جامعة جنوب الوادى

كلية التربية بالغردقة

الفرقة الثالثة عام رياضيات Math

المادة : (7) Applied (Electrostatic)

Chapter (1)

Vector calculas

1-1 Introduction :

Vectors are introduced in physics and mathematics courses , primarily in the Cartesian coordinates system . Although cylindrical may be found in calculus texts The spherical coordinates system is seldom presented . All three coordinate systems must be used in electromagnetic .

In this chapter we study the concepts of vector functions of one or more scalar variables and their applications and also study a vector differential operators and various derivatives of vector functions .

1-2Vector function of a single variable :

If to each value of scalar variable t , in certain interval $[a\,,b]$, there corresponds by any law what is over , a unique value of a variable vector $\vec{r}$, then $\vec{r}$ is called a vector function of the scalar variable t defined in the interval $[a\,,b]$. If $\vec{r}$ is a vector function of scalar variable t , then we write $\vec{r} = \overrightarrow{f(t)}$, where $\overrightarrow{f(t)}$ indicates the law of correspondence .

Examples :

(1)- The function $\vec{r} = a\cos t\ \underline{i} + b\sin t\ \underline{j} + 0\underline{k}$ is a vector equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, which represents a circle when $a = b$.

(2)- The function $\vec{r} = a\,t^2 \; \underline{i} + 2a\,t \; \underline{j} + 0\underline{k}$ is a vector equation of the parabola $y^2 = 4\,ax$.

1-3Limit of a vector function:

A vector function $\overrightarrow{f(t)}$ is said to have a limit L as t tends to a , if for a given $\epsilon > 0$, however small it may be , there exists a $\delta > 0$, such that

$\left|\overrightarrow{f(t)} - L\right| < \epsilon$ such $0 < |t-a| \leq \delta$. This fact , we express symbolically as , $\lim_{t\to a} \overrightarrow{f(t)} = \mathrm{L}$.

Properties of a limit:

Let $\overrightarrow{f(t)} = f_1(t)\underline{i} + f_2(t)\underline{j} + f_3(t)\underline{k}$ & $\overrightarrow{g(t)} = g_1(t)\underline{i} + g_2(t)\underline{j} + g_3(t)\underline{k}$

Be two vector functions , $\varphi(t)$ be a scalar function of t , and

$$\vec{L} = L_1\underline{i} + L_2\underline{j} + L_3\underline{k} \;\; \& \; \vec{M} = M_1\underline{i} + M_2\underline{j} + M_3\underline{k}$$

As two constant vector such that :

$$\lim_{t\to a} \overrightarrow{f(t)} = \vec{\mathrm{L}} \quad , \quad \lim_{t\to a} \overrightarrow{g(t)} = \vec{\mathrm{M}} \quad \text{and} \quad \lim_{t\to a} \varphi(t) = \mathrm{l}$$

Then:

$$(i) \quad \lim_{t\to a} f_1(\mathrm{t}) = \mathrm{L}_1, \lim_{t\to a} f_2(\mathrm{t}) = \mathrm{L}_3, \lim_{t\to a} f_1(\mathrm{t}) = \mathrm{L}_3$$

$$(ii) \quad \lim_{t\to a} \left[\overrightarrow{f(t)} \pm \overrightarrow{g(t)}\right] = \vec{\mathrm{L}} \pm \vec{\mathrm{M}}$$

$$(iii) \quad \lim_{t\to a}\left[\overrightarrow{f(t)}.\ \overrightarrow{g(t)}\right] = \vec{L}\,.\ \vec{M}$$

$$(iv) \quad \lim_{t\to a}\left[\overrightarrow{f(t)} \wedge \ \overrightarrow{g(t)}\right] = \vec{L}\ \wedge\ \vec{M}$$

$$(v) \quad \lim_{t\to a}\varphi(t)\overrightarrow{f(t)} = l\,\vec{L} \ , \ (vi) \quad \lim_{t\to a}\left|\overrightarrow{f(t)}\right| = \left|\vec{L}\right| \ .$$

1-4 Continuity of a vector function:

A vector function $\overrightarrow{f(t)}$ is said to be continuous at $t = a$ if :

(i) $\overrightarrow{f(a)}$ is defined (ii) $\lim_{t\to a}\overrightarrow{f(t)}$ exists (iii) $\lim_{t\to a}\overrightarrow{f(t)} = \overrightarrow{f(a)}$

A vector function $\overrightarrow{f(t)}$ is said to be continuous in the interval $[a,b]$ if it is continuous for every value of t in $[a,b]$.

Remarks :

(i) If $\overrightarrow{f(t)}$ be continuous , then $f_1(t), f_2(t)$ *and* $f_3(t)$ are also continuous scalar functions and conversely is right .

(ii) If $\overrightarrow{f(t)}$ and $\overrightarrow{g(t)}$ be to continuous vector functions and let $\varphi(t)$ be to continuous scalar function of t then :

(a) $\overrightarrow{f(t)} + \overrightarrow{g(t)}$ (b) $\overrightarrow{f(t)}.\overrightarrow{g(t)}$ (c) $\overrightarrow{f(t)} \wedge \overrightarrow{g(t)}$ (d) $\varphi(t)\overrightarrow{f(t)}$

Are also continuous .

1-5 Derivative of a vector function:

$[a]$ Derivative

Let $\overrightarrow{f(t)}$ be to vector function then:

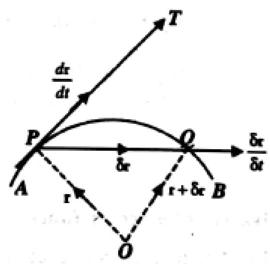
$$\lim_{t \to a} \frac{\overrightarrow{f(t+\delta t)} - \overrightarrow{f(t)}}{\delta(t)} = \lim_{t \to a} \frac{\overrightarrow{\delta(t)}}{\delta(t)}$$

If it exists , is called the differential coefficient of $\overrightarrow{f(t)}$ with respect to t , and is denoted by $\frac{d\vec{f}}{dt}$ or $\overrightarrow{f'(t)}$.

A vector function $\overrightarrow{f(t)}$ is said to be differentiable if it has a differential coefficient for all values of t belongs to its interval of definition .

$[b]$ Geometrical interpretation of derivative:

Let $\vec{r} = \overrightarrow{f(t)}$ be a continuous and single valued vector function of the scalar variable t .



Let O be the origin . Let P &Q be tow neighboring points on a continuous curve . Corresponding to the values t and $t + \delta t$ of the scalar variable so that

$\overrightarrow{OP} = \vec{r}$ and $\overrightarrow{OQ} = \vec{r} + \overrightarrow{\delta r}$, therefore

$$\overrightarrow{PQ} = position\ vector\ of\ \ Q - position\ vector\ of\ \ P$$

$$= \left(\vec{r} + \overrightarrow{\delta r}\right) - \vec{r}\ and = \overrightarrow{\delta r}$$

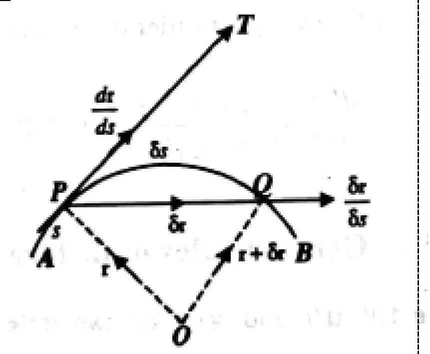
Thus $\dfrac{\overrightarrow{\delta(t)}}{\delta(t)} = \dfrac{\overrightarrow{PQ}}{\delta(t)}$

When $Q \to P \to 0$, the chord $\overrightarrow{PQ} \to$ tangent $\overrightarrow{PT}$ to the curve at P , thus geometrically , the derivative $\dfrac{\overrightarrow{dr}}{dt}$ of a vector function represents a vector whose direction is that of the tangent $\overrightarrow{PT}$

To the curve AB at P in the sense of increasing t of the slope of the tangent at P .

$[c]$ <u>Unit tangent vector to the curve :</u>

Let P &Q be tow neighboring points on a curve . Let A be any fixed on it and s &$s + \delta s$ be the arc lengths measured along the curve from A to P and from A to Q respectively .



Let $r = f(s)$ be a continuous and single valued scalar function of the scalar variable s . Let O be the origin of reference and let $\overrightarrow{OP} = \vec{r}$ and $\overrightarrow{OQ} = \vec{r} + \overrightarrow{\delta r}$. Therefore

$$\overrightarrow{PQ} = \left(\vec{r} + \overrightarrow{\delta r}\right) - \vec{r} = \overrightarrow{\delta r}$$

Thus $\frac{\overrightarrow{\delta r}}{\delta s} = \frac{\overrightarrow{PQ}}{\delta s}$, when $Q \to P$, $\delta s \to 0$, the chord $\overrightarrow{PQ} \to \overrightarrow{PT}$, the tangent to the curve at P .

Thus geometrically $\lim_{\delta s \to 0} \frac{\overrightarrow{\delta r}}{\delta s} = \frac{\overrightarrow{dr}}{ds}$ represents a vector whose direction is that of the tangent $\overrightarrow{PT}$ to the curve AB at P in the sense of increasing s . Further :

$$\left| \frac{d\vec{r}}{ds} \right| = \lim_{\delta s \to 0} \left| \frac{\overrightarrow{\delta r}}{\delta s} \right| = \lim_{Q \to P} \frac{\left|\overrightarrow{\delta r}\right|}{\text{arc PQ}} = \lim_{Q \to P} \frac{\text{chord PQ}}{\text{arc PQ}} = 1 \quad .$$

Thus $\frac{d\vec{r}}{ds}$ is a unit vector along the tangent $\overrightarrow{PT}$ at P in the direction of increasing s , and we shall denote it by $\underline{t}$ or $\hat{t}$.

That is $\frac{d\vec{r}}{ds}$.

$[d]$ **Successive derivatives:**

In general $\frac{d\vec{r}}{dt}$ is a function of t and if it possesses a derivative, then the derivative $\frac{d}{dt}\left(\frac{d\vec{r}}{dr}\right)$ denoted by $\frac{d^2\vec{r}}{dt^2}$.

Similarly , the higher derivatives of $\vec{r}$ is defended as :

$$\frac{d^n\vec{r}}{dt^n} = \frac{d}{dt}\left(\frac{d^{n-1}\vec{r}}{dt^{n-1}}\right) \quad , \quad \text{for all} \quad n \geq 2 \, .$$

$[e]$ General rules of differentiation:

If $\overrightarrow{u(t)}$ & $\overrightarrow{v(t)}$ be two differential vector functions of the scalar t , and $\varphi(t)$ be a differentiable function of t , then :

(i) $\frac{d}{dt}(\vec{u} \pm \vec{v}) = \frac{d\vec{u}}{dt} \pm \frac{d\vec{v}}{dt}$

(ii) $\frac{d}{dt}(\vec{u}\,.\,\vec{v}) = \vec{u}\,.\,\frac{d\vec{v}}{dt} \pm \vec{v}\,.\,\frac{d\vec{u}}{dt}$

(iii) $\frac{d}{dt}(\vec{u} \wedge \vec{v}) = \vec{u} \wedge \frac{d\vec{v}}{dt} \pm \frac{d\vec{u}}{dt} \wedge \vec{v}$

(iv) $\frac{d}{dt}(\varphi\,\vec{u}) = \varphi\frac{d\vec{u}}{dt} \pm \frac{d\varphi}{dt}\,\vec{u}$

Examples :

(1)-Show that the derivative of a vector of constant magnitude is perpendicular to the vector ,or show that the necessary and sufficient condition for the vector $\overrightarrow{v(t)}$ to have a constant magnitude is $\overrightarrow{v(t)}\,.\,\frac{d\overrightarrow{v(t)}}{dt} = 0$.

The solution :

Let $\overrightarrow{v(t)}$ be a vector of constant magnitude $v(t)$. then :

$$\frac{dv(t)}{dt} = 0 \Leftrightarrow \frac{d|v(t)|}{dt} = 0 \Leftrightarrow \frac{d|v(t)|^2}{dt} = 0 \Leftrightarrow \frac{d\left(\overrightarrow{v(t)}\,.\overrightarrow{v(t)}\right)}{dt} = 0$$

$$\Leftrightarrow \frac{d}{dt}\left(\overrightarrow{v(t)}\,.\overrightarrow{v(t)}\right) = 0 \;\Leftrightarrow\; \overrightarrow{v(t)}.\frac{d}{dt}\,\overrightarrow{v(t)} + \frac{d}{dt}\,\overrightarrow{v(t)}\,.\overrightarrow{v(t)}$$

By rule (ii) $\Leftrightarrow 2\,\overrightarrow{v(t)}.\frac{d}{dt}\,\overrightarrow{v(t)} = 0 \;\Leftrightarrow\; \overrightarrow{v(t)}.\frac{d}{dt}\,\overrightarrow{v(t)} = 0$.

Thus , the derivative of a vector of constant magnitude is perpendicular to the vector .

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(2)-If $\overrightarrow{v(t)}$ be the differential vector function of the scalar t , prove that $\frac{d}{dt}\left(\overrightarrow{v(t)} \wedge \frac{d\overrightarrow{v(t)}}{dt}\right) = \overrightarrow{v(t)} \wedge \frac{d^2\overrightarrow{v(t)}}{dt^2}$.

The solution :

$\frac{d}{dt}\left(\overrightarrow{v(t)} \wedge \frac{d\overrightarrow{v(t)}}{dt}\right) = \overrightarrow{v(t)} \wedge \frac{d}{dt}\left(\frac{d\overrightarrow{v(t)}}{dt}\right) + \frac{d\overrightarrow{v(t)}}{dt} \wedge \frac{d\overrightarrow{v(t)}}{dt}$ by rule (iii) $= \overrightarrow{v(t)} \wedge \frac{d}{dt}\left(\frac{d\overrightarrow{v(t)}}{dt}\right) + \vec{0} = \overrightarrow{v(t)} \wedge \frac{d}{dt}\left(\frac{d\overrightarrow{v(t)}}{dt}\right)$ (since $\vec{A} \wedge \vec{A} = \vec{0}$)

$$= \overrightarrow{v(t)} \wedge \frac{d^2}{dt^2}\left(\overrightarrow{v(t)}\right) \ .$$

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(3)-Prove that the necessary and sufficient condition for the vector $\overrightarrow{v(t)}$ to have a constant direction is $\overrightarrow{v(t)} \wedge \frac{d\overrightarrow{v(t)}}{dt} = 0$

<u>The solution :</u>

Let $\overrightarrow{v(t)} = v(t)\,\underline{t}$ where $\underline{t}$ is a unit vector in the direction of the vector $\overrightarrow{v(t)}$. Then :

$$\overrightarrow{v(t)} \wedge \frac{d\overrightarrow{v(t)}}{dt} = \vec{0} \Leftrightarrow v\,\underline{t} \wedge \frac{dv\,\underline{t}}{dt} = \vec{0} \Leftrightarrow v\,\underline{t} \wedge \left(\frac{dv}{dt}\underline{t} + v\frac{d\underline{t}}{dt}\right) = \vec{0} \quad \Leftrightarrow v\,\underline{t} \wedge \frac{dv}{dt}\underline{t} + v\,\underline{t} \wedge v\frac{d\underline{t}}{dt} = \vec{0} \Leftrightarrow v\,\underline{t} \wedge \underline{t}\frac{dv}{dt} + v^2\underline{t} \wedge \frac{d\underline{t}}{dt} = \vec{0}$$

$$\Leftrightarrow v\left(\vec{0}\right)\frac{dv}{dt} + v^2\underline{t} \wedge \frac{d\underline{t}}{dt} = \vec{0} \ \Leftrightarrow v^2\underline{t} \wedge \frac{d\underline{t}}{dt} = \vec{0} \Leftrightarrow \frac{d\underline{t}}{dt} = \vec{0}$$

$\left(\text{this result becausethat } \underline{t} \neq \vec{0}\right) \Leftrightarrow$

$\underline{t}$ is of constant direction $\Leftrightarrow \vec{v}$ is of constant direction .

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(4)-Prove that the necessary and sufficient condition for the vector $\overrightarrow{f(t)}$ to be constant is $\frac{d\overrightarrow{f(t)}}{dt} = 0$.

<u>The solution :</u>

Let $\overrightarrow{f(t)}$ be a constant vector . Then we have $\overrightarrow{f(t+\delta t)} = \overrightarrow{f(t)}$

so that

$$\frac{d\overrightarrow{f(t)}}{dt} = \lim_{\delta t \to 0} \frac{\overrightarrow{f(t+\delta t)} - \overrightarrow{f(t)}}{\delta t} = \lim_{\delta t \to 0} \frac{\vec{0}}{\delta t} = \vec{0}$$

Conversely . Let $\overrightarrow{f(t)} = f_{1\underline{i}} + f_{2\underline{j}} + f_{3\underline{k}}$ and $\frac{d\overrightarrow{f(t)}}{dt} = \vec{0}$ .then:

$$\frac{df_1}{dt}\,\underline{i} + \frac{df_2}{dt}\,\underline{j} + \frac{df_3}{dt}\,\underline{k} = \vec{0} \Longrightarrow \frac{df_1}{dt} = 0 \,, \frac{df_2}{dt} = 0 \,, \frac{df_3}{dt} = 0 \,.$$

$$\Longrightarrow f_1 = constant \,, f_2 = constant \,, f_3 = constant$$

$$\Longrightarrow \overrightarrow{f(t)} = constant \,.$$

1-6 Scalar and vector point function:

In this section we propose to study two types of functions . One is a scalar function while the other is a vector function .

$[a]$ Scalar point function:

If to each point $P(x,y,z)$ of a region R , there exists a definite scalar denoted by $\varphi(P)$ or $\varphi(x,y,z)$, then φ is said to be scalar point function for the region R .

The set of all points of the region R together with the set of all values of the scalar function φ be is said to be a scalar field R .

Example:

The temperature of a body at any instant , density of a body and potential due to gravitationally matter are examples of scalar point function .

$[b]$ Vector point function:

If to each point $P(x,y,z)$ of a region R , there exists a definite vector denoted by $\vec{f}(P)$ or $\vec{f}(x,y,z)$, then $\vec{f}$ is said to be vector point function for the region R .

The set of all points of the region R together with the set of all values of the vector function $\vec{f}(P)$ is said to be a vector field R .

Example:

The velocity of a moving fluid at any instant , and the gravitational intensity of force are examples of vector point function .

1-7 Vector differential operator $\vec{\nabla}$:

Vector differential operator $\vec{\nabla}$ (read as del or nabla) is defined as :

$$\vec{\nabla}= \frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k} \equiv \underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z} \quad .$$

The operator $\vec{\nabla}$ serves a vector differential operator .

$[a]$ Gradient of a scalar point function:

Let $\varphi(x, y, z)$ be a continuously differential scalar function .

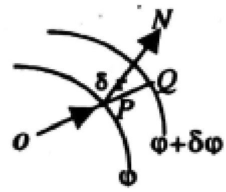
The gradient of φ , denoted by $\vec{\nabla}\varphi$ or simply $grad\varphi$ is defined as :

$$grad\varphi = \vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\underline{i} + \frac{\partial\varphi}{\partial y}\underline{j} + \frac{\partial\varphi}{\partial z}\underline{k} \quad .$$

The $\vec{\nabla}\varphi$ is <u>vector</u> . If C is a constant , then $\vec{\nabla}C\varphi = C\vec{\nabla}\varphi$.

<u>Geometrical significance of grad of scalar point function:</u>

If φ is a scalar point function , then $grad\varphi$ is a vector normal to the surface $\varphi(x, y, z) = C$, and has A magnitude equals to the rate of change of φ along this normal .



$[b]$ **<u>Divergence of a vector point function:</u>**

The divergence of a vector point function $\vec{f}(x, y, z) = f_x\underline{i} + f_y\underline{j} + f_z\underline{k}$ is denoted by $\vec{\nabla}.\vec{f}$, or simply $div\,\vec{f}$, as :

$$div\,\vec{f} = \vec{\nabla}.\vec{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$$

The $div\,\vec{f}$ is <u>scalar</u> . If C is a constant , then $\vec{\nabla}.C\vec{f} = C\overrightarrow{\nabla}.\vec{f}$.

Physical significance of div (in electrostatic):

$div\,\vec{f}$ represents the amount of electric flux v per unit volume per unit time . Generally the divergence is roughly a measure of a vector $field'$ increasing in the direction it points.

But more accurately a measure of that $field'$ tendency to converge on or repel from a point .

If the flux v entering any element of space is the same as that leaving it (that is $div\,\vec{f} = 0$) everywhere , then such a point function is called a solenoid vector function .

$[c]$ Curl of a vector point function:

The curl of a vector point function

$\vec{f}\,(x, y, z) = f_x\underline{i} + f_y\underline{j} + f_z\underline{k}$ is denoted by $\vec{\nabla} \wedge \vec{f}$, or simply $curl\,\vec{f}$, as :

$$curl\,\vec{f} = \vec{\nabla} \wedge \vec{f} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

The $curl\,\vec{f}$ is vector. If C is a constant, then $\vec{\nabla} \wedge C\vec{f} = C(\vec{\nabla}\,\overrightarrow{\wedge\, f})$.

Physical significance of curl (in electrostatic):

In vector calculus , the curl (or rotor) is a vector operator that describes the rotation of a vector field .The direction of the curl is the axis of rotation ,as determined by the right-hand role, and the magnitude of the curl is the magnitude of the rotation .

$[d]$ Some properties for the vector differential operator $\vec{\nabla}$:

Let $\vec{A}$ & $\vec{B}$ are two differentiable vector functions of the , and $\phi \& \psi$ are two differentiable scalar functions , and If $\alpha \& \beta$ as two arbitrary constants , then :

(1) $\vec{\nabla}\,(\alpha\phi \pm \beta\psi) = \alpha\vec{\nabla}\phi \pm \beta\vec{\nabla}\psi \quad ,\vec{\nabla}\alpha = \vec{\nabla}\beta = \vec{0}$.

(2) $\vec{\nabla}\,(\phi\,\psi) = \phi\vec{\nabla}\phi + \psi\vec{\nabla}\phi$.

(3) $\vec{\nabla}\left(\frac{\phi}{\psi}\right) = \left(\psi\vec{\nabla}\phi - \phi\vec{\nabla}\psi\right)/\psi^2$

$$(4)\vec{\nabla}(\vec{A}.\vec{B}) = (\vec{B}.\vec{\nabla})\vec{A} + (\vec{A}.\vec{\nabla})\vec{B} + \vec{B}\wedge(\vec{\nabla}\wedge\vec{A}) + \vec{A}\wedge(\vec{\nabla}\wedge\vec{B})$$

$$(5)\vec{\nabla}.(\propto\vec{A} + \beta\vec{B}) = \propto(\vec{\nabla}.\vec{A}) + \beta(\vec{\nabla}.\vec{B})$$

$$(6)\ \vec{\nabla}.(\phi\vec{A}) = (\vec{\nabla}\phi).\vec{A} + \phi(\vec{\nabla}.\vec{A}) \quad .$$

$$(7)\ \vec{\nabla}\wedge(\phi\vec{A}) = (\vec{\nabla}\phi)\wedge\vec{A} + \phi(\vec{\nabla}\wedge\vec{A}) \quad .$$

$$(8)\vec{\nabla}\wedge(\propto\vec{A} + \beta\vec{B}) = \propto(\vec{\nabla}\wedge\vec{A}) + \beta(\vec{\nabla}\wedge\vec{B})$$

$$(9)\vec{\nabla}.(\vec{A}\wedge\vec{B}) = \vec{B}.(\vec{\nabla}\wedge\vec{A}) - \vec{A}.(\vec{\nabla}\wedge\vec{B})$$

$$(10)\vec{\nabla}\wedge(\vec{A}\wedge\vec{B}) = \vec{B}.(\vec{\nabla}\wedge\vec{A}) - \vec{B}(\vec{\nabla}.\vec{A}) - (\vec{A}.\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}.\vec{B})$$

$$(11)\vec{\nabla}.(\vec{\nabla}\phi) = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

Where $\left(\nabla^2 = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right)$ is called Laplace operator.

$$(12)\vec{\nabla}\wedge(\vec{\nabla}\phi) = \vec{0}$$

$$(13)\vec{\nabla}.(\vec{\nabla}\wedge\vec{A}) = 0$$

$$(14)\vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{A}) = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \nabla^2\vec{A}$$

Examples : Calculate :

(i)-$\vec{\nabla} f(\mathrm{r})$ (ii)-$\vec{\nabla} . \vec{\mathrm{r}}$ (iii)-$\vec{\nabla} \wedge \vec{\mathrm{r}}$

(iv)-$\vec{\nabla}.\left(\vec{\mathrm{r}} f(\mathrm{r})\right)$ (v)-$\vec{\nabla} \wedge \left(\vec{\mathrm{r}} f(\mathrm{r})\right)$

Where $\vec{\mathrm{r}} = \mathrm{x}\underline{i} + y\underline{j} + z_z\underline{k}$

The solution :

(i)- It is clear that $f(\mathrm{r}) = f\left(\sqrt{x^2+y^2+z^2}\right) = f(x, y, z)$

Then $\vec{\nabla} f(\mathrm{r}) = \frac{\partial \mathrm{f(r)}}{\partial x}\underline{i} + \frac{\partial \mathrm{f(r)}}{\partial y}\underline{j} + \frac{\partial \mathrm{f(r)}}{\partial z}\underline{k}$

But $\frac{\partial \mathrm{f(r)}}{\partial x} = \frac{df}{dr}\ \frac{\partial \mathrm{r}}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}}\ f' = \frac{x}{r}\ f'$

Similarly $\frac{\partial \mathrm{f(r)}}{\partial y} = \frac{y}{r}\ f'$ & $\frac{\partial \mathrm{f(r)}}{\partial z} = \frac{z}{r}\ f'$

(ii)- $\vec{\nabla} . \vec{\mathrm{r}} = \left(\frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}\right) . \left(\mathrm{x}\underline{i} + y\underline{j} + z_z\underline{k}\right)$

$$= \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) = 3$$

(iii)- $$\vec{\nabla} \wedge \vec{r} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{0}$$

(iv)- $$\vec{\nabla}.\left(\vec{r}f(r)\right) = \frac{\partial}{\partial x}\left(xf(r)\right) + \frac{\partial}{\partial y}\left(yf(r)\right) + \frac{\partial}{\partial z}\left(zf(r)\right)$$

$$= f(r)\frac{\partial x}{\partial x} + x\frac{\partial f}{\partial x} + f(r)\frac{\partial y}{\partial y} + y\frac{\partial f}{\partial y} + f(r)\frac{\partial z}{\partial z} + z\frac{\partial f}{\partial z}$$

$$= 3\,f(r) + \frac{(x^2 + y^2 + z^2)}{r}f', \quad from\ \ (i)$$

$$= 3\,f(r) + rf'$$

(v)- $$\vec{\nabla} \wedge \left(\vec{r}f(r)\right) = f(r)\,\vec{\nabla} \wedge\ \vec{r} + \vec{\nabla}\left(f(r)\right) \wedge\ \vec{r}$$

$$= f(r)\,\vec{0} + \overrightarrow{r_0}(f') \wedge\ \vec{r}\ ,\ \ from\ \ (i)\&(ii)$$

$$= \ \vec{0} + \vec{0} = \vec{0}\ ,\ \ since\ \ \overrightarrow{r_0} \parallel \vec{r}$$

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Chapter (2)

Vector Integration

1-1 Introduction :

Let $\vec{r} = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}$, be the position vector of a point $P(x, y. z)$.

For all values of $t \in [a, b]$. The point $P.$ describes the curve C .

The curve C is called smooth curve if $\vec{r}(t)$ possesses a continuous first derivative (not equal to zero vector) for all $t \in [a, b]$.

A curve which is made up of finite numbers of smooth curves is called piecewise smooth curve . A curve is said to be closed curve if its initial and terminal points are same .

Throughout this chapter we shall consider only smooth curves unless otherwise mentioned .

<u>Definition</u> :A closed smooth curve which does not intersect itself anywhere is known as simple closed curve .

Examples : circle , ellipse

Simple closed curve   Not a simple closed curve

Definition : A region is said to be simply connected if any closed curve

lying entirely within the region can be constructed (or shrunk) continuously for a

point without any portion of the curve passing out of the region .

A region which is not simply connected is called multiply connected region .

Examples : Regions inside the circle , cubes , sphere , …. , are simply connected regions .

Definition : A surface $r = f(u, v)$ is said to be smooth if it is possesses continuous first order partial derivatives .

Throughout this chapter we shall consider only smooth surfaces unless otherwise mentioned .

1-2 Line Integral :

Let C be a smooth curve given by $\vec{r} = \vec{f}(t)$.

$\vec{r} = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}$, be the position vector of a point $P(x, y. z)$.

For all values of $t \in [a, b]$. The point P. describes the curve C .

Let $\vec{F}(r)$ be a continuous vector point function on C . .

Let A be a fixed point on C and S be the length of the curve from A to any point $P(x, y, z)$ on C . Then we have $\frac{d\vec{r}}{ds}$ is the unit vector tangent to the curve at P . Thus , the component of $\vec{F}(r)$ along the tangent at P is $\vec{F} . \frac{d\vec{r}}{ds}$.

It is clearly a function of S for any point on the curve . Then :

$$\int_c \overrightarrow{F} . \frac{d\vec{r}}{ds} \quad \text{or} \quad \int_c \overrightarrow{F} . d\vec{r}$$

Is called the tangent line integral of $\vec{F}(r)$ along C .

Observations on line integral :

(1) Since the integrand of the above tangential line integral is scalar ,then it is the

ordinary line integral of elementary calculus .

(2) If C is a closed curve , then we denote the above tangential line integral by putting a circle on the integral sign as : $\oint_c \vec{F} . d\vec{r}$.

(3) If C is a join of finite smooth curves $C_1 , C_2 , \dots \dots , C_n$, then :

$$\oint_c \vec{F} . d\vec{r} = \oint_{c_1} \vec{F} . d\vec{r} + \oint_{c_2} \vec{F} . d\vec{r} + \dots \dots + \oint_{c_n} \vec{F} . d\vec{r} = \sum_{i=1}^{n} \oint_{c_i} \vec{F} . d\vec{r} \quad .$$

(4) If $\vec{F} = F_1(x, y, z)\underline{i} + F_2(x, y, z)\underline{j} + F_3(x, y, z)\underline{k}$, then :

$$\int_c \overrightarrow{\mathrm{F}} \,.\, d\vec{r} = \int_c \left(F_1\underline{i} + F_2\underline{j} + F_3\underline{k}\right) \,.\left(dx\underline{i} + dy\underline{j} + dz\underline{k}\right)$$

$$= \int_c (F_1 dx + F_2 dy + F_3 dz)$$

(5) The line integral $\int_c \overrightarrow{F} \,.\, d\vec{r}$ can be also be written as $\int_c \overrightarrow{F} \,.\, \frac{d\vec{r}}{\mathrm{dt}}\ \mathrm{dt}$.

(6) The other types line integrals are $\int_c \overrightarrow{F} \wedge d\vec{r}$ and $\int_c \varphi\, d\vec{r}$.

(7) If $\vec{F}$ is the force acting on a particle to displace along the curve C , then

$\int_c \overrightarrow{F} \,.\, d\vec{r}$, represents physically the total work done during the displacement from A to B .

(8) If $\vec{F}$ is the velocity of a fluid particle along the curve C , then $\oint_c \vec{F} \,.\, d\vec{r}$, is called the circulation around the curve .

(9) If the circulation $\oint_c \vec{F} \,.\, d\vec{r} = 0$, around every closed curve C in the region R then C , then $\vec{F}$ is called irrotational in R .

<u>Examples</u> :

(1) Evaluate $\int_{(0,0)}^{(1,2)} \vec{F} \,.\, d\vec{r}$ if $\vec{F} = 3xy\underline{i} - y^2\underline{j}$ along the curve C :

$y = 2x^2$ on the plane xy .

<u>The solution :</u>

$I = \int_{(0,0)}^{(1,2)} \vec{F}.d\vec{r} = \int_{(0,0)}^{(1,2)} \left(3xy\underline{i} - y^2\underline{j}\right).\left(dx\underline{i} + dy\underline{j}\right) = \int_{(0,0)}^{(1,2)} (3xydx - y^2 dy)$

Along the line $y = 2x^2$ that is $(dy = 4xdx)$, we get :

$$I = \int_0^1 3x\,(2x^2)\,dx - (2x^2)^2(4xdx)$$

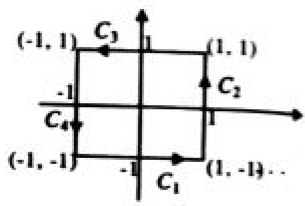
$$= \int_0^1 (6\,x^3 - 16\,x^5)\,dx = \left[\frac{3x^3}{2} - \frac{8x^6}{3}\right]_0^1 = \frac{3}{2} - \frac{8}{3} = -\frac{7}{6} \quad .$$

----

(2) Evaluate $\int_c (x^2 + xy)dx + (x^2 + y^2)dy$, where C is the square formed by the lines $y = \pm 1$ & $x = \pm 1$.

The solution :

I = $\int_c (x^2 + xy)dx + (x^2 + y^2)dy =$
$\sum_{i=1}^{4} \int_{c_i} (x^2 + xy)dx + (x^2 + y^2)dy$



Equation to c_1 is $y = -1$ $(\therefore dy = 0)$.

Hence :

$\int_{c_1} (x^2 + xy)dx + (x^2 + y^2)dy = \int_{-1}^{1} \big(x^2 + x(-1)\big)dx + (x^2 + (-1)^2)(0)$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{-1}^{1} = \frac{2}{3} \quad .$$

Equation to c_2 is $x = 1$ $(\therefore dx = 0)$. Hence :

$$\int_{c_2} (x^2+xy)dx+(x^2+y^2)dy=\int_{-1}^{1}(1+y^2)dy=\left[y+\frac{y^3}{3}\right]_{-1}^{1} = \frac{8}{3} \ .$$

Equation to $c_3 \ is \ y=1 \ (\therefore dy=0)$. Hence :

$$\int_{c_3}(x^2+xy)dx+(x^2+y^2)dy=\int_{1}^{-1}(x^2+x)dx=\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_{1}^{-1} = -\frac{2}{3}.$$

Equation to $c_4 \ is \ x=-1 \ (\therefore dx=0)$. Hence :

$$\int_{c_4} (x^2+xy)dx+(x^2+y^2)dy=\int_{1}^{-1}(1+y^2)dy==\left[y+\frac{y^3}{3}\right]_{-1}^{1}$$

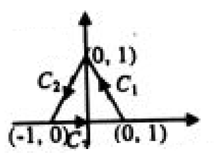
$$=-\frac{8}{3} \ .$$

Substation these result we get :

$$\mathrm{I}=\frac{2}{3}+\frac{8}{3}-\frac{2}{3}-\frac{8}{3}= 0 \ .$$

--

(3) Compute the line integral $\int_c y^2\,dx-x^2dy$, about the triangle whose vertices are $(1{,}0)$, $(0{,}1)$ & $(-1{,}0)$.



<u>The solution :</u>

$$\mathrm{I}=\int_c y^2\,dx-x^2dy=\sum_{i=1}^{3}\int_{c_i} y^2\,dx-x^2dy$$

On c_1 we have $\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1} \Rightarrow \frac{y-0}{x-1} = \frac{1-0}{0-1} \Rightarrow \frac{y}{x-1} = \frac{1}{-1}$

$$\Rightarrow \ y = -x+1 \Rightarrow (\therefore dy = -dx).$$

Hence :

$$\therefore \ \int_{c_1} y^2\,dx - x^2 dy = \int_1^0 (-x+1)^2(-dx) - x^2(-dx)$$

$$= \int_1^0 (2x^2-2x+1)dx = \left[\frac{2}{3}x^3 - x^2 + x\right]_1^0 = \left((0) - \left(\frac{2}{3} - 1 + 1\right)\right) = -\frac{2}{3}$$

On c_2 we have $\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1} \Rightarrow \frac{y-1}{x-0} = \frac{0-1}{-1-0} \Rightarrow \frac{y-1}{x} = 1$

$$\Rightarrow \ y = x+1 \Rightarrow (\therefore dy = dx).$$

Hence :

$$\therefore \ \int_{c_2} (x+1)^2\,dx - x^2 dx = \int_0^{-1} (2x+1)\ \mathrm{dx} = [x^2+x]_0^{-1} = 0$$

On c_3 we have $c_3\ is\ \frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1} \Rightarrow \frac{y-0}{x+1} = \frac{0-0}{1+1} \Rightarrow \frac{y}{x+1} = 0$

$$\Rightarrow \ y = 0 \Rightarrow (\therefore dy = 0).$$

Hence :

$$\therefore \ \int_{c_3} (0)^2\,dx - x^2(0) = 0$$

Substation these result we g $\mathrm{I} = -\frac{2}{3} + 0 + 0 = -\frac{2}{3}$

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(4) If $\vec{F} = (3x^2 + 6y)\underline{i} - 14yz\underline{j} + 20\, xz^2\underline{k}$, then evaluate $\int_c \overrightarrow{F} \,.\, d\vec{r}$:

From (0,0,0) to (1,1,1) along the path $x = t, y = t^2, z = t^3$.

The solution :

On the path $x = t, y = t^2, z = t^3$ we have :

$$dx = dt \quad , \quad dy = 2tdt \quad and \quad dz = 3\,t^2 dt$$

Also $x = 0 \ \ to\, x = 1 \Longrightarrow \ t = 0 \ \ to\, t = 1$

Thus

$$\int_c \overrightarrow{F} \,.\, d\vec{r} = \int_c (3\mathrm{x}^2 + 6\mathrm{y})\mathrm{dx} - 14\ \mathrm{yz\ dy} + 20\ \mathrm{xz}^2\mathrm{dz}$$

$$I = \int_0^1 (3t^2 + 6t^2\,)\,dt - 14(t^2)(t^3)(2\,tdt) + 20(t)(t^6)(3t^2 dt)$$

$$I = \int_0^1 (9t^2 - 28\,t^6 + 60\,t^9\,)\,dt = [3t^3 - 4t^7 + 6\,t^{10}]_0^1 = 5$$

Exercises:

(1) Evaluate $\int_c \overrightarrow{F} \,.\, d\vec{r}$, in the following cases :

(i) $\vec{F} = (3xy)\underline{i} - y^2\underline{j}$ $,where\ \ C\ is\ y = 2x^2 from$ (0,0) to (1,2) .

(ii) $\vec{F} = (x^2 + y^2)\underline{i} -$
$2xy\underline{j}$ $,where\ \ C\ is\ the\ rectangular\ in\ the\ xy -$.

$plane\ bounded\ by\ \ y = 0\,, x = a\,, y = b and\ x = 0$.

(iii) $\vec{F} = (2x + y)\underline{i} - (3y - xxy)\underline{j}$ $, where\ \ C\ is\ the\ cu\ xrve\ \ in\ the\ xy -$

$plane\ of\ the\ straight\ line from$ $(0,0)$ to $(2,0)$ to $(3,2)$.

(2) Evaluate $\int_c (\mathrm{xy} + \mathrm{z}^2) dx$ $, where\ \ C\ is\ arc\ of\ the\ helix$

$x = \cos t\,, y = \sin t\,, z = t\ which\ jions\ \ (1,0,0)\ and\ \ (-1,0,\pi)$

1-3 Surface Integral :

Let by $\vec{r} = \vec{f}(x, y)$ be a smooth surface by S ,and by $\vec{F}(r)$ is a continuous vector point function . Let $\underline{n}$ be unit vector outer normal to the surface S , then the integral :

$$\text{Evaluate} \quad \int_c \overrightarrow{F}\,.\underline{n}\ dS \quad \text{or} \quad \iint_S \vec{F}\,.\underline{n}\, dS$$

Is called the surface integral or normal integral of $\vec{F}(r)$ over the region S .

Observations on surface integral :

(1) The other type of line integral are

$$\int_S \overrightarrow{\mathrm{F}} \wedge \mathrm{d}\vec{\mathrm{S}} \quad , \quad \int_S \varphi\, \mathrm{d}\vec{\mathrm{S}} \quad , \quad \int_S \overrightarrow{\mathrm{F}}\ \mathrm{dS}$$

(2) If $\vec{F} = F_x(x, y, z)\underline{i} + F_y(x, y, z)\underline{j} + F_z(x, y, z)\underline{k}$, then :

$$\int_c \overrightarrow{\mathrm{F}}\,.\,\mathrm{d}\vec{\mathrm{S}} = \iint_S F_x\ dydz + F_y\ dxdz + F_z\ dxdy$$

(3) If S is a closed surface then the surface integral is denoted by

$\oint_S \vec{F} \cdot \overrightarrow{dS}$

(4) If $\vec{F}$ represents the velocity of a fluid particle , then the total outward

flux of $\vec{F}$ across a closed surface S is the surface integral $\oint_S \vec{F} \cdot \overrightarrow{dS}$.

Further , if $\oint_S \vec{F} \cdot \overrightarrow{dS} = 0$, across every closed surface S in a region R ,

then $\vec{F}$ is called solenoidal vector point function in R .

(5) Surface integral can be used in estimation of gravitational field , electric

force and magnetic force .

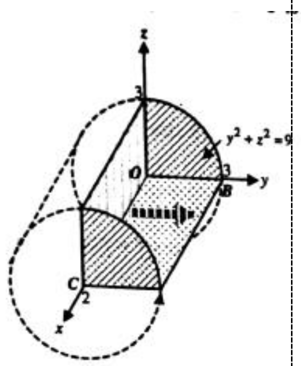
Example :

Evaluate $\int_S \vec{F} \cdot \underline{n} \; dS$, where $\vec{F} = 2x^2\underline{i} - y^2\underline{j} + 4zx\underline{k}$ and S is the surface $y^2 + z^2 = 9$, bounded by $x = 0 \; and \; x = 2$ in the first octant .

The solution

Surface S is projected along and $xy - plane$ is and $OCAB$,the normal to the surface $\varphi = y^2 + z^2 - 9 = 0$ is

$$\underline{n} = \frac{\vec{\nabla}\varphi}{|\vec{\nabla}\varphi|} = \frac{2y\underline{j}+2z\underline{k}}{\sqrt{4y^2+4z^2}} = \frac{y\underline{j}+z\underline{k}}{\sqrt{y^2+z^2}}$$



$$\int_S \vec{F} \cdot \underline{n} \; dS = \iint_S \vec{F} \cdot \underline{n} \frac{dx\,dy}{|\underline{n} \cdot \underline{k}|} \qquad (1)$$

$$\underline{n} \cdot \underline{k} == \frac{y\underline{j}+z\underline{k}}{\sqrt{y^2+z^2}} \cdot \underline{k} = \frac{z}{\sqrt{y^2+z^2}}$$

$$\overrightarrow{F}\,.\,\underline{n} = \left(2x^2\underline{i} - y^2\underline{j} + 4zx\underline{k}\right)\,.\left(\frac{y\underline{j}+z\underline{k}}{\sqrt{y^2+z^2}}\right) = \frac{-y^3+4z^2x}{\sqrt{y^2+z^2}}$$

Substituting in (1) , we get :

$$\int_S \overrightarrow{F}\,.\,\underline{n}\ dS = \iint_S \frac{-y^3+4z^2x}{z}\ dx\,dy$$

$$= \iint_S \frac{-y^3+4x(9-y^2)}{\sqrt{9-y^2}}\ dx\,dy$$

$$= \int_0^3\int_0^2 \frac{-y^3+4x(9-y^2)}{\sqrt{9-y^2}}\ dx\,dy$$

$$= \int_0^3 \left[\frac{-xy^3+2x^2(9-y^2)}{\sqrt{9-y^2}}\right]_{x=0}^{x=2}\ dy$$

$$= \int_0^3 \frac{-2y^3+8(9-y^2)}{\sqrt{9-y^2}}\ dy \qquad (2)$$

Putting $y = 3\sin\theta$,so that $dy = 3\cos\theta\ d\theta$, (2) reduces to $= \int_0^{\frac{\pi}{2}}(-6\,sin^3\theta + 72\,cos^2\theta)d\theta = -6\left[\frac{2}{3} + 72\frac{1}{2}\left(\frac{\pi}{2}\right)\right] = -(4+108\pi)$

<u>1-4Green' theorem in a plane:</u>

<u>Statement</u> : If R is a closed region in $xy - plane$ bounded by a simple closed curve C and if and $P(x,y)$ and $Q(x,y)$ are continuous function having continuous partial derivatives in R , then:

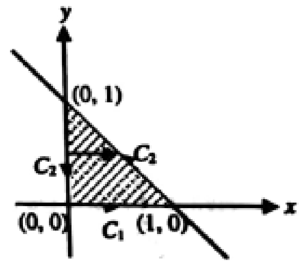
$$\oint P\,dx + Q\,dy = \iint_R \left(Q_x - P_y\right)dxdy \ \text{ where } \ R\ Q_x = \frac{\partial Q}{\partial x}\ ,P_y = \frac{\partial P}{\partial y}$$

<u>Examples</u> :

Verify Green theorem for $\oint(3x - 8\,y^2)\,dx + (4\,y - 6xy)\,dy$ where C

is the boundary of the region bounded by $x = 0 \; and \; y = 0 \; and \; x + y = 1$

The solution



Here $P = 3x - 8\,y^2 \; and \; Q = 4\,y - 6xy$

$\therefore \; P_y = -16\,y \; and \; Q_x = -6y$

Now : $C = c_1 + c_2 + c_3$ **where**

$c_1 = OA \quad : y = 0 \;, \Rightarrow \; dy = 0 \;,$

$c_2 = AB \quad : y = -x + 1 \;, \Rightarrow \; dy = -dx$ and

$c_3 = BO \quad : x = 0 \;, \Rightarrow \; dx = 0$

$\therefore \; I = \int_{c_1} + \int_{c_2} + \int_{c_3}$

$$= \int_0^1 (3x)\,dx + \int_1^0 (3x - 8(-x+1)^2)\,dx$$

$$+\big(4(-x+1) - 6x(-x+1)\big)(-dx) + \int_1^0 (4y)\,dy$$

$$= 3\int_0^1 x\,dx + \int_1^0 (-14x^2 + 29x - 12)\,dx + 4\int_0^1 y\,dy$$

$$= \frac{3}{2} - \frac{7}{6} - 2 = -\frac{5}{3} = L.H.S \qquad (1)$$

Further $\because \; P_y = -16\,y \; and \; Q_x = -6y$

Hence $\iint_R \left(Q_x - P_y\right) dxdy = \iint_R (-6y + 16y)\,dxdy$

$$= 10\int_{y=0}^{y=1}\int_{x=0}^{x=-y+1} y\,dx\,dy = 10\int_0^1 y\,[x]_0^{-y+1}\,dy$$

$$= 10\int_0^1 y(-y+1)\,dy = 10\int_0^1 y(-y^2+y)\,dy$$

$$= \left[-\frac{y^3}{3}+\frac{y^2}{2}\right]_0^1 = 10\left[-\frac{1}{3}+\frac{1}{2}\right] = \frac{10}{6}[-1] = -\frac{5}{3} = R.H.s \qquad (2)$$

From $\frac{y^3}{3}+\frac{y^2}{2}$ (1) & (2) ,we see that the theorem is verified .

Exercises:

(1) Verify Green' theorem for $\oint (xy+y^2)dx + x^2\,dy$ where C is determined by $x = y^2 \; and \; y = x^2$

(2) Verify Green' theorem for the scalar line integral of $\vec{F} = (x^2-y^2)\underline{i} + 2xy\underline{j}$ over the rectangular region bounded by the

$x = 0\,, y = 0, x = a \; and \; y = b$

1-5 Stoke' theorem in a plane:

Statement : Let S be an open surface bounded by a simple closed curve C and if $\vec{F} = F_x\underline{i} + F_y\underline{j} + F_z\underline{k}$, be any continuously differentiable vector point function then :

$\oint_C \overrightarrow{F} \,.\overrightarrow{dr} = \int_S curl\;\overrightarrow{F} \,.\underline{n}\; dS$ (*) Where $\underline{n}$ is the unit external normal vector at any point on S .

Note : Stoke' theorem is another relation between a line integral and a surface integral .

Observations on Stoke' theorem:

(1) writing $\vec{r} = x\underline{i} + y\underline{j} + z\underline{k}$ so that $\overrightarrow{dr} = dx\underline{i} + dy\underline{j} + dz\underline{k}$ and since the unit vector $\underline{n}$ can be written as : $\underline{n} = \cos\alpha\,\underline{i} + \cos\beta\,\underline{j} + \cos\gamma\,\underline{k}$,then

The relation $(*)$ reduces to

$$\oint_C \mathrm{F_x} + \mathrm{F_y} + \mathrm{F_z} = \int_S \begin{vmatrix} \underline{\mathrm{i}} & \underline{\mathrm{j}} & \underline{\mathrm{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathrm{F_x} & \mathrm{F_y} & \mathrm{F_z} \end{vmatrix} .\underline{n}\ dS$$

$$\int_S \left[\left(\frac{\partial \mathrm{F_z}}{\partial y} - \frac{\partial \mathrm{F_y}}{\partial z}\right)\cos\alpha + \left(\frac{\partial \mathrm{F_x}}{\partial z} - \frac{\partial \mathrm{F_z}}{\partial x}\right)\cos\beta + \left(\frac{\partial \mathrm{F_y}}{\partial x} - \frac{\partial \mathrm{F_x}}{\partial y}\right)\cos\gamma\right] dS$$

(2) Let $,\vec{F} = P\underline{i} + Q\underline{j}$ be a vector function which is continuously differentiable in a region S of $xy - plane$ bounded by a closed curve C .Then :

$$\oint_C \vec{F}\,.\overrightarrow{dr} = \oint_C \left(P\underline{i} + Q\underline{j}\right).\left(dx\underline{i} + dy\underline{j}\right) = \oint_C \mathrm{Pdx} + \mathrm{Qdy} \qquad (1)$$

And

$$\text{Let} \quad ,curl\,\vec{F} = \begin{vmatrix} \underline{\mathrm{i}} & \underline{\mathrm{j}} & \underline{\mathrm{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathrm{P} & \mathrm{Q} & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\underline{k} \qquad \left(\frac{\partial}{\partial z} = 0\right)$$

Hence $\int_S curl\,\overrightarrow{F}\,.\underline{n}\;dS = \int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\underline{k}\;\;.\underline{k}\;\,dx\,dy$ (∗)

$$= \int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\;dx\,dy \qquad (2)$$

$$\left(since\ \underline{n} = \underline{k}\ is\ a\ uint\ normal\ to\ xy - plane\right)$$

Expressions (1) & (2) implies that the Stoke theorem reduces to Green

Theorem in this case . Hence Green theorem

In a plane is referred to as Stoke theorem (that is Green theorem is particular case

of Stoke theorem in a plane) .

Exercises:

(1) Verify Stoke theorem for $\vec{F} = (x^2 + y^2)\underline{i} - 2xy\underline{j}$ taking around the rectangular whose vertices are $(-a, 0), (\,a, 0)\,, (a, b)\,, (-a, b)$.

1-6 Gauss divergence'theorem:

Statement : If $\vec{F}$ is a continuously differentiable vector point function in

the region E bounded by the closed surface S then :

$$\oint_S \overrightarrow{F}\,.\underline{n}\;dS = \int_E div\,\overrightarrow{F}\;dV$$

Where $\underline{n}$ is the unit external normal vector at any point on S .

Note : This theorem is a relation between a surface integral and volume integral.

Example :

Verify Gauss divergence 'theorem $\vec{F} = (x^2 - y^2)\underline{i} + (y^2 - zx)\underline{j} + (z^2 - xy)\underline{k}$,

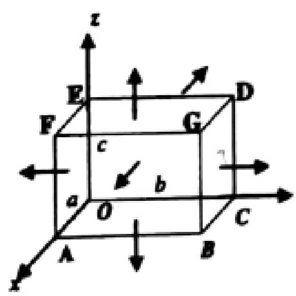
Taken over the rectangular parallelepiped $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$

$\oint (3x - 8\,y^2)\,dx + (4\,y - 6xy)\,dy$ where C

is the boundary of the region bounded by $x = 0$ *and* $y = 0$ *and* $x + y = 1$

The solution

Substituting In the relation :



$$\oint_S \overrightarrow{F} \cdot \underline{n}\;dS = \int_E div\,\overrightarrow{F}\;dV$$

We see That

$$div\,\overrightarrow{F} = 2x + 2y = 2z = 2(x + y + z)$$

$$\therefore \int_E div\,\overrightarrow{F}\;dV == \int_0^c \int_0^b \int_0^a 2(x + y + z)\quad dx\,dy\,dz$$

$$= 2\int_0^c \int_0^b \int_0^a \left[\frac{x^2}{2} + (y + z)x\right]_0^a \quad dy\,dz$$

$$= 2\int_0^c \int_0^b \left[\frac{a^2}{2} + (y + z)a\right] \quad dy\,dz$$

$$= 2a \int_0^c \int_0^b \left[\frac{a}{2} + (y+z)\right] \, dy \, dz$$

$$= 2a \int_0^c \left[\frac{a}{2} y + \left(\frac{y^2}{2} + zy\right)\right]_0^b dz$$

$$= \frac{2ab}{2} \int_0^c [a + (b + 2z)] \, dz$$

$$= \frac{2ab}{2} [az + (bz + z^2)]_0^c$$

$$= ab[ac + (bc + c^2)]$$

$$= abc[a + b + c] = R.H.S \qquad (1)$$

On the surface $S = S_1 + S_2 + S_3 + S_4 + S_5 + S_6$, we have :

For $S_1 = OABC$: will be $z = 0\,, \underline{n} = -\underline{k}$, then :

$$I_1 = \int_{S_1} \vec{F} \,.\, \underline{n} \; dS = -\int_{S_1} (z^2 - xy) \; dx \, dy$$

$$= -\int_{S_1} (-xy) \; dx \, dy = -\int_0^b \int_0^a (-xy) \; dx \, dy = \int_0^b \left[\frac{x^2}{2}\right]_0^a y \, dy$$

$$= \int_0^b \frac{a^2}{2} y dy = \frac{a^2}{2} \int_0^b y \, dy = \frac{a^2}{2} \left[\frac{y^2}{2}\right]_0^b = \frac{a^2 b^2}{4} \; .$$

Similarly on $S_2 = FGDE$: will be $z = c\,, \underline{n} = \underline{k}$, then : $I_2 = \mathrm{abc}^2$

And on $S_3 = OCDE$: will be $x = 0\,, \underline{n} = -\underline{i}$, then : $I_3 = \frac{\mathrm{b}^2 \mathrm{c}^2}{4}$

And on $S_4 = ABGF$: will be $x = a\,, \underline{n} = \underline{i}$, then : $I_4 = a^2 bc - \frac{\mathrm{b}^2 \mathrm{c}^2}{4}$

And on $S_5 = OAFE$: will be $y = 0\,, \underline{n} = -\underline{j}$, then : $I_5 = \frac{\mathrm{a}^2 \mathrm{c}^2}{4}$

And on $S_6 = BCDG$: will be $y = c\ ,\underline{n} = \underline{j}$, then : $I_6 = ab^2c - \frac{a^2c^2}{4}$

From all above we see that

$I = \sum_{i=1}^{6} I_i = abc(a + b + c)$ (2)

From (1) & (2) , we get that the theorem is verified .

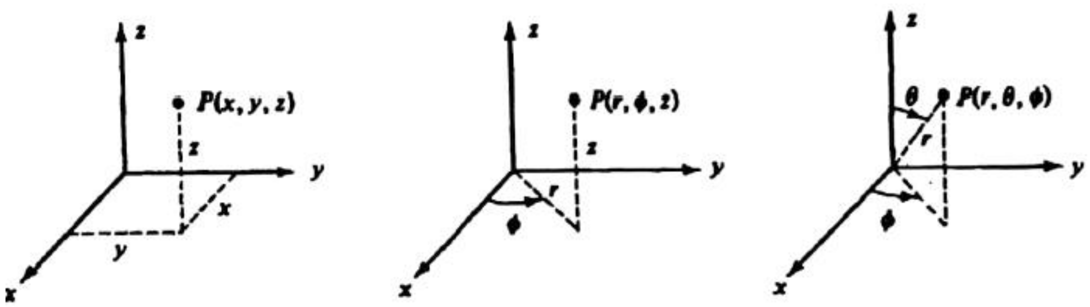
Chapter (3)

Coordinate systems

3-1 The type of coordinates :

A problems which has cylindrical or spherical symmetry could be expressed in the familiar Cartesian coordinate system . However , the solution fail to show the symmetry and in most cases would be needlessly complex .Therefore throughout this course , in addition to the Cartesian system , the circular cylindrical and the spherical coordinate systems ,will be used . All three will examined together in order to illustrate the similarities and differences .

A point P is described by three coordinates , in Cartesian $(x, y. z)$, in circular cylindrical $(\rho, \varphi. z)$, and in spherical $(r, \theta. \varphi)$, as shown in fig. (1). The angel φ is the same angle in both the cylindrical and spherical systems ,but in different order . The z coordinate is the same in both the cylindrical and Cartesian systems in the same order . In the cylindrical coordinate ρ is measures the distance from the z -axis while r in spherical coordinate measures the distance from the origin to that point .



(a) Cartesian (b) Cylindrical (c) Spherical

The component forms of a vector in three systems are

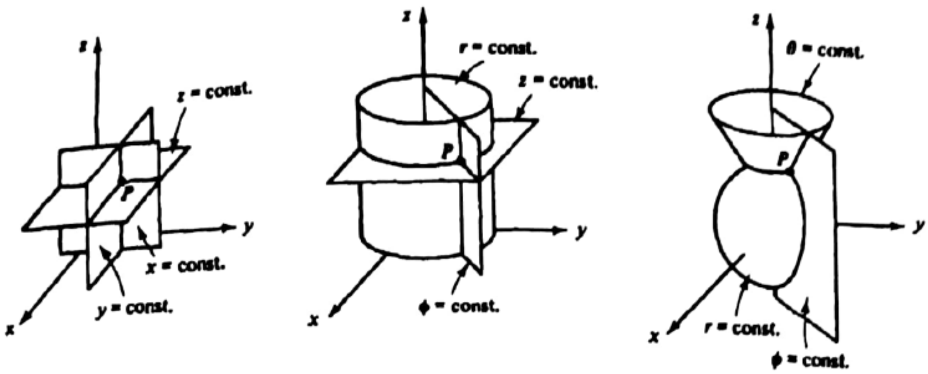
$$\vec{A} = A_x \underline{i} + A_y \underline{j} + A_z \underline{k} \quad , \qquad \text{(Cartesian)}$$

$$\vec{A} = A_\rho \underline{\rho_0} + A_\varphi \underline{\varphi_0} + A_z \underline{k} \quad , \qquad \text{(cylindrical)}$$

$$\vec{A} = A_r \underline{r_0} + A_\theta \underline{\theta_0} + A_\varphi \underline{\varphi_0} \quad , \qquad \text{(Spherical)}$$

It should be noted that the components $A_x, A_\rho, A_\theta, \dots\dots, etc$, are not generally constants but more often are functions of the coordinates in that particular system

,and the $\underline{i}, \underline{\rho_0}, \underline{\theta_0}, \dots\dots, etc$ are unit vectors described in the fig. (2) below



3-2 Differential Volume , Surface and line Elements :

There are relatively few problems in electrostatic an electromagnetic that can be solved without some sort of integration-along a curve , over a surface, or throughout a volume . Hence the corresponding differential elements must be clearly understood .

When the coordinates of point$P(x, y, z)$are expanded to$(x + dx, y + dy, z + dz)$

Or $(\rho + d\rho, \varphi + d\varphi, z + dz)$ or $(r + dr, \theta + d\theta , \varphi + d\varphi)$, a differential volume dv is formed. To the first order in infinitesimal quantities the differential volume is ,in all three coordinate system , a rectangular box . The value of dv in each system is given in fig. (3) .

From fig. (3) may also be read the areas of the surface elements that bound the differential volume . For instance ,in spherical coordinates , the differential surface element perpendicular to $\underline{r_0}$ is

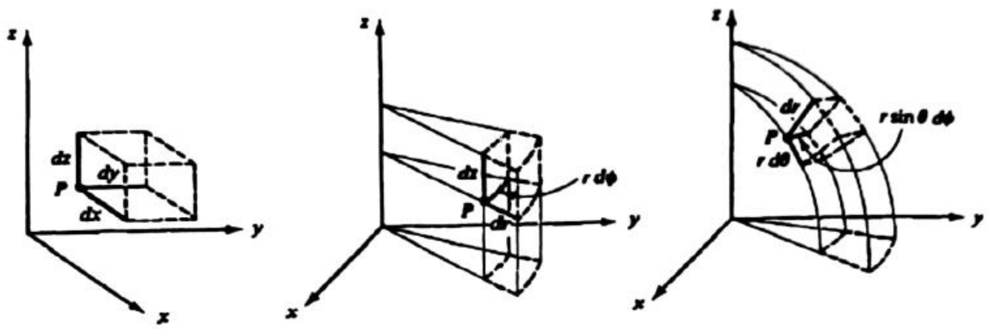
$$ds = (r\, d\theta)(r \sin\theta\, d\varphi) = r^2 \sin\theta\ d\theta\, d\varphi$$

The differential line element , dl is the diagonal through P . Thus :

$$dl^2 = dx^2 + dy^2 + dz^2 \quad , \qquad \text{(Cartesian)}$$

$$dl^2 = d\rho^2 + r^2 d\varphi^2 + dz^2 \quad , \qquad \text{(cylindrical)}$$

$$dl^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2, \qquad \text{(Spherical)}$$



$dv = dx\,dy\,dz$ (a) Cartesian

$dv = r\,dr\,d\phi\,dz$ (b) Cylindrical

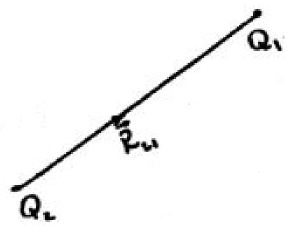
$dv = r^2 \sin\theta\, dr\, d\theta\, d\phi$ (c) Spherical

Chapter (4)

Coulomb Forces

4-1 Coulomb' Law:

There is a force between tow charges which is directly proportional to the charge magnitudes and inversely proportional to the square of the separation distance .



This coulomb law , in vector form stated as :

$$\vec{F} = \frac{Q_1 Q_2}{4\pi\epsilon d^2}\underline{a} \qquad (1)$$

Where $\underline{a}$ is a unit vector in the direction

of $\overrightarrow{R_{21}}$ which is the vector from Q_2 to Q_1 and $Q_2 d = \left|\overrightarrow{R_{21}}\right|$.

ϵ is the permittivity of the medium , with the units $C^2/N^2.m^2$, or , equivalently , Farads per meter (F/m) , where , the force $\vec{F}$ is Newton

(N) , the distance is in meters (m) and the desired unit of charge is the Coulomb (C)

, those are in the Rational SI units . For free space or vacuum we see That :

$$\epsilon = \epsilon_0 = 8.854 \times 10^{-12}\ F/m \cong (10^{-9}/36\pi)\ F/m$$

For media other than free space $\epsilon = \epsilon_0 C_r$, where C_r is the permittivity or dielectric constant .

Free space is to be assumed in all problems and examples as well as the approximate value for ϵ_0 , unless there is a statement to contrary .

Because C is a rather large , charges are often given in :

Micro coulomb $\mu C = 10^{-6} C$

nano coulomb $nC = 10^{-9} C$

pico coulomb $pC = 10^{-12} C$

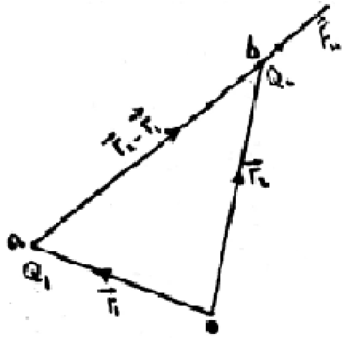
In equation (1) , the force $\vec{F} = \overrightarrow{F_{21}}$ means the force produced by charge the

Q_2on the charge Q_1 ,so the inverse is $\overrightarrow{F_{12}} = -\overrightarrow{F_{21}}$ and $\overrightarrow{R_{12}} = -\overrightarrow{R_{21}}$.

The equation (1) can be rewritten , by refers the vectors w.r.t. to reference of coordinates system $(oxyz)$ for example .

This can be shown as in the front figure

To be



$$\overrightarrow{F_{21}} = \frac{Q_1Q_2}{4\pi\epsilon|\vec{r}_1-\vec{r}_2|^3}\left(\frac{\vec{r}_1-\vec{r}_2}{|\vec{r}_1-\vec{r}_2|}\right) = \frac{Q_1Q_2}{4\pi\epsilon|\vec{r}_1-\vec{r}_2|^2}(\vec{r}_1-\vec{r}_2) \qquad (2)$$

Note that if there is a n charges $Q_1, Q_2, \dots \dots \dots, Q_n$ which have the position vectors $\vec{r}_1, \vec{r}_2, \dots \dots .., \vec{r}_n$,the force on the charge Q_1 with position vector $\vec{r}_1$ is

$$\vec{F} = \sum_{i=1}^{n} \frac{Q_1Q_2}{4\pi\epsilon|r-\vec{r}_i|^2}(r-\vec{r}_i) \qquad (3)$$

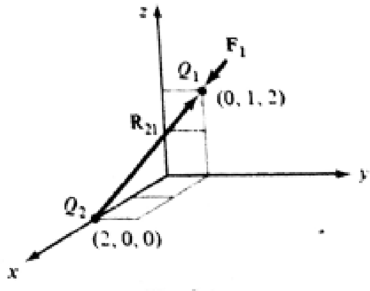
Examples :

(1) Find the force on the charge $Q_1 = 20\ \mu C$,due to charge $Q_2 = -300\ \mu C$,where Q_1 is at $(0,1,2)\ m$ while Q_2 is at $(2,0,0)\ m$.

The solution

Referring to the figure

$\overrightarrow{R_{21}} = -2\underline{i} + \underline{j} + 2\underline{k}$



$\vec{a} = \frac{\overrightarrow{R_{21}}}{|R_{21}|} = \frac{1}{3}\left(-2\underline{i} + \underline{j} + 2\underline{k}\right)$

Then

$\overrightarrow{F_{21}} = \frac{(20\times10^{-6})(-300\times10^{-6})}{4\pi(10^{-9}/36\pi)(3)^2}\left(\frac{-2\underline{i}+\underline{j}+2\underline{k}}{3}\right)$

$= 6\left(\frac{-2\underline{i}+\underline{j}+2\underline{k}}{3}\right) N = \left(4\underline{i} - 2\underline{j} - 4\underline{k}\right) N$.

The force magnitude is 6 N and its direction is such that $Q_1$is attracted to Q_2 .

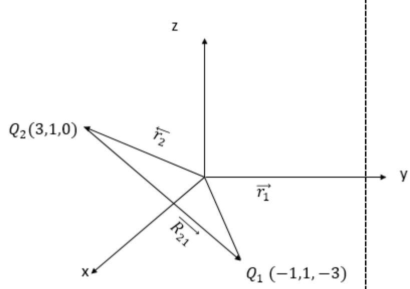
----

(2) Tow point charges $Q_1 = 50\ \mu C$ and $Q_2 = 10\ \mu C$ are located at $(-1,1,-3)\ m$ and $(3,1,0)\ m$ respectively .Find the force on the charge Q_1 .

The solution

Referring to the figure

$\overrightarrow{R_{21}} = -4\underline{i} - 3\underline{k}$



$\vec{a} = \frac{\overrightarrow{R_{21}}}{|R_{21}|} = \frac{-4\underline{i}-3\underline{k}}{5}$

Then

$\overrightarrow{F_{21}} = \frac{Q_1Q_2}{4\pi\epsilon|\overrightarrow{R}_{21}|^2}\ \vec{a} =$

$\frac{(50\times10^{-6})(10\times10^{-6})}{4\pi(10^{-9}/36\pi)(5)^2}\left(\frac{-4\underline{i}-3\underline{k}}{5}\right)$

$= (0.18)\left(-0.8\underline{i} - 0.6\underline{k}\right) N$.

The force magnitude is $0.18\ N$ and its direction is given by the unit vector $Q_1$is $= -0.8\underline{i} - 0.6\underline{k}$.

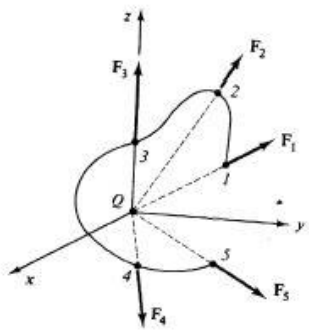
==

Exercises:

(1) Find the force on $100\ \mu C$ charge at $(0,0,3)\ m$ as a result of existence of four like charges of $20\ \mu C$ which located on $x\ and\ y$ at $\pm 4\ m$.

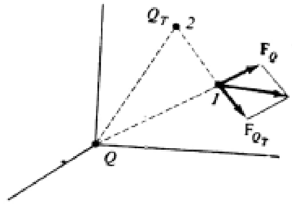
(2) A point charge $Q = 300\ \mu C$ located at $(1,-1,-3)\ m$ experiences a force $\vec{F} = \left(8\underline{i} - 8\underline{j} + 4\underline{k}\right) N$ due to a point charge Q_2 at $(3,-3,-2)\ m$. Find Q_2 . (3) Find the force on a point charge of $50\ \mu C$ at $(0,0,5)\ m$ due to a point charge of $500\ \pi\ \mu C$ that is uniformly distributed over the circular disk $r < 5\ m$, $z = 0\ m$.

In the region around an isolated point charge there is a spherically symmetrically force field. This is made evident when charge Q is fixed at The origin ,as in Fig. (1) and a second charge



Q_T , is moved about in the region . At each location a force acts along the line joining the tow charges directed away from the origin if the charges are of like sign. This can be expressed in spherical coordinates by

$$\vec{F} = \frac{Q_1 Q_T}{4\pi\epsilon_0 r^2}\underline{a_r} \qquad (4)$$



It should be noted that unless $Q_T \ll Q$

The symmetrically field at Q is disturbed by Q_T .

At location 1 in Fig (2),the force $\overrightarrow{F_1}$ is seen to be the vector sum $\overrightarrow{F_1} = \overrightarrow{F_Q} + \overrightarrow{F_{Q_T}}$

This should come as no surprise , since if Q has a force field so also must Q_T .

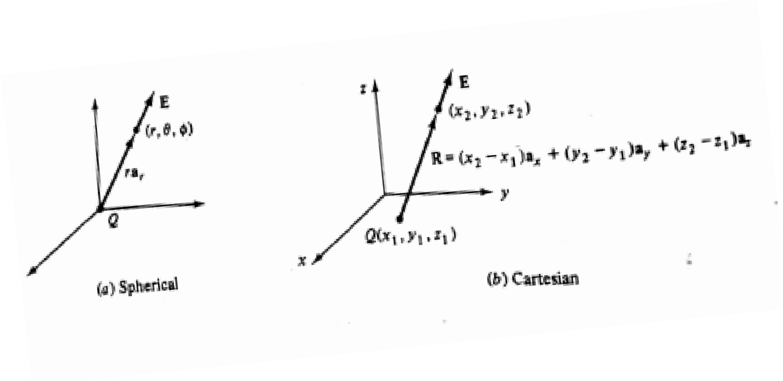
When the tow charges are in same region , the resulting field will of necessity

be the point-by point vector sum of the two fields .This is the superposition principle for coulomb forces , it extends to any number of charges .

4-2 Electric Field Intensity:

Suppose that ,in the above situation ,the test charge Q_T is sufficiently small as so not to disturb significantly the field of the fixed charge Q.Then the electric field intensity , $\vec{E}$, due to Q is defined to be the force per unit charge on Q_T :

$$\vec{E} = \frac{1}{Q_T}\overrightarrow{F_T} = \frac{Q}{4\pi\epsilon_0 r^2}\underline{a_r} \qquad (5)$$



The expression for $\vec{E}$ is in spherical coordinates with origin at the location of Q

(fig. $(3\ a)$) . It may be transformed to other coordinate system . In an arbitrary Cartesian coordinate system

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 R^2}\underline{a_R} \qquad (6)$$

Where the separation vector $\vec{R}$ is as given in (fig. $(3\ b)$) .

The units of $\vec{E}$ are Newton per coulomb (N/C) of the equivalent Volts per meter (V/m) .

<u>4-3 Charge Distributions:</u>

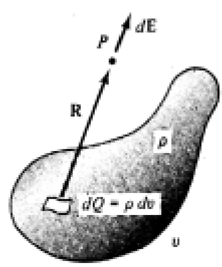
(1) <u>Volume charge</u>

When charge is distributed throughout a specified volume, each charge element contributes to the electric field at an external point .

A summation or integration is then required to

obtain the total electric field .

It is useful to consider continuous



(in fact differentiable) charge distribution

and to define charge density by $\rho_v = \frac{dQ}{dv}$ (C/m^3) , then $dQ = \rho dv$

with reference to volume v in Fig (4) , each differential charge dQ produces

a differential electric field :

$$d\vec{E} = \frac{dQ}{4\pi\epsilon_0 R^2} \underline{a_R}$$

At the observation point P . Assuming that the only charge in the region is

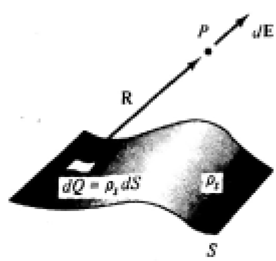
contained within the volume, the total electric field at $P$is obtained by integration

over the volume is :

$$\vec{E} = \int_v \frac{\rho_v}{4\pi\epsilon_0 R^2} dv \, \underline{a_R} \qquad (7)$$

(2) <u>Sheet charge</u>

When charge is distributed over a specified surface or sheet , each differential charge element dQ on the sheet results in a differential electric field :



$$d\vec{E} = \frac{dQ}{4\pi\epsilon_0 R^2} \underline{a_R}$$

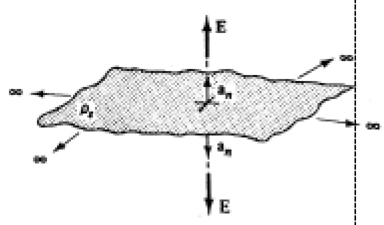
at the point P see the Fig (5) . If the charge density is ρ_s (C/m^2) and if no other charge is present in the region , then the total electric field at P is

$$\vec{E} = \int_s \frac{\rho_s}{4\pi\epsilon_0 R^2}\, ds\, \underline{a_R}$$

If charge is distributed with uniform density ρ_s (C/m^2) over an infinite plane ,

Then the field is given by :

$$\vec{E} = \frac{\rho_s}{2\epsilon_0}\, \underline{a_n} \qquad (8)$$



This field is of constant magnitude and has

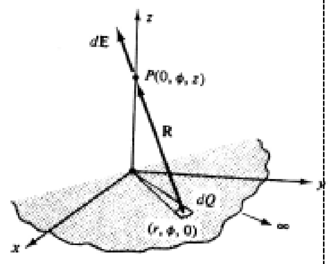
Mirror symmetry about the plane charge , and

the derivation of last equation by use the cylindrical

coordinates system , with the charge in the $z = 0$ plane as shown in Fig (7)

$$d\,\vec{E} = \frac{\rho_s\, r dr\, d\varphi}{4\pi\epsilon_0 (r^2+z^2)}\left(\frac{-\underline{a_r} + z\underline{a_z}}{\sqrt{r^2+z^2}}\right)$$

Symmetry about the $z-$ axis results in cancellation of redial components



$$\vec{E} = \int_0^{2\pi}\int_0^{\infty} \frac{\rho_s\, rzdr\, d\varphi}{4\pi\epsilon_0 (r^2+z^2)^{3/2}}\ \underline{a_z} \qquad \vec{E} = \frac{\rho_s\, z}{2\epsilon_0}\left[\frac{-1}{\sqrt{r^2+z^2}}\right]_0^{\infty} \underline{a_z} = \frac{\rho_s\, z}{2\epsilon_0}\ \underline{a_z} = \frac{\rho_s\, z}{2\epsilon_0}\ \underline{k}\ .$$

This result is for points above the xy plane . Below the xy plane the unit

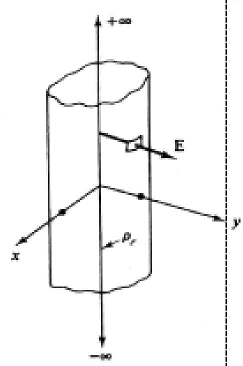
vector changes to $-\underline{a_z} = -\underline{k}$.

The generalized form may be written using unit normal vector $\underline{a_n}$ as

$$\vec{E} = \frac{\rho_s}{2\epsilon_0}\ \underline{a_n}$$

This eclectic field is everywhere normal to the plane of the charge and its magnitude is independent of the distance from the plane .

(3) <u>Infinite line charge:</u>



If charge is distributed with uniform density ρ_l (C/m) along an infinite straight line –which will be chosen as the $z-$ axis , then the field is given by

$$\vec{E} = \frac{\rho_l}{2\epsilon_0 r}\ \underline{a_r}$$

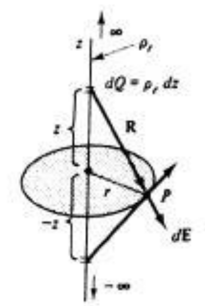
This is in cylindrical coordinates see Fig (8) This field has cylindrical symmetry

and is inversely proportional to the first power of the distance from the line charge .

For derivation of this form of $\vec{E}$, we will use cylindrical coordinates see Fig (9)

At P $$d\,\vec{E} = \frac{dQ}{4\pi\epsilon_0 R^2}\left(\frac{r\underline{a_r} - r\underline{a_z}}{\sqrt{r^2+z^2}}\right)$$

Since for every dQ at z , there is another chargeddQ at $-z$,then the z component will canceled .Thus



$$\vec{E} = \int_{-\infty}^{\infty} \frac{\rho_l\, r\, dz}{4\pi\epsilon_0 (r^2+z^2)^{3/2}}\, \underline{a_r}$$

From Fig (9) we see that, $tan\,\theta = z/r$,$\therefore z = r\tan\theta$,which tends to

$$dz = r\ sec^2\theta\, d\,\theta$$

Then , the field $\vec{E}$ will be

$$\vec{E} = \frac{\rho_l}{4\pi\epsilon_0}\int_{-\pi/2}^{\pi/2} \frac{r^2\, sec^2\theta\, d\,\theta}{(r^2 + r^2 tan^2\theta)^{3/2}}\, \underline{a_r}$$

$$= \frac{\rho_l}{4\pi\epsilon_0}\int_{-\pi/2}^{\pi/2} \frac{r^2\, sec^2\theta\, d\,\theta}{r^3(1 + tan^2\theta)^{3/2}}\, \underline{a_r}$$

$$= \frac{\rho_l}{4\pi\epsilon_0 r}\int_{-\pi/2}^{\pi/2} \frac{sec^2\theta\, d\,\theta}{sec^3\theta}\, \underline{a_r} = \frac{\rho_l}{4\pi\epsilon_0 r}\int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta\, \underline{a_r}$$

$$= \frac{\rho_l}{4\pi\epsilon_0 r}[\sin\,\theta]_{-\pi/2}^{\pi/2}\, \underline{a_r} = \frac{\rho_l}{4\pi\epsilon_0 r}[1+1]\, \underline{a_r} = \frac{\rho_l}{2\pi\epsilon_0 r}\, \underline{a_r}\ .$$

<u>Examples</u> :

(1) A plane $y = 3\ m$ contains a uniform charge distribution of density

$\rho_s = \frac{10^{-8}}{6\pi} 20 \ \mathrm{C/m^2}$, determine $\vec{E}$ at all points .

The solution

For $y > 3\,m$:

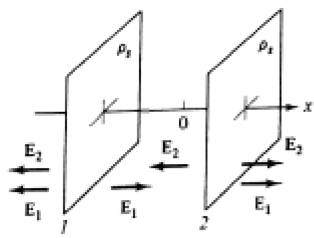
$$\vec{E} = \frac{\rho_s}{2\epsilon_0}\,\underline{a_n} = \frac{(10^{-8}/6\pi)}{2(10^{-9}/36\pi)}\,\underline{j} = 30\,\underline{j} \quad \mathrm{V/m}$$

---

(2) Tow infinite uniform sheets of charge , each with density ρ_s , are located at $x = \pm a$, determine $\vec{E}$ in all regions .

The solution

Only parts of the tow sheets results of charge are in the front figure .



Both sheets result in $\vec{E}$ fields that are directed along $x-$ axis , independent of the distance , then :

$$\vec{E} = \vec{E}_1 + \vec{E}_2 \begin{cases} \frac{\rho_s}{2\epsilon_0}(-\underline{i}) + \frac{\rho_s}{2\epsilon_0}(-\underline{i}) & : x < -a \\ \frac{\rho_s}{2\epsilon_0}(\underline{i}) + \frac{\rho_s}{2\epsilon_0}(-\underline{i}) & : -a < x < a \\ \frac{\rho_s}{2\epsilon_0}(\underline{i}) + \frac{\rho_s}{2\epsilon_0}(\underline{i}) & : x > a \end{cases} = \begin{cases} -\frac{\rho_s}{\epsilon_0}\,\underline{i} & : x < -a \\ 0 & : |x| < a \\ \frac{\rho_s}{\epsilon_0}(\underline{i}) & : x > a \end{cases}$$

---

(3) Find $\vec{E}$ in example (2) in case of the sheet -1- has a density ρ_s , while the sheet -2- has a density $-\rho_s$.

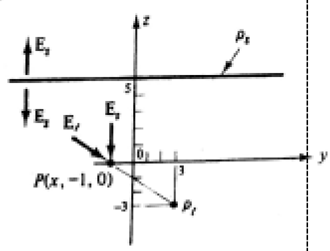
The solution

$$\vec{E} = \begin{cases} 0 & : x < -a \\ \dfrac{\rho_s}{\epsilon_0}\ (\underline{i}) & : |x| < a \\ 0 & : x > a \end{cases}$$

----

(4) A uniform sheet charge with $\rho_s = \frac{1}{3\pi}\ \mathrm{NC/m^2}$ is located at $z = 5m$ and a uniform line charge with $\rho_l = \frac{-25}{9}\ \mathrm{NC/m}$ which paths through the point $(x, 3, -3)$ m and parallel to $x -$ axis . Find $\vec{E}$ at the point $(x, -1, 0)$ m .

The solution



The two charge configuration are parallel to $x -$ axis .Hence the view in the figure is taken looking at the yz plane from positive x.

Due to the sheet charge $\rho_s = \frac{1}{3\pi}\ \mathrm{NC} = \frac{1}{3\pi}\ 10^{-6}\mathrm{C}$

$$(\vec{E}_s)_P = \frac{\rho_s}{2\epsilon_0}\ \underline{a_n} = \frac{(10^{-6}/3\pi)}{2\left((1/36\pi)10^{-6}\right)}\ (-\underline{k})$$

$$= (10^{-6}/3\pi)(36\pi/2)(10^6)(-\underline{k}) = -6\ \underline{k}\ \mathrm{V/m}$$

Due to the line charge $\rho_l = \frac{-25}{9}\ \mathrm{NC/m} \quad = \frac{-25}{9}\ 10^{-6}\ \mathrm{C/m}$

$$(\vec{E}_l)_P = \frac{\rho_l}{2\pi\epsilon_0 r}\ \underline{a_r} = \frac{\left(\frac{-25}{9}\ 10^{-6}\right)}{2\pi\left((10^{-6}/36\pi)5\right)}\left(\frac{-4\underline{j}+3\underline{k}}{5}\right)\frac{36}{9}$$

$$= 2\left(4\underline{j} - 3\underline{k}\right) = 8\underline{j} - 6\underline{k}$$

Then the total electric field is

$$\vec{E} = (\vec{E}_s)_P + (\vec{E}_l)_P = -8\underline{j} - 12\underline{k} \quad \mathrm{V/m} \quad .$$

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Exercises:

(1) Determine $\vec{E}$ at $(2,0,2)$ m due to three standard charge distributions as follows :a uniform sheet at $x = 0\ m$ with $\rho_{s1} = \frac{1}{3\pi}\ \mathrm{NC/m^2}$, a uniform sheet at $x = 4\ m$ with $\rho_{s2} = \frac{-1}{3\pi}\ \mathrm{NC/m^2}$ and a uniform line at $x = 6\ m, y = 0\ m$ with $\rho_l = -2\ \mathrm{NC/m}$.

(2) Determine $\vec{E}$ at $(2,0,0)$ m due to a charge distributed along the $z-$ axis Between $z = \pm 5\ m$ with a uniform density $\rho_l = 20\,\mathrm{NC/m}$ in Cartesian coordinates, then in cylindrical coordinates .

(3) Determine $\vec{E}$ at $(2,0,0)$ m due to a charge distributed from $z = 5\ m$ along the $z-$ axis to ∞ and from $-\infty$ to $z = -5\ m$ with a uniform density $\rho_l = 20\,\mathrm{NC/m}$ in both Cartesian coordinates, and cylindrical coordinates .

(4) What will happen if the charge configuration of problem (2) & (3) are superimposed .

(5) Find the electric field intensity $\vec{E}$ at $(0, \varphi, h)$ m in cylindrical coordinates due to uniformly charged disk $r \leq a\ m, z = 0\ m$. what is result if $a \longrightarrow \infty$.

Chapter (5)

Electric flux and Gauss'Law

5-1Net charge in a region:

With charge density defined before , it is possible to obtain the net charge contained in a specified volume by integration .From :

$$dQ = \rho\, dv \qquad C$$

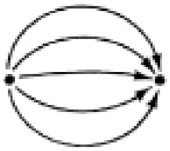
it follows that

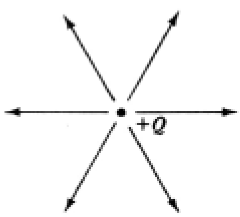
$$Q = \int_v \rho\, dv \qquad C$$

Of course, ρneeds not be constant through the volume v .

5-2Electric flux:

By definition electric flux ,ψ ,originates on positive charge and terminates on negative charge.In the absence of negative charge, the flux terminates at infinity. Also by definition , one coulomb of electric charge gives rise to one coulomb of electric flux .Hence :$\psi = Q \qquad C$

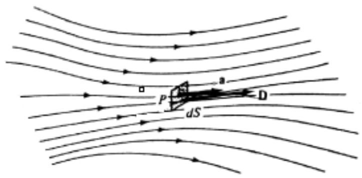
+Q  −Q



In the above Fig (a) the flux leaves Q_+ and terminates on Q_- . This assum that the tow charges of equal magnitude . the case of positive charge with no negative charge in the region is illustrated in Fig (b) . Hence thefluxlines are equally spaced throughout the solid angle , and reach out toward infinity .

5-3flux Density:

While the electric flux ψ is a scalar quantity , the density of electric flux $\vec{D}$, is a vector field which takes its direction from the lines of flux .



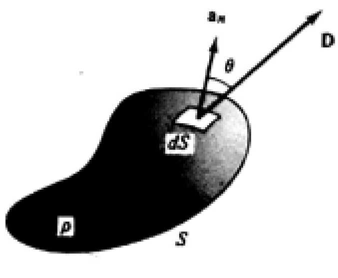
If the neighborhood of point P the lines of flux have the direction of unit vector $\vec{a}$, see the front Fig. ,and if an amount of flux $d\psi$ crosses the differential area ds which is normal to $\vec{a}$,then the direction flux density at P isS

$$\vec{D} = \frac{d\psi}{ds}\vec{a}\,C/m^2$$

Which means that the density of electric flux $\vec{D}$ is numbers of the flux lines cross the unit area in normal direction (**<u>this is the definition of the density $\vec{D}$</u>**)

5-4Gauss' Law:

A volume charge distribution of density ρ C/m^2 is shown enclosed by surface s as in Fig. beside .



Since each coulomb of charge Q has , by definition ,

One coulomb of flux ψ , it follows that the net flux

Crossing the closed surface s is an exact measure of the net charge enclosed.

However the density $\vec{D}$ may vary in magnitude and direction from point to point of s , on general ,$\vec{D}$ makes an angle φ with the normal , the differential flux crossing ds is given by :

$$d\psi = D\cos\theta = \vec{D} \,.\, ds\,\vec{a} = \vec{D} \,.\, \overrightarrow{ds}$$

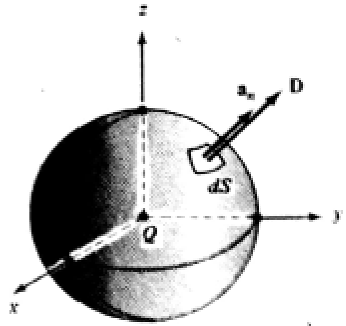
Where $\overrightarrow{ds}$ is the vector surface element , of magnitude ds and direction $\vec{a}$ .The unit vector $\vec{a}$ is always taken to point out of s , so that $d\psi$ is the amount of flux passing from interior of s to the exterior of s through ds .

Integration of the above expression for $d\psi$ over the closed surface s gives :

$$\psi = \int_{s} \vec{D} \,.\, \overrightarrow{ds} = Q \;.$$

This is Gauss' Law , which states that the total flux out of a closed surface is equal to the net charge within that surface . It will be seen that a great deal of valuable information can be obtained from the application of Gauss' Law without actually carrying out the integration .

5-5Relation between the flux denisty $\overrightarrow{D}$and Electric field intensity $\overrightarrow{E}$:



Consider a point charge Q (assumed positivef

For simplicity) at the origin as shown in the front

Figure. If this is enclosed by a spherical surface of

Radius r , then , by symmetry , $\vec{D}$ duetoQ is of

Constant magnitude over the surface and is everywhere normal to the surface .

Gauss' Law then gives :

$$Q = \int_s \vec{D} \,.\, \overrightarrow{ds} = D \oint_s ds = D(4\pi r^2) \quad (\theta = 0) \ .$$

From which $D = Q/4\pi r^2$, therefore

$$\vec{D} = \frac{Q}{4\pi r^2}\vec{a} = \frac{Q}{4\pi r^2}\overrightarrow{a_r}$$

But, from (3) , the electric field intensity $\vec{E}$ due to point charge Q is

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\overrightarrow{a_r}$$

It follows that

$$\vec{D} = \epsilon_0 \vec{E}$$

More generally, for any electric field in an E isotropic medium of permittivity ϵ

Then

$$\vec{D} = \epsilon\,\vec{E} \ .$$

Thus $\vec{D}\ and\ \ \vec{E}$ fields will have exactly the same form , since they defers only by a factor which is a constant of the medium .

While the electric field$\vec{E}$ due to a charge configuration is a function of the permittivity ϵ ,the flux density$\vec{D}$ is not . In problems involving multiple dielectrics a distinct advantage will be found in first obtaining $\vec{D}$,then converting to $\vec{E}$ within each dielectric .

<u>Note:</u> Special Gaussian surface:

The spherical surface used in derivation of section (5) was a special Gaussian surface in that it satisfied the following defining conditions :

$1 -$ The surface is closed .

$2 -$At each point of surface $\vec{D}$ either normal or tangential to thesurface .

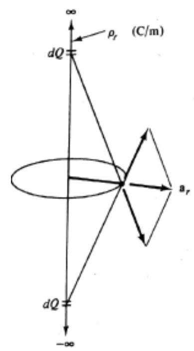
$3 - \vec{D}$ has the same value at all points of the surface where$\vec{D}$ is normal.

<u>Examples</u> :

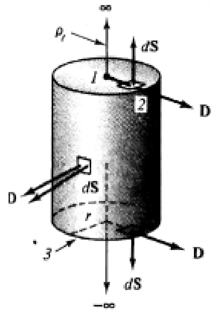
(1) Use a special Gaussian surface to find$D$due to a uniform line charge $\rho_l(C/m)$

<u>The solution :</u>

Takethe line charge as the a special $z - axis$ of cylindrical coordinates (see the Fig (1). By cylindrical symmetry , $\vec{D}$ can only have an r



component , and this component can only depends on r . Thus , the special Gaussian surface for this problem is a closed circular cylinder whose axis is the $z - axis$ (see the Fig (2).



Appling Gauss' Law :

$$Q = \int_{s_1} \vec{D} \,.\, \overrightarrow{ds} + \int_{s_2} \vec{D} \,.\, \overrightarrow{ds} + \int_{s_3} \vec{D} \,.\, \overrightarrow{ds} \ .$$

Over surfaces s_1 , s_2 ,we note that $\vec{D} \& \overrightarrow{ds}$ are orthogonal and so the integrals vanish $(\theta = \pi/2)$ .Over surface s_3 ,we note that $\vec{D} \& \overrightarrow{ds}$ are parallel (or antiparallel $(\theta = 0)$, if ρ_l is negative) and D is constant because r is constant. Thus

$$Q = \int_{s_2} \vec{D} \,.\, \overrightarrow{ds} = D \int_{s_3} ds = D(2\pi r L) \ .$$

Where $L$is thelength of cylinder .But the enclosed charge is $Q = \rho_l L$. Hence :

$$D = \rho_l / 2\pi r \text{ and } \quad \vec{D} = (\rho_l / 2\pi r\,) \overrightarrow{a_r} \ .$$

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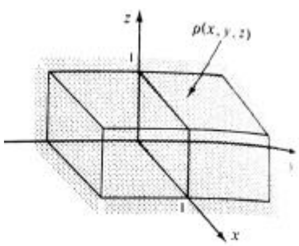
(2) Find the charge in the volume defined by $0 \le x\,,\ y\,,\ z\ \le 1\, m$ if $\rho = 30\, x^2 y \ \ (\mu C/m^2)$. What change occurs for the limit ,$-1 \le \ y \ \le 0\, m$?

The solution :

Since $dQ = \rho\, dv$

$$\therefore\ Q = \int_0^1 \int_0^1 \int_0^1 30\, x^2 y \ \ dx\, dy\, dz \ \ = 5\mu C$$



For the change in the limit on y

$$Q = \int_0^1 \int_{-1}^0 \int_0^1 30\, x^2 y \;\; dx\, dy\, dz \;\; = -5\mu C$$

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(3) Find the charge in the volume defined by $1 \le r \le 2\ m$ in spherical coordinates if $\rho = 30\ x^2 y\ \ (\mu C/m^2)$.

$\rho = \frac{5\cos^2\theta}{r^4} (C/m^3)$.

The solution :

By integration we get :

$$Q = \rho\, dv = \int_0^{2\pi} \int_0^{\pi} \int_0^2 \frac{5\cos^2\theta}{r^4} r^2 \sin\theta\, dr\, d\theta\, d\varphi \;\; = 5\pi\ C \quad .$$

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(4) Three point charge $Q_1 = 30\ n\ C$, $Q_2 = 150\ n\ C$ &$Q_3 = -70\ n\ C$ are enclosed by surface s .What net flux crosses s .

The solution :

Since the electric flux was defined as originates on positive charges and terminates on the negative charges ,the part of the flux from the positive chargesterminates on the negative charges.

$$\psi_{net} = Q_{net} = 30 + 150 - 70 = 110\ nC \quad .$$

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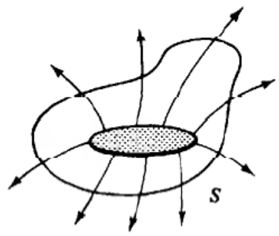
(5) What net flux crosses the closed surface s shown in the Fig. , which contains a charge distribution in the form of a plane disk of radius $4\ m$ with a density $\rho_s = \frac{(\sin^2\varphi)}{2r} (C/m^2)$? .

The solution :

$\psi = Q = \int_0^{2\pi} \int_0^4 \frac{(\sin^2\varphi)}{2r}\ r\ dr\ d\varphi = 2\pi\ C$.



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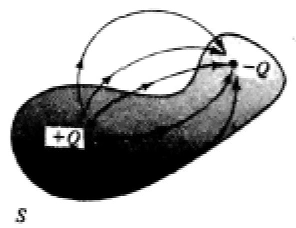
(6) Two point charges of the same magnitude but opposite signs are enclosedby a surface s. Can flux ψ cross the surface ? .

The solution :

While the flux can cross the surface as shown

In the Fig. , the net flux out of vector s will be

Zero so long as the charges are of the same magnitude .



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(7) A circular disk of radius $4\ m$ with a charge density $\rho_s = 12 \sin\varphi\ (\mu C/m^2)$ isenclosed by surface s . What net flux crosses s? .

The solution :

$\psi = Q = \int_0^{2\pi} \int_0^4 \sin\varphi\ r\ dr\ d\varphi = 0\ \mu\ C$.

Since the disk contains equal amounts of positive and negative charge $Q_1 \sin(\varphi+\pi) = -\sin\varphi$. not net flux cross s .

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(8) Give an electric flux $\vec{D} = 2x\,\underline{i} + 3\,\underline{j}(\mathrm{C/m^2})$, determine the net flux crossing the surface of cube $2m$ on an edge centered at the origin .(the edge of the cube are parallel to the coordinate axis) .

The solution :

$$\psi = \int_s \vec{D}\,.\,\overrightarrow{ds} = \int_{x=1}\left(2\,\underline{i} + 3\,\underline{j}\right).\left(ds\underline{i}\right) + \int_{x=-1}\left(-2\,\underline{i} + 3\,\underline{j}\right).\left(-ds\underline{i}\right) .$$

$$+\int_{y=1}\left(2x\,\underline{i} + 3\,\underline{j}\right).\left(ds\underline{j}\right) + \int_{y=-1}\left(2x\,\underline{i} + 3\,\underline{j}\right).\left(-ds\underline{j}\right)$$

$$+\int_{z=1}\left(2x\,\underline{i} + 3\,\underline{j}\right).\left(ds\underline{k}\right) + \int_{z=-1}\left(2x\,\underline{i} + 3\,\underline{j}\right).\left(-ds\underline{k}\right)$$

$$\therefore \quad \psi = \int_s \vec{D}\,.\,\overrightarrow{ds} = 2\int_{x=1} ds + 2\int_{x=-1} ds \;.$$

$$+3\int_{y=1} ds - 3\int_{y=-1} ds + 0 + 0 = 16\ C$$

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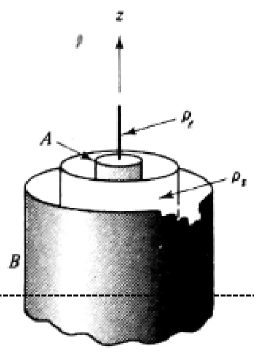
(9) A uniform line charge of $\sigma_l = 3\ (\mu\mathrm{C/m})$ lies along the $z-$axis , and a concentric circular cylinder of radius $2\ m$ has $\sigma_s = (-1.5/4\pi)(\mu\mathrm{C/m^2})$.Both distributions are infinite in extent with $z-$axis . Use Gauss' Lawto find $\vec{D}$ in all regions.

The solution :

1 $-$Using the special Gaussian surface A As Shawn in the Fig. ,and proceeding as in example (1), we get :



$$Q = \sigma_l l \Longrightarrow \vec{D} = \frac{\sigma_l l}{2\pi r}\overrightarrow{a_r} \qquad 0 < r < 2$$

2 −Using the special Gaussian surface B

$$Q = Q_1 + Q_2 = \sigma_l l + \sigma_s(2\pi r l) \qquad r > 2$$

$$= \left(\sigma_l + \sigma_s(2\pi r)\right)\, l \qquad r > 2$$

From Gauss' Law , we get :

$$Q = \int_s \vec{D} \cdot \overrightarrow{ds} = D\oint_s ds = D(2\pi r\, l) \ .$$

$$\therefore \left(\sigma_l + \sigma_s(2\pi r)\right)\, l = \int_s \vec{D} \cdot \overrightarrow{ds} = D\oint_s ds = D(2\pi r\, l) \ .$$

$$\therefore D = \frac{\left(\sigma_l + \sigma_s(2\pi r)\right)}{2\pi r}$$

$$\therefore \vec{D} = \frac{\left(\sigma_l + \sigma_s(2\pi r)\right)}{2\pi r}\overrightarrow{a_r} \qquad r > 2$$

Now for all regions we have :

$$\therefore \vec{D} = \begin{cases} \dfrac{\sigma_l l}{2\pi r}\overrightarrow{a_r} & 0 < r < 2 \\ \dfrac{\left(\sigma_l + \sigma_s(2\pi r)\right)}{2\pi r}\overrightarrow{a_r} & r > 2 \end{cases}$$

And for numerical data we have :

$$\vec{D} = \begin{cases} \dfrac{0.477}{r}\overrightarrow{a_r} & 0 < r < 2 \\ \dfrac{0.239}{r}\overrightarrow{a_r} & r > 2 \end{cases}$$