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(ODEs II)

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الفصل الدراسي الثاني

Ordinary Differential Equations II

Linear Differential Equation of Second Order Part 1

LINEAR DIFFERENTIAL EQUATION OF SECOND ORDER

The general form

$$y'' + p(x)\, y' + q(x)\, y = r(x)$$

Homogeneous if $r(x) = 0$.

Non-homogeneous if $r(x) \neq 0$.

p, q : co-efficient of the equation.

EXAMPLE

Homogeneous linear equation

$$(1 - x^2)y'' - 2xy' + 6y = 0$$

Non-homogeneous linear equation

$$y'' + 4y = e^{-x} \sin x$$

Non-linear differential equations:

$$x\,(y''\, y + y'^2) + 2y'\, y = 0$$

$$y'' = \sqrt{y'^2 + 1} \qquad y''^2 - y'^2 - 1 = 0$$

SOLUTION

A solution of second order differential equation (linear or non-linear) is a function y (x) that has first and second derivatives, y' (x) and y'' (x) and satisfies the given differential equation for all x in a given interval.

EXAMPLE

Let Differential Equation: $y'' - y = 0$

Two solutions: $y = e^{x}$ and $y = e^{-x}$ How?

$y = e^{x} \Rightarrow y' = e^{x},\ y'' = e^{x} \quad \therefore\ y'' - y = 0$

$y = e^{-x} \Rightarrow y' = -e^{-x},\ y'' = e^{-x} \quad \therefore\ y'' - y = 0$

ae^{x} and be^{-x} are also solution

$y = ae^{x} + be^{-x}$ is also a solution

Is ..... a solution of ....

CHECK ans

$y'' - y = 0$

$y = ae^{x} + be^{-x},$

$\Rightarrow y' = ae^{x} - be^{-x},$

$y'' = ae^{x} + be^{-x}$

Hence

$y'' - y = ae^{x} + be^{-x} - (ae^{x} + be^{-x}) = 0$

FUNDAMENTAL THEOREM

For a homogeneous linear differential equation

$$y'' + p(x)\,y' + q(x)\,y = 0$$

any linear combination of two solution of this equation is again a solution. In other words

If y_1 and y_2 are two solution of

$$y'' + p(x)\,y' + q(x)\,y = 0$$

Then

$$y = c_1 y_1 + c_2 y_2$$

is also a solution, where c_1 and c_2 are any arbitrary constants.

Def

GENERAL SOLUTION

For a second order homogeneous linear equation

$$y'' + p(x)\,y' + q(x)\,y = 0$$

a general solution will be of the form

$$y = c_1 y_1 + c_2 y_2$$

Def IVP

INITIAL VALUE PROBLEM

An initial value problem will have equation

$$y'' + p(x)\,y' + q(x)\,y = 0$$

and two initial conditions

ICs → $y(x_0) = k_0$ and $y'(x_0) = k_1$

Using these two conditions the particular solution from general solution can be obtained

EXAMPLE

Solve the initial value problem

$$y'' - y = 0, \quad y(0) = 4 \text{ and } y'(0) = -2$$

SOLUTION

$y'' - y = 0$ $\qquad e^{x}$ and e^{-x} are two solutions

The general solution: $\quad y = c_1 e^{x} + c_2 e^{-x}$

The initial conditions: $\quad y(0) = c_1 + c_2 = 4$

$y' = c_1 e^{x} - c_2 e^{-x} \qquad \therefore\ y'(0) = c_1 - c_2 = -2$

$\Rightarrow\ c_1 = 1,\ c_2 = 3$

Particular solution: $\quad y = e^{x} + 3e^{-x}$

$y_1 = e^{x}$ and $y_2 = 4e^{x}$,

The general solution: $\quad y = c_1 e^{x} + 4c_2 e^{x}$

$$y' = c_1 e^{x} + 4c_2 e^{x}$$

$$y(0) = c_1 + 4c_2 = 4$$

$$y'(0) = c_1 + 4c_2 = -2$$

Inconsistent equations

using two initial conditions we cannot find c_1 and c_2.

$$\frac{y_1}{y_2} = \frac{e^{x}}{4e^{x}} = \frac{1}{4}$$

constant

$$\frac{y_1}{y_2} = \frac{e^{x}}{e^{-x}} = e^{2x}$$

BASIS OR FUNDAMENTAL SYSTEM

- A general solution of an equation

$$y'' + p(x)\,y' + q(x)\,y = 0$$

is of the form $y = c_1 y_1 + c_2 y_2$

with y_1 and y_2 not being proportional solution and c_1 and c_2 are arbitrary constants.

These y_1 and y_2 are called a basis or fundamental system of the given equation.

PARTICULAR SOLUTION

- A particular solution of

$$y'' + p(x)\,y' + q(x)\,y = 0$$

can be obtained if specific values of c_1 and c_2 are assigned in

$$y = c_1 y_1 + c_2 y_2$$

PROPORTIONAL

- The two solution y_1 and y_2 are called proportional if

$$\frac{y_1}{y_2} = k, \quad \text{a constant.}$$

LINEAR INDEPENDENCE

Two functions $y_1(x)$ and $y_2(x)$ are called linearly independent if

$$k_1 y_1 + k_2 y_2 = 0 \quad \Rightarrow \quad k_1 = 0,\ k_2 = 0\ .$$

LINEAR DEPENDENCE

y_1 and y_2 are called linearly dependent if for some k_1 and k_2 not both zero

$$k_1 y_1 + k_2 y_2 = 0$$

REFORMULATED DEFINITION OF BASIS

A linearly independent pair of solutions y_1 and y_2 for

$$y'' + p(x)y' + q(x)y = 0$$

is called a basis or fundamental system.

In our previous example e^x and e^{-x} form the basis.

Prove that {...,...} linearly independent

How to obtain a Basis if one solution is known.

Reduction of order Technique

Let y_1 be a non-zero solution of

$$y'' + p(x)y' + q(x)y = 0$$

Then another solution y_2 linearly independent of y_1

$$y_2(x) = u(x)y_1(x)$$

How to get y_2 ?

substitute y_2 and its derivatives in given equation i.e. $y_2(x) = u(x)y_1(x)$

$$y_2' = u'y_1 + uy_1', \quad y_2'' = u''y_1 + 2u'y_1' + uy_1''$$

$$y'' + p(x)y' + q(x)y = 0$$

$$u''y_1 + 2u'y_1' + uy_1'' + p(u'y_1 + uy_1') + quy_1 = 0$$

$$u''y_1 + u'(2y_1' + py_1) + u(y_1'' + py_1' + qy_1) = 0$$

$y_1'' + py_1' + qy = 0$ Since y_1 is a solution

$u''y_1 + u'(2y_1' + py_1) = 0$ an equation in u'' & u'

Let $u' = U, u'' = U'$

$$\Rightarrow U' + \left(\frac{2y_1'}{y_1} + p\right)U = 0$$

Separation of variable and integration

$$U' + \left(\frac{2y_1'}{y_1} + p\right)U = 0 \quad \Rightarrow \frac{dU}{U} = -\left(\frac{2y_1'}{y_1} + p\right)dx$$

$$\ln U = -2\ln y_1 - \int p\,dx \quad \Rightarrow \quad U = \frac{1}{y_1^2}e^{-\int p\,dx}$$

$$u = \int U\,dx = \int \frac{1}{y_1^2}e^{-\int p\,dx}\,dx$$

$\frac{y_2}{y_1} = u$ is not a constant

Since, $U = u' \neq 0$

EXAMPLE 1

Find the general solution of differential equation

$$(1 - x^2)y'' - 2xy' + 2y = 0$$

using the fact that $y_1(x) = x$ is a solution of this equation

SOLUTION

First check,

$$y_1 = x, \qquad y_1' = 1, \qquad y_1'' = 0$$

$$(1 - x^2)y'' - 2xy' + 2y = 0$$

$$-2x + 2x = 0$$

In order to find ....

Let

$$y_2(x) = u(x)y_1(x)$$

$$(1-x^2)y'' - 2xy' + 2y = 0$$

$$y'' - \frac{2x}{1-x^2}y' + \frac{2}{1-x^2}y = 0 \quad \Rightarrow p(x) = -\frac{2x}{1-x^2}$$

Second solution: $y_2 = u(x).x$

$$u = \int \frac{1}{y_1^2} e^{-\int p(x)dx} dx$$

$$-\int p(x)dx = -\ln(1-x^2) \qquad \therefore e^{-\int p(x)dx} = \frac{1}{1-x^2}$$

$$\therefore \quad u = \int \frac{1}{y_1^2} e^{-\int p(x)dx} dx \qquad = \int \frac{1}{x^2(1-x^2)} dx$$

$$= \int \frac{1}{x^2} dx + \int \frac{1}{1-x^2} dx$$

$$= \int \frac{1}{x^2} dx + \frac{1}{2}\int \frac{1}{1-x} dx + \frac{1}{2}\int \frac{1}{1+x} dx$$

$$= -\frac{1}{x} + \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$\therefore \quad y_2(x) = x\,u(x) \qquad = -1 + \frac{1}{2}x\ln\left(\frac{1+x}{1-x}\right)$$

General solution

$$y = c_1 x + c_2\left(-1 + \frac{1}{2}x\ln\left(\frac{1+x}{1-x}\right)\right)$$

EXAMPLE 2

Find the basis of solution for differential equation

$$x^2y'' - x\,y' + y = 0$$

SOLUTION

$$x^2y'' - x\,y' + y = 0$$

$y_1 = x$ is solution, check

$y_1 = x,\quad y_1' = 1,\quad y_1'' = 0 \quad \Rightarrow \quad -x + x = 0$

$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 0 \qquad \Rightarrow p(x) = -\frac{1}{x}$

$u = \int \frac{1}{y_1^2} e^{-\int p(x)dx} dx \;= \int \frac{1}{x^2} e^{\ln x} dx \;= \int \frac{x}{x^2} dx$

$= \int \frac{1}{x} dx \;= \ln x \quad \Rightarrow y_2 = x \ln x$

$\therefore$ basis of solution is { x, x lnx}.

$$y'' + p(x)\,y' + q(x)\,y = 0$$

The standard form $\quad y'' + a\,y' + by = 0$

How To solve? $\quad y = e^{\lambda x}$

Substituting y, y' and y''

$\lambda^2 e^{\lambda x} + a\,\lambda\, e^{\lambda x} + b\, e^{\lambda x} = 0 \quad \Rightarrow (\lambda^2 + a\lambda + b)\, e^{\lambda x} = 0$

$e^{\lambda x} \neq 0 \;\forall x \qquad \Rightarrow \lambda^2 + a\lambda + b = 0$

Characteristic Equation

CHARACTERISTIC EQUATION

The quadratic equation, thus obtained

$$\lambda^2 + a\lambda + b = 0$$

is called the Characteristic Equation or auxiliary equation of differential equation

$$y'' + a\,y' + by = 0$$

$$\lambda^2 + a\lambda + b = 0$$

- Case I : two real roots if $a^2 - 4b > 0$
- Case II : a real double root if $a^2 - 4b = 0$
- Case III : complex conjugate roots if $a^2 - 4b < 0$.

CASE I: Two distinct real roots λ_1 & λ_2

$y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$

$$\because \frac{y_1}{y_2} = e^{(\lambda_1 - \lambda_2)x} \neq \text{constant.}$$

General solution:

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

CASE II: Real double root λ

$a^2 - 4b = 0, \Rightarrow$ so $\lambda = -a/2,$

$$y_1 = e^{\lambda x} = e^{-ax/2} \qquad y_2 = uy_1$$

$$y'' + a y' + by = 0$$

$$u = \int \frac{1}{y_1^2} e^{-\int a dx} dx = \int \frac{e^{-ax}}{e^{-ax}} dx = x, \therefore y_2 = xy_1 = xe^{-ax/2}$$

General solution: $y = c_1 e^{-ax/2} + c_2 x e^{-ax/2}$

$$= (c_1 + c_2 x) e^{-ax/2}$$

CASE III: Complex roots

When $a^2 - 4b < 0$. The Roots:

$$\lambda_1 = -\frac{a}{2} + \frac{1}{2}\sqrt{a^2 - 4b},$$

$$\lambda_2 = -\frac{a}{2} - \frac{1}{2}\sqrt{a^2 - 4b}$$

$\lambda_1 = s + it$ and $\lambda_2 = s - it$

$$e^{\lambda_1 x} = e^{(s+it)x} = e^{sx}.(\cos tx + i \sin tx)$$

and $$e^{\lambda_2 x} = e^{(s-it)x} = e^{sx}.(\cos tx - i \sin tx)$$

$$y_1 = e^{-\frac{a}{2}x} \cos tx, \qquad y_2 = e^{-\frac{a}{2}x} \sin tx$$

where, $$t = \frac{1}{2}\sqrt{4b - a^2}$$

General solution:

$$y = c_1 e^{-\frac{a}{2}x} \cos tx + c_2 e^{-\frac{a}{2}x} \sin tx$$

Summary of Steps

To find the general solution of

$$y'' + a y' + by = 0$$

- Write down the characteristic equation

$$\lambda^2 + a\lambda + b = 0$$

Let λ_1 and λ_2 be its roots

$$= \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

- If $a^2 - 4b > 0$

i.e. if λ_1 & λ_2 are distinct real numbers

The general solution:

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

- If $a^2 - 4b = 0$

i.e. $\lambda_1 = \lambda_2 = -a/2$ is double real root

The general solution

$$y = (c_1 + c_2 x) e^{-\frac{a}{2}x}$$

- If $a^2 - 4b < 0$

i.e. the roots λ_1 & λ_2 are complex conjugate

The general solution is

$$y = c_1 e^{-\frac{a}{2}x} \cos\beta x + c_2 e^{-\frac{a}{2}x} \sin\beta x$$

$\lambda^2 + a\lambda + b = 0$

where $\beta = \frac{1}{2}\sqrt{4b - a^2}$

EXAMPLE 1

Find the general solution of

$$y'' - 9y' + 20y = 0$$

SOLUTION

Diff. Equation: $y'' - 9y' + 20y = 0$

The characteristic equation:

$$\lambda^2 - 9\lambda + 20 = 0$$

Factorization: $(\lambda - 5)(\lambda - 4) = 0$

Roots: $\lambda_1 = 4, \lambda_2 = 5$

Hence the general solution:

$$y = c_1 e^{4x} + c_2 e^{5x}$$

EXAMPLE 2

Solve the initial value problem

$$y'' + y' - 6y = 0$$

$$y(0) = 10, \ y'(0) = 0$$

SOLUTION

Diff. Equation: $y'' + y' - 6y = 0$

The characteristic equation:

$$\lambda^2 + \lambda - 6 = 0$$

Factorization: $(\lambda + 3)(\lambda - 2) = 0$

Roots: $\lambda_1 = -3, \lambda_2 = 2$

The general solution: $y(x) = c_1 e^{-3x} + c_2 e^{2x}$

$$y(x) = c_1e^{-3x} + c_2e^{2x}$$

$$y'(x) = -3c_1e^{-3x} + 2c_2e^{2x}$$

Initial Conditions: $y(0) = 10,\ y'(0) = 0$

$$\therefore\ y(0) = c_1 + c_2 = 10$$

$$y'(0) = -3c_1 + 2c_2 = 0$$

$$\therefore\ c_1 = 4,\ c_2 = 6$$

The solution: $y(x) = 4e^{-3x} + 6e^{2x}$

EXAMPLE 3

Q1 (a)

Solve the initial value problem

$$y'' - 4y' + 4y = 0$$

$$y(0) = 2,\ y'(0) = 1$$

SOLUTION

Differential Equation:

$$y'' - 4y' + 4y = 0$$

The characteristic equation:

$$\lambda^2 - 4\lambda + 4 = 0 \quad \Rightarrow (\lambda - 2)^2 = 0$$

double root: $\lambda = 2$

General solution:

$$y(x) = (c_1 + c_2x)e^{2x}$$

$$y(x) = (c_1 + c_2 x)e^{2x}$$

$$y'(x) = 2c_1 e^{2x} + 2c_2 x e^{2x} + c_2 e^{2x}$$

Initial conditions: $y(0) = 2, \quad y'(0) = 1$

$\therefore \quad y(0) = c_1 = 2,$

$y'(0) = 2c_1 + c_2 = 1 \quad \Rightarrow c_2 = -3,$

Particular solution:

$$y(x) = (2 - 3x)e^{2x}$$

EXAMPLE 4

Give the general solution of differential equation

$$4y'' + 4y' + 10y = 0$$

SOLUTION

Differential Equation: $4y'' + 4y' + 10y = 0$

The characteristic equation:

$$4\lambda^2 + 4\lambda + 10 = 0$$

Roots: $\lambda = \dfrac{-4 \pm \sqrt{16 - 160}}{8} \qquad = \dfrac{-4 \pm 12i}{8}$

$$\Rightarrow \lambda_1 = -\frac{1}{2} + \frac{3i}{2} \qquad \lambda_2 = -\frac{1}{2} - \frac{3i}{2}$$

The general solution:

$$y(x) = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{3}{2}\right)x + c_2 \sin\left(\frac{3}{2}\right)x\right)$$

or

$$y(x) = c_1 e^{-\frac{x}{2}} \cos\left(\frac{3}{2}\right)x + c_2 e^{-\frac{x}{2}} \sin\left(\frac{3}{2}\right)x$$

Chapter 2

Linear Differential Equation of Second Order Part 2

EXISTENCE AND UNIQUENESS THEOREM FOR INITIAL VALUE PROBLEM

If $p(x)$ and $q(x)$ are continuous functions on some interval I and x_0 is in I, for differential equation $y'' + p(x)y' + q(x)y = 0$ and two initial conditions $y(x_0) = k_0$ and $y'(x_0) = k_1$ then this IVP has a unique solution $y(x)$ on interval I.

WRONSKIAN

$$y'' + p(x)y' + q(x)y = 0$$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

Linear Independence

The two solution y_1 and y_2 will be linearly independent if $W(y_1, y_2) \neq 0$ some $x \in I$.

Some Properties of Wronskian

Given two functions $f(x)$ and $g(x)$ that are differentiable on some interval I

1) if $W(f, g)(x_0) \neq 0$. for some x_0 in I, then $f(x)$ and $g(x)$ are linearly independent on I

2) If $f(x)$ and $g(x)$ are linearly dependent on I then $W(f, g) = 0 \ \forall \ x$ in I.

Abel's Theorem

If $y_1(x)$ and $y_2(x)$ are two solutions to

$$y'' + p(x)\,y' + q(x)\,y = 0$$

then the Wronskian of two solutions is

$$W(y_1, y_2)(x) = W(y_1, y_2)(x_0)\, e^{-\int_{x_0}^{x} p(x)dx}$$

for some x_0.

Proof

$$y'' + p\,y' + q\,y = 0$$

∵ y_1 & y_2 are two solution

∴ $y_1'' + py_1' + qy_1 = 0$ & $y_2'' + py_2' + qy_2 = 0$

$$\Rightarrow (y_2'' y_1 - y_1'' y_2) + p\,(y_2' y_1 - y_1' y_2) = 0$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

$$W' = y_1' y_2' + y_1 y_2'' - y_1' y_2' - y_1'' y_2 = y_1 y_2'' - y_1'' y_2$$

$$(y_2'' y_1 - y_1'' y_2) + p\,(y_1 y_2' - y_2 y_1') = 0$$

$$\Rightarrow W' + pW = 0$$

$$W(x) = W(x_0)\, e^{-\int_{x_0}^{x} p(x)dx}$$

EXAMPLE

Find the general solution of

$$y'' - 2y' + y = 0$$

SOLUTION

The characteristic equation:

$$\lambda^2 - 2\lambda + 1 = 0$$

Double root: $\lambda = 1$

The two solutions: $y_1 = e^x$, $y_2 = xe^x$

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix}$$

$$= e^{2x} + xe^{2x} - xe^{2x} \quad = e^{2x} \neq 0$$

Hence y_1 and y_2 are linearly independent.

$$\frac{y_1}{y_2} = \frac{1}{x}$$

To be sure

Hence general solution:

$$y = c_1 e^x + c_2 xe^x$$

NON HOMOGENEOUS SECOND ORDER LINEAR DIFFERENTIAL EQUATION

NON HOMOGENEOUS SECOND ORDER LINEAR DIFFERENTIAL EQUATION

$$y'' + p(x)\,y' + q(x)\,y = r(x) \neq 0$$

The corresponding homogeneous equation

$$y'' + p(x)\,y' + q(x)\,y = 0$$

THEOREM

Suppose $Y_1(x)$ and $Y_2(x)$ are two solution of

$$y'' + p(x)\,y' + q(x)\,y = r(x)$$

and $y_1(x)$, $y_2(x)$ are fundamental set of solution of the corresponding homogenous differential equation

$$y'' + p(x)\,y' + q(x)\,y = 0$$

Then, $Y_1(x) - Y_2(x)$ is its the solution and can be written as

It satisfy

$$Y_1(x) - Y_2(x) = c_1 y_1(x) + c_2 y_2(x)$$

PROOF

$$(Y_1(x) - Y_2(x))'' = Y''_1(x) - Y_2''(x),$$

$$(Y_1(x) - Y_2(x))' = Y_1'(x) - Y_2'(x)$$

$$y'' + p(x)\,y' + q(x)\,y = 0$$

$$Y''_1(x) - Y_2''(x) + p(x)(Y_1'(x) - Y_2'(x)) + q(x)(Y_1(x) - Y_2(x))$$

$$= Y''_1(x) + p(x)Y_1'(x) + q(x)Y_1(x) - (Y_2''(x) + p(x)Y_2'(x) + q(x)Y_2(x)) = r(x) - r(x) = 0$$

$\therefore$ $Y_1(x) - Y_2(x)$ is solution

$\therefore$ $Y_1 - Y_2 = c_1 y_1 + c_2 y_2$ for some c_1 & c_2

$y(x)$ and $y_p(x)$

$$y'' + p(x)\, y' + q(x)\, y = r(x)$$

$$y(x) - y_p(x) = c_1 y_1(x) + c_2 y_2(x).$$

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

General Solution

$$y(x) = y_h(x) + y_p(x)$$

THE MAIN RESULT

GENERAL SOLUTION

$$y'' + p(x)\, y' + q(x)\, y = r(x)$$

$$y(x) = y_h(x) + y_p(x)$$

$y_h(x)$ is the general solution of

$$y'' + p(x)\, y' + q(x)\, y = 0$$

And $y_p(x)$ is a particular solution of

$$y'' + p(x)\, y' + q(x)\, y = r(x)$$

PARTICULAR SOLUTION

A particular solution of

$$y'' + p(x)\, y' + q(x)\, y = r(x)$$

is a solution obtained from

$$y(x) = y_h(x) + y_p(x)$$

by giving specific values to arbitrary constants c_1 and c_2 in $y_h(x)$.

The General Solution of Non-Homogeneous Equation Includes All Solutions

if p (x), q (x) and r (x) are continuous on some open interval I, Then every solution of $y'' + p(x)\,y' + q(x)\,y = r(x)$ on I is obtained by giving suitable values to the arbitrary constants in general solution
$y(x) = y_h(x) + y_p(x)$.

No Singular Solution!!

Summary

PRACTICAL CONCLUSION

To solve non homogeneous equation

$$y'' + p(x)\,y' + q(x)\,y = r(x)$$

or an initial value problem .

- Solve the homogeneous equation

$$y'' + p(x)\,y' + q(x)\,y = 0$$

- Find a particular solution y_p

How to obtain this solution??

BASIC TECHNIQUE TO SOLVE NON HOMOGENEOUS EQUATION

- ☐ Method To Find y_p

EXAMPLE

Motivation example

Find the general solution of

$$y'' + 3y' + 2y = 12e^x$$

SOLUTION

Corresponding homogeneous equation:

$$y'' + 3y' + 2y = 0$$

Characteristic equation: $\lambda^2 + 3\lambda + 2 = 0$

Roots: $\lambda = -1 , -2$

$$\therefore \quad y_h = c_1 e^{-x} + c_2 e^{-2x}$$

Differential Equation: $y'' + 3y' + 2y = 12e^x$

Particular solution y_p : Let, $y_p = A e^x$

$$\Rightarrow \; y_p' = A e^x \text{ and } y_p'' = A e^x$$

Substitute in given equation:

$$A e^x (1 + 3 + 2) = 12e^x \Rightarrow 6Ae^x = 12e^x \Rightarrow A = 2$$

$\therefore \; y_p = 2e^x$ General Solution:

$$y(x) = y_h(x) + y_p(x) = c_1 e^{-x} + c_2 e^{-2x} + 2e^x$$

Two methods to find y_p

- Method of undetermined Coefficient
- Method of variation of parameters

METHOD OF UNDETERMINED COEFFICIENT

- Applicable: Equations with constant coefficient and r (x) is of special form
  - r (x) = e ^{ax} or polynomial
  - r (x) = cos (bx)

METHOD OF UNDETERMINED COEFFICIENT

Cases for r(x)....

Terms in r(x)	Choice of y_p
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n (n=0,1,\dots)$	$K_n x^n + K_{n-1} x^{n-1} + \dots K_1 x + K_0$
$k\cos\omega x$ $k\sin\omega x$	$A \sin \omega x + B \cos \omega x$
$ke^{\alpha x}\cos\omega x$ $ke^{\alpha x}\sin\omega x$	$e^{\alpha x}(A \sin \omega x + B \cos \omega x)$

RULES OF THE METHOD

BASIC RULE

$$y'' + p(x)\, y' + q(x)\, y = r(x)$$

- If $r(x)$ is one of the function in the first column of the table .
- Choose corresponding function y_p in second column
- Find the value of undetermined coefficient by putting y_p and its derivatives in

$$y'' + p(x)\, y' + q(x)\, y = r(x)$$

MODIFICATION RULE

- If any term in the choice for y_p is also in the solution y_h of corresponding homogeneous equation

$$y'' + p(x)\, y' + q(x)\, y = 0$$

- Multiply the choice y_p by x or
- By x^2, if the solution corresponds to double root of characteristic equation of homogeneous equation.

SUM RULE

- If $r(x)$ is sum of different functions given in column one of the table, then choose y_p the sum of corresponding functions in second column.

EXAMPLES

EXAMPLE 1

Solve the non homogeneous equation

$$y'' + 4y = 8x^2$$

SOLUTION

$$y'' + 4y = 8x^2$$

Corresponding homogeneous equation:

$$y'' + 4y = 0$$

Characteristic equation: $\lambda^2 + 4 = 0$

Roots: $\lambda = \pm 2i$

$$y_h = c_1\cos 2x + c_2\sin 2x$$

$r(x) = 8x^2$

$\because$ $r(x) = 8x^2$ $\quad y_p = k_2x^2 + k_1x + k_0$

$\Rightarrow y_p' = 2k_2x + k_1 \quad \Rightarrow y_p'' = 2k_2, \quad y'' + 4y = 8x^2$

Substituting: $2k_2 + 4k_2x^2 + 4k_1x + 4k_0 = 8x^2$

Equating the coefficients:

$4k_2 = 8, \quad 4k_1 = 0, \quad 4k_0 + 2k_2 = 0$

$\Rightarrow k_2 = 2, \quad k_1 = 0, \quad k_0 = -1$

$\Rightarrow y_p = 2x^2 - 1$

$y(x) = c_1\cos 2x + c_2\sin 2x + 2x^2 - 1$

- One can try with kx^2 only and see that it fails.

Try!

Lec 7

EXAMPLE 2

Solve initial value problem

$$y'' + 2y' + y = e^{-x}$$
$$y(0) = -1,\ y'(0) = 1$$

IVP

SOLUTION

Differential equation: $y'' + 2y' + y = e^{-x}$

Corresponding homogeneous equation:

$$y'' + 2y' + y = 0$$

Characteristic equation: $\lambda^2 + 2\lambda + 1 = 0$

Double Roots: $\lambda = -1, -1$ ✓

$$y_h = (c_1 + c_2 x)e^{-x}$$

$r(x) = e^{-x}$ $\quad\quad y_p = C e^{-x}$

Page 26 Mod Rule

$$y_p = Ax^2 e^{-x}$$

$$y_p' = 2Axe^{-x} - Ax^2e^{-x}$$

$$y_p'' = 2Ae^{-x} - 2Axe^{-x} - 2Axe^{-x} + Ax^2e^{-x}$$

Substituting in: $y'' + 2y' + y = e^{-x}$

$$2Ae^{-x} - 4Axe^{-x} + Ax^2e^{-x} + 2(2Axe^{-x} - Ax^2e^{-x}) + Ax^2e^{-x} = e^{-x}$$

$$\Rightarrow 2Ae^{-x} = e^{-x} \quad \Rightarrow 2A = 1 \quad \Rightarrow A = \frac{1}{2}$$

General Solution: $y(x) = (c_1 + c_2 x)e^{-x} + \frac{1}{2}x^2e^{-x}$

The particular solution

Initial Condition: $y(0) = -1,\ y'(0) = 1$

$$y(x) = (c_1 + c_2 x)e^{-x} + \frac{1}{2}x^2 e^{-x}$$

$$y(0) = -1 \qquad \Rightarrow c_1 = -1$$

$$y'(x) = -(c_1 + c_2 x)e^{-x} - \frac{1}{2}x^2 e^{-x} + c_2 e^{-x} + x e^{-x}$$

$$y'(0) = 1 \quad \Rightarrow -c_1 + c_2 = 1 \quad \Rightarrow c_2 = 0$$

$$y(x) = -e^{-x} + \frac{1}{2}x^2 e^{-x}$$

EXAMPLE 3

Find the general solution of

$$y'' - 3y' - 4y = 3e^{2x} + 2\sin x - 8e^{-x}$$

SOLUTION

Corresponding homogeneous equation:

$$y'' - 3y' - 4y = 0$$

Characteristic equation: $\lambda^2 - 3\lambda - 4 = 0$

Roots: $\lambda = -1,\ \lambda = 4$

$$y_h = c_1 e^{-x} + c_2 e^{4x}$$

$$r(x) = 3e^{2x} + 2\sin x - 8e^{-x}$$

The choice of y_p :

$$y_p = Ae^{2x} + B_1\sin x + B_2\cos x + D\,x\,e^{-x}$$

$$\Rightarrow\ y_p' = 2Ae^{2x} + B_1\cos x - B_2\sin x - Dxe^{-x} + De^{-x}$$

$$y_p'' = 4Ae^{2x} - B_1\sin x - B_2\cos x + Dxe^{-x} - De^{-x} - De^{-x}$$

Substituting: $y'' - 3y' - 4y = 3e^{2x} + 2\sin x - 8e^{-x}$

$$-6Ae^{2x} - (5B_1 - 3B_2)\sin x - (5B_2 + 3B_1)\cos x - 5De^{-x}$$

$$= 3e^{2x} + 2\sin x - 8e^{-x}$$

Equating the coefficients: $-6A = 3 \Rightarrow A = -1/2$

$$5B_1 - 3B_2 = -2,\ 5B_2 + 3B_1 = 0 \Rightarrow B_1 = -5/17,\ B_2 = 3/17$$

$$5D = 8 \quad \Rightarrow D = 8/5$$

$$y_p = -\frac{1}{2}e^{2x} - \frac{5}{17}\sin x + \frac{3}{17}\cos x + \frac{8}{5}xe^{-x}$$

General Solution:

$$y(x) = c_1e^{-x} + c_2e^{4x} - \frac{1}{2}e^{2x} - \frac{5}{17}\sin x + \frac{3}{17}\cos x + \frac{8}{5}xe^{-x}$$

METHOD OF VARIATION OF PARAMETERS

- No special assumption on p (x), q (x) and r (x)

Non Homogeneous Equation:

$$y'' + p(x)\, y' + q(x)\, y = r(x)$$

with p(x),q(x) and r(x) being arbitrary and continuous functions on I.

associated homogeneous equation:

$$y'' + p(x)\, y' + q(x)\, y = 0$$

The basis of solution: $\{y_1, y_2\}$

$$y_h = c_1 y_1 + c_2 y_2$$

$$y'' + p(x)\, y' + q(x)\, y = r(x)$$

Particular solution

$$y_p = u_1(x) y_1(x) + u_2(x) y_2(x)$$

Substitute y_p, y_p', and y_p''

we get the system

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1' y_1' + u_2' y_2' = r(x)$$

$$y'' + p(x)\,y' + q(x)\,y = r(x)$$

Particular solution

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$

Substitute y_p, y_p', and y_p''

we get the system

$$u_1'y_1 + u_2'y_2 = 0$$

$$u_1'y_1' + u_2'y_2' = r(x)$$

Solution: $$u_1(x) = -\int \frac{y_2(x)r(x)}{W(y_1, y_2)}dx$$

$$u_2(x) = \int \frac{y_1(x)r(x)}{W(y_1, y_2)}dx$$

$W(y_1,y_2)$ is the Wronskian of fundamental solution y_1 and y_2.

$$y_p = -y_1(x)\int \frac{y_2(x)r(x)}{W(y_1, y_2)}dx + y_2(x)\int \frac{y_1(x)r(x)}{W(y_1, y_2)}dx$$

EXAMPLE

Find the general solution of

$$y'' - 4y' + 4y = \frac{e^{2x}}{x}$$

SOLUTION

The associated homogeneous equation:

$$y'' - 4y' + 4y = 0$$

Characteristic equation: $\lambda^2 - 4\lambda + 4 = 0$

Double roots: $\lambda = 2, 2$

Basis of solution: $y_1(x) = e^{2x}$, $y_2(x) = xe^{2x}$

$$W(y_1, y_2) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = e^{4x} \neq 0$$

$$y_h = c_1 e^{2x} + c_2 x e^{2x}$$

and for y_p $\qquad r(x) = \dfrac{e^{2x}}{x}$

$$u_1 = -\int \frac{y_2(x) r(x)}{W(y_1, y_2)} dx = -\int \frac{xe^{2x} \cdot \frac{e^{2x}}{x}}{e^{4x}} dx = -x$$

$$u_2 = \int \frac{y_1(x) r(x)}{W(y_1, y_2)} dx = \int \frac{e^{2x} \cdot \frac{e^{2x}}{x}}{e^{4x}} dx = \ln x$$

$$y_p = u_1(x) y_1(x) + u_2(x) y_2(x)$$

$$y_p = -xe^{2x} + \ln(x) xe^{2x}$$

General solution:

$$y(x) = c_1 e^{2x} + c_2 x e^{2x} - xe^{2x} + x \ln x e^{2x}$$

$$y(x) = (c_1 + (c_2 - 1 + \ln x)\, x) e^{2x}$$

Chapter 3

Higher Order Linear Differential Equations

Differential equations of order n (>1)

$$F(x,y,y',\ldots\ldots,y^{(n)})=0 \qquad y^{(n)}=\frac{d^n y}{dx^n}$$

n^{th} order linear differential equation

Standard form

$$y^{(n)}+p_{n-1}(x)y^{(n-1)}+\ldots\ldots+p_1(x)y'+p_0(x)y=r(x)$$

Non homogeneous

$$y^{(n)}+p_{n-1}(x)y^{(n-1)}+\ldots\ldots+p_1(x)y'+p_0(x)y=0$$

Homogeneous

Solution

A solution of n^{th} order differential equation on some open interval I is function y(x), which is differentiable n times on I and the function and its derivatives satisfy the equation.

Superposition Principle Or Linearity Principle

For the homogeneous linear differential equation

$$y^{(n)}+p_{n-1}(x)y^{(n-1)}+\ldots\ldots+p_1(x)y'+p_0(x)y=0$$

the sums and constant multiples of solutions on some open interval I are again the solution of this, in other words, The linear combination of solutions of homogeneous linear equation

$$y^{(n)}+p_{n-1}(x)y^{(n-1)}+\ldots\ldots+p_1(x)y'+p_0(x)y=0$$

are also solution of the same equation.

The Basis, General Solution And Particular Solution

$$y^{(n)}+p_{n-1}(x)y^{(n-1)}+\ldots\ldots+p_1(x)y'+p_0(x)y=0$$

General solution:

$$y(x)=c_1y_1+c_2y_2+\ldots.+c_ny_n$$

y_1, y_2, ...y_n are linearly independent solutions.

The basis or fundamental system of solutions

Particular solution:

specific values to constants c_1, c_2, ..., c_n.

Example

The functions $y_1 = x$, $y_2 = 3x$, $y_3 = x^2$ are linearly dependent on any interval I.

Solution

$c_1y_1+c_2y_2+\ldots.+c_ny_n$ Linear combination

$c_1y_1+c_2y_2+\ldots.+c_ny_n=0$

Some non zero constants c_1, c_2, ...c_n

functions y_1,.......y_n are linearly dependent

Let, $c_1 = 3$, $c_2 = -1$, $c_3 = 0$

$3y_1 - 1.y_2 + 0.\ y_3 = 3x - 3x = 0\ \forall\ x$ in I

$\therefore$ y_1, y_2 and y_3 are linearly dependent

Linearly Independent Function

The n functions $y_1, y_2, \ldots, y_n$ are linearly independent if there does not exist any non zero constants such that the linear combination $c_1y_1+c_2y_2+\ldots+c_ny_n=0$

Or

$$c_1y_1+c_2y_2+\ldots+c_ny_n=0$$

If and only if $$c_1=c_2=\ldots\ldots=c_n=0$$

WORNSKIAN

The Wronskian for n functions $y_1, y_2, \ldots\ldots y_n$

$$W(y_1,\ldots\ldots y_n)=\begin{vmatrix} y_1 & y_2 & - & - & y_n \\ y_1' & y_2' & - & - & y_n' \\ - & - & - & - & - \\ - & - & - & - & - \\ y_1^{(n-1)} & y_2^{(n-1)} & - & - & y_n^{(n-1)} \end{vmatrix}$$

$W(y_1,\ldots\ldots y_n)\neq 0$ Linearly independent

$W(y_1,\ldots\ldots y_n)=0$ Linearly dependent

Example

$$y_1=x,\quad y_2=3x,\quad y_3=x^2$$

$$\therefore\ W=\begin{vmatrix} x & 3x & x^2 \\ 1 & 3 & 2x \\ 0 & 0 & 2 \end{vmatrix}=2(3x-3x)=0,\quad \forall x$$

$\therefore$ y_1, y_2 and y_3 are linearly dependent

Example

Find the solution of $y^{(4)} - 5y'' + 4y = 0$

Solution

$$y = e^{\lambda x} \qquad \Rightarrow y' = \lambda e^{\lambda x}, \qquad y'' = \lambda^2 e^{\lambda x},$$

$$y^{(3)} = \lambda^3 e^{\lambda x}, \qquad y^{(4)} = \lambda^4 e^{\lambda x}$$

Substitution: $\lambda^4 e^{\lambda x} - 5\lambda^2 e^{\lambda x} + 4e^{\lambda x} = 0$

$$\Rightarrow (\lambda^4 - 5\lambda^2 + 4)e^{\lambda x} = 0 \Rightarrow \lambda^4 - 5\lambda^2 + 4 = 0 \quad \because e^{\lambda x} \neq 0, \ \forall x$$

Characteristic Equation   **Factorization**

$$\lambda^4 - 4\lambda^2 - \lambda^2 + 4 = 0 \quad \Rightarrow \lambda^2(\lambda^2 - 4) - (\lambda^2 - 4) = 0$$

$$\Rightarrow (\lambda^2 - 1)(\lambda^2 - 4) = 0 \quad \Rightarrow (\lambda - 1)(\lambda + 1)(\lambda - 2)(\lambda + 2) = 0$$

Roots: $\lambda = 1, -1, 2, -2$

Solutions: $e^{x}, e^{-x}, e^{2x}, e^{-2x}$

$$W = \begin{vmatrix} e^{x} & e^{-x} & e^{2x} & e^{-2x} \\ e^{x} & -e^{-x} & 2e^{2x} & -2e^{-2x} \\ e^{x} & e^{-x} & 4e^{2x} & 4e^{-2x} \\ e^{x} & -e^{-x} & 8e^{2x} & -8e^{-2x} \end{vmatrix} \neq 0. \qquad (144) \neq 0.$$

Basis: $y_1 = e^{x}, y_2 = e^{-x}, y_3 = e^{2x}, y_4 = e^{-2x}$

General solution

$$y = c_1 e^{x} + c_2 e^{-x} + c_3 e^{2x} + c_4 e^{-2x}$$

Initial Value Problem

The differential equation:

$$y^{(n)}+p_{n-1}(x)y^{(n-1)}+\ldots\ldots+p_1(x)y'+p_0(x)y=0$$

and n initial conditions:

$$y(x_0)=k_0,\ y'(x_0)=k_1,\ \ldots\ldots\ldots,\ y^{(n-1)}(x_0)=k_{n-1}$$

Existence And Uniqueness Theorem For Initial Value Problem

If $p_0(x)$, $p_1(x)$, ..., $p_{n-1}(x)$ are continuous functions on some open interval I and $x_0 \in$ I, then the initial value problem

$$y^{(n)}+p_{n-1}(x)y^{(n-1)}+\ldots\ldots+p_1(x)y'+p_0(x)y=0$$

and $y(x_0)=k_0$, $y'(x_0)=k_1$, ..., $y^{(n-1)}(x_0)=k_{n-1}$ has a unique solution y(x) on I.

Existence of A General Solution

Theorem:

Let the coefficients $p_0(x)$, $p_1(x)$, ..., $p_{n-1}(x)$ in

$$y^{(n)}+p_{n-1}(x)y^{(n-1)}+\ldots\ldots+p_1(x)y'+p_0(x)y=0$$

are continuous on some interval I, then every solution y = y (x) of this equation is of the form $y(x)=c_1y_1+c_2y_2+\ldots+c_ny_n$

y_1, y_2, ...y_n are basis of solutions and c_1, c_2, ...c_n are arbitrary constants.

General solution includes all solutions.

Higher Order Homogenous Linear Equation With Constant Coefficients

nth order homogeneous linear equation:

$$y^{(n)}+a_{n-1}y^{(n-1)}+......+a_1y'+a_0y=0$$

Constants

$$y=e^{\lambda x} \Rightarrow y'=\lambda e^{\lambda x},\ y''=\lambda^2 e^{\lambda x},......,\ y^{(n)}=\lambda^n e^{\lambda x}$$

Substitution:

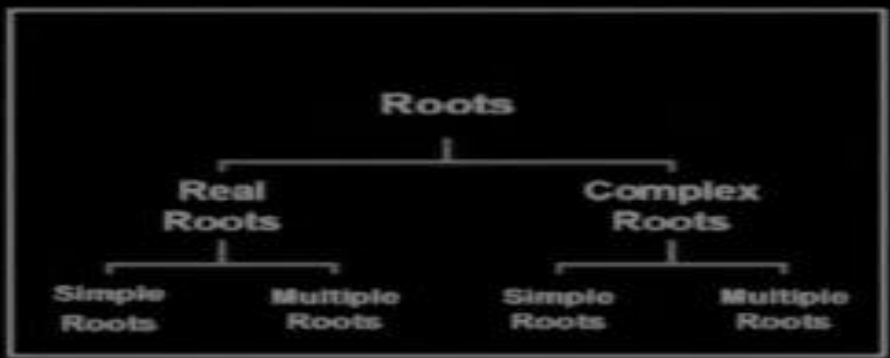
$$\lambda^n e^{\lambda x}+a_{n-1}\lambda^{n-1}e^{\lambda x}+......+a_1\lambda e^{\lambda x}+a_0e^{\lambda x}=0$$

$$\Rightarrow (\lambda^n+a_{n-1}\lambda^{n-1}+......+a_1\lambda+a_0)e^{\lambda x}=0$$

Characteristic Equation:

$$\lambda^n+a_{n-1}\lambda^{n-1}+......+a_1\lambda+a_0=0$$

n Roots



Four Cases:

1. Simple Real roots
2. Simple complex roots
3. Multiple Real roots
4. Multiple complex roots

Case 1: Simple Real Roots

Characteristic equation

$$\lambda^n + a_{n-1}\lambda^{n-1} + \dots\dots + a_1\lambda + a_0 = 0$$

$\lambda_1, \lambda_2, \dots\dots \lambda_n$ real and distinct

n solutions:

$$y_1 = e^{\lambda_1 x}, \quad y_2 = e^{\lambda_2 x}, \dots\dots, \quad y_n = e^{\lambda_n x}$$

linearly independent

General Solution:

$$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \dots\dots\dots + c_n e^{\lambda_n x}$$

Linear Independence

$$W = \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} & . & . & e^{\lambda_n x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} & . & . & \lambda_n e^{\lambda_n x} \\ . & . & . & . & . \\ . & . & . & . & . \\ \lambda_1^{n-1} e^{\lambda_1 x} & \lambda_2^{n-1} e^{\lambda_2 x} & . & . & \lambda_n^{n-1} e^{\lambda_n x} \end{vmatrix}$$

$$= e^{(\lambda_1+\lambda_2+\dots+\lambda_n)x} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{vmatrix} \neq 0$$

$$= e^{(\lambda_1+\lambda_2+\dots+\lambda_n)x}(-1)^{n(n-1)/2}\, V$$

$$V = \prod_{i<j}(\lambda_i - \lambda_j)$$

Case 2: Simple Complex Roots

Characteristic equation

$$\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0$$

conjugate pair

$$\lambda_1 = \alpha + i\beta, \quad \lambda_2 = \alpha - i\beta,$$

Corresponding linearly independent solution

$$y_1 = e^{\alpha x}\cos\beta x, \qquad y_2 = e^{\alpha x}\sin\beta x$$

Case 3: Multiple Real Roots

Characteristic equation

$$\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0$$

Let λ^* be a roots with multiplicity k

k Linearly independent solutions:

$$y_1 = e^{\lambda^* x}, \quad y_2 = xe^{\lambda^* x}, \dots\dots, \quad y_k = x^{k-1}e^{\lambda^* x}$$

Case 4: Multiple Complex Roots

Characteristic equation

$$\lambda^n + a_{n-1}\lambda^{n-1} + \dots\dots + a_1\lambda + a_0 = 0$$

Let, $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$

multiplicity k

2k Linearly independent solutions:

$$y_1 = e^{\alpha x}\cos\beta x,\ y_2 = xe^{\alpha x}\cos\beta x, \dots\dots, y_k = x^{k-1}e^{\alpha x}\cos\beta x,$$

$$y_{k+1} = e^{\alpha x}\sin\beta x,\ y_{k+2} = xe^{\alpha x}\sin\beta x, \dots\dots, y_{2k} = x^{k-1}e^{\alpha x}\sin\beta x$$

n Linearly independent solutions:

$$y_1, y_2, \dots\dots\dots, y_n$$

General Solution:

$$y(x) = c_1y_1 + c_2y_2 + \dots\dots\dots + c_ny_n$$

Examples

Example 1: Solve $y^{(4)} - 16y = 0$

Solution

Characteristic Equation: $\lambda^4 - 16 = 0$

Factorization: $(\lambda-2)(\lambda+2)(\lambda^2+4) = 0$

Roots: $\lambda = -2, 2, -2i, 2i$

4 Linearly Independent Solutions:

$$y_1 = e^{-2x},\quad y_2 = e^{2x},\quad y_3 = \cos 2x,\quad y_4 = \sin 2x$$

General Solution:

$$y(x) = c_1e^{-2x} + c_2e^{2x} + c_3\cos 2x + c_4\sin 2x$$

WORNSKIAN

$$W = \begin{vmatrix} e^{-2x} & e^{2x} & \cos 2x & \sin 2x \\ -2e^{-2x} & 2e^{2x} & -2\sin 2x & 2\cos 2x \\ 4e^{-2x} & 4e^{2x} & -4\cos 2x & -4\sin 2x \\ -8e^{-2x} & 8e^{2x} & 8\sin 2x & -8\cos 2x \end{vmatrix}$$

$$= 1024 \cos 2x \sin 2x$$

Example: 2

Solve the initial value problem

$$y''' - y'' - y' + y = 0, \quad y(0) = 2,\ y'(0) = 1,\ y''(0) = 0$$

Solution

Characteristic Equation: $\lambda^3 - \lambda^2 - \lambda + 1 = 0$

Factorization: $\lambda^3 - \lambda^2 - \lambda + 1 = \lambda^2(\lambda - 1) - (\lambda - 1)$

$= (\lambda^2 - 1)(\lambda - 1) \quad = (\lambda - 1)^2(\lambda + 1)$

Roots: $\lambda = -1, 1, 1$ 3 Independent Solutions:

$y_1 = e^{-x}, \quad y_2 = e^{x}, \quad y_3 = xe^{x}$

General Solution: $y(x) = c_1 e^{-x} + c_2 e^{x} + c_3 x e^{x}$

WORNSKIAN

$$W = \begin{vmatrix} e^{-x} & e^{x} & xe^{x} \\ -e^{-x} & e^{x} & xe^{x}+e^{x} \\ e^{-x} & e^{x} & xe^{x}+2e^{x} \end{vmatrix}$$

$$= e^{x}\begin{vmatrix} 1 & 1 & x \\ -1 & 1 & x+1 \\ 1 & 1 & x+2 \end{vmatrix} = 4e^{x} \neq 0$$

Particular Solution:

Initial Conditions: $y(0) = 2,\ y'(0) = 1,\ y''(0) = 0$

General Solution: $y(x) = c_1e^{-x} + c_2e^{x} + c_3xe^{x}$

$$y'(x) = -c_1e^{-x} + c_2e^{x} + c_3xe^{x} + c_3e^{x}$$

$$y''(x) = c_1e^{-x} + c_2e^{x} + c_3xe^{x} + c_3e^{x} + c_3e^{x}$$

$y(0) = c_1 + c_2 = 2,\quad y'(0) = -c_1 + c_2 + c_3 = 1,$

$y''(0) = c_1 + c_2 + 2c_3 = 0 \quad \Rightarrow \quad c_1 = 0, c_2 = 2, c_3 = -1$

Solution of IVP: $y(x) = 2e^{x} - xe^{x} \quad = (2-x)e^{x}$

Higher Order Non Homogeneous Linear Equations

Higher Order Non Homogeneous Linear Differential Equations

The standard form

$$y^{(n)}+p_{n-1}(x)y^{(n-1)}+\ldots\ldots+p_1(x)y'+p_0(x)y=r(x)$$

corresponding homogeneous equation

$$y^{(n)}+p_{n-1}(x)y^{(n-1)}+\ldots\ldots+p_1(x)y'+p_0(x)y=0$$

General Solution: $y = y_h + y_p$

EXISTENCE AND UNIQUENESS OF SOLUTION

$$y^{(n)}+p_{n-1}(x)y^{(n-1)}+\ldots\ldots+p_1(x)y'+p_0(x)y=r(x)$$

if the coefficients $p_k(x)$, $k = 0,\ldots\ldots n-1$ and $r(x)$ are continuous on I, then general solution exists and includes all solutions.

Initial Value Problem (IVP):

$$y^{(n)}+p_{n-1}(x)y^{(n-1)}+\ldots\ldots+p_1(x)y'+p_0(x)y=r(x)$$

n initial conditions:

$$y(x_0) = k_0,\ \ y'(x_0)= k_1,\ \ldots\ldots,\ y^{(n-1)}(x_0) = k_{n-1},\ \ x_0 \in I$$

IVP has unique solution.

Methods to find y_p, the particular solution of

$$y^{(n)}+p_{n-1}(x)y^{(n-1)}+\ldots\ldots+p_1(x)y'+p_0(x)y=r(x)$$

- Method of undetermined coefficients
- Method of Variation of Parameters

METHOD OF UNDETERMINED COEFFICIENTS

n^{th} order linear equation has constant coefficients and r (x) is of special form

- r (x) = e^{ax} or polynomial
- r (x) = cos (bx), sin (bx)

METHOD OF UNDETERMINED COEFFICIENT

Terms in r(x)	Choice of y_p
$ke^{\gamma x}$	$C e^{\gamma x}$
$kx^n (n=0,1,\ldots)$	$K_n x^n + K_{n-1}x^{n-1}+\ldots K_1 x+K_0$
$k\cos\omega x$ $k\sin\omega x$	$A\sin\omega x + B\cos\omega x$
$ke^{\alpha x}\cos\omega x$ $ke^{\alpha x}\sin\omega x$	$e^{\alpha x}(A\sin\omega x + B\cos\omega x)$

RULES FOR THE METHOD OF UNDETERMINED CO-EFFICIENTS

BASIC RULE

$$y^{(n)}+p_{n-1}(x)y^{(n-1)}+\ldots\ldots+p_1(x)y'+p_0(x)y=r(x)$$

- If r(x) is one of the function in the first column of the table .
- Choose corresponding function y_p in second column
- Find the value of undetermined coefficient by putting y_p and its derivatives in

$$y^{(n)}+p_{n-1}(x)y^{(n-1)}+\ldots\ldots+p_1(x)y'+p_0(x)y=r(x)$$

MODIFICATION RULE

- If any term in the choice for y_p in also in the solution y_h of corresponding homogeneous equation of

$$y^{(n)}+p_{n-1}(x)y^{(n-1)}+\ldots\ldots+p_1(x)y'+p_0(x)y=r(x)$$

- Multiply the choice y_p by x^k, where k is the smallest positive integer such that no terms of $x^k\, y_p(x)$ is a solution of

$$y^{(n)}+p_{n-1}(x)y^{(n-1)}+\ldots\ldots+p_1(x)y'+p_0(x)y=0$$

SUM RULE

- If r(x) is sum of different functions given in column one of the table, then choose y_p the sum of corresponding functions in second column.

Example

Solve the initial value problem

$$y^{(3)}+3y''+3y'+y=30e^{-x}, \quad y(0)=3, y'(0)=-3, y''(0)=-47$$

Solution

Corresponding Homogeneous Equation:

$$y^{(3)}+3y''+3y'+y=0$$

Characteristic Equation: $\lambda^3+3\lambda^2+3\lambda+1=0$

Roots: $\lambda=-1, -1, -1$ triple roots

Three Linearly Independent Solutions:

$$y_1=e^{-x}, \quad y_2=xe^{-x}, \quad y_3=x^2e^{-x}$$

$$y_h(x)=c_1e^{-x}+c_2xe^{-x}+c_3x^2e^{-x}$$

$$y_h(x)= c_1e^{-x}+c_2xe^{-x}+c_3x^2e^{-x}$$

Particular solution:

$$y_p = Ax^3e^{-x} \Rightarrow y_p'= A(3x^2-x^3)e^{-x},$$

$$y_p''= A(6x-3x^2-3x^2+x^3)e^{-x} = A(6x-6x^2+x^3)e^{-x}$$

$$y_p^{(3)}= A[6-12x+3x^2-(6x-6x^2+x^3)]e^{-x}$$
$$= A[6-18x+9x^2-x^3]e^{-x}$$

Substitution: $y^{(3)}+3y''+3y'+y=30e^{-x}$

$$A[6-18x+9x^2-x^3]e^{-x}+3A(6x-6x^2+x^3)e^{-x}$$
$$+3A(3x^2-x^3)e^{-x}+Ax^3e^{-x}=30e^{-x}$$

$$\Rightarrow 6Ae^{-x}=30e^{-x} \Rightarrow 6A= 30 \Rightarrow A= 5 \Rightarrow y_p= 5x^3e^{-x}$$

General Solution: $y = y_h + y_p$

$$y(x)= c_1e^{-x}+c_2xe^{-x}+c_3x^2e^{-x}+5x^3e^{-x}$$

Solution of IVP

Initial Conditions: $y(0)=3, y'(0)=-3, y''(0)=-47$

$$y(x)= (c_1+c_2x+c_3x^2+5x^3)e^{-x}$$

$$y'(x)= -(c_1+c_2x+c_3x^2+5x^3)e^{-x}+(c_2+2c_3x+15x^2)e^{-x}$$

$$y''(x)= -((c_2-c_1)+(2c_3-c_2)x+(15-c_3)x^2-5x^3)e^{-x}$$
$$+((2c_3-c_2)+2(15-c_3)x-15x^2)e^{-x}$$

$$\Rightarrow y(0)= c_1= 3 \quad y'(0)= -c_1+c_2= -3 \Rightarrow c_2=0$$

$$y''(0)= 2c_3-2c_2+c_1= -47 \Rightarrow 2c_3+c_1= -47 \Rightarrow c_3= -25$$

$$\Rightarrow c_1= 3, c_2= 0, c_3= -25$$

Solution of IVP: $y(x)= (3-25x^2+5x^3)e^{-x}$

METHOD OF VARIATION OF PARAMETERS

Method of Variation of Parameters

$$y^{(n)}+p_{n-1}(x)y^{(n-1)}+\ldots\ldots+p_1(x)y'+p_0(x)y=r(x)$$

Basis of solutions: $y_1, \ldots\ldots, y_n$

Associated homogeneous equation

$$y^{(n)}+p_{n-1}(x)y^{(n-1)}+\ldots\ldots+p_1(x)y'+p_0(x)y=0$$

The particular solution

$$y_p(x)=u_1(x)y_1(x)+u_2(x)y_2(x)+\ldots\ldots+u_n(x)y_n(x)$$

$$u_1'(x)y_1(x)+u_2'(x)y_2(x)+\ldots\ldots+u_n'(x)y_n(x)=0$$

$$u_1'(x)y_1'(x)+u_2'(x)y_2'(x)+\ldots\ldots+u_n'(x)y_n'(x)=0$$

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$$u_1'(x)y_1^{(n-1)}(x)+u_2'(x)y_2^{(n-1)}(x)+\ldots+u_n'(x)y_n^{(n-1)}(x)=0$$

The determinant of this system is Wronskian of $y_1, \ldots\ldots, y_n$

Solution: $u_i'(x) = \frac{W_i(x)}{W(x)} r(x), \quad i = 1, \ldots\ldots, n$

W (x) = Wronskian

$W_i(x)$ = the determinant obtained from W (x) by replacing i^{th} column to (0,……., 0, 1)

$$y_p = y_1(x)\int \frac{W_1(x)}{W(x)} r(x)dx + y_2(x)\int \frac{W_2(x)}{W(x)} r(x)dx + \ldots$$

$$\ldots + y_n(x)\int \frac{W_n(x)}{W(x)} r(x)dx$$

$$\therefore y_p(x) = \sum_{i=1}^{n} y_i(x)\int \frac{W_i(x)}{W(x)} r(x)dx$$

Example

Find the general Solution of

$$y''' + y' = \tan x, \quad -\pi/2 < x < \pi/2$$

Solution

Given differential equation: $y''' + y' = \tan x$

Associated homogeneous equation: $y''' + y' = 0$

Characteristic Equation: $\lambda^3 + \lambda = 0$

Factorization: $\lambda(\lambda^2 + 1) = 0$ Roots: $\lambda = 0, \lambda = \pm i$

Fundamental System of Solution:

$$y_1 = 1,\ y_2 = \cos x,\ y_3 = \sin x$$

Hence $y_h(x) = c_1 + c_2 \cos x + c_3 \sin x$

Particular solution y_p

Method of variation of parameters

$$\therefore W = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = \sin^2 x + \cos^2 x = 1$$

$$W_1 = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 1 & -\cos x & -\sin x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$W_2 = \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & 1 & -\sin x \end{vmatrix} = -\cos x$$

$$W_3 = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & 1 \end{vmatrix} = -\sin x$$

$W(x)=1 \quad W_1(x)=1 \quad W_2(x)=-\cos x \quad W_3(x)=-\sin x$

$r(x)=\tan x$

Thus

$$\therefore u_1(x) = \int \frac{W_1(x)}{W(x)} r(x)dx \quad = \int \tan x dx \quad = \ln(\sec x)$$

$$u_2(x) = \int \frac{W_2(x)}{W(x)} r(x)dx \quad = \int -\cos x \tan x \, dx$$

$$= -\int \sin x dx \quad = \cos x$$

$$u_3(x) = \int \frac{W_3(x)}{W(x)} r(x)dx \quad = \int -\sin x \tan x dx$$

$$= \cos x \tan x - \int \cos x \sec^2 x dx$$

$$= \sin x - \int \sec x dx \quad = \sin x - \ln(\sec x + \tan x)$$

$$u_1 = \ln(\sec x) \quad u_2 = \cos x \quad u_3 = \sin x - \ln(\sec x + \tan x)$$

$$\because y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x) + u_3(x) y_3(x)$$

$$u_1 = \ln(\sec x) \quad u_2 = \cos x \quad u_3 = \sin x - \ln(\sec x + \tan x)$$

$$y_1 = 1, \ y_2 = \cos x, \ y_3 = \sin x$$

$$\therefore y_p(x) = \ln(\sec x) + \cos^2 x + \sin^2 x - \sin x . \ln(\sec x + \tan x)$$

$$\therefore y_p(x) = \ln(\sec x) + 1 - \sin x . \ln(\sec x + \tan x)$$

General Solution: $y(x) = y_h + y_p$

$$y(x) = c_1^* + c_2 \cos x + c_3 \sin x + \ln(\sec x) - \sin x . \ln(\sec x + \tan x)$$

$$c_1^* = c_1 + 1$$