Mathematics of finance and investment

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PART ONE

SIMPLE INTEREST

CHAPTER 1

AMOUNT AND PRESENT VALUE

Simple interest is defined as the product of principal, rate, and time. This definition leads to the simple interest formula.

$$I = Prt \qquad (1)$$

I = simple interest in dollars (or other monetary unit)

P = principal in dollars (or other monetary unit consistent with interest)

r = interest rate or percent of the principal that is to be paid per unit of time

t = time in units that correspond to the rate

In this text, we consider the dollar as the unit of monetary exchange.

Note that r and t must be consistently stated. That is, if the rate is an annual rate, the time must be stated in years; or if the rate is a monthly rate, the time must be stated in months,

Example 1

A bank pays 8% per annum on savings accounts. A person opens an account with a deposit of $300 on January 1. How much interest will the person receive on April 1?

Solution:

Since the rate is an annual rate, the time must be expressed in years. The time is 3 months or $\frac{3}{12} = \frac{1}{4}$ year. We also have $P = 300$ and $r = .08$.

$$I = 300 \times .08 \times \frac{1}{4} = \$\ 6.00$$

Example 2

The interest paid on a loan of $500 for 2 months was $12.50. What was the interest rate?

Solution

We substitute $I = 12.50$, $P = 500$, and $t = \frac{1}{6}$.

$$500 \times r \times \frac{1}{6} = 12.50$$

$$r = \frac{6\times12.50}{500} = 0.15 = 15\%$$

Example 3

A person gets $63.75 every 6 months from an investment that pays 6% interest. How much money is invested?

Solution

Substituting $I = 63.75$ $r = .06$, and $t = \frac{1}{2}$, we obtain

$$P \times .06 \times \frac{1}{2} = 63.75$$

$$P = \$2125.00$$

Example 4

How long will it take $5000 to earn $50 interest at 6%?

Solution

Substituting $I = 50$, $P = 5000$, and $r = .06$, we have

$$5000 \times .06 \times t = 50$$

$$t = \frac{50}{5000\times .06} = \frac{1}{6}$$ year, or 2 months

Example 5

A woman borrows \$2000 from a credit union. Each month she is to pay \$100 on the principal. She also pays interest at the rate of 1% a month on the unpaid balance at the beginning of the month. Find the total interest.

Solution

Note that the rate is a monthly rate. The first month’s interest is

$$I = 2000 \times .01 \times 1 = \$20.00$$

The total payment for the first month is \$120, and the new unpaid balance is \$1900. For the second month the interest is

$$I = 1900 \times .01 \times 1 = \$19.00$$

Since the unpaid balance decreases \$100 a month, the interest decreases \$1.00 each month. After 19 payments the debt is down to \$100, and the last interest payment is

$$I = 100 \times .01 \times 1 = \$1.00$$

The interest payments re \$20, \$18,…\$2, \$1and the total interest the sum of these values.

$$\text{Total interest} = 20 + 19 + 18 + \ldots + 2 + 1$$

The values in this sum form an arithmetic progression, a set of numbers each of which can be obtained by adding a specific constant to its predecessor. In this case, each number is found by adding – 1 to the previous number. The sum of an arithmetic progression is given by

$$\text{Sum} = \frac{n}{2}\ (t_1 + t_n)$$

Where n is the number of terms and t_1 are the first and last interest payments. Substituting, we have

$$\text{Total interest} = \frac{20}{2}\ (20.00 + 1.00) = \$210.00$$

AMOUNT

The sum of the principal and the interest is called the amount, designated by the symbol S. this definition leads to the formula

$$S = P + I$$

$$= P + Prt$$

Factoring, we have

$$S = P\ (1 + rt) \qquad (2)$$

Example

A man borrows \$350 for 6 months at 15%. What amount must he repay?

Solution

Substituting $P = 350,\ r = .15,\ and\ t = \frac{1}{2}$ in formula (2), we find that

$$S = 350\ (1 + .15 \times \frac{1}{2}) = 350\ (1.075) = \$376.25$$

This problem could have been worked by getting the simple interest and adding it to the principal.

$$I = Prt = 350 \times .15 \times \frac{1}{2} = \$26.25$$

$$S = P + I = \$350.00 + 26.25 = \$376.25$$

You can solve many problems in the mathematics of finance by more than one method. look for the easiest way, thereby reducing both labor and the risk of numerical errors. working a problem more.

EXACT AND ORDINARY INTEREST

When the time is in days and the rate is an annual rate, it is necessary to convert the days to a fractional part of a year when substituting in the simple interest formulas. Interest computed using a divisor of 360 is called ordinary interest. when the divisor is 365 or 366, the result is known as exact interest.

For a given rate of interest, a denominator of 360 results in a borrower paying more interest in dollars that would be the case if 365 or 366 were used. on individual loans the difference may not be large. when many borrowers each pay a little more, however, the total difference is a substantial sum. this increased revenue makes the 360-day year popular with lenders.

Example

Figure the ordinary and exact interest and exact interest on 60-day loan of $300 if the rate is 15%

Solution

Substituting P =300 and R =.15 in formula (1), we have

$$\text{Ordinary interest} = 300 \times .15 \times \frac{60}{360} = \$7.50$$

$$\text{Exact interest} = 300 \times .15 \times \frac{60}{365} = \$7.40$$

Note that ordinary interest is greater than exact interest and is easier to compute when the work must be done without a calculator.

EXACT AND APPROXIMATE TIME

There are two ways to compute the number of days between calendar dates. the more common method is the exact method, which includes all days except the first. another method is to add the number of days in each month during the term of the loan, not counting the first day but counting the last one. the approximate method is based on the assumption that all the full months contain 30 days. to this number is added the exact number of days that remain in the term of the loan.

Example

Find the exact and approximate time between March 5 and September 28.

Solution

We find that September 28 is the 271st day in the year and March 5 is the 64th day. therefore, the exact time is 271 – 64 = 207 days., we can set up a table as follows:

March	26(31-5
April	30
May	31
June	30
July	31
August	31
September	28
Total	207 days

To get the approximate time, we count the number of months from March 5 to September 5. then we can figure that 6 × 30 = 180 days. to this result we add the 23 days from September 5 to September 28, for total of 203 days.

PRESENT VALUE

To find the amount of a principal invested at simple interest, we use the formula $S=P(1+rt)$. If we know the amount and want to obtain the principal, we solve the formula for P.

$$P = \frac{S}{1+rt} \qquad (3)$$

` ***Example***

If money is worth 5%, what is the present value of $105 due in 1 year?

Solution

Substituting $S = 105$ $r= .05$, and $t = 1$ in formula (3), we find that

$$P = \frac{105}{1+.05\times 1} = \$100.00$$

Hence $100 invested now at 5% should amount to $105 in a year. Substituting in the amount formula verifies this.

$$S = 100\ (1+ .05 \times 1) = \$105.00$$

Calculating the present value of a sum due in the future is called **discounting.** When the simple interest formula is used to find the present value, the difference between the amount and the present value is called **simple discount.** Note that the simple discount on the future amount is the same as the simple interest on the principal or present value.

Chapter 2

Bank Discount

Bank Discount is the charge for many loans is based on the final amount rather than on the present value. This charge is called **bank discount**, or simply discount. The money that the borrower receives is called **bank proceeds**, or just **proceeds**. The rate percentage used in computing the discount is called the **bank discount rate** or **discount rate**.

As an illustration of a bank discount transaction, consider the case of a person who wants to borrow \$100 for a year from a lender who uses a discount rate of 6%. The lender will take 6% of \$100 from the \$100 and give the borrower \$94. Thus the computation of bank discount is exactly the same as the computation of simple interest except that it is based on the amount rather than the present value. This realization leads us to the bank discount formula:

$$D = Sdt \qquad (4)$$

D = the bank discount in dollars

S= the amount or maturity value

d= the discount rate per unit of time expressed as a decimal

t = the time in units that correspond to the rate

Since the proceeds or present value of the loan is the difference between the amount and the discount, we can say that

$$P = S – D = S – Sdt$$

Factoring, we have

$$P = S\ (1\text{-}dt) \qquad (5)$$

Solving this expression for S results in a formula that is useful when a borrower wants a certain amount of cash and the problem is to determine the maturity of the loan to be discounted.

$$S = \frac{P}{1-dt} \qquad (6)$$

Example 1

A woman borrows \$600 for 6 months from a lender who uses a discount rate of 10%. What is the discount and how much money does the borrower get?

Solution

Substituting $\boldsymbol{S}$ = 600, d = .10, and $t = \frac{1}{2}$ in formula (4), we have

$$D = 600 \times .10 \times \frac{1}{2} = \$30.00$$

Since the proceeds represent the difference between the maturity value and the discount, the borrower well get 600.00 -30.00 = \$570.00.

Example 2

A man wants to get \$2000 cash with the loan to be repaid in 6 months. If the borrows the money from a bank that charges a 12 % discount rate, what size loan should he ask for?

Solution

Substituting P = 2000, d= .12, and $t = \frac{1}{2}$ in formula (6) we obtain

$$S = \frac{2000}{1-.12\times\frac{1}{2}} = \frac{2000}{.94} = \$2127.66$$

$D = 2127.66 \times .012 \times \frac{1}{2} = \$127.66$

Proceeds = 2127.66 – 127.66 = \$2000.00

Bank discount is sometimes called interest in advance because it is based on the future amount rather than on the present value. In cases where the amount is known, bank discount, requiring only multiplication. is easier to compute that discount at a simple interest rate, which requires division. A given bank discount rate result in a larger money return to the lender than the same simple interest rate. For these reasons, bank discount is commonly used to discount sums of money for period of time off a year or less.

Example 3

Find the present value of \$100 due in 1 year: at a simple interest rate of $12\frac{1}{2}\%$.

Solution

For a simple interest rate of $12\frac{1}{2}\%$, we substitute $S = 100$. R = .125, and $t = 1$

$$P = \frac{100}{1+.125\times 1} + \frac{100}{1.125} = \$88.89$$

For a bank discount rate of $12\frac{1}{2}\%$, we substitute $S = 100$, $d = .125$,

$$D = 100 \times .125 \times 1 = \$12.50$$

$$\text{Present value or proceeds} = 100 - 12.50 = \$87.50$$

Not that the present value at $12\frac{1}{2}\%$, bank discount is \$1.39 less for the same maturity value that if it were based on a $12\frac{1}{2}\%$, interest rate.

INTEREST RATE EQUIVALENT TO A BANK DISCOUNT RATE

As the previous example shows, the present value at a given discount rate is less that the present value based on the same interest rate. For comparison purposes, it is desirable to be able to determine the interest rate that is equivalent to a given discount rate. **Coupon equivalent** is a common expression for the simple interest rate equivalent to a bank discount rate. The reason is that some securities bear coupons that are clipped and cashed to get the periodic interest payments. The dollar value of each coupon is based on simple interest

A discount rate and an interest rate are equivalent if the two rates result in the same present value for a given amount due in the future. To obtain the relationship between *r* and *d*, all we have to do is get the present value of an amount *S* due in the future by formula (3) and set it equal to the present value of the same amount *S* by formula (5).

$$\frac{S}{1+rt} = S(1-dt)$$

Dividing both sides by *S*, we obtain

$$\frac{1}{1+rt} = 1 - dt$$

Inverting both sides, we have

$$1 + rt\frac{1}{1-dt}$$

Subtracting 1 from both sides and simplifying. We find that

$$rt = \frac{1}{1-dt} - 1 = \frac{1-1+dt}{1-dt} = \frac{1}{1-dt}$$

Dividing both sides by *t* yields

$$r = \frac{d}{1-dt} \qquad (7)$$

in a similar way we find that the discount rate corresponding to a given interest rate is

$$d = \frac{r}{1-rt} \qquad (8)$$

***Example* 1**

A bank discount a \$200 note due in a year using a bank discount rate of 12%. What interest rate is the bank getting?

Solution

Substituting d = .12 and $t = 1$ in formula (7), we find have

$$r = \frac{.12}{1-.12\times 1} = \frac{12}{.88} = .1364 = 13.64\%$$

***Example* 2**

A lender charges a discount rate of $13\frac{1}{2}$% for discounting a note for \$600 due in 2 months. What is the equivalent interest rate?

Solution

Substituting d = .135 and $t = \frac{1}{6}$ in formula (7), we find that

$$r = \frac{.135}{1-.135\times\frac{1}{6}} = \frac{.135}{.9775} = .1381 = 13.81\%$$

***Example* 3**

To earn an interest rate of 11% on a 6 months' loan, a lender should charge what discount rate?

Solution

Substituting d = .11 and $t = \frac{1}{2}$ in formula (8), we obtain

$$d = \frac{.11}{1-.11\times\frac{1}{2}} = \frac{.11}{1.055} = .1043 = 10.43\%$$

PROMISSORY NOTES

Bank discount is used to determine the value of a promissory note at a stated point in time. The proceeds of a note are found as follows:

1- Find the maturity value of the note. This is the face value if it is non-interest-bearing. If the note is interest-bearing, the maturity value is the face value plus interest at the stated rate for the term of the note, the time from the date of the note the maturity date.
2- Discount the maturity value using the discount rate from the date the note is discounted to the maturity date.
3- Subtract the discount from the maturity value.

Example 1

On July 28, 1996, John, A. Blalock discounts the note in Figure 2-1 at a bank that charges a discount rate of 12%. Find the proceeds.

Solution

Date	**Day of year**
Maturity date, December 15, 1996	350 (leap year)
Discount date, July 28, 1996	210
Discount period	140

Substituting S = 3000, d = .12, and t = 140/360 in formula (4), we have

$$D = 3000 \times .12 \times \frac{140}{360} = \$140.00$$

$$\text{Proceeds} = 3000.00 - 140.00 = \$2860.00$$

Solution

First, to get the maturity value, we substitute P = 3000, r = .11, and t = $\frac{1}{2}$ in formula (1).

$$I = 3000 \times .11 \times \frac{1}{2} = \$165.00$$

$$S = 3000.00 + 165.00 = \$3165.00$$

The elapsed time from July 28 to December 15 is 140 days. Substituting S = 3145.00, d = 0.12, and t = 140/360 in formula (4), we obtain

$$D = 3156.00 \times .12 \times \frac{140}{360} = \$147.70$$

$$\text{Proceeds} = 3165.00 - 147.70 = \$3017.30$$

Chapter 3

EQUATIONS OF VALUE

Sometimes it is desirable to replace one or more obligations with one or more payments at different times that will be equivalent in value to the original obligations. For example, suppose you owe the same creditor $200 due now and $106 due in a year. You have the cash available to settle all the obligations. However, it would be foolish to make cash payment of $306 because part of the debt is not due now. You will want thee creditor to make some allowance for the early payment of the $106 due in a year. If you are using simple interest to reach a settlement, you and your creditor must agree on a rate. Suppose you decide to use 6%. Then we simply figure the present value at 6% of $106 due in a year.

$$P = \frac{106}{1+.06\times 1} = \$100$$

We can add this to the $200 due now to get a total cash settlement of $300. *Note that two sums of money cannot be added until they have been brought to the same point in time.*

In the previous example, suppose that you did not have the cash ad asked the creditor to allow you to settle all thee debts in a year. You must expect to pay interest on the $200 due now. If the agreed rate is 6%, the $200 would amount to $212 in a year. Because this sum is at the same point in time as thee $106, the two can be added, making thee total debt $318 *at that time*. The three alternatives we have discussed can be summarized as follows:

$200 now and $106 in a year
$300 now
$318 in a year

Are equivalent in value if money is worth 6%

Always bring obligations to the same point using the specified rate before combining them. This common point is called a **focal date**, or **comparison date**. When everything has been brought to a focal date, and **equation of value** can be set up and unknown quantities determined. If any obligations are interest bearing, maturity values must be determined before the obligations are moved to the focal date.

If obligations are put on one side of the time line and payments on the other side, it is easy to tell them apart when setting up the equation of value.

In simple interest problems the answer will very slightly depending on the location of the focal date. we will see that in compound interest problems the location of the focal date does not affect the answer. You should recognize that the focal date is simply an arbitrary point in time that enables all obligations and payments to be brought to the same time and at equation of value to be obtained.

There are only two ways to move money – backward and forward. Look at any time diagram. If a sum is to be moved forward, use an amount formula.

Example 1

A person owes $200 due in 6 months and $300 due in 1 year. The creditor will accept a cash settlement of both debts using a simple interest rate of 18% and putting the focal date now. Determine the size of the cash settlement.

Solution

A time diagram with the original debts on one side of the line and the new obligations on the other side will make it easy to obtain the correct equation

To get the \$200 to the focal date, we discount it for 6 months. The \$300 must be discounted for 1 year. Since the unknown payment is at the focal date, it equals simply \$*x*. now that everything is at the same point in time, we set up the equation of value:

$$x = \frac{200}{1+.18\times\frac{1}{2}} + \frac{300}{1+.18\times1} = 183.49 + 254.24 = \$437.73$$

Alternate Solution

We could also do this problem by bringing the values to the focal date separately and then combining them.

Value now of \$200 in 6 months = $\frac{200}{1+.18\times\frac{1}{2}} = \frac{200}{1.09} = \$183.49$

Value now of \$300 in 1 year = + $\frac{300}{1+.18\times1} = \frac{300}{1.18} = \underline{254.24}$

Value now for both debts = \$437.73

***Example* 2**

Solve the preceding problem using 12 months hence as the focal date.

Solution

The \$200 must be accumulated for 6 months, the \$300 is at the focal date, and the unknown payment must be accumulated for 1 year. Then we have the equation of value:

$$x(1 + .18 \times 1) = 200(1 + .18 \times \tfrac{1}{2}) + 300$$

$$1.18\, x = 218 + 300 = 518$$

$$x = \frac{518}{1.18} = \$438.98$$

This example shows that in simple interest problems the answer varies slightly with the location of the focal date. Thus both parties should agree on both the rate and the focal date.

***Example* 3**

A person owes $1000 due in 1 year with interest at 14%. Two equal payments in 3 and 9 months, respectively, will be used to discharge this obligation. What will be the size of these payments if the person and the creditor agree to use an interest rate of 14% and a focal date 1 year hence?

Solution

First, find thee maturity value of the $1000 debt.

$$S = 1000\,(1 + .14 \times 1) = \$1140$$

No further use will be made of the original debt of $1000. Now make a time diagram, move everything to the focal date, and set up the equation of value.

$$x\,(1 + .14 \times \tfrac{3}{4}) + x\,(1+.14 \times \tfrac{1}{4}) = 1140$$

$$1.105\, x + 1.035\, x = 1140$$

$$2.14x = 1140$$

$$x = \frac{1140}{2.14} = \$532.71$$

***Example* 4**

A person borrows \$2000 at 15% interest on June 1, 1996. The debt will be repaid with two equal payments, one on December 1, 1996 and the other on June 1, 1997. Put the focal date on June 1, 1996 and find the size of the payments. Use approximate time and ordinary interest.

Solution

The \$2000 is already at the focal date. The unknown payments are brought back using formula (3). Setting up an equation of value and using reciprocals to simplify computations, we have.

$$\frac{x}{1+.15\times\frac{1}{2}}+\frac{x}{1+.15\times 1}=2000$$

$$\frac{x}{1.075}+\frac{x}{1.15}=2000$$

$$.930233+.869565x=2000$$

$$1.799798x=2000$$

$$x=\$1111.24$$

***Example* 5**

Work example 4 with the focal date on June 1, 1997

Solution

The equation of value is

$$x(1+.15\times\frac{1}{2})+x=2000\,(1+.15\times 1)$$

$$1.075\,x+x=2300$$

$$2.075\,x=2300$$

$$x=\frac{2300}{2.075}$$

Note that a different focal date resulted in simpler computations and slightly different answer.

Example 6

A person borrowed \$6000 on September 15, 1992 agreeing to pay \$2000 on January 15, 1993, and \$2000 on May 15, 1993. If the interest rate was 18%, how much was paid on September 15, 1993 to settle the debt? Use approximate time and ordinary interest. In this problem the logical place for the focal date is September 15, 1993, the final settlement date.

Solution

Using formula (2), we carry forward the debt and two payments to the focal date. Then we write the equation of value.

$$6000(1 + .18 \times 1) = 2000(1 + .18 \times \frac{2}{3}) + 2000(1 + .18 \times \frac{1}{3}) + x$$

$$7080 = 2240 + 2120 + x$$

$$x = 7080 - 2240 - 2120 = \$2720$$

Part two

COMPOUND INTEREST

Chapter 1

AMOUNT AT COMPOUND INTEREST

In most business transactions the practice is to quote an annual interest rate and the frequency of conversion. From this information the rate per period is determined. This 6% compounded (or converted) semiannually means that 3% interest will be earned every 6 months. The quoted annual rate is called the nominal rate and is indicated by the symbol j. the number of interest conversion periods per year is indicated by the symbol m. the equation relating j, i, and m is $i = j/m$ or $j = im$ The symbol $j_{(m)}$. Means a nominal rate j converted m times a year. When no conversion period is stated in a problem, assume that the interest is compounded annually.

In this table shows a few examples of quoted or nominal rates and the corresponding conversion periods per year and rate per period.

Quoted on Nominal Rate (j)	Conversion Periods per Year (m)	Rate Per Period (j =j/m)	
		Percent	Decimal
6% compounded annually	1	6	.06
6% compounded semiannually	2	3	.03

6% compounded quarterly	4	$1\frac{1}{2}$	.015
6% compounded monthly	12	$\frac{1}{2}$	.005
9% compounded quarterly	4	$2\frac{1}{4}$	.0225
9% compounded semiannually	2	$4\frac{1}{2}$	.045
9% compounded monthly	12	$\frac{3}{4}$	.0075

To eliminate the tedious period-by-period computations used in the example at the beginning of this chapter, we invest P dollars for n periods at a rate of *i* per period and derive a formula for the final amount. A numerical example is used to illustrate the steps. In this example p = \$1000, $i = .04$, and $n = 2$. The principal at the end of the first period will be indicated by p_1; at the end of the second period, by p_2; and so on.

Original

Principal	P	\$1000
Interest	pi	$1000 \times .04$
$P1$	$P + Pi = (1 + i)$	$1000 + 1000 \times .04 = 1000\ (1.04)$
Interest	$P + (1 + i)i$	$= 1000(1.04).04$
$P2$	$P + (1 + i) + P(1+i)i$	$=1000(1.04) + 1000(1.04).04$
	$= P(1 + i)(1 + i)$	$= 1000(1.04)(1.04)^2$
	$= P(1 + i)^2$	$= 1000(1.04)^2$

If there were more periods, each new principal would be |(1 + i) times the preceding value. At the end of n periods the final amount, for which the symbol is S, would equal thee original principal times $(1 + i)^n$. this results in the basic formula for compound interest,

$$S = P\,(1 + i)^n$$

S = the amount at compound interest

P = the principal

i = the rate per conversion period

N = the number of conversion periods

The factor $(1 + i)^n$ is called the accumulation factor, or amount of 1, and is sometimes designated by the symbol s.

Note that this formula has the same “ingredients” as thee formulas for amount with simple interest and amount at bank discount– namely, amount (S), present value (P), rate (i), and time (n). They are applied differently in the different formulas. We can see that the amount for 8% is not halfway between the corresponding values at 6% and 10%. On the other hand, with simple interest the value for 8% will be halfway between the 6% and 10% values. The computation in compound interest are somewhat more difficult mathematically, but today we are provided with the machinery to make the computations easy.

The numerical value of $(1 + i)^n$ can be computed by successive multiplication, by logarithms, or by the binomial theorem. In practical business usage the numerical value is usually obtained from previously computed tables, or by using an electronic calculator or a computer. In this book, values of $(1 + i)^n$ for common interest rates will be found in the "amount of 1" common of Table 2 or Table 3. The columns of data in Table 2 have descriptive headings that tell what the various columns are in a few simple words. At the bottom of each column of data symbols are given to help you select the correct factor for a problem.

You can easily appreciate the usefulness of Table 2 when you consider how much time it saves. Carrying a sum of money forward by getting the simple interest for each period and adding it to the principal means two arithmetic operations for each period. This calculation involves a lot of work and many chances to make mistakes if many periods are involved. Table 2 makes it possible to carry money forward 20, 30, or more periods just as easily as making one simple interest computation. Figure 3-3 shows how to get the correct factor from Table 2. Table 3 is an extension of Table 2.

If you have access to a calculator with exponential capability (a key denoted y^x or x^y), you can perform these computations even more easily, you are by bound by the limitation of any table, but only by the limitations of the calculator. The amount of 1 can be calculated for any rate and any number of periods. The problems throughout this book can be worked using such a calculator. Some examples and exercises are beyond the extent of the babbles given in the book; they are indicated by a (C). On such a calculator, it is only necessary to enter the value of

1 + *i* as the base and n as the exponent, and then let the electronics do the work.

FINDING *n*

When a sum of money earns interest from one date to another, the elapsed time must be obtained and converted into interest conversion periods or the value of n. one way to get elapsed time is to subtract the earlier calendar date from the later one.

***Example** 1*

How many semiannual conversion periods are there from June 1, 1989, to December 1, 1994?

- Solution

The dates are put in tabular form to simplify subtraction. After subtracting, we multiply the number of years by the periods per year and divide the number of months by the months in a period. We add these results to get the number of periods.

Year	**Month**	**Day**
1994	12	1
1989	-6	-1
5 years	6 months	0 days

$n = (5 \times 2) + 6/6 = 10 + 1 = 11$

HOW MANY PLACES?

The factors in Table 2 and Table 3 are given to 10 decimal places, making them adequate for handling large sums of money. In the examples worked in the tables published by the Financial Publishing Company, the complete factor is used and the answer is rounded to the scent. If a calculator is available, as it usually is in a business, this method is the sensible one to use in working a problem since it ensures maximum accuracy without any troublesome questions about the number of decimal places to use to get answers correct to the nearest cent.

Since many students have to do calculations manually, we use simple sums in most problems to minimize outline arithmetic. To father conserve your time, we give answers that were obtained by rounding the factors to the same number of decimal places as there are digits in the sum of money given in thee problem, including the cents. Thus a principal of $25.000 would mean round the factor to 7 decimal places, while 4 places would be used for a principal of $25. This practice usually gives an answer that is the same as, or within a penny or two of, what would be obtained using the entire factor. Thus, the reader who uses a calculator may get answers that differ slightly from those in this book.

Example 1

Find the compound amounts of $25, $25000, and $2.5000.000 invested at 6% converted quarterly for 5 years.

Solution

$25(1.015)^{20} = 25 \times 1.3469 = \$33.67$

$2500(1.015)^{20} = 2500 \times 1.34855 = \$3367.14$

$2{,}500{,}000(1.015)^{20} = 2{,}500{,}000\times 1.346855007 = \$3.367.137.52$

Example 2

Banks and other savings institutions offer certificates of deposit and other savings instruments that permit deferral of interest payment until the deposit matures or is redeemed. This practice makes possible the automatic reinvestment of interest at the rate paid on the original investment. This rate is usually higher than the rate paid by the institution on its passbook accounts.

A person aged 60 put $10.000 in a deferred account paying 8% converted quarterly. The account is to mature in 5 years. Find the amount at that time.

Solution

Substituting $P = 10.000$, $i = .02$, and $n = 20$

$P = 10.000(1.02)^{20} = 10.000 \times 1.4859474 = \$14.859.47$

Table 2 and 3 list compound interest factors for many rates. The Financial Compound Interest And Annuity Tables published by the Financial Publishing Company give many more. Any set of tables, however, will be able to include only a limited number of values for rates. If you use a calculator with exponential capability, you can work with any rate.

Example 3

A bank pays 7.8% compound quarterly on savings accounts. A woman put $5000 into such an account on July 1, 1990. Find the amount in the account on January 1, 1995.

Solution

In this case $i = \frac{.078}{4} = .0195$ and $n = 18$

$$S = 5000(1.0195)^{18} = \$7078.48$$

Example 4

A depositor planned to leave $2000 in a savings and loan association paying 5% compounded semiannually for a period of 5 years. At the end of $2\ \frac{1}{2}$ years the depositor had to withdraw $1000. What amount will be in the account at the end of the original 5-year period?

Solution

Substituting $P = 2.92$, $i = .09$, and $n = 15$ in formula (9), we obtain

$$S = 2.92\ (1.09)^{15} = 2.92 \times 3.642 = \$10.63$$

Example 5

During the period 1970 – 1980, the population of a city increased 8%. If the population was 500.000 in 1980, what is the estimated population for 2000 assuming that the same rate of growth continues?

Solution

The conversion period is the basic unit of time in compound interest problems. In financial problems this period is usually 1 year or less. In organic problems it may be many years or it may be a few seconds or less. In this problem it is a decade, so for 20 years, $n = 2$. Substituting in formula (9), we have.

$S = 500.000(1.08)^2 = 500.000 \times 1.166400 = 583.200$

Effective Interest Rates

To put different rates and frequencies of conversion on a comparable basis, we determine the **effective rate**, r. the effective rate is the rate converted annually that will produce the same amount of interest per year as the nominal rate *J* converted m time per year.

If the nominal rate is 6% converted annually, the effective rate will

Also be 6% but if the nominal rate is 6% converted semiannually, the amount of \$1 at the end of one year will be $(1.03)^2$ =\$1.0609. this calculation is simply the accumulation factor for a rate per period of 3% and two periods. The interest on \$1 for 1 year is than \$ 1.0609- 1.000=\$.0609 this result is equivalent to an annual rate of 6.09%

Thus 6.09% converted annually would result in the same amount of interest as 6% converted semiannually. The computations used to get the effective rate of 6.09% in this case can be summarized as follows

$1.0609 = (1.03)^2$

$r = 1.0609 - 1 = 0.609 = 6.09\%$

To obtain an equation to find the effective rate in general, we assume that the effective rate produces the same amount *S* from the given principal ***p*** in 1 year as the compound interest. We than have

$\boldsymbol{S} = p(1+rt) = p(1+r)$

And

$S = p(1+i)^m$

Setting the value of *S* equal

$P(1+r) = p(1+i)^m$

Divide both sides by p:

$1+r \ \ =(1+i)^m$

$r = (1+i)^m - 1$

Notice that the effective rate is determined only by the nominal rate and the frequency of compounding. The effective rate does not depend on the amount of the principal.

Example1

Find the effective rate of interest equivalent to 8% converted semiannually

Solution

The rate per period ***i*** =4% or .04 and ***m***=2. In getting the effective

rate it is usually sufficiently accurate to take the amount -of- 1 factor from table 2 to four decimal places. Substituting in formula (10), we have.

$$\boldsymbol{r} = (1.04)^2 - 1$$

$$= 1.0816 - 1.000 = .0816 = 8.16\%$$

Thus 8.16 compounded annually will produce the same amount of Interest at 8% compounded semiannually.

Example 2

At 7% compounded semiannually \$2000 will amount to how muc 3 years and 5 months?

Solution

The total time in this is 6 whole periods and 5 months left over the compound amount at the end of the 6 whole periods is

$$\boldsymbol{S} = 2000(1.035)^6 = 2000 \times 1.229255 = \$2458.51$$

The simple interest for the remaining 5 months is

$$2458.51 \times 5/12 \times .07 = \$71.71$$

Therefore, the amount at the end of 3 years and 5 months is

$$2458.51 + 71.71 = \$2530.22$$

Chapter 2

present value at Compound interest

The present value is defined as the principal that will amount given sum at the specified future date.

The difference between the future amount and its present value is the **Compound discount**.

To find the present value of a future amount we simply solv compound interest formula for P by dividing both sides by $(1+i)^n$:

$$P=\frac{s}{(1+i)n}$$

With the aid of a calculator this equation can be used to find P. When the work must be done by hand it is more convenient to use negative exponents to express the right hand side we give the equation in both forms

$$P=S(1+i)^{-n}=\frac{s}{(1+i)n} \qquad (11)$$

P=the principal or present value

S=the amount due in the future

I= the rate per period

N=the number of periods

The quantity

$$\frac{1}{(1+i)n}$$

Or $(1+i)^{-n}$ is called the discount factor present worth of 1 or **Present value of 1**.

Example 1

Find the present value of $5000 due in 4 years if money is Worth 8% compounded semiannually.

Solution

Substituting ***S***=500, **i**=.04, and n=8 in formula (11) and using The present –worth-of-1 factor from table2 we have

$$P=5000(1.04)^{-8}=5000\times .730690=\$3653.45$$

This statement means that if $3653.45 had been put at interest for 4 years at 8% compounded semiannually the amount would be %5000 a good exercise for students is to carry money backward and forward in this way to show that the results do check.

Example 2

Find the present value of $7500 due in 4 years if money is worth 14% compounded monthly.

Solution

In this case we must use table 3 we let ***S*** =7500, i= $\frac{14}{12}$%, and N=48

$$P = 7500 \div (1+\frac{14}{12})^{48} = \frac{7500}{1.745007} = \$4297.98$$

An alternative method is to compute

$(1/1.745006919) = .5730636305.$

$P=7500 \times .5730636305=\$4297.98$

Using a calculator with exponential capability allows for the use of any interest rate.

Example 3

How much must be invested in an account paying 8.4% compounded monthly in order to accumulate to $1500
In 5 years?

Solution

The rate, i=8.4%/12=.7% is not found in table 2 0r 3.
However, using a calculator with S=1500,i=.007
And n=60 we have

$$P=15000(1.007)^{-60}=\$9870.13$$

When the time involves a part of a conversion period, we bring *S* back for the minimum number of periods that includes the given time and then compute the simple interest on the principal up to the point at which the present value is wanted.

Example 4

A note with a maturity value of $1000 is due in 3 years and 8 months what is its present value at 6% compou semiannually?

Solution

The original investment will earn interest for 7 periods at 3% then simple interest for the last two months.

The value of the investment at 3 years,6 months is obtained Using the simple interest present value formula (3)

$$P= \frac{s}{1+rt} = \frac{1000}{1+.06\times\frac{1}{6}} = \$990.09$$

Now use the compound interest present value formula (11) To obtain the value of P.

$$P=990.09(1.03)^{-7}=990.09\times.81309=\$805.04$$

One such problems where it is necessary to move money backward and forward or to keep track of several sums of money, it is good practice to make a time diagram that shows the dollar value of each inflow and outflow of money at the point in time where the flow occurs figure3-6 is a diagram for this problem.

In the preceding examples the maturity value of the obligation was known and it was necessary only to discount this amount. Many times, however, the original debt will be interest-bearing And the maturity value will not be given. Then the maturity value at the given interest rate must be determined finally present value of this maturity value is obtained by discounting the rate specified for discounting. The two rates may be, frequently are different.

Example 5

On August 5 1985 Mr. Kane loaned Mr. hill \$2000 at 12 % converted semiannually. Ms. Hill gave Mr. Kane a note promising to repay the loan with accumulated interest in 6 years. on February ,5 ,1989, Mr. Kane sold the note to a buyer, who charged an interest rate of 16% converted semiannually for discounting. How much did Mr. Kane get?

Solution

We must first determine the maturity value of the debt

$$S=2000(1.06)^{12}=2000\times 2.012196=\$4024.39$$

Now the original time and the original interest rate have no more bearing on the problem. All that the buyer of the note is interested in is the maturity value, how long the period of time is until collection can be made and what the note is worth at the specific discount rate.

The note matures on august 5 ,1991. From February 5 ,1989 to this maturity date is $2\frac{1}{2}$ years, or 5 periods of 6 months Each. therefore, the value of the not at 16% compounded Semiannually is

$P=4024.39(1.08)^{-5}=4024.39\times680583=\$2738.95$

Note that the period for discounting is from the time the buyer buys the note to the maturity date. The time at which the original debt was contracted does not enter into this part of the problem excepts to establish the maturity date a time diagram will prevent many of the common errors
made in dealing with this type of problem.

Example 6

A person can buy a piece of property for $4500 cash or for $2000 down and $3000 in 3 years if the person has money earing 6% converted semiannually, which is the better Purchase plan and by how much now?

Solution

We get the present value of $3000 due in 3 years at 6% compounded semiannually.

$P=3000(1.03)^{-6}=3000\times.837484=\$2512.45$

Adding this amount to the $2000 down payment make the present value of the time payment plan $4512.45. By paying $4500 cash, the buyer saves $12.45 now.

Example 7

A piece of property can be purchased for $2850 cash or for $3000 in 12 months. Which is the better plan for the buyer if

Money is worth 7% compounded quarterly? Find the cash (present value) equivalent of the savings made by adopting The better plan.

Solution

We can compare alternative purchase plans by bringing all payments to the same points in time and seeing which plan is better. While the comparison point can be anywhere in time, it is usually best to make the comparison on a present value basis. This method gets all proposals on a "now" basis the Point at which the decision is to be made.

Putting the focal date now and using formula (11) to Discount the $3000 we have

$P=3000(1.0175)^{-4}=3000\times.932959=\$2798.88$

The 2850$ cash payment is simply 1850$ at the focal date. Since the present value of the 3000$ due in a year is less than the cash payment. It is better to pay later. The cash equivalent of the savings is:

2850 – 2798,88 = 51,12$

A different rate of interest could lead to a different decision. If the buyer’s money was earning only 5% compounded quarterly, the present value of 3000$ would be

$$P = 3000(1.0125)^{-4} = 3000 \times .951524 = 28\,54{,}57\$$$

In this case it is better to pay cash and save 2854.58 – 2850.00 = 4.57$ on a cash basis.

EXTENSION OF THE TABLES

One advantage of using a calculator with exponential capability is that there is no practical limitation on rates or numbers of periods. However, for someone using the tables, if the number of period in the problem exceeds the number given, the table can be extended using the law of exponents, $a^m \times a^n = a^{m+n}$. Applying this law in revers order to the amount-of-1 factor, we get $(1 + i)^{m+n} = (1 + i)^m \times (1 + i)^n$.note that whereas the individual exponents add up to the total exponent, we obtain the required factor by multiplying the individual factors.

Example 1

It was reported shortly after the united states bicentennial celebration in 1976 that the mayor of pique, Ohio, was unhappy with the amount of money the city was able to afford for it's festivities. He then contributed 100$ to a mayor's centennial fund of pique to accumulate to accumulate money for the nation's tercentennial in 2076. Find how much his contribution will be worth in 2076 at 5% converted annually.

Solution

We note that n is 100 and the table goes only to 60. However, $(1.05)^{60} \times (1.05)^{40} = (1.05)^{100}$.using these factors, we have

S = $100 \times 18{,}67919 \ \times 7.03999 = 13{,}150.13\$$

Alternate solution

When n is easily divisible by table values, the work can be simplified by using the same factor more than once. In this example $(1.05)^{50} \times (1.05)^{50} = (1.05)^{100}$. Using this fact, we obtain

S = 100$\times$ $11.46740 \ \times 11.46740 = 13{,}150{,}13\$$

The required multiplication may be done in any order

Example 2

In his will a college alumnus appointed a trust company to handle his estate. The company was instructed to set aside a sum in a separate account sufficient to pay his alma matter 250,000$ at the end of 50 years. What sum should the company deposit in the separate account if it earns 8% converted quarterly?

Solution

Substituting S = 250,000$, i = .02, and n = 200 in formula (11), we have

P = 250,000$(1 \ + \ 1.02)^{-200}$

To use table 2 the present-worth-of-1 factor must be broken down into factors with powers no larger than 60. One solution would be

$$250{,}000(1.02)^{-60}\,(1.02)^{-60}\,(1.02)^{-60}\,(1.02)^{-20}$$

Since we are dealing with a large sum of money for a long time, complete factors were substituted and calculations made on calculator.

P = $250{,}000 \times .03047822665 \times .3047822665 \times .3047822665 \times .06729713331 = 4763.27\$$

Alternate solution

Dividing 200 into 4 equal parts, we find that

$$(1.02)^{-200} = [(1.02)^{-50}]^{4}$$

Example 3

Suppose the interest in example 2 was compounded monthly. Find the necessary deposit.

Solution

P = $250{,}000\left(1+\frac{.08}{12}\right)^{-600} = 250{,}000\left[\left(1+\frac{.08}{12}\right)^{-60}\right]^{10}$

$= 250{,}000 \;〚(.6712104444)〛 \hat{}10 = 4640.10\$$

monthly to amount to 100,000$ in 50 years?

Chapter 3

EQUATIONS OF VALUE

In business transaction it is often necessary to exchange one set of obligation for another set of different amounts due at different times. To do this, it is necessary to bring all the obligations to common date called a **focal date**. Then we set up an **equation of value** in which all the original obligations at the focal date equal all the new obligations at the focal date. this procedure is based on the fact that we can find the value of any sum of money at any time by accumulating it at com-pound interest if we take it into the future, or discounting it if we bring it back in time. The equation of value should be thoroughly understood because it is the most effective way to solve many investment problems, particularly the more complicated ones.

Example 1

A person owes 20,000$ due in a 1 year and 30,000$ due in 2 years. The lender agrees to the settlement of both obligations with a cash payment. Before the problem can be worked, the tow parties must agree on an interest rate or value of money to be used in setting up the equation of value. In this case we assume that the lender specifies that

10% compounded semiannually will be used. If this is as much as or more than the borrower can get elsewhere, then the wise decision is to use the cash to cancel the debts. Assume that the borrower is satisfied with this rate of interest, and determine the size of the cash payment.

Solution

The first thing to decide is the location of the focal date. In compound interest problems the answer will be the same regardless of the location of the focal date. Therefore, the only consideration is reducing our work to a minimum. If there is only one unknown payment, putting the focal date at the time that payment is made will eliminate division to obtain the solution.

Most students find a sketch helpful in getting times correct in focal date problems. If the original obligations are put on one side of the time scale and the payment on the other side, the equation of value can be quickly and correctly determined from the sketch.

20,000$ must be discounted for 2 periods and the 30,000$ for 4 periods to transfer them to the focal date. Because the focal date is at the point at which the unknown payment is made, we can set up the equation of value:

$$x = 20{,}000(1.05)^{-2} + 30{,}000\ (1.05)^{-4}$$

$$= 20{,}000 \times\ .9070295 + 30{,}000\ \times\ .8227025$$

$$= 18{,}140.59 + 24{,}681.07 = 42{,}821.66\$$$

Thus, a cash payment of 42,821.66$ will equitably settle debts of 20,000$ and 30,000$ due in 1 and 2 years if money is worth 10% compounded semiannually.

Because the focal date method of solving problems is so useful in the mathematics of finance, you should both understand and have confidence in this method. To show that this is a fair and correct answer to this problem, let us consider it in other way. Suppose that the borrower had invested 42,821,66$ at 10% compounded semiannually, the rate the borrower and the lender agreed on. At the end of 1 year this would amount to $42{,}821.66\$(1.05)^2$, or 47,210,88$. If the borrower deducts the 20,000$ now owed, a balance of 27,210,88$ remains. Carrying this amount forward for another year results in an amount of $27{,}210{,}88(1.05)^2$, or 30,000$. You should check several problems in this way until you are convinced of the utility and the accuracy of the focal date and the equation of value.

Example 2

A person owes 50,000$ due now. The lender agrees to settle this obligation with 2 equal payments in 1 and 2 years, respectively. Find the size of the payments if the settlement is based 9%.

Solution

Since there are 2 unknown payments, it will not be possible to avoid division in the final step. If the focal date is put at 2 years, the coefficient will be as simple as possible and the work will be less than if the focal date were put anywhere else. A sketch shows what factors are needed to get everything to the focal date.

Since the two payments are equal, we designate each of them by x. the value of the first one at the focal date is x(1.09). since the second one is already at the focal date, its value at the point is x.

We now set up an equation of value in which the payments carried to the focal date equal the original obligation at the focal date.

$$X(1.09) + x = 50{,}000(1.09)^2$$

$$2.09x = 50{,}000 \times 1.1881 = \$59{,}405$$

$$x = \frac{59{,}405}{2.09} = \$28{,}423.44$$

Thus, payments of $28,423.44 in 1 and 2 years will equitably settle a debt of $50,000 due now if money is worth 9%. Sometimes students think that this answer should be accumulated or discounted from the focal date to the time the payment is made. This assumption is incorrect. The equation of value gives the size of the payment at the time the payment is made.

Example 3

A piece of property is sold for $50.000. The buyer pays $20,000 cash, and signs a non-interest-bearing note for $10,000 due in 1 year and a second non-interest-bearing note for $10,000 due in 2 years. If the seller charges 10% compounded annually, what non-interest-bearing note due in 3 years will pay off the debt?

Solution

Carrying the $30,000 balance and the payments to the focal date, we get the equation of value,

x = 30,000$(1.10)^3$ _ 10,000$(1.10)^2$ _ 10,000(1.10)

= 30,000 × 1.331 _ (10,000 × 1.21) _ (10,000 × 1.10)

= 39,930 _ 12,100 _ 11,000

= $16,830

This problem can also be worked or cheeked using United States Rule.

Original debt	$30,000
Interest for 1 year	3,000

Amount due in 1 year	33,000
Deduct first payment	10,000
New balance	23,000
Interest for 1 year	2,300
Amount due in 2 years	25,300
Deduct second payment	10,000
New balance	15,300
Interest for 1 year	1,530
Amount due in 3 years	$16,830

Example 4

If any obligations bear interest, we must first compute the maturity values. A person owes $20,000 due in 3 years with interest at 10% compounded quarterly, and $10,000 due in 5 years with interest at 8%. If money is worth 9%, what single payment 6 years hence will be equivalent to the original obligations?

Solution

First, we obtain the maturity values of the debts:

$$20{,}000(1.025)^{12} = 20{,}000 \times 1.3448888 = \$26{,}897.78$$

$$10{,}000(1.08)^{5} = 10{,}000 \times 1.4693281 = \$14{,}693.28$$

Now we sketch the problem and set up the equation of value.

$x = 26{,}897.78(1.09)^3 + 13{,}693.28(1.09)$

$= (26{,}897.78 \times 1.295029) + (14{,}693.28 \times 1.09)$

$= 34{,}833.41 + 16{,}015.68$

$= \$50{,}849.09$

Chapter 4

ORDINARY ANNUITIES

TYPES OF ANNUITES

Annuities are divided into annuities certain and contingent annuities an **annuity certain** is one for which the payments being and end at fixed times. In most cases the payments on a home from an annuity certain because the payments start on a fixed date and continue until the required number has been made. Even if the buyer of the home dies, any outstanding debt on the home must be paid.

A **contingent annuity** is one for which the date of the first or last payment, or both, depends on some event. Pension, social security and many life insurance policies are examples of contingent annuities.

The **payment interval**, or **rent period**, is the length of time between successive payments. Payments may be made annually semiannually, monthly, weekly, or at any fixed interval. The term of an annuity is the time between the start of the first rent period and the end of the last rend period. **The periodic rent** is the size of each payment in dollars and cents.

In this chapter we work with the **ordinary annuity**, for which the periodic payments are made at the end of each period. In the next chapter discuss the **annuity due**, which has the payments made at the beginning of each period.

AMOUNT OF AN ORDINARY ANNUITY

The final value or amount of an annuity is the sum of all periodic payments and the compound interest in them accumulated to the end of the term. In the case of an ordinary annuity, this amount will be the value of the annuity on the date of the last payment.

Example

Starting 1 year from now, a person deposits 500$ a year in an account paying 6% compounded annually. What amount has accumulated just after the 4th deposit is made?

Solution

A sketch is always helpful in visualizing annuity problems. In case we put the focal date at 4 years because we want the amount of all the payments at that time. Starting with the last payment and accumulating all of them to the focal date, we have

4th payment	500,00$
3rd payment 500(1,06)	530,00
2nd payment500$(1,06)^2$	561,80
1st payment500 $(1,06)^3$	595,51

Amount of the annuity 2187.31$

AMOUNT FORMULA

If there are many payments, the preceding method for obtaining the amount would require too much work. Now we derive a general formula for the amount of an annuity of n payments of 1$ each and a rate o I per period. As in the preceding example, we accumulate each payment to the end of the term. At this point the last payment will be 1$, since it has had no time to earn interest. The next-to-the-last payment will be 1(1+i) or simply (1+i+, because it has earned interest for 1 period. The payment before this will amount to$(1+i)^2$, and so on. The first payment will amount to $(1+i)^{n-1}$, because it has earned interest for 1 period less than the number of payments.

Starting with the last payment and writing the sum of all the payments accumulated to the end of the annuity, we have

$$S_n = 1+(1+i)+(1+i)^2+\ldots+(1+i)^{n-2}+(1+i)^{n-1}$$

The expression on the right is a geometric progression in which the first term is 1, the common ratio is (1+i), and the number of terms is n. in algebra it’s shown that the sum of the terms in a geometric progression is

$$S = a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(r^n-1)}{r-1}$$

Substituting values, a =1 and r= (1 +i) from the annuity problem into this general formula, we get

$$S_n = \frac{1\{(1+i)^n-1\}}{(1+i)-1} = \frac{(1+i)^n-1}{i}$$

This factor is also called the amount of an annuity of 1 per period. If the periodic rent is R$ per period instead of 1$, we indicate the total amount of the payments by the symbol S_n. All we have to do to obtain the amount of a annuity of R$ per period is to multiply R by s_n . thus the basic formula for the amount of an ordinary annuity is

$$S_n = Rs_n = R\frac{(1+i)^n-1}{i}$$

S_n = amount of an ordinary annuity of n payments

R = periodic payment or rent

S_n = amount of 1 per period for n periods at the rate I per period

Students who use the tables or a calculator with programmed financial functions will find the first form of the equation, S_n =Rs_n, to be more useful. Those who use of the more common calculators with exponential capability will prefer the second form.

Students using a calculator to compute the factor will need to work out the quantity$\{(1+i)^n$-1} /i. Again, the advantage is that they will not be limited by any set of tables. If the tables are to be used, many people prefer to work with j(m), the nominal annual rate converted n times a year. They use the marginal index to get on the right page of the compound interest and annuity tables. Numerical values of s_n are in the second column of factors in table 2.

Example 1

Starting 1 year from now, a person deposits 500$ a year in an account paying 6% interest compounded annually. What amount is in the account just after the 4th deposit is made?

Solution

Substituting R=500, n =4, and I = 6%, we have

$$S4 = 500 s_n 6\% = 500 \times 4.37462 = 2187,31\$$$

Alternate solution

$$S = 500\frac{(1.06)^4-1}{.06} = 500\frac{1.26247696-1}{.06} = 2187.31\$$$

This problem is the same as the one that took several multiplication and an addition when each deposit was handled separately.

Example 2

A woman deposits 200$ at the end of each 3 months in a bank that pays 5 % converted quarterly. How much will she have to her credit at the end of 10 years?

Solution

Substituting R= 200, n = 400, and I = 1 ¼%, we have

S40 = 200s_{40}1 ¼% = 200× 51,58956 = 10,297,91$

The deposits total 200× 40 = 8000$. Thus the total compound interest is 2297.91$

PRESENT VALUE OF AN ORDINARY ANNUITY

The present value of an annuity is the sum of the present value of all the payments of the annuity. To get the present value, we assume an annuity of n payments of 1$ each and a rate of I per period. Then we discount each payment to the beginning of the annuity. The sum of these discounted values is designated by the symbol a_n.

Writing the sum of all the payments discounted to the beginning of the annuity, we have

$$a_n = (1 + i)^{-1} + (1 + i)^{-2} + \ldots + (1 + i)^{n-1} + (1 + i)^{-n}$$

The expression on the right is a geometric progression in which the first term is $(1 + I)^{-1}$, the common ratio is $(1 + I)^{-1}$, and the number of terms is n. substituting these values in the formula for the sum of a geometric progression, we have

$$a_n = \frac{(1+i)^{-1}\{[(1+i)^{-1}]^n - 1\}}{(1+i)^{-1} - 1}$$

Multiplying numerator and denominator by (1+ i) gives

$$a_n = \frac{(1+i)^{-n} - 1}{1-(1+i)} = \frac{(1+i)^{-n} - 1}{1-1-i} = \frac{(1+i)^{-n} - 1}{-i}$$

$$= \frac{1-(1+i)^{-n}}{i} = \frac{1-v^n}{i}$$

Values of a_n are given in the “present worth of 1 per period” column of table 2. If the periodic rent is $R per period instead of $1, we indicate the present value of the annuity by the symbol An. To get the value of An in dollars, all we need to do is to multiply R by a_n. Thus the basic formula for the present value of an ordinary annuity is

$$\text{An} = \text{R}a_n = \frac{1-(1+i)^{-n}}{i} = \frac{1-v^n}{i} \qquad (15)$$

An = present value of an ordinary annuity of n payments

R = periodic payment or rent

a_n = present worth of 1 per periodic for n periods at the rate i per period

v^n = present worth of 1 $[another\ symbol\ for\ (1+i)^{-n}]$

Note that we have given the formulas for the amount and present value of an annuity is two forms – one using the factors s_n and a_n, the other using the exponential form as obtained is the original derivation. The form using s_n and a_n is preferable for the person using the tables or financial calculators. The exponential form can be used with the more common scientific calculator. Student who wish to use the exponential form should also become familiar with the tables because some types of problems to be encountered later are more easily solved using the tables.

Example 1

A donor wants to provide a \$3000 scholarship every year for 4 years with the first to be awarded 1 year from now. If the school can get 9% return on its investment, how much money should the donor give now?

Solution

Substituting in the present value formula,

$$A4 = 3000a_4 = 3000\times 3.239720 = \$9719.16$$

Note that \$12,000 will actually be paid from this gift.

From the annuity problems worked thus far, you should recognize that the annuity formulas, like the compound interest formulas, make it possible to shift financial obligations from one point in time to another point. For further emphasis we restate in summary from the basic principles underlying the mathematics of finance:

1. Except in cash transactions, the date on which a given sum of money has a certain dollar value must be specified.
2. Once the dollar value on a certain date has been specified the dollar value on any other date can be determined by using the stated interest rate and the appropriate formula.
3. Two or more items must not be equated unless they have been brought to the same point in time.

Example 2

Wilson agrees to pay smith $1000 at the end of each year for 5 years. If money is worth 7%, what is the cash equivalent of this debt? If Wilson does not make any payments until the end of 5 years, how much should be paid at that time if this single payment is to be equivalent to the original payments using an interest rate of 7%?

Solution

The original debt is $1000 at the end of each year. Thus, as each payment is made, it has a dollar value of $1000. If a payment is made before it is due, its dollar value will be less than $1000. If the entire debt is settled immediately, smith will have use of the money from 1 to 5 years sooner than would otherwise have been the case. Therefore, smith will be willing to write off the entire debt if the present value of the payments is received. This situation is expressed by formula (15)

$$A5 = 1000s_{5\ 7\%} = 1000\times 4.100196 = \$4100.20$$

If payment is made after is due date, the dollar value will be increased by the interest. If all the debt is settled at the end pf 5 years, Smith will expect the amount of the payment at that time. This set of events is given by formula (14):

$$S5 = 1000s_{5\ 7\%} = 1000\times 5.750739 = \$5750.74$$

Thus $4100.20 now is equivalent in value to the five $1000 payments at the end of each year is money is worth 7%. Likewise, $5750.74 in 5 years is equivalent to the original obligation. As a check on our work, let us see if $4100.20 now is equivalent to $5750.74 in 5 years at 7%. Substituting in formula (9), we have

$$4100.20(1.07)^5 = 4100.20\times 1.402552 = \$5750.74$$

FINDING n

In an annuity problem, n is number of payments. When a problem is sated in terms of calendar dates, students often find that getting n is the most difficult step in solving the problem. The safest approach is to make a time diagram and a careful analysis of each problem.

In determining the value of n, it is important to remember that n is the number of periods from the beginning of the first period to the end of the last period. You must be careful to distinguish between the date of the first payment and the date of the beginning of the first period of the annuity. In an ordinary annuity the payments are made at the end of each period. The first payment does not occur at the beginning of the first period but at the end. Therefore, it is necessary to read each problem very carefully in order to understand whether the date given is that of the first payment or of the beginning of the annuity. The number of period from the start of an ordinary annuity until the last payment is equal to n. However, the number of periods from the date of the first payment to the date of the last payment is 1 less than n, so to compute n in this case you must find only the number of periods from the first payment to the last payment and then add 1.

Example 1

An ordinary annuity starts on June 1, 1988, and ends on December 1, 1993. Payments are made every 6 months. Find n.

Solution

By the definition of an ordinary annuity, a payment is made at the end of each period

The elapsed time is found from the starting and terminal dates.

Year	Month	Day
1993	12	1
-1998	- 6	- 1
5 years	6 months	0 days

Number of time periods = (5×2) + 6/6 = 10 +1 = 11

Since there is a payment at the end of the each period, n=11.

Example 2

An annuity has payments made every 3 months with the first payment on November 15, 1986, and the last payment on August 15, 1994. Find the number of payments.

Solution

When the dates of the first and last payments are given, special care must be used in getting n. We start with a time diagram

Subtracting the starting date from the terminal date gives the elapsed time.

Year	Month	Day
3	20	

1994	8	15
-1986	-11	- 15
7 years	9 month	0 days

Number of time periods = (7×4) + 9/3 = 28 +3 = 31

Since the problem gives the dates of the first and last payments. n = 32, 1 greater than number of periods from first to last payment.

EXTENSION OF TABLES

In some problems the number of payments is greater than can be found directly in the tables. You can solve such problems by dividing the annuity into parts and then accumulating or discounting the amount or present value of each part of the annuity to the desired point in time.

Example 1

Find the amount of an annuity of $100 at the end of each month for 30 years at 6% converted monthly.

Solution

There are 360 payments, so we divide the annuity into 2 annuities of 180 payments each. The last 180 payments from an ordinary annuity that has an amount of

$s_{180} = 100 s_{180}$ ½% = 100×290.81871 = $29,081.87

The amount of the first 180 payments just after the 180th payments is also $29,081,87. At the time of the 180th payment, this single sum is equivalent is value to the 180 payments. From here on, all 180 payments can be moved at one time simply by moving the single sum. To find the value of the first 180 payments at the end of the term, we take this equivalent single sum forward at compound interest.

$S = 29{,}081.87 \times (1.005)^{180}$

$= 29{,}081.87 \times 2.4540936 = \$71{,}369.63$

Adding the 2 amounts gives a total amount of$100,451.50

Alternate solution

Of course, using a calculator does not subject one to the limitations of the available tables. Using a calculator

$$s_{360} = 100\frac{(1.005)^{360} \quad 1}{.005} = \$100{,}451.50$$

Example 2

Find the present value of an annuity of $100 at the end of each month for 30 years at 6% converted monthly.

Solution

We divide the annuity into 2 annuities of 180 payments each. The first 180 payments form an ordinary annuity that has a present value of

A180 = $100a_{180}$ ½ % = 100 × 118.50351 = $11,850.35

At a point in time 1 period before the 181^{st} payment, the value of the last 180 payments is also $11,850.35. Now we replace payments 181 to 360 with this single sum. To get the value of the last 180 payments at the beginning of the term, we discount the equivalent single sum.

P = 11,850.35$(1.005)^{-180}$

= 11,850.35 × .4074824 = $4828.81

Adding the two results gives a total present value of $16,679.16.

Alternate solution

Again, a calculator can be used to solve the problem.

A360 = $100\frac{1-(1.005)^{-360}}{.005}$ = $16,679.16

When possible, the simplest solution for splitting payments is to divide the annuity into equal parts as was done in these examples. If we had 325 payments, we could separate them into annuities of 180 and 145 payments or any other combination that would total 325 with each part 180 or less.

Example 3

Check the answer to example 1 and 2.

Solution

Since both examples involve the same annuity, we check by taking the present value from example 2 forward 30 years at 6% converted monthly.

S = 16,679.16$(1.005)^{360}$ = 16,679.16$(1.005)^{180}(1.005)^{180}$

= 16,679.16$(2.4540936)^{2}$ = $100,451.50

The result is the same as the amount derived in example 1.

PERIODIC PAYMENT OF AN ANNUITY

In practical business problems the amount or present value of an annuity is frequently known, and the periodic payment is to be determined. This determination can be made by solving the amount and present value formulas for R.

Solving formula , s_n = Rs_n, for R, we have

$$R = \frac{s_n}{s_n} = s_n \times \frac{1}{s_n} = S_n \frac{i}{(1+i)^n - 1} \qquad (16)$$

R = periodic payment or rent

S_n = amount of annuity of n payments

$\frac{1}{s_n}$ = periodic deposit that will grow to 1$ in n payments

Using the previous formula when the future amount is known. Solving formula (15), A_n = Ra_n, for R, we have the following formula:

$$R = \frac{A_n}{a_n} = A_n \times \frac{1}{a_n} = A_n \frac{i}{1-(1+i)^{-n}}$$

R = periodic payment or rent

A_n = present value of annuity of n payments

$\frac{1}{a_n}$ = periodic payment necessary to pay off a loan of 1 $ in n payments

Use the previous formula when the present value is known.

While R could be found by diving $S_n\, or\, A_n$ by the correct factor, problems of this type occur so frequently in practice that the reciprocals of these factors have been determined. The numerical values of $1/s_n$ are giving in the "sinking fund" column (third column of factors) of table 2 and $1/a_n$ in the " partial payment" column (sixth column of factors). This availability of values means that the periodic rent can be determined by multiplying as indicated in the final 2 forms of formulas above. When setting up problems, the rate per period is often included a subscript. If in doubt about which formula to use, draw a time diagram. If the payment precede the single sum, it is an amount; use formula (16) and the " sinking fund" column. If the payments follow the single sum, it is a present value; use formula (17) and the "partial payment" column.

Again we have equations in more than one form. The first is a division form and will be used with table 3 and a simple calculator. Because these problems occur frequently and division is more difficult to perform by hand than multiplication, table 2 provides the numerical values of 1/s_n (sinking fund) and 1/a_n (partial payment). In problems in which table 2 is used, the second form is used in calculating the periodic payments. finally, if you are using a scientific calculator, you can apply the last form for any rate and number of periods.

Example 1

If money is worth 5% compounded semiannually, how much must a person save every 6 months to accumulate 3000$ in 4 years?

Solution

A time diagram shows that the 3000$ is an amount in the future. Substituting S8 = 3000, n = 8, and I = 2 ½% in formula (16), we have

Form 1: R = $\frac{s_n}{s_n}$ =$\frac{3000}{s_8\ 2{,}5\%}$ = $\frac{3000}{8{,}7361159004}$ =343,40$

Form 2: R = s_n $\times \frac{1}{s} = 3000 \times \frac{1}{s_8\ 2{,}5\%}$ = 3000 × .114467 = 343,40$

Form 3: R = s_n $\frac{i}{(1+i)^n-1} = 3000\frac{.025}{(1{,}025)^8-1}$ =343,40%

Note that 8 payments of 343,40$ total 2747,20$. The balance needed to produce an amount of 3000$ comes from the accumulated interest on each payment from the time it is made to the end of the 4 years.

Example 2

A couple wants to buy a new automobile costing 9000$. Their down payment, including trade-in, is 1500$. The reminder is to be paid in monthly installments over 4 years at 12% converted monthly. Find the size of the monthly payment.

Solution

A time diagram shows that 7500$ is the present value. Substituting A48 =7500, n = 48, and I = .01 in formula (17), we use all three forms:

Form 1: R = $(7500\ )/(a\_48\ 1\%) = \ 7500/37{,}9739595 =$ $197{,}50\$$

Form 2: R = 7500 × $\frac{1}{a_{48}\ 1\%}$ =7500 × .0263338354 = 197,50$

Form 3: R = 7500 × $\frac{.01}{1-(1{,}01)^{-48}}$ =197,50$

Example 3

A family wants to buy a home costing 100,000$. If they put 15,000$ down and get a 30-year mortgage at 13% with monthly payments, how much will they be required to pay monthly?

Solution

There are 360 payments, so we cannot use table 2. Applying table 3 and form 1 with n= 360, A360 =85,000$, and I = $\frac{13}{12}$%, we get

$$R = \frac{85,000}{a_{360}} = \frac{85,000}{90,3996054} = 940,27\$$$

Note that 360 payments of 940,27$ amount to 338,497,20$, making the total interest 253,497,20$

It is interesting to see that if the family could borrow the 85,000$ at 12% compounded monthly, the payment would be

$$R = \frac{85,000}{97,2183311} = 874,32\$$$

Hence a reduction of 1 percentage point in the rate results in a savings of 65,95$ every month. All of this difference is interest. Over 30 years the saving amount to 360 × 65,95$ = 23,742$. This sum is not, of course, equivalent to a total cash saving of the amount since the savings are spread over 360 months at the rate of 65,95$ per month. The equivalent cash saving will depend on the value of money to the family. For example, if they can invest at 6% converted monthly, the equivalent cash saving would be

$65{,}95a_{360}$ = $65{,}95 \times 166{,}7916144 = 10{,}999{,}91\$$

Whether we make the comparison on the basis of the total saving in interest or on the equivalent cash value of the saving, there is no doubt that a person should shop around to find the most reasonable rate before borrowing money. An apparently small difference in the rate may translate into the saving of thousands of dollars.

Another thing to note in the previous example is that with 12% rate, the interest totaled just under 230,000$. Interest costs on a home loan can be reduced by making a larger down payment. Savings on interest provide an incentive for people to save money.

EXTENTION OF TABLES

In using either of the first two forms of formula (16) and (17), you are limited to the rates and terms given in the tables. When dealing with a number of periods not included in the tables, use the methods de-scribed in section 4,7 and divide.

To reduce arithmetic, examples and exercises involving extension of tables use factors rounded to the same number of decimal places as there are digits in the sum of money. When a calculator is available, it is good practice to use complete factors and round the final answer as suggested in section 3,6.

Example 1

How much must be deposited each quarter in an account paying 6% converted quarterly to accumulate 10,000$ in 20 years?

Solution

S = 10,000, I = 1 ½ %, and n = 80

$$R = \frac{10{,}000}{s_{80}\ 1\ ½\%}$$

We get s_{80} by using two annuities of 40 payments each.

Amount of last 40 payments =s_{40} =54.2678939

Amount of first 40 payments = $s_{40}\ (1{,}015)^{40}$

= 54,2678939 × 1,8140184 = 984429581

s_{80} = 152.7108520

If only one problem is to be solved, we can obtain R by diving 10,000 by s_{80}.

$$R = \frac{10{,}000}{152{,}7108520} = 65{,}48\$$$

Example 2

To help finance a purchase of a mobile home, a couple borrows 30,000$. The loan is to be repaid in quarterly payments over 25 years. If the rate is 10% converted quarterly, find the size of the quarterly payments.

Solution

Substituting

$A_{n=30,000,\ \ n=100,\ \ and\ i=2\ ½\%\ in\ formula\ (17),we\ have}$

$$R = \frac{30,000}{a_{100}\ 2\ ½\%}$$

We get a_{100} using 2 annuities of 50 payments each. All payments are discounted to the present.

Present value of first 50 payments = a_{50} = 28,3623117

Present value of last 50 payments =$a_{50}\ (1.025)^{-50}$

= 28,3623117× .2909422 =8,2517934

a_{100} = 36,6141051

$$R = \frac{30,000}{36,6141051} = 819,36\$$$

FINDING THE TERM OF AN ANNUITY

Some problems specify the amount or present value, the size of the payments, and the rate. The number of payments is what needs to be determined. When an integral number of payments is not exactly equivalent to the original amount or present value, one of the following procedures is used in practice. First, the last regular payment can be increased by a sum that will make the payments equivalent to the amount or present value. The second alternative is that a smaller concluding payment can be made one period after the last full payment.

In this book we follow the second procedure unless the problem states that the last full payment is to be increased. Sometimes when a certain amount of money is to be accumulated, a smaller concluding payment will not be required because the interest after the last full payment will equal or exceed the balance needed.

Example 1

A woman wants to accumulate 5000$ by making payments of 1000$ at the end of each year. If she gets 5% on her money, how many regular payments will she make and what will be the size of the last payment?

Solution

Using formula (14) for the amount of an annuity, we obtain

$$1000 s_n 5\% = 5000$$

$$s_n\ 5\% = \frac{5000}{1000} = 5.0000$$

We look in table 2 under 5% for the factor 5.0000 in the "amount of 1 per period" column. We find that the factor is 4.310125 for 4 periods and 5.525631 for 5 periods. Therefore the woman will have to make 4 deposits of 1000$ and a fifth smaller deposit to be determined. To find the size of this last deposits, we use an equation of value.

Instead of taking the four 1000$ payments separately to the focal date, we get the amount of these four payments at the end of 4 years.

S40 = $1000s\_4\ 5\% = 1000 \times 4,310125 = 4310,13\$$

Now we can take this amount to the focal date using simple interest in the equation of value:

4310,13(1,05) + x = 5000

X = 5000 – 4525,64

= 474,36$

Thus, if the woman deposits 1000$ at the end of each year for 4 years and 474,36$ at the end of the fifth year, she will have exactly 5000$ in her account. In problem like this one, note that the last deposit will never be larger than the others.

Alternate solution

It is also possible to get the 4 payments to the focal date by looking up s_5 and subtracting 1 to allow for the fact that no full payment is made at the end. Then the amount becomes

1000(5,525631 -1) = 4525,63$

Hence a balance of 474,37$ will have to be deposited at the end of the fifth year.

The 1-cent difference in the answers is due to rounding.

This second method will be discussed under annuities due in the next chapter.

Example 2

Do example 1 again assuming that the woman wants to accumulate 7000$

Solution

Substituting in formula (14), we find that

$$1000 s_n\, 5\% = 7000$$

$$s_n 5\% \ = 7.0000$$

The second column in table 2 shows that 7,0000 lies between 6 and 7 payments. the amount of 6 full payments is

$$S\_6 = 1000 \times 6{,}8019113 = 6801{,}91\$$$

Carrying this amount forward for 1 year at simple interest, we obtain

$$6801{,}91(1.05) = 7142.01\$$$

No smaller concluding payment is required

Example 3

A woman dies and leaves her husband an estate worth 50,000$. Instead of receiving the full bequest in cash, he is to get monthly payments of 1,000$. How many monthly payments will he receive and what smaller payment 1 month after the last regular payment will settle the estate if interest is at 6% compounded monthly?

Solution

Using formula (15) for the present value of an annuity, we have

$$1000a_n \text{ ½\%} = 50{,}000$$

Solving for a_n, we obtain

$$a_n \text{ ½\%} = 50$$

Looking in table 2 under ½% for the factor 50, we find that $a_{57=49,49031\ and\ a_{58}}$= 50,23911. Therefore the widower will receive 57 payments of 100$ and a 58th payment that is smaller. To find the size of the concluding payment, we set up an equation of value.

$$S\_57 = 1000s\_57 \text{ ½\%} = 1000 \times 65{,}763609 = 65{,}763{,}61\$$$

Now we take this amount and the 5000$ to the focal date and set up the equation of value.

$$65{,}763{,}61(1{,}005) + x = 50{,}000(1{,}005)^{58}$$

$$X = 50{,}000 \times 1{,}3354621 - 65{,}763{,}61(1{,}005)$$

$$= 66{,}773{,}11 – 66{,}092{,}43 = 680{,}68\$$$

yment.

Chapter 5

OTHER ANNUITIES CERTAIN

ANNUITY FUNCTUINS

The amount of 1 per-period function gives the amount of n periodic deposits of 1$ at the time of the last deposit. The symbol for this function is s_n. The present worth of 1 per-period function gives the present worth of n payments of 1$ at a point in time one period before the first payment. The symbol for this function is a_n.

Ordinary annuities have the payments made at the end of the time periods. The amount of an ordinary annuity is found by simply substituting the value of s_n in formula, S_n =Rs_n. The present value of an ordinary annuity is found by substituting the value of a_n in formula, A_n=Ra_n.

Now we consider other types of annuities that are evaluated. To solve these annuities, we modify or combine the ordinary annuity functions. Since this is a simple matter, publishers of compounded interest tables print only numerical values of the ordinary annuity functions.

We should understand exactly what is meant by the tabled values of s_n and a_n.

The value of an annuity may be needed at any point in time from several periods before the first payment to several periods after the last payment.

Annuity Due

An annuity due is one in which the payments are made at the beginning of the payment interval, the first payment being due at once. Insurance premiums and property rentals are examples of annuities due. To find the amount and the present value formulas for the annuity due, we modify the formulas already derived from the ordinary annuity so that the ordinary annuity tables can be used with simple modifications.

Note that the diagram starts with a payment and ends one period after the last, or nth, payment.

Our sketch shows that if we obtain the amount of an ordinary annuity of n+1 payment, we will have the amount of the corresponding annuity due except that we will have included a final payment that is not actually made at the end of the last period of the annuity due. Therefore, all we have to do is subtract that last payment to permit us to use the ordinary annuity tables to get the amount of an annuity due.

The procedure leads to the following formula:

$$S_n \text{ (due)} = \mathrm{R}s_{n+1} - \mathrm{R}$$

$$= \mathrm{R}(s_{n+1} - 1)$$

$$=R\left(\frac{(1+i)^{n+1}-1}{i}-1\right)$$

$$=R\frac{(1+i)^{n+1}-(1+i)}{i}$$

$$=R\frac{(1+i)^{n}-1}{i}(1+i)$$

$$=Rs_n\ (1+i)$$

$$S_n(\text{due}) = R(s_{n+1}-1)$$

$$=Rs_n(1+i) = R\frac{(1+i)^{n}-1}{i}\ (1+i)$$

Again we have a factor form of the equation using the factors as well as an exponential form for those using a scientific calculator.

The rather complex appearing quantity that is to be multiplied by R in the first form is quite simple to get. First we add 1 to the number of payments in the annuity due and use the resulting value when we go the tables. For example, if there are 10 payments, we look up the value for 11 periods. Then we subtract 1 from the factor. For an interest rate of 2% and 11 periods, the factor is 12, 1687(correct to four decimal places). To get the amount of an annuity due of 10 payments, we would use 11, 1687 for the factor. All these steps can be done without writing.

To get the present value of an annuity due, we again make a sketch with the corresponding ordinary annuity.

We match present value of an annuity due; we again make a sketch with the corresponding ordinary annuity. We match the annuity due with an ordinary annuity so that we can get numerical factors from the compound interest and annuity tables. If we obtain the present value of an ordinary annuity of n – 1 payment, we will have the present value of the corresponding annuity due except that we will not have included a payment that is made at the beginning of the term of the annuity due. Therefore all we have to do is add this first payment. This procedure leads to the following formula for the present value of an annuity due:

$$\text{An} = \text{R}a_{n+1} + \text{R}$$

$$= \text{R}(a_{n+1} + 1)$$

$$= \text{R}[(1\ \_\ 〖(1+i)〗\ \hat{}\,(-(n-1)))/i+1]$$

$$= \text{R}\frac{\left((1+i)\_(1+i)^{(n-1}\right)}{i}$$

$$= \text{R}\frac{1^{-n}_{(1+i)}}{i}\,(1+\text{i})$$

$$= \text{R}a_n(1+\text{i})$$

$$\text{An(due)} = \text{R}(a\_(n-i)+1)$$

$$= \text{R}a_n(1+\text{i}) = \text{R}\frac{1^{-n}_{(1+i)}}{i}\,(1+\text{i})$$

Again, it is simple to use the ordinary interest tables. First we subtract 1 from the number of payments in the annuity due and use the resulting value when we go to the table. For example, if there are 8 payments in an annuity due, we look up the value for 7. Then we add 1 to the factor. For an interest rate of 6% and 7 periods, the factor is 5.58238. Adding 1 to this factor, we obtain 6.58238 as the present value of an annuity due of 8 payments at a rate of 6%.

Example 1

An investment of $200 is made at the beginning of each year for 10 years. If interest is 6% effective, how much will the investment be worth at the end of 10 years?

Solution

We use all three forms of formula (18) with R= 200, n = 10, and i = 6%.

First form: S_n(due) = 200(S_(11 – 1)=200×13.97164=$2794.33

Second form: Sn(due) = 200s_{10}(1.06) =200×13.18079×1.06

=$2794.33

Third form: s_n (due) = $200\frac{(1.06)^{10}-1}{.06}$ (1.06) = $2794.33

Example 2

A student wants to have $2500 for a trip after graduation 4 years from now. How must she invest at the beginning of each year starting now if she gets 5% compounded annually on her saving?

Solution

Substituting in formula and solving for R, we have

$$2500= R(s_5 \_ 1) = 4.525631R$$

$$R= \frac{2500}{4.525631} = \$552.41$$

Alternate solution

We can check the answer by using an equation of value with the focal date at the time of the last deposit. The 4 payments are brought to the focal date as ordinary annuity, and the $600 is brought to the focal date by discounting it for 1 period. See figure 5-5.

$$Rs_4 =2500(1+i)^{-1}$$

$$R \quad = 2500(1.05)^1 \times \frac{1}{s_4}$$

$$= 2500 \times .952381 \times .232012 = \$552.41$$

This method requires an additional multiplication but no division.

We can also work this problem using the second form:

$$2500= Rs_4(1.05)$$

$$= R\times 4.310125\times 1.05$$

$$2500 = 4.52563R$$

$$R = \frac{2500}{4.52563} = \$552.41$$

Example 3

The premium on a life insurance policy is \$60 a quarter, payable in advance. Find the cash equivalent of year's premiums if the insurance company charges 6% converted quarterly for the privilege of paying a smaller amount every three month instead of all at once for the year.

Solution

Substituting R= 60, and i = 1 ½% into all three forms of formula (19), we have

First form: An(due) =60(a_(3) + 1)= 60×3.9122= \$234.73

Second form: An(due) = 60a_4(1.015) = 60×3.8544×1.015 =\$234.73

Third form: An(due) = $60\frac{1-(1.015)^{-4}}{.015}$ (1.015)= \$234.73

Although paying by the quarter is convenient, the saving from paying annually can be substantial.

DEFERRED ANNUITY:

A deferred annuity is one in which the first payment is made not at the beginning or end of the first period, but at the same later date. When the first payment is made at the end of 10 periods, the annuity is said to be deferred 9 periods. Similarly, an annuity that is deferred for 12 periods will have the first payment at the end of 13 periods. It is important to understand that the interval of deferment ends I period before the first payment. In practice attention must be given to this point because in same problems the interval of deferment is given, in others the time of the first payment. When the time of the first payment is given, you must determine the interval of deferment before substituting in the present value formula. For example, if payments are made quarterly and the first payment is made in 4 years, the interval of deferment is 15 periods.

Figure 5-9 shows a deferred annuity of n payments that is deferred for m periods.

To obtain a formula for the present value of a deferred annuity, we begin by assuming that a payment is made at the end of each period during the interval of deferment. If this were the case , we would have an ordinary annuity od m+n payments and the present value would be Ra$\sqrt{m}+n$. This value, of course, is too large because it includes the assumed payments during the interval of deferment. These assumed payments from an ordinary annuity, and their present value is Ra $\sqrt{m}$. Therefore, to get the present value of a deferred annuity, all we have to do is subtract the present value of the m assumed payments from the present value of the m + n payments.

$$\text{An(def)} = \text{R}a_{m+n} - \text{R}a_m$$

$$= \text{R}(a_{m+n} - a_m)$$

$$= \left[\frac{1 - (1+i)^{-(m+n)}}{i} - \frac{1 - (1+i)^{-m}}{i}\right]$$

$$= \text{R}\,\frac{(1+i)^{-m} - (1+i)^{-(m+n)}}{i}$$

$$= \text{R}\,\frac{1 - (1+i)^{-n}}{i}(1+i)^{-m}$$

$$= \text{R}a_n(1+i)^{-m}$$

$$\text{A}_{\text{n}}\text{(def)} = \text{R}(a_{m+n} - a_m) = \text{R}a_n(1+i)^{-m}$$

$$= \text{R}\frac{(1+i)^{-m} - (1+i)^{-(m+n)}}{i}$$

To get the present value of a deferred annuity, from the first we look up a_{m+n} and a_m in the present worth of 1 per period column of table 2, subtract the smaller value from the larger, and multiply this difference by the periodic payment.

The present value of a deferred annuity can also be obtained by using formula (15) for the present value of an ordinary annuity to get the value of the payments 1 period before the first payment is made. Then this equivalent single sum can be discounted for the m periods in the interval of deferment, and we have

$$A_n(\text{def}) = Ra_n(1+i)^{-m}$$

This from of the deferred annuity formula is useful when logarithms are used table 3 to get the present value of the deferred annuity using the second from and the fact that $(1+i)^{-m} = 1/(1+i)^m$. You can use the last from if you have access to a scientific calculator.

Example 1

Find the present value of a deferred annuity of \$500 a year for 10 years that is deferred 5 years. Money is worth 6%

Solution

We solve using all three forms, substituting n=10 = 5,R=500,and I =6%.

First form: $A_{10}(\text{def}) = 500(a_{15} - a_5)$

$$=500(9.71225\_4.21236)$$

$$=500\times 5.49989$$

$$=\$2749.95$$

Second form: $A_{10}(\text{def}) = 500a_{10}(1.06)^{-5}$

$=500\times7.36009\times.74726$

$=\$2749.95$

Third form: $A_{10}(def) = 500\frac{(1.05)^{-5}-(1.06)^{-15}}{.06}$

$=\$2749.94$

The 1-cent difference is due to rounding.

Example 2

Find the present value of an annuity od $50 every 3 months for 5 years if the first payment is made in 3 years. Money is worth 5% converted quarterly.

Solution

Here n=20 and i ¼% . To find the interval of deferment, we drop back one period from the first payment, so m=11.

$$A_{20}(def) = 50(a_{11+20} \_ a_{11})$$

$$=50(25.5693\_10.2178) = \$767.58$$

Example 3

A woman inherits $20,000. Instead of taking the cash, she invests the money at 8% converted quarterly with the understanding that she will receive 20 equal quarterly payments with the first payment to be made in 5 years. Find the size of payments.

Solution

The interval of deferment is 19 periods, the present value of the annuity is \$20.000, and i is 2%. Substituting in formula (20) we have

$$20.000 = R(a_{19+20} - a_{19})$$

$$= R(26.9025888 - 15.6784620)$$

$$= 11.2241268R$$

$$\mathbf{R = \frac{20.000}{11.2241268}}$$

$$= \$1781.88$$

Alternate Solution

Using the alternate solution for deferred annuities, we find that

$$20.000 = Ra_{20}\,(1.02)^{-19}$$

$$R = 20.000(1.02)^{19} \times \frac{1}{a_{20}}$$

$$= 20.000 \times 1.4568112 \times .0611567$$

$$= \$1781.88$$

Chapter 6

Amortization and sinking Funds

AMORTIZATION OF A DEBT

We consider the amortization method. When a debt is repaid by this method, a series of periodic payments, usually equal in amount, pay the interest outstanding at the time payments are made and repay a part of the principal. As the principal is gradually reduced in this way, the interest on the unpaid balance decrease. In other words, as time goes on, an increasing portion of the periodic payments is available to reduce the debt. Sooner or later most of the students who study this material will purchase a car, home, or other item on time and will amortize the debt.

FINDING THE PAYMENT

When a debt is amortized by equal payments at equal intervals, the debt becomes the present value of annuity. We determine the size of the payments by the methods used to get the periodic rent in the annuity problems in the preceding chapter.

Example 1

A family buys a $60.000 home and pays $10.000 down. They get a 26 years mortgage for the balance. If the lender charges 12% converted monthly, what is the size of the monthly payment?

Solution

Here we have an ordinary annuity with An= $50,000, n= 300, and i = 1%. From formula (17) we find

$$R= \frac{50.000}{a_{300}1\%}$$

$$= \frac{50.000}{94.9465513}$$

$$= \$526.612$$

When a debt is amortized, the common practice is to round up any fraction od a cent. Thus the payment in example 1 would be rounded to $526,62 if the rounding were to the next highest cent. This ensures complete amortization in the time specified. If each payment were low by a fraction of a cent, there would be a small outstanding debt after the specified number of payments had been made. Instead of rounding up to the cent, a lender may round up to the dime, the dollar, or any other unit of money. This practice does not injure the borrower because the higher periodic payment will be offset by a smaller concluding payment.

Example 2

A debt of $10,000 bearing interest at 12% compounded semiannually is to be amortized in 20 semiannual payments.

Find the concluding payment if the semiannual payment is rounded up to the: (a) cent; (b) dime; (c) dollar.

Solution

Substituting A20 = 10.000, i = 6%, and n= 20 in formula (17), we have:

$$R= 10.000\times \frac{1}{a_{20}\ 6\%}$$

$$=10.000\times .0871846 = \$871.846$$

We can see that the simplest method of solution is to set up an equation of value with the focal date at the 20th payment. Then the concluding payment is simply the difference between the \$10.000 carried to the focal date and the amount of an annuity due of 19 regular payments. Formula takes the \$10.000 to the focal date and formula brings forward the 19 payments.

(a) If the payment is rounded up to the cent, R=\$871.85.

$$X= 10.000(1.06)^{20}\ \_\ 871.85(s_{20}\text{-}1)$$

$$=10.000\times 3.20713547\quad \_871.85\times 35.78559$$

$$= 32{,}071.35\ \_\ 31{,}199.67$$

$$= \$871.68$$

(b) If the payment is rounded up to the dime, R= \$871.90.

$$X= 32{,}071.35\ \_\ 871.90\times 35.78559= \$869.89$$

(c) If we round up to the dollar, R= \$872.00.

X= 32,071.35 _ 872×35.78559 = $866.32

OUTSTANDING PRINCIPAL

knowing the principal outstanding at a certain time is often important. The borrower may want to pay off the rest of debt in a lump sum, or the lender may want to sell the debt that is still owed. To calculate the principal outstanding at any time, we can find the value of the remaining payments at that time.

Example 1

To pay off an $80,000 debt, a man got a 20-year loan at 12% converted monthly. How much does he still owe after he has paid on it for 5 years?

Solution

The monthly payment is $880.87. To find the outstanding principal, we use an equation of value and compare the original debt with the payments that have been made. We put the focal date at the 60th payment and bring everything to this point. See figure 6-2.

X= $80.000(1.01)^{60}$ _ $880.87s_{60}$

= 80.000×1.81669670 _ 880.87×81.6696699

= 145,335.75 _ 71,940.30= $73,395.38

Thus far the principal has been reduced by 80.000 _ 73,395.38 = $6604.62, although the man has paid $52,852.20 (60 payments ×$880.87). During the early years of a long term amortization program, a large part of the money paid goes for interest. Recognition of this fact leads to a practical point: the buyer of a home should try to borrow form a lender who offers the lowest rate of interest and who will permit the borrower to pay off the mortgage more quickly than the contract requires without charging a fee. Frequently, a borrower will find that an increase in income or an inheritance will allow the payment of more per month on the mortgage than the original terms specified. Since the borrower probably cannot invest this money at as high a rate as that being paid on the mortgage, it is to the borrowers advantage to use it to reduce the debt. It is disconcerting to discover that a small print provision, which the borrower had not read, in the contract imposes a penalty for this advance payment.

Alternate solution

In this problem we can get a good check on our answer by finding the present value of the remaining 60 payments, which form an ordinary annuity.

$$A_{180} = 880.87\, a_{180} 1\%$$

$$= 880.87 \times 83.3216640 = \$73,395.55$$

This result differs by 17 cents from the preceding result because the monthly payment was rounded up to the next cent. Carrying the monthly payment to several decimal places would make the two results closer, but there is no reason to go to this trouble, since the payment must be rounded in practical problems.

We could not use this method od checking if the payment had been rounded up to the dime or to the dollar because then the concluding payment would have differed more from the other payments.

Example 2

On May 15,1992, a woman borrowed \$25,000 at 15% converted monthly. She planned to repay the debt in equal monthly payments over 15 years with the first payment on June 15,1992. The 12 payments during 1994 reduced the principal by how much? What was the total interest paid in 1994?

Solution

We use formula to find the monthly payment.

$$R= 25{,}000 \times \frac{.0125}{1\_(1.0125)^{-180}} = \$349.90$$

The total reduction in principal in 1994 will be the difference between the outstanding principal in December 1993 and December 1994. A time diagram shows that the outstanding principal must be determined after the 19th and the 31 st payments. This determination requires two equations of value using these points as focal dates.

Outstanding on 12/15/93:

$25{,}000(1.0125)^{19}$ _ 349.90 s_{19} =$24,203.50

Outstanding on 12/15/94

$25{,}000(1.0125)^{31}$ _ 349.90s_{31} = $\frac{23{,}594.48}{\$609.02}$

To find the total interest paid in 1994, we subtract the amount applied to principal from the total payments, which equal 12 × 349.90=$4198.80. Then we find that the total interest paid is 4198.80 _ 609.02= $3589.78.

OWNER'S EQUITY

The owner's equity starts with the down payment and is gradually increased by that part of each monthly payment over and above the interest. On a long term loan, equity in a home is built up slowly at first because of interest charges. The amount of the owner's equity can be found at any time by subtracting the outstanding principal from the original loan and adding the down payment to this difference.

We should recognize that owner's equity calculated in this way makes no allowance for increases or decreases in the value of the property. If a neighborhood or a piece of property deteriorates, an owner may not recover the investment when the real estate is sold. On the other hand, if the property increases in value, the calculated equity is less than the true value of the owner's investment. A long period of rising prices and favorable taxation of capital gains have resulted in attractive financial returns on many pieces of real estate.

Example

A person buys a 62,500$ home and pays 12,500$ down. The homeowner gets a 25-year, 12%loan for the balance. Monthly the payments are 526,62$. What is the owner's equity after 10 years?

Solution

The outstanding principal after the 120th payment is found by putting a focal date at that time and taking the original debt and the payments to that point.

Outstanding principal= $50{,}000(1{,}01)^{120}$ _ $526{,}62s_{120}$

=165,019,34 -121,142,97 =43,876,37$

Addition to owner's equity = 50,000 -43,876,37 = 6123,63$

At the end of 10 years the homeowner has paid a total of 120 × 526,62 =63,149,40 to get 6123,63$ worth of house. After 10 years the owner's equity is 12,500,00 + 5123,63 = 18,623 ,63$.

AMORTIZATION SCHEDULE

When debts are repaid using the amortization method, it’s important to know what portion of each payment goes for interest and how much is applied to reducing the principal. In the following example we see that interest can be a substantial part of the periodic payment.

Example 1

A person buys a home and gets a 25-year, 12%mortgage for 50,000$. The monthly payments are 526,62$. The buyer entering the lending institution to make the first monthly payment probably thinks that a house is being bought with it. How much house is being purchased with the payment?

Solution

The interest rate is 1% a month. The interest for the first month will be 500$. Deducing this amount from the monthly payment means that the homeowner gets only 26,62$worth of house with the payment of 526,62$. Of course, things get brighter immediately. After the first payment the unpaid balance drops to 50,000-26,62 =49,973,38$. The second month’s interest is figured on this principal and amounts to 499,73$. With the second payment the buyer gets more house , 27 cents more as a matter of fact. As the payments are continued, more and more of each payment will be applied to the debt. when the debt gets down to approximately 20,000$ (this reduction will take about 21 years), the interest will be only about 200$and most of the payment will go for house.

A person who buys a home on amortization plan should get a periodic statement from the lender showing how much interest has been paid and the amount of reduction of the mortgage. The buyer could also obtain an amortization schedule that shows the exact financial breakdown at any time. The interest column will show the buyer how much interest can be deducted in a calendar year when computing income tax. The same columns will show the lender how much must be reported as income. Since there is no space here to show an amortization schedule for 300 periods, we use a shorter problem. The schedule for this problem will include all the steps necessary to construct any amortization schedule.

Example 2

A debt of 5000$ is to be amortized by 5quarterly payments made at 3-month intervals. If interest is charged at the rate of 12% convertible quarterly, find the periodic payment and construct an amortization schedule. Round the payment up to the cent.

Solution

Substituting in formula, we have

$R=5000\times \frac{1}{a_{5\ 3\%}} = 1091,775\$= 1091,78\$$

The final payment equals the preceding loan balance plus interest on this balance for the last period. When determining the periodic payment, lenders customarily round up.

Amortization schedule

Payment number	Total payment	Payment on interest .03 × (5)	Payment on principal (2)-(3)	Balance of loan (5)-(4)
1	1091.78$	150.00$	941,78$	5000,00$
2	1091.78	121,75	970,03	4058,22
3	1091,78	92,65	999,13	3088,19
4	1091,78	62,67	1029,11	2089,06
5	1091,75	31,80	1059,95	1059,95
	5458,87$	458,87$	5000,00$	—

SUMMERY OF LECTUERS

Simple interest formula

$$I= Prt$$

I= simple interest in dollars (or the monetary unit)

P= Principal in dollars…………….

R= interest rate

T= time in unit that cores pond to the rate.

Note :

That r and t must be consistently stated. That is, if the rate is an annual rate, the time must be stated in years; or if the rate is monthly rate, the time must be stated in months.

Simple interest Application:

Saving accounts.

Open accounts

Certificates of deposit.

Promissory note:

Face Value:
Is the amount of money shown on the note, this amount is usually given in figures and in words.

Rate:
Is the percent at which interest is to be calculated؟

Payee:
Is the person of firm to whom the note is due?

Maturity date:

Is the date the note is due? This is calculated from the date and term of the note.

Maturity value:

Is the face value plus interest if any?

- If no interest rate is given in a note then

Maturity value = face value

No.1 A bank pays 8% per annual on savings accounts. A person opens an account with a deposit of $ 300 on January 1, how much interest will the person receive on April 1?

Solution:

$$I = Prt$$

$$I= 300 \times .08 \times \frac{1}{4} = \$6.00$$

Since the rate is an annual rate, the time must be expressed in years. The time is 3 months or $\frac{3}{12}=\frac{1}{4}$

No.2 The interest paid on a loan of $ 500 for 2 months was $12.50 what was the interest rate?

Solution:

$$500 \times r \times \frac{1}{6} = 12.50$$

$$r = \frac{6 \times 12.50}{500} = .15 = 15\%$$

No.3 A person gets $ 63.75 every 6 month from an investment that pays 6% interest rate, How much money is invested?

Solution:

$$I = prt$$

$$63.75 = p \times .06 \times \frac{1}{2}$$

$$P = \frac{63.75 \times 2}{.06} = \$2125.00$$

No.4 How long will it take $ 5000 to earn $ 50 interest at 6%?

$$I = 50$$

$$P = 5000$$

$$r = .06$$

$$I = prt$$

Formula:

$$50 = 5000 \times .06 \times t$$

$$t = \frac{50}{5000 \times .06} = \frac{1}{6}\, year\ or\ 2\ months$$

No.5 A woman borrows $ 2000 from a credit union. Each month she is to pay $ 100 on the principal. She also pays interest at the rate of 1% a month on the unpaid balance at the beginning of the month. Find the total interest.

Solution:

Note that the rate is monthly rate. The first month's interest is: I=2000× .01 × 1 = 20.00.

The total payment for the first month is \$ 120, and the new unpaid balance is \$ 1900. for the second month the interest is: I=1900× .01× 1=19.00

Since the unpaid balance decrease \$ 100 a month, the interest decreases \$ 1.00 each month After 19 payments the debt is down to \$ 100, and the least interest payment is

I= 100 × .01 × 1=\$1.00

Total interest = 20+19+18+…. +2+1

$$Sum = \frac{n}{2}(t_1 + t_n)$$

$$Sum = total\ \text{interest} = \frac{20}{2}(20+1)\$210.00$$

- Amount: الجملة

Amount is the sum of the principal and the interest -S-

S=P+I S=P+ prt S=P(1+rt): القانون الأساسي للجملة

- A man borrows \$300 for 6 months at 15%. What amount must be he repay.

P= 350 R = .15 t= ½

S=P(1+rt) S=350(1+15× ½) =376.25

ويمكن الحل بطريقة أخرى

$$I = Prt \quad = 350 \times .15 \times \tfrac{1}{2} = 26.25$$

$$S = P + I \quad = 350 + 26.25 = 376.25$$

Exercises:

- A woman borrows $ 30.000 to buy a home the interest rate is 12% and the monthly payment is $308.59. How much of the first payment goes to interest and How much to principal?
- If a person invests $ 3000 at 10%, how long will it take him to get 75.00 interest? (3 month)
- A man borrows 95. six months later he repays the loan, principal and interest, with a payment of 100. what interest rate did he pay? (10.53%)
- A person borrowed $800 from a credit union that charges 1% per month on the outstanding balance of a loan. Every month for 8 month the person paid $100 on the principal plus interest on the balance at the beginning of the month. Find the total interest

Determining the number of days

There are two methods:

- 1- Exact method.
- 2- Approximate method.

- Each year has 365 days except leap years which have 366 days.
- It is assumed that each of 12 months in a year has 30 days. (360).

Example 1:

Find the exact time and the approximate time from June 24, 1998 to September 27, 1998.

Solution

a) The exact time:

June …………… 6 days (Remainder 30- 24 = 6)

July …………… 31 days, Aug……..31 days sept….. 27
6+31+31+27=95 days.

b) The approximate time.

6+30+30+27=93

Example 2:

Find the Exact time and the approximate time from November 14, 1996 to April 24, 1997.

Solution:

a) 16 + 31 + 31+ 28 + 31+ 24 = 161 days.

b) 16 + 30 + 30+ 30 + 30 + 24 = 160

وبطريقة أخرى

Arrange the two given dates in the following from by the order of year, and then subtract:

	Year	Month	Day
Ending date	1997	4	24
Beginning date	1996	11	14
	0	5	10

The approximate time = 5× 30+10=160

ملحوظة هامة

Leap years are those years evenly divisible by 4, such as 1988, 1992 and 1996, is which the month of February has 29 days instead of 28 as in other years, But the last year of any century, although it is divisible by 4, is not a leap year unless it is divisible by 400. thus, 1700, 1800 and 1900 were not leap years, but 2000.

Example 3:

Find the Exact time January 22, 1996 to March 12, 1996.

Solution:

January	9	days (Remainder, 31-22 =9)
February	29	days (Leap years)
March	12	days

50 days

(Not counting the first day but counting the last)

Exercise

(A)- find the simple interest in each of the following problems.

Principal	Interest Rate	Time
1- $326.70	16%	15 months
2- 268.80	15%	1 years, 5 months
3- 200	5% a 3 month	2 years, 6 months

(B)- Determine number of days in each of the following problem by (a) the exact time method and (b) the approximate time method.

1- February 17, 1997 to march 10, 1998.

2- December 16, 1999 to April 9. 2000.

3- October 10, 1998 to May 7, 1999.

4- January 6, 1997 to January 30, 98.

5- ………………………………. 30, 97.

6- October 10, 1899 to May 7, 1900.

Ordinary and Exact Interest:

The exact and approximate methods of determining the number of days provide four possible ways to express a number of days as a fraction of a year.

$$\left.\begin{array}{l} 1-\dfrac{Exact\,time}{360}(Ban\,\text{ker}'s\ Rule) \\ 2-\dfrac{Approximate\,time}{360} \end{array}\right\} for\ computing\ ordinary\,\text{interest}$$

$$\left.\begin{array}{l} 3-\dfrac{Exact\,time}{365} \\ 4-\dfrac{Approximate\,time}{365} \end{array}\right\} Computing\ Exact\,\text{interest}$$

Ordinary interest is the value of interest computed by using 360 as the divisor in the time factor.

$$I = pr(\frac{t}{360})$$

Exact interest is result, when 365 is used as the divisor in the time factor.

$$I_e = pr(\frac{t}{365})$$

Example 1:

Find the ordinary interest on \$ 1,392 at 6% for 70 days.

Solution:

$$I = pr(\frac{t}{360}) \quad = 1,392 \times \frac{9}{100} \times \frac{70}{360} = 24.36$$

Example 2:

Find the exact interest on $ 500 at 18% for 30 days.

Solution:

$$I_e = pr(\frac{t}{365}) \quad = 500 \times \frac{18}{100} \times \frac{30}{365} = 7.40$$

The relationship Between I and I_e

Use of the ordinary interest method always gives a larger interest value than the exact interest method.

$$\frac{I}{\mathrm{I_e}} = \frac{73}{72}$$

$$proof$$

$$\frac{I}{\mathrm{I_e}} = \frac{\mathrm{Pr}t}{360} \div \frac{\mathrm{Pr}t}{365} = \frac{365}{360} = \frac{73}{72}$$

وبنفس الطريقة:

$$\frac{I_e}{\mathrm{I}} = \frac{72}{73} \qquad \text{Then} \quad \mathrm{I} = \mathrm{I_e} + \mathrm{I_e}(\frac{1}{72})$$

$$= \mathrm{I_e}(1 + \frac{1}{72})$$

$$\mathrm{I_e} = \mathrm{I}\,(1 - \frac{1}{73})$$

Example 3:

For each of the following ordinary interests (I), find exact interest (I_e).

Solution:

36.50 2.19 58.40

$$I_e = I(1-\frac{1}{73}) \quad = 36.5\,(1\text{-}\frac{1}{73})$$

$$= 36.5 - 36.5 \times \frac{1}{73} = 36.5 - .5 = 36$$

Example 4:

For each of the following exact interests find the ordinary interest.

Solution:

36 5.04 43.20

$$I = I_e(1-\frac{1}{72}) \quad = 36\,(1\text{-}\frac{1}{72}) = 36 + \frac{36}{72} = 36.5$$

ملاحظات عامة

$$S = p(1+rt) \qquad \mathrm{r} = \frac{\mathrm{I}}{\mathrm{pt}}$$

$$I = prt \qquad \mathrm{t} = \frac{\mathrm{I}}{\mathrm{pr}}$$

تمارين هامة

On May 24, 1999, Joan Harrison borrowed $ 650 and agreed to repay loan together with interest at 12% in 90 days. what amount must she repay? On what date.

Solution:

$$I = 650 \times \frac{12}{100} \times \frac{90}{36} = \$19.50$$

$$S = 650 + 19.50 = \$669.50$$

$$\begin{array}{ll}
May\ldots\ldots\ldots\ldots\ldots\ldots\ldots & 7 \text{ days}(31-24) \\
\text{June}\ldots\ldots\ldots\ldots\ldots\ldots\ldots & 30 \text{ days} \\
\text{July}\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \frac{31}{68} \text{ days} \\
\text{August} & \frac{22}{90} \text{ days}
\end{array}$$

The amount, 669.5, must repay on August 22.1999.

- A man borrowed $ 1,350 and paid $ 1,372 after four months. What was the interest rate charged for the debt?

Present value, and simple discount

Finding the principal

The principal may be obtained in the following two ways:

First way:

By the formula $I = prt$ $\qquad$ $\mathrm{P} = \frac{\mathrm{I}}{\mathrm{rt}}$

Example:

A woman receives $ 300 interest in three months from an investment that pays 12% interest what is the principal she has invested?

Solution:

$$\textit{Substituting}\ \mathrm{I} = 300 \qquad \mathrm{r} = 12\% \qquad \mathrm{t} = \frac{3}{12}$$

$$300 = p \times \frac{12}{100} \times \frac{1}{4}$$

$$p = \frac{300}{.12 \times \frac{1}{4}} = 300 \times \frac{100}{3} = \$10.000$$

Second way:

S= p+I = P(1+ rt)

When both sides of the formula divided by (I+rt)

$$\frac{S}{(1+rt)} = \frac{P\cancel{(1+rt)}}{\cancel{(1+rt)}}$$

$$P = \frac{S}{(I + rt)}$$

$$P = \frac{S}{Amount \text{ of } 1}$$

Present value is value at the time of investment, such as the principal, or at any time before the maturity date (due date).

Example:

How much money must Ali invest today at 12% simple interest if he is to receive \$ 1,416. the amount, in 1.5 year.

Solution:

$$P = \frac{1{,}416}{1 + \frac{12}{100} \times 1.5} = \$1{,}200$$

The present value of 1,416 that is due in 1.5 years and includes 12% interest is \$ 1,200.

Simple Discount:

Process of finding the present value of give amount that is due on a future date and include a simple interest is called discounting at simple interest.

Simple Discount:

Is the difference between the amount and its present value.

1- Discounting A non- Interest- Bearing Debt:

Example:

What is present value of 3,248 that is due at the end of two months if the interest rate is 9%? What is the simple discount?

Solution:

$$P = \frac{S}{1+rt}$$

$$\mathrm{P} = \frac{3{,}248}{1+\frac{9}{100}\times\frac{1}{6}} = \frac{3{,}248}{1+.015} = 3200(\mathit{present}\ \text{value})$$

$$\mathrm{I} = 3{,}248 - 3{,}200 = 48(\text{simple discount})$$

Simple discount on the amount, 3,248 is the same as the simple interest on the present value 3200.

1- Discounting an interest-Bearing Debt:

To find the present value of an interest-bearing debt, take the following steps:

Step (1): Find the maturity value (the amount) according the original interest rate. Use the formula S=p(1+rt) where S is the maturity value and P is the original debt.

Step (2): Find the present value (the value on the date of discount) of the maturity value according to the interest to the interest rate for discounting and the discount period.

Discount period:

Is the period from the date of discount to the maturity date.

Use the formula: $P = \frac{S}{1+rt}$

Where P is the present value and S is the maturity value

Note:

The value of r and t in this step are often different from the value of r and t in step (1).

Example 1:

A man borrowed $ 1,000 on May 1. 1999 and agreed to repay the money plus 8% interest in six months. two months after the money was borrowed the creditor agreed to settle the debt by discounting it at the simple interest rate of 9%.

How much did the creditor receive when he discounted the debt?

Solution:

Step (1): Find the maturity value of the debt according to the original of the debt.

P=1,000 r =8% $t = \frac{6}{12} = \frac{1}{2} ya$

$$S = P(1+rt) = 1000(1+\frac{8}{100}\times\frac{1}{2}) = s1,040$$

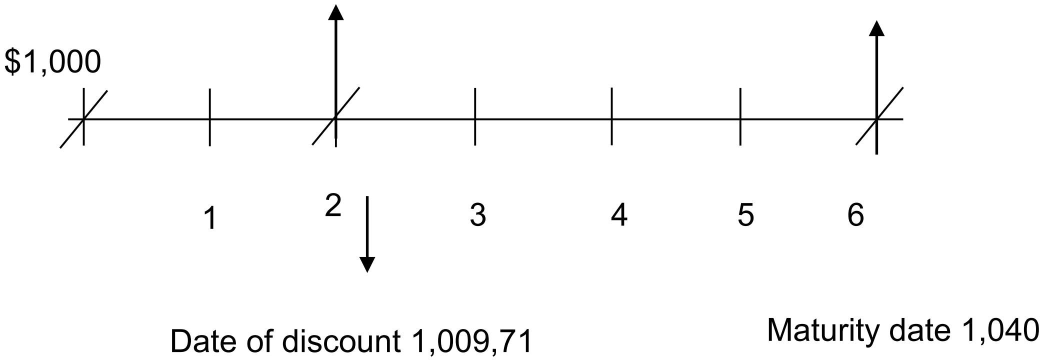
Step (2): Find the present value of the maturity value according to the discounting term.

$$t = \frac{4}{12} \, or \, 6 - 2 = 4$$

S =1,040 r = 9%

$$P \frac{S}{1+rt} = \frac{1,040}{1+\frac{9}{100}\times\frac{1}{3}} = \frac{1,040}{1,030} = 1,009,71$$

Value on July 1, 1999



Bank Discount (simply discount) proceeds:

Is the money that the borrower receives from the bank

Bank Discount rate:

The rate percentage used in computing the discount bank

D= s d t Formula 1

D = the bank discount.

S = the amount or maturity value.

d= the discount rate per unit of time expressed as a decimal.

t= the time in units that correspond to the rate.

Since the proceeds or present value of the difference between the amount and the discount, we can say that.

P= s-D = s - s d t = S (1- d t)

P=s (1- d t) Formula 2

$$S = \frac{P}{1-dt}$$ Formula 3

Bank discount is sometimes called interest <u>in advance </u>because it is based on the future rather than on the present value. If the amount is known, bank discount, requiring only multiplication, is easier to compute than discount at simple interest rate, which requires division. A given bank discount rate result in a <u>larger</u> money return to the lender than the same simple interest rate, for these reason, Bank discount is commonly used to discount sums of money for period of time of a year or less.

Example 1:

Find the present value of $100 for 1 year: at a simple interest rate of 12.5%, at a bank discount rate of 11.5%.

Solution:

$$P = \frac{100}{1+.125\times 1} = 88.89$$

$$P = s(1-dt) = 100(1-.125\times 1) = 88.5$$

Example 2:

Adell promises to repay a bank loan of $700 at the end of 60 days?

a) If the bank charges 18% interest in advance what is the discount.

b) How much docs Adell receive?

Solution:

$$a)\ \mathrm{D} = \mathrm{sdt} = 700 \times .18 \times \frac{1}{6} = 21$$

$$b)\ \mathrm{P} = \mathrm{S} - \mathrm{D} = 700 - 21 = 679$$

Example 3:

What is the final amount of a bank charges $ 2 interest in a advance for three months at 8%?

Solution:

$$\mathrm{D} = \mathrm{sdt}. \qquad 2 = \mathrm{S} \times .08 \times \frac{1}{4}$$

$$S = 2 \times \frac{100}{2} = \$100.$$

Example 4:

Adell wants $ 5000 in cash as the proceeds of loan a bank that charge a 16% discount. What is the loan that Adell must pay on the maturity date? (3m)

Solution:

$P = S(1 - dt)$ $\qquad\qquad$ $\mathrm{S} = 5208.33$

$$\mathrm{S} = \frac{\mathrm{P}}{1 - \mathrm{dt}}$$

Example 5:

A man received a 30- day of 550 from a bank. The proceeds were 544.5 what is the discount rate?

Solution:

$D = S - P$ $\qquad\qquad$ $\mathrm{D} = 5.50$

$\mathrm{D} = \mathrm{s\ d\ t}$ $\qquad$ $550 = 550.\mathrm{d}(\frac{1}{12})$ $\quad$ $\mathrm{d} = 12\%$

Example 6:

A man borrow $ 800 from a bank. The charged 15% interest in advance. He received 770 from the loan. When will the loan be due?

Solution:

$D = S - P = 30$ $\qquad\qquad$ $\mathrm{D} = \mathrm{s}\ d\ t$

$$30 = 800 \times \frac{15}{100} \times t$$

$$t = \frac{1}{4} = 90 \text{ days after the date of borrowing}$$

- Promissory notes:

Bank discount is used to determine the value of a promissory note at a stated point in time. The proceeds of a note are found as follows:

1- Find the maturity value of the note. This is the face value if it is non- interest- bearing if the notes is interest- bearing, the maturity value is the face value plus interest at the stated rate for the term of the note, the time from date of the note to the maturity date.

2- Discount the maturity value using the discount rate from the date the note is discounted to the maturity date.

3- Subtract the discount from the maturity value.

- The following information is found in the promissory note:

Face value, Date of the note, term of the note, interest rate, maturity date, maturity value.

- Promissory notes or simply notes have two parties: the maker, who make the promise to pay, and the payee, to whom the promise is made.

Example 1:

A three-month, non- interest- bearing note dated on march 2, 1999 was discount in a bank on April 3 at 12% the proceeds were 1,470 find the face value of note?

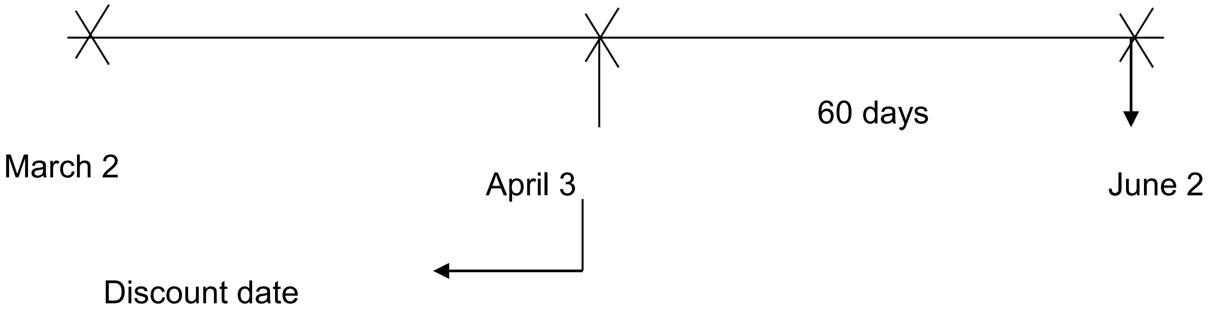
Solution:

$\underline{Non-\text{int}\,erest-bearing}$ $\quad$ $\text{Maturity value} = \text{S}$

$\text{P} = 1{,}470$ $\quad$ the due is june 2

$\downarrow$

$\Pr oceeds$ $\quad$ $\text{d} = 12\%$ $\quad$ $\text{t} = \frac{60}{360} = \frac{1}{6}\,year$

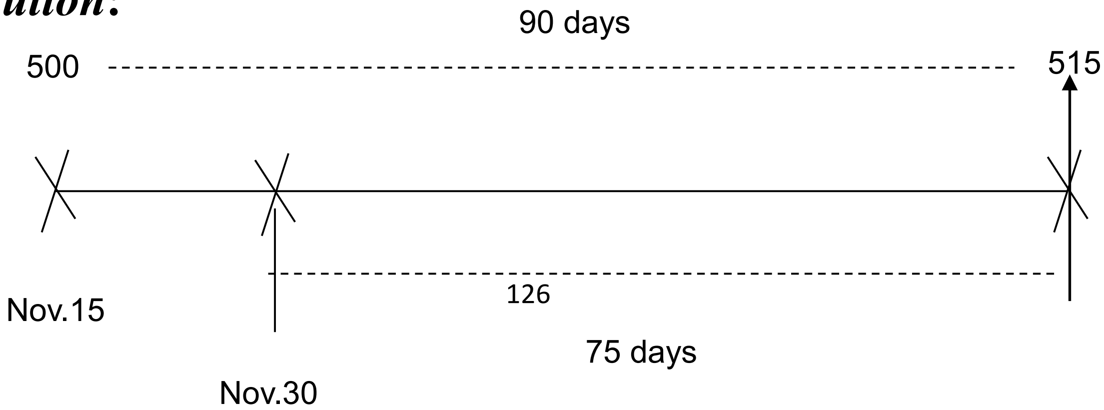


$$S = \frac{P}{1+dt} = \frac{1{,}470}{1-\frac{12}{100}\times\frac{1}{6}} = 1500$$

Example 2:

Adell had a note for $500 with an interest rate of 12%. The note was dated November 15, 1998 and the maturity date was 90 days after date. On November 30 1998, he took the note to this bank, which discounted it at a discount rate of 14% How much did he receive from bank?

***Solution*:**



$$S = 500\left[1 + \frac{12}{100} \times \frac{1}{4}\right] = 515$$

$$D = sdt = 515 \times \frac{14}{100} \times \frac{75}{360} = 15.02$$

$$P = S - D = 515 - 15.02 = 499.98$$

لاحظ الاختلاف بين معدل الفائدة ومعدل الخصم

14% 12%

Relationship Between Bank Discount rate and Simple interest rate:

$$r = \frac{d}{1 - dt} \qquad \mathrm{d} = \frac{\mathrm{r}}{1 + \mathrm{rt}}$$

- When the simple interest rate (r) and the bank discount rate (d) are the same the discount computed by the bank discount method is greater than computed. By the simple discount method.
- The present value at given discount rate is less than the present value based on the same interest rate.
- When the simple discount (I) and the Bank discount (d) are

the simple discount method is greater than the computed by the bank discount method.

Example 1:

A bank discounts a 75- day note at 9%. What is the equivalent simple interest rate earned by Bank?

Solution:

$$r = \frac{d}{1-dt} = \frac{.09}{1-.09\times\frac{75}{360}} = 9.17\%$$

Example 2:

At what rate should a bank discount a 60 – day note if the bank is to earn simple interest equivalent to 8%?

Solution:

$$d = \frac{r}{1+rt} = \frac{.08}{1+.08\times\frac{1}{6}} = \frac{3}{38} = .07897$$

$$= 7.897\% \text{ or rounded to } 7.89\%$$

تمارين هامة

Exercise:

1- Omnia signs a three- month note for 1.200 the Bank charges 6% interest in advance. How much does omnia receive as the proceeds? If she borrows the proceeds at simple interest and repays $ 1,200 at the simple interest rate that she must pay?

2- Hazem receives $ 3,450 in cash from a bank on April 1 and greed to pay $ 3,600 on August 29 for the loan. What is the bank discount? What is the equivalent simple interest rate?

Equivalent value involving simple interest

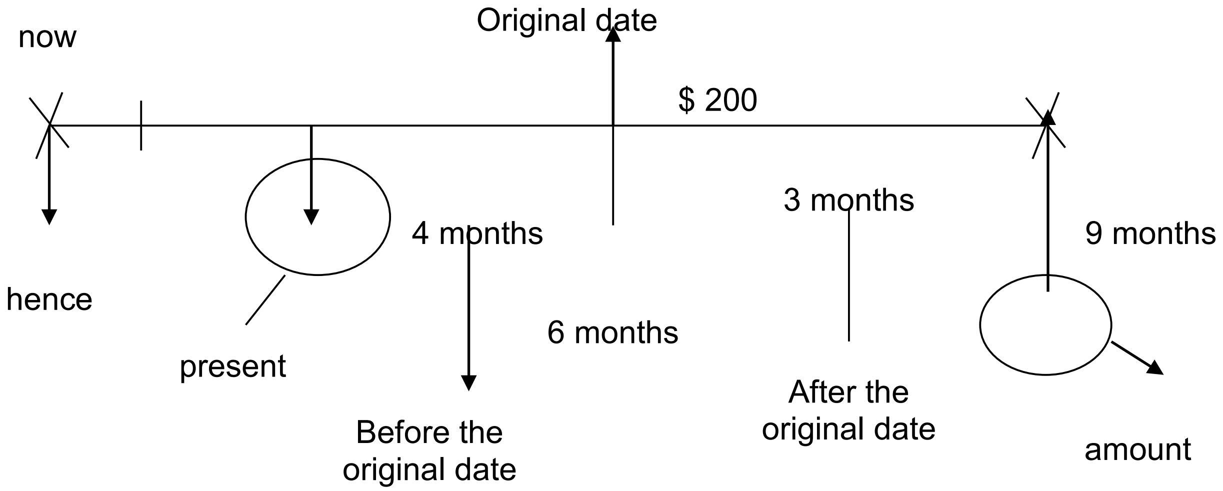
principle equation:

In focal date or in comparison date

- Values of old debt = values of new debt.
- Present value for old debt = present value for new debt.
- Amount value for old debt = Amount value for new debt

Example 1:

A debt of $ 200 is due in six months if the rate of interest is 15%, what is the value of the debt if it is paid (a) two months hence? (b) six months hence? (c) Nine months hence?



$$a)\ \mathrm{P} = \frac{200}{1+.15\times\frac{1}{3}} = \frac{200}{1.05} = 190.48$$

B) If the debt of $200 is paid in six months hence, the payment is $200, unchanged.

C) Three months after the due date, the required equivalent value is more than $ 200, thus, the amount formula.

$$S = p(1+rt) \qquad \mathrm{S} = 200(1+.15\times\frac{1}{4}) = 207.50$$

Example 2:

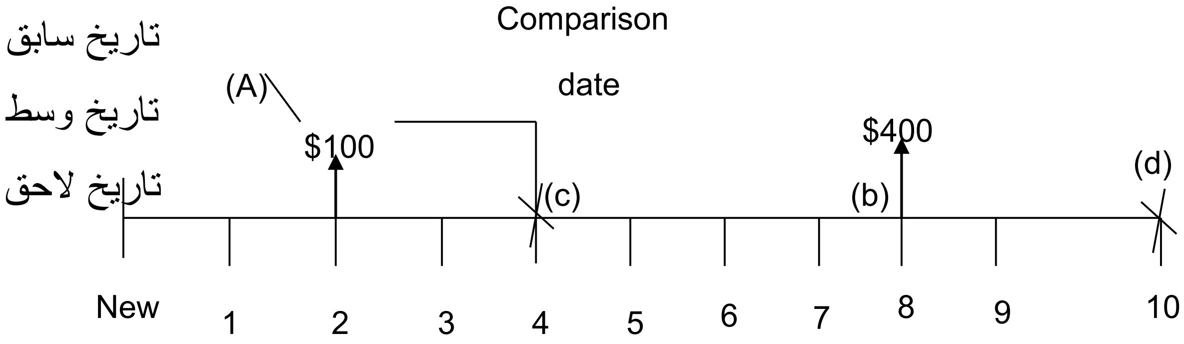
A man owes (1) $ 100, due in two months, and (2) $ 400, due in light months. His creditor has agreed to settle the debts by two equal payment in four months, respectively, find the size of each payment if the rate of interest is 6% and the comparison date in four months hence.

Solution:

Let x represent each equal payment. The value are of the comparison date are computed as follows:

New debt = old debt

(C) (d) (a) (b)



a) : The value of old debt of $100 becomes $101 on the comparison date are computed as follows:

$$S = 100\left[1 + .06 \times \frac{1}{6}\right] = 101$$

$$b) : \mathrm{P} = \frac{400}{1 + (.06 \times \frac{1}{3})} = 392.16$$

c) : The value of the first new debt, which is due in four month, does not change and is X, since the comparison date is also in four months.

d) The value of the second new debt, which is due in ten month, become $\frac{x}{1.03}$ on the comparison date. It is computed a follow.

$$P = \frac{x}{1 + .06 \times \frac{1}{2}} = \frac{x}{1.03}$$

The equation of value (المعادلة الاساسية) based on the comparison date is given below:

New debts = old debts ⟶ Equation of value

$$x+\frac{x}{1.03}=101+392.16$$

$$x(1+\frac{1}{1.03})=493.16 \qquad x=\$250.22$$

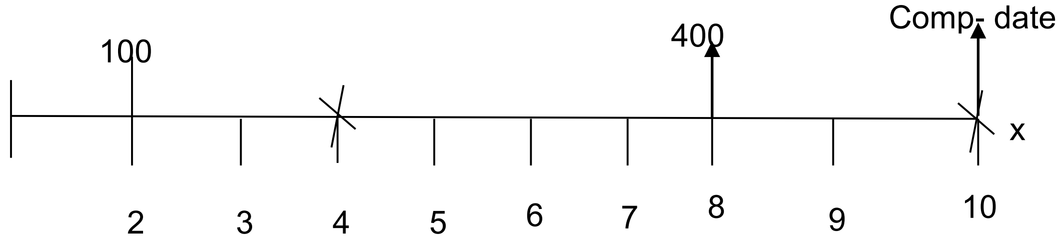
The two original debts may be discharged by paying $250.22 in four months and 250 in ten months. On the comparison date

new debts = $250.22+\frac{250.22}{1+.06x\frac{1}{2}}$ = 493.16

Exercise:

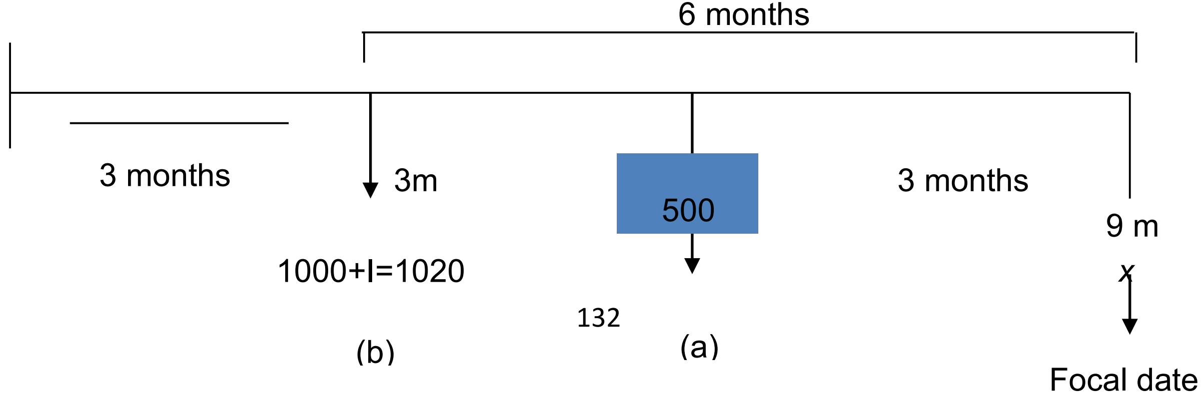
1- In the above example, what is the size of each payment if the comparison date is set ten months hence?

X= 250.25



2- Ali owes (a) $ 500 due in six months and (b) $ 1.000 plus 8% interest due in three months. If money is worth 6%, what single payment nine months hence will be equivalent to the two original?

Solution:



X=1,050.6+507.5=1,558.1

Equivalent time: المدة المكافئة

$$t = \frac{D_1t_1 + D_2t_2 + D_3t_3.........}{D}$$

Where:

T = Equivalent time.

D_1 = the first debts due in t_1 days.

D_2 = the second debts due in t_2 days.

$D=D_1+D_2+D_3$……………

Example 1:

A Store purchased merchandise in the following amounts; $200, due in three months; 500, due in four months; and 100, due in six months. What will the equated date (تاريخ الاستحقاق المتوسط)be if a single payment of 800 discharge the three debts? Assume that the interest rate is 18%.

Solution:

$$t = \frac{200 \times 3 + 500 \times 4 + 100 \times 6}{800} = \frac{3200}{800} = months$$

If there is no interest (r=0) the balance, 800 probably can be paid at any time; that is, there is no equated date.

Example 2:

The debt 200, 300, and 500 are due in 10 days, 20 days, and 30 days, respectively. If the payment plans are changed to pay 100, 200. and 300 in 10 days. 20 days and 30 days, when will a single payment of 400 discharge the balance?

Solution:

$$t = \frac{200\times10+300\times20+500\times30}{1{,}000}$$

$$T_4 = \frac{D_1t_1 + D_2t_2 + D_3t_3 - (P_1T_1 + P_2T_2 + P_3T_3)}{P_4}$$

$$T_4 = 23days$$

Example 3:

A man made the following purchases at the omer company September 25, 400 ; October 25, 300 ; and December 4, 500. He made the following payment: October 19, 200 and November 5, 100. what is the equated date on which he may make a single payment of 900 to discharge the balance?

Solution:

Let September 25 be the present. Then

D_1= 400, t_1=0 (due at present)

D_2= 300, t_2=30 days (from 9/25 to 10/25)

D_3= 500, t_3= 70 days (from 9/25 to 12/4)

P_1= 200, T_1= 24 days (from 9/25 to 10/19)

P_2= 100, T_2= 41 days (from 9/25 to 11/5)

D= 400+300+500= 1,200

P_3=D-(P_1+P_2)=1,200-(200+100)

$$T_3 = \frac{(400\times 0)+(300\times 30)+(500\times 70)-\left[(200\times 24)+(100\times 41)\right]}{900}$$

T_3=39 days. From September 25 or November 3.

The debt is probably settled after December 4, the last purchase date.

Compound Interest

Compound Amount formula

in many business transactions, the interest is computed annually, semiannually, quarterly, monthly, daily.

$S= P(1+i)^n$

S= the amount at compound interest

P= the principal.

i= the rate per conversion periods.

n= the number of conversion periods.

The factor $(1+i)^n$ is called the accumulation factor or amount 1.

Note:

1- the numerical value of $(1+i)^n$ can be computed by logarithms or by the binomial theorem, in practical business usage the numerical value is usually obtained from <u>previously computed tables</u> جدول محسوب مسبقاً

2- if there <u>is only one conversion</u> period compound interest is the same as the simple interest.

Table:

N	Amount of 1 عشر أرقام بعد العلامة	Amount of 1 per period	Present of 1	Present Of 1 per period
1	1,005…..	1,……..	,99…...	•
2	•	•	•	•
3	•	•	•	•
.	•	•	•	•
.	•	•	•	•
.	•	•	•	•
.	•	•	•	•
.	•	•	•	•
60	•	•	•	•

Rate 1%	Period	Amount of 1	Amount of 1 per period	Sinking
Annually 1%				
Semian 2%				

Quarterly **4%**
Monthly **12%**

Example 1:

Find the amount of $ 1.500 invested at 12% compounded quarterly and due at the end of 4.1/4 years.

Solution:

$S=p(1+i)^{n}=1.500(1+3\%)^{17}$

$^{=}$ $1,500(1.652848)=2,479.272$

Example 2:

A note having a face value of $ 1,000 and bearing interest at 16% compound quarterly will mature in 10.5- years. What is maturity value?

Solution:

$S=p(1+i)^{n}=1,000(1+4\%)^{42}$

$^{=}$ $1,000(5.192784)=5,192.784$

Example 3:

Find the compound amount when the principal is $ 1,000 the interest tare is 24% compound monthly, and the term is 10 years.

Solution:

$S=p(1+i)^n=1,000(1+2\%)^{120}$

$= 1,000(1+2\%)^{60}\times\ (1+2\%)^{60}$

من
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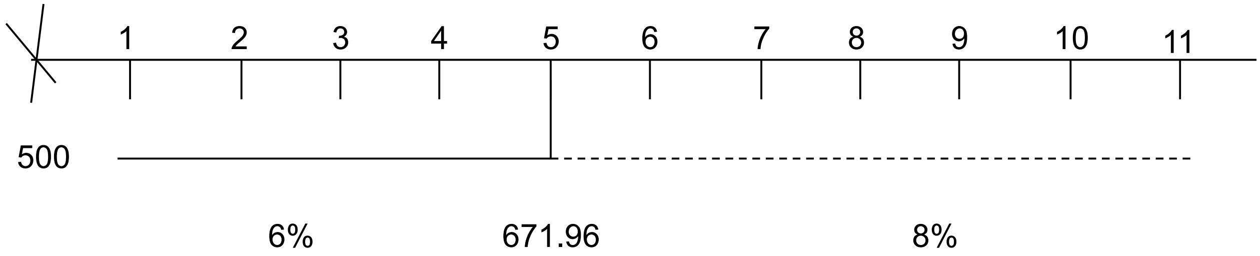
$= 1,000(1+2\%)^{30}\times\ (1+2\%)^{30}\times\ (1+2\%)^{30}\times\ (1+2\%)^{30}$

$= 10,765.16$

Example 4:

If the principal is 500 and the interest is 6% compounded semiannually for the first five years and 8% compounded quarterly for the next six years, what is the compound amount at the end of the 11th year?

Solution:



$S= 500(1+3\%)^{10} =671.96$

$S= 671.96(1+2\%)^{24}=671.96\times 1.60844=1,080.81$

$S= 500(1+3\%)^{10}(1+2\%)^{24}=1,080.81$

Example 5:

Find the compound amount if \$1,000 is invested at 8.5%

compound quarterly for 2.5 years.

Solution:

S=p(1+i)^{n}=1,000(1+2 $\frac{1}{8}$ %)^{10}

$$(1+2\frac{1}{4}\%)^{10}=1.24920343 \qquad from the table$$

$$(1+2\frac{1}{8}\%)^{10}= \qquad x$$

$$(1+2\%)^{10}=\frac{1.21899442}{0.0302090} \quad from the table$$

We use interpolation method:

$$(1+2\frac{1}{8})^{10}=(1+2\%)^{10}+.03020901\times\frac{1}{2}$$

$$=1.21899442+.015104505$$

$$=1.234098925$$

$$S=1{,}000(1.234098925)=1{,}234.10$$

- The conversion periods include a fractional part.

Example 6:

Find the compound amount and the compound interest when 1,000 is invested for three years and two months at 6% compounded semiannually

Solution:

<u>Methods A</u>: the compound amount at the end of the sixth

conversion period is:

$S=1000(1+3\%)^6 =1000(1.1940521)=1,194.05$ the simple interest for the remaining period (two months) is:

$1,194.05\times 6\% \times \frac{2}{12} \quad =11.94$

The final compound amount is

$1,194.05+11.94=1,205.99$

The compound interest = 1,205.99 – 1000=205.99

<u>Methods B</u>:

$S=P(1+i)^n$

$S= 1000(1+3\%)^6 \quad =1000(1+3\%)^6 (1+3\%)$

$$= 1000\underset{\text{table}}{\underset{\downarrow}{(1.194052)}}\ \underset{\text{table}}{\underset{\downarrow}{(1.009902)}} = 1,205.88$$

The compound interest = 1,205.88-1000=205.88

<u>Notice</u> that the interest obtained A is greater than that calculated in method B when the investment time is a fraction of the conversion period use of the compound interest method always gives less interest than the simple interest methods.

The effective interest rates:

Effective rate: r **Normal rate: J**

$r = (1+i)^m - 1$ $\qquad$ $i = \frac{J}{m}$

M= number of converted times a year interest equivalent to 8% converted semiannually.

Solution:

i= 4% m=2 semiannually.

$r = (1+i)^m - 1 = (1+4\%)^2 - 1$

$= 1.0816-1=8.16\%$

Exercise

1- find the effective rate of interest equivalent to a tare of 15.5% compound.

a) Semiannually.

b) Quarterly.

c) Monthly.

2- A man invested 400 for ten years. The interest rate is 8% compounded annually for the first six years and 13% compound monthly for the next four years. What is the amount at the end of ten years?

3- find the amount and the interest to 5000 is invested at 13% compound monthly for five years and $7\frac{1}{3}$ months.

Two Problems

$$\begin{tikzcd}
& \text{Two Problems} \arrow[dl, dash] \arrow[dr, dash] & \\
\text{المعدل} \quad 1\frac{1}{12} & & 67\frac{1}{3} \quad \text{المدة}
\end{tikzcd}$$

ملاحظات على حل التمرين الأخير:- المدة أولاً:

$$S = P(1+i)^2 \qquad\qquad S = 5000(1+1\frac{1}{12}\%)^{67\frac{1}{3}}$$

$$\begin{tikzcd}
& & (1+1\frac{1}{12}\%)^{67\frac{1}{3}} \arrow[dl] \arrow[dr] & & \\
& (1+1\frac{1}{12}\%)^{67} \arrow[dl] \arrow[d] & & (1+1\frac{1}{12}\%)^{68} \arrow[d] \arrow[dr] & \\
(1+1\%)^{67} & (1+2\%)^{67} & & (1+1\%)^{68} & (1+2\%)^{68}
\end{tikzcd}$$

ويستكمل الحل

ويمكن الحل بطريقة أخري

$$\begin{tikzcd}
& & (1+1\frac{1}{12}\%)^{67\frac{1}{3}} \arrow[dl] \arrow[dr] & & \\
& (1+1\%)^{67\frac{1}{3}} \arrow[dl] \arrow[d] & & (1+2\%)^{67\frac{1}{3}} \arrow[d] \arrow[dr] & \\
(1+1\%)^{67} & (1+1\%)^{68} & & (1+2\%)^{67} & (1+2\%)^{68}
\end{tikzcd}$$

ويستكمل الحل

Present Value at Compound interest:

1- ……….. Is defined as the principal that will amount to given sum at the specified future date.

2- ………… the deference between the future amount and its present value.

$$P = \frac{S}{(1+i)^n} = S(1+i)^{-n} = Sx\frac{1}{(1+i)^n}$$

P= the principal or present value.

S= the amount due in the future.

i= the rate per period.

n= the number of period.

$\frac{1}{(1+i)^n} or (1+i)^{-n}$ Is called the discount factor, present worth of 1 or present value of 1

- Discount factor for common interest rates are given in the “present worth of 1” column

Example 1:

Find the present value of 1000 due at the end of 4 ¼ years if money is worth 12% compounded quarterly.

Solution:

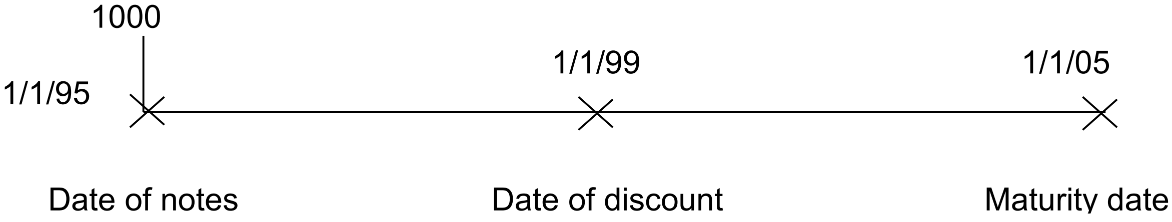
$$P = \frac{S}{(1+i)^n} = S \times \frac{1}{(1+i)^n}$$

$$P = 1000 \times \frac{1}{(1+3\%)^{17}} = 1000 \times .60501645$$

Example 2:

A note of 1,000 dated January 1 1995 at 6% comp. quarterly for ten years dis. On Jan. 1 1999 what are the proceeds and the compounded discount if the note was discounted at 8% compounded semiannually?

Solution:



- **<u>Step 1</u>**: find the maturity value on Jan. 1 2005

$S=P(1+i)^n= 1000(1+1 \frac{1}{2} \%)^{40} = 1000\times 1.814018=1,814.02$

- **<u>Step 2</u>**: find the present value (4 الجدول الخانة)as of Jan.1 99 the date of discount.

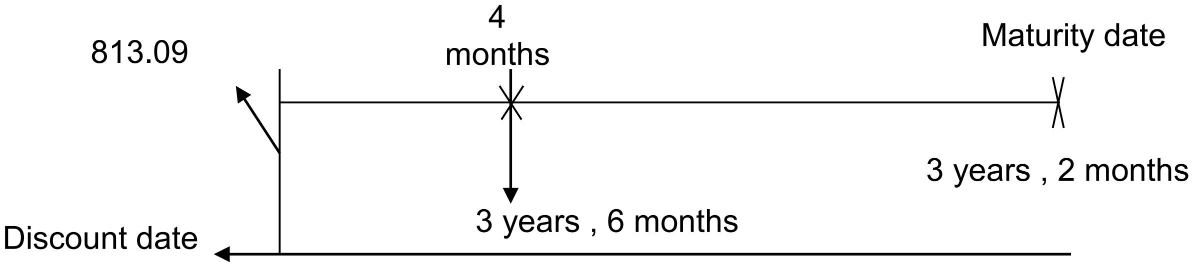
$P=s \times(1+4\%)^{-12}= 1,814.02 \times.624597=1,133.03$

Compound discount = 1,814.02- 1,133.03=680.99

Example 3:

A non- interest- bearing note of 1.000 is discount at 6% compound semiannually for three years and two months. Find the proceeds and the compound discount.

Solution:



Method A: $P=S(1+i)^{-n}=1000(1+3\%)^{-7}$

$P=1000\times .813092 = 813.09$

Simple interest = 813.09× 6% × $\frac{4}{12}$ =16.26 the proceeds = 813.09 + 16.26=829.35

Method B: $P=S(1+i)^{-n}$ $P=1000(1+3\%)^{-6\frac{1}{3}}$

$$P=1000(1+3\%)^{-6}(1+3\%)^{-\frac{1}{3}}=1000\times\frac{(1+3\%)^{-6}}{(1+3\%)^{\frac{1}{3}}}$$

$$=1000\times\frac{.837484}{1.009902}=829.27$$

The proceeds in methods A are larger than those method B. Such a condition is always true when the discount time includes a fraction of the conversion period.

Finding the interest rate: Use of table

$$(1+i)^n = \frac{S}{P}$$

Example 4:

If 1,000 will accumulate to 5,054.47 in 17 years, what is the interest rate compounded annually?

Solution:

$$(1+i)^{17} = \frac{S}{P} = \frac{5{,}054.47}{1{,}000} = 5.05447$$

In the n= 17 of table, the entry 5.05447 is found in the 10% column. Thus I =10%.

Example 5:

At what nominal interest rate compounded semiannually for ten years will 300 accumulate to 890?

$$(1+i)^{20} = \frac{890}{300} = 2.9667$$

There is no entry 2.9667 in the row in table

I	$(1+i)^{20}$		
6%	3.2071	(1)	nominal rate
x	2.9667	(2)	
5 ½ %	2.9178	(3)	J= m x i
(2) – (3)	x -5 ½ %	.0489	

(1) – (3) ½ % .2893

$$x - 5\frac{1}{2}\% = \frac{1}{2}\% \frac{.0489}{.2893} = .000845$$

$$x = 5\frac{1}{2}\% + .000845 = .055 + .000845 = .055845$$

i = .055845 (interest rate per semiannual per semiannual period

the nominal interest rate is .055845×2 =.111690 = 11.16% or rounded to 11.17% annual

تمريـــــــن هـــــــــام

Bank A offer its depositors an interest rate of 6% compounded monthly, while Bank B gives its depositors an interest rate of 6 ½ % compounded semiannually. Which of the two Bank makes the better offer?

Solution:

The effectives rate based on the interest rate of Bank A is:

$$f = (1 + 6\%/12)^{12} - 1 = (1 + \frac{1}{2}\%)^{12} - 1 = 1.0617 - 1 = .0617 \, or \, 6.17\%$$

The effective rate based on the interest rate of Bank B is:

$$f = (1 + 6\frac{1}{2}/2)^{2} - 1 = (1 + 3\frac{1}{4})^{2} - 1 = 1.06661 - 1 = .0661 or \, 6.61\%$$

البنك B افضل

تمرين هام

How long will it take 1,000 to accumulate to the amount of 1,105 at 24% compound monthly

Solution:

$$(1+i)^n = \frac{S}{P} \qquad (1+2\%)^n = \frac{1{,}105}{1{,}000} = 1.105$$

In the 2% column in table, the value 1.105 is between the entries 1.10408080 (where n=5) and 1.26242(where n=6).

5< n< 6-month n=6

تمرين هام

Find the present value of 5000 due in 5 years if money is worth 13% compound monthly.

Solution:

$$\begin{tikzcd} & (1+1\%)^{-60} \\ P = 5000(1+1\frac{1}{12}\%)^{-60} \arrow[ur, dash] \arrow[dr, dash] & \\ & (1+2\%)^{-60} \end{tikzcd}$$

$(1+1\%)^{-60} = .550449$ (1)

$(1+2\%)^{-60} = .304782$ (2)

بطرح المعادلة (2) من المعادلة (1) تحصل على الفرق المقابل لـ 1%

(الفرق في القيمة الحالية)

$=.550449 - .304782 = .245667$

الفرق المقابل لمعدل $\frac{1}{12}$% = الفرق المقابل لـ 1% * $\frac{1}{12}$

$$.02047225 = \; = .245667 \times \frac{1}{12}$$

الفرق المقابل لمعدل $1\frac{1}{12}$% =

الفرق المقابل لمعدل 1% – الفرق المقابل لمعدل $\frac{1}{12}$

$.02047225 - .550449 =$

$.5299767 =$

P= 5000× .5299767= 2649.88

Equivalent values involving comp. interest

in comparison date.

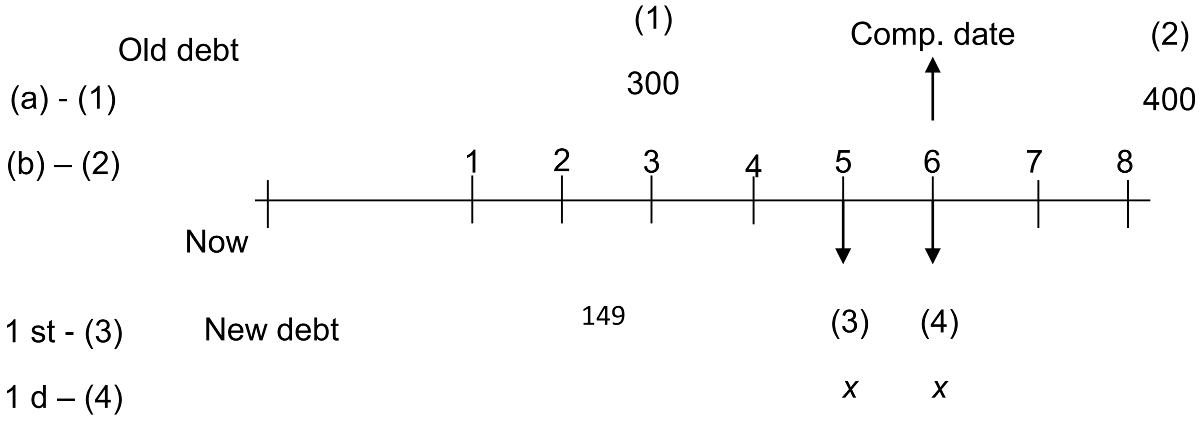
Old debts = New debts

Equation of value

Example 1:

A man owes (a) 300 due in three years and (b) 400 due in eight years. He and discredited have agreed to settle the debts by two equal payment in five and six years respectively. Find the size of each payment if money is worth 6% compounded semiannually.

***Solution*:**



$$300(1+3\%)^{6}+400(1+3\%)^{-4}=\times(1+3\%)^{2}+x$$

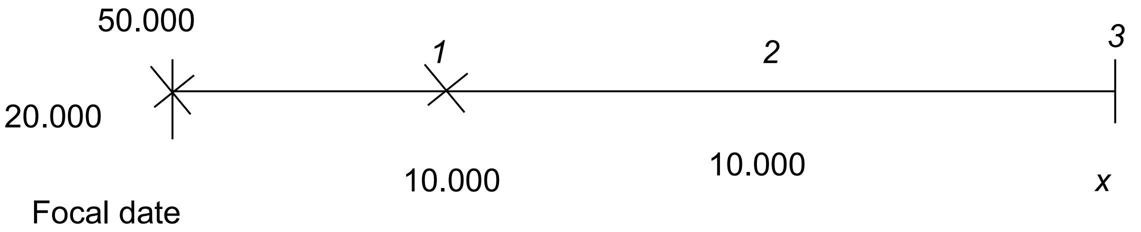
$$\begin{aligned}&300(1.9405)+400(.88849)=x(1.0609)+x\\&713.62=2.0609x\\&x=346.27\end{aligned}$$

Example 2:

A car is sold for 50,000 the buyer pays 20,000 cash, and signs a non- interest- bearing note for 10,000 due in I year and a second non- interest note for 10,000 due in 2 year. If the seller charges 10% compounded annually, what non- interest- bearing note due in 3 years will pay off the debt?

Solution:

old debts = New debts in focal date



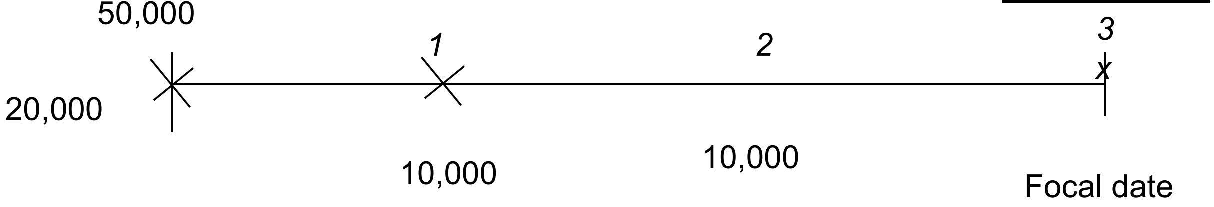
$30,000=10,000(1+,1)^{-1}+10,000(1+,1)^{-2}+X(1+,1)^{-3}$

$x\ (1+,1)^{-3}=30,000-\ [10,000(1+,1)^{-1}+\ 10,000(1+,1)^{-2}]$

من جدول القيمة الحالية

ويستكمل الحل

طريقة أخرى:



old debts = New debts   in focal date

$30,000(1+,1^{)3}=10.000(1+,1)^{2}+10.000(1+,1)^{1}+x$

$x\ =30,000(1+,1)^{3}\ -\ [10,000(1+,1)^{2}+\ 10,000(1+,1)^{1}]$

من جدول الجملة

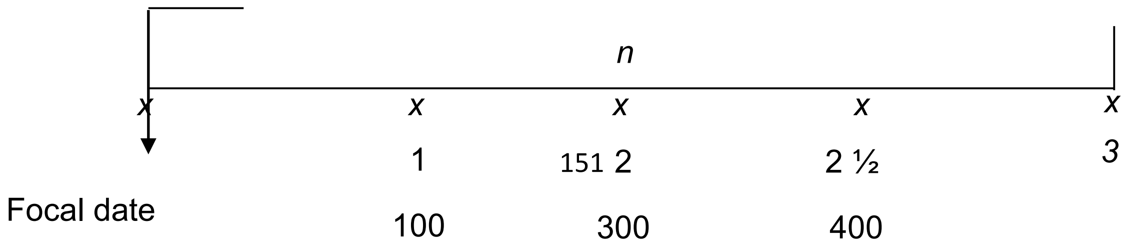
ويستكمل الحل

Equated date: it the date on which a single payment equal to the sum of a set of obligation is made to discharge the obligation.

Example 2:

Ali borrowed some money from Medhat as follows: (a) \$100 due in one year, (b) \$300 due in two years, and (c) \$400 in two and half years. If money is worth 4% compounded semiannually, when can Ali discharge all of his debts single payment of \$800.

Solution:



New debts = old debts in comp. date

$$800(1+2\%)^{-n} = 100(1+2\%)^{-2} + 300(1+2)^{-4} + 400(1+2\%)^{-5}$$

$$(1+2\%)^{-n} = \frac{735.564}{800} = .9195 \quad \text{interpolation}$$

From table:

n	$(1+2)^{-n}$					
4	.9238	(1)	(2)- (3)	x-5	.0138	(4)
X	.9195	(2)	(1)- (3)	-1	.0181	(5)
5	.9057	(3)				

$$x-5 = -1\times\frac{.0138}{.0181} = -1\times(\frac{138}{181}) = -.7624$$

$$x = 5-.7624 = 4.2476$$

Thus. n= 4.2376 (semiannually period) or two years and 43 day (.2376 × 180)

Ordinary Annuities: (Amount at the end of each period)

<u>Amount formula:</u>

$$Sn = Rs\overline{n|} = R\frac{(1+i)^n - 1}{i}$$

Sn = *amount* of ordinary annuity of n payment

R = Periodic payment or rent

$S\overline{n|}$ = amount of 1 per period for n periods at the rate per period

Numerical Values of S n are in the second column of factors in Table من العمود 2 من الجدول

Example 1:

If $ 20 is deposited at the end of each month for three years in a fund that earns 6% interest compounded monthly, what will be the final value at the end of the three- year term? What is the total interest?

Solution:

R= 20 per month $i = \frac{6\%}{12} = \frac{1}{2}\%$ per period

n= 3 × 12 = 36 monthly

Final value = 20 × $S\overline{36|}$½ % = 20 × 39.361=786.72

من الجدول (العمود الثاني)

<u>Present value of an Annuity:</u>

$$An = R\ a\overline{n|} = R\frac{1-(1+i)^{-n}}{i} = R\frac{1-vn}{i}$$

An= Present of an ordinary annuity of n payment.

R = Periodic payment or rent

An= present worth of 1 per period for n periods at the rate I per period

Vn = present worth of 1 $(1+i)^{-n}$

Example 1:

What is the cash value of a car that can be bought for $1,000 down and $500 a month for 36 months if money is worth 12% compounded monthly?

Solution: (ordinary annuity)

$An = 500\ \overline{36}|1\% = 500 \times 30.1075 = 15{,}053.70$

↓

من الجدول (العمود الخامس)

The cash price of car = 1,000 + 15,053.70=16,053.70

Example 2:

Find the present value of an annuity of $200 payable at the end of each year for 20 year if money is worth 5.2% compounded annually.

Solution:

$An = R\ \overline{an}|_i = 200\ a\ \overline{20}|_{5.2\ \%}$ interpolation

i	$a\ \overline{20}|_i$	
5 ½ % = 5.5 %	11.950382	(1)
5.2%	x	(2)

	5%	12.462210	(3)
(2) – (3)	.2%	x -12.462210	(4)
(1) – (3)	½ %	- .511828 ⟶	(5)

$$x-12.462210=-.511828\times\frac{2}{5}=-.2047312$$

$$x=12.462210\ -.2047312=12.2574788$$

$$\text{An}=200\times 12.25748=2{,}451.50$$

ويمكن الحل باستخدام المعادلة الأساسية والآلة الحاسبة:

$$An=R.\frac{1-(1+i)^{-n}}{i}=200\times\frac{1(i+5.2\%)-20}{5.2\%}$$

حل المثال السابق على أساس الجملة وتأكد من الفرق.

Example 3:

The amount of an annuity ten year is $10,000 find the size of the quarterly payment if the interest rate is 8% compounded quarterly.

Solution:

$$R=\frac{Sn}{S\,\overline{n|}\%}\qquad R=\frac{10{,}000}{S\ \overline{40|}2\%}=\frac{10{,}000}{60.40198}=165$$

من عمود جملة الدفعة من الجدول

ويمكن الحل باستخدام عمود القيمة الحالية للدفعة من الجدول:

$$R = S_n\left(\frac{1}{\overline{an}|} - i\right)$$

$$R = 10.000\left(\frac{1}{a\overline{40}|2\%} - 2\%\right) = 10.000(.0365558 - .02)$$

$$= 10.000 \times .0165558 = 165$$

Example 4:

At what nominal tare compounded quarterly will an annuity of 150 payable at the end of each quarter amount to $6,600 in eight years?

Solution:

$$S\,\overline{n}| = 6,600 \qquad \mathrm{R} = 150 \qquad \mathrm{n} = 32 \qquad \mathrm{i} = ?$$

$$6.600 = 150 \times \overline{32}|\,\mathrm{i}\% \qquad \mathrm{S}\,\overline{32}|\ \mathrm{i}\% = \frac{6,600}{150} = 44$$

Follow the line for n = 32 in table(2عمود) to find the value of 44 or the two values closest to 44.

$$1\frac{7}{8}\% < i < 2\%$$

	i	$S\,\overline{32}\rceil\ i$	
	2%	44.2270	(1)
	x	44.0000	(2)
	$1\frac{7}{8}\%$	43.3079	(3)
(2) - (3)	$x - 1\frac{7}{8}\%$	.6921	(4)
(1) - (3)	$\frac{1}{8}\%$	.9191	(5)

$$x - 1\frac{7}{8}\% = \frac{1}{8}\%\left(\frac{.6921}{.9191}\right) = \frac{1}{800} \times \frac{6921}{9191} = .00094$$

$$x = 1\frac{7}{8}\% + .00094 = .01969$$

The desired value of i is = 1.969% per quarterly.

The nominal interest rate = .01969 × 4 = 7.88%

Example 5:

A man borrows $40,000 and agrees to repay it by paying $2000 at the end of each quarter. If the interest charged is 8% compounded quarterly long will he has to pay?

Solution:

An = 40,000   R= 2000   i = 2%   n = ? Quarters

An= R $a\,\overline{n}\rceil i$ 40,000 = 2000 × $a\,\overline{n}\rceil\, 2\%$

$$a_{\overline{n}|}2\% = \frac{40000}{2000} = 20$$

In the 2% column of table, find the two values closest to 20. the finding are as follows:

When n = 25 the entry = 19.52345

When n = 26 the entry = 20.12104376

$$n = 26 quarters = \frac{26}{4} years.$$

Example 6:

Ali signed a ten- month non interest- bearing note for $5000. He was offered the privilege of discharging the obligation by making 10 equal monthly payments of 488 payable at the end of each month; if he can invest his money at 5 ½ % compounded monthly, should he accept the offer?

Solution:

$$\frac{1}{2}\% < i < \frac{2}{3}\% \; no \text{ interpolation}$$

Annuity Due: هام

An annuity due is an annuity for which the periodic payment are made at the beginning of each payment interval.

- Amount of an annuity Due.

$Sn\,(due) = R\, s_{\overline{n + 1}|\,i\%} - R \longrightarrow$ 1 بدلالة الدفعة العادية (لا يوجد جدول خاص بها)

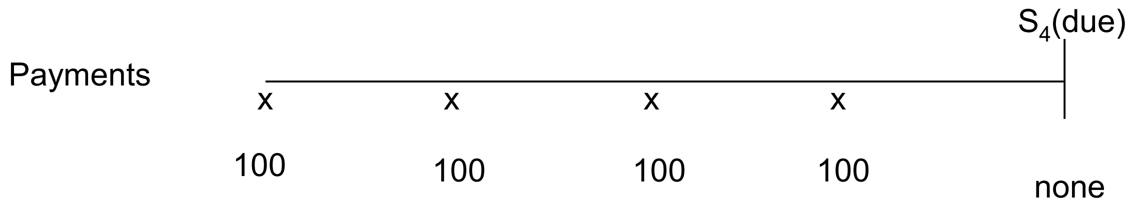
$Sn\,(due) = R\,(s_{\overline{n + 1}|\,i\%} - 1)$ 2

$Sn\,(due) = R\, s_{\overline{n}|\,i\%}(1 + i\%)$ 3

Example 1:

What is the amount of an annuity due for one year if each payment is $100 payable at the beginning of each quarter and the interest rate is 4% compounded quarterly?

Solution:



$S_4(due) \quad = 100\, S_{\overline{4+1}|}\, i\% - 100$

$= 100\,(5.10101) - 100 = 410.10$

<u>Or</u>

$S_4(due) \quad = 100\,(5.10101 - 1) = 100 \times 4.10101 = 410.10$

<u>Or</u>

$S4(due) = 100(S_{\overline{4}|}1\%) \times (1 + 1\%)$

$S4(due) = 100\,(4.06040)\,(1.01) = 100(4.10101) = 410.10$

<u>- Present value of an annuity Due.</u>

$An\,(due) = R\, a_{\overline{n - 1}|\,i} + R$

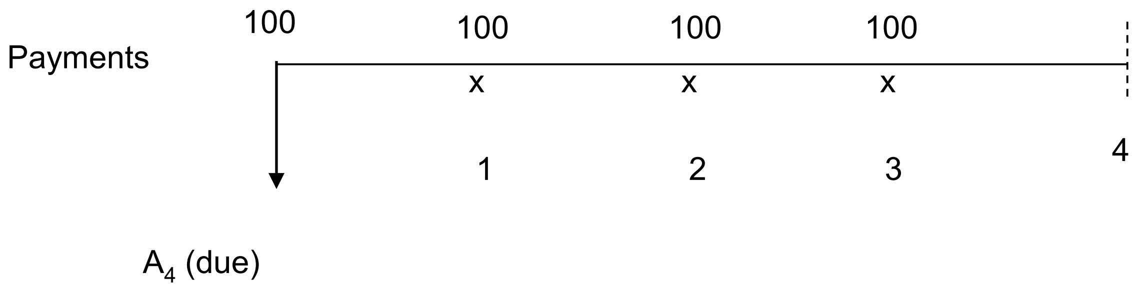
A_n (due)= $R\,(a_{\overline{n-1|}\,i} + 1)$

A_n (due)= $R\, a_{\overline{n|}\,i}\,(1 + i\,\%)$

Example 2:

What is the present value of an annuity due if the size of each payment is \$100 payable at the beginning of each quarter for one year and the interest rate is 4% compounded quarterly?

Solution:



A_4 (due)= $100(a_{\overline{4-1|}}\,1\%) + 100 = 100(a\ 3\ i\%) + 100$

A_4 (due)= $\underset{\downarrow}{100\ (2.94099)} + 100 = 394.10$

Or العمود الخامس

A_4 (due)= $100(a_{\overline{4-1|}}\,1\% + 1) = 100\ (2.94099+1)$

$= 100\ (3.94099) = 394.10$

Or

A_4 (due) = $100.\ a\ 4\ i\% \times (1+1\%)$

$= 100 \times 3.90196 \times 1.01$

$= 100 \times 3.94099 = 394.10$

Relationship Between the amount and the present value of an Annuity Due.

S_n (due) = A_n (due) $(1+i)^n$ 1

A_n (due) = S_n (due) $(1+i)^{-n}$ 2

Example:

If the amount of an annuity is \$410.10 what is the present value of the annuity due. (where i = 1%, n=4

Solution:

A_4 (due) = S_n (due) $(1+i)^{-n}$

= S_4 (due) $(1+1\%)^{-4}$

= 410.10 (.96098) = 394.10

استنتاجـــــــــــــات هـــــــــــــامة

$$S_n(due) = R\frac{(1+i)^n - 1}{i}(1+i)$$

بدلالة جدول الجملة لمبلغ

$$A_n(due) = R\frac{1-(1+i)^{-n}}{i}(1+i)$$

Other types of problems in an annuity due

Example 1:

A man wishes to receive \$2000 five years from now. How must he invest at the beginning of each year if the first payment starts now and the interest is 9% compounded annually?

Solution:

$S_n(\text{due}) = 2000$ $i = 9\%$ $n = 5$ $R = ?$

$S_n(\text{due}) = R\,(S_{\overline{n+1}|\,i} - 1)$

$2000 \quad = R(S_{\overline{5+1}|\,9\%} - 1)$

$2000 \quad = R(7.523334-1)$

$$R = \frac{2000}{6.523} = 307$$

Example:

A washing machine that sells for $600 can be bought under terms of 20 equal monthly payment, starting now. If money is worth 21% compounded monthly, what is the size of each payment?

Solution:

$$R = 35.20$$

Example:

At what nominal rate compounded annually will an annuity due of 1000 payable at the beginning of each year for four years, amount to 4,500

***Solution*:**

S_n (due) = 4,500 R=1000 n=4 i =?

$S_n \text{ (due)} = R\, S_{\overline{n+1}|i} - R$

$4.500 = 1.00\, S_{\overline{4+1}|i} - 1,000$

$$S_{\overline{5}|i} = \frac{4500 + 1,000}{1,000} = 5.5$$

$$\text{in table} \quad S_{\overline{5}|4\frac{1}{2}\%} = 5.470709$$

i = J

$$S_{\overline{5}|5\%} = 5.525631 \qquad 4\frac{1}{2}\% < i < 5\%$$

Example:

What is the nominal rate compounded quarterly if the present value of an annuity of \$300 payable at the beginning of each quarter for six years is 5,800?

***Solution*:**

A_n (due) 5,800 R=300 n=6×4=24 i =?

$A_n \text{ (due)} = R\, a_{\overline{n-1}|i} + R$

$5,800 = 300\, a_{\overline{24-1}|i} + 300$

$$a_{\overline{23}|i} = \frac{5,800 - 300}{300} = 18.333$$

$$\text{in table} \quad a_{\overline{23}|1\frac{3}{4}\%} = 18.801247$$

$$a_{\overline{23}|2\%} = 18.2922 \qquad 1\frac{3}{4}\% < i < 2\%$$

7%< nominal rate < 8%   ويستكمل الحل

Example:

If 100 is deposited at the beginning of each monthly, How many months will be required for the deposits to amount to at least 7,400? (9%)

Solution:

$$S_n(due) = 7{,}400 \qquad \text{R} = 100 \qquad \text{i} = \frac{9\%}{12} = \frac{3}{4}\%$$

$$S_n(due) = R\,S\,\overline{\text{n}+1}|\ \text{- R}$$

$$7.400 = 100\,\text{S}\,\overline{\text{n}+1}|\text{-}100$$

$$S\,\overline{n+1}|\frac{3}{4}\% = \frac{7.400+100}{100} = 75$$

$$\textit{in}\text{ the table }\ \text{S}\,\overline{\text{n}+1}|\tfrac{3}{4}\% \left\langle \begin{array}{l} \text{S}\,\overline{59}|\tfrac{3}{4}\%\text{ is} \\ 73.870111 \\ \text{S}\,\overline{60}|\tfrac{3}{4}\%\ \textit{is} \\ 75.424136 \end{array}\right.$$

$$\text{n}+1 = 60$$

$$\text{n} = 60\text{ - }1 = \textcircled{59}$$

$$\text{n} = 59\text{ months}$$

Example:

A man bought $4,250 boat and agreed to pay for it in installments of $500 at the beginning of every six months starting on the date of purchase. If the interest charged is 12% compounded semiannually, how long will it take the man to pay for the boat?

Solution:

$A_n(due) = 4{,}250 \qquad \mathrm{R} = 500 \qquad \mathrm{i} = 64. \qquad \mathrm{n} = ?$

$A_n(due) = R\,a\,\overline{\mathrm{n}-1}|_{\mathrm{i}} + \mathrm{R} \qquad \frac{12}{2}\%$

$4{,}250 = 500\ \mathrm{a}\ \overline{\mathrm{n}-1}|_{6\%} + 500$

$\mathrm{a}\ \overline{\mathrm{n}-1}|_{6\%} = \frac{4{,}250 - 500}{500} 7.5$

10< n-1 < 11

n-1=11 n= 12 ولا تستخدم طريقة النسبة والتناسب

Deferred Annuity

- Deferred Annuity is an annuity starts on a future date.
- The period between now and the beginning of the term of the annuity is called the Period of deferment.
- An annuity of $100 payable quarterly for six payments with the first payment to be made at the end of the third quarter is deferred annuity.

The Amount of a Deferred Annuity:

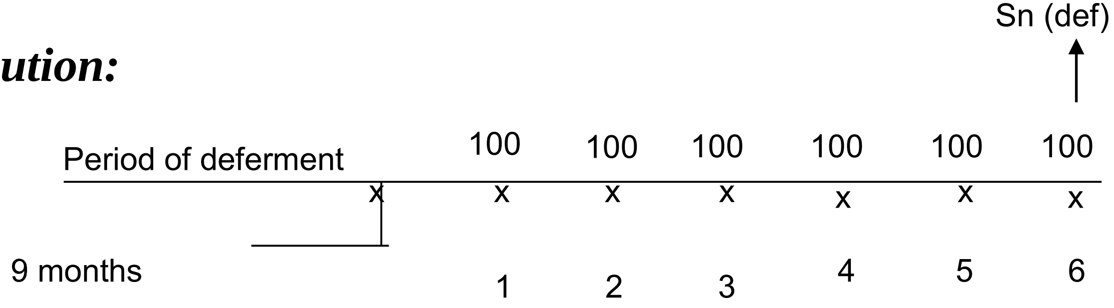
- The amount of a deferred Annuity is the final value at the end of the term of the annuity.
- The amount of the deferred annuity is the same as the amount of the ordinary annuity.

$$\boxed{Sn(def) = Sn = Rs\ n_{i}}$$

Example 1:

Find the amount of annuity of $100 payable at the end of each quarter for six payments. The interest rate is 6% compounded quietly. The first payment is due at the end of nine months.

Solution:



R= 100 $\quad$ i = 1 ½ % $\quad$ n=6

$Sn\ (def) = R\,S_{\overline{n|}\,i} = 100\,S_{\overline{6|}\,1\frac{1}{2}\%} = 622.96$

The present value of a Deferred Annuity

- The present value of a deferred annuity is the value at the beginning of the period of deferment, not at the beginning of the term of the ordinary annuity.
- The present value of a deferred annuity denoted by An (def), may be found by either of the following two method.

Method A:-

$An\ (def) = A_{d+n} - A_{d} = R\,a_{\overline{d+n|}\,i} - R\,a_{\overline{d|}\,i}$

$An\ (def) = R\,(a_{\overline{d+n|}\,i} - a\ \ d_{i})$

Example 1:

Find the present value of an annuity of % 100 payable at the end of each quarter for six payments. The interest rate is 6%

compounded quietly the first payment is due at the end of 6 month.

Solution:

R= 100 i= 1 ½ % n= 6 d= (2 quarters or 6 m)

$A_{d+n} = A_{2+6} = A_8 = R\, a_{\overline{8}|1\frac{1}{2}\%} = 100\, a_{\overline{8}|1\frac{1}{2}\%}$

$A_d = A_2 = R\, a_{\overline{2}|1\frac{1}{2}\%} = 100\, a_{\overline{2}|1\frac{1}{2}\%}$

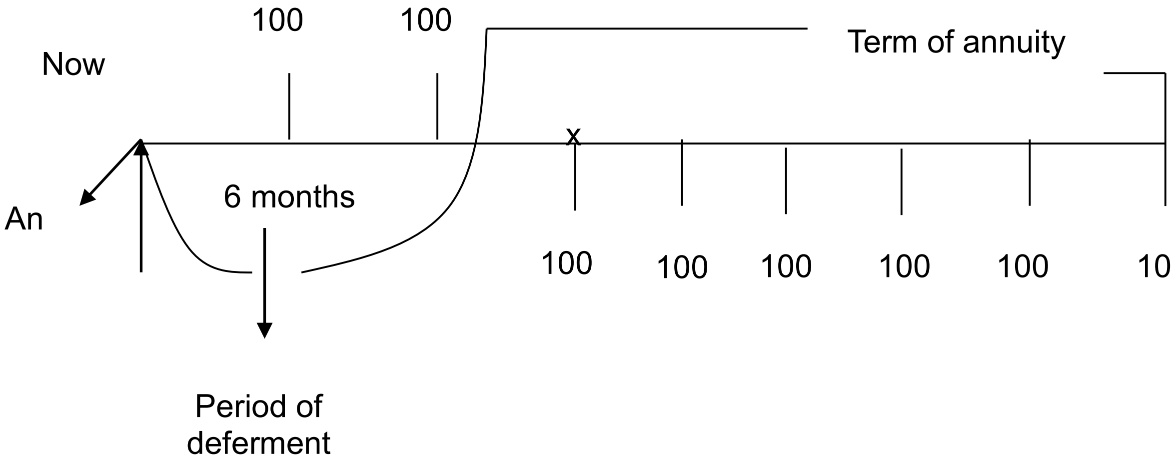
$A_6(\text{def}) = A_8 - A_2 = 100\, a_{\overline{8}|1\frac{1}{2}\%} - 100\, a_{\overline{2}|1\frac{1}{2}\%}$

$= 100\ (7.48592) - 100\ (1.95588)$

$= 748.592 - 195.588 = 553.004$

Example is diagrammed as follows to illustrate

<u>Method A:-</u>



<u>Method B:-</u>

$A_n(\text{def}) = A_n\ (1+i)^{-d} = R\, a_{\overline{n}|i}\ (1+i)^{-d}$

$A_n = A_6 = 100\ a_{\overline{6}|1\frac{1}{2}\%} = 100\ (5.69719) = \$\ 569.72$

$A_n(def) = A_n\ (1+i)^{-d} = 569.72\ (1+1\ \frac{1}{2}\ \%)^{-2}$

$= 569.72\ (.97066)\ = 553$

Example:

If \$ 50 is deposited at the end of each month and the interest rate is 6% compounded monthly, How many month will be required for the deposits to equal a present value of \$ 4,500? The first deposit id made at the end of six months.

Solution:

$A_n(def) = 4.500 \qquad R= 50 \quad i= \frac{1}{2}\ \% \qquad d=5$ or5 payments

n = ? (month or payment)

$4{,}500= 50\ a_{\overline{5+n}|\frac{1}{2}\%} - 50\ a_{\overline{5}|\frac{1}{2}\%}$

$= 50\ a_{\overline{5+n}|\frac{1}{2}\%} - 50\ (4.9259)$

$= 50\ a_{\overline{5+n}|\frac{1}{2}\%} - 246.295$

$$a_{\overline{5+n}|\frac{1}{2}\%} = \frac{4.500+246.295}{50} = 94.925$$

In the ½ % of table find the first entry greater than 94.9259.

$A_{\overline{130}|}\frac{1}{2}\ \% = 95.4216$

Thus $5+n = 130 \qquad n= 130- 50 = 125$ (month)

استهلاك القروض وتخصيص الأموال (هـــــام)

Amortization and sinking founds

Amortization of Debt:-

1- The debt is amortized by equal Payment at equal intervals:

Example:

A debt of 5000 is to be Amortized by 5 quarterly payments made at 3 month interval. If interest is charged at the rate of 12% convertible quarterly, find the periodic payment and construct an amortization schedule.

We can use this formula:

$$R = A_n \times \frac{1}{a_{\overline{n}|\mathrm{i}}}$$

R= equal payment (periodic payment).

A_n= Present value of an annuity of n payments.

$\frac{1}{a\overline{n}|i}$ = periodic payment necessary to pay off a loan of 1 in n payment.

from table (third column)

$$R = 5000 \times \frac{1}{a_{\overline{5}|3\%}} = 5000 \times .2183254 = 1091.78$$

Amortization Schedule:

(1) Payment number	(2) Total payment	(3) Payment on interest .03 x(5)	(4) Payment on principal (2)-(3)	(5) Balance of loan (5)- (4)
1	1091.78	150.00	941.78	4058.22
2	1091.78	121.75	970.03	3088.19
3	1091.78	9265	999.13	2089.06
4	1091.78	62.67	1029.11	1059.95
5	1091.78	31.80	1059.95	-
	5458.87	458.87	5000	

Column 3:

$$150 = 5000 \times \frac{3}{100} = 150$$

$$121.75 = 4058.22 \times \frac{3}{100} = 121.75$$

$$92.65 = 3088.19 \times \frac{3}{100} = 92.65$$

وهكذا

Column 4:

$$941.78 = 1091.78 - 150 = 941.78$$

$$970.03 = 1091.78 - 121.70 = 970.03$$ **وهكذا**

Column 5:

$$4058.22 = 5000 - 941.78 = 4058.22$$

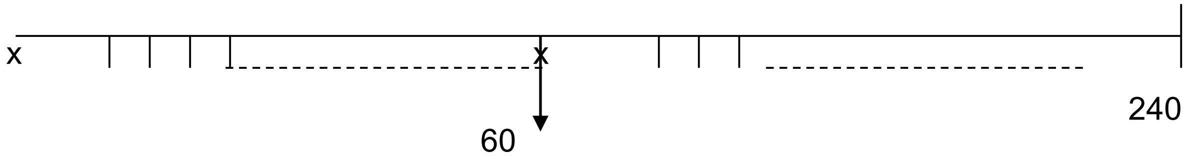
تمريـــــــــن هـــــــــام

To pay off an s 80,000 debt, a man got a 20- year loan at 12% converted monthly. How much does he still owe after he has paid on it for 5 years?

Solution:

$$\textit{monthly}\ \text{payment} = \frac{80,000}{a\,\overline{240}|_{1\%}} = \frac{80,000}{90.8194} = 880.87$$

Focal date at the 60 the payments that have been made.



$$\text{outstanding principal} = 80,000(1.01)^{60} - 880.87\ S\ 60_{i\%}$$
$$= \$73,395.38$$

هل يمكن أن يعطي هذه الأرقام ؟

$A\ \overline{5}|_{i\%}$= ……. **$A\ \overline{60}|_{i\%}$= ……** **$a\ \overline{180}|_{i\%}$= ……**

تمرين هام

A debt of 6000 is to be amortized by payment of 300 at the end of every month interest charged is 12% compounded monthly. Find:

a) The number of payments.

المتبقي من الأصل (رصيد القرض)

b) The outstanding principal after the 15th payment.

c) The outstanding principal after each payment is made.

Solution:

$$R = A_n \times \frac{1}{a_{\overline{n}|i\%}}$$

$$a)\ a_{\overline{n}|i\%} = \frac{6000}{300} = 20$$

$$22 < n < 23 \qquad n = 23$$

$$\text{b)} = 6000(1+1\%)^{15} - 300\,S_{\overline{15}|i\%}$$

$$= 6000(1.160968) - 300 \times 13.865051$$

$$= 6965.8 - 4159.51 = 2806.2$$

c) Construct Amortization Schedule

Example:

A debt of 6,000 is to be discharged by payment of 1,000 at the end of every month interest charged is 12% compounded monthly. Find:

a) The number of payments.

b) The outstanding principal after each payment is made.

c) The interest included in each payment.

d) The principal included in each payment, and.

e) The size of the final payment and the total of the payments.

Solution:

Method A:- without constructing an Amortization schedule.

$$A_n = R . a_{\overline{n}|\, i} \qquad 6000 = 1000 .\ a_{\overline{n}|\, i\%}$$

$$a_{\overline{n}|\, i\%} = \frac{6000}{300} = 6 \qquad \textit{from}\ \text{table}$$

$$6 < n < 7$$

$$\textit{by}\ \text{interpolat ion} \qquad \text{n} = 6.219276$$

b)- The outstanding principal after each payment is made. Only the outstanding principal after the fifth payment (or at the beginning of the sixth payment interval) is computed her for illustration.

Step (1): find the value of the original debt on the date of the fifth payment as if no payment were made previously use the formula $S= P(1+i)^n$

$S= 6000\ (1+1\%)^5 = 6,306.06$

Step (2): find the value of the five payments up to that date as if the payment were invested use the formula $Sn= R.S\ n_{\ i\%}= 1000\times S\ 5_{\ i\%} = 5,101.00$

Step (3): find the difference between the results of step (1) and step (2) 6,306.06- 5,101.00= 1205.06

c) the interest and principal include in each payment is computed here for illustration. Interest for the sixth payment period = outstanding principal after the fifth payment is made multiplied by 1% = 1,205.06 x 1/100 = 12.05.

d)- Principal include in sixth payment = sixth payment – interest for the six-payment period = 1,000-12.05=987.95

e)- The final payment of 219.28 and the total payment of 6,219.28. They may be obtained by any one of the following three methods without constructing a schedule.

Method (1): - Assume the debtor paid 1,000 (which the same amount as the other payment) on the seventh payment date the over payment المدفوع بالزيادة after the seventh payments is 780.72 and is computed as follows [see the method] used for (b)المطلوب.

$S_7= 1.000\ S_{\overline{7}|\,i\%} = 7,213.535$ [Amount of payment on the seventh payment date $S= 6,000\ (1+1\%)^7 = 6,433.812$ (Amount of the debt on the seventh payment date.

Over payment المدفوع بالزيادة= 7,213.535 – 6,432.812

= 780.72

The final payment = 1,000- 780.72= 219.28

The total of the payments = 6000 + 219.28= 6,219.28

Method(2):- find the outstanding principal after the sixth payment (the last of the equal payment of 1,000 and then add the interest on the outstanding principal to obtain the final payment.

$S= 6000(1+1\%)^6 = 6,369.12$ (Amount of the debt)

$S_6= 1,000\ S_{\overline{6}|i\%} = 6,152.015$ (Amount of payments).

Outstanding principal after the sixth payment = 6,369.12 – 6,151.015 = 217.106

The interest for the seventh period = 217.11× 1/100= 2.17

The final payment = 217.11+2.17=219.28

Method(3):- total of payment number of payment (n) x the size of each payment (R)

By interpolation (n)= 6.2192769

Total of payment = 6.219276 x 1000= 6,219.28

The final payment = 6,219.28- 6000= 219.28.

Amortization By the Add- on interest Method.

Example:

A man borrowed 1,200 from a bank and agreed to make 12 equal monthly payments the first installment the first installment being payable one month from the date of borrowing. The bank computed the monthly payment by the “add on interest method” at the advertised interest rate of 6% per year. Find:

a) The size of each payment.

b) The actual annual interest rate charged by bank.

Solution:

a) Simple interest used

$$I = \text{Pr}t \qquad \text{I} = 1{,}200 \times \frac{6}{100} \times 1 = 72$$

$$Amount = 1{,}200 + 72 = 1272$$

$$Size \text{ of payment} = \frac{1272}{12} = 106.$$

b) $A_n = R.a_{\overline{n}|i\%}$

$1200 = 106 \times a_{\overline{12}|i\%}$

$$a_{\overline{12}|i\%} = \frac{1200}{106} = 11.3208$$

$$\frac{7}{8}\% < i < 1\%$$

$$by \text{ interpolation i} = .91\%$$

$$\text{nominal rate} = .91 \times 12 = 10.92\%$$

All periodic payment Except the final payment are Equal.

<u>Method B</u>:

Constricting Amortization schedule

(1) period-month interval	(2) Outstanding principal beginning each period (2) - (5)	(3) Interest at end of period (2) x1%	(4) Payment at end of each period (4) - (3)	(5) Portion principal reduce by each payment
1	6,000	60.00	1,000	940.00
2	5,060	50.60	1,000	40.949.
3	4,110.60	41.11	1,000	958.89
4	3,151.71	31.52	1,000	968
5	2,183.23	21.83	1,000	978.17
6	1,205.06	12.05	1,000	987.95
7	217.11	2.17	219.28	217.11
Total		219.28		6000

Sinking Funds: when a sum of money will be needed at some future date, a good practice is to accumulate systematically a fund that will equal the sum desired at the time it is needed money accumulated in this way is called a sinking fund.

Sinking funds are used to:

1- pay off debts.

2- provide money for pure chase of new equipment.

Example:

A person wants to have $ 7500 to purchase a new car in 3 years. How much be deposited every 6 months is an account paying 6% converted semiannually.

$$R = Sn \times \frac{1}{s_{\overline{n}|i}} \textit{ from } \text{table column3}$$

$$\text{R} = 7500 \times \frac{1}{\text{S}_{\overline{6}|3\%}} = 7500 \times .154597 = 1159.49$$

Sinking fund schedule:

(1) period	(2) Amount in fund at start of period	(3) Interest Earned on sinking fund. During period	(4)Deposit at End of Period	Amount fund at end of period
1	-	-	1159.49	1159.49
2	1159.49	34.78	1159.49	2353.76
3	2353.76	70.61	1159.49	3583.86
4	3583.86	107.52	1159.49	4850.87
5	4850.87	145.53	1159.49	6155.89
6	6155.89	184.68	1159.49	7500
		543.12	6956.88	7500

Column 2:

- **Nothing**

1159.49 = Deposit at End of period

2353.76 = 1159.49 + 1159.49+ 34.78=

3583.86 = 2353.76 + 1159.49+ 70.61=

وهكذا

Column 3:

$$1159.49 \times \frac{3}{100} = 34.78$$

$$2353.76 \times \frac{3}{100} = 70.61$$

وهكذا

EXSMS FOR PRIOR YEARS

South Valley University Faculty of commerce	**Date:6/6/2022 Time Allowed:3 hours**
Mathematics of finance and investment	
Second Year	**English teaching section**

Question One (60 Points)

<u>Choose the correct answer</u>

1- On June25, 2021 Hazem borrowed $ 5000 and agreed to repay loan together with interest rate at 10% in 90 days. the amount must he repay:

a- 5525 b-5125 c-5225 d-5325

2- The present value of $5075 that is due at the end of two months if the interest rate 9% is :

a- 4900 b-5250 c-5000 d-5240

3- Simple discount of $5075 that is due at the end of two months if the interest rate 9% is:

a- 57 b-70 c-60 d-75

4- The present value of $500 for 1 year at a bank discount rate of 10%:

a- 450 b-540 c-460 d-470

5- The final amount of a bank charge $200 interest in an advance for three months at 8%:

a- 11000 b-12000 c-11500 d-10000

6- A man borrows $ 10000 from a bank. The charged 10% interest in advance. He received $9750, when will the loan be due after the date of borrowing:

a- 70 day b-80 days c-85 days d-90 days

7- A three month, non- interest- bearing note dated on march 2, 2021 was discount in a bank on April 3 at 12% the proceeds were $9800, the face value of note is :

a- 10000 b-10100 c-10200 d-9500

8- A bank discounts a 75- day note at 9%, the equivalent simple interest rate earned by bank is :

a- 8.17% b- 9.17% c- 9% d- 8%

9- The rate should a bank discount a 60 day note if the bank is to earn simple interest equivalent to 8% is:

a- 8.89% b- 7.89% c- 8% d- 7%

10- A man received a 30day of $10000 from a bank. The proceeds were $9900, the discount rate is:

a- 13% b- 10% c- 9% d- 12%

11- A man wants $5000 in cash as the proceeds of loan a bank that charge a 16% discount. the loan that must be paid on the maturity date after three month is:

a- 5208.33 b- 4208.33 c- 4000 d- 5000

12- A bank pays 8% per annual on savings accounts. A person opens an account with a deposit of $ 3000 on January 1, the interest will the person receive on April 1, is:

a-$60 b-$70 c-$90 d-$50

13- A person gets $ 200 every 6 months from an investment that pays 10% interest rate, money invested:

a- 4100 b-4000 c-4200 d-4500

14- A woman borrows $ 2000 from a credit union. Each month she is to pay $ 100 on the principal. She also pays interest at the rate of 1% a month on the unpaid balance at the beginning of the month. the total interest is:

a- 200 b- 240 c- 120 d- 210

15- A man borrows $35000 for 6 months at 15%. the amount must be he repay is:

a- 35000 b-37625 c-36725 d-32760

16- The Exact time from November 14, 1996 to April 24, 1997 is :

a- 161 b-160 c-159 d-162

17- The approximate time from November14, 1996 to April 24, 1997 is:

a- 162 b-163 c-158 d-160

18- The ordinary interest on $ 5000 at 10% for 90 days is:

a- 125 b-152 c-130 d-135

19- The exact interest on $ 10000 at 10% for 30 days:

a- 82.19 b-72.19 c-92 d-80

20- Find the amount of $1500 invested at 12% compounded quarterly and due at the end of 4.1/4 years

a- 2479.272 b-2497.272 c-2797.272 d-1479.272

21- Find the present value of $1000 due at the end of 4 ¼ years if money is worth 12% compounded quarterly

a- 605.016 b-506.016 c-1605.016 d-1000

22- A non- interest- bearing note of $1000 is discount at 6% compound semiannually for three years and two months. Find the proceeds

a- 892.35 b-792.35 c-829.35 d- =929.35

23- If $50 is deposited at the end of each month for three years in a fund that earns 6% interest compounded monthly, what will be the final value at the end of the three- year term?

a- 1968.05 b-1698.05 c-1986.05 d-1800

24-What is the amount of an annuity due for one year if each payment is $100 payable at the beginning of each quarter and the interest rate is 4% compounded quarterly?

a- 510.10 b-610.10 c-310.10 d-410.10

25 -Find the amount of annuity of $100 payable at the end of each quarter for six payments. The interest rate is 6% compounded quarterly. The first payment is due at the end of nine months.

a- 622.96 b-626.96 c-526.96 d-462.96

26- What is the present value of an annuity due if the size of each payment is $100 payable at the beginning of each quarter for one year and the interest rate is 4% compounded quarterly?

a- 349.099 b- 394.099 c-294.099 d-449.099

27- What is the cash value of a car that can be bought for $1,000 down and $500 a month for 36 month if money is worth 12% compounded monthly?

a- 16053.75 b- 15053.75 c-17053.75 d- 14053.75

28- How much money must Ali invest today at 10% simple interest if he is to receive $1150 after 1.5year?

a- 1200 b-1000 c -1100 d-1300

29-A debt of $ 200 is due in six month, if the rate of interest is 15%, what is the value of the debt if it is paid after two months hence?

a- 200 b-180.48 c-190.48 d-170.48

30-A Store purchased merchandise in the following amounts; $200, due in three months; 500, due in four months; and 100, due in six months. What will the equated date be if a single payment of 800 discharge the three debts?

a- 3month - 6month 7- 5month d- 4month

Question Two (40 points)

State whether each of the following statements is true or false

1- Deferred Annuity is an annuity starts on a future date

2- The period between now and the end of the term of the annuity is called the Period of deferment

3- An annuity of $500 payable quarterly for six payments with the first payment to be made at the end of the third quarter is deferred annuity.

4- The amount of a deferred Annuity is the final value at the beginning of the term of the annuity

5- The amount of the deferred annuity is the same as the amount of the ordinary annuity.

6- The present value of a deferred annuity is the value at the end of the period of deferment.

7- Simple interest is defined as the product principal, rate, and time.

8- If no interest rate is given in a note then maturity value equal face value

9- Since the rate is a monthly rate, the time must be expressed in years

10- Ordinary interest is the value of interest computed by using 365 as the divisor in the time factor.

11- Simple Discount is the difference between the amount and its present value.

12- Discount period is the period from the date of discount to the maturity date.

13- Proceeds is the money that the borrower receives from the bank

14- Bank discount is sometimes called interest in advance because it is based on the future rather than on the present value

15- A given bank discount rate result in a larger money return to the lender than the same simple interest rate

16- Bank discount is commonly used to discount sums of money for period of time of a year or less.

17- The present value at given discount rate is more than the present value based on the same interest rate.

18- There is deferent between focal date and comparison date

19- Amount of old debt = Present value of new debt in focal date

20- If there is only one conversion period compound interest is the same as the simple interest.

(NOTE)

$(1+3\%)^{17} = 1.652848$

$(1+3\%)^{-7} = 0.813092$

$S\,\overline{36}|\tfrac{1}{2}\% = 39.361$

$S\,\overline{5}|1\% = 5.10101$

$S\,\overline{6}|1\tfrac{1}{2}\% = 6.2296$

$A\,\overline{3}|\;1\% = 2.94099$

$A\,\overline{36}|1\% = 30.1075$

Good Luck
Prof. Attia Gallol

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