Calculus II

First Year

Second Semester

Chapter

1

Definite Integrals

1 Definite Integral

Sometimes in geometrical and other applications of integral calculus it becomes necessary to find the difference in the values of an integral of a function $f(x)$ for two given values of the variable x, say a and b. This difference is called the *definite integral* of $f(x)$ from a to b or between the *limits* a and b.

This definite integral is denoted by

$$\int_a^b f(x)\,dx$$

and is read as "*the integral of $f(x)$ with respect to x between the limits a and b*".

It is often written thus:

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a),$$

where $F(x)$ is an integral of $f(x)$, $F(b)$ is the value of $F(x)$ at $x = b$, and $F(a)$ is the value of $F(x)$ at $x = a$.

The number a is called the *lower limit* and the number b, the *upper limit* of integration. The interval (a, b) is called the *range of integration.*

Fundamental Theorem of Integral Calculus: Let $f \in \mathbf{R}[a, b]$ and let ϕ be a differentiable function on $[a, b]$ such that $\phi'(x) = f(x)$ for all $x \in [a, b]$. Then

$$\int_a^b f(x)\, dx = \phi(b) - \phi(a).$$

2 Fundamental Properties of Definite Integrals

Property 1: *We have* $\int_a^b f(x)\, dx = \int_a^b f(t)\, dt$, *i.e., the value of a definite integral does not change with the change of variable of integration (also called '**argument**') provided the limits of integration remain the same.*

Proof: Let $\int f(x)\, dx = F(x)$; then $\int f(t)\, dt = F(t)$.

Now $$\int_a^b f(x)\, dx = [F(x)]_a^b = F(b) - F(a), \quad \text{...(1)}$$

and $$\int_a^b f(t)\, dt = [F(t)]_a^b = F(b) - F(a), \quad \text{...(2)}$$

From (1) and (2), we see that $\int_a^b f(x)\, dx = \int_a^b f(t)\, dt$.

Property 2: *We have* $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$, *i.e., interchanging the limits of a definite integral does not change the absolute value but changes only the sign of the integral.*

Proof : Let $\int f(x)\, dx = F(x)$. Then

$$\int_a^b f(x)\, dx = [F(x)]_a^b = F(b) - F(a) \quad \text{...(1)}$$

Also $$-\int_b^a f(x)\, dx = -[F(x)]_b^a = -[F(a) - F(b)] = F(b) - F(a). \quad \text{...(2)}$$

From (1) and (2), we see that $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$.

Property 3: *We have* $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$

Proof: Let $\int f(x)\, dx = F(x)$.

Then the R.H.S.

$$= [F(x)]_a^c + [F(x)]_c^b = \{F(c) - F(a)\} + \{F(b) - F(c)\}$$

$$= F(b) - F(a) = \int_a^b f(x)\, dx = \text{L.H.S.}$$

Note 1: This property also holds true even if the point c is exterior to the interval (a, b).

Note 2: In place of one additional point c, we can take several points. Thus

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^{c_3} f(x)\,dx + \ldots$$
$$+ \int_{c_{r-1}}^{c_r} f(x)\,dx + \ldots + \int_{c_n}^{b} f(x)\,dx.$$

Property 4: *We have* $\int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx.$

Proof: Let $I = \int_0^a f(x)\,dx.$

Put $x = a - t$, so that $dx = -\,dt$.

When $x = 0, t = a$ and when $x = a, t = 0$.

$$\therefore \quad I = \int_a^0 f(a - t)\,(-\,dt) = \int_0^a f(a - t)\,dt, \qquad \text{[by property 2]}$$

$$= \int_0^a f(a - x)\,dx. \qquad \text{[by property 1]}$$

Property 5: $\int_{-a}^{a} f(x)\,dx = 0$ *or* $\int_0^a f(x)\,dx$, *according as* $f(x)$ *is an odd or an even function of* x.

Proof: Odd and even functions. A function $f(x)$ is said to be

(i) *an odd function* of x if $f(-x) = -f(x)$,

(ii) *an even function* of x if $f(-x) = f(x)$.

Now $$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx, \text{ by property 3.} \qquad \ldots(1)$$

Let $u = \int_{-a}^{0} f(x)\,dx$. In the integral u, put $x = -\,t$ so that $dx = -\,dt$.

Also $t = a$, when $x = -\,a$ and $t = 0$ when $x = 0$.

$$\therefore \quad u = \int_a^0 f(-t)\,(-\,dt) = \int_0^a f(-t)\,dt, \qquad \text{[by property 2]}$$

$$= \int_0^a f(-x)\,dx, \qquad \text{[by property 1]}$$

$$= -\int_0^a f(x)\,dx, \text{ if } f(x) \text{ is an odd function of } x,$$

or $$= \int_0^a f(x)\,dx, \text{ if } f(x) \text{ is an even function of } x.$$

$\therefore$ from (1), we get

$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0,$$

if $f(x)$ is an odd function of x

and $$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx,$$

if $f(x)$ is an even function of x.

Property 6: $\displaystyle\int_{0}^{2a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$, *if* $f(2a-x) = f(x)$

and $\displaystyle\int_{0}^{2a} f(x)\,x = 0$, *if* $f(2a-x) = -f(x)$.

Proof : We have $\displaystyle\int_{0}^{2a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{a}^{2a} f(x)\,dx$

$$= \int_{0}^{a} f(x)\,dx - \int_{a}^{0} f(2a-y)\,dy,$$

[putting $x = 2a - y$ in the second integral and changing the limits]

$$= \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(2a-y)\,dy,$$

interchanging the limits in the second integral

$$= \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(2a-x)\,dx,$$

changing the argument from y to x in the second integral

$$= 2\int_{0}^{a} f(x)\,dx, \text{ if } f(2a-x) = f(x)$$

or $$= 0, \text{ if } f(2a-x) = -f(x).$$

Corollary: $\displaystyle\int_{0}^{2a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(2a-x)\,dx.$

Remember:

(i) $\displaystyle\int_{-\pi/2}^{\pi/2} f(\sin x)\,dx = 2\int_{0}^{\pi/2} f(\sin x)\,dx$ or $= 0$

as if, $f(\sin x)$ is an *even* or an *odd* function respectively.

(ii) $\displaystyle\int_{0}^{\pi} f(\sin x)\,dx = 2\int_{0}^{\pi/2} f(\sin x)\,dx,$ [by property 6, because $\sin(\pi - x) = \sin x$]

(iii) $\displaystyle\int_{-\pi/2}^{\pi/2} f(\cos x)\,dx = 2\int_{0}^{\pi/2} f(\cos x)\,dx,$ [by property 5]

(iv) $\int_0^{\pi} f(\cos x)\, dx = 2\int_0^{\pi/2} f(\cos x)\, dx$ or $= 0$,

as if, f (cos x) is an *even* or an *odd* function respectively.

(v) $\int_0^{\pi/2} f(\sin x)\, dx = \int_0^{\pi/2} f\left\{\sin\left(\frac{1}{2}\pi - x\right)\right\} dx,$ [by property 4]

$$= \int_0^{\pi/2} f(\cos x)\, dx.$$

(vi) $\int_0^{\pi} \sin^m x \cos^n x\, dx = 2\int_0^{\pi/2} \sin^m x \cos^n x\, dx$ or $= 0$,

according as n is an even or an odd integer, (by property 6).

Illustrative Examples

Example 1: *Evaluate* $\int_0^{\pi} \cos^{2n} x\, dx$.

Solution: We have $\int_0^{\pi} \cos^{2n} x\, dx = 2\int_0^{\pi/2} \cos^{2n} x\, dx,$

$\left[\because \int_0^{2a} f(x)\, dx = 2\int_0^{a} f(x)\, dx \text{ if } f(2a - x) = f(x).\right.$

Here taking $f(x) = \cos^{2n} x$, we see that

$\left. f(\pi - x) = \cos^{2n}(\pi - x) = (-\cos x)^{2n} = \cos^{2n} x = f(x)\right]$

$$= 2 \cdot \frac{(2n-1)(2n-3)\ldots\ldots 3.1}{2n(2n-2)(2n-4)\ldots\ldots 4.2} \cdot \frac{\pi}{2}, \text{ by Walli's formula}$$

$$= \frac{(2n-1)(2n-3)\ldots 3.1}{2^n . n!} \cdot \pi.$$

Example 2: *Evaluate* $\int_0^{\pi} \theta \sin^3 \theta\, d\theta$.

Solution: Let $I = \int_0^{\pi} \theta . \sin^3 \theta\, d\theta.$ ...(1)

Then $I = \int_0^{\pi} (\pi - \theta) \sin^3 (\pi - \theta)\, d\theta,$

$\left[\because \int_0^{a} f(x)\, dx = \int_0^{a} f(a - x)\, dx, \text{ refer prop. 4}\right]$

$$= \int_0^{\pi} (\pi - \theta) \sin^3 \theta\, d\theta. \quad \ldots(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi} [\theta \sin^3 \theta + (\pi - \theta) \sin^3 \theta]\, d\theta = \int_0^{\pi} (\theta + \pi - \theta) \sin^3 \theta\, d\theta$$

$$= \int_0^{\pi} \pi \sin^3 \theta\, d\theta = \pi \int_0^{\pi} \sin^3 \theta\, d\theta$$

$$= 2\pi \int_0^{\pi/2} \sin^3 \theta\, d\theta, \text{ by a property of definite integrals; refer prop. 6}$$

$$= 2\pi \cdot \frac{2}{3.1} .1, \quad \text{by Walli's formula}$$

$$= 4\pi/3.$$

$\therefore \qquad I = \frac{2}{3} \pi .$

Example 3: *Prove without performing integration that*

$$\int_{-a}^{2a} \frac{x\, dx}{x^2 + p^2} = \int_{a}^{2a} \frac{x\, dx}{x^2 + p^2} .$$

Solution: We have

$$\int_{-a}^{2a} \frac{x\, dx}{x^2 + p^2} = \int_{-a}^{a} \frac{x\, dx}{x^2 + p^2} + \int_{a}^{2a} \frac{x\, dx}{x^2 + p^2} . \qquad ...(1)$$

But if $f(x) = \dfrac{x}{x^2 + p^2}$, then $f(-x) = \dfrac{-x}{x^2 + p^2} = -f(x)$.

Therefore $f(x)$ is an odd function of x.

$$\therefore \qquad \int_{-a}^{a} \frac{x\, dx}{x^2 + p^2} = 0.$$

So from (1), we get $\displaystyle\int_{-a}^{2a} \frac{x\, dx}{x^2 + p^2} = \int_{a}^{2a} \frac{x\, dx}{x^2 + p^2} .$

Example 4: *Evaluate* $\displaystyle\int_0^{\pi} \frac{x\, dx}{a^2 \cos^2 x + b^2 \sin^2 x} .$

(Kumaun 2012)

Solution: Let $\displaystyle I = \int_0^{\pi} \frac{x\, dx}{a^2 \cos^2 x + b^2 \sin^2 x} . \qquad ...(1)$

Then $$I = \int_0^{\pi} \frac{(\pi - x)\, dx}{a^2 \cos^2 (\pi - x) + b^2 \sin^2 (\pi - x)},$$

$$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx\right]$$

$$= \int_0^{\pi} \frac{(\pi - x)\, dx}{a^2 \cos^2 x + b^2 \sin^2 x} . \qquad ...(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi} \frac{x + (\pi - x)}{a^2 \cos^2 x + b^2 \sin^2 x}\, dx = \pi \int_0^{\pi} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$$

$$= 2\pi \int_0^{\pi/2} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x},$$

by a property of definite integrals, refer prop. 6.

$\therefore$ $$I = \pi \int_0^{\pi/2} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \pi \int_0^{\pi/2} \frac{\sec^2 x\, dx}{a^2 + b^2 \tan^2 x},$$

dividing the numerator and the denominator by $\cos^2 x$.

Now put $b \tan x = t$. Then $b \sec^2 x\, dx = dt$.

Also when $x = 0$, $t = 0$ and when $x \to \pi/2, t \to \infty$.

$\therefore$ $$I = \frac{\pi}{b} \int_0^{\infty} \frac{dt}{a^2 + t^2} = \frac{\pi}{b} \cdot \frac{1}{a} \left[\tan^{-1} \frac{t}{a} \right]_0^{\infty}$$

$$= \frac{\pi}{ab} [\tan^{-1} \infty - \tan^{-1} 0] = \frac{\pi}{ab} \left[\frac{\pi}{2} - 0 \right] = \frac{\pi^2}{2ab}.$$

Example 5: *Evaluate* $\int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \sin x \cos x}\, dx.$

Solution: Let $I = \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \sin x \cos x}\, dx.$

Then $$I = \int_0^{\pi/2} \frac{\cos\left(\frac{1}{2}\pi - x\right) - \sin\left(\frac{1}{2}\pi - x\right)}{1 + \sin\left(\frac{1}{2}\pi - x\right) \cos\left(\frac{1}{2}\pi - x\right)}\, dx,$$ [Refer prop. 4]

$$= \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \cos x \sin x}\, dx = -\int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \sin x \cos x}\, dx = -I.$$

$\therefore$ $2I = 0$ or $I = 0$.

Example 6: *Evaluate* $\int_0^{\pi} \frac{x\, dx}{1 + \sin x}.$

Solution: Let $I = \int_0^{\pi} \frac{x\, dx}{1 + \sin x} = \int_0^{\pi} \frac{(\pi - x)\, dx}{1 + \sin(\pi - x)},$ [Refer prop. 4]

$$= \int_0^{\pi} \frac{(\pi - x)}{1 + \sin x}\, dx = \int_0^{\pi} \frac{\pi}{1 + \sin x}\, dx - \int_0^{\pi} \frac{x}{1 + \sin x}\, dx$$

$$= \pi \int_0^{\pi} \frac{1}{1 + \sin x}\, dx - I.$$

$\therefore$ $$2I = \pi \int_0^{\pi} \frac{dx}{1 + \sin x} = 2\pi \int_0^{\pi/2} \frac{dx}{1 + \sin x},$$ [Refer prop. 6] or $$I = \pi \int_0^{\pi/2} \frac{dx}{1+\sin x} = \pi \int_0^{\pi/2} \frac{dx}{1+\sin\left(\frac{1}{2}\pi - x\right)},$$ [Refer prop. 4]

$$= \pi \int_0^{\pi/2} \frac{dx}{1+\cos x} = \pi \int_0^{\pi/2} \frac{dx}{2\cos^2 \frac{1}{2}x} = \pi \int_0^{\pi/2} \frac{1}{2}\sec^2 \frac{1}{2} x \, dx$$

$$= \pi \left[\tan \frac{1}{2} x\right]_0^{\pi/2} = \pi \left[\tan \frac{1}{4}\pi - \tan 0\right] = \pi(1-0) = \pi.$$

Example 7: *Show that* $\int_0^{\pi/2} \log \sin x \, dx = -\frac{1}{2}\pi \log 2$ *or* $\frac{1}{2}\pi \log \frac{1}{2}$.

Solution: Let $I = \int_0^{\pi/2} \log \sin x \, dx.$ ...(1)

Then $I = \int_0^{\pi/2} \log \sin\left(\frac{1}{2}\pi - x\right) dx,$ $\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$$= \int_0^{\pi/2} \log \cos x \, dx. \quad ...(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \log \sin x \, dx + \int_0^{\pi/2} \log \cos x \, dx$$

$$= \int_0^{\pi/2} \log(\sin x \cos x) \, dx$$ **(Note)**

$$= \int_0^{\pi/2} \log\left\{\frac{\sin 2x}{2}\right\} dx = \int_0^{\pi/2} (\log \sin 2x - \log 2)\, dx$$

$$= \int_0^{\pi/2} \log \sin 2x \, dx - \int_0^{\pi/2} \log 2 \, dx$$

$$= \int_0^{\pi/2} \log \sin 2x \, dx - (\log 2)\,[x]_0^{\pi/2}$$

$$= \int_0^{\pi/2} \log \sin 2x \, dx - \frac{\pi}{2}\log 2.$$

Now put $2x = t$, so that $2\,dx = dt$. Also $t = 0$ when $x = 0$ and $t = \pi$ when $x = \frac{1}{2}\pi$.

$$\therefore \quad 2I = \frac{1}{2}\int_0^{\pi} \log \sin t \, dt - \frac{\pi}{2}\log 2$$

$$= \frac{1}{2}\int_0^{\pi} \log \sin x \, dx - \frac{\pi}{2}\log 2,$$ [Refer prop. 1]

$$= \frac{1}{2}.2\int_0^{\pi/2} \log \sin x \, dx - \frac{\pi}{2}\log 2,$$ [Refer prop. 6]

$$= I - \frac{1}{2}\pi \log 2.$$

Therefore $2I - I = -\frac{1}{2}\pi \log 2$

or $I = -\frac{1}{2}\pi \log 2 = \frac{1}{2}\pi \log (2)^{-1} = \frac{1}{2}\pi \log \frac{1}{2}$.

Example 8: *Show that* $\int_0^{\pi/2} x \cot x \, dx = \frac{1}{2}\pi \log 2$.

Solution: Let $I = \int_0^{\pi/2} x \cot x \, dx$. Integrating by parts taking cot x as the second function, we get

$$I = [x \log \sin x]_0^{\pi/2} - \int_0^{\pi/2} 1 . \log \sin x \, dx$$

$$= \left[\frac{\pi}{2} \log 1 - \lim_{x \to 0} x \log \sin x\right] - \int_0^{\pi/2} \log \sin x \, dx$$

$$= 0 - \lim_{x \to 0} x \log \sin x - \int_0^{\pi/2} \log \sin x \, dx.$$

Now $\lim_{x \to 0} x \log \sin x = \lim_{x \to 0} \frac{\log \sin x}{1/x}$ $\left[\text{form } \frac{\infty}{\infty}\right]$

$$= \lim_{x \to 0} \frac{(1/\sin x)\cos x}{-1/x^2} = \lim_{x \to 0} \frac{-x^2 \cos x}{\sin x} \qquad \left[\text{form } \frac{\infty}{\infty}\right]$$

$$= \lim_{x \to 0} \frac{-2x \cos x + x^2 \sin x}{\cos x} = \frac{0}{1} = 0.$$

$\therefore$ $I = 0 - \int_0^{\pi/2} \log \sin x \, dx = -\int_0^{\pi/2} \log \sin x \, dx.$

Now let $u = \int_0^{\pi/2} \log \sin x \, dx.$

Then proceeding as in Example 7, we have $u = -\frac{1}{2}\pi \log 2.$

$\therefore$ $I = -u = \frac{1}{2}\pi \log 2.$

Example 9: *Show that* $\int_0^{\pi/4} \log (1 + \tan \theta) \, d\theta = \frac{\pi}{8} \log 2.$

Solution: Let $I = \int_0^{\pi/4} \log (1 + \tan \theta) \, d\theta.$

Then $I = \int_0^{\pi/4} \log \left\{1 + \tan\left(\frac{1}{4}\pi - \theta\right)\right\} d\theta,$ $\left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$

$$= \int_0^{\pi/4} \log \left[1 + \frac{(1 - \tan \theta)}{(1 + \tan \theta)}\right] d\theta = \int_0^{\pi/4} \log \left\{\frac{2}{1 + \tan \theta}\right\} d\theta$$

$$= \int_0^{\pi/4} \log 2 \,.\, d\theta - \int_0^{\pi/4} \log (1 + \tan \theta)\, d\theta$$

$$= \log 2 \,.\, [\theta]_0^{\pi/4} - I.$$

$\therefore$ $$2I = \frac{1}{4} \pi \log 2 \quad \text{or} \quad I = \frac{1}{8} \pi \log 2 .$$

Example 10: *Show that* $\displaystyle\int_0^{\pi/2} \frac{\sin x \, dx}{\sin x + \cos x} = \frac{\pi}{4}.$

(Lucknow 2014)

Solution: Let $\displaystyle I = \int_0^{\pi/2} \frac{\sin x \, dx}{\sin x + \cos x}$. ...(1)

Then $$I = \int_0^{\pi/2} \frac{\sin\left(\frac{1}{2}\pi - x\right)}{\sin\left(\frac{1}{2}\pi - x\right) + \cos\left(\frac{1}{2}\pi - x\right)}$$ [Refer prop. 4]

$$= \int_0^{\pi/2} \frac{\cos x \, dx}{\cos x + \sin x}. \quad ...(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\sin x \, dx}{\sin x + \cos x} + \int_0^{\pi/2} \frac{\cos x \, dx}{\sin x + \cos x}$$

$$= \int_0^{\pi/2} \left[\frac{\sin x}{\sin x + \cos x} + \frac{\cos x}{\sin x + \cos x}\right] dx$$

$$= \int_0^{\pi/2} 1 \,.\, dx = [x]_0^{\pi/2} = \frac{\pi}{2}.$$

$\therefore$ $$I = \frac{1}{4}\pi .$$

Comprehensive Exercise 1

Evaluate the following integrals :

1. (i) $\displaystyle\int_0^{\pi} \cos^6 x \, dx.$ (ii) $\displaystyle\int_0^{\pi} \sin^3 x \, dx.$

2. (i) $\displaystyle\int_{-1}^{1} \frac{x^2 \sin^{-1} x}{\sqrt{(1 - x^2)}} dx.$ (ii) $\displaystyle\int_{-a}^{a} x \sqrt{(a^2 - x^2)}\, dx.$

(iii) $\displaystyle\int_{-1}^{1} \frac{x \sin^{-1} x}{\sqrt{(1 - x^2)}} dx.$

3. (i) $\displaystyle\int_0^{\pi} \frac{dx}{a + b \cos x}.$ (ii) $\displaystyle\int_0^{2\pi} \frac{dx}{a + b \cos x + c \sin x}.$

§ 7. Methods of Integration.

There are various methods of integration by which we can reduce the given integral to one of the fundamental or known integrals. Following are the four principal methods of integration :

(*i*) *Integration by substitution,*
(*ii*) *Integration by parts,*
(*iii*) *Integration by decomposition into sum,*
(*iv*) *Integration by successive reduction.*

(i) Integration by substitution. A change in the variable of integration often reduces an integral to one of the fundamental integrals. The method in which we change the variable to some other variable is called the **Method of substitution.**

Let $I = \int f(x)\,dx$; then by differentiation w.r.t. x, we have

$\dfrac{dI}{dx} = f(x)$. Now put $x = \phi(t)$, so that $\dfrac{dx}{dt} = \phi'(t)$.

Then $\frac{dI}{dt}=\frac{dI}{dx}\cdot\frac{dx}{dt}=f(x).\phi'(t)=f\{\phi(t)\}\,\phi'(t)$, for $x=\phi(t)$.

This gives, $I=\int f\{\phi(t)\}.\phi'(t)\,dt$.

Rule to remember.

To evaluate $\int f\{\phi(x)\}.\phi'(x)\,dx$,

put $\phi(x)=t$ and $\phi'(x)\,dx=dt$,

where $\phi'(x)$ is the differential coefficient of $\phi(x)$ w.r.t. x.

Important. The success of the method of substitution depends on choosing the substitution $x=\phi(t)$ so that the new integrand $f\{\phi(t)\}.\phi'(t)$ is of a form whose integral is known. This is done by guess rather than in according with some rule. However, try to put that expression of x equal to t whose differential coefficient is multiplied with dx.

Solved Examples

Ex. 1. *Evaluate* $\int x\sec^2 x^2\,dx$.

Sol. Put $x^2=t$, so that $2x\,dx=dt$ or $x\,dx=\frac{1}{2}dt$.

$\therefore\quad \int x\sec^2 x^2\,dx=\frac{1}{2}.\int\sec^2 t\,dt=\frac{1}{2}\tan t=\frac{1}{2}\tan x^2$.

Ex. 2. *Evaluate* $-\int\frac{\cos^{-1}x}{\sqrt{(1-x^2)}}dx$.

Sol. Put $\cos^{-1}x=t$, so that $-\frac{1}{\sqrt{(1-x^2)}}dx=dt$.

$\therefore$ the given integral $=\int t\,dt=\frac{1}{2}t^2=\frac{1}{2}(\cos^{-1}x)^2$.

Ex. 3. *Evaluate* $\int[1/(cx+d)^4]\,dx$.

Sol. Put $cx+d=t$; then $c\,dx=dt$, or $dx=dt/c$.

Thus the given integral

$$=\int(cx+d)^{-4}dx=\int t^{-4}\frac{dt}{c}=\frac{1}{c}\int t^{-4}dt=\frac{1}{c}\cdot\frac{t^{-3}}{-3}$$

$$=-\frac{1}{3c\,(cx+d)^3}.$$

Ex. 4. *Evaluate* $\int 20^{5x}dx$.

Sol. Put $5x=t$; then $5dx=dt$ or $dx=\frac{1}{5}dt$.

$$\therefore\quad I=\int 20^{5x}dx=\int\frac{1}{5}\cdot 20^t\,dt$$

$$=\frac{1}{5}\frac{20^t}{\log 20}=\frac{1}{5}\frac{20^{5x}}{\log 20}.$$

Ex. 5. *Evaluate* $\int\frac{dx}{\sqrt{\{1-(cx+d)^2\}}}$.

Sol. Put $cx+d=t$, so that $c\,dx=dt$. .

Hence the given integral

$$= \int \frac{1}{c} \frac{dt}{\sqrt{(1-t^2)}} = \frac{1}{c} \sin^{-1} t = \frac{1}{c} \sin^{-1} (cx + d).$$

Ex. 6. *Evaluate* $\int \{1/(c^2 + b^2y^2)\}\, dy$.

Sol. Put $by = t$; $\therefore$ $b\, dy = dt$ or $dy = (1/b)\, dt$.

$$\therefore \quad I = \frac{1}{b} \int \frac{dt}{c^2 + t^2} = \frac{1}{bc} \tan^{-1} \frac{t}{c} = \frac{1}{bc} \tan^{-1} \left(\frac{by}{c}\right).$$

Ex. 7. *Evaluate* $\int \{(\cos ax)/(\sin^2 ax)\}\, dx$.

Sol. We have $I = \displaystyle\int \frac{\cos ax\, dx}{\sin^2 ax} = \int \frac{\cos ax\, dx}{\sin ax \sin ax}$

$= \int \cot ax . \operatorname{cosec} ax\, dx$.

Now put $ax = t$; $\therefore$ $a dx = dt$ or $dx = (1/a)\, dt$.

Then $I = \dfrac{1}{a} \int \cot t \operatorname{cosec} t\, dt = -\dfrac{1}{a} \operatorname{cosec} t = -\dfrac{1}{a} \operatorname{cosec} ax$.

Ex. 8. *Evaluate* $\displaystyle\int -\frac{\operatorname{cosec}^2 x}{\sqrt{(\cot^2 x - 16)}}\, dx$.

Sol. Put $\cot x = t$, so that $-\operatorname{cosec}^2 x\, dx = dt$.

Hence $I = \displaystyle\int \frac{dt}{\sqrt{(t^2 - 16)}} = \cosh^{-1} \frac{t}{4} = \cosh^{-1} \left(\frac{\cot x}{4}\right)$.

Ex. 9. *Evaluate* $\int e^x \cos e^x\, dx$.

Sol. Put $e^x = t$, so that $e^x\, dx = dt$.

$\therefore$ $I = \int e^x \cos e^x\, dx = \int \cos t\, dt = \sin t = \sin e^x$.

Ex. 10. *Evaluate* $\int \sin^2 x \cos x\, dx$.

Sol. Put $\sin x = t$, so that $\cos x\, dx = dt$.

$\therefore$ $I = \int \sin^2 x \cos x\, dx = \int t^2\, dt = \frac{1}{3} t^3 = \frac{1}{3} \sin^3 x$.

Ex. 11. *Evaluate* $\int [4x^3/(1 + x^8)]\, dx$.

Sol. Put $x^4 = t$ so that $4x^3\, dx = dt$.

$\therefore$ $I = \int [4x^3/(1 + x^8)]\, dx = \int [1/(1 + t^2)]\, dt = \tan^{-1} t$

$= \tan^{-1} x^4$.

§ 8. Three important forms of integrals.

1. $\displaystyle\int \frac{\mathbf{f}'(\mathbf{x})}{\mathbf{f}(\mathbf{x})}\, \mathbf{dx} = \log \mathbf{f}(\mathbf{x})$.

Put $f(x) = t$; differentiating we have $f'(x)\, dx = dt$.

$$\therefore \quad \int \frac{f'(x)}{f(x)}\, dx = \int \frac{dt}{t} = \log t = \log f(x).$$

Thus **(Remember)**

the integral of a fraction whose numerator is the exact derivative of its denominator is equal to the logarithm of its denominator.

For example $\int \frac{4x^3}{1+x^4}\,dx = \log(1+x^4)$,

as in this case numerator is the exact derivative of the denominator.

Similarly $\int \frac{e^x}{1+e^x}\,dx = \log(1+e^x)$.

Integrals of tan x, cot x, sec x and cosec x.

(i) $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{(-\sin x)}{\cos x}\,dx$,

adjusting the numerator as the exact diff. coeffi. of the denominator

$= -\log\cos x = \log(\cos x)^{-1} = \log(\sec x)$.

(ii) Similarly, $\int \cot x\,dx = \int \frac{\cos x\,dx}{\sin x} = \log(\sin x)$.

(iii) $\int \operatorname{cosec} x\,dx = \int \frac{dx}{\sin x} = \int \frac{dx}{2\sin\frac{1}{2}x\cos\frac{1}{2}x} = \int \frac{\sec^2\frac{1}{2}x\,dx}{2\tan\frac{1}{2}x}$,

[dividing Nr. and Dr. by $\cos^2\frac{1}{2}x$]

$= \log(\tan\frac{1}{2}x)$, ($\because \frac{1}{2}\sec^2\frac{1}{2}x$ is the diff. coeffi. of $\tan\frac{1}{2}x$].

(iv) $\int \sec x\,dx = \int \operatorname{cosec}(\frac{1}{2}\pi + x)\,dx$.

Now proceeding as in the case of $\int \operatorname{cosec} x\,dx$, we have

$$\int \sec x\,dx = \log\tan\left(\frac{x}{2}+\frac{\pi}{4}\right).$$

***Alternative Method.**

$$\int \sec x\,dx = \int \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}\,dx,$$

[We have multiplied the Nr. and Dr. both by $(\sec x + \tan x)$]

$= \int \frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x}$, [Here Nr. is the diff. coeffi. of Dr.]

$= \log(\sec x + \tan x)$.

****2.** $\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}$, **when** $n \neq -1$.

(Power formula)

Putting $f(x) = t$, so that $f'(x)\,dx = dt$, we get

$$\int [f(x)]^n f'(x)\,dx = \int t^n\,dt = \frac{t^{n+1}}{n+1}, \text{ (for } n \neq -1) = \frac{[f(x)]^{n+1}}{n+1}.$$

Thus **Remember :** *If the integrand consists of the product of a constant power of a function $f(x)$ and the derivative $f'(x)$ of $f(x)$, to obtain the integral we increase the index by unity and then divide by the increased index.* This is known as **Power formula.** The students are advised to have a lot of practice of applying this formula.

3. $\int \mathbf{f}'(\mathbf{ax+b})\,\mathbf{dx} = \dfrac{\mathbf{f}(\mathbf{ax+b})}{\mathbf{a}}$.

Solved Examples

Ex. 1. *Integrate* **(i)** $\dfrac{ax+b}{ax^2+2bx+c}$. **(ii)** $\dfrac{ax^{n-1}}{x^n+b}$.

Sol. (i) Let $I = \displaystyle\int \frac{ax+b}{ax^2+2bx+c}\,dx$.

[Put $ax^2 + 2bx + c = t$, so that $(2ax + 2b)\,dx = dt$].

$\therefore \quad I = \dfrac{1}{2}\displaystyle\int \frac{dt}{t} = \frac{1}{2}\log t = \frac{1}{2}\log(ax^2+2bx+c)$.

(ii) Let $I = \displaystyle\int \frac{ax^{n-1}\,dx}{x^n+b} = \frac{1}{n}\int \frac{dt}{t}$,

[Putting $x^n + b = t$, so that $nx^{n-1}\,dx = dt$]

$= (a/n)\,.\log(t) = (a/n)\,.\log(x^n+b)$.

Ex. 2. *Integrate* **(i)** $\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, **(ii)** $\dfrac{10x^9 + 10^x.\log_e 10}{10^x + x^{10}}$.

Sol. (i) Let $I = \displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$.

Now putting $e^x + e^{-x} = t$, so that $(e^x - e^{-x})\,dx = dt$,
we have $I = \int (1/t)\,dt = \log t = \log(e^x + e^{-x})$.

(ii) Here $I = \displaystyle\int \frac{10x^9 + 10^x.\log_e 10}{10^x + x^{10}}\,dx$.

Now putting $10^x + x^{10} = t$, and $(10^x \log_e 10 + 10x^9)\,dx = dt$,
we have $I = \int (1/t)\,dt = \log t = \log(10^x + x^{10})$.

Ex. 3. *Integrate* **(i)** $\dfrac{cosec^2 x}{1 + cot\,x}$, **(ii)** $\dfrac{1}{(1+x^2)\,tan^{-1}x}$.

Sol. (i) Here $I = \int \{\operatorname{cosec}^2 x/(1+\cot x)\}\,dx$.

Putting $1 + \cot x = t$, so that $-\operatorname{cosec}^2 x\,dx = dt$,
we have $I = -\int (1/t)\,dt = -\log t = -\log(1+\cot x)$.

(ii) Here $I = \displaystyle\int \frac{dx}{(1+x^2)\tan^{-1}x}$.

Putting $\tan^{-1}x = t$, so that $[1/(1+x^2)]\,dx = dt$, we have

$I = \int (1/t)\,dt = \log t = \log(\tan^{-1}x)$.

Ex. 4. *Integrate*

(i) $\dfrac{sin\,x}{a + b\,cos\,x}$, **(ii)** $\dfrac{sin\,x\,cos\,x}{a\,cos^2 x + b\,sin^2 x}$.

Sol. (i) Here $I = \int \frac{\sin x \, dx}{a + b \cos x} = -\frac{1}{b} \int \frac{-b \sin x}{a + b \cos x} dx$.

Putting $a + b \cos x = t$, so that $-b \sin x \, dx = dt$, we have

$$I = (-1/b) \int (1/t) \, dt$$
$$= -(1/b) . \log(t) = -(1/b) . \log(a + b \cos x).$$

(ii) Let $I = \int \left(\frac{\sin x \cos x}{a \cos^2 x + b \sin^2 x} \right) dx$.

Put $a \cos^2 x + b \sin^2 x = t$,

so that $(-2a \cos x \sin x + 2b \sin x \cos x) \, dx = dt$,

or $\{2(b - a) \sin x \cos x\} \, dx = dt$.

We have $I = \frac{1}{2(b-a)} \int \frac{dt}{t} = \frac{1}{2(b-a)} \log t$

$$= \frac{1}{2(b-a)} \log(a \cos^2 x + b \sin^2 x).$$

Ex. 5. *Integrate*

(i) $\frac{\sin x}{1 + \cos^2 x}$, **(ii)** $\frac{\sin(a + b \log x)}{x}$.

Sol. (i) Here $I = \int \frac{\sin x}{1 + \cos^2 x} dx$.

Putting $\cos x = t$, so that $-\sin x \, dx = dt$, we have

$$I = -\int dt/(1 + t^2) = -\tan^{-1} t = -\tan^{-1}(\cos x).$$

(ii) Here $I = \int [\{\sin(a + b \log x)\}/x] \, dx$.

Putting $a + b \log x = t$, so that $(b/x) \, dx = dt$, we have

$$I = \frac{1}{b} \int \sin t \, dt = -\frac{1}{b} \cos t = -\frac{1}{b} \cos(a + b \log x).$$

Ex. 6. *Integrate*

(i) $\frac{1}{x \cos^2(1 + \log x)}$ (ii) $\frac{1}{x(1 + \log x)^m}$.

Sol. (i) Here $I = \int dx/\{x \cos^2(1 + \log x)\}$.

Putting $1 + \log x = t$, so that $(1/x) \, dx = dt$, we have

$$I = \int dt/\cos^2 t = \int \sec^2 t \, dt = \tan t = \tan(1 + \log x).$$

(ii) Here $I = \int dx/\{x(1 + \log x)^m\}$.

Putting $1 + \log x = t$, so that $(1/x) \, dx = dt$, we have

$$I = \int \frac{dt}{t^m} = \frac{t^{-m+1}}{-m+1} = \frac{(1 + \log x)^{-m+1}}{(1-m)}$$

$$= \frac{1}{(1-m)} (1 + \log x)^{1-m}.$$

Ex. 7. *Integrate* **(i)** $\frac{e^{\tan^{-1} x}}{1 + x^2}$ **(Meerut 1986 S, 88)**

(ii) $\dfrac{\sin(\tan^{-1} x)}{1+x^2}$.

Sol. (i) Putting $\tan^{-1} x = t$, so that $[1/(1+x^2)]\,dx = dt$, we have

$$I = \int \frac{e^{\tan^{-1} x}}{1+x^2}\,dx = \int e^t . dt = e^t = e^{\tan^{-1} x}.$$

(ii) Putting $\tan^{-1} x = t$ so that $[1/(1+x^2)]\,dx = dt$, we have

$$I = \int \frac{\sin(\tan^{-1} x)}{1+x^2}\,dx = \int \sin t\,dt = -\cos t = -\cos(\tan^{-1} x).$$

Ex. 8. *Integrate* (i) $x \cos^3 x^2 . \sin x^2$ (ii) $x^3 \tan^4 x^4 . \sec^2 x^4$.

Sol. (i) Here $I = \int x \cos^3 x^2 \sin x^2\,dx$.

First, putting $x^2 = t$, so that $2x\,dx = dt$, we have

$$I = \tfrac{1}{2} \int \cos^3 t \sin t\,dt.$$

Now putting $\cos t = u$, so that $-\sin t\,dt = du$, we have

$$I = -\tfrac{1}{2} \int u^3\,du = -\tfrac{1}{2}\frac{u^4}{4} = -\frac{1}{8}u^4$$

$$= -\tfrac{1}{8}\cos^4 t = -\tfrac{1}{8}\cos^4 x^2. \qquad [\because t = x^2]$$

Note. The students should also solve this problem by making the single substitution $\cos x^2 = t$.

(ii) Here $I = \int x^3 \tan^4 x^4 \sec^2 x^4\,dx$.

Putting $\tan x^4 = t$, so that $(\sec^2 x^4) . 4x^3\,dx = dt$, we have

$$I = \int \frac{t^4}{4} \cdot dt = \tfrac{1}{4}\left(\frac{t^5}{5}\right) = \frac{(\tan x^4)^5}{20}.$$

Ex. 9. *Integrate* $\dfrac{x^3 \tan^{-1} x^4}{1+x^8}$.

Sol. Putting $\tan^{-1} x^4 = t$ so that $\dfrac{1}{1+x^8} \cdot 4x^3\,dx = dt$, we have

$$I = \int \frac{x^3 \tan^{-1} x^4}{1+x^8}\,dx = \tfrac{1}{4}\int t\,dt = \tfrac{1}{8}t^2 = \tfrac{1}{8}(\tan^{-1} x^4)^2.$$

Ex. 10. *Evaluate* $\displaystyle\int \frac{e^x(1+x)}{\sin^2(xe^x)}\,dx$. **(Meerut 1982)**

Sol. Putting $xe^x = t$ so that $(e^x + xe^x)\,dx = dt$

or $e^x(1+x)\,dx = dt$, we have

$$I = \int \frac{dt}{\sin^2 t} = \int \operatorname{cosec}^2 t\,dt = -\cot t = -\cot(xe^x).$$

Ex. 11. *Integrate* (i) $\dfrac{1}{(e^x+1)}$, (ii) $\dfrac{1}{(e^x-1)}$.

(Meerut 1972)

Sol. (i) Here $I = \int \frac{dx}{e^x + 1} = \int \frac{e^{-x}}{1 + e^{-x}} dx,$

[Multiplying Nr. and Dr. both by e^{-x}]

$= - \int \frac{-e^{-x}}{1 + e^{-x}} dx = - \log(1 + e^{-x}),$

[∵ Nr. is diff. coeffi. of Dr.]

$= - \log \left(\frac{1 + e^x}{e^x} \right) = - [\log(1 + e^x) - \log e^x]$

$= x \log e - \log(1 + e^x) = x - \log(1 + e^x).$

(ii) Similarly $\int \frac{dx}{e^x - 1} = \log(1 - e^{-x}).$

Ex. 12. *Integrate* (i) $\frac{\cot x}{\log(\sin x)}$, (ii) $\frac{\tan x}{\log(\sec x)}$.

Sol. (i) Here $\frac{d}{dx}(\log \sin x) = \frac{1}{\sin x} \cos x = \cot x.$

$\therefore \quad I = \int \frac{\cot x \, dx}{\log \sin x} = \log(\log \sin x),$

[∵ Nr. is diff. coeffi. of Dr.]

(ii) Similarly, we have

$$\int \frac{\tan x \, dx}{\log \sec x} = \log(\log \sec x).$$

****Ex. 13.** *Integrate* (i) $\sqrt{(1 + \sin x)}$, (ii) $\frac{1}{\sqrt{(1 + \sin x)}}$

(Meerut 1984 P)

Sol. (i) We have

$I = \int \sqrt{(1 + \sin x)} \, dx = \int \sqrt{\{1 - \cos(\frac{1}{2}\pi + x)\}} \, dx$

$= \int \sqrt{[2 \sin^2(\frac{1}{4}\pi + \frac{1}{2}x)]} \, dx = \sqrt{2} \int \sin(\frac{1}{2}x + \frac{1}{4}\pi) \, dx.$

Now putting $\frac{1}{2}x + \frac{1}{4}\pi = t$, so that $\frac{1}{2} dx = dt$ or $dx = 2\,dt$, we have

$I = \sqrt{2} \int 2 \sin t \, dt = -2\sqrt{2} \cos t = -2\sqrt{2} \cos(\frac{1}{2}x + \frac{1}{4}\pi).$

(ii) Here $I = \int \frac{dx}{\sqrt{(1 + \sin x)}} = \frac{1}{\sqrt{2}} \int \operatorname{cosec}(\frac{1}{2}x + \frac{1}{4}\pi) \, dx,$

[Proceeding as in part (i)]

Now putting $\frac{1}{2}x + \frac{1}{4}\pi = t$ so that $dx = 2\,dt$, we have

$I = \frac{2}{\sqrt{2}} \int \operatorname{cosec} t \, dt = \sqrt{2} \log \tan(\frac{1}{2}t)$

$= \sqrt{2} \log \tan(\frac{1}{4}x + \frac{1}{8}\pi).$

Ex. 14. *Integrate* (i) $\frac{\sin x}{\sqrt{(1 + \sin x)}}$ **(Meerut 1985)**

(ii) $\sqrt{(1-\cos x)}$.

Sol. **(i)** Here $I = \int \frac{\sin x\, dx}{\sqrt{(1+\sin x)}} = \int \frac{(1+\sin x) - 1}{\sqrt{(1+\sin x)}} dx$

$= \int \sqrt{(1+\sin x)}\, dx - \int \frac{1}{\sqrt{(1+\sin x)}} dx$

$= -2\sqrt{2} \cos(\frac{1}{2}x + \frac{1}{4}\pi) - \sqrt{2} \log \tan(\frac{1}{4}x + \frac{1}{8}\pi)$,

[Proceeding as in Ex. 13 parts (i) and (ii)].

(ii) Here $I = \int \sqrt{(1-\cos x)}\, dx = \sqrt{2} \int \sin \frac{1}{2}x\, dx$

$= -\sqrt{2} \frac{\cos(x/2)}{1/2} = -2\sqrt{2} \cos(x/2)$.

Ex. 15. *Integrate*

(i) $\frac{\sec x}{a + b \tan x}$ **(ii)** $\frac{\sec x}{\sqrt{3} + \tan x}$ **(Meerut 1989)**

Sol. **(i)** Let $I = \int \frac{\sec x}{a + b \tan x} dx = \int \frac{dx}{a \cos x + b \sin x}$.

Now let $a = r \sin \phi$ and $b = r \cos \phi$. This gives

$r = \sqrt{(a^2 + b^2)}$, and $\phi = \tan^{-1}(a/b)$.

$\therefore \quad I = \int \frac{dx}{r \sin(x + \phi)} = \frac{1}{\sqrt{(a^2+b^2)}} \int \operatorname{cosec}(x+\phi)\, dx$

$= \frac{1}{\sqrt{(a^2+b^2)}} \log \tan(\frac{1}{2}x + \frac{1}{2}\phi)$

$= \frac{1}{\sqrt{(a^2+b^2)}} \log \tan\left(\frac{1}{2}x + \frac{1}{2}\tan^{-1}\frac{a}{b}\right)$.

(ii) We have $\int \frac{\sec x\, dx}{\sqrt{3} + \tan x} = \int \frac{\sec x\, dx}{\sqrt{3} + (\sin x/\cos x)}$

$= \int \frac{dx}{\sqrt{3}\cos x + \sin x} = \int \frac{dx}{2[(\sqrt{3}/2)\cos x + \frac{1}{2}\sin x]}$

$= \frac{1}{2} \int \frac{dx}{\sin(x + \frac{1}{3}\pi)} = \frac{1}{2} \int \operatorname{cosec}(x + \frac{1}{3}\pi)\, dx$

$= \frac{1}{2} \log \tan[\frac{1}{2}x + (\pi/6)]$.

Ex. 16. *Integrate*

(i) $\frac{\cos 2x}{\sin x}$, **(ii)** $\frac{\cos 2x}{\cos x}$.

Sol. **(i)** $I = \int \frac{\cos 2x}{\sin x} dx = \int \frac{1 - 2\sin^2 x}{\sin x} dx$

[$\because \cos 2x = 1 - 2\sin^2 x$]

$= \int (\operatorname{cosec} x - 2 \sin x)\, dx = \log \tan(\frac{1}{2}x) + 2 \cos x$.

(ii) Here $I = \int \frac{\cos 2x}{\cos x} dx = \int \frac{2\cos^2 x - 1}{\cos x} dx$,

$[\because \cos 2x = 2\cos^2 x - 1]$

$= \int (2\cos x - \sec x)\, dx = 2\sin x - \log \tan\left(\frac{1}{2}x + \frac{1}{4}\pi\right)$.

Ex. 17. *Integrate* $1/(1 + 3\sin^2 x)$.

Sol. Dividing Nr. and Dr. by $\cos^2 x$, we have

$$I = \int \frac{dx}{1 + 3\sin^2 x} = \int \frac{\sec^2 x\, dx}{\sec^2 x + 3\tan^2 x}$$

$$= \int \frac{\sec^2 x\, dx}{(1 + \tan^2 x) + 3\tan^2 x}$$

$$= \int \frac{\sec^2 x\, dx}{1 + 4\tan^2 x}.$$

Now putting $2\tan x = t$ so that $2\sec^2 x\, dx = dt$, we have

$$I = \frac{1}{2}\int \frac{dt}{1 + t^2} = \frac{1}{2}\tan^{-1} t = \frac{1}{2}\tan^{-1}(2\tan x).$$

Ex. 18. *Evaluate the following integrals* :

(i) $\int \frac{d\theta}{a^2 \sin^2\theta + b^2\cos^2\theta}$ **(Meerut 1985 P)**

(ii) $\int \frac{dx}{4\sin^2 x + 5\cos^2 x}$ **(Meerut 1989)**

Sol. **(i)** Dividing Nr. and Dr. by $\cos^2\theta$, we have

$$I = \int \frac{d\theta}{a^2\sin^2\theta + b^2\cos^2\theta} = \int \frac{\sec^2\theta\, d\theta}{a^2\tan^2\theta + b^2}$$

$$= \frac{1}{a^2}\int \frac{\sec^2\theta\, d\theta}{\tan^2\theta + (b^2/a^2)}$$

$$= \frac{1}{a^2}\int \frac{dt}{t^2 + (b^2/a^2)}$$ [Putting $t = \tan\theta$, so that $dt = \sec^2\theta\, d\theta$]

$$= \frac{1}{a^2}\cdot\frac{a}{b}\tan^{-1}\left(\frac{t}{b/a}\right) = \frac{1}{ab}\tan^{-1}\left(\frac{a}{b}\tan\theta\right).$$

(ii) Let $I = \int \frac{dx}{4\sin^2 x + 5\cos^2 x}$

$= \int \frac{\sec^2 x\, dx}{4\tan^2 x + 5}$, dividing the Nr. and Dr. by $\cos^2 x$

$= \int \frac{dt}{4t^2 + 5}$, putting $\tan x = t$ so that $\sec^2 x\, dx = dt$

$$= \frac{1}{4}\int \frac{dt}{t^2 + (\sqrt{5}/2)^2} = \frac{1}{4}\frac{1}{(\sqrt{5}/2)}\tan^{-1}\frac{t}{\sqrt{5}/2}$$

$$= \frac{1}{2\sqrt{5}} \tan^{-1} \frac{2t}{\sqrt{5}} = \frac{1}{2\sqrt{5}} \tan^{-1} \frac{2 \tan x}{\sqrt{5}}.$$

Ex. 19. *Evaluate* $\int [(\cos x)/(a^2 + b^2 \sin^2 x)]\, dx$.

Sol. Putting $b \sin x = t$ so that $b \cos x\, dx = dt$, we have

$$I = \int \frac{\cos x\, dx}{a^2 + b^2 \sin^2 x} = \frac{1}{b} \int \frac{dt}{a^2 + t^2} = \frac{1}{b} \cdot \frac{1}{a} \tan^{-1} \left(\frac{t}{a}\right)$$

$$= \frac{1}{ab} \tan^{-1} \left(\frac{b \sin x}{a}\right).$$

Ex. 20. *Evaluate* $\int \frac{\cot (\log x)}{x} dx$.

Sol. Putting $\log x = t$ so that $(1/x)\, dx = dt$, we have the given integral

$$I = \int \cot t\, dt = \log \sin t = \log \{\sin (\log x)\}.$$

§ 9. Some more standard Integrals.

(i) To evaluate $\int [1/\sqrt{(a^2 + x^2)}]\, dx$.

For complete solution of this problem see § 6, page 7. The result is

$$\int \frac{dx}{\sqrt{(a^2 + x^2)}} = \sinh^{-1} \left(\frac{x}{a}\right) = \log \{x + \sqrt{(x^2 + a^2)}\}.$$

(ii) To evaluate $\int \frac{dx}{\sqrt{(a^2 - x^2)}}$.

For complete solution of this problem see § 6, page 7. The result is

$$\int \frac{dx}{\sqrt{(a^2 - x^2)}} = \sin^{-1} \left(\frac{x}{a}\right).$$

(iii) To evaluate $\int \frac{dx}{\sqrt{(x^2 - a^2)}}$.

Put $x = a \cosh \theta$ so that $dx = a \sinh \theta\, d\theta$.

Then the given integral $= \int \frac{dx}{\sqrt{(x^2 - a^2)}} = \int \frac{a \sinh \theta\, d\theta}{\sqrt{(a^2 \cosh^2 \theta - a^2)}}$

$$= \int \frac{a \sinh \theta\, d\theta}{a \sqrt{(\cosh^2 \theta - 1)}} = \int \frac{\sinh \theta\, d\theta}{\sinh \theta} = \int d\theta = \theta = \cosh^{-1} (x/a)$$

$$= \log \left[\frac{x}{a} + \sqrt{\left\{\left(\frac{x}{a}\right)^2 - 1\right\}}\right] = \log \left\{\frac{x + \sqrt{(x^2 - a^2)}}{a}\right\}$$

$$= \log \{x + \sqrt{(x^2 - a^2)}\} - \log a = \log \{x + \sqrt{(x^2 - a^2)}\},$$

because the constant term $- \log a$ may be added to the constant of integration c which we usually do not write.

Thus $\int \frac{dx}{\sqrt{(x^2 - a^2)}} = \cosh^{-1} (x/a) = \log \{x + \sqrt{(x^2 - a^2)}\}.$

(iv) To evaluate $\int \sqrt{(x^2 + a^2)}\, dx$.

Put $x = a \sinh\theta$ so that $dx = a\cosh\theta\, d\theta$.

Then the given integral $= \int \sqrt{(a^2 \sinh^2\theta + a^2)}\,.\,a\cosh\theta\, d\theta$

$= \int a^2 \cosh^2\theta\, d\theta = \int \frac{1}{2} a^2 (1 + \cosh 2\theta)\, d\theta,$

$[\because \cosh 2\theta = 2\cosh^2\theta - 1]$

$= \frac{1}{2} a^2 \int (1 + \cosh 2\theta)\, d\theta = \frac{1}{2} a^2 [\theta + \frac{1}{2} \sinh 2\theta]$

$= \frac{1}{2} a^2 [\theta + \sinh\theta \cosh\theta],$ $[\because \sinh 2\theta = 2 \sinh\theta \cosh\theta]$

$= \frac{1}{2} a^2 [\theta + \sinh\theta \sqrt{(1 + \sinh^2\theta)}]$

$$= \frac{a^2}{2}\left[\sinh^{-1}\frac{x}{a} + \frac{x}{a}\sqrt{\left(1 + \frac{x^2}{a^2}\right)}\right]$$

$$= \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + \frac{a^2}{2}\cdot\frac{x}{a^2}\sqrt{(a^2 + x^2)}.$$

Thus $\int \sqrt{(x^2 + a^2)}\, dx = \frac{x}{2}\sqrt{(x^2 + a^2)} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)$

$$= \frac{x}{2}\sqrt{(x^2 + a^2)} + \frac{a^2}{2}\log\{x + \sqrt{(x^2 + a^2)}\}.$$

(v) To evaluate $\int \sqrt{(a^2 - x^2)}\, dx$. **(Kashmir 1983)**

Put $x = a\sin\theta$ so that $dx = a\cos\theta\, d\theta$.

Then the given integral $= \int a\cos\theta\,.\,a\cos\theta\, d\theta$

$= \int a^2\cos^2\theta\, d\theta = \frac{1}{2}a^2 \int 2\cos^2\theta\, d\theta = \frac{1}{2}a^2 \int (1 + \cos 2\theta)\, d\theta$

$= \frac{1}{2}a^2 (\theta + \frac{1}{2}\sin 2\theta) = \frac{1}{2}a^2(\theta + \sin\theta\cos\theta)$

$= \frac{1}{2}a^2 [\theta + \sin\theta\sqrt{(1 - \sin^2\theta)}]$

$$= \frac{1}{2}a^2 \sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}a^2\cdot\frac{x}{a}\sqrt{\left(1 - \frac{x^2}{a^2}\right)}.$$

Thus $\int \sqrt{(a^2 - x^2)}\, dx = \frac{1}{2}x\sqrt{(a^2 - x^2)} + \frac{1}{2}a^2\sin^{-1}(x/a).$

(vi) To evaluate $\int \sqrt{(x^2 - a^2)}\, dx$.

Put $x = a\cosh\theta$ so that $dx = a\sinh\theta\, d\theta$.

Then the given integral $= \int \sqrt{(a^2\cosh^2\theta - a^2)}\, a\sinh\theta\, d\theta$

$= \int a^2\sinh^2\theta\, d\theta = \int \frac{1}{2}a^2(\cosh 2\theta - 1)\, d\theta,$

$[\because 2\sinh^2\theta = \cosh 2\theta - 1]$

$= \frac{1}{2}a^2[\frac{1}{2}\sinh 2\theta - \theta] = \frac{1}{2}a^2[\sinh\theta\cosh\theta - \theta]$

$= \frac{1}{2}a^2[\sqrt{(\cosh^2\theta - 1)}\cosh\theta - \theta]$

$$= \frac{1}{2}a^2\left[\sqrt{\left(\frac{x^2}{a^2} - 1\right)}\cdot\frac{x}{a} - \cosh^{-1}\frac{x}{a}\right].\quad\left[\because \cosh\theta = \frac{x}{a}\right]$$

Thus $\int \sqrt{(x^2-a^2)}\,dx = \frac{1}{2}x\sqrt{(x^2-a^2)} - \frac{1}{2}a^2\cosh^{-1}(x/a)$

$$= \frac{1}{2}x\sqrt{(x^2-a^2)} - \frac{1}{2}a^2\log\{x+\sqrt{(x^2-a^2)}\}.$$

Solved Examples

Ex. 1. *Evaluate* $\int \frac{e^y\,dy}{\sqrt{(1+e^{2y})}}$.

Sol. Put $e^y = t$ so that $e^y\,dy = dt$.

$\therefore$ the given integral

$$= \int \{1/\sqrt{(1+t^2)}\}\,dt = \sinh^{-1} t = \sinh^{-1}(e^y).$$

Ex. 2. *Evaluate* $\int \cos x\sqrt{(4-\sin^2 x)}\,dx$.

Sol. Put $\sin x = t$ so that $\cos x\,dx = dt$.

Then the given integral $= \int \sqrt{(4-t^2)}\,dt = \int \sqrt{(2^2-t^2)}\,dt$

$$= \frac{1}{2}t\sqrt{(4-t^2)} + \frac{2^2}{2}\sin^{-1}(t/2)$$

$$= \frac{1}{2}\sin x.\sqrt{(4-\sin^2 x)} + 2\sin^{-1}(\tfrac{1}{2}\sin x).$$

Ex. 3. *Evaluate* $\int \sec x\tan x\sqrt{(\sec^2 x+1)}\,dx$.

Sol. Put $\sec x = t$ so that $\sec x\tan x\,dx = dt$.

Then the given integral $= \int \sqrt{(t^2+1)}\,dt$

$$= \frac{1}{2}t\sqrt{(t^2+1)} + \frac{1}{2}\sinh^{-1} t$$

$$= \frac{1}{2}\sec x\sqrt{(\sec^2 x+1)} + \frac{1}{2}\sinh^{-1}(\sec x).$$

Ex. 4. *Evaluate* $\int \frac{x}{\sqrt{(x^4+4)}}\,dx$.

Sol. Put $x^2 = t$; $\therefore$ $2x\,dx = dt$.

$\therefore$ the given integral $= \frac{1}{2}\int \frac{1}{\sqrt{(t^2+4)}}\,dt$

$$= \frac{1}{2}\sinh^{-1}(t/2) = \frac{1}{2}\sinh^{-1}(x^2/2).$$

Ex. 5. *Evaluate* $\int \{x^2/\sqrt{(x^6-9)}\}\,dx$.

Sol. Put $x^3 = t$, $\therefore$ $3x^2\,dx = dt$.

$\therefore$ the given integral $= \frac{1}{3}\int \frac{dt}{\sqrt{(t^2-9)}}$

$$= \frac{1}{3}\cosh^{-1}(t/3) = \frac{1}{3}\cosh^{-1}(x^3/3).$$

Ex. 6. *Evaluate* $\int x\sqrt{(x^4+9)}\,dx$.

Sol. Put $x^2 = t$; $\therefore$ $2x\,dx = dt$.

$\therefore$ the given integral $= \frac{1}{2}\int \sqrt{(t^2+3^2)}\,dt$

$$= \frac{1}{2}\left[\frac{1}{2}t\sqrt{(t^2+9)} + \frac{9}{2}\sinh^{-1}(t/3)\right]$$

$$= \frac{1}{4}x^2\sqrt{(x^4+9)} + \frac{9}{4}\sinh^{-1}(x^2/3).$$

Ex. 7. *Evaluate* $\int x^2 \sqrt{(x^6-1)}\, dx$.

Sol. Put $x^3 = t$; $\therefore$ $3x^2\, dx = dt$.

$\therefore$ the given integral $= \frac{1}{3}\int \sqrt{(t^2-1)}\, dt$

$$= \frac{1}{3}\left[\frac{t}{2}\sqrt{(t^2-1)} - \frac{1}{2}\cosh^{-1}\left(\frac{t}{1}\right)\right]$$

$$= \frac{1}{3}\left[\frac{x^3}{2}\sqrt{(x^6-1)} - \frac{1}{2}\cosh^{-1} x^3\right]$$

$$= \tfrac{1}{6}x^3\sqrt{(x^6-1)} - \tfrac{1}{6}\cosh^{-1} x^3.$$

§ 10. Integral of the product of two functions.

Integration by parts. Let u and v be two functions of x. Then we have from differential calculus

$$\frac{d}{dx}(uv) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx} \qquad \text{...(1)}$$

Integrating both sides of (1) with respect to x, we have

$$uv = \int u \cdot \frac{dv}{dx}\, dx + \int v \cdot \frac{du}{dx}\, dx.$$

By transposition, we have

$$\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}\, dx. \qquad \text{...(2)}$$

Now put $u = f_1(x)$ and $v = \int f_2(x)\, dx$, so that $\dfrac{dv}{dx} = f_2(x)$.

Then from (2), we have

$$\int \mathbf{f_1(x)\, f_2(x)\, dx = f_1(x) . \int f_2(x)\, dx}$$

$$\mathbf{- \int \left[\left\{\frac{d}{dx} f_1(x)\right\} \cdot \int f_2(x)\, dx\right] dx}$$

i.e., the integral of the product of two functions

= first function × integral of second function

− integral of {diff. coeffi. of first function × Integral of second function}.

Note 1. Care must be taken in choosing the first function and the second function. Obviously we must take that function as the second function whose integral is well known to us. Thus to evaluate $\int x \log x\, dx$ we shall take x as the second function because we so far do not know the integral of $\log x$. But to evaluate $\int x \sin x\, dx$ we must take $\sin x$ as the second function and x as the first function. Here if we take x as the second function, then the new integral will become more complicated. Thus to evaluate integrals of the type $\int x^2 e^x\, dx$ $\int x^3 \cos x\, dx$ etc., the function of the type x^n must be taken as the first function. In certain cases we can take unity (*i.e.*, 1) as the second function. Thus to evaluate $\int \log x \, dx$ we shall take 1 as the second function. To evaluate $\int e^x \sin x \, dx$ we can take either e^x or $\sin x$ as the second function.

Note 2. The formula of integration by parts can be applied more than once if necessary.

Note 3. Integration by parts as applied to the functions of the type $e^x [f(x) + f'(x)]$.

Let $I = \int e^x [f(x) + f'(x)] \, dx = \int e^x f(x) \, dx + \int e^x f'(x) \, dx$.

Integrating the first integral by parts regarding e^x as the 2nd function, we have

$$I = [f(x) e^x - \int f'(x) e^x dx] + \int f'(x) e^x dx = e^x f(x).$$

[Note that we have left the other integral unchanged because the last two integrals cancel each other].

§ 11. Successive integration by parts.

If u is a function of the type $a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$, where n is a positive integer, the following formula for successive integration by parts can be applied. While writing this formula the successive differential coefficients of u have been denoted by u', u'', u''' etc., while the successive integrals of v have been denoted by v_1, v_2, v_3 etc. Thus

$$\int uv \, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots.$$

The process of successive integration by parts will be continued till on being differentiated successively the differential coefficient of u becomes zero. The following examples will make the process clear.

Example 1. *Evaluate* $\int x^5 e^x \, dx$.

Sol. Here e^x will be successively integrated and x^5 will be successively differentiated. Thus applying successive integration by parts, the given integral

$$I = x^5 e^x - (5x^4) e^x + (20x^3) e^x - (60x^2) e^x + (120x) e^x - 120 e^x,$$

the process of successive integration by parts terminates because the differential coefficient of 120 is zero.

$$\therefore \quad I = e^x (x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120).$$

Example 2. *Evaluate* $\int x^4 \sin x \, dx$.

Sol. Applying successive integration by parts, the given integral

$$I = x^4 . (-\cos x) - (4x^3) . (-\sin x) + (12x^2) . (\cos x) - (24x) . (\sin x) + (24) . (-\cos x)$$

$$= -x^4 \cos x + 4x^3 \sin x + 12x^2 \cos x - 24x \sin x - 24 \cos x.$$

Remark. While applying successive integration by parts the successive differential coefficients and the successive integrals must at the first stage be put within brackets.

Example 3. *Evaluate* $\int x^3 e^{-x} dx$.

Sol. Applying successive integration by parts, the given integral

$$I = (x^3) \cdot (-e^{-x}) - (3x^2) \cdot (e^{-x}) + (6x) \cdot (-e^{-x}) - (6) \cdot (e^{-x})$$
$$= -x^3 e^{-x} - 3x^2 e^{-x} - 6xe^{-x} - 6e^{-x}.$$
$$= -(x^3 + 3x^2 + 6x + 6)\, e^{-x}.$$

§ 12. Integrals of $e^{ax} \cos bx$ and $e^{ax} \sin bx$.

Let $I = \int e^{ax} \sin bx\, dx$.

Integrating by parts taking $\sin bx$ as the second function, we get

$$I = -\frac{e^{ax} \cos bx}{b} - \int a e^{ax} \left(-\frac{\cos bx}{b}\right) dx$$
$$= -\frac{e^{ax} \cos bx}{b} + \frac{a}{b} \int e^{ax} \cos bx\, dx.$$

Again integrating by parts taking $\cos bx$ as the second function, we get

$$I = -\frac{e^{ax} \cos bx}{b} + \frac{a}{b}\left[\frac{e^{ax} \sin bx}{b} - \int a e^{ax} \frac{\sin bx}{b} dx\right]$$

or $$I = -\frac{e^{ax} \cos bx}{b} + \frac{a}{b^2} e^{ax} \sin bx - \frac{a^2}{b^2} \int e^{ax} \sin bx\, dx$$

or $$I = \frac{e^{ax}}{b^2}(a \sin bx - b \cos bx) - \frac{a^2}{b^2} I. \qquad [\because \int e^{ax} \sin bx\, dx = I]$$

Transposing the term $-\frac{a^2}{b^2} I$ to the left hand side, we get

$$\left(1 + \frac{a^2}{b^2}\right) I = \frac{e^{ax}}{b^2}(a \sin bx - b \cos bx)$$

or $$\frac{1}{b^2}(a^2 + b^2)\, I = \frac{1}{b^2} e^{ax} (a \sin bx - b \cos bx).$$

$$\therefore \quad I = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx).$$

Thus $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx)$.

Similarly $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx)$.

Alternative forms of $\int e^{ax} \sin bx\, dx$ and $\int e^{ax} \cos bx\, dx$.

We have $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx)$.

Put $a = r \cos \theta$ and $b = r \sin \theta$. Then

$$r = \sqrt{(a^2 + b^2)} \quad \text{and} \quad \theta = \tan^{-1}(b/a).$$

Now we have

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{r^2}(r\cos\theta \sin bx - r\sin\theta\cos bx)$$

$$= \frac{e^{ax}}{r}\sin(bx-\theta).$$

Thus $\int e^{ax} \sin bx\, dx = \dfrac{e^{ax}}{\sqrt{(a^2+b^2)}} \sin\left(bx - \tan^{-1}\dfrac{b}{a}\right)$.

Similarly $\int e^{ax} \cos bx\, dx = \dfrac{e^{ax}}{\sqrt{(a^2+b^2)}} \cos\left(bx - \tan^{-1}\dfrac{b}{a}\right)$.

Solved Examples

Ex.1. *Integrate* **(i)** $x \log x$, **(ii)** $(\log x)/x^2$, **(iii)** $x^n \log x$.

Sol. **(i)** Here x should be taken as the second function because the integral of $\log x$ cannot be easily written down.

We have $\int x \log x\, dx = \int (\log x)\, x\, dx$

$= (\log x) \cdot \frac{1}{2}x^2 - \int (1/x) \cdot \frac{1}{2}x^2\, dx$, (Integrating by parts)

$= \frac{1}{2}x^2 \log x - \frac{1}{2}\int x\, dx = \frac{1}{2}x^2 \log x - \frac{1}{2} \cdot (x^2/2)$

$= \frac{1}{4}x^2 \log x^2 - \frac{1}{4}x^2 \log e$, $[\because \log e = 1]$ **Note**

$= \frac{1}{4}x^2 \log (x^2/e)$.

(ii) We have $\int [(\log x)/x^2]\, dx = \int (\log x)(1/x^2)\, dx$

$= (\log x)(-1/x) - \int (1/x).(-1/x)\, dx$,

[Integrating by parts taking $1/x^2$ as the second function]

$= -(1/x)\log x - (1/x) = -(1/x)(\log x + \log e)$ **Note**

$= -(1/x)\log(x e)$.

(iii) We have $\int x^n \log x\, dx = \int (\log x).x^n\, dx$

$$= (\log x) \cdot \frac{x^{n+1}}{n+1} - \int \frac{1}{x} \cdot \frac{x^{n+1}}{n+1}\, dx,$$

[Integrating by parts taking x^n as the second function]

$$= (\log x) \cdot \frac{x^{n+1}}{n+1} - \int \frac{x^n}{n+1}\, dx$$

$$= (\log x) \cdot \frac{x^{n+1}}{n+1} - \frac{x^{n+1}}{(n+1)^2}.$$

Ex. 2. *Integrate* **(i)** $\tan^{-1} x$, **(ii)** $\cot^{-1} x$, **(iii)** $\sin^{-1} x$.

Sol. **(i)** As there is only one function here, unity should be taken as the 2nd function. We have $\int \tan^{-1} x\, dx = \int (\tan^{-1} x).1\, dx$.

Integrating by parts regarding 1 as the second function, we have

$\int (\tan^{-1} x).1\, dx = (\tan^{-1} x).x - \int \{1/(1+x^2)\}\, x\, dx$

$$= x\tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2}\, dx = x \tan^{-1} x - \frac{1}{2}\log(1+x^2),$$

($\because$ Nr. $2x$ is the diff. coeffi. of the Dr. $1+x^2$)

(ii) We have $\int \cot^{-1} x \, dx = \int (\cot^{-1} x) . 1 \, dx$

$= (\cot^{-1} x) . x - \int \{-1/(1+x^2)\} . x \, dx,$

[Integrating by parts taking unity as the second function]

$= x \cot^{-1} x + \frac{1}{2} \int \{2x/(1+x^2)\} \, dx = x \cot^{-1} x + \frac{1}{2} \log (1+x^2).$

(iii) We have $\int \sin^{-1} x \, dx = \int (\sin^{-1} x) . 1 \, dx$

$= (\sin^{-1} x) . x - \int \{1/\sqrt{(1-x^2)}\} \, x \, dx$

$= x \sin^{-1} x + \frac{1}{2} \int (1-x^2)^{-1/2} (-2x) \, dx$ **Note**

$= x \sin^{-1} x + \frac{1}{2} \dfrac{(1-x^2)^{1/2}}{1/2},$

[By power formula; see § 8 on page 12]

$= x \sin^{-1} x + (1-x^2)^{1/2}.$

Ex. 3. *Integrate* (i) $x^2 e^{mx}$, (ii) $x^2 \sin x$, (iii) $x^2 \cos 2x$.

Sol. (i) Integrating by parts taking e^{mx} as the second function, we have $\int x^2 e^{mx} \, dx = x^2 \cdot \dfrac{e^{mx}}{m} - \displaystyle\int 2x \cdot \dfrac{e^{mx}}{m} dx$

$= \dfrac{x^2}{m} e^{mx} - \dfrac{2}{m} \int x . e^{mx} \, dx = \dfrac{x^2}{m} e^{mx} - \dfrac{2}{m} \left\{ x \cdot \dfrac{e^{mx}}{m} - \displaystyle\int 1 . \dfrac{e^{mx}}{m} dx \right\},$

[Again integrating by parts]

$= \dfrac{x^2}{m} e^{mx} - \dfrac{2}{m^2} x \, e^{mx} + \dfrac{2}{m^3} e^{mx}$

$= e^{mx} m^{-3} (m^2 x^2 - 2mx + 2).$

(ii) Integrating by parts taking $\sin x$ as second function, we have

$\int x^2 \sin x \, dx = -x^2 \cos x - \int 2x (-\cos x) \, dx$

$= -x^2 \cos x + 2 \int x \cos x \, dx$

$= -x^2 \cos x + 2x \sin x - 2 \int \sin x \, dx$

$= -x^2 \cos x + 2x \sin x + 2 \cos x.$

Similarly $\int x^2 \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin x.$

(iii) Integrating by parts taking $\cos 2x$ as second function, we have

$\int x^2 \cos 2x \, dx = x^2 . (\frac{1}{2} \sin 2x) - \int 2x . \frac{1}{2} \sin 2x \, dx$

$= \frac{1}{2} x^2 \sin 2x - \int x \sin 2x \, dx$

$= \frac{1}{2} x^2 \sin 2x - \{x . \frac{1}{2} (-\cos 2x) + \frac{1}{2} \int \cos 2x \, dx\},$

(Again integrating by parts taking $\sin 2x$ as the 2nd function)

$= \frac{1}{2} x^2 \sin 2x + \frac{1}{2} x \cos 2x - \frac{1}{2} \int \cos 2x \, dx$

$= \frac{1}{2} x^2 \sin 2x + \frac{1}{2} x \cos 2x - \frac{1}{4} \sin 2x$

$= \frac{1}{2}(x^2 - \frac{1}{2}) \sin 2x + \frac{1}{2} x \cos 2x.$

Ex. 4. *Integrate* **(i)** $\log x$, **(M.U. 81)**

(ii) $(\log x)^2$, **(iii)** $x^n (\log x)^2$.

Sol. (i) As there is only one function here, unity should be taken as the second function. We have $\int \log x \, dx = \int (\log x) . 1 \, dx$

$= (\log x) . x - \int (1/x) . x \, dx = x \log x - \int 1 . dx = x \log x - x$

$= x (\log x - 1) = x \log (x/e).$

(ii) We have $\int (\log x)^2 dx = \int (\log x)^2 . 1 \, dx$

$= (\log x)^2 . x - \int \{(2 \log x) . (1/x)\} . x \, dx,$

[Integrating by parts taking 1 as the second function]

$= x (\log x)^2 - 2 \int (\log x) . 1 \, dx$

$$= x (\log x)^2 - 2 \left[(\log x) . x - \int \frac{1}{x} \cdot x \, dx \right],$$

[Again integrating by parts taking 1 as the 2nd function]

$= x (\log x)^2 - 2x \log x + 2 \int 1 . dx$

$= x (\log x)^2 - 2x \log x + 2x.$

(iii) We have $\int x^n (\log x)^2 dx = \int (\log x)^2 . x^n \, dx$

$$= (\log x)^2 \frac{x^{n+1}}{n+1} - \int \left\{ (2 \log x) . \frac{1}{x} \right\} \frac{x^{n+1}}{n+1} dx,$$

[Integrating by parts taking x^n as the second function]

$$= \frac{1}{(n+1)} x^{n+1} (\log x)^2 - \frac{2}{n+1} \int (\log x) . x^n \, dx$$

$$= \frac{1}{(n+1)} x^{n+1} (\log x)^2 - \frac{2}{(n+1)} \left[(\log x) . \frac{x^{n+1}}{n+1} - \int \frac{1}{x} \cdot \frac{x^{n+1}}{n+1} \cdot dx \right],$$

[Again integrating by parts taking x^n as the second function]

$$= \frac{1}{(n+1)} x^{n+1} . (\log x)^2 - \frac{2}{(n+1)^2} (\log x) . x^{n+1} + \frac{2}{(n+1)^2} \int x^n \, dx$$

$$= \frac{1}{(n+1)} x^{n+1} (\log x)^2 - \frac{2}{(n+1)^2} (\log x) . x^{n+1} + \frac{2}{(n+1)^3} x^{n+1}$$

$$= x^{n+1} \left[\frac{(\log x)^2}{(n+1)} - \frac{2 \log x}{(n+1)^2} + \frac{2}{(n+1)^3} \right].$$

Ex. 5. *Integrate* **(i)** $e^x \sin x$ **(ii)** $e^x \cos x$,

(iii) $e^{2x} \sin x$, **(iv)** $e^{3x} \cos 4x$.

Sol. **(i)** Integrating by parts taking $\sin x$ as the second function, we have

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx.$$

Now taking $\cos x$ as the second function we again apply integration by parts to the integral on the right hand side. Thus, we get

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx.$$

Transposing the last term on the right hand side to the left and dividing by 2, we get

$$\int e^x \sin x \, dx = \tfrac{1}{2} e^x (\sin x - \cos x).$$

(ii) Here $\int e^x \cos x \, dx = e^x . \sin x - \int e^x \sin x \, dx$,

[Integrating by parts taking $\cos x$ as the second function]

$$= e^x \sin x - [e^x . (-\cos x) - \int e^x (-\cos x) \, dx],$$

[Again integrating by parts taking $\sin x$ as the second function]

$$= e^x (\sin x + \cos x) - \int e^x \cos x \, dx.$$

Transposing the last term on the right hand side to the left and dividing by 2, we get

$$\int e^x \cos x \, dx = \tfrac{1}{2} e^x (\sin x + \cos x).$$

(iii) We have

$$\int e^{2x} \sin x \, dx = e^{2x} . (-\cos x) - \int 2e^{2x} . (-\cos x) \, dx,$$

[Integrating by parts taking $\sin x$ as the second function]

$$= -e^{2x} \cos x + 2 \int e^{2x} \cos x \, dx$$

$$= -e^{2x} \cos x + 2 [e^{2x} . \sin x - \int 2e^{2x} \sin x \, dx]$$

$$= -e^{2x} \cos x + 2e^{2x} \sin x - 4 \int e^{2x} \sin x \, dx.$$

Transposing and dividing by 5, we get

$$\int e^{2x} \sin x \, dx = \tfrac{1}{5} e^{2x} [2 \sin x - \cos x].$$

(iv) Integrating by parts taking $\cos 4x$ as the second function, we have

$$\int e^{2x} \cos 4x \, dx + e^{3x} . \left(\frac{\sin 4x}{4}\right) - \int (3e^{3x}) . \left(\frac{\sin 4x}{4}\right) dx$$

$$= \tfrac{1}{4} e^{3x} \sin 4x - \tfrac{3}{4} \int e^{3x} \sin 4x \, dx$$

$$= \frac{1}{4} e^{3x} \sin 4x - \frac{3}{4} \left[e^{3x} . \left(-\frac{\cos 4x}{4}\right) - \int 3e^{3x} . \left(-\frac{\cos 4x}{4}\right) dx \right]$$

$$= \tfrac{1}{4} e^{3x} \sin 4x + \tfrac{3}{16} e^{3x} \cos 4x - \tfrac{9}{16} \int e^{3x} \cos 4x \, dx.$$

Transposing the last term on the right hand side to the left, we have

$$\left(1 + \tfrac{9}{16}\right) \int e^{3x} \cos 4x \, dx = \tfrac{1}{4} e^{3x} \sin 4x + \tfrac{3}{16} e^{3x} \cos 4x$$

or $$\tfrac{25}{16} \int e^{3x} \cos 4x \, dx = \tfrac{1}{4} e^{3x} \sin 4x + \tfrac{3}{16} e^{3x} \cos 4x$$

or $$\int e^{3x} \cos 4x \, dx = \tfrac{4}{25} e^{3x} \sin 4x + \tfrac{3}{25} e^{3x} \cos 4x$$

$= \frac{1}{25} e^{3x} (4 \sin 4x + 3 \cos 4x)$.

Alternative solution. $\int e^{3x} \cos 4x \, dx$ is of the form $\int e^{ax} \cos bx \, dx$, where $a = 3$ and $b = 4$.

Now $\int e^{ax} \cos bx \, dx = \dfrac{e^{ax}}{\sqrt{(a^2 + b^2)}} \cos \left(bx - \tan^{-1} \dfrac{b}{a} \right)$.

[See § 12, page 24]

$\therefore \quad \int e^{3x} \cos 4x \, dx = \dfrac{e^{3x}}{\sqrt{(9 + 16)}} \cos \left(4x - \tan^{-1} \dfrac{4}{3} \right)$

$= \dfrac{e^{3x}}{5} \cos (4x - \tan^{-1} \frac{4}{3})$.

Ex. 6. *Evaluate* $\int e^x (n \cos x + \sin x) \, dx$. **(Meerut 1977, 86)**

Sol. $\int e^x (n \cos x + \sin x) \, dx = n \int e^x \cos x \, dx + \int e^x \sin x \, dx$.

Now using results of Ex. 5 (i), and (ii), we get the required value of the given integral

$= \frac{1}{2} n e^x (\cos x + \sin x) + \frac{1}{2} e^x (\sin x - \cos x)$.

Ex. 7 (a). *Evaluate* $\displaystyle\int \frac{x e^x}{(x + 1)^2} dx$.

(Kashmir 1983; Delhi 80; Meerut 82, 84 P, 90)

Sol. We have $\displaystyle\int \frac{x e^x}{(x + 1)^2} dx = \int (x e^x) \frac{1}{(x + 1)^2} dx$.

Integrating by parts taking $\dfrac{1}{(x + 1)^2}$ as the second function and $x e^x$ as the first function, we have

$$\int \frac{x e^x}{(x + 1)^2} dx = (x e^x) \left(- \frac{1}{x + 1} \right) - \int (e^x + x e^x) \left(- \frac{1}{x + 1} \right) dx,$$

$\left[\text{Note that the integral of } \dfrac{1}{(x + 1)^2} \text{ is } - \dfrac{1}{x + 1} \right]$

$$= - \frac{x e^x}{x + 1} + \int e^x (x + 1) \cdot \frac{1}{x + 1} dx$$

$$= - \frac{x e^x}{x + 1} + \int e^x \, dx = - \frac{x e^x}{x + 1} + e^x$$

$$= e^x \left[1 - \frac{x}{x + 1} \right] = e^x \frac{x + 1 - x}{x + 1} = \frac{e^x}{x + 1}.$$

Alternative solution.

We have $\displaystyle\int \frac{x e^x}{(x + 1)^2} dx = \int e^x \frac{(x + 1) - 1}{(x + 1)^2} dx$

$$= \int e^x \left[\frac{1}{x + 1} - \frac{1}{(x + 1)^2} \right] dx$$

$= \int e^x [f(x) + f'(x)]\, dx$, where $f(x) = \dfrac{1}{x+1}$

$= \int e^x f(x)\, dx + \int e^x f'(x)\, dx$

$= e^x f(x) - \int e^x f'(x)\, dx + \int e^x f'(x)\, dx$,

applying integration by parts to the first integral taking e^x as the second function

$= e^x f(x) = e^x \dfrac{1}{x+1}$.

Ex. 7 (b). *Evaluate* $\displaystyle\int e^x \frac{(x^2+1)}{(x+1)^2}\, dx$.

Sol. We have $\displaystyle\int e^x \frac{(x^2+1)}{(x+1)^2}\, dx$

$$= \int e^x \frac{(x^2-1)+2}{(x+1)^2}\, dx = \int e^x \frac{(x+1)(x-1)+2}{(x+1)^2}\, dx$$

$$= \int e^x \left[\frac{x-1}{x+1} + \frac{2}{(x+1)^2}\right] dx$$

$$= \int e^x \left[\frac{(x+1)-2}{x+1} + \frac{2}{(x+1)^2}\right] dx$$

$$= \int e^x \left[\left\{1 - \frac{2}{x+1}\right\} + \frac{2}{(x+1)^2}\right] dx$$

$= \int e^x [f(x) + f'(x)]\, dx$, where $f(x) = 1 - \dfrac{2}{x+1}$ so that $f'(x) = \dfrac{2}{(x+1)^2}$

$$= e^x f(x) = e^x \left[1 - \frac{2}{x+1}\right] = e^x \frac{x-1}{x+1}.$$

Ex. 7 (c). *Evaluate* $\displaystyle\int e^x \frac{(1-x)^2}{(1+x^2)^2}\, dx$.

Sol. We have $\displaystyle\int e^x \frac{(1-x)^2}{(1+x^2)^2}\, dx = \int e^x \frac{(1+x^2)-2x}{(1+x^2)^2}\, dx$

$$= \int e^x \left[\frac{1}{1+x^2} - \frac{2x}{(1+x^2)^2}\right] dx$$

$= \int e^x [f(x) + f'(x)]\, dx$, where $f(x) = \dfrac{1}{1+x^2}$

and $f'(x) = \dfrac{-2x}{(1+x^2)^2}$

$$= e^x f(x) = e^x \cdot \frac{1}{1+x^2}.$$

Ex. 7 (d). *Evaluate* $\int e^x \frac{x^2 + 3x + 3}{(x + 2)^2} dx$.

Sol. We have $\int e^x \frac{x^2 + 3x + 3}{(x + 2)^2} dx$

$$= \int e^x \frac{(x + 2)(x + 1) + 1}{(x + 2)^2} dx$$

$$= \int e^x \left[\frac{x + 1}{x + 2} + \frac{1}{(x + 2)^2}\right] dx$$

$= \int e^x [f(x) + f'(x)] dx$, where

$f(x) = \frac{x + 1}{x + 2} = \frac{(x + 2) - 1}{x + 2} = 1 - \frac{1}{x + 2}$ so that $f'(x) = \frac{1}{(x + 2)^2}$

$$= e^x \frac{x + 1}{x + 2}.$$

Ex. 7 (e). *Evaluate* $\int e^x \frac{x^3 - x + 2}{(x^2 + 1)^2} dx$.

Sol. We have $\int e^x \frac{x^3 - x + 2}{(x^2 + 1)^2} dx$

$$= \int e^x \frac{(x^2 + 1)(x + 1) - x^2 - 2x + 1}{(x^2 + 1)^2} dx$$

$$= \int e^x \left[\frac{x + 1}{x^2 + 1} + \frac{1 - x^2 - 2x}{(x^2 + 1)^2}\right] dx$$

$= \int e^x [f(x) + f'(x)] dx$, where $f(x) = \frac{x + 1}{x^2 + 1}$ so that

$$f'(x) = \frac{1.(x^2 + 1) - 2x(x + 1)}{(x^2 + 1)^2} = \frac{1 - x^2 - 2x}{(x^2 + 1)^2}$$

$$= e^x f(x) = e^x \frac{x + 1}{x^2 + 1}.$$

Ex. 7 (f). *Evaluate* $\int \frac{\log x}{(1 + \log x)^2} dx$.

Sol. Put $\log x = t$. $\therefore x = e^t$ and $dx = e^t dt$.

Then the given integral

$$I = \int \frac{t e^t}{(1 + t)^2} = \int e^t \frac{(1 + t) - 1}{(1 + t)^2} dt$$

$$= \int e^t \left[\frac{1}{1 + t} - \frac{1}{(1 + t)^2}\right] dt$$

$= \int e^t [f(t) + f'(t)] dt$, where $f(t) = \frac{1}{1 + t}$

$$= e^t f(t) = e^t \frac{1}{1+t} = \frac{x}{1+\log x}.$$

Ex. 8 (a). *Evaluate* $\int \frac{e^x (1 + \sin x)}{1 + \cos x} dx$.

Sol. The given integral $I = \int \frac{e^x dx}{1+\cos x} + \int \frac{e^x \sin x \, dx}{(1+\cos x)}$

$$= \int \frac{e^x dx}{2\cos^2 (x/2)} + \int \frac{e^x 2 \sin (x/2) \cos (x/2)}{2\cos^2 (x/2)} dx$$

$$= \tfrac{1}{2} \int e^x \sec^2 (x/2)\, dx + \int e^x \tan (x/2)\, dx$$

$$= \int e^x \left[\tan \frac{x}{2} + \frac{1}{2} \sec^2 \frac{x}{2}\right] dx.$$

Since $\frac{d}{dx} (\tan \frac{1}{2}x) = \frac{1}{2} \sec^2 \frac{x}{2}$, therefore this *integral is of the type* $\int$ **e**^{x} **[f (x) + f ' (x)] dx**.

To evaluate this integral, integrating $e^x \tan (x/2)$ by parts regarding e^x as the second function, we get

$$I = e^x \tan \frac{x}{2} - \int \frac{1}{2} e^x . \sec^2 \frac{x}{2} dx + \int \frac{1}{2} e^x \sec^2 \frac{x}{2} dx = e^x \tan \frac{x}{2},$$

because the last two integrals cancel each other.

Ex. 8 (b). *Evaluate* $\int e^x \frac{1 - \sin x}{1 - \cos x} dx$. **(Meerut 1980, 83, 87)**

Sol. We have

$$\int e^x \frac{1-\sin x}{1-\cos x} dx = \int e^x \left[\frac{1}{1-\cos x} - \frac{\sin x}{1-\cos x}\right] dx$$

$$= \int e^x \left[\frac{1}{2\sin^2 \frac{1}{2}x} - \frac{2 \sin \frac{1}{2}x \cos \frac{1}{2}x}{2 \sin^2 \frac{1}{2}x}\right] dx$$

$$= \int e^x [\tfrac{1}{2} \operatorname{cosec}^2 \tfrac{1}{2}x - \cot \tfrac{1}{2}x]\, dx$$

$$= \int e^x [(-\cot \tfrac{1}{2}x) + \tfrac{1}{2} \operatorname{cosec}^2 \tfrac{1}{2}x]\, dx$$

$$= \int e^x [f(x) + f'(x)]\, dx, \text{ where}$$

$$f(x) = -\cot \tfrac{1}{2}x \text{ so that } f'(x) = \tfrac{1}{2} \operatorname{cosec}^2 \tfrac{1}{2}x$$

$$= e^x f(x) = e^x (-\cot \tfrac{1}{2}x) = -e^x \cot \tfrac{1}{2}x.$$

Ex. 8 (c). *Evaluate* $\int e^x \frac{2 + \sin 2x}{1 + \cos 2x} dx$.

Sol. We have $\int e^x \frac{2 + \sin 2x}{1 + \cos 2x} dx$

$$= \int e^x \left[\frac{2}{1+\cos 2x} + \frac{\sin 2x}{1+\cos 2x}\right] dx$$

$$= \int e^x \left[\frac{2}{2\cos^2 x} + \frac{2\sin x \cos x}{2\cos^2 x}\right] dx$$

$= \int e^x [\sec^2 x + \tan x]\, dx$

$= \int e^x [\tan x + \sec^2 x]\, dx$

$= \int e^x [f(x) + f'(x)]\, dx$, where $f(x) = \tan x$

$= e^x f(x) = e^x \tan x$.

Ex. 8 (d). *Evaluate* $\int e^x (\cot x + \log \sin x)\, dx$.

Sol. We have $\int e^x (\cot x + \log \sin x)\, dx$

$= \int e^x (\log \sin x + \cot x)\, dx$

$= \int e^x [f(x) + f'(x)]\, dx$, where

$f(x) = \log \sin x$ so that $f'(x) = (1/\sin x) \cos x = \cot x$

$= e^x f(x) = e^x \log \sin x$.

Ex. 8 (e). *Evaluate* $\int e^x [\log(\sec x + \tan x) + \sec x]\, dx$.

Sol. The given integral

$I = \int e^x \sec x\, dx + \int e^x \log(\sec x + \tan x)\, dx$

$= e^x \log(\sec x + \tan x) - \int e^x \log(\sec x + \tan x)\, dx$

$+ \int e^x \log(\sec x + \tan x)\, dx$,

(applying integration by parts to the first integral taking e^x as the second function and keeping the second integral as it is)

$= e^x \log(\sec x + \tan x)$.

Ex. 8 (f). *Evaluate* $\int e^x \left[\frac{1 + \sqrt{(1-x^2)} \sin^{-1} x}{\sqrt{(1-x^2)}}\right] dx$.

Sol. The given integral

$$I = \int e^x \left[\frac{1}{\sqrt{(1-x^2)}} + \sin^{-1} x\right] dx$$

$$= \int e^x \left[\sin^{-1} x + \frac{1}{\sqrt{(1-x^2)}}\right] dx$$

$= \int e^x [f(x) + f'(x)]\, dx$, where

$f(x) = \sin^{-1} x$ so that $f'(x) = \dfrac{1}{\sqrt{(1-x^2)}}$

$= e^x f(x) = e^x \cdot \dfrac{1}{\sqrt{(1-x^2)}}$.

Ex. 9. *Evaluate* $\int \frac{x \sin^{-1} x}{\sqrt{(1-x^2)}}\, dx$.

Sol. Put $\sin^{-1} x = t$ (or $x = \sin t$); $\therefore \dfrac{1}{\sqrt{(1-x^2)}} dx = dt$.

$$\therefore \int \frac{x \sin^{-1} x}{\sqrt{(1-x^2)}} dx = \int (\sin t) . t\, dt = \int t \sin t\, dt$$
$$= t.(-\cos t) - \int 1.(-\cos t).dt = -t\cos t + \sin t$$
$$= -t\sqrt{(1-\sin^2 t)} + \sin t$$
$$= -\sin^{-1} x . \sqrt{(1-x^2)} + x.$$

Ex. 10. *Evaluate* $\displaystyle\int \frac{x^2 \tan^{-1} x}{1+x^2} dx.$ **(Meerut 1990 S)**

Sol. Put $\tan^{-1} x = t$ (or $x = \tan t$); $\therefore \dfrac{1}{1+x^2} dx = dt$.

$$\therefore \int \frac{x^2 \tan^{-1} x}{1+x^2} dx = \int t \tan^2 t\, dt = \int t(\sec^2 t - 1)\, dt \quad \textbf{[Note]}$$
$$= \int t \sec^2 t\, dt - \int t\, dt$$
$$= t \tan t - \int \tan t\, dt - \tfrac{1}{2} t^2$$
$$= t \tan t - \log \sec t - \tfrac{1}{2} t^2 = t \tan t - \log\sqrt{(1+\tan^2 t)} - \tfrac{1}{2} t^2$$
$$= x \tan^{-1} x - \log\sqrt{(1+x^2)} - \tfrac{1}{2}(\tan^{-1} x)^2, \quad [\because x = \tan t].$$

Ex. 11. *Evaluate* $\displaystyle\int \frac{\sin^{-1} x\, dx}{(1-x^2)^{3/2}}.$

Sol. Put $\sin^{-1} x = t$, *i.e.*, $x = \sin t$ so that $dx = \cos t\, dt$.

$$\therefore \int \frac{\sin^{-1} x\, dx}{(1-x^2)^{3/2}} = \int \frac{t}{\cos^3 t} \cdot \cos t\, dt = \int t . \sec^2 t\, dt$$
$$= t \tan t - \int 1 . \tan t\, dt$$
$$= t \tan t + \log \cos t, \quad \left[\because \int \tan t\, dt = -\log \cos t\right]$$
$$= t \frac{\sin t}{\cos t} + \log \cos t = t \frac{\sin t}{\sqrt{(1-\sin^2 t)}} + \log\{\sqrt{(1-\sin^2 t)}\}$$
$$= (\sin^{-1} x) \cdot \frac{x}{\sqrt{(1-x^2)}} + \log\sqrt{(1-x^2)}, \quad [\because x = \sin t]$$
$$= \frac{x \sin^{-1} x}{\sqrt{(1-x^2)}} + \tfrac{1}{2}\log(1-x^2).$$

Ex. 12. *Find* $\displaystyle\int \frac{2x \sin^{-1}(x^2)}{\sqrt{(1-x^4)}} dx.$

Sol. Put $\sin^{-1} x^2 = t$, so that $\dfrac{2x\, dx}{\sqrt{(1-x^4)}} = dt$.

$\therefore$ the given integral $= \int t\, dt = \tfrac{1}{2} t^2 = \tfrac{1}{2}(\sin^{-1} x^2)^2$.

Ex. 13. *Evaluate* $\displaystyle\int \frac{x \tan^{-1} x}{(1+x^2)^{3/2}} dx.$

Sol. Put $\tan^{-1} x = t$, so that $x = \tan t$. Also $dx = \sec^2 t\, dt$.

$\therefore$ the given integral $= \displaystyle\int \frac{(\tan t)\, t \sec^2 t\, dt}{(1+\tan^2 t)^{3/2}} = \int \frac{t \tan t \sec^2 t\, dt}{\sec^3 t}$

$$= \int \frac{t \tan t}{\sec t}\, dt = \int t \sin t\, dt.$$

Now integrating by parts regarding $\sin t$ as the second function, the given integral $= -t \cos t + \int \cos t\, dt = -t \cos t + \sin t$

$$= -\frac{\tan^{-1} x}{\sqrt{(1+x^2)}} + \frac{x}{\sqrt{(1+x^2)}} = \frac{x - \tan^{-1} x}{\sqrt{(1+x^2)}}.$$

Ex. 14. *Evaluate* $\int x^3 e^{x^2} dx$.

Sol. The given integral $I = \int x^2 . e^{x^2} . x\, dx$.

Put $x^2 = t$ so that $2x\, dx = dt$ or $x\, dx = \frac{1}{2} dt$.

$\therefore$ the given integral $I = \frac{1}{2} \int t . e^t\, dt$.

Now integrating by parts regarding e^t as the 2nd function, we have

$$I = \tfrac{1}{2} t . e^t - \tfrac{1}{2} \int e^t\, dt = \tfrac{1}{2} t . e^t - \tfrac{1}{2} e^t$$

$$= \tfrac{1}{2} e^t (t-1) = \tfrac{1}{2} e^{x^2} (x^2 - 1).$$

Ex. 15. *Evaluate* $\displaystyle\int \frac{x^2\, dx}{(x \sin x + \cos x)^2}$. **(Delhi 1979; Meerut 84 S)**

Sol. Let $\displaystyle I = \int \frac{x^2}{(x \sin x + \cos x)^2} dx$

$$= \int x^2 (x \sin x + \cos x)^{-2} dx.$$

Here $\dfrac{d}{dx}(x \sin x + \cos x) = \sin x + x \cos x - \sin x = x \cos x$.

So we adjust the given integral in the form

$$I = \int \frac{x^2}{x \cos x} \{(x \sin x + \cos x)^{-2} (x \cos x)\}\, dx$$

$$= \int \left(\frac{x}{\cos x}\right) \{(x \sin x + \cos x)^{-2} (x \cos x)\}\, dx.$$

Now by power formula, the integral of

$(x \sin x + \cos x)^{-2} (x \cos x)$ is $\{(x \sin x + \cos x)^{-1}\}/(-1)$

i.e., $-1/(x \sin x + \cos x)$. So applying to I integration by parts taking $(x \sin x + \cos x)^{-2} (x \cos x)$ as the second function, we get

$$I = \left(\frac{x}{\cos x}\right)\left(-\frac{1}{x \sin x + \cos x}\right) - \int \left[\left\{\frac{d}{dx}\left(\frac{x}{\cos x}\right)\right\} \cdot \left\{-\frac{1}{x \sin x + \cos x}\right\}\right] dx$$

Chapter

1

Integration by Partial Fractions

1.1 Rational Fractions

A fraction whose numerator and denominator are both rational and algebraic functions is defined as a rational algebraic fraction or simply a rational fraction.

Thus, $$\frac{f(x)}{\phi(x)} = \frac{a_0 x^m + a_1 x^{m-1} + \ldots + a_{m-1} x + a_m}{b_0 x^n + b_1 x^{n-1} + \ldots + b_{n-1} x + b_n},$$

in which $a_0, a_1, \ldots, a_m, b_0, b_1, \ldots, b_n$ are constants and m and n are positive integers, is a rational algebraic fraction.

If degree $f(x) <$ degree $\phi(x)$, then $\frac{f(z)}{\phi(x)}$ is called a proper rational fraction.

If degree $f(x) \geq$ degree $\phi(x)$, then $\frac{f(x)}{\phi(x)}$ is called an improper rational fraction.

If $\frac{f(x)}{\Phi(x)}$ is an improper rational fraction, then by dividing $f(x)$ by $\phi(x)$, we can express $\frac{f(x)}{\phi(x)}$ as the sum of a polynomial and a proper rational fraction.

1.2 Partial Fractions

Any proper rational fraction $f(x)/\phi(x)$ can be expressed as the sum of rational fractions, each having a simple factor of $\phi(x)$. Each such fraction is called a partial fraction and the process of obtaining them is called the decomposition or resolution of the given fraction into partial fractions.

The resolution of $f(x)/\phi(x)$ into partial fractions will depend upon the nature of factors of $\phi(x)$. According to these factors, we obtain the corresponding partial fractions. The following table gives an idea what kind of partial fractions are to be taken for what kind of factors in the denominator :

	Factor in the denominator	Form of the partial fraction
(i)	$(x-a)$	$\dfrac{A}{(x-a)}$
(ii)	$(x-a)^2$	$\dfrac{A}{(x-a)}+\dfrac{B}{(x-a)^2}$
(iii)	$(x-a)^3$	$\dfrac{A}{(x-a)}+\dfrac{B}{(x-a)^2}+\dfrac{C}{(x-a)^3}$
(iv)	(ax^2+bx+c)	$\dfrac{Ax+B}{ax^2+bx+c}$
(v)	$(ax^2+bx+c)^2$	$\dfrac{Ax+B}{cx^2+bx+c}+\dfrac{Cx+D}{(ax^2+bx+c)^2}$

Note: There are as many constants to be determined as the degree of the denominator.

We explain the method of partial fraction decomposition through some examples.

Illustrative Examples

Example 1: *Resolve* $\dfrac{(x-1)}{(x-3)(x-2)}$ *into partial fractions.*

Solution: Let $\dfrac{x-1}{(x-3)(x-2)}=\dfrac{A}{(x-3)}+\dfrac{B}{(x-2)}=\dfrac{A(x-2)+B(x-3)}{(x-3)(x-2)}$.

Clearly $\quad x-1\equiv A(x-2)+B(x-3)$. ...(1)

Comparing the coefficients of x and the constant terms on both sides of (1), we get

$$1=A+B \qquad \text{...(2)}$$

and
$$-1=-2A-3B \qquad \text{...(3)}$$

Solving (2) and (3), we get $A = 2, B = -1$.

$$\therefore \qquad \frac{(x-1)}{(x-3)(x-2)} = \frac{2}{(x-3)} - \frac{1}{(x-2)}.$$

Note: An easy way to find the constants A and B etc. corresponding to linear non-repeated factors is like this : The factor below A is $(x-3)$. The equation $x-3=0$ gives $x=3$. Now suppress $(x-3)$ in the given fraction $\frac{(x-1)}{(x-3)(x-2)}$ and put $x=3$ in the remaining fraction $\frac{(t-1)}{(x-2)}$ to get A. Thus, $A = \frac{3-1}{3-2} = 2$.

Similarly $\qquad B = \frac{2-1}{2-3} = -1.$

Example 2: *Resolve* $\frac{x^3-6x^2+10x-2}{x^2-5x+6}$ *into partial fractions.*

Solution: Here since numerator is not of a lower degree than the denominator, we first divide the numerator by the denominator.

We have $\qquad \frac{x^3-6x^2+10x-2}{x^2-5x+6} = x-1+\frac{(-x+4)}{x^2-5x+6}.$

Now let $\qquad \frac{-x+4}{x^2-5x+6} = \frac{-x+4}{(x-3)(x-2)} = \frac{A}{(x-3)} + \frac{B}{(x-2)}.$

Then $\qquad -x+4 \equiv A(x-2) + B(x-3). \qquad \ldots(1)$

Putting $x=3$ in (1), we get $A=1$.

Putting $x=2$ in (1), we get $B=-2$.

$$\therefore \qquad \frac{-x+4}{x^2-5x+6} = \frac{1}{(x-3)} - \frac{2}{(x-2)}.$$

Hence, $\qquad \frac{x^3-6x^2+10x-2}{x^2-5x+6} = x-1+\frac{1}{(x-3)} - \frac{2}{(x-2)}.$

Example 3: *Resolve* $\frac{16}{(x-2)(x+2)^2}$ *into partial fractions.*

Solution: Let $\qquad \frac{16}{(x-2)(x+2)^2} = \frac{A}{(x-2)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}.$

Then $\qquad 16 \equiv A(x+2)^2 + B(x+2)(x-2) + C(x-2). \qquad \ldots(1)$

Putting $x=2$ in (1), we get $A=1$.

Comparing the coefficients of x^2 and constant terms on both sides of (1), we get

$$A + B = 0 \quad \text{and} \quad 4A - 4B - 2C = 16.$$

These give $\qquad B = -1, C = -4$.

Hence, $$\frac{16}{(x-2)(x+2)^2}=\frac{1}{(x-2)}-\frac{1}{(x+2)}-\frac{4}{(x+2)^2}.$$

Example 4: *Resolve* $\dfrac{2x-1}{(x+1)(x^2+2)}$ *into partial fractions.*

Solution: Let $$\frac{2x-1}{(x+1)(x^2+2)}=\frac{A}{(x+1)}+\frac{Bx+C}{(x^2+2)}.$$

Then $$2x-1\equiv A(x^2+2)+(Bx+C)(x+1). \qquad \ldots(1)$$

Putting $x=-1$ in (1), we get $A=-1$.

Comparing the coefficients of x^2 and x on both sides of (1), we get

$$A+B=0 \quad \text{and} \quad B+C=2.$$

These give $B=-A=1, C=2-B=1$.

$$\therefore \quad \frac{3x-1}{(x+1)(x^2+2)}=-\frac{1}{(x+1)}+\frac{x+1}{x^2+2}.$$

Example 5: *Resolve* $\dfrac{(2x-3)}{(x-1)(x^2+1)^2}$ *into partial fractions.*

Solution: Let $$\frac{(2x-3)}{(x-1)(x^2+1)^2}=\frac{A}{(x-1)}+\frac{Bx+C}{(x^2+1)}+\frac{Dx+E}{(x^2+1)^2}.$$

Then $$2x-3\equiv A(x^2+1)^2+(Bx+C)(x-1)(x^2+1)+(Dx+E)(x-1). \qquad \ldots(1)$$

Putting $x=1$ in (1), we get $A=-\dfrac{1}{4}$.

Comparing the coefficients of x^4, x^3, x^2 and x on both sides of (1), we get

$$A+B=0, \quad C-B=0, \quad 2A+B-C+D=0 \quad \text{and} \quad -B+C-D+E=2.$$

Putting $A=-\dfrac{1}{4}$ and solving these equations, we get

$$B=\frac{1}{4}, \; C=\frac{1}{4}, \quad D=\frac{1}{2} \quad \text{and} \quad E=\frac{5}{2}.$$

$$\therefore \quad \frac{(2x-3)}{(x-1)(x^2+1)^2}=-\frac{1}{4(x-1)}+\frac{(x+1)}{4(x^2+1)}+\frac{(x+5)}{2(x^2+1)^2}.$$

Example 6: *Resolve* $\dfrac{x^2+x+1}{(x-1)^4}$ *into partial fractions.*

Solution: Let $(x-1)=y$. Then $x=(y+1)$.

$$\therefore \quad \frac{x^2+x+1}{(x-1)^4}=\frac{(y+1)^2+(y+1)+1}{y^4}=\frac{y^2+3y+3}{y^4}$$

$$=\frac{1}{y^2}+\frac{3}{y^3}+\frac{3}{y^4}=\frac{1}{(x-1)^2}+\frac{3}{(x-1)^3}+\frac{3}{(x-1)^4}.$$

1.3 Integration of Rational Fractions by Partial Fraction

We can use the method of partial fraction decomposition to integrate rational fractions. The following examples illustrate the procedure.

Illustrative Examples

Example 7: *Evaluate :* $\int \frac{(x+1)\,dx}{x^3 + x^2 - 6x}$.

Solution: Here $\frac{x+1}{x^3 + x^2 - 6x} = \frac{x+1}{x\,(x-2)\,(x+3)}$

$$\equiv \frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x+3)}, \text{(say)}.$$

To find A suppress x in the given fraction and put $x = 0$ in the remaining fraction.

Thus, $A = \frac{0+1}{(0-2)\,(0+3)} = -\frac{1}{6}$.

To find B suppress $(x-2)$ in the given fraction and put $x = 2$ in the remaining fraction.

Thus, $B = \frac{2+1}{2\,(2+3)} = \frac{3}{10}$.

Similarly $C = \frac{-3+1}{-3\,(-3-2)} = -\frac{2}{15}$.

Thus, $\frac{x+1}{x\,(x-2)\,(x+3)} = -\frac{1}{6x} + \frac{3}{10\,(x-2)} - \frac{2}{15\,(x+3)}$.

Obviously $\int \frac{(x+1)\,dx}{x\,(x-2)\,(x+3)} = -\int \frac{1.\,dx}{6x} + \int \frac{3\,dx}{10\,(x-2)} - \int \frac{2\,dx}{15\,(x+3)}$

$$= -\frac{1}{6}\log|x| + \frac{3}{10}\log|x-2| - \frac{2}{15}\log|x+3| + c.$$

Example 8: *Evaluate* $\int \frac{x^3}{(x-1)\,(x-2)\,(x-3)}\,dx$.

Solution: Here since the numerator is not of a lower degree than the denominator, we divide the numerator by the denominator till the remainder is of lesser degree than the denominator. We orally see that the quotient is 1.

We need not find out the actual value of the remainder because ultimately we have to break the fraction into partial fractions. Note that the denominators of the partial fractions depend only upon the denominator of the given fraction. So let

$$\frac{x^3}{(x-1)\,(x-2)\,(x-3)} \equiv 1 + \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}.$$

We have $A = \dfrac{1^3}{(1-2)(1-3)} = \dfrac{1}{2}$, $B = \dfrac{2^3}{(2-1)(2-3)} = -8$,

and $C = \dfrac{3^3}{(3-1)(3-2)} = \dfrac{27}{2}$.

$\therefore$ $$\frac{x^3}{(x-1)(x-2)(x-3)} = 1 + \frac{1}{2(x-1)} - \frac{8}{(x-2)} + \frac{27}{2(x-3)}.$$

Hence, $$\int \frac{x^3\, dx}{(x-1)(x-2)(x-3)}$$
$$= \int 1 \,.\, dx + \int \frac{dx}{2(x-1)} - \int \frac{8\, dx}{(x-2)} + \int \frac{27\, dx}{2(x-3)}$$
$$= x + \frac{1}{2} \log|x-1| - 8 \log|x-2| + \frac{27}{2} \log|x-3| + c.$$

Example 9: *Evaluate* $\displaystyle\int \frac{x^2}{(x^2+2)(x^2+3)}\, dx.$

Solution: Let $y = x^2$.

Then $$\frac{x^2}{(x^2+2)(x^2+3)} = \frac{y}{(y+2)(y+3)} \equiv \frac{A}{(y+2)} + \frac{B}{(y+3)}, \text{(say)}.$$

We have $A =$ the value of $\dfrac{y}{y+3}$, when y is -2,

$= -2$

and $B =$ the value of $\dfrac{y}{y+2}$, when y is -3,

$= 3.$

Thus, $$\frac{x^2}{(x^2+2)(x^2+3)} = \frac{-2}{y+2} + \frac{3}{y+3} = \frac{-2}{x^2+2} + \frac{3}{x^2+3}.$$

$\therefore$ $$\int \frac{x^2}{(x^2+2)(x^2+3)}\, dx = -2 \int \frac{dx}{x^2+2} + 3 \int \frac{dx}{x^2+3}$$
$$= -2 \cdot \frac{1}{\sqrt{2}} \tan^{-1} \frac{x}{\sqrt{2}} + 3 \cdot \frac{1}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} + c$$
$$= -\sqrt{2} \tan^{-1} \frac{x}{\sqrt{2}} + \sqrt{3} \tan^{-1} \frac{x}{\sqrt{3}} + c.$$

Note: In the above example, the substitution was made only for the partial fraction part and not for the integration part.

Example 10: *Evaluate :* $\displaystyle\int \frac{8}{(x+2)(x^2+4)}\, dx.$

Solution: Let $\dfrac{8}{(x+2)(x^2+4)} \equiv \dfrac{A}{x+2} + \dfrac{Bx+C}{x^2+4}$

or $$8 \equiv A(x^2+4) + (Bx+C)(x+2). \quad \ldots(1)$$

Then $A =$ the value of $\dfrac{8}{x^2+4}$, when x is $-2 = 1$.

Comparing the coefficients of x^2 and x on both sides of (1), we get

$A + B = 0$ and $2B + C = 0 \Rightarrow B = -A = -1, C = -2B = 2.$

Thus, $$\frac{8}{(x+2)(x^2+4)} = \frac{1}{x+2} + \frac{(-x+2)}{x^2+4}.$$

$\therefore$ $$\int \frac{8}{(x+2)(x^2+4)}\,dx = \int \frac{1}{x+2}\,dx + \int \frac{(-x+2)}{x^2+4}\,dx$$

$$= \int \frac{1}{x+2}\,dx - \int \frac{x}{x^2+4}\,dx + 2\int \frac{dx}{x^2+4}$$

$$= \log|x+2| - \frac{1}{2}\log|x^2+4| + 2\cdot\frac{1}{2}\tan^{-1}\frac{x}{2} + c$$

$$= \log|x+2| - \frac{1}{2}\log|x^2+4| + \tan^{-1}\frac{x}{2} + c.$$

Example 11: *Evaluate* $\displaystyle\int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}\,dx.$

Solution: We have $\dfrac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = \dfrac{(y+1)(y+2)}{(y+3)(y+4)}$, where $y = x^2$.

Now let $\dfrac{(y+1)(y+2)}{(y+3)(y+4)} = 1 + \dfrac{A}{y+3} + \dfrac{B}{y+4}$, resolving into partial fractions.

We have $A = \dfrac{(-3+1)(-3+2)}{(-3+4)} = 2, \quad B = \dfrac{(-4+1)(-4+2)}{(-4+3)} = -6.$

$\therefore$ $$\frac{(y+1)(y+2)}{(y+3)(y+4)} = 1 + \frac{2}{y+3} - \frac{6}{y+4}.$$

$\therefore$ the given integral $I = \displaystyle\int \left[1 + \frac{2}{x^2+3} - \frac{6}{x^2+4}\right] dx$

$$= \int dx + 2\int \frac{dx}{x^2+3} - 6\int \frac{dx}{x^2+4}$$

$$= x + 2\,.\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - 6\,.\frac{1}{2}\tan^{-1}\frac{x}{2}$$

$$= x + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - 3\tan^{-1}\left(\frac{x}{2}\right).$$

Example 12: *Integrate* $x / \{(x-1)^3 (x-2)\}$.

Solution: Putting $(x-1) = y$ or $x = y+1$, we get

$$\frac{x}{(x-1)^3\,(x-2)}=\frac{y+1}{y^3\,(y+1-2)}=\frac{y+1}{y^3\,(y-1)}=\frac{1+y}{y^3\,(-1+y)},$$

[Note that we have arranged the Nr. and the Dr. in ascending powers of y]

$$=\frac{1}{y^3}\left[-1-2y-2y^2+\frac{2y^3}{-1+y}\right], \text{by actual division}$$

$$=-\frac{1}{y^3}-\frac{2}{y^2}-\frac{2}{y}+\frac{2}{(y-1)}$$

$$=-\frac{1}{(x-1)^3}-\frac{2}{(x-1)^2}-\frac{2}{(x-1)}+\frac{2}{(x-2)}. \qquad [\because y=x-1]$$

Hence the required integral of the given fraction

$$=-\int\frac{dx}{(x-1)^3}-\int\frac{2\,dx}{(x-1)^2}-\int\frac{2\,dx}{(x-1)}+\int\frac{2\,dx}{(x-2)}$$

$$=\frac{1}{x\,(x-1)^2}+\frac{2}{(x-1)}-2\log(x-1)+2\log(x-2).$$

Comprehensive Exercise 1

Integrate the following :

1. $(x^2+1)/(x^2-1)$.
2. $x^2/\{(x+1)(x-2)(x+3)\}$
3. $x^2/\{(x-1)(3x-1)(3x-2)\}$.
4. $x/\{(x-a)(x-b)(x-c)\}$.
5. $\{(x-a)(x-b)(x-c)\}/\{(x-\alpha)(x-\beta)(x-\gamma)\}$.
6. $(x^2+x+2)/\{(x-2)(x-1)\}$.
7. $\displaystyle\int\frac{dx}{(x-1)^2\,(x^2+4)}$.
8. $\displaystyle\int\frac{(x^2+x+1)\,dx}{(x+1)^2\,(x+2)}$.
9. $\displaystyle\int\frac{dx}{x^3\,(x-1)^2\,(x+1)}$.
10. $(x^2+2)/\{(x-1)(x-2)^3\}$.
11. $(3x+1)/\{(x-1)^3\,(x+1)\}$.
12. $\displaystyle\int\frac{dx}{x\,(x^n+1)}$.

Answers 1

1. $x+\log\dfrac{x-1}{x+1}$
2. $\dfrac{9}{10}\log(x+3)+\dfrac{4}{15}\log(x-2)-\dfrac{1}{6}\log(x+1)$

3. $\frac{1}{2}\log(x-1)+\frac{1}{18}\log(3x-1)-\frac{4}{9}\log(3x-2)$

4. $\Sigma\left[\frac{a\log(x-a)}{(a-b)(a-c)}\right]$

5. $x+\Sigma\left[\frac{(\alpha-a)(\alpha-b)(\alpha-c)}{(\alpha-\beta)(\alpha-\gamma)}\log(x-\alpha)\right]$

6. $x+4\log\{(x-2)^2/(x-1)\}$

7. $-\frac{2}{25}\log(x-1)-\frac{1}{5(x-1)}+\frac{1}{25}\log(x^2+4)-\frac{3}{50}\tan^{-1}\frac{x}{2}$

8. $\log\frac{(x+2)^3}{(x+1)^2}-\frac{1}{x+1}$

9. $2\log x-\frac{1}{x}-\frac{1}{2x^2}-\frac{7}{4}\log(x-1)-\frac{1}{2(x-1)}-\frac{1}{4}\log(x+1)$

10. $-\frac{3}{(x-2)^2}+\frac{2}{(x-2)}+3\log(x-2)-3\log(x-1)$

11. $\frac{-1}{(x-1)^2}-\frac{1}{2(x-1)}+\frac{1}{4}\log\frac{x+1}{x-1}$

12. $\frac{1}{n}\log|x^n|-\frac{1}{n}\log|x^n+1|+c$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. If degree of $f(x)$ < degree of $\phi(x)$, then the fraction $\frac{f(x)}{\phi(x)}$ is called

   (a) proper fraction (b) improper fraction
   (c) unit fraction (d) None of these

2. If the function has a factor in the denominator $(x-a)^3$, then the form of the partial fraction is

   (a) $\frac{A}{(x-a)^2}+\frac{B}{(x-a)^3}$ (b) $\frac{A}{(x-a)}+\frac{Bx+C}{(x-a)^2}$
   (c) $\frac{A}{(x-a)}+\frac{B}{(x-a)^2}+\frac{C}{(x-a)^3}$ (d) None of these

3. After resolving the function $\frac{(x-1)}{(x-3)(x-2)}$ into partial fractions, we get the value of A and B as

   (a) $A=2, B=1$ (b) $A=-2, B=1$
   (c) $A=2, B=-1$ (d) None of these

Chapter

2

Integration of Rational Functions

2.1 Integration of $1/(ax^2 + bx + c)$

To evaluate such integrals put the denominator in the form $a\{(x+\alpha)^2 \pm \beta^2\}$ and then integrate.

Illustrative Examples

Example 1: *Integrate* $1/(9x^2 - 12x + 8)$.

Solution: We have $\displaystyle\int \frac{dx}{8x^2 - 12x + 8} = \frac{1}{9}\int \frac{dx}{x^2 - \frac{4}{3}x + \frac{8}{9}}$,

making the coeff. of x^2 in the denominator as 1

$$= \frac{1}{9}\int \frac{dx}{\left(x^2 - \frac{4}{3}x + \frac{4}{9} + \frac{8}{9} - \frac{4}{9}\right)} = \frac{1}{9}\int \frac{dx}{\left(x - \frac{2}{3}\right)^2 + \frac{4}{9}}$$

$$= \frac{1}{9}\int \frac{dx}{\left(x - \frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2}$$

$$= \frac{1}{9} \cdot \frac{3}{2} \cdot \tan^{-1} \frac{\left(x - \frac{2}{3}\right)}{2/3} = \frac{1}{6} \tan^{-1} \frac{3x - 2}{2}.$$

Example 2: *Evaluate* $\int_0^1 \{1/(1 - x + x^2)\} dx$.

Solution: Dr. $= 1 - x + x^2 = \left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$.

$$\therefore \quad \int_0^1 \frac{dx}{1 - x + x^2} = \int_0^1 \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{2}{\sqrt{3}} \left[\tan^{-1} \left(\frac{x - \frac{1}{2}}{\sqrt{3}/2}\right)\right]_0^1$$

$$= \frac{2}{\sqrt{3}} \left[\tan^{-1} \left(\frac{2x - 1}{\sqrt{3}}\right)\right]_0^1 = \frac{2}{\sqrt{3}} \left[\tan^{-1} \left(\frac{1}{\sqrt{3}}\right) - \tan^{-1} \left(-\frac{1}{\sqrt{3}}\right)\right]$$

$$= \frac{2}{\sqrt{3}} \left[\tan^{-1} \left(\frac{1}{\sqrt{3}}\right) + \tan^{-1} \left(\frac{1}{\sqrt{3}}\right)\right], \qquad [\because \tan^{-1}(-x) = -\tan^{-1} x]$$

$$= \frac{4}{\sqrt{3}} \tan^{-1} \left(\frac{1}{\sqrt{3}}\right) = \frac{4}{\sqrt{3}} \cdot \frac{\pi}{6} = \frac{2\pi}{3\sqrt{3}}.$$

Comprehensive Exercise 1

1. Integrate $1/(2x^2 + x + 1)$.
2. Evaluate $\int \frac{dx}{2x^2 + 3x + 5}$.
3. Integrate $1/(2x^2 + x - 1)$.
4. Integrate $1/(x^2 - 3x + 2)$.
5. Evaluate $\int \frac{x}{x^4 + x^2 + 1} dx$.

Answers 1

1. $\frac{2}{\sqrt{7}} \tan^{-1} \left(\frac{4x + 1}{\sqrt{7}}\right)$
2. $\frac{2}{\sqrt{31}} \cdot \tan^{-1} \left[\frac{4x + 3}{\sqrt{31}}\right]$

3. $\frac{1}{3}\log\{(2x-1)/(x-1)\}$

4. $\log\left[\frac{x-2}{x-1}\right]$

5. $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right)$

2.2 Integration of $(px+q)/(ax^2+bx+c)$

To integrate such integrals break the given fraction into two fractions such that in one the numerator is the differential coefficient of the denominator, and in the other the numerator is merely a constant. Thus

$$\int \frac{(px+q)\,dx}{ax^2+bx+c} = \int \frac{(p/2a)(2ax+b)+q-\{(pb)/2a\}}{ax^2+bx+c}\,dx$$

$$= \frac{p}{2a}\int \frac{2ax+b}{ax^2+bx+c}\,dx + \int \frac{q-\{(pb)/(2a)\}}{ax^2+bx+c}\,dx$$

$$= \frac{p}{2a}\log(ax^2+bx+c) + \int \frac{q-\{(pb)/(2a)\}}{ax^2+bx+c}\,dx\,.$$

The 2nd integral can now be easily evaluated.

Illustrative Examples

Example 3: *Integrate $x/(x^2+x-6)$.*

Solution: Let $I = \int \frac{x}{x^2+x-6}\,dx$.

Here $\frac{d}{dx}$(denominator) $= \frac{d}{dx}(x^2+x-6) = 2x+1$.

$$\therefore \qquad I = \int \frac{\frac{1}{2}(2x+1)-\frac{1}{2}}{x^2+x-6}\,dx = \frac{1}{2}\int \frac{2x+1}{x^2+x-6}\,dx - \frac{1}{2}\int \frac{dx}{x^2+x-6}$$

$$= \frac{1}{2}\log(x^2+x-6) - \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{2}\right)^2-6-\frac{1}{4}}$$

$$= \frac{1}{2}\log(x^2+x-6) - \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{2}\right)^2-\frac{25}{4}}$$

$$= \frac{1}{2}\log(x^2+x-6) - \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{2}\right)^2-\left(\frac{5}{2}\right)^2}$$

$$= \frac{1}{2} \log (x^2 + x - 6) - \frac{1}{2} \cdot \frac{1}{2 \cdot \left(\frac{5}{2}\right)} \log \frac{x + \frac{1}{2} - \frac{5}{2}}{x + \frac{1}{2} + \frac{5}{2}}$$

$$= \frac{1}{2} \log (x^2 + x - 6) - \frac{1}{10} \log \frac{x - 2}{x + 3}.$$

Example 4: *Integrate* $(3x + 1) / (2x^2 - 2x + 3)$.

Solution: Here $\frac{d}{dx} (2x^2 - 2x + 3) = 4x - 2$.

$$\therefore \qquad I = \int \frac{3x + 1}{2x^2 - 2x + 3} dx = \int \frac{\frac{3}{4} (4x - 2) + 1 + \frac{3}{2}}{(2x^2 - 2x + 3)} dx \qquad \textbf{(Note)}$$

$$= \frac{3}{4} \int \frac{4x - 2}{2x^2 - 2x + 3} dx + \frac{5}{2} \int \frac{1}{2x^2 - 2x + 3} dx$$

$$= \frac{3}{4} \log (2x^2 - 2x + 3) + \frac{5}{2.2} \int \frac{dx}{x^2 - x + (3/2)}$$

$$= \frac{3}{4} \log (2x^2 - 2x + 3) + \frac{5}{4} \int \frac{dx}{\left(x - \frac{1}{2}\right)^2 + (3/2) - (1/4)}$$

$$= \frac{3}{4} \log (2x^2 - 2x + 3) + \frac{5}{4} \int \frac{dx}{\left(x - \frac{1}{2}\right)^2 + (\sqrt{5}/2)^2}$$

$$= \frac{3}{4} \log (2x^2 - 2x + 3) + \frac{5}{4} \frac{1}{(\sqrt{5}/2)} \tan^{-1} \left\{ \frac{x - \frac{1}{2}}{(\sqrt{5}/2)} \right\}$$

$$= \frac{3}{4} \log (2x^2 - 2x + 3) + \frac{\sqrt{5}}{2} \tan^{-1} \left(\frac{2x - 1}{\sqrt{5}} \right).$$

Comprehensive Exercise 2

1. Integrate $3x / (x^2 - x - 2)$.
2. Integrate $(5x - 2) / (1 + 2x + 3x^2)$.
3. Integrate $x^2 / (x^4 + x^2 + 1)$.
4. Evaluate $\int_0^1 \frac{(x - 3)\, dx}{x^2 + 2x - 4}$.

2.5 Integration of Rational Functions by Substitution

The integration of rational functions by substitution is explained by the following examples.

Illustrative Examples

Example 7: *Integrate* $(x^2+1)/(x^4+1)$.

Solution: Let $I=\int \frac{x^2+1}{x^4+1}\,dx$.

Here both the numerator and the denominator do not contain odd powers of x. Also the numerator is of degree 2 and the denominator is of degree 4. So dividing the numerator and the denominator by x^2, we get

$$I=\int \frac{1+(1/x^2)}{x^2+(1/x^2)}\,dx$$

$$=\int \frac{1+(1/x^2)}{[x-(1/x)]^2+2}\,dx, \qquad \left[\text{Note that } \frac{d}{dx}\left\{x-\frac{1}{x}\right\}=1+\frac{1}{x^2}\right].$$

Now put $x-(1/x)=t$ so that $\{1+(1/x^2)\}\,dx=dt$.

$$\therefore \qquad I=\int \frac{dt}{t^2+2}=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)=\frac{1}{\sqrt{2}}\tan^{-1}\left\{\frac{x-(1/x)}{\sqrt{2}}\right\}$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right).$$

Example 8: *Integrate* $(x^2-1)/(x^4+x^2+1)$.

Solution: We have $I=\int \frac{x^2-1}{x^4+x^2+1}\,dx$, [Note the form of the integrand]

$$=\int \frac{1-(1/x^2)}{x^2+1+(1/x^2)}\,dx,$$

dividing the numerator and the denominator by x^2

$$=\int \frac{1-(1/x^2)}{\{x+(1/x)\}^2-1}\,dx. \qquad \left[\text{Note that } \frac{d}{dx}\{x+(1/x)\}=1-(1/x^2)\right]$$

Now put $x+(1/x)=t$, so that $\{1-(1/x^2)\}\,dx=dt$.

$$\therefore \qquad I=\int \frac{dt}{t^2-1}=\frac{1}{2}\log\frac{t-1}{t+1}$$

$$= \frac{1}{2} \log \frac{x + (1/x) - 1}{x + (1/x) + 1}$$

$$= \frac{1}{2} \log \frac{x^2 - x + 1}{x^2 + x + 1}.$$

Example 9: *Integrate* $x^2 / (x^4 + a^4)$.

Solution: We have $I = \int \frac{x^2}{x^4 + a^4} dx = \int \frac{1}{\{x^2 + (a^4 / x^2)\}} dx$,

dividing the numerator and the denominator by x^2

$$= \frac{1}{2} \int \frac{\{1 - (a^2 / x^2)\} + \{1 + (a^2 / x^2)\}}{x^2 + (a^4 / x^2)} dx$$

$$= \frac{1}{2} \int \left\{ \frac{1 - (a^2 / x^2)}{\{x + (a^2 / x)\}^2 - 2a^2} + \frac{1 + (a^2 / x^2)}{\{x - (a^2 / x)\}^2 + 2a^2} \right\} dx.$$

In the first integral, put

$$\{x + (a^2 / x)\} = t \text{ so that } \{1 - (a^2 / x^2)\}\, dx = dt,$$

and in the second integral, put

$$x - (a^2 / x) = z \text{ so that } \{1 + (a^2 / x^2)\}\, dx = dz.$$

$$\therefore \qquad I = \frac{1}{2} \left[\int \frac{dt}{t^2 - 2a^2} + \int \frac{dz}{z^2 + 2a^2} \right]$$

$$= \frac{1}{2} \left[\frac{1}{2a\sqrt{2}} \log \frac{t - a\sqrt{2}}{t + a\sqrt{2}} + \frac{1}{a\sqrt{2}} \tan^{-1} \frac{z}{a\sqrt{2}} \right]$$

$$= \frac{1}{4a\sqrt{2}} \log \left[\frac{\{x + (a^2 / x) - a\sqrt{2}\}}{\{x + (a^2 / x) + a\sqrt{2}\}} \right] + \frac{1}{2a\sqrt{2}} \tan^{-1} \frac{\{x - (a^2 / x)\}}{a\sqrt{2}}$$

$$= \frac{\sqrt{2}}{8a} \log \left\{ \frac{x^2 - \sqrt{2}ax + a^2}{x^2 + \sqrt{2}ax + a^2} \right\} + \frac{\sqrt{2}}{4a} \tan^{-1} \left\{ \frac{x^2 - a^2}{\sqrt{2}ax} \right\}.$$

Example 10: *Integrate* $1 / \{x (x^5 + 1)\}$.

Solution: We have $I = \int \frac{1}{x (x^5 + 1)} dx = \int \frac{x^{5-1}}{x^5 (x^5 + 1)} dx$. **(Note)**

Now put $x^5 = t$ so that $5x^{5-1}\, dx = dt$.

$\therefore$ required integral $I = \frac{1}{5} \int \frac{dt}{t(t+1)} = \frac{1}{5} \int \left[\frac{1}{t} - \frac{1}{(t+1)} \right] dt$

$$= (1/5) [\log t - \log (t+1)] = (1/5) . \log \{t / (t+1)\}$$

$$= (1/5) . \log \{x^5 / (x^5 + 1)\}, \qquad [\because t = x^5]$$

Example 11: *Evaluate* $\int \frac{\sin x}{\sin 4x}\,dx$.

Solution: We have $I = \int \frac{\sin x}{\sin 4x}\,dx = \int \frac{\sin x\,dx}{2\sin 2x\cos 2x}$

$$= \int \frac{\sin x\,dx}{4\sin x\cos x\cos 2x} = \frac{1}{4}\int \frac{dx}{\cos x\cos 2x} = \frac{1}{4}\int \frac{\cos x\,dx}{\cos 2x\cos^2 x}$$

$$= \frac{1}{4}\int \frac{\cos x\,dx}{(1-\sin^2 x)(1-2\sin^2 x)}. \qquad \textbf{(Note)}$$

Now put $\sin x = t$ so that $\cos x\,dx = dt$.

$$\therefore \qquad I = \frac{1}{4}\int \frac{dt}{(1-t^2)(1-2t^2)} = \frac{1}{4}\int \frac{dt}{(t^2-1)(2t^2-1)}$$

$$= \frac{1}{4}\int \left[\frac{1}{(t^2-1)} - \frac{2}{(2t^2-1)}\right] dt \text{, resolving into partial fractions}$$

$$= \frac{1}{4}\int \frac{dt}{(t^2-1)} - \frac{1}{4}\int \frac{dt}{t^2 - \left(\frac{1}{2}\right)}$$

$$= \frac{1}{4}\cdot\frac{1}{2}\log\left(\frac{t-1}{t+1}\right) - \frac{1}{4}\cdot\frac{1}{2\,.\,(1/\sqrt{2})}\log\left[\frac{t-(1/\sqrt{2})}{1+(t/\sqrt{2})}\right] \qquad \textbf{(Note)}$$

$$= \frac{1}{8}\log\left(\frac{t-1}{t+1}\right) - \frac{1}{4\sqrt{2}}\log\left(\frac{t\sqrt{2}-1}{t\sqrt{2}+1}\right)$$

$$= \frac{1}{8}\log\left(\frac{\sin x-1}{\sin x+1}\right) - \frac{1}{4\sqrt{2}}\log\left(\frac{\sqrt{2}\sin x-1}{\sqrt{2}\sin x+1}\right).$$

Example 12: *Evaluate* $\int_0^{\pi/4} \sqrt{(\cot\theta)}\,d\theta$. **(Meerut 1982 S; Delhi 74)**

Solution: Let $I = \int_0^{\pi/4} \sqrt{(\cot\theta)}\,d\theta$.

Put $\cot\theta = z^2$ so that $-\operatorname{cosec}^2\theta\,d\theta = 2z\,dz$

or $d\theta = \frac{-2z\,dz}{\operatorname{cosec}^2\theta} = \frac{-2z\,dz}{1+\cot^2\theta} = \frac{-2z\,dz}{1+z^4}$.

Also when $\theta = 0$, $z = \infty$ and when $\theta = \pi/4$, $z = 1$.

$$\therefore \qquad I = \int_\infty^1 \frac{z(-2z)\,dz}{z^4+1} = \int_1^\infty \frac{2z^2}{z^4+1}\,dz, \qquad \left[\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right]$$

$$= \int_1^\infty \frac{2}{z^2+(1/z^2)}\,dz,$$

dividing the numerator and the denominator by z^2

$$= \int_1^\infty \frac{\{1 + (1/z^2)\} + \{1 - (1/z^2)\}}{z^2 + (1/z^2)}\, dz$$

$$= \int_1^\infty \frac{\{1 + (1/z^2)\}\, dz}{\{z - (1/z)\}^2 + 2} + \int_1^\infty \frac{\{1 - (1/z^2)\}\, dz}{\{z + (1/z)\}^2 - 2}.$$

In the first integral put $z - (1/z) = t$ so that $[1 + (1/z^2)]\, dz = dt$. The corresponding limits for t are 0 to ∞. In the second integral put $z + (1/z) = u$ so that $[1 - (1/z^2)]\, dz = du$. The limits for u are from 2 to ∞. Hence

$$I = \int_0^\infty \frac{dt}{t^2 + 2} + \int_2^\infty \frac{du}{u^2 - 2}$$

$$= \frac{1}{\sqrt{2}} \left[\tan^{-1} \frac{t}{\sqrt{2}}\right]_0^\infty + \frac{1}{2\sqrt{2}} \left[\log \frac{u - \sqrt{2}}{u + \sqrt{2}}\right]_2^\infty$$

$$= \frac{1}{\sqrt{2}} [\tan^{-1} \infty - \tan^{-1} 0] + \frac{1}{2\sqrt{2}} \left[\lim_{u \to \infty} \log \frac{u - \sqrt{2}}{u + \sqrt{2}} - \log \frac{2 - \sqrt{2}}{2 + \sqrt{2}}\right]$$

$$= \frac{1}{\sqrt{2}} \left(\frac{\pi}{2} - 3\right) + \frac{1}{2\sqrt{2}} \left[\lim_{u \to \infty} \log \left\{\frac{1 - (\sqrt{2}/u)}{1 + (\sqrt{2}/u)}\right\} - \log \left\{\frac{\sqrt{2}\,(\sqrt{2} - 1)}{\sqrt{2}\,(\sqrt{2} + 1)}\right\}\right]$$

$$= \frac{\pi}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \left[\log 1 - \log \frac{\sqrt{2} - 1}{\sqrt{2} + 1}\right]$$

$$= \frac{\pi}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} \log \left\{\frac{(\sqrt{2} - 1)(\sqrt{2} - 1)}{(\sqrt{2} + 1)(\sqrt{2} - 1)}\right\} = \frac{\pi}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} \log (\sqrt{2} - 1)^2$$

$$= \frac{\pi}{2\sqrt{2}} - \frac{1}{\sqrt{2}} \log (\sqrt{2} - 1) = \frac{\pi\sqrt{2}}{4} - \frac{\sqrt{2}}{2} \log (\sqrt{2} - 1).$$

Example 13: *Integrate* $1/(\sin x + \sin 2x)$.

Solution: We have $I = \int \frac{dx}{\sin x + \sin 2x} = \int \frac{dx}{\sin x + 2 \sin x \cos x}$

$$= \int \frac{dx}{\sin x\,(1 + 2\cos x)}$$

$$= \int \frac{\sin x\, dx}{\sin^2 x\,(1 + 2\cos x)} = \int \frac{\sin x\, dx}{(1 - \cos^2 x)(1 + 2\cos x)}. \qquad \textbf{(Note)}$$

Now putting $\cos x = t$, so that $-\sin x\, dx = dt$, we get

$$I = -\int \frac{dt}{(1 - t^2)(1 + 2t)} = -\int \frac{dt}{(1 - t)(1 + t)(1 + 2t)}$$

$$= -\int \left[\frac{1}{6(1 - t)} - \frac{1}{2(1 + t)} + \frac{4}{3(1 + 2t)}\right] dt, \qquad \text{[by partial fractions]}$$

$$= \frac{1}{6} \log (1 - t) + \frac{1}{2} \log (1 + t) - \frac{2}{3} \log (1 + 2t)$$

$$= \frac{1}{6} \log (1 - \cos x) + \frac{1}{2} \log (1 + \cos x) - \frac{2}{3} \log (1 + 2 \cos x).$$

Example 14: *Evaluate* $\int_0^{\pi/4} \frac{dx}{\cos^4 x + \cos^2 x \sin^2 x + \sin^4 x}$.

Solution: Let $I = \int_0^{\pi/4} \frac{dx}{\cos^4 x + \cos^2 x \sin^2 x + \sin^4 x}$

$$= \int_0^{\pi/4} \frac{\sec^4 x \, dx}{1 + \tan^2 x + \tan^4 x},$$

dividing the numerator and the denominator by $\cos^4 x$

$$= \int_0^{\pi/4} \frac{(1 + \tan^2 x) \sec^2 x \, dx}{\tan^4 x + \tan^2 x + 1}, \qquad [\because 1 + \tan^2 x = \sec^2 x].$$

Now put $\tan x = t$ so that $\sec^2 x \, dx = dt$.

Also when $x = 0$, $t = \tan 0 = 0$ and when $x = \pi / 4$, $t = \tan \frac{1}{4} \pi = 1$.

$$\therefore \quad I = \int_0^1 \frac{1 + t^2}{t^4 + t^2 + 1} dt, \qquad \text{[Note the form of the integrand]}$$

$$= \int_0^1 \frac{[1 + (1 / t^2)] \, dt}{t^2 + 1 + (1 / t^2)},$$

dividing the numerator and the denominator by t^2

$$= \int_0^1 \frac{[1 + (1 / t^2)] \, dt}{\{t - (1 / t)\}^2 + 3}. \qquad \textbf{(Note)}$$

Now put $t - (1 / t) = y$ so that $\{1 + (1 / t^2)\} \, dt = dy$.

Also when $t = 0$, $y = -\infty$ and when $t = 1$, $y = 0$.

$$\therefore \quad I = \int_{-\infty}^0 \frac{dy}{y^2 + 3} = \frac{1}{\sqrt{3}} \left[\tan^{-1} \frac{y}{\sqrt{3}} \right]_{-\infty}^0$$

$$= (1 / \sqrt{3}) [\tan^{-1} 0 - \tan^{-1} (-\infty)]$$

$$= \frac{1}{\sqrt{3}} \left[0 - \left(-\frac{1}{2} \pi \right) \right] = \frac{\pi}{2\sqrt{3}} = \frac{\sqrt{3}}{6} \pi.$$

Example 15: *Evaluate* $\int x^2 \log (1 - x^2) \, dx$ *and deduce that*

$$\frac{1}{1.5} + \frac{1}{2.7} + \frac{1}{3.9} + \ldots = \frac{8}{2} - \frac{2}{3} \log 2.$$

Solution: Integrating by parts regarding x^2 as the second function, we get

$$\int \{\log(1-x^2)\}.x^2\,dx = \{\log(1-x^2)\}\cdot\frac{x^3}{3} - \int \frac{-2x\,.\,x^3}{(1-x^2)\,3}\,dx$$

$$= \frac{1}{3}x^3\log(1-x^2) + \frac{2}{3}\int \frac{1-(1-x^4)}{(1-x^2)}\,dx \qquad \textbf{(Note)}$$

$$= \frac{1}{3}x^3\log(1-x^2) + \frac{2}{3}\int \frac{dx}{1-x^2} - \frac{2}{3}\int (1+x^2)\,dx$$

$$= \frac{1}{3}x^3\log(1+x) + \frac{1}{3}x^3\log(1-x) + \frac{1}{3}\log\{(1+x)/(1-x)\} - \frac{2}{3}\{x+(x^3/3)\}$$

$$= \frac{1}{3}(x^3+1)\log(1+x) + \frac{1}{3}(x^3-1)\log(1-x) - \frac{2}{3}\{x+(x^3/3)\}.$$

$$\therefore \quad \int_0^1 x^2\log(1-x^2)\,dx = \left[\frac{1}{3}(x^3+1)\log(1+x) + \frac{1}{3}(x^3-1)\log(1-x) - \frac{2}{3}\{x+(x^3/3)\}\right]_0^1$$

$$= \frac{2}{3}\log 2 - \frac{8}{9}. \qquad ...(1)$$

Note that $\lim_{x\to 1}(x^3-1)\log(1-x)$

$$= \lim_{x\to 1}(x^2+x+1)\cdot\lim_{x\to 1}(x-1)\log(1-x)$$

$$= 3\cdot\lim_{x\to 1}\frac{\log(1-x)}{1/(x-1)}, \qquad \left[\text{form } \frac{\infty}{\infty}\right]$$

$$= 3\cdot\lim_{x\to 1}\frac{-1/(1-x)}{-1/(x-1)^2} = 3\cdot\lim_{x\to 1}(1-x) = 0.$$

Again $\displaystyle\int_0^1 x^2\log(1-x^2)\,dx = \int_0^1 x^2\left(-x^2-\frac{x^4}{2}-\frac{x^6}{3}-\ldots\right)dx$

$$= -\int_0^1\left(x^4+\frac{x^6}{2}+\frac{x^8}{3}+\ldots\right)dx$$

$$= -\left[\frac{x^5}{5}+\frac{x^7}{2.7}+\frac{x^9}{3.9}+\ldots\right]_0^1 = -\left[\frac{1}{1.5}+\frac{1}{2.1}+\frac{1}{3.9}+\ldots\right] \qquad ...(2)$$

Equating the two values of the given integral from (1) and (2), we get

$$\frac{1}{1.5}+\frac{1}{2.7}+\frac{1}{3.9}+\ldots = \frac{8}{9}-\frac{2}{3}\log 2.$$

Example 16: *Evaluate* $\displaystyle\int_0^1 \frac{x^2+1}{x^4+x^2+1}\,dx$, *and deduce that*

$$1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \ldots = \frac{\pi}{2\sqrt{3}}.$$

Solution: Let $I = \int \frac{x^2 + 1}{x^4 + x^2 + 1}\, dx = \int \frac{1 + (1/x^2)}{x^2 + 1 + (1/x^2)}\, dx.$

Now putting $x - (1/x) = z$ so that $\{1 + (1/x^2)\}\, dx = dz$

and $x^2 + 1 + (1/x^2) = \{x - (1/x)\}^2 + 3 = z^2 + 3$, we get

$$I = \int \frac{dz}{z^2 + 3} = \frac{1}{\sqrt{3}} \tan^{-1} \frac{z}{\sqrt{3}} = \frac{1}{\sqrt{3}} \tan^{-1} \frac{\{x - (1/x)\}}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} \frac{x^2 - 1}{x\sqrt{3}}.$$

$\therefore$
$$\int_0^1 \frac{x^2 + 1}{x^4 + x^2 + 1} = \frac{1}{\sqrt{3}} \left[\tan^{-1} \frac{x^2 - 1}{x\sqrt{3}} \right]_0^1$$

$$= \frac{1}{\sqrt{3}} [\tan^{-1} 0 - \tan^{-1}(-\infty)] = \frac{1}{\sqrt{3}} \left[0 - \left(-\frac{1}{2} \right) \pi \right] = \frac{\pi}{2\sqrt{3}} \quad \ldots(1)$$

Again
$$\int_0^1 \frac{1 + x^2}{1 + x^2 + x^4}\, dx = \int_0^1 \frac{(1 - x^4)\, dx}{(1 + x^2 + x^4)(1 - x^2)} \quad \textbf{(Note)}$$

$$= \int_0^1 \frac{1 - x^4}{1 - x^6}\, dx = \int_0^1 (1 - x^4)(1 - x^6)^{-1}\, dx$$

$$= \int_0^1 (1 - x^4)(1 + x^6 + x^{12} + \ldots)\, dx$$

$$= \int_0^1 (1 - x^4 + x^6 - x^{10} + x^{12} - \ldots)\, dx$$

$$= \left[x - \frac{x^5}{5} + \frac{x^7}{7} - \frac{x^{11}}{11} + \frac{x^{13}}{13} - \ldots \right]_0^1$$

$$= 1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \ldots \quad \ldots(2)$$

Comparing (1) and (2), we get

$$1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \ldots = \frac{\pi}{2\sqrt{3}}.$$

Comprehensive Exercise 4

1. Integrate $1/(1 + 3e^x + 2e^{2x})$.

2. Integrate $\dfrac{x^2-1}{x^4+1}$.

3. Integrate $\dfrac{x^2+1}{x^4-x^2+1}$.

4. Integrate $1/(x^4+8x^2+9)$.

5. Evaluate $\displaystyle\int \frac{dx}{x^4+x^2+1}$.

6. Integrate $1/\{x(x^2+1)^3\}$.

7. Integrate $1/(x^4+a^4)$.

8. Integrate $1/(x^4+1)$.

9. Integrate $1/\{x(x^n+1)\}$.

10. Evaluate $\displaystyle\int_1^2 \frac{dx}{x(1+2x)^2}$.

11. Evaluate $\displaystyle\int_0^{\pi/4} \sqrt{(\tan\theta)}\, d\theta$.

12. Evaluate $\displaystyle\int_0^{\pi/2} \frac{\cos x\, dx}{(1+\sin x)(2+\sin x)}$.

13. Evaluate $\displaystyle\int_0^{\pi/2} \frac{\cos x\, dx}{(1+\sin x)(2+\sin x)(3+\sin x)}$.

14. Integrate $(1+\sin x)/\{\sin x(1+\cos x)\}$.

15. Integrate $1/\{\sin x(3+\cos^2 x)\}$.

16. Integrate $\sec x/(1+\operatorname{cosec} x)$.

17. Show that

$$\int_0^\infty \frac{x^2\, dx}{(x^2+a^2)(x^2+b^2)(x^2+c^2)} = \frac{\pi}{2(a+b)(b+c)(c+a)}.$$

18. Show that the sum of the infinite series

$$\frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2b} - \frac{1}{a+3b} + \dots, (a>0, b>0)$$

can be expressed in the form $\displaystyle\int_0^1 \frac{t^{a-1}\, dt}{1+t^b}$ and hence prove that

$$1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \frac{1}{13} - \frac{1}{16} + \dots = \frac{1}{3}\left[\frac{\pi}{\sqrt{3}} + \log 2\right].$$

Answers 4

1. $\log(1+e^x) - 2\log(1+2e^x) + x$

2. $\dfrac{1}{2\sqrt{2}} \log \dfrac{x^2 - \sqrt{2}\,x + 1}{x^2 + \sqrt{2}\,x + 1}$

3. $\tan^{-1}\{x - (1/x)\}$

4. $\dfrac{1}{6\sqrt{14}} \tan^{-1} \dfrac{x^2 - 3}{x\sqrt{14}} - \dfrac{1}{6\sqrt{2}} \tan^{-1} \dfrac{x^2 + 3}{x\sqrt{2}}$

5. $\dfrac{1}{2\sqrt{3}} \tan^{-1} \dfrac{x^2 - 1}{\sqrt{3}x} - \dfrac{1}{4} \log \dfrac{x^2 - x + 1}{x^2 + x + 1}$

6. $\dfrac{1}{4(x^2+1)^2} + \dfrac{1}{2(x^2+1)} + \dfrac{1}{2} \log \dfrac{x^2}{x^2+1}$

7. $\dfrac{1}{2a^3\sqrt{2}} \tan^{-1} \left(\dfrac{x^2 - a^2}{xa\sqrt{2}} \right) - \dfrac{1}{4a^3\sqrt{2}} \log \dfrac{x^2 - \sqrt{2}ax + a^2}{x^2 + \sqrt{2}ax + a^2}$

8. $\dfrac{1}{2\sqrt{2}} \tan^{-1} \dfrac{x^2 - 1}{x\sqrt{2}} - \dfrac{1}{4\sqrt{2}} \log \dfrac{x^2 - x\sqrt{2} + 1}{x^2 + x\sqrt{2} + 1}$

9. $\dfrac{1}{n} \log \left(\dfrac{x^n}{x^n + 1} \right)$

10. $-\dfrac{2}{15} + \log\left(\dfrac{6}{5}\right)$

11. $\dfrac{\pi\sqrt{2}}{4} + \dfrac{\sqrt{2}}{2} \log(\sqrt{2} - 1)$

12. $\log(4/3)$

13. $\dfrac{5}{2}\log 2 - \dfrac{3}{2}\log 3$

14. $-\dfrac{1}{2}\log\left(\cot \dfrac{1}{2}x\right) + \dfrac{1}{4}\sec^2 \dfrac{1}{2}x + \tan\dfrac{1}{2}x$

15. $\dfrac{1}{4}\log\left(\tan\dfrac{1}{2}x\right) - \{1/(4/\sqrt{3})\}\tan^{-1}\{(\cos x)/\sqrt{3}\}$

16. $\dfrac{1}{4}\log\{(1+\sin x)/(1-\sin x)\} + 1/\{2(1+\sin x)\}$.

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. To evaluate the integral of $1/(ax^2+bx+c)$, we put the denominator in the form

(a) $b\{(x+\alpha)^2 \pm \beta^2\}$ (b) $a\{(x+\alpha)^2 \pm \beta^2\}$

(c) $c\{(x+\alpha)^2 \pm \beta^2\}$ (d) None of these

2. The value of $\int \frac{dx}{(9x^2-12x+8)}$ is

(a) $\frac{1}{6}\tan^{-1}\frac{3x-2}{2}$ (b) $6\tan^{-1}\frac{3x-2}{2}$

(c) $\frac{1}{6}\tan^{-1}\frac{3x+2}{2}$ (d) None of these

3. The value of $\int_0^{\pi/2} \frac{\cos x\,dx}{(1+\sin x)(2+\sin x)}$ is

(a) $\log 4$ (b) $\log\frac{3}{4}$

(c) $\log\frac{4}{3}$ (d) $\log 3$

Fill in the Blank(s)

Fill in the blanks "......" so that the following statements are complete and correct.

1. The value of the integral $\int \frac{x^2+1}{x^4-x^2+1}\,dx$ is ....

2. The value of the integral $\int \frac{dx}{x(x^5+1)}$ is ....

3. The value of the integral $\int \frac{dx}{(2x^2+x-1)}$ is ....

True or False

Write 'T' for true and 'F' for false statement.

1. The integral $\int \frac{dx}{(x^2+k)^n}$ is evaluated by the method of successive reduction.

2. The value of the $\int \frac{(x^2-1)}{(x^4+x^2+1)}\,dx$ is $2\log\frac{x^2-x+1}{x^2+x+1}$.

3. The value of the $\int \frac{dx}{\{x(x^n+1)\}}$ is $\frac{1}{n}\log\left(\frac{x^n}{x^n+1}\right)$.

Answers

Multiple Choice Questions

1. (b) **2.** (a) **3.** (c)

Fill in the Blank(s)

1. $\tan^{-1}\left\{x-\frac{1}{x}\right\}$

2. $\frac{1}{5}\log\left\{\frac{x^5}{x^5+1}\right\}$

3. $\frac{1}{3}\log\left\{\frac{(2x-1)}{(x-1)}\right\}$

True or False

1. T **2.** F **3.** T

□

Chapter

3

Integration of Irrational Functions

3.1 Integration by Rationalization

In many problems rationalization is brought about by multiplying a similar quantity both in numerator and denominator. Sometimes this quantity may differ in sign.

Illustrative Examples

Example 1: *Evaluate* $\int \sqrt{\left(\frac{1+x}{1-x}\right)}\, dx.$

Solution: Multiplying the numerator and the denominator by $\sqrt{(1+x)}$,we have the given integral

$$= \int \frac{\sqrt{(1+x)}}{\sqrt{(1-x)}} \cdot \frac{\sqrt{(1+x)}}{\sqrt{(1+x)}}\, dx = \int \frac{(1+x)}{\sqrt{(1-x^2)}}\, dx$$

$$= \int \frac{1.\, dx}{\sqrt{(1-x^2)}} + \int \frac{x.\, dx}{\sqrt{(1-x^2)}} = \sin^{-1} x + \int \frac{x\, dx}{\sqrt{(1-x^2)}}$$

$$= \sin^{-1} x - \frac{1}{2} \int (1-x^2)^{-1/2} (-2x)\, dx$$

$$= (\sin^{-1} x) - \frac{1}{2}\{(1-x^2)^{1/2}\}/(1/2), \quad \text{[By power formula]}$$

$$= \sin^{-1} x - \sqrt{(1-x^2)}.$$

Example 2: *Integrate* $1/[x+\sqrt{(x^2-1)}]$.

Solution: Rationalizing the denominator, we have

$$\int \frac{dx}{[x+\sqrt{(x^2-1)}]} = \int \frac{\{x-\sqrt{(x^2-1)}\}}{x^2-(x^2-1)}\,dx$$

$$= \int [x-\sqrt{(x^2-1)}]\,dx = \int x\,dx - \int \sqrt{(x^2-1)}\,dx$$

$$= \frac{1}{2}x^2 - \frac{1}{2}x\sqrt{(x^2-1)} + \frac{1}{2}\log\{x+\sqrt{(x^2-1)}\}.$$

3.2 Integration of $\dfrac{1}{(ax+b)\sqrt{(cx+d)}}$.

In such problems, put $cx+d=t^2$, so that $c\,dx = 2t\,dt$; then the fraction reduces to a form which can be easily integrated.

Illustrative Examples

Example 3: *Integrate* $1/[(x+2)\sqrt{(x+1)}]$.

Solution: Put $(x+1)=t^2$, so that $dx = 2t\,dt$.

$$\therefore \quad \int \frac{dx}{(x+2)\sqrt{(x+1)}} = \int \frac{2t\,dt}{(t^2+1).t} = 2\int \frac{dt}{t^2+1}$$

$$= 2\tan^{-1} t = 2\tan^{-1}[\sqrt{(x+1)}].$$

Example 4: *Evaluate* $\displaystyle\int \frac{x^2\,dx}{(x-1)\sqrt{(x+2)}}$.

Solution: Put $(x+2)=t^2$, so that $dx=2t\,dt$. Also $x=t^2-2$.

$$\therefore \quad \int \frac{x^2\,dx}{(x-1)\sqrt{(x+2)}} = \int \frac{(t^2-2)^2.2\,t\,dt}{(t^2-3).t} = 2\int \frac{t^4-4t^2+4}{t^2-3}\,dt$$

$$= 2\int [t^2-1+\{1/(t^2-3)\}]\,dt,$$

dividing the numerator by the denominator

$$\Rightarrow \qquad \frac{1}{\sqrt{(n+1)}} < \frac{1}{\sqrt{n}} \quad \Rightarrow \quad u_{n+1} < u_n \text{ for all } n.$$

Again $\qquad \lim u_n = \lim \dfrac{1}{\sqrt{n}} = 0.$

Hence by Leibnitz's test, the given series $\Sigma\,[(-1)^{n-1}/\sqrt{n}]$ is convergent.

Now $\Sigma \left| \dfrac{(-1)^{n-1}}{\sqrt{n}} \right| = \Sigma \dfrac{1}{\sqrt{n}}$ is divergent because $\Sigma\,(1/n^p)$ is divergent if $p \le 1$ and here $p = \dfrac{1}{2}\cdot$

Hence the given series is semi-convergent or conditionally convergent.

Example 43: *Examine the convergence and absolute convergence of the series*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, n}{n^2+1}\cdot \qquad \textbf{(Kashi 2013)}$$

Solution: Obviously the given series is an alternating series

$$u_1 - u_2 + u_3 - u_4 + \ldots, u_n > 0 \text{ for all } n.$$

Here $u_n = \dfrac{n}{n^2+1} > 0$ for all n.

Also $u_{n+1} - u_n = \dfrac{n+1}{(n+1)^2+1} - \dfrac{n}{n^2+1} = \dfrac{-n^2-n+1}{(n^2+1)\,[(n+1)^2+1]} < 0$ for all n.

Thus $u_{n+1} < u_n$ for all n.

Again $\quad \lim u_n = \lim \dfrac{n}{n^2+1} = \lim \dfrac{1}{n\,[1+(1/n^2)]} = 0.$

Hence by Leibnitz's test, the given series converges.

Now we shall test the given series for absolute convergence.

Consider the series $\Sigma\, u_n'$ of positive terms, where

$$u_n' = \left| \frac{(-1)^{n+1}\, n}{n^2+1} \right| = \frac{n}{n^2+1} = \frac{1}{n\,[1+(1/n^2)]}\cdot$$

Take $v_n = \dfrac{1}{n}\cdot$ Then $\lim \dfrac{u_n'}{v_n} = \lim \dfrac{1}{1+(1/n^2)} = 1$ which is finite and non-zero.

Hence by comparison test $\Sigma u_n'$ and $\Sigma\, v_n$ are either both convergent or both divergent. But for v_n, $p = 1$ so that $\Sigma\, v_n$ is divergent. Hence $\Sigma\, u_n'$ is divergent.

Hence the given series is not absolutely convergent *i.e.*, it is conditionally convergent.

Example 44: *Show that the series* $\Sigma\,(-1)^n\,[\sqrt{(n^2+1)} - n]$ *is conditionally convergent.*

Solution: The given series is an alternating series $\Sigma\,(-1)^n\, u_n, u_n > 0$ for all n.

Here $\qquad u_n = \sqrt{(n^2+1)} - n = \dfrac{[\sqrt{(n^2+1)} - n]\,[\sqrt{(n^2+1)} + n]}{\sqrt{(n^2+1)} + n}$

$$= 2\left[\frac{1}{3}t^3 - t + \{1/(2\sqrt{3})\}\log\{(t-\sqrt{3})/(t+\sqrt{3})\}\right]$$

$$= 2\left[\frac{(x+2)^{3/2}}{3} - \sqrt{(x+2)} + \frac{1}{2\sqrt{3}}\log\frac{\sqrt{(x+2)}-\sqrt{3}}{\sqrt{(x+2)}+\sqrt{3}}\right].$$

3.3 Integration of $1/\{(ax^2 + bx + c)\sqrt{(Ax + B)}\}$.

Such fractions are integrated by putting $Ax + B = t^2$.

Illustrative Examples

Example 5: *Integrate* $1/\{(x^2 - 4)\sqrt{(x+1)}\}$.

Solution: Put $x + 1 = t^2$, so that $dx = 2t\,dt$. Also $x^2 = t^2 - 1$.

$$\therefore \quad \int \frac{dx}{(x^2-4)\sqrt{(x+1)}} = \int \frac{2t\,dt}{\{(t^2-1)^2 - 2^2\}\,t}$$

$$= 2\int \frac{dt}{(t^2-1+2)(t^2-1-2)} = 2\int \frac{dt}{(t^2+1)(t^2-3)}$$

$$= \frac{1}{2}\int\left[\frac{1}{(t^2-3)} - \frac{1}{(t^2+1)}\right] dt \text{, by partial fractions}$$

$$= \frac{1}{2}\int \frac{dt}{t^2-3} - \frac{1}{2}\int \frac{dt}{t^2+1}$$

$$= \frac{1}{2}\cdot\frac{1}{2\sqrt{3}}\log\left\{\frac{t-\sqrt{3}}{t+\sqrt{3}}\right\} - \frac{1}{2}\tan^{-1} t$$

$$= \frac{1}{4\sqrt{3}}\log\left\{\frac{\sqrt{(x+1)}-\sqrt{3}}{\sqrt{(x+1)}+\sqrt{3}}\right\} - \frac{1}{2}\tan^{-1}\{\sqrt{(x+1)}\}.$$

Comprehensive Exercise 1

1. Integrate $\sqrt{x}/(1+x)$.
2. Integrate $\sqrt{[(x-1)/(x+1)]}$.
3. Evaluate $\displaystyle\int \frac{dx}{\sqrt{(1+x)} + \sqrt{x}}$.
4. Evaluate $\displaystyle\int \frac{dx}{\sqrt{(x+a)} + \sqrt{(x+b)}}$.
5. Evaluate $\displaystyle\int \frac{1}{x}\sqrt{\left(\frac{x-1}{x+1}\right)}\,dx$.

6. Evaluate $\int \frac{dx}{(x+2)\sqrt{(x+3)}}$.

7. Evaluate $\int \frac{dx}{(x+2)\sqrt{(x-1)}}$.

8. Integrate $1/[(x-3)\sqrt{(x+2)}]$.

9. Integrate $1/[(2x+1)\sqrt{(4x+3)}]$.

10. Evaluate $\int \frac{x+1}{(x-1)\sqrt{(x+2)}}\,dx$.

11. Integrate $x^3/\{(x-1)\sqrt{(x-2)}\}$.

12. Integrate $1/\{(x^2+1)\sqrt{x}\}$.

13. Integrate $1/\{x^2\sqrt{(x+1)}\}$.

14. Integrate $(x+2)/\{(x^2+3x+3)\sqrt{(x+1)}\}$.

Answers 1

1. $2\sqrt{x} - 2\tan^{-1}\sqrt{x}$
2. $\sqrt{(x^2-1)} - \cosh^{-1} x$
3. $\frac{2}{3}\cdot(1+x)^{3/2} - \frac{2}{3}x^{3/2}$
4. $\frac{2}{3}\frac{1}{(b-a)}[(x+b)^{3/2} - (x+a)^{3/2}]$
5. $\cosh^{-1} x - \sec^{-1} x$
6. $\log \frac{\sqrt{(x+3)}-1}{\sqrt{(x+3)}+1}$
7. $\frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{\sqrt{(x-1)}}{\sqrt{3}}\right]$
8. $\frac{1}{\sqrt{5}}\log\left\{\frac{\sqrt{(x+2)}-\sqrt{5}}{\sqrt{(x+2)}+\sqrt{5}}\right\}$
9. $\frac{1}{2}\log\frac{\sqrt{(4x+3)}-1}{\sqrt{(4x+3)}+1}$
10. $2\left[\sqrt{(x+2)} + \frac{1}{\sqrt{3}}\log\frac{\sqrt{(x+2)}-\sqrt{3}}{\sqrt{(x+2)}+\sqrt{3}}\right]$
11. $\frac{2}{5}(x-2)^{5/2} + \frac{10}{3}(x-2)^{3/2} - 6(x-2)^{1/2} + 22\tan^{-1}\{\sqrt{(x-2)}\}$
12. $\frac{1}{\sqrt{2}}\tan^{-1}\frac{x-1}{\sqrt{(2x)}} - \frac{1}{2\sqrt{2}}\log\left[\frac{x-\sqrt{(2x)}+1}{x+\sqrt{(2x)}+1}\right]$

13. $-\dfrac{\sqrt{(x+1)}}{x}+\dfrac{1}{2}\log\left\{\dfrac{\sqrt{(x+1)}+1}{\sqrt{(x+1)}-1}\right\}$

14. $\dfrac{2}{\sqrt{3}}\tan^{-1}\left[\dfrac{x}{\sqrt{\{3(x+1)\}}}\right]$

3.4 Integration of $1/\sqrt{(ax^2+bx+c)}$

We can express ax^2+bx+c as $a\{x^2+(b/a)x+(c/a)\}$

or $$a\left\{\left(x+\frac{b}{2a}\right)^2+\frac{c}{a}-\frac{b^2}{4a^2}\right\}$$

or $$a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right].$$

This is of the form $a\{(x+\alpha)^2\pm\beta^2\}$.

Thus the given integral can be reduced to one of the standard forms

$$\int\frac{dx}{\sqrt{(x^2+a^2)}},\int\frac{dx}{(x^2-a^2)}\text{ or }\int\frac{dx}{\sqrt{(a^2-x^2)}}.$$

So it can be easily evaluated.

Illustrative Examples

Example 6: *Integrate* $1/\sqrt{(2+x-3x^2)}$.

Solution: We have $\displaystyle\int\frac{dx}{\sqrt{(2+x-3x^2)}}=\frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{\left\{\frac{2}{3}-\left(x^2-\frac{1}{3}x\right)\right\}}}$

$$=\frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{\left\{\frac{2}{3}+\frac{1}{36}-\left(x^2-\frac{1}{3}x+\frac{1}{36}\right)\right\}}}=\frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{\left\{\frac{25}{36}-\left(x-\frac{1}{6}\right)^2\right\}}}$$

$$=\frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{\left\{\left(\frac{5}{6}\right)^2-\left(x-\frac{1}{6}\right)^2\right\}}}$$

$$=\frac{1}{\sqrt{3}}\sin^{-1}\left\{\frac{\left(x-\frac{1}{6}\right)}{5/6}\right\}=\frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{6x-1}{5}\right).$$

Note: Remember that $\displaystyle\int\frac{dx}{\sqrt{\{a^2-(x-b)^2\}}}=\sin^{-1}\left(\frac{x-b}{a}\right)$.

Example 7: *Integrate* $1/\sqrt{(1-x-x^2)}$.

Solution: We have $\displaystyle\int \frac{dx}{\sqrt{(1-x-x^2)}} = \int \frac{dx}{\sqrt{\{1-(x^2+x)\}}}$

$$= \int \frac{dx}{\sqrt{\left\{1-\left(x+\frac{1}{2}\right)^2+\frac{1}{4}\right\}}}$$

$$= \int \frac{dx}{\sqrt{\left\{\frac{5}{4}-\left(x+\frac{1}{2}\right)^2\right\}}} = \sin^{-1}\left\{\frac{x+\frac{1}{2}}{\frac{1}{2}\sqrt{5}}\right\} = \sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right).$$

3.5 Integration of $\sqrt{(ax^2+bx+c)}$

$\sqrt{(ax^2+bx+c)}$ can be integrated by reducing ax^2+bx+c to the form $a\{(x+\alpha)^2 \pm \beta^2\}$. The given integral can then be easily evaluated by applying one of the following standard results.

$$\int \sqrt{(x^2+a^2)}\,dx = \frac{1}{2}x\,(x^2+a^2) + \frac{1}{2}a^2 \sinh^{-1}(x/a)$$

$$= \frac{1}{2}x\sqrt{(x^2+a^2)} + \frac{1}{2}a^2 \log\{x+\sqrt{(x^2+a^2)}\};$$

$$\int \sqrt{(x^2-a^2)}\,dx = \frac{1}{2}x\sqrt{(x^2-a^2)} - \frac{1}{2}a^2 \cosh^{-1}(x/a)$$

$$= \frac{1}{2}x\sqrt{(x^2-a^2)} - \frac{1}{2}a^2 \log[x+\sqrt{(x^2-a^2)}];$$

and $$\int \sqrt{(a^2-x^2)}\,dx = \frac{1}{2}x\sqrt{(a^2-x^2)} + \frac{1}{2}a^2 \sin^{-1}(x/a).$$

Note that in each result the sign before a^2 in the second term is the same as the sign before a^2 in the expression under the radical sign.

Illustrative Examples

Example 8: *Integrate* $\sqrt{(x^2-x+1)}$.

Solution: We have $\displaystyle\int \sqrt{(x^2-x+1)}\,dx = \int \left\{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}\right\} dx$, [form $\displaystyle\int (x^2+a^2)\,dx$]

$$= \frac{1}{2}\left(x-\frac{1}{2}\right)\sqrt{\left\{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}\right\}} + \frac{1}{2}\cdot\left(\frac{3}{4}\right)\sinh^{-1}\left\{\left(x-\frac{1}{2}\right)\Big/\left(\frac{1}{2}\sqrt{3}\right)\right\}$$

$$= \frac{1}{2}\left(x-\frac{1}{2}\right)\sqrt{(x^2-x+1)} + \frac{3}{8}\sinh^{-1}\{(2x-1)/\sqrt{3}\}.$$

Example 9: *Integrate* $\sqrt{(4-3x-2x^2)}$.

Solution: We have $\int \sqrt{(4-3x-2x^2)}\,dx = \sqrt{2}\int \sqrt{(2-\frac{3}{2}x-x^2)}\,dx$

$$= \sqrt{2}\int \sqrt{\{2-(x^2+\tfrac{3}{2}x)\}}\,dx = \sqrt{2}\int \sqrt{\{2+\tfrac{9}{16}-(x^2+\tfrac{3}{2}x+\tfrac{9}{16})\}}\,dx$$

$$= \sqrt{2}\int \sqrt{\left\{\frac{41}{16}-\left(x+\frac{3}{4}\right)^2\right\}}\,dx, \qquad \left[\text{form} \int \sqrt{(a^2-x^2)}\,dx\right]$$

$$= \sqrt{2}\int \sqrt{\left(\frac{41}{16}-t^2\right)}\,dt, \text{ putting } x+\frac{3}{4}=t \text{ so that } dx=dt$$

$$= \sqrt{2}\cdot\frac{t}{2}\sqrt{\left(\frac{41}{16}-t^2\right)}+\sqrt{2}\cdot\frac{41}{32}\sin^{-1}\{t/(\sqrt{41}/4)\}$$

$$= \sqrt{2}\cdot\frac{x+\frac{3}{4}}{2}\sqrt{\left\{\frac{41}{16}-\left(x+\frac{3}{4}\right)^2\right\}}+\sqrt{2}\cdot\frac{41}{32}\sin^{-1}\left\{\frac{x+\frac{3}{4}}{\sqrt{41}/4}\right\}$$

$$= \frac{4x+3}{8}\sqrt{(4-3x-2x^2)}+\frac{41\sqrt{2}}{32}\sin^{-1}\left(\frac{4x+3}{\sqrt{41}}\right).$$

Comprehensive Exercise 2

1. Integrate $1/\sqrt{(2x^2-x+2)}$.
2. Integrate $1/\sqrt{(2x^2+3x+4)}$.
3. Integrate $\dfrac{1}{\sqrt{(x^2+x+1)}}$.
4. Integrate $1/\sqrt{(4+3x-2x^2)}$.
5. Integrate $1/\sqrt{(3x-x^2-2)}$.
6. Evaluate $\int \sqrt{\{(x-1)(2-x)\}}\,dx$.

Answers 2

1. $\dfrac{1}{\sqrt{2}}\sinh^{-1}\left[\dfrac{4x-1}{\sqrt{15}}\right]$
2. $\dfrac{1}{\sqrt{2}}\sinh^{-1}\left\{\dfrac{4x+3}{\sqrt{23}}\right\}$
3. $\sinh^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right)$
4. $\dfrac{1}{\sqrt{2}}\sin^{-1}\left(\dfrac{4x-3}{\sqrt{41}}\right)$
5. $\sin^{-1}(2x-3)$
6. $\dfrac{1}{4}(2x-3)\sqrt{(3x-x^2-2)}+\dfrac{1}{8}\sin^{-1}(2x-3)$

Chapter

5

Reduction Formulae (For Trigonometric Functions)

5.1 Reduction Formulae

A reduction formula is a formula which connects an integral, which cannot otherwise be evaluated, with another integral of the same type but of lower degree. It is generally obtained by applying the rule of integration by parts.

5.2 Reduction Formulae for $\int \sin^n x\, dx$ and $\int \cos^n x\, dx$, n being a +ive integer

(a) Let $I_n = \int \sin^n x\, dx$ or $I_n = \int \sin^{n-1} x \sin x\, dx$. **(Note)**

Integrating by parts regarding sin x as the 2nd function, we have

$$I_n = \sin^{n-1} x\,.(-\cos x) - \int (n-1) \sin^{n-2} x\,.\cos x\,.(-\cos x)\, dx$$

$$= -\sin^{n-1} x\,.\cos x + (n-1) \int \sin^{n-2} x\,.\cos^2 x\, dx$$

$$= -\sin^{n-1} x \,.\cos x + (n-1)\int \sin^{n-2} x \,.(1-\sin^2 x)\, dx \qquad \textbf{(Note)}$$

$$= -\sin^{n-1} x \,.\cos x + (n-1)\int \sin^{n-2} x\, dx - (n-1)\int \sin^n x\, dx$$

$$= -\sin^{n-1} x \,.\cos x + (n-1)\int \sin^{n-2} x\, dx - (n-1)\, I_n.$$

Transposing the last term to the left, we have

$$I_n (1+n-1) = -\sin^{n-1} x \,.\cos x + (n-1)\, I_{n-2},$$

$$[\because I_{n-2} = \int \sin^{n-2} x\, dx]$$

or $$nI_n = -\sin^{n-1} x \cos x + (n-1)\, I_{n-2}$$

or $$I_n = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}\,.$$

$\therefore$ $$\int \sin^n x\, dx = -\frac{1}{n}\sin^{n-1} x \,.\, \cos x + \frac{n-1}{n}\int \sin^{n-2} x\, dx.$$

(Bundelkhand 2008; Agra 2014)

(b) Let $I_n = \int \cos^n x\, dx$ or $I_n = \int \cos^{n-1} x \,.\cos x\, dx.$

Integrating by parts regarding $\cos x$ as the 2nd function, we have

$$I_n = \cos^{n-1} x \,.\sin x - \int (n-1) \cos^{n-2} x \,.(\sin x) \,.\sin x\, dx$$

$$= \cos^{n-1} x \,.\sin x + (n-1)\int \cos^{n-2} x \,.\sin^2 x\, dx$$

$$= \cos^{n-1} x \,.\sin x + (n-1)\int \cos^{n-2} x\, (1-\cos^2 x)\, dx$$

$$= \cos^{n-1} x \,.\sin x + (n-1)\int \cos^{n-2} x\, dx - (n-1)\int \cos^n x\, dx$$

$$= \cos^{n-1} x \sin x + (n-1)\, I_{n-2} - (n-1)\, I_n.$$

Transposing the last term to the left, we have

$$I_n (1+n-1) = \cos^{n-1} x \,.\sin x + (n-1)\, I_{n-2}$$

or $$n\, I_n = \cos^{n-1} x \,.\sin x + (n-1)\, I_{n-2}\,.$$

$\therefore$ $$\int \cos^n x\, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\, dx.$$

5.3 Walli's Formula

To evaluate $\int_0^{\pi/2} \sin^n x\, dx$ and $\int_0^{\pi/2} \cos^n x\, dx$.

Proceeding as in the previous article, we have

$$\int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$$

$$\therefore \quad \int_0^{\pi/2} \sin^n x\,dx = -\left[\frac{\sin^{n-1} x \cos x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

$$= 0 + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx. \qquad ...(1)$$

Putting $(n-2)$ in place of n in (1), we have

$$\int_0^{\pi/2} \sin^{n-2} x\,dx = \frac{n-3}{n-2}\int_0^{\pi/2} \sin^{n-4} x\,dx.$$

Substituting this value in (1), we have

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\int_0^{\pi/2} \sin^{n-4} x\,dx$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\int_0^{\pi/2} \sin^{n-6} x\,dx\,. \qquad ...(2)$$

Now two cases arise *viz*,, n is even or odd.

Case I: When n is odd.

In this case by the repeated application of the reduction formula (1), the last integral of (2) is

$$\int_0^{\pi/2} \sin x\,dx = [-\cos x]_0^{\pi/2} = 1.$$

Hence when n is odd, from (2), we have

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{2}{3}\int_0^{\pi/2} \sin x\,dx$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\cdots\cdot\frac{2}{3}\cdot 1$$

$$= \frac{(n-1)(n-3)\ldots\ldots 4.2}{n(n-2)\ldots 3.1}\cdot 1.$$

Case II. When n is even.

In this case the last integral of (2) is

$$\int_0^{\pi/2} \sin^0 x\,dx = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2}\cdot$$

Hence when n is even, from (2), we have

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\ldots\frac{3}{4}\cdot\frac{1}{2}\int_0^{\pi/2} \sin^0 x\,dx$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{(n-1)(n-3)\ldots 3.1}{n(n-2)\ldots 4.2} \cdot \frac{\pi}{2}.$$

If we evaluate $\int_0^{\pi/2} \cos^n x\, dx$, we get the same results.

$\therefore$ $$\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx.$$ **(Note)**

Note: Walli's formula is applicable only when the limits are from 0 to $\frac{1}{2}\pi$.

Illustrative Examples

Example 1: *Establish a reduction formula for* $\int \sin^n (2x)\, dx$.

Solution: Let $I_n = \int \sin^n (2x)\, dx$ or $I_n = \int \sin^{n-1} (2x) \sin (2x)\, dx$.

Integrating by parts regarding $\sin 2x$ as the 2nd function, we have

$$I_n = \sin^{n-1}(2x)\left[-\frac{1}{2}\cos 2x\right] - \int \{(n-1)\sin^{n-2} 2x \cdot \cos 2x .2\}.\left(-\frac{1}{2}\cos 2x\right) dx$$

$$= -\frac{1}{2}\sin^{n-1} 2x .\cos 2x + (n-1)\int \sin^{n-2} 2x .\cos^2 2x\, dx$$

$$= -\frac{1}{2}\sin^{n-1} 2x .\cos 2x + (n-1)\int \sin^{n-2} 2x .(1-\sin^2 2x)\, dx$$

$$= -\frac{1}{2}\sin^{n-1} 2x .\cos 2x + (n-1)\int \sin^{n-2} 2x\, dx - (n-1)\int \sin^n 2x\, dx$$

$$= -\frac{1}{2}\sin^{n-1} 2x .\cos 2x + (n-1) I_{n-2} - (n-1) I_n.$$

Transposing the last term to the left, we have

$$n I_n = -\frac{1}{2}\sin^{n-1} 2x .\cos 2x + (n-1) I_{n-2}$$

or $$I_n = -\frac{\sin^{n-1} 2x .\cos 2x}{2n} + \frac{n-1}{n} I_{n-2},$$ is the reduction formula.

Example 2: *Prove that* $\int_0^{\pi/2} \sin^{2m} x\, dx = \frac{(2m)!}{\{2^m . m!\}^2} \cdot \frac{\pi}{2}$.

Solution: Here $2m$ is even. Hence from article 5.3 (Case II), we get

$$\int_0^{\pi/2} \sin^{2m} x\, dx = \frac{(2m-1)(2m-3)\ldots 3.1}{(2m)(2m-2)\ldots 4.2} \cdot \frac{\pi}{2}$$ **(Walli's formula)**

$$= \frac{2m(2m-1)(2m-2)\ldots 3.2.1}{\{2m(2m-2)\ldots.4.2\}^2} \cdot \frac{\pi}{2}.$$

[Multiplying Nr. & Dr. by $2m(2m-2)(2m-4)\ldots 4.2$]

$$= \frac{(2m)!}{\{2^m . m(m-1)(m-2)\ldots\ldots 2.1\}^2} \cdot \frac{\pi}{2}$$

$$= \frac{(2m)!}{\{2^m . m!\}^2} \cdot \frac{\pi}{2}.$$

Example 3: *Evaluate* $\int_0^{2a} \frac{x^{9/2}\, dx}{\sqrt{(2a-x)}}$.

Solution: Put $x = 2a \sin^2 \theta$, so that $dx = 2a.2 \sin\theta \cos\theta\, d\theta$.

Also when $x = 0$, $\sin^2 \theta = 0$ *i.e.*, $\theta = 0$

and when $x = 2a$, $\sin^2 \theta = 1$ *i.e.*, $\theta = \pi/2$.

Then $$\int_0^{2a} \frac{x^{9/2}\, dx}{\sqrt{(2a-x)}} = \int_0^{\pi/2} \frac{(2a\sin^2\theta)^{9/2} . 4a \sin\theta\cos\theta\, d\theta}{\sqrt{(2a - 2a\sin^2\theta)}}$$

$$= \int_0^{\pi/2} \frac{(2a)^{9/2} . 4a \sin^{10}\theta . \cos\theta\, d\theta}{(2a)^{1/2} . \cos\theta}$$

$$= (2a)^4 . 4a \int_0^{\pi/2} \sin^{10}\theta\, d\theta$$

$$= 64\, a^5 \cdot \frac{9}{10} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{63\, a^5\, \pi}{8}.$$

5.4 Reduction Formulae for $\int \tan^n x\, dx$ and $\int \cot^n x\, dx$

(a) We have $\int \tan^n x\, dx = \int \tan^{n-2} x . \tan^2 x\, dx$ **(Note)**

$$= \int \tan^{n-2} x . (\sec^2 x - 1)\, dx$$

$$= \int \tan^{n-2} x . \sec^2 x\, dx - \int \tan^{n-2} x\, dx$$

$$= \frac{(\tan x)^{n-2+1}}{n-2+1} - \int \tan^{n-2} x\, dx$$

or $$\int \tan^n x\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx,$$

which is the required reduction formula.

Application: *Evaluate* $\int \tan^4 x\, dx$.

Putting $n = 4$ in the above reduction formula, we have

$$\int \tan^4 x\,dx = \frac{1}{3}\tan^3 x - \int \tan^2 x\,dx = \frac{1}{3}\tan^3 x - \int (\sec^2 x - 1)\,dx$$

$$= \frac{1}{3}\tan^3 x - \tan x + x.$$

(b) We have $\int \cot^n x\,dx = \int \cot^{n-2} x\,.\cot^2 x\,dx$

$$= \int \cot^{n-2} x\,.(\operatorname{cosec}^2 x - 1)\,dx$$

$$= \int \cot^{n-2} x\,.\operatorname{cosec}^2 x\,dx - \int \cot^{n-2} x\,dx$$

$$= -\frac{(\cot x)^{n-1}}{n-1} - \int \cot^{n-2} x\,dx$$

or $$\boldsymbol{\cot^n x\,dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\,dx,}$$

which is the required reduction formula.

Application: Putting $n = 5$ in the above reduction formula and applying it repeatedly, we have

$$\int \cot^5 x\,dx = -\frac{1}{4}\cot^4 x - \int \cot^3 x\,dx$$

$$= -\frac{1}{4}\cot^4 x - [-\frac{1}{2}\cot^2 x - \int \cot x\,dx]$$

$$= -\frac{1}{4}\cot^4 x + \frac{1}{2}\cot^2 x + \int \cot x\,dx$$

$$= -\frac{1}{4}\cot^4 x + \frac{1}{2}\cot^2 x + \log \sin x.$$

5.5 Reduction Formulae for $\int \sec^n x\,dx$ and $\int \operatorname{cosec}^n x\,dx$

(Bundelkhand 2011)

(a) We have $I_n = \int \sec^n x\,dx = \int \sec^{n-2} x\,.\sec^2 x\,dx.$ **(Note)**

Integrating by parts regarding $\sec^2 x$ as the 2nd function, we have

$$I_n = \sec^{n-2} x \tan x - \int (n-2)\sec^{n-3} x \sec x \tan^2 x\,dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x(\sec^2 x - 1)\,dx \qquad \textbf{(Note)}$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx.$$

Transposing the term containing $\int \sec^n x\,dx$ to the left, we have

$$(n-2+1)\int \sec^{n} x\,dx = \sec^{n-2} x \tan x + (n-2)\int \sec^{n-2} x\,dx$$

or $$(n-1)\int \sec^{n} x\,dx = \sec^{n-2} x \tan x + (n-2)\int \sec^{n-2} x\,dx.$$

Dividing both sides by $(n-1)$, we have

$$\int \sec^{n} x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx,$$

which is the required reduction formula.

(b) To find the reduction formula for $\int \operatorname{cosec}^{n} x\,dx$, proceed exactly in the same way as in part (a). Thus, we get

$$\int \operatorname{cosec}^{n} x\,dx = -\frac{\operatorname{cosec}^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1}\int \operatorname{cosec}^{n-2} x\,dx,$$

as the required reduction formula for $\int \operatorname{cosec}^{n} x\,dx$.

5.6 Reduction Formula for $\int \sin^{m} x \cos^{n} x\,dx$

(Kanpur 2014)

Let $$I_{m,n} = \int \sin^{m} x \cos^{n} x\,dx$$

$$= \int \sin^{m} x \cos^{n-1} x \cos x\,dx = \int \cos^{n-1} x\,.(\sin^{m} x \cos x)\,dx.$$

Integrating by parts taking $\sin^{m} x \cos x$ as the second function, we get

$$I_{m,n} = \frac{\sin^{m+1} x}{m+1}\cos^{n-1} x + \frac{n-1}{m+1}\int \sin^{m+1} x \cos^{n-2} x \sin x\,dx$$

$$= \frac{\sin^{m+1} x}{m+1}\cos^{n-1} x + \frac{n-1}{m+1}\int \sin^{m} x \cos^{n-2} x \sin^{2} x\,dx$$

$$= \frac{\sin^{m+1} x}{m+1}\cdot\cos^{n-1} x + \frac{n-1}{m+1}\int \sin^{m} x \cos^{n-2} x\,.(1-\cos^{2} x)\,dx$$

$$= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1}\int \sin^{m} x \cos^{n-2} x\,dx - \frac{n-1}{m+1} I_{m,n}.$$

Transposing the last term to the left, we have

$$I_{m,n}\left(1+\frac{n-1}{m+1}\right) = \frac{\sin^{m+1} x.\cos^{n-1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

or $$I_{m,n}\left(\frac{m+n}{m+1}\right) = \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}.$$

Thus the required reduction formula is

$$I_{m,n} = \frac{\sin^{m+1} x \cdot \cos^{n-1} x}{m+n} + \frac{(n-1)\, I_{m,n-2}}{m+n}.$$

Note: If we write $I_{m,n} = \int \sin^m x \cos^n x \, dx$

$$= \int \sin^{m-1} x \cdot (\cos^n x \sin x)\, dx,$$

then integrating by parts regarding $\cos^n x \sin x$ as the 2nd function, the reduction formula can be obtained as

$$\boldsymbol{I_{m,n} = -\frac{\sin^{m-1} x \cdot \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}}.$$

Similarly other four reduction formulae for $\int \sin^m x \cos^n x \, dx$ may be obtained as

$$I_{m,n} = -\frac{\sin^{m+1} x \cos^{n+1} x}{n+1} + \frac{m+n+2}{n+1} I_{m,n+2}.$$

[To obtain this reduction formula put $(n+2)$ in place of n in the reduction formula obtained in 5.6 and adjust the result accordingly]

$$I_{m,n} = \frac{\sin^{m+1} x \cos^{n+1} x}{m+1} + \frac{m+n+2}{m+1} I_{m+2,n}$$

$$I_{m,n} = -\frac{\sin^{m+1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} I_{m-2,n+2}$$

$$I_{m,n} = \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} I_{m+2,n-2}.$$

[This reduction formula has been obtained in 5.6 at the stage we applied integration by parts]

Illustrative Examples

Example 4: *Evaluate* $\int \frac{d\theta}{\sin^4 \frac{1}{2}\theta}$.

Solution: We have $\int \frac{d\theta}{\sin^4 \frac{1}{2}\theta} = \int \operatorname{cosec}^4 \frac{\theta}{2}\, d\theta$

$$= 2\int \operatorname{cosec}^4 x \, dx, \text{ putting } \theta = 2x.$$

But $\int \operatorname{cosec}^n x \, dx = -\frac{\operatorname{cosec}^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1}\int \operatorname{cosec}^{n-2} x \, dx.$

[Derive this formula here]

Putting $n = 4$, we get

$$\int \operatorname{cosec}^4 x\,dx = -\frac{\operatorname{cosec}^2 x \cot x}{3} + \frac{2}{3}\int \operatorname{cosec}^2 x\,dx$$

$$= -\frac{1}{3}\operatorname{cosec}^2 x \cot x + \frac{2}{3}(-\cot x).$$

Hence the given integral

$$= 2\int \operatorname{cosec}^4 x\,dx = -\frac{2}{3}\operatorname{cosec}^2 x \cot x - \frac{4}{3}\cot x$$

$$= -\frac{2}{3}\operatorname{cosec}^2 \frac{1}{2}\theta \cot \frac{1}{2}\theta - \frac{4}{3}\cot \frac{1}{2}\theta. \qquad [\because \ x = \theta/2]$$

Example 5: *Evaluate* $\int (1 + x^2)^{3/2}\,dx.$

Solution: Put $x = \tan\theta$, so that $dx = \sec^2\theta\,d\theta$.

Then $\int (1 + x^2)^{3/2}\,dx = \int \sec^2\theta \sec^3\theta\,d\theta = \int \sec^5\theta\,d\theta.$

Now we shall form a reduction formula for $\int \sec^n\theta\,d\theta.$

Proceeding as in article 5.5 (a), we get

$$\int \sec^n\theta\,d\theta = \frac{\sec^{n-2}\theta \tan\theta}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}\theta\,d\theta.$$

$$\therefore \quad \int \sec^5\theta\,d\theta = \frac{1}{4}\sec^3\theta \tan\theta + \frac{3}{4}\int \sec^3\theta\,d\theta$$

$$= \frac{1}{4}\sec^3\theta\tan\theta + \frac{3}{4}\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta\,d\theta\right]$$

$$= \frac{1}{4}\sec^3\theta\tan\theta + \frac{3}{8}\sec\theta\tan\theta + \frac{3}{8}\log(\sec\theta + \tan\theta)$$

$$= \frac{1}{4}[(1 + x^2)^{3/2} . x] + \frac{3}{8}x(1 + x^2)^{1/2} + \frac{3}{8}\log\{x + \sqrt{(1 + x^2)}\}.$$

Comprehensive Exercise 1

Evaluate the following integrals :

1. $\int \sin^6 x\,dx.$

2. $\int_0^{\pi/2} \sin^6 x\,dx.$

3. $\int_0^{\pi/2} \cos^9 x\,dx.$

4. $\int_0^{\pi/2} \cos^{10} x\,dx.$

5. $\int_0^{\pi/4} \tan^5\theta\,d\theta.$

6. $\int_0^{a} x^5 (2a^2 - x^2)^{-3}\,dx.$

7. $\int \sec^3 x \, dx.$

8. $\int_0^{\pi/4} \sec^3 x \, dx.$

9. $\int_0^a (a^2 + x^2)^{5/2} \, dx.$

10. $\int_0^{\pi/4} \sin^2 \theta \cos^4 \theta \, d\theta.$

11. $\int \tan^6 x \, dx.$

12. Show that $\int_0^a \frac{x^4}{\sqrt{(a^2 - x^2)}} dx = \frac{3a^4 \pi}{16}.$

13. If $I_n = \int_0^{\pi/4} \tan^n x \, dx$, show that $I_n + I_{n-2} = \frac{1}{n-1}$, and deduce the value of I_5.

(Kanpur 2005, 12; Avadh 06, 11; Bundelkhand 06; Purvanchal 14)

14. If $I_n = \int_0^{\pi/4} \tan^n x \, dx$, prove that $n(I_{n-1} + I_{n+1}) = 1$. **(Kanpur 2005; Avadh 06)**

Answers 1

1. $-\frac{1}{6}\sin^5 x \cos x - \frac{5}{24}\sin^3 x \cos x - \frac{5}{16}\sin x \cos x + \frac{5}{16}x$

2. $\frac{5\pi}{32}$ **3.** $\frac{128}{315}$ **4.** $\frac{63\pi}{512}$

5. $\frac{1}{2}\left[\log 2 - \frac{1}{2}\right]$ **6.** $\frac{1}{2}\left[\log 2 - \frac{1}{2}\right]$

7. $\frac{1}{2}\sec x \tan x + \frac{1}{2}\log(\sec x + \tan x)$

8. $\frac{1}{2}\sqrt{2} + \frac{1}{2}\log(\sqrt{2} + 1)$ **9.** $\frac{a^6}{48}\left[67\sqrt{2} + 15 \log \tan\left(\frac{3}{8}\pi\right)\right]$

10. $\frac{1}{48} + \frac{\pi}{64}$ **11.** $\frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x$

13. $\frac{1}{2}\left(\log 2 - \frac{1}{2}\right)$

5.7 Gamma Function

The definite integral $\int_0^{\infty} e^{-x} x^{n-1} \, dx$ is called the *second Eulerian integral* and is denoted by the symbol $\Gamma(n)$ [read as Gamma n].

Chapter

2

Improper Integrals (Infinite Integrals)

1 Some Definitions

1. **Infinite Interval:** The interval whose length (range) is infinite is said to be an *infinite interval*. Thus the intervals (a, ∞), $(-\infty, b)$ and $(-\infty, \infty)$ are infinite intervals.

2. **Bounded Functions:** A function $f(x)$ is said to be *bounded* over the interval I if there exist two real numbers a and b $(b > a)$ such that

$$a \le f(x) \le b \text{ for all } x \in I.$$

A function $f(x)$ is said to be unbounded at a point, if it becomes infinite at that point. Thus the function

$$f(x) = x / \{(x-1)(x-2)\}$$

is unbounded at each of the points $x = 1$ and $x = 2$.

3. **Monotonic functions:** A real valued function f defined on an interval I is said to be **monotonically** increasing if

$$x > y \Rightarrow f(x) > f(y) \ \forall \ x, y \in I$$

and **monotonically decreasing** if

$$x > y \Rightarrow f(x) < f(y) \ \forall \ x, y \in I.$$

A function f defined on an interval I is said to be a monotonic function if it is either monotonically decreasing or monotonically increasing on I.

For example the function f defined by $f(x) = \sin x$ is monotonically increasing in the interval $0 \le x \le \frac{1}{2}\pi$ and monotonically decreasing in the interval $\frac{1}{2}\pi \le x \le \pi$.

4. **Proper Integral:** The definite integral $\int_a^b f(x)\, dx$ is said to be a *proper integral* if the range of integration is finite and the integrand $f(x)$ is bounded. The integral $\int_0^{\pi/2} \sin x\, dx$ is a proper integral. Also $\int_0^1 \frac{\sin x}{x}\, dx$ is an example of a proper integral because $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

5. **Improper Integrals:** The definite integral $\int_a^b f(x)\, dx$ is said to be an *improper integral* if (*i*) the interval (a, b) is not finite (*i.e.*, is infinite) and the function $f(x)$ is bounded over this interval; or (*ii*) the interval (a, b) is finite and $f(x)$ is not bounded over this interval; or (*iii*) neither the interval (a, b) is finite nor $f(x)$ is bounded over it.

6. **Improper integrals of the first kind or infinite integrals:** A definite integral $\int_a^b f(x)\, dx$ in which the range of integration is infinite (*i.e.*, either $b = \infty$ or $a = -\infty$ or both) and the integrand $f(x)$ is bounded, is called an improper integral of the first kind or an infinite integral. Thus $\int_0^\infty \frac{dx}{1+x^2}$ is an improper integral of the first kind since the upper limit of integration is infinite and the integrand $1/(1+x^2)$ is bounded. Similarly $\int_{-\infty}^0 e^x\, dx$ is an example of an improper integral of the first kind because here the lower limit of integration is infinite. Also $\int_{-\infty}^\infty \frac{dx}{1+x^2}$ is an improper integral of the first kind.

In case the **interval (a, b) is infinite and the integrand $f(x)$ is bounded**, we define

(*i*) $\int_a^\infty f(x)\, dx = \lim_{x \to \infty} \int_a^x f(x)\, dx,$

provided that the limit exists finitely *i.e.*, the limit is equal to a definite real number.

(*ii*) $\int_{-\infty}^b f(x)\, dx = \lim_{x \to \infty} \int_{-x}^b f(x)\, dx,$

provided that the limit exists finitely.

(*iii*) $\int_{-\infty}^\infty f(x)\, dx = \lim_{x_1 \to \infty} \int_{-x_1}^c f(x)\, dx + \lim_{x_2 \to \infty} \int_c^{x_2} f(x)\, dx$

provided that both these limits exist finitely.

7. **Improper integrals of the second kind:** A definite integral $\int_a^b f(x)\, dx$ in which the range of integration is finite but the integrand $f(x)$ is unbounded at one or more points of the interval $a \le x \le b$, is called an improper integral of the second kind.

Thus $\int_0^4 \frac{dx}{(x-2)(x-3)}$

and $\int_0^1 \frac{1}{x^2}\,dx$ are improper integrals of the second kind.

In the case of the definite integral

$$\int_a^b f(x)\,dx,$$

if the range of integration (a, b) is finite and the integrand **$f(x)$ is unbounded at one or more points of the given interval,** we define the value of the integral as follows :

(i) If $f(x)$ is unbounded at $x = b$ only *i.e.*, if $f(x) \to \infty$ as $x \to b$ only, then we define

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \int_a^{b-\varepsilon} f(x)\,dx,$$

provided that the limit exists finitely. Here ε is a small positive number.

(ii) If $f(x) \to \infty$ as $x \to a$ only, then we define

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^{b} f(x)\,dx,$$

provided that the limit exists finitely.

(iii) If $f(x) \to \infty$ as $x \to c$ only, where $a < c < b$, then we define

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \int_a^{c-\varepsilon} f(x)\,dx + \lim_{\varepsilon' \to 0} \int_{c+\varepsilon'}^{b} f(x)\,dx,$$

provided that both these limits exist finitely.

(iv) If $f(x)$ is unbounded at both the points a and b of the interval (a, b) and is bounded at each other point of this interval, we write

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

where $a < c < b$ and the value of the integral exists only if each of the integrals on the right hand side exists.

2 Convergence of Improper Integrals

When the limit of an improper integral as defined above, is a definite finite number, we say that the given integral is **convergent** and the value of the integral is equal to the value of that limit. When the limit is ∞ or $-\infty$, the integral is said to be **divergent** *i.e.*, the value of the integral does not exist.

In case the limit is neither a definite number nor ∞ or $-\infty$, the integral is said to be **oscillatory** and in this case also the value of the integral does not exist *i.e.*, the integral is not convergent. We can define the convergence of the infinite integral $\int_a^\infty f(x)\,dx$ as follows :

Definition: *The integral* $\int_a^{\infty} f(x)\,dx$ *is said to converge to the value* I*, if for any arbitrarily chosen positive number* ε*, however small but not zero, there exists a corresponding positive number* N *such that*

$$\left| \int_a^b f(x)\,dx - I \right| < \varepsilon \text{ for all values of } b \geq N.$$

Similarly we can define the convergence of an integral, when the lower limit is infinite, or when the integrand becomes infinite at the upper or lower limit.

Illustrative Examples

Example 1: *Discuss the convergence of the following integrals by evaluating them*

(*i*) $\int_1^{\infty} \frac{dx}{\sqrt{x}}$, (*ii*) $\int_1^{\infty} \frac{dx}{x^{3/2}}$.

Solution: (i) We have

$$\int_1^{\infty} \frac{dx}{\sqrt{x}} = \lim_{x \to \infty} \int_1^x \frac{dx}{\sqrt{x}}, \text{ (By def.)}$$

$$= \lim_{x \to \infty} \int_1^x x^{-1/2}\,dx = \lim_{x \to \infty} \left[\frac{x^{1/2}}{1/2} \right]_1^x$$

$$= \lim_{x \to \infty} [2\sqrt{x} - 2] = \infty.$$

Thus the limit does not exist finitely and therefore the given integral is divergent (*i.e.*, the integral does not exist).

(ii) We have

$$\int_1^{\infty} \frac{dx}{x^{3/2}} = \lim_{x \to \infty} \int_1^x \frac{dx}{x^{3/2}}, \quad \text{(By def.)}$$

$$= \lim_{x \to \infty} \int_1^x x^{-3/2}\,dx = \lim_{x \to \infty} \left[\frac{x^{-1/2}}{-1/2} \right]_1^x = \lim_{x \to \infty} \left[-\frac{2}{\sqrt{x}} \right]_1^x$$

$$= \lim_{x \to \infty} \left[-\frac{2}{\sqrt{x}} + 2 \right] = 2.$$

Thus the limit exists and is unique and finite; therefore the given integral is convergent and its value is 2.

Example 2: *Test the convergence of* $\int_0^{\infty} e^{-mx}\,dx$, $(m > 0)$.

Solution: We have $\int_0^{\infty} e^{-mx}\,dx = \lim_{x \to \infty} \int_0^{\infty} e^{-mx}\,dx$, (by def.)

$$= \lim_{x \to \infty} \left[\frac{e^{-mx}}{-m} \right]_0^x = \lim_{x \to \infty} \left\{ -\frac{1}{m} (e^{-mx} - 1) \right\}$$

$$= -\frac{1}{m} [0 - 1] = \frac{1}{m}.$$

Thus the limit exists and is unique and finite, therefore the given integral is convergent.

Example 3: *Test the convergence of* $\int_0^{\infty} \frac{4a\,dx}{x^2 + 4a^2}$.

Solution: We have $\int_0^{\infty} \frac{4a\,dx}{x^2 + 4a^2} = \lim_{x \to \infty} \int_0^x \frac{4a\,dx}{x^2 + (2a)^2}$, (By def.)

$$= \lim_{x \to \infty} \left[4a \cdot \frac{1}{2a} \tan^{-1} \frac{x}{2a}\right]_0^x = 2 \lim_{x \to \infty} \left[\tan^{-1} \frac{x}{2a}\right]_0^x$$

$$= 2 \,.\, \lim_{x \to \infty} \left[\tan^{-1} \frac{x}{2a} - 0\right] = 2 \,.\, [\tan^{-1} \infty] = 2 \,.\, \frac{\pi}{2} = \pi.$$

Thus the limit exists and is unique and finite; therefore the given integral is convergent.

Example 4: *Test the convergence of* (*i*) $\int_{-\infty}^{0} e^{x}\,dx$; (*ii*) $\int_{-\infty}^{0} e^{-x}\,dx$.

Solution: (i) We have

$$\int_{-\infty}^{0} e^x\,dx = \lim_{x \to \infty} \int_{-x}^{0} e^{x}\,dx, \quad \text{(By def.)}$$

$$= \lim_{x \to \infty} [e^x]_{-x}^0 = \lim_{x \to \infty} [1 - e^{-x}] = [1 - 0] = 1.$$

Thus the limit exists and is unique and finite; therefore the given integral is convergent.

(ii) We have $\int_{-\infty}^{0} e^{-x}\,dx = \lim_{x \to \infty} \int_{-x}^{0} e^{-x}\,dx$, (By def.)

$$= \lim_{x \to \infty} \left[\frac{e^{-x}}{-1}\right]_{-x}^{0} = -\lim_{x \to \infty} [e^0 - e^{x}] = \infty.$$

Thus the limit does not exist finitely and therefore the given integral is divergent(*i.e.*, the integral does not exist).

Example 5: *Test the convergence of* $\int_{-\infty}^{\infty} \frac{dx}{1 + x^2}$.

(Kanpur 2008; Gorakhpur 11)

Solution: We have

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \int_{-\infty}^{0} \frac{dx}{1 + x^2} + \int_0^{\infty} \frac{dx}{1 + x^2}$$

$$= \lim_{x \to \infty} \int_{-x}^{0} \frac{dx}{1 + x^2} + \lim_{x \to \infty} \int_0^x \frac{dx}{1 + x^2}$$

$$= \lim_{x \to \infty} [\tan^{-1} x]_{-x}^{0} + \lim_{x \to \infty} [\tan^{-1} x]_0^x$$

$$= \lim_{x \to \infty} [0 - \tan^{-1} (-x)] + \lim_{x \to \infty} [\tan^{-1} x - 0]$$

$$= -(-\pi / 2) + \pi / 2 = \pi.$$

Thus the limit exists and is unique and finite; therefore the given integral is convergent.

Example 6: *Evaluate* $\int_0^1 \frac{dx}{\sqrt{x}}$.

(Gorakhpur 2010)

Solution: In the given integral, the integrand $1/\sqrt{x}$ becomes infinite at the lower limit $x=0$. Therefore we have

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon \to 0} \int_{0+\varepsilon}^1 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon \to 0} [2\sqrt{x}]_{\varepsilon}^1$$

$$= \lim_{\varepsilon \to 0} [2 - 2\sqrt{\varepsilon}] = 2.$$

Hence the given integral is convergent and its value is 2.

Example 7: *Evaluate* $\int_0^1 \frac{dx}{\sqrt{(1-x)}}$.

Solution: Here the integrand *i.e.*, $1/\sqrt{(1-x)}$ becomes unbounded *i.e.*, infinite at the upper limit (*i.e.*, $x=1$).

$$\therefore \quad \int_0^1 \frac{dx}{\sqrt{(1-x)}} = \lim_{\varepsilon \to 0} \int_0^{1-\varepsilon} \frac{dx}{\sqrt{(1-x)}}$$

$$= \lim_{\varepsilon \to 0} [-2\sqrt{(1-x)}]_0^{1-\varepsilon} = \lim_{\varepsilon \to 0} [-2\sqrt{\varepsilon} + 2] = 2,$$

which is a definite real number. Hence the given integral is convergent and its value is 2.

Example 8: *Evaluate* $\int_{-1}^1 \frac{dx}{x^2}$.

Solution: Here the integrand becomes infinite at $x=0$ and $-1 < 0 < 1$.

$$\therefore \quad \int_{-1}^1 \frac{dx}{x^2} = \lim_{\varepsilon \to 0} \int_{-1}^{-\varepsilon} \frac{dx}{x^2} + \lim_{\varepsilon' \to 0} \int_{\varepsilon'}^1 \frac{dx}{x^2}$$

$$= \lim_{\varepsilon \to 0} \left[-\frac{1}{x}\right]_{-1}^{-\varepsilon} + \lim_{\varepsilon' \to 0} \left[-\frac{1}{x}\right]_{\varepsilon'}^1$$

$$= \lim_{\varepsilon \to 0} \left[\frac{1}{\varepsilon} - 1\right] + \lim_{\varepsilon' \to 0} \left[-1 + \frac{1}{\varepsilon'}\right].$$

Since both the limits do not exist finitely, therefore the integral does not exist and is divergent.

Comprehensive Exercise 1

Evaluate the following integrals and discuss their convergence :

1. $\int_1^\infty \frac{dx}{x}$.
2. $\int_3^\infty \frac{dx}{(x-2)^2}$.
3. $\int_0^\infty e^{2x}\, dx$.
4. $\int_0^\infty \frac{dx}{(1+x)^{2/3}}$.
5. $\int_{-\infty}^0 \sinh x\, dx$.
6. $\int_{-\infty}^0 \cosh x\, dx$.

7. $\int_0^{\infty} \cos x \, dx.$

8. $\int_{-\infty}^{\infty} e^{-x} \, dx.$

9. $\int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2}.$

10. $\int_0^1 \frac{dx}{x^3}.$

11. $\int_0^1 \frac{dx}{1 - x}.$

12. $\int_{-1}^{1} \frac{dx}{x^{2/3}}.$ **(Gorakhpur 2011)**

Answers 1

1. ∞, divergent
2. 1, convergent
3. ∞, divergent
4. ∞, divergent
5. $-\infty$, divergent
6. ∞, divergent
7. Oscillates and so not convergent
8. ∞, divergent
9. π, convergent
10. ∞, divergent
11. ∞, divergent
12. 6, convergent

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. The integral $\int_1^{\infty} \frac{dx}{\sqrt{x}}$ is

(a) convergent
(b) divergent
(c) uniformly convergent
(d) none of these

2. The integral $\int_{-\infty}^{0} e^x \, dx$ is

(a) convergent
(b) divergent
(c) uniformly convergent
(d) none of these

3. Value of the integral $\int_{-\infty}^{\infty} \frac{dx}{1 + x^2}$ is

(a) $\pi / 2$
(b) $-\pi / 2$
(c) π
(d) $-\pi$

4. Value of the integral $\int_0^1 \frac{dx}{\sqrt{(1 - x)}}$ is

(a) 2
(b) -2
(c) 1
(d) -1

Fill in the Blank(s)

Fill in the blanks "......" so that the following statements are complete and correct.

1. The definite integral $\int_a^b f(x)\,dx$ is said to be a ...... if the range of integration (a, b) is finite and the integrand $f(x)$ is bounded over (a, b).
2. The definite integral $\int_a^b f(x)\,dx$ is said to be an improper integral if the interval (a, b) is finite and $f(x)$ is not ...... over this interval.
3. The definite integral $\int_a^b f(x)\,dx$ is said to be an ...... if the interval (a, b) is not finite and $f(x)$ is bounded over (a, b).

True or False

Write 'T' for true and 'F' for false statement.

1. A definite integral $\int_a^b f(x)\,dx$ in which the range of integration (a, b) is finite but the integrand $f(x)$ is unbounded at one or more points of the interval $a \le x \le b$, is called an improper integral of the second kind.
2. The integral $\int_0^\infty \frac{dx}{1+x^2}$ is an improper integral of the second kind.
3. The integral $\int_0^4 \frac{dx}{(x-2)(x-3)}$ is an improper integral of the first kind.

Answers

Multiple Choice Questions

1. (b) 2. (a) 3. (c) 4. (a)

Fill in the Blank(s)

1. proper integral 2. bounded
3. improper integral of the first kind

True or False

1. *T* 2. *F* 3. *T*

□

Chapter

2

Infinite Series

2.1 Infinite Series

An expression of the form $u_1 + u_2 + \ldots + u_n + \ldots$ *in which every term is followed by another according to some definite law is called a* **series.**

The series is called a **finite series**, if the number of terms is *finite*. Symbolically, the finite series $u_1 + u_2 + \ldots + u_n$ having n terms is denoted by $\sum_{r=1}^{n} u_r$.

The series is called an infinite series, if the number of terms is infinite. Symbolically, the infinite series $u_1 + u_2 + \ldots + u_n + \ldots$ is denoted by $\sum_{n=1}^{\infty} u_n$ or simply by $\Sigma\, u_n$.

Since we are going to deal with infinite series only, therefore we shall simply use the term '**series**' *to denote an infinite series.*

2.2 Convergence and Divergence of Series

Convergent Series : **(Kashi 2014)**

A series Σu_n is said to be convergent if S_n, the sum of its first n terms, tends to a definite finite limit S as n tends to infinity.

We write $S = \lim_{n \to \infty} S_n$.

The finite limit S to which S_n tends is called the sum of the series.

Divergent Series: *A series Σu_n is said to be divergent if S_n, the sum of its first n terms, tends to either $+\infty$ or $-\infty$ as n tends to infinity,*

i.e., if $\lim_{n \to \infty} S_n = \infty$ *or* $-\infty$.

Oscillatory Series: *A series Σu_n is said to be an oscillatory series if S_n, the sum of its first n terms, neither tends to a definite finite limit nor to $+\infty$ or $-\infty$ as n tends to ∞.*

The series is said to *oscillate finitely,* if the value of S_n as $n \to \infty$ fluctuates within a finite range. It is said to *oscillate infinitely,* if S_n tends to infinity and its sign is alternately positive and negative.

Sequence of Partial Sums of a Series :

If S_n denotes the sum of the first n terms of the series Σu_n, so that

$$S_n = u_1 + u_2 + \ldots + u_n,$$

then S_n is called the **partial sum** of the first n terms of the series and the sequence $< S_n > = < S_1, S_2, \ldots, S_n, \ldots >$ is called the **sequence of partial sums** of the given series. We can define the convergent, divergent and oscillatory series in terms of the sequence of partial sums.

Definition: *A series Σu_n is said to be convergent, divergent or oscillatory according as the sequence $< S_n >$ of its partial sums is convergent, divergent or oscillatory.*

If the sequence $< S_n >$ of partial sums of a series Σu_n converges to S then S is said to be the sum of the series Σu_n.

Note: Since the limits for infinite series will be taken as $n \to \infty$, so throughout this chapter we shall write $\lim_{n \to \infty}$ as 'lim' only.

Illustration 1:

The series $1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \ldots + \left(\frac{2}{3}\right)^{n-1} + \ldots$ is convergent.

Here the given series is a geometric series with common ratio $2/3 < 1$.

$$\therefore \quad S_n = \frac{1.\{1 - (2/3)^n\}}{1 - (2/3)} = 3\{1 - (2/3)^n\}.$$

Now, $\lim S_n = \lim 3\{1 - (2/3)^n\} = 3(1 - 0)$ $[\because 2/3 < 1]$

$= 3$, a definite finite number.

Consequently the given series is convergent.

Illustration 2:

The series $1 + 2 + 3 + \ldots + n + \ldots$ is divergent.

Here, $S_n = 1 + 2 + 3 + \ldots + n = \frac{1}{2} n(n+1)$.

$\therefore \quad \lim S_n = \lim \frac{1}{2} n(n+1) = \infty$.

Consequently the given series is divergent.

Illustration 3:

The series $2 - 2 + 2 - 2 + \ldots$ is oscillatory.

Here, $S_n = 0$ if n is even,

$\quad = 2$, if n is odd.

Therefore, the sequence $< S_n >$ of partial sums of the series, and consequently the given series, is oscillatory.

Below we give some results which will be found useful and can be easily proved.

1. The nature of a series remains unaltered if
   (i) the signs of all the terms are changed;
   (ii) a finite number of terms are added or omitted;
   (iii) each term of the series is multiplied or divided by the same fixed number c which is not zero.
2. If Σu_n converges to A and Σv_n converges to B, then $\Sigma (u_n + v_n)$ converges to $A + B$.
3. If Σu_n converges to A and $c \in \mathbf{R}$, then Σcu_n converges to cA.
4. If Σu_n converges to A and Σv_n converges to B and $p, q \in \mathbf{R}$, then $\Sigma (pu_n + qv_n)$ converges to $pA + qB$.
5. If Σu_n diverges and $c \in \mathbf{R}$, $c \neq 0$, then Σcu_n diverges.
6. If Σu_n and Σv_n are two divergent series having all terms positive, then $\Sigma (u_n + v_n)$ also diverges.

2.3 A Necessary Condition for Convergence

For a series Σu_n to be convergent, it is necessary that $\lim u_n = 0$.

Or *For every convergent series Σu_n, we must have* $\lim u_n = 0$.

Let the series Σu_n be convergent. Let S_n denote the sum of n terms of the series Σu_n.

Then $\quad S_n = u_1 + u_2 + \ldots + u_n$ and $S_{n-1} = u_1 + u_2 + \ldots + u_{n-1}$.

$$\therefore \quad u_n = S_n - S_{n-1}. \quad \ldots(1)$$

Since the series Σu_n is convergent, therefore, S_n and S_{n-1} both will tend to the same finite limit, say S, as $n \to \infty$.

Taking limits of both sides of (1), we get

$$\lim u_n = \lim S_n - \lim S_{n-1} = S - S = 0.$$

Hence for a convergent series, it is necessary that $\lim u_n = 0$.

Note: It is to be noted that the above condition is only necessary but not sufficient for a series to be convergent *i.e.,* if $\lim u_n = 0$, then the series Σu_n may or may not be convergent.

For example, consider the series

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots + \frac{1}{\sqrt{n}} + \ldots$$

Here $u_n = \frac{1}{\sqrt{n}}$, so that $\lim u_n = \lim \frac{1}{\sqrt{n}} = 0$. But the series does not converge as shown below.

We have $S_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \ldots + \frac{1}{\sqrt{n}} = \frac{n}{\sqrt{n}} = \sqrt{n}$,

i.e., $S_n > \sqrt{n}$, which tends to infinity as n tends to infinity. Hence the series is divergent.

Again consider the geometric series $\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots$, for which $\lim u_n = \lim \left(\frac{1}{2}\right)^n = 0$ and the series is convergent.

Thus if $u_n \to 0$, we cannot say anything about the behaviour of the series but if u_n does not tend to zero, the series definitely does not converge. The more useful form of the above test is as follows:

If a series Σu_n be such that u_n does not tend to zero as n tends to infinity, then the series does not converge.

2.4 Cauchy's General Principle of Convergence for Series

Sometimes it is either impossible or difficult to find the sequence of partial sums of a given series and yet we want to know whether the series converges or not. Now we shall establish a fundamental principle, for dealing with the convergence of such series, known as *Cauchy's general principle of convergence.*

Theorem: *A necessary and sufficient condition for a series Σu_n to converge is that for each $\varepsilon > 0$, there exists a positive integer m, such that*

$$|u_{m+1} + u_{m+2} + \ldots + u_n| < \varepsilon \text{ for all } n > m$$

Or $$|u_{p+1} + u_{p+2} + \ldots + u_q| < \varepsilon \text{ for all } q \geq p \geq m$$

Or $$|u_{n+1} + u_{n+2} + \ldots + u_{n+p}| < \varepsilon \text{ for all } n \geq m,\ p > 0.$$

Proof: Let $< S_n >$ be the sequence of partial sums of the series Σu_n. The series Σu_n will converge, iff the sequence $< S_n >$ of its partial sums converges. By Cauchy's general principle of convergence for sequences, we know that a necessary and sufficient condition for the convergence of $< S_n >$ is that for each $\varepsilon > 0$, there exists $m \in \mathbf{N}$ such that

$$|S_n - S_m| < \varepsilon \text{ for all } n > m$$

i.e., $$|u_{m+1} + u_{m+2} + \ldots + u_n| < \varepsilon \text{ for all } n > m.$$

Hence the result.

Illustrative Examples

Example 1: *Discuss the convergence of a geometric series.*

Solution: Consider the geometric series

$$a + ax + ax^2 + ax^3 + \ldots + ax^{n-1} + \ldots . \qquad \ldots(1)$$

Let S_n be the sum of first n terms of the seires (1).

$\therefore \quad S_n = \dfrac{a(1-x^n)}{1-x}$ if $x < 1$ and $S_n = \dfrac{a(x^n - 1)}{x-1}$ if $x > 1$.

Case I: When $|x| < 1$ *i.e.*, $-1 < x < 1$.

If $|x| < 1$, then $x^n \to 0$ as $n \to \infty$.

$$\therefore \quad \lim S_n = \lim \frac{a(1-x^n)}{1-x} = \frac{a(1-0)}{1-x} = \frac{a}{1-x},$$

which is a definite finite number and therefore the series is convergent.

Case II: When $x = 1$.

If $x = 1$, then each term of the series (1) is a.

$\therefore \quad S_n = a + a + \ldots$ to n terms $= na$.

$\therefore \quad \lim S_n = \infty$ or $-\infty$ according as a is positive or negative. Hence the series is divergent.

Case III: When $x > 1$.

If $x > 1$, then $x^n \to \infty$ as $n \to \infty$.

$\therefore \quad \lim S_n = \lim \dfrac{a(x^n - 1)}{x-1} = \infty$ or $-\infty$ according as $a >$ or < 0.

Hence the series is divergent.

Case IV: When $x = -1$.

If $x = -1$, then the series (1) becomes $a - a + a - a + \ldots .$

The sum of n terms of the series is a or 0 according as n is odd or even.

Hence the series is an oscillatory series, the oscillation being finite.

Case V: When $x < -1$.

If $x < -1$, then $-x > 1$.

Let $r = -x$, then $r > 1$ and so $r^n \to \infty$ as $n \to \infty$.

Now $\quad S_n = \dfrac{a(1-x^n)}{1-x} = \dfrac{a\{1-(-r)^n\}}{1-(-r)} \qquad [\because x = -r]$

$$= \frac{a(1+r^n)}{1+r} \text{ or } \frac{a(1-r^n)}{1+r}, \text{according as } n \text{ is odd or even.}$$

$\therefore$ in this case $\lim S_n$ is ∞ or $-\infty$ according as n is odd or even, provided $a > 0$ and if $a < 0$ the results are reversed.

Therefore in this case the series is an oscillatory series, the oscillation being infinite.

Hence a geometric series whose common ratio is x is convergent if $|x| < 1$, divergent if $x \geq 1$ and oscillatory if $x \leq -1$.

Example 2: *Prove that the series* $\Sigma \dfrac{1}{4^n}$ *converges to* $\dfrac{1}{3}\cdot$

Solution: Here $S_n = \dfrac{1}{4} + \dfrac{1}{4^2} + \dfrac{1}{4^3} + \ldots + \dfrac{1}{4^n} = \dfrac{\dfrac{1}{4}\left\{1 - \left(\dfrac{1}{4}\right)^n\right\}}{1 - \dfrac{1}{4}} = \dfrac{1}{3}\left(1 - \dfrac{1}{4^n}\right)\cdot$

$$\therefore \quad \lim S_n = \lim \frac{1}{3}\left(1 - \frac{1}{4^n}\right) = \frac{1}{3}\cdot \qquad \left[\because \lim \frac{1}{4^n} = 0\right]$$

$\therefore$ the sequence $< S_n >$ converges to $\dfrac{1}{3}$ and hence Σu_n converges to $\dfrac{1}{3}\cdot$

Example 3: *Test the convergence of the series*

$$\log_e 2 + \log_e \frac{3}{2} + \log_e \frac{4}{3} + \log_e \frac{5}{4} + \ldots.$$

Solution: Here, $S_n = \log_e 2 + \log_e \dfrac{3}{2} + \log_e \dfrac{4}{3} + \ldots + \log_e \left(\dfrac{n+1}{n}\right)$

$$= \log_e \left\{2 \cdot \frac{3}{2} \cdot \frac{4}{3} \ldots \frac{n+1}{n}\right\} = \log_e (n+1).$$

$$\therefore \quad \lim S_n = \lim \log (n+1) = \log \infty = \infty.$$

Hence the given series is divergent.

Example 4: *Show that the series*

$$\sqrt{\left(\frac{1}{4}\right)} + \sqrt{\left(\frac{2}{6}\right)} + \ldots + \sqrt{\left[\frac{n}{2(n+1)}\right]} + \ldots$$

does not converge.

Solution: Here,

$$u_n = \sqrt{\left[\frac{n}{2(n+1)}\right]} = \frac{1}{\sqrt{2}} \sqrt{\left(\frac{n}{n+1}\right)} = \frac{1}{\sqrt{2}} \cdot \left[\frac{1}{1 + (1/n)}\right]^{1/2}.$$

$$\therefore \quad \lim_{n \to \infty} u_n = \frac{1}{\sqrt{2}} \neq 0.$$

Hence the given series does not converge.

Example 5: *Show that the series* $\Sigma \frac{1}{n}$ *does not converge.*

Solution: Let the given series converge. Then for $\varepsilon = \frac{1}{4}$, by Cauchy's general principle of convergence, we can find a positive integer m such that

$$\frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{n} < \frac{1}{4} \text{ for all } n > m.$$

Taking $n = 2m$, we see that

$$\frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{n} = \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{2m}$$

$$> m \cdot \frac{1}{2m} = \frac{1}{2}.$$

Thus we get a contradiction. Hence the given series does not converge.

2.5 Series of Positive Terms

If Σu_n *is a series of positive terms then* $u_n > 0$ *for all* $n \in \mathbf{N}$.

The important aspect of this series is that its sequence of partial sums is increasing.

We have $S_n = u_1 + u_2 + \ldots + u_n$, then $S_n - S_{n-1} = u_n$.

Since $u_n > 0$ for all n, therefore we get $S_n - S_{n-1} > 0$ for all n, *i.e.*, $S_n > S_{n-1}$ for all n *i.e.*, the sequence $< S_n >$ is a monotonically increasing sequence.

Now a monotonic sequence can either converge or diverge but cannot oscillate. Hence, we have only two possibilities for a series of positive terms, either the series converges or it diverges.

We give some fundamental results for series of positive terms.

Theorem 1: *A series* Σu_n *of positive terms converges iff there exists a number* K *such that*

$$u_1 + u_2 + \ldots + u_n < K \text{ for all } n.$$

Proof: First, suppose that there exists a number K such that

$$u_1 + u_2 + \ldots + u_n < K, \ \forall n \quad i.e., \quad S_n < K, \ \forall n.$$

This shows that the sequence $< S_n >$ of partial sums of the series Σu_n is bounded above. Also, the sequence $< S_n >$ is an increasing sequence, since the series Σu_n is of positive terms. We know that every bounded monotonic sequence converges. Therefore $< S_n >$ converges and hence Σu_n converges.

Conversely, we assume that Σu_n converges. Then, the sequence $< S_n >$ of partial sums of the series converges. We know that every convergent sequence is bounded. Therefore $< S_n >$ is bounded and hence there exist real numbers k and K such that $k < S_n < K$, for all n.

It gives $S_n < K$ *i.e.*, $u_1 + u_2 + \ldots + u_n < K$, for all n.

Note: In the light of the above theorem, we conclude that to show that a series of positive terms converges, it is sufficient to show that the sequence of its partial sums is bounded. On the other hand, to show that a series of positive terms diverges, we have to show that the sequence of its partial sums is not bounded, *i.e.*, for any real number A, there exists a positive integer m such that $S_m > A$.

Theorem 2: *A series of positive terms is divergent if each term after a fixed stage is greater than some fixed positive number.*

Proof: Let each term of the series be greater than a fixed positive number. We can assume so because the convergence or divergence of the series is not affected by omitting a finite number of terms.

So let Σu_n be the given series of positive terms and let $u_n > k$ (a fixed positive number) for all n.

Now $S_n = u_1 + u_2 + \ldots + u_n > nk.$

But $\lim nk = \infty$.

$\therefore \quad \lim S_n = \infty$.

Hence the series Σu_n is divergent.

Corollary: *A series of positive terms is divergent if* $\lim u_n > 0$.

Proof: Let $\lim u_n = l$, where $l > 0$. Then for a given $\varepsilon > 0$, there exists a positive integer m such that

$$|u_n - l| < \varepsilon, \text{ for all } n \geq m$$

i.e., $$l - \varepsilon < u_n < l + \varepsilon, \text{ for all } n \geq m.$$

Let $l - \varepsilon = a$. Then a is a fixed positive number because ε can be taken as small as we please. For example take $\varepsilon = \frac{1}{2} l$.

Thus $u_n > a$ for all $n \geq m$. Hence the given series is divergent.

Theorem 3: *If each term of a series Σu_n of positive terms, does not exceed the corresponding term of a convergent series Σv_n of positive terms, then Σu_n is convergent. While, if each term of Σu_n exceeds (or equals) the corresponding term of a divergent series of positive terms, then Σu_n is divergent.*

Proof: Let $u_n \leq v_n$ for all n.

Let S_n and S_n' be the sums of first n terms of the two series Σu_n and Σv_n respectively.

Then $S_n = u_1 + u_2 + \ldots + u_n$ and $S_n' = v_1 + v_2 + \ldots + v_n$.

Since $u_n \leq v_n \;\; \forall \; n$, therefore, $S_n \leq S_n'$.

But Σv_n is convergent, therefore $S_n' \to S'$ (a finite quantity) as $n \to \infty$.

$\therefore \quad \lim S_n \leq S'$ (a finite quantity).

$\therefore \quad S_n$ itself tends to a finite limit as $n \to \infty$.

Hence the series Σu_n is convergent.

Now if $u_n \geq v_n$, for all n, then $S_n \geq S_n'$.

But Σv_n is divergent, therefore $S_n' \to \infty$ as $n \to \infty$ and hence $S_n \to \infty$ as $n \to \infty$. Consequently Σu_n is divergent.

2.6 The Auxiliary Series $\Sigma 1/n^p$

The infinite series

$$\Sigma \frac{1}{n^p} \text{ i.e., } \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \ldots$$

is convergent if $p > 1$ *and divergent if* $p \leq 1$. **(Kumaun 2001; Avadh 05; Kanpur 07; Kashi 13; Rohilkhand 14; Agra 14)**

Proof:

Case I: Let $p > 1$. Since the terms of the given series are all positive, we can group them as we like. Hence we write the given series

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots = \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \left(\frac{1}{8^p} + \frac{1}{9^p} + \ldots + \frac{1}{15^p}\right) + \ldots \qquad \ldots(1)$$

Now since $p > 1$,

$\therefore \quad 3 > 2 \Rightarrow 3^p > 2^p \Rightarrow 1/3^p < 1/2^p$.

$\therefore \quad \dfrac{1}{2^p} + \dfrac{1}{3^p} < \dfrac{1}{2^p} + \dfrac{1}{2^p}$

or $\quad \dfrac{1}{2^p} + \dfrac{1}{3^p} < \dfrac{2}{2^p}$.

Similarly $\quad \dfrac{1}{4^p} + \dfrac{1}{5^p} + \dfrac{1}{6^p} + \dfrac{1}{7^p} < \dfrac{1}{4^p} + \dfrac{1}{4^p} + \dfrac{1}{4^p} + \dfrac{1}{4^p} = \dfrac{4}{4^p}$,

$\quad \dfrac{1}{8^p} + \dfrac{1}{9^p} + \ldots + \dfrac{1}{15^p} < \dfrac{8}{8^p}$,

and so on.

Thus we observe that on being grouped as mentioned in (1), the given series is term by term

$$< \frac{1}{1^p} + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \ldots$$

But the series on the R.H.S. of the above inequality is a geometric series and is convergent since its common ratio is $2/2^p = 1/2^{p-1}$ which is less than 1 as $p > 1$.

Thus the given series on being grouped as in (1) is term by term less than a convergent series.

Consequently the given series is convergent when $p > 1$.

Case II: Let $p = 1$. Then we group terms of the given series as

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \ldots + \frac{1}{16}\right) + \ldots \quad \ldots(2)$$

Now as $3 < 4$, so $\frac{1}{3} > \frac{1}{4}$ or $\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4}$

or $\frac{1}{3} + \frac{1}{4} > \frac{2}{4}$ *i.e.*, $\frac{1}{3} + \frac{1}{4} > \frac{1}{2}$.

Similarly, $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$,

$\frac{1}{9} + \frac{1}{10} + \ldots + \frac{1}{16} > \frac{1}{2}$, and so on.

Thus we observe that on being grouped as in (2), the given series is term by term

$$> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots. \quad \ldots(3)$$

The series on the R.H.S. of (3) is divergent as the sum of the first n terms of the series

$$= 1 + (n-1) \cdot \frac{1}{2} = \frac{1}{2}(n+1), \text{ which tends to infinity as } n \to \infty.$$

Thus the given series on being grouped as in (2) is term by term greater than a divergent series.

Consequently the given series is divergent when $p = 1$.

Case III: Let $p < 1$. Then

$$\frac{1}{n^p} > \frac{1}{n} \text{ for } n = 2, 3, 4, \ldots.$$

In this case the given series

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots$$

is term by term greater than the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots,$$

which is a divergent series, as proved in case II.

Consequently the given series is divergent when $p < 1$.

Hence the proof is complete.

Now we shall give some tests to know whether the given series of positive terms is convergent or divergent without actually finding out the sum of its n terms.

2.7 Comparison Test

Theorem: First form: *Let Σu_n and Σv_n be two series of positive terms such that $u_n < K v_n$ for all n, where K is a fixed positive number. Then if Σv_n converges, so does Σu_n, and if Σu_n diverges, then Σv_n also diverges.*

Proof: Since $u_n < Kv_n$ for all n,

$$\therefore \quad u_1 + u_2 + \ldots + u_n < K(v_1 + v_2 + \ldots + v_n), \quad \forall \quad n. \qquad \ldots(1)$$

Now if Σv_n converges, then there must exist a positive real number A, such that

$$v_1 + v_2 + \ldots + v_n < A, \quad \forall \ n. \qquad \ldots(2)$$

From (1) and (2), we get

$$u_1 + u_2 + \ldots + u_n < K A, \quad \forall \ n.$$

Thus the sequence of partial sums of the series Σu_n is bounded above and hence Σu_n converges.

To prove the other result, we assume that Σu_n diverges. Then for any positive real number B, there must exist a positive integer m such that

$$u_1 + u_2 + \ldots + u_n > BK, \text{ for all } n > m. \qquad \ldots(3)$$

From (1) and (3), we get

$$v_1 + v_2 + \ldots + v_n > B, \text{ for all } n > m.$$

Hence the series Σv_n diverges.

Second form: *Let Σu_n and Σv_n be two series of positive terms and let k and K be positive real numbers such that*

$$kv_n < u_n < Kv_n, \text{ for all } n.$$

Then the series Σu_n and Σv_n converge or diverge together.

Proof: From $kv_n < u_n < Kv_n$, for all n, we get

$$kv_n < u_n \quad \text{or} \quad v_n < \left(\frac{1}{k}\right) u_n, \text{ for all } n.$$

Now applying the result proved in the first form of the comparison test, we conclude that

(i) if Σu_n converges, then Σv_n also converges.

(ii) if Σv_n diverges, then Σu_n also diverges.

Again, applying the result of the first form of the comparison test for the inequality $u_n < Kv_n$, we conclude that

(iii) if Σv_n converges, then Σu_n also converges.

(iv) if Σu_n diverges, then Σv_n also diverges.

The desired result now follows from (i), (ii), (iii) and (iv).

Third form: *Let Σu_n and Σv_n be two series of positive terms and let K be a positive number such that $u_n < Kv_n$ for all $n > m$, m being a fixed positive integer. Then if the series Σv_n be convergent, then the series Σu_n is also convergent and if the series Σu_n is divergent, then the series, Σv_n is also divergent.*

Proof: The above result follows from the result of the first form of the comparison test because the convergence or the divergence of a series remains unaffected by omitting a finite number of terms of the series.

Fourth form: *Let Σu_n and Σv_n be two series of positive terms and let k and K be positive real numbers such that $kv_n < u_n < Kv_n$ for all $n > m$, m being a fixed positive integer. Then the series Σu_n and Σv_n converge or diverge together.*

Proof: Since the omission of a finite number of terms of a series has no effect on its convergence or divergence, therefore,

(i) the series $u_1 + u_2 + \ldots$ and the series $u_{m+1} + u_{m+2} + \ldots$ converge or diverge together ;

and (ii) the series $v_1 + v_2 + \ldots$ and the series $v_{m+1} + v_{m+2} + \ldots$ converge or diverge together.

Again, $kv_n < u_n < Kv_n$ for all $n > m \Rightarrow kv_{m+p} < u_{m+p} < Kv_{m+p}$ for all $p \in \mathbf{N}$, therefore, by the result of the second form of the comparison test, we have

(iii) the series $u_{m+1} + u_{m+2} + \ldots$ and the series $v_{m+1} + v_{m+2} + \ldots$ converge or diverge together.

Hence from (i), (ii) and (iii), we conclude that the series Σu_n and Σv_n converge or diverge together.

Fifth form: (Important from the point of view of application to the solution of problems): *Let Σu_n and Σv_n be two series of positive terms such that*

$$\lim_{n \to \infty} \frac{u_n}{v_n} = l \text{ (finite and non-zero)};$$

then both the series converge or diverge together i.e., the two series Σu_n and Σv_n are either both convergent or both divergent.

Proof: We have $\dfrac{u_n}{v_n} > 0$ for all n, therefore

$$\lim_{n \to \infty} \frac{u_n}{v_n} \geq 0 \quad i.e., \quad l \geq 0.$$

Since $l \neq 0$ (given), therefore, $l > 0$.

Choose $\varepsilon > 0$ in such a way that $l - \varepsilon > 0$.

Now $\lim_{n \to \infty} \dfrac{u_n}{v_n} = l \Rightarrow$ there exists $m \in \mathbf{N}$ such that

$$l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon, \text{ for all } n > m. \qquad \ldots(1)$$

Since $v_n > 0 \ \forall \ n$, hence multiplying (1) throughout by v_n, we get

$$(l - \varepsilon) v_n < u_n < (l + \varepsilon) v_n, \text{ for all } n > m. \qquad \ldots(2)$$

Now if Σv_n is convergent then $\Sigma (l + \varepsilon) v_n$ is also convergent. In this case from (2), we see that Σu_n is term by term less than a convergent series $\Sigma (l + \varepsilon) v_n$ except possibly for a finite number of terms. Therefore the series Σu_n is also convergent.

Again if $\Sigma\, v_n$ is divergent then $\Sigma\,(l-\varepsilon)\, v_n$ is also divergent. In this case from (2), we see that $\Sigma\, u_n$ is term by term greater than a divergent series $\Sigma\,(l-\varepsilon)\, v_n$ except possibly for a finite number of terms. Therefore the series $\Sigma\, u_n$ is also divergent. Hence the series $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

Sixth form: *Let $\Sigma\, u_n$ and $\Sigma\, v_n$ be two series of positive terms such that*

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}},\ \textit{for all } n \geq m.$$

Then $\Sigma\, v_n$ converges $\Rightarrow \Sigma\, u_n$ converges and $\Sigma\, u_n$ diverges $\Rightarrow \Sigma\, v_n$ diverges.

Proof: We have $\dfrac{u_n}{u_{n+1}} > \dfrac{v_n}{v_{n+1}}$, for all $n \geq m$.

Putting $n = m+1, m+2, \ldots, n-1$ in the above inequality, we get

$$\frac{u_{m+1}}{u_{m+2}} > \frac{v_{m+1}}{v_{m+2}},\quad \frac{u_{m+2}}{u_{m+3}} > \frac{v_{m+2}}{v_{m+3}},\quad \ldots,\quad \frac{u_{n-1}}{u_n} > \frac{v_{n-1}}{v_n}.$$

Multiplying the corresponding sides of these inequalities, we get

$$\frac{u_{m+1}}{u_n} > \frac{v_{m+1}}{v_n}, \text{ for all } n > m,$$

i.e.,

$$u_n < \left(\frac{u_{m+1}}{v_{m+1}}\right) v_n, \text{ for all } n > m.$$

Now the result follows from the third form.

Note 1: From the point of view of applications, the third and the fifth forms of the comparison test are the most useful.

Note 2: The geometric series $\Sigma\, \dfrac{1}{r^n}$ and the auxiliary series $\Sigma\, \dfrac{1}{n^p}$ will play a prominent role for comparison.

Working rule for applying comparison test:

The v_n-method: Comparison test is usually applied when the nth term u_n of the given series $\Sigma\, u_n$ contains the powers of n only which may be positive or negative, integral or fractional. The auxiliary series $\Sigma\,(1/n^p)$ is chosen as the series $\Sigma\, v_n$. From article 2.6, we know that $\Sigma\,(1/n^p)$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Now the question arises that how to choose v_n? For applying comparison test, it is necessary that $\lim \dfrac{u_n}{v_n}$ should be finite and non-zero. It will be so if we take $v_n = \dfrac{1}{n^{p-q}}$, where p and q are respectively the highest indices of n in the denominator and numerator of u_n when it is in the form of a fraction. If u_n, can be expanded in ascending powers of $1/n$, then to get v_n, we should retain only the lowest power of $1/n$. After making a proper choice of v_n, we find $\lim (u_n / v_n)$ which should come out to be finite and non-zero. Then the series Σu_n and Σv_n are either both convergent or both divergent. The whole procedure will be clear from the examples that follow article 2.8.

Illustrative Examples

Example 6: *Test for convergence the series*

$$1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\ldots+\frac{1}{n^n}+\ldots\ldots.$$

Solution: Since $n^n > 2^n$ for all $n > 2$, therefore, $\frac{1}{n^n} < \frac{1}{2^n}$.

Here $\quad u_n = \frac{1}{n^n}$. Let $v_n = \frac{1}{2^n}$.

Since $u_n < v_n$ for all $n > 2$ and Σv_n is a convergent series (a geometric series with common ratio $\frac{1}{2}$), therefore, by the comparison test, the given series converges.

Example 7: *Test for convergence the series whose nth terms are*

(i) $\frac{\sqrt{n}}{n^2+1}$ **(Kumaun 2002; Kanpur 06; Meerut 13B; Agra 14)**

(ii) $\frac{(2n^2-1)^{1/3}}{(3n^3+2n+5)^{1/4}}$ **(Kanpur 2009; Meerut 13)**

(iii) $\frac{n^p}{(1+n)^q}$.

Solution: (i) Here $u_n = \frac{\sqrt{n}}{n^2+1}$.

Take $\quad v_n = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$,

i.e., the auxiliary series is $\Sigma v_n = \Sigma \frac{1}{n^{3/2}}$.

Now $$\lim \frac{u_n}{v_n} = \lim \left\{\frac{\sqrt{n}}{n^2+1} . n^{3/2}\right\} = \lim \frac{n^2}{n^2+1}$$

$$= \lim \frac{1}{1+(1/n^2)} = 1, \text{which is finite and non-zero.}$$

Since the auxiliary series $\Sigma v_n = \Sigma (1/n^{3/2})$ is convergent $\left(p=\frac{3}{2}>1\right)$, therefore, by comparison test the given series Σu_n is also convergent.

(ii) Here $u_n = \dfrac{(2n^2 - 1)^{1/3}}{(3n^3 + 2n + 5)^{1/4}} = \dfrac{n^{2/3}\,(2 - 1/n^2)^{1/3}}{n^{3/4}\,(3 + 2/n^2 + 5/n^3)^{1/4}}$

$$= \frac{1}{n^{1/12}} \cdot \frac{(2 - 1/n^2)^{1/3}}{(3 + 2/n^2 + 5/n^3)^{1/4}} \cdot$$

Take $v_n = \dfrac{1}{n^{1/12}} \cdot$

Then $\dfrac{u_n}{v_n} = \dfrac{(2 - 1/n^2)^{1/3}}{(3 + 2/n^2 + 5/n^3)^{1/4}} \cdot$

$\therefore$ $\lim \dfrac{u_n}{v_n} = \dfrac{2^{1/3}}{3^{1/4}}$, which is finite and non-zero.

Hence, by comparison test, Σu_n and Σv_n are either both convergent or both divergent.

But the auxiliary series Σv_n is divergent because $p = 1/12 < 1$. Hence Σu_n is also divergent.

(iii) Here $u_n = \dfrac{n^p}{(n+1)^q} \cdot$

Take $v_n = \dfrac{n^p}{n^q} = \dfrac{1}{n^{q-p}} \cdot$

Now $\lim \dfrac{u_n}{v_n} = \lim \left\{ \dfrac{n^p}{(n+1)^q} \cdot n^{q-p} \right\} = \lim \dfrac{1}{(1 + 1/n)^q} = 1$, which is finite and non-zero.

Therefore, by comparison test, Σu_n and Σv_n are either both convergent or both divergent.

But the auxiliary series $\Sigma v_n = \Sigma \dfrac{1}{n^{q-p}}$ is convergent if $q - p > 1$ *i.e.* if $p - q + 1 < 0$ and divergent if $q - p \le 1$ *i.e.* if $p - q + 1 \ge 0$.

Hence by comparison test the given series Σu_n is convergent if $p - q + 1 < 0$ and divergent if $p - q + 1 \ge 0$.

Example 8: *Test for convergence the series whose nth terms are*

(*i*) $\dfrac{1}{1 + 1/n}$ **(Avadh 2012)**

(*ii*) $\sin \dfrac{1}{n}$ **(Kanpur 2012)**

(*iii*) $\tan^{-1} \dfrac{1}{n} \cdot$ **(Kanpur 2008)**

Solution: (i) Here $u_n = \dfrac{1}{1 + 1/n} \cdot$

We have, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \dfrac{1}{1+(1/n)} = 1$, which is > 0.

$\therefore$ the given series is divergent.

(ii) Here, $u_n = \sin\dfrac{1}{n} = \dfrac{1}{n} - \dfrac{1}{3\,!}\dfrac{1}{n^3} + \dfrac{1}{5\,!}\dfrac{1}{n^5} - \ldots\ldots$.

Take $v_n = 1/n$, since the lowest power of $1/n$ in u_n is $1/n$. The auxiliary series $\Sigma\, v_n = \Sigma\,(1/n)$ is divergent as here $p = 1$.

Now $$\lim \frac{u_n}{v_n} = \lim\left(1 - \frac{1}{3\,!}\cdot\frac{1}{n^2} + \frac{1}{5\,!}\cdot\frac{1}{n^4} - \ldots\right) = 1,$$

which is finite and non-zero.

Hence by comparison test the given series is divergent.

(iii) Here, $u_n = \tan^{-1}\left(\dfrac{1}{n}\right) = \dfrac{1}{n} - \dfrac{1}{3n^3} + \dfrac{1}{5n^5} - \ldots$

$$\left[\because \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right]$$

The lowest power of $1/n$ in u_n is $1/n$. Therefore, to apply the comparison test, the auxiliary series is taken as $\Sigma\, v_n = \Sigma\,(1/n)$.

Now, $\lim \dfrac{u_n}{v_n} = \lim\left(1 - \dfrac{1}{3n^2} + \dfrac{1}{5n^4} - \ldots\right) = 1$, which is finite and non-zero.

But the auxiliary series $\Sigma\, v_n = \Sigma\,(1/n)$ is divergent as here $p = 1$.

Hence by comparison test the given series is divergent.

Example 9: *Test the convergence of the series*

$$\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \frac{5}{4^p} + \ldots\ldots$$

(Kumaun 2000; Avadh 10)

Solution: Here $u_n = \dfrac{n+1}{n^p}$. Take $v_n = \dfrac{n}{n^p} = \dfrac{1}{n^{p-1}}$.

Now $$\lim \frac{u_n}{v_n} = \lim\left(\frac{n+1}{n^p}\,.\,n^{p-1}\right) = \lim\left(1 + \frac{1}{n}\right) = 1,$$

which is finite and non-zero.

Hence by comparison test $\Sigma\, u_n$ and $\Sigma\, v_n$ are either both convergent or both divergent.

But the auxiliary series $\Sigma\, v_n = \Sigma\,\dfrac{1}{n^{p-1}}$ is convergent if $p - 1 > 1$ *i.e.,* $p > 2$, and divergent if $p - 1 \le 1$ *i.e.* if $p \le 2$.

Hence the given series $\Sigma\, u_n$ is convergent if $p > 2$ and divergent if $p \le 2$.

Example 10: *Test the convergence of the following series*

(i) $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\dots$ **(Avadh 2014)**

(ii) $\frac{1}{1+\sqrt{2}}+\frac{2}{1+2\sqrt{3}}+\frac{3}{1+3\sqrt{4}}+\dots$

Solution: (i) Omitting the first term, if the given series is denoted by Σu_n, then

$$\Sigma u_n=\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\dots=\Sigma\frac{n^n}{(n+1)^{n+1}}\cdot$$

Here, $u_n=\frac{n^n}{(n+1)^{n+1}}\cdot$ Take $v_n=\frac{n^n}{n^{n+1}}=\frac{1}{n}\cdot$

Now
$$\lim\frac{u_n}{v_n}=\lim\left\{\frac{n^n}{(n+1)^{n+1}}\cdot n\right\}$$
$$=\lim\left\{\frac{1}{(1+1/n)^n\cdot(1+1/n)}\right\}=\frac{1}{e},\quad\left[\because \lim\left(1+\frac{1}{n}\right)^n=e\right]$$

which is finite and non-zero.

But the auxiliary series $\Sigma v_n=\Sigma(1/n)$ is divergent as here $p=1$. Hence by comparison test the given series is divergent.

(ii) Here, $u_n=\frac{n}{1+n\sqrt{(n+1)}}\cdot$

Take $v_n=\frac{n}{n\sqrt{n}}=\frac{1}{n^{1/2}}\cdot$

Now
$$\lim\frac{u_n}{v_n}=\lim\left\{\frac{n}{1+n\sqrt{(n+1)}}\cdot n^{1/2}\right\}$$
$$=\lim\left\{\frac{1}{1/n^{3/2}+\sqrt{(1+1/n)}}\right\}=1,$$ which is finite and non-zero.

Since the auxiliary series $\Sigma v_n=\Sigma(1/n^{1/2})$ is divergent as here $p=1/2<1$, therefore, by comparison test the given series is divergent.

Example 11: *Test the following series for convergence whose nth terms are given by*

(i) $(n^3+1)^{1/3}-n$ **(Meerut 2013)**

(ii) $\sqrt{(n^4+1)}-\sqrt{(n^4-1)}$.

(Kanpur 2006; Avadh 06, 14; Meerut 13B; Kashi 14)

Solution: (i) Here, $u_n=(n^3+1)^{1/3}-n=(n^3)^{1/3}(1+1/n^3)^{1/3}-n$

$$= n\left[\left(1+\frac{1}{n^3}\right)^{1/3}-1\right] = n\left[1+\frac{1}{3}\cdot\frac{1}{n^3}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2\,!}\cdot\frac{1}{n^6}+\ldots-1\right]$$

$$= \frac{1}{3n^2}-\frac{1}{9n^5}+\ldots.$$

Taking the lowest power of $1/n$ in u_n, the auxiliary series is given by

$$\Sigma v_n = \Sigma (1/n^2).$$

Now $$\lim \frac{u_n}{v_n} = \lim \left\{\left(\frac{1}{3n^2}-\frac{1}{9n^5}+\ldots\right).\, n^2\right\} = \lim \left(\frac{1}{3}-\frac{1}{9n^3}+\ldots\right) = \frac{1}{3},$$

which is finite and non-zero.

Since the auxiliary series $\Sigma v_n = \Sigma (1/n^2)$ is convergent as here $p = 2 > 1$, therefore by comparison test the given series Σu_n is also convergent.

(ii) Here $u_n = \sqrt{(n^4+1)} - \sqrt{(n^4-1)}$

$$= n^2\,[(1+1/n^4)^{1/2} - (1-1/n^4)^{1/2}]$$

$$= n^2\left[\left\{1+\frac{1}{2}\cdot\frac{1}{n^4}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2\,!}\cdot\frac{1}{n^8}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3\,!}\cdot\frac{1}{n^{12}}+\ldots\right\}\right.$$

$$\left.-\left\{1-\frac{1}{2}\cdot\frac{1}{n^4}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2\,!}\cdot\frac{1}{n^8}-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3\,!}\cdot\frac{1}{n^{12}}+\ldots\right\}\right]$$

$$= n^2\left[2\left\{\frac{1}{2n^4}+\frac{1}{16n^{12}}+\ldots\right\}\right] = \frac{1}{n^2}+\frac{1}{8n^{10}}+\ldots.$$

The lowest power of $1/n$ in u_n is $1/n^2$. Therefore to apply the comparison test we take the auxiliary series as $\Sigma v_n = \Sigma 1/n^2$, which is convergent as $p = 2 > 1$.

Now $$\lim \frac{u_n}{v_n} = \lim \left[\left\{\frac{1}{n^2}+\frac{1}{8n^{10}}+\ldots\right\}.n^2\right]$$

$$= \lim \left\{1+\frac{1}{8n^8}+\ldots\right\} = 1, \text{ which is finite and non-zero.}$$

Therefore, by comparison test, Σu_n and Σv_n converge or diverge together. Since Σv_n is convergent, therefore, Σu_n is also convergent.

Alternate solution: We have $u_n = \sqrt{(n^4+1)} - \sqrt{(n^4-1)}$

$$= \frac{[\sqrt{(n^4+1)}-\sqrt{(n^4-1)}]\,[\sqrt{(n^4+1)}+\sqrt{(n^4-1)}]}{\sqrt{(n^4+1)}+\sqrt{(n^4-1)}}$$

$$= \frac{(n^4+1)-(n^4-1)}{\sqrt{(n^4+1)}+\sqrt{(n^4-1)}} = \frac{2}{\sqrt{(n^4+1)}+\sqrt{(n^4-1)}}$$

$$= \frac{1}{n^2} \cdot \frac{2}{\sqrt{[1+(1/n^4)]}+\sqrt{[1-(1/n^4)]}} \cdot$$

Take $\quad v_n = \dfrac{1}{n^2} \cdot$

Then $\quad \dfrac{u_n}{v_n} = \dfrac{2}{\sqrt{[1+(1/n^4)]}+\sqrt{[1-(1/n^4)]}} \cdot$

$\therefore \quad \lim_{n\to\infty} \dfrac{u_n}{v_n} = 1$ which is finite and non-zero.

Hence by comparison test $\Sigma\, u_n$ and $\Sigma\, v_n$ are either both convergent or both divergent.

But for $v_n = \dfrac{1}{n^2} = \dfrac{1}{n^p}$, $p = 2 > 1$.

$\therefore \quad \Sigma\, v_n$ is convergent and hence $\Sigma\, u_n$ is also convergent.

Example 12: *Test for convergence of the following series :*

(i) $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{(a+b/n)}} \cdot$ *(ii)* $\sum\limits_{n=1}^{\infty} \dfrac{1}{2^n+3^n} \cdot$ *(iii)* $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^3}\left(\dfrac{n+2}{n+3}\right)^n$.

Solution: (i) Here, $u_n = \dfrac{1}{n^{(a+b/n)}} = \dfrac{1}{n^a \,.\, n^{b/n}} \cdot$ Let $v_n = \dfrac{1}{n^a} \cdot$

Now $\quad \lim \dfrac{u_n}{v_n} = \lim \left[\dfrac{1}{n^a \,.\, n^{b/n}} \,.\, n^a\right] = \lim \left(\dfrac{1}{n^{b/n}}\right)$

$$= \lim \frac{1}{(n^{1/n})^b} = \frac{1}{(1)^b} \qquad \left[\because \lim_{n\to\infty} n^{1/n} = 1\right]$$

$= 1,$ which is finite and non-zero.

We know that $\Sigma\, v_n = \Sigma\,(1/n^a)$ is convergent if $a > 1$ and divergent if $a \le 1$.

Hence by comparison test the given series $\Sigma\, u_n$ is convergent if $a > 1$ and divergent if $a \le 1$.

(ii) Here, $u_n = \dfrac{1}{2^n+3^n} = \dfrac{1}{3^n\left[1+\left(\dfrac{2}{3}\right)^n\right]} \cdot$

Take $\quad v_n = \dfrac{1}{3^n} \cdot$

We know that $\Sigma\, v_n = \Sigma\,(1/3^n)$ is a geometric series with common ratio $1/3 < 1$, hence it is convergent.

Now $\lim \dfrac{u_n}{v_n} = \lim \dfrac{1}{1+\left(\dfrac{2}{3}\right)^n} = 1, \qquad [\because \lim r^n = 0, 0 < r < 1]$

which is finite and non-zero.

Hence by comparison test the given series $\Sigma\, u_n$ is convergent.

(iii) Here, $u_n = \dfrac{1}{n^3}\left(\dfrac{n+2}{n+3}\right)^n.$

Take $v_n = \dfrac{1}{n^3} \cdot$ Then $\Sigma\, v_n = \Sigma\, \dfrac{1}{n^3}$ is convergent as $p = 3 > 1$.

Now $\dfrac{u_n}{v_n} = \dfrac{1}{n^3}\left(\dfrac{n+2}{n+3}\right)^n \cdot n^3 = \left(\dfrac{n+2}{n+3}\right)^n = \dfrac{n^n\left(1+\dfrac{2}{n}\right)^n}{n^n\left(1+\dfrac{3}{n}\right)^n} = \dfrac{\left(1+\dfrac{2}{n}\right)^n}{\left(1+\dfrac{3}{n}\right)^n}.$

We know that $\lim \left(1+\dfrac{x}{n}\right)^n = e^x$.

$\therefore \quad \lim \dfrac{u_n}{v_n} = \dfrac{e^2}{e^3} = \dfrac{1}{e}$, which is finite and non-zero.

Hence by comparison test, $\Sigma\, u_n$ is convergent.

Example 13: *Test for convergence the series*

$$\frac{1}{1+2^{-1}} + \frac{2}{1+2^{-2}} + \frac{3}{1+2^{-3}} + \frac{4}{1+2^{-4}} + \ldots$$

Solution: Here, $u_n = \dfrac{n}{1+2^{-n}} \cdot$

$\therefore \quad \lim u_n = \lim \dfrac{n}{1+\left(\dfrac{1}{2}\right)^n} = \infty,$

which is > 0. Also $\Sigma\, u_n$ is a series of positive terms.
Hence the given series $\Sigma\, u_n$ is divergent.

Comprehensive Exercise 1

Test for convergence the following series :

1. (i) $\dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \dfrac{1}{4 \cdot 5} + \dfrac{1}{5 \cdot 6} + \ldots\ldots$

(ii) $\dfrac{1 \cdot 2}{3^2 \cdot 4^2} + \dfrac{3 \cdot 4}{5^2 \cdot 6^2} + \dfrac{5 \cdot 6}{7^2 \cdot 8^2} + \ldots\ldots$

(Kumaun 2002; Meerut 12B)

(iii) $\dfrac{1}{1 \cdot 2 \cdot 3} + \dfrac{3}{2 \cdot 3 \cdot 4} + \dfrac{5}{3 \cdot 4 \cdot 5} + \ldots\ldots$

(Avadh 2011; Meerut 12)

(iv) $\dfrac{(1+a)(1+b)}{1.2.3} + \dfrac{(2+a)(2+b)}{2.3.4} + \dfrac{(3+a)(3+b)}{3.4.5} + \ldots\ldots$

2. (i) $\dfrac{1}{\sqrt{1}+\sqrt{2}}+\dfrac{1}{\sqrt{2}+\sqrt{3}}+\dfrac{1}{\sqrt{3}+\sqrt{4}}+\ldots\ldots$

(ii) $\dfrac{\sqrt{2}-1}{3^3-1}+\dfrac{\sqrt{3}-1}{4^3-1}+\dfrac{\sqrt{4}-1}{5^3-1}+\dfrac{\sqrt{5}-1}{6^3-1}+\ldots\ldots$

(iii) $1+\dfrac{1}{2\cdot 2^{1/100}}+\dfrac{1}{3\cdot 3^{1/100}}+\dfrac{1}{4\cdot 4^{1/100}}+\ldots\ldots$

3. (i) $\sum\sqrt{\left(\dfrac{n}{n^5+2}\right)}.$

(ii) $\sum\dfrac{1}{(2n-1)^p}.$

(iii) $\sum\left(\dfrac{1}{\sqrt{n}}\sin\dfrac{1}{n}\right).$

(iv) $\sum\cos\dfrac{1}{n}.$ **(Kanpur 2007)**

4. (i) $\Sigma\,[\sqrt{(n+1)}-\sqrt{n}].$

(ii) $\Sigma\,[\sqrt{(n^2+1)}-n].$

(iii) $\Sigma\,[\sqrt{(n^3+1)}-\sqrt{n^3}].$

(iv) $\Sigma\,[\sqrt{(n^4+1)}-n^2].$

5. (i) $\sqrt{\left(\dfrac{1}{2^3}\right)}+\sqrt{\left(\dfrac{2}{3^3}\right)}+\sqrt{\left(\dfrac{3}{4^3}\right)}+\ldots$

(ii) The series whose nth term is $\dfrac{1}{n}\sin\dfrac{1}{n}$. **(Kanpur 2005)**

Answers 1

1. (i) Convergent (ii) Convergent
(iii) Convergent (iv) Divergent

2. (i) Divergent (ii) Convergent
(iii) Convergent

3. (i) Convergent
(ii) Convergent if $p>1$, divergent if $p\le 1$
(iii) Convergent (iv) Divergent

4. (i) Divergent (ii) Divergent
(iii) Convergent (iv) Convergent

5. (i) Divergent (ii) Convergent

2.8 Cauchy's Root Test

Theorem 1: *Let Σu_n be a series of positive terms such that* $\lim u_n^{1/n} = l$. *Then*

(*i*) Σu_n *converges, if* $l < 1$; (*ii*) Σu_n *diverges, if* $l > 1$;

(*iii*) *the test fails and the series may either converge or diverge, if* $l = 1$.

(*Here $u_n^{1/n}$ stands for positive nth root of u_n*).

(Kumaun 2001; Kanpur 04, 07; Avadh 06; Meerut 12)

Proof: Since $u_n > 0$, for all n, and $(u_n)^{1/n}$ stands for positive nth root of u_n, $\lim u_n^{1/n} = l \geq 0$.

Since $\lim u_n^{1/n} = l$, therefore for $\varepsilon > 0$ there exists a positive integer m, such that

$$|u_n^{1/n} - l| < \varepsilon, \text{ for all } n > m,$$

i.e., $$l - \varepsilon < u_n^{1/n} < l + \varepsilon, \text{ for all } n > m,$$

i.e., $$(l - \varepsilon)^n < u_n < (l + \varepsilon)^n, \text{ for all } n > m. \quad \ldots(1)$$

(i) Let $l < 1$.

Choose $\varepsilon > 0$, such that $r = l + \varepsilon < 1$.

Then $0 \leq l < r < 1$.

From (1), we get $u_n < (l + \varepsilon)^n$ for all $n > m$ *i.e.,* $u_n < r^n$ for all $n > m$.

Since Σr^n is a geometric series with common ratio r less than unity, Σr^n is convergent. Therefore, by comparison test, Σu_n is convergent.

(ii) Let $l > 1$.

Choose $\varepsilon > 0$, such that $r = l - \varepsilon > 1$.

From (1), we get $(l - \varepsilon)^n < u_n$ for all $n > m$ *i.e.,* $u_n > r^n$ for all $n > m$.

Since Σr^n is a geometric series with common ratio greater than unity, Σr^n is divergent. Therefore, by comparison test, Σu_n is divergent.

(iii) Let $l = 1$.

Consider the series Σu_n, where $u_n = 1/n$.

Then $$u_n^{1/n} = \left(\frac{1}{n}\right)^{1/n}, \text{ so that } \lim u_n^{1/n} = 1. \text{ [Note that } \lim n^{1/n} = 1].$$

Since $\Sigma (1/n)$ diverges, hence, we observe that if

$\lim u_n^{1/n} = 1$, the series Σu_n may diverge.

Now, consider the series Σu_n, where $u_n = 1/n^2$.

In this case also, $\lim u_n^{1/n} = 1$.

Since $\Sigma (1/n^2)$ converges, hence, we observe that if $\lim u_n^{1/n} = 1$, the series Σu_n may converge.

Thus the above two examples show that Cauchy's root test fails to decide the nature of the series when $l = 1$.

Note 1: In general the Root test is used when powers are involved.

Another form of Cauchy's Root Test: The root test can also be stated in the form given below :

A series Σu_n of positive terms is convergent if for every value of $n \geq m$, m being finite, $(u_n)^{1/n}$ is less than a fixed number which is less than unity.

The series is divergent if $(u_n)^{1/n} \geq 1$ for every value of $n \geq m$.

Proof: **Case 1:** Given $(u_n)^{1/n} < r, \;\forall n \geq m$ where r is a fixed positive number such that $r < 1$.

$\therefore \qquad u_n < r^n$, for all $n \geq m$.

Since Σr^n is a geometric series with common ratio r less than unity, Σr^n is convergent. Therefore, by comparison test, Σu_n is convergent.

Case 2: Given $(u_n)^{1/n} \geq 1, \forall \; n \geq m$.

$\therefore \qquad u_n \geq 1, \forall \; n \geq m.$

Omitting the first $m - 1$ terms of the series because it will not affect the convergence or divergence of the series, we have

$$u_n \geq 1, \forall \; n \in \mathbf{N}$$

$\Rightarrow \qquad S_n = u_1 + \ldots + u_n \geq n \;\Rightarrow\; \lim S_n = \infty$

$\Rightarrow \qquad$ the series is divergent.

Theorem 2: *Let Σu_n be a series of positive terms such that $u_n^{1/n} \to \infty$ as $n \to \infty$. Then Σu_n diverges.*

Proof: Let $r > 1$. Since $u_n^{1/n} \to \infty$ as $n \to \infty$, therefore, there exists a positive integer m such that $u_n^{1/n} > r$ for all $n \geq m \;\Rightarrow\; u_n > r^n$ for all $n \geq m$.

For $r > 1$, the geometric series Σr^n is divergent.

Hence, by comparison test, Σu_n is divergent.

Some important limits to be remembered:

1. $\lim\limits_{n \to \infty} n^{1/n} = 1$.
2. $\lim\limits_{n \to \infty} \dfrac{\log n}{n} = 0$.
3. $\lim\limits_{n \to \infty} \left(1 + \dfrac{x}{n}\right)^n = e^x$.
4. $\lim\limits_{n \to \infty} \left(1 + \dfrac{x}{n}\right)^p = 1$, if p is finite *i.e.*, if p is a fixed real number.
5. $\lim\limits_{n \to \infty} \left(1 + \dfrac{x}{n}\right)^{n+p} = e^x$, if p is finite.

6. $$\lim_{n\to\infty} \frac{a_0 n^p + a_1 n^{p-1} + a_2 n^{p-2} + \ldots + a_{p-1} n + a_p}{b_0 n^q + b_1 n^{q-1} + b_2 n^{q-2} + \ldots + b_{q-1} n + b_q}$$

$$= \lim_{n\to\infty} \frac{n^p [a_0 + a_1 (1/n) + a_2 (1/n)^2 + \ldots.]}{n^q [b_0 + b_1 (1/n) + b_2 (1/n)^2 + \ldots]}$$

$$= \begin{cases} a_0 / b_0, \text{ if } p = q \\ 0, \text{ if } q > p \\ \infty, \text{ if } p > q \text{ and } a_0 > 0, b_0 > 0. \end{cases}$$

Illustrative Examples

Example 14: *Assuming that $n^{1/n} \to 1$ as $n \to \infty$, show by applying Cauchy's nth root test that the series $\sum_{n=1}^{\infty} (n^{1/n} - 1)^n$ converges.*

Solution: Here, $u_n = (n^{1/n} - 1)^n$.

$\therefore \quad u_n^{1/n} = n^{1/n} - 1.$

$\therefore \quad \lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} (n^{1/n} - 1) = 0 < 1.$

Hence, by Cauchy's root test, the given series converges.

Example 15: *Test the convergence of the following series*

(*i*) $\Sigma \left(1 + \frac{1}{n}\right)^{-n^2}$ (*ii*) $\Sigma \frac{x^n}{n!}$

(*iii*) $\frac{1^3}{3} + \frac{2^3}{3^2} + \frac{3^3}{3^3} + \frac{4^3}{3^4} + \ldots + \frac{n^3}{3^n} + \ldots$

Solution: (i) Here $u_n = \left(1 + \frac{1}{n}\right)^{-n^2}$.

$\therefore \quad u_n^{1/n} = \left(1 + \frac{1}{n}\right)^{-n}$.

$\therefore \quad \lim u_n^{1/n} = \lim \frac{1}{\left(1 + \frac{1}{n}\right)^n} = \frac{1}{e} < 1.$ $[\because 2 < e < 3]$

Hence by Cauchy's root test the given series is convergent.

(ii) Here $u_n = \frac{x^n}{n!}$.

$\therefore \quad u_n^{1/n} = \frac{x}{(n!)^{1/n}}$.

$\therefore \quad \lim u_n^{1/n} = \lim \frac{x}{(n!)^{1/n}} = \lim \left[\frac{n}{(n!)^{1/n}} \cdot \frac{x}{n}\right] = \lim \left[\frac{(n^n)^{1/n}}{(n!)^{1/n}} \cdot \frac{x}{n}\right]$

$$= \lim \left[\left(\frac{n^n}{n\,!} \right)^{1/n} \cdot \frac{x}{n} \right] = e \,.\, \lim \frac{x}{n} \qquad \left[\because \lim \left(\frac{n^n}{n\,!} \right)^{1/n} = e \right]$$

$$= e \,.\, 0 = 0 < 1.$$

Hence by Cauchy's root test, the given series is convergent.

(iii) Here $u_n = \dfrac{n^3}{3^n}$.

$\therefore \qquad u_n^{1/n} = \dfrac{n^{3/n}}{3}$.

$\therefore \qquad \lim u_n^{1/n} = \lim \dfrac{1}{3} n^{3/n} = \dfrac{1}{3} \lim (n^{1/n})^3 = \dfrac{1}{3} \cdot 1 < 1.$

Hence by Cauchy's root test the given series is convergent.

Example 16: *Test the convergence of the series*

$$\left(\frac{2^2}{1^2} - \frac{2}{1} \right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2} \right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3} \right)^{-3} + \ldots$$

(Kumaun 2001; Meerut 13B)

Solution: Here $u_n = \left[\dfrac{(n+1)^{n+1}}{n^{n+1}} - \dfrac{n+1}{n} \right]^{-n}$.

$$\therefore \qquad u_n^{1/n} = \left[\left(\frac{n+1}{n} \right)^{n+1} - \frac{n+1}{n} \right]^{-1} = \left(\frac{n+1}{n} \right)^{-1} \left[\left(\frac{n+1}{n} \right)^{n} - 1 \right]^{-1}$$

$$= \left(1 + \frac{1}{n} \right)^{-1} \left[\left(1 + \frac{1}{n} \right)^{n} - 1 \right]^{-1}.$$

$$\therefore \qquad \lim u_n^{1/n} = (1+0)^{-1} \, (e-1)^{-1} = \frac{1}{e-1} < 1. \qquad [\because 2 < e < 3]$$

Hence by Cauchy's root test the given series is convergent.

Example 17: *Test for convergence* $\Sigma\, 3^{-n-(-1)^n}$.

Solution: Here $u_n = 3^{-n-(-1)^n} = \begin{cases} 3^{-n} \,.\, 3^{-1}, & \text{if } n \text{ is even} \\ 3^{-n} \,.\, 3, & \text{if } n \text{ is odd.} \end{cases}$

$$\therefore \qquad u_n^{1/n} = \begin{cases} 3^{-1} \,.\, 3^{-1/n} = \dfrac{1}{3} \cdot \dfrac{1}{3^{1/n}}, & \text{if } n \text{ is even} \\ 3^{-1} \,.\, 3^{1/n} = \dfrac{1}{3} \,.\, 3^{1/n}, & \text{if } n \text{ is odd.} \end{cases}$$

$$\therefore \qquad \lim u_n^{1/n} = \frac{1}{3} < 1. \qquad [\because \lim a^{1/n} = 1 \text{ if } a > 0]$$

Hence by Cauchy's root test the given series is convergent.

Example 18: *Test for convergence* $\Sigma \left(\frac{n+1}{n+2}\right)^n . x^n, \ (x > 0).$ **(Meerut 2013)**

Solution: Here $u_n = \left(\frac{n+1}{n+2}\right)^n x^n.$

$\therefore \qquad u_n^{1/n} = \frac{n+1}{n+2} \cdot x.$

$\therefore \qquad \lim u_n^{1/n} = \lim \left[\frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)} \cdot x\right] = x.$

$\therefore$ By Cauchy's root test, Σu_n converges if $x < 1$ and Σu_n diverges if $x > 1$.

For $x = 1$, the test fails. When $x = 1$, $u_n = \left(\frac{n+1}{n+2}\right)^n$.

$\therefore \qquad \lim u_n = \lim \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n} = \frac{e}{e^2} = \frac{1}{e} > 0.$

$\therefore$ The series Σu_n diverges when $x = 1$.

Hence the given series converges if $x < 1$ and diverges if $x \geq 1$.

Example 19: *Test for convergence* $\Sigma \frac{1}{(\log n)^n}$.

Solution: Here $u_n = \frac{1}{(\log n)^n}$.

$\therefore \qquad u_n^{1/n} = \frac{1}{\log n}$.

$\therefore \qquad \lim u_n^{1/n} = \lim \frac{1}{\log n} = 0$, which is < 1.

Hence by Cauchy's root test the given series is convergent.

Comprehensive Exercise 2

Test for convergence the following series :

1. (i) $\sum \frac{1}{n^{1+(1/n)}}$. **(Kanpur 2008; Avadh 12)**

   (ii) $\sum \left(1+\frac{1}{n}\right)^{n^2}$.

2. (i) $\sum \left(\frac{n}{n+1}\right)^{n^2}$. (Avadh 2013; Kashi 14)

(ii) $\sum \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$.

3. (i) $2+\frac{3}{2}x+\frac{4}{3}x^2+\frac{5}{4}x^3+\dots$, where $x>0$.

(ii) $\sum_{n=1}^{\infty} \frac{3n+1}{4n+3} x^n, x>0$.

4. $\frac{2}{1^2}x+\frac{3^2}{2^3}x^2+\frac{4^3}{3^4}x^3+\dots+\frac{(n+1)^n}{n^{n+1}}x^n+\dots$

5. (i) $1+\frac{x}{2}+\frac{x^2}{3^2}+\frac{x^3}{4^3}+\dots\infty,\ x>0$.

(ii) $x+2x^2+3x^3+4x^4+\dots$.

(iii) $\frac{1}{2}+\left(\frac{2}{3}\right)x+\left(\frac{3}{4}\right)^2 x^2+\left(\frac{4}{5}\right)^3 x^3+\dots\infty, x>0$.

Answers 2

1. (i) Divergent
   (ii) Divergent
2. (i) Convergent
   (ii) Convergent
3. (i) Convergent if $x<1$ and divergent if $x\geq 1$
   (ii) Convergent if $x<1$ and divergent if $x\geq 1$
4. Convergent if $x<1$ and divergent if $x\geq 1$
5. (i) Convergent
   (ii) Convergent if $x<1$ and divergent if $x\geq 1$
   (iii) Convergent if $x<1$ and divergent if $x\geq 1$

2.9 D' Alembert's Ratio Test

(Avadh 2003, 05; Kanpur 05; Meerut 12B; Kashi 14)

Theorem 1: *Let Σu_n be a series of positive terms such that* $\lim \frac{u_{n+1}}{u_n}=l$. *Then*

(i) Σu_n converges if $l<1$ *(ii) Σu_n diverges if $l>1$*

and (iii) the test fails to decide the nature of the series if $l=1$.

Proof: Since $u_n > 0$, for all n, therefore

$$\frac{u_{n+1}}{u_n} > 0 \Rightarrow \lim \frac{u_{n+1}}{u_n} = l \geq 0.$$

Since $\lim \frac{u_{n+1}}{u_n} = l$, therefore, for $\varepsilon > 0$, there exists a positive integer m such that

$$\left| \frac{u_{n+1}}{u_n} - l \right| < \varepsilon, \text{ for all } n \geq m$$

i.e., $$l - \varepsilon < \frac{u_{n+1}}{u_n} < l + \varepsilon, \text{ for all } n \geq m.$$

Putting $n = m, m+1, \ldots, n-1$ in succession in the above inequality and multiplying the corresponding sides of the $(n - m)$ inequalities thus obtained, we get

$$(l - \varepsilon)^{n-m} < \frac{u_n}{u_m} < (l + \varepsilon)^{n-m} \text{ for all } n > m$$

i.e., $$(l - \varepsilon)^n \frac{u_m}{(l - \varepsilon)^m} < u_n < (l + \varepsilon)^n \frac{u_m}{(l + \varepsilon)^m} \text{ for all } n > m. \quad \ldots(1)$$

(i) Let $l < 1$.

Choose $\varepsilon > 0$ such that $r = l + \varepsilon < 1$.

Then $0 \leq l < r < 1$.

From (1), we get $u_n < \left(\frac{u_m}{r^m} \right) r^n$ for all $n > m$

i.e., $u_n < \alpha r^n$ for all $n > m$ where $\alpha = \frac{u_m}{r^m} \in \mathbf{R}^+$.

Since Σr^n is a geometric series with common ratio less than unity, Σr^n is convergent. Hence by comparison test, Σu_n is convergent.

(ii) Let $l > 1$.

Choose $\varepsilon > 0$ such that $r = l - \varepsilon > 1$.

From (1), we get $\frac{u_m}{r^m} r^n < u_n$, for all $n > m$

i.e., $u_n > \beta r^n$, for all $n > m$ where $\beta = \frac{u_m}{r^m} \in \mathbf{R}^+$.

Since Σr^n is a geometric series with common ratio greater than unity, Σr^n is divergent. Therefore, by comparison test, Σu_n is divergent.

(iii) Let $l = 1$.

Consider the series Σu_n where $u_n = 1/n^2$.

Here $$\lim \frac{u_{n+1}}{u_n} = \lim \frac{n^2}{(n+1)^2} = \lim \frac{1}{\left(1 + \frac{1}{n}\right)^2} = 1.$$

Since the series $\Sigma\,(1/n^2)$ converges, we observe that if $l = 1$, the series may be convergent.

Now, consider the series $\Sigma\, u_n$, where $u_n = 1/n$.

Here $\qquad \lim \dfrac{u_{n+1}}{u_n} = \lim \dfrac{n}{n+1} = \lim \dfrac{1}{1+\dfrac{1}{n}} = 1\cdot$

Since the series $\Sigma\,(1/n)$ diverges, we observe that if $l = 1$, the series may be divergent.

Thus the above two examples show that the test fails to decide the nature when $l = 1$.

Note 1: Taking the reciprocals, the ratio test can also be stated in the form given below.

The series $\Sigma\, u_n$ of positive terms is convergent if lim $\dfrac{u_n}{u_{n+1}} > 1$ and divergent if lim $\dfrac{u_n}{u_{n+1}} < 1$.

If lim $\dfrac{u_n}{u_{n+1}} = 1$, the test fails.

We shall usually apply the ratio test in this form which will in the later part of this chapter be more convenient for further investigation in case the ratio test fails.

The ratio test is generally applied when the nth term of the series involves factorials, products of several factors, or combinations of powers and factorials.

Another form of D' Alembert's Ratio Test: The ratio test can also be stated in the form given below :

An infinite series of positive terms is convergent if from and after some term the ratio of each term to the preceding term is less than a fixed number which is less than unity.

The series is divergent if the above ratio is greater than or equal to unity.

Proof: Case 1: It is given that

$$\frac{u_{n+1}}{u_n} < r \text{ for all } n \geq m, \qquad \text{...(1)}$$

where r is a fixed positive number such that $r < 1$.

To prove $\Sigma\, u_n$ is convergent.

Putting $n = m, m+1, \ldots, n-1$ in succession in (1) and multiplying the corresponding sides of the $n - m$ inequalities thus obtained, we get

$$\frac{u_{m+1}}{u_m} \cdot \frac{u_{m+2}}{u_{m+1}} \cdot \frac{u_{m+3}}{u_{m+2}} \cdot \ldots \cdot \frac{u_n}{u_{n-1}} < r^{n-m}$$

$$\Rightarrow \qquad \frac{u_n}{u_m} < r^{n-m} \quad \Rightarrow \quad u_n < \frac{u_m}{r^m}\, r^n$$

$$\Rightarrow \qquad u_n < \alpha\, r^n, \text{ for all } n > m \text{ where } \alpha = \frac{u_m}{r^m} \in \mathbf{R}^+.$$

Since $\Sigma\, r^n$ is a geometric series with common ratio less than unity, $\Sigma\, r^n$ is convergent. Hence by comparison test, $\Sigma\, u_n$ is also convergent.

Case 2: It is given that

$$\frac{u_{n+1}}{u_n} \geq 1 \text{ for all } n \geq m. \qquad \text{...(2)}$$

Putting $n = m, m+1, \ldots, n-1$ in succession in (2) and multiplying the corresponding sides of the $n - m$ inequalities thus obtained, we get

$$\frac{u_n}{u_m} \geq 1 \quad \Rightarrow \quad u_n \geq u_m \text{ for all } n > m.$$

Omitting the first m terms of the series because it will not affect the convergence or divergence of the series, we have

$$u_n \geq u_m \text{ for all } n \in \mathbf{N}$$

$\Rightarrow$ $S_n = u_1 + \ldots + u_n \geq n\, u_m$

$\Rightarrow$ $\lim S_n = \infty$

$\Rightarrow$ the series is divergent.

Theorem 2: *Let $\Sigma\, u_n$ be a series of positive terms such that $\dfrac{u_{n+1}}{u_n} \to \infty$ as $n \to \infty$. Then $\Sigma\, u_n$ diverges.*

Proof: Since $\dfrac{u_{n+1}}{u_n} \to \infty$ as $n \to \infty$, therefore, there exists a positive integer m such that

$$\frac{u_{n+1}}{u_n} > 2 \text{ for all } n \geq m \quad \textit{i.e.,} \quad u_{n+1} > 2\, u_n \text{ for all } n \geq m.$$

Replacing n by $m, m+1, m+2, \ldots, n-1$ and multiplying the $(n - m)$ inequalities, we get

$$u_n > 2^{n-m} \,.\, u_m \text{ for all } n > m$$

i.e., $$u_n > \left(\frac{u_m}{2^m}\right) 2^n \text{ for all } n > m.$$

Since the geometric series $\Sigma\, 2^n$ diverges, hence, by comparison test $\Sigma\, u_n$ diverges.

Note: In a similar manner it can be proved that $\Sigma\, u_n$ is convergent if

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \infty.$$

2.10 Remarks on the Ratio Test

It should be noted that D' Alembert's ratio test does not tell us anything about the convergence of the series $\Sigma\, u_n$ if we only know that $\dfrac{u_n}{u_{n+1}} > 1\, \forall\, n$.

If $u_n = \dfrac{1}{n}$, then $\dfrac{u_n}{u_{n+1}} = 1 + \dfrac{1}{n} > 1$ for all n while the series $\Sigma\, u_n$ is divergent. Also, for the convergence of the series $\Sigma\, u_n$ it is not necessary that $\dfrac{u_n}{u_{n+1}}$ should have a definite limit. For a change in the order of the terms of a series of positive terms may affect the value of $\lim \dfrac{u_n}{u_{n+1}}$ but it does not affect the convergence of the series.

For example, let us consider the series

$$1 + x + x^2 + x^3 + \ldots \text{ where } 0 < x < 1. \qquad \ldots(1)$$

Changing the order of terms, the series becomes

$$x + 1 + x^3 + x^2 + x^5 + x^4 + \ldots \qquad \ldots(2)$$

Since the series (1) is convergent, therefore, the series (2) is also convergent. But in the series (2), the ratio u_n / u_{n+1} is alternately x and $1/x^3$ and consequently $\lim\,(u_n/u_{n+1})$ is not definite.

In comparison with Cauchy's root test, D'Alembert's ratio test is more useful since it is easier to apply than the root test because generally u_n / u_{n+1} is a simpler fraction than u_n. However **the root test is stronger than the ratio test**. To be more precise, whenever the ratio test indicates the nature of the series, the root test does too. But sometimes the ratio test does not apply while the root test succeeds.

Illustrative Examples

Example 20: *Test for convergence the following series :*

(*i*) $1 + \dfrac{2^p}{2\,!} + \dfrac{3^p}{3\,!} + \dfrac{4^p}{4\,!} + \ldots$ **(Bundelkhand 2006)**

(*ii*) $\dfrac{1}{1+2} + \dfrac{2}{1+2^2} + \dfrac{3}{1+2^3} + \ldots$

Solution: (i) Here $u_n = \dfrac{n^p}{n\,!}$.

$$\therefore \quad u_{n+1} = \frac{(n+1)^p}{(n+1)\,!}.$$

Now $$\frac{u_n}{u_{n+1}} = \frac{n^p}{n\,!} \cdot \frac{(n+1)\,!}{(n+1)^p} = \frac{(n+1)\,n^p}{(n+1)^p} = \frac{n+1}{(1+1/n)^p}.$$

$$\therefore \quad \lim \frac{u_n}{u_{n+1}} = \lim \frac{n+1}{(1+1/n)^p} = \infty,$$

which is > 1 for all values of p.

Hence by ratio test the series $\Sigma\, u_n$ is convergent.

(ii) Here $u_n = \dfrac{n}{1+2^n}$.

$\therefore \quad u_{n+1} = \dfrac{n+1}{1+2^{n+1}}$.

Now $\dfrac{u_n}{u_{n+1}} = \dfrac{n}{1+2^n} \cdot \dfrac{1+2^{n+1}}{n+1} = \dfrac{n \,.\, 2^{n+1}\,(1+1/2^{n+1})}{2^n\,(1+1/2^n)\,.\,n\,(1+1/n)}$

$$= \frac{2\,(1+1/2^{n+1})}{(1+1/2^n)\,(1+1/n)}.$$

$\therefore \quad \lim \dfrac{u_n}{u_{n+1}} = 2 \cdot \dfrac{(1+0)}{(1+0)\,(1+0)} = 2$, which is > 1.

Therefore, by ratio test, the given series converges.

Example 21: *Test for convergence the series whose nth term is*

(*i*) $\dfrac{n^3+a}{2^n+a}$, (*ii*) $\dfrac{n\,!}{n^n}$, **(Purvanchal 2014)** (*iii*) $\sqrt{\left\{\dfrac{2^n-1}{3^n-1}\right\}}$.

Solution: (i) Here $u_n = \dfrac{n^3+a}{2^n+a}$, $u_{n+1} = \dfrac{(n+1)^3+a}{2^{n+1}+a}$.

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{n^3+a}{2^n+a} \cdot \frac{2^{n+1}+a}{(n+1)^3+a}$$

$$= \frac{n^3\,(1+a/n^3)\,.\,2^{n+1}\,(1+a/2^{n+1})}{2^n\,(1+a/2^n)\,.\,n^3\,\{(1+1/n)^3+a/n^3\}}$$

$$= 2 \cdot \frac{(1+a/n^3)\,(1+a/2^{n+1})}{(1+a/2^n)\,\{(1+1/n)^3+a/n^3\}}.$$

Now $\lim \dfrac{u_n}{u_{n+1}} = 2 \cdot \dfrac{(1+0)\,(1+0)}{(1+0)\,\{(1+0)^3+0\}} = 2$, which is > 1.

Therefore, by ratio test the given series converges.

(ii) Here $u_n = \dfrac{n\,!}{n^n}$ so that $u_{n+1} = \dfrac{(n+1)\,!}{(n+1)^{n+1}}$.

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{n\,!}{n^n} \,.\, \frac{(n+1)^{n+1}}{(n+1)\,!} = \frac{(n+1)^{n+1}}{n^n\,.\,(n+1)} = \left(1+\frac{1}{n}\right)^n.$$

$\therefore \quad \lim \dfrac{u_n}{u_{n+1}} = \lim \left(1+\dfrac{1}{n}\right)^n = e$, which is > 1.

Therefore, by ratio test the given series converges.

(iii) Here $u_n = \sqrt{\left(\frac{2^n - 1}{3^n - 1}\right)}, u_{n+1} = \sqrt{\left(\frac{2^{n+1} - 1}{3^{n+1} - 1}\right)}$.

Now $$\frac{u_n}{u_{n+1}} = \sqrt{\left(\frac{2^n - 1}{3^n - 1} \cdot \frac{3^{n+1} - 1}{2^{n+1} - 1}\right)} = \sqrt{\left\{\frac{2^n (1 - 1/2^n) . 3^n (3 - 1/3^n)}{3^n (1 - 1/3^n) . 2^n (2 - 1/2^n)}\right\}}$$

$$= \sqrt{\left\{\frac{(1 - 1/2^n)(3 - 1/3^n)}{(1 - 1/3^n)(2 - 1/2^n)}\right\}}.$$

$\therefore$ $\lim \frac{u_n}{u_{n+1}} = \sqrt{\left(\frac{3}{2}\right)}$, which is > 1.

Therefore, by ratio test the given series converges.

Example 22: *Show that the series*

$$1 + \frac{\alpha + 1}{\beta + 1} + \frac{(\alpha + 1)(2\alpha + 1)}{(\beta + 1)(2\beta + 1)} + \frac{(\alpha + 1)(2\alpha + 1)(3\alpha + 1)}{(\beta + 1)(2\beta + 1)(3\beta + 1)} + ...$$

converges if $\beta > \alpha > 0$ *and diverges if* $\alpha \geq \beta > 0$ $[\alpha > 0, \beta > 0]$.

Solution: Here,

$$u_n = \frac{(\alpha + 1)(2\alpha + 1)......[(n-1)\alpha + 1]}{(\beta + 1)(2\beta + 1)......[(n-1)\beta + 1]},$$

so that $$u_{n+1} = \frac{(\alpha + 1)(2\alpha + 1)......[(n-1)\alpha + 1](n\alpha + 1)}{(\beta + 1)(2\beta + 1)......[(n-1)\beta + 1](n\beta + 1)}.$$

Now $$\frac{u_n}{u_{n+1}} = \frac{n\beta + 1}{n\alpha + 1} = \frac{\beta + 1/n}{\alpha + 1/n}.$$

$\therefore$ $$\lim \frac{u_n}{u_{n+1}} = \lim \frac{\beta + 1/n}{\alpha + 1/n} = \frac{\beta}{\alpha}.$$

Hence by ratio test the series is convergent if $\frac{\beta}{\alpha} > 1$ *i.e.,* if $\beta > \alpha > 0$, divergent if $\frac{\beta}{\alpha} < 1$, *i.e.,* if $\alpha > \beta > 0$, and the test fails if $\frac{\beta}{\alpha} = 1$ *i.e.,* if $\beta = \alpha$.

When $\beta = \alpha$, then the given series becomes

$$1 + 1 + 1 + ...$$

S_n = the sum of n terms of this series = n.

Since $\lim S_n = \infty$, hence the series is divergent.

Thus the given series is convergent if $\beta > \alpha > 0$ and divergent if $\alpha \geq \beta > 0$.

Example 23: *Test for convergence the following series* :

(*i*) $1 + 3x + 5x^2 + 7x^3 + ...$ (*ii*) $1 + \frac{x}{2^2} + \frac{x^2}{3^2} + \frac{x^3}{4^2} + ...$

Solution: (i) Here $u_n = (2n - 1)\, x^{n-1}$, $u_{n+1} = (2n + 1)\, x^n$.

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{(2\,n - 1)\, x^{n-1}}{(2n + 1)\, x^n} = \frac{(2 - 1/n)}{(2 + 1/n)} \cdot \frac{1}{x} \cdot$$

$$\therefore \quad \lim \frac{u_n}{u_{n+1}} = \frac{2}{2} \cdot \frac{1}{x} = \frac{1}{x} \cdot$$

Hence by ratio test the series is convergent if $1/x > 1$ *i.e.* if

$$1 > x \quad \text{or} \quad x < 1,$$

the series is divergent if $1/x < 1$, *i.e.* if $x > 1$ and the test fails if $1/x = 1$ *i.e.* if $x = 1$.

When $x = 1$, then the given series becomes

$$1 + 3 + 5 + 7 + \ldots$$

S_n = sum of n terms of this series $= \dfrac{n}{2}(1 + 2n - 1) = n^2$.

Since $\lim S_n = \infty$, hence this series is divergent.

Thus the given series converges if $x < 1$ and diverges if $x \geq 1$.

(ii) Here $u_n = \dfrac{x^{n-1}}{n^2}$, so that $u_{n+1} = \dfrac{x^n}{(n+1)^2} \cdot$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{x^{n-1}}{n^2} \cdot \frac{(n+1)^2}{x^n} = \left(1 + \frac{1}{n}\right)^2 \cdot \frac{1}{x} \cdot$$

$$\therefore \quad \lim \frac{u_n}{u_{n+1}} = \lim \left(1 + \frac{1}{n}\right)^2 \cdot \frac{1}{x} = \frac{1}{x}.$$

Hence by ratio test the series converges if $1/x > 1$ *i.e.* if $x < 1$, diverges if $1/x < 1$ *i.e.* if $x > 1$ and the test fails if $1/x = 1$ *i.e.* if $x = 1$.

When $x = 1$, then $u_n = 1/n^2$. We know that $\Sigma\,(1/n^2)$ is convergent because here $p = 2 > 1$.

Thus the given series converges if $x \leq 1$ and diverges if $x > 1$.

Example 24: *Test for convergence the series whose nth term is*

(i) $\dfrac{1}{x^n + x^{-n}}$, *(ii)* $\dfrac{a^n}{x^n + a^n} \cdot$

Solution: (i) Here $u_n = \dfrac{1}{x^n + x^{-n}} = \dfrac{x^n}{x^{2n} + 1}$, $u_{n+1} = \dfrac{x^{n+1}}{x^{2\,(n+1)} + 1} \cdot$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{x^n}{x^{2n} + 1} \cdot \frac{x^{2\,(n+1)} + 1}{x^{n+1}} = \frac{x^{2n+2} + 1}{x^{2n} + 1} \cdot \frac{1}{x} \cdot$$

Now (u_n / u_{n+1}) can be found only if we know that

$$x < 1 \quad \text{or} \quad x > 1.$$

Let $\quad x < 1.$

Then $\lim \dfrac{u_n}{u_{n+1}} = \lim \left[\dfrac{x^{2n+2}+1}{x^{2n}+1} \cdot \dfrac{1}{x}\right]$

$= \dfrac{1}{x}$. $\quad [\because \lim x^{2n+2} = 0 = \lim x^{2n} \text{ if } x < 1]$

But if $x < 1$, then $1/x > 1$.

$\therefore$ if $x < 1$, we have $\lim (u_n / u_{n+1}) > 1$ and hence by ratio test the series converges in this case.

Now let $x > 1$.

Then $\lim \dfrac{u_n}{u_{n+1}} = \lim \left[\dfrac{x^{2n+2}+1}{x^{2n}+1} \cdot \dfrac{1}{x}\right] = \lim \left[\dfrac{x^{2n+2}(1+1/x^{2n+2})}{x^{2n}(1+1/x^{2n})} \cdot \dfrac{1}{x}\right]$

$= \lim \left[x \dfrac{(1+1/x^{2n+2})}{(1+1/x^{2n})}\right]$

$= x \quad [\because \lim 1/x^{2n+2} = 0 \text{ if } x > 1]$

$\therefore$ if $x > 1$, we have $\lim (u_n / u_{n+1}) = x$ *i.e.* > 1 and hence by ratio test the series is convergent in this case also.

Again, if $x = 1$, then $u_n = \dfrac{1}{1+1} = \dfrac{1}{2}$,

i.e., the series becomes $\dfrac{1}{2} + \dfrac{1}{2} + \dfrac{1}{2} + \ldots$

S_n = sum of its n terms $= \dfrac{1}{2} \cdot n$.

Since $\lim S_n = \infty$, hence, the series is divergent if $x = 1$.

Thus the given series is convergent if $x > 1$ or $x < 1$ and divergent if $x = 1$.

(ii) Here $u_n = \dfrac{a^n}{x^n + a^n}, u_{n+1} = \dfrac{a^{n+1}}{x^{n+1} + a^{n+1}}$.

$\therefore \quad \dfrac{u_n}{u_{n+1}} = \dfrac{a^n}{x^n + a^n} \cdot \dfrac{x^{n+1} + a^{n+1}}{a^{n+1}} = \dfrac{x^{n+1} + a^{n+1}}{a(x^n + a^n)}$.

Let $x > a$.

Then $\lim \dfrac{u_n}{u_{n+1}} = \lim \dfrac{x^{n+1} + a^{n+1}}{a(x^n + a^n)} = \lim \dfrac{x^{n+1}[1 + (a/x)^{n+1}]}{ax^n [1 + (a/x)^n]}$

$= \lim \dfrac{x}{a} \dfrac{[1 + (a/x)^{n+1}]}{[1 + (a/x)^n]} = \dfrac{x}{a}$, which is > 1 as $x > a$.

Hence by ratio test the given series converges if $x > a$.

Let $x < a$.

Then $\lim \dfrac{u_n}{u_{n+1}} = \lim \dfrac{a^{n+1}\,[1+(x/a)^{n+1}]}{a\,.\,a^n\,[1+(x/a)^n]} = \lim \dfrac{[1+(x/a)^{n+1}]}{[1+(x/a)^n]} = 1.$

$\therefore$ the ratio test fails in this case.

But in this case, $\lim u_n = \lim \dfrac{a^n}{x^n + a^n} = \lim \dfrac{a^n}{a^n\,[1+(x/a)^n]} = 1$, which is > 0.

$\therefore$ the given series diverges if $x < a$.

Now, if $x = a$, the series is $\dfrac{1}{2} + \dfrac{1}{2} + \ldots$, which diverges.

Hence the given series is convergent if $x > a$ and divergent if $x \le a$.

Example 25: *Test for convergence the following series*

(*i*) $\dfrac{1}{2\sqrt{1}} + \dfrac{x^2}{3\sqrt{2}} + \dfrac{x^4}{4\sqrt{3}} + \dfrac{x^6}{5\sqrt{4}} + \ldots$

(*ii*) $x + \dfrac{3}{5}x^2 + \dfrac{8}{10}x^3 + \dfrac{15}{17}x^4 + \ldots + \dfrac{n^2-1}{n^2+1}x^n + \ldots.$

(Avadh 2012)

Solution: (i) Here $u_n = \dfrac{x^{2n-2}}{(n+1)\sqrt{n}}$, $u_{n+1} = \dfrac{x^{2n}}{(n+2)\sqrt{(n+1)}}\cdot$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{x^{2n-2}}{(n+1)\sqrt{n}} \cdot \frac{(n+2)\sqrt{(n+1)}}{x^{2n}} = \frac{(1+2/n)}{(1+1/n)}\sqrt{\left(1+\frac{1}{n}\right)} \cdot \frac{1}{x^2}\cdot$$

$$\therefore \quad \lim \frac{u_n}{u_{n+1}} = \frac{1}{1}\,.\sqrt{1}\,.\frac{1}{x^2} = \frac{1}{x^2}\cdot$$

$\therefore$ by ratio test the given series is convergent if $1/x^2 > 1$ *i.e.*, if $x^2 < 1$, divergent if $1/x^2 < 1$ *i.e.*, if $x^2 > 1$ and the test fails if $x^2 = 1$.

When $x^2 = 1$, we have $u_n = \dfrac{1}{(n+1)\sqrt{n}}\cdot$ Take $v_n = \dfrac{1}{n\sqrt{n}}\cdot$

$$\therefore \quad \lim \frac{u_n}{v_n} = \lim \frac{n\sqrt{n}}{(n+1)\sqrt{n}} = \lim \frac{1}{(1+1/n)} = 1,$$

which is finite and non-zero. Hence by comparison test $\Sigma\, u_n$ and $\Sigma\, v_n$ are either both convergent or both divergent.

Since $\Sigma\, v_n = \Sigma\,(1/n^{3/2})$ is convergent as $p = 3/2 > 1$, therefore the given series is also convergent if $x^2 = 1$.

Thus the given series is convergent if $x^2 \le 1$ and divergent if $x^2 > 1$.

(ii) Here $u_n = \dfrac{n^2-1}{n^2+1}x^n$, $u_{n+1} = \dfrac{(n+1)^2-1}{(n+1)^2+1}x^{n+1}.$

$\therefore \qquad \dfrac{u_n}{u_{n+1}} = \dfrac{n^2 - 1}{n^2 + 1} x^n \cdot \dfrac{(n+1)^2 + 1}{(n+1)^2 - 1} \cdot \dfrac{1}{x^{n+1}}$

$\qquad\qquad = \dfrac{1 - 1/n^2}{1 + 1/n^2} \cdot \dfrac{1 + 2/n + 2/n^2}{1 + 2/n} \cdot \dfrac{1}{x} \cdot$

$\therefore \qquad \lim \dfrac{u_n}{u_{n+1}} = \dfrac{1}{x} \cdot$

$\therefore$ by ratio test the given series is convergent if $1/x > 1$ *i.e.*, if $x < 1$, divergent if $1/x < 1$ *i.e.,* if $x > 1$ and the test fails if $x = 1$.

When $x = 1$, $u_n = \dfrac{n^2 - 1}{n^2 + 1} = \dfrac{1 - 1/n^2}{1 + 1/n^2} \cdot$

$\therefore \qquad \lim u_n = \lim \dfrac{1 - 1/n^2}{1 + 1/n^2} = 1$, which is > 0.

$\therefore \qquad$ the given series is divergent if $x = 1$.

Thus the given series is convergent if $x < 1$ and divergent if $x \geq 1$.

Comprehensive Exercise 3

Test for convergence the following series :

1. (i) $2x + \dfrac{3x^2}{8} + \dfrac{4x^3}{27} + \ldots + \dfrac{(n+1)\, x^n}{n^3} + \ldots$

(Kumaun 2003; Kanpur 11; Meerut 12,12B)

(ii) $\dfrac{x^2}{2\sqrt{1}} + \dfrac{x^3}{3\sqrt{2}} + \dfrac{x^4}{4\sqrt{3}} + \ldots, x > 0.$

(iii) $1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots, x > 0.$

2. $\dfrac{1^2 \cdot 2^2}{1!} + \dfrac{2^2 \cdot 3^2}{2!} + \dfrac{3^2 \cdot 4^2}{3!} + \ldots$

3. (i) $1 + \dfrac{x}{2} + \dfrac{x^2}{5} + \dfrac{x^3}{10} + \ldots + \dfrac{x^n}{n^2 + 1} + \ldots$

(ii) $1 + \dfrac{2}{5} x + \dfrac{6}{9} x^2 + \dfrac{14}{17} x^3 + \ldots + \dfrac{2^n - 2}{2^n + 1} x^{n-1} + \ldots$

4. $\frac{x}{1\cdot 2}+\frac{x^2}{2\cdot 3}+\frac{x^3}{3\cdot 4}+\dots\dots.$

5. $\frac{1}{3}+\frac{1.2}{3.5}+\frac{1.2.3}{3.5.7}+\frac{1.2.3.4}{3.5.7.9}+\dots$

6. (i) $\sum \frac{n!\,3^n}{n^n}.$ (ii) $\sum \frac{n^3}{(n-1)!}.$

7. (i) $\sum \left(\frac{3n-1}{2^n}\right)$ (ii) $\sum \left(\frac{x^n}{x+n}\right).$

8. (i) $\sum \frac{x^n}{a+\sqrt{n}}$ (ii) $\sum \frac{n^n}{n!}.$

9. Test for convergence the series whose n th term is

(i) $\frac{n^2(n+1)^2}{n!}$ (ii) $\frac{2^n-1}{3^n+1}$

(iii) $\frac{\sqrt{n}}{\sqrt{(n^2+1)}}x^n, (x>0)$ (iv) $\frac{n^3-1}{n^3+1}x^n, (x>0)$

(v) $\sqrt{\left(\frac{n-1}{n^3+1}\right)}x^n, (x>0)$ (vi) $\frac{3^n-2}{3^n+1}x^{n-1}, (x>0).$

(vii) $\frac{x^n}{x^n+a^n}, x>0, a>0.$

10. Examine the convergence of the series

$$\frac{1}{1^p}+\frac{x}{3^p}+\frac{x^2}{5^p}+\frac{x^3}{7^p}+\dots$$

Answers 3

1. (i) Convergent if $x \le 1$ and divergent if $x > 1$
   (ii) Convergent if $x \le 1$ and divergent if $x > 1$
   (iii) Convergent for all real values of x
2. Convergent
3. (i) Convergent if $x \le 1$ and divergent if $x > 1$
   (ii) Convergent if $x < 1$ and divergent if $x \ge 1$
4. Convergent if $x \le 1$ and divergent if $x > 1$
5. Convergent
6. (i) Divergent (ii) Convergent
7. (i) Convergent (ii) Convergent if $x < 1$ and divergent if $x \ge 1$

8. (i) Convergent if $x < 1$ and divergent if $x \geq 1$

(ii) Divergent

9. (i) Convergent

(ii) Convergent

(iii) Convergent if $x < 1$ and divergent if $x \geq 1$

(iv) Convergent if $x < 1$ and divergent if $x \geq 1$

(v) Convergent if $x < 1$ and divergent if $x \geq 1$

(vi) Convergent if $x < 1$ and divergent if $x \geq 1$

(vii) Convergent if $x < a$ and divergent if $x \geq a$

10. Convergent if $x < 1$ and divergent if $x > 1$

In case $x = 1$, then convergent if $p > 1$ and divergent if $p \leq 1$.

2.11 Cauchy's Condensation Test

(Avadh 2012)

Theorem: *If the function $f(n)$ is positive for all positive integral values of n and continually decreases as n increases, then the two infinite series*

$$f(1) + f(2) + f(3) + \dots + f(n) + \dots$$

and

$$a\,f(a) + a^2 f(a^2) + a^3 f(a^3) + \dots + a^n f(a^n) + \dots$$

are either both convergent or both divergent, a being a positive integer greater than unity.

Proof: The terms in the series $\Sigma f(n)$ can be arranged as

$$\{f(1) + f(2) + f(3) + \dots + f(a)\} + \{f(a+1) + f(a+2) + \dots + f(a^2)\} + \{f(a^2+1) + f(a^2+2) + \dots + f(a^3)\} + \dots$$
$$\dots + \{f(a^m+1) + f(a^m+2) + \dots + f(a^{m+1})\} + \dots \quad \dots(1)$$

The terms in the $(m+1)$ th group are

$$f(a^m+1) + f(a^m+2) + \dots + f(a^{m+1}). \quad \dots(2)$$

The number of terms in this group is $(a^{m+1} - a^m)$ *i.e.,* $a^m(a-1)$. Also $f(a^{m+1})$ is the smallest term in this group since the terms go on decreasing.

$$\therefore \quad f(a^m+1) + f(a^m+2) + \dots + f(a^{m+1}) > a^m(a-1)\,f(a^{m+1})$$

or

$$f(a^m+1) + f(a^m+2) + \dots + f(a^{m+1}) > \frac{a-1}{a}\{a^{m+1} f(a^{m+1})\}. \quad \dots(3)$$

Putting $m = 0, 1, 2, 3, \dots$ successively in (3), we have

$$f(2) + f(3) + \dots + f(a) > \frac{a-1}{a}\{a\,f(a)\}$$

$$f(a+1) + f(a+2) + \dots + f(a^2) > \frac{a-1}{a}\{a^2 f(a^2)\}$$

$$f(a^2+1)+f(a^2+2)+\dots+f(a^3)>\frac{a-1}{a}\{a^3 f(a^3)\}$$

... ...

... ...

Adding all the above inequalities, we get

$$\Sigma f(n)-f(1)>\frac{a-1}{a}\Sigma[a^n f(a^n)].$$

This shows that if the series $\Sigma\, a^n f(a^n)$ is divergent, so also is the series $\Sigma f(n)$.
Again, each term of the $(m+1)$ th group given by (2) is less than $f(a^m)$. Hence, we have

$$f(a^m+1)+f(a^m+2)+\dots+f(a^{m+1})$$
$$<f(a^m)+f(a^m)+\dots+f(a^m)=a^m(a-1)f(a^m)$$

i.e. $$f(a^m+1)+f(a^m+2)+\dots+f(a^{m+1})<(a-1)\{a^m f(a^m)\}. \quad \dots(4)$$

Putting $m=0,1,2,3,\dots$ successively in (4), we have

$$f(2)+f(3)+\dots+f(a)<(a-1)\{f(1)\}$$
$$f(a+1)+f(a+2)+\dots+f(a^2)<(a-1)\{a f(a)\}$$
$$f(a^2+1)+f(a^2+2)+\dots+f(a^3)<(a-1)\{a^2 f(a^2)\}$$

... ...

... ...

Adding all these inequalities, we get

$$\Sigma f(n)-f(1)<(a-1)f(1)+(a-1)\Sigma a^n f(a^n).$$

This shows that if $\Sigma\, a^n f(a^n)$ is convergent, so also is $\Sigma f(n)$.

Note: For the validity of the above theorem it is sufficient if $f(n)$ be positive and constantly decreases for values of n greater than a fixed positive integer r.

2.12 The Auxiliary Series $\Sigma \dfrac{1}{n(\log n)^p}$

Theorem: *The series*

$$1+\frac{1}{2(\log 2)^p}+\frac{1}{3(\log 3)^p}+\dots+\frac{1}{n(\log n)^p}+\dots$$

is convergent if $p>1$ *and divergent if* $p\le 1$.

Proof: Case 1: Let $p\le 0$.

Then $\dfrac{1}{n(\log n)^p}\ge\dfrac{1}{n}$ for all $n\ge 2$.

Since the series $\Sigma(1/n)$ is divergent, therefore by comparison test $\Sigma\dfrac{1}{n(\log n)^p}$ is also divergent.

Case 2: Let $p > 0$. Let $f(n) = \dfrac{1}{n(\log n)^p}$.

Obviously $f(n) > 0$ for all $n \geq 2$.

Now the given series $\Sigma \dfrac{1}{n(\log n)^p} = \Sigma f(n)$.

Since $< n(\log n)^p >$ is an increasing sequence, therefore $< f(n) >$ is a decreasing sequence. Hence by Cauchy's condensation test given in article 2.11, the series $\Sigma f(n)$ is convergent or divergent according as the series $\Sigma a^n f(a^n)$ is convergent or divergent.

Now $\quad a^n f(a^n) = \dfrac{a^n}{a^n(\log a^n)^p} = \dfrac{1}{(n \log a)^p} = \dfrac{1}{(\log a)^p} \cdot \dfrac{1}{n^p}$.

Since $\dfrac{1}{(\log a)^p}$ is a constant, hence the series $\Sigma a^n f(a^n)$ is convergent or divergent according as the series $\Sigma (1/n^p)$ is convergent or divergent. But the series $\Sigma 1/n^p$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Hence by Cauchy's condensation test the given series is also convergent if $p > 1$ and divergent if $p \leq 1$.

Illustrative Examples

Example 26: *Test the convergence of the following series :*

(*i*) $\dfrac{1}{\log 2} + \dfrac{1}{\log 3} + \dfrac{1}{\log 4} + \ldots$ $\qquad$ (*ii*) $\dfrac{\log 2}{2} + \dfrac{\log 3}{3} + \dfrac{\log 4}{4} + \ldots$

(*iii*) $\dfrac{1}{(\log 2)^p} + \dfrac{1}{(\log 3)^p} + \ldots + \dfrac{1}{(\log n)^p} + \ldots$

Solution: (i) Here $f(n) = \dfrac{1}{\log n} > 0$ for all $n \geq 2$. Also $f(n)$ decreases continually as n increases.

Now $a^n f(a^n) = \dfrac{a^n}{\log(a^n)} = \dfrac{a^n}{n \log a}$, a being taken as some positive integer > 1.

Consider the series $\Sigma a^n f(a^n) = \Sigma \{ a^n / (n \log a) \} = \Sigma v_n$, (say).

Here $v_n = \dfrac{a^n}{n \log a}$, so that $v_{n+1} = \dfrac{a^{n+1}}{(n+1) \log a}$.

$$\therefore \quad \frac{v_n}{v_{n+1}} = \frac{n+1}{n} \cdot \frac{1}{a} = \left(1 + \frac{1}{n}\right) \cdot \frac{1}{a}.$$

$$\therefore \quad \lim \frac{v_n}{v_{n+1}} = \frac{1}{a} \text{ which is} < 1 \text{ as by our choice } a > 1.$$

$\therefore$ by ratio test the series $\Sigma v_n = \Sigma a^n f(a^n)$ is divergent.

Consequently by Cauchy's condensation test the given series

$$\Sigma f(n) = \frac{1}{\log 2} + \frac{1}{\log 3} + \ldots,$$ is also divergent.

(ii) Here $f(n) = \dfrac{\log n}{n} > 0$ for all $n \geq 2$.

Also $f(n)$ decreases continually as n increases.

Now $a^n f(a^n) = a^n \left(\dfrac{\log a^n}{a^n} \right) = n \log a$, a being taken as some + ive integer > 1.

Now the series $\Sigma a^n f(a^n) = \Sigma (n \log a) = \log a \,.\, \Sigma n$
is divergent because the series Σn is divergent.

Hence by Cauchy's condensation test the given series $\Sigma f(n) = \Sigma \dfrac{\log n}{n}$ is also divergent.

(iii) If $p \leq 0$, the given series is obviously divergent. So let us consider the case when $p > 0$. Here $f(n) = \dfrac{1}{(\log n)^p} > 0$ for all $n \geq 2$.

Also $f(n)$ decreases continually as n increases.

Now $a^n f(a^n) = \dfrac{a^n}{(\log a^n)^p} = \dfrac{a^n}{n^p (\log a)^p}$, a being taken > 1.

Consider the series $\Sigma a^n f(a^n) = \Sigma \dfrac{a^n}{n^p (\log a)^p} = \Sigma v_n$, say.

Here $v_n = \dfrac{a^n}{n^p (\log a)^p}$, so that $v_{n+1} = \dfrac{a^{n+1}}{(n+1)^p (\log a)^p}$.

$$\therefore \quad \frac{v_n}{v_{n+1}} = \frac{a^n}{n^p (\log a)^p} \cdot \frac{(n+1)^p (\log a)^p}{a^{n+1}} = \left(1 + \frac{1}{n}\right)^p \cdot \frac{1}{a}.$$

$$\therefore \quad \lim \frac{v_n}{v_{n+1}} = \frac{1}{a}$$ which is < 1 as $a > 1$.

$\therefore$ by ratio test the series $\Sigma v_n = \Sigma a^n f(a^n)$ is divergent.

Therefore by Cauchy's condensation test the given series $\Sigma f(n)$ is also divergent.

Example 27: *Test for convergence the following series*

(*i*) $$\frac{(\log 2)^2}{2^2} + \frac{(\log 3)^2}{3^2} + \frac{(\log 4)^2}{4^2} + \ldots + \frac{(\log n)^2}{n^2} + \ldots$$

(*ii*) $$\frac{1}{(2 \log 2)^p} + \frac{1}{(3 \log 3)^p} + \ldots + \frac{1}{(n \log n)^p} + \ldots$$

Solution: (i) Here we can take the first term of the series as

$$\frac{(\log 1)^2}{1^2}$$ because $\log 1 = 0$.

$$\therefore \quad u_n = n\text{th term of the series} = \frac{(\log n)^2}{n^2} = f(n), \text{ say.}$$

It is positive for all $n \geq 2$ and decreases continually as n increases.

Now $a^n\ f\ (a^n) = \dfrac{a^n\ (\log a^n)^2}{(a^n)^2} = \dfrac{a^n\ n^2\ (\log a)^2}{(a^n)^2} = \dfrac{n^2\ (\log a)^2}{a^n}$, a being taken to be a +ive integer > 1.

Consider the series $\Sigma\, a^n\ f\ (a^n) = \Sigma \{ n^2\ (\log a)^2\ /\ a^n \} = \Sigma v_n$, (say).

Here $\quad v_n = \dfrac{n^2\ (\log a)^2}{a^n},\ v_{n+1} = \dfrac{(n+1)^2\ (\log a)^2}{a^{n+1}}.$

$$\therefore \quad \frac{v_n}{v_{n+1}} = \frac{n^2\ (\log a)^2}{a^n} \cdot \frac{a^{n+1}}{(n+1)^2\ (\log a)^2} = \frac{a}{(1+1/n)^2}.$$

$$\therefore \quad \lim \frac{v_n}{v_{n+1}} = \lim \frac{a}{(1+1/n)^2} = a > 1 \text{ since by our choice } a > 1.$$

$\therefore$ by ratio test the series $\Sigma\, v_n$ is convergent.

Hence by Cauchy's condensation test the given series $\Sigma\ f\ (n)$ is also convergent.

(ii) If $p \leq 0$, obviously the given series is divergent. So it remains to discuss the case when $p > 0$.

When $p > 0$, we have $f\ (n) = \dfrac{1}{(n \log n)^p} > 0$ for all $n \geq 2$ and it decreases continually as n increases.

Now $a^n\ f\ (a^n) = \dfrac{a^n}{(a^n\ \log a^n)^{\ p}} = \dfrac{1}{a^{n(p-1)}\ .\ n^p\ (\log a)^p}$, a to be taken > 1.

Case I: Let $p > 1$. Then $a^{n(p-1)} > 1$ as $a > 1$.

$$\therefore \quad a^n\ f\ (a^n) = \frac{1}{a^{n(p-1)} \cdot n^p\ (\log a)^p} < \frac{1}{(\log a)^p} \cdot \frac{1}{n^p} \qquad \ldots(1)$$

Now $1/(\log a)^p$ is a fixed positive real number and the series $\Sigma\,(1/n^p)$ is convergent because $p > 1$.

Hence from (1), by comparison test (second form) given in article 2.7, the series $\Sigma\ a^n\ f\ (a^n)$ is convergent.

Now by Cauchy's condensation test it follows that the given series $\Sigma\ f\ (n)$ is also convergent.

Case II: Let $p \leq 1$. Then $a^{n(p-1)} \leq 1$ as $a > 1$.

$$\therefore \quad a^n f\ (a^n) \geq \frac{1}{(\log a)^p} \cdot \frac{1}{n^p}. \qquad \ldots(2)$$

Now $1/(\log a)^p$ is a fixed +ive real number and the series $\Sigma(1/n^p)$ is divergent, p being ≤ 1.

Hence from (2), by comparison test the series $\Sigma\ a^n f\ (a^n)$ is divergent.

Now by Cauchy's condensation test it follows that the given series $\Sigma f(n)$ is also divergent.

Hence the given series is convergent if $p > 1$ and divergent if $p \leq 1$.

2.13 Raabe's Test

Theorem: *The series Σu_n of positive terms is convergent or divergent according as*

$$\lim \left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) \right\} > 1 \text{ or } < 1.$$

Proof: Case I: Let $\lim \left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) \right\} = k$, where $k > 1$.

Choose a number p such that $k > p > 1$.

Compare the series Σu_n with the auxiliary series $\Sigma v_n = \Sigma \frac{1}{n^p}$, which is convergent since $p > 1$.

By article 2.7, sixth form of comparison test, Σu_n is convergent if after some particular term

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$$

or
$$\frac{u_n}{u_{n+1}} > \frac{1/n^p}{1/(n+1)^p} = \left(\frac{n+1}{n} \right)^p = \left(1 + \frac{1}{n} \right)^p$$

$$= 1 + p \cdot \frac{1}{n} + \frac{p(p-1)}{2!} \cdot \frac{1}{n^2} + \ldots$$

or
$$n \left(\frac{u_n}{u_{n+1}} - 1 \right) > p + \frac{p(p-1)}{2!} \cdot \frac{1}{n} + \ldots \qquad \ldots(1)$$

If n be taken sufficiently large the L.H.S and R.H.S. of (1) respectively approach k and p. Also k is greater than p. Therefore (1) is satisfied for sufficiently large values of n. Hence Σu_n is convergent if

$$\lim \left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) \right\} > 1.$$

Case II: Let $\lim \left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) \right\} = l$, where $l < 1$.

Choose a number p such that $l < p < 1$.

Compare the series Σu_n with the auxiliary series $\Sigma v_n = \Sigma (1/n^p)$ which is divergent since $p < 1$.

The series Σu_n is divergent if after some particular term

$$\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}},$$ [By article 2.7, sixth form of comparison test]

or $$n\left(\frac{u_n}{u_{n+1}} - 1\right) < p + \frac{p(p-1)}{2!}\cdot\frac{1}{n} + \ldots \qquad \ldots(2)$$

(Proceeding as in case I)

If n be taken sufficiently large the L.H.S. and R.H.S. of (2) respectively approach l and p. Also $l < p$. Thus (2) is satisfied for sufficiently large values of n. Hence $\Sigma\, u_n$ is divergent if

$$\lim\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} < 1.$$

However, if $\lim n\left(\frac{u_n}{u_{n+1}} - 1\right) = 1$, the Raabe's test fails.

Note: Rabbe's test is to be applied when D'Alembert's ratio test fails.

Illustrative Examples

Example 28: *Test the convergence of the series*

(*i*) $\Sigma \frac{n!\,x^n}{3.5.7\ldots.(2n+1)}$ (*ii*) $\sum_{n=1}^{\infty} \frac{1}{1+\log n}$.

Solution: (i) Here $u_n = \frac{n!\,x^n}{3.5.7\ldots(2n+1)}$

so that $$u_{n+1} = \frac{(n+1)!\,x^{n+1}}{3.5.7\ldots.(2n+1)(2n+3)}.$$

Now $$\frac{u_n}{u_{n+1}} = \frac{(2n+3)}{(n+1)!}\cdot\frac{n!}{x} = \frac{2n+3}{n+1}\cdot\frac{1}{x} = \left(\frac{2+3/n}{1+1/n}\right)\cdot\frac{1}{x}.$$

$$\therefore \quad \lim\frac{u_n}{u_{n+1}} = \lim\left(\frac{2+3/n}{1+1/n}\right)\cdot\frac{1}{x} = \frac{2}{x}.$$

Hence by D' Alembert's ratio test the series converges if $\frac{2}{x} > 1$ *i.e.,* if $x < 2$ and diverges if $2/x < 1$ *i.e.,* if $x > 2$ and the test fails when $2/x = 1$ *i.e.,* when $x = 2$.
In case $x = 2$, we apply Raabe's test.
When $x = 2$, $\frac{u_n}{u_{n+1}} = \frac{2n+3}{2(n+1)}$.

$$\therefore \quad n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left(\frac{2n+3}{2n+2} - 1\right) = \frac{n}{2(n+1)} = \frac{1}{2(1+1/n)}.$$

$\therefore \quad \lim n\left(\dfrac{u_n}{u_{n+1}}-1\right)=\lim \dfrac{1}{2\,(1+1/n)}=\dfrac{1}{2}<1.$

Hence by Raabe's test $\Sigma\, u_n$ is divergent if $x=2$.

Thus the given series $\Sigma\, u_n$ is convergent if $x<2$ and divergent if $x \geq 2$.

(ii) Here $\quad u_n=\dfrac{1}{1+\log n};\ u_{n+1}=\dfrac{1}{1+\log (n+1)}.$

Now
$$\frac{u_n}{u_{n+1}}=\frac{1+\log (n+1)}{1+\log n}$$
$$=\frac{1+\log \{n\,(1+1/n)\}}{1+\log n}=\frac{1+\log n+\log (1+1/n)}{1+\log n}$$
$$=\frac{\log (en)+\log (1+1/n)}{\log (en)}=1+\frac{1}{\log (en)}\log (1+1/n)$$
$$=1+\frac{1}{\log (en)}\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\right)$$
$$=1+\frac{1}{n\log (en)}-\frac{1}{2n^2\log (en)}+\ldots.$$

$\therefore \quad \lim \dfrac{u_n}{u_{n+1}}=1$, and so the ratio test fails.

Now we apply Raabe's test. We have
$$\lim n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim n\left[\frac{1}{n\log (en)}-\frac{1}{2n^2\log (en)}+\ldots\right]$$
$$=\lim\left[\frac{1}{\log (en)}-\frac{1}{2n\log (en)}+\ldots\right]=0<1.$$

Hence by Raabe's test the given series is divergent.

Example 29: *Test the convergence of the series*

(i) $\dfrac{1^2}{4^2}+\dfrac{1^2\,.\,5^2}{4^2\,.\,8^2}+\dfrac{1^2\,.\,5^2\,.\,9^2}{4^2\,.\,8^2\,.\,12^2}+\dfrac{1^2\,.\,5^2\,.\,9^2\,.\,13^2}{4^2\,.\,8^2\,.\,12^2\,.\,16^2}+\ldots$ **(Meerut 2013)**

(*ii*) $1+\dfrac{3}{7}x+\dfrac{3\,.\,6}{7\,.\,10}x^2+\dfrac{3\,.\,6\,.\,9}{7\,.\,10\,.\,13}x^3+\dfrac{3\,.\,6\,.\,9\,.\,12}{7\,.\,10\,.\,13\,.\,16}x^4+\ldots$ **(Meerut 2013B)**

Solution: (i) Here $u_n=\dfrac{1^2\,.\,5^2\,.\,9^2\,\ldots\ldots(4n-3)^2}{4^2\,.\,8^2\,.\,12^2\,\ldots\ldots(4n)^2}.$

[Note that the nth term of the sequence $1^2, 5^2, 9^2, \ldots$ is
$$\{1+(n-1)\,4\}^2 \ \textit{i.e.,}\ (4n-3)^2$$
and the nth term of the sequence
$$4^2, 8^2, 12^2, \ldots \text{ is } \{4+(n-1)\,4\}^2 \ \textit{i.e.,}\ (4n)^2\ ].$$

Then $$u_{n+1} = \frac{1^2 \,.\, 5^2 \,.\, 9^2 \ldots\ldots (4n-3)^2 \,.\, (4n+1)^2}{4^2 \,.\, 8^2 \,.\, 12^2 \ldots\ldots (4n)^2 \,.\, (4n+4)^2}.$$

Now $$\frac{u_n}{u_{n+1}} = \frac{(4n+4)^2}{(4n+1)^2} = \frac{(4+4/n)^2}{(4+1/n)^2}.$$

∴ $$\lim \frac{u_n}{u_{n+1}} = 1, \text{ so that the ratio test fails.}$$

Now we apply Raabe's test. We have

$$\lim n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim n\left[\frac{(4n+4)^2}{(4n+1)^2} - 1\right] = \lim n\left[\frac{24n+15}{(4n+1)^2}\right]$$

$$= \lim \left\{\frac{24+15/n}{(4+1/n)^2}\right\} = \frac{24}{4^2} = \frac{3}{2}, \text{ which is} > 1.$$

Hence by Raabe's test the series is convergent.

(ii) Omitting the first term of the series, we have

nth term of the sequence 3, 6, 9, ... is $3 + (n-1)\,3 = 3n$

and nth term of the sequence 7, 10, 13, ... is $7 + (n-1)\,3 = 3n + 4$.

∴ $$u_n = \frac{3 \,.\, 6 \,.\, 9 \ldots\ldots 3n}{7 \,.\, 10 \,.\, 13 \ldots\ldots (3n+4)}\, x^n,$$

and $$u_{n+1} = \frac{3 \,.\, 6 \,.\, 9 \ldots\ldots 3n \,.\, (3n+3)}{7 \,.\, 10 \,.\, 13 \ldots\ldots (3n+4)\,(3n+7)}\, x^{n+1}.$$

∴ $$\frac{u_n}{u_{n+1}} = \left(\frac{3n+7}{3n+3}\right) \cdot \frac{1}{x} = \left(\frac{3+7/n}{3+3/n}\right) \cdot \frac{1}{x}.$$

∴ $$\lim \frac{u_n}{u_{n+1}} = \lim \left(\frac{3+7/n}{3+3/n}\right) \cdot \frac{1}{x} = \frac{3}{3} \cdot \frac{1}{x} = \frac{1}{x}.$$

Hence by ratio test, the series is convergent if $1/x > 1$ *i.e.,* if $x < 1$, divergent if $1/x < 1$ *i.e.,* if $x > 1$ and the test fails if $1/x = 1$ *i.e.,* if $x = 1$.

If $x = 1$, then $\dfrac{u_n}{u_{n+1}} = \dfrac{3n+7}{3n+3}$.

Now we apply Raabe's test. We have

$$\lim n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim n\left(\frac{3n+7}{3n+3} - 1\right) = \lim \frac{4n}{3n+3}$$

$$= \lim \frac{4}{3+3/n} = \frac{4}{3}, \text{ which is} > 1.$$

Hence the series is convergent when $x = 1$.

Thus the given series is convergent if $x \le 1$ and divergent if $x > 1$.

Example 30: *Test for convergence the following series*

$$1 + a + \frac{a(a+1)}{1 \cdot 2} + \frac{a(a+1)(a+2)}{1 \cdot 2 \cdot 3} + \dots$$

Solution: Leaving the first term, we have

$$u_n = \frac{a(a+1)(a+2)\dots(a+n-1)}{1 \cdot 2 \cdot 3 \dots n},$$

and then $$u_{n+1} = \frac{a(a+1)(a+2)\dots(a+n-1)(a+n)}{1 \cdot 2 \cdot 3 \dots n(n+1)}.$$

Now $$\frac{u_n}{u_{n+1}} = \frac{n+1}{a+n} = \frac{1+1/n}{a/n+1}.$$

$\therefore$ $$\lim \frac{u_n}{u_{n+1}} = 1,$$ so that the ratio test fails.

Now we apply Raabe's test. We have

$$\lim n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim n\left(\frac{n+1}{a+n} - 1\right) = \lim \frac{n(1-a)}{a+n}$$

$$= \lim \frac{(1-a)}{1+a/n} = 1 - a.$$

Hence by Raabe's test, the given series is convergent if $1 - a > 1$ *i.e.*, if $a < 0$, divergent if $1 - a < 1$ *i.e.*, if $a > 0$ and the test fails if $1 - a = 1$ *i.e.*, if $a = 0$.

In case $a = 0$, the given series becomes $1 + 0 + 0 + 0 + \dots$

The sum of n terms of this series is always 1. Therefore the series is convergent if $a = 0$.

Thus the given series is convergent if $a \le 0$ and divergent if $a > 0$.

Comprehensive Exercise 4

Test for convergence the following series :

1. $1 + \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \dots$

2. $1 + \frac{1}{2} \cdot \frac{x^2}{4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^4}{8} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} \cdot \frac{x^6}{12} + \dots$

3. $x^2 + \frac{2^2}{3 \cdot 4}x^4 + \frac{2^2 \cdot 4^2}{3 \cdot 4 \cdot 5 \cdot 6}x^6 + \frac{2^2 \cdot 4^2 \cdot 6^2}{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}x^8 + \dots$ **(Kanpur 2014)**

4. (i) $1 + \frac{x}{1} + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \dots$ **(Kashi 2014)**

(ii) $\frac{x}{1} + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \dots$

(iii) $\frac{x}{1} + \frac{1}{2} \cdot \frac{x^2}{3} + \frac{1.3}{2.4} \cdot \frac{x^3}{5} + \frac{1.3.5}{2.4.6} \cdot \frac{x^4}{7} + \ldots, x > 0.$

5. $\frac{a}{a+3} + \frac{a(a+2)}{(a+3)(a+5)} x + \frac{a(a+2)(a+4)}{(a+3)(a+5)(a+7)} x^2 + \ldots$

6. $\sum \frac{4.7.\ldots.(3n+1)}{1.2.\ldots.n} x^n.$

7. Apply Cauchy's condensation test to discuss the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n \log n (\log \log n)^p}$.

Answers 4

1. Convergent if $x < 1$ and divergent if $x \geq 1$
2. Convergent if $x^2 \leq 1$ and divergent if $x^2 > 1$
3. Convergent if $x^2 \leq 1$ and divergent if $x^2 > 1$
4. (i) Convergent if $x^2 \leq 1$ and divergent if $x^2 > 1$
   (ii) Convergent if $x^2 \leq 1$ and divergent if $x^2 > 1$
   (iii) Convergent if $x \leq 1$ and divergent if $x > 1$
5. Convergent if $x \leq 1$ and divergent if $x > 1$
6. Convergent if $x < 1/3$ and divergent if $x \geq 1/3$
7. Convergent if $p > 1$ and divergent if $p \leq 1$

2.14 Logarithmic Test

Theorem: *The series Σu_n of positive terms is convergent or divergent according as*

$$lim \left\{ n \; log \frac{u_n}{u_{n+1}} \right\} > 1 \; or < 1.$$

Proof: First suppose that

$$\lim \left\{ n \log \frac{u_n}{u_{n+1}} \right\} = k, \text{ where } k > 1.$$

Choose a number p such that $k > p > 1$.

Compare the given series with the auxiliary series Σv_n where $v_n = 1/n^p$, which is convergent as $p > 1$.

The series Σu_n is convergent if after some particular term

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}},$$ [by article 2.7, sixth form of comparison test.]

or $$\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p} = \left(1 + \frac{1}{n}\right)^p$$

or $$\log \frac{u_n}{u_{n+1}} > \log \left(1 + \frac{1}{n}\right)^p = p \log \left(1 + \frac{1}{n}\right) = p\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\right]$$

or $$n \log \frac{u_n}{u_{n+1}} > p - \frac{p}{2n} + \frac{p}{3n^2} - \ldots \qquad \ldots(1)$$

If n is taken sufficiently large the L.H.S. and R.H.S. of (1) respectively approach k and p. Also $k > p$.

Thus (1) is satisfied for sufficiently large values of n. Hence the series Σu_n is convergent if

$$\lim \left\{n \log \frac{u_n}{u_{n+1}}\right\} > 1.$$

Similarly, it can be proved that Σu_n is divergent if

$$\lim \left\{n \log \frac{u_n}{u_{n+1}}\right\} < 1.$$

[The procedure of proof will be the same as given in the proof of Raabe's test]

However, if $\lim \left\{n \log \frac{u_n}{u_{n+1}}\right\} = 1$, the test fails.

Note: This test is an alternative to Raabe's test and is to be applied when D'Alembert's ratio test fails and when either

(i) n occurs as an exponent in $\frac{u_n}{u_{n+1}}$ so that it is not convenient to find $\frac{u_n}{u_{n+1}} - 1$

(ii) taking logarithm of $\frac{u_n}{u_{n+1}}$ makes the evaluation of limits easier.

Illustrative Examples

Example 31: *Test for convergence the series*

$$1 + \frac{2\,x}{2\,!} + \frac{3^2\ x^2}{3\,!} + \frac{4^3\ x^3}{4\,!} + \frac{5^4\ x^4}{5\,!} + \ldots\ldots$$

(Kashi 2013;Avadh14)

Solution: Here $u_n = \frac{n^{n-1}}{n\,!} x^{n-1}$, $u_{n+1} = \frac{(n+1)^n}{(n+1)\,!} x^n$.

Now $$\frac{u_n}{u_{n+1}} = \frac{n^{n-1}}{n\,!} \frac{(n+1)\,!}{(n+1)^n} \cdot \frac{1}{x} = \frac{n^{n-1}\ (n+1)}{(n+1)^n} \cdot \frac{1}{x}$$

$$= \left(\frac{n}{n+1}\right)^{n-1} \cdot \frac{1}{x} = \frac{1}{(1+1/n)^{n-1}} \frac{1}{x}$$

$$= \frac{1}{(1+1/n)^n} \cdot (1+1/n) \cdot \frac{1}{x} \cdot$$

$$\therefore \quad \lim \frac{u_n}{u_{n+1}} = \lim \left\{\frac{(1+1/n)}{(1+1/n)^n} \cdot \frac{1}{x}\right\} = \frac{1}{ex} \cdot \qquad [\because \lim (1+1/n)^n = e]$$

$\therefore$ by ratio test the series Σu_n converges if $1/ex > 1$ *i.e.,* if $x < 1/e$, diverges if $1/ex < 1$ *i.e.,* if $x > 1/e$ and the test fails if $1/ex = 1$ *i.e.* if $x = 1/e$.

Now if $x = 1/e$, $\dfrac{u_n}{u_{n+1}} = \dfrac{e(1+1/n)}{(1+1/n)^n} \cdot$ Applying log test, we get

$$\lim \left(n \log \frac{u_n}{u_{n+1}}\right) = \lim \left[n \log \left\{\frac{e(1+1/n)}{(1+1/n)^n}\right\}\right]$$

$$= \lim n \left[\log e + \log (1+1/n) - n \log (1+1/n)\right]$$

$$= \lim n \left[1 + \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} \dots\right) - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} \dots\right)\right]$$

$$= \lim n \left[\left(1 + \frac{1}{2}\right) \cdot \frac{1}{n} + \left(-\frac{1}{2} - \frac{1}{3}\right)\frac{1}{n^2} + \dots\right]$$

$$= \lim \left[\frac{3}{2} - \frac{5}{6n} + \dots\right] = \frac{3}{2} \quad i.e., > 1.$$

Therefore the series Σu_n converges when $x = 1/e$.
Hence the given series is convergent if $x \le 1/e$ and divergent if $x > 1/e$.

Example 32: *Test for convergence the series*

$$\frac{(a+x)}{1!} + \frac{(a+2x)^2}{2!} + \frac{(a+3x)^3}{3!} + \dots$$

Solution: Here $u_n = \dfrac{(a+nx)^n}{n!}$, $u_{n+1} = \dfrac{[a+(n+1)x]^{n+1}}{(n+1)!}$.

Now
$$\frac{u_n}{u_{n+1}} = \frac{(a+nx)^n}{n!} \cdot \frac{(n+1)!}{[a+(n+1)x]^{n+1}} = \frac{(a+nx)^n (n+1)}{[a+(n+1)x]^{n+1}}$$

$$= \frac{(nx)^n (a/nx+1)^n \, n(1+1/n)}{(n+1)^{n+1} x^{n+1} [a/(n+1)x+1]^{n+1}}$$

$$= \frac{1}{x} \cdot \frac{n^{n+1} (1+a/nx)^n (1+1/n)}{n^{n+1} (1+1/n)^{n+1} [1+a/(n+1)x]^{n+1}}$$

$$= \frac{1}{x} \cdot \frac{\left[1 + \frac{(a/x)}{n}\right]^n}{\left[1+\frac{1}{n}\right]^n \left[1 + \frac{(a/x)}{n+1}\right]^{n+1}} \cdot$$

$$\therefore \quad \lim \frac{u_n}{u_{n+1}} = \frac{1}{x} \frac{e^{a/x}}{e.\, e^{a/x}} \qquad \left[\because \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x\right]$$

$$= \frac{1}{ex}.$$

Hence by ratio test the given series is convergent if $1/e\,x > 1$ *i.e.*, if $x < 1/e$, divergent if $1/ex < 1$ *i.e.*, if $x > 1/e$ and the test fails if $1/ex = 1$ *i.e.*, if $x = 1/e$.

If $x = 1/e$, $\quad \dfrac{u_n}{u_{n+1}} = \dfrac{e\left(1 + \dfrac{ea}{n}\right)^n}{\left(1 + \dfrac{1}{n}\right)^n \left[1 + \dfrac{ae}{n+1}\right]^{n+1}}.$

Applying logarithmic test, we get

$$\lim \left(n \log \frac{u_n}{u_{n+1}}\right) = \lim n \log \left[\frac{e\left(1 + \frac{ea}{n}\right)^n}{\left(1 + \frac{1}{n}\right)^n \left\{1 + \frac{ae}{n+1}\right\}^{n+1}}\right]$$

$$= \lim n \left[\log e + n \log \left(1 + \frac{ea}{n}\right) - n \log \left(1 + \frac{1}{n}\right) - (n+1) \log \left(1 + \frac{ae}{n+1}\right)\right]$$

$$= \lim n \left[1 + n\left(\frac{ea}{n} - \frac{e^2 a^2}{2n^2} + \frac{e^3 a^3}{3n^3} - \dots\right) - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \dots\right)\right.$$
$$\left. - (n+1)\left\{\frac{ea}{n+1} - \frac{e^2 a^2}{2(n+1)^2} + \dots\right\}\right]$$

$$= \lim n \left[\left(-\frac{e^2 a^2}{2} + \frac{1}{2}\right)\frac{1}{n} + \frac{e^2 a^2}{2(n+1)} + \left(\frac{e^3 a^3}{3} - \frac{1}{3}\right)\frac{1}{n^2} + \dots\right]$$

$$= \lim \left[\left(-\frac{e^2 a^2}{2} + \frac{1}{2}\right) + \frac{e^2 a^2}{2(1 + 1/n)} + \left(\frac{e^3 a^3 - 1}{3n}\right) + \dots\right]$$

$$= -\frac{e^2 a^2}{2} + \frac{1}{2} + \frac{e^2 a^2}{2} = \frac{1}{2}, \text{ which is } < 1.$$

$\therefore$ the series is divergent if $x = 1/e$.

Thus the given series is convergent if $x < 1/e$ and divergent if $x \geq 1/e$.

Comprehensive Exercise 5

Test for convergence the following series :

1. $x^2 (\log 2)^q + x^3 (\log 3)^q + x^4 (\log 4)^q + \dots$

2. $\frac{1}{(\log 2)^p} + \frac{1}{(\log 3)^p} + \ldots + \frac{1}{(\log n)^p} + \ldots$

3. (i) $x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \frac{4^4 x^4}{4!} + \frac{5^5 x^5}{5!} + \ldots$

(ii) $1 + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \frac{4^4 x^4}{4!} + \frac{5^5 x^5}{5!} + \ldots$

4. $1 + \frac{1!}{2} x + \frac{2!}{3^2} x^2 + \frac{3!}{4^3} x^3 + \frac{4!}{5^4} x^4 + \ldots$ **(Kanpur 2014)**

Answers 5

1. Convergent if $x < 1$ and divergent if $x \geq 1$
2. Divergent for all values of p
3. (i) Convergent if $x < 1/e$ and divergent if $x \geq 1/e$

   (ii) Convergent if $x < 1/e$ and divergent if $x \geq 1/e$
4. Convergent if $x < e$ and divergent if $x \geq e$

2.15 De Morgan's and Bertrand's Test

Theorem: *The series Σu_n of positive terms is convergent or divergent according as*

$$\lim \left[\left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) - 1 \right\} \log n \right] > 1 \text{ or } < 1.$$

Proof: Let $\lim \left[\left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) - 1 \right\} \log n \right] = k$, where $k > 1$.

Take a number p such that $k > p > 1$.

Compare the series Σu_n with the auxiliary series Σv_n, where $v_n = \frac{1}{n (\log n)^p}$, which is convergent as $p > 1$.

The series Σu_n is convergent if after some particular term

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}, \qquad \text{[By article 2.7, sixth form of comparison test]}$$

i.e., $$\frac{u_n}{u_{n+1}} > \frac{1}{n (\log n)^p} \cdot (n+1) \{ \log (n+1) \}^p, \qquad \left[\because \ v_n = \frac{1}{n (\log n)^p} \right]$$

i.e., $$\frac{u_n}{u_{n+1}} > \left(\frac{n+1}{n} \right) \left[\frac{\log \{ n (1 + 1/n) \}}{\log n} \right]^p$$

i.e., $$\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)\left[\frac{\log n + \log(1+1/n)}{\log n}\right]^p$$

i.e., $$\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)\left[\frac{\log n + \frac{1}{n} - \frac{1}{2n^2} + \dots}{\log n}\right]^p$$

i.e., $$\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)\left[1 + \frac{1}{n \log n} - \frac{1}{2n^2 \log n} + \dots\right]^p$$

i.e., $$\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)\left[1 + \frac{p}{n \log n} + \dots\right]$$

i.e., $$\frac{u_n}{u_{n+1}} > 1 + \frac{1}{n} + \frac{p}{n \log n} + \dots$$

i.e., $$n\left(\frac{u_n}{u_{n+1}} - 1\right) > 1 + \frac{p}{\log n} + \dots$$

i.e., $$n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1 > \frac{p}{\log n} + \dots$$

i.e., $$\left[n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right] \log n > p + \text{terms containing } n \text{ or } \log n$$

in the denominator. ...(1)

Now as n becomes sufficiently large the L.H.S. and R.H.S. of (1) respectively approach k and p. Also $k > p$.

Thus (1) is satisfied for sufficiently large values of n.

Hence the series $\Sigma\, u_n$ is convergent if

$$\lim \left[\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\} \log n\right] > 1.$$

Similarly, it can be proved as in the case of Raabe's test that $\Sigma\, u_n$ is divergent if

$$\lim \left[\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\} \log n\right] < 1.$$

Note: This test is to be applied when both D' Alembert's ratio test and Raabe's test fail.

2.16 An Alternative to Bertrand's Test

Theorem: *The series $\Sigma\, u_n$ of positive terms is convergent or divergent according as*

$$lim\left[\left(n\ log\ \frac{u_n}{u_{n+1}} - 1\right) log\ n\right] > 1\ or < 1.$$

Proof: Let $\lim\left[\left(n \log \frac{u_n}{u_{n+1}} - 1\right) \log n\right] = k$, where $k > 1$.

Take a number p such that $k > p > 1$.

Compare the given series $\Sigma\, u_n$ with the auxiliary series $\Sigma\, v_n$ where $v_n = \frac{1}{n\,(\log\ n)^{\,p}}$,

which is convergent since $p > 1$. The series $\Sigma\, u_n$ is convergent if after some particular term

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}},$$ by article 2.7, sixth form of comparison test

i.e. $$\frac{u_n}{u_{n+1}} > 1 + \frac{1}{n} + \frac{p}{n \log n} + \ldots$$ (Proceeding as in article 2.15)

i.e. $$\log \frac{u_n}{u_{n+1}} > \log\left\{1 + \left(\frac{1}{n} + \frac{p}{n \log n} + \ldots\right)\right\}$$

i.e. $$\log \frac{u_n}{u_{n+1}} > \left(\frac{1}{n} + \frac{p}{n \log n} + \ldots\right) - \frac{1}{2}\left(\frac{1}{n} + \frac{p}{n \log n} + \ldots\right)^2 + \ldots$$

i.e. $$n \log \frac{u_n}{u_{n+1}} > n\left[\frac{1}{n} + \frac{p}{n \log n} - \frac{1}{2n^2} + \ldots\right]$$

i.e. $$n \log \frac{u_n}{u_{n+1}} > 1 + \frac{p}{\log n} - \frac{1}{2n} + \ldots$$

i.e., $$n \log \frac{u_n}{u_{n+1}} - 1 > \frac{p}{\log n} - \frac{1}{2n} + \ldots$$

i.e., $$\left(n \log \frac{u_n}{u_{n+1}} - 1\right) \log n > p - \frac{1}{2}\left\{\frac{\log n}{n}\right\} + \ldots \qquad \ldots(1)$$

Now as n becomes sufficiently large the L.H.S. and R.H.S. of (1) respectively approach k and p. Also $k > p$. Thus (1) is satisfied for sufficiently large values of n. Hence the series $\Sigma\, u_n$ is convergent if

$$\lim\left[\left\{n \log \frac{u_n}{u_{n+1}} - 1\right\} \log n\right] > 1.$$

Similarly, it can be proved that $\Sigma\, u_n$ is divergent if

$$\lim\left[\left\{n \log \frac{u_n}{u_{n+1}} - 1\right\} \log n\right] < 1.$$

Note: This test is to be applied when the log test of article 2.14 fails *i.e.*, when $\lim \frac{u_n}{u_{n+1}} = 1$ and also $\lim n \log \frac{u_n}{u_{n+1}} = 1$.

Illustrative Examples

Example 33: *Test for convergence the following series :*

(*i*) $\frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} x + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} x^2 + \ldots$ **(Bundelkhand 2014)**

(*ii*) $1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha (\alpha+1) \beta (\beta+1)}{1 \cdot 2 \cdot \gamma (\gamma+1)} x^2 + \frac{\alpha (\alpha+1)(\alpha+2) \beta (\beta+1)(\beta+2)}{1 \cdot 2 \cdot 3 \cdot \gamma (\gamma+1)(\gamma+2)} x^3 + \ldots$

Solution: (i) Here $u_n = \frac{1^2 \cdot 3^2 \ldots (2n-1)^2}{2^2 \cdot 4^2 \ldots (2n)^2} x^{n-1}$,

and $$u_{n+1} = \frac{1^2 \cdot 3^2 \ldots (2n-1)^2 \cdot (2n+1)^2}{2^2 \cdot 4^2 \ldots (2n)^2 (2n+2)^2} x^n.$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} \cdot \frac{1}{x} = \left\{\frac{2+2/n}{2+1/n}\right\}^2 \cdot \frac{1}{x}.$$

$$\therefore \quad \lim \frac{u_n}{u_{n+1}} = \lim \left[\left\{\frac{2+2/n}{2+1/n}\right\}^2 \cdot \frac{1}{x}\right] = \frac{2^2}{2^2} \cdot \frac{1}{x} = \frac{1}{x}.$$

$\therefore$ by ratio test the given series Σu_n is convergent if $1/x > 1$ *i.e.*, $x < 1$, divergent if $1/x < 1$ *i.e.*, $x > 1$ and the test fails if $1/x = 1$ *i.e.*, $x = 1$.

When $x = 1$, we have

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2}.$$

$$\therefore \quad n\left\{\frac{u_n}{u_{n+1}} - 1\right\} = n\left\{\frac{(2n+2)^2}{(2n+1)^2} - 1\right\} = \frac{n(4n+3)}{(2n+1)^2} = \frac{4+3/n}{(2+1/n)^2}.$$

$$\therefore \quad \lim n\left\{\frac{u_n}{u_{n+1}} - 1\right\} = \lim \frac{4+3/n}{(2+1/n)^2} = \frac{4}{2^2} = 1.$$

$\therefore$ Raabe's test also fails when $x = 1$ and so we shall now apply De Morgan's test.

Now $$n\left\{\frac{u_n}{u_{n+1}} - 1\right\} - 1 = \frac{n(4n+3)}{(2n+1)^2} - 1 = \frac{-n-1}{(2n+1)^2}.$$

$$\therefore \quad \lim \left[\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\} \log n\right]$$

$$= \lim \left[\left\{ \frac{-n-1}{(2n+1)^2} \right\} \log n \right] = \lim \left[\frac{-1-1/n}{(2+1/n)^2} \cdot \frac{\log n}{n} \right]$$

$$= \frac{-1}{2^2} \,.\, 0 = 0 < 1. \qquad \left[\textbf{Note} \text{ that } \lim \frac{\log n}{n} = 0 \right]$$

$\therefore$ by De Morgan's test $\Sigma\, u_n$ is divergent when $x = 1$.

Hence the given series $\Sigma\, u_n$ is convergent if $x < 1$ and divergent if $x \geq 1$.

(ii) Omitting the first term, we have

$$u_n = \frac{\alpha\,(\alpha+1)\,(\alpha+2)\ldots(\alpha+n-1)\,\beta\,(\beta+1)\,(\beta+2)\ldots\ldots(\beta+n-1)}{1\,.\,2\ldots n\,.\,\gamma\,(\gamma+1)\,(\gamma+2)\ldots(\gamma+n-1)}\, x^n,$$

$$u_{n+1} = \frac{\alpha\,(\alpha+1)\ldots(\alpha+n-1)\,(\alpha+n)\,\beta\,(\beta+1)\ldots(\beta+n-1)\,(\beta+n)}{1\,.\,2\ldots n\,(n+1)\,.\,\gamma\,(\gamma+1)\ldots(\gamma+n-1)\,(\gamma+n)}\, x^{n+1}.$$

Now $\dfrac{u_n}{u_{n+1}} = \dfrac{(n+1)\,(\gamma+n)}{(\alpha+n)\,(\beta+n)} \cdot \dfrac{1}{x} = \dfrac{(1+1/n)\,(\gamma/n+1)}{(\alpha/n+1)\,(\beta/n+1)} \cdot \dfrac{1}{x} \cdot$

$\therefore \lim \dfrac{u_n}{u_{n+1}} = \dfrac{1\,.\,1}{1\,.\,1} \cdot \dfrac{1}{x} = \dfrac{1}{x}$ so that by ratio test the series is convergent if $1/x > 1$ *i.e.*, $x < 1$ and divergent if $1/x < 1$ *i.e.*, $x > 1$ and the test fails if $1/x = 1$ *i.e.*, $x = 1$.

When $x = 1$, we have

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)\,(\gamma+n)}{(\alpha+n)\,(\beta+n)} = \frac{n^2 + (\gamma+1)\,n + \gamma}{n^2 + (\alpha+\beta)\,n + \alpha\beta} \cdot$$

$$\therefore \quad n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left[\frac{n^2 + (\gamma+1)\,n + \gamma}{n^2 + (\alpha+\beta)\,n + \alpha\beta} - 1\right]$$

$$= \frac{n\,\{(\gamma+1-\alpha-\beta)\,n + (\gamma-\alpha\beta)\}}{n^2 + (\alpha+\beta)\,n + \alpha\beta}$$

$$= \frac{(\gamma+1-\alpha-\beta) + (\gamma-\alpha\beta)/n}{1 + (\alpha+\beta)/n + \alpha\beta/n^2} \cdot$$

$$\therefore \quad \lim n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{\gamma+1-\alpha-\beta}{1} = \gamma+1-\alpha-\beta \cdot$$

$\therefore$ if $x = 1$, then by Raabe's test, the series is convergent if $\gamma+1-\alpha-\beta > 1$ *i.e.*, if $\gamma > \alpha+\beta$, divergent if $\gamma+1-\alpha-\beta < 1$ *i.e.*, if $\gamma < \alpha+\beta$, and the test fails if $\gamma+1-\alpha-\beta = 1$ *i.e.*, if $\gamma = \alpha+\beta$.

When $\gamma = \alpha+\beta$, we have

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{n\,\{n + \alpha + \beta - \alpha\beta\}}{n^2 + (\alpha+\beta)\,n + \alpha\beta} \cdot$$

Now we shall apply De Morgan's test.

We have

$$\lim\left[\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n\right]=\lim\left[\left\{\frac{n\,(n+\alpha+\beta-\alpha\beta)}{n^2+(\alpha+\beta)\,n+\alpha\beta}-1\right\}\log n\right]$$

$$=\lim\left[\frac{-\alpha\beta n-\alpha\beta}{n^2+(\alpha+\beta)\,n+\alpha\beta}\cdot\log n\right]$$

$$=\lim\left[\frac{-\alpha\beta\,(1+1/n)}{1+(\alpha+\beta)/n+\alpha\beta/n^2}\cdot\frac{\log n}{n}\right]$$

$$=\frac{-\alpha\beta}{1}\cdot 0=0,\text{ which is}<1.\qquad\left[\textbf{Note}\text{ that }\lim\frac{\log n}{n}=0\right]$$

∴ by De-Morgan's test the series is divergent if $\gamma=\alpha+\beta$.

Thus the given series is convergent if $x<1$, divergent if $x>1$ and for $x=1$, the series is convergent if $\gamma>\alpha+\beta$ and divergent if $\gamma\le\alpha+\beta$.

Example 34: *Test for convergence the series*

$$1^p+\left(\frac{1}{2}\right)^p+\left(\frac{1.3}{2.4}\right)^p+\left(\frac{1.3.5}{2.4.6}\right)^p+\ldots$$

Solution: Omitting the first term 1^p, we have

$$u_n=\left[\frac{1.3.5.\ldots.(2n-1)}{2.4.6\ldots.(2n)}\right]^p,$$

and then $u_{n+1}=\left[\dfrac{1.3.5\ldots(2n-1)\,(2n+1)}{2.4.6\ldots(2n)\,(2n+2)}\right]^p$.

Now $\dfrac{u_n}{u_{n+1}}=\left(\dfrac{2n+2}{2n+1}\right)^p=\left(\dfrac{1+1/n}{1+1/2n}\right)^p$.

∴ $\lim\dfrac{u_n}{u_{n+1}}=\left(\dfrac{1}{1}\right)^p=1$ *i.e.*, the ratio test fails.

Now we apply logarithmic test.

We have $\log\dfrac{u_n}{u_{n+1}}=\log\left(\dfrac{2n+2}{2n+1}\right)^p=\log\left(\dfrac{1+1/n}{1+1/2n}\right)^p$

$$=p\,[\log(1+1/n)-\log(1+1/2n)]$$

$$=p\left[\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\right)-\left(\frac{1}{2n}-\frac{1}{2\,.\,2^2n^2}+\frac{1}{3\,.\,2^3n^3}-\ldots\right)\right]$$

$$=p\left[\left\{1-\frac{1}{2}\right\}\frac{1}{n}-\frac{1}{2}\cdot\left\{1-\frac{1}{4}\right\}\frac{1}{n^2}+\frac{1}{3}\left\{1-\frac{1}{8}\right\}\frac{1}{n^3}-\ldots\right]$$

$$= p\left[\frac{1}{2n} - \frac{3}{8n^2} + \frac{7}{24n^3} - \dots\right].$$

$$\therefore \qquad n \log \frac{u_n}{u_{n+1}} = p\left[\frac{1}{2} - \frac{3}{8n} + \frac{7}{24n^2} - \dots\right].$$

$\therefore \ \lim n \log \dfrac{u_n}{u_{n+1}} = \dfrac{p}{2}$, so that the series is convergent if $p/2 > 1$ *i.e.*, if $p > 2$, divergent if $p/2 < 1$ *i.e.*, if $p < 2$ and the test fails if $p/2 = 1$ *i.e.*, if $p = 2$.

If $p = 2$, we have

$$n \log \frac{u_n}{u_{n+1}} = 2\left[\frac{1}{2} - \frac{3}{8n} + \frac{7}{24n^2} - \dots\right] = 1 - \frac{3}{4n} + \frac{7}{12n^2} - \dots .$$

$$\therefore \qquad \lim\left[\left(n \log \frac{u_n}{u_{n+1}} - 1\right)\log n\right] = \lim\left[\left\{-\frac{3}{4n} + \frac{7}{12n^2} - \dots\right\}.\log n\right]$$

$$= \lim\left[\left\{-\frac{3}{4} + \frac{7}{12n} - \dots\right\}\cdot\frac{\log n}{n}\right] = \left\{-\frac{3}{4}\right\} .0 = 0, \text{ which is} < 1.$$

Hence by Alternative to Bertrand's test given in article 2.16, the series is divergent when $p = 2$.

Thus the given series is convergent if $p > 2$ and divergent if $p \le 2$.

Comprehensive Exercise 6

Test for convergence the following series :

1. $1 + \dfrac{2^2}{3^2} + \dfrac{2^2.4^2}{3^2.5^2} + \dfrac{2^2.4^2.6^2}{3^2.5^2.7^2} + \dots$ **(Kashi 2013; Meerut 13)**

2. $\dfrac{1^2}{2^2} + \dfrac{1^2.3^2}{2^2.4^2} + \dfrac{1^2.3^2.5^2}{2^2.4^2.6^2} + \dots$ **(Kumaun 2003)**

3. $\dfrac{a}{b} + \dfrac{a(a+1)}{b(b+1)} + \dfrac{a(a+1)(a+2)}{b(b+1)(b+2)} + \dots$

4. $1 + \dfrac{a(1-a)}{1^2} + \dfrac{(1+a)a(1-a)(2-a)}{1^2.2^2} + \dfrac{(2+a)(1+a)a(1-a)(2-a)(3-a)}{1^2.2^2.3^2} + \dots$

5. $1 + \dfrac{\alpha}{1.\beta}x + \dfrac{\alpha(\alpha+1)^2}{1.2\beta(\beta+1)}x^2 + \dfrac{\alpha(\alpha+1)^2(\alpha+2)^2}{1.2.3\beta(\beta+1)(\beta+2)}x^3 + \dots$

6. $\left\{\dfrac{1}{2.4}\right\}^{2/3} + \left\{\dfrac{1.3}{2.4.6}\right\}^{2/3} + \left\{\dfrac{1.3.5}{2.4.6.8}\right\}^{2/3} + \dots$

7. $x + x^{1+1/2} + x^{1+1/2+1/3} + x^{1+1/2+1/3+1/4} + \dots$

Answers 6

1. Divergent
2. Divergent
3. Convergent if $b - a > 1$ and divergent if $b - a \leq 1$
4. Divergent
5. Convergent if $x < 1$, divergent if $x > 1$ and when $x = 1$ then convergent if $\beta > 2\alpha$ and divergent if $\beta \leq 2\alpha$
6. Divergent
7. Convergent if $x < 1/e$ and divergent if $x \geq 1/e$

2.17 Summary of Tests

Let the given series of positive terms be Σu_n. Then to test the series for convergence we proceed as follows :

1. **Find lim u_n:** (a) If $\lim u_n > 0$, the series is divergent.

   (b) If $\lim u_n = 0$, then the series may or may not be convergent. In this case we apply further tests to decide the nature of the series.

2. **If lim $u_n = 0$** and u_n can be arranged as an algebraic fraction in n, then usually **comparison test** should be applied.

3. **If n occurs as an exponent in** u_n and $\lim (u_n)^{1/n}$ can be easily evaluated, then **Cauchy's root test** should be applied.

4. **Cauchy's condensation test** is generally applied when u_n involves $\log n$.

   In case all the above tests are not applicable then we adopt the following scheme of testing in the given order.

5. **D' Alembert's ratio test:** For this we find $\lim \dfrac{u_n}{u_{n+1}}\cdot$ The series is convergent or divergent according as this limit is > 1 or < 1. In case this limit is equal to 1 (unity), this test fails. Then we proceed to apply either test 6 (a) or 6 (b) or 6(c) given below depending upon the nature of u_n and u_n / u_{n+1}.

6. **(a) Comparison test:** In some cases when D' Alembert's ratio test fails, the convergence of the series may be decided by comparison test.

   (b) Raabe's test: For this we find $\lim n\left(\dfrac{u_n}{u_{n+1}} - 1\right)$. The series is convergent or divergent according as this limit is > 1 or < 1. In case the limit is equal to 1, this test fails and we apply test 7 (a).

(c) **Logarithmic test:** If $\frac{u_n}{u_{n+1}} - 1$ cannot be evaluated easily while $\log \frac{u_n}{u_{n+1}}$ can be easily evaluated then we apply logarithmic test. Here we find $\lim \left(n \log \frac{u_n}{u_{n+1}} \right)$. If this limit > 1, the series is convergent and if this limit < 1, the series is divergent. In case the limit $= 1$, this test fails and we apply test 7 (b).

7. (a) **De Morgan's and Bertrand's test:**

Find $\lim \left[\left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) - 1 \right\} \log n \right]$.

The series is convergent or divergent according as this limit is > 1 or < 1.

Note: When this test is applied, we shall generally find that the limit comes out to be equal to zero and since $0 < 1$, the series is divergent.

(b) **Alternative to Bertrand's test:**

To apply this test we find $\lim \left[\left(n \log \frac{u_n}{u_{n+1}} - 1 \right) \log n \right]$.

The series is convergent or divergent according as this limit is > 1 or < 1.

2.18 Kummer's Test

Theorem: *Let Σu_n and $\Sigma (1/d_n)$ be two series of positive terms and let $v_n = d_n (u_n / u_{n+1}) - d_{n+1}$. Then*

(*i*) *if a fixed positive number k can be found so that after a certain stage, say for $n \geq m$, $v_n \geq k$, the series, Σu_n is convergent;*

(*ii*) *if $v_n \leq 0$ for $n \geq m$ and $\Sigma (1/d_n)$ is divergent, Σu_n is divergent.*

Proof: (i) From the condition given in the statement of the theorem for Σu_n to be convergent, we have for $n \geq m$, where m is a fixed +ive integer,

$$v_n \geq k > 0 \quad i.e., \quad d_n \left(\frac{u_n}{u_{n+1}} \right) - d_{n+1} \geq k$$

i.e.,
$$d_n u_n - d_{n+1} u_{n+1} \geq k u_{n+1} \qquad \ldots(1)$$

[**Note** that u_{n+1} is positive]

Replacing n by $m, m+1, m+2, \ldots, n-1$ in succession in (1), we get

$$d_m u_m - d_{m+1} u_{m+1} \geq k u_{m+1},$$
$$d_{m+1} u_{m+1} - d_{m+2} u_{m+2} \geq k u_{m+2},$$
$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$
$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$
$$d_{n-1} u_{n-1} - d_n u_n \geq k u_n.$$

Adding the corresponding sides of these inequalities, we get

$$d_m\, u_m - d_n\, u_n \geq k\,(u_{m+1} + u_{m+2} + \ldots + u_n)$$

or $$u_{m+1} + u_{m+2} + \ldots + u_n \leq \frac{1}{k}\,(d_m u_m - d_n\, u_n) \qquad [\because\ k > 0]$$

or $$u_{m+1} + u_{m+2} + \ldots + u_n < \frac{1}{k}\, d_m u_m \qquad \ldots(2)$$

or $$S_n - S_m < \frac{1}{k}\, d_m u_m,$$

where $S_n = u_1 + \ldots + u_m + u_{m+1} + \ldots + u_n = S_m + u_{m+1} + \ldots + u_n$

or $$S_n < S_m + \frac{1}{k}\, d_m u_m, \text{ using (2).}$$

Since S_n is less than a fixed number, hence the series $\Sigma\, u_n$ is convergent.

(ii) We have $v_n \leq 0$ for $n \geq m$ (given)

i.e., $$d_n \left(\frac{u_n}{u_{n+1}}\right) - d_{n+1} \leq 0 \text{ for } n \geq m$$

i.e., $d_n u_n \leq d_{n+1}\, u_{n+1}$, for $n \geq m$.

Putting $n = m, m + 1, m + 2, \ldots, n - 1$ in succession, we have

$$d_m u_m \leq d_{m+1} u_{m+1} \leq d_{m+2}\, u_{m+2} \leq \ldots\ldots \leq d_n u_n$$

i.e., $$d_m u_m \leq d_n u_n$$

or $$u_n \geq (d_m u_m) / d_n.$$

Now $d_m u_m$ is a fixed number and the series $\Sigma\,(1/d_n)$ is divergent (given), hence, by comparison test the series $\Sigma\, u_n$ is also divergent.

2.19 Gauss's Test

Theorem: *Let $\Sigma\, u_n$ be a series of positive terms and u_n / u_{n+1} can be expressed in the form*

$$\frac{u_n}{u_{n+1}} = 1 + \frac{a}{n} + \frac{b_n}{n^p},$$

where $p > 1$ and $|b_n| <$ a fixed number k or (in particular) b_n tends to a finite limit as $n \to \infty$, then $\Sigma\, u_n$ converges if $a > 1$ and diverges if $a \leq 1$.

Proof: It is given that

$$\frac{u_n}{u_{n+1}} = 1 + \frac{a}{n} + \frac{b_n}{n^p} \quad i.e., \quad n\left(\frac{u_n}{u_{n+1}} - 1\right) = a + \frac{b_n}{n^{p-1}}.$$

$\therefore$ $$\lim n\left(\frac{u_n}{u_{n+1}} - 1\right) = a. \qquad [\because\ p > 1 \text{ and } |b_n| < k]$$

Hence by Raabe's test $\Sigma\, u_n$ converges if $a > 1$ and diverges if $a < 1$. The test fails if $a = 1$ and then we apply Kummer's test to find the convergence of $\Sigma\, u_n$.

When $a = 1$, we have

$$\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n} + \frac{b_n}{n^p}.$$

Since the series $\frac{1}{2 \log 2} + \frac{1}{3 \log 3} + \ldots + \frac{1}{n \log n} + \ldots$ is divergent, we take $d_n = n \log n$ for all $n \geq 2$ in Kummer's test. Then we have

$$v_n = d_n \frac{u_n}{u_{n+1}} - d_{n+1}$$

$$= (n \log n).\left(1 + \frac{1}{n} + \frac{b_n}{n^p}\right) - (n+1) \log (n+1)$$

$$= n\left(1 + \frac{1}{n}\right) \log n + \frac{b_n}{n^{p-1}} \log n - (n+1) \log (n+1)$$

$$= (n+1) \{ \log n - \log (n+1) \} + \frac{b_n}{n^{p-1}} \log n$$

$$= (n+1) \log \left(\frac{n}{n+1}\right) + \frac{b_n}{n^{p-1}} \log n$$

$$= (n+1) \log \left(1 - \frac{1}{n+1}\right) + \frac{\log n}{n^{p-1}} \cdot b_n$$

$$= (n+1)\left[-\frac{1}{n+1} - \frac{1}{2(n+1)^2} - \ldots\right] + \frac{\log n}{n^{p-1}} \cdot b_n$$

$$= \left[-1 - \frac{1}{2(n+1)} - \ldots\right] + \frac{\log n}{n^{p-1}} \cdot b_n \qquad \ldots(1)$$

Now $\lim \frac{\log n}{n^{p-1}} = 0$ as $p > 1$ and $< b_n >$ is a bounded sequence because $|b_n| < k$.

$$\therefore \qquad \lim \left[\frac{\log n}{n^{p-1}} \cdot b_n\right] = 0 \cdot \qquad \ldots(2)$$

Hence taking limit when $n \to \infty$, we get from (1) with the help of (2) $\lim v_n = -1 + 0 = -1$, which shows that after a certain stage $v_n < 0$. Also $\Sigma\ (1/d_n)$ *i.e.*, $\Sigma\ (1/n \log n)$ is divergent. Hence by Kummer's test the series $\Sigma\ u_n$ is divergent. Thus the series $\Sigma\ u_n$ is convergent if $a > 1$ and divergent if $a \leq 1$.

2.20 Cauchy-Maclaurin's Integral Test

Improper integrals: Integrals of the form $\int_a^\infty f(x)\, dx$ where $a \in \mathbf{R}$ are called improper integrals.

Let $F(t) = \int_a^t f(x)\, dx$ for $a \leq t < \infty$.

If $\lim_{t \to \infty} F(t)$ exists and is equal to $l \in \mathbf{R}$, the improper integral $\int_a^\infty f(x)\, dx$ is said to **converge** to l, otherwise it is called a **divergent integral**.

Theorem: *Let $f(x)$ be a non-negative monotonically decreasing integrable function on* $[1, \infty[$. *Then the series* $\sum_{n=1}^{\infty} f(n)$ *and the improper integral* $\int_1^{\infty} f(x)\, dx$ *converge or diverge together i.e., the series* $\Sigma f(n)$ *converges or diverges according as the integral* $\int_1^{\infty} f(x)\, dx$ *tends to a finite limit or diverges to* ∞ *as* $n \to \infty$.

Proof: Since $f(x)$ is non-negative on $[1, \infty[$, therefore

$$f(x) \geq 0 \quad \forall x \geq 1$$

i.e., the series $\sum_{n=1}^{\infty} f(n)$ is of non-negative terms.

For any $x \in [1, \infty[$, we can find $n \in \mathbf{N}$ such that $n \leq x \leq n+1$.

Since f is monotonically decreasing on $[1, \infty[$, therefore, we have

$$f(n) \geq f(x) \geq f(n+1) \quad \text{if} \quad n \leq x \leq n+1.$$

$$\therefore \quad \int_n^{n+1} f(n)\, dx \geq \int_n^{n+1} f(x)\, dx \geq \int_n^{n+1} f(n+1)\, dx$$

or
$$f(n) \geq \int_n^{n+1} f(x)\, dx \geq f(n+1) \quad \ldots(1)$$

Putting $n = 1, 2, \ldots, (n-1)$ in (1) in succession and then adding all the results, we get

$$f(1) + f(2) + \ldots + f(n-1) \geq \int_1^2 f(x)\, dx + \int_2^3 f(x)\, dx + \ldots + \int_{n-1}^{n} f(x)\, dx$$

$$\geq f(2) + f(3) + \ldots + f(n). \quad \ldots(2)$$

Let
$$s_n = f(1) + f(2) + \ldots + f(n)$$

and
$$I_n = \int_1^2 f(x)\, dx + \int_2^3 f(x)\, dx + \ldots + \int_{n-1}^{n} f(x)\, dx = \int_1^n f(x)\, dx.$$

Then (2) can be written as

$$s_n - f(n) \geq I_n \geq s_n - f(1)$$

or
$$-f(n) \geq I_n - s_n \geq -f(1)$$

or
$$f(n) \leq s_n - I_n \leq f(1). \quad \ldots(3)$$

The result (3) is true for all $n \in \mathbf{N}$.

Let $u_n = s_n - I_n$ for all $n \in \mathbf{N}$.

Now
$$u_{n+1} - u_n = (s_{n+1} - I_{n+1}) - (s_n - I_n) = (s_{n+1} - s_n) - (I_{n+1} - I_n)$$

$$= f(n+1) - \int_n^{n+1} f(x)\, dx \leq 0, \text{ using (1).}$$

$\therefore \quad u_{n+1} \leq u_n$ for all $n \in \mathbf{N}$

i.e., $< u_n >$ is a monotonically decreasing sequence.

Also by (3), $u_n \geq f(n) \geq 0$ for all n and hence $< u_n >$ is bounded below. Thus the sequence $< u_n >$ is convergent *i.e.,* $< u_n >$ tends to a finite limit as $n \to \infty$.

Since $s_n = u_n + I_n$ and $< u_n >$ is convergent, it follows that the sequences $< s_n >$ and $< I_n >$ converge or diverge together. Consequently the series $\Sigma f(n)$ and the integral $\int_1^\infty f(x)\,dx$ converge or diverge together.

Note: The series $\Sigma f(n)$ and the integral $\int_a^\infty f(x)\,dx$ converge or diverge together for $a \geq 1$.

Illustrative Examples

Example 35: *Test the convergence of the series*

$$\frac{1^2}{2^2} + \frac{1^2 . 3^2}{2^2 . 4^2} + \frac{1^2 . 3^2 . 5^2}{2^2 . 4^2 . 6^2} + \ldots\ldots$$

Solution: Here, $u_n = \dfrac{1^2 . 3^2 . 5^2 \ldots (2n-1)^2}{2^2 . 4^2 . 6^2 \ldots (2n)^2}$

and $$u_{n+1} = \frac{1^2 . 3^2 . 5^2 \ldots (2n-1)^2 \ (2n+1)^2}{2^2 . 4^2 . 6^2 \ldots (2n)^2 \ (2n+2)^2}.$$

Now $$\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2}.$$

$\therefore$ $$\lim \frac{u_n}{u_{n+1}} = 1.$$

Hence D'Alembert's ratio test fails. Now we apply Raabe's test. We have

$$\lim n \left(\frac{u_n}{u_{n+1}} - 1 \right) = \lim n \left\{ \frac{(2n+2)^2}{(2n+1)^2} - 1 \right\} = \lim \frac{n(4n+3)}{(2n+1)^2} = 1$$

i.e. Raabe's test also fails. Now we apply Gauss test. We can write

$$\frac{u_n}{u_{n+1}} = \left(1 + \frac{1}{n}\right)^2 \left(1 + \frac{1}{2n}\right)^{-2}$$

$$= \left(1 + \frac{2}{n} + \frac{1}{n^2}\right)\left(1 - 2 . \frac{1}{2n} + 3 . \frac{1}{4n^2} - \ldots\right)$$

$$= 1 + \frac{1}{n} - \frac{1}{4n^2} + \ldots = 1 + \frac{1}{n} + \frac{1}{n^2}\left(-\frac{1}{4} + \ldots\right)$$

$$= 1 + \frac{a}{n} + \frac{b_n}{n^2} \text{ where } b_n \to -\frac{1}{4} \text{ as } n \to \infty.$$

Since $a = 1$, hence by Gauss's test, the series Σu_n is divergent.

Example 36: *Test for convergence the series*

$$1 + \frac{2^2}{3^2} + \frac{2^2 . 4^2}{3^2 . 5^2} + \frac{2^2 . 4^2 . 6^2}{3^2 . 5^2 . 7^2} + \ldots\ldots$$

Solution: Here, $u_n = \dfrac{2^2 . 4^2 . 6^2 ......(2n-2)^2}{3^2 . 5^2 . 7^2 ......(2n-1)^2}$

and $$u_{n+1} = \frac{2^2 .4^2 .6^2 ......(2n-2)^2 \ (2n)^2}{3^2 5^2 .7^2 ......(2n-1)^2 \ (2n+1)^2}.$$

$\therefore$ $$\lim \frac{u_n}{u_{n+1}} = \lim \frac{(2n+1)^2}{(2n)^2} = 1 \text{ so that ratio test fails.}$$

Now we apply Raabe's test. We have

$$\lim n \left(\frac{u_n}{u_{n+1}} - 1 \right) = \lim n \left\{ \frac{(2n+1)^2}{4n^2} - 1 \right\} = \lim \frac{n\,(4n+1)}{4n^2} = 1.$$

Hence Raabe's test also fails. Now we apply Gauss test. We can write

$$\frac{u_n}{u_{n+1}} = \frac{4n^2 + 4n + 1}{4n^2} = 1 + \frac{1}{n} + \frac{1}{4n^2} = 1 + \frac{a}{n} + \frac{b_n}{n^p}.$$

Here $a = 1, b_n = \dfrac{1}{4}, p = 2 > 1$. Consequently the series is divergent by Gauss's test.

Example 37: *Show by Cauchy's integral test that the series*

$$\sum_{n=2}^{\infty} \frac{1}{n\,(\log n)^p}$$

converges if $p > 1$ and diverges if $0 < p \le 1$.

Solution: Let $f(x) = \dfrac{1}{x\,(\log x)^p}$, $p > 0$ and $x \in [2, \infty[$. Then $f(x) > 0$ and is monotonically decreasing for $2 \le x < \infty$.

Let $$I_n = \int_2^n \frac{1}{x\,(\log x)^p}\, dx .$$

Then $$I_n = \left[\frac{(\log x)^{1-p}}{1-p} \right]_2^n, \text{ when } p \ne 1$$

$$= \frac{1}{1-p} \left[(\log n)^{1-p} - (\log 2)^{1-p} \right]$$

and when $p = 1$, we have $I_n = [\log \log x]_2^n = \log \log n - \log \log 2.$

Hence when $n \to \infty$, we have

$$\lim_{n \to \infty} \int_2^n f(x)\, dx = \infty, \text{ if } p \le 1, \quad \text{and} \quad = -\frac{1}{1-p} (\log 2)^{1-p} \text{ if } .$$

Thus the integral $\int_2^{\infty} f(x)\, dx$ converges if $p > 1$ and diverges if $0 < p \le 1$.

Hence by Cauchy's integral test the series

$$\sum_{n=2}^{\infty} f(n) = \sum_{n=2}^{\infty} \frac{1}{n\,(\log n)^p}$$

converges if $p > 1$ and diverges if $0 < p \le 1$.

Example 38: *Show by integral test that the series* $\Sigma \dfrac{1}{n^p}$ $(p > 0)$ *converges if* $p > 1$ *and diverges if* $0 < p \leq 1$. **(Agra 2014)**

Solution: Let $f(x) = \dfrac{1}{x^p}$, $p > 0$ and $x \in [1, \infty[$.

Then $f(x) > 0$ and is monotonically decreasing on $[1, \infty[$.

Let
$$I_n = \int_1^n \frac{1}{x^p}\,dx = \int_1^n x^{-p}\,dx$$
$$= \begin{cases} \dfrac{n^{1-p}}{1-p} - \dfrac{1}{1-p}, & \text{if } p \neq 1 \\ \log n, & \text{if } p = 1. \end{cases}$$

Now, when $n \to \infty$, $n^{1-p} = \dfrac{1}{n^{p-1}} \to 0$ if $p > 1$, $n^{1-p} \to \infty$ if $p < 1$ and $\log n \to \infty$.

$$\therefore \quad \lim_{n \to \infty} I_n = -\frac{1}{1-p} = \frac{1}{p-1} \text{ if } p > 1 \text{ and } \lim I_n = \infty \text{ if } p \leq 1.$$

Thus the integral $\int_1^\infty f(x)\,dx$ converges if $p > 1$ and diverges if $p \leq 1$ and hence by Cauchy's integral test the series $\Sigma \dfrac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Example 39: *Show that* $\displaystyle\lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n\right)$ *exists.*

Solution: Let $f(x) = \dfrac{1}{x}$ where $1 \leq x < \infty$.

Then $f(x) > 0$ and monotonically decreasing on $[1, \infty[$.

Take $s_n = f(1) + f(2) + \ldots + f(n) = 1 + \dfrac{1}{2} + \ldots + \dfrac{1}{n}$,

and
$$I_n = \int_1^n f(x)\,dx = \int_1^n \frac{1}{x}\,dx = \log n.$$

Then proceeding as in article 2.20 and thus here using condition (3) of article 2.20, we get

$$f(n) \leq s_n - I_n \leq f(1) \text{ for all } n, \quad \text{or} \quad 0 < \frac{1}{n} \leq s_n - I_n \leq 1 \text{ for all } n.$$

This shows that the sequence $< s_n - I_n >$ is bounded below.

Also as shown in article 2.20, $< s_n - I_n >$ is a monotonically decreasing sequence and hence it converges.

We call the limit of this sequence **Euler's constant** and denote it by γ which is $0 \cdot 577$ approximately.

Hence $\lim \left\{1 + \dfrac{1}{2} + \ldots + \dfrac{1}{n} - \log n\right\}$ exists and is equal to γ, where γ is called Euler's constant.

Comprehensive Exercise 7

1. Apply integral test to check the convergence of the following series :

(i) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, (ii) $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$, (iii) $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$.

2. Apply Cauchy's integral test to prove the convergence of the series

(i) $\Sigma \frac{1}{n^2+1}$, (ii) $\Sigma \frac{1}{n(n+1)}$, (iii) $\Sigma \frac{1}{n\sqrt{(n^2-1)}}$.

3. If $f(x)$ is positive and monotonically decreasing when $x \geq 1$, then prove that the sequence whose nth term is

$$f(1) + f(2) + \ldots + f(n) - \int_1^n f(x)\, dx$$

converges to a finite limit.

Answers 7

1. (i) Divergent (ii) Convergent (iii) Convergent

2.21 Alternating Series

So far we have mainly dealt with series of positive terms. We have seen that a series of positive terms either converges or diverges and cannot oscillate. But a series which contains an infinite number of positive and an infinite number of negative terms may either converge or diverge or oscillate.

Alternating Series: Definition: *A series whose terms are alternately positive and negative is called an* **alternating series.** Thus an alternating series is of the form

$$u_1 - u_2 + u_3 - u_4 + \ldots + (-1)^{n-1} u_n + \ldots$$

where $u_n > 0$ for all n. It is denoted as

$$\sum_{n=1}^{\infty} (-1)^{n-1} u_n .$$

The following are some examples of an alternating series.

(i) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots$

(ii) $1 - \frac{3}{4} + \frac{4}{6} - \frac{5}{8} + \frac{6}{10} - \frac{7}{12} + \ldots + \frac{(-1)^{n-1}(n+1)}{2n} + \ldots$

(iii) $1 - \frac{2}{\log 2} + \frac{3}{\log 3} - \frac{4}{\log 4} + \ldots$

Theorem: Alternating Series Test (Leibnitz's Test): *An infinite series $\Sigma (-1)^{n-1} u_n$ in which the terms are alternately positive and negative is convergent if each term is numerically less than the preceding term and lim $u_n = 0$.*

Symbolically, the alternating series

$$u_1 - u_2 + u_3 - u_4 + \ldots + (-1)^{n-1} u_n + \ldots, \ (u_n > 0 \text{ for all } n)$$

converges if

(*i*) $u_{n+1} \le u_n$ *for all n i.e.,* $u_1 \ge u_2 \ge u_3 \ge u_4 \ge \ldots$

and (*ii*) *lim* $u_n = 0$ *i.e.,* $u_n \to 0$ *as* $n \to \infty$.

Proof: Let $S_n = u_1 - u_2 + u_3 - u_4 + \ldots + (-1)^{n-1} u_n$ so that $< S_n >$ is the sequence of partial sums of the given series.

We shall prove the theorem in **two steps**.

(i) First we shall prove that the subsequences $< S_{2n} >$ and $< S_{2n+1} >$ of the sequence $< S_n >$ converge to the same limit, say S.

We have $\quad S_{2n} = u_1 - u_2 + \ldots + u_{2n-1} - u_{2n}$

and $\quad S_{2n+2} = u_1 - u_2 + \ldots + u_{2n-1} - u_{2n} + u_{2n+1} - u_{2n+2}$.

$\therefore \ S_{2n+2} - S_{2n} = u_{2n+1} - u_{2n+2} \ge 0$ for all n because it is given that $u_{n+1} \le u_n$ for all n.

$\therefore \ S_{2n+2} \ge S_{2n}$ for all n and so the sequence $< S_{2n} >$ is monotonically increasing.

Again for all n,

$$S_{2n} = u_1 - [(u_2 - u_3) + (u_4 - u_5) + \ldots + (u_{2n-2} - u_{2n-1}) + u_{2n}]$$

$= u_1$ – some positive number because $u_2 - u_3, \ldots, u_{2n-2} - u_{2n-1}, u_{2n}$ are all positive

$\le u_1$.

Thus $S_{2n} \le u_1$ for all n and so the sequence $< S_{2n} >$ is bounded above.

Since the sequence $< S_{2n} >$ is monotonically increasing and bounded above, therefore it converges. Let lim $S_{2n} = S$.

Now $S_{2n+1} = S_{2n} + u_{2n+1}$.

$$\therefore \quad \lim S_{2n+1} = \lim S_{2n} + \lim u_{2n+1}$$

$$= S + 0 \qquad [\because \lim u_n = 0]$$

$$= S.$$

$\therefore$ the sequence $< S_{2n+1} >$ also converges to S.

Thus the subsequences $< S_{2n} >$ and $< S_{2n+1} >$ of the sequence $< S_n >$ converge to the same limit S.

(ii) Now we shall show that the sequence $< S_n >$ also converges to S.

Let $\varepsilon > 0$ be given. Since the sequences $< S_{2n} >$ and $< S_{2n+1} >$ both converge to S, therefore there exist +ive integers m_1 and m_2 such that

$$|S_{2n} - S| < \varepsilon \text{ for all } n \ge m_1$$

and $\quad |S_{2n+1} - S| < \varepsilon$ for all $n \ge m_2$.

Let $m = \max(m_1, m_2)$.

Then $\quad |S_n - S| < \varepsilon$ for all $n \geq 2m$.

$\therefore \quad$ the sequence $< S_n >$ converges to S.

Hence the given series $\Sigma\, (-1)^{n-1}\; u_n$ converges.

Note 1: The above test is equally applicable to the series $\Sigma\,(-1)^n\; u_n, u_n > 0$ for all n, provided both the conditions (i) and (ii) are satisfied.

Note 2: If in the case of an alternating series

$$u_1 - u_2 + u_3 - u_4 + \ldots\ldots \; (u_n > 0 \text{ for all } n),$$

the terms continually decrease, we cannot say that the series is convergent unless $\lim u_n = 0$. Because if $\lim u_n \neq 0$, then $\lim S_{2n}$ and $\lim S_{2n+1}$ will differ and so the series will not be convergent. Such a series is an oscillatory series.

For example, consider the series

$$2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \ldots.$$

Here the terms are alternately positive and negative and each term is numerically less than the preceding term because

$$2 > \frac{3}{2} > \frac{4}{3} > \frac{5}{4} > \ldots\ldots.$$

But here $\lim u_n = \lim \dfrac{n+1}{n} = \lim \left(1 + \dfrac{1}{n}\right) = 1 \neq 0$. Hence the given series is not convergent. As a matter of fact it is an oscillatory series.

Illustrative Examples

Example 40: *Show that the series* $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ *converges.* **(Avadh 2012)**

Solution: The given series is an alternating series

$$u_1 - u_2 + u_3 - u_4 + \ldots + (-1)^{n-1} u_n + \ldots, (u_n > 0 \text{ for all } n).$$

Here $u_n = 1/n > 0$ for all n.

We have $u_{n+1} - u_n = \dfrac{1}{n+1} - \dfrac{1}{n} = \dfrac{n-n-1}{n\,(n+1)} = \dfrac{-1}{n\,(n+1)} < 0$ for all n.

Thus $u_{n+1} < u_n$ for all n *i.e.*, each term is numerically less than the preceding term.

Also $\quad \lim u_n = \lim \dfrac{1}{n} = 0.$

Hence by Leibnitz's test for alternating series, the given series is convergent.

Example 41: *Show that the following series are convergent.*

(*i*) $1^{-p} - 2^{-p} + 3^{-p} - \ldots$ *when* $p > 0$.

(*ii*) $\dfrac{1}{x+1} - \dfrac{1}{x+2} + \dfrac{1}{x+3} - \ldots$ *except when x is a negative integer.*

Solution: (i) The given series is an alternating series

$$u_1 - u_2 + u_3 - u_4 + \ldots + (-1)^{n-1} u_n + \ldots, (u_n > 0 \text{ for all } n).$$

Here $u_n = 1/n^p > 0$ for all n.

Also since $p > 0$, we have $\dfrac{1}{1^p} > \dfrac{1}{2^p} > \dfrac{1}{3^p} > \ldots$.

Thus $u_{n+1} < u_n$ for all n.

Also $\lim u_n = \lim \dfrac{1}{n^p} = 0$, since $p > 0$.

Hence by alternating series test the given series is convergent for $p > 0$.

(ii) The given series is

$$\frac{1}{x+1} - \frac{1}{x+2} + \frac{1}{x+3} - \ldots, \; x \text{ is not a –ive integer.}$$

If $x > -1$, then the terms are alternately positive and negative from the beginning. If $x < -1$, excluding –ive integers, then the terms are *ultimately* alternating in sign. Since the removal of a finite number of terms does not affect the convergence of the series, therefore we may assume the series to be alternating in sign in both the cases. Obviously $u_1 > u_2 > u_3 > u_4 > \ldots$ *i.e.*, each term of the series is numerically less than the preceding term.

Also $\lim u_n = \lim \dfrac{1}{x+n} = 0$.

Hence by alternating series test, the given series is convergent.

Comprehensive Exercise 8

1. Examine the convergence of the series

$$\frac{1}{1.2} - \frac{1}{3.4} + \frac{1}{5.6} - \frac{1}{7.8} + \ldots.$$

2. Show that the series

$$\frac{\log 2}{2^2} - \frac{\log 3}{3^2} + \frac{\log 4}{4^2} - \ldots \text{ converges.}$$

3. Examine the convergence of the series

$$\sum_{n=1}^{\infty} (-1)^n \left[\frac{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}}{n} \right].$$

4. Test the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \left[\frac{1}{n} + \frac{(-1)^{n+1}}{\sqrt{n}} \right]$.

Answers 8

1. Convergent
3. **Convergent**
4. Divergent

2.22 Absolute Convergence and Conditional Convergence

(Meerut 2012B)

Absolutely Convergent Series:

Definition: *A series Σu_n is said to be* **absolutely convergent** *if the series $\Sigma |u_n|$ is convergent.*

If Σu_n is a series of positive terms, then Σu_n and $\Sigma |u_n|$ are the same series and so if Σu_n is convergent, it is also absolutely convergent. Hence for a series of positive terms the concepts of convergence and absolute convergence are the same.

But if a series Σu_n contains an *infinite* number of positive and an infinite number of negative terms, then Σu_n is absolutely convergent only if the series $\Sigma |u_n|$ obtained from Σu_n by making all its terms positive is convergent.

For example the series

$$\Sigma u_n = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \ldots$$

is absolutely convergent. Here we see that the series

$$\Sigma |u_n| = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$$

is an infinite geometric series of positive terms with common ratio $\frac{1}{2}$ which is < 1 and so it is convergent. Hence the given series Σu_n is absolutely convergent.

Non-absolutely convergent or semi-convergent or conditionally convergent series:

Definition: *A series Σu_n is said to be* **semi-convergent** *or* **conditionally convergent** *or* **non-absolutely convergent** *if Σu_n is convergent but $\Sigma |u_n|$ is divergent.*

For example, consider the series

$$\Sigma u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots.$$

It is an alternating series in which each term is numerically less than the preceding term and $\lim u_n = \lim (1/n) = 0$. Hence by alternating series test, Σu_n is a convergent series.

But the series $\Sigma |u_n| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ is the series $\Sigma (1/n^p)$, for $p = 1$, and we know that it is divergent. Thus here Σu_n is convergent while $\Sigma |u_n|$ is divergent.

Hence Σu_n is a semi-convergent or conditionally convergent or non-absolutely convergent series.

Tests for absolute convergence: To test the absolute convergence of the series Σu_n, we have to simply test the convergence of the series $\Sigma \mid u_n \mid$ which is a series of positive terms. Hence the various tests given for the series of positive terms are precisely the tests which we are to apply to check the absolute convergence of the series Σu_n. We have to simply replace u_n by $|u_n|$ in these tests. For example by Cauchy's root test, the series Σu_n is absolutely convergent if $\lim |u_n|^{1/n} < 1$. Similarly by D'Alembert's ratio test the series Σu_n is absolutely convergent if

$$\lim \frac{|u_n|}{|u_{n+1}|} = \lim \left| \frac{u_n}{u_{n+1}} \right| > 1.$$

Similarly comparison test or other tests may be used.

However these tests cannot give any information about the conditional convergence of the series.

2.23 Some Important Theorems on Absolutely Convergent Series

Theorem 1: *Every absolutely convergent series is convergent but the converse is not necessarily true i.e., convergence need not imply absolute convergence.*

Theorem 2: *In an absolutely convergent series, the series formed by its positive terms alone is convergent and the series formed by its negative terms alone is convergent.*

Theorem 3: Re-arrangement of terms of an absolutely convergent series: *If the terms of an absolutely convergent series are re-arranged the series remains convergent and its sum unaltered.*

Or The sum of an absolutely convergent series is independent of the order of terms.

Illustrative Examples

Example 42: *Show that the series*

$$\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \dots$$ **(Meerut 2012)**

is conditionally convergent.

Solution: The given series is an alternating series

$$u_1 - u_2 + u_3 - \dots + (-1)^{n-1} u_n + \dots, (u_n > 0 \text{ for all } n).$$

Here $u_n = \dfrac{1}{\sqrt{n}} > 0$ for all n.

Also for all n, $\sqrt{(n+1)} > \sqrt{n}$