جامعة جنوب الوادى

كلية التربية بالغردقة

الفرقة الرابعة عام رياضيات Math

المادة : (pure 15)

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الفصل الدراسى الثانى 2022-2023

Chapter VI

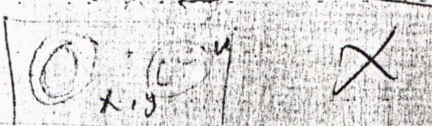
انفصال بديهية

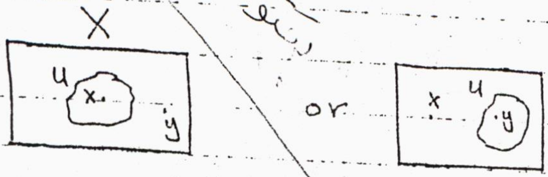
Separation Axioms

Many properties of a topological space depend upon the distributive of the open sets in the space. Every concept in topology is defined in terms of open sets, in order to make non-trivial and interesting statements about a space, it is necessary that the space possess a fairly rich collection of open sets. In this chapter we sall study various degrees of such richness. We shall defined a number of related conditions all of which assert the existence of open sets which will contain something but which will also exclude something else. For this reason, the conditions are known as separation axioms called T_i, $i = 0, 1, 2, 3, \dots$

1. T_0 - space:

Definition: A topological space X is said to be satisfy the T_0 - axiom, or is said to be a T_0 - space if given any two distinct points in X, there exists an open set which contains one of them but not the other. i.e. $\forall x, y \in X$, $\exists U \subset X$ an open set such that either $x \in U$, $y \in X - U$ or $y \in U$ and $x \in X - U$.





Example 1 Let $X=\{a,b\}$ and $\tau=\{X,\phi,\{a\}\}$ then (X,τ) is T_0, because $\{a\}$ is open set containing a, but $b \notin \{a\}$.

Example 2 Let $X=\{a,b,c\}$ and $\tau=\{X,\phi,\{a\},\{a,b\}\}$. Then (X,τ) is a T_0-space.

Example 3 Let $X=\{a,b,c\}$ and $\tau=\{X,\phi,\{a\},\{b,c\}\}$ then (X,τ) is not T_0, because for b, c in X there is open set $\{b,c\}$, $b,c\in\{b,c\}$.

Example 4 Every discrete space is T_0-space.

Example 5 Every indiscrete space containing more than one point is not T_0-space.

Example 6 Let τ be the topology on R whose members are ϕ, R and all sets of the form (a,∞) for $a\in R$. For $x,y\in R$ with $x<y$, there exists an open set containing y but not x (for example (x,∞)), but there no open set which contains x but not y i.e. (R,τ) is T_0-space. (T_0)

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<u>Proposition 1</u>. Every subspace of a T_0-space is T_0-space. [illegible]

~~proof~~ Left to the reader.

$(X, \tau) \to (A, \tau_A)$

<u>2. T_1-Space:</u> let $x, y \in A$, $x \neq y$

$\Rightarrow \exists\, U \in \tau$, $x \in U$ or $y \in U$

<u>Definition 2.</u> $\Rightarrow x \in U \cap A$ or $x \in M \cap A$

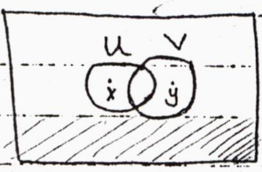
$M \in \tau_A$ [illegible]

A space X is said to be a T_1-space if for every two distinct points x and $y \in X$, there exists an open set containing x but not y (and hence also another open set containing y but not x)

i.e.

$\forall x, y \in X$, $x \neq y \Rightarrow \exists\; U, V \in \tau$ such that $x \in U$, $y \in X - U$ and $x \notin V$, $y \in V$.

X



It is clear that

$$T_1 \Rightarrow T_0$$

but the converse may be not true in general, as is illustrated by the following example:

<u>Example 1</u>. Let $X = \{a, b\}$ and $\tau = \{X, \phi, \{a\}\}$, then (X, τ) is a T_0-space but not T_1-space, since X is the only open set containing b, but also containing a.

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Remark. A space (R, τ) defined as above (in page 28) shows that is converse is false.

The following theorem gives a very simple characterization of T_1-spaces.

Theorem 1. Let X be a topological space. Then, the following are equivalent:

(1) The space X is a T_1-space.

(2) For every $x \in X$, the singleton $\{x\}$ is closed i.e. $\overline{\{x\}} = \{x\}$

(3) For every $x \in X$, $\{x\}' = \phi$.

Proof

(1) $\Rightarrow$ (2):

Let $x \in X$ be any point. We show that $\{x\}^c$ is open. Let $y \in \{x\}^c \Rightarrow x \neq y$. Since X is T_1-space, there exists an open set U_y containing y such that $x \notin U_y$. Hence $y \in U_y \subset \{x\}^c \Rightarrow \{x\}^c$ is a nbd of y $\forall y \in \{x\}^c$ $\Rightarrow \{x\}^c$ is open $\Rightarrow \{x\}$ is closed.

(2) $\Rightarrow$ (3): Let $x \in X$ be any point, by (2) $\{x\}$ is closed $\Rightarrow \overline{\{x\}} = \{x\}$. Since $\overline{\{x\}} = \{x\} \cup \{x\}' \xRightarrow{\text{ص ١٢ (٧)}}$ $\{x\} \cup \{x\}' = \{x\} \Rightarrow \{x\}' = \phi$ or $\{x\}' = \{x\}$, but $x \notin \{x\}' \Rightarrow \{x\}' = \phi$.

$(3) \Rightarrow (1)$:

Let $x, y \in X$, $x \neq y$. By (3) $\{x\}' = \phi$, $U_y \cap \{x\} = \phi$

$y \notin \{x\}' \Rightarrow \exists$ open set U_y containing y such that $x \notin U_y$. therefor X is T_1-space.

Remark: Since the finite unions of closed sets is closed. Then we have:

Theorem 2:

(1) Every finite T_1-space is a discrete

~~(2) Every metric~~ space is T_1-space

(3) The Cofinite space is a T_1-space

Proof

(1) Since every finite subset is closed. Hence every subset of X is open i.e. every subset of X is both open and closed, then X is a discrete space

~~(2)~~ Since every finite subset of a metric space is closed, then every metric space is T_1

(3) left to the reader. Let $x \in X$ and $\{x\}$

Theorem 3: The property of being a T_1-space is a topological property.

Definition 3 A property P of a topological space X is said to be hereditary if every subspace of X has also P.

Theorem 4 The property of being a T_1-space is hereditary.

Proof: Let (X, τ) be a T_1-space and (A, τ_A) be a subspace of (X, τ). We show that (A, τ_A) is also a T_1-space. Let $x, y \in A$ with $x \neq y$. Since X is T_1-space $\Rightarrow \exists$ open set U in X containing x and $y \notin U \Rightarrow V = A \cap U$ is open set in A containing x and $y \notin V \Rightarrow (A, \tau_A)$ is T_1-space.

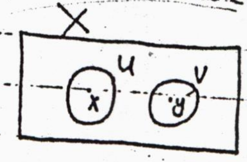
3. T_2-Space (Hausdorff):

Definition 4

A topological space X is said to be T_2-space (or a Hausdorff space) if for every distinct point $x, y \in X$ there exist disjoint open sets U, V in X such that $x \in U$ and $y \in V$

i.e.

$\forall\ x, y \in X,\ x \neq y \Rightarrow U, V \in \tau$ such that $x \in U$ and $y \in V$ $\boxed{\text{and } U \cap V = \phi}$



It is clear that:

$$T_2 \Rightarrow T_1 \Rightarrow T_0$$

but the converse is not true, in general, by the following example:

Example 1 An infinte Cofinite topological space X is T_1-space, but not T_2.

If $x, y \in X$, then $U_x = \{y\}^c$ contains x but not y and U_x is open set. Also $U_y = \{x\}^c$ is open set containing y but not x. Thus U_x and U_y are two open sets containing x and y respectively. Therefore X is a T_1-space. Now

$$U_x \cap U_y = \{y\}^c \cap \{x\}^c = \{x, y\}^c \neq \phi.$$

Therefore X is not T_2-space.

Or by the following:

Let $x, y \in X$ be any two points and U, V be any open sets containing x and y respectively such that $U \cap V = \phi \Rightarrow U \subset V^c$. Since X is cofinite $\Rightarrow V^c$ is finite $\Rightarrow U \not\subset V^c$. This contradicts with $U \cap V = \phi$.

Example 2 The usual topological space (R, τ) where τ generated by the closed interval $[a, b]$ is T_2-space. Let $a, b \in R$ with $a \neq b$, say $a < b$. Then for $G = (a-1, a]$ and $H = (a, b]$ we have $a \in G$, $b \in H$.

and $G \cap H = \phi$

Hence (R, τ) is T_2-space.

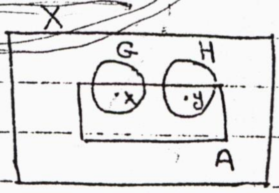
Example 3. Every discrete space is T_2-space, while an indiscrete space with more than one point can never be a T_2-space.

T_2-Space

Theorem 1. The Hausdorffness is a hereditary property

Proof Let (X, τ) be a Hausdorff space and (A, τ_A) be a subspace of (X, τ). Let $x, y \in A$ with $x \neq y$. Since X is Hausdorff $\Rightarrow \exists\, G, H \in \tau$ such that $x \in G$, $y \in H$ and $G \cap H = \phi$. Then $x \in G \cap A \in \tau_A$, $y \in H \cap A \in \tau_A$ and $(G \cap A) \cap (H \cap A) = A \cap (G \cap H) = A \cap \phi = \phi$.

$\Rightarrow (A, \tau_A)$ is a T_2-space.



Theorem 2 The Hausdorffness property is a topological property.

Proof. Let $f : X \longrightarrow Y$ be a homeomorphism mapping from a T_2-space X onto a space Y. We will show

that Y is also a T_2-space. For this purpose let $x, y \in Y$, $x \neq y$. Since f is surjective $\Rightarrow$ $f^{-1}(x) \neq f^{-1}(y)$. Since X is T_2-space $\Rightarrow \exists$ open sets U, V in X such that $f^{-1}(x) \in U$, $f^{-1}(y) \in V$ and $U \cap V = \phi$. Since f is bijective and open $\Rightarrow$ $f(U)$, $f(V)$ are open sets in Y such that $x \in f(U)$, $y \in f(V)$ and
$f(U) \cap f(V) = f(U \cap V) = f(\phi) = \phi$.
Therefore Y is a T_2-space.

Theorem 3: Every metric space is a T_2-space.

Proof Let $x, y \in X$ with $x \neq y$. Then by the definition of a metric space $d(x, y) = \epsilon > 0$. Consider the open spheres $G = S(x, \frac{1}{3}\epsilon)$ and $H = S(y, \frac{1}{3}\epsilon)$ with centered at x and y. We show that $G \cap H = \phi$. If $p \in G \cap H$, then $d(x, p) < \frac{1}{3}\epsilon$ and $d(y, p) < \frac{1}{3}\epsilon$. By the triangle inequity we have.

$$d(x, y) \leq d(x, p) + d(p, y) < \frac{1}{3}\epsilon + \frac{1}{3}\epsilon = \frac{2}{3}\epsilon$$

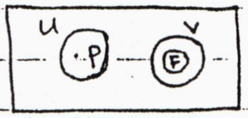
But this contradicts with $d(x, y) < \epsilon$. Hence H and G are disjoint open sets. Therefore X is T_2-space.

proof

4. Regular space: (T_3)

Definition 5: A space X is said to be regular if for each closed set F in X and a point $p \notin F$, there exist disjoint open sets U and V in X such that $p \in U$, $F \subseteq V$.

i.e.



$\forall$ F closed and $p \in F^c \Longrightarrow$

$\exists$ $U, V \in \tau$ such that $p \in U$, $F \subseteq V$ and $\boxed{U \cap V} = \phi$.

Definition 6 A regular T_1-space is called a T_3-space

i.e.

$$T_3 = T_1 + \text{regular}$$

ليس شرط

A regular space need not be a T_1-space as is illustrated by the following example:

Example 1: Let $X = \{a, b, c\}$ with the topology
$\tau = \{X, \phi, \{a\}, \{b, c\}\}$

It obvious that the collection of all closed sets in X is:

$\mathcal{F} = \{\phi, X, \{b, c\}, \{a\}\}$ [illegible]

Then (X, τ) is regular space, but not T_1-space, because $b, c \in X$, $b \neq c$ and there is no open set which containing one of them.

Example 2 Every indiscrete space X has at least two points is regular, but not T_1-space.

 $T_3 = T_1 + \text{Regular}$

الدرس - $\boxed{11}$

Example 3 An infinite cofinite space is T_1, but not regular.

Theorem 1 $\boxed{\text{Every}}$ T_3-space is Hausdorff

i.e.

$T_3 \Rightarrow T_2$

$T_3 \Rightarrow T_2$

Proof Let $x, y \in X$, $\boxed{x \neq y}$. Since X is $T_3 \Rightarrow X$ is T_1-space $\Longrightarrow \{x\}$ is closed, $y \notin \{x\}$. Since X is regular $\Longrightarrow \exists$ open sets U, V such that $\{x\} \subset U$, $y \in V$ and $U \cap V = \phi$. Hence X is a T_2-space.

Theorem 2 Regularity is a hereditary property

Proof Suppose (X, τ) is a regular space and (A, τ_A) is a subspace of (X, τ). Let $y \in A$ and D be a closed subset of A not containing y. Then D is of the form $C \cap A$ where C is a closed set of X. Note that $y \notin C$ for other wise $y \in D$. Hence by regularity of X, there exist open sets U, V in X such that $y \in U$, $C \subset V$ and $U \cap V = \phi$. Let $G = U \cap A$, $H = V \cap A$. Then G and H are open in the relative topology on A. Also, $y \in G$, $D \subset H$ and $G \cap H = \phi$. Thus the space (A, τ_A) is regular.

Theorem 3 A topological space X is regular iff for every $x \in X$ and every open set U containing x, there exists an open set V containing x such that $x \in V \subset \overline{V} \subset U$.

Proof Let X be regular space and U be an open set containing x $\Rightarrow$ U^c is closed set not containing x. Since X is regular $\Rightarrow$ $\exists$ open sets V, W in X such that $x \in V$, $U^c \subset W$ and $V \cap W = \phi$

$W^c \subset U$ $\quad$ $U \supset W^c$

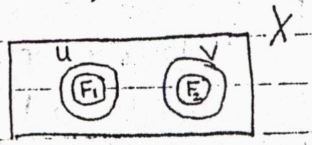
$\Rightarrow V \subset W^c$, $W^c \subset U \Rightarrow \overline{V} \subset \overline{W^c} = W^c \subset U$ $\Rightarrow V \subset W^c$

$\Rightarrow x \in V \subset \overline{V} \subset U$.

Conversely, let $x \in X$ and F be closed set in X such that $x \notin F$. Then $x \in F^c$ and F^c is open set, by hypothesis, $\exists$ open set V such that $x \in V \subset \overline{V} \subset F^c$. $\Rightarrow x \in V$, $F \subset \overline{V}^c$. Since $\overline{V}^c$ is an open set and $V \cap \overline{V}^c = \phi \Rightarrow X$ is regular space.

5. Normal space

Definition 7 A topological space is said to be normal if for every two disjoint closed sets F_1, F_2 in X, there exist two disjoint open sets U, V such that $F_1 \subset U$, $F_2 \subset V$.



Normal space

i.e

$\forall F_1, F_2 \in \mathcal{F} \Rightarrow \exists U, V \in \tau$ such that $F_1 \subset U$, $F_2 \subset V$, $U \cap V = \phi$, $F_1 \cap F_2 = \phi$

Normal space

closed $U^c \subset W$ = $\overline{U^c} \subset W$

closure $\quad$ $U \supset W^c$

Definition 8: A topological space is said to be T_4-space if it is normal and T_1
i.e.

T_4-space = normal + T_1

A normal space need not be regular space or T_1 (i.e. normal $\not\Rightarrow$ regular, T_1) as is illustrated by the following example:

Example 1 Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a,b\}\}$. The closed sets in X is:

$\mathcal{F} = \{\phi, X, \{b,c\}, \{a,c\}, \{c\}\}$. Closed

If F_1 and F_2 are disjoint closed subsets of X, then one of them, say F_1, must be the empty set ϕ. Hence X and ϕ are the only disjoint closed sets, $\phi \subset \phi$ and $F_2 \subset X$. Hence (X, τ) is normal. On other hand (X, τ) is not a T_1-space, since the singleton set $\{a\}$ is not closed. Furthemore (X, τ) is not a regular space, since $a \notin \{c\}$ and X is the only open set containing $\{c\}$ and a.

Theorem 1 T_4-space $\Rightarrow$ T_3-space

Proof Let X be a T_4-space, $p \in X$ and $F \subset X$ be a closed set such that $p \notin F$. Since X is T_4-space $\Rightarrow X$ is T_1-space $\Rightarrow \{p\}$ is closed set $\Rightarrow F, \{p\}$ two disjoint closed sets. Since X is normal $\Rightarrow \exists$ open

$F_1 = F \cap A$ closed

closed in X

sets U, V in X such that $F \subset U$, $\{P\} \subset V$ and $U \cap V = \phi$. Then X is regular since X is regular and T_1-space $\Rightarrow$ X is T_3-space.

The following example shows that the property of a being normal space is not hereditary in general.

Example 2 Let $X = \{a, b, c, d\}$ with the topology $\tau = \{X, \phi, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$. Then the closed sets in X are ϕ, X, $\{b,c,d\}$, $\{c,d\}$, $\{b,d\}$, $\{d\}$ $\Rightarrow$ X, ϕ the only disjoint closed sets $\Rightarrow$ **X is normal**. Now if $A = \{a,b,c\}$, then $\tau_A = \{A, \phi, \{a\}, \{a,b\}, \{a,c\}\}$. Then (A, τ_A) is not normal because $\{b\}$, $\{c\}$ are two disjoint closed sets in A has ~~no~~ disjoint nbds. But if A is closed, say $A = \{b,c,d\}$, in X, then $\tau_A = \{A, \phi, \{b\}, \{c\}, \{b,c\}\}$ and (A, τ_A) is normal.

~~$\{c,d\}$, $\{b,d\}$, $\{d\}$~~

Theorem 2 Every closed subspace of a normal space is normal.

Proof Let (A, τ_A) be a closed subspace of a normal space (X, τ) and F_1, F_2 are disjoint closed sets in A. Since A is closed in X, then F_1, F_2 are disjoint closed sets in X. But X is normal $\Rightarrow \exists\, U_1, U_2 \in \tau$ such that $F_1 \subset U_1$, $F_2 \subset U_2$ and $U_1 \cap U_2 = \phi$.

Now, $F_1 \subset U_1 \cap A = V_1$, $F_2 \subset U_2 \cap A = V_2$, V_1, $V_2 \in \tau_A$ and $V_1 \cap V_2 = (U_1 \cap A) \cap (U_2 \cap A) = (U_1 \cap U_2) \cap A = \phi \cap A = \phi$. Thus (A, τ_A) is normal.

<u>Theorem 3</u> A topological space is normal iff for every closed set F and open set U containing F, there exist an open set V such that $F \subset V \subset \bar{V} \subset U$.

<u>proof</u> Since $F \subset U \Rightarrow F$, U^c are disjoint closed sets. Since X is normal $\Rightarrow \exists$ two open sets V, W such that $F \subset V$, $U^c \subset W$ and $V \cap W = \phi \Rightarrow$ $F \subset V$, $W^c \subset U$, $V \subset W^c$ since W^c is closed $\Rightarrow F \subset V$, $W^c \subset U$, $\bar{V} \subset \overline{W^c} = W^c \Rightarrow F \subset V \subset \bar{V} \subset U$.

Conversely, let F_1, F_2 be two disjoint closed sets i.e. $F_1 \cap F_2 = \phi \Rightarrow F_1 \subset F_2^c$. Since F_2^c is an open set containing the closed set F_1, by hypothesis, $\exists$ open set V such that $F_1 \subset V \subset \bar{V} \subset F_2^c \Rightarrow F_1 \subset V$, $F_2 \subset \bar{V}^c$ since V, $\bar{V}^c$ two disjoint ~~closed~~ sets, then X is normal.

We conclude this Chapter by the following implications

$$T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$$

but the converse may be not true in general.

let f : is open mapping $A \subset X$ $\Rightarrow A^{\circ}$ is open in X $\Rightarrow f(A^{\circ})$ is open in Y

$f(A^{\circ}) = (f(A^{\circ}))^{\circ}$

sin $A^{\circ} \subset A \Rightarrow f(A^{\circ}) \subset f(A)$

$(f(A^{\circ}))^{\circ} \subset (f(A))^{\circ}$

$f(A^{\circ}) \subset f(A)^{\circ}$

Chapter VI

Compactness

1. Covers.

Definition! Let (X, τ) be a topological space and $\mathcal{G} = \{G_i \ i \in I\}$ be a class of subsets of X. Then

(1) $\mathcal{G}$ is said to be cover of X if:

$$X = \cup\{G_i : G_i \in \mathcal{G}\} = \bigcup_{G_i \in \mathcal{G}} G_i$$

(2) A cover of X is said to be open cover if $\mathcal{G} \subset \tau$ i.e. each member A_i of $\mathcal{G}$ is open

(3) $\mathcal{G}$ is called cover of a set $A \subset X$ if:

$$A \subset \cup\{A_i : A_i \in \mathcal{G}\}$$

(4) If there exists a finite number of $\mathcal{G}$ as $G_1, G_2, \dots, G_n$ such that $X = G_1 \cup G_2 \cup \dots \cup G_n$ or $A \subset G_1 \cup G_2 \cup \dots \cup G_n$, we say that $\mathcal{G}$ has a finite subcover.

(5) If $\mathcal{G}$ is a cover of X, then $\forall x \in X \ \exists G_i \in \mathcal{G}$ such that $x \in G_i$

-2-

2. Compact spaces:

Definition 2 A topological space (X, τ) is said to be *compact* if every open cover of X has a finite subcover.

That is, X is compact provided each open cover of X admits a finite subcover of X.

Example 1 Every finite topological space is compact.

Solution: Let X be any finite set and (X, τ) a topology on X, say $X = \{x_1, x_2, \dots, x_n\}$. If $\{G_i\}$ is an open cover of X, then each point in X belongs to one of members of $\{G_i\}$, say,

$$x_1 \in G_{i_1}, \; x_2 \in G_{i_2}, \; \dots, \; x_n \in G_{i_n}$$

Then

$$X = G_{i_1} \cup G_{i_2} \cup \dots \cup G_{i_n}$$

Example 2 Every indiscrete space is compact since $\{X\}$ is open coverd itself.

Example 3 Every infinite discrete space $(X, \mathcal{D})$ is not compact.

Solution. Consider a class $\mathcal{G} = \{\{p\} : p \in X\}$, then $\mathcal{G}$ is an open cover of X, because $X = \cup\{\{p\} : p \in X\}$ and $\{p\}$ is open $\forall\, p \in X$. $\mathcal{G}$ has not finite subcover.

Example 4 Every cofinite topological space is compact.

Solution Let $(X, \mathcal{C})$ be an cofinite topological space. If X is finite, then by example (1) X is compact. suppose that X is infinite and let $\mathcal{G} = \{G_i : G_i \in \mathcal{C}\}$ be an open cover of X. If $G_0 \in \mathcal{G} \Rightarrow G_0^c$ is finite i.e. $G_0^c = \{a_1, a_2, \dots, a_n\}$. Since $\mathcal{G}$ is a cover of $X \Rightarrow \exists\, G_{i_k} \in \mathcal{G}$ such that $a_k \in G_{i_k}$. Hence $G_0^c \subset G_{i_1} \cup G_{i_2} \cup \dots \cup G_{i_m} \Rightarrow$ $X = G_0 \cup G_0^c \subset G_0 \cup G_{i_1} \cup G_{i_2} \cup \dots \cup G_i \Rightarrow \mathcal{G}$ has a finite subcover of X. Therefore, X is compact.

Example 5 The usual topological space is not compact

Solution Let $\mathcal{G} = \{(-n, n) : n \in N\} = \{(-1,1), (-2,2), \dots\}$ $\Rightarrow R = \bigcup_{n=1}^{\infty} (-n, n) \Rightarrow \mathcal{G}$ is an open cover of R. observe that $\mathcal{G}$ has no finite subcover of R.

3. Finite intersection property (F.I.P):

Definition 3 A class $\mathcal{G}$ of subsets of X is called satisfy the finite intersection property if for any finite subclass $\{G_1, G_2, \dots, G_n\}$ of $\mathcal{G}$, the intersection

$$G_1 \cap G_2 \cap \dots \cap G_n \neq \phi$$

and $\mathcal{G}$ has a non-empty intersection if,

$$\cap \{G_i : G_i \in \mathcal{G}\} \neq \phi$$

Example 1 Consider the following class of open intervals:

$$\mathcal{G} = \{(0, \tfrac{1}{n}) : n \in N\} = \{(0,1), (0,\tfrac{1}{2}), (0,\tfrac{1}{3}), \dots\}$$

Then $\mathcal{G}$ has the finite intersection property, since

$$(0, a_1) \cap (0, a_2) \cap \cdots \cap (0, a_n) = (0, a)$$

where $a = \min\{a_1, a_2, \cdots, a_n\}$.

While $\mathcal{G}$ itself has the empty intersection i.e

$$\cap\{(0, \tfrac{1}{n}) : n \in N\} = \phi.$$

Example 2. Consider the following class

$$\mathcal{G} = \{(-\infty, z] : z \in Z\}$$
$$= \{\cdots (-\infty, -1], (-\infty, 0], (-\infty, 1], \cdots\}$$

Note that $\mathcal{G}$ has the empty intersection, because

$$\cap\{G_n : G_n \in \mathcal{G}\} = \phi,$$

but any finite subclass of $\mathcal{G}$ has the non-empty intersection i.e $\mathcal{G}$ has the finite intersection property, but has no non-empty intersection.

Theorem 1. A topological space X is compact iff every class $\mathcal{G}$ of closed sets satisfies the F.I.P., has the non-empty intersection.

Proof. Let X be a compact space and $\mathcal{G} = \{F_i : F_i$ is closed in $X\}$ be a class of closed subsets which has the F.I.P. Suppose that $\cap\{F_i : i \in I\} = \phi$. Then, $X = X - \bigcap_i F_i = \bigcup_i (X - F_i)$. Thus $\{F_i^c : i \in I\}$ is an open cover of X and must contain a finite subcover $\{F_{i_j}^c : j = 1, 2, \cdots, n\}$ of X, since X is compact. Hence $X = \bigcup_{j=1}^{n} F_{i_j}^c = X - \bigcap_{j=1}^{n} F_{i_j}$ which implies that

$$\bigcap_{j=1}^{n} F_{ij} = \phi$$

This contradicts that $\mathcal{G}$ has the F.I.P.

Therefore

$$\cap \{F_i : F_i \in \mathcal{G}\} \neq \phi$$

Conversely, suppose that any class of closed sets in X which has the F.I.P., has a non-empty intersection and let X is not compact. Then, $\exists$ open cover of X which has no finite subcover $\Rightarrow$ for every finite subcover $\{G_{i_1}, G_{i_2}, \dots, G_{i_n}\}$ of a cover $\mathcal{G} = \{G_i : i \in I\}$ we have

$$X \neq G_{i_1} \cup G_{i_2} \cup \dots \cup G_{i_n}$$

$$\Rightarrow X - \bigcup_{j=1}^{n} G_{ij} \neq \phi \quad \forall n \in N$$

$$\Rightarrow \bigcap_{j=1}^{n} (X - G_{ij}) \neq \phi \quad \forall n \in N$$

$\Rightarrow$ the class $\{X - G_i : G_i \in \mathcal{G}\}$ of closed sets of X has the F.I.P.. Then by hypothesis,

$$\cap \{X - G_i : G_i \in \mathcal{G}\} \neq \phi$$

$\Rightarrow$ $- \cup \{G_i : G_i \in \mathcal{G}\} \neq \phi$ and this contradicts that $\mathcal{G}$ is a cover of X. Therefore X is compact.

With the above terminology, we can now state the notion of compactness in terms of the closed sets as:

(i) X is compact

(ii) For any class $\{F_i\}$ of closed subsets of X with

$\cap F_i = \Phi$ implies $\{F_i\}$ contains a finite subclass $\{F_{i_1}, F_{i_2}, \ldots, F_{i_n}\}$ with $F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_n} = \Phi$

Corollary 1. A topological space X is compact iff for every class of subsets $\mathcal{A} = \{A_i : A_i \subset X\}$ of X which satisfying the F.I.P., the intersection

$$\cap \{\overline{A} : A \in \mathcal{A}\} \neq \Phi$$

4. Compact sets

Definition (4). A subset K of a topological space X is compact if every open cover $\mathcal{V}$ of K has a finite subcover

i.e. $\forall \mathcal{V}$, $K \subset \cup \mathcal{V} \Rightarrow \exists V_1, V_2, \ldots, V_n \in \mathcal{V}$ such that $K \subset V_1 \cup V_2 \cup \cdots \cup V_n$

Example 1. Every finite subset of a topological space is compact.

Example 2. show that $(0,1)$ is not compact.

Solution: Consider the usual topological space $(\mathbb{R}, \mathcal{U})$ and $\mathcal{V} = \{(\frac{1}{n+2}, \frac{1}{n}) : n \in \mathbb{N}\}$

$= \{(\frac{1}{3}, 1), (\frac{1}{4}, \frac{1}{2}), \ldots\}$

Then $\mathcal{U}$ is an open cover of the open interval $(0,1)$, while $\mathcal{U}$ has no finite subcover of $(0,1)$, because, if for any finite subclass of $\mathcal{U}$, say,

$$\mathcal{U}' = \{(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\}$$

and $\epsilon = \min\{a_1, a_2, \ldots, a_n\}$, $\epsilon > 0$. Then

$$(a_1, b_1) \cup (a_2, b_2) \cup \cdots \cup (a_n, b_n) \subset (\epsilon, 1)$$

but,

$$(0,1) = (0, \epsilon] \cup (\epsilon, 1)$$

and $(0, \epsilon]$, $(\epsilon, 1)$ are disjoint.

Therefore, $\mathcal{U}'$ is not cover of $(0,1)$ and so $(0,1)$ is not compact.

Theorem (2) (Heine Borel)

Every cover of a closed bounded interval $[a,b]$, $a, b \in R$ has a finite subcover.

i.e. every closed bounded interval is compact.

Theorem 3 Let (X, τ) be a topological space and $A \subset X$. Then A is compact iff A is compact with respect to the relative topology on A.

Proof. Let $A \subset X$ be a compact subset of X and $\{G_i\}$ be an τ_A-open cover of A i.e. $A \subset \bigcup_i G_i$ and $G_i \in \tau_A$. By the definition of the relative topology on A, then for each i, there exists an open set $H_i \in \tau$ such that $G_i = A \cap H_i$. Hence

$$A \subset \bigcup_i G_i \subset \bigcup_i H_i$$

Therefore $\{H_i\}$ is a τ- open cover of A. Since A is compact, $\{H_i\}$ admits ~~a~~ finite subfamily $\{H_{i_1}, H_{i_2}, \ldots, H_{i_n}\}$ such that

$$A \subset H_{i_1} \cup H_{i_2} \cup \cdots \cup H_{i_n}$$

Then

$$\begin{aligned} A &\subset A \cap (H_{i_1} \cup H_{i_2} \cup \cdots \cup H_{i_n}) \\ &= (A \cap H_{i_1}) \cup (A \cap H_{i_2}) \cup \cdots \cup (A \cap H_{i_n}) \\ &= G_{i_1} \cup G_{i_2} \cup \cdots \cup G_{i_n} \end{aligned}$$

So, $\{G_i\}$ has a finite subcover $\{G_{i_1}, G_{i_2}, \ldots, G_{i_n}\}$ of A. Hence A is compact with respect to τ_A.

Conversely, let $\{H_i\}$ be an τ- open cover of A, by open sets in X, put $G_i = H_i \cap A$, then

$$A \subset \bigcup_i H_i$$

which implies

$$A \subset A \cap (\bigcup_i H_i) = \bigcup_i (A \cap H_i) = \bigcup_i G_i$$

i.e. $\{G_i\}$ is a τ_A - open cover of A. Since A is compact with respect to τ_A, $\exists$ $\{G_{i_1}, \ldots, G_{i_n}\}$ a finite subfamily such that

$$A \subset \bigcup_{j=1}^{n} G_{i_j} = \bigcup_{j=1}^{n} (A \cap H_{i_j}) \subset \bigcup_{j=1}^{n} H_{i_j}$$

Thus $\{H_i\}$ has a finite subcover of A and A is compact.

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Example 3 Since every subspace of an indiscrete space is indiscrete space. By theorem (3) we have every subset of an indiscrete space is compact.

Example 4 Similarly, every subset of a cofinite space is compact.

Theorem 4 Every closed subset of a compact space is compact.

Proof Let F be a closed subset of a compact space X. Let $\{G_i\}$ be an open cover of F by open subsets of X i.e.

$$F \subset \bigcup_i G_i$$

Then $X = (\bigcup_i G_i) \cup F^c$,
that is

$$\mathcal{G}^* = \{G_i\} \cup \{F^c\}$$

is an open cover of X, since F^c is open.
Since X is compact, hence $\mathcal{G}^*$ admits a finite subcover of X, say

$$X = (\bigcup_{j=1}^{n} G_{i_j}) \cup \{F^c\} \quad , \quad G_{i_j} \in \{G_i\}$$

But F and F^c are disjoint, then

$$F \subset G_{i_1} \cup G_{i_2} \cup \cdots \cup G_{i_n}$$

i.e. any open cover of F contains a finite subcover.

5. Compactness and Separation axioms:

In this artical we study the links between the concepts of compactness and the separation axioms

Theorem 5. Let X be a Hausdorff space, $p \in X$ and F a compact subset of X not containing p (i.e $p \notin F$), then there exist open sets U, V such that $p \in U$, $F \subset V$ and $U \cap V = \phi$.

Proof Let $x \in F \Rightarrow x \neq p$. Since X is T_2-space $\Rightarrow \exists\ U_x, V_x$ open sets such that $x \in U_x$, $p \in V_x$ and $U_x \cap V_x = \phi$. The family $\{U_x : x \in F\}$ is an open cover of F. Since F is compact, there is a finite subcover $\{U_{x_1}, U_{x_2}, \ldots, U_{x_n}\}$ of F i.e.

$$F \subset U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_n} = U$$

If we take $V = V_{x_1} \cap V_{x_2} \cap \cdots \cap V_{x_n} \Rightarrow V$ is open set containing p and

$$U \cap V = (U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_n}) \cap (V_{x_1} \cap V_{x_2} \cap \cdots \cap V_{x_n})$$

$$= (U_{x_1} \cap V_{x_1}) \cup (U_{x_2} \cap V_{x_2}) \cup \cdots \cup (U_{x_n} \cap V_{x_n})$$

$$= \phi \cup \phi \cdots \cup \phi$$

$$= \phi.$$

Then U, V are open subsets, $F \in U$, $p \in U$ and $U \cap V = \phi$.

Corollary 2: A compact subset in a Hausdorff space, is closed.

Proof Suppose X is a T_2-space and F is a compact subset of X. Then by theorem 5, for any $p \notin F$ there exist open sets U, V such that $p \in U$, $F \subset V$ and $U \cap V = \phi$. In particular, $U \cap F = \phi$ and hence $U \subset X - F$. Then $p \in U \subset X - F$, i.e. $X - F$ is a nbd of each of its points. So $X - F$ is open and F is closed.

Theorem 6. Every compact Hausdorff space is a T_3-space.

proof Let X be a compact, Hausdorff space. Then every closed subset of X is compact and so the space X is regular by theorem 5. Since X is also T_1 (being T_2) the result follows.

Theorem 7 If F is a compact set in a regular space X and if U is open set containing F, there exists an open set V such that $F \subset V \subset \overline{V} \subset U$ (i.e. U contains a closed nbd of F)

Proof. Since U is open and $F \subset U \Rightarrow U$ is nbd of each $p \in F$. Then for each $p \in F$ $\exists$ open set V_p such that $p \in V_p \subset \overline{V_p} \subset U$ (because X is regular).

Thus, $\mathcal{V} = \{V_p : p \in F\}$ is an open cover of F. Since F is compact $\Rightarrow \exists\, V_{p_1}, V_{p_2}, \dots, V_{p_n} \in \mathcal{V}$ such that $F \subset V_{p_1} \cup V_{p_2} \cup \dots \cup V_{p_n} = V$ since

$$\overline{V} = \overline{V_{p_1} \cup V_{p_2} \cup \dots \cup V_{p_n}} = \overline{V}_{p_1} \cup \overline{V}_{p_2} \cup \dots \cup \overline{V}_{p_n}.$$ Then

$$F \subset V \subset \overline{V} = \overline{V}_{p_1} \cup \overline{V}_{p_2} \cup \dots \cup \overline{V}_{p_n} \subset U.$$

Corollary 3 Every compact regular space is normal

Proof Let F_1, F_2 be two disjoint closed sets in X. Since $F_1 \cap F_2 = \phi \Rightarrow F_2 \subset F_1^c \Rightarrow F_1^c$ is open set containing F_2. Since F_2 is closed in a compact space $\Rightarrow F_2$ is compact. Since X is regular and F_2 is compact contained in the open set F_1^c, by theorem 7, there exists open V such that

$$F_2 \subset V \subset \overline{V} \subset F_1^c$$

$\Rightarrow \quad F_2 \subset V, \quad F_1 \subset \overline{V}^c = U \Rightarrow U, V$ are open sets such that $F_1 \subset U$, $F_2 \subset V$ and

$$U \cap V = \overline{V}^c \cap V = \phi$$

Therefore, X is normal.

6. Compactness and continuity

Theorem 8:

The continuous image of a compact space is compact.

Proof. Let $f:(X,\tau_1) \longrightarrow (Y,\tau_2)$ be a continuous mapping from a compact space X into a space Y. Let $\mathcal{G} = \{G_i : G_i \in \tau_2\}$ be an open cover of $f(X)$. Since f is continuous $\Rightarrow f^{-1}(G_i)$ is open in X $\forall G_i \in \mathcal{G}$.
Since $f(X) = \cup\{G_i : G_i \in \mathcal{G}\}$

$$\Rightarrow X = f^{-1}(f(X)) = f^{-1}(\cup\{G_i : G_i \in \mathcal{G}\})$$
$$= \cup\{f^{-1}(G_i) : G_i \in \mathcal{G}\}$$

Hence the family $\{f^{-1}(G_i) : G_i \in \mathcal{G}\}$ is an open cover of X. But X is compact, there exists $f^{-1}(G_{i_1}), f^{-1}(G_{i_2}), \dots, f^{-1}(G_{in})$ such that

$$X = \cup\{f^{-1}(G_{ij}) : j=1,2,\dots n\}$$

$$\Rightarrow f(X) = f(\cup\{f^{-1}(G_{ij}) : j=1,2,\dots n\})$$
$$= \cup\{f f^{-1}(G_{ij}) : j=1,2,\dots,n\}$$
$$\subset \cup\{G_{ij} : j=1,2,\dots n\}$$

Then $\mathcal{G}$ has a finite subcover of $f(X)$. So, $f(X)$ is compact.

Coroliary 4. Every continuous map from a compact space into a T_2- space is closed.

Proof. suppose $f: X \longrightarrow Y$ is continuous, where X is compact and Y is Hausdorff. Let F be a closed subset of F. Then F is compact by theorem 4 and so $f(F)$

is compact by theorem 8. But Y is a Hausdorff, then $f(F)$ is closed by corollary (4). Hence the image of closed sets in X is closed in Y i.e. the map f is closed.

Theorem 9. Let $f: X \longrightarrow Y$ be a bijective continuous mapping. If X is compact and Y is Hausdorff, then X and Y are homeomorphic i.e. f is homeomorphism.

proof. Since f is a bijective and continuous. We claim that f is open. Let G be an open set in X. Then $X - G$ is closed and hence $f(X - G)$ is closed in Y by the corollary 4. But

$$f(X - G) = Y - f(G)$$

because f is bijective. So $f(G)$ is open in Y. Thus f is a continuous, open bijective and hence a homeomorphism.

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