Chapter 1

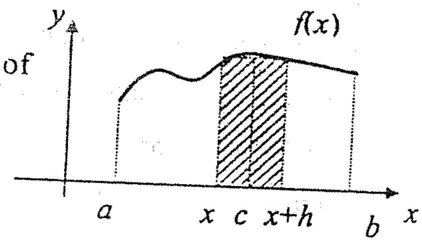
Indefinite Integration

1.1 Introduction

In the course of differential calculus, we studied the mathematical process of finding the derivative of a function, and we considered various applications of derivatives. In this course we will study another branch of calculus, called integral calculus. In differential calculus, the tangent problem led us to formulate, in terms of limits, the idea of a derivative, which later turned out to be applicable, through velocities and other rates of change, to a variety of applied problems.

In integral calculus, the area problem will lead us to formulate, again in terms of limits, the idea of an integral, which will later be used to find volumes, lengths of curves, work, and forces. The area problem states that: Given a function that is continuous and nonnegative in an interval

$[a, b]$, it is required to find the area of the region bounded by the graph of the curve, the interval $[a, b]$ on the x-axis and the vertical lines $x = a$ and $x = b$. The major development of solving this area problem was made independently by I. Newton and G. Leibniz in 1675. They discovered that areas could be obtained by reversing the process of differentiation. Their idea is that, to find the area $A(x)$ of the region bounded by the graph of a nonnegative and continuous function $f(x)$ on the interval $[a, x]$ and the x-axis (where x is any point on the x-axis), we first find the derivative of the area function $A(x)$, then we use the value of the derivative $A'(x)$ to determine $A(x)$ itself.



It is clear from the definition of the derivative that

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

But, it can be seen from the opposite figure that

$$\frac{A(x+h) - A(x)}{h} \approx \frac{h.f(c)}{h} = f(c),$$

where c is a point between and $x + h$, and when $h \to 0, c \to x$. This implies that

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h \to 0} f(c) = f(x).$$

The above result means that $A(x)$, which we are looking for, is a function whose derivative is $f(x)$.

The above discussion shows that there is a strong connection between the two branches of calculus. Namely, the differential and integral calculus. This connection is shown via the Fundamental Theorems of Calculus. These theorems greatly simplify the solution of many mathematical problems.

1.2 Antiderivative

We already discussed methods of finding derivatives of functions in the course of differential calculus. We will now turn our attention towards reversing the operation of differentiation. Given the derivative of a function, we are looking to find the function itself. This process is called ***antidifferentiation***. For example, if the derivative of the function is $3x^2$, we know that the function would be

$F(x) = x^3$ because $\frac{d}{dx}\left(x^3\right) = 3x^2$. But the function could also be $F(x) = x^3 + 4$ because $\frac{d}{dx}\left(x^3 + 4\right) = 3x^2$. It is clear that any function of the form $F(x) = x^3 + C$, where C is a constant, will have $F'(x) = 3x^2$ as its derivative. Thus, we say that the ***antiderivative*** of $f(x) = 3x^2$ is $F(x) = x^3 + C$, where C is an ***arbitrary constant.***

It is easily seen that

If $f(x) = x^4$, then $F(x) = \frac{x^5}{5} + C$

and,

If $f(x) = x^7$, then $F(x) = \frac{x^8}{8} + C$

In general,

If $f(x) = x^n$, then $F(x) = \frac{x^{n+1}}{n+1} + C$, for $n \neq -1$

Example (1): What are the antiderivatives of $f(x) = x^{11}$, $g(x) = x^{1/3}$ and $h(x) = x^{-1/2}$?

Solution: Using the above formula, we get

$$F(x)=\frac{x^{12}}{12}+C,$$

$$G(x)=\frac{x^{4/3}}{\frac{4}{3}}+C=\frac{3}{4}x^{4/3}+C,$$

and,

$$H(x)=\frac{x^{1/2}}{1/2}+C=2x^{1/2}+C.$$

From the above discussion we reach the following definition of the antiderivative of a function.

Definition 1.1 (Antiderivative): The differentiable function $F(x)$ is called an ***antiderivative*** of $f(x)$ on a given interval, if $F'(x)=f(x)$, on that interval.

The following theorem gives the relation between different antiderivatives of a given function.

Theorem 1.1: If $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$, then there is a constant C such that

$$F(x)-G(x)=C$$

(Two antiderivatives of a function can differ only by a constant.)

For example,

$$F(x)=x^3+2, \qquad G(x)=x^3, \qquad H(x)=x^3-6$$

are all antiderivatives of the function $f(x)=3x^2$.

1.3 The Indefinite Integral

The process of finding an antiderivative is called ***integration***. The function that results when integration takes place is called ***indefinite integral***, or more simply, an ***integral***. We denote the indefinite integral of a function $f(x)$ by $\int f(x)dx$. The symbol $\int$ is called the ***integral sign*** and the function $f(x)$ is called the ***integrand***. The dx in the indefinite integral means that $\int f(x)\,dx$ is the integral of $f(x)$ with respect to the variable x just as the symbol $df(x)/dx$ means the derivative of $f(x)$ with respect to x. Thus we have the following definition

<u>Definition</u> 1.2 (Indefinite integral):

If $F(x)$ is any antiderviative of a given function $f(x)$

i.e. $$F'(x) = f(x)$$

then, the indefinite integral of $f(x)$ with respect to the variable x is given by

$$\int f(x)\,dx = F(x) + C$$

The constant C is called "***the constant of integration***".

For example, using this notation,

$$\int 3x^2\,dx = x^3 + C$$

where C is the constant of integration.

1.4 Basic Integration Rules

Since integration is the inverse operation of differentiation so we have the following trivial rule.

<u>Rule 1</u>

$$\int \frac{df}{dx}\,dx = \int df = f(x) + C,$$

$$\frac{d}{dx}\int f(x)dx = f(x).$$

That is to say, the two symbols "$\frac{d}{dx}$" and "$\int$" are opposite operators, any one of them cancels the other.

Also, as shown earlier, the derivative of the product of a constant and a function is the product of the constant and the derivative of the function. A similar rule applies to indefinite integral. Moreover, since derivative of sums or differences are found term by term, indefinite integrals can also be found term by term. This is described by the following two rules.

__Rule 2__ **(Scalar Multiplication Rule)** The constant factor can be taken outside the integral sign. That is:
For any constant k, $\int kf(x)\,dx = k\int f(x)\,dx$.

__Example__ (1)

$$\int 3x^5\,dx = 3\int x^5\,dx = 3\left(\frac{x^6}{6} + C_1\right) = \frac{1}{2}x^6 + C$$

__Rule 3__ **(Sum or Difference Rule):** The indefinite integral of the algebraic sum (or difference) of two functions equals the sum of their integrals:

$$\int[f(x)\pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Example (2)

$$\int\left(x-x^4\right)dx=\int x\,dx-\int x^4\,dx=\frac{x^2}{2}-\frac{x^5}{5}+C$$

Example (3)

$$\int\left(21x^{3/4}+8x^3\right)dx=\int 21x^{3/4}\,dx+\int 8x^3\,dx$$

$$=21\cdot\frac{x^{7/4}}{7/4}+8\cdot\frac{x^4}{4}+C=12x^{7/4}+2x^4+C$$

Example (4)

$$\int(x-3)^2\,dx=\int\left(x^2-6x+9\right)dx$$

$$=\frac{x^3}{3}-6\cdot\frac{x^2}{2}+9x+C=\frac{x^3}{3}-3x^2+9x+C$$

1.5 Table of Famous Integrals

The following table summarizes the integral formulas for some elementary functions .

The Integrand	The Integral
$x^n, \quad n \neq -1$	$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$
$1/x$	$\int \frac{1}{x} \, dx = \ln\|x\| + C$
e^x	$\int e^x \, dx = e^x + C$
a^x	$\int a^x \, dx = \frac{a^x}{\ln a} + C$
$\cos x$	$\int \cos x \, dx = \sin x + C$
$\sin x$	$\int \sin x \, dx = -\cos x + C$
$\sec^2 x$	$\int \sec^2 x \, dx = \tan x + C$
$\csc^2 x$	$\int \csc^2 x \, dx = -\cot x + C$
$\sec x \tan x$	$\int \sec x \tan x \, dx = \sec x + C$
$\csc x \cot x$	$\int \csc x \cot x \, dx = -\csc x + C$
$\cosh x$	$\int \cosh x \, dx = \sinh x + C$
$\sinh x$	$\int \sinh x \, dx = \cosh x + C$
$\text{sech}^2 x$	$\int \text{sech}^2 x \, dx = \tanh x + C$

The Integrand	The Integral
$\operatorname{csch}^2 x$	$\int \operatorname{csch}^2 x\, dx = -\coth x + C$
$\operatorname{sech} x \tanh x$	$\int \operatorname{sech} x \tanh x\, dx = -\operatorname{sech} x + C$
$\operatorname{csch} x \coth x$	$\int \operatorname{csch} x \coth x\, dx = -\operatorname{csch} x + C$
$\dfrac{1}{\sqrt{1-x^2}}$	$\int \dfrac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C = -\cos^{-1} x + C$
$\dfrac{1}{1+x^2}$	$\int \dfrac{1}{1+x^2} dx = \tan^{-1} x + C = -\cot^{-1} x + C$
$\dfrac{1}{x\sqrt{x^2-1}}$	$\int \dfrac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C = -\csc^{-1} x + C$
$\dfrac{1}{\sqrt{1+x^2}}$	$\int \dfrac{1}{\sqrt{1+x^2}} dx = \sinh^{-1} x + C$
$\dfrac{1}{\sqrt{x^2-1}}$	$\int \dfrac{1}{\sqrt{x^2-1}} dx = \cosh^{-1} x + C$
$\dfrac{1}{1-x^2}, \lvert x\rvert < 1$	$\int \dfrac{1}{1-x^2} dx = \tanh^{-1} x + C$
$\dfrac{1}{1-x^2}, \quad \lvert x\rvert > 1$	$\int \dfrac{1}{1-x^2} dx = \coth^{-1} x + C$

The Integrand	The Integral
$\dfrac{1}{x\sqrt{1-x^2}},\ \lvert x\rvert<1$	$\displaystyle\int\frac{1}{x\sqrt{1-x^2}}\,dx = -\operatorname{sech}^{-1}x + C$
$\dfrac{1}{x\sqrt{1+x^2}}$	$\displaystyle\int\frac{1}{x\sqrt{1+x^2}}\,dx = -\operatorname{csch}^{-1}x + C$

Using rules 1,2 and 3 together with the above table we are able to solve the following example:

Example (1): Find each of the following integrals

(i) $\displaystyle\int\left(-\frac{4}{x} + 4e^x\right)dx$ (ii) $\displaystyle\int\left(\frac{1}{\sqrt{1-x^2}} + \frac{1}{x\sqrt{1-x^2}}\right)dx$

(iii) $\displaystyle\int\frac{10x^2 - 1}{\sqrt{x}}\,dx$ (iv) $\displaystyle\int\left(\sec^2 x + \cos x - 3^x\right)dx$

(v) $\displaystyle\int\left[\frac{3}{\sqrt[3]{x}} + \frac{1}{2\sqrt{x}} + x\sqrt[4]{x}\right]dx$

Solution:

(i) $\displaystyle\int\left(-\frac{4}{x} + 4e^x\right)dx = -4\ln\lvert x\rvert + 4e^x + C$

(ii) $\int\left(\frac{1}{\sqrt{1-x^2}}+\frac{1}{x\sqrt{1-x^2}}\right)dx=\sin^{-1}x-\text{sech}^{-1}x+C$

(iii) $\int\frac{10x^2-1}{\sqrt{x}}dx=\int\left(\frac{10x^2}{\sqrt{x}}-\frac{1}{\sqrt{x}}\right)dx$

$$=\int\left(10x^{3/2}-x^{-1/2}\right)dx$$

$$=10\frac{x^{5/2}}{5/2}-\frac{x^{1/2}}{1/2}+C$$

$$=4x^{5/2}-2\sqrt{x}+C$$

(iv) $\int\left(\sec^2 x+\cos x-3^x\right)dx=\tan x+\sin x-\frac{3^x}{\ln 3}+C$

(v) $\int\left[\frac{3}{\sqrt[3]{x}}+\frac{1}{2\sqrt{x}}+x\sqrt[4]{x}\right]dx=\int\left[3x^{-1/3}+\frac{1}{2}x^{-1/2}+x^{5/4}\right]dx$

$$=3\int x^{-1/3}dx+\frac{1}{2}\int x^{-1/2}dx+\int x^{5/4}dx$$

$$=3\frac{x^{2/3}}{2/3}+\frac{1}{2}\frac{x^{1/2}}{1/2}+\frac{x^{9/4}}{9/4}+C$$

$$=\frac{9}{2}x^{2/3}+x^{1/2}+\frac{4}{9}x^2\sqrt[4]{x}+C$$

$$=\frac{9}{2}\sqrt[3]{x^2}+\sqrt{x}+\frac{4}{9}x^2\sqrt[4]{x}+C$$

Rule 4

If $$\int f(x)\,dx = F(x) + C,$$

then,

(a) $\int f(ax)dx = \frac{1}{a} F(ax) + C$

(b) $\int f(x+b)dx = F(x+b) + C$

(c) $\int f(ax+b)dx = \frac{1}{a} F(ax+b) + C$

Applying Rules 1,2,3 and 4 we can solve the following examples:

Example (2):

$$\int co3x dx = \frac{1}{3}\sin 3x + C$$

$$\int e^{7x} dx = \frac{1}{7} e^{7x} + C$$

$$\int \sin(x+5)dx = -\cos(x+5) + C,$$

$$\int e^{x-1} dx = e^{x-1} + C$$

$$\int \sin(3x-1)dx = -\frac{1}{3}\cos(3x-1) + C,$$

$$\int e^{5x+2} dx = \frac{1}{5} e^{5x+2} + C,$$

$$\int \frac{dx}{4x-7} = \frac{1}{4}\ln|4x-7| + C.$$

Example (3): If $\frac{d^2y}{dx^2} = 4\pi(\cos 2x - \sin 2x)$, find $\frac{dy}{dx}$, then find y.

Solution: Since $\int \frac{df}{dx} dx = f(x) + C$ then

$$\frac{dy}{dx} = \int \frac{d^2y}{dx^2} dx = \int 4\pi(\cos 2x - \sin 2x) dx$$
$$= 2\pi(\sin 2x + \cos 2x) + C_1,$$

and,

$$y = \int \frac{dy}{dx} dx = \int ((2\pi(\sin 2x + \cos 2x)) + C_1) dx$$
$$= \pi(-\cos 2x + \sin 2x) + C_1 x + C_2,$$

where, C_1 and C_2 are constants.

Using Rule 4 we get the following more general table of basic integrals.

$$\int (ax+b)^n \, dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + C, \qquad n \neq -1$$

$$\int \frac{1}{(ax+b)} \, dx = \frac{1}{a} \ln|(ax+b)| + C$$

$$\int e^{(ax+b)} \, dx = \frac{1}{a} e^{(ax+b)} + C$$

$$\int A^{(ax+b)} dx = \frac{1}{a}\frac{A^{(ax+b)}}{\ln A} + C$$

$$\int \cos(ax+b) \, dx = \frac{1}{a} \sin(ax+b) + C$$

$$\int \sin(ax+b) \, dx = -\frac{1}{a} \cos(ax+b) + C$$

$$\int \sec^2(ax+b) \, dx = \frac{1}{a} \tan(ax+b) + C$$

$$\int \csc^2(ax+b) \, dx = -\frac{1}{a} \cot(ax+b) + C$$

$$\int \sec(ax+b)\tan(ax+b) \, dx = \frac{1}{a} \sec(ax+b) + C$$

$$\int \csc(ax+b)\cot(ax+b) \, dx = -\frac{1}{a} \csc(ax+b) + C$$

$$\int \cosh(ax+b)\,dx = \frac{1}{a}\sinh(ax+b) + C$$

$$\int \sinh(ax+b)\,dx = \frac{1}{a}\cosh(ax+b) + C$$

$$\int \operatorname{sech}^2(ax+b)\,dx = \frac{1}{a}\tanh(ax+b) + C$$

$$\int \operatorname{csch}^2(ax+b)\,dx = -\frac{1}{a}\coth(ax+b) + C$$

$$\int \operatorname{sech}(ax+b)\tanh(ax+b)\,dx = -\frac{1}{a}\operatorname{sech}(ax+b) + C$$

$$\int \operatorname{csch}(ax+b)\coth(ax+b)\,dx = -\frac{1}{a}\operatorname{csch}(ax+b) + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\left(\frac{x}{a}\right)+C$$

$$\int \frac{1}{a^2-x^2}\,dx=\frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)+C,\ |x|<a$$

$$\int \frac{1}{a^2-x^2}\,dx=\frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right)+C,\ |x|>a,$$

$$\int \frac{1}{x\sqrt{a^2-x^2}}\,dx=-\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{x}{a}\right)+C,\qquad |x|<a$$

$$\int \frac{1}{x\sqrt{a^2+x^2}}\,dx=-\frac{1}{a}\operatorname{csch}^{-1}\left(\frac{x}{a}\right)+C$$

Using the above table we can solve the following examples:

<u>Example</u> (4): Find each of the following integrals

(i) $\int e^{(3x-4)}\,dx$ (ii) $\int \frac{1}{\sqrt{4-x^2}}\,dx$

(iii) $\int \frac{1}{9+16x^2}\,dx$ (iv) $\int \frac{1}{(3x+7)}\,dx$

(v) $\int \operatorname{sech}(2x-3)\tanh(2x-3)\,dx$

Solution

(i) $\int e^{(3x-4)}\,dx = \frac{1}{3}e^{(3x-4)} + C$

(ii) $\int \frac{1}{\sqrt{4-x^2}}\,dx = \sin^{-1}\left(\frac{x}{2}\right) + C$

(iii) $\int \frac{1}{9+16x^2}\,dx = \frac{1}{16}\int \frac{1}{9/16+x^2}\,dx$

$= \frac{1}{16}\int \frac{1}{(3/4)^2+x^2}\,dx$

$= \frac{1}{16}\,\frac{1}{(3/4)}\tan^{-1}\left(\frac{x}{3/4}\right) + C$

$= \frac{1}{12}\tan^{-1}\left(\frac{4x}{3}\right) + C$

(iv) $\int \frac{1}{(3x+7)}\,dx = \frac{1}{3}\ln|3x+7| + C$

(v) $\int \text{sech}(2x-3)\tanh(2x-3)dx = -\frac{1}{2}\text{sech}(2x-3) + C$

1.6 Determination of the Constant of Integration

In order to find the value of the constant of integration we need an auxiliary condition to be satisfied, as in the following example:

Example (1): Find the equation of the curve whose slope at the point (x,y) is $3x^2$ if the curve is required to pass through the point $(1,-1)$.

Solution: The slope of the curve at any point (x,y) is $\frac{dy}{dx}$.

But,

$$\frac{dy}{dx} = 3x^2$$

Integrating both sides with respect to x, we get

$$\int \frac{dy}{dx}\,dx = \int 3x^2\,dx$$

or

$$y = x^3 + C$$

This, last equation is the equation of the curve passing through a general point (x,y).

Now, if the curve has to pass through the point (1,−1) we must have that

$$y = x^3 + C$$
$$-1 = 1 + C,$$

from which we get

$$C = -2.$$

Therefore, the equation of the required curve is

$$y = x^3 - 2$$

Exercise 1.1

Integrate each of the following functions with respect to x:

1. $\left(2\sqrt{x}+\frac{4}{\sqrt{x}}\right)^3$ 2. $\sin(5x-1)$ 3. $\cosh 6x$

4. $\sec^2(6x-5)$ 5. $6^x 5^{2x}$ 6. $\frac{1}{4x+5}$

7. $(2x+7)^{-7}$ 8. $\sqrt{2x+5}$ 9. $e^{-2(-3x+8)}dx$

10. If $y''=12t^2-6t+5$, find y' and y given that $y=y'=5$ when $t=0$, ($y'=\frac{dy}{dt}$).

11. Find the equation of the curve whose slope is $-12x^3$ and passing through the point $(1,8)$. Find y when $x=3$.

12. Evaluate each of the following integrals:

(a) $\int\frac{1}{x\sqrt{a^2+x^2}}dx$ (b) $\int\frac{1}{\sqrt{14-x^2}}dx$

(c) $\int \operatorname{csch}(3x+1)\coth(3x+1)dx$ (d) $\int \csc^2(4x-9)dx$

Chapter (2)

Methods of Integration

Chapter 2

Methods of Integration

2.1 Introduction

In this chapter we shall develop techniques for obtaining indefinite integrals of more complicated functions. The most important integration techniques that will be considered here are the following:

Integration by Substitution (Change of Variables), Integration of Trigonometric Functions, Integration by Removing Roots, Integration by Parts, Integration by Reduction, Integration using Partial Fractions, Miscellaneous Method.

Before introducing these techniques we have the following important rules:

2.2 Two Important Rules

Using the chain rule for the derivatives, we get

$$\frac{d}{dx}[f(x)]^{n+1} = (n+1)[f(x)]^n f'(x) \qquad (1)$$

Integrate both sides of (1) with respect to x, we obtain

$$[f(x)]^{n+1} = (n+1)\int [f(x)]^n \cdot f'(x)\,dx \qquad (2)$$

which implies

Rule 1

$$\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + C, n \neq -1 \qquad (3)$$

Remark: The integration of a function raised to a given power, multiplied by the derivative of this function is computed easily from (3).

For example,

$$\int (1+\sin x)^2 \cos x\,dx = \frac{1}{3}(1+\sin x)^3 + C,$$

$$\int \frac{(\ln x)^4}{x}dx = \int (\ln x)^4 \cdot \left(\frac{1}{x}\right) dx = \frac{1}{5}(\ln x)^5 + C$$

Also, since $\frac{d}{dx}\ln|f(x)| = \frac{1}{f(x)}$ then we have:

Rule 2

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$$

Using the above two rules we can solve the following examples:

Example (1): Evaluate each of the following integrals

(i) $\int (x^3+1)^9 x^2\, dx$ (ii) $\int x\sqrt{7x^2+12}\, dx$

(iii) $\int \frac{\sin 2x}{\sqrt{1+5\cos 2x}}dx$ (iv) $\int \frac{x}{x^2+5}dx$

(v) $\int \frac{1}{x \ln x}dx$ (vi) $\int \tan x\, dx$

(vii) $\int \coth x\, dx$ (viii) $\int \sec x\, dx$

(ix) $\int \frac{\tan^{-1} x}{1+x^2}dx$ (x) $\int \frac{48\,\text{csch}6x\,\coth 6x}{(1+\text{csch}6x)^3}$

(xi) $\int \left(\ln x + \frac{1}{\ln x}\right)\frac{1}{x}dx$ (xii) $\int \sqrt{\sin x}\,\cos x\, dx$

(xiii) $\int (6x-1)\sqrt{3x^2-x+5}\, dx$ (ivx) $\int \frac{6x-1}{3x^2-x+5}dx$

Solution

(i) $\int (x^3+1)^9 x^2\,dx = \frac{1}{3}\int (x^3+1)^9 3x^2\,dx$ $\quad f(x)=(x^3+1)$

$f'(x)=3x^2$

then by Rule 1

$$\int (x^3+1)^9 x^2\,dx = \frac{1}{3}\cdot\frac{1}{10}(x^3+1)^{10}+C$$

(ii) $\int x\sqrt{7x^2+12}\,dx = \frac{1}{14}\int \sqrt{7x^2+12}\cdot 14x\,dx$

$$= \frac{1}{14}\int (7x^2+12)^{\frac{1}{2}}\cdot 14x\,dx \qquad f(x)=7x^2+12$$

$$f'(x)=14x$$

$$= \frac{1}{14}\cdot\frac{2}{3}(7x^2+12)^{\frac{3}{2}}+C$$

(iii) $\int \frac{\sin 2x}{\sqrt{1+5\cos 2x}}dx = \int (1+5\cos 2x)^{-\frac{1}{2}}\sin 2x\,dx$

$$= \frac{-1}{10}\int (1+5\cos 2x)^{-\frac{1}{2}}(-10\sin 2x)$$

$$= \frac{-2}{10}(1+5\cos 2x)^{\frac{1}{2}}+C$$

(iv) $\int \frac{x}{x^2+5}dx = \frac{1}{2}\int \frac{2x}{x^2+5}dx$ $\quad f(x)=x^2+5$

$f'(x)=2x$

then by Rule 2

$$\int \frac{x}{x^2+5}dx = \frac{1}{2}\ln(x^2+5)+C$$

(v) $\int \frac{1}{x \ln x} dx = \int \frac{1/x}{\ln x} dx$

$f(x) = \ln x$

$f'(x) = \frac{1}{x}$

then by Rule 2

$$\int \frac{1}{x \ln x} dx = \ln|\ln x| + C$$

(vi) $\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{-\sin x}{\cos x} dx$

$$= -\ln|\cos x| + C = \ln|\sec x| + C$$

(vii) $\int \coth x\, dx = \int \frac{\cosh x}{\sinh x} dx = \ln|\sinh x| + C$

(viii) $\int \sec x \cdot dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} dx$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx = \ln|\sec x + \tan x| + C$$

(ix) $\int \frac{\tan^{-1} x}{1 + x^2} dx = \int \tan^{-1} x \cdot \frac{1}{1 + x^2} dx = \frac{1}{2}\left(\tan^{-1} x\right)^2 + C$

(x) $\int \frac{48 \operatorname{csch} 6x \coth 6x}{(1 + \operatorname{csch} 6x)^3}$

$$= -8 \int (1 + \operatorname{csch} 6x)^{-3} \cdot (-6 \operatorname{csch} 6x \coth 6x) dx$$

$$= \frac{-8}{-2}(1 + \operatorname{csch} 6x)^{-2} + C = \frac{4}{(1 + \operatorname{csch} 6x)^2} + C$$

(xi) $\int\left(\ln x+\frac{1}{\ln x}\right)\frac{1}{x}dx=\int(\ln x)\frac{1}{x}dx+\int\frac{1/x}{\ln x}dx$

$$=\frac{1}{2}(\ln x)^2+\ln\left|\ln x\right|+C$$

(xii) $\int\sqrt{\sin x}\,\cos x\,dx=\int(\sin x)^{\frac{1}{2}}\cos x\,dx=\frac{2}{3}(\sin x)^{\frac{3}{2}}+C$

(xiii) $\int(6x-1)\sqrt{3x^2-x+5}\,dx=\int\left(3x^2-x+5\right)^{\frac{1}{2}}(6x-1)dx$

$$=\frac{2}{3}\left(3x^2-x+5\right)^{\frac{3}{2}}+C$$

(ivx) $\int\frac{6x-1}{3x^2-x+5}\,dx=\ln\left|3x^2-x+5\right|+C$

2.3 Integration by Simple Substitution (Change of Variables)

Consider the integral

$$\int f(x)\,dx \qquad (1)$$

Sometimes it is difficult to evaluate this integral directly, so we introduce a new variable to get an easier integral or we get directly one of the standard integrals. Of course, dx must also be replaced by the appropriate differential.

Let us replace the variable x by an appropriate function of another variable t, say. For example, let

$$x = \varphi(t)$$

then,

$$dx = \varphi'(t)\,dt$$

Using these substitutions in (1), we get

$$\int f(x)dx = \int f(\varphi(t))\varphi'(t)dt, \qquad (2)$$

which is assumed to be an easier integral, and so we can evaluate it directly. The following examples illustrate the above idea:

<u>Example</u> **(1):** Evaluate the following integrals

(i) $\int \frac{dx}{x\sqrt{1-(\ln x)^2}}$ (ii) $\int \frac{x^2}{5-x^6}\,dx$

(iii) $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$ (iv) $\int x^2\sqrt{x+2}\,dx$

(v) $\int \frac{\cos x}{\sin^2 x - 2\sin x + 1}\,dx$ (vi) $\int x e^{3x^2}\,dx$

(vii) $\int \frac{\sinh(\ln x)}{x}\,dx$ (viii) $\int x^2 \cos x^3\,dx$

Solution

(i) **For evaluating the integral**

$$\int \frac{dx}{x\sqrt{1-(\ln x)^2}},$$

let $\ln x = t \Rightarrow \frac{1}{x}dx = dt$, so that we obtain

$$\int \frac{dx}{x\sqrt{1-(\ln x)^2}} = \int \frac{1}{\sqrt{1-(\ln x)^2}} \cdot \frac{dx}{x}$$

$$= \int \frac{1}{\sqrt{1-t^2}} dt = \sin^{-1} t + C$$

$$= \sin^{-1}(\ln x) + C$$

(ii) For the integral $\int \frac{x^2}{5-x^6}dx = \int \frac{x^2}{5-(x^3)^2}dx$

let, $x^3 = t \Rightarrow 3x^2 dx = dt$, so, we obtain

$$\int \frac{x^2}{5-x^6}dx = \frac{1}{3}\int \frac{3x^2}{5-x^6}dx = \frac{1}{3}\frac{3x^2}{5-(x^3)^2}dx$$

$$= \frac{1}{3}\int \frac{1}{5-t^2}dt = \frac{1}{3}\cdot\frac{1}{\sqrt{5}}\tanh^{-1}\left(\frac{t}{\sqrt{5}}\right) + C$$

$$= \frac{1}{3\sqrt{5}}\tanh^{-1}\left(\frac{x^3}{\sqrt{5}}\right) + C$$

(iii) For the integral $\int \frac{e^x}{\sqrt{1-e^{2x}}}dx = \int \frac{e^x}{\sqrt{1-(e^x)^2}}dx$,

let, $e^x = t \Rightarrow \underline{e^x dx = dt}$, so, we get

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}dx = \int \frac{1}{\sqrt{1-t^2}}dt$$

$$=\sin^{-1} t + C = \sin^{-1}(e^x) + C$$

(iv) For the integral $\int x^2\sqrt{x+2}\,dx$,

let, $\sqrt{x+2} = t \Rightarrow x+2 = t^2 \Rightarrow dx = 2tdt$,

Then,

$$\int x^2\sqrt{x+2}\,dx = \int (t^2-2)^2\, 2t^2\, dt$$

$$= \int (t^4 - 4t^2 + 4)\, 2t^2\, dt$$

$$= \int 2(t^6 - 4t^4 + 4t^2)\, dt$$

$$= 2[\frac{t^7}{7} - 4\frac{t^5}{5} + 4\frac{t^3}{3}] + C$$

$$= \frac{2}{7}(x+2)^{7/2} - \frac{8}{5}(x+2)^{5/2} + \frac{8}{3}(x+2)^{3/2} + C$$

The above integral can be solved by using another substitution as follows:

let, $x+2=t \Rightarrow dx = dt, \quad x = t-2$

Then,

$$\int x^2\sqrt{x+2}\,dx = \int (t-2)^2 t^{1/2}\,dt$$

$$= \int (t^2 - 4t + 4)t^{1/2}\,dt$$

$$= \int (t^{5/2} - 4t^{3/2} + 4t^{1/2})\,dt$$

$$= \frac{t^{7/2}}{7/2} - 4\frac{t^{5/2}}{5/2} + 4\frac{t^{3/2}}{3/2} + C$$

$$= \frac{2}{7}(x+2)^{7/2} - \frac{8}{5}(x+2)^{5/2} + \frac{8}{3}(x+2)^{3/2} + C$$

(v) For the integral $\int \frac{\cos x}{\sin^2 x - 2\sin x + 1}\,dx$, we use the substitution

$$\sin x = t \quad \Rightarrow \quad \cos x\,dx = dt$$

So we have

$$\int \frac{\cos x}{\sin^2 x - 2\sin x + 1}\,dx = \int \frac{1}{t^2 - 2t + 1}\,dt$$

$$= \int \frac{1}{(t-1)^2}\,dt = \int (t-1)^{-2}\,dt = \frac{(t-1)^{-1}}{-1} + C$$

$$= \frac{-1}{(t-1)} + C = \frac{-1}{(\sin x - 1)} + C$$

The above integral can be evaluated directly as follows:

$$\int \frac{\cos x}{\sin^2 x - 2\sin x + 1} dx = \int \frac{\cos x}{(\sin x - 1)^2} dx$$

$$\int (\sin x - 1)^{-2} \cos x dx = \frac{(\sin x - 1)^{-1}}{-1} + C$$

$$= \frac{-1}{(\sin x - 1)} + C$$

(vi) For the integral $\int x e^{3x^2} dx$ we use the substitution

$t = 3x^2 \Rightarrow dt = 6x\,dx$ then

$$\int x e^{3x^2} dx = \int \frac{1}{6} e^t dt = \frac{1}{6} e^t + C = \frac{1}{6} e^{3x^2} + C$$

(vii) For the integral $\int \frac{\sinh(\ln x)}{x} dx$ we use the substitution

$t = \ln x \Rightarrow dt = \frac{1}{x} dx$ then

$$\int \frac{\sinh(\ln x)}{x} dx = \int \sinh t\, dt = \coth t + C = \cosh(\ln x) + C)$$

(viii) For the integral $\int x^2 \cos x^3\, dx$ we use the substitution

$t = x^3 \Rightarrow dt = 3x^2\, dx$ then

$$\int x^2 \cos x^3\, dx = \int \frac{1}{3} \cos t\, dt = \frac{1}{3} \sin t + C = \frac{1}{3} \sin x^3 + C$$

Exercises (2)

Evaluate each of the following integrals:

(1) $\int x(2x^2+3)^{10}\,dx$

(2) $\int x^2\sqrt[3]{3x^3+7}\,dx$

(3) $\int \frac{(1+\sqrt{x})^3}{\sqrt{x}}\,dx$

(4) $\int \sqrt{x}\cos\sqrt{x^3}\,dx$

(5) $\int \tan x\sec^2 x\,dx$

(6) $\int \sqrt[4]{2x+5}\,dx$

(7) $\int \frac{dx}{\sqrt{4-5x}}$

(8) $\int \frac{\sin 2x}{\sqrt{1-\cos 2x}}\,dx$

(9) $\int x\cot x^2\csc x^2\,dx$

(10) $\int \left(1+\frac{1}{x}\right)^2\frac{1}{x^2}\,dx$

(11) $\int \sin x(1+\cos x)^2\,dx$

(12) $\int \sin^3 x\cos x\,dx$

(13) $\int \frac{\tanh^{-1}x}{1-x^2}\,dx$

(14) $\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}\,dx$

(15) $\int \frac{\cos x}{\sqrt{9-\sin^2 x}}\,dx$

(16) $\int \frac{1}{x\sqrt{x^6-1}}\,dx$

(17) $\int \frac{dx}{\sqrt{e^{2x}-25}}$

(18) $\int \frac{dx}{\sqrt{81+16x^2}}$

(19) $\int e^{\sin x}\cos x\,dx$

(20) $\int \frac{(2^x+1)^2}{2^x}\,dx$

(21) $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$ (22) $\int \frac{x^3}{\sqrt[3]{x^2+4}}dx$

(23) $\int \sqrt{\frac{1}{x}-\frac{1}{\sqrt{x}}}\,dx$ (24) $\int \sqrt{\sqrt{x}-1}\,dx$

2.4 Integration of Trigonometric Functions

The following trigonometric identities are useful for evaluating some integrals involving trigonometric functions:

$\sin^2 x + \cos^2 x = 1$	$\cos^2 x = \frac{1+\cos 2x}{2}$
$\tan^2 x + 1 = \sec^2 x$	$\sin^2 x = \frac{1-\cos 2x}{2}$
$1 + \cot^2 x = \csc^2 x$	
$\sin x \cos y = \frac{1}{2}\left[\sin(x-y)+\sin(x+y)\right]$	
$\cos x \cos y = \frac{1}{2}\left[\cos(x-y)+\cos(x+y)\right]$	
$\sin x \sin y = \frac{1}{2}\left[\cos(x-y)-\cos(x+y)\right]$	

(a) Integration of $\int \sin^n x dx$ & $\int \cos^n x dx$ where n is a positive even integer

For these types of integrals we proceed as in the following examples:

Example (1): Evaluate $\int \sin^2 x\, dx$, $\int \cos^2 x\, dx$ and $\int \sin^4 x dx$

Solution:

$$\int \sin^2 x\, dx = \int \frac{1}{2}(1 - \cos 2x)dx = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C$$

$$\int \cos^2 x\, dx = \int \frac{1}{2}(1 + \cos 2x)dx = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C$$

$$\int \sin^4 x\, dx = \int (\sin^2 x)^2 dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right)dx$$

$$= \frac{1}{4}\left(x - 2\frac{\sin 2x}{2} + \frac{1}{2}\left(x + \frac{\sin 4x}{4}\right)\right) + C$$

$$=\frac{1}{4}\left(\frac{3}{2}x-\sin 2x+\frac{\sin 4x}{8}\right)+C$$

Example (2): Find $\int (2 + 3\cos 2x)^2\, dx$

Solution:

$$\int (2+3\cos 2x)^2\, dx = \int (4+12\cos 2x+9\cos^2 2x)\, dx$$

$$= 4x+6\sin 2x+\frac{9}{2}\int (1+\cos 4x)\, dx$$

$$= 4x+6\sin 2x+\frac{9}{2}x+\frac{9}{8}\sin 4x+C$$

$$= \frac{17}{2}x+6\sin 2x+\frac{9}{8}\sin 4x+C$$

Example (3): Find $\int \frac{dx}{\sin^2 x\cos^2 x}$

Solution:

$$\int \frac{dx}{\sin^2 x\cos^2 x} = \int \frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos^2 x}dx = \int \frac{dx}{\cos^2 x}+\int \frac{dx}{\sin^2 x}$$

$$= \int (\sec^2 x+\operatorname{cosec}^2 x)dx$$

$$= \tan x-\cot x+C$$

Notice that the above integral can be evaluated by another method as follows:

$$\int \frac{dx}{\sin^2 x\cos^2 x} = \int \sec^2 x \csc^2 x dx = \int \sec^2 x(1+\cot^2 x)dx$$

$$= \int (\sec^2 x + \sec^2 x \cot^2 x)dx$$

$$= \int (\sec^2 x + \csc^2 x)dx$$

$$= \tan x - \cot x + C$$

(b) Integration of $\int \sin^n x dx$ & $\int \cos^n x dx$ where n is a positive odd integer

In this case, we follow the procedure shown in the following examples:

<u>Example</u> (1): Evaluate $\int \sin^3 x\, dx$, $\int \cos^5 x\, dx$

<u>Solution</u>

Separate one $\cos x$ to get

$$\int \sin^3 x\, dx = \int \sin^2 x \cdot \sin x\, dx = \int (1-\cos^2 x)\sin x\, dx$$

$$(\text{put } t = \cos x) \Rightarrow dt = -\sin x\, dx$$

$$= \int (1-t^2)(-dt) = -t + \frac{1}{3}t^3 + C = -\cos x + \frac{1}{3}\cos^3 x + C$$

$$\int \cos^5 x\,dx = \int \cos^4 x \cdot \cos x\,dx$$

$$= \int (1-\sin^2 x)^2 \cos x\,dx, \quad (\text{put } t = \sin x)$$

$$= \int (1 - 2t^2 + t^4)\,dt$$

$$= t - \frac{2}{3}t^3 + \frac{1}{5}t^5 + C$$

$$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$$

(c) Integration of $\int \sin^n x \cos^m x dx$ where at least one of n or m is a positive odd integer

We have the following two cases:

(a) If n is odd, use $\sin^2 x = 1 - \cos^2 x$

(b) If m is odd, use $\cos^2 x = 1 - \sin^2 x$

Example (1): Evaluate $\int \sin^2 x \cos^3 x\,dx$

Solution:

$$\int \sin^2 x \cos^3 x\,dx = \int \sin^2 x \cos^2 x \cdot \cos x\,dx$$

$$= \int \sin^2 x (1 - \sin^2 x) \cos x\,dx$$

$$= \int (\sin^2 x - \sin^4 x) \cos x\,dx = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$$

(d) Integration of $\int \sin^n x \cos^m x\, dx$ where n and m are positive even integers

We express each of $\sin x$ and $\cos x$ in terms of the double angle trigonometric identities, so that we reduce this type of integrals into integrals of different powers of $\sin x$ and $\cos x$ as in the following examples:

Example (1): Evaluate $\int \sin^2 x \cos^4 x\, dx$

Solution:

$$\int \sin^2 x \cos^4 x\, dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$=\frac{1}{8}\int \left(1-\cos^2 2x\right)\left(1+\cos 2x\right)dx$$

$$=\frac{1}{8}\int \left(1+\cos 2x-\cos^2 2x-\cos^3 2x\right)dx$$

$$=\frac{1}{8}\int \left(1+\cos 2x-\frac{1+\cos 4x}{2}-\cos^2 2x\cos 2x\right)dx$$

$$=\frac{1}{8}\int \left(\frac{1}{2}+\cos 2x-\frac{1}{2}\cos 4x-\left(1-\sin^2 2x\right)\cos 2x\right)dx$$

$$=\frac{1}{8}\int\left(\frac{1}{2}-\frac{1}{2}\cos 4x+\sin^2 2x\cos 2x\right)dx$$

$$=\frac{1}{8}\left(\frac{1}{2}x-\frac{1}{2}\frac{\sin 4x}{4}+\frac{1}{3\cdot 2}\sin^3 2x\right)+C$$

$$=\frac{1}{8}\left(\frac{1}{2}x-\frac{1}{8}\sin 4x+\frac{1}{6}\sin^3 2x\right)+C$$

(e) Integration of product of trigonometric functions with different arguments

(i) $\int \sin ax\cos bx\,dx=\frac{1}{2}\int[\sin(a-b)x+\sin(a+b)x]dx$

$$=-\frac{1}{2}\left[\frac{\cos(a-b)x}{(a-b)}+\frac{\cos(a+b)x}{(a+b)}\right]+C$$

(ii) $\int\cos ax\cos bx dx=\frac{1}{2}\int[\cos(a-b)x+\cos(a+b)x]dx$

$$=\frac{1}{2}\left[\frac{\sin(a-b)x}{(a-b)}+\frac{\sin(a+b)x}{(a+b)}\right]+C$$

(iii) $\int\sin ax\sin bx\,dx=\frac{1}{2}\int[\cos(a-b)x-\cos(a+b)x]dx$

$$=\frac{1}{2}\left[\frac{\sin(a-b)x}{(a-b)}-\frac{\sin(a+b)x}{(a+b)}\right]+C$$

Example (1): Find each of the following integrals

(i) $I_1 = \int \sin 4x \cos 5x dx$, (ii) $I_2 = \int \sin 7x \sin 4x dx$,

(iii) $I_3 = \int \cos 5x \cos 3x dx$.

Solution:

(i) $I_1 = \int \sin 4x \cos 5x dx = \frac{1}{2}\int (\sin 9x - \sin x) dx$

$$= \frac{1}{2}\left[\frac{-1}{9}\cos 9x + \cos x\right] + C = \frac{-1}{18}\cos 9x + \frac{1}{2}\cos x + C$$

(ii) $I_2 = \int \sin 7x \sin 4x dx = \frac{1}{2}\int (\cos 3x - \cos 11x) dx$

$$= \frac{1}{2}\left[\frac{1}{3}\sin 3x - \frac{1}{11}\sin 11x\right] + C = \frac{1}{6}\sin 3x - \frac{1}{22}\sin 11x + C$$

(iii) $I_3 = \int \cos 5x \cos 3x dx = \frac{1}{2}\int (\cos 2x + \cos 8x) dx$

$$= \frac{1}{2}\left[\frac{1}{2}\sin 2x + \frac{1}{8}\sin 8x\right] + C = \frac{1}{16}\sin 8x + \frac{1}{4}\sin 2x + C$$

2.5 Evaluating Integrals of the Form $\int R(\sin x,\ \cos x)\,dx$

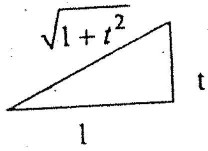
where, $R(\sin x,\ \cos x)$ is a rational function of $\sin x$ and $\cos x$, which may take one of the following forms:

$$\int \frac{dx}{a \pm b\sin x},\quad \int \frac{dx}{a \pm b\cos x},\quad \int \frac{dx}{a + \sin x + \cos x},$$

$$\int \frac{dx}{a\sin x + b\cos x},\quad \int \frac{dx}{a + b\sec x}$$

To solve any of the above integrals we use the substitution

$$\tan\frac{x}{2} = t \quad \Rightarrow \quad x = 2\tan^{-1} t \Rightarrow$$



Then,

$$dx = \frac{2}{1+t^2}dt,$$

$$\sin\frac{x}{2} = \frac{t}{\sqrt{1+t^2}}, \qquad \cos\frac{x}{2} = \frac{1}{\sqrt{1+t^2}},$$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2t}{1+t^2},$$

$$\cos x = 2\cos^2\frac{x}{2} - 1 = \frac{1-t^2}{1+t^2},$$

Example (1): Find $\int \frac{dx}{5+4\cos x}$

Solution: Put $t=\tan\frac{x}{2}$, then,

$$dx=\frac{2}{1+t^2}dt,\ \cos x=\frac{1-t^2}{1+t^2}.$$

Substituting we get

$$\int\frac{dx}{5+4\cos x}=\int\frac{\frac{2dt}{(1+t^2)}}{5+4\frac{(1-t^2)}{1+t^2}}=\int\frac{2}{5(1+t^2)+4(1-t^2)}dt$$

$$=\int\frac{2}{9+t^2}dt=\frac{2}{3}\tan^{-1}\frac{t}{3}+C=\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right)+C$$

Example (2): Evaluate $\int\frac{dx}{3\sin x+4\cos x}$

Solution: Put $t=\tan\frac{x}{2}$, then,

$$dx=\frac{2}{1+t^2}dt,\ \sin x=\frac{2t}{1+t^2},\ \cos x=\frac{1-t^2}{1+t^2},$$

substituting we get

$$\int \frac{dx}{3\sin x + 4\cos x} = \int \frac{\frac{2dt}{1+t^2}}{3\frac{2t}{1+t^2} + 4\frac{1-t^2}{1+t^2}} = \int \frac{2\,dt}{6t + 4 - 4t^2}$$

$$= \frac{1}{2}\int \frac{dt}{\frac{6}{4}t + 1 - t^2} = -\frac{1}{2}\int \frac{dt}{t^2 - \frac{3}{2}t - 1}$$

$$= -\frac{1}{2}\int \frac{dt}{\left(t - \frac{3}{4}\right)^2 - \frac{9}{16} - 1} = \frac{1}{2}\int \frac{dt}{\frac{25}{16} - \left(t - \frac{3}{4}\right)^2}$$

$$= \frac{1}{2}\left(\tanh^{-1}\frac{t - \frac{3}{4}}{\frac{5}{4}}\right) \times \frac{4}{5} + C = \frac{2}{5}\tanh^{-1}\left(\frac{4t-3}{5}\right) + C$$

$$= \frac{2}{5}\tanh^{-1}\left(\frac{4\tan\left(\frac{x}{2}\right) - 3}{5}\right) + C$$

2.6 Integration of Hyperbolic Functions

The hyperbolic integrals are computed with the aid of the following hyperbolic identities

$\cosh^2 x - \sinh^2 x = 1$	$\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$
$1 - \tanh^2 x = \operatorname{sech}^2 x$	$\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$
$\coth^2 x - 1 = \operatorname{csch}^2 x$	

$$\sinh x \cosh y = \frac{1}{2}\left[\sinh(x+y) + \sinh(x-y)\right]$$

$$\cosh x \cosh y = \frac{1}{2}\left[\cosh(x+y) + \cosh(x-y)\right]$$

$$\sinh x \sinh y = \frac{1}{2}\left[\cosh(x+y) - \cosh(x-y)\right]$$

Using the above identities and following the order in which integration of trigonometric functions are computed, we can easily evaluate integrals involving hyperbolic functions as illustrated in the following examples:

1. $\int \cosh x dx = \sinh x + C$
2. $\int \sinh x dx = \cosh x + C$
3. $\int \tanh x\, dx = \int \frac{\sinh x}{\cosh x} dx = \ln|\cosh x| + C$
4. $\int \coth x\, dx = \int \frac{\cosh x}{\sinh x} dx = \ln|\sinh x| + C$
5. $\int \sec h x dx = \int \frac{1}{\cosh x} \cdot \frac{\cosh x}{\cosh x} dx$

$= \int \frac{\cosh x}{1+\sinh^2 x} dx$, using the substitution $t = \sinh x$ we get

$= \int \frac{dt}{1+t^2} = \tan^{-1} t + C = \tan^{-1}(\sinh x) + C$

Notice that the above integral can be evaluated by two other methods (try to find them). Similarly, we can deal with $\int \csc h x dx$.

Example (1): Evaluate each of the following integrals

1. $\int \cosh^2 x dx = \int \frac{1 + \cosh 2x}{2} dx = \frac{1}{2}\left(x + \frac{\sinh 2x}{2}\right) + C$

2. $\int \sinh^2 x\, dx = \int \frac{\cosh 2x - 1}{2} dx = \frac{1}{2}\left(\frac{\sinh 2x}{2} - x\right) + C$

3. $\int \operatorname{sec}h^2 x dx = \tanh x + C$

4. $\int \operatorname{csc}h^2 x dx = -\coth x + C$

5. $\int \tanh^2 x dx = \int \left(1 - \operatorname{sec}h^2 x\right) dx = x - \tanh x + C$

6. $\int \coth^2 x dx = \int \left(\operatorname{csc}h^2 x + 1\right) dx = -\coth x + x + C$

Example (2):

(i) $\int \cosh^3 x \, dx$ (ii) $\int \sinh^3 x \cosh^2 x \, dx$

Solution

(i) $\int \cosh^3 x \, dx = \int \cosh^2 x \cosh dx$

$$= \int \left(1 + \sinh^2 x\right) \cosh dx$$

$$= \sinh x + \frac{1}{3} \sinh^3 x + C$$

(ii) $\int \sinh^3 x \cosh^2 x \, dx = \int \sinh^2 x \cosh^2 x \cdot \sinh x \, dx$

$$= \int \left(\cosh^2 x - 1\right) \cosh^2 x \cdot \sinh x dx$$

$$= \int \left(\cosh^4 x - \cosh^2 x\right) \cdot \sinh x dx$$

$$= \frac{1}{5} \cosh^5 x - \frac{1}{3} \cosh^3 x + C$$

Example (3): Find $\int (5-2\sinh 3x)^2\,dx$

Solution:

$$\int(5-2\sinh 3x)^2\,dx = \int(25-20\sinh 3x+4\sinh^2 3x)dx$$
$$= 25x-\frac{20}{3}\cosh 3x+2\int(\cosh 6x-1)dx$$
$$= 25x-\frac{20}{3}\cosh 3x+\frac{1}{3}\sinh 6x-2x+C$$
$$= 23x-\frac{20}{3}\cosh 3x+\frac{1}{3}\sinh 6x+C$$

Example (4): Find $\int (2\operatorname{sech}x-\tanh x)^2\,dx$

Solution:

$$\int (2\operatorname{sech}x-\tanh x)^2\,dx =$$
$$= \int(4\operatorname{sech}^2 x-4\operatorname{sech}x\tanh x+\tanh^2 x)dx$$
$$= \int(4\operatorname{sech}^2 x-4\operatorname{sech}x\tanh x+1-\operatorname{sech}^2 x)dx$$
$$= \int(3\operatorname{sech}^2 x-4\operatorname{sech}x\tanh x+1)dx$$
$$= 3\tanh x+4\operatorname{sech}x+x+C$$

Example (5): Find

(i) $I_1=\int\cosh 8x\sinh 6x\,dx$ (ii) $I_2=\int\sinh 6x\sinh 8x\,dx$

(iii) $I_3 = \int \cosh 3x \cosh 6x \, dx$

Solution:

(i) $I_1 = \int \sinh 6x \cosh 8x dx$

$$= \frac{1}{2} \int [\sinh 14x + \sinh(-2x)] \, dx$$

$$= \frac{1}{2} \left[\frac{1}{14} \cosh 14x - \frac{1}{2} \cosh 2x \right] + C$$

$$= \frac{1}{28} \cosh 14x - \frac{1}{4} \cosh 2x + C$$

(ii) $I_2 = \int \sinh 6x \sinh 8x dx$

$$= \frac{1}{2} \int [\cosh 14x - \cosh 2x] dx$$

$$= \frac{1}{2} \left[\frac{1}{14} \sinh 14x - \frac{1}{2} \sinh 2x \right] + C$$

$$= \frac{1}{28} \sinh 14x - \frac{1}{4} \sinh 2x + C$$

(iii) $I_3 = \int \cosh 3x \cosh 6x dx$

$$= \frac{1}{2} \int [\cosh 9x + \cosh 3x] dx$$

$$= \frac{1}{2} \left[\frac{1}{9} \sinh 9x + \frac{1}{3} \sinh 3x \right] + C$$

$$= \frac{1}{18} \sinh 9x + \frac{1}{6} \sinh 3x + C$$

Exercises (3)

Evaluate each of the following integrals:

(1) $\int \sin^2 2x\,dx$

(2) $\int \cos^7 x\,dx$

(3) $\int \sin^5 x \cos^2 x\,dx$

(4) $\int \sin^4 x \cos^2 x\,dx$

(5) $\int \sqrt{\sin x}\cos^3 x\,dx$

(6) $\int \frac{\cos^3 x}{\sqrt{\sin x}}\,dx$

(7) $\int \sin 5x \cos 3x\,dx$

(8) $\int \cos x \cos 5x\,dx$

(9) $\int \sin 5x \sin 3x\,dx$

(10) $\int \left(1+\sqrt{\cos x}\right)^2 \sin x\,dx$

(11) $\int \frac{\cos x}{2-\sin x}\,dx$

(12) $\int \frac{dx}{2+\sin x}$

(13) $\int \frac{dx}{3+2\cos x}$

(14) $\int \frac{1}{1+\sin x+\cos x}\,dx$

(15) $\int \frac{1}{\tan x+\sin x}\,dx$

(16) $\int \frac{dx}{\sin x-\sqrt{3}\cos x}$

(17) $\int \sinh^4 x\,dx$

(18) $\int \cosh^5 2x\,dx$

(19) $\int \left(1+\sinh^2 x\right)\cosh^3 x\,dx$

(20) $\int \sinh^4 x \cosh^4 x\,dx$

(21) $\int \sinh 2x \cosh 3x\,dx$

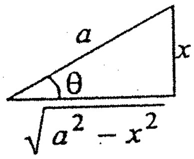
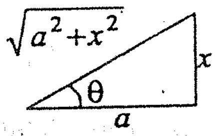
(22) $\int \sinh 3x \sinh 7x\,dx$

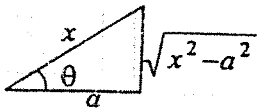
2.7 Integration by Removing Roots

Sometimes the integrands of certain integrals involve radical expressions in the forms

$$\sqrt{a^2 - x^2}, \quad \sqrt{a^2 + x^2}, \quad \sqrt{x^2 - a^2}$$

Each of such integrals can be evaluated by using a suitable trigonometric or hyperbolic substitution. The following table shows some of these substitutions:

Form	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta,$ $dx = a\cos\theta d\theta$ 	$1 - \sin^2\theta = \cos^2\theta$ $a^2 - x^2 = a^2 - a^2\sin^2\theta$ $= a^2\cos^2\theta$ $\boxed{\sqrt{a^2 - x^2} = a\cos\theta}$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta,$ $dx = a\sec^2\theta d\theta$ 	$1 + \tan^2\theta = \sec^2\theta$ $a^2 + x^2 = a^2 + a^2\tan^2\theta$ $= a^2\sec^2\theta$ $\boxed{\sqrt{a^2 + x^2} = a\sec\theta}$

$\sqrt{x^2-a^2}$	$x = a\sec\theta,$ $dx = a\sec\theta\tan\theta d\theta$ 	$\sec^2\theta - 1 = \tan^2\theta$ $x^2 - a^2 = a^2\sec^2\theta - a^2$ $= a^2\tan^2\theta$ $\boxed{\sqrt{x^2-a^2} = a\tan\theta}$

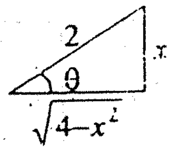
Other substitutions can also be used in order to remove the roots. These substitutions are listed in the following table.

Form	Substitution	Identity
$\sqrt{a^2-x^2}$	$x = a\tanh\theta,$ $dx = a\text{sech}^2\theta\, d\theta$	$1-\tanh^2\theta = \text{sech}^2\theta$ $a^2 - x^2 = a^2 - a^2\tanh^2\theta$ $= a^2\text{sech}^2\theta$ $\boxed{\sqrt{a^2-x^2} = a\,\text{sech}\theta}$
$\sqrt{a^2+x^2}$	$x = a\sinh\theta,$ $dx = a\cosh d\theta$	$1+\sinh^2\theta = \cosh^2\theta$ $a^2 + x^2 = a^2 + a^2\sinh^2\theta$ $= a^2\cosh^2\theta$ $\boxed{\sqrt{a^2+x^2} = a\cosh\theta}$

$\sqrt{x^2-a^2}$	$x=a\cosh\theta$, $dx=a\sinh\theta d\theta$	$\cosh^2\theta-1=\sinh^2\theta$ $x^2-a^2=a^2\cosh^2\theta-a^2$ $=a^2\sinh^2\theta$ $\boxed{\sqrt{x^2-a^2}=a\sinh\theta}$

It must be noticed that each of these substitutions reduces the integral to an easier one that can be evaluated directly.

Example (1): Find $\int\frac{\sqrt{4-x^2}}{x^2}dx$.



Solution: Put

$$x=2\sin\theta \Rightarrow dx=2\cos\theta d\theta,$$

and,

$$4-x^2=4-4\sin^2\theta=4(1-\sin^2\theta)=4\cos^2\theta$$

So we have

$$\int\frac{\sqrt{4-x^2}}{x^2}dx=\int\frac{2\cos\theta}{4\sin^2\theta}\cdot 2\cos\theta\, d\theta$$

$$=\int\frac{\cos\theta}{\sin^2\theta}\cos\theta\, d\theta=\int\frac{\cos^2\theta}{\sin^2\theta}d\theta$$

$$= \int \cot^2 \theta \, d\theta = \int \left(\csc^2 \theta - 1 \right) d\theta$$

$$= -\cot\theta - \theta + C$$

$$= -\frac{\sqrt{4-x^2}}{x} - \sin^{-1}\left(\frac{x}{2}\right) + C$$

Example (2): Find $\int \frac{1}{x^2\sqrt{5-x^2}} \, dx$.

Solution: Put $x = \sqrt{5}\sin\theta \Rightarrow dx = \sqrt{5}\cos\theta d\theta$,

and,

$$5 - x^2 = 5 - 5\sin^2\theta = 5\left(1 - \sin^2\theta\right) = 5\cos^2\theta$$

So we have

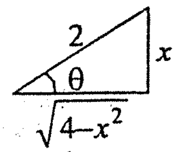
$$\int \frac{1}{x^2\sqrt{5-x^2}} dx = \int \frac{1}{5\sqrt{5}\sin^2\theta\cos\theta} \cdot \sqrt{5}\cos\theta d\theta$$

$$= \frac{1}{5}\int \frac{1}{\sin^2\theta} d\theta = \frac{1}{5}\int \csc^2 \theta d\theta = -\frac{1}{5}\cot\theta + C$$

$$= -\frac{1}{5}\cot(\sin^{-1}\frac{x}{\sqrt{5}}) + C = -\frac{\sqrt{5-x^2}}{5x} + C$$

Example (3): Evaluate $\int x^3\sqrt{4-x^2} \, dx$

Solution: Put $x = 2\sin\theta \Rightarrow dx = 2\cos\theta \, d\theta$



and $\quad 4-x^2 = 4-4\sin^2\theta$

$$= 4\left(1-\sin^2\theta\right) = 4\cos^2\theta$$

so we have

$$\int x^3\sqrt{4-x^2}\ dx = \int 8\sin^3\theta\sqrt{4\cos^2\theta}\cdot 2\cos\theta\ d\theta$$

$$= \int 32\sin^3\theta\cos^2\theta\ d\theta = 32\int \sin^2\theta\cos^2\theta\cdot\sin\theta\ d\theta$$

$$= 32\int\left(1-\cos^2\theta\right)\cos^2\theta\ \sin\theta\ d\theta$$

$$= 32\int\left(\cos^2\theta-\cos^4\theta\right)\sin\theta\ d\theta$$

$$= 32\left[\frac{1}{3}\cos^3\theta-\frac{1}{5}\cos^5\theta\right]+C$$

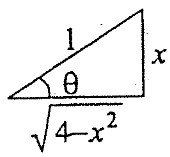
$$= 32\left[\frac{1}{3}\cos^3\left(\sin^{-1}\left(\frac{x}{2}\right)\right)-\frac{1}{5}\cos^5\left(\sin^{-1}\left(\frac{x}{2}\right)\right)\right]+C$$

$$= 32\left[\frac{1}{3}\left(\frac{\sqrt{4-x^2}}{2}\right)^3-\frac{1}{5}\left(\frac{\sqrt{4-x^2}}{2}\right)^5\right]+C$$

Example (4): Evaluate $\int x^4\sqrt{1-x^2}\ dx$

Solution: Put $\quad x=\sin\theta \quad \Rightarrow \quad dx=\cos\theta\ d\theta$

so we have



$$\int x^4\sqrt{1-x^2}\ dx = \int\sin^4\theta\sqrt{1-\sin^2\theta}\cdot\cos\theta\ d\theta$$

$$= \int\sin^4\theta\cos^2\theta\ d\theta = \int\left(\sin^2\theta\right)^2\cos^2\theta\ d\theta$$

$$= \int \left[\frac{1}{2}(1-\cos 2\theta)\right]^2 \left[\frac{1}{2}(1+\cos 2\theta)\right] d\theta$$

$$= \frac{1}{8}\int \left(1 - 2\cos 2\theta + \cos^2 2\theta\right)\left(1+\cos 2\theta\right) d\theta$$

$$= \frac{1}{8}\int \left(1 - \cos 2\theta - \cos^2 2\theta + \cos^3 2\theta\right) d\theta$$

$$= \frac{1}{8}\int \left[1 - \cos 2\theta - \frac{1}{2}(1+\cos 4\theta) + \cos^2 2\theta \cos 2\theta\right] d\theta$$

$$= \frac{1}{8}\int \left[1 - \cos 2\theta - \frac{1}{2} - \frac{1}{2}\cos 4\theta + \left(1 - \sin^2 2\theta\right)\cos 2\theta\right] d\theta$$

$$= \frac{1}{8}\left[\frac{1}{2}\theta - \frac{1}{2}\sin 2\theta - \frac{1}{8}\sin 4\theta + \frac{1}{2}\sin 2\theta - \frac{1}{6}\sin^3 2\theta\right] + C$$

$$= \frac{1}{8}\left[\frac{1}{2}\sin^{-1} x - \frac{1}{8}\sin\left(4\sin^{-1} x\right) - \frac{1}{6}\sin^3\left(2\sin^{-1} x\right)\right] + C$$

Example (5): Find $\int \frac{\left(1-x^2\right)^{3/2}}{x^6} dx$

Solution: Put

$$x = \sin\theta \Rightarrow dx = \cos\theta d\theta,$$

and,

$$1 - x^2 = 1 - \sin^2\theta = \cos^2\theta$$

So we have

$$\int \frac{(1-x^2)^{3/2}}{x^6} dx = \int \frac{\cos^3\theta}{\sin^6\theta} \cdot \cos\theta d\theta$$

$$= \int \frac{\cos^4\theta}{\sin^6\theta} d\theta = \int \cot^4\theta \csc^2\theta d\theta$$

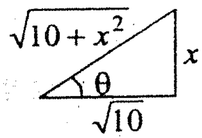
$$= -\frac{1}{5}\cot^5\theta + C = -\frac{1}{5}\cot^5(\sin^{-1}x) + C$$

$$= -\frac{1}{5}\left(\frac{\sqrt{1-x2}}{x}\right)^5 + C$$

Example (6): Evaluate $\int \frac{1}{\sqrt{\left(10+x^2\right)^3}} dx$.

Solution: Put $x=\sqrt{10}\tan\theta \Rightarrow dx=\sqrt{10}\sec^2\theta d\theta$,

i.e.



$$10 + x^2 = 10 + 10\tan^2\theta$$

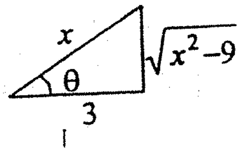
$$= 10\left(1+\tan^2\theta\right) = 10\sec^2\theta$$

So we have

$$\int \frac{1}{\sqrt{\left(10+x^2\right)^3}} dx = \int \frac{\sqrt{10}\sec^2\theta}{\sqrt{\left(10\sec^2\theta\right)^3}} d\theta = \int \frac{\sqrt{10}\sec^2\theta}{(10)^{3/2}\sec^3\theta} d\theta$$

$$= \frac{1}{10}\int \frac{1}{\sec\theta} d\theta = \frac{1}{10}\int \cos\theta d\theta = \frac{1}{10}\sin\theta + C = \frac{1}{10}\frac{x}{\sqrt{10+x^2}} + C$$

Example (7): Evaluate $\int \frac{1}{\sqrt{x^2-9}}dx$.



Solution: Put $x = 3\sec\theta \;\Rightarrow\; dx = 3\sec\theta\tan\theta\, d\theta,$

and,

$$x^2-9 = 9\sec^2\theta - 9 = 9(\sec^2\theta - 1) = 9\tan^2\theta$$

Thus we have

$$\int \frac{1}{\sqrt{x^2-9}}dx = \int \frac{3\sec\theta\tan\theta}{3\tan\theta}d\theta = \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C$$

$$= \ln\left|\frac{x+\sqrt{x^2-9}}{3}\right| + C = \ln\left|x+\sqrt{x^2-9}\right| + C$$

Example (8): Evaluate $\int \frac{\sqrt{x^2-9}}{x}dx$

Solution: Put

$$x = 3\sec\theta \;\Rightarrow\; dx = 3\sec\theta\tan\theta\, d\theta,$$

then

$$\int \frac{\sqrt{x^2-9}}{x}\,dx = \int \frac{3\tan\theta}{3\sec\theta}(3\sec\theta\tan\theta)d\theta$$
$$= \int 3\tan^2\theta\,d\theta = 3\int(\sec^2\theta - 1)d\theta$$
$$= 3(\tan\theta - \theta) + C = \sqrt{x^2-9} - 3\sec^{-1}\frac{x}{3} + C$$

Example (9): Evaluate $\int\sqrt{a^2+x^2}\,dx$

Solution: Let $x = a\sinh\theta \Rightarrow dx = a\cosh\theta d\theta$,

and,

$$a^2 + x^2 = a^2 + a^2\sinh^2\theta$$
$$= a^2(1+\sinh^2\theta) = a^2\cosh^2\theta$$

So we have

$$\int\sqrt{a^2+x^2}\,dx = a^2\int\sqrt{1+\sinh^2\theta}\,.\cosh\theta\,d\theta$$
$$= a^2\int\cosh^2\theta\,d\theta$$
$$= a^2\int\frac{1}{2}(\cosh 2\theta + 1)\,d\theta$$
$$= \frac{1}{2}a^2\left[\frac{\sinh 2\theta}{2} + \theta\right] + C \qquad (1)$$

Since, $\sinh\theta = \frac{x}{a}$, then,

$$\cosh\theta = \sqrt{1+\left(\frac{x}{a}\right)^2},$$

$$\sinh 2\theta = 2\sinh\theta\cosh\theta = 2\,\frac{x\sqrt{a^2+x^2}}{a^2}$$

Substitute in (1), we get

$$\int\sqrt{a^2+x^2}\,dx = \frac{1}{2}a^2\left[\frac{x\sqrt{a^2+x^2}}{a^2} + \sinh^{-1}\left(\frac{x}{a}\right)\right]+C$$

Notice that the above example can be solved using trigonometric substitution. But this will be done after introducing the method of integration by parts as we shall see in the next section.

Exercises (4)

Evaluate each of the following integrals:

(1) $\int \frac{dx}{x\sqrt{4-x^2}}$

(2) $\int \frac{\sqrt{4-x^2}}{x^2}\,dx$

(3) $\int \frac{dx}{x^2\sqrt{9+x^2}}$

(4) $\int \frac{dx}{x^3\sqrt{x^2-25}}$

(5) $\int \frac{1}{\left(x^2-1\right)^{3/2}}$

(6) $\int \frac{dx}{\left(36+x^2\right)^2}$

(7) $\int \frac{x^3}{\sqrt{9x^2+49}}\,dx$

(8) $\int \frac{dx}{x\sqrt{25x^2+16}}$

(9) $\int \frac{x^2}{\left(1-9x^2\right)^{3/2}}\,dx$

(10) $\int \frac{\left(4+x^2\right)^2}{x^3}\,dx$

(11) $\int \frac{3x-5}{\sqrt{1-x^2}}\,dx$

(12) $\int \sqrt{25+x^2}\,dx$

(13) $\int \frac{x^2}{\sqrt{x^2-1}}\,dx$

(14) $\int \frac{\sqrt{x^2-4}}{x}\,dx$

(15) $\int \frac{dx}{x^2\sqrt{x^2-25}}$

(16) $\int x^2\sqrt{9-x^2}\,dx$

2.8 Integration by Parts

2.8.1 Standard Formula of Integration by Parts

For most differentiation rules, introduced in the previous course, there are corresponding integration rules. For instance, the chain rule for differentiation corresponds to the substitution rule for integration. The integration rule that corresponds to the product rule for differentiation is the rule of integration by parts. In order to see this correspondence let u and v be continuously differentiable functions of x, then

$$\frac{d}{dx}(uv)=uv'+vu'$$

Integrate both sides with respect to x, we get

$$uv=\int\left(uv'+vu'\right)dx$$

or

$$uv=\int uv'dx+\int vu'dx$$

This last equation can be rearranged in the form

$$\int uv'\,dx=uv-\int vu'\,dx,$$

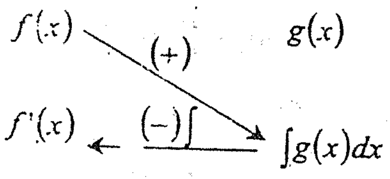
or $$\int u\,dv = uv - \int v\,du \qquad (1)$$

The above formula is called the ***integration by parts*** formula, which is used to evaluate the integral of a product of two functions. It shows that the integral $\int uv'dx$ is reduced to another integral $\int vu'\,dx$ which is supposed to be simpler than the one we started with.

For a product of two functions, which is usually applied for, the integration by parts formula can be expressed in the form:

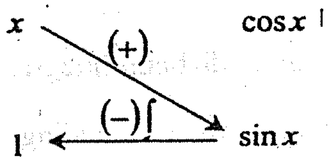
$$\begin{aligned}\int f(x)g(x)dx &= \left(\int f(x)dx\right)g(x) - \int\left[\int f(x)dx\right]g'(x)dx \\ &= \left(\int g(x)dx\right)f(x) - \int\left[\int g(x)dx\right]f'(x)dx\end{aligned}$$

This formula can be tabulated as follows:



Example (1): Find $\int x\cos x\,dx$.

Solution:



Thus, we have

$$\int x \cos x \, dx = x \sin x - \int (\sin x) \, dx$$
$$= x \sin x - (-\cos x) + C = x \sin x + \cos x + C$$

Remark: The main reason for using integration by parts is to obtain a simpler integral than the one we started with. Thus, in Example 1 we started with $\int x \cos x \, dx$ and expressed it in terms of the simpler integral $\int \sin x \, dx$. If we had chosen $f(x) = \cos x$ and $g(x) = x$, then $f'(x) = -\sin x$ and $\int g(x) \, dx = \frac{x^2}{2}$, so the integration by parts formula gives

$$\int x \cos x \, dx = \left(\frac{x^2}{2}\right) \cos x - \int \frac{x^2}{2} - \sin x \, dx$$
$$= \frac{1}{2} x^2 \cos x + \frac{1}{2} \int x^2 \sin x \, dx$$

But $\int x^2 \sin x\,dx$ is a more difficult integral than the one we started with. Therefore, when deciding to choose the function to be differentiated and that to be integrated, we have to choose them in such a way that the resulting integral is easier than the given one.

Example (2): Evaluate each of the following integrals

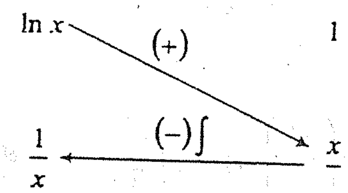
(i) $\int \ln x\,dx$ (ii) $\int \tan^{-1} x\,dx$

Solution:

(i) $\int \ln x\,dx = \int 1\ \ln x\,dx = x \ln x - \int x \cdot \frac{1}{x}\,dx$

$$= x \ln x - \int dx = x \ln x - x + C$$

or using the tabular form we get

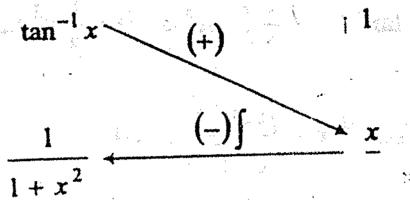


Then,

$$\int \ln x\,dx = (\ln x)(x) - \int (x)\left(\frac{1}{x}\right)dx = (\ln x)(x) - \int dx$$

$$= x \ln x - x + C$$

(ii) Again,.



Then,

$$\int \tan^{-1} x \, dx = (\tan^{-1} x)(x) - \int (x)\left(\frac{1}{1+x^2}\right) dx$$

$$= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} dx$$

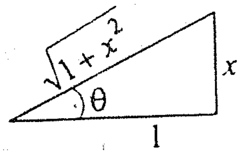
$$= x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C$$

Example (3): Evaluate $\int x^2 \tan^{-1} x \, dx$

Solution:

put $x = \tan\theta$

$dx = \sec^2\theta \, d\theta$



put $t = \cos\theta$

$dt = -\sin\theta \, d\theta$

$$\int x^2 \tan^{-1} x \, dx$$

$$= \frac{1}{3} x^3 \tan^{-1} x - \int \frac{1}{3} x^3 \cdot \frac{1}{1+x^2} dx$$

$$= \frac{1}{3} x^3 \tan^{-1} x - \frac{1}{3} \int \frac{x^3}{1+x^2} dx$$

$$= \frac{1}{3} x^3 \tan^{-1} x - \frac{1}{3} \int \frac{\tan^3\theta}{\sec^2\theta} \cdot \sec^2\theta \, d\theta$$

$$=\frac{1}{3}x^3\tan^{-1}x-\frac{1}{3}\int\tan^3\theta\,d\theta=\frac{1}{3}x^3\tan^{-1}x-\frac{1}{3}\int\frac{\sin^3\theta}{\cos^3\theta}\,d\theta$$

$$=\frac{1}{3}x^3\tan^{-1}x-\int\frac{\sin^2\theta}{\cos^3\theta}\sin\theta\,d\theta$$

$$=\frac{1}{3}x^3\tan^{-1}x-\int\frac{1-\cos^2\theta}{\cos^3\theta}\sin\theta\,d\theta=\frac{1}{3}x^3\tan^{-1}x+\int\frac{1-t^2}{t^3}\,dt$$

$$=\frac{1}{3}x^3\tan^{-1}x-\int\left(\frac{1}{t}-\frac{1}{t^3}\right)dt=\frac{1}{3}x^3\tan^{-1}x-\ln|t|+\frac{1}{2t^2}+C$$

$$=\frac{1}{3}x^3\tan^{-1}x-\ln|\cos\theta|+\frac{1}{2}\sec^2\theta+C$$

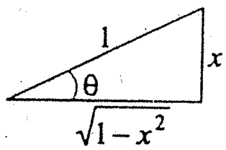
$$=\frac{1}{3}x^3\tan^{-1}x-\ln\left|\cos\left(\tan^{-1}x\right)\right|+\frac{1}{2}\sqrt{1+x^2}+C$$

Example (4): Evaluate $\int x\sin^{-1}x\,dx$

Solution:

put $x=\sin\theta$

$dx=\cos\theta\,d\theta$



$$\int x\sin^{-1}x\,dx=\frac{1}{2}x^2\sin^{-1}x-\frac{1}{2}\int x^2\frac{1}{\sqrt{1-x^2}}\,dx$$

$$=\frac{1}{2}x^2\sin^{-1}x-\frac{1}{2}\int\sin^2\theta\frac{1}{\cos\theta}\cdot\cos\theta\,d\theta$$

$$=\frac{1}{2}x^2\sin^{-1}x-\frac{1}{2}\int\sin^2\theta\,d\theta=\frac{1}{2}x^2\sin^{-1}x-\frac{1}{4}\int(1-\cos2\theta)d\theta$$

$$=\frac{1}{2}x^2\sin^{-1}x-\frac{1}{4}\left[\theta-\frac{1}{2}\sin2\theta\right]+C$$

$$= \frac{1}{2}x^2 \sin^{-1} x - \frac{1}{4}\left[\sin^{-1} x - \frac{1}{2}\sin\left(2\sin^{-1} x\right)\right] + C$$

Example (5): Evaluate $\int \sec^3 x\, dx$

Solution: The given integral may be written as

$$\int \sec^3 x\, dx = \int \sec x \cdot \sec^2 x\, dx$$
$$= \sec x \tan x - \int (\tan x)\,(\sec x \tan x)\,dx$$
$$= \sec x \tan x - \int \sec x \tan^2 x\, dx$$
$$= \sec x \tan x - \int \sec x\left(\sec^2 x - 1\right)dx$$
$$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$
$$= \sec x \tan x - \int \sec^3 x\,dx + \ln\left|\sec x + \tan x\right| + C$$

From which we get

$$2\int \sec^3 x\, dx = \sec x \tan x + \ln\left|\sec x + \tan x\right| + C$$

Thus,

$$\boxed{\int \sec^3 x\, dx = \frac{1}{2}\left[\sec x \tan x + \ln\left|\sec x + \tan x\right|\right] + C}$$

Notice that this last integral is very important because it appears quite frequently in applications.

Example (6): Evaluate $\int\sqrt{a^2+x^2}\,dx$

Solution: Put $x=a\tan\theta \Rightarrow dx=a\sec^2\theta d\theta$,

then

$$\int\sqrt{a^2+x^2}\,dx=\int a\sec\theta\cdot a\sec^2\theta\,d\theta$$

$$=a^2\int\sec^3\theta\,d\theta=\frac{1}{2}a^2\left[\sec\theta\tan\theta+\ln\left|\sec\theta+\tan\theta\right|\right]$$

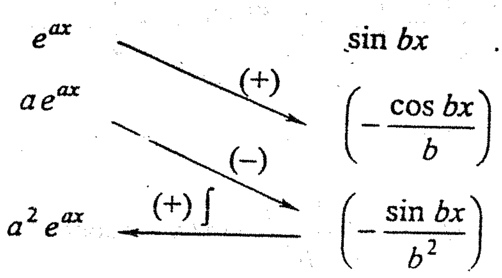
$$=\frac{a^2}{2}\left[\frac{x\sqrt{a^2+x^2}}{a^2}+\ln\left|\frac{\sqrt{a^2+x^2}}{a}\right|+\frac{x}{a}\right]+C$$

Example (7): Find each of the following integrals

(i) $\int e^{ax}\sin bx\,dx$ (ii) $\int e^{ax}\cos bx\,dx$

Solution:

(i) From the tabular formula of integration by parts we obtain



From this pattern we find the given integral as

$$I=\int e^{ax}\sin bx\,dx=\left(e^{ax}\right)\left(-\frac{\cos bx}{b}\right)-\left(ae^{ax}\right)\left(-\frac{\sin bx}{b^2}\right)$$
$$+\int\left(a^2e^{ax}\right)\left(-\frac{\sin bx}{b^2}\right)dx$$

or,

$$I=-\frac{1}{b}e^{ax}\cos bx+\frac{a}{b^2}e^{ax}\sin bx-\frac{a^2}{b^2}\int e^{ax}\sin bx\,dx$$

or,

$$I=-\frac{1}{b}e^{ax}\cos bx+\frac{a}{b^2}e^{ax}\sin bx-\frac{a^2}{b^2}I$$

From which we obtain

$$I\left(1+\frac{a^2}{b^2}\right)=-\frac{1}{b}e^{ax}\cos bx+\frac{a}{b^2}e^{ax}\sin bx$$

or,

$$I\left(\frac{a^2+b^2}{b^2}\right) = \frac{a}{b^2}e^{ax}\sin bx - \frac{1}{b}e^{ax}\cos bx$$

Thus, we have

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$$

(ii) Similarly, we can prove that

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C$$

Tabular Formula of Integration by Parts

Consider the integral

$$\int f(x)g(x)dx \qquad (2)$$

We can evaluate this integral by the use of the tabular formula of the integration by parts, especially if one of the two functions is a power function. In fact, the tabular formula of the integration by parts is a successive application of the main formula. This tabular formula evaluates the integral of two functions as illustrated by the following pattern

Repeated differentiation		Repeated integration
$f(x)$		$g(x)$
	$(+)$ ↘	
$f'(x)$		$\int g(x)\,dx$
	$(-)$ ↘	
$f''(x)$		$\int\left[\int g(x)\,dx\right]dx$
	$(+)$ ↘	
$f'''(x)$		$\int\left[\int\left[\int g(x)\,dx\right]dx\right]dx$
$\vdots$		$\vdots$

If the function $f(x)$ becomes zero after a finite number of differentiation this table terminates and the given integral is then evaluated. If the two functions do not vanish with differentiation, then we terminate the given integral with another integral as in the case of integration of the functions

$$\int e^{ax}\cos x\,dx, \int e^{ax}\sin x\,dx, \int \sec^3 x\,dx, \cdots.$$

Example (1): Evaluate $\int x^3\cos 2x\,dx$

Solution: Using the tabular formula of integration by parts, we get

x^3		$\cos 2x$
	$(+)$	$\left(\dfrac{\sin 2x}{2}\right)$
$3x^2$		
	$(-)$	$\left(-\dfrac{\cos 2x}{4}\right)$
$6x$		
	$(+)$	$\left(-\dfrac{\sin 2x}{8}\right)$
6		
	$(-)$	$\left(\dfrac{\cos 2x}{16}\right)$
$\boxed{0}$		

From this pattern we find the given integral as

$$\int x^3 \cos 2x \, dx = \left(x^3\right)\left(\frac{\sin 2x}{2}\right) - \left(3x^2\right)\left(-\frac{\cos 2x}{4}\right)$$

$$+ (6x)\left(-\frac{\sin 2x}{8}\right) - (6)\left(\frac{\cos 2x}{16}\right) + C$$

or,

$$\int x^3 \cos 2x \, dx = \frac{1}{2}x^3 \sin 2x + \frac{3}{4}x^2 \cos 2x - \frac{3}{4}x \sin 2x$$

$$-\frac{3}{8}\cos 2x + C$$

2.9 Integration by Successive Reduction

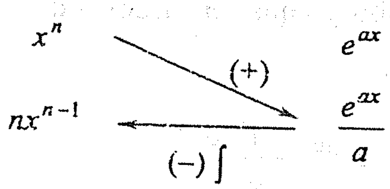
Integration by successive reduction is one way of simplifying complicated integrals. The basic idea of this method is to obtain a recurrence formula for the given integral. By this we mean, to obtain a formula expressing the original integral I_n, say, in terms of a lower order integral I_{n-1} or I_{n-2}, say. We explain this idea by the following examples:

Example (1): Find a reduction formula for the integral

$$I_n = \int x^n e^{ax} dx,$$

where, n is a positive integer. Hence, find $\int x^3 e^{ax} dx$.

Solution: Integrating by parts, we get



$$I_n = \int x^n e^{ax} dx = \left(x^n\right)\left(\frac{e^{ax}}{a}\right) - \frac{n}{a}\int x^{n-1} e^{ax} dx$$

Hence,

$$I_n = \frac{1}{a} x^n e^{ax} - \frac{n}{a} I_{n-1}$$

Substitute in the reduction formula to evaluate I_3; we get respectively,

$$n = 3 \Rightarrow I_3 = \frac{1}{a} x^3 e^{ax} - \frac{3}{a} I_2$$

$$n = 2 \Rightarrow I_2 = \frac{1}{a} x^2 e^{ax} - \frac{2}{a} I_1$$

$$n = 1 \Rightarrow I_1 = \frac{1}{a} x e^{ax} - \frac{1}{a} I_0$$

From the given integral we can get I_0 by setting $n = 0$, so we have

$$I_0 = \int x^0 e^{ax} dx = \int e^{ax} dx = \frac{e^{ax}}{a}$$

Substitute in the above equations from bottom to top to find I_3, thus we obtain

$$I_1 = \frac{1}{a} x e^{ax} - \frac{1}{a} \cdot \frac{e^{ax}}{a}$$

$$I_2 = \frac{1}{a} x^2 e^{ax} - \frac{2}{a}\left(\frac{1}{a} x e^{ax} - \frac{1}{a} \cdot \frac{e^{ax}}{a}\right)$$

$$= \frac{1}{a} x^2 e^{ax} - \frac{2}{a^2} x e^{ax} + \frac{2}{a^3} e^{ax},$$

and thus we have

$$I_3 = \frac{1}{a}x^3e^{ax} - \frac{3}{a}\left(\frac{1}{a}x^2e^{ax} - \frac{2}{a^2}xe^{ax} + \frac{2}{a^3}e^{ax}\right)$$

$$= \frac{1}{a}x^3e^{ax} - \frac{3}{a^2}x^2e^{ax} + \frac{6}{a^3}xe^{ax} - \frac{6}{a^4}e^{ax} + C$$

$$= e^{ax}\left(\frac{x^3}{a} - \frac{3x^2}{a^2} + \frac{6x}{a^3} - \frac{6}{a^4}\right) + C$$

Example (2): Find a reduction formula for the following integrals, and hence evaluate the indicated integral

(i) $I_n = \int \left(\sin^{-1} x\right)^n dx, \quad I_4$

(ii) $I_m = \int \left(\cos^{-1} x\right)^m dx, \quad I_5$

Solution:

(i) Put $t = \sin^{-1} x$,

then, we have $\sin t = x$ and $dx = \cos t\, dt$. Substitute in the given integral we get

$$I_n = \int t^n \cos t\, dt.$$

Integrate by parts two times, we get

t^n	(+) ↘	$\cos t$
nt^{n-1}	(−) ↘	$\sin t$
$n(n-1)t^{n-2}$	← (+)	$-\cos t$

From the above table we obtain

$$I_n = t^n \sin t + nt^{n-1} \cos t - \int n(n-1)t^{n-2} \cos t \, dt$$

which implies that

$$I_n = t^n \sin t + nt^{n-1} \cos t - n(n-1)I_{n-2}$$
$$= x\left(\sin^{-1} x\right)^n + n\left(\sin^{-1} x\right)^{n-1} \sqrt{1-x^2} - n(n-1)I_{n-2}$$

To evaluate I_4, we substitute successively in the derived reduction formula. Therefore, we get

$$n = 4 \quad \Rightarrow \quad I_4 = t^4 \sin t + 4t^3 \cos t - 4 \cdot 3I_2,$$
$$n = 2 \quad \Rightarrow \quad I_2 = t^2 \sin t + 2t \cos t - 2 \cdot 1I_0,$$

But

$$I_0 = \int \cos t \, dt = \sin t$$

Thus we have

$$I_2 = t^2 \sin t + 2t \cos t - 2 \sin t,$$

and,

$$I_4 = t^4 \sin t + 4t^3 \cos t - 12\left(t^2 \sin t + 2t \cos t - 2 \sin t\right)$$

$$= t^4 \sin t + 4t^3 \cos t - 12t^2 \sin t - 24t \cos t + 24 \sin t + C$$

$$= x\left(\sin^{-1} x\right)^4 + 4\sqrt{1-x^2}\left(\sin^{-1} x\right)^3 - 12x\left(\sin^{-1} x\right)^2$$

$$-24\sqrt{1-x^2}\left(\sin^{-1} x\right) + 24x + C$$

(ii) We leave this part as an exercise.

Example (3): Evaluate $I_n = \int \sec^n x\, dx$

Solution:

Case (1): n is a positive even integer

In this case the integral can be evaluated directly as follows:

Let $n = 2m$, then

$$\int \sec^n x\, dx = \int \sec^{2m} x\, dx = \int \sec^{2m-2} \sec^2 x\, dx$$

$$= \int (1 + \tan^2 x)^{m-1} \sec^2 x\, dx$$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$, then

$$\int \sec^n x\, dx = \int (1+t^2)^{m-1} dt.$$

We then apply the binomial theorem and integrate term by term. For a numerical example, let us evaluate,

$$\int \sec^8 x dx = \int \sec^6 x \sec^2 x dx \quad (t = \tan x \Rightarrow dt = \sec^2 x dx)$$

$$= \int (1 + \tan^2 x)^3 \sec^2 x dx$$

$$= \int (1 + t^2)^3 dt = (1 + 3t^2 + 3t^4 + t^6) dt$$

$$= t + t^3 + \frac{3}{5} t^5 + \frac{1}{7} t^7 + C$$

$$= \tan x + \tan^3 x + \frac{3}{5} \tan^5 x + \frac{1}{7} \tan^7 x + C$$

Case (2): **n is a positive odd integer**

In this case we get a reduction formula as follows:

Let

$$I_n = \int \sec^n x dx = \int \sec^{n-2} x \sec^2 x dx$$

$$= (\sec^{n-2} x)(\tan x) - \int (\tan x)(n-2)\sec^{n-3} x \sec x \tan x dx$$

$$= (\tan x)\sec^{n-2} x - (n-2)\int \sec^{n-2} x \tan^2 x dx$$

$$= (\tan x)\sec^{n-2} x - (n-2)\left(\int \sec^{n-2} x(\sec^2 x - 1)dx\right)$$

$$= (\tan x)\sec^{n-2} x - (n-2)\int \sec^n x dx + (n-2)\int \sec^{n-2} x dx$$

$$= (\tan x)\sec^{n-2} x - (n-2)I_n + (n-2)I_{n-2}$$

$$= \frac{1}{n-1}\left[(\tan x)\sec^{n-2} x + (n-2)I_{n-2}\right],$$

which is a reduction formula for I_n; namely

$$I_n = \frac{1}{n-1}\left[(\tan x)\sec^{n-2} x + (n-2)I_{n-2}\right]$$

For a numerical example let us evaluate, $I_7 = \int \sec^7 x dx$.

$$I_7 = \int \sec^7 x dx$$

$$I_7 = \frac{1}{6}\left[(\tan x)(\sec^5 x) + (5)I_5\right]$$

$$I_5 = \frac{1}{4}\left[(\tan x)(\sec^3 x) + (3)I_3\right]$$

$$I_3 = \frac{1}{2}\left[(\tan x)(\sec \ x) + I_1\right]$$

$$I_1 = \int \sec \ x dx = \ln|\tan x + \sec x| + C$$

Thus we have

$$I_7 = \frac{1}{6}(\tan x)(\sec^5 x) + \frac{5}{24}(\tan x)(\sec^3 x) +$$

$$+ \frac{15}{48}(\tan x)(\sec x) + \frac{15}{48}\ln|\tan x + \sec x| + C$$

Example (4): Evaluate $\int \tan^n x \, dx$.

Solution:

Case (1): n is an odd positive integer

In this case the integral can be evaluated directly as follows:

Put $n = 2m + 1$,

then,

$$\int \tan^n x dx = \int \tan^{2m+1} x dx$$
$$= \int \tan^{2m} x \tan x dx = \int (\sec^2 x - 1)^m \tan x dx$$
$$= \int \frac{(\sec^2 x - 1)^m}{\sec x} \tan x \sec x dx$$
$$= \int \frac{(t^2 - 1)^m}{t} dt. \qquad (t = \sec x)$$

Then, applying the binomial theorem and integrate term by term to obtain the final result. For a numerical example, let us evaluate:

Example (5): Evaluate $\int \tan^7 x dx$

Solution:

$$\int \tan^7 x dx = \int \tan^6 x \tan x dx = \int \frac{(\sec^2 x - 1)^3}{\sec x} \tan x \sec x dx$$
$$= \int \frac{(t^2 - 1)^3}{t} dt, \qquad \text{where } t = \sec x$$

$$\int \tan^7 x dx = \int \frac{(t^6 - 3t^4 + 3t^2 - 1)}{t} dt$$

$$= \frac{\sec^6 x}{6} - \frac{3}{4}\sec^4 x + \frac{3}{2}\sec^2 x - \ln|\sec x| + C$$

Case (2): n is an even positive integer

In this case we get a reduction formula as follows:

Let

$$I_n = \int \tan^n x dx = \int (\tan^{n-2} x)(\sec^2 x - 1) dx$$

$$= \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x dx$$

$$= \frac{\tan^{n-1} x}{n-1} - I_{n-2},$$

which is the required reduction formula. As a numerical example, let us have the following:

Example (6): Evaluate $I_6 = \int \tan^6 x dx$

Solution:

$$I_6 = \int \tan^6 x dx = \frac{\tan^5 x}{5} - I_4,$$

$$I_4 = \frac{\tan^3 x}{3} - I_2,$$

$$I_2 = \tan x - I_0,$$

$$I_0 = \int \tan^0 x dx = \int 1 \, dx = x$$

Thus, we have

$$I_6 = \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + C$$

Integrals of the form $\int \sec^m x \tan^n x dx$

Case (1): m is an even integer

In this case we separate one $\sec^2 x$ from $\sec^m x$ and then proceed as in the following examples:

Example (7): Evaluate the following integral

$$\int \sec^6 x \tan^6 x \, dx .$$

Solution:

$$\begin{aligned}
\int \sec^6 x \tan^6 x \, dx &= \int \sec^4 x \tan^6 x \, \underline{\sec^2 x} \, dx \\
&= \int (\sec^2 x)^2 \tan^6 x \, \underline{\sec^2 x} \, dx \\
&= \int (1 + \tan^2 x)^2 \tan^6 x \, \underline{\sec^2 x} \, dx \\
&= \int (\tan^6 x + 2\tan^8 x + \tan^{10} x) \, \underline{\sec^2 x} \, dx \\
&= \frac{1}{7}\tan^7 x + \frac{2}{9}\tan^9 x + \frac{1}{11}\tan^{11} x + C
\end{aligned}$$

Example (8): Evaluate the following integral

$$\int \sec^4 x \tan^3 x \, dx.$$

Solution:

$$\begin{aligned}
\int \sec^4 x \tan^3 x \, dx &= \int \sec^2 x \tan^3 x \underline{\sec^2 x \, dx} \\
&= \int \sec^2 x \tan^3 x \underline{\sec^2 x} \, dx \\
&= \int (1+\tan^2 x)\tan^3 x \underline{\sec^2 x} \, dx \\
&= \int (\tan^3 x + \tan^5 x) \underline{\sec^2 x} \, dx \\
&= \frac{1}{4}\tan^4 x + \frac{1}{6}\tan^6 x + C
\end{aligned}$$

Case (2): **n is an odd integer**

In this case we separate one $\sec x \tan x$ from $\sec^m x \tan^n x$ and then proceed as in the following example:

Example (9): Evaluate the following integral

$$\int \sec^3 x \tan^3 x \, dx.$$

Solution:

$$\begin{aligned}
\int \sec^3 x \tan^3 x \, dx &= \int \underline{\sec^2} x \tan^2 x \underline{\sec x \tan x \, dx} \\
&= \int \sec^2 x (\sec^2 x - 1) \underline{\sec x \tan x \, dx} \\
&= \int (\sec^4 x - \sec^2 x) \underline{\sec x \tan x \, dx} \\
&= \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 + C
\end{aligned}$$

Case (3): m is an odd integer and n is an even integer

In this case we express the whole integral in terms of $\int \sec^k x dx$ and proceed as in the case of integration of the form $\int \sec^k x dx$.

Example (10): Evaluate the following integral

$$\int \sec x \tan^2 x \, dx .$$

Solution:

$$\int \sec x \tan^2 x \, dx = \int \sec x (\sec^2 x - 1) \, dx$$

$$= \int (\sec^3 x - \sec x) \, dx$$

$$= \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) - \ln|\sec x + \tan x| + C$$

$$= \frac{1}{2}(\sec x \tan x - \ln|\sec x + \tan x|) + C.$$

Reduction formulas involving two parameters

Example (11): Find a reduction formula for

$$I_{m,n} = \int x^m (\ln x)^n \, dx .$$

Solution: , Using integration by parts we directly obtain

$$I_{m,n} = \int x^m (\ln x)^n \, dx$$

$$= (\ln x)^n \left(\frac{x^{m+1}}{m+1}\right) - \int \left(\frac{x^{m+1}}{m+1}\right) n(\ln x)^{n-1} \cdot \frac{1}{x} dx$$

$$= (\ln x)^n \left(\frac{x^{m+1}}{m+1}\right) - \frac{n}{m+1} \int x^m (\ln x)^{n-1} dx$$

$$= \frac{1}{m+1} x^{m+1} (\ln x)^n - \frac{n}{m+1} I_{m,n-1}$$

Hence, the required reduction formula is

$$I_{m,n} = \frac{1}{m+1} x^{m+1} (\ln x)^n - \frac{n}{m+1} I_{m,n-1}.$$

<u>Example</u> (12): Find a reduction formula for

$$I_{n,m} = \int x^n \sin mx dx.$$

<u>Solution</u>:

$$I_{n,m} = \int x^n \sin mx dx$$

$$= x^n \left(\frac{-\cos mx}{m}\right) - \int \left(\frac{-\cos mx}{m}\right) n x^{n-1} dx$$

$$= x^n \left(\frac{-\cos mx}{m}\right) + \frac{n}{m} \int x^{n-1} \cos mx dx$$

$$= \frac{-\cos mx}{m} x^n + \frac{n}{m}\left[x^{n-1} \frac{\sin mx}{m} - \int \left(\frac{\sin mx}{m}\right)(n-1)x^{n-2} dx\right]$$

$$= \frac{-\cos mx}{m} x^n + \frac{n}{m^2} x^{n-1} \sin mx - \frac{n(n-1)}{m^2} \int x^{n-2} \sin mx dx$$

$$= \frac{-\cos mx}{m} x^n + \frac{n}{m^2} x^{n-1} \sin mx - \frac{n(n-1)}{m^2} I_{n-2,m}$$

or,

$$I_{n,m} = \frac{-\cos mx}{m} x^n + \frac{n}{m^2} x^{n-1} \sin mx - \frac{n(n-1)}{m^2} I_{n-2,m}$$

Example (13): Find a reduction formula for

$I_{m,n} = \int \cos^m x \sin nx dx$, ($m$ and n are positive integrs)

Solution:

$$I_{m,n} = \int \cos^m x \sin n x\, dx$$

$$= \cos^m x\left(-\frac{\cos nx}{n}\right) - \int \left(-\frac{\cos nx}{n}\right) m \cos^{m-1} x(-\sin x) dx$$

$$= -\frac{1}{n} \cos nx \cos^m x - \frac{m}{n} \int \cos^{m-1} x \sin x \cos nx dx$$

But,

$$\sin(n-1)x = \sin nx \cos x - \cos nx \sin x.$$

Then,

$$\sin x \cos nx = \sin nx \cos x - \sin(n-1)x$$

Therefore,

$$I_{m,n} = -\frac{1}{n}\cos nx \cos^m x - \frac{m}{n}\int \cos^{m-1} x \sin nx \cos x dx$$

$$+\frac{m}{n}\int \cos^{m-1} x \sin(n-1)dx$$

$$= -\frac{1}{n}\cos nx \cos^m x - \frac{m}{n} I_{m,n} + \frac{m}{n} I_{m-1,n-1}$$

or

$$I_{m,n} = \frac{1}{m+n}\left[-\cos nx \cos^m x + m I_{(m-1,n-1)}\right]$$

Exercises (5)

Evaluate each of the following integrals:

(1) $\int x^2 e^{3x}\,dx$

(2) $\int x\cos 5x\,dx$

(3) $\int x\sec x\tan x\,dx$

(4) $\int \sqrt{x}\ln x\,dx$

(5) $\int x\csc^2 x\,dx$

(6) $\int \sin(\ln x)\,dx$

(7) $\int \frac{x^3}{\sqrt{1+x^2}}\,dx$

(8) $\int x(2x+3)^{99}\,dx$

(9) $\int \frac{x^5}{\sqrt{1-x^3}}\,dx$

(10) $\int (\ln x)^2\,dx$

(11) $\int (x+4)\cosh 4x\,dx$

(12) $\int \cos\sqrt{x}\,dx$

(13) $\int \frac{x\,e^x}{(x+1)^2}\,dx$

(14) $\int \sec^{-1}\sqrt{x}\,dx$

(15) $\int \ln(x^2+1)\,dx$

(16) $\int \sqrt{x}\tan^{-1}\sqrt{x}\,dx$

(17) Find a reduction formula for

(i) $\int x^n \cos mx\,dx$

(ii) $\int x^n \sinh mx\,dx$

(iii) $\int \sec^m x\tan^n x\,dx$

(iv) $\int \frac{dx}{(x^2+a^2)^n}$

2.10 Integrals Involving Quadratic Polynomial Functions

We have three cases to consider

(a) Integrals of the Form

$$\int \sqrt{ax^2+bx+c}\ dx$$

In this case we first take the coefficient of x^2 outside the square root sign and then complete the square of the quantity under the square root sign to get one of the three standard integrals $\int k\sqrt{L^2-t^2}\,dt$, $\int k\sqrt{L^2+t^2}\,dt$ or $\int k\sqrt{t^2-L^2}\,dt$ which have been treated previously. The following example clarify this case:

<u>Example</u> (1): Evaluate the integrals

(i) $\int \sqrt{x^2-4x+13}\ dx$ (ii) $\int \sqrt{6-4x-2x^2}\ dx$

Solution

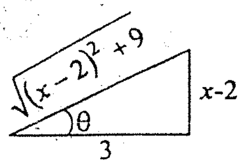
(i) $\int \sqrt{x^2-4x+13}\ dx = \int \sqrt{(x-2)^2-4+13}\ dx$

$$= \int \sqrt{(x-2)^2+9}\ dx$$

Put,

$$(x-2)=3\tan\theta \quad \Rightarrow \quad \begin{cases} dx = 3\sec^2\theta\, d\theta \\ (x-2)^2+9 = 9\sec^2\theta \end{cases}$$

Therefore, we have



$$\int\sqrt{x^2-4x+13}\,dx = \int\sqrt{(x-2)^2+9}\,dx$$

$$=\int 3\sec\theta \cdot 3\sec^2\theta d\theta = 9\int\sec^3\theta d\theta$$

$$=9\cdot\frac{1}{2}\left[\sec\theta\tan\theta+\ln(\sec\theta+\tan\theta)\right]+C$$

$$=\frac{9}{2}\left[\sec\theta\tan\theta+\ln(\sec\theta+\tan\theta)\right]+C$$

$$=\frac{9}{2}\left[\frac{\sqrt{(x-2)^2+9}}{3}\,\frac{(x-2)}{3}+\ln\left|\frac{\sqrt{(x-2)^2+9}}{3}+\frac{(x-2)}{3}\right|\right]+C$$

$$=\frac{1}{2}\left[(x-2)\sqrt{(x-2)^2+9}+9\ln\left|\sqrt{(x-2)^2+9}+(x-2)\right|\right]+C$$

(ii) Since,

$$6-4x-2x^2 = -2\left(x^2+2x-3\right)$$

$$=-2\left((x+1)^2-1-3\right) = -2\left[(x+1)^2-4\right]$$

$$=2\left[4-(x+1)^2\right]$$

Then,

$$\int\sqrt{6-4x-2x^2}\,dx=\sqrt{2}\int\sqrt{4-(x+1)^2}\,dx$$

Using the substitution

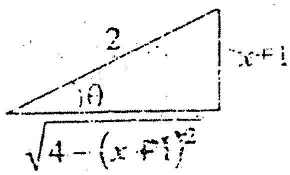
$$(x+1)=2\sin\theta \quad\Rightarrow\quad \begin{cases} dx=2\cos\theta\,d\theta \\ 4-(x+1)^2=4\cos^2\theta \end{cases}$$

We obtain,

$$\int\sqrt{6-4x-2x^2}\,dx=\sqrt{2}\int\sqrt{4-(x+1)^2}\,dx$$

$$=\sqrt{2}\int 2\cos\theta\times 2\cos\theta\,d\theta=4\sqrt{2}\int\cos^2\theta\,d\theta$$

$$=4\sqrt{2}\int\frac{1+\cos 2\theta}{2}\,d\theta$$



$$=2\sqrt{2}\left(\theta+\frac{\sin 2\theta}{2}\right)+C$$

$$=2\sqrt{2}(\theta+\sin\theta\cos\theta)+C$$

$$=2\sqrt{2}\left(\sin^{-1}\left(\frac{x+1}{2}\right)+\left(\frac{x+1}{2}\right)\left(\frac{\sqrt{4-(x+1)^2}}{2}\right)\right)+C$$

b) Integrals of the Forms

$$\int\frac{dx}{ax^2+bx+c},\qquad \int\frac{dx}{\sqrt{ax^2+bx+c}}$$

These forms are reduced to the standard integrals by completing the square of the expression in the denominator.

Example (2): Evaluate the integrals

(i) $\displaystyle\int \frac{1}{2x^2-8x+5}\,dx$ (ii) $\displaystyle\int \frac{1}{\sqrt{x^2+2x+5}}\,dx$

Solution:

(i) Since,

$$2x^2-8x+5 = 2\left(x^2-4x+\frac{5}{2}\right) = 2\left((x-2)^2-4+\frac{5}{2}\right)$$

$$= 2\left[(x-2)^2-\frac{3}{2}\right]$$

Then,

$$\int \frac{1}{2x^2-8x+5}\,dx = \frac{1}{2}\int \frac{1}{(x-2)^2-\frac{3}{2}}\,dx$$

$$= -\frac{1}{2}\cdot\frac{1}{\sqrt{3/2}}\tanh^{-1}\left(\frac{x-2}{\sqrt{3/2}}\right)+C$$

$$= -\frac{1}{\sqrt{6}}\tanh^{-1}\left(\frac{x-2}{\sqrt{3/2}}\right)+C$$

(ii) For the integral $\displaystyle\int \frac{1}{\sqrt{x^2+2x+5}}\,dx$, we complete the square of the quantity in the denominator to get

$$x^2+2x+5 = \left((x+1)^2-1+5\right) = (x+1)^2+4$$

Thus we obtain

$$\int \frac{1}{\sqrt{x^2+2x+5}}dx = \int \frac{1}{\sqrt{(x+1)^2+4}}dx = \sinh^{-1}\left(\frac{x+1}{2}\right)+C$$

(c) Integrals of the Form

$$\int \frac{Ax+B}{ax^2+bx+c}dx, \qquad \int \frac{Ax+B}{\sqrt{ax^2+bx+c}}dx$$

Each of these integrals is reduced to two integrals; in the first integral the numerator will be the derivative of the denominator or we shall have the integral of a function multiplied by its derivative. The other integral is of the previous form, i.e. we get a standard integration by completing the square of the quantity in the denominator.

Example (3): Evaluate the integrals

(i) $\int \frac{3x-6}{x^2+4x+5}dx$ (ii) $\int \frac{x+4}{\sqrt{2x+x^2}}dx$

Solution:

(i) For the first integral, we have

$$\int\frac{3x-6}{x^2+4x+5}dx=3\int\frac{x-2}{x^2+4x+5}dx$$

$$=\frac{3}{2}\int\frac{2x-4}{x^2+4x+5}dx=\frac{3}{2}\int\frac{2x+4-4-4}{x^2+4x+5}dx$$

$$=\frac{3}{2}\int\frac{2x+4}{x^2+4x+5}dx+\frac{3}{2}\int\frac{-8}{(x+2)^2+1}dx$$

$$=\frac{3}{2}\ln\left|x^2+4x+5\right|-12\tan^{-1}(x+2)+C$$

(ii) $$\int\frac{x+4}{\sqrt{2x+x^2}}dx=\frac{1}{2}\int\frac{2x+8}{\sqrt{2x+x^2}}dx$$

$$=\frac{1}{2}\int\frac{2x+2-2+8}{\sqrt{2x+x^2}}dx$$

$$=\frac{1}{2}\int\frac{2x+2}{\sqrt{2x+x^2}}dx+\frac{1}{2}\int\frac{6}{\sqrt{x^2+2x+1-1}}dx$$

$$=\frac{1}{2}\int\left(2x+x^2\right)^{-1/2}\cdot(2x+2)dx+3\int\frac{1}{\sqrt{(x+1)^2-1}}dx$$

$$=\frac{1}{2}\frac{\left(2x+x^2\right)^{1/2}}{1/2}+3\cosh^{-1}(x+1)+C$$

$$=\sqrt{2x+x^2}+3\cosh^{-1}(x+1)+C$$

(d) Integrals of the Form

$$\int \sqrt{\frac{ax+b}{Ax+B}}\, dx$$

In this case, multiply the integrand by $\dfrac{\sqrt{ax+b}}{\sqrt{ax+b}}$, so we get

$$\int \sqrt{\frac{ax+b}{Ax+B}}\, dx = \int \frac{ax+b}{\sqrt{Lx^2+Mx+N}}\, dx,$$

which is the integral studied in the previous case.

Example (4): Evaluate $\int \sqrt{\dfrac{x+3}{x+2}}\, dx$

Solution:

$$\int \sqrt{\frac{x+3}{x+2}} dx = \int \sqrt{\frac{x+3}{x+2}} \cdot \frac{\sqrt{x+3}}{\sqrt{x+3}}\, dx = \int \frac{x+3}{\sqrt{(x+2)(x+3)}} dx$$

$$= \int \frac{x+3}{\sqrt{x^2+5x+6}} dx$$

$$= \frac{1}{2}\left[\int \frac{2x+5}{\sqrt{x^2+5x+6}} dx + \int \frac{dx}{\sqrt{x^2+5x+6}} \right]$$

$$= \sqrt{x^2+5x+6} + \frac{1}{2} \int \frac{dx}{\sqrt{x^2+5x+6}}$$

$$= \sqrt{x^2 + 5x + 6} + \frac{1}{2}\int \frac{dx}{\sqrt{\left(x + \frac{5}{2}\right)^2 - \frac{1}{4}}}$$

$$= \sqrt{x^2 + 5x + 6} + \frac{1}{2}\cosh^{-1} 2\left(x + \frac{5}{2}\right) + C$$

$$= \sqrt{x^2 + 5x + 6} + \frac{1}{2}\cosh^{-1}(2x + 5) + C.$$

Example (5): Evaluate $\int \sqrt{\frac{1-x}{2x+3}}\, dx, \qquad -\frac{3}{2} < x < 1$

Solution:

$$\int \sqrt{\frac{1-x}{2x+3}}\, dx = \int \frac{(1-x)}{\sqrt{-2x^2 - x + 3}}\, dx$$

The first derivative of $-2x^2 - x + 3$ is $-4x - 1$, thus we have

$$\int \sqrt{\frac{1-x}{2x+3}}\, dx = \frac{1}{4}\int \frac{-4x+4}{\sqrt{-2x^2 - x + 3}} dx$$

$$= \frac{1}{4} \times 2\sqrt{-2x^2 - x + 3} + \frac{1}{4}\int \frac{5}{\sqrt{-2\left(x^2 + \frac{x}{2} - \frac{3}{2}\right)}} dx$$

$$=\frac{1}{2}\sqrt{-2x^2-x+3}+\int\frac{\frac{5}{4}\sqrt{2}}{\sqrt{-\left(\left(x+\frac{1}{4}^2\right)-\frac{1}{16}-\frac{3}{2}\right)}}dx$$

$$=\frac{1}{2}\sqrt{-2x^2-x+3}+\int\frac{\frac{5}{4}\sqrt{2}}{\sqrt{\frac{25}{16}-\left(x+\frac{1}{4}\right)^2}}dx$$

$$=\frac{1}{2}\int\sqrt{-2x^2-x+3}+\frac{5}{4\sqrt{2}}\sin^{-1}\frac{\left(x+\frac{1}{4}\right)}{5/4}+C$$

$$=\frac{1}{2}\int\sqrt{-2x^2-x+3}+\frac{5}{4\sqrt{2}}\sin^{-1}\frac{(4x+1)}{5}+C$$

Exercises (6)

Evaluate each of the following integrals:

(1) $\int \frac{x\,dx}{x^2-4x+8}$

(2) $\int \frac{dx}{\sqrt{5-4x-2x^2}}$

(3) $\int \frac{dx}{x^2-4x+13}$

(4) $\int \frac{dx}{\sqrt{2x-x^2}}$

(5) $\int \frac{x\,dx}{x^2+6x+10}$

(6) $\int \frac{dx}{\sqrt{x^2-6x+10}}$

(7) $\int \sqrt{3-2x-x^2}\,dx$

(8) $\int \frac{e^x\,dx}{\sqrt{1+e^x+e^{2x}}}$

(9) $\int \frac{dx}{\left(x^2+6x+13\right)^{3/2}}$

(10) $\int \sqrt{x(6-x)}\,dx$

(11) $\int \frac{\cos x\,dx}{\sin^2 x-\sin x-2}$

(12) $\int \frac{\sin 2x\,dx}{\sin^2 x-2\sin x-8}$

(13) $\int \frac{\sin x\,dx}{5\cos x+\cos^2 x}$

(14) $\int \sqrt{\frac{x+2}{x+3}}\,dx$

2.12 Using Partial Fraction Decomposition to Evaluate Integral of Rational Functions

Consider we want to evaluate integrals of the form

$$\int \frac{a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_m x^m}{b_0 + b_1 x + b_2 x^2 + \ldots\ldots + b_n x^n}\, dx,$$

<u>Case</u> (1): If the degree of the numerator is less than the degree of the denominator we proceed as in the following examples:

<u>Example</u> (5): Evaluate $\int \frac{8x+3}{x^2 - 2x - 3} dx$.

<u>Solution</u>: we decompose the expression $\frac{8x+3}{x^2 - 2x - 3}$ as in example (2), section (2.11) to get

$$\frac{8x+3}{x^2 - 2x - 3} = \frac{5}{4(x+1)} + \frac{27}{4(x-3)},$$

then,

$$\int \frac{8x+3}{x^2-2x-3}dx = \frac{5}{4}\int \frac{1}{1+x}dx + \frac{27}{4}\int \frac{1}{x-3}dx$$

$$= \frac{5}{4}\ln|x+1| + \frac{27}{4}\ln|x-3| + C$$

Example (6): Evaluate $\int \frac{6x^2+x+10}{x^3-3x-2}dx$

Solution: In example (3), section (2.11), we have seen that

$$\frac{6x^2+x+10}{x^3-3x-2} = \frac{-5}{(x+1)^2} + \frac{2}{(x+1)} + \frac{4}{(x-2)}$$

then,

$$\int \frac{6x^2+x+10}{x^3-3x-2}dx = -5\int \frac{1}{(x+1)^2}dx + 2\int \frac{1}{(x+1)}dx + 4\int \frac{1}{(x-2)}dx$$

$$= -5\frac{(x+1)^{-1}}{-1} + 2\ln|x+1| + 4\ln|x-2| + C$$

$$= \frac{5}{x+1} + 2\ln|x+1| + 4\ln|x-2| + C$$

Example (7): Evaluate $\int \frac{7x^2+8}{x^3-1}dx$

Solution: In example (4), section (2.11), we have seen that

$$\frac{7x^2+8}{x^3-1}=\frac{5}{x-1}+\frac{2x-3}{x^2+x+1}$$

$$\therefore \int\frac{7x^2+8}{x^3-1}dx=5\int\frac{1}{x-1}dx+\int\frac{2x-3}{x^2+x+1}dx$$

$$=5\ln(x-1)+\int\frac{2x+1-4}{x^2+x+1}dx$$

$$=5\ln(x-1)+\int\frac{2x+1}{x^2+x+1}dx-4\int\frac{1}{x^2+x+1}dx$$

$$=5\ln(x-1)+\ln(x^2+x+1)-4\int\frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}$$

$$=\ln|x-1|^5+\ln(x^2+x+1)-4\times\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right)+C$$

$$=\ln|x-1|^5+\ln(x^2+x+1)-\frac{8}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C.$$

<u>Case</u> (2): If the degree of numerator m is greater than the degree of denominator n we divide using the prolongation method of division to express the integrand as the sum of a polynomial and a proper rational function as we have done in example (1), section (2.11). Then

$$\int \frac{a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_m x^m}{b_0 + b_1 x + b_2 x^2 + \ldots\ldots + b_n x^n} dx$$

$$= \int (a'_0 + a'_1 x + a'_2 x^2 + \ldots\ldots + a'_p x^p) dx$$

$$+ \int \frac{C_0 + C_1 x + C_2 x^2 + \ldots\ldots + C_q x^q}{b_0 + b_1 x + b_2 x^2 + \ldots\ldots + b_n x^n} dx,$$

where $p + q = m$ and $q < n$. Then, the first part can be integrated without difficulties, but for the second part we follow the method of partial fractions mentioned above and then integrate each fraction separately.

Example (8): Evaluate $\int \frac{x^4 - 5x - 7}{x^2 - 4} dx$

Solution: From example (1), we see that

$$\frac{x^4 - 5x - 7}{x^2 - 4} = x^2 + 4 + \frac{-5x + 9}{x^2 - 4}$$

Then,

$$\int \frac{x^4 - 5x - 7}{x^2 - 4} dx = \int (x^2 + 4) dx + \int \frac{-5x + 9}{x^2 - 4} dx$$

But,

$$\frac{-5x+9}{x^2-4}=\frac{A}{x-2}+\frac{B}{x+2}$$

$$-5x+9=A(x+2)+B(x-2)$$

$$-5=A+B\,,\quad 9=2A-2B,$$

then,

$$A=-\frac{1}{4},\quad B=-\frac{19}{4}.$$

Hence,

$$\frac{-5x+9}{x^2-4}=\frac{-1}{4(x-2)}+\frac{-19}{4(x+2)},$$

and

$$\int\frac{x^4-5x-7}{x^2-4}dx=\frac{x^3}{3}+4x-\frac{1}{4}\int\frac{1}{x-2}dx-\frac{19}{4}\int\frac{1}{x+2}dx$$

$$=\frac{x^3}{3}+4x-\frac{1}{4}\ln|x-2|-\frac{19}{4}\ln|x+2|+C$$

Example (9): Evaluate $\int\frac{e^x}{\left(e^{2x}+2e^x+1\right)\left(e^x+2\right)}dx$

Solution: We first use the substitution

$$e^x=u\Rightarrow e^{2x}=u^2,\ e^x dx=du$$

Then,

$$\int \frac{e^x}{(e^{2x}+2e^x+1)(e^x+2)}dx = \int \frac{du}{(u^2+2u+1)(u+2)}$$

$$= \int \frac{du}{(u+1)^2(u+2)}$$

Using partial fractions decomposition we get

$$\frac{1}{(u+1)^2(u+2)} = \frac{A}{(u+1)^2} + \frac{B}{(u+1)} + \frac{C}{(u+2)}$$

Then,

$$1 = A(u+2) + B(u+2)(u+1) + C(u+1)^2$$

Put u = −1: we get $A = -1$.

Put u = −2: we get $C = 1$

Put u = 0: we get $1 = A + 2B + C \Rightarrow 1 = -1 + 2B + 1 \Rightarrow B = \frac{1}{2}$

Hence,

$$\frac{1}{(u+1)^2(u+2)} = \frac{-1}{(u+1)^2} + \frac{1/2}{(u+1)} + \frac{1}{(u+2)}$$

and,

$$\int \frac{1}{(u+1)^2(u+2)}du = \int\left(\frac{-1}{(u+1)^2}+\frac{1/2}{(u+1)}+\frac{1}{(u+2)}\right)du$$

$$=\frac{1}{u+1}+\frac{1}{2}\ln|u+1|+\ln|u+2|+C$$

Hence,

$$\int \frac{e^x}{\left(e^{2x}+2e^x+1\right)\left(e^x+2\right)}dx$$

$$=\frac{1}{e^x+1}+\frac{1}{2}\ln\left|e^x+1\right|+\ln\left|e^x+2\right|+C$$

Exercises (7)

Evaluate each of the following integrals:

(1) $\int \frac{dx}{x^2+x-2}$

(2) $\int \frac{2x+4}{x^3-2x^2}\,dx$

(3) $\int \frac{x^2+x-2}{3x^3-x^2+3x-1}\,dx$

(4) $\int \frac{3x^2-10}{x^2-4x+4}\,dx$

(5) $\int \frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2}\,dx$

(6) $\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}\,dx$

(7) $\int \frac{2x^2+3}{x(x-1)^2}\,dx$

(8) $\int \frac{dx}{x^3+x}$

(9) $\int \frac{2x^5-x^3-1}{x^3-4x}\,dx$

(10) $\int \frac{x^2+x-16}{(x+1)(x-3)^2}\,dx$

(11) $\int \frac{x^3+3x^2+x+9}{(x^2+1)(x^2+3)}\,dx$

(12) $\int \frac{x^2}{(x+2)^3}\,dx$

(13) $\int \frac{\cos\theta\,d\theta}{\sin^2\theta+4\sin\theta-5}$

(14) $\int \frac{e^t}{e^{2t}-4}\,dt$

(15) $\int \frac{x^5}{(x^2+4)^2}\,dx$

(16) $\int \frac{x^6-x^3+1}{x^4+9x^2}\,dx$

Review Exercises on Chapter (2)

1. Evaluate each of the following integrals:

$\int x^2\sqrt{7+3x}\,dx$, $\int\sqrt{x-4x^2}\,dx$, $\int e^{\pi x}\sin^{-1}(e^{\pi x})\,dx$

$\int\frac{3x}{4x-1}\,dx$, $\int\sqrt{x^2-3}\,dx$, $\int\frac{\sqrt{x^2+5}}{x^2}\,dx$,

$\int\frac{dx}{x^2\sqrt{x^2-2}}$, $\int\frac{\sqrt{3-x^2}}{x}\,dx$, $\int x^3\ln x\,dx$,

$\int\frac{\ln x}{\sqrt{x}}\,dx$ $\int e^{-2x}\sin 3x\,dx$ $\int\frac{\sin^2(\ln x)}{x}\,dx$

$\int\frac{dx}{x^2+4x-5}$ $\int\sqrt{3-2x-x^2}\,dx$ $\int\frac{x\,dx}{\sqrt{5+4x-x^2}}$

$\int\frac{x\,dx}{x^2+6x+13}$ $\int\sin 3x\sin 2x\,dx$ $\int\sin 2x\cos 5x\,dx$

$\int\cos 3x\cos 2x\,dx$ $\int\sin^2 x\cos^2 x\,dx$ $\int\sin^2 x\cos^4 x\,dx$

$\int\sin^3 2x\cos^2 2x\,dx$ $\int\frac{\sin x}{\cos^8 x}\,dx$ $\int\tan^5 x\,dx$

$\int\tan^5 x\sec x\,dx$ $\int\tan^5 x\sec^4 x\,dx$ $\int\tan^4 x\sec^4 x\,dx$

$\int\frac{dx}{x^2+x-2}$ $\int\frac{2x+4}{x^3-2x^2}\,dx$ $\int\frac{dx}{1+e^x}$

$\int \frac{x^2 + x - 2}{3x^3 - x^2 + 3x - 1} dx$

$\int \frac{\cos\theta}{\sin^2\theta + 4\sin\theta - 5} d\theta$

$\int \frac{5x^3 - 3x^2 + 7x - 3}{(x^2 + 1)^2}$

$\int \frac{x^{2/3}}{x + 1} dx$

$\int \frac{e^t}{e^{2t} - 4} dt$

$\int \frac{e^t}{e^{2t} - 4e^t + 3} dt$

$\int \frac{1 + \sqrt{x}}{1 - \sqrt{x}} dx$

$\int \frac{\ln x \, dx}{x\sqrt{4\ln x - 1}}$

$\int e^x \sqrt{3 - 4e^{2x}} \, dx$

$\int x \sin 3x \, dx$

$\int e^{-\sqrt{x}} \, dx$

$\int \frac{e^{4x}}{(4 - 3e^{2x})^2} dx$

$\int \frac{\cos x}{\sqrt{2 - \sin^2 x}} dx$

$\int \frac{x^3}{(3 + x^2)^{5/2}} dx$

$\int \frac{e^x \, dx}{\sqrt{1 + e^x + e^{2x}}}$

$\int \sqrt{1 + e^x} \, dx$

$\int \frac{dx}{1 + \sin x}$

$\int \sqrt{x(4 - x)} \, dx$

$\int \frac{dx}{4\sin x - 3\cos x}$

$\int \frac{dx}{1 - \sin x + \cos x}$

$\int \frac{\cos x \, dx}{2 - \cos x}$

2. Derive a reduction formula in part (a) and use it to evaluate the integral in (b)

(i) (a) $\int x^n e^x dx$, (b) $\int x^3 e^x dx$

(ii) (a) $\int \tan^n x dx$, (b) $\int \tan^6 x dx$

(iii) (a) $\int (\ln x)^n dx$, (b) $\int (\ln x)^5 dx$

(iv) (a) $\int \sin^n x \cos^m x dx$, (b) $\int \sin^6 x \cos^4 x dx$

Chapter (3)

Definite and Improper Integrals

Chapter 3

Definite and Improper Integrals

Definite integration is a fundamental concept of mathematical analysis. It is a powerful tool in mathematics, physics, mechanics and other disciplines. It is used for calculation of areas of regions bounded by curves, arc lengths, volumes, work, velocity, and others.

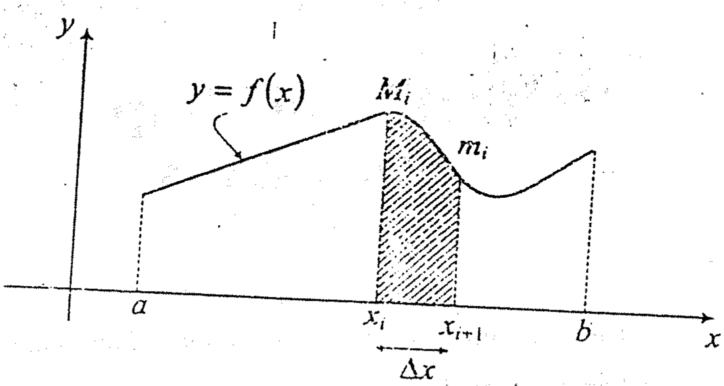
3.1 Definition and Geometric Interpretation of Definite Integrals

Let $f(x)$ be a continuous nonnegative function defined on the interval $[a, b]$. Divide the interval $[a, b]$ into n-equal subintervals by the points:

$$x_0 = a,\ x_1,\ x_2,\ \cdots,\ x_n = b,$$

and let

$$\Delta x = x_{i+1} - x_i, \qquad i = 0, 1, \cdots, n-1.$$



Let M_i and m_i be the maximum and minimum values of the function $f(x)$ in the interval $[x_i, x_{i+1}]$. Define the lower sums L_n by:

$$L_n = \sum_{i=0}^{n-1} m_i \, \Delta x,$$

and the upper sums U_n by:

$$U_n = \sum_{i=0}^{n-1} M_i \, \Delta x.$$

If we take limits as $n \to \infty$ $(\Delta x \to 0)$ we get

$$U = \lim_{n\to\infty} U_n, \qquad L = \lim_{n\to\infty} L_n$$

These two limits exist and they are called the upper and lower integrals of the function $f(x)$ over the interval $[a, b]$, respectively. If $f(x)$ is continuous and nonnegative on $[a, b]$, then U = L. In this case we define the definite integral of $f(x)$ over $[a, b]$ as follows:

$$\int_a^b f(x)\,dx = U = L$$

We shall also define,

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

<u>**Remark:**</u> If $f(x) \geq 0$ on $[a, b]$, then

$$\int_a^b f(x)\,dx = A,$$

where A is the area bounded by the curve $y = f(x)$, the x-axis, and the two vertical lines $x = a$ and $x = b$.

3.2 Basic Rules on Definite Integration

Rule 1 (Scalar Multiplication Rule)

The constant factor may be taken outside the sign of the definite integral. That is

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx, \text{ for any constant } k,$$

For example, $\int_0^\pi 3\sin x\,dx = 3\int_0^\pi \sin x\,dx$.

Rule 2 (Sum and Difference Rule)

The definite integral of an algebraic sum (difference) of several functions is equal to the algebraic sum (difference) of the integrals of the summands. Thus, in the case of two terms

$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

For example, $\int_a^b [x + \sin x]\,dx = \int_a^b x\,dx + \int_a^b \sin x\,dx$.

Rule 3

If the function $f(x) \geq 0$ on the interval $[a, b]$ then,

$$\int_a^b f(x)\,dx \geq 0$$

Rule 4

If on the interval $[a,\ b]$ where $(a < b)$, the function $f(x)$ and $g(x)$ satisfy the condition $f(x) \le g(x)$ then,

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

Rule 5

If m and M are the smallest and greatest values of the function $f(x)$ on the interval $[a,\ b]$ and $(a \le b)$ then,

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

The above rule is usualy used to estimate the value of the integral without calculating it as seen in the following example:

Example (1): Evaluate the upper and lower values of the integral $\int_0^{2\pi} \frac{dx}{\sqrt{10+6\sin x}}$.

Solution: By using rule (5),

$$m = \min_{0 \le x \le 2\pi} \frac{1}{\sqrt{10+6\sin x}} = \left.\frac{1}{\sqrt{10+6\sin x}}\right|_{x=\pi/2} = 0.25$$

$$M = \max_{0 \le x \le 2\pi} \frac{1}{\sqrt{10 + 6\sin x}} = \frac{1}{\sqrt{10 + 6\sin x}}\bigg|_{x=3\pi/2} = 0.5$$

Thus we have

$$2\pi(0.25) \le \int_0^{2\pi} \frac{dx}{\sqrt{10 + 6\sin x}} \le 2\pi(0.5)$$

$$\frac{\pi}{2} \le \int_0^{2\pi} \frac{dx}{\sqrt{10 + 6\sin x}} \le \pi$$

Rule 6

If the two integration limits are equal then the corresponding integral vanishes, i. e.

$$\int_a^a f(x)\,dx = 0$$

Rule 7

For any three numbers a, b and c we have that

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

For example, $\int_0^{\pi} \sin x\,dx = \int_0^{\pi/2} \sin x\,dx + \int_{\pi/2}^{\pi} \sin x\,dx.$

Rule 8

If the function $f(x)$ is even ($f(-x) = f(x)$), then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx\,,$$

Rule 9

If the function $f(x)$ is odd ($f(-x) = -f(x)$), then

$$\int_{-a}^{a} f(x)\,dx = 0\,,$$

Remark: Rules 8 and 9 can be written in the following compact form

$$\int_{-a}^{a} f(x)\;dx = \begin{cases} 2\int_{0}^{a} f(x)\;dx & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd} \end{cases}$$

Rule 10

$$\int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(a-x)\,dx$$

Rule 11

$$\int_{0}^{2a} f(x)\,dx = \begin{cases} 2\int_{0}^{a} f(x)\,dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$$

Example (2): Find

(i) $\int_{-3}^{3} x^5 dx$, (ii) $\int_{-2}^{2} x^4 dx$, (iii) $\int_{-\pi}^{\pi} \sin^7 x \; dx$

Solution:

(i) Since x^5 is an odd function, then

$$I_1 = \int_{-3}^{3} x^5 dx = 0$$

(ii) Since x^4 is an even function, then

$$I_2 = \int_{-2}^{2} x^4 dx = 2\int_{0}^{2} x^4 dx = 2 \frac{x^5}{5} \Bigg|_0^2 = \frac{64}{5}.$$

(iii) Since $\sin^7 x$ is an odd function, then (by rule 9)

$$I_3 = \int_{-\pi}^{\pi} \sin^7 x dx = 0.$$

3.3 The Fundamental Theorems of Integral Calculus

In the previous section we introduced the properties of definite integration without showing how to calculate it. In this section, in addition to giving the relation between differentiation and definite integration and the relation between definite and indefinite integration we show how to calculate definite integration in terms of indefinite integration. The following theorems are called the fundamental theorem of calculus because, as early mentioned, they establish the relationship between differentiation and integration.

Theorem 3.1: if $f(x)$ is continuous on $[a, b]$ and $F(x)$ is defined on $[a, b]$ by

$$F(x) = \int_a^x f(t)\,dt\,,$$

then $F(x)$ is differentiable on $[a, b]$ and

$$\frac{dF(x)}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = f(x), \qquad a < x < b\,.$$

Example (1): Find $S'(x)$ of the Fresnel function $S(x) = \int_0^x \sin(\frac{\pi t^2}{2})dt$. This function appears in the study of differaction of light waves and recently in the design of highways.

Solution: Applying Theorem (3.1), we get

$$S'(x) = \sin(\frac{\pi x^2}{2}).$$

The above theorem has the following generalization.

Theorem (3.2): If $f(x)$ is continuous function on $[a, b]$ and $u(x)$ and $v(x)$ are differentable functions of x whose values lie in $[a, b]$ and let $y = \int_{u(x)}^{v(x)} f(t)dt$, then

$$\frac{dy}{dx} = \frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = f(v(x))v'(x) - f(u(x))u'(x)$$

Example (2): Find $\frac{dy}{dx}$ if:

(i) $y = \int_0^x \sin t \, dt$ (ii) $y = \int_{2x}^{x^2} \cos t \, dt$

Solution:

(i) Using Theorem (3.1), we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\int_0^x \sin t \, dt\right) = \sin x$$

(ii) The direct application of Theorem (3.2) yields

$$\frac{dy}{dx} = \frac{d}{dx}\left(\int_{2x}^{x^2} \cos t \, dt\right)$$
$$= (\cos(x^2)).(2x) - (\cos(2x)).(2) = 2\,x \cos x^2 - 2\cos 2x$$

Example (3): Find the equation of the tangent line to the curve $y = F(x)$ at the point on the curve, where $x = 1$ and

$$F(x) = \int_1^{x^2} \frac{\sin(\frac{\pi}{2}t^2)}{e^{t-1}} dt.$$

Solution:

At $x = 1$ $y(1) = F(1) = \int_1^1 \frac{\sin(\frac{\pi}{2}t^2)}{e^{t-1}} dt = 0$.

Then, the slope m of the tangent line to the curve $y = F(x)$ is given by

$$y'(1) = \frac{d}{dx}F(x)\Big|_{x=1} = \frac{\sin(\frac{\pi}{2}x^4)}{e^{x^4}-1}(2x)\Bigg|_{x=1} = 2.$$

Hence, the equation of the tangent line at the point (1,0) is

$$y - y_1 = m(x - x_1)$$

or,

$$y - 0 = 2(x - 1)$$

Hence, the required equation is

$$y = 2(x - 1)$$

Example (4): Find $f(4)$ if $\int_0^x f(t)dt = x\cos(\pi x)$.

Solution: We have that

$$\frac{d}{dx}\left(\int_0^x f(t)\,dt\right) = f(x)$$

$$= \frac{d}{dx}(x\cos(\pi x)) = \cos(\pi x) - (\pi)\,x\sin(\pi x),$$

from which we get

$$f(x) = \cos(\pi x) - (\pi)x\sin(\pi x)$$

Therefore,

$$f(4) = \cos(4\pi) - 4\pi\sin(4\pi) = 1$$

The following theorem shows how to calculate definite integration in terms of indefinite integration

Theorem (3.3): (*The First Fundamental Theorem of Calculus*). If $f(x)$ is a continuous function on $[a,\ b]$ and $F(x)$ is any antiderivative of $f(x)$ on $[a,\ b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

The following example is a direct application of the above theorem.

Example (5): Find

(i) $\int_0^2 x^3\,dx,$ (ii) $\int_0^{\pi/2} (6\cos 4x + 7)dx$ (iii) $\int_1^4 \frac{1+\sqrt{x}}{x^2}dx$

Solution:

(i) $I_1 = \int_0^2 x^3\,dx = \left.\frac{x^4}{4}\right|_0^2 = \frac{16}{4} - \frac{0}{4} = 4$

(ii) $I_2 = \int_0^{\pi/2} (6\cos 4x + 7)dx = \left[\frac{6}{4}\sin 4x + 7x\right]_0^{\pi/2}$

$$= \left[\frac{3}{2}\sin 2\pi + 7\left(\frac{\pi}{2}\right)\right] - \left[\frac{3}{2}\sin 0 + 7(0)\right] = \frac{7\pi}{2}$$

(iii) $I_3 = \int_1^4 \frac{(1+\sqrt{x})}{x^2}\,dx = \int_1^4 x^{-2}\,dx + \int_1^4 x^{-3/2}\,dx$

$$= -\left.\frac{1}{x}\right|_1^4 + \left[\frac{x^{-1/2}}{-1/2}\right]_1^4$$

$$= -\left(\frac{1}{4} - 1\right) - 2\left(\frac{1}{\sqrt{4}} - 1\right) = \frac{3}{4} + 1 = \frac{7}{4}$$

3.4 Integration by Substitution

The following theorem is analogous to rule (2) (the rule of integration by substitution) of section 2.2

Theorem (3.4): Given the integral $\int_a^b f(x)dx$, where $f(x)$ is continuous on the interval $[a,\ b]$ and let $x = \varphi(t)$. Suppose that the function $\varphi(t)$ satisfies the following conditions:

(I) The value of $\varphi(t)$ varies from a to b when t varies from α to β so that $\varphi(\alpha) = a,\ \varphi(\beta) = b$ and all intermediate values of $\varphi(t)$ are in $[a,\ b]$.

(II) The derivative $\varphi'(t)$ of $\varphi(t)$ is a continuous function on the closed interval $[\alpha,\ \beta]$.

Then,

$$\int_a^b f(x)dx = \int_\alpha^\beta f[\varphi(t)]\,\varphi'(t)\,dt$$

In the following examples the above conditions are automatically satisfied, therefore we shall not verify them.

Example (1): Find $I = \int_0^a \sqrt{a^2 - x^2}\,dx, \quad (a > 0)$

Solution: By the substitution

$$x = a \sin t \quad \Rightarrow \quad dx = a \cos t\, dt,$$

we get the following new limits.

When $x = 0 \Rightarrow 0 = a\sin t \Rightarrow t = 0$;

when $x = a \Rightarrow a = a\sin t \Rightarrow 1 = \sin t \Rightarrow t = \frac{\pi}{2}$.

Then,

$$I = \int_0^a \sqrt{a^2 - x^2}\, dx = a^2 \int_0^{\pi/2} \cos^2 t\, dt$$

$$= a^2 \int_0^{\pi/2} \frac{1 + \cos 2t}{2}\, dt$$

$$= \frac{a^2}{2}\left[\, t\Big|_0^{\pi/2} + \frac{1}{2}\sin 2t\, \Big|_0^{\pi/2}\right] = \frac{\pi a^2}{4}$$

Example (2): Find $I = \int_0^{\ln 2} \sqrt{e^x - 1}\, dx$

Solution: Put

$$t = \sqrt{e^x - 1} \Rightarrow t^2 = e^x - 1 \Rightarrow e^x = t^2 + 1$$

Then,

$$e^x dx = 2t dt \Rightarrow dx = \frac{2t}{1 + t^2} dt$$

when $x = 0$, $t = \sqrt{e^0 - 1} = 0$,

and when $x = \ln 2$, $t = \sqrt{e^{\ln 2} - 1} = \sqrt{2-1} = 1$.

Hence,

$$I = \int_0^{\ln 2} \sqrt{e^x - 1}\, dx = \int_0^1 \frac{2t^2}{1+t^2}\, dt$$

$$= 2\int_0^1 \frac{(1+t^2)-1}{1+t^2}\, dt = 2\int_0^1 (1 - \frac{1}{1+t^2})\, dt$$

$$= 2t \Big|_0^1 - 2\tan^{-1} t \Big|_0^1 = 2 - \frac{\pi}{2}$$

3.5 Integration by Parts

Theorem (3.5): Let the functions $u = u(x)$ and $v = v(x)$ have continuous derivatives $u'(x)$ and $v'(x)$ on the interval $[a,\ b]$. Then,

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Example (1): Evaluate each of the following integrals:

(i) $I_1 = \int_0^1 xe^x dx$ (ii) $I_2 = \int_0^1 \tan^{-1} x dx$

(iii) $I_3 = \int_1^e \frac{\ln x}{x^2} dx$

Solution:

Using integration by parts we have that

(i) $I_1 = \int_0^1 xe^x dx = xe^x\Big|_0^1 - \int_0^1 e^x dx$

$$= xe^x\Big|_0^1 - e^x\Big|_0^1 = e - e + 1 = 1$$

(ii) $I_2 = \int_0^1 \tan^{-1} x dx = x\tan^{-1} x\Big|_0^1 - \int_0^1 \frac{x}{1+x^2} dx$

$= \frac{\pi}{4} - \frac{1}{2}\ln(1+x^2)\Big|_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2.$

(iii) $I_3 = -\frac{1}{x}\ln x\Big|_1^e + \int_1^e \frac{dx}{x^2}$

$= \frac{\ln x}{x}\Big|_1^e - \frac{1}{x}\Big|_1^e = (\frac{-\ln e}{e} - \frac{1}{e}) - (-\frac{\ln 1}{1} - \frac{1}{1})$

$= 1 - \frac{2}{e}.$

Exercises (1)

Find the value of each of the following integrals:

(1) $\int_0^1 x\, e^{-5x}\, dx$ (2) $\int_{\sqrt{e}}^{e} \frac{\ln x}{x^2}\, dx$

(3) $\int_0^{1/2} \sin^{-1} x\, dx$ (4) $\int_0^{\pi} (x + x\cos x) dx$

(5) $\int_0^{\pi/4} \sin^4 x\, dx$ (6) $\int_0^{\pi/2} \cos^6 x\, dx$

(7) $\int_{0}^{\pi/2} \tan^3\left(\frac{x}{2}\right) dx$

(8) $\int_{0}^{1/4} \sec \pi x \, \tan \pi x \, dx$

(9) $\int_{0}^{1} \sqrt{1+x^2} \, dx$

(10) $\int_{\sqrt{2}}^{2} \frac{\sqrt{2x^2-4}}{x} \, dx$

(11) $\int_{0}^{3} \frac{x^3}{\left(3+x^2\right)^{5/2}} \, dx$

(12) $\int_{1}^{3} \frac{dx}{x^4\sqrt{x^2+3}}$

(13) $\int_{2}^{3} \frac{x \, dx}{(x-1)^6}$

(14) $\int_{0}^{25} \frac{dx}{\sqrt{4+\sqrt{x}}}$

(15) $\int_{0}^{1} \frac{x-1}{x^2+x+1} \, dx$

(16) $\int_{0}^{\pi/4} \cos x \, \cos 5x \, dx$

(17) Find the value of x if

(i) $\int_{2}^{x} \frac{dt}{t(4-t)} = \frac{1}{2}$

(ii) $\int_{1}^{x} \frac{dt}{t\sqrt{2t-1}} = 1$