf{r}}_\nu}{\partial q_\alpha} = \frac{\partial T}{\partial q_\alpha} \tag{2}$$

$$\sum_\nu m_\nu \dot{\mathbf{r}}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha} = \frac{\partial T}{\partial \dot{q}_\alpha} \tag{3}$$

Then substituting (2) and (3) in (1), we find

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = \Phi_\alpha \tag{4}$$

The quantity $$p_\alpha = \frac{\partial T}{\partial \dot{q}_\alpha} \tag{5}$$

is called the *generalized momentum* or *conjugate momentum* associated with the generalized coordinate q_α.

11.11. Suppose that the forces acting on a system of particles are derivable from a potential function V, i.e. suppose that the system is conservative. Prove that if $L = T - V$ is the Lagrangian function, then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0$$

If the forces are derivable from a potential V, then [see Problem 11.7],

$$\Phi_\alpha = \frac{\partial W}{\partial q_\alpha} = -\frac{\partial V}{\partial q_\alpha}$$

Since the potential, or potential energy is a function of only the q's [and possibly the time t],

$$\frac{\partial L}{\partial \dot{q}_\alpha} = \frac{\partial}{\partial \dot{q}_\alpha}(T - V) = \frac{\partial T}{\partial \dot{q}_\alpha}$$

Then from Problem 11.10,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = -\frac{\partial V}{\partial q_\alpha} \quad \text{or} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0$$

11.12. (a) Set up the Lagrangian for a simple pendulum and (b) obtain an equation describing its motion.

(a) Choose as generalized coordinate the angle θ made by string OB of the pendulum and the vertical OA [see Fig. 11-4]. If l is the length of OB, then the kinetic energy is

$$T = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m(l\dot{\theta})^2 = \tfrac{1}{2}ml^2\dot{\theta}^2 \qquad (1)$$

where m is the mass of the bob.

The potential energy of mass m [taking as reference level a horizontal plane through the lowest point A] is given by

$$V = mg(OA - OC) = mg(l - l\cos\theta)$$
$$= mgl(1 - \cos\theta) \qquad (2)$$

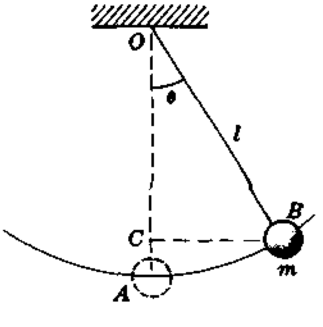

Fig. 11-4

Thus the Lagrangian is

$$L = T - V = \tfrac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta) \qquad (3)$$

(b) Lagrange's equation is
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \qquad (4)$$

From (3),
$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta, \qquad \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \qquad (5)$$

Substituting these in (4), we find

$$ml^2\ddot{\theta} + mgl\sin\theta = 0 \quad \text{or} \quad \ddot{\theta} + \frac{g}{l}\sin\theta = 0 \qquad (6)$$

which is the required equation of motion [compare Problem 4.23, page 102].

11.13. A mass M_2 hangs at one end of a string which passes over a fixed frictionless non-rotating pulley [see Fig. 11-5 below]. At the other end of this string there is a non-rotating pulley of mass M_1 over which there is a string carrying masses m_1 and m_2. (a) Set up the Lagrangian of the system. (b) Find the acceleration of mass M_2.

Let X_1 and X_2 be the distances of masses M_1 and M_2 respectively below the center of the fixed pulley. Let x_1 and x_2 be the distances of masses m_1 and m_2 respectively below the center of the movable pulley M_1.

Since the strings are fixed in length,

$$X_1 + X_2 = \text{constant} = a, \qquad x_1 + x_2 = \text{constant} = b$$

Then by differentiating with respect to time t,

$$\dot{X}_1 + \dot{X}_2 = 0 \quad \text{or} \quad \dot{X}_2 = -\dot{X}_1$$

and

$$\dot{x}_1 + \dot{x}_2 = 0 \quad \text{or} \quad \dot{x}_2 = -\dot{x}_1$$

Thus we have

Velocity of $M_1 = \dot{X}_1$

Velocity of $M_2 = \dot{X}_2 = -\dot{X}_1$

Velocity of $m_1 = \frac{d}{dt}(X_1 + x_1) = \dot{X}_1 + \dot{x}_1$

Velocity of $m_2 = \frac{d}{dt}(X_1 + x_2) = \dot{X}_1 + \dot{x}_2 = \dot{X}_1 - \dot{x}_1$

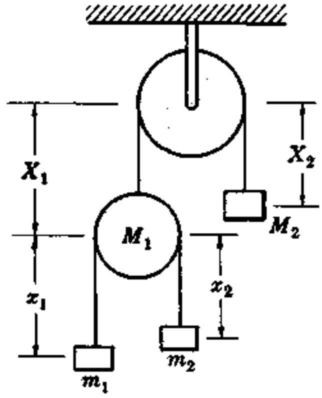

Fig. 11-5

Then the total kinetic energy of the system is

$$T = \tfrac{1}{2}M_1\dot{X}_1^2 + \tfrac{1}{2}M_2\dot{X}_1^2 + \tfrac{1}{2}m_1(\dot{X}_1 + \dot{x}_1)^2 + \tfrac{1}{2}m_2(\dot{X}_1 - \dot{x}_1)^2 \tag{1}$$

The total potential energy of the system measured from a horizontal plane through the center of the fixed pulley as reference is

$$\begin{aligned} V &= -M_1gX_1 - M_2gX_2 - m_1g(X_1 + x_1) - m_2g(X_1 + x_2) \\ &= -M_1gX_1 - M_2g(a - X_1) - m_1g(X_1 + x_1) - m_2g(X_1 + b - x_1) \end{aligned} \tag{2}$$

Then the Lagrangian is

$$\begin{aligned} L &= T - V \\ &= \tfrac{1}{2}M_1\dot{X}_1^2 + \tfrac{1}{2}M_2\dot{X}_1^2 + \tfrac{1}{2}m_1(\dot{X}_1 + \dot{x}_1)^2 + \tfrac{1}{2}m_2(\dot{X}_1 - \dot{x}_1)^2 \\ &\quad + M_1gX_1 + M_2g(a - X_1) + m_1g(X_1 + x_1) + m_2g(X_1 + b - x_1) \end{aligned} \tag{3}$$

Lagrange's equations corresponding to X_1 and x_1 are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}_1}\right) - \frac{\partial L}{\partial X_1} = 0, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0 \tag{4}$$

From (3) we have

$$\frac{\partial L}{\partial X_1} = M_1g - M_2g + m_1g + m_2g = (M_1 - M_2 + m_1 + m_2)g$$

$$\frac{\partial L}{\partial \dot{X}_1} = M_1\dot{X}_1 + M_2\dot{X}_1 + m_1(\dot{X}_1 + \dot{x}_1) + m_2(\dot{X}_1 - \dot{x}_1) = (M_1 + M_2 + m_1 + m_2)\dot{X}_1 + (m_1 - m_2)\dot{x}_1$$

$$\frac{\partial L}{\partial x_1} = m_1g - m_2g = (m_1 - m_2)g$$

$$\frac{\partial L}{\partial \dot{x}_1} = m_1(\dot{X}_1 + \dot{x}_1) - m_2(\dot{X}_1 - \dot{x}_1) = (m_1 - m_2)\dot{X}_1 + (m_1 + m_2)\dot{x}_1$$

Thus equations (4) become

$$(M_1 + M_2 + m_1 + m_2)\ddot{X}_1 + (m_1 - m_2)\ddot{x}_1 = (M_1 - M_2 + m_1 + m_2)g$$

$$(m_1 - m_2)\ddot{X}_1 + (m_1 + m_2)\ddot{x}_1 = (m_1 - m_2)g$$

Solving simultaneously, we find

$$\ddot{X}_1 = \frac{(M_1 - M_2)(m_1 + m_2) + 4m_1m_2}{(M_1 + M_2)(m_1 + m_2) + 4m_1m_2}\,g$$

$$\ddot{x}_1 = \frac{2M_2(m_1 - m_2)}{(M_1 + M_2)(m_1 + m_2) + 4m_1m_2}\,g$$

Then the downward acceleration of mass M_2 is constant and equal to

$$\ddot{X}_2 = -\ddot{X}_1 = \frac{(M_2 - M_1)(m_1 + m_2) - 4m_1m_2}{(M_1 + M_2)(m_1 + m_2) + 4m_1m_2}\,g$$

11.14. Use Lagrange's equations to set up the differential equation of the vibrating masses of Problem 8.1, page 197.

Refer to Figs. 8-7 and 8-8 of page 197. The kinetic energy of the system is

$$T = \tfrac{1}{2}m\dot{x}_1^2 + \tfrac{1}{2}m\dot{x}_2^2 \tag{1}$$

Since the stretches of springs AP, PQ and QB of Fig. 8-8 are numerically equal to x_1, $x_2 - x_1$ and x_2 respectively, the potential energy of the system is

$$V = \tfrac{1}{2}\kappa x_1^2 + \tfrac{1}{2}\kappa(x_2 - x_1)^2 + \tfrac{1}{2}\kappa x_2^2 \tag{2}$$

Thus the Lagrangian is

$$L = T - V = \tfrac{1}{2}m\dot{x}_1^2 + \tfrac{1}{2}m\dot{x}_2^2 - \tfrac{1}{2}\kappa x_1^2 - \tfrac{1}{2}\kappa(x_2 - x_1)^2 - \tfrac{1}{2}\kappa x_2^2 \tag{3}$$

Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0 \tag{4}$$

Then since

$$\frac{\partial L}{\partial x_1} = -\kappa x_1 + \kappa(x_2 - x_1) = \kappa(x_2 - 2x_1), \qquad \frac{\partial L}{\partial \dot{x}_1} = m\dot{x}_1$$

$$\frac{\partial L}{\partial x_2} = -\kappa(x_2 - x_1) - \kappa x_2 = \kappa(x_1 - 2x_2), \qquad \frac{\partial L}{\partial \dot{x}_2} = m\dot{x}_2$$

equations (4) become

$$m\ddot{x}_1 = \kappa(x_2 - 2x_1), \qquad m\ddot{x}_2 = \kappa(x_1 - 2x_2) \tag{5}$$

agreeing with those obtained in Problem 8.1, page 197.

11.15. Use Lagrange's equations to find the differential equation for a compound pendulum which oscillates in a vertical plane about a fixed horizontal axis.

Let the plane of oscillation be represented by the xy plane of Fig. 11-6, where O is its intersection with the axis of rotation and C is the center of mass.

Suppose that the mass of the pendulum is M, its moment of inertia about the axis of rotation is $I_0 = MK^2$ [K = radius of gyration], and distance $OC = h$.

If θ is the instantaneous angle which OC makes with the vertical axis through O, then the kinetic energy is $T = \tfrac{1}{2}I_0\dot{\theta}^2 = \tfrac{1}{2}MK^2\dot{\theta}^2$. The potential energy relative to a horizontal plane through O is $V = -Mgh\cos\theta$. Then the Lagrangian is

$$L = T - V = \tfrac{1}{2}MK^2\dot{\theta}^2 + Mgh\cos\theta$$

Since $\partial L/\partial\theta = -Mgh\sin\theta$ and $\partial L/\partial\dot{\theta} = MK^2\dot{\theta}$, Lagrange's equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

i.e.,

$$MK^2\ddot{\theta} + Mgh\sin\theta = 0 \quad \text{or} \quad \ddot{\theta} + \frac{gh}{K^2}\sin\theta = 0$$

Compare Problem 9.24, page 237.

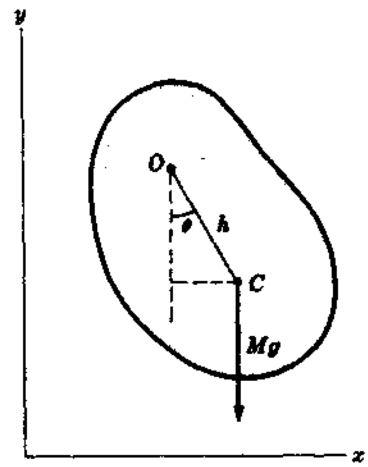

Fig. 11-6

11.16. A particle of mass m moves in a conservative force field. Find (a) the Lagrangian function, (b) the equations of motion in cylindrical coordinates (ρ, ϕ, z) [see Problem 1.147, page 32].

(a) The total kinetic energy $T = \tfrac{1}{2}m[\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2]$. The potential energy $V = V(\rho, \phi, z)$. Then the Lagrangian function is

$$L = T - V = \tfrac{1}{2}m[\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2] - V(\rho, \phi, z)$$

(b) Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\rho}}\right) - \frac{\partial L}{\partial \rho} = 0, \quad \text{i.e.} \quad \frac{d}{dt}(m\dot{\rho}) - \left(m\rho\dot{\phi}^2 - \frac{\partial V}{\partial \rho}\right) = 0 \quad \text{or} \quad m(\ddot{\rho} - \rho\dot{\phi}^2) = -\frac{\partial V}{\partial \rho}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0, \quad \text{i.e.} \quad \frac{d}{dt}(m\rho^2\dot{\phi}) + \frac{\partial V}{\partial \phi} = 0 \quad \text{or} \quad m\frac{d}{dt}(\rho^2\dot{\phi}) = -\frac{\partial V}{\partial \phi}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0, \quad \text{i.e.} \quad \frac{d}{dt}(m\dot{z}) + \frac{\partial V}{\partial z} = 0 \quad \text{or} \quad m\ddot{z} = -\frac{\partial V}{\partial z}$$

11.17. Work Problem 11.16 if the particle moves in the xy plane and if the potential depends only on the distance from the origin.

In this case V depends only on ρ and $z = 0$. Then Lagrange's equations in part (b) of Problem 11.16 become

$$m(\ddot{\rho} - \rho\dot{\phi}^2) = -\frac{\partial V}{\partial \rho}, \qquad \frac{d}{dt}(\rho^2\dot{\phi}) = 0$$

These are the equations for motion in a central force field obtained in Problem 5.3, page 122.

LAGRANGE'S EQUATIONS FOR NON-HOLONOMIC SYSTEMS

11.18. Derive Lagrange's equations (15), page 284, for non-holonomic constraints.

Assume that there are m constraint conditions of the form

$$\sum_\alpha A_\alpha\, dq_\alpha + A\, dt = 0, \quad \sum_\alpha B_\alpha\, dq_\alpha + B\, dt = 0, \quad \ldots \tag{1}$$

where $m < n$, the number of coordinates q_α.

As in Problem 11.10, page 288, we have

$$Y_\alpha \equiv \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = \sum_\nu m_\nu \ddot{\mathbf{r}}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha} \tag{2}$$

If $\delta\mathbf{r}_\nu$ are virtual displacements which satisfy the instantaneous constraints [obtained by considering that time t is a constant], then

$$\delta\mathbf{r}_\nu = \sum_\alpha \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}\,\delta q_\alpha \tag{3}$$

Now the virtual work done is

$$\delta W = \sum_\nu m_\nu \ddot{\mathbf{r}}_\nu \cdot \delta\mathbf{r}_\nu = \sum_\nu \sum_\alpha m_\nu \ddot{\mathbf{r}}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}\,\delta q_\alpha = \sum_\alpha Y_\alpha\,\delta q_\alpha \tag{4}$$

Now since the virtual work can be written in terms of the generalized forces Φ_α as

$$\delta W = \sum_\alpha \Phi_\alpha\,\delta q_\alpha \tag{5}$$

we have by subtraction of (4) and (5),

$$\sum_\alpha (Y_\alpha - \Phi_\alpha)\,\delta q_\alpha = 0 \tag{6}$$

Since the δq_α are not all independent, we cannot conclude that $Y_\alpha = \Phi_\alpha$ which would lead to Lagrange's equations as obtained in Problem 11.10.

From (1), since t is constant for instantaneous contraints, we have the m equations

$$\sum_\alpha A_\alpha\,\delta q_\alpha = 0, \quad \sum_\alpha B_\alpha\,\delta q_\alpha = 0, \quad \ldots \tag{7}$$

Multiplying these by the m Lagrange multipliers $\lambda_1, \lambda_2, \ldots$ and adding, we have

$$\sum_\alpha (\lambda_1 A_\alpha + \lambda_2 B_\alpha + \cdots)\,\delta q_\alpha = 0 \tag{8}$$

Subtraction of (6) and (8) yields

$$\sum_\alpha (Y_\alpha - \Phi_\alpha - \lambda_1 A_\alpha - \lambda_2 B_\alpha - \cdots)\,\delta q_\alpha = 0 \tag{9}$$

Now because of equations (7) we can solve for m of the quantities δq_α [say $\delta q_1, \ldots, \delta q_m$] in terms of the remaining δq_α [say $\delta q_{m+1}, \ldots, \delta q_n$]. Thus in (9) we can consider $\delta q_1, \ldots, \delta q_m$ as dependent and $\delta q_{m+1}, \ldots, \delta q_n$ as independent.

Let us arbitrarily set the coefficients of the dependent variables equal to zero, i.e.,

$$Y_\alpha - \Phi_\alpha - \lambda_1 A_\alpha - \lambda_2 B_\alpha - \cdots = 0, \qquad \alpha = 1, 2, \ldots, m \tag{10}$$

Then there will be left in the sum (9) only the independent quantities δq_α and since these are arbitrary it follows that their coefficients will be zero. Thus

$$Y_\alpha - \Phi_\alpha - \lambda_1 A_\alpha - \lambda_2 B_\alpha - \cdots = 0, \qquad \alpha = m+1, \ldots, n \tag{11}$$

Equations (2), (10) and (11) thus lead to

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = \Phi_\alpha + \lambda_1 A_\alpha + \lambda_2 B_\alpha + \cdots \qquad \alpha = 1, 2, \ldots, n \tag{12}$$

as required. These equations together with (1) lead to $n+m$ equations in $n+m$ unknowns.

11.19. Derive equations (16), page 284, for conservative non-holonomic systems.

From Problem 11.18,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = \Phi_\alpha + \lambda_1 A_\alpha + \lambda_2 B_\alpha + \cdots \tag{1}$$

Then if the forces are derivable from a potential, $\Phi_\alpha = -\partial V/\partial q_\alpha$ where V does not depend on $\dot{q}_\alpha$. Thus (1) can be written

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = \lambda_1 A_\alpha + \lambda_2 B_\alpha + \cdots \tag{2}$$

where $L = T - V$.

11.20. A particle of mass m moves under the influence of gravity on the inner surface of the paraboloid of revolution $x^2 + y^2 = az$ which is assumed frictionless [see Fig. 11-7]. Obtain the equations of motion.

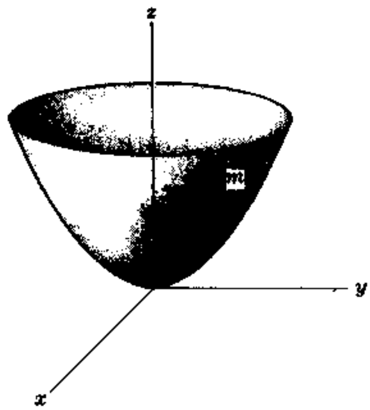

Fig. 11-7

By Problem 11.16, the Lagrangian in cylindrical coordinates is given by

$$L = \tfrac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) - mgz \tag{1}$$

Since $x^2 + y^2 = \rho^2$, the constraint condition is $\rho^2 - az = 0$ so that

$$2\rho\,\delta\rho - a\,\delta z = 0 \tag{2}$$

If we call $q_1 = \rho$, $q_2 = \phi$, $q_3 = z$ and compare (2) with the equations (7) of Problem 11.18, we see that

$$A_1 = 2\rho, \quad A_2 = 0, \quad A_3 = -a \tag{3}$$

since only one constraint is given. Lagrange's equations [see Problem 11.19] can thus be written

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = \lambda_1 A_\alpha \qquad \alpha = 1, 2, 3$$

i.e., $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\rho}}\right) - \frac{\partial L}{\partial \rho} = 2\lambda_1\rho, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = -\lambda_1 a$$

Using (1), these become

$$m(\ddot{\rho}-\rho\dot{\phi}^2) \;=\; 2\lambda_1\rho \tag{4}$$

$$m\frac{d}{dt}(\rho^2\dot{\theta}) \;=\; 0 \tag{5}$$

$$m\ddot{z} \;=\; -mg-\lambda_1 a \tag{6}$$

We also have the constraint condition

$$2\rho\dot{\rho}-a\dot{z} \;=\; 0 \tag{7}$$

The four equations (4), (5), (6) and (7) enable us to find the four unknowns ρ, ϕ, z, λ_1.

11.21. (*a*) Prove that the particle of Problem 11.20 will describe a horizontal circle in the plane $z = h$ provided that it is given an angular velocity whose magnitude is $\omega = \sqrt{2g/a}$.

(*b*) Prove that if the particle is displaced slightly from this circular path it will undergo oscillations about the path with frequency given by $(1/\pi)\sqrt{2g/a}$.

(*c*) Discuss the stability of the particle in the circular path.

(*a*) The radius of the circle obtained as the intersection of the plane $z = h$ with the paraboloid $\rho^2 = az$ is

$$\rho_0 \;=\; \sqrt{ah} \tag{1}$$

Letting $z = h$ in equation (6) of Problem 11.20, we find

$$\lambda_1 \;=\; -mg/a \tag{2}$$

Then using (1) and (2) in equation (4) of Problem 11.20 and calling $\dot{\phi} = \omega$, we find $m(-\rho_0\omega^2) = 2(-mg/a)\rho_0$ or $\omega^2 = 2g/a$ from which

$$\omega \;=\; \sqrt{2g/a} \tag{3}$$

The period and frequency of the particle in this circular path are given respectively by

$$P_1 \;=\; 2\pi\sqrt{\frac{a}{2g}} \qquad \text{and} \qquad f_1 \;=\; \frac{1}{2\pi}\sqrt{\frac{2g}{a}} \tag{4}$$

(*b*) From equation (5) of Problem 11.20, we find

$$\rho^2\dot{\phi} \;=\; \text{constant} \;=\; A \tag{5}$$

Assuming that the particle starts with angular speed ω, we find $A = ah\omega$ so that

$$\dot{\phi} \;=\; ah\omega/\rho^2 \tag{6}$$

Since the vibration takes place very nearly in the plane $z = h$, we find by letting $z = h$ in equation (6) of Problem 11.20 that

$$\lambda_1 \;=\; -mg/a \tag{7}$$

Using (6) and (7) in equation (4) of Problem 11.20, we find

$$\ddot{\rho} - a^2h^2\omega^2/\rho^3 \;=\; -2g\rho/a \tag{8}$$

Now if the path departs slightly from the circle, then ρ will depart slightly from ρ_0. Thus we are led to make the transformation

$$\rho \;=\; \rho_0 + u \tag{9}$$

in (8), where u is small compared with ρ_0. Then (8) becomes

$$\ddot{u} - \frac{a^2h^2\omega^2}{(\rho_0+u)^3} \;=\; -\frac{2g}{a}(\rho_0+u) \tag{10}$$

But to a high degree of approximation,

$$\frac{1}{(\rho_0+u)^3} \;=\; \frac{1}{\rho_0^3(1+u/\rho_0)^3} \;=\; \frac{1}{\rho_0^3}\left(1+\frac{u}{\rho_0}\right)^{-3} \;=\; \frac{1}{\rho_0^3}\left(1-\frac{3u}{\rho_0}\right)$$

by the binomial theorem, where we have neglected terms involving $u^2, u^3, \ldots$. Using the values of ρ_0 and ω given by (1) and (3) respectively, (10) becomes

$$\ddot{u} + (8g/a)u = 0 \tag{5}$$

whose solution is $u = \epsilon_1 \cos\sqrt{8g/a}\,t + \epsilon_2 \sin\sqrt{8g/a}\,t$. Thus

$$\rho = \rho_0 + u = \sqrt{ah} + \epsilon_1 \cos\sqrt{8g/a}\,t + \epsilon_2 \sin\sqrt{8g/a}\,t$$

It follows that if the particle is displaced slightly from the circular path of radius $\rho_0 = \sqrt{ah}$, it will undergo oscillations about the path with frequency

$$f_2 = \frac{1}{2\pi}\sqrt{\frac{8g}{a}} = \frac{1}{\pi}\sqrt{\frac{2g}{a}} \tag{6}$$

or period

$$P_2 = \pi\sqrt{\frac{a}{2g}} \tag{7}$$

It is interesting that the period of oscillation in the circular path given by *(4)* is twice the period of oscillation about the circular path given by *(7)*.

(c) Since the particle tends to return to the circular path when it is displaced slightly from it, the motion is one of *stability*.

11.22. Discuss the physical significance of the Lagrange multipliers $\lambda_1, \lambda_2, \ldots$ in Problem 11.18.

In case there are no constraints the equations of motion are by Problem 11.10,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = \Phi_\alpha$$

In case there are constraints the equations are by Problem 11.18,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = \Phi_\alpha + \lambda_1 A_\alpha + \lambda_2 B_\alpha + \cdots$$

It follows that the terms $\lambda_1 A_\alpha + \lambda_2 B_\alpha + \cdots$ correspond to the generalized forces associated with constraints.

Physically, the Lagrange multipliers are associated with the constraint forces acting on the system. Thus when we determine the Lagrange multipliers we are essentially taking into account the effect of the constraint forces without actually finding these forces explicitly.

LAGRANGE'S EQUATIONS WITH IMPULSIVE FORCES

11.23. Derive the equations *(20)*, page 285.

For the case where forces are finite we have by Problem 11.10,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = \Phi_\alpha \tag{1}$$

where

$$\Phi_\alpha = \sum_\nu \mathbf{F}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha} \tag{2}$$

Integrating both sides of *(1)* with respect to t from $t = 0$ to $t = \tau$,

$$\int_0^\tau \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) dt - \int_0^\tau \frac{\partial T}{\partial q_\alpha}\, dt = \int_0^\tau \Phi_\alpha\, dt \tag{3}$$

so that

$$\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_{t=\tau} - \left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_{t=0} - \int_0^\tau \frac{\partial T}{\partial q_\alpha}\, dt = \sum_\nu \left\{\left(\int_0^\tau \mathbf{F}_\nu\, dt\right) \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}\right\} \tag{4}$$

Taking the limit as $\tau \to 0$, we have

$$\lim_{\tau\to 0}\left\{\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_{t=\tau} - \left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_{t=0}\right\} - \lim_{\tau\to 0}\int_0^\tau \frac{\partial T}{\partial q_\alpha}\, dt = \sum_\nu \left\{\left(\lim_{\tau\to 0}\int_0^\tau \mathbf{F}_\nu\, dt\right) \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}\right\}$$

or
$$\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_2 - \left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_1 \;=\; \sum_\nu \mathcal{J}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha} \;=\; \mathcal{T}_\alpha$$

using $\lim_{\tau\to 0}\int_0^\tau \frac{\partial T}{\partial q_\alpha}\,dt = 0$ since $\frac{\partial T}{\partial q_\alpha}$ is finite, and $\lim_{\tau\to 0}\int_0^\tau \mathbf{F}_\nu\,dt = \mathcal{J}_\nu$.

11.24. A square $ABCD$ formed by four rods of length $2l$ and mass m hinged at their ends, rests on a horizontal frictionless table. An impulse of magnitude $\mathcal{J}$ is applied to the vertex A in the direction AD. Find the equations of motion.

After the square is struck, its shape will in general be a rhombus [Fig. 11-8].

Suppose that at any time t the angles made by sides AD (or BC) and AB (or CD) with the x axis are θ_1 and θ_2 respectively, while the coordinates of the center M are (x, y). Thus x, y, θ_1, θ_2 are the generalized coordinates.

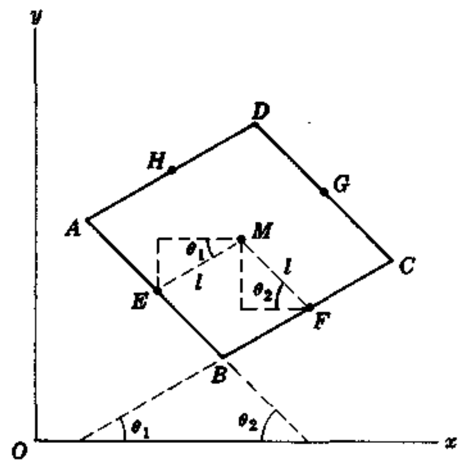

Fig. 11-8

From Fig. 11-8 we see that the position vectors of the centers E, F, G, H of the rods are given respectively by

$$\mathbf{r}_E = (x - l\cos\theta_1)\mathbf{i} + (y - l\sin\theta_1)\mathbf{j}$$
$$\mathbf{r}_F = (x + l\cos\theta_2)\mathbf{i} + (y - l\sin\theta_2)\mathbf{j}$$
$$\mathbf{r}_G = (x + l\cos\theta_1)\mathbf{i} + (y + l\sin\theta_1)\mathbf{j}$$
$$\mathbf{r}_H = (x - l\cos\theta_2)\mathbf{i} + (y + l\sin\theta_2)\mathbf{j}$$

The velocities of E, F, G and H at any time are given by

$$\mathbf{v}_E = \dot{\mathbf{r}}_E = (\dot{x} + l\sin\theta_1\,\dot{\theta}_1)\mathbf{i} + (\dot{y} - l\cos\theta_1\,\dot{\theta}_1)\mathbf{j}$$
$$\mathbf{v}_F = \dot{\mathbf{r}}_F = (\dot{x} - l\sin\theta_2\,\dot{\theta}_2)\mathbf{i} + (\dot{y} - l\cos\theta_2\,\dot{\theta}_2)\mathbf{j}$$
$$\mathbf{v}_G = \dot{\mathbf{r}}_G = (\dot{x} - l\sin\theta_1\,\dot{\theta}_1)\mathbf{i} + (\dot{y} + l\cos\theta_1\,\dot{\theta}_1)\mathbf{j}$$
$$\mathbf{v}_H = \dot{\mathbf{r}}_H = (\dot{x} + l\sin\theta_2\,\dot{\theta}_2)\mathbf{i} + (\dot{y} + l\cos\theta_2\,\dot{\theta}_2)\mathbf{j}$$

The kinetic energy of a rod such as AB is the same as the kinetic energy of a particle of mass m located at its center of mass E plus the kinetic energy of rotation about an axis through E perpendicular to the xy plane. Since the angular velocity has magnitude $\dot{\theta}_2$ and the moment of inertia of a rod of length $2l$ about its center of mass is $I_{AB} = \frac{1}{3}ml^2$, the total energy of rod AB is

$$T_{AB} = \tfrac{1}{2}m\dot{r}_E^2 + \tfrac{1}{2}I_{AB}\dot{\theta}_2^2$$

Similarly, the total kinetic energies of rods BC, CD and AD are

$$T_{BC} = \tfrac{1}{2}m\dot{\mathbf{r}}_F^2 + \tfrac{1}{2}I_{BC}\dot{\theta}_1^2, \quad T_{CD} = \tfrac{1}{2}m\dot{\mathbf{r}}_G^2 + \tfrac{1}{2}I_{CD}\dot{\theta}_2^2, \quad T_{AD} = \tfrac{1}{2}m\dot{\mathbf{r}}_H^2 + \tfrac{1}{2}I_{AD}\dot{\theta}_1^2$$

Thus the total kinetic energy is [using the fact that $I = I_{AB} = I_{BC} = I_{CD} = \frac{1}{3}ml^2$]

$$\begin{aligned} T &= T_{AB} + T_{BC} + T_{CD} + T_{AD} \\ &= \tfrac{1}{2}m(\dot{\mathbf{r}}_E^2 + \dot{\mathbf{r}}_F^2 + \dot{\mathbf{r}}_G^2 + \dot{\mathbf{r}}_H^2) + I(\dot{\theta}_1^2 + \dot{\theta}_2^2) \\ &= \tfrac{1}{2}m(4\dot{x}^2 + 4\dot{y}^2 + 2l^2\dot{\theta}_1^2 + 2l^2\dot{\theta}_2^2) + \tfrac{1}{3}ml^2(\dot{\theta}_1^2 + \dot{\theta}_2^2) \\ &= 2m(\dot{x}^2 + \dot{y}^2) + \tfrac{4}{3}ml^2(\dot{\theta}_1^2 + \dot{\theta}_2^2) \end{aligned}$$

Let us assume that initially the rhombus is a square at rest with its sides parallel to the coordinate axes and its center located at the origin. Then we have

$$x = 0,\ y = 0,\quad \theta_1 = \pi/2,\ \theta_2 = 0,\quad \dot{x} = 0,\ \dot{y} = 0,\quad \dot{\theta}_1 = 0,\ \dot{\theta}_2 = 0$$

If we use the notation $(\ )_1$ and $(\ )_2$ to denote quantities before and after the impulse is applied, we have

$$\left(\frac{\partial T}{\partial \dot{x}}\right)_1 = (4m\dot{x})_1 = 0 \qquad \left(\frac{\partial T}{\partial \dot{y}}\right)_1 = (4m\dot{y})_1 = 0$$
$$\left(\frac{\partial T}{\partial \dot{\theta}_1}\right)_1 = (\tfrac{8}{3}ml^2\dot{\theta}_1)_1 = 0 \qquad \left(\frac{\partial T}{\partial \dot{\theta}_2}\right)_1 = (\tfrac{8}{3}ml^2\dot{\theta}_2)_1 = 0$$

$$\left(\frac{\partial T}{\partial \dot{x}}\right)_2 = (4m\dot{x})_2 = 4m\dot{x} \qquad \left(\frac{\partial T}{\partial \dot{y}}\right)_2 = (4m\dot{y})_2 = 4m\dot{y}$$

$$\left(\frac{\partial T}{\partial \dot{\theta}_1}\right)_2 = (\tfrac{8}{3}ml^2\dot{\theta}_1)_2 = \tfrac{8}{3}ml^2\dot{\theta}_1 \qquad \left(\frac{\partial T}{\partial \dot{\theta}_2}\right)_2 = \tfrac{8}{3}ml^2\dot{\theta}_2 = \tfrac{8}{3}ml^2\dot{\theta}_2$$

Then

$$\left(\frac{\partial T}{\partial \dot{x}}\right)_2 - \left(\frac{\partial T}{\partial \dot{x}}\right)_1 = \mathcal{F}_x \quad \text{or} \quad 4m\dot{x} = \mathcal{F}_x \tag{1}$$

$$\left(\frac{\partial T}{\partial \dot{y}}\right)_2 - \left(\frac{\partial T}{\partial \dot{y}}\right)_1 = \mathcal{F}_y \quad \text{or} \quad 4m\dot{y} = \mathcal{F}_y \tag{2}$$

$$\left(\frac{\partial T}{\partial \dot{\theta}_1}\right)_2 - \left(\frac{\partial T}{\partial \dot{\theta}_1}\right)_1 = \mathcal{F}_{\theta_1} \quad \text{or} \quad \tfrac{8}{3}ml^2\dot{\theta}_1 = \mathcal{F}_{\theta_1} \tag{3}$$

$$\left(\frac{\partial T}{\partial \dot{\theta}_2}\right)_2 - \left(\frac{\partial T}{\partial \dot{\theta}_2}\right)_1 = \mathcal{F}_{\theta_2} \quad \text{or} \quad \tfrac{8}{3}ml^2\dot{\theta}_2 = \mathcal{F}_{\theta_2} \tag{4}$$

where for simplicity we have now removed the subscript $(\ )_2$.

To find $\mathcal{F}_x, \mathcal{F}_y, \mathcal{F}_{\theta_1}, \mathcal{F}_{\theta_2}$ we note that

$$\mathcal{F}_\alpha = \sum_\nu \mathcal{J}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha} \tag{5}$$

where $\mathcal{J}_\nu$ are the impulsive forces. We thus have

$$\mathcal{F}_x = \mathcal{J}_A \cdot \frac{\partial \mathbf{r}_A}{\partial x} + \mathcal{J}_B \cdot \frac{\partial \mathbf{r}_B}{\partial x} + \mathcal{J}_C \cdot \frac{\partial \mathbf{r}_C}{\partial x} + \mathcal{J}_D \cdot \frac{\partial \mathbf{r}_D}{\partial x} \tag{6}$$

$$\mathcal{F}_y = \mathcal{J}_A \cdot \frac{\partial \mathbf{r}_A}{\partial y} + \mathcal{J}_B \cdot \frac{\partial \mathbf{r}_B}{\partial y} + \mathcal{J}_C \cdot \frac{\partial \mathbf{r}_C}{\partial y} + \mathcal{J}_D \cdot \frac{\partial \mathbf{r}_D}{\partial y} \tag{7}$$

$$\mathcal{F}_{\theta_1} = \mathcal{J}_A \cdot \frac{\partial \mathbf{r}_A}{\partial \theta_1} + \mathcal{J}_B \cdot \frac{\partial \mathbf{r}_B}{\partial \theta_1} + \mathcal{J}_C \cdot \frac{\partial \mathbf{r}_C}{\partial \theta_1} + \mathcal{J}_D \cdot \frac{\partial \mathbf{r}_D}{\partial \theta_1} \tag{8}$$

$$\mathcal{F}_{\theta_2} = \mathcal{J}_A \cdot \frac{\partial \mathbf{r}_A}{\partial \theta_2} + \mathcal{J}_B \cdot \frac{\partial \mathbf{r}_B}{\partial \theta_2} + \mathcal{J}_C \cdot \frac{\partial \mathbf{r}_C}{\partial \theta_2} + \mathcal{J}_D \cdot \frac{\partial \mathbf{r}_D}{\partial \theta_2} \tag{9}$$

Now from Fig. 11-8 we find the position vectors of A, B, C, D given by

$$\mathbf{r}_A = (x - l\cos\theta_1 - l\cos\theta_2)\mathbf{i} + (y - l\sin\theta_1 + l\sin\theta_2)\mathbf{j}$$

$$\mathbf{r}_B = (x - l\cos\theta_1 + l\cos\theta_2)\mathbf{i} + (y - l\sin\theta_1 - l\sin\theta_2)\mathbf{j}$$

$$\mathbf{r}_C = (x + l\cos\theta_1 + l\cos\theta_2)\mathbf{i} + (y + l\sin\theta_1 - l\sin\theta_2)\mathbf{j}$$

$$\mathbf{r}_D = (x + l\cos\theta_1 - l\cos\theta_2)\mathbf{i} + (y + l\sin\theta_1 + l\sin\theta_2)\mathbf{j}$$

Since the impulsive force at A is initially in the direction of the positive y axis, we have

$$\mathcal{J}_A = \mathcal{J}\mathbf{j} \tag{10}$$

Thus equations (6)-(9) yield

$$\mathcal{F}_x = 0, \quad \mathcal{F}_y = \mathcal{J}, \quad \mathcal{F}_{\theta_1} = -\mathcal{J}l\cos\theta_1, \quad \mathcal{F}_{\theta_2} = \mathcal{J}l\cos\theta_2 \tag{11}$$

Then equations (1)-(4) become

$$4m\dot{x} = 0, \quad 4m\dot{y} = \mathcal{J}, \quad \tfrac{8}{3}ml^2\dot{\theta}_1 = -\mathcal{J}l\cos\theta_1, \quad \tfrac{8}{3}ml^2\dot{\theta}_2 = \mathcal{J}l\cos\theta_2 \tag{12}$$

11.25. Prove that the kinetic energy developed immediately after application of the impulsive forces in Problem 11.24 is $T = \mathcal{J}^2/2m$.

From equations (12) of Problem 11.24, we have

$$\dot{x} = 0, \quad \dot{y} = \frac{\mathcal{J}}{4m}, \quad \dot{\theta}_1 = -\frac{3\mathcal{J}}{8ml}\cos\theta_1, \quad \dot{\theta}_2 = \frac{3\mathcal{J}}{8ml}\cos\theta_2$$

Substituting these values in the kinetic energy obtained in Problem 11.24, we find

$$T = \frac{\mathcal{J}^2}{8m} + \frac{3\mathcal{J}^2}{8m}(\cos^2\theta_1 + \cos^2\theta_2) \qquad (1)$$

But immediately after application of the impulsive forces, $\theta_1 = \pi/2$ and $\theta_2 = 0$ approximately. Thus (1) becomes $T = \mathcal{J}^2/2m$.

MISCELLANEOUS PROBLEMS

11.26. In Fig. 11-9, AB is a straight frictionless wire fixed at point A on a vertical axis OA such that AB rotates about OA with constant angular velocity ω. A bead of mass m is constrained to move on the wire. (a) Set up the Lagrangian. (b) Write Lagrange's equations. (c) Determine the motion at any time.

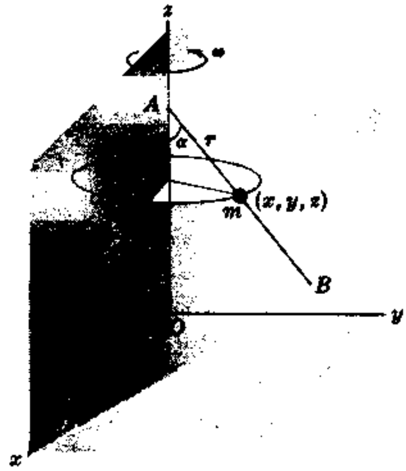

Fig. 11-9

(a) Let r be the distance of the bead from point A of the wire at time t. The rectangular coordinates of the bead are then given by

$$x = r\sin\alpha\cos\omega t$$
$$y = r\sin\alpha\sin\omega t$$
$$z = h - r\cos\alpha$$

where it is assumed that at $t = 0$ the wire is in the xz plane and that the distance from O to A is h.

The kinetic energy of the bead is

$$\begin{aligned} T &= \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\ &= \tfrac{1}{2}m\{(\dot{r}\sin\alpha\cos\omega t - \omega r\sin\alpha\sin\omega t)^2 \\ &\qquad + (\dot{r}\sin\alpha\sin\omega t + \omega r\sin\alpha\cos\omega t)^2 + (-\dot{r}\cos\alpha)^2\} \\ &= \tfrac{1}{2}m(\dot{r}^2 + \omega^2 r^2\sin^2\alpha) \end{aligned}$$

The potential energy, taking the xy plane as reference level, is $V = mgz = mg(h - r\cos\alpha)$. Then the Lagrangian is

$$L = T - V = \tfrac{1}{2}m(\dot{r}^2 + \omega^2 r^2\sin^2\alpha) - mg(h - r\cos\alpha)$$

(b) We have
$$\frac{\partial L}{\partial r} = m\omega^2 r\sin^2\alpha + mg\cos\alpha, \qquad \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

and Lagrange's equation is $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{r}}\right) - \dfrac{\partial L}{\partial r} = 0$ or

$$m\ddot{r} - (m\omega^2 r\sin^2\alpha + mg\cos\alpha) = 0$$

i.e.,
$$\ddot{r} - (\omega^2\sin^2\alpha)r = g\cos\alpha \qquad (1)$$

(c) The general solution of equation (1) with the right hand side replaced by zero is

$$c_1 e^{(\omega\sin\alpha)t} + c_2 e^{-(\omega\sin\alpha)t}$$

Since the right hand side of (1) is a constant, a particular solution is $\dfrac{-g\cos\alpha}{\omega^2\sin^2\alpha}$. Thus the general solution of (1) is

$$r = c_1 e^{(\omega\sin\alpha)t} + c_2 e^{-(\omega\sin\alpha)t} - \frac{g\cos\alpha}{\omega^2\sin^2\alpha} \qquad (2)$$

This result can also be written in terms of hyperbolic functions as

$$r = c_3\cosh(\omega\sin\alpha)t + c_4\sinh(\omega\sin\alpha)t - \frac{g\cos\alpha}{\omega^2\sin^2\alpha} \qquad (3)$$

11.27. Suppose that in Problem 11.26 the bead starts from rest at A. How long will it take to reach the end B of the wire assuming that the length of the wire is l?

Since the bead starts from rest at $t = 0$, we have $r = 0,\ \dot{r} = 0$ at $t = 0$. Then from equation (2) of Problem 11.26,

$$c_1 + c_2 = \frac{g \cos\alpha}{\omega^2 \sin^2\alpha} \qquad \text{and} \qquad c_1 - c_2 = 0$$

Thus $c_1 = c_2 = \dfrac{g\cos\alpha}{2\omega^2\sin^2\alpha}$ and (2) of Problem 11.26 becomes

$$r = \frac{g\cos\alpha}{2\omega^2\sin^2\alpha}\{e^{(\omega\sin\alpha)t} + e^{-(\omega\sin\alpha)t}\} - \frac{g\cos\alpha}{\omega^2\sin^2\alpha} \tag{1}$$

or

$$r = \frac{g\cos\alpha}{\omega^2\sin^2\alpha}\{\cosh(\omega\sin\alpha)t - 1\} \tag{2}$$

which can also be obtained from equation (3) of Problem 11.26. When $r = l$, (2) yields

$$\cosh(\omega\sin\alpha)t = 1 + \frac{l\omega^2\sin^2\alpha}{g\cos\alpha}$$

so that the required time is

$$t = \frac{1}{\omega\sin\alpha}\cosh^{-1}\left(1 + \frac{l\omega^2\sin^2\alpha}{g\cos^2\alpha}\right)$$

$$= \frac{1}{\omega\sin\alpha}\ln\left\{\left(1 + \frac{l\omega^2\sin^2\alpha}{g\cos^2\alpha}\right) + \sqrt{\left(1 + \frac{l\omega^2\sin^2\alpha}{g\cos^2\alpha}\right)^2 - 1}\right\}$$

11.28. A double pendulum [see Problem 11.1(*c*) and Fig. 11-3, page 285] vibrates in a vertical plane. (*a*) Write the Lagrangian of the system. (*b*) Obtain equations for the motion.

(*a*) The transformation equations given in Problem 11.2, page 286,

$$x_1 = l_1\cos\theta_1 \qquad y_1 = l_1\sin\theta_1$$
$$x_2 = l_1\cos\theta_1 + l_2\cos\theta_2 \qquad y_2 = l_1\sin\theta_1 + l_2\sin\theta_2$$

yield

$$\dot{x}_1 = -l_1\dot{\theta}_1\sin\theta_1 \qquad \dot{y}_1 = l_1\dot{\theta}_1\cos\theta_1$$
$$\dot{x}_2 = -l_1\dot{\theta}_1\sin\theta_1 - l_2\dot{\theta}_2\sin\theta_2 \qquad \dot{y}_2 = l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2$$

The kinetic energy of the system is

$$T = \tfrac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2)$$
$$= \tfrac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \tfrac{1}{2}m_2[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)]$$

The potential energy of the system [taking as reference level a plane at distance $l_1 + l_2$ below the point of suspension of Fig. 11-3] is

$$V = m_1g[l_1 + l_2 - l_1\cos\theta_1] + m_2g[l_1 + l_2 - (l_1\cos\theta_1 + l_2\cos\theta_2)]$$

Then the Lagrangian is

$$\begin{aligned} L &= T - V \\ &= \tfrac{1}{2}m_1l_1^2\dot{\theta}_1^2 + \tfrac{1}{2}m_2[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)] \\ &\quad - m_1g[l_1 + l_2 - l_1\cos\theta_1] - m_2g[l_1 + l_2 - (l_1\cos\theta_1 + l_2\cos\theta_2)] \end{aligned} \tag{1}$$

(*b*) The Lagrange equations associated with θ_1 and θ_2 are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right) - \frac{\partial L}{\partial\theta_1} = 0, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right) - \frac{\partial L}{\partial\theta_2} = 0 \tag{2}$$

From (1) we find

$$\begin{aligned}
\partial L/\partial \theta_1 &= -m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1 - \theta_2) - m_1 g l_1 \sin\theta_1 - m_2 g l_1 \sin\theta_1 \\
\partial L/\partial \dot\theta_1 &= m l_1^2 \dot\theta_1 + m_2 l_1^2 \dot\theta_1 + m_2 l_1 l_2 \dot\theta_2 \cos(\theta_1 - \theta_2) \\
\partial L/\partial \theta_2 &= m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1 - \theta_2) - m_2 g l_2 \sin\theta_2 \\
\partial L/\partial \dot\theta_2 &= m_2 l_2^2 \dot\theta_2 + m_2 l_1 l_2 \dot\theta_1 \cos(\theta_1 - \theta_2)
\end{aligned}$$

Thus equations (2) become

$$m_1 l_1^2 \ddot\theta_1 + m_2 l_1^2 \ddot\theta_1 + m_2 l_1 l_2 \ddot\theta_2 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot\theta_2 (\dot\theta_1 - \dot\theta_2) \sin(\theta_1 - \theta_2)$$
$$= -m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1 - \theta_2) - m_1 g l_1 \sin\theta_1 - m_2 g l_1 \sin\theta_1$$

and

$$m_2 l_2^2 \ddot\theta_2 + m_2 l_1 l_2 \ddot\theta_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot\theta_1 (\dot\theta_1 - \dot\theta_2) \sin(\theta_1 - \theta_2)$$
$$= m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1 - \theta_2) - m_2 g l_2 \sin\theta_2$$

which reduce respectively to

$$(m_1 + m_2) l_1^2 \ddot\theta_1 + m_2 l_1 l_2 \ddot\theta_2 \cos(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot\theta_2^2 \sin(\theta_1 - \theta_2) = -(m_1 + m_2) g l_1 \sin\theta_1 \quad (3)$$

and

$$m_2 l_2^2 \ddot\theta_2 + m_2 l_1 l_2 \ddot\theta_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot\theta_1^2 \sin(\theta_1 - \theta_2) = -m_2 g l_2 \sin\theta_2 \quad (4)$$

11.29. Write the equations of Problem 11.28 for the case $m_1 = m_2 = m$ and $l_1 = l_2 = l$.

Letting $m_1 = m_2$, $l_1 = l_2$ in equations (3) and (4) of Problem 11.28 and simplifying, they can be written

$$2l\,\ddot\theta_1 + l\,\ddot\theta_2 \cos(\theta_1 - \theta_2) + l\dot\theta_2^2 \sin(\theta_1 - \theta_2) = -2g \sin\theta_1 \quad (1)$$

$$l\,\ddot\theta_1 \cos(\theta_1 - \theta_2) + l\,\ddot\theta_2 - l\dot\theta_1^2 \sin(\theta_1 - \theta_2) = -g \sin\theta_2 \quad (2)$$

11.30. Obtain the equations of Problem 11.29 for the case where the oscillations are assumed to be small.

Using the approximations $\sin\theta = \theta$, $\cos\theta = 1$ and neglecting terms involving $\dot\theta^2\theta$, the equations (1) and (2) of Problem 11.29 become

$$\begin{aligned}
2l\,\ddot\theta_1 + l\,\ddot\theta_2 &= -2g\theta_1 \\
l\,\ddot\theta_1 + l\,\ddot\theta_2 &= -g\theta_2
\end{aligned}$$

11.31. Find the (a) normal frequencies and (b) normal modes corresponding to the small oscillations of the double pendulum.

(a) Let $\theta_1 = A_1 \cos\omega t$, $\theta_2 = A_2 \cos\omega t$ [or $A_1 e^{i\omega t}$, $A_2 e^{i\omega t}$] in the equations of Problem 11.30. Then they can be written

$$\left.\begin{aligned}
2(g - l\omega^2)A_1 - l\omega^2 A_2 &= 0 \\
-l\omega^2 A_1 + (g - l\omega^2)A_2 &= 0
\end{aligned}\right\} \quad (1)$$

In order for A_1 and A_2 to be different from zero, we must have the determinant of the coefficients equal to zero, i.e.,

$$\begin{vmatrix} 2(g - l\omega^2) & -l\omega^2 \\ -l\omega^2 & g - l\omega^2 \end{vmatrix} = 0$$

or $l^2\omega^4 - 4lg\omega^2 + 2g^2 = 0$. Solving, we find

$$\omega^2 = \frac{4lg \pm \sqrt{16l^2g^2 - 8l^2g^2}}{2l^2} = \frac{(2 \pm \sqrt{2})g}{l}$$

or

$$\omega_1^2 = \frac{(2+\sqrt{2})g}{l}, \quad \omega_2^2 = \frac{(2-\sqrt{2})g}{l} \quad (2)$$

Thus the normal frequencies are given by

$$f_1 = \frac{\omega_1}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{(2+\sqrt{2})g}{l}} \quad \text{and} \quad f_2 = \frac{\omega_2}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{(2-\sqrt{2})g}{l}} \tag{3}$$

(b) Substituting $\omega^2 = \omega_1^2 = (2+\sqrt{2})g/l$ in equations (1) of Part (a) yields

$$A_2 = -\sqrt{2}\,A_1 \tag{4}$$

This corresponds to the normal mode in which the bobs are moving in *opposite directions.*

Substituting $\omega^2 = \omega_2^2 = (2-\sqrt{2})g/l$ in equations (1) of Part (a) yields

$$A_2 = \sqrt{2}\,A_1 \tag{5}$$

This corresponds to the normal mode in which the bobs are moving in the *same directions.*

11.32. (a) Set up the Lagrangian for the motion of a symmetrical top [see Problem 10.25, page 268] and (b) obtain the equations of motion.

(a) The kinetic energy in terms of the Euler angles [see Problem 10.24, page 268] is

$$T = \tfrac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) = \tfrac{1}{2}I_1(\dot\phi^2\sin^2\theta + \dot\theta^2) + \tfrac{1}{2}I_3(\dot\phi\cos\theta + \dot\psi)^2 \tag{1}$$

The potential energy is $$V = mgl\cos\theta \tag{2}$$

as seen from Fig. 10-18, page 269, since distance $OC = l$ and the height of the center of mass C above the xy plane is therefore $l\cos\theta$. Thus

$$L = T - V = \tfrac{1}{2}I_1(\dot\phi^2\sin^2\theta + \dot\theta^2) + \tfrac{1}{2}I_3(\dot\phi\cos\theta + \dot\psi)^2 - mgl\cos\theta \tag{3}$$

(b)

$$\partial L/\partial\theta = I_1\dot\phi^2\sin\theta\cos\theta + I_3(\dot\phi\cos\theta + \dot\psi)(-\dot\phi\sin\theta) + mgl\sin\theta$$

$$\partial L/\partial\dot\theta = I_1\dot\theta$$

$$\partial L/\partial\phi = 0$$

$$\partial L/\partial\dot\phi = I_1\dot\phi\sin^2\theta + I_3(\dot\phi\cos\theta + \dot\psi)\cos\theta$$

$$\partial L/\partial\psi = 0$$

$$\partial L/\partial\dot\psi = I_3(\dot\phi\cos\theta + \dot\psi)$$

Then Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = 0, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\phi}\right) - \frac{\partial L}{\partial\phi} = 0, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\psi}\right) - \frac{\partial L}{\partial\psi} = 0$$

or $$I_1\ddot\theta - I_1\dot\phi^2\sin\theta\cos\theta + I_3(\dot\phi\cos\theta + \dot\psi)\dot\phi\sin\theta - mgl\sin\theta = 0 \tag{4}$$

$$\frac{d}{dt}[I_1\dot\phi\sin^2\theta + I_3(\dot\phi\cos\theta + \dot\psi)\cos\theta] = 0 \tag{5}$$

$$\frac{d}{dt}[I_3(\dot\phi\cos\theta + \dot\psi)] = 0 \tag{6}$$

11.33. Use the results of Problem 11.32 to obtain agreement with the equation of (a) Problem 10.29(b), page 270, and (b) Problem 10.27(a), page 270.

(a) From equations (5) and (6) of Problem 11.32 we obtain on integrating,

$$I_1\dot\phi\sin^2\theta + I_3(\dot\phi\cos\theta + \dot\psi)\cos\theta = \text{constant} = K \tag{1}$$

$$\dot\phi\cos\theta + \dot\psi = A \tag{2}$$

Using (2) in (1), we find $$I_1\dot\phi\sin^2\theta + I_3A\cos\theta = K$$

(b) Using (2) in equation (4) of Problem 11.32, we find

$$I_1\ddot\theta - I_1\dot\phi^2\sin\theta\cos\theta + I_3A\dot\phi\sin\theta = mgl\sin\theta$$

11.34. Derive Euler's equations of motion for a rigid body by use of Lagrange's equations.

The kinetic energy in terms of the Euler angles is [see Problem 10.24, page 268]

$$\begin{aligned} T &= \tfrac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) \\ &= \tfrac{1}{2}I_1(\dot\phi \sin\theta \sin\psi + \dot\theta\cos\psi)^2 + \tfrac{1}{2}I_2(\dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi)^2 + \tfrac{1}{2}I_3(\dot\phi\cos\theta + \dot\psi)^2 \end{aligned}$$

Then

$$\begin{aligned} \partial T/\partial\psi &= I_1(\dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi)(\dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi) \\ &\quad + I_2(\dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi)(-\dot\phi\sin\theta\sin\psi - \dot\theta\cos\psi) \\ &= I_1\omega_1\omega_2 + I_2(\omega_2)(-\omega_1) = (I_1 - I_2)\omega_1\omega_2 \end{aligned}$$

$$\partial T/\partial\dot\psi = I_3(\dot\phi\cos\theta + \dot\psi) = I_3\omega_3$$

Then by Problem 11.10, page 288, Lagrange's equation corresponding to ψ is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\psi}\right) - \frac{\partial T}{\partial\psi} = \Phi_\psi$$

or

$$I_3\dot\omega_3 + (I_2 - I_1)\omega_1\omega_2 = \Phi_\psi \tag{1}$$

This is Euler's third equation of *(22)*, page 256. The quantity Φ_ψ represents the generalized force corresponding to a rotation ψ about an axis and physically represents the component Λ_3 of the torque about this axis [see Problem 11.102].

The remaining equations

$$I_1\dot\omega_1 + (I_3 - I_2)\omega_2\omega_3 = \Lambda_1 \tag{2}$$

$$I_2\dot\omega_2 + (I_1 - I_3)\omega_3\omega_1 = \Lambda_2 \tag{3}$$

can be obtained from symmetry considerations by permutation of the indices. They are not directly obtained by using the Lagrange equations corresponding to θ and ϕ but can indirectly be deduced from them [see Problem 11.79].

11.35. A bead slides without friction on a frictionless wire in the shape of a cycloid [Fig. 11-10] with equations

$$x = a(\theta - \sin\theta), \quad y = a(1 + \cos\theta)$$

where $0 \leqq \theta \leqq 2\pi$. Find (*a*) the Lagrangian function, (*b*) the equation of motion.

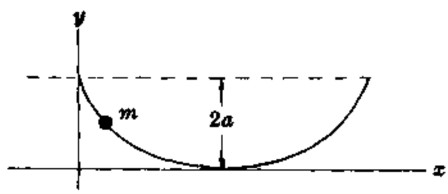

Fig. 11-10

(*a*) Kinetic energy $= T = \tfrac{1}{2}m(\dot x^2 + \dot y^2)$

$$\begin{aligned} &= \tfrac{1}{2}ma^2\{[(1-\cos\theta)\dot\theta]^2 + [-\sin\theta\,\dot\theta]^2\} \\ &= ma^2(1-\cos\theta)\dot\theta^2 \end{aligned}$$

Potential energy $= V = mgy = mga(1+\cos\theta)$

Then

Lagrangian $= L = T - V = ma^2(1-\cos\theta)\dot\theta^2 - mga(1+\cos\theta)$

(*b*) $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial\dot\theta}\right) - \dfrac{\partial L}{\partial\theta} = 0$, i.e. $\dfrac{d}{dt}[2ma^2(1-\cos\theta)\dot\theta] - [ma^2\sin\theta\,\dot\theta^2 + mga\sin\theta] = 0$

or

$$\frac{d}{dt}\{(1-\cos\theta)\dot\theta\} - \tfrac{1}{2}\sin\theta\,\dot\theta^2 - \frac{g}{2a}\sin\theta = 0$$

which can be written

$$(1-\cos\theta)\ddot\theta + \tfrac{1}{2}\sin\theta\,\dot\theta^2 - \frac{g}{2a}\sin\theta = 0$$

11.36. (*a*) Show that the equation of motion obtained in part (*b*) of Problem 11.35 can be written

$$\frac{d^2u}{dt^2} + \frac{g}{4a}u = 0 \quad \text{where } u = \cos(\theta/2)$$

and thus (b) show that the bead oscillates with period $2\pi\sqrt{4a/g}$.

(a) If $u = \cos(\theta/2)$, then

$$\frac{du}{dt} = -\tfrac{1}{2}\sin(\theta/2)\dot{\theta}, \qquad \frac{d^2u}{dt^2} = -\tfrac{1}{2}\sin(\theta/2)\ddot{\theta} - \tfrac{1}{4}\cos(\theta/2)\dot{\theta}^2$$

Thus $\dfrac{d^2u}{dt^2} + \dfrac{g}{4a}u = 0$ is the same as

$$-\tfrac{1}{2}\sin(\theta/2)\ddot{\theta} - \tfrac{1}{4}\cos(\theta/2)\dot{\theta}^2 + \frac{g}{4a}\cos(\theta/2) = 0$$

which can be written as

$$\ddot{\theta} + \tfrac{1}{2}\cot(\theta/2)\dot{\theta}^2 - \frac{g}{2a}\cot(\theta/2) = 0 \tag{1}$$

Since
$$\cot(\theta/2) = \frac{\cos(\theta/2)}{\sin(\theta/2)} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} = \frac{\sin\theta}{1-\cos\theta}$$

it follows that equation (1) is the same as that obtained in Problem 11.35(b).

(b) The solution of the equation is

$$u = \cos(\theta/2) = c_1\cos\sqrt{4a/g}\,t + c_2\sin\sqrt{4a/g}\,t$$

from which we see that $\cos(\theta/2)$ returns to its original value after a time $2\pi\sqrt{4a/g}$ which is the required period. Note that this period is the same as that of a simple pendulum with length $l = 4a$.

An application of this is the *cycloidal pendulum*. See Problem 4.86, page 112.

11.37. Obtain equations for the rolling sphere of Problem 9.42, page 244 by use of Lagrange's equations.

Refer to Fig. 9-33 in which ϕ and ψ represent generalized coordinates. Since the sphere of radius $CP = a$ rolls without slipping on the sphere of radius $OP = b$, we have

$$b\,d\phi/dt = a\,d\psi/dt \quad\text{or}\quad b\dot{\phi} = a\dot{\psi}$$

which shows that if $\phi = 0$ when $\psi = 0$, then

$$b\phi = a\psi \tag{1}$$

Thus ϕ and ψ [and therefore $d\phi$ and $d\psi$ or $\delta\phi$ and $\delta\psi$] are not independent.

The kinetic energy of the rolling sphere is

$$\begin{aligned} T &= \tfrac{1}{2}m(a+b)^2\dot{\phi}^2 + \tfrac{1}{2}I\omega^2 \\ &= \tfrac{1}{2}m(a+b)^2\dot{\phi}^2 + \tfrac{1}{2}(\tfrac{2}{5}ma^2)(\dot{\phi}+\dot{\psi})^2 \end{aligned}$$

using the fact that $I = \frac{2}{5}ma^2$ is the moment of inertia of the sphere about a horizontal axis through its center of mass.

The potential energy of the rolling sphere [taking the horizontal plane through O as reference level] is

$$V = mg(a+b)\cos\phi$$

Thus the Lagrangian is

$$L = T - V = \tfrac{1}{2}m(a+b)^2\dot{\phi}^2 + \tfrac{1}{5}ma^2(\dot{\phi}+\dot{\psi})^2 - mg(a+b)\cos\phi \tag{2}$$

We use Lagrange's equations (16), page 284, for non-holonomic systems. From (1) we have

$$b\,\delta\phi - a\,\delta\psi = 0 \tag{3}$$

so that if we call $q_1 = \phi$ and $q_2 = \psi$ and compare with equation (7) of Problem 11.18, page 293, we find

$$A_1 = b, \quad A_2 = -a \tag{4}$$

Thus equations (16), page 284, become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} = \lambda_1 b \tag{5}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\psi}}\right) - \frac{\partial L}{\partial\psi} = -\lambda_1 a \tag{6}$$

Substitution of (2) into (5) and (6) yields

$$m(a+b)^2\ddot{\phi} + \tfrac{2}{5}ma^2(\ddot{\phi}+\ddot{\psi}) - mg(a+b)\sin\phi \;=\; \lambda_1 b \tag{7}$$

$$\tfrac{2}{5}ma^2(\ddot{\phi}+\ddot{\psi}) \;=\; -\lambda_1 a \tag{8}$$

Substituting $\psi = (b/a)\phi$ [from (1)] into (7) and (8), we find

$$m(a+b)^2\ddot{\phi} + \tfrac{2}{5}ma^2(1+b/a)\ddot{\phi} - mg(a+b)\sin\phi \;=\; \lambda_1 b \tag{9}$$

$$\tfrac{2}{5}ma^2(1+b/a)\ddot{\phi} \;=\; -\lambda_1 a \tag{10}$$

Now from (10) we have $\qquad \lambda_1 \;=\; -\tfrac{2}{5}m(a+b)\ddot{\phi}$

and using this in (9) it becomes after simplifying and solving for $\ddot{\phi}$,

$$\ddot{\phi} \;=\; \frac{5g}{7(a+b)}\sin\phi$$

This is the same equation as that of (2) in Problem 9.42, page 244, with $\phi = \pi/2 - \theta$. To find the required angle at which the sphere falls off, see Problem 11.104.

11.38. (*a*) Solve the equations of motion obtained in Problem 11.24, page 296, and (*b*) interpret physically.

(*a*) From the first of equations (12) in Problem 11.24 we have

$$x \;=\; \text{constant} \;=\; 0 \tag{1}$$

since $x = 0$ at $t = 0$. Similarly, from the second of equations (12) we have

$$y \;=\; \frac{\mathcal{J}}{4m}t \tag{2}$$

since $y = 0$ at $t = 0$.

From the third of equations (12) we find on separating the variables,

$$\sec\theta_1\, d\theta_1 \;=\; -\frac{3\mathcal{J}}{8ml}\,dt$$

or on integrating, $\qquad \ln\cot\left(\frac{\pi}{4}-\frac{\theta_1}{2}\right) \;=\; -\frac{3\mathcal{J}t}{8ml} + c_1$

i.e., $\qquad \tan\left(\frac{\pi}{4}-\frac{\theta_1}{2}\right) \;=\; c_2 e^{3\mathcal{J}t/8ml}$

Thus since $\theta_1 = \pi/2$ at $t = 0$, we have $c_2 = 0$. This means that for all time we must have $\theta_1 = \pi/2$.

From the fourth of equations (12) in Problem 11.24 we have similarly,

$$\sec\theta_2\, d\theta_2 \;=\; \frac{3\mathcal{J}}{8ml}\,dt$$

or on integrating, $\qquad \ln\cot\left(\frac{\pi}{4}-\frac{\theta_2}{2}\right) \;=\; \frac{3\mathcal{J}t}{8ml} + c_3$

i.e., $\qquad \tan\left(\frac{\pi}{4}-\frac{\theta_2}{2}\right) \;=\; c_4 e^{-3\mathcal{J}t/8ml}$

Now when $t = 0$, $\theta_2 = 0$ so that $c_4 = 1$. Then

$$\tan\left(\frac{\pi}{4}-\frac{\theta_2}{2}\right) \;=\; e^{-3\mathcal{J}t/8ml} \qquad \text{or} \qquad \theta_2 \;=\; \frac{\pi}{2} - 2\tan^{-1}(e^{-3\mathcal{J}t/8ml})$$

(*b*) Equations (1) and (2) show that the center moves along the y axis with constant speed $\mathcal{J}/4m$. The rods AD and BC are always parallel to the y axis while rods AB and CD slowly rotate until finally [$t \to \infty$] the rhombus collapses, so that all four rods will be on the y axis.

Supplementary Problems

GENERALIZED COORDINATES AND TRANSFORMATION EQUATIONS

11.39. Give a set of generalized coordinates needed to completely specify the motion of each of the following: (a) a bead constrained to move on a circular wire; (b) a particle constrained to move on a sphere; (c) a compound pendulum [see page 228]; (d) an Atwood's machine [see Problem 3.22, page 76]; (e) a circular disk rolling on a horizontal plane; (f) a cone rolling on a horizontal plane.

11.40. Write transformation equations for the motion of a triple pendulum in terms of a suitable set of generalized coordinates.

11.41. A particle moves on the upper surface of a frictionless paraboloid of revolution whose equation is $x^2 + y^2 = cz$. Write transformation equations for the motion of the particle in terms of a suitable set of generalized coordinates.

11.42. Write transformation equations for the motion of a particle constrained to move on a sphere.

CLASSIFICATION OF MECHANICAL SYSTEMS

11.43. Classify each of the following according as they are (i) scleronomic or rheonomic, (ii) holonomic or non-holonomic, and (iii) conservative or non-conservative:

(a) a horizontal cylinder of radius a rolling inside a perfectly rough hollow horizontal cylinder of radius $b > a$;

(b) a cylinder rolling [and possibly sliding] down an inclined plane of angle α;

(c) a sphere rolling down another sphere which is rolling with uniform speed along a horizontal plane;

(d) a particle constrained to move along a line under the influence of a force which is inversely proportional to the square of its distance from a fixed point and a damping force proportional to the square of the instantaneous speed.

Ans. (a) scleronomic, holonomic, conservative
(b) scleronomic, non-holonomic, conservative
(c) rheonomic, non-holonomic, conservative
(d) scleronomic, holonomic, non-conservative

WORK, KINETIC ENERGY AND GENERALIZED FORCES

11.44. Prove that if the transformation equations are given by $\mathbf{r}_\nu = \mathbf{r}_\nu(q_1, q_2, \ldots, q_n)$, i.e. do not involve the time t explicitly, then the kinetic energy can be written as

$$T = \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} a_{\alpha\beta}\,\dot{q}_\alpha\,\dot{q}_\beta$$

where $a_{\alpha\beta}$ are functions of the q_α.

11.45. Discuss Problem 11.44 in case the transformation equations depend explicitly on the time t.

11.46. If $F(\lambda x, \lambda y, \lambda z) = \lambda^n F(x, y, z)$ where λ is a parameter, then F is said to be a *homogeneous function of order n*. Determine which (if any) of the following functions are homogeneous, giving the order in each case:

(a) $x^2 + y^2 + z^2 + xy + yz + xz$
(b) $3x - 2y + 4z$
(c) $xyz + 2xy + 2xz + 2yz$
(d) $(x + y + z)/x$
(e) $x^3 \tan^{-1}(y/x)$
(f) $4 \sin xy$
(g) $(x + y + z)/(x^2 + y^2 + z^2)$

Ans. (a) homogeneous of order 2, (b) homogeneous of order 1, (c) non-homogeneous, (d) homogeneous of order zero, (e) homogeneous or order 3, (f) non-homogeneous, (g) homogeneous of order -1.

11.47. If $F(x, y, z)$ is homogeneous of order n [see Problem 11.46], prove that

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z} = nF$$

This is called *Euler's theorem on homogeneous functions.*

[*Hint.* Differentiate both sides of the identity $F(\lambda x, \lambda y, \lambda z) = \lambda^n F(x, y, z)$ with respect to λ and then place $\lambda = 1$.]

11.48. Generalize the result of Problem 11.47.

11.49. Prove that if the transformation equations do not depend explicitly on time t, and T is the kinetic energy, then

$$\dot{q}_1 \frac{\partial T}{\partial \dot{q}_1} + \dot{q}_2 \frac{\partial T}{\partial \dot{q}_2} + \cdots + \dot{q}_n \frac{\partial T}{\partial \dot{q}_n} = 2T$$

Can you prove this directly without the use of Euler's theorem on homogeneous functions [Problem 11.47]?

LAGRANGE'S EQUATIONS

11.50. (a) Set up the Lagrangian for a one dimensional harmonic oscillator and (b) write Lagrange's equations. *Ans.* (a) $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}\kappa x^2$, (b) $m\ddot{x} + \kappa x = 0$

11.51. (a) Set up the Lagrangian for a particle of mass m falling freely in a uniform gravitational field and (b) write Lagrange's equations.

11.52. Work Problem 11.51 in case the gravitational force field varies inversely as the square of the distance from a fixed point O assuming that the particle moves in a straight line through O.

11.53. Use Lagrange's equations to describe the motion of a particle of mass m down a frictionless inclined plane of angle α.

11.54. Use Lagrange's equations to describe the motion of a projectile launched with speed v_0 at angle α with the horizontal.

11.55. Use Lagrange's equations to solve the problem of the (a) two-dimensional and (b) three-dimensional harmonic oscillator.

11.56. A particle of mass m is connected to a fixed point P on a horizontal plane by a string of length l. The plane rotates with constant angular speed ω about a vertical axis through a point O of the plane, where $OP = a$. (a) Set up the Lagrangian of the system. (b) Write the equations of motion of the particle.

11.57. The rectangular coordinates (x, y, z) defining the position of a particle of mass m moving in a force field having potential V are given in terms of spherical coordinates (r, θ, ϕ) by the transformation equations

$$x = r \sin\phi \cos\theta, \quad y = r \sin\phi \sin\theta, \quad z = r \cos\phi$$

Use Lagrange's equations to set up the equations of motion.

Ans.
$$m[\ddot{r} - r\dot{\phi}^2 - r\dot{\theta}^2 \cos^2\phi] = -\frac{\partial V}{\partial r}$$

$$m\left[\frac{d}{dt}(r^2\dot{\phi}) + r^2\dot{\theta}^2 \sin\phi \cos\phi\right] = -\frac{\partial V}{\partial \theta}$$

$$m\frac{d}{dt}(r^2\dot{\theta} \sin^2\phi) = -\frac{\partial V}{\partial \phi}$$

11.58. Work Problem 11.56 if the particle does not necessarily move in a straight line through O.

11.59. Work Problem 4.23, page 102, by use of Lagrange's equations.

LAGRANGE'S EQUATIONS FOR NON-HOLONOMIC SYSTEMS

11.60. (a) Work Problem 11.20, page 293, if the paraboloid is replaced by the cone $x^2 + y^2 = c^2z^2$. (b) What modification must be made to Problem 11.21, page 294, in this case?

11.61. Use the method of Lagrange's equations for non-holonomic systems to solve the problem of a particle of mass m sliding down a frictionless inclined plane of angle α.

11.62. Work Problem 3.74, page 82 by using the method of Lagrange's equations for non-holonomic systems.

LAGRANGE'S EQUATIONS WITH IMPULSIVE FORCES

11.63. A uniform rod of length l and mass M is at rest on a horizontal frictionless table. An impulse of magnitude $\mathcal{J}$ is applied to one end A of the rod and perpendicular to it. Prove that (a) the velocity given to end A is $4\mathcal{J}/M$, (b) the velocity of the center of mass is $\mathcal{J}/M$ and (c) the rod rotates about the center of mass with angular velocity of magnitude $6\mathcal{J}/Ml$.

11.64. In Fig. 11-11, AB and BC represent two uniform rods having the same length l and mass M smoothly hinged at B and at rest on a horizontal frictionless plane. An impulse is applied at C normal to BC in the direction indicated in Fig. 11-11 so that the initial velocity of point C is $\mathbf{v}_0$. Find (a) the initial velocities of points A and B and (b) the magnitudes of the initial angular velocities of AB and BC about their centers of mass.

Ans. (a) $\mathbf{v}_0/7, -2\mathbf{v}_0/7$; (b) $3v_0/7l, -9v_0/7l$

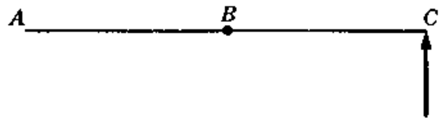

Fig. 11-11

11.65. Prove that the total kinetic energy developed by the system of Problem 11.64 after the impulse is $\frac{1}{7}Mv_0^2$.

11.66. A square of side a and mass M, formed from 4 uniform rods which are smoothly hinged at their edges, rests on a horizontal frictionless plane. An impulse is applied at a vertex in a direction of the diagonal through the vertex so that the vertex is given a velocity of magnitude v_0. Prove that the rods move about their centers of mass with angular speed $3v_0/4a$.

11.67. (a) If $\mathcal{J}$ is the magnitude of the impulse applied to the vertex in Problem 11.66, prove that the kinetic energy developed by the rods is given by $5\mathcal{J}^2/4M$. (b) What is this kinetic energy in terms of v_0? (c) Does the direction of the impulse make any difference? Explain.

11.68. In Problem 11.24, page 296, suppose that the impulse is applied at the center of one of the rods in a direction which is perpendicular to the rod. Prove that the kinetic energy developed is $\mathcal{J}^2/8m$.

MISCELLANEOUS PROBLEMS

11.69. A particle of mass m moves on the inside of a smooth hollow hemisphere of radius a having its vertex on a horizontal plane. With what horizontal speed must it be projected so that it will remain in a horizontal circle at height h above the vertex?

11.70. A particle of mass m is constrained to move inside a thin hollow frictionless tube [see Fig. 11-12] which is rotating with constant angular velocity ω in a horizontal xy plane about a fixed vertical axis through O. Using Lagrange's equations, describe the motion.

11.71. Work Problem 11.70 if the xy plane is vertical.

11.72. A particle of mass m moves in a central force field having potential $V(r)$ where r is the distance from the force center. Using spherical coordinates, (a) set up the Lagrangian and (b) determine the equations of motion. Can you deduce from these equations that the motion takes place in a plane [compare Problem 5.1, page 121]?

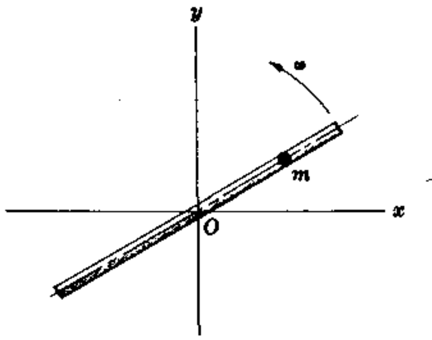

Fig. 11-12

11.73. A particle moves on a frictionless horizontal wire of radius a, acted upon by a resisting force which is proportional to the instantaneous speed. If the particle is given an initial speed v_0, find the position of the particle at any time t.

Ans. $\theta = (mv_0/\kappa)(1 - e^{-\kappa t/ma})$ where θ is the angle which a radius drawn to m makes with a fixed radius such that $\theta = 0$ at $t = 0$, and κ is the constant of proportionality.

11.74. Work Problem 11.73 if the resisting force is proportional to the square of the instantaneous speed.

Ans. $\theta = \dfrac{m}{\kappa a}\ln\left(\dfrac{m + \kappa v_0 t}{m}\right)$

11.75. A spherical pendulum is fixed at point O but is otherwise free to move in any direction. Write equations for its motion.

11.76. Work Problem 9.29, page 239, by use of Lagrange's equations.

11.77. Work Problem 11.20 if the paraboloid of revolution is replaced by the elliptic paraboloid $az = bx^2 + cy^2$ where a, b, c are positive constants.

11.78. Prove that the generalized force corresponding to the angle of rotation about an axis physically represents the component of the torque about this axis.

11.79. (a) Obtain Lagrange's equations corresponding to θ and ϕ in Problem 11.34, page 302, and show that these are not the same as equations (2) and (3) of that problem. (b) Show how to obtain equations (2) and (3) of Problem 11.34 from the Lagrange equations of (a).

11.80. Two circular disks, of radius of gyrations K_1, K_2 and masses m_1, m_2 respectively, are suspended vertically on a wire of negligible mass [see Fig. 11-13]. They are set into motion by twisting one or both of the disks in their planes and then releasing. Let θ_1 and θ_2 be the angles made with some specified direction.

(a) Prove that the kinetic energy is

$$T = \tfrac{1}{2}(m_1K_1^2\dot{\theta}_1^2 + m_2K_2^2\dot{\theta}_2^2)$$

(b) Prove that the potential energy is

$$V = \tfrac{1}{2}[\tau_1\theta_1^2 + \tau_2(\theta_2 - \theta_1)^2]$$

where τ_1 and τ_2 are torsion constants, i.e. the torques required to rotate the disks through one radian.

(c) Set up Lagrange's equations for the motion.

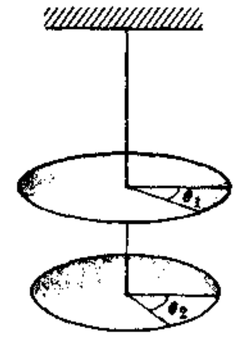

Fig. 11-13

11.81. Solve the vibrating system of Problem 11.80, finding (a) the normal frequencies and (b) the normal modes of vibration.

11.82. Generalize the results of Problem 11.80 and 11.81 to 3 or more disks.

11.83. (a) Prove that if $m_1 \neq m_2$ and $l_1 \neq l_2$ in the double pendulum of Problem 11.28, then the normal frequencies for small oscillations are given by $\omega/2\pi$ where

$$\omega^2 = \frac{(m_1+m_2)(l_1+l_2) \pm \sqrt{(m_1+m_2)[m_1(l_1-l_2)^2 + m_2(l_1+l_2)^2]}}{2l_1l_2m_1}\,g$$

(b) Discuss the normal modes corresponding to the frequencies in (a).

11.84. Examine the special case $l_1 = l_2$, $m_1 \neq m_2$ in Problem 11.83.

11.85. Use Lagrange's equations to describe the motion of a sphere of radius a rolling on the inner surface of a smooth hollow hemisphere of radius $b > a$.

11.86. A particle on the inside surface of a frictionless paraboloid of revolution $az = x^2 + y^2$ at a height H_1 above the vertex is given a horizontal velocity v_0. Find the value of v_0 in order that the particle oscillate between the planes $z = H_1$ and $z = H_2$. *Ans.* $v_0 = \sqrt{2gH_2}$

11.87. Find the period of the oscillation in Problem 11.86.

11.88. A sphere of radius a is given an initial velocity v_0 up a frictionless inclined plane of angle α in a direction which is not along the line of greatest slope. Prove that its center describes a parabola.

11.89. A bead of mass m is constrained to move on a frictionless horizontal circular wire of radius a which is rotating at constant angular speed ω about a fixed vertical axis passing through a point on the wire. Prove that relative to the wire the bead oscillates like a simple pendulum.

11.90. If a particle of mass m and charge e moves with velocity $\mathbf{v}$ in an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$, the force acting on it is given by

$$\mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

In terms of a scalar potential Φ and a vector potential $\mathbf{A}$ the fields can be expressed by the relations

$$\mathbf{E} = -\nabla\Phi - \partial\mathbf{A}/\partial t, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Prove that the Lagrangian defining the motion of such a particle is

$$L = \tfrac{1}{2}mv^2 + e(\mathbf{A} \cdot \mathbf{v}) - e\Phi$$

11.91. Work Problem 10.86, page 278, by use of Lagrange's equations.

11.92. A uniform rod of length l and mass M has its ends constrained to move on the circumference of a smooth vertical circular wire of radius $a > l/2$ which rotates about a vertical diameter with constant angular speed ω. Obtain equations for the motion of the rod.

11.93. Suppose that the potential V depends on $\dot{q}_\nu$ as well as q_ν. Prove that the quantity

$$T + V - \sum \dot{q}_\nu \frac{\partial V}{\partial \dot{q}_\nu}$$

is a constant.

11.94. Use Lagrange's equations to set up and solve the two body problem as discussed in Chapter 5 [see for example page 121.]

11.95. Find the acceleration of the 5 gm mass in the pulley system of Fig. 11-14. *Ans.* $71g/622$

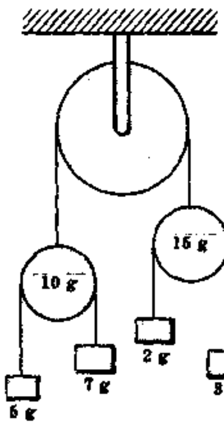

Fig. 11-14

11.96. A circular cylinder of radius a having radius of gyration K with respect to its center, moves down an inclined plane of angle α. If the coefficient of friction is μ, use Lagrange's equations to prove that the cylinder will roll without slipping if $\mu < \dfrac{K^2}{a^2 + K^2}\tan\alpha$. Discuss the cases where μ does not satisfy this inequality.

11.97. Use Lagrange's equations to solve Problem 8.27, page 213.

11.98. Describe the motion of the rods of Problem 11.64 at any time t after the impulse has been applied.

11.99. In Fig. 11-15, AB represents a frictionless horizontal plane having a small opening at O. A string of length l which passes through O has at its ends a particle P of mass m and a particle Q of equal mass which hangs freely. The particle P is given an initial velocity of magnitude v_0 at right angles to string OP when the length $OP = a$. Let r be the instantaneous distance OP while θ is the angle between OP and some fixed line through O.

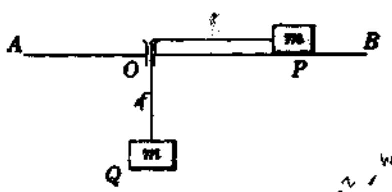

Fig. 11-15

(a) Set up the Lagrangian of the system.

(b) Write a differential equation for the motion of P in terms of r.

(c) Find the speed of P at any position.

Ans. (a) $L = \tfrac{1}{2}m[2\dot{r}^2 + r^2\dot{\theta}^2] + mg(l - r)$

(b) $\ddot{r} = a^2v_0^2/r^3 - g$

(c) $\dot{r} = \sqrt{2av_0^2 + 2g(a - r) - 2a^2v_0^2/r}$

11.100. Work Problem 11.99 if the masses of particles P and Q are m_1 and m_2 respectively.

11.101. Prove that if $v_0 = \sqrt{ag}$ the particle P of Problem 11.99 remains in stable equilibrium in the circle $r = a$ and that if it is slightly displaced from this equilibrium position it oscillates about this position with simple harmonic motion of period $2\pi\sqrt{2a/3g}$.

11.102. Prove that the quantity Φ_ψ in Problem 11.34, page 302, physically represents the component Λ_3 of the torque.

11.103. Describe the motion of the system of (*a*) Problem 11.63 and (*b*) Problem 11.66 at any time t after the impulse has been applied.

11.104. Show how to find the angle at which the sphere of Problem 11.37, page 303, falls off.

11.105. (*a*) Set up the Lagrangian for the triple pendulum of Fig. 11-16.

(*b*) Find the equations of motion.

11.106. Obtain the normal frequencies and normal modes for the triple pendulum of Problem 11.105 assuming small oscillations.

11.107. Work Problems 11.105 and 11.106 for the case where the masses and lengths are unequal.

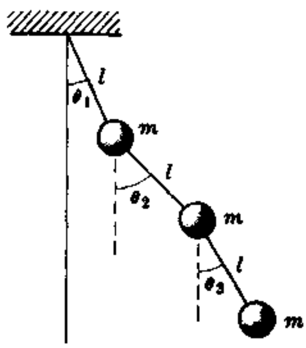

Fig. 11-16

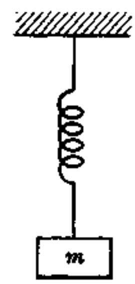

Fig. 11-17

11.108. A vertical spring [Fig. 11-17] has constant κ and mass M. If a mass m is placed on the spring and set into motion, use Lagrange's equations to prove that the system will move with simple harmonic motion of period $2\pi\sqrt{(M+3m)/3\kappa}$.

Chapter 12 HAMILTONIAN THEORY

HAMILTONIAN METHODS

In Chapter 11 we investigated a formulation of mechanics due to Lagrange. In this chapter we investigate a formulation due to Hamilton known collectively as *Hamiltonian methods* or *Hamiltonian theory.* Although such theory can be used to solve specific problems in mechanics, it develops that it is more useful in supplying fundamental postulates in such fields as quantum mechanics, statistical mechanics and celestial mechanics.

THE HAMILTONIAN

Just as the *Lagrangian function,* or briefly the *Lagrangian,* is fundamental to Chapter 11, so the *Hamiltonian function,* or briefly the *Hamiltonian,* is fundamental to this chapter.

The Hamiltonian, symbolized by H, is defined in terms of the Lagrangian L as

$$H = \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha - L \tag{1}$$

and must be expressed as a function of the generalized coordinates q_α and generalized momenta p_α. To accomplish this the generalized velocities $\dot{q}_\alpha$ must be eliminated from (1) by using Lagrange's equations [see Problem 12.3, for example]. In such case the function H can be written

$$H(p_1, \ldots, p_n, q_1, \ldots, q_n, t) \tag{2}$$

or briefly $H(p_\alpha, q_\alpha, t)$, and is also called the *Hamiltonian of the system.*

HAMILTON'S EQUATIONS

In terms of the Hamiltonian, the equations of motion of the system can be written in the symmetrical form

$$\left.\begin{aligned} \dot{p}_\alpha &= -\frac{\partial H}{\partial q_\alpha} \\ \dot{q}_\alpha &= \frac{\partial H}{\partial p_\alpha} \end{aligned}\right\} \tag{3}$$

These are called *Hamilton's canonical equations,* or briefly *Hamilton's equations.* The equations serve to indicate that the p_α and q_α play similar roles in a general formulation of mechanical principles.

THE HAMILTONIAN FOR CONSERVATIVE SYSTEMS

If a system is conservative, the Hamiltonian H can be interpreted as the total energy (kinetic and potential) of the system, i.e.,

$$H = T + V \tag{4}$$

Often this provides an easy way for setting up the Hamiltonian of a system.

IGNORABLE OR CYCLIC COORDINATES

A coordinate q_α which does not appear explicitly in the Lagrangian is called an *ignorable* or *cyclic coordinate*. In such case

$$\dot{p}_\alpha = \frac{\partial L}{\partial q_\alpha} = 0 \tag{5}$$

so that p_α is a constant, often called a *constant of the motion.*

In such case we also have $\partial H/\partial q_\alpha = 0$.

PHASE SPACE

The Hamiltonian formulation provides an obvious symmetry between the p_α and q_α which we call *momentum* and *position coordinates* respectively. It is often useful to imagine a space of $2n$ dimensions in which a *representative point* is indicated by the $2n$ coordinates

$$(p_1, \ldots, p_n, q_1, \ldots, q_n) \tag{6}$$

Such a space is called a $2n$ dimensional *phase space* or a *pq* phase space.

Whenever we know the state of a mechanical system at time t, i.e. we know all position and momentum coordinates, then this corresponds to a particular point in phase space. Conversely, a point in phase space specifies the state of the mechanical system. While the mechanical system moves in the physical 3 dimensional space, the representative point describes some path in the phase space in accordance with equations (*3*).

LIOUVILLE'S THEOREM

Let us consider a very large collection of conservative mechanical systems having the same Hamiltonian. In such case the Hamiltonian is the total energy and is constant, i.e.,

$$H(p_1, \ldots, p_n, q_1, \ldots, q_n) = \text{constant} = E \tag{7}$$

which can be represented by a surface in phase space.

Let us suppose that the total energies of all these systems lie between E_1 and E_2. Then the paths of all these systems in phase space will lie between the two surfaces $H = E_1$ and $H = E_2$ as indicated schematically in Fig. 12-1.

Since the systems have different initial conditions, they will move along different paths in the phase space. Let us imagine that the initial points are contained in region $\mathcal{R}_1$ of Fig. 12-1 and that after time t these points occupy region $\mathcal{R}_2$. For example, the representative point corresponding to one particular system moves from point A to point B. From the choice of $\mathcal{R}_1$ and $\mathcal{R}_2$ it is clear that the number of representative points in them are the same. What is not so obvious is the following theorem called *Liouville's theorem.*

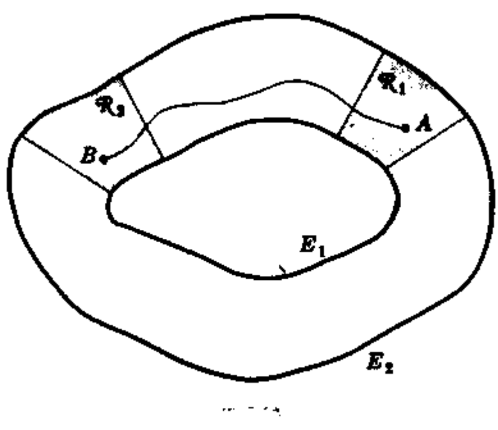

Fig. 12-1

Theorem 12.1*: *Liouville's Theorem. The $2n$ dimensional volumes of $\mathcal{R}_1$ and $\mathcal{R}_2$ are the same, or if we define the number of points per unit volume as the density then the density is constant.

We can think of the points of $\mathcal{R}_1$ as particles of an incompressible fluid which move from $\mathcal{R}_1$ to $\mathcal{R}_2$ in time t.

THE CALCULUS OF VARIATIONS

A problem which often arises in mathematics is that of finding a curve $y = Y(x)$ joining the points where $x = a$ and $x = b$ such that the integral

$$\int_a^b F(x, y, y')\,dx \tag{8}$$

where $y' = dy/dx$, is a maximum or minimum, also called an *extremum* or *extreme value.* The curve itself is often called an *extremal.* It can be shown [see Problem 12.6] that a necessary condition for (*8*) to have an extremum is

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0 \tag{9}$$

which is often called *Euler's equation.* This and similar problems are considered in a branch of mathematics called the *calculus of variations.*

HAMILTON'S PRINCIPLE

The obvious similarity of (*9*) to Lagrange's equations leads one to consider the problem of determining the extremals of

$$\int_{t_1}^{t_2} L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)\,dt \tag{10}$$

or briefly, $$\int_{t_1}^{t_2} L\,dt$$

where $L = T - V$ is the Lagrangian of a system.

We can show that a necessary condition for an extremal is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0 \tag{11}$$

which are precisely Lagrange's equations. The result led Hamilton to formulate a general variational principle known as

Hamilton's Principle. A conservative mechanical system moves from time t_1 to time t_2 in such a way that

$$\int_{t_1}^{t_2} L\,dt \tag{12}$$

sometimes called the *action integral,* has an extreme value.

Because the extreme value of (*12*) is often a minimum, the principle is sometimes referred to as *Hamilton's principle of least action.*

The fact that the integral (*12*) is an extremum is often symbolized by stating that

$$\delta \int_{t_1}^{t_2} L\,dt = 0 \tag{13}$$

where δ is the variation symbol.

CANONICAL OR CONTACT TRANSFORMATIONS

The ease in solution of many problems in mechanics often hinges on the particular generalized coordinates used. Consequently it is desirable to examine transformations from one set of position and momentum coordinates to another. For example if we call q_α and p_α the old position and momentum coordinates while Q_α and P_α are the new position and momentum coordinates, the transformation is

$$P_\alpha = P_\alpha(p_1, \ldots, p_n, q_1, \ldots, q_n, t), \quad Q_\alpha = Q_\alpha(p_1, \ldots, p_n, q_1, \ldots, q_n, t) \tag{14}$$

denoted briefly by

$$P_\alpha = P_\alpha(p_\alpha, q_\alpha, t), \quad Q_\alpha = Q_\alpha(p_\alpha, q_\alpha, t) \tag{15}$$

We restrict ourselves to transformations called *canonical* or *contact transformations* for which there exists a function $\mathcal{H}$ called the Hamiltonian in the new coordinates such that

$$\dot{P}_\alpha = -\frac{\partial \mathcal{H}}{\partial Q_\alpha}, \quad \dot{Q}_\alpha = \frac{\partial \mathcal{H}}{\partial P_\alpha} \tag{16}$$

In such case we often refer to Q_α and P_α as *canonical coordinates.*

The Lagrangians in the old and new coordinates are $L(p_\alpha, q_\alpha, t)$ and $\mathcal{L}(P_\alpha, Q_\alpha, t)$ respectively. They are related to the Hamiltonians $H(p_\alpha, q_\alpha, t)$ and $\mathcal{H}(P_\alpha, Q_\alpha, t)$ by the equations

$$H = \sum p_\alpha \dot{q}_\alpha - L, \quad \mathcal{H} = \sum P_\alpha \dot{Q}_\alpha - \mathcal{L} \tag{17}$$

where the summations extend from $\alpha = 1$ to n.

CONDITION THAT A TRANSFORMATION BE CANONICAL

The following theorem is of interest.

Theorem 12.2. The transformation

$$P_\alpha = P_\alpha(p_\alpha, q_\alpha, t), \quad Q_\alpha = Q_\alpha(p_\alpha, q_\alpha, t) \tag{18}$$

is canonical if

$$\sum p_\alpha \, dq_\alpha - \sum P_\alpha \, dQ_\alpha \tag{19}$$

is an exact differential.

GENERATING FUNCTIONS

By Hamilton's principle the canonical transformation (*14*) or (*15*) must satisfy the conditions that $\int_{t_1}^{t_2} L\,dt$ and $\int_{t_1}^{t_2} \mathcal{L}\,dt$ are both extrema, i.e. we must simultaneously have

$$\delta \int_{t_1}^{t_2} L\,dt = 0 \quad \text{and} \quad \delta \int_{t_1}^{t_2} \mathcal{L}\,dt = 0 \tag{20}$$

These will be satisfied if there is a function $\mathcal{G}$ such that

$$\frac{d\mathcal{G}}{dt} = L - \mathcal{L} \tag{21}$$

See Problem 12.11. We call $\mathcal{G}$ a *generating function.*

By assuming that $\mathcal{G}$ is a function, which we shall denote by $\mathcal{S}$, of the old position coordinates q_α and the new momentum coordinates P_α as well as the time t, i.e.,

$$\mathcal{G} = \mathcal{S}(q_\alpha P_\alpha, t) \tag{22}$$

we can prove that [see Problem 12.13]

$$p_\alpha = \frac{\partial \mathcal{S}}{\partial q_\alpha}, \quad Q_\alpha = \frac{\partial \mathcal{S}}{\partial P_\alpha}, \quad \mathcal{H} = \frac{\partial \mathcal{S}}{\partial t} + H \tag{23}$$

where

$$\dot{P}_\alpha = -\frac{\partial \mathcal{H}}{\partial Q_\alpha}, \quad \dot{Q}_\alpha = \frac{\partial \mathcal{H}}{\partial P_\alpha} \tag{24}$$

Similar results hold if the generating function is a function of other coordinates [see Problem 12.12].

THE HAMILTON-JACOBI EQUATION

If we can find a canonical transformation leading to $\mathcal{H} \equiv 0$, then we see from (24) that P_α and Q_α will be constants [i.e., P_α and Q_α will be ignorable coordinates]. Thus by means of the transformation we are able to find p_α and q_α and thereby determine the motion of the system. The procedure hinges on finding the right generating function. From the third equation of (23) we see by putting $\mathcal{H} \equiv 0$ that this generating function must satisfy the partial differential equation

$$\frac{\partial \mathcal{S}}{\partial t} + H(p_\alpha, q_\alpha, t) = 0 \tag{25}$$

or

$$\frac{\partial \mathcal{S}}{\partial t} + H\left(\frac{\partial \mathcal{S}}{\partial q_\alpha}, q_\alpha, t\right) = 0 \tag{26}$$

This is called the *Hamilton-Jacobi equation.*

SOLUTION OF THE HAMILTON-JACOBI EQUATION

To accomplish our aims we need to find a suitable solution of the Hamilton-Jacobi equation. Now since this equation contains a total of $n+1$ independent variables, i.e. $q_1, q_2, \ldots, q_n$ and t, one such solution called the *complete solution,* will involve $n+1$ constants. Omitting an arbitrary additive constant and denoting the remaining n constants by $\beta_1, \beta_2, \ldots, \beta_n$ [none of which is additive] this solution can be written

$$\mathcal{S} = \mathcal{S}(q_1, q_2, \ldots, q_n, \beta_1, \beta_2, \ldots, \beta_n, t) \tag{27}$$

When this solution is obtained we can then determine the old momentum coordinates by

$$p_\alpha = \frac{\partial \mathcal{S}}{\partial q_\alpha} \tag{28}$$

Also, if we identify the new momentum coordinates P_α with the constants β_α, then

$$Q_\alpha = \frac{\partial \mathcal{S}}{\partial \beta_\alpha} = \gamma_\alpha \tag{29}$$

where γ_α, $\alpha = 1, \ldots, n$ are constants.

Using these we can then find q_α as functions of β_α, γ_α and t, which gives the motion of the system.

CASE WHERE HAMILTONIAN IS INDEPENDENT OF TIME

In obtaining the complete solution of the Hamilton-Jacobi equation, it is often useful to assume a solution of the form

$$\mathcal{S} = S_1(q_1) + S_2(q_2) + \cdots + S_n(q_n) + F(t) \tag{30}$$

where each function on the right depends on only one variable [see Problems 12.15 and 12.16]. This method, often called the method of *separation of variables*, is especially useful when the Hamiltonian does not depend explicitly on time. We then find that $F(t) = -Et$, and if the time independent part of $\mathcal{S}$ is denoted by

$$S = S_1(q_1) + S_2(q_2) + \cdots + S_n(q_n) \tag{31}$$

the Hamilton-Jacobi equation (*26*) reduces to

$$H\left(\frac{\partial S}{\partial q_\alpha}, q_\alpha\right) = E \tag{32}$$

where E is a constant representing the total energy of the system.

The equation (*32*) can also be obtained directly by assuming a generating function S which is independent of time. In such case equations (*23*) and (*24*) are replaced by

$$p_\alpha = \frac{\partial S}{\partial q_\alpha}, \quad Q_\alpha = \frac{\partial S}{\partial P_\alpha}, \quad \mathcal{H} = H = E \tag{33}$$

where

$$\dot{P}_\alpha = -\frac{\partial \mathcal{H}}{\partial Q_\alpha}, \quad \dot{Q}_\alpha = \frac{\partial \mathcal{H}}{\partial P_\alpha} \tag{34}$$

PHASE INTEGRALS. ACTION AND ANGLE VARIABLES

Hamiltonian methods are useful in the investigation of mechanical systems which are periodic. In such case the projections of the motion of the representative point in phase space on any $p_\alpha q_\alpha$ plane will be closed curves C_α. The line integral

$$J_\alpha = \oint_{C_\alpha} p_\alpha \, dq_\alpha \tag{35}$$

is called a *phase integral* or *action variable.*

We can show [see Problems 12.17 and 12.18] that

$$S = S(q_1, \ldots, q_n, J_1, \ldots, J_n) \tag{36}$$

where

$$p_\alpha = \frac{\partial S}{\partial q_\alpha}, \quad Q_\alpha = \frac{\partial S}{\partial J_\alpha} \tag{37}$$

It is customary to denote the new coordinates Q_α by w_α so that equations (*37*) are replaced by

$$p_\alpha = \frac{\partial S}{\partial q_\alpha}, \quad w_\alpha = \frac{\partial S}{\partial J_\alpha} \tag{38}$$

Thus Hamilton's equations become [see equations (*33*) and (*34*)]

$$\dot{J}_\alpha = -\frac{\partial \mathcal{H}}{\partial w_\alpha}, \quad \dot{w}_\alpha = \frac{\partial \mathcal{H}}{\partial J_\alpha} \tag{39}$$

where $\mathcal{H} = E$ in this case depends only on the constants J_α. Then from the second equation in (*39*),

$$w_\alpha = f_\alpha t + c_\alpha \tag{40}$$

where f_α and c_α are constants. We call w_α *angle variables*. The frequencies f_α are given by

$$f_\alpha = \frac{\partial \mathcal{H}}{\partial J_\alpha} \tag{41}$$

See Problems 12.19 and 12.20.

Solved Problems

THE HAMILTONIAN AND HAMILTON'S EQUATIONS

12.1. If the Hamiltonian $H = \sum p_\alpha \dot{q}_\alpha - L$, where the summation extends from $\alpha = 1$ to n, is expressed as a function of the coordinates q_α and momenta p_α, prove *Hamilton's equations*,

$$\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}, \qquad \dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}$$

regardless of whether H (*a*) does not or (*b*) does contain the variable time t explicitly.

(*a*) *H does not contain t explicitly.*

Taking the differential of $H = \sum p_\alpha \dot{q}_\alpha - L$, we have

$$dH = \sum p_\alpha \, d\dot{q}_\alpha + \sum \dot{q}_\alpha \, dp_\alpha - \sum \frac{\partial L}{\partial q_\alpha} dq_\alpha - \sum \frac{\partial L}{\partial \dot{q}_\alpha} d\dot{q}_\alpha \tag{1}$$

Then using the fact that $p_\alpha = \partial L/\partial \dot{q}_\alpha$ and $\dot{p}_\alpha = \partial L/\partial q_\alpha$, this reduces to

$$dH = \sum \dot{q}_\alpha \, dp_\alpha - \sum \dot{p}_\alpha \, dq_\alpha \tag{2}$$

But since H is expressed as a function of p_α and q_α, we have

$$dH = \sum \frac{\partial H}{\partial p_\alpha} dp_\alpha + \sum \frac{\partial H}{\partial q_\alpha} dq_\alpha \tag{3}$$

Comparing (*2*) and (*3*) we have, as required,

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}, \qquad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$$

(*b*) *H does contain t explicitly.*

In this case equations (*1*), (*2*) and (*3*) of part (*a*) are replaced by the equations

$$dH = \sum p_\alpha \, d\dot{q}_\alpha + \sum \dot{q}_\alpha \, dp_\alpha - \sum \frac{\partial L}{\partial q_\alpha} dq_\alpha - \sum \frac{\partial L}{\partial \dot{q}_\alpha} d\dot{q}_\alpha - \frac{\partial L}{\partial t} dt \tag{4}$$

$$dH = \sum \dot{q}_\alpha \, dp_\alpha - \sum \dot{p}_\alpha \, dq_\alpha - \frac{\partial L}{\partial t} dt \tag{5}$$

$$dH = \sum \frac{\partial H}{\partial p_\alpha} dp_\alpha + \sum \frac{\partial H}{\partial q_\alpha} dq_\alpha + \frac{\partial H}{\partial t} dt \tag{6}$$

Then comparing (*5*) and (*6*), we have

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}, \qquad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}, \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

12.2. If the Hamiltonian H is independent of t explicitly, prove that it is (*a*) a constant and is (*b*) equal to the total energy of the system.

(*a*) From equation (*2*) of Problem 12.1 we have

$$\frac{dH}{dt} = \sum \dot{q}_\alpha \dot{p}_\alpha - \sum \dot{p}_\alpha \dot{q}_\alpha = 0$$

Thus H is a constant, say E.

(*b*) By Euler's theorem on homogeneous functions [see Problem 11.47, page 305],

$$\sum \dot{q}_\alpha \frac{\partial T}{\partial \dot{q}_\alpha} = 2T$$

where T is the kinetic energy. Then since $p_\alpha = \partial L/\partial \dot{q}_\alpha = \partial T/\partial \dot{q}_\alpha$ [assuming the potential V does not depend on $\dot{q}_\alpha$], we have $\sum p_\alpha \dot{q}_\alpha = 2T$. Thus as required,

$$H = \sum p_\alpha \dot{q}_\alpha - L = 2T - (T - V) = T + V = E$$

12.3. A particle moves in the xy plane under the influence of a central force depending only on its distance from the origin. (a) Set up the Hamiltonian for the system. (b) Write Hamilton's equations of motion.

(a) Assume that the particle is located by its polar coordinates (r, θ) and that the potential due to the central force is $V(r)$. Since the kinetic energy of the particle is $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$, the Lagrangian is

$$L = T - V = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \qquad (1)$$

We have $$p_r = \partial L/\partial \dot{r} = m\dot{r}, \qquad p_\theta = \partial L/\partial \dot{\theta} = mr^2\dot{\theta} \qquad (2)$$

so that $$\dot{r} = p_r/m, \qquad \dot{\theta} = p_\theta/mr^2 \qquad (3)$$

Then the Hamiltonian is given by

$$\begin{aligned} H &= \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha - L = p_r\dot{r} + p_\theta\dot{\theta} - \{\tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)\} \\ &= p_r\left(\frac{p_r}{m}\right) + p_\theta\left(\frac{p_\theta}{mr^2}\right) - \left\{\tfrac{1}{2}m\left(\frac{p_r^2}{m^2} + r^2\cdot\frac{p_\theta^2}{m^2r^4}\right) - V(r)\right\} \qquad (4) \\ &= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r) \end{aligned}$$

Note that this is the total energy expressed in terms of coordinates and momenta.

(b) Hamilton's equations are $\qquad \dot{q}_\alpha = \partial H/\partial p_\alpha, \quad \dot{p}_\alpha = -\partial H/\partial q_\alpha$

Thus $$\dot{r} = \partial H/\partial p_r = p_r/m, \qquad \dot{\theta} = \partial H/\partial p_\theta = p_\theta/mr^2 \qquad (5)$$

$$\dot{p}_r = -\partial H/\partial r = p_\theta^2/mr^3 - V'(r), \qquad \dot{p}_\theta = -\partial H/\partial \theta = 0 \qquad (6)$$

Note that the equations (5) are equivalent to the corresponding equations (3).

PHASE SPACE AND LIOUVILLE'S THEOREM

12.4. Prove Liouville's theorem for the case of one degree of freedom.

We can think of the mechanical system as being described in terms of the motion of representative points through an element of volume in phase space. In the case of a mechanical system with one degree of freedom, we have a two dimensional (p, q) phase space and the volume element reduces to an area element $dp\,dq$ [Fig. 12-2].

Let $\rho = \rho(p, q, t)$ be the density of representative points, i.e. the number of representative points per unit area as obtained by an appropriate limiting procedure. Since the speed with which representative points enter through AB is $\dot{q}$, the number of representative points which enter through AB per unit time is

$$\rho\dot{q}\,dp \qquad (1)$$

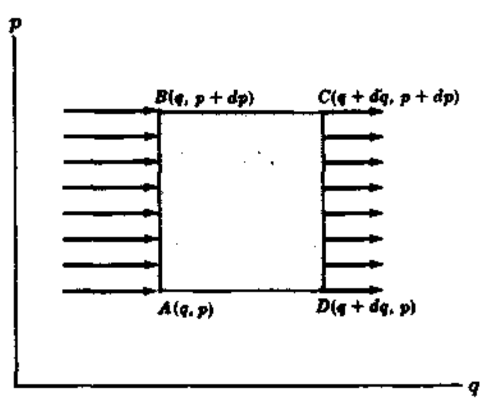

Fig. 12-2

The number of representative points which leave through CD is

$$\left\{\rho\dot{q} + \frac{\partial}{\partial q}(\rho\dot{q})\,dq\right\} dp \qquad (2)$$

Thus the number which remain in the element is (1) minus (2), or

$$-\frac{\partial}{\partial q}(\rho\dot{q})\,dp\,dq \tag{3}$$

Similarly the number of representative points which enter through AD and leave through BC are respectively

$$\rho\dot{p}\,dq \qquad \text{and} \qquad \left\{\rho\dot{p} + \frac{\partial}{\partial p}(\rho\dot{p})\,dp\right\}dq$$

Thus the number which remain in the element is

$$-\frac{\partial}{\partial p}(\rho\dot{p})\,dp\,dq \tag{4}$$

The increase in representative points is thus [adding (3) and (4)]

$$-\left\{\frac{\partial(\rho\dot{q})}{\partial q} + \frac{\partial(\rho\dot{p})}{\partial p}\right\}dp\,dq$$

Since this is equal to $\frac{\partial\rho}{\partial t}dp\,dq$, we must have

$$\frac{\partial\rho}{\partial t} + \left\{\frac{\partial(\rho\dot{q})}{\partial q} + \frac{\partial(\rho\dot{p})}{\partial p}\right\} = 0$$

or

$$\frac{\partial\rho}{\partial t} + \rho\frac{\partial\dot{q}}{\partial q} + \frac{\partial\rho}{\partial q}\dot{q} + \rho\frac{\partial\dot{p}}{\partial p} + \frac{\partial\rho}{\partial p}\dot{p} = 0 \tag{5}$$

Now by Hamilton's equations $\dot{p} = -\partial H/\partial q$, $\dot{q} = \partial H/\partial p$ so that

$$\frac{\partial\dot{p}}{\partial p} = -\frac{\partial^2 H}{\partial p\,\partial q}, \qquad \frac{\partial\dot{q}}{\partial q} = \frac{\partial^2 H}{\partial q\,\partial p}$$

Thus since we suppose that the Hamiltonian has continuous second order derivatives, it follows that $\partial\dot{p}/\partial p = -\partial\dot{q}/\partial q$. Using this in (5), it becomes

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial q}\dot{q} + \frac{\partial\rho}{\partial p}\dot{p} = 0 \tag{6}$$

But this can be written

$$d\rho/dt = 0 \tag{7}$$

which shows that the density in phase space is constant and thus proves Liouville's theorem.

12.5. Prove Liouville's theorem in the general case.

In the general case the element of volume in phase space is

$$dV = dq_1 \cdots dq_n\,dp_1 \cdots dp_n$$

In exactly the same manner as in Problem 12.4 the increase of representative points in dV is found to be

$$-\left\{\frac{\partial(\rho\dot{q}_1)}{\partial q_1} + \cdots + \frac{\partial(\rho\dot{q}_n)}{\partial q_n} + \frac{\partial(\rho\dot{p}_1)}{\partial p_1} + \cdots + \frac{\partial(\rho\dot{p}_n)}{\partial p_n}\right\}dV$$

and since this is equal to $\frac{\partial\rho}{\partial t}dV$, we must have

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho\dot{q}_1)}{\partial q_1} + \cdots + \frac{\partial(\rho\dot{q}_n)}{\partial q_n} + \frac{\partial(\rho\dot{p}_1)}{\partial p_1} + \cdots + \frac{\partial(\rho\dot{p}_n)}{\partial p_n} = 0$$

or

$$\frac{\partial\rho}{\partial t} + \sum_{\alpha=1}^{n}\frac{\partial(\rho\dot{q}_\alpha)}{\partial q_\alpha} + \sum_{\alpha=1}^{n}\frac{\partial(\rho\dot{p}_\alpha)}{\partial p_\alpha} = 0$$

This can be written as

$$\frac{\partial\rho}{\partial t} + \sum_{\alpha=1}^{n}\left(\frac{\partial\rho}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial\rho}{\partial p_\alpha}\dot{p}_\alpha\right) + \sum_{\alpha=1}^{n}\rho\left(\frac{\partial\dot{q}_\alpha}{\partial q_\alpha} + \frac{\partial\dot{p}_\alpha}{\partial p_\alpha}\right) = 0 \tag{1}$$

Now by Hamilton's equations $\dot{p}_\alpha = -\partial H/\partial q_\alpha$, $\dot{q}_\alpha = \partial H/\partial p_\alpha$ so that

$$\frac{\partial \dot{p}_\alpha}{\partial p_\alpha} = -\frac{\partial^2 H}{\partial p_\alpha\,\partial q_\alpha}, \qquad \frac{\partial \dot{q}_\alpha}{\partial p_\alpha} = \frac{\partial^2 H}{\partial q_\alpha\,\partial p_\alpha}$$

Hence $\partial\dot{p}_\alpha/\partial p_\alpha = -\partial\dot{q}_\alpha/\partial q_\alpha$ and (1) becomes

$$\frac{\partial \rho}{\partial t} + \sum_{\alpha=1}^{n}\left(\frac{\partial \rho}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial \rho}{\partial p_\alpha}\dot{p}_\alpha\right) = 0 \tag{2}$$

i.e.,

$$d\rho/dt = 0 \tag{3}$$

or ρ = constant.

Note that we have used the fact that if $\rho = \rho(q_1, \ldots, q_n, p_1, \ldots, p_n, t)$ then

$$\frac{d\rho}{dt} = \sum_{\alpha=1}^{n}\left(\frac{\partial \rho}{\partial q_\alpha}\frac{dq_\alpha}{dt} + \frac{\partial \rho}{\partial p_\alpha}\frac{dp_\alpha}{dt}\right) + \frac{\partial \rho}{\partial t} = \sum_{\alpha=1}^{n}\left(\frac{\partial \rho}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial \rho}{\partial p_\alpha}\dot{p}_\alpha\right) + \frac{\partial \rho}{\partial t}$$

CALCULUS OF VARIATIONS AND HAMILTON'S PRINCIPLE

12.6. Prove that a necessary condition for $I = \int_a^b F(x, y, y')\,dx$ to be an extremum [maximum or minimum] is $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$.

Suppose that the curve which makes I an extremum is given by

$$y = Y(x), \qquad a \leqq x \leqq b \tag{1}$$

Then

$$y = Y(x) + \epsilon\,\eta(x) = Y + \epsilon\eta \tag{2}$$

where ϵ is independent of x, is a neighboring curve through $x = a$ and $x = b$ if we choose

$$\eta(a) = \eta(b) = 0 \tag{3}$$

The value of I for this neighboring curve is

$$I(\epsilon) = \int_a^b F(x, Y + \epsilon\eta, Y' + \epsilon\eta')\,dx \tag{4}$$

This is an extremum for $\epsilon = 0$. A necessary condition that this be so is that $\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = 0$. But by differentiation under the integral sign, assuming this is valid, we find

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = \int_a^b \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\right) dx = 0$$

which can be written on integrating by parts as

$$\int_a^b \frac{\partial F}{\partial y}\eta\,dx + \left.\frac{\partial F}{\partial y'}\eta\right|_a^b - \int_a^b \eta\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)dx$$

$$= \int_a^b \eta\left\{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right\}dx = 0$$

where we have used (3). Since η is arbitrary, we must have

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad \text{or} \qquad \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

which is called *Euler's* or *Lagrange's* equation. The result is easily extended to the integral

$$\int_a^b F(x, y_1, y_1', y_2, y_2', \ldots, y_n, y_n')\,dx$$

and leads to the *Euler's* or *Lagrange's* equations

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'_\alpha}\right) - \frac{\partial F}{\partial y_\alpha} = 0 \qquad \alpha = 1, 2, \ldots, n$$

By using a Taylor series expansion we find from (4) that

$$I(\epsilon) - I(0) = \epsilon \int_a^b \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\right) dx + \text{higher order terms in } \epsilon^2, \epsilon^3, \text{etc.} \tag{5}$$

The coefficient of ϵ in (5) is often called the *variation* of the integral and is denoted by

$$\delta \int_a^b F(x, y, y')\, dx$$

The fact that $\int_a^b F(x, y, y')\, dx$ is an extremum is thus indicated by

$$\delta \int_a^b F(x, y, y')\, dx = 0$$

12.7. Discuss the relationship of Hamilton's principle with Problem 12.6.

By identifying the function $F(x, y, y')$ with the Lagrangian $L(t, q, \dot{q})$ where x, y and y' are replaced by $t, q, \dot{q}$ respectively, we see that a necessary condition for the action integral

$$\int_{t_1}^{t_2} L\, dt \tag{1}$$

to be an extremum [maximum or minimum] is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{2}$$

Since we have already seen that (2) describes the motion of a particle, it follows that such motion can also be achieved by requiring that (1) be an extremum, which is Hamilton's principle.

For systems involving n degrees of freedom we consider the integral (1) where

$$L = L(t, q_1, \dot{q}_1, q_2, \dot{q}_2, \ldots, q_n, \dot{q}_n)$$

which lead to the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0 \qquad \alpha = 1, 2, \ldots, n$$

12.8. A particle slides from rest at one point on a frictionless wire in a vertical plane to another point under the influence of gravity. Find the total time taken.

Let the shape of the wire be indicated by curve C in Fig. 12-3 and suppose that the starting and finishing points are taken to be the origin and the point $A(x_0, y_0)$ respectively.

Let $P(x, y)$ denote any position of the particle which we assume has mass m. From the principle of conservation of energy, if we choose the horizontal line through A as reference level, we have

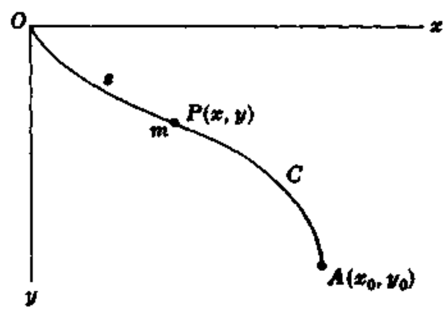

Fig. 12-3

Potential energy at O + kinetic energy at O = potential energy at P + kinetic energy at P

or
$$mgy_0 + 0 = mg(y_0 - y) + \tfrac{1}{2}m(ds/dt)^2$$

where ds/dt is the instantaneous speed of the particle at time t. Then

$$ds/dt = \pm\sqrt{2gy} \tag{1}$$

If we measure the arc length s from the origin, then s increases as the particle moves. Thus ds/dt is positive, so that $ds/dt = \sqrt{2gy}$ or $dt = ds/\sqrt{2gy}$.

The total time taken to go from $y = 0$ to $y = y_0$ is

$$\tau = \int_0^\tau dt = \int_{y=0}^{y_0} \frac{ds}{\sqrt{2gy}}$$

But $(ds)^2 = (dx)^2 + (dy)^2$ or $ds = \sqrt{1+y'^2}\,dx$. Thus the required time is

$$\tau = \frac{1}{\sqrt{2g}} \int_{y=0}^{y_0} \frac{\sqrt{1+y'^2}}{\sqrt{y}}\,dx \tag{2}$$

12.9. If the particle of Problem 12.8 is to travel from point O to point A in the least possible time, show that the differential equation of the curve C defining the shape of the wire is $1 + y'^2 + 2yy'' = 0$.

A necessary condition for the time τ given by equation (2) of Problem 12.8 to be a minimum is that

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0 \tag{1}$$

where

$$F = (1+y'^2)^{1/2}\,y^{-1/2} \tag{2}$$

Now

$$\partial F/\partial y' = (1+y'^2)^{-1/2}\,y'\,y^{-1/2}, \qquad \partial F/\partial y = -\tfrac{1}{2}(1+y'^2)^{1/2}\,y^{-3/2}$$

Substituting these in (1), performing the indicated differentiation with respect to x and simplifying, we obtain the required differential equation.

The problem of finding the shape of the wire is often called the *brachistochrone problem.*

12.10. (a) Solve the differential equation in Problem 12.9 and thus (b) show that the required curve is a *cycloid*.

(a) Since x is missing in the differential equation, let $y' = u$ so that

$$y'' = \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = \frac{du}{dy}y' = u\frac{du}{dy}$$

Then the differential equation becomes

$$1 + u^2 + 2yu\frac{du}{dy} = 0 \qquad \text{or} \qquad \frac{2u\,du}{1+u^2} + \frac{dy}{y} = 0$$

Integration yields

$$\ln(1+u^2) + \ln y = \ln b \qquad \text{or} \qquad (1+u^2)y = b$$

where b is a constant. Thus

$$u = y' = \frac{dy}{dx} = \sqrt{\frac{b-y}{y}}$$

since the slope must be positive. Separating the variables and integrating, we find

$$x = \int \sqrt{\frac{y}{b-y}}\,dy + c$$

Letting $y = b\sin^2\theta$, this can be written

$$\begin{aligned} x &= \int \sqrt{\frac{b\sin^2\theta}{b\cos^2\theta}} \cdot 2b\sin\theta\cos\theta\,d\theta + c \\ &= 2b\int \sin^2\theta\,d\theta + c = b\int (1-\cos 2\theta)\,d\theta + c = \tfrac{1}{2}b(2\theta - \sin 2\theta) + c \end{aligned}$$

Thus the parametric equations of the required curve are

$$x = \tfrac{1}{2}b(2\theta - \sin 2\theta) + c, \qquad y = b\sin^2\theta = \tfrac{1}{2}b(1-\cos 2\theta)$$

Since the curve must pass through the point $x = 0, y = 0$, we have $c = 0$. Then letting

$$\phi = 2\theta, \qquad a = \tfrac{1}{2}b \tag{1}$$

the required parametric equations are

$$x = a(\phi - \sin\phi), \qquad y = a(1-\cos\phi) \tag{2}$$

(b) The equations (2) are parametric equations of a *cycloid* [see Fig. 12-4]. The constant a must be determined so that the curve passes through point A. The cycloid is the path taken by a fixed point on a circle as it rolls along a given line [see Problem 12.89].

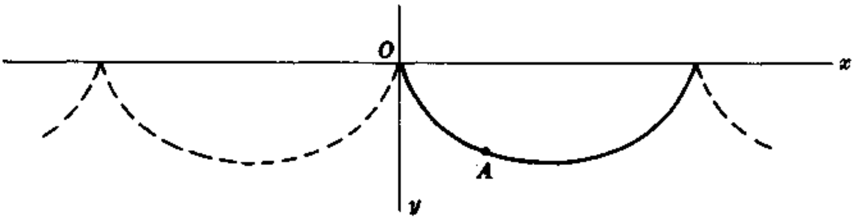

Fig. 12-4

CANONICAL TRANSFORMATIONS AND GENERATING FUNCTIONS

12.11. Prove that a transformation is canonical if there exists a function $\mathcal{G}$ such that $d\mathcal{G}/dt = L - \mathcal{L}$.

The integrals $\int_{t_1}^{t_2} L\,dt$ and $\int_{t_1}^{t_2} \mathcal{L}\,dt$ must simultaneously be extrema so that their variations are zero, i.e.,

$$\delta \int_{t_1}^{t_2} L\,dt = 0 \quad \text{and} \quad \delta \int_{t_1}^{t_2} \mathcal{L}\,dt = 0$$

Thus by subtraction,
$$\delta \int_{t_1}^{t_2} (L - \mathcal{L})\,dt = 0$$

This can be accomplished if there exists a function $\mathcal{G}$ such that
$$L - \mathcal{L} = d\mathcal{G}/dt$$

since in such case
$$\delta \int_{t_1}^{t_2} \frac{d\mathcal{G}}{dt}dt = \delta\{\mathcal{G}(t_2) - \mathcal{G}(t_1)\} = 0$$

The function $\mathcal{G}$ is called a *generating function.*

12.12. Suppose that the generating function is a function $\mathcal{T}$ of the old and new position coordinates q_α and Q_α respectively as well as the time t, i.e. $\mathcal{T} = \mathcal{T}(q_\alpha, Q_\alpha, t)$. Prove that

$$p_\alpha = \frac{\partial \mathcal{T}}{\partial q_\alpha}, \quad P_\alpha = -\frac{\partial \mathcal{T}}{\partial Q_\alpha}, \quad \mathcal{H} = \frac{\partial \mathcal{T}}{\partial t} + H \quad \text{where} \quad \dot{P}_\alpha = -\frac{\partial \mathcal{H}}{\partial Q_\alpha}, \quad \dot{Q}_\alpha = \frac{\partial \mathcal{H}}{\partial P_\alpha}$$

By Problem 12.11,
$$\begin{aligned} \frac{d\mathcal{T}}{dt} &= L - \mathcal{L} = \sum p_\alpha \dot{q}_\alpha - H - \left\{\sum P_\alpha \dot{Q}_\alpha - \mathcal{H}\right\} \\ &= \sum p_\alpha \dot{q}_\alpha - \sum P_\alpha \dot{Q}_\alpha + \mathcal{H} - H \end{aligned}$$

or
$$d\mathcal{T} = \sum p_\alpha\, dq_\alpha - \sum P_\alpha\, dQ_\alpha + (\mathcal{H} - H)\,dt \qquad (1)$$

But if $\mathcal{T} = \mathcal{T}(q_\alpha, Q_\alpha, t)$, then
$$d\mathcal{T} = \sum \frac{\partial \mathcal{T}}{\partial q_\alpha} dq_\alpha + \sum \frac{\partial \mathcal{T}}{\partial Q_\alpha} dQ_\alpha + \frac{\partial \mathcal{T}}{\partial t} dt \qquad (2)$$

Comparing (1) and (2), we have as required
$$p_\alpha = \frac{\partial \mathcal{T}}{\partial q_\alpha}, \quad P_\alpha = -\frac{\partial \mathcal{T}}{\partial Q_\alpha}, \quad \mathcal{H} - H = \frac{\partial \mathcal{T}}{\partial t}$$

The equations
$$\dot{P}_\alpha = -\frac{\partial \mathcal{H}}{\partial Q_\alpha}, \quad \dot{Q}_\alpha = \frac{\partial \mathcal{H}}{\partial P_\alpha}$$

follow from the fact that $\mathcal{H}$ is the Hamiltonian in the coordinates P_α, Q_α so that Hamilton's equations hold as in Problem 12.1.

12.13. Let $\mathcal{S}$ be a generating function dependent only on q_α, P_α, t. Prove that

$$p_\alpha = \frac{\partial \mathcal{S}}{\partial q_\alpha}, \quad Q_\alpha = \frac{\partial \mathcal{S}}{\partial P_\alpha}, \quad \mathcal{H} = \frac{\partial \mathcal{S}}{\partial t} + H \qquad \text{where} \qquad \dot{P}_\alpha = -\frac{\partial \mathcal{H}}{\partial Q_\alpha}, \quad \dot{Q}_\alpha = \frac{\partial \mathcal{H}}{\partial P_\alpha}$$

From Problem 12.12, equation (1), we have

$$\begin{aligned} d\mathcal{T} &= \sum p_\alpha \, dq_\alpha - \sum P_\alpha \, dQ_\alpha + (\mathcal{H} - H)\, dt \\ &= \sum p_\alpha \, dq_\alpha - d\left\{\sum P_\alpha Q_\alpha\right\} + \sum Q_\alpha \, dP_\alpha + (\mathcal{H} - H)\, dt \end{aligned}$$

or

$$d\left(\mathcal{T} + \sum P_\alpha Q_\alpha\right) = \sum p_\alpha \, dq_\alpha + \sum Q_\alpha \, dP_\alpha + (\mathcal{H} - H)\, dt \tag{1}$$

i.e.,

$$d\mathcal{S} = \sum p_\alpha \, dq_\alpha + \sum Q_\alpha \, dP_\alpha + (\mathcal{H} - H)\, dt \tag{2}$$

where

$$\mathcal{S} = \mathcal{T} + \sum P_\alpha Q_\alpha \tag{3}$$

But since $\mathcal{S}$ is a function of q_α, P_α, t,

$$d\mathcal{S} = \sum \frac{\partial \mathcal{S}}{\partial q_\alpha} dq_\alpha + \sum \frac{\partial \mathcal{S}}{\partial P_\alpha} dP_\alpha + \frac{\partial \mathcal{S}}{\partial t} dt \tag{4}$$

Comparing (2) and (4),

$$p_\alpha = \frac{\partial \mathcal{S}}{\partial q_\alpha}, \quad Q_\alpha = \frac{\partial \mathcal{S}}{\partial P_\alpha}, \quad \mathcal{H} = \frac{\partial \mathcal{S}}{\partial t} + H$$

The results

$$\dot{P}_\alpha = -\frac{\partial \mathcal{H}}{\partial Q_\alpha}, \quad \dot{Q}_\alpha = \frac{\partial \mathcal{H}}{\partial P_\alpha}$$

follow as in Problem 12.12, since $\mathcal{H}$ is the Hamiltonian.

12.14. Prove that the transformation $P = \frac{1}{2}(p^2 + q^2)$, $Q = \tan^{-1}(q/p)$ is canonical.

Method 1.

Let the Hamiltonians in the coordinates p, q and P, Q be respectively $H(p, q)$ and $\mathcal{H}(P, Q)$ so that $H(p, q) = \mathcal{H}(P, Q)$. Since p, q are canonical coordinates,

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p} \tag{1}$$

But

$$\dot{p} = \frac{\partial p}{\partial P}\dot{P} + \frac{\partial p}{\partial Q}\dot{Q}, \quad \dot{q} = \frac{\partial q}{\partial P}\dot{P} + \frac{\partial q}{\partial Q}\dot{Q} \tag{2}$$

$$\frac{\partial H}{\partial q} = \frac{\partial \mathcal{H}}{\partial P}\frac{\partial P}{\partial q} + \frac{\partial \mathcal{H}}{\partial Q}\frac{\partial Q}{\partial q}, \quad \frac{\partial H}{\partial p} = \frac{\partial \mathcal{H}}{\partial P}\frac{\partial P}{\partial p} + \frac{\partial \mathcal{H}}{\partial Q}\frac{\partial Q}{\partial p} \tag{3}$$

From the given transformation equations we have

$$\frac{\partial P}{\partial p} = p, \quad \frac{\partial P}{\partial q} = q, \quad \frac{\partial Q}{\partial p} = \frac{-q}{p^2 + q^2}, \quad \frac{\partial Q}{\partial q} = \frac{p}{p^2 + q^2}$$

Also, differentiating the transformation equations with respect to P and Q respectively, we find

$$1 = p\frac{\partial p}{\partial P} + q\frac{\partial q}{\partial P}, \quad 0 = \left(p\frac{\partial q}{\partial P} - q\frac{\partial p}{\partial P}\right)\Big/(p^2 + q^2)$$

$$0 = p\frac{\partial p}{\partial Q} + q\frac{\partial q}{\partial Q}, \quad 1 = \left(p\frac{\partial q}{\partial Q} - q\frac{\partial p}{\partial Q}\right)\Big/(p^2 + q^2)$$

Solving simultaneously, we find

$$\frac{\partial p}{\partial P} = \frac{p}{p^2 + q^2}, \quad \frac{\partial q}{\partial P} = \frac{q}{p^2 + q^2}, \quad \frac{\partial p}{\partial Q} = -q, \quad \frac{\partial q}{\partial Q} = p \tag{4}$$

Then equations (1) and (2) become

$$\dot{p} = \frac{p}{p^2+q^2}\dot{P} - q\dot{Q}, \qquad \dot{q} = \frac{q}{p^2+q^2}\dot{P} + p\dot{Q} \tag{5}$$

$$\frac{\partial H^*}{\partial q} = q\frac{\partial \mathcal{H}}{\partial P} + \frac{p}{p^2+q^2}\frac{\partial \mathcal{H}}{\partial Q}, \qquad \frac{\partial H}{\partial p} = p\frac{\partial \mathcal{H}}{\partial P} - \frac{q}{p^2+q^2}\frac{\partial \mathcal{H}}{\partial Q} \tag{6}$$

Thus from equations (1), (5) and (6) we have

$$\frac{p}{p^2+q^2}\dot{P} - q\dot{Q} = -q\frac{\partial \mathcal{H}}{\partial P} - \frac{p}{p^2+q^2}\frac{\partial \mathcal{H}}{\partial Q}$$

$$\frac{q}{p^2+q^2}\dot{P} + p\dot{Q} = p\frac{\partial \mathcal{H}}{\partial P} - \frac{q}{p^2+q^2}\frac{\partial \mathcal{H}}{\partial Q}$$

Solving these simultaneously we find

$$\dot{P} = -\frac{\partial \mathcal{H}}{\partial Q}, \qquad \dot{Q} = \frac{\partial \mathcal{H}}{\partial P} \tag{7}$$

which show that P and Q are canonical and that the transformation is therefore canonical.

Method 2.

By Theorem 12.2, page 314, the transformation is canonical if

$$\sum p_\alpha\, dq_\alpha - \sum P_\alpha\, dQ_\alpha \tag{8}$$

is an exact differential. In this case (8) becomes

$$p\,dq - P\,dQ = p\,dq - \tfrac{1}{2}(p^2+q^2)\left(\frac{p\,dq - q\,dp}{p^2+q^2}\right)$$

$$= \tfrac{1}{2}(p\,dq + q\,dp) = d(\tfrac{1}{2}pq)$$

an exact differential. Thus the transformation is canonical.

THE HAMILTON-JACOBI EQUATION

12.15. (a) Write the Hamiltonian for the one dimensional harmonic oscillator of mass m (b) Write the corresponding Hamilton-Jacobi equation. (c) Use the Hamilton-Jacob method to obtain the motion of the oscillator.

(a) **Method 1.**

Let q be the position coordinate of the harmonic oscillator, so that $\dot{q}$ is its velocity. Sinc the kinetic energy is $T = \frac{1}{2}m\dot{q}^2$ and the potential energy is $V = \frac{1}{2}\kappa q^2$, the Lagrangian is

$$L = T - V = \tfrac{1}{2}m\dot{q}^2 - \tfrac{1}{2}\kappa q^2 \tag{1}$$

The momentum is $$p = \partial L/\partial\dot{q} = m\dot{q} \tag{2}$$

so that $$\dot{q} = p/m \tag{3}$$

Then the Hamiltonian is

$$H = \sum p_\alpha \dot{q}_\alpha - L = p\dot{q} - (\tfrac{1}{2}m\dot{q}^2 - \tfrac{1}{2}\kappa q^2)$$
$$= \tfrac{1}{2}p^2/m + \tfrac{1}{2}\kappa q^2 \tag{4}$$

Method 2.

By Problem 12.2, since the Hamiltonian is the same as the total energy for conservativ systems,

$$H = \tfrac{1}{2}m\dot{q}^2 + \tfrac{1}{2}\kappa q^2 = \tfrac{1}{2}m(p/m)^2 + \tfrac{1}{2}\kappa q^2 = \tfrac{1}{2}p^2/m + \tfrac{1}{2}\kappa q^2$$

(*b*) Using $p = \partial \mathcal{S}/\partial q$ and the Hamiltonian of part (*a*), the Hamilton-Jacobi equation is [see equation (*26*), page 315]

$$\frac{\partial \mathcal{S}}{\partial t} + \frac{1}{2m}\left(\frac{\partial \mathcal{S}}{\partial q}\right)^2 + \tfrac{1}{2}\kappa q^2 = 0 \tag{5}$$

(*c*) Assume a solution to (*5*) of the form

$$\mathcal{S} = S_1(q) + S_2(t) \tag{6}$$

Then (*5*) becomes

$$\frac{1}{2m}\left(\frac{dS_1}{dq}\right)^2 + \tfrac{1}{2}\kappa q^2 = -\frac{dS_2}{dt} \tag{7}$$

Setting each side equal to the constant β, we find

$$\frac{1}{2m}\left(\frac{dS_1}{dq}\right)^2 + \tfrac{1}{2}\kappa q^2 = \beta, \qquad \frac{dS_2}{dt} = -\beta$$

whose solutions, omitting constants of integration, are

$$S_1 = \int \sqrt{2m(\beta - \tfrac{1}{2}\kappa q^2)}\, dq, \qquad S_2 = -\beta t \tag{8}$$

so that (*6*) becomes

$$\mathcal{S} = \int \sqrt{2m(\beta - \tfrac{1}{2}\kappa q^2)}\, dq - \beta t \tag{9}$$

Let us identify β with the new momentum coordinate P. Then we have for the new position coordinate,

$$Q = \frac{\partial \mathcal{S}}{\partial \beta} = \frac{\partial}{\partial \beta}\left\{\int \sqrt{2m(\beta - \tfrac{1}{2}\kappa q^2)}\, dq - \beta t\right\}$$

$$= \frac{\sqrt{2m}}{2}\int \frac{dq}{\sqrt{\beta - \tfrac{1}{2}\kappa q^2}} - t$$

But since the new coordinate Q is a constant γ,

$$\frac{\sqrt{2m}}{2}\int \frac{dq}{\sqrt{\beta - \tfrac{1}{2}\kappa q^2}} - t = \gamma$$

or on integrating,

$$\sqrt{m/\kappa}\,\sin^{-1}(q\sqrt{\kappa/2\beta}) = t + \gamma$$

Then solving for q,

$$q = \sqrt{2\beta/\kappa}\,\sin\sqrt{\kappa/m}\,(t + \gamma) \tag{10}$$

which is the required solution. The constants β and γ can be found from the initial conditions.

It is of interest to note that the quantity β is physically equal to the total energy E of the system [see Problem 12.92(*a*)]. The result (*9*) with $\beta = E$ illustrates equation (*31*) on page 316.

12.16. Use Hamilton-Jacobi methods to solve Kepler's problem for a particle in an inverse square central force field.

The Hamiltonian is

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{K}{r} \tag{1}$$

Then since $p_r = \partial \mathcal{S}/\partial r$, $p_\theta = \partial \mathcal{S}/\partial \theta$, the Hamilton-Jacobi equation is

$$\frac{\partial \mathcal{S}}{\partial t} + \frac{1}{2m}\left\{\left(\frac{\partial \mathcal{S}}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \mathcal{S}}{\partial \theta}\right)^2\right\} - \frac{K}{r} = 0 \tag{2}$$

Let

$$\mathcal{S} = S_1(r) + S_2(\theta) + S_3(t) \tag{3}$$

Then (2) becomes $$\frac{1}{2m}\left\{\left(\frac{dS_1}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{dS_2}{d\theta}\right)^2\right\} - \frac{K}{r} = -\frac{dS_3}{dt}$$

Setting both sides equal to the constant β_3, we find

$$dS_3/dt = -\beta_3 \tag{4}$$

$$\frac{1}{2m}\left\{\left(\frac{dS_1}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{dS_2}{d\theta}\right)^2\right\} - \frac{K}{r} = \beta_3 \tag{5}$$

Integration of (4) yields, apart from a constant of integration,

$$S_3 = -\beta_3 t$$

Multiply both sides of (5) by $2mr^2$ and write it in the form

$$\left(\frac{dS_2}{d\theta}\right)^2 = r^2\left\{2m\beta_3 + \frac{2mK}{r} - \left(\frac{dS_1}{dr}\right)^2\right\}$$

Then since one side depends only on θ while the other side depends only on r, it follows that each side is a constant. Thus

$$dS_2/d\theta = \beta_2 \quad \text{or} \quad S_2 = \beta_2\theta \tag{6}$$

and $$r^2\left\{2m\beta_3 + \frac{2mK}{r} - \left(\frac{dS_1}{dr}\right)^2\right\} = \beta_2^2$$

or $$\frac{dS_1}{dr} = \sqrt{2m\beta_3 + 2mK/r - \beta_2^2/r^2} \tag{7}$$

on taking the positive square root. Then

$$S_1 = \int \sqrt{2m\beta_3 + 2mK/r - \beta_2^2/r^2}\, dr \tag{8}$$

Thus $$\mathcal{S} = \int \sqrt{2m\beta_3 + 2mK/r - \beta_2^2/r^2}\, dr + \beta_2\theta - \beta_3 t \tag{9}$$

Identifying β_2 and β_3 with the new momenta P_r and P_θ respectively, we have

$$Q_r = \frac{\partial\mathcal{S}}{\partial\beta_2} = \frac{\partial}{\partial\beta_2}\int \sqrt{2m\beta_3 + 2mK/r - \beta_2^2/r^2}\, dr + \theta = \gamma_1$$

$$Q_\theta = \frac{\partial\mathcal{S}}{\partial\beta_3} = \frac{\partial}{\partial\beta_3}\int \sqrt{2m\beta_3 + 2mK/r - \beta_2^2/r^2}\, dr - t = \gamma_2$$

since Q_r and Q_θ are constants, say γ_1 and γ_2. On performing the differentiations with respect to β_2 and β_3, we find

$$\int \frac{\beta_2\, dr}{r^2\sqrt{2m\beta_3 + 2mK/r - \beta_2^2/r^2}} = \theta - \gamma_1 \tag{10}$$

$$\int \frac{m\, dr}{\sqrt{2m\beta_3 + 2mK/r - \beta_2^2/r^2}} = t + \gamma_2 \tag{11}$$

The integral in (10) can be evaluated by using the substitution $r = 1/u$, and after integrating we find as the equation of the orbit,

$$r = \frac{\beta_2^2/mK}{1 - \sqrt{1 + 2\beta_3\beta_2^2/mK^2}\cos(\theta + \pi/2 - \gamma_1)} \tag{12}$$

The constant β_3 can be identified with the energy E [see Problem 12.92(b)], thus illustrating equation (31), page 316. If $E = \beta_3 < 0$, the orbit is an ellipse; if $E = \beta_3 = 0$, it is a parabola; and if $E = \beta_3 > 0$, it is a hyperbola. This agrees with the results of Chapter 5.

The equation (11) when integrated yields the position as a function of time.

PHASE INTEGRALS AND ANGLE VARIABLES

12.17. Let $\mathcal{S}$ be a complete solution of the Hamilton-Jacobi equation containing the n constants $\beta_1, \ldots, \beta_n$. Let $J_\alpha = \oint p_\alpha \, dq_\alpha$. Prove that the J_α are functions of the β_α only.

We have
$$\mathcal{S} = S_1(q_1, \beta_1, \ldots, \beta_n) + \cdots + S_n(q_n, \beta_1, \ldots, \beta_n) - \beta_1 t \tag{1}$$
where the constant $\beta_1 = E$, the total energy. Now
$$p_\alpha = \frac{\partial \mathcal{S}}{\partial q_\alpha} = \frac{dS_\alpha}{dq_\alpha} \tag{2}$$
Thus
$$J_\alpha = \oint p_\alpha \, dq_\alpha = \oint \frac{dS_\alpha}{dq_\alpha} \, dq_\alpha \tag{3}$$
But in this integration q_α is integrated out, so that the only quantities remaining are the constants $\beta_1, \ldots, \beta_n$. Thus we have the n equations
$$J_\alpha = J_\alpha(\beta_1, \ldots, \beta_n) \qquad \alpha = 1, \ldots, n \tag{4}$$
Using (4) we can solve for $\beta_1, \ldots, \beta_n$ in terms of $J_1, \ldots, J_n$ and express (1) in terms of the J_α.

12.18. (*a*) Suppose that the new position and momentum coordinates are taken to be w_α and J_α respectively. Prove that if $\mathcal{H}$ is the new Hamiltonian,
$$\dot{J}_\alpha = -\partial\mathcal{H}/\partial w_\alpha, \qquad \dot{w}_\alpha = \partial\mathcal{H}/\partial J_\alpha$$

(*b*) Deduce from (*a*) that
$$J_\alpha = \text{constant} \qquad \text{and} \qquad w_\alpha = f_\alpha t + c_\alpha$$
where f_α and c_α are constants and $f_\alpha = \partial\mathcal{H}/\partial J_\alpha$.

(*a*) By Hamilton's equations for the canonical coordinates Q_α, P_α,
$$\dot{P}_\alpha = -\partial\mathcal{H}/\partial Q_\alpha, \qquad \dot{Q}_\alpha = \partial\mathcal{H}/\partial P_\alpha \tag{1}$$
Then since the new position and momentum coordinates are taken as $Q_\alpha = w_\alpha$ and $P_\alpha = J_\alpha$, these equations become
$$\dot{J}_\alpha = -\partial\mathcal{H}/\partial w_\alpha, \qquad \dot{w}_\alpha = \partial\mathcal{H}/\partial J_\alpha \tag{2}$$

(*b*) Since $\mathcal{H} = E$, the new Hamiltonian depends only on the J_α and not on the w_α. Thus from (2) we have
$$\dot{J}_\alpha = 0, \qquad \dot{w}_\alpha = \text{constant} = f_\alpha \tag{3}$$
where $f_\alpha = \partial\mathcal{H}/\partial J_\alpha$. From (3) we find, as required,
$$J_\alpha = \text{constant}, \qquad w_\alpha = f_\alpha t + c_\alpha \tag{4}$$
The quantities J_α are called *action variables* while the corresponding integrals
$$\oint p_\alpha \, dq_\alpha = J_\alpha \tag{5}$$
where the integration is performed over a complete cycle of the coordinate q_α, are called *phase integrals*. The quantities w_α are called *angle variables*.

12.19. (*a*) Let Δw_α denote the change in w_α corresponding to a complete cycle in the particular coordinate q_r. Prove that
$$\Delta w_\alpha = \begin{cases} 1 & \text{if } \alpha = r \\ 0 & \text{if } \alpha \neq r \end{cases}$$

(*b*) Give a physical interpretation to the result in (*a*).

(*a*)

$$\Delta w_\alpha \;=\; \oint \frac{\partial w_\alpha}{\partial q_r}\,dq_r \;=\; \oint \frac{\partial}{\partial q_r}\left(\frac{\partial S}{\partial J_\alpha}\right)dq_r \;=\; \oint \frac{\partial}{\partial J_\alpha}\left(\frac{\partial S}{\partial q_r}\right)dq_r$$

$$= \frac{\partial}{\partial J_\alpha}\oint \frac{\partial S}{\partial q_r}\,dq_r \;=\; \frac{\partial J_r}{\partial J_\alpha} \;=\; \begin{cases}1 & \text{if } \alpha = r\\ 0 & \text{if } \alpha \neq r\end{cases}$$

where we have used the fact that $w_\alpha = \partial S/\partial J_\alpha$ [see Problems 12.17 and 12.18] and have assumed that the order of differentiation and integration is immaterial.

(*b*) From (*a*) it follows that w_α changes by one when q_α goes through a complete cycle but that there is no change when any other q goes through a complete cycle. It follows that q_α is a periodic function of w_α of period one. Physically this means that the f_α in equation (*4*) of Problem 12.18 are *frequencies*.

12.20. Determine the frequency of the harmonic oscillator of Problem 12.15.

A complete cycle of the coordinate q [see equation (*10*), Problem 12.15] consists in the motion from $q = -\sqrt{2\beta/\kappa}$ to $q = +\sqrt{2\beta/\kappa}$ and back to $q = -\sqrt{2\beta/\kappa}$. Then the action variable is

$$J \;=\; \oint p\,dq \;=\; 2\int_{-\sqrt{2\beta/\kappa}}^{\sqrt{2\beta/\kappa}} \sqrt{2m(\beta - \tfrac{1}{2}\kappa q^2)}\,dq \;=\; 4\int_0^{\sqrt{2\beta/\kappa}} \sqrt{2m(\beta - \tfrac{1}{2}\kappa q^2)}\,dq$$

$$= 2\pi\beta\sqrt{m/\kappa}$$

Thus $$\beta \;=\; E \;=\; \frac{J}{2\pi}\sqrt{\frac{\kappa}{m}} \;=\; \mathcal{H} \qquad \text{and} \qquad f \;=\; \frac{\partial \mathcal{H}}{\partial J} \;=\; \frac{1}{2\pi}\sqrt{\frac{\kappa}{m}}$$

12.21. Determine the frequency of the Kepler problem [see Problem 12.16].

A complete cycle of the coordinate r consists in the motion from $r = r_{\min}$ to $r_{\max}$ and back to $r = r_{\min}$, where $r_{\min}$ and $r_{\max}$ are the minimum and maximum values of r given by the zeros of the quadratic equation [see equation (*10*), Problem 12.16]

$$2m\beta_3 + 2mK/r - \beta_2^2/r^2 \;=\; 0 \tag{1}$$

We then have from equations (*6*) and (*7*) of Problem 12.16,

$$J_\theta \;=\; \oint p_\theta\,d\theta \;=\; \oint \frac{\partial \mathcal{S}}{\partial\theta}\,d\theta \;=\; \oint \frac{dS_2}{d\theta}\,d\theta \;=\; \int_0^{2\pi}\beta_2\,d\theta \;=\; 2\pi\beta_2 \tag{2}$$

$$J_r \;=\; \oint p_r\,dr \;=\; \oint \frac{\partial \mathcal{S}}{\partial r}\,dr \;=\; \oint \frac{dS_1}{dr}\,dr \;=\; 2\int_{r_{\min}}^{r_{\max}} \sqrt{2m\beta_3 + 2mK/r - \beta_2^2/r^2}\,dr$$

$$= 2\pi mK/\sqrt{-2m\beta_3} \;-\; 2\pi\beta_2 \tag{3}$$

From (*2*) and (*3*) we have on elimination of β_2,

$$J_\theta + J_r \;=\; 2\pi mK/\sqrt{-2m\beta_3} \tag{4}$$

Since $\beta_3 = E$, (*4*) yields

$$E \;=\; -\frac{2\pi^2 mK^2}{(J_\theta + J_r)^2} \qquad \text{so that} \qquad \mathcal{H} \;=\; -\frac{2\pi^2 mK^2}{(J_\theta + J_r)^2}$$

Then the frequencies are

$$f_\theta \;=\; \frac{\partial \mathcal{H}}{\partial J_\theta} \;=\; \frac{4\pi^2 mK^2}{(J_\theta + J_r)^3}, \qquad f_r \;=\; \frac{\partial \mathcal{H}}{\partial J_r} \;=\; \frac{4\pi^2 mK^2}{(J_\theta + J_r)^3}$$

Since these two frequencies are the same, i.e. there is only one frequency, we say that the system is *degenerate*.

MISCELLANEOUS PROBLEMS

12.22. A particle of mass m moves in a force field of potential V. Write (a) the Hamiltonian and (b) Hamilton's equations in spherical coordinates (r, θ, ϕ).

(a) The kinetic energy in spherical coordinates is

$$T = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2) \tag{1}$$

Then the Lagrangian is

$$L = T - V = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2) - V(r, \theta, \phi) \tag{2}$$

We have

$$p_r = \partial L/\partial \dot{r} = m\dot{r}, \quad p_\theta = \partial L/\partial \dot{\theta} = mr^2\dot{\theta}, \quad p_\phi = \partial L/\partial \dot{\phi} = mr^2 \sin^2\theta\, \dot{\phi} \tag{3}$$

and

$$\dot{r} = \frac{p_r}{m}, \quad \dot{\theta} = \frac{p_\theta}{mr^2}, \quad \dot{\phi} = \frac{p_\phi}{mr^2 \sin^2\theta} \tag{4}$$

The Hamiltonian is given by

$$\begin{aligned} H &= \sum p_\alpha \dot{q}_\alpha - L \\ &= p_r\dot{r} + p_\theta\dot{\theta} + p_\phi\dot{\phi} - \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) + V(r, \theta, \phi) \\ &= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2 \sin^2\theta} + V(r, \theta, \phi) \end{aligned} \tag{5}$$

where we have used the results of equations (4).

We can also obtain (5) directly by using the fact that for conservative systems the Hamiltonian is the total energy, i.e. $H = T + V$.

(b) Hamilton's equations are $\dot{q}_\alpha = \dfrac{\partial H}{\partial p_\alpha}$, $\dot{p}_\alpha = -\dfrac{\partial H}{\partial q_\alpha}$. Then from part (a),

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}, \quad \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}, \quad \dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2 \sin^2\theta}$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} + \frac{p_\phi^2}{mr^3 \sin^2\theta} - \frac{\partial V}{\partial r}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2 \cos\theta}{mr^2 \sin^3\theta} - \frac{\partial V}{\partial \theta}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = -\frac{\partial V}{\partial \phi}$$

12.23. A particle of mass m moves in a force field whose potential in spherical coordinates is $V = -(K \cos\theta)/r^2$. Write the Hamilton-Jacobi equation describing its motion.

By Problem 12.22 the Hamiltonian is

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) - \frac{K\cos\theta}{r^2} \tag{1}$$

Writing $p_r = \dfrac{\partial \mathcal{S}}{\partial r}$, $p_\theta = \dfrac{\partial \mathcal{S}}{\partial \theta}$, $p_\phi = \dfrac{\partial \mathcal{S}}{\partial \phi}$, the required Hamilton-Jacobi equation is

$$\frac{\partial \mathcal{S}}{\partial t} + \frac{1}{2m}\left\{\left(\frac{\partial \mathcal{S}}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \mathcal{S}}{\partial \theta}\right)^2 + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial \mathcal{S}}{\partial \phi}\right)^2\right\} - \frac{K\cos\theta}{r^2} = 0 \tag{2}$$

12.24. (a) Find a complete solution of the Hamilton-Jacobi equation of Problem 12.23 and (b) indicate how the motion of the particle can be determined.

(a) Letting $\mathcal{S} = S_1(r) + S_2(\theta) + S_3(\phi) - Et$ in equation (2) of Problem 12.23, it can be written

$$\frac{1}{2m}\left(\frac{dS_1}{dr}\right)^2 + \frac{1}{2mr^2}\left(\frac{dS_2}{d\theta}\right)^2 + \frac{1}{2mr^2\sin^2\theta}\left(\frac{dS_3}{d\phi}\right)^2 - \frac{K\cos\theta}{r^2} = E \qquad (1)$$

Multiplying equation (1) by $2mr^2$ and rearranging terms,

$$r^2\left(\frac{dS_1}{dr}\right)^2 - 2mEr^2 = -\left(\frac{dS_2}{d\theta}\right)^2 - \frac{1}{\sin^2\theta}\left(\frac{dS_3}{d\phi}\right)^2 + 2mK\cos\theta$$

Since the left side depends only on r while the right side depends on θ and ϕ, it follows that each side must be a constant which we shall call β_1. Thus

$$r^2\left(\frac{dS_1}{dr}\right)^2 - 2mEr^2 = \beta_1 \qquad (2)$$

and

$$-\left(\frac{dS_2}{d\theta}\right)^2 - \frac{1}{\sin^2\theta}\left(\frac{dS_3}{d\phi}\right)^2 + 2mK\cos\theta = \beta_1 \qquad (3)$$

Multiplying equation (3) by $\sin^2\theta$ and rearranging terms,

$$\left(\frac{dS_3}{d\phi}\right)^2 = 2mK\sin^2\theta\cos\theta - \beta_1\sin^2\theta - \sin^2\theta\left(\frac{dS_2}{d\theta}\right)^2 \qquad (4)$$

Since the left side depends only on ϕ while the right side depends only on θ each side must be a constant which we can call β_2. However, since

$$p_\phi = \frac{\partial\mathcal{S}}{\partial\phi} = \frac{dS_3}{d\phi} \qquad (5)$$

we can write $\beta_2 = p_\phi^2$. This is a consequence of the fact that ϕ is a cyclic or ignorable coordinate. Then (4) becomes

$$2mK\sin^2\theta\cos\theta - \beta_1\sin^2\theta - \sin^2\theta\left(\frac{dS_2}{d\theta}\right)^2 = p_\phi^2 \qquad (6)$$

By solving equations (2), (6) and (5), we obtain

$$S_1 = \int\sqrt{2mE + \beta_1/r^2}\,dr, \quad S_2 = \int\sqrt{2mK\cos\theta - p_\phi^2\csc^2\theta - \beta_1}\,d\theta, \quad S_3 = p_\phi\phi$$

where we have chosen the positive square roots and omitted arbitrary additive constants. The complete solution is

$$\mathcal{S} = \int\sqrt{2mE + \beta_1/r^2}\,dr + \int\sqrt{2mK\cos\theta - p_\phi^2\csc^2\theta - \beta_1}\,d\theta + p_\phi\phi - Et$$

(b) The required equations of motion are found by writing

$$\frac{\partial\mathcal{S}}{\partial\beta_1} = \gamma_1, \quad \frac{\partial\mathcal{S}}{\partial E} = \gamma_2, \quad \frac{\partial\mathcal{S}}{\partial p_\phi} = \gamma_3$$

and then solving these to obtain the coordinates r, θ, ϕ as functions of time using initial conditions to evaluate the arbitrary constants.

12.25. If the functions F and G depend on the position coordinates q_α, momenta p_α and time t, the *Poisson bracket* of F and G is defined as

$$[F, G] = \sum_\alpha\left(\frac{\partial F}{\partial p_\alpha}\frac{\partial G}{\partial q_\alpha} - \frac{\partial F}{\partial q_\alpha}\frac{\partial G}{\partial p_\alpha}\right)$$

Prove that (a) $[F, G] = -[G, F]$, (b) $[F_1 + F_2, G] = [F_1, G] + [F_2, G]$, (c) $[F, q_r] = \partial F/\partial p_r$, (d) $[F, p_r] = -\partial F/\partial q_r$.

(a) $$[F, G] = \sum_\alpha\left(\frac{\partial F}{\partial p_\alpha}\frac{\partial G}{\partial q_\alpha} - \frac{\partial F}{\partial q_\alpha}\frac{\partial G}{\partial p_\alpha}\right) = -\sum_\alpha\left(\frac{\partial G}{\partial p_\alpha}\frac{\partial F}{\partial q_\alpha} - \frac{\partial G}{\partial q_\alpha}\frac{\partial F}{\partial p_\alpha}\right) = -[G, F]$$

This shows that the Poisson bracket does not obey the *commutative law of algebra.*

(b)
$$[F_1+F_2, G] = \sum_\alpha \left\{\frac{\partial(F_1+F_2)}{\partial p_\alpha}\frac{\partial G}{\partial q_\alpha} - \frac{\partial(F_1+F_2)}{\partial q_\alpha}\frac{\partial G}{\partial p_\alpha}\right\}$$
$$= \sum_\alpha \left(\frac{\partial F_1}{\partial p_\alpha}\frac{\partial G}{\partial q_\alpha} - \frac{\partial F_1}{\partial q_\alpha}\frac{\partial G}{\partial p_\alpha}\right) + \sum_\alpha \left(\frac{\partial F_2}{\partial p_\alpha}\frac{\partial G}{\partial q_\alpha} - \frac{\partial F_2}{\partial q_\alpha}\frac{\partial G}{\partial p_\alpha}\right)$$
$$= [F_1, G] + [F_2, G]$$

This shows that the Poisson bracket obeys the *distributive law of algebra.*

(c)
$$[F, q_r] = \sum_\alpha \left(\frac{\partial F}{\partial p_\alpha}\frac{\partial q_r}{\partial q_\alpha} - \frac{\partial F}{\partial q_\alpha}\frac{\partial q_r}{\partial p_\alpha}\right) = \frac{\partial F}{\partial p_r}$$

since $\partial q_r/\partial q_\alpha = 1$ for $\alpha = r$ and 0 for $\alpha \neq r$, while $\partial q_r/\partial p_\alpha = 0$ for all α. Since r is arbitrary, the required result follows.

(d)
$$[F, p_r] = \sum_\alpha \left(\frac{\partial F}{\partial p_\alpha}\frac{\partial p_r}{\partial q_\alpha} - \frac{\partial F}{\partial q_\alpha}\frac{\partial p_r}{\partial p_\alpha}\right) = -\frac{\partial F}{\partial q_r}$$

since $\partial p_r/\partial q_\alpha = 0$ for all α, while $\partial p_r/\partial p_\alpha = 1$ for $\alpha = r$ and 0 for $\alpha \neq r$. Since r is arbitrary, the required result follows.

12.26. If H is the Hamiltonian, prove that if f is any function depending on position, momenta and time, then
$$\frac{df}{dt} = \frac{\partial f}{\partial t} + [H, f]$$

$$df = \frac{\partial f}{\partial t}dt + \sum_\alpha \left(\frac{\partial f}{\partial q_\alpha}dq_\alpha + \frac{\partial f}{\partial p_\alpha}dp_\alpha\right) \tag{1}$$

or
$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_\alpha \left(\frac{\partial f}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial f}{\partial p_\alpha}\dot{p}_\alpha\right) \tag{2}$$

But by Hamilton's equations,
$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}, \quad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha} \tag{3}$$

Then (2) can be written
$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_\alpha \left(\frac{\partial f}{\partial q_\alpha}\frac{\partial H}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial H}{\partial q_\alpha}\right) = \frac{\partial f}{\partial t} + [H, f]$$

Supplementary Problems

THE HAMILTONIAN AND HAMILTON'S EQUATIONS

12.27. A particle of mass m moves in a force field of potential V. (a) Write the Hamiltonian and (b) Hamilton's equations in rectangular coordinates (x, y, z).

Ans. (a) $H = (p_x^2 + p_y^2 + p_z^2)/2m + V(x, y, z)$
(b) $\dot{x} = p_x/m$, $\dot{y} = p_y/m$, $\dot{z} = p_z/m$, $\dot{p}_x = -\partial V/\partial x$, $\dot{p}_y = -\partial V/\partial y$, $\dot{p}_z = -\partial V/\partial z$

12.28. Use Hamilton's equations to obtain the motion of a particle of mass m down a frictionless inclined plane of angle α.

12.29. Work the problem of small oscillations of a simple pendulum by using Hamilton's equations.

12.30. Use Hamilton's equations to obtain the motion of a projectile launched with speed v_0 at angle α with the horizontal.

12.31. Using Hamilton's equations, work the problem of the harmonic oscillator in (*a*) one dimension, (*b*) two dimensions, (*c*) three dimensions.

12.32. Work Problem 3.27, page 78 by using Hamilton's equation.

PHASE SPACE AND LIOUVILLE'S THEOREM

12.33. Explain why the path of a phase point in phase space which represents the motion of a system of particles can never cross itself.

12.34. Carry out the details in the proof of Liouville's theorem for the case of two degrees of freedom.

CALCULUS OF VARIATIONS AND HAMILTON'S PRINCIPLE

12.35. Use the methods of the calculus of variations to find that curve connecting two fixed points in a plane which has the shortest length.

12.36. Prove that if the function F in the integral $\int_a^b F(x, y, y')\,dx$ is independent of x, then the integral is an extremum if $F - y'F_{y'} = c$ where c is a constant.

12.37. Use the result of Problem 12.36 to solve (*a*) Problem 12.9, page 322, (*b*) Problem 12.35.

12.38. It is desired to revolve the curve of Fig. 12-5 having endpoints fixed at $P(x_1, y_1)$ and $Q(x_2, y_2)$ about the x axis so that the area I of the surface of revolution is a minimum.

(*a*) Show that $I = 2\pi \int_{x_1}^{x_2} y\sqrt{1 + y'^2}\,dx$.

(*b*) Obtain the differential equation of the curve.

(*c*) Prove that the required curve is a catenary.

Ans. (*b*) $yy'' = 1 + (y')^2$

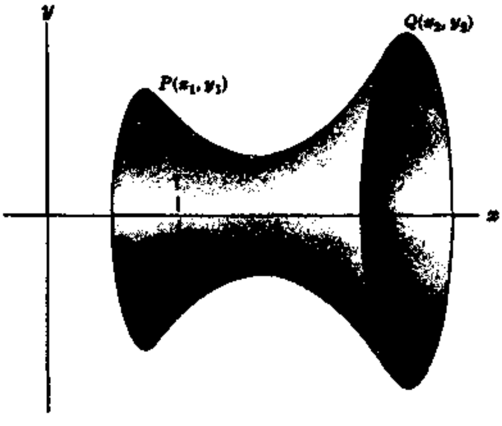

Fig. 12-5

12.39. Two identical circular wires in contact are placed in a soap solution and then separated so as to form a soap film. Explain why the shape of the soap film surface is related to the result of Problem 12.38.

12.40. Use Hamilton's principle to find the motion of a simple pendulum.

12.41. Work the problem of a projectile by using Hamilton's principle.

12.42. Use Hamilton's principle to find the motion of a solid cylinder rolling down an inclined plane of angle α.

CANONICAL TRANSFORMATIONS AND GENERATING FUNCTIONS

12.43. Prove that the transformation $Q = p$, $P = -q$ is canonical.

12.44. Prove that the transformation $Q = q \tan p$, $P = \ln \sin p$ is canonical.

12.45. (*a*) Prove that the Hamiltonian for a harmonic oscillator can be written in the form $H = \frac{1}{2}p^2/m + \frac{1}{2}\kappa q^2$.

(*b*) Prove that the transformation $q = \sqrt{2P/\sqrt{\kappa m}}\,\sin Q$, $p = \sqrt{2P\sqrt{\kappa m}}\,\cos Q$ is canonical.

(*c*) Express the Hamiltonian of part (*a*) in terms of P and Q and show that Q is cyclic.

(*d*) Obtain the solution of the harmonic oscillator by using the above results.

12.46. Prove that the generating function giving rise to the canonical transformation in Problem 12.45(*b*) is $S = \frac{1}{2}\sqrt{\kappa m}\, q^2 \cot Q$.

12.47. Prove that the result of two or more successive canonical transformations is also canonical.

12.48. Let $\mathcal{U}$ be a generating function dependent only on Q_α, p_α, t. Prove that

$$P_\alpha \;=\; -\frac{\partial \mathcal{U}}{\partial Q_\alpha}, \quad q_\alpha \;=\; -\frac{\partial \mathcal{U}}{\partial p_\alpha}, \quad \mathcal{H} \;=\; \frac{\partial \mathcal{U}}{\partial t} + H$$

12.49. Let $\mathcal{V}$ be a generating function dependent only on the old and new momenta p_α and P_α respectively and the time t. Prove that

$$q_\alpha \;=\; -\frac{\partial \mathcal{V}}{\partial p_\alpha}, \quad Q_\alpha \;=\; \frac{\partial \mathcal{V}}{\partial P_\alpha}, \quad \mathcal{H} \;=\; \frac{\partial \mathcal{V}}{\partial t} + H$$

12.50. Prove that the generating function $\mathcal{U}$ of Problem 12.48 is related to the generating function $\mathcal{T}$ of Problem 12.12 by $\mathcal{U} \;=\; \mathcal{T} - \sum p_\alpha q_\alpha$.

12.51. Prove that the generating function $\mathcal{V}$ of Problem 12.49 is related to the generating function $\mathcal{T}$ Problem 12.12 by $\mathcal{V} \;=\; \mathcal{T} + \sum P_\alpha Q_\alpha - \sum p_\alpha q_\alpha$.

THE HAMILTON-JACOBI EQUATION

12.52. Use the Hamilton-Jacobi method to determine the motion of a particle falling vertically in a uniform gravitational field.

12.53. (*a*) Set up the Hamilton-Jacobi equation for the motion of a particle sliding down a frictionless inclined plane of angle α. (*b*) Solve the Hamilton-Jacobi equation in (*a*) and thus determine the motion of the particle.

12.54. Work the problem of a projectile launched with speed v_0 at angle α with the horizontal by using Hamilton-Jacobi methods.

12.55. Use Hamilton-Jacobi methods to describe the motion and find the frequencies of a harmonic oscillator in (*a*) 2 dimensions, (*b*) 3 dimensions.

12.56. Use Hamilton-Jacobi methods to arrive at the generating function of Problem 12.46.

PHASE INTEGRALS AND ANGLE VARIABLES

12.57. Use the method of phase integrals and angle variables to find the frequency of a simple pendulum of length l, assuming that oscillations are small. *Ans.* $\frac{1}{2\pi}\sqrt{\frac{g}{l}}$

12.58. Find the frequencies of (*a*) a 2 dimensional harmonic oscillator, (*b*) a 3 dimensional harmonic oscillator.

12.59. Obtain the frequency of small oscillations of a compound pendulum by using phase integrals.

12.60. Two equal masses m connected by equal springs to fixed walls at A and B are free to slide in a line on a frictionless plane AB [see Fig. 12-6]. Using phase integrals determine the frequencies of the normal modes.

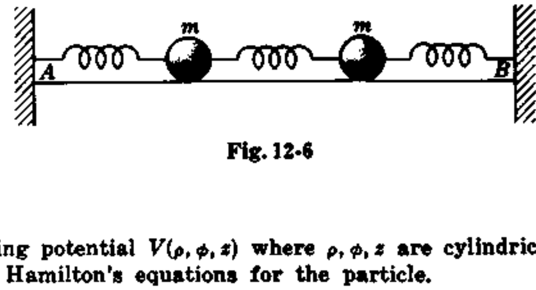

Fig. 12-6

12.61 Discuss Problem 12.57 if oscillations are not assumed small.

MISCELLANEOUS PROBLEMS

12.62. A particle of mass m moves in a force field having potential $V(\rho, \phi, z)$ where ρ, ϕ, z are cylindrical coordinates. Give (*a*) the Hamiltonian and (*b*) Hamilton's equations for the particle.

Ans. (*a*) $H \;=\; (p_\rho^2 + p_\phi^2/\rho^2 + p_z^2)/2m + V(\rho, \phi, z)$

(*b*) $\dot{\rho} = p_\rho/m,\ \dot{\phi} = p_\phi/m r^2,\ \dot{z} = p_z/m,\ \dot{p}_\rho = p_\phi^2/m\rho^3 - \partial V/\partial\rho,\ \dot{p}_\phi = -\partial V/\partial\phi,\ \dot{p}_z = -\partial V/\partial z$

12.63. A particle of mass m which moves in a plane relative to a fixed set of axes has a Hamiltonian given by the total energy. Find the Hamiltonian relative to a set of axes which rotates at constant angular velocity ω relative to the fixed axes.

12.64. Set up the Hamiltonian for a double pendulum. Use Hamilton-Jacobi methods to determine the normal frequencies for the case of small vibrations.

12.65. Prove that a necessary condition for $I = \int_{t_1}^{t_2} F(t, x, \dot{x}, \ddot{x})\,dt$ to be an extremum is that

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial F}{\partial \ddot{x}}\right) = 0$$

Can you generalize this result?

12.66. Work Problem 3.22, page 76, by Hamiltonian methods.

12.67. A particle of mass m moves on the inside of a frictionless vertical cone having equation $x^2 + y^2 = z^2 \tan^2 \alpha$. (a) Write the Hamiltonian and (b) Hamilton's equations using cylindrical coordinates.

Ans. (a) $H = \dfrac{p_\rho^2 \sin^2 \alpha}{2m} + \dfrac{p_\phi^2}{2m\rho^2} + mg\rho \cot \alpha$

(b) $\dot{\rho} = \dfrac{p_\rho \sin^2 \alpha}{m}, \quad \dot{p}_\rho = \dfrac{p_\phi^2}{m\rho^3} - mg \cot \alpha$

12.68. Use the results of Problem 12.67 to prove that there will be a stable orbit in any horizontal plane $z = h > 0$, and find the frequency in this orbit.

12.69. Prove that the product of a position coordinate and its canonically conjugate momentum must have the dimension of *action* or energy multiplied by time, i.e. ML^2T^{-1}.

12.70. Perform the integration of equation *(10)* of Problem 12.16 and compare with the solution of the Kepler problem in Chapter 5.

12.71. Verify the integration result *(3)* of Problem 12.21.

12.72. Prove that Euler's equation *(9)*, page 313, can be written as

$$y'' \frac{\partial^2 F}{\partial y'^2} + y' \frac{\partial^2 F}{\partial y' \partial y} + \frac{\partial^2 F}{\partial y' \partial x} - \frac{\partial F}{\partial y} = 0$$

12.73. A man can travel by boat with speed v_1 and can walk with speed v_2. Referring to Fig. 12-7, prove that in order to travel from point A on one side of a river bank to a point B on the other side in the least time he must land his boat at point P where angles θ_1 and θ_2 are such that

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2}$$

Discuss the relationship of this result to the refraction of light in the theory of optics.

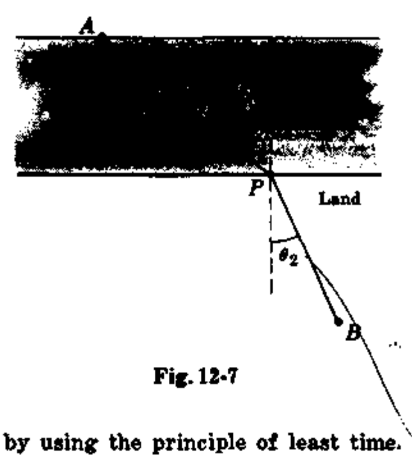

Fig. 12-7

12.74. Prove that if a particle moves under no external forces, i.e. it is a free particle, then the principle of least action becomes one of least time. Discuss the relationship of this result to Problem 12.73.

12.75. Derive the condition for reflection of light in optical theory by using the principle of least time.

12.76. It is desired to find the shape of a curve lying in a plane and having fixed endpoints such that its moment of inertia about an axis perpendicular to the plane and passing through a fixed origin is a minimum.

(a) Using polar coordinates (r, θ), show that the problem is equivalent to minimizing the integral

$$I = \int_{r=r_1}^{r_2} r^2\sqrt{1 + r^2(d\theta/dr)^2}\, dr$$

where the fixed endpoints of the wire are (r_1, θ_1), (r_2, θ_2).

(b) Write Euler's equation, thus obtaining the differential equation of the curve.

(c) Solve the differential equation obtained in (b) and thus find the equation of the curve.

Ans. (c) $r^3 = c_1 \sec(3\theta - c_2)$ where c_1 and c_2 are determined so that the curve passes through the fixed points.

12.77. Use the Hamilton-Jacobi method to set up the equations of motion of a spherical pendulum.

12.78. Use Hamilton-Jacobi methods to solve Problems 11.20, page 293, and 11.21, page 294.

12.79. If $[F, G]$ is the Poisson bracket [see Problems 12.25 and 12.26], prove that

(a) $[F_1F_2, G] = F_1[F_2, G] + F_2[F_1, G]$

(b) $\dfrac{\partial}{\partial t}[F, G] = \left[\dfrac{\partial F}{\partial t}, G\right] + \left[F, \dfrac{\partial G}{\partial t}\right]$

(c) $\dfrac{d}{dt}[F, G] = \left[\dfrac{dF}{dt}, G\right] + \left[F, \dfrac{dG}{dt}\right]$

12.80. Prove that (a) $[q_\alpha, q_\beta] = 0$, (b) $[p_\alpha, p_\beta] = 0$, (c) $[p_\alpha, q_\beta] = \delta_{\alpha\beta}$

where $\delta_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}$ is called the *Kronecker delta.*

12.81. Evaluate $[H, t]$ where H is the Hamiltonian and t is the time. Are H and t canonically conjugate variables? Explain.

12.82. Prove *Jacobi's identity* for Poisson brackets

$$[F_1, [F_2, F_3]] + [F_2, [F_3, F_1]] + [F_3, [F_1, F_2]] = 0$$

12.83. Illustrate Liouville's theorem by using the one dimensional harmonic oscillator.

12.84. (a) Is the Lagrangian of a dynamical system unique? Explain.

(b) Discuss the uniqueness of the generalized momenta and Hamiltonian of a system.

12.85. (a) Set up the Hamiltonian for a string consisting of N particles [see Problem 8.29, page 215]

(b) Use Hamilton-Jacobi methods to find the normal modes and frequencies.

12.86. Prove that the Poisson bracket is invariant under a canonical transformation.

12.87. Prove that Liouville's theorem is equivalent to the result $\partial\rho/\partial t = [\rho, H]$.

12.88. (a) Let $Q_\alpha = \sum_{\mu=1}^{n} a_{\alpha\mu} q_\mu$, $P_\alpha = \sum_{\mu=1}^{n} b_{\alpha\mu} p_\mu$ where $a_{\alpha\mu}$ and $b_{\alpha\mu}$ are given constants and $\alpha = 1, 2, \ldots, n$. Prove that the transformation is canonical if and only if $b_{\alpha\mu} = \Delta_{\alpha\mu}/\Delta$ where Δ is the determinant.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

and $\Delta_{\alpha\mu}$ is the cofactor of the element $a_{\alpha\mu}$ in this determinant.

(b) Prove that the conditions in (a) are equivalent to the condition $\sum P_\alpha Q_\alpha = \sum p_\alpha q_\alpha$.

12.89. Prove that the path taken by a fixed point on a circle as it rolls along a given line is a cycloid.

12.90. (a) Express as an integral the total potential energy of a uniform chain whose ends are suspended from two fixed points. (b) Using the fact that for equilibrium the total potential energy is a minimum, use the calculus of variations to show that the equation of the curve in which the chain hangs is a *catenary* as in Problem 7.32, page 186. [*Hint.* Find the minimum of the integral subject to the constraint condition that the total length of the chain is a given constant.]

12.91. Use the methods of the calculus of variations to find the closed plane curve which encloses the largest area.

12.92. Prove that the constants (a) β in Problem 12.15 and (b) β_3 in Problem 12.16 can be identified with the total energy.

12.93. If the theory of relativity is taken into account in the motion of a particle of mass m in a force field of potential V, the Hamiltonian is given by

$$H = \sqrt{p^2c^2 + m^2c^4} + V$$

where c is the speed of light. Obtain the equations of motion for this particle.

12.94. Use Hamiltonian methods to solve the problem of a particle moving in an inverse cube force field.

12.95. Use spherical coordinates to solve Kepler's problem.

12.96. Suppose that m of the n coordinates $q_1, q_2, \ldots, q_n$ are cyclic [say the first m, i.e. $q_1, q_2, \ldots, q_m$]. Let

$$\mathcal{R} = \sum_{\alpha=1}^{m} c_\alpha \dot{q}_\alpha - L \qquad \text{where} \qquad c_\alpha = \partial L/\partial \dot{q}_\alpha$$

Prove that for $\alpha = m+1, \ldots, n$ $$\frac{d}{dt}\left(\frac{\partial \mathcal{R}}{\partial \dot{q}_\alpha}\right) = \frac{\partial \mathcal{R}}{\partial q_\alpha}$$

The function $\mathcal{R}$ is called *Routh's function* or the *Routhian.* By using it a problem involving n degrees of freedom is reduced to one involving $n - m$ degrees of freedom.

12.97. Using the properties $$\delta L = \frac{\partial L}{\partial y}\delta y + \frac{\partial L}{\partial y'}\delta y', \qquad (\delta y)' = \delta y'$$

of the variational symbol δ [see Problem 12.6] and assuming that the operator δ can be brought under the integral sign, show how Lagrange's equations can be derived from Hamilton's principle.

12.98. Let $P = P(p,q)$, $Q = Q(p,q)$. Suppose that the Hamiltonian expressed in terms of p,q and P,Q are given by $H = H(p,q)$ and $\mathcal{H} = \mathcal{H}(P,Q)$ respectively. Prove that if

$$\dot{q} = \partial H/\partial p, \qquad \dot{p} = -\partial H/\partial q$$

then $$\dot{Q} = \partial \mathcal{H}/\partial P, \qquad \dot{P} = -\partial \mathcal{H}/\partial Q$$

provided that the *Jacobian determinant* [or briefly *Jacobian*]

$$\frac{\partial(P,Q)}{\partial(p,q)} = \begin{vmatrix} \partial P/\partial p & \partial P/\partial q \\ \partial Q/\partial p & \partial Q/\partial q \end{vmatrix} = 1$$

Discuss the connection of the results with Hamiltonian theory.

12.99. (a) Set up the Hamiltonian for a solid cylinder rolling down an inclined plane of angle α.
(b) Write Hamilton's equations and deduce the motion of the cylinder from them.
(c) Use Hamilton-Jacobi methods to obtain the motion of the cylinder and compare with part (b).

12.100. Work Problem 7.22, page 180, by Hamilton-Jacobi methods.

12.101. Write (*a*) the Hamiltonian and (*b*) Hamilton's equations for a particle of charge e and mass m moving in an electromagnetic field [see Problem 11.90, page 309].

Ans. (*a*) $H = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 + e\Phi$

(*b*) $\mathbf{v} = \frac{1}{m}(\mathbf{p} - e\mathbf{A}), \quad \dot{\mathbf{p}} = -e\nabla\Phi + e\nabla(\mathbf{A}\cdot\mathbf{v})$

12.102. (*a*) Obtain the Hamilton-Jacobi equation for the motion of the particle in Problem 12.101. (*b*) Use the result to write equations for the motion of a charged particle in an electromagnetic field.

12.103. (*a*) Write the Hamiltonian for a symmetrical top and thus obtain the equations of motion. (*b*) Compare the results obtained in (*a*) with those of Chapter 10.

12.104. Prove Theorem 12.2, page 314.

12.105. An atom consists of an electron of charge $-e$ moving in a central force field $\mathbf{F}$ about a nucleus of charge Ze such that

$$\mathbf{F} = -\frac{Ze^2\mathbf{r}}{r^3}$$

where $\mathbf{r}$ is the position vector of the electron relative to the nucleus and Z is the atomic number. In Bohr's quantum theory of the atom the phase integrals are integer multiples of *Planck's constant* h, i.e.,

$$\oint p_r\,dr = n_1 h, \qquad \oint p_\theta\,d\theta = n_2 h$$

Using these equations, prove that there will be only a discrete set of energies given by

$$E_n = -\frac{2\pi^2 m Z^2 e^4}{n^2 h^2}$$

where $n = n_1 + n_2 = 1, 2, 3, 4, \ldots$ is called the *orbital quantum number*.