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(Numerical and Complex Analysis)

استاذ المقرر : د. اسماعيل جاد امين

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الفصل الدراسي الثاني

Numerical Analysis

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Chapter 1

Initial-Value Problems

1.1 Introduction

It's well known that many differential equation, specially the nonlinear type, has no analytical solution, therefore the numerical methods arise for such cases.
In the current chapter, we will present some of those methods for the ordinary differential equation of order one that has the following form:

$$y' = \frac{dy}{dx} = f(x, y), \qquad y(a) = y_0, \text{ and } x \in [a, b]. \tag{1.1}$$

Equation (1.1), that has a given initial value, is well known as an **Initial value problem**. In this equation the function $f(x, y)$ in the right hand side has to be continuous function in its domain. Before we present the numerical methods for such type of equation, we shall present some preliminaries that has to be verified from the mathematical analysis point of view.

Definition 1.1.1 — Lipschitz condition. A function $f(x, y)$ is said to be Lipschitz in the variable y at a region I with $I = \{(x, y), a \leq x \leq b, c \leq y \leq d\}$, if there exist a constant $L > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2| \; \forall c \leq y_1 \leq y_2 \leq d.$$

Theorem 1.1.1 — Existence and uniqueness of the solution . The initial value problem

$$y' = f(x,y) \text{ on } [a,b], y(x_o) = y_0$$

has a unique solution in the interval $[a,b]$ if

1. $f(x,y)$ in continuous with respect to x,y
2. the function f in Lipschitz in the variable y.

The initial value problem

$$y' = f(x,y), y(x_o) = y_0$$

has a unique solution if $f(x,y)$ is differentiable with respect to y, and $|f_y(x,y)| \leq L$ in the region $I = \{(x,y), a \leq x \leq b, c \leq y \leq d\}$.

■ **Example 1.1** Verify that the following initial value problem

$$y' = \frac{1}{2}(x+y), \qquad y(0) = 1$$

has a unique solution. ■

Solution.

$$\begin{aligned} &f(x,y) = \tfrac{1}{2}(x+y) \\ &f_y(x,y) = \tfrac{1}{2} \\ &|f(x,y_1) - f(x,y_2)| \leq |f_y(x,y)(y_1-y_2)| = \tfrac{1}{2}|y_1-y_2| \end{aligned} \tag{1.2}$$

this means that $f(x,y)$ verifies the Lipschitz condition and it is a polynomial of order one, thus it's a continuous in x,y. Therefore, this initial value problem has a unique solution. ■

Ⓡ **Second order differential equation:** The second order differential equation with two initial conditions can be converted to two equations from the first order, for instance,

$$\begin{aligned} &y'' - xy' - x^2y^2 = x^3 \\ &y(0) = 1, y'(0) = 2 \end{aligned}$$

using $y' = z$, then, we can rewritten that equation in the following system

$$\begin{aligned} &y' = z \\ &z' = xz + x^2y^2 + x^3 \\ &y(0) = 1, z(0) = 2 \end{aligned}$$

equivalent to the following form,

$$\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} z \\ xz + xz + x^3 \end{pmatrix}, \begin{pmatrix} y(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

This remark could be generalized to n order differential equation with n initial conditions in the form

$$\begin{aligned} &a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x) \\ &y^{(m)}(0) = y_0^{(m)}, m = 0, 1, 2, \cdots, n-1 \end{aligned}$$

where, $a_o, a_1, \ldots, a_n$ are functions of x, y only. The resulting system will be n equations from the first order as

$$Y' = F(x, Y), Y(0) = Y_0$$

1.2 The numerical solution for the initial value problem

Using the different numerical method, we are able to find an approximate value for the function $y(x)$ at the points $x_1, x_2, x_3, \ldots, x_n$ which divided the interval $[a, b]$ into equal partitions. During this course, we will present the numerical methods for solving a system of first order differential equation as well as the higher order system of Ordinary Differential Equations(ODEs).

The known methods that is used to solve the ODEs could be classified into two main type, namely

- One-step methods
- Multi-step methods

In the one-step methods, the value of the function at a point is determined using only it's value at the previous point. On the other hand, for the multi-step methods it is calculated using the value of the function at many points that which are known from the previous steps.
Some of those methods(one-step methods)that used to solve the first order ODEs are:

- Picard method
- Taylor method
- Runge-Kutta method

1.3 Single-step methods or One-step methods

As its mentioned above, in those method the value of the functions is estimated at a point using only its value at the previous point, thus using only one value to estimate the value of the function at another point.

1.3.1 Picard method

One of the one-step method that is used to solve the ODEs of first order and this method depends on the integration of the function as we will see later.
Let $y' = f(x, y)$ with the initial condition is $y(x_0) = y_0$ and we need to find the value of the function at $x_0 + h$ i.e. $y(x_0 + h)$ such that

$$y' = f(x, y), \qquad y(x_0) = y_0 \tag{1.3}$$

integrating the above equation from x_0 to x, we have

$$\int_{x_0}^{x} \frac{dy}{dx} dx = \int_{x_0}^{x} f(x, y) dx$$

$$y = y_0 + \int_{x_o}^{x} f(x, y) dx \tag{1.4}$$

then, the first approximation y_1 for y can be obtained by substituting y_0 instead of y in the right hand side of the last equation, i.e.,

$$y_1 = y_0 + \int_{x_o}^{x} f(x, y_0)dx, \tag{1.5}$$

the second approximation y_2, can be obtained also by substituting y_1 instead of y in the right hand side of equation (1.4), i.e.,

$$y_2 = y_0 + \int_{x_o}^{x} f(x, y_1)dx, \tag{1.6}$$

continuing with a similar way, then we cab obtained the following repeated relations

$$\boxed{y_{n+1} = y(x_0) + \int_{x_o}^{x} f(x, y_n(x))dx,} \tag{1.7}$$

and this repeated relation can be stopped whenever the following condition holds

$$|y_{n+1} - y_n| \leq \epsilon, \tag{1.8}$$

where, ϵ is a small positive constant.

■ **Example 1.2** Using Picard method, find an approximate value of y at $x = 0.2$ if

$$y' = x - y, \qquad y(0) = 1$$

■

Solution.

$$\begin{aligned} y_{n+1} &= y(x_0) + \int_{x_o}^{x} f(x, y_n(x))dx \\ &= y(x_0) + \int_{x_o}^{x} (x - y_n))dx \\ &= 1 + \int_{x_o}^{x} (x - y_n)dx, \qquad n = 0, 1, 2, \ldots \end{aligned}$$

$$y_1(x) = 1 + \int_0^x (x-1)dx = 1 - x + \frac{x^2}{2}$$

$$y_2(x) = 1 + \int_0^x \left[x - \left(1 - x + \frac{x^2}{2}\right)\right] dx = 1 - x + x^2 - \frac{x^3}{6}$$

$$y_3(x) = 1 + \int_0^x \left[x - \left(1 - x + x^2 - \frac{x^3}{6}\right)\right] dx = 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{24}$$

$$\begin{aligned} y_4(x) &= 1 + \int_0^x \left[x - \left(1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{24}\right)\right] dx \\ &= 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{120} \end{aligned}$$

$$y_5(x) = 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{60} + \frac{x^6}{720}$$

at $x = 0.2$, we have

$$y_0 = 1, \qquad y_1 = 0.2, \qquad y_2 = 0.83867, \qquad y_3 = 0.83740,$$

$$y_4 = 0.83746, \qquad y_5 = 0.83746,$$

thus,

$$y(0.2) = 0.83746$$

■

■ **Example 1.3** Using Picard method, find the solution of the following initial value problem

$$y' = y, \qquad y(0) = 1,$$

(Note that, the analytical solution is $y(x) = e^x$) ■

Solution.

$$\begin{aligned} y_{n+1} &= y(x_0) + \int_{x_o}^{x} f(x, y_n(x))dx \\ &= y_0 + \int_{x_o}^{x} y_n dx \end{aligned}$$

$$y_1(x) = 1 + \int_0^x dx = 1 + x$$

$$y_2(x) = 1 + \int_0^x (1 + x)dx = 1 + x + \frac{x^2}{2}$$

$$\cdots\cdots\cdots\cdots$$

$$y_n(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}$$

■

■ **Example 1.4** Using Picard method, find the solution of the following initial value problem

$$\frac{dy}{dx} = xe^y, \qquad y(0) = 0,$$

then, find $y(0.1), y(0.2), y(1)$
(Note that, the analytical solution is $y(x) = -\ln\left[1 - \frac{x^2}{2}\right]$) ■

Solution.

$$
\begin{aligned}
y_{n+1} &= y(x_0) + \int_{x_o}^{x} f(x, y_n(x))dx \\
&= y_0 + \int_0^x x e^{y_n} dx \\
&= 0 + \int_0^x x e^{y_n} dx, \qquad n = 0, 1, 2, \ldots
\end{aligned}
$$

$$
y_1(x) = 0 + \int_0^x x e^0 dx = \frac{x^2}{2}
$$

$$
y_2(x) = 0 + \int_0^x \left[x\left(e^{\frac{x^2}{2}}\right)\right] dx = e^{\frac{x^2}{2}} - 1
$$

$$
y(x) = e^{\frac{x^2}{2}} - 1
$$

$$
\begin{aligned}
y(0.1) &= 0.0050125 \\
y(0.1) &= 0.0202013 \\
y(1) &= 0.6487213
\end{aligned}
$$

■

■ **Example 1.5** Using Picard method, find an approximate value for y at $x = 0.1, 0.2, 0.3$ assuming that

$$
\frac{dy}{dx} = 1 + xy, \qquad y(0) = 1
$$

■

Solution. we use the repeated relations for the Picard which are

$$
\begin{aligned}
y_{n+1} &= y_0 + \int_{x_o}^{x} f(x, y_n(x))dx \\
y_1(x) &= y_0 + \int_{x_0}^{x} f(x, y_0)dx = 1 + \int_{x_0}^{x} (1 + xy_0)dx \\
&= 1 + \int_{0}^{x} (1 + x)dx = 1 + x + \frac{x^2}{2} \\
y_2(x) &= y_0 + \int_{x_0}^{x} f(x, y_1)dx \\
&= 1 + \int_{0}^{x} (1 + xy_1)dx \\
&= 1 + \int_{0}^{x} \Big[1 + x\big(1 + x + \frac{x^2}{2}\big)\Big]dx \\
&= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} \\
y_3(x) &= y_0 + \int_{x_0}^{x} f(x, y_2)dx \\
&= 1 + \int_{0}^{x} (1 + xy_2)dx \\
&= 1 + \int_{0}^{x} \Big[1 + x\big(1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}\big)\Big]dx \\
&= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48} \\
y_4(x) &= y_0 + \int_{x_0}^{x} f(x, y_3)dx \\
&= 1 + \int_{0}^{x} (1 + xy_3)dx \\
&= 1 + \int_{0}^{x} \Big[1 + x\big(1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}\big)\Big]dx \\
&= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48} + \frac{x^7}{120} + \frac{x^8}{284}
\end{aligned}
$$

- First, in order to obtain the solution at $x = 0.1$, we put $x = 0.1$ in the above relations, then we have

$$y_1 = 1.105, \qquad y_2 = 1.1053458, \qquad y_3 = 1.3551897, \qquad y_4 = 1.355192$$

- Similarly, in order to obtain the solution at $x = 0.2$, we put $x = 0.2$ in the above relations, then we have

$$y_1 = 1.22, \qquad y_2 = 1.2228667, \qquad y_3 = 1.2228894, \qquad y_4 = 1.2228895$$

thus, $y(0.2) = 1.223$

- Also, in order to obtain the solution at $x = 0.3$, we put $x = 0.3$ in the above relations, then we have

$$y_1 = 1.345, \qquad y_2 = 1.35550125, \qquad y_3 = 1.3551897, \qquad y_4 = 1.355192$$

thus, $y(0.3) = 1.355$ ■

(R) **(Disadvantage of this method)** Due to the integration that exist in this method it is considered non practical method. Also, it might be difficult to perform a programming code for it.

1.3.2 Taylor serious method

This method depends in the derivatives of the function. Suppose that $y(x)$ is a solution for equation (1.1), then $y(x)$ can be written using Taylor expansion around the point $x = x_0$ as follows

$$y(x) = y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!}y_0'' + \cdots + \frac{(x - x_0)^n}{n!}y_0^{(n)} + R_{n+1}$$

where,

$$R_{n+1} = \frac{(x - x_0)^{n+1}}{(n + 1)!}y^{(n+1)}(\eta), \qquad \eta \in (x_0, x)$$

putting $h = (x - x_0)$ then we can rewrite $y(x)$ as

$$y(x) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \cdots + \frac{h^n}{n!}y_0^{(n)} + R_{n+1} \qquad (1.9)$$

with

$$R_{n+1} = \frac{h^{n+1}}{(n+1)!} y^{(n+1)}(\eta), \qquad \eta \in (x_0, x_0 + h)$$

Now, in order to obtain the solution we have to determine the following derivatives

$$y'(x_0), y''(x_0), y'''(x_0)$$

that can by performed as

$$\begin{aligned} y'(x) = f(x,y) \Rightarrow y''^{(x)} &= f'(x,y) \\ &= f_x(x,y) + f_y(x,y)y' \\ &= f_x(x,y) + f_y(x,y)f \end{aligned} \tag{1.10}$$

similarly, for all the other higher order derivatives. thus all the derivatives is going to be a function of $f(x,y)$ and the derivatives of $f(x,y)$. Now, from (1.10) into (1.9), we have

$$\begin{aligned} y(x_0 + h) =& y_0 + hf_0 + \frac{h^2}{2!}(f_x + f_y f)_{(x_0,y_0)} + \\ & \frac{h^3}{3!}(f_{xx} + 2f_{xy} + f_{yy}f^2 + f_x f_y + f_y^2 f)_{(x_0,y_0)} + \dots \end{aligned} \tag{1.11}$$

and the error in this case takes the following form

$$\text{Error} = \frac{h^{n+1} y^{(n+1)(\eta)}}{(n+1)!}, \quad 0 < \eta < h$$

The following are the needed steps for performing the current method:

- First: to obtain $y(x_1)$, we have to compute the following derivatives; $y'(x_0), y''(x_0), y'''(x_0), \dots$ such that
  - y' is $f(x,y)$ from the ODE,
  - y'' can be obtained by by performing the derivative of y' with respect to x,
  -y''' can be obtained by by performing the derivative of y'' with respect to x and so on - This should be done each time with substituting x with x_0 , thus we can write the following

  $$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \dots$$

  Doing so means that we have calculated $y(x_1)$ $(x_1 = x_0 + h)$.

- Second: to obtain $y(x_2)$, we have to compute the following derivatives; $y'(x_1), y''(x_1), y'''(x_1), \ldots$. Therefore, we can write the following such that

$$y_2 = y(x_2) = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \ldots$$

where, $x_2 = x_1 + h$

- Third: to obtain $y(x_3)$, we have to compute the following derivatives; $y'(x_2), y''(x_2), y'''(x_2), \ldots$. Therefore, we can write the following such that

$$y_3 = y(x_3) = y_2 + hy_2' + \frac{h^2}{2!}y_2'' + \frac{h^3}{3!}y_2''' + \ldots$$

where, $x_3 = x_2 + h$

- Finally: we can easily repeat the above steps several times till we obtain a value for $y_n = y(x_n)$ at the points $n = 0, 1, 2, 3, \ldots, x_n = x_0 + nh$, and we have

$$y_n = y(x_n) = y_{n-1} + hy_{n-1}' + \frac{h^2}{2!}y_{n-1}'' + \frac{h^3}{3!}y_{n-1}''' + \ldots$$

■ **Example 1.6** Using Taylor method to find the solution of the following ODEs;

$$\frac{dy}{dx} = x - y, \qquad y(0) = 1, \qquad h = 0.2$$

■

Solution. It's easily to write;

$$\begin{aligned} y &= y(x) & y(0) &= 1 \\ y' &= f(x,y) = x - y & y'(0) &= -1 \\ y'' &= 1 - y' & y''(0) &= 2 \\ y''' &= -y'' & y'''(0) &= -2 \\ y^{iv} &= -y''' & y^{iv}(0) &= 2 \\ y^{v} &= -y^{ive} & y^{v}(0) &= -2 \end{aligned} \tag{1.12}$$

then, substituting from (1.12) in the following relations

$$y_1 = y(x_1) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \dots$$

leads to

$$\begin{aligned} y(0.2) = y_1 =& 1 + (0.2)(-1) + \frac{((0.2)^2)}{2!}(2) + \frac{(0.2)^3}{3!}(-2) \\ &+ \frac{(0.2)^4}{4!}(2) + \frac{(0.2)^5}{5!}(-2) + \dots \end{aligned} \tag{1.13}$$

thus, $y(0.2) = y_1 = 0.83746$ ■

■ **Example 1.7** Find the solution of the following ODE

$$\frac{dy}{dx} = x + y, \quad y(0) = 2$$

then, find $y(0.1), y(0.2)$. ■

Solution.

$$\begin{aligned} &y = y(x), &&y'(x) = x + y, &&y'' = 1 + y', \\ &y'''(x) = y'', &&y^{iv} = y''', &&y^{v} = y^{iv}, \ \dots \end{aligned} \tag{1.14}$$

First: In order to calculate $y(0.1)$, we plug in $x = 0.1$ in the right hand side of relations (1.14), then

$$\begin{aligned} &y(0) = 2, &&y'(0) = 0 + 2, &&y''(0) = 1 + 2 = 3, \\ &y'''(0) = 3, &&y^{iv}(0) = 3, &&y^{v}(0) = 3, \dots \end{aligned} \tag{1.15}$$

thus,

$$y(x_1) = y_1 = y_0 + hy'(0) + \frac{h^2}{2!}y''(0) + \frac{h^3}{3!}y'''(0) + \frac{h^4}{4!}y^{iv}(0) + \frac{h^5}{5!}y^{v}(0) + \dots,$$

$$h = x_1 - x_0 = 0.1 - 0 = 0.1$$

$$y_1 = y(0.1) = 2 + (0.1)(2) + \frac{(0.1)^2}{2!}(3) + \frac{(0.1)^3}{3!}(3) + \frac{(0.1)^4}{4!}(3) + \frac{(0.1)^5}{5!}(3)$$

$$y_1 = y(0.1) = 2.2$$

Second: in order to calculate $y(0.2)$, we plug in $x = 0.2$ in the right hand side of relations (1.14), then

$$y(0.1) = 2.2, \qquad y'(0.1) = 0.1 + 2.2, \qquad y''(0.1) = 1 + 2.3 = 3.3,$$
$$y'''(0.1) = 3.3, \qquad y^{iv}(0.1) = 3.3, \qquad y^{v}(0.1) = 3.3, \ldots \tag{1.16}$$

thus,

$$y(x_2) = y_2 = y(x_1) + hy'(x_1) + \frac{h^2}{2!}y''(x_1) + \frac{h^3}{3!}y'''(x_1) + \frac{h^4}{4!}y^{iv}(x_1) + \frac{h^5}{5!}y^{v}(x_1) + \ldots,$$

$$h = x_2 - x_1 = 0.2 - 0.1 = 0.1$$

$$y_2 = y(0.2) = 2.2 + (0.1)(2.3) + \frac{(0.1)^2}{2!}(3.3) + \frac{(0.1)^3}{3!}(3.3) + \frac{(0.1)^4}{4!}(3.3) + \frac{(0.1)^5}{5!}(3.3)$$

$$y_2 = y(0.2) = 2.21551275$$

■

■ **Example 1.8** Using Taylor method, find the solution for the following ODE

$$\frac{dy}{dx} = x^2 + y^2, \quad y(0) = 1.$$

■

$$\begin{aligned}
&f(x, y) = x^2 + y^2, \; x_0 = 0, \; y_0 = 1, && \\
&y = y(x), && y(0) = 1, \\
&y' = f(x, y) = x^2 + y^2 && y'(0) = 1 \\
&y'' = 2x + 2yy' && y''(0) = 2 \\
&y''' = 2 + 2yy'' + 2(y')^2 && y'''(0) = 8
\end{aligned} \tag{1.17}$$

then using

$$y(x) = y_0 + (x - x_0)y'(x_o) + \frac{(x - x_0)^2}{2!}y''(x_0) + \ldots,$$

we conclude that

$$y(x) = 1 + x + x^2 + \frac{8}{3!}x^3$$

(R) **(Disadvantage of this method)** It is clear that this method is non practical method due to the various differentiations that one have to compute during the solution. There for we present here some other methods that we can practically deal with.

Lec2

1.3.3 Normal Euler method

This method is driven from Taylor method assuming that $h << 1$ in the Taylor expansion. Assuming so leads to the possibility of taking only three terms in the Taylor expansion, thus,

$$y(x) = y(x_0+h) = y(x_0)+hy'(x_0)+\frac{h^2}{2!}y''(\xi), \; x_0 < \xi < x_0+h \quad (1.18)$$

The third term in the above equation represents the error in the method and it becomes very small whenever h is small enough, thus,

$$\text{Error} = E = \frac{y''(\xi)h^2}{2} = O(h^2) \quad (1.19)$$

Equation (1.18) represents the solution at a point $x = x_0 + h$ with the given solution at $x = x_0$ i.e., $y(x_0)$ is given as an initial value. Similarly, we can find the solution at $x = x_0 + 2h$ and repeating this steps we can find also the solution at $x = x_0 + (n-1)h$. Thus, the normal Euler can take the following form;

$$y_{n+1} = y_n + hy'_n + O(h^2)$$

Also, since

$$y'_n = f(x_n, y_n),$$

then, Euler formula can be rewritten as

$$y_{n+1} = y_n + hf(x_n, y_n), \qquad E = \frac{h^2}{2}y''(\xi), \quad x_n < \xi < x_{n+1} \quad (1.20)$$

■ **Example 1.9** Find the solution of the following ODE

$$\frac{dy}{dx} = x + y, \quad y(0) = 1, \text{in the interval } [0, 0.1] \text{ taking } h = 0.02.$$

■

Solution. Using the repeated relations (1.20)

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$$y(0.02) = y_1 = y_0 + hf(x_0, y_0) = 1 + (0.02)(0 + 1) = 1.02$$

$$y(0.04) = y_2 = y_1 + hf(x_1, x_1) = 1.02 + (0.02)(0.02 + 1.02) = 1.0408$$

$$y(0.06) = y_3 = y_2 + hf(x_2, x_2) = 1.0408 + (0.02)(0.04 + 1.0408) = 1.0624$$

$$y(0.08) = y_4 = y_3 + hf(x_3, x_3) = 1.0048$$

$$y(0.1) = y_5 = y_4 + hf(x_4, x_4) = 1.1081$$

■

R The analytical solution for the ODE in the previous example at $x = 0.1$ is 1.1103, hence the numerical error is

$$E = 1.1103 - 1.1081 = 0.0022$$

1.3.4 A modified Euler method

The modified Euler method is driven also from Taylor serious with an extra term compare to the normal Euler method, i.e.,

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2}y''_n \tag{1.21}$$

Since, $y''_n = \frac{y'_{n+1} - y'_n}{h}$ (from the usual definition of the first derivative of a function). Substituting in (1.21) for the value of y''_n, we have

$$\begin{aligned} y_{n+1} &= y_n + hy^prime_n + \frac{h^2}{2}\Big(\frac{y'_{n+1} - y'_n}{h}\Big) \\ &= y_n + h\Big(y'_n + \frac{1}{2}y'_{n+1} - \frac{1}{2}y'_n\Big) \\ &= y_n + \frac{h}{2}(y'_n + y'_{n+1}) \end{aligned} \tag{1.22}$$

Hence, the final form of the modified Euler method is

$$y_{n+1} = y_n + \frac{h(y'_n + y'_{n+1})}{2}, \tag{1.23}$$

where, $y'_n = f({}_n, y_n),\ y'_{n+1} = f(x_{n+1}, y_{n+1})$

R Determining y'_{n+1}, that appears in the right hand side of equation (1.23), depends on the value of y_{n+1}, that is still unknown, therefore, the steps of for solving such case using the modified Euler method are

- Determine y_{n+1}, using the normal Euler method.
- Use the previous value to compute the value of y'_{n+1} such that
$$y'_{n+1} = f(x_{n+1}, y_{n+1})$$
- Substitute for the values of y_n, y'_n, y'_{n+1} in the right hand side of equation (1.23) in order to obtain a value for y_{n+1} which is now obtained by the modified Euler method, thus,this method is called Predictor Corrector method, i.e.,

$$\begin{aligned} y_1^{(P)} &= y_0 + hy'_0 = y_0 + hf\left(x_0, y_0\right) \\ y_1^{(C)} &= y_0 + \tfrac{h}{2}\left(y'_0 + y'_1\right) = y_0 + \tfrac{h}{2}\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(P)}\right)\right] \end{aligned}$$

■ **Example 1.10** Find the numerical solution of the following ODEs

$$\frac{dy}{dx} = x^2 + y,\ \ y(0) = 1$$

at $x = 0.2$, considering $h = 0.1$, using the modified Euler method. ■

Solution.

$$\begin{aligned} y_1^{(P)} &= y_0 + hf\left(x_0, y_0\right) = 1 + (0.1)(0 + 1) = 1.1 \\ y_1^{(C)} &= y_0 + \frac{h}{2}\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(P)}\right)\right] \\ &= 1 + \frac{0.1}{2}\left\{(0 + 1) + \left[(0.1)^2 + 1.1\right]\right\} = 1.1055 \\ y(0.1) &= 1.1055 \end{aligned}$$

$$\begin{aligned}
y_2^{(P)} &= y_1 + hf\left(x_1, y_1^{(C)}\right) \\
&= 1.1055 + (0.1)\left[(0.1)^2 + 1.1055\right] \\
&= 1.22605 \\
y_2^{(C)} &= y_1 + \frac{h}{2}\left[f\left(x_1, y_1\right) + f\left(x_2, y_2^{(P)}\right)\right] \\
&= 1.1055 + \frac{0.1}{2}\left\{\left[(0.1)^2 + 1.1055\right] + \left[(0.2)^2 + 1.22605\right]\right\} \\
&= 1.224577 \\
y_2^{(C)} &= 1.224577
\end{aligned}$$

■

■ **Example 1.11** Using the modified Euler method, find the solution of the following ODEs

$$\frac{dy}{dx} = x + y, \ \ y(0) = 1$$

at $x = 0.04$, considering $h = 0.02$. ■

Solution.

$$\begin{aligned}
y_1^{(P)} &= y_0 + hf\left(x_0, y_0\right) = 1 + (0.02)(0 + 1) = 1.02 \\
y_1^{(C)} &= y_0 + \frac{h}{2}\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(P)}\right)\right] \\
&= 1 + \frac{0.02}{2}\left\{(0 + 1) + [(0.02) + 1.02]\right\} = 1.0204 \\
y(0.02) &= 1.0204
\end{aligned}$$

$$
\begin{aligned}
y_2^{(P)} &= y_1 + hf\left(x_1, y_1^{(C)}\right) \\
&= 1.0204 + 0.02(0.02 + 1.0204) = 1.041208 \\
&= 1.041208 \\
y_2^{(C)} &= y_1 + \frac{h}{2}\left[f\left(x_1, y_1\right) + f\left(x_2, y_2^{(P)}\right)\right] \\
&= 1.0204 + \frac{0.02}{2}\{[(0.02) + 1.0204] + [(0.04) + 1.041208]\} \\
&= 1.0416 \\
y_2^{(C)} &= 1.0416
\end{aligned}
$$

■

■ **Example 1.12** Use the modified Euler method to find the solution of the following ODEs

$$y' = x + y,\ y(0) = 2$$

for obtaining the value of $y(0.2)$ using the step size $h = 0.025$. ■

Solution. First, obtaining y_{n+1}, which means $y(0.2)$, using the modified Euler method. we apply the following repeated relations

$$
\begin{aligned}
&y_{n+1} = y_n + hf\left(x_n, y_n\right) \\
&y(0.025) = y_1 = y_0 + hf\left(x_0, y_0\right) \\
&\qquad = 2 + (0.025)[(0)(4)] = 2 \\
&y(0.05) = y_2 = y_1 + hf\left(x_1, y_1\right) \\
&\qquad = 2 + (0.025)[(0)(4)] = 2 \\
&y(0.075) = y_3 = y_2 + hf\left(x_2, y_2\right) \\
&\qquad = 2 + (0.025)[(0)(4)] = 2 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&y(0.100) = y(0.125) = y(0.150) = y(0.175) = y(0.200) = 2 \\
&y_{n+1} = y(0.2) = 2, \\
&y'_{n+1} = -\left(x_{n+1}\right)\left(y_{n+1}^2\right) = -(0.2)(4) = -0.8
\end{aligned}
\tag{1.24}
$$

Now, we use the relation of the modified Euler equation, that is

$$
\begin{aligned}
&y_{n+1} = y_n + \tfrac{h(y'_n + y'_{n+1})}{2}, y_n = y(0.175) = 2\\
&y'_n = -(0.175)(4) = -0.700\\
&y_{n+1} = y(0.2) = -0.7 + \tfrac{0.025(-0.7-0.8)}{2} = -0.7187
\end{aligned}
$$

■

1.3.5 Runge-Kutta method

It's one of the most important methods for solving the differential equations, which can be driven using Taylor expansion and the order of this method depends on how many terms are considered from the Taylor expansion, thus we have the following types of the method

Runge-Kutta method of second order (RK2)

It is used to obtain the solution of a differential equation of the form

$$\frac{dy}{dx} = f(x, y), \;\; y(x_0) = y_0, \tag{1.25}$$

and it can be driven as follows; assume

$$
\begin{aligned}
y_{n+1} &= y_n + ak_1 + bk_2,\\
k_1 &= hf(x_n, y_n),\\
k_2 &= hf(x_n + \alpha h, y_n + \beta k_1),
\end{aligned} \tag{1.26}
$$

where, a, b, α, β are constants that can be determined with the following way;
-Using the Taylor expansion for eq. (1.25) at a point x_n, we have

$$y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2!} f\prime(x_n, y_n) + O(h^3), \tag{1.27}$$

where,

$$f\prime(x_n, y_n) = \frac{df_n}{dx} = \left(f_x + f_y \frac{dy}{dx}\right)_n = (f_x + f_y f)_n$$

Now, substituting in the above equation about the value of $f\prime(x_n, y_n)$, we have

$$y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2!}(f_x + f_y f)_n + O(h^3), \qquad (1.28)$$

the term k_2 which is used in RK2 can be rewritten in the following form (using Taylor expansion for a two variable function)

$$\begin{aligned} k_2 &= hf\left(x_n + \alpha h, y_n + \beta k_1\right) \\ &= hf\left(x_n, y_n\right) + \alpha h^2 f_x\left(x_n, y_n\right) + \beta h k_1 f_y\left(x_n, y_n\right) \\ &= h\left(f_n + \alpha h f_x + \beta k_1 f_y\right)_n = h\left(f_n + \alpha h f_x + \beta h f_y f\right)_n \text{ (since } k_1 = hf) \end{aligned}$$

Substituting in (1.25) for the value of k_2, we have

$$y_{n+1} = y_n + ahf\left(x_n, y_n\right) + bh\left(f + \alpha h f_x + \beta h f_y\right)_n,$$

which can be rewritten as,

$$y_{n+1} = y_n + (a+b)hf\left(x_n, y_n\right) + h^2\left(\alpha b f_x + \beta b f f_y\right)_n,$$

Thus,

$$a + b = 1, \ \alpha b = \frac{1}{2}, \ \beta b = \frac{1}{2}$$

This equation has three relations in four variables, therefore the solution of is infinite number in which one can pick any value for one of the variables to get the other three variables. Also, this equation can be rewritten in the following form

$$b(\alpha - \beta) = 0, \ b \neq 0 \implies \alpha - \beta = 0 \implies \alpha = \beta$$

Now,

- choose $\alpha = \beta = \frac{1}{2}$, leads to $a = 0, b = 1$, which is incorrect as we should have $(a \neq 0)$.

- choose $\alpha = \beta = 1$, leads to $a = b = \frac{1}{2}$

then substituting for the values of a, b, α, β,then we achieve to our goal i.e.,

$$
\begin{aligned}
y_{n+1} &= y_n + a k_1 + b k_2, \text{ with} \\
k_1 &= h f(x_n, y_n), \\
k_2 &= h f(x_n + \alpha h, y_n + \beta k_1),
\end{aligned}
\tag{1.29}
$$

that is RK2.

■ **Example 1.13** Use RK2 method to find the solution of the following ODEs

$$\frac{dy}{dx} = x^2 + y^2, \;\; y(2) = -1$$

at $x = 2.3$ using the step size $h = 0.1$. ■

Solution.

$$f(x, y) = x^2 + y^2$$

$$y_1 = y_0 + \frac{k_1 + k_2}{2},$$

$$k_1 = hf(x_0, y_0) = hf(2, -1) = (0.1)(4 + 1) = 0.5$$

$$k_2 = hf(x_0+h, y_0+k_1) = hf(2+0.1, -1+0.5) = hf(2.1, -0.5) = 0.466$$

$$y_1 = -1 + \frac{1}{2}(0.5 + 0.466) = -0.517$$

$$y_2 = y_1 + \frac{k_1 + k_2}{2},$$

$$
\begin{aligned}
k_1 &= hf\,(x_1, y_1) \\
&= hf\,(x_1, y_1) = hf_1 = hf(2.1, -0.517) \\
&= (0.1)\left[(2.1)^2 + (-0.517)^2\right] = 0.468, \\
k_2 &= hf(x_1 + h, y_1 + k_1) \\
&= (0.1)\left[(2.2)^2 + (-0.049)^2\right] = 0.484
\end{aligned}
$$

$$
\begin{aligned}
y_2 &= -0.517 + \frac{1}{2}(0.468 + 0.484) = -0.041 \\
y_3 &= y_2 + \frac{k_1 + k_2}{2} \\
k_1 &= hf(x_2, y_2) = hf_2 = hf(2.2, -0.041) \\
&= (0.1)\Big[(2.2)^2 + (-0.041)^2\Big] = 0.484, \\
k_2 &= hf(2.2 + 0.1, -0.041 + 0.484) = hf(2.3, 0.443) \\
&= (0.1)\Big[(2.3)^2 + (0.443)^2\Big] = 0.548 \\
y_3 &= -0.041 + \frac{1}{2}(0.484 + 0.548) = 0.475
\end{aligned}
$$

■

■ **Example 1.14** Let

$$\frac{dy}{dx} = x^2 - y, \;\; y(0) = 1$$

find $y(0.1), y(0.2)$ using RK2. ■

Solution.

$$f(x, y) = x^2 - y$$

$$x_0 = 0, y_0 = 1 \Longrightarrow f(x_0, y_0) = -1,$$

Now, the RK2 method is

$$k_1 = hf(x_0, y_0) = ()0.1(0 - 1) = -0.1$$

$$k_2 = hf(x_0 + h, y_0 + k_1) = hf(0.1, 0.9) = (0.1)((0.1)^2 - 0.9) = -0.089$$

$$K = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(-0.1 + 0.089) = -0.0945$$

$$y_1 = y(0.1) = y_0 + K = 1 - 0.0945 = 0.9055$$

For computing $y(0.2)$ we take $(x_1, y_1) = (0.1, 0.9055)$ instead of (x_0, y_0), then we repeat the method again

$$
\begin{aligned}
k_1 &= hf\ (x_1, y_1) \\
&= h(x_1^2 - y_1) = (0.1)\Big[(0.1)^2 - 0.905\Big] = -0.08955, \\
k_2 &= hf(x_0 + h, y_0 + k_1) \\
&= (0.1)\Big[(0.2)^2 - 0.81595\Big] = 0.077595
\end{aligned}
$$

$$K = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(-0.08955 - 0.077595) = -0.0835725$$
$$y_2 = y(0.2) = y_1 + k = 0.9055 - 0.0835725 = 0.821975$$

■

Rung-Kutta of fourth order (RK4)

This method is considered one of the most popular method as its is more accurate compare to the Rung-Kutta of second order method. This ,method could be driven in a similar way to that of RK2 increasing. It takes the following form

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where,

$$\begin{aligned} k_1 &= hf(x_n, y_n), \\ k_2 &= hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_1), \\ k_3 &= hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_2), \\ k_4 &= hf(x_n + h, y_n + k_3), \end{aligned}$$

■ **Example 1.15** Use the RK4 in order to solve the following ODE

$$\frac{dy}{dx} = x + y \quad y(0) = 1$$

at $x = 0.1$, using,$h = 0.1$. ■

Solution.

$$k_1 = hf(x_n, y_n) = hf(0,1) = 0.1(0+1) = 0.1$$

$$k_2 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right)$$

$$= hf\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\right)$$

$$= hf(0.05, 1.05)$$

$$= 0.1(0.05 + 1.05) = 0.11$$

$$k_3 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2\right)$$

$$= 0.1 \tag{1.30}$$

$$= 0.1 \quad 0.05 + 1.055 = 0.11050$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

$$= 0.1f(0.1, 1.11050) = 0.12105$$

$$y_{n+1} = y_n + \frac{1}{6}k_1 + 2k_2 + 2k_3 + k_4$$

$$y(0.1) = 1.0 + \frac{1}{6}(0.1) + 0.22 + 0.221 + 0.1205$$

$$= 1.11034$$

■

■ **Example 1.16** From the following ODE

$$\frac{dy}{dx} = x^2 - y, \; y(0) = 1$$

find $y(0.1)$, $y(0.2)$, using RK4. ■

Solution.

$$k_1 = hf\left(x_n, y_n\right) = hf(0,1) = 0.1(0-1) = -0.1$$

$$k_2 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right)$$

$$= hf\left(0.05, 0.98\right)$$

$$= 0.1(0.05^2 - 0.95) = 0.09475$$

$$k_3 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2\right)$$

$$= 0.1f(0.05^2, 0.952625) = -0.0950125$$

$$k_4 = hf\left(x_n + h, y_n + k_3\right)$$

$$= 0.1f(0.1^2 - 0.0950125) = 0.0894987$$

$$K = \frac{1}{6}k_1 + 2k_1 + 2k_2 + k_3$$

$$= \frac{1}{6}[-0.1 + 2 - 0.09475$$

$$+ 2 - 0.0950125 - 0.0894987]$$

$$= -0.0948372$$

$$y_1 = y(0.1) = y_o + K = 1 - 0.0948372 = 0.9051627.$$

Now, to compute $y(0.2)$ we take $(x_1, y_1) = (0.1, 0.9051627)$ instead of (x_0, y_0) and repeat the method to get the following

$$k_1 = hf\left(x_1, y_1\right) = hf(0.1, 0.9051627)$$

$$= 0.1\left[0.1^2 - 0.9051627\right] = -0.0895162$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = hf(0.15, 0.8604046)$$

$$= 0.1\left[0.15^2 - 0.8604046\right] = -0.837904$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = hf(0.15, 0.8632674)$$

$$= 0.1\left[0.15^2 - 0.8632674\right] = -0.0840767$$

$$k_4 = hf \quad x_1 + h, y_1 + k_3 = hf(0.2, 0.8210859)$$

$$= 0.1\left[0.2^2 - 0.8210859\right] = -0.0781085$$

$$K = \frac{1}{6}k_1 + 2k_1 + 2k_2 + k_3$$

$$= \frac{1}{6}[-0.0895162 + 2 - 0.0837904$$

$$+ 2 - 0.0840767 - 0.0781085]$$

$$= -0.0838931$$

$$y_2 = y(0.2) = y_1 + K$$

$$= 0.9051627 - 0.0838931$$

$$= 0.08212695$$

■

■ **Example 1.17** Suppose we have the following ODE

$$\frac{dy}{dx} = x^2 - y, \ y(0) = 1,$$

find $y(0.1)$, $y(0.2)$, using RK2. ■

Solution.

$$\begin{aligned}
k_1 &= hf(x_0, y_0) = 0.1[0-1] = -0.1 \\
k_2 &= hf(x_0 + h, y_0 + k_1) \\
&= hf0.1, 0.9 = 0.1\left[0.1^2 - 0.9\right] \\
&= -0.089 \\
K &= \frac{1}{2}k_1 + k_2 = \frac{1}{2} - 0.1 + 0.089 = -0.0945 \\
y_1 &= y(0.1) = y_0 + k = 1 - 0.0945 = 0.9055
\end{aligned}$$

Then, to compute $y(0.2)$, we take $(x_1, y_1) = (0.1, 0.9055)$ instead of (x_0, y_0) and repeat the method to get the following

$$\begin{aligned}
k_1 &= hf(x_1, y_1) = hx_1^2 - y_1 \\
&= 0.1\left[0.1^2 - 0.905\right] = -0.08955 \\
k_2 &= hf(x_0 + h, y_0 + k_1) \\
&= hf0.2, 0.81595 = 0.1\left[0.2^2 - 0.81595\right] \\
&= -0.077595 \\
K &= \frac{1}{2}k_1 + k_2 = \frac{1}{2} - 0.08955 - 0.077595 = -0.0835725 \\
y_2 &= y(0.2) = y_1 + k = 0.9055 - 0.0835725 = 0.821975
\end{aligned}$$

■

■ **Example 1.18** Use RK2 to solve the following ODE

$$\frac{dy}{dx} = y - x, \;\; y(0) = 2,$$

at $x = 0.2$, using $h = 0.1$. ■

Solution.

$$k_1 = hf(x_0, y_0) = hf(0, 2)$$

$$= 0.2[2 - 0] = 0.4$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = hf(0.1, 2.2)$$

$$= 0.2[2.2 - 0.1] = 0.42$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = hf(0.1, 2.21)$$
$$= 0.2[2.21 - 0.1] = 0.422$$
$$k_4 = hf \quad x_0 + h, y_0 + k_3 = hf(0.2, 2.422)$$
$$= 0.2[2.422 - 0.2] = 0.4644$$
$$y(0.2) = y_0 + \frac{1}{6}k_1 + 2k_2 + 2k_3 + k_4$$
$$= 2 + [0.4 + 20.42 + 20.422 + 0.4644]$$
$$= 2.4247266$$

■

Exercise 1.1 Use RK4 to find the values of $y(0.1), y(0.2), y(0.3)$ of the following ODE
$$\frac{dy}{dx} = xy + y^2,\ y(0) = 1,$$

Chapter 2

Numerical solution for systems of ordinary differential equation

2.1 Solving differential systems of first order

The general form of system of ordinary differential equation from the first order is

$$\left.\begin{aligned} y_1' &= f_1\left(x, y_1, y_2, \cdots, y_n\right) \\ y_2' &= f_2\left(x, y_1, y_2, \cdots, y_n\right) \\ &\vdots \qquad \vdots \\ y_n' &= f_n\left(x, y_1, y_2, \cdots, y_n\right) \\ \text{with,} & \\ y_1(x_0) = \alpha_1, & y_2(x_0) = \alpha_2, \ldots, y_n(x_0) = \alpha_n \end{aligned}\right\} \tag{2.1}$$

All the methods mentioned in the previous chapter for solving an equation from the initial value problem type can be used to solve system of ordinary differential equation as in (2.1). We are going to show how those methods can be extended to solve a system of ODEs. During our discussion, we are going to focus our attention to a system of two equations and in order to make the picture more clear we will use alternative notations as;

$$y' = f(x, y, z), z' = \phi(x, y, z)$$

be a system formed of two equations with the following two initial conditions

$$y(x_0) = y_0, z(x_0) = z_0$$

2.1.1 Picard method

Suppose

$$\begin{cases} y' = f(x, y, z) \\ z' = \phi(x, y, z) \\ \text{with the initial condition} \\ y(x_0) = y_0, \;\; z(x_0) = z_0 \end{cases} \tag{2.2}$$

the first approximation y_1, z_1 can be obtained in a similar way to that of the one differential equation, i.e.,

$$\begin{array}{l} y_1 = y_0 + \int_{x_0}^{x} f\left(x, y_0, z_0\right) dx \\ z_1 = z_0 + \int_{x_0}^{x} \phi\left(x, y_0, z_0\right) dx, \end{array}$$

the second approximation is

$$\begin{array}{l} y_2 = y_0 + \int_{x_0}^{x} f\left(x, y_1, z_1\right) dx \\ z_2 = z_0 + \int_{x_0}^{x} \phi\left(x, y_1, z_1\right) dx \end{array}$$

and, so on

■ **Example 2.1** Use Picard method to find an approximate vale for y, z to solve

$$\frac{dy}{dx} = z, \;\; \frac{dz}{dx} = x^3(y + z)$$

with the initial conditions $y(0) = 1, z(0) = \frac{1}{2}$. ■

Solution. Since,

$$\begin{array}{l} \frac{dy}{dx} = f(x, y, z) = z \\ \frac{dz}{dx} = \phi(x, y, z) = x^3(y + z) \\ y = y_0 + \int_{x_0}^{x} f(x, y, z)dx \\ z = z_0 + \int_{x_0}^{x} \phi(x, y, z)dx \end{array}$$

The first approximation is

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0, z_0)\, dx = 1 + \int_0^x (1/2) dx = 1 + \tfrac{x}{2}$$

$$z_1 = z_0 + \int_{x_0}^{x} \phi(x, y_0, z_0)\, dx = \tfrac{1}{2} + \int_0^x x^3 \left(1 + \tfrac{1}{2}\right) dx$$
$$= \tfrac{1}{2} + \tfrac{3x^4}{8},$$

the second approximation is

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1, z_1)\, dx = 1 + \int_0^x \left(\frac{1}{2} + \frac{3x^4}{8}\right) dx$$
$$= 1 + \frac{x}{2} + \frac{3x^4}{40}$$

$$z_2 = z_0 + \int_{x_0}^{x} \phi(x, y_1, z_1)\, dx = \frac{1}{2} + \int_0^x x^3 \left(1 + \frac{x}{2} + \frac{1}{2} + \frac{3x^4}{8}\right) dx$$
$$= \frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64},$$

and, the third approximation is

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2, z_2)\, dx = 1 + \int_0^x \left(\frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64}\right) dx$$
$$= 1 + \frac{x}{2} + \frac{3x^4}{40} + \frac{x^6}{60} + \frac{x^9}{192}$$

$$z_3 = z_0 + \int_{x_0}^{x} \phi(x, y_2, z_2)\, dx$$
$$= \frac{1}{2} + \int_0^x x^3 \left(1 + \frac{x}{2} + \frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64}\right) dx$$
$$= \frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64} + \frac{7x^9}{360} + \frac{x^{12}}{256}$$

therefore, at $x = 0.1$ we have

$$y_1 = 1.05, \qquad y_2 = 1.500008, \qquad y_3 = 1.500008$$
$$z_1 = 0.5000375, \qquad z_2 = 0.5000385, \qquad z_3 = 0.5000385$$

■

2.1.2 Taylor method

Let $y(x), z(x)$ be the solution of the system (2.1), then by Taylor expansion of $y(x), z(x)$ around the point $x = x_0$, we have

$$\begin{cases} y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots \\ \\ z_1 = z_0 + hz_0' + \frac{h^2}{2!}z_0'' + \frac{h^3}{3!}z_0''' + \cdots \end{cases} \tag{2.3}$$

in order to obtain the solution, we have to determine the values of $y_0', y_0'', y_0''', \ldots$, also the values of $z_0', z_0'', z_0''', \ldots$, which can be done by differentiating $y' = f(x, y, z), z' = \phi(x, y, z)$ with respect to x, then substituting in (2.3), we have y_1, z_1 in the first step.
Similarly, in the second step we have

$$\begin{aligned} y_2 &= y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \cdots \\ \\ z_2 &= z_1 + hz_1' + \frac{h^2}{2!}z_1'' + \frac{h^3}{3!}z_1''' + \cdots \end{aligned} \tag{2.4}$$

where, y_1, z_1 and all its derivatives we obtained n the previous step. Repeating this, we will be able to obtain the values for the other steps

■ **Example 2.2** Using Taylor method, find the solution for

$$\begin{aligned} \frac{dy}{dx} &= x + z, y(0) = 2 \\ \frac{dz}{dx} &= x - y^2, \ z(0) = 1 \end{aligned}$$

at the point $x = 0.2$ with $h = 0.1$. ■

Solution. Since,

$$y' = x + z, \quad y(0) = 2$$

$$z' = x - y^2, \quad z(0) = 1$$

we can evaluate the following derivatives

$$\begin{aligned} y' &= x + z \\ y'' &= 1 + z' \\ y''' &= z'' \end{aligned}$$

$$\begin{aligned} z' &= x - y^2 \\ z'' &= 1 - 2yy' \\ z''' &= -2\left[yy'' + y'^2\right] \end{aligned}$$

then, we use Taylor series to obtain y_1, z_1 as

$$y_1 = y_0 + hy_0' + \tfrac{h^2}{2!}y_0'' + \tfrac{h^3}{3!}y_0''' + \cdots$$
$$z_1 = z_0 + hz_0' + \tfrac{h^2}{2!}z_0'' + \tfrac{h^3}{3!}z_0''' + \cdots$$

at

$$x_0 = 0, y_0 = 2, \quad z_0 = 1, h = 0.1$$

we get,

$$y_0' = x_0 + z_0 = 1, \qquad z_0' = x_0 - y_0^2 = -4$$
$$y_0'' = 1 + z_0' = 1 - 4 = -3, \quad z_0'' = 1 - 2y_0y_0' = 1 - 2(2)(1) = -3$$
$$y_0''' = z_0'' = -3, \qquad z_0''' = -2\left[y_0y_0'' + y_0'^2\right] = -2\left[2(-3) + 1^2\right] = 10$$

substituting with those values in the Taylor series we get

$$y_1 = 2 + (0.1)(1) + \frac{(0.1)^2}{2!}(-3) + \frac{(0.1)^3}{3!}(-3) + \cdots$$
$$= 2 + 0.1 - 0.015 - 0.0005 = 2.0845$$

$$z_1 = 1 + (0.1)(-4) + \frac{(0.1)^2}{2!}(-3) + \frac{(0.1)^3}{3!}(10) + \cdots$$
$$= 1 - 0.4 - 0.015 + 0.001667 = 0.5867$$
$$y(0.1) = 2.0845$$
$$z(0.1) = 0.5867$$

Similarly, for obtaining $y(0.2), z(0.2)$, we can write

$$y_2 = y_1 + hy_1' + \tfrac{h^2}{2!}y_1'' + \tfrac{h^3}{3!}y_1''' + \cdots$$
$$z_2 = z_1 + hz_1' + \tfrac{h^2}{2!}z_1'' + \tfrac{h^3}{3!}z_1''' + \cdots$$

at,

$$x_1 = 0.1, \quad y_1 = 2.0845, \quad z_1 = 0.5867$$

we get,

$$y_1' = x_1 + z_1 = 0.06867, \qquad z_1' = x_1 - y_1^2 = -4.2451403$$
$$y_1'' = 1 + z_1' = -3.2451403, \quad z_1'' = 1 - 2y_1y_1' = -1.8628523$$
$$y_1''' = z_1'' = -1.8628523, \qquad z_1''' = -2\left[y_1y_1'' + y_1'^2\right] = 12.585876$$

thus,

$y_2(0.2)$

$$\begin{aligned} y_2 &= 2.0845 + (0.1)(0.6867) + \frac{(0.1)^2}{2!}(-3.2451403) \\ &+ \frac{(0.1)^3}{3!}(-1.8628523) + \cdots \\ &= 2.1366338 \end{aligned}$$

$$\begin{aligned} z_2 &= 0.5867 + (0.1)(-4.2451403) + \frac{(0.1)^2}{2!}(-1.8628523) \\ &+ \frac{(0.1)^3}{3!}(12.585876) + \cdots \\ &= 0.1549693 \end{aligned}$$

■

2.1.3 Runge-kutta method

Let,

$$\frac{dy}{dx} = f_1(x, y, z), \quad \frac{dz}{dx} = f_2(x, y, z)$$

with the initial conditions

$$y(x_0) = y_0, \quad z(x_0) = z_0$$

The solution of the previous system using RK2, takes the following form

$$\begin{aligned} y_{n+1} &= y_n + \tfrac{1}{2}(k_1 + k_2) \\ z_{n+1} &= z_n + \tfrac{1}{2}(l_1 + l_2) \end{aligned}$$

where,

$$\begin{aligned} k_1 &= hf_1(x, y, z), & l_1 &= hf_2(x, y, z) \\ k_2 &= hf_1(x + h, y + k_1, z + l_1), & l_2 &= hf_2(x + h, y + k_1, z + l_1) \end{aligned}$$

The solution of the previous system using RK4, takes the following form

$$\begin{aligned} y_{n+1} &= y_n + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ z_{n+1} &= z_n + \tfrac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) \end{aligned}$$

where,

$$k_1 = hf(x, y, z),\ l = hf_2(x, y, z)$$

$$k_2 = hf_1\left(x + \tfrac{h}{2}, y + \tfrac{k_1}{2}, z + \tfrac{l_1}{2}\right)$$

$$l_2 = hf_2\left(x + \tfrac{h}{2}, y + \tfrac{k_1}{2}, z + \tfrac{l_1}{2}\right)$$

$$k_3 = hf_1\left(x + \tfrac{h}{2}, y + \tfrac{k_2}{2}, z + \tfrac{l_2}{2}\right),$$

$$l_3 = hf_2\left(x + \tfrac{h}{2}, y + \tfrac{k_2}{2}, z + \tfrac{l_2}{2}\right)$$

$$k_4 = hf_1\left(x + h, y + k_3, z + l_3\right)$$

$$l_4 = hf_2\left(x + h, y + k_3, z + l_3\right)$$

■ **Example 2.3** Using Rung-Kutta 4th find the solution for

$$\tfrac{dy}{dx} = yz + x, y(0) = 1$$

$$\tfrac{dz}{dx} = xz + y,\ z(0) = -1$$

and then find $y(0.2), z(0.2)$ ■

Solution. since

$$f_1(x, y, z) = yz + x, \quad f_2(x, y, z) = xz + y$$

$$x_0 = 0, y_0 = 1, z_0 = -1$$

$$k_1 = hf_1\left(x_0, y_0, z_0\right) = (0.1)[(1)(-1) + 0] = -0.1$$

$$l_1 = hf_2\left(x_0, y_0, z_0\right) = (0.1)[(0)(-1) + 1] = 0.1$$

$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = hf_1(0.05, 0.95, -0.95)$$

$$= (0.1)[(0.95)(-0.95) + 0.05] = -0.08525$$

$$l_2 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = hf_2(0.05, 0.95, -0.95)$$

$$= (0.1)[(0.05)(-0.95) + 0.95] = 0.09025$$

$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$= hf_1(0.05, 0.957375, -0.954875)$$

$$= (0.1)[(0.957375)(-0.954875) + 0.05] = -0.0864173$$

$$l_3 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$=hf_2(0.05, 0.957375, -0.954875)$$

$$=(0.1)[(0.05)(-0.954875) + 0.957375] = -0.0909631$$

$$k_4 = hf_1\left(x + h, y + k_3, z + l_3\right)$$

$$= hf_1(0.1, 0.9135827, -0.9090369)$$

$$= (0.1)[(0.9135827)(-0.9090369) + 0.1]$$

$$= -0.073048$$

$$l_4 = hf_2\left(x + h, y + k_3, z + l_3\right)$$

$$= hf_2(0.1, 0.9135827, -0.9090369)$$

$$= (0.1)[(0.1)(-0.9090369) + 0.9135827]$$

$$= 0.822679$$

$$k = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$= \frac{1}{6}[0.1 + 2(-0.08525) + 2(-0.0864173) - 0.073048]$$

$$= -0.0860637$$

$$l = \frac{1}{6}\left(l_1 + 2l_2 + 2l_3 + l_4\right)$$

$$= \frac{1}{6}[0.1 + 2(0.09025) + 2(0.0909631) - 0.0822679]$$

$$= -0.0907823$$

$$y_1 = y(0.1) = y_0 + k = 1 - 0.0860637 = 0.9139363$$

$$z_1 = z(0.1) = z_0 + l = -1 + 0.0907823 = -0.9092176$$

$$x_1 = 0.1, y_1 = 0.9139363, z_1 = -0.9092176$$

and, to get $y(0.2), z(0.2)$, we perform the following

$$k_1 = hf_1\left(x_1, y_1, z_1\right) = h\left(y_1 z_1 + x_1\right) = -0.0730966$$

$$l_1 = hf_2\left(x_1, y_1, z_1\right) = h\left(x_1 z_1 + y_1\right) = -0.08230145$$

$$k_2 = hf_1\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}, z_1 + \frac{l_1}{2}\right)$$

$$= hf_1(0.15, 0.877388, -0.8680669)$$

$$= (0.1)[(0.877388)(-0.8680669) + 0.15] = -0.0611631$$

$$l_2 = hf_2\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}, z_1 + \frac{l_1}{2}\right)$$

$$= hf_2(0.15, 0.877388, -0.8680669)$$

$$= (0.1)[(0.15)(-0.8680669) + 0.877388] = 0.0747177$$

$$k_3 = hf_1\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}, z_1 + \frac{l_2}{2}\right)$$

$$= hf_1(0.15, 0.8833547, -0.8718587)$$

$$= (0.1)[(0.8833547)(-0.8718587) + 0.15] = -0.062016$$

$$l_3 = hf_2\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}, z_1 + \frac{l_2}{2}\right)$$

$$= hf_2(0.15, 0.8833547, -0.8718587)$$

$$= (0.1)[(0.15)(-0.8718587) + 0.8833547] = 0.0750851$$

$$k_4 = hf_1\left(x + h, y + k_3, z + l_3\right)$$

$$= hf_1(0.2, 0.8519203, -0.8341324)$$

$$= (0.1)[(0.8519203)(-0.8341324) + 0.2]$$

$$= -0.0510614$$

$$l_4 = hf_2\left(x + h, y + k_3, z + l_3\right)$$

$$= hf_2(0.2, 0.8519203, -0.8341324)$$

$$= (0.1)[(0.2)(-0.8341324) + 0.8519203]$$

$$= 0.0685093$$

$$k = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$= \frac{1}{6}[-0.0730966 + 2(-0.0611631)$$

$$+ 2(-0.062016) - 0.0510614]$$

$$= -0.0617527$$

$$
\begin{aligned}
l &= \frac{1}{6}\left(l_1 + 2l_2 + 2l_3 + l_4\right) \\
&= \frac{1}{6}[0.08230145 + 2(-0.0747177) \\
&+ 2(0.0750851) + 0.0685093] \\
&= 0.0750693 \\
y_2 &= y(0.2) = y_1 + k = 0.9139363 - 0.0617527 \\
&= 0.8521836 \\
z_2 &= z(0.2) = z_1 + l = -0.9092176 + 0.0750693 \\
&= -0.8341482
\end{aligned}
$$

■

2.2 Ordinary differential equation of higher order

The generalized form of ordinary differential equation of n order is

$$y^{(n)} = f(x, y, y', y'', y''', \ldots, y^{(n-1)}) \tag{2.5}$$

and the initial values are

$$y(x_0) = \alpha_0, y'(x_0) = \alpha_1, y''(x_0) = \alpha_2, \ldots, y^{(n-1)}(x_0) = \alpha_{n-1}.$$

This equation could be solved after converting it into a system of ordinary differential equation of first order that had been discussed before.
In order to convert equation (2.5) into a system of ordinary differential

equation of first order, we let

$$\begin{aligned} y_1 &= y, \\ y_2 &= y' \\ y_3 &= y'' \\ &\vdots \\ y_n &= y^{(n-1)}, \end{aligned} \tag{2.6}$$

differentiating this system, we have

$$\begin{aligned} y_1' &= y' = y_2, \\ y_2' &= y'' = y_3 \\ y_3' &= y''' = y_4 \\ &\vdots \\ y_n' &= y^{(n)} = f(x, y_1, y_2, y_3, y_4, \ldots, y_n), \end{aligned} \tag{2.7}$$

This means that high order differential equation has been converted into a system of first order. Here, it will be enough to solve a second order differential equation using the previous mentioned methods.

2.2.1 Picard method for solving a second order differential equation

Consider the second order ordinary differential equation

$$y'' = f(x, y, y') \tag{2.8}$$

with the initial conditions

$$y(x_0) = y_0 = \alpha_0, y'(x_0) = \alpha_1$$

we write this equation in a form of system of first order which can be done by letting

$$y' = z, \qquad z' = y'' = f(x, y, z)$$

■ **Example 2.4** Using Picard method, find the solution of the following second order differential equation

$$\begin{aligned} &y'' + 2xy' + y = 0 \\ &y(0) = 0.5,\ y'(0) = 0.1 \end{aligned} \tag{2.9}$$

at $x = 0.1$. ■

Solution. let

$$y' = z \Rightarrow y'' = z' = \frac{dz}{dx}$$

thus, eq. (2.9) reads

$$\frac{dz}{dx} + 2xz + y = 0 \ \Rightarrow \frac{dz}{dx} = -(2xz + y)$$

This means that eq. (2.9) can be rewritten in the follow system form

$$\begin{aligned} y' &= z, \\ z' &= -(2xz + y) \end{aligned}$$

with the following initial conditions

$$y(0) = y_0 = 0.5, z(0) = z_0 = 0.1$$

let

$$y' = f(x, y, z) = z, \qquad z' = \phi(x, y, z) = -(2xz + y)$$

Using Picard method, we get

$$\begin{aligned} y &= y_o + \textstyle\int_{x_0}^{x} f(x, y, z)dx \\ z &= z_0 + \textstyle\int_{x_0}^{x} \phi(x, y, z)dx \end{aligned}$$

The first approximation reads

$$\begin{aligned}
y_1 &= y_0 + \int_{x_0}^{x} f\left(x, y_0, z_0\right) dx \\
&= 0.5 + \int_{x_0}^{x} z_0 dx = 0.5 + \int_{x_0}^{x} (0.1) dx \\
&= 0.5 + (0.1)x \\
z_1 &= z_0 + \int_{x_0}^{x} \phi\left(x, y_0, z_0\right) dx \\
&= 0.1 - \int_{x_0}^{x} \left(2xz_0 + y_0\right) dx = 0.1 - \int_{x_0}^{x} (0.2x + 0.5) dx \\
&= 0.1 - (0.5)x - (0.1)x^2
\end{aligned}$$

the second approximation is

$$\begin{aligned}
y_2 &= y_0 + \int_{x_0}^{x} f\left(x, y_1, z_1\right) dx \\
&= 0.5 + \int_{x_0}^{x} z_1 dx = 0.5 + \int_{x_0}^{x} \left(0.1 - (0.5)x - (0.1)x^2\right) dx \\
&= 0.5 + (0.1)x - \frac{(0.5)x^2}{2} - \frac{(0.1)x^3}{3} \\
z_2 &= z_0 + \int_{x_0}^{x} \phi\left(x, y_1, z_1\right) dx \\
&= 0.1 - \int_{x_0}^{x} \left(2xz_1 + y_1\right) dx \\
&= 0.1 - \int_{x_0}^{x} \left[\left(2x\left(0.1 - 0.5x - 0.1x^2\right) + (0.5 + 0.1x)\right] dx \\
&= 0.1 - (0.5)x - \frac{(0.3)x^2}{2} - \frac{(2.5)x^3}{6} + \frac{(0.2)x^4}{4}
\end{aligned}$$

and, the third approximation is

$$\begin{aligned}
y_3 &= y_0 + \int_{x_0}^{x} f(x, y_2, z_2)\, dx \\
&= 0.5 + \int_{x_0}^{x} z_2 dx = 0.5 + \int_{x_0}^{x} \left[0.1 - 0.5x + \frac{0.3}{2}x^2 - \frac{2.5}{6}x^3 + \frac{0.1}{4}x^4\right] dx \\
&= 0.5 + (0.1)x - \frac{(0.5)x^2}{2} - \frac{(0.1)x^3}{3} + \frac{x^4}{12} + \frac{(0.1)x^5}{10} \\
z_3 &= z_0 + \int_{x_0}^{x} \phi(x, y_2, z_2)\, dx \\
&= 0.1 - \int_{x_0}^{x} (2xz_2 + y_2)\, dx \\
&= 0.1 - (0.5)x - \frac{(0.3)x^2}{2} - \frac{(2.5)x^3}{6} + \frac{(0.2)x^4}{4} + \frac{2x^5}{15} + \frac{(0.1)x^6}{6}
\end{aligned}$$

Now, at $x = 0.1$, we have

$$y_1 = 0.51,\ y_2 = 0.50746667,\ y_3 = 0.50745933,$$

Thus, $y(0.1) = 0.5075$. ■

2.2.2 Taylor method

Suppose we have the following second order differential equation

$$y'' = f(x, y, y')$$

with the initial conditions

$$y(x_0) = y_0 = \alpha_0, y'(x_0) = \alpha_1$$

this equation can be converted into

$$\begin{aligned}
&y' = z, \\
&z' = f(x, y, z) \Rightarrow y'' = z' = f(x, y, z)
\end{aligned}$$

with the initial conditions

$$y(x_0) = y_0, y'(x_0) = z_0$$

Now, using Taylor expansion for the last two equation, we have

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!} z_0'' + \frac{h^3}{3!} z_0''' + \ldots$$

$$\begin{aligned} y_1 &= y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots \\ &= y_0 + hz_0 + \frac{h^2}{2!} z_0' + \frac{h^3}{3!} z_0'' + \ldots \end{aligned}$$

where, z_0', z_0'', z_0''' can be obtained be differentiating the second equation of the system.
With a similar way, we can get the second approximation of y_2, z_2 as

$$\begin{aligned} z_2 &= z_1 + hz_1' + \frac{h^2}{2!} z_1'' + \frac{h^3}{3!} z_1''' + \cdots \\ y_2 &= y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots \\ &= y_1 + hz_1 + \frac{h^2}{2!} z_1' + \frac{h^3}{3!} z_1'' + \cdots \end{aligned}$$

where, y_1, z_1 are know at this stage from the previous iterations. Finally, using the same manner, we can get approximate values for the other intervals.

■ **Example 2.5** Using Taylor expansion at $x = 0.1, 0.2$, find the solution of the following second order differential equation

$$\begin{aligned} & y'' - x(y')^2 + y^2 = 0 \\ & y(0) = 1, \; y'(0) = 0 \end{aligned} \tag{2.10}$$

at $x = 0.1$. ■

Solution. Putting

$$y' = z \Rightarrow y'' = z'$$

Therefore, the differential equation takes the following form

$$\begin{cases} y' = z \\ z' = xz^2 - y^2 \end{cases} \tag{2.11}$$

with the initial conditions

$$\begin{aligned} y(0) &= y_0 = 1 \\ z(0) &= z_0 = 0 \end{aligned} \tag{2.12}$$

Using Taylor expansion

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!}z_0'' + \frac{h^3}{3!}z_0''' + \cdots$$
$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} + \cdots$$

from the first equation, we have

$$z' = xz^2 - y^2, \qquad y'' = z'$$
$$z'' = z^2 + 2xzz' - 2yy', \qquad y''' = z''$$
$$z''' = 2zz' + 2\left[xPz' + x\left(z'\right)^2 + zz'\right]$$
$$- 2\left[yy'' + \left(y'\right)^2\right], \qquad y^{iv} = z'''$$

thus,

$$z_0' = x_0z_0^2 - y_0^2 = (0)(0)^2 - (1)^2 = -1$$
$$z_0'' = z_0^2 + 2x_0z_0z_0' - 2y_0y_0'$$
$$= (0)^2 + 2(0)(0)(-1) - 2(1)(0) = 0$$
$$z_0''' = 2z_0z_0' + 2\left[x_0z_0z_0' + x_0\left(z_0'\right)^2 + z_0z_0'\right]$$
$$- 2\left[y_0y_0'' + \left(y_0'\right)^2\right]$$
$$= 2(0)(-1) + 2\left[(0)(0)(-1) + (0)(-1)^2 + (0)(-1)\right]$$
$$- 2\left[(1)(-1) + (0)^2\right] = 2$$

substituting into the two equations of the system, we get

$$z_1 = 0 + (0.1)(-1) + \frac{(0.1)^2}{2!}(0) + \frac{(0.1)^3}{3!}(2) + \cdots$$
$$= -0.0997$$
$$y_1 = y(0.1) = 1 + (0.1)(0) + \frac{(0.1)^2}{2!}(-1) + \frac{(0.1)^3}{3!}(0) + \frac{(0.1)^4}{4!}(2) + \cdots$$
$$= 0.9950083 \approx 0.995$$

$$y_2 = y(0.2) = y_1 + hy_1' + \frac{h^2}{2!}y_1' + \frac{h^3}{3!}y_1''' + \frac{h^4}{4!}y_1^{iv} + \cdots$$
$$= y_1 + hz_1 + \frac{h^2}{2!}z_1' + \frac{h^3}{3!}z_1'' + \frac{h^4}{4!}z_1''' + \cdots$$

thus,

$$y_1 = 0.995, \quad z_1 = -0.0997 \tag{2.13}$$
$$z_1' = x_1 z_1^2 - y_1^2 = (0.1)(-0.0997) - (0.995)^2$$
$$= -0.9890309$$
$$z_1'' = z_1^2 + 2x_1 z_1 z_1' - 2y_1 y_1' = -0.1687416$$

then,

$$y_2 = 0.995 + \frac{(0.1)}{1!}(-0.0997) + \frac{(0.1)^2}{2!}(-0.9890309)$$
$$+ \frac{(0.1)^3}{3!}(-0.1687416) + \cdots = 0.9801129 \approx 0.9801$$
$$z_2 = z_1 + \frac{h}{1!}z_1' + \frac{h^2}{2!}z_1'' + \frac{h^3}{3!}z_1''' + \cdots$$
$$= -0.0997 + \frac{(0.1)}{1!}(-0.0997) + \frac{(0.1)^2}{2!}(-0.9890309)$$
$$+ \frac{(0.1)^3}{3!}(-0.1687416) = -0.1145871$$

■

2.2.3 Runge-Kutta

Suppose we have the following second order differential equation

$$y'' = f(x, y, y')$$

with the initial conditions

$$y(x_0) = y_0 = \alpha_0, y'(x_0) = \alpha_1$$

let

$$y' = z \Rightarrow y'' = z'$$

this equation now is converted into two equations from the first order as

$$\begin{aligned}&y' = z = f_1(x, y, z)\\&y'' = z' = f_2(x, y, z)\\&y(x_0) = y_0,\ z(x_0) = z_0\end{aligned}$$

that can be solved numerically using Rung-Kutta method.

■ **Example 2.6** Using Runge-Kutta of fourth order method (RK4), find the solution of the following second order differential equation

$$\begin{aligned}&y'' = xy' - y\\&y(0) = 3,\ y'(0) = 0\end{aligned} \tag{2.14}$$

at $x = 0.1$. ■

Solution. Suppose

$$\begin{aligned}&y' = z = f_1(x, y, z)\\&z' = xz - y = f_2(x, y, z)\\&y(0) = 3, z(0) = 0\end{aligned}$$

here,

$$x_0 = 0, y_0 = 3, z_0 = 0$$

Using RK4

$$\begin{aligned}
k_1 &= hf_1\left(x_0, y_0, z_0\right) = h\left(z_0\right) = (0.1)(0) = 0\\
l_1 &= hf_2\left(x_0, y_0, z_0\right) = h\left(x_0 z_0 - y_0\right)\\
&= (0.1)[(0)(0) - 3] = -0.3\\
k_2 &= hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = hf_1(0.05, 3, -0.15)\\
&= (0.1)(-0.15) = -0.015\\
l_2 &= hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = hf_2(0.05, 3, -0.15)\\
&= (0.1)[(0.05)(-0.15) - 3] = 0.030075
\end{aligned}$$

$$
\begin{aligned}
k_3 &= hf_1\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2}\right)\\
&= hf_1(0.05, 2.9925, -0.150375)\\
&= (0.1)(-0.150375) = -0.0150375\\
l_3 &= hf_2\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2}\right)\\
&= hf_2(0.05, 2.9925, -0.150375)\\
&= (0.1)[(0.05)(-0.150375) - 2.9925] = -0.03000018\\
k_4 &= hf_1\left(x+h, y+k_3, z+l_3\right)\\
&= hf_1(0.1, 2.9849624, -0.3000018)\\
&= (0.1)(-0.3000018) = -0.03000018\\
l_4 &= hf_2\left(x+h, y+k_3, z+l_3\right)\\
&= hf_2(0.1, 2.9849624, -0.3000018)\\
&= (0.1)[(0.1)(-0.3000018) - 2.9849624]\\
&= -0.3014962\\
&= \frac{1}{6}\left(k_1+2k_2+2k_3+k_4\right)\\
&= \frac{1}{6}[0+2(-0.015)+2(-0.0150375)-0.03000018]\\
&= -0.0150125\\
l &= \frac{1}{6}\left(l_1+2l_2+2l_3+l_4\right)\\
&= \frac{1}{6}[-0.3+2(-0.30075)+2(-0.3000018)-0.3014962]\\
&= -0.3004999\\
y_1 &= y(0.1) = y_0+k = 3-0.0150125 = 2.9849875\\
z_1 &= z(0.1) = z_0+l = 0-0.3004999 = -0.3004999
\end{aligned}
$$

■

Chapter 3

Multi-step methods

3.1 Introduction

In the previous chapters, we have studied the one-step methods which require the information of the solution at only one point, say; $x = x_0$, to obtain the value of the solution at $x = x_{n+1}$. On the other hand, the multi-step methods require the information of the solution at many points to obtain the final solution and those methods need the computation of $y(x), y'(x)$ at the points $x_0, x_1, x_2, \ldots, x_n$. Moreover, they depend on the integration of the differential equation.

3.2 Adam's Bashforth method

This method is used to solve the differential equation of the following form

$$y' = f(x, y), \qquad y(x_0) = y_0 \tag{3.1}$$

by integrating the two sides of the above equation from x_n to x_{n+1}, we have

$$\int_{x_n}^{x_{n+1}} dy = \int_{x_n}^{x_{n+1}} f(x, y)dx$$

or,

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f(x, y)dx$$

in order to perform the integration of the right hand side of the above equation, we approximate the function $f(x, y)$ in the form of a polynomial of second order using the Newton backward difference form, i.e.,

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} \Big[f_n + q\nabla f_n + \frac{q(q+1)}{2!}\nabla^2 f_n + \frac{q(q+1)(q+2)}{3!}\nabla^3 f_n + \dots \Big] dx$$

using the following change of variables

$$\begin{aligned} &x = x_n + qh \Rightarrow dx = hdq \\ &x = x_n \Rightarrow q = 0, \\ &x = x_{n+1} \Rightarrow q = 1, (\text{since } x_{n+1} - x_n = h) \end{aligned}$$

then the previous integration reads

$$\begin{aligned} y_{n+1} &= y_n + h\int_0^1 \Big[f_n + q\nabla f_n + \frac{q(q+1)}{2!}\nabla^2 f_n \\ &\quad + \frac{q(q+1)(q+2)}{3!}\nabla^3 f_n + \dots \Big] dq \\ y_{n+1} &= y_n + h\Big[q f_n + \frac{q}{2}\nabla f_n + \frac{(q^3/3) + (q^2/2)}{2!}\nabla^2 f_n \Big]_0^1 \end{aligned}$$

from which, we get

$$y_{n+1} = y_n + h\Big[f_n + \frac{1}{2}\nabla f_n + \frac{5}{12}\nabla^2 f_n]$$

then, substituting for $\nabla f_n, \nabla^2 f_n$, we have

$$\begin{aligned} &\nabla f_n = f_n - f_{n-1} \\ &\nabla^2 f_n = f_n - 2f_{n-1} + f_{n-2} \\ &y_{n+1} = y_n + h\left[f_n + \tfrac{1}{2}\left(f_n - f_{n-1}\right) + \tfrac{5}{12}\left(f_n - 2f_{n-1} + f_{n-2}\right)\right] \\ &y_{n+1} = y_n + \tfrac{h}{12}\left(23f_n - 16f_{n-1} + 5f_{n-2}\right), n \geq 2 \end{aligned}$$

this equation represents the Adam's Bashforth method for solving a differential equation of first order at a certain point.

■ **Example 3.1** Using Adam's Bashforth method, find the solution of the following differential equation

$$y' = y^2, \; y(0) = 1, \; h = 0.1 \tag{3.2}$$

then, find $y(0.3)$. ■

Solution. The Adam's Bashforth method of order three is

$$y_{n+1} = y_n + \frac{h}{12}(23f_n - 16f_{n-1} + 5f_{n-2}), \qquad n \geq 2$$

this means that we need to know the value of the function at three constituting points, one of those needed values can be obtained from the initial condition while the other two values can be computed using one of the one-step methods.
In this example, we choose the Taylor method as a one-step method, i.e.,

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2!}y''_n + \frac{h^3}{3!}y'''_n + \ldots$$

where,

$$\begin{aligned}
y'_n &= -y_n^2 \\
y''_n &= -2y_n y'_n = -2y_n\left(-y_n^2\right) = 2y_n^3 \\
y''_n &= 6y_n^2 y'_n = 6y_n^2\left(-y_n^2\right) = -6y_n^4 \\
\therefore y_{n+1} &= y_n + h\left(-y_n^2\right) + \tfrac{h^2}{2!}\left(2y_n^3\right) + \tfrac{h^3}{3!}\left(-6y_n^4\right) + \cdots \\
y_1 &= y_0 - hy_0^2 + h^2y_0^3 - h^3y_0^4 \\
&= 1 - (0.1)(1)^2 + (0.1)^2(1)^3 - (0.1)^3(1)^4 = 0.909 \\
y'_0 &= -y_1^2 \Rightarrow y'_1 = -(0.909)^2 = -0.826281 \\
\therefore f_1 &= -0.826281
\end{aligned}$$

$$\begin{aligned} y_2 &= y_1 - hy_1^2 + h^2y_1^3 - h^3y_1^4 \\ &= 0.909 - (0.1)(0.909)^2 + (0.1)^2(0.909)^3 - (0.1)^3(0.909)^4 \\ &= 0.833200055 \\ \therefore y_2' &= -y_2^2 \Rightarrow y_2' = -(0.833200055)^2 \\ &= -0.69422233 \\ \therefore f_2 &= -0.69422233 \end{aligned}$$

Now, using Adam's Bashforth method, we have

$$\begin{aligned} y_3 &= y_2 + \frac{0.1}{12}\left(23f_2 - 16f_1 + 5f_0\right) \\ &= 0.83300054 + \frac{0.1}{12}[23(-0.69422233) \\ &\quad - 16(-0.826281) + 5(-1)] = 0.7686449074 \end{aligned}$$

■

3.3 Adam's Maulton method

This method is one of the multi-step method and its difference compare to the Adam's Bashforth method is that it is an implicit method i.e., the expected method is corrected in the same step before moving to the next step.
Consider the following differential equation

$$y' = f(x, y), \qquad y(x_0) = y_0$$

Then, integrating the above equation from x_n to x_{n+1} leads to

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f(x, y)dx$$

and, in order to integrate the right hand side of that equation, we approximate the function $f(x, y)$ as a polynomial using Newton formula of backward interpolation.

$$\begin{aligned} y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} \Bigg[& f_{n+1} + q\nabla f_{n+1} + \frac{q(q+1)}{2!}\nabla^2 f_{n+1} \\ & + \frac{q(q+1)(q+2)}{3!}\nabla^3 f_{n+1} + \cdots \Bigg] dx \end{aligned}$$

Now, using the following relation

$$
\begin{aligned}
&x = x_{n+1} + qh \Rightarrow dx = hdq \\
&x = x_n \Rightarrow q = -1 \\
&x = x_{n+1} \Rightarrow q = 0, (\text{ since } x_{n+1} - x_n = h)
\end{aligned}
$$

we get,

$$
\begin{aligned}
y_{n+1} = y_n + h \int_{-1}^{0} \Bigg[f_{n+1} + q\nabla f_{n+1} + \frac{q(q+1)}{2!}\nabla^2 f_{n+1} \\
+ \frac{q(q+1)(q+2)}{3!}\nabla^3 f_{n+1} + \cdots \Bigg] dq
\end{aligned}
$$

Performing the previous integration, we have

$$
y_{n+1} = y_n + h \left[q f_{n+1} + \frac{q^2}{2}\nabla f_{n+1} + \frac{(q^3/3) + (q^2/2)}{2!}\nabla^2 f_{n+1} \right]_{-1}^{0}.
$$

Substituting the valued of $\nabla f_{n+1}, \nabla^2 f_{n+1}$

$$
\begin{aligned}
&\nabla f_{n+1} = f_{n+1} - f_n \\
&\nabla^2 f_{n+1} = f_{n+1} - 2f_n + f_{n-1}
\end{aligned}
$$

we get,

$$
y_{n+1} = y_n + h\Big[f_{n+1} - \frac{1}{2}(f_{n+1} - f_n) - \frac{1}{12}(f_{n+1} - 2f_n + f_{n-1}) \Big]
$$

which concludes the following formula

$$
y_{n+1} = y_n + \frac{h}{12}\Big[5f_{n+1} + 8f_n - f_{n-1} \Big], \quad n \geq 1 \tag{3.3}
$$

that is the Adam's Maulton method.

■ **Example 3.2** Using Adam's Maulton method, find $y(0.4)$ for the following differential equation

$$
y' = x + y, \; y(0) = 1, \; h = 0.1 \tag{3.4}
$$

■

Solution. In order to determine $y(0.4)$, using Adam's Maulton method, by eq. (3.3)

$$y_4 = y_3 + \frac{h}{12}\Big[5f_4 + 8f_3 - f_2\Big]$$

and, to determine f_4, it is required to use an explicit method; let's say Adam's Bashforth method i.e.,

$$y_4 = y_3 + \frac{h}{12}(23f_3 - 16f_2 + 5f_1)$$

also,

$$y_3 = y_2 + \frac{h}{12}(23f_2 - 16f_1 + 5f_0)$$

the question now is to obtain f_1 and f_2, that can be obtained with the help of one-step method, for instance, RK4

$$\begin{aligned}
y_1 &= y_0 + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
k_1 &= hf(x_0, y_0) = h[x_0 + y_0] = (0.1)(1) = 0.1 \\
k_2 &= hf\left(x_0 + \tfrac{h}{2}, y_0 + \tfrac{k_1}{2}\right) = (0.1)f(0.05, 1.05) \\
&= (0.1)[0.05 + 1.05] = 0.11 \\
k_3 &= hf\left(x_0 + \tfrac{h}{2}, y_0 + \tfrac{k_2}{2}\right) = hf(0.05, 1.055) \\
&= (0.1)[0.05 + 1.055] = 0.11050 \\
k_4 &= hf\left(x_0 + h, y_0 + k_3\right) = hf(0.1, 1.1105) \\
&= (0.1)[0.1 + 1.1105] = 0.12105
\end{aligned}$$

thus,

$$\begin{aligned}
y_1 =& y_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) \\
&= 1.0 + \frac{1}{6}[0.1 + 0.22 + 0.221 + 0.12105] \\
&= 1.11034
\end{aligned}$$

Similarly, we can use RK4 again to obtain $y_2 = 1.2428$

$$\begin{aligned}
y' &= f(x, y) = x + y \\
f_1 &= x_1 + y_1 = 0.1 + 1.1034 \\
&= 1.21034
\end{aligned}$$

substituting, f_1, f_2, we obtain the value of y_3

$$
\begin{aligned}
y_3 &= y_2 + \frac{h}{12}(23f_2 - 16f_1 + 5f_0) \\
&= 1.2428 + \frac{0.1}{12}[23(1.4428) - 16(1.21034) + 5(1)] \\
&= 1.399624667 \\
f_3 &= x_3 + y_3 = 0.3 + 1.399624667 \\
&= 1.699625
\end{aligned}
$$

then, substituting for f_3, y_3 we have

$$
\begin{aligned}
y_4^{(P)} &= y_3 + \frac{h}{12}(23f_3 - 16f_2 + 5f_1) \\
&= 1.39962447 + \frac{0.1}{12}[23(1.699635) - 16(1.4428) + 5(1.21034)] \\
&= 1.583443599 \\
f_4 &= x_4 + y_4^{(P)} = 0.4 + 1.583443899 \\
&= 1.98344
\end{aligned}
$$

then, we have

$$
\begin{aligned}
y_4^{(C)} &= 1.399624667 + \frac{0.1}{12}[5(1.98344) + 8(1.699625) - 1.4425] \\
&= 1.58385045
\end{aligned}
$$

R We can obtain the value of y_3, using RK4 instead of using Adam's Bashforth method.

■

3.4 Milne's method

One of the multi-step method and it is different from the previous methods in the following issues; (1) The expected value at a certain step is corrected before moving to the next step and, (2) It's required to know the values of the function $f(x, y)$ at four constitutive points i.e., we need to know y at $x_n, x_{n-1}, x_{n-2}, x_{n-3}$ to evaluate y at x_{n+1}. Consider the following differential equation

$$y' = f(x, y), \; y(x_0) = y_0 \tag{3.5}$$

integrating this equation from x_{n-3} to x_{n+1}, we get

$$\int_{x_{n-3}}^{x_{n+1}} dy = \int_{x_{n-3}}^{x_{n+1}} f(x, y)dx$$

As in Adam's method, we approximate $f(x, y)$ by a polynomial of second order using Newton formula for backward interpolation, then we can write,

$$y_{n+1} - y_{n-3} = \int_{x_{n-3}}^{x_{n+1}} \Big(f_n + q\nabla f_n + \frac{q(q+1)}{2!}\nabla^2 f_n + E \Big)$$

where,

$$E = \frac{q(q+1)(q+2)}{3!} h^3 f^{(3)}(\xi), \; x_{n-3} \le \xi \le x_{n+1}$$

using the following relation

$$\begin{aligned} &x = x_{n+1} + qh \Rightarrow dx = hdq \\ &x = x_{n-3} \Rightarrow q = -3 \\ &x = x_{n+1} \Rightarrow q = 1, (\text{ since } x_{n+1} - x_n = h) \end{aligned}$$

we get,

$$y_{n+1} = y_{n-3} + h \int_{-3}^{1} \left[f_n + q\nabla f_n + \frac{q(q+1)}{2!}\nabla^2 f_{n+1} + E \right] dq$$

performing the integration for the variable q, we get

$$y_{n+1} = y_{n-3} + 4h\Big(f_n - \nabla f_n + \frac{2}{3}\nabla^2 f_n \Big) + O(h^5)$$

Substitution for the vales of $\nabla f_n, \nabla^2 f_n$, we have

$$y_{n+1} = y_{n-3} + \frac{4h}{3}\Big(2f_n - f_{n-1} + 2f_{n-2} \Big) + O(h^5)$$

Note that, the value of y_{n+1} obtained from the above equation is called the predicted vale which denoted by $y_{n+1}^{(P)}$ and in order to correct or enhance this value, we may use Simpson rule for integration. Integrate (3.5) from x_{n-1} to x_{n+1} and change the limits of the integration as done before, we have

$$y_{n+1} = y_{n-1} + h \int_{-1}^{1} \left[f_n + q\Delta f_n + \frac{q(q+1)}{2!}\Delta^2 f_n + \cdots \right] dq$$

substituting for Δf_n and $\Delta^2 f_n$, we have

$$y_{n+1} = y_{n-1} + \frac{h}{3}\left(f_{n-1} + 4f_n + f_{n+1}\right) + O(h^5)$$

which is called the corrected value and is denoted by $y_{n+1}^{(C)}$.

R For the purpose of applying the above method, it's required to know four values of the function and in case of they are not known, we may use any method of the one-step methods.

■ **Example 3.3** let

$$\frac{dy}{dx} = \frac{1}{x+y},$$

$$y(0) = 2,\ y(0.2) = 2.0933,\ y(0.4) = 2.1755,\ y(0.6) = 2.2493$$

find $y(0.8)$ using Milne's method. ■

Solution.

$$y_{n+1}^{(P)} = y_{n-3} + \frac{4h}{3}\left(2y'_n - y'_{n-1} + 2y'_{n-2}\right)$$

since,

$$x_0 = 0,\ x_1 = 0.2,\ x_3 = 0.6,\ h = 0.2,$$

$$y_0 = 2,\ y_1 = 2.0933,\ y_2 = 2.1755,\ y_3 = 2.2493$$

Now, we have

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left(2y'_3 - y'_2 + 2y'_1\right)$$

and,

$$y_1' = \frac{1}{x_1 + y_1} = \frac{1}{0.2 + 2.0933} = 0.4360528$$

$$y_2' = \frac{1}{x_2 + y_2} = \frac{1}{0.4 + 2.1755} = 0.3882741$$

$$y_3' = \frac{1}{x_3 + y_3} = \frac{1}{0.6 + 2.2493} = 0.3509633$$

thus,

$$\begin{aligned} y_4^{(P)} &= 2 + \frac{4(0.2)}{3}\left(2(0.3509633) - (0.3882741) + 2(0.4360528)\right) \\ &= 2.3162022 \end{aligned}$$

Now, for the corrected values, we have

$$y_{n+1}^{(C)} = y_{n-1} + \frac{h}{3}\left(y_{n-1}' + 4y_n' + y_{n+1}'\right)$$

for the current case, we have $n = 3$ i.e.,

$$y_4^{(C)} = y_2 + \frac{h}{3}\left(y_2' + 4y_3' + y_4'\right)$$

and,

$$\begin{aligned} y_4^{(P)} &= 2.3162022, x_4 = 0.8 \\ y_4' &= \frac{1}{x_4 + y_4^{(P)}} = \frac{1}{0.8 + 2.3162022} = 0.3209034 \end{aligned} \tag{3.6}$$

thus,

$$\begin{aligned} y_4^{(C)} &= 2.1755 + \frac{0.2}{3}[0.3882741 + 4(0.3509633) + 0.3209034] \\ &= 2.3163687 \\ \therefore\ y(0.8) &= y_4 = 2.3164 \end{aligned}$$

■

■ **Example 3.4** Find the solution of the following differential

$$\frac{dy}{dx} = (x + y)y,\ y(0) = 1,\ h = 0.1 \tag{3.7}$$

using Milne's method to obtain $y(0.4)$. compute y at $x = 0.1, 0.2, 0.3$ using RK4 ■

Solution. First, we compute $y(0.1)$, $y(0.2)$ $y(0.3)$ using RK4, this computations are left to the reader, which lead to

$$y(0.1) = 1.11689, \; y(0.2) = 1.27739, \; y(0.3) = 1.50412,$$

$$\begin{aligned} x_0 &= 0, & y_0 &= 1 \\ x_1 &= 0.1, & y_1 &= 1.11689 \\ x_2 &= 0.2, & y_2 &= 1.27739 \\ x_3 &= 0.3, & y_3 &= 1.1.50412 \end{aligned} \tag{3.8}$$

Since,

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left(2y_3' - y_2' + 2y_1'\right)$$

and,

$$\begin{aligned} y' &= (x+y)y \\ y_1' &= (x_1 + y_1)\, y_1 = (0.1 + 1.11689)(1.11689) = 1.3591323 \\ y_2' &= (x_2 + y_2)\, y_2 = (0.2 + 1.27739)(1.27739) = 1.8872032 \\ y_3' &= (x_3 + y_3)\, y_3 = (0.3 + 1.50412)(1.50412) = 2.713613 \end{aligned} \tag{3.9}$$

Thus,

$$y_4^{(P)} = 1 + \frac{4(0.1)}{3}\left(2(2.713613) - 1.8872032 + 2(1.3591323)\right) = 1.8344383$$

Now, the corrected value reads,

$$y_4^{(C)} = y_2 + \frac{h}{3}\left(y_2' + 4y_3' + y_4'\right)$$

$$y_4' = (x_4 + y_4^{(P)})y_4^{(P)} = (0.4 + 1.8344383)(1.8344383) = 4.0989392$$

Thus,

$$y_4^{(C)} = 1.27739 + \frac{(0.1)}{3}\left(1.8872.32 + 4(2.713613) + 4.0989392\right) = 1.8387431$$

■

■ **Example 3.5** Find for the following differential equation

$$\frac{dy}{dx} = (x+y), \; y(0) = 1, \; h = 0.1 \tag{3.10}$$

the value of $y(0.5)$. ■

Solution. Since Milne's method for the predicted value reads

$$y_{n+1}^{(P)} = y_{n-3} + \frac{4h}{3}\left(2y'_n - y'_{n-1} + 2y'_{n-2}\right)$$

we have to determine the value of y at four points, so we use RK for this purpose, and we have the following results,

x	y	$y' = f(x,y) = x + y$
0	$y_{n-3} = 1$	$f_{n-3} = 1$
0.1	$y_{n-2} = 1.11$	$f_{n-2} = 1.210$
0.2	$y_{n-1} = 1.242$	$f_{n-1} = 1.442$
0.3	$y_n = 1.399$	$f_n = 1.699$

Therefore,

$$y_4^{(P)} = 1 + \frac{4(0.1)}{3}\left[2(1.699) - (1.442) + 2(1.210)\right] = 1.58364$$

Now, to compute $y_{n+1}^{(C)}$ we need to find f_{n+1}

$$f_{n+1} = f(x_{n+1}, y_{n+1}^{(P)}) = f(0.4, 1.584) = 1.984$$

and since,

$$y_4^{(C)} = y_2 + \frac{h}{3}(y'_2 + 4y'_3 + y'_4)$$

we have,

$$y_4^{(C)} = 1.242 + \frac{(0.1)}{3}[1.984 + 4(1.699) + 1.442] = 1.58364$$

Note that, $y_{n+1}^{(P)}$, $y_{n+1}^{(C)}$ have the same value i.e. there is no enhancement in the value of y. Now, we have the values of f ready and we do not have to use RK again. Thus,

$$y_{n+1}^{(P)} = y(0.5) = 2.29742$$

$$y_{n+1}^{(C)} = y(0.5) = 2.29742$$

■

Chapter 4

Boundary Value Problems

This chapter is devoted for the following items:

4.1 The Finite Difference Method for Linear Problems

4.2 Solution of the Discretized Problem

Boundary Value Problems

The Finite Difference Method for Linear Problems

In the previous chapters, we have considered the **initial value problems** for ordinary differential equations which has the following form

$y'(t) = f(t, y) \quad , t \geq a$

$y(a) = \alpha$

In many problems, however, there will be conditions on the solution given at more than one point. For a single first order equation $y'(t) = f(t, y)$, data at one point completely determines the solution so that if conditions at more than one point are given, either higher order equations or systems of equations must be treated.

Consider the second-order equation

$y''(t) = f(t, y, y'), 0 \leq t \leq 1$ (1)

With the boundary conditions:

$y(0) = \alpha, y(1) = \beta$ (2)

Equations (1) and (2) define a two-point **boundary value problem**.

[MCQ]The problem:[$y''(t) = f(t, y, y'), 0 \leq t \leq 1$] is… for ordinary differential equations. boundary value problem- **initial value problems-…**
[MCQ]The problem:[$y'(t) = f(t, y) \quad , t \geq a, y(a) = \alpha, y(0) = \alpha, y(1) = \beta$] is… for ordinary differential equations. **boundary value problem-** initial value problems-…

If the function *f* of Eq. (1) is nonlinear in either y(t) or y'(t), the **boundary value problem** is nonlinear. Nonlinear boundary value problems are more difficult to solve, and we shall not consider them.

In this chapter we treat only linear problems, in which Eq. (1) may be written in the form

$y''(t) = b(t)y'(t) + c(t)y(t) + d(t), 0 \leq t \leq 1$ (3)

where b, c, and d are given functions of t. The boundary conditions that we consider first will be of form (2). Later, we shall treat other types of boundary conditions.

Equations (3) and (2) define a linear two-point boundary-value problem for the unknown function y, and our task is to develop procedures to approximate the solution. We will assume that the problem has a unique solution that is at least two times continuously differentiable.

We first consider the special case of (3) in which b(x) = 0, so we have the following example:

Example

Consider the following boundary value problem

$\boldsymbol{y''(t) = c(t)y(t) + d(t), 0 \leq t \leq 1}$

with the conditions,

$\boldsymbol{y(0) = \alpha, y(1) = \beta}$

Use finite difference approximation to obtain y'' (t).

Obtain the resulting tridaigonal system. Find the coefficient matrix when c(t)=0.

Solution:

We will assume that c(t) > 0 for 0 < t < 1; this is a sufficient condition for the problem (4), (2) to have a unique solution.

To begin the numerical solution we divide the interval [0,1] into a number of equal subintervals of length h, as shown in Figure 3.1.

To obtain numerical solution for this problem , we divide the interval [0,1] into n+1 sub-interval using the points
$t_i = t_0 + ih = ih \,;\, t_0 = 0 \,, h = \frac{1}{n+1}$, i= 1,2,…..,n+1

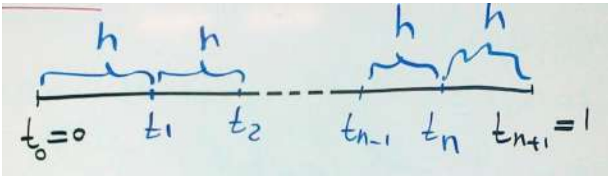


Figure 1: Grid Points

Using difference method to approximate $\boldsymbol{y''(t)}$

$$\boldsymbol{y''(t_i)} = \frac{\boldsymbol{y(t_{i+1})} - 2\boldsymbol{y(t_i)} + \boldsymbol{y(t_{i-1})}}{h^2} \tag{4}$$

Where $t_{i+1} = t_i + h$

First , we write the given problem at $t = t_i$

$$y''(t_i) = c(t_i)y(t_i) + d(t_i), \qquad i = 1,2, \dots, n \tag{5}$$

$$\mathrm{y}(t_0) = \mathrm{y}(0) = \alpha \qquad , \mathrm{y}(t_{n+1}) = \mathrm{y}(1) = \beta \tag{6}$$

substituting from (4), Eq (5) becomes

$$\frac{y(t_{i+1}) - 2y(t_i) + y(t_{i-1})}{h^2} = c(t_i)y(t_i) + d(t_i) \tag{7}$$

For simplicity , we write $y(t_i) = y_i$

$$\boldsymbol{\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = c_i y_i + d_i \qquad , i = 1, 2, \dots, n} \tag{8}$$

$$\boldsymbol{y_0 = \alpha, \qquad y_{n+1} = \beta} \tag{9}$$

For (8) multiplying on $\boldsymbol{h^2}$

$$\boldsymbol{y_{i+1} - 2y_i + y_{i-1} = h^2 c_i y_i + h^2 d_i}$$

$$\boldsymbol{y_{i+1} - 2y_i - h^2 c_i y_i + y_{i-1} = h^2 d_i}$$

This equation can be rearranged to have the following scheme for all he values of i

$$y_{i+1} - (2 + h^2 c_i) y_i + y_{i-1} = h^2 d_i \,, \quad i = 1, 2, \dots, n \tag{10}$$

$$y_0 = \alpha \quad , y_{n+1} = \beta \tag{9}$$

$$\begin{aligned}
i = 1 \qquad & y_2 - \left(2 + h^2 c_1\right) y_1 + y_0 = h^2 d_1 \\
i = 2 \qquad & y_3 - \left(2 + h^2 c_2\right) y_2 + y_1 = h^2 d_2 \\
& \quad \vdots \\
i = n-1 \qquad & y_n - \left(2 + h^2 c_{n-1}\right) y_{n-1} + y_{n-2} = h^2 d_{n-1} \\
i = n \qquad & y_{n+1} - (2 + h^2 c_n) y_n + y_{n-1} = h^2 d_n
\end{aligned} \tag{11}$$

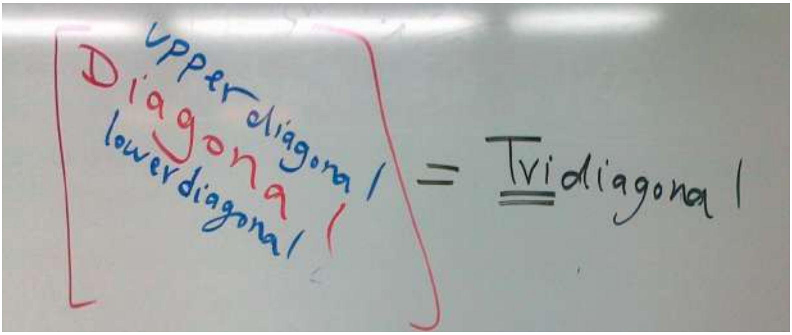
$$\left.\begin{aligned}
y_2 - \left(2 + h^2 c_1\right) y_1 + \alpha &= h^2 d_1 \\
y_3 - \left(2 + h^2 c_2\right) y_2 + y_1 &= h^2 d_2 \\
&\vdots \\
y_n - \left(2 + h^2 c_{n-1}\right) y_{n-1} + y_{n-2} &= h^2 d_{n-1} \\
\beta - (2 + h^2 c_n) y_n + y_{n-1} &= h^2 d_n
\end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned}
y_2 - \left(2 + h^2 c_1\right) y_1 &= h^2 d_1 - \alpha \\
y_3 - \left(2 + h^2 c_2\right) y_2 + y_1 &= h^2 d_2 \\
&\vdots \\
y_n - \left(2 + h^2 c_{n-1}\right) y_{n-1} + y_{n-2} &= h^2 d_{n-1} \\
-\left(2 + h^2 c_n\right) y_n + y_{n-1} &= h^2 d_n - \beta
\end{aligned}\right\} \tag{13}$$

In matrix form AY=B

$$\begin{bmatrix}
-(2 + h^2 c_2) & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -(2 + h^2 c_2) & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & \vdots & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & -(2 + h^2 c_2) & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -(2 + h^2 c_2)
\end{bmatrix}
\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix}
=
\begin{bmatrix} h^2 d_1 - \alpha \\ h^2 d_2 \\ \vdots \\ h^2 d_{n-1} \\ h^2 d_n - \beta \end{bmatrix} \tag{14}$$

The coefficient matrix A is tridiagonal Eq.4 is the resulting tridiagonal system which we must solve to obtain the numerical solution.



$$\begin{bmatrix} -(2+h^2c_2) & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -(2+h^2c_2) & 0 & 0 & 0 & 0 & 0 & 0 \\ & & \vdots & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & -(2+h^2c_2) & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -(2+h^2c_2) \end{bmatrix}$$

When c(t) = 0 in the given problem the the coefficient matrix A is

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & \vdots & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

This is an important matrix which arises in many contexts, as we shall see. Matrices of the form (3.1.9) or (3.1.10) are called *tridiagonal* since only the three main diagonals of the matrix have non-zero elements. Tridiagonal matrices arise in a variety of applications in addition to the two-point boundary value problems of this chapter.

[MCQ]The resulting algebraic system of applying difference methods in approximating BVP of ODE's is
Tridiagonal-diagonal-…

Example
Consider the boundary value problem
$y''(t) = 2\,, 0 \le t \le 1$ **(1)**
With the conditions,
$y(0) = 0, y(1) = 1$ **(2)**

Use n=3 with difference approximation to y'' (t).
Obtain the resulting tridaigonal system.

Answer:

Consider the BVP $\boldsymbol{y''(t) = 2\,, 0 \le t \le 1}$ **(1)**

$\boldsymbol{y(0) = 0, y(1) = 1}$ **(2)**

$t_i = 0 + ih = ih\,;\ h = \dfrac{1}{n+1}$

With n=3➔ h = $\frac{1}{4}$: step between points

$t_0 = 0$	$t_1 = \frac{1}{4}$	$t_2 = \frac{1}{2}$	$t_3 = \frac{3}{4}$	$t_4 = 1$
$y_0 = 0$	$y_1 = ?$	$y_2 = ?$	$y_3 = ?$	$y_4 = 1$

0 0.25 0.5 0.75 1

$$y''(t) = 2$$

by integration

$$y'(t) = 2\int dt + c = 2t + c$$

by integration

$$y(t) = 2\int t dt + ct + d$$

$y(t) = t^2 + ct + d$

$y(0) = 0$ ⟹ 0 = 0 + 0 + d ⟹ d = 0

$y(1) = 1$ ⟹ 1 = 1 + c ⟹ c = 0

Hence the exact solution

$y(t) = t^2$

$$h = \frac{b-a}{n+1} = \frac{1-0}{4} = \frac{1}{4}$$

$y''(t) = 2$ ➔$y''(t_i) = \frac{y(t_{i+1}) - 2y(t_i) + y(t_{i-1})}{h^2}$ (3)

$\frac{y(t_{i+1}) - 2y(t_i) + y(t_{i-1})}{h^2} = 2$, h = $\frac{1}{4}$

So

$$16\,[y(t_{i+1}) - 2y(t_i) + y(t_{i-1})] = 2$$

Subsitute in the given Eq.(1) , divide by 16

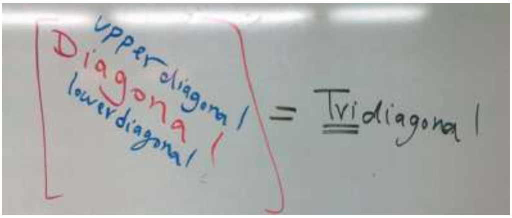
$$\mathrm{y_{i+1}} - 2\mathrm{y_i} + \mathrm{y_{i-1}} = \frac{2}{16} = \frac{1}{8}$$

$$\mathrm{y_{i+1}} - 2\mathrm{y_i} + \mathrm{y_{i-1}} = \frac{1}{8} \quad ;\mathrm{i} = 1, 2,3 \qquad (4)$$

The resulting tridiagonal system is

$$
\left.\begin{aligned}
y_2 - 2y_1 &= \frac{1}{8} && (i)\\
y_3 - 2y_2 + y_1 &= \frac{1}{8} && (ii)\\
1 - 2y_3 + y_2 &= \frac{1}{8} && (iii)
\end{aligned}\right\}
$$

$$
\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}
$$



[MCQ]The resulting algebraic system of applying difference methods in approximating BVP of ODE's is Tridiagonal-diagonal-…
[MCQ]The resulting algebraic system of applying difference approximation to $y''(t)$ and backward formula for $y'(t)$ for the BVP:$\boldsymbol{y''(t) = 2 \,, 0 \le t \le 1}$, $y(0) = 0, y(1) = 1$. with $h = \frac{1}{4}$ is $\boldsymbol{a_{13}y_3 + a_{12}y_2 + a_{11}y_1} = \mathrm{b}_1, \boldsymbol{a_{23}y_3 + a_{22}y_2 + a_{21}y_1} = \mathrm{b}_2, \boldsymbol{a_{33}y_3 + a_{32}y_2 + a_{31}y_1} = \mathrm{b}_3$ Answer the following 9 questions: 1) $\boldsymbol{a_{13}} = \left[0, 1, -2, \frac{1}{8}\right]$ (2) $\boldsymbol{a_{12}} = \left[0, 1, -2, \frac{1}{8}\right]$ (3) $\boldsymbol{a_{11}} = \left[0, 1, -2, \frac{1}{8}\right]$ 4) $\boldsymbol{a_{23}} = \left[0, 1, -2, \frac{1}{8}\right]$ (5) $\boldsymbol{a_{22}} = \left[0, 1, -2, \frac{1}{8}\right]$ (6) $\boldsymbol{a_{21}} = \left[0, 1, -2, \frac{1}{8}\right]$ 7) $\boldsymbol{a_{33}} = \left[0, 1, -2, \frac{1}{8}\right]$ (8) $\boldsymbol{a_{32}} = \left[0, 1, -2, \frac{1}{8}\right]$ (9) $\boldsymbol{a_{31}} = \left[0, 1, -2, \frac{1}{8}\right]$

This is the resulting system of the equation which defines the unknowns y_1, y_2 and y_3

Solution of the Discretized Problem

In the previous section we saw that the use of finite difference discretization of the two-point boundary value problem C.1.3) led to a system of linear equations. The exact form of this system depends on the boundary conditions, but in all the cases we considered, except periodic boundary conditions, the system was of the tridiagonal form

$$
\begin{bmatrix}
a_{11} & a_{12} & & & \\
a_{21} & a_{22} & a_{23} & & \\
 & a_{32} & \ddots & \ddots & \\
 & & \ddots & & a_{n-1,n} \\
 & & & a_{n,n-1} & a_{nn}
\end{bmatrix}
\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}
=
\begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}. \qquad (3.2.1)
$$

Gaussian Elimination
We will solve the system (3.2.1) by the Gaussian elimination method. This method, along with several variants, will be considered in detail for general linear systems in the next chapter

[Q] Use **Gaussian Elimination** method to obtain the numerical solution for the resulting system of the previous problem

$$y_2 - 2y_1 = \frac{1}{8} \qquad (i)$$

$$y_3 - 2y_2 + y_1 = \frac{1}{8} \qquad (ii)$$

$$1 - 2y_3 + y_2 = \frac{1}{8} \qquad (iii)$$

To solve this system

2(i) + (ii) ⟹ Eliminating y_2

2(i) ⟹ $2y_2 - 4y_1 = \frac{2}{8}$

(ii) ⟹ $y_3 - 2y_2 + y_1 = \frac{1}{8}$

$y_3 - 3y_1 = \frac{3}{8}$ ⟶ (iv)

(ii) + 2(iii) ⟹ Eliminating y_2

2(iii) ⟹ $\frac{16}{8} - 4y_3 + 2y_2 = \frac{2}{8}$

(ii) ⟹ $y_3 - 2y_2 + y_1 = \frac{1}{8}$

$-3y_3 + y_1 = \frac{-13}{8}$ ⟶ (v)

$$y_3 - 3y_1 = \frac{3}{8} \qquad \text{(iv)}$$

$$-3y_3 + y_1 = \frac{-13}{8} \qquad \text{(v)}$$

3(v) + (iv) ⟹ Eliminating y_3

3(v) ⟹ $-9y_3 + 3y_1 = \frac{-39}{8}$

(ii) ⟹ $y_3 - 3y_1 = \frac{3}{8}$

$-8y_3 = \frac{-36}{8} = \frac{18}{4} = \frac{9}{2}$

$$y_3 = \frac{-9}{-2(8)} = \frac{9}{16}$$

In (v)

$y_1 = \frac{-13}{8} + 3y_3 = \frac{-13}{8} + 3(\frac{9}{16}) = \frac{-26}{16} + \frac{27}{16} = \frac{1}{16}$

$y_0 = 0 = y(t_0) = y(0)$

$y_1 = \frac{1}{16} \cong y(t_1) = y\left(\frac{1}{4}\right)$

$y_2 = \frac{1}{4} \cong y(t_2) = y\left(\frac{1}{2}\right)$

$y_3 = \frac{9}{16} \cong y(t_3) = y\left(\frac{1}{2}\right)$

$y_4 = 1 = y(t_4) = y(1)$

Exact solution

So a= b = 0

$y_e(t) = t^2$

t	y_a	y_e	Error
0	0	0	0
$\frac{1}{4}$	$y_1 = \frac{1}{16}$	$\frac{1}{16}$	0
$\frac{1}{2}$	$y_2 = \frac{1}{4}$	$\frac{1}{4}$	0
$\frac{3}{4}$	$y_3 = \frac{9}{16}$	$\frac{9}{16}$	0
1	1	1	0

[MCQ]Consider the application of Gaussian Elimination method for solving the resulting algebraic system of applying difference approximation tocetain BVP:

$$y_2 - 2y_1 = \frac{1}{8}, y_3 - 2y_2 + y_1 = \frac{1}{8}, 1 - 2y_3 + y_2 = \frac{1}{8}.$$

Answer the following 3 questions:

1) $y_1 = \left[0, \frac{1}{16}, \frac{1}{4}, \frac{9}{16}\right]$ (2) $y_2 = \left[0, \frac{1}{16}, \frac{1}{4}, \frac{9}{16}\right]$ (3) $y_3 = \left[0, \frac{1}{16}, \frac{1}{4}, \frac{9}{16}\right]$

Home Work

[1] Consider the boundary value problem

$y''(t) + y'(t) = 2(1 + t), 0 \le t \le 1$ **(1)**

With the conditions,

$y(0) = 0, y(1) = 1$ **(2)**

Use $h = \frac{1}{4}$ with difference approximation to y'' (t) and forward formula for y'. Obtain the resulting tridaigonal system.

[1](b) Use Gaussian Elimination method to obtain the numerical solution for the resulting system of the previous problem. If you know that the exact solution is $y(t) = t^2$ then obtain the numerical error.

[2a] Consider the boundary value problem

$$y''(t) + ty'(t) - 2y(t) = 2$$

With the conditions,

$$y(0) = 0, y(1) = 1$$

Use n=3 with difference approximation of y'' (t) and central difference approximation of y' (t). Obtain the resulting tridaigonal system.

[2b] Use Gaussian Elimination method to obtain the numerical solution for the resulting system of the previous problem. If you know that the exact solution is $y(t) = t^2$ then obtain the numerical error.

[3a] Consider the boundary value problem

$$y''(t) + 5ty'(t) - 3y(t) = 7t^2 + 2$$

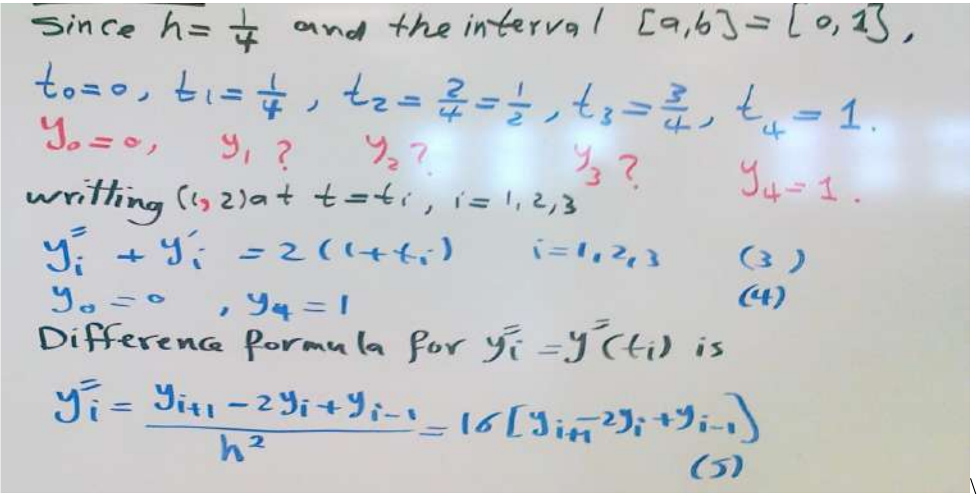
With the conditions,

$$y(0) = 0, y(1) = 1$$

Use n=3 with difference approximation to y'' (t) and y' (t). Obtain the resulting tridaigonal system.

[3b] Use Gaussian Elimination method to obtain the numerical solution for the resulting system of the previous problem. If you know that the exact solution is $y(t) = t^2$ then obtain the numerical error.

Answer of [1a]



\

backward formula

$$y_i' = \frac{y_i - y_{i-1}}{h} = 4[y_i - y_{i-1}] \qquad (6)$$

Inserting (5), (6) in (3),

$$16[y_{i+1} - 2y_i + y_{i-1}] + 4[y_i - y_{i-1}] = 2(1 + t_i)$$
$$16y_{i+1} - 32y_i + 16y_{i-1} + 4y_i - 4y_{i-1} = (2 + 2t_i)$$
$$16y_{i+1} + (4y_i - 32y_i) + (16y_{i-1} - 4y_{i-1}) = (2 + 2t_i)$$

$$16y_{i+1} - 28y_i + 12y_{i-1} = 2 + 2t_i \quad , i = 1, 2, 3 \qquad \textbf{(7)}$$
$$i = 1 ==> 16y_2 - 28y_1 + 12y_0 = 2 + 2t_1 = 2 + 2 * \frac{1}{4}$$
$$i = 2 ==> 16y_3 - 28y_2 + 12y_1 = 2 + 2 * \frac{2}{4}$$
$$i = 3 ==> 16y_4 - 28y_3 + 12y_2 = 2 + 2 * \frac{3}{4}$$

y_0=0, $y_4 = 1$

$$16y_2 - 28y_1 = 2 + 2t_1 = 2\frac{1}{2}$$
$$16y_3 - 28y_2 + 12y_1 = 3$$
$$-28y_3 + 12y_2 = 3\frac{1}{2} - 16$$

the resulting tridaigonal system **is**

$$16y_2 - 28y_1 = \frac{5}{2}$$
$$16y_3 - 28y_2 + 12y_1 = 3$$
$$-28y_3 + 12y_2 = -\frac{25}{2}$$

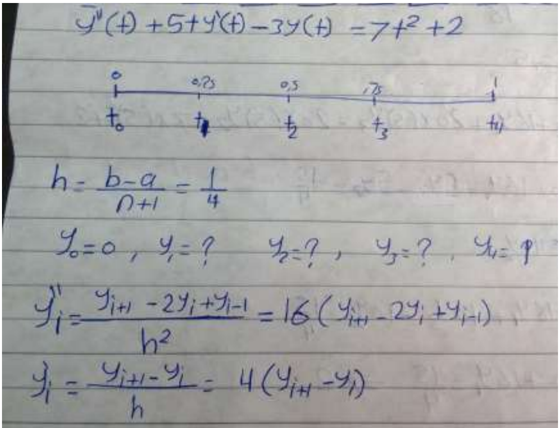
[MCQ]The resulting algebraic system of applying difference approximation to $y''(t)$ and backward formula for $y'(t)$ for the BVP:$y''(t) + y'(t) = 2(1+t)\,, 0 \le t \le 1,\ \ y(0) = 0, y(1) = 1$. with $h = \frac{1}{4}$ is

$\boldsymbol{a_{13}y_3} + \boldsymbol{a_{12}y_2} + \boldsymbol{a_{11}y_1} = \mathrm{b}_1, \boldsymbol{a_{23}y_3} + \boldsymbol{a_{22}y_2} + \boldsymbol{a_{21}y_1} = \mathrm{b}_2, \boldsymbol{a_{33}y_3} + \boldsymbol{a_{32}y_2} + \boldsymbol{a_{31}y_1} = \mathrm{b}_3$

Answer the following 9 questions:

1) $\boldsymbol{a_{13}} = [0, \mathbf{16}, \mathbf{-28}, \mathbf{12}]$	**(2)** $\boldsymbol{a_{12}} = [\mathbf{0}, 16, \mathbf{-28}, \mathbf{12}]$	**(3)** $\boldsymbol{a_{11}} = [\mathbf{0}, \mathbf{16}, -28, \mathbf{12}]$
4) $\boldsymbol{a_{23}} = [\mathbf{0}, 16, \mathbf{-28}, \mathbf{12}]$	**(5)** $\boldsymbol{a_{22}} = [\mathbf{0}, \mathbf{16}, -28, \mathbf{12}]$	**(6)** $\boldsymbol{a_{21}} = [\mathbf{0}, \mathbf{16}, \mathbf{-28}, 12]$
7) $\boldsymbol{a_{33}} = [\mathbf{0}, \mathbf{16}, -28, \mathbf{12}]$	**(8)** $\boldsymbol{a_{32}} = [\mathbf{0}, \mathbf{16}, \mathbf{-28}, 12]$	**(9)** $\boldsymbol{a_{31}} = [0, \mathbf{16}, \mathbf{-28}, \mathbf{12}]$

<u>Answer of [3a]</u>



$y''(t) + 5t\,y'(t) - 3y(t) = 7t^2 + 2$

0, 0.25, 0.5, .75, 1

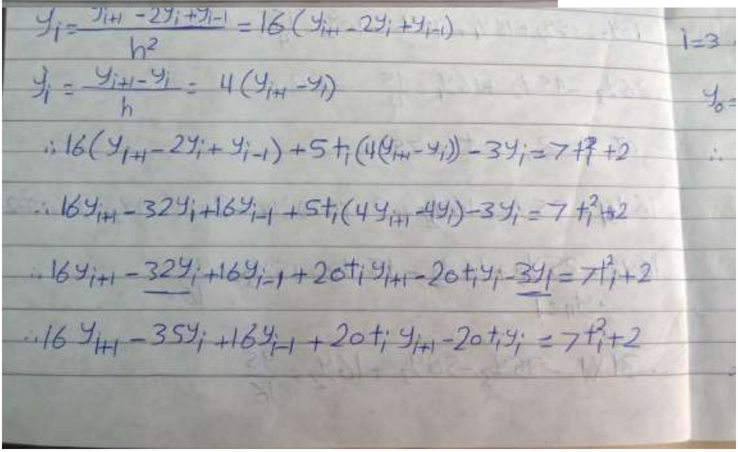
t_0, t_1, t_2, t_3, t_4

$h = \frac{b-a}{n+1} = \frac{1}{4}$

$y_0 = 0\,,\ y_1 = ?\quad y_2 = ?\,,\ y_3 = ?\,,\ y_4 = ?$

$y''_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = 16(y_{i+1} - 2y_i + y_{i-1})$

$y'_i = \frac{y_{i+1} - y_i}{h} = 4(y_{i+1} - y_i)$



$y''_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = 16(y_{i+1} - 2y_i + y_{i-1})$

$y'_i = \frac{y_{i+1} - y_i}{h} = 4(y_{i+1} - y_i)$

$\therefore 16(y_{i+1} - 2y_i + y_{i-1}) + 5t_i(4(y_{i+1} - y_i)) - 3y_i = 7t_i^2 + 2$

$\therefore 16y_{i+1} - 32y_i + 16y_{i-1} + 5t_i(4y_{i+1} - 4y_i) - 3y_i = 7t_i^2 + 2$

$\therefore 16y_{i+1} - 32y_i + 16y_{i-1} + 20t_i y_{i+1} - 20t_i y_i - 3y_i = 7t_i^2 + 2$

$\therefore 16y_{i+1} - 35y_i + 16y_{i-1} + 20t_i y_{i+1} - 20t_i y_i = 7t_i^2 + 2$

i=3

$y_0 =$

$i=1 \quad , t_1 = 0.25$

$\therefore 16y_2 - 35y_1 + 16y_0 + 20\times(0.25)y_2 - 20\times(0.25)y_1 = 7(0.25)^2 + 2$

$\therefore 16y_2 - 35y_1 + 16\times(0) + 5y_2 - 5y_1 = \frac{39}{16}$

$21y_2 - 40y_1 = \frac{39}{16} \Rightarrow 1$

$i=2, \quad t_2 = .5$

$i=2, \quad t_2 = .5$

$16y_3 - 35y_2 + 16y_1 + 20\times(.5)y_3 - 20\times(.5)y_2 = 7\times(.5)^2 + 2$

~~$16y_3 - 35y_2 + 16y_1 + 9y_3 - 5y_2 = \frac{15}{4}$~~

~~16~~ 20 ~~$y_3 - 35$~~ $40y_2$

$16y_3 - 35y_2 + 16y_1 + 10y_3 - 10y_2 = \frac{15}{4}$

$26y_3 - 45y_2 + 16y_1 = \frac{15}{4} \Rightarrow 2$

at $i=3, \quad t = 0.75$

$16y_3 - 35y_2 + 16y_1 + 10y_3 - 10y_2 = \frac{15}{4}$

$26y_3 - 45y_2 + 16y_1 = \frac{15}{4} \Rightarrow 2$

at $i=3, \quad t = 0.75$

$16y_4 - 35y_3 + 16y_2 + 20\times(.75)y_4 - 20\times(.75)y_3 = 7\times(.75)^2 + 2$

$16y_4 - 35y_3 + 16y_2 + 15y_4 - 15y_3 = \frac{95}{16}$

$\therefore y_4 = 1$

$\therefore 31\times 1$ ~~$-35y_3$~~ $-50y_3 + 16y_2 = \frac{95}{16}$

$16y_2 - 50y_3 = \frac{95}{16} - 31 = -\frac{401}{16} \Rightarrow 3$

resulting tridagional system

$$\begin{bmatrix} -40 & 21 & 0 \\ 16 & -45 & 26 \\ 0 & 16 & -50 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \frac{39}{16} \\ \frac{15}{4} \\ -\frac{401}{16} \end{bmatrix}$$

Answer of [3b]

$12y_2 - 40y_1 = \frac{39}{16} \Rightarrow 1$ الجزء الثاني من الحل هو حساب y_1, y_2, y_3 باستخدام الآلة الحاسبة

$26y_3 - 45y_2 + 16y_1 = \frac{15}{4} \Rightarrow 2$

$16y_2 - 50y_3 = -\frac{401}{16}$

$y_1 = -0,018$

$y_2 = 0,142$

$y_3 = 0,546$

t	y	y_e	error
0	0 ~~0,018~~	0	0
,25	~~0,142~~ 0,018	1/16	0,044
0,5	0,142	1/4	0,108
0,75	0,546	9/16	0,016
1	1	1	0

Complex Analysis

1. Complex Numbers

1.0. Introduction

We assume that the reader is familiar with the properties of the real number system **R**. We observe that in the real number system the equation $x^2 + 1 = 0$ has no solution. This leads to the definition of complex numbers in which equations of the form $x^2 + a = 0$, where $a > 0$, have solutions. In this chapter we develop the basic algebraic and geometric properties of the complex number system C.

1.1. Complex Numbers

Definition. A complex number z is of the form $x + iy$ where x and y are real numbers and i is an imaginary unit with the property $i^2 = -1$. x and y are called the real and imaginary parts of z and we write $x = \text{Re } z$ and $y = \text{Im } z$.

If $x = 0$ the complex number z is called *purely imaginary*. If $y = 0$ then z is *real*.

Two complex numbers are said to be equal iff they have the same real parts and the same imaginary parts.

Let C denote the set of all complex numbers.

Thus $C = \{x + iy / x, y \in \mathbf{R}\}$.

Definition. We define **addition and multiplication** in C as follows

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

Theorem 1.1. C is a field under addition and multiplication defined above

Proof. Obviously $z_1 + z_2$ and $z_1 z_2 \in C$.

Since addition of real numbers is associative and commutative it follows that addition in C is also associative and commutative.

$0 = 0 + i0$ is the additive identity and the additive inverse of $z = x + iy$ is $(-x) + i(-y)$. Hence $(C, +)$ is an abelian group.

Let $z_1, z_2 \in C^*$, the set of non zero complex numbers. Then $z_1 = x_1 + iy_1$ where x_1 and y_1 are not simultaneously zero and $z_2 = x_2 + iy_2$ where x_2 and y_2 are not simultaneously zero.

We claim that $z_1 z_2 \in C^*$.

We have $z_1 z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$.

Suppose $z_1 z_2 = 0$.

$$\text{Then} \quad x_1x_2 - y_1y_2 = 0 \qquad \ldots (1)$$

$$\text{and} \quad x_1y_2 + x_2y_1 = 0 \qquad \ldots (2)$$

Multiplying (1) by y_2 and (2) by x_2 and subtracting we get $y_1(y_2^2 + x_2^2) = 0$.

$\therefore$ Either $y_1 = 0$ or $y_2^2 + x_2^2 = 0$.

$\therefore$ Either $y_1 = 0$ or ($y_2 = 0$ and $x_2 = 0$).

Similarly either $x_1 = 0$ or ($y_2 = 0$ and $x_2 = 0$).

Thus ($x_1 = 0$ and $y_1 = 0$) or ($x_2 = 0$ and $y_2 = 0$).

$\therefore$ $z_1 = 0$ or $z_2 = 0$ which is a contradiction. Hence $z_1 z_2 \in C^*$.

It can be easily verified that multiplication is associative and commutative.

$1 + i0 \in C$ is the multiplicative identity element.

Let $z = x + iy$ be a non zero complex number.

$\therefore$ Either $x \neq 0$ or $y \neq 0$. Hence $x^2 + y^2 > 0$.

$$\text{Now,} \quad \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)}$$

$$= \frac{x-iy}{x^2+y^2}$$

$$= \left[\frac{x}{x^2+y^2}\right] + i\left[\frac{-y}{x^2+y^2}\right].$$

Thus $\frac{1}{z} \in C^*$ and it is the multiplicative inverse of z.

Further it can easily be verified that $z_1(z_2+z_3) = z_1z_2 + z_1z_3$ for all complex numbers $z_1, z_2, z_3 \in C$.

Hence $(C, +, .)$ is a field.

Remark 1. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2 \neq 0$, then

$$\frac{z_1}{z_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\,\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}.$$

Remark 2. It is important to note that there is no order structure in the complex number system so that we cannot compare two complex numbers.

Remark 3. The complex number $a + ib$ can also be represented by the ordered pair of real numbers (a, b).

Exercises.

1. Verify the following.

   (i) $(\sqrt{2} - i) - i(1 + \sqrt{2}i) = 2(\sqrt{2} - i)$

   (ii) $(1+i)^3 = -2 + 2i$

   (iii) $(1-i)^4 = -4$

   (iv) $(2, -3)(-2, 1) = (-1, 8)$

   (v) $(3, 1)(3, -1)\left(\frac{1}{5}, \frac{1}{10}\right) = (2, 1)$

   (vi) $\frac{1+i}{1-i} = i$

   (vii) $\frac{1-i}{1+i} = -i$

   (viii) $\left[\frac{1+i}{1-i}\right]^5 - \left[\frac{1-i}{1+i}\right]^5 = 2i.$

2. Show that each of the two numbers $z = 1 \pm i$ satisfies the equation $z^2 - 2z + 2 = 0$.

1.2. Conjugation and Modulus

Let $z = x + iy$ be a complex number. Then the complex number $x - iy$ is called the conjugate of z and it is denoted by $\bar{z}$.

The mapping $f : C \to C$ defined by $f(z) = \bar{z}$ is called the complex conjugation.

Note. 1. z is real iff $z = \bar{z}$.

2. $\bar{\bar{z}} = z$. → $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

3. $z + \bar{z} = 2\,\text{Re}\,z$ so that $x = \frac{z + \bar{z}}{2}$.

4. $z - \bar{z} = 2i\,\text{Im}\,z$ so that $y = \frac{z - \bar{z}}{2i}$.

5. $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

6. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

The following theorem is an application of the above properties of conjugation.

Theorem 1.2. If α is a root of the polynomial equation

$$f(z) = a_0 z^n + a_1 z^{n-1} + \ldots\ldots + a_{n-1} z + a_n = 0$$

where $a_0, a_1, \ldots\ldots, a_n \in \mathbb{R}$ and $a_0 \neq 0$ then $\overline{\alpha}$ is also a root of $f(z) = 0$.

(i.e.) The non-real roots of a polynomial equation with real coefficients occur in conjugate pairs.

Proof. Since α is a root of $f(z) = 0$ we have $f(\alpha) = 0$

Hence $a_0\alpha^n + a_1\alpha^{n-1} + \ldots\ldots + a_{n-1}\alpha + a_n = 0$

$\therefore$ $\overline{a_0\alpha^n + a_1\alpha^{n-1} + \ldots\ldots + a_{n-1}\alpha + a_n} = \overline{0}.$

$\therefore$ $\overline{a_0}\,\overline{\alpha}^n + \overline{a_1}\,\overline{\alpha}^{n-1} + \ldots\ldots + \overline{a}_{n-1}\overline{\alpha} + \overline{a}_n = \overline{0}.$

$\therefore$ $a_0\overline{\alpha}^n + a_1\overline{\alpha}^{n-1} + \ldots\ldots + a_{n-1}\overline{\alpha} + a_n = 0.$ (since a_i is real).

$\therefore$ $a_0(\overline{\alpha})^n + a_1(\overline{\alpha})^{n-1} + \ldots\ldots + a_{n-1}(\overline{\alpha}) + a_n = 0.$

$\therefore$ $f(\overline{\alpha}) = 0$ so that $\overline{\alpha}$ is also a root of $f(z) = 0$.

Definition. Let $z = x + iy$ be a complex number. The modulus or absolute value of z denoted by $|z|$ is defined by $|z| = \sqrt{x^2 + y^2}$.

Remark. $|z|$ represents the distance between $z = (x, y)$ and the origin $O = (0, 0)$.

Theorem 1.3. (i) $|z| \geq 0$ and $|z| = 0$ iff $z = 0$

(ii) $z\overline{z} = |z|^2$

(iii) $|z_1 z_2| = |z_1||z_2|$

(iv) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ provided $z_2 \neq 0$.

(v) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}\,(z_1\overline{z}_2)$

(vi) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\mathrm{Re}\,(z_1\overline{z}_2)$

(vii) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$.

Proof. The proof is straight forward and is left as an exercise.

Solved Problems.

Problem 1. Find the absolute value of $\dfrac{(1+3i)(1-2i)}{3+4i}$.

Solution.

$$\left|\frac{(1+3i)(1-2i)}{3+4i}\right| = \frac{|1+3i||1-2i|}{|3+4i|}$$

$$= \frac{\sqrt{10}\sqrt{5}}{5}$$

$$= \sqrt{2}$$

Problem 2. Find the condition under which the equation $az + b\overline{z} + c = 0$ in one complex unknown has exactly one solution and compute that solution.

Solution. $az + b\overline{z} + c = 0$ ... (1)

Taking conjugate we have

$$\overline{a}\,\overline{z} + \overline{b}z + \overline{c} = 0 \quad \ldots (2)$$

Eliminating $\overline{z}$ between (1) and (2) we get

$$z(a\overline{a} - b\overline{b}) = b\overline{c} - \overline{a}c.$$

$$\therefore \quad z(|a|^2 - |b|^2) = b\overline{c} - \overline{a}c.$$

Hence, if $|a| \neq |b|$ the given equation has unique solution and the solution is given by

$$z = \frac{b\overline{c} - \overline{a}c}{|a|^2 - |b|^2}$$

Problem 3. If z_1 and z_2 are two complex numbers prove that $\left|\dfrac{z_1 - z_2}{1 - \overline{z}_1 z_2}\right| = 1$ if either $|z_1| = 1$ or $|z_2| = 1$. What exception must be made if $|z_1| = 1$ and $|z_2| = 1$.

Solution. Suppose $|z_1| = 1$. Hence $|\overline{z}_1| = 1$ and $z_1\overline{z}_1 = |z_1|^2 = 1$.

Now
$$\left|\frac{z_1 - z_2}{1 - \overline{z}_1 z_2}\right| = \left|\frac{z_1 - z_2}{z_1\overline{z}_1 - \overline{z}_1 z_2}\right|$$

$$= \left|\frac{z_1 - z_2}{\overline{z}_1(z_1 - z_2)}\right|$$

$$= \frac{1}{|z_1|}$$

$$= 1$$

Similarly if $|\overline{z}_2| = 1$ we have $\left|\dfrac{z_1 - z_2}{1 - \overline{z}_1 z_2}\right| = 1.$

If $|z_1| = 1$ and $|z_2| = 1$, then the result is true provided $1 - \overline{z}_1 z_2 \neq 0$.

i.e., if $z_1 - z_1\overline{z}_1 z_2 \neq 0.$

i.e., if $z_1 \neq |z_1|^2 z_2$.

i.e., if $z_1 \neq z_2$.

Exercises.

1. Find the modulus of

(i) $\frac{(2-i)(1+i)}{1-i}$ (ii) $\frac{2+i}{4i+(1+i)^2}$

2. Find the conjugate of

(i) $\frac{1+2i}{1-(1-i)^2}$ (ii) $\frac{5+2i}{5-2i}-\frac{3-4i}{4+3i}+\frac{1}{i}$

(iii) $\frac{1+i}{1-i}$ (iv) $\bar{z}+3i$

3. Prove that $|2z-3\bar{z}|^2 = x^2+25y^2$.
4. Find the cartesian form of the equation

$$|z-4i|^2+|z+4i|^2=34.$$

5. If z_1 and z_2 are complex numbers show that

$$|z_1-z_2|^2+|z_1+z_2|^2=2(|z_1|^2+|z_2|^2).$$

6. If z is a complex number such that $z\bar{z}=1$ compute $|1+z|^2+|1-z|^2$.

Answers.

[1]. (i) $\sqrt{5}$ (ii) $\frac{\sqrt{5}}{6}$ [2]. (i) 1 (ii) $-i$ (iii) $-i$ (iv) $z-3i$

[4]. $x^2+y^2=1$ [6]. 4

1.3. Inequalities

In this section we shall prove some important inequalities which will be of constant use. Since there is no order relation in the set of complex numbers the inequalities are among real numbers associated with any complex number.

Theorem 1.4. For any three complex numbers z, z_1 and z_2

(i) $-|z|\le \text{Re } z\le |z|$

(ii) $-|z|\le \text{Im } z\le |z|$

(iii) $|z_1+z_2|\le |z_1|+|z_2|$ (Triangle inequality)

(iv) $|z_1-z_2|\ge ||z_1|-|z_2||$

Proof. Let $z=x+iy$. Hence $|z|=\sqrt{x^2+y^2}$

Now $-\sqrt{x^2+y^2}\le x\le \sqrt{x^2+y^2}$ and $-\sqrt{x^2+y^2}\le y\le\sqrt{x^2+y^2}$.

$\therefore$ $-|z|\le \text{Re } z\le |z|$ and $-|z|\le \text{Im } z\le |z|$ proving (i) and (ii).

(iii) $|z_1+z_2|^2=(z_1+z_2)\overline{(z_1+z_2)}$ (since $|z|^2=z\bar{z}$)

$$\begin{aligned}
&=(z_1+z_2)(\bar{z}_1+\bar{z}_2)\\
&=z_1\bar{z}_1+z_1\bar{z}_2+\bar{z}_1z_2+z_2\bar{z}_2\\
&=|z_1|^2+(z_1\bar{z}_2+\bar{z}_1z_2)+|z_2|^2\\
&=|z_1|^2+2\,\text{Re }(z_1\bar{z}_2)+|z_2|^2 \quad (\text{since } z+\bar{z}=2\,\text{Re } z)\\
&\le |z_1|^2+2|z_1\bar{z}_2|+|z_2|^2 \quad (\text{since Re }(z_1\bar{z}_2)\le |z_1\bar{z}_2|)\\
&=|z_1|^2+2|z_1||\bar{z}_2|+|z_2|^2\\
&=|z_1|^2+2|z_1||z_2|+|z_2|^2 \ (\text{since } |z_2|=|\bar{z}_2|)\\
&=(|z_1|+|z_2|)^2.
\end{aligned}$$

Thus $|z_1+z_2|^2\le (|z_1|+|z_2|)^2$.

$\therefore$ $|z_1+z_2|\le |z_1|+|z_2|$.

(iv) Writing $z_1=(z_1-z_2)+z_2$ and $z_2=(z_2-z_1)+z_1$ and using (iii)

we get $|z_1-z_2|+|z_2|\ge |z_1|$ and $|z_1-z_2|+|z_1|\ge |z_2|$

Hence $|z_1-z_2|\ge |z_1|-|z_2|$ and $|z_1-z_2|\ge |z_2|-|z_1|=-(|z_1|-|z_2|)$.

Hence $-|z_1-z_2|\le |z_1|-|z_2|\le |z_1-z_2|$ so that $||z_1|-|z_2||\le |z_1-z_2|$.

Thus $|z_1-z_2|\ge ||z_1|-|z_2||$.

Note. For any complex numbers $z_1, z_2, \dots, z_n$ we have

$$|z_1+z_2+\cdots+z_n|\le |z_1|+|z_2|+\cdots+|z_n|.$$

Exercise.

1. If z_1 and z_2 are two complex numbers show that

(i) $||z_1|-|z_2||\le |z_1+z_2|\le |z_1|+|z_2|$

(ii) $||z_1|-|z_2||\le |z_1-z_2|\le |z_1|+|z_2|$.

1.4. Square Root

In this section we describe an explicit method of finding the square root of a complex number.

Let $a + ib$ be a given complex number. Let $x + iy$ be a square root of $a + ib$. Then $(x + iy)^2 = a + ib$.

$$\therefore \quad x^2 - y^2 + 2ixy = a + ib.$$

$$\therefore \quad x^2 - y^2 = a \qquad \ldots (1)$$

$$2xy = b \qquad \ldots (2)$$

Hence $\quad (x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = a^2 + b^2$

$$\therefore \quad x^2 + y^2 = \sqrt{a^2 + b^2} \qquad \ldots (3)$$

From (1) and (3) we get

$$x^2 = \frac{1}{2}\left(a + \sqrt{a^2 + b^2}\right) \text{ and } y^2 = \frac{1}{2}\left(-a + \sqrt{a^2 + b^2}\right).$$

$$\therefore \quad x = \pm\left[\frac{1}{2}(a + \sqrt{a^2 + b^2})\right]^{1/2} \text{ and } y = \pm\left[\frac{1}{2}(-a + \sqrt{a^2 + b^2})\right]^{1/2} \qquad \ldots (4)$$

Now from (2) we notice that if $b > 0$, x and y must be of same sign and if $b < 0$, x and y must be of opposite signs.

Thus choosing appropriate signs for x and y we get two square roots of $a + ib$.

If $b = 0$ then z is real and we get two square roots as $\pm\sqrt{a}$ provided $a > 0$ and $\pm i\sqrt{-a}$ provided $a < 0$.

Solved problem.

Problem 1. Find the square roots of $1 + i$.

Solution. Let $x + iy$ be a square root of $1 + i$. Then $(x + iy)^2 = 1 + i$.

Hence $\quad x^2 - y^2 = 1 \qquad \ldots (1)$

and $\quad 2xy = 1 \qquad \ldots (2)$

Now $\quad (x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2$

$\qquad = 2.$ (Using (1) and (2))

$$\therefore \quad x^2 + y^2 = \sqrt{2} \qquad \ldots (3)$$

From (1) and (3) we get

$$x^2 = \frac{1}{2}(1 + \sqrt{2}) \text{ and } y^2 = \frac{1}{2}(-1 + \sqrt{2})$$

$$\therefore \quad x = \pm\left[\frac{1}{2}(1 + \sqrt{2})\right]^{1/2} \text{ and } y = \pm\left[\frac{1}{2}(-1 + \sqrt{2})\right]^{1/2}$$

From (2) we notice that x and y are of same sign. Hence the two square roots of $1 + i$ are given by $\left[\frac{1}{2}(1 + \sqrt{2})\right]^{1/2} + i\left[\frac{1}{2}(-1 + \sqrt{2})\right]^{1/2}$

and $-\left[\frac{1}{2}(1 + \sqrt{2})\right]^{1/2} - i\left[\frac{1}{2}(-1 + \sqrt{2})\right]^{1/2}$.

1.5. Geometrical Representation of Complex Numbers

We can represent any complex number $x + iy$ by a point (x, y) in $\mathbf{R} \times \mathbf{R}$. The plane $\mathbf{R} \times \mathbf{R}$ representing the complex numbers in this way is called the complex plane.

The x-axis is referred to as the **real axis** and the y-axis is referred to as the imaginary axis. With this representation modulus of z represents the distance between z and the origin.

The complex number $z = x + iy$ can also be represented by the vector $\overrightarrow{OP}$ where $P = (x, y)$.

Polar form of a complex number

Consider any non zero complex number $z = x + iy$.

Let (r, θ) denote the polar coordinates of the point (x, y).

Hence $x = r\cos\theta$ and $y = r\sin\theta$.

$$\therefore \quad z = r(\cos\theta + i\sin\theta).$$

We notice that $r = |z| = \sqrt{x^2 + y^2}$ which is the **magnitude** of the complex number and θ is called the amplitude **or argument** of z and is denoted by arg z or *amp* z.

We note that the value of arg z is not unique. If $\theta = \arg z$ then $\theta + 2n\pi$ where n is any integer is also a value of arg z. The value of arg z lying in the range $(-\pi, \pi]$ is called the **principal value** of arg z.

Theorem 1.5. If z_1 and z_2 are any two non zero complex numbers then

(i) $\arg z_1 = \arg \overline{z_1}$

(ii) $\arg z_1 z_2 = \arg z_1 + \arg z_2$

(iii) $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$.

Proof. (i) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$.

$$\therefore \quad \bar{z}_1 = r_1(\cos\theta_1 - i\sin\theta_1)$$
$$= r_1[\cos(-\theta_1) + i\sin(-\theta_1)].$$

Hence $\arg \bar{z}_1 = -\theta_1 = -\arg z_1$

(ii) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Now $z_1z_2 = r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$
$$= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$
$$= r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)].$$

$$\therefore \quad \arg z_1z_2 = \theta_1 + \theta_2 = \arg z_1 + \arg z_2.$$

(iii) $$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$
$$= \left(\frac{r_1}{r_2}\right)\frac{(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}$$
$$= \left(\frac{r_1}{r_2}\right)\frac{(\cos\theta_1 + i\sin\theta_1)[\cos(-\theta_2) + i\sin(-\theta_2)]}{\cos^2\theta_2 + \sin^2\theta_2}$$
$$= \left(\frac{r_1}{r_2}\right)[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)] \text{ by (ii).}$$

$$\therefore \quad \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2.$$
$$= \arg z_1 - \arg z_2.$$

Remark 1. Let z_1 and z_2 be two non zero complex numbers represented by the points P and Q respectively in the complex plane. Then the sum $z_1 + z_2$ is represented by the point R which is the fourth vertex of the parallelogram constructed with OP and OQ as adjacent sides. This is a consequence of the parallelogram law for vector addition.

Remark 2. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two non zero complex numbers represented by the points P and Q respectively in the complex plane. Then the complex number z_1z_2 has modulus r_1r_2 and amplitude $\theta_1+\theta_2$. Hence the point R representing the complex number z_1z_2 is obtained as follows. Rotate the line segment OP through an angle θ_2 in the anticlockwise direction giving the line segment OP'. On the ray we choose R such that $OR = r_1r_2$.

Theorem 1.6. Let $z = r(\cos\theta + i\sin\theta)$ be any non zero complex number and n be any integer. Then $z^n = r^n(\cos n\theta + i\sin n\theta)$.

Proof. We first prove this result for positive integers by induction on n.

The result is obviously true when $n = 1$.

Suppose the result is true for $n = m$.

Hence $z^m = r^m(\cos m\theta + i\sin m\theta)$

Now $$z^{m+1} = z^m z = r^m(\cos m\theta + i\sin m\theta)r(\cos\theta + i\sin\theta)$$
$$= r^{m+1}[\cos(m+1)\theta + i\sin(m+1)\theta].$$

Hence the result is true for $n = m+1$.

Hence $z^n = r^n(\cos n\theta + i\sin n\theta)$ for all positive integers n.

The result is obviously true if $n = 0$.

Now, $$z^{-1} = \frac{1}{z} = \frac{1}{r(\cos\theta + i\sin\theta)}$$
$$= \frac{1}{r}\frac{\cos\theta - i\sin\theta}{(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}$$
$$= r^{-1}[\cos(-\theta) + i\sin(-\theta)]$$

$\therefore$ The result is true for $n = -1$. Hence it follows that the result is true for all negative integers.

Hence $z^n = r^n(\cos n\theta + i\sin n\theta)$ for all $n \in \mathbb{Z}$.

Corollary. (De-Movire's theorem)

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Solved Problem.

Problem 1. For any three distinct complex numbers z, a, b the principal value of $\arg\left(\frac{z-a}{z-b}\right)$ represents the angle between the line segment joining z and a and the line segment joining z and b, taken in the appropriate sense.

Solution. Let A, B, P be the points in the complex plane representing the complex numbers a, b, z respectively. Then the complex number $z-a$, $z-b$ are represented by the vectors $\overrightarrow{AP}$ and $\overrightarrow{BP}$ respectively. Hence the principal value of arg $\left[\frac{z-a}{z-b}\right]$ gives the angle between the line segment AP and BP taken in the appropriate sense.

Exercises

1. Find one value of arg z, when

(i) $z = \frac{-2}{1+i\sqrt{3}}$ (ii) $z = i$ (iii) $z = (\sqrt{3}-i)^6$

Answers.

(i) $\frac{2\pi}{3}$ (ii) $\frac{\pi}{2}$ (iii) $\frac{\pi}{2}$

1.6. n^{th} Roots of Complex Numbers

De-Moivre's theorem can be used to calculate the n^{th} roots of any non zero complex number.

Theorem 1.7. Let n be positive integer. Then any non zero complex number has n distinct n^{th} roots.

Proof. Let $z = r(\cos\theta + i\sin\theta)$ be a non zero complex number.

Let $\omega = \rho(\cos\varphi + i\sin\varphi)$ be an n^{th} root of z.

Then $\omega^n = z$.

$$\therefore \quad \rho^n(\cos n\varphi + i\sin n\varphi) = r(\cos\theta + i\sin\theta).$$

$$\therefore \quad \rho^n = r \text{ and } n\varphi = \theta + 2k\pi \text{ where } k \in \mathbb{Z}.$$

$$\therefore \quad \rho = r^{1/n} \text{ and } \varphi = \frac{\theta}{n} + \frac{2k\pi}{n} \text{ where } k \in \mathbb{Z}.$$

However, only the values $k = 0, 1, 2, \ldots\ldots, (n-1)$ will give different values of ω. Hence z has n distinct n^{th} roots and they are given by

$$\omega = r^{1/n}\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right] \text{ where } k = 0, 1, 2, \ldots, n-1.$$

Corollary. The n^{th} roots of unity are given by

$$\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right) \text{ where } k = 0, 1, 2, \ldots\ldots, n-1.$$

Proof. When $z = 1$ we have $r = 1$ and $\theta = 0$. Hence the result follows.

Remark 1. Let $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. Then the n n^{th} roots of unity are given by $1, \omega, \omega^2, \ldots\ldots, \omega^{n-1}$.

Remark 2. If z is any non zero complex number and z_0 is any one n^{th} root of z then the set of all n^{th} roots of z are given by

$$z_0,\ z_0\,\omega,\ z_0\,\omega^2,\ \ldots\ldots,\ z_0\,\omega^{n-1}.$$

Remark 3. Since $1, \omega, \omega^2, \ldots\ldots\ \omega^{n-1}$ are the roots of the equation $z^n = 1$ we have $\omega^n = 1$ and $1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$.

Exercises.

1. Find all the values of (i) $(-i)^{1/3}$ (ii) $(2i)^{1/2}$ and exhibit them geometrically.
2. Find all the values of $\left[\frac{1}{2} + \frac{i\sqrt{3}}{2}\right]^{3/4}$. Hence prove that the product of the values is 1.

Answers. (i) $i, (\pm\sqrt{3}-i)/2$ (ii) $\pm(1+i)$

1.7. Circles and Straight Lines

Equation of circles and straight lines in the complex plane can be expressed in terms of z and $\bar{z}$.

General equation of circles. Equation of the circle with centre a and radius r is given by $|z-a| = r$.

$$\text{(ie)} \quad (z-a)(\bar{z}-\bar{a}) = r^2.$$

$$\text{(ie)} \quad z\bar{z} - a\bar{z} - \bar{a}z + a\bar{a} - r^2 = 0.$$

This equation is of the form $z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + \beta = 0$ where β is a real number.

Further any equation of the above form can be rewritten as

$|z+\alpha|^2 = \alpha\bar{\alpha} - \beta$ and hence represents a circle provided $\alpha\bar{\alpha} - \beta > 0$.

It represents a circle with centre $-\alpha$ and radius $\sqrt{\alpha\bar{\alpha} - \beta}$

Thus the general equation of a circle is given by

$$z\bar{z}+\bar{\alpha}z+\alpha\bar{z}+\beta=0 \text{ where } \beta \text{ is real and } \alpha\bar{\alpha}-\beta>0.$$

General equation of straight lines. To find the general equation of the straight line passing through a and b we note that $\arg\left[\frac{z-a}{z-b}\right]$ represents the angle between the lines joining a to z and b to z where z is any point on the line joining a and b. (refer problem 1 in 1.5)

$\therefore$ If z, a, b are collinear then $\arg\left[\frac{z-a}{z-b}\right]=0$ or π.

$\therefore$ $\frac{z-a}{z-b}$ is real. Hence $\frac{z-a}{z-b}=\overline{\left[\frac{z-a}{z-b}\right]}$.

$\therefore$ $\frac{z-a}{z-b}=\frac{\bar{z}-\bar{a}}{\bar{z}-\bar{b}}$.

$\therefore$ $(\bar{a}-\bar{b})z-(a-b)\bar{z}+(a\bar{b}-\bar{a}b)=0$.

$\therefore$ $(\bar{a}-\bar{b})z-(a-b)\bar{z}+2i\operatorname{Im}(a\bar{b})=0$. (since $z-\bar{z}=2i\operatorname{Im} z$)

$\therefore$ $i(\bar{a}-\bar{b})z-i(a-b)\bar{z}-2\operatorname{Im}(a\bar{b})=0$.

This equation is of the form $\bar{\alpha}z+\alpha\bar{z}+\beta=0$ where $\alpha\neq 0$ and β is real.

Further any equation of the above form represents a straight line. This can be easily seen by changing the above equation into cartesian form.

$\therefore$ The general equation of a straight line is given by

$$\bar{\alpha}z+\alpha\bar{z}+\beta=0 \quad \text{where} \quad \alpha\neq 0 \text{ and } \beta \text{ is real.}$$

Theorem 1.8. Equation of the line joining a and b is

$$(\bar{a}-\bar{b})z+(b-a)\bar{z}+(a\bar{b}-\bar{a}b)=0.$$

Theorem 1.9. If a and b are two distinct complex numbers where $b\neq 0$, then the equation $z=a+tb$ where t is a real parameter represents a straight line passing through the point a and parellel to b.

Proof. Let z be any point on the line passing through a and parallel to b.

The vectors represented by $z-a$ and b are parallel.

Hence $z-a=tb$ for some real number t.

Hence $z=a+tb$, which is the equation of the required straight line

Definition. Two points P and Q are called **reflection points** for a given straight line l iff l is the perpendicular bisector of the segment PQ.

Theorem 1.10. Two points z_1 and z_2 are reflection points for the line

$$\bar{\alpha}z+\alpha\bar{z}+\beta=0 \text{ iff } \bar{\alpha}z_1+\alpha\bar{z}_2+\beta=0.$$

Proof. Let z_1 and z_2 be reflection points for the straight line

$$\bar{\alpha}z+\alpha\bar{z}+\beta=0. \quad \ldots (1)$$

$\therefore$ For any point z on the line we have $|z-z_1|=|z-z_2|$.

$\therefore$ $|z-z_1|^2=|z-z_2|^2$.

$\therefore$ $(z-z_1)(\bar{z}-\bar{z}_1)=(z-z_2)(\bar{z}-\bar{z}_2)$

$\therefore$ $z(\bar{z}_2-\bar{z}_1)+\bar{z}(z_2-z_1)+z_1\bar{z}_1-z_2\bar{z}_2=0 \quad \ldots (2)$

Since the equation is true for any point z on the given line it may be regarded as the equation of the given line.

$\therefore$ From (1) and (2) we get

$$\frac{\bar{\alpha}}{\bar{z}_2-\bar{z}_1}=\frac{\alpha}{z_2-z_1}=\frac{\beta}{z_1\bar{z}_1-z_2\bar{z}_2}=k \quad \text{(say)}$$

$\therefore$ $\alpha=k(z_2-z_1)$; $\bar{\alpha}=k(\bar{z}_2-\bar{z}_1)$ and $\beta=k(z_1\bar{z}_1-z_2\bar{z}_2)$.

$\therefore$ $\bar{\alpha}z_1+\alpha\bar{z}_2+\beta=k[z_1(\bar{z}_2-\bar{z}_1)+\bar{z}_2(z_2-z_1)+(z_1\bar{z}_1-z_2\bar{z}_2)]$
$=0$.

Conversely, suppose $\bar{\alpha}z_1+\alpha\bar{z}_2+\beta=0. \quad \ldots (3)$

Subtracting (3) from (1) we get $\bar{\alpha}(z-z_1)+\alpha(\bar{z}-\bar{z}_2)=0$.

(i.e) $\bar{\alpha}(z-z_1)=-\alpha(\bar{z}-\bar{z}_2)$.

$\therefore$ $|\bar{\alpha}||z-z_1|=|\alpha||\bar{z}-\bar{z}_2|$.

$\therefore$ $|z-z_1|=|\bar{z}-\bar{z}_2|=|\overline{z-z_2}|=|z-z_2|$ for any point z on the given line.

$\therefore$ z_1 and z_2 are reflection points for the line (1).

Definition. Two points P and Q are said to be **inverse points with respect to a circle** with centre O and radius r if Q lies on the ray OP and $OP.OQ=r^2$.

Theorem 1.11. z_1 and z_2 are inverse points with respect to a circle

$$z\bar{z}+\bar{\alpha}z+\alpha\bar{z}+\beta=0 \text{ iff } z_1\bar{z}_2+\bar{\alpha}z_1+\alpha\bar{z}_2+\beta=0.$$

Proof. Suppose z_1 and z_2 are inverse points with respect to the circle

$$z\bar{z}+\bar{\alpha}z+\alpha\bar{z}+\beta=0. \qquad \ldots (1)$$

(1) can be rewritten as $|z+\alpha|^2=\alpha\bar{\alpha}-\beta$.

$\therefore$ The centre of the circle is $-\alpha$ and radius is $\sqrt{(\alpha\bar{\alpha}-\beta)}$.

Since z_1 and z_2 are inverse points with respect to (1) we have

$$\arg(z_1+\alpha)=\arg(z_2+\alpha) \qquad \ldots (2)$$

$$\text{and } |z_1+\alpha||z_2+\alpha|=\alpha\bar{\alpha}-\beta \qquad \ldots (3)$$

$$\therefore \arg(z_1+\alpha)\overline{(z_2+\alpha)}=\arg(z_1+\alpha)+\arg\overline{(z_2+\alpha)}$$
$$=\arg(z_1+\alpha)-\arg(z_2+\alpha)$$
$$=0 \text{ (by 2)}$$

$\therefore$ $(z_1+\alpha)\overline{(z_2+\alpha)}$ is a positive real number.

Hence using (3) we get $(z_1+\alpha)\overline{(z_2+\alpha)}=\alpha\bar{\alpha}-\beta$.

$$\therefore \quad z_1\bar{z}_2+\bar{\alpha}z_1+\alpha\bar{z}_2+\beta=0$$

Converse can be similarly proved.

Note 1. Let z_1, z_2, z_3 and z_4 be four distinct points which are either con-cyclic or collinear. Then $\arg\dfrac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$ is either 0 or π depending on the relative positions of the points.

Hence $\dfrac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$ is purely real.

Note 2. The equation

$$pz\bar{z}+\bar{\alpha}z+\alpha\bar{z}+\beta=0 \qquad \ldots (1)$$

where p and β are real and $\alpha\bar{\alpha}-p\beta\geq 0$ can be taken as the joint equation of the family of circles and straight line.

When $p\neq 0$ it represents a circle.

When $p=0$ it represents a straight line.

Further z_1 and z_2 are *inverse points* or *reflection points* w.r.t (1) iff $pz_1\bar{z}_2+\bar{\alpha}z_1+\alpha\bar{z}_2+\beta=0$.

Solved problems

Problem 1. Prove that the equation $\left|\dfrac{z-z_1}{z-z_2}\right|=\lambda$ where λ is a nonnegative parameter represents a family of circles such that z_1 and z_2 are inverse points for every member of the family.

Solution. $\left|\dfrac{z-z_1}{z-z_2}\right|=\lambda$.

$$\Rightarrow \left(\frac{z-z_1}{z-z_2}\right)\left(\frac{\bar{z}-\bar{z}_1}{\bar{z}-\bar{z}_2}\right)=\lambda^2.$$

$$\Rightarrow (1-\lambda^2)z\bar{z}+(\bar{z}_2\lambda^2-\bar{z}_1)z+(z_2\lambda^2-z_1)\bar{z}+(z_1\bar{z}_1-\lambda^2z_2\bar{z}_2)=0 \qquad (1)$$

$\therefore$ (1) represents a circle when $\lambda\neq 1$.

Using theorem 1.11. Note 2, it can be verified that z_1 and z_2 are inverse points w.r.t. (1).

When $\lambda=1$ the given equation represents a straight line which is the perpendicular bisector of the line segment joining z_1 and z_2.

Clearly z_1 and z_2 are reflection points for this line.

Problem 2. Prove that $\arg\left(\dfrac{z-a}{z-b}\right)=\mu$ where μ is a real parameter, represents a family of circles every member of which passes through a and b.

Solution. For any fixed value μ, $\arg\left(\dfrac{z-a}{z-b}\right)=\mu$ is the locus of a point z such that the angle between the lines joining a to z and b to z is μ.

Clearly this locus is the arc of a circle passing through a and b. The remaining part of the circle is represented by the equation $\arg\left(\dfrac{z-a}{z-b}\right)=\mu+\pi$. Hence the result follows.

Exercises:

1. Show that the inverse point of any point α with respect to the unit circel $|z|=1$ is $\dfrac{1}{\bar{\alpha}}$
2. Find the inverse point of $-i$ with respect to the circle
$$2z\bar{z}+(i-1)z-(i+1)\bar{z}=0$$
3. Prove that the equation of the circle passing through three points z_1, z_2, z_3 is given by

$$\frac{(z-z_1)(z_3-z_2)}{(z-z_2)(z_3-z_1)} = \frac{(\bar{z}-\bar{z}_1)(\bar{z}_3-\bar{z}_2)}{(\bar{z}-\bar{z}_2)(\bar{z}_3-\bar{z}_1)}$$

(Hint: If z is any point on the circle then $\dfrac{(z-z_1)(z_3-z_2)}{(z-z_2)(z_3-z_1)}$ is purely real.

4 Find the equation of the circle passing through the points $1,\ i,\ 1+i$

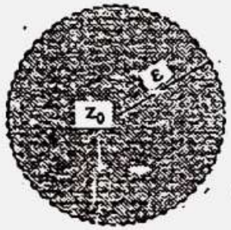
Answers: 2. $\frac{2}{5}+\frac{1}{5}i$ 4. $2z\bar{z}+(i-1)z-(i+1)\bar{z}=0$

1.8. Regions in the Complex Plane

We have seen that the distance between two points z_1 and z_2 in the complex plane is $|z_1-z_2|$. Hence the set C of complex numbers becomes a *metric space* with the metric d defined by $d(z_1, z_2) = |z_1 - z_2|$. Therefore we can talk about *neighbourhood, interior point, open set, closed set, limit point, connected set* etc in the complex plane.

We now recall the relevant concepts with particular reference to the complex plane.

Definition. Let z_0 be any complex number. Let ε be a positive real number. Then the set of all points z satisfying $|z-z_0|<\varepsilon$ is called a **neighbourhood** of z_0 and is represented by $N_\varepsilon(z_0)$ or $S(z_0, \varepsilon)$.



Thus $N_\varepsilon(z_0) = \{z / |z-z_0| < \varepsilon\}$.

We observe that $|z-z_0|<\varepsilon$ represents the interior of the circle with centre z_0 and radius ε.

Note. $|z-z_0| \le \varepsilon$ represents the set of points on and inside the circle with centre z_0 and radius ε and is called the **closed circular disc** with centre z_0 and radius ε.

Definition. Let $S \subseteq C$. Let $z_0 \in S$. Then z_0 is said to be an **interior point** of S if there exists a neighbourhood $N_\varepsilon(z_0)$ such that $N_\varepsilon(z_0) \subseteq S$.

S is called an **open set** if every point of S is an interior point of S.

Definition. Let $S \subseteq C$. Let $z_0 \in C$. Then z_0 is called a **limit point** of S if every neighbourhood of z_0 contains infinitely many points of S.

S is called a **closed set** if it contains all its limit points.

Remark. It can be verified that a set S is closed iff its complement $C-S$ is open.

Definition. Let $S \subseteq C$. Let $z_0 \in C$. Then z_0 is called a **boundary point** of S if z_0 is a limit point of both S and $C-S$. Thus z_0 is a boundary point of S iff every neighbourhood of z_0 contains infinitely many points of S and infinitely many points of $C-S$.

Definition. Let $S \subseteq C$. Then S is called a **bounded set** if there exists a real number k such that $|z| \le k$ for all $z \in S$.

Definition. Let $S \subseteq C$. Then S is called a **connected set** if every pair of points in S can be joined by a polygon which lies in S.

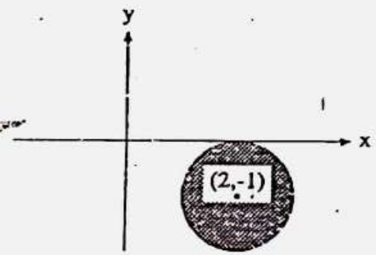
Definition. A nonempty open connected subset of C is called a region in C.

Example 1. Let $D = \{z / |z-2+i| \le 1\}$.

(ie) D is the set of all complex numbers satisfying $|z-(2-i)| \le 1$.

Clearly D represents the closed disc with centre $2-i$ and radius 1.

Clearly D is a connected and bounded set. We observe that the points which lie on the circle $|z-(2-i)|=1$ are not interior points of D. Hence D is not open. Hence D is not a region.



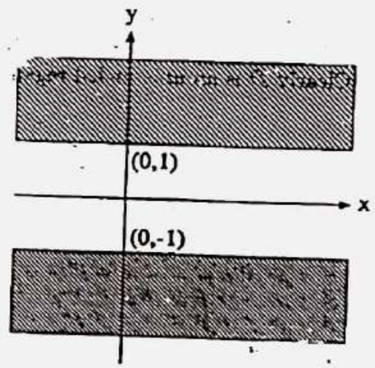
Example 2. Let $D = \{z / |\operatorname{Im} z| > 1\}$

Let $z = x+iy$.

$\therefore \quad D = \{z/|y|>1\}$
$= \{z/y>1 \text{ or } y<-1\}$
$= \{z/y>1\} \cup \{z/y<-1\}$.

Clearly D is the union of two half planes and it is unbounded as shown in the figure.

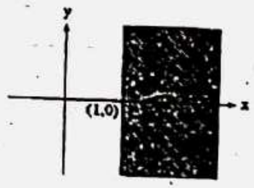
Obviously if z_1 is any complex number with $\operatorname{Im} z_1 > 1$ and z_2 is any complex number with $\operatorname{Im} z_2 < -1$ then z_1 and z_2 can not be joined by a polygon entirely lying in D. Hence D is not connected.



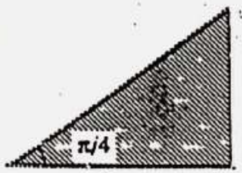
Hence D is not a region.

Example 3. Let $D = \{z/\operatorname{Re} z > 1\}$. Then D is a region in C.

Here D is the half plane as shown in the figure.

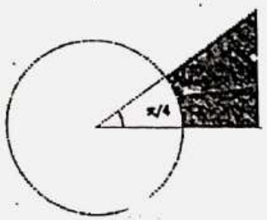


Example 4. $D = \{z/0 < \arg z < \pi/4\}$ is a region in C.



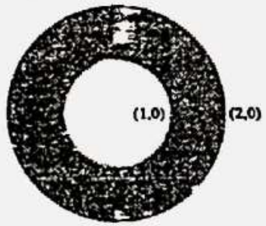
Example 5.
Let $D = \{z/0 < \arg z < \pi/4$ and $|z| > 1\}$

D is as shown in the figure. Clearly D is an unbounded region in C.



Example 6. Let $D = \{z/1 < |z| < 2\}$.

D is the region bounded by the circles $|z| = 1$ and $|z| = 2$. Such a region is called an annulus or annular region.



Exercises.

1. For each of the following subsets of C sketch the set and determine whether it is a region.

   (a) $|z - 2 + i| \leq 1$ (b) $|2z + 3| > 4$
   (c) $\operatorname{Im} z > 1$ (d) $|\operatorname{Im} z| > 1$
   (e) $|z| > 0,\ 0 \leq \arg z \leq \pi/4$
   (f) $|z - 4| \geq |z|$
   (g) $0 < |z - z_0| < \delta$ where z_0 is a fixed point and δ is a positive number.

2. Sketch each of the following subsets of C determined by the given conditions.

   (a) $|z - 1 + i| = 1$ (b) $|z + i| \leq 3$
   (c) $|z - i| = |z + i|$ (d) $\operatorname{Re}(\bar{z} - i) = 2$
   (e) $\operatorname{Re} z \geq 0$ (f) $|\operatorname{Re} z| + |\operatorname{Im} z| = 1$
   (g) $\operatorname{Im} z \geq 0$ (h) $\operatorname{Re} z + \operatorname{Im} z = 0$
   (i) $1 < |z| < 2$ (j) $\frac{3}{2} < |z - 1| < \frac{5}{2}$

3. Let r be a positive constant and let z_0 be a fixed complex number. Show that the equation of the circle with centre at $-z_0$ and radius r may be written as

$$|z|^2 + 2\operatorname{Re}(\overline{z_0}\, z) + |z_0|^2 = r^2.$$

4. If the points z_1, z_2, z_3 are the vertices of an equilateral triangle prove that $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$.

5. If z is a variable point and $\operatorname{Re}\left(\frac{z-4}{z-2i}\right) = 0$ prove that the locus of z is a circle.

1.9. The Extended Complex Plane

The function $w = 1/z$ defines a one-one mapping of the complex plane onto itself with two notable exceptions. The point $z = 0$ has no image and the point $w = 0$ has no pre-image. To rectify this situation we extend the complex plane by adding a symbol ∞ called the point at infinity. Its connection with the other complex numbers is established by defining

(i) $a + \infty = \infty + a = \infty$ for all $a \in C \cup \{\infty\}$.
(ii) $a\,\infty = \infty\, a = \infty$ for all $a \in C - \{0\}$ and for $a = \infty$.
(iii) $\frac{a}{\infty} = 0$ for all $a \in C - \{0\}$.
(iv) $\frac{a}{0} = \infty$ for all $a \in C - \{0\}$.

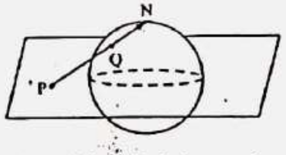
$C \cup \{\infty\}$ is called **the extended complex plane.**

We assume that in the extended complex plane every straight line passes through ∞.

The extended complex plane can be represented by points on a sphere. In the three dimensional Euclidean space with coordinates (x, y, z), we identify the x-y plane with the complex plane.

Let S denote the unit sphere with centre origin given by the equation $x^2 + y^2 + z^2 = 1$.

Let $N = (0, 0, 1)$; $P = (x, y, 0)$.



Let the line NP intersect the sphere S again at Q. Then Q is called **the steriographic projection** of P on the sphere and is taken as the point representing the complex number $z = x + iy$. In this way every point on the sphere except N represents a unique complex number. We assign the point N to represent ∞. Thus we get a one-one correspondence between the points on the sphere S and the extended complex plane.

Solved problems.

Problem 1. Find the complex number represented by the given point $Q = (x_1, x_2, x_3) \in S$ where S is the unit sphere with centre origin.

Solution. Let $N = (0, 0, 1)$ and $Q = (x_1, x_2, x_3)$.
The equation of the line joining N and Q is given by

$$\frac{x}{x_1} = \frac{y}{x_2} = \frac{1 - z}{1 - x_3}.$$

To find the point of intersection $P = (x, y, 0)$ of this line with the x-y plane, we put $z = 0$.

$$\therefore \quad x = \frac{x_1}{1 - x_3} \text{ and } y = \frac{x_2}{1 - x_3}.$$

Hence $z = x + iy = \dfrac{1}{1 - x_3}(x_1 + ix_2)$.
This gives the complex number represented by $Q = (x_1, x_2, x_3)$.

Problem 2. Find the point $Q = (x_1, x_2, x_3)$ on the sphere S that represents the complex number $z = x + iy$.

Solution. Let $P = (x, y, 0)$ and $N = (0, 0, 1)$.
Since N, P and Q are collinear, as in problem 1, we have

$$z = x + iy = \frac{1}{1 - x_3}(x_1 + ix_2) \quad \ldots (1)$$

$$\therefore \quad |z|^2 = \frac{x_1^2 + x_2^2}{(1 - x_3)^2}$$

$$= \frac{1 - x_3^2}{(1 - x_3)^2} \quad (\text{since } x_1^2 + x_2^2 + x_3^2 = 1).$$

$$= \frac{1 + x_3}{1 - x_3}.$$

Hence $x_3 = \dfrac{|z|^2 - 1}{|z|^2 + 1}$.

$$\text{Also, from (1) } x_1 = x(1 - x_3) = \frac{z + \bar{z}}{2}\left[1 - \frac{|z|^2 - 1}{|z|^2 + 1}\right]$$

$$= \frac{z + \bar{z}}{|z|^2 + 1}.$$

$$\text{Similarly } x_2 = \frac{z - \bar{z}}{i(|z|^2 + 1)}.$$

Thus the complex number z is represented by the point $Q = (x_1, x_2, x_3)$ where

$$x_1 = \frac{z + \bar{z}}{|z|^2 + 1};\ x_2 = \frac{z - \bar{z}}{i(|z|^2 + 1)};\ x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}.$$

Exercises.

1. Find the points on the sphere S corresponding to the complex numbers (i) i (ii) $-i$ (iii) $1 + i$ (iv) $1 - i$ (v) $2 - 3i$.

2. Find the complex numbers represented by the points $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$, $(1,0,0)$ and $(0,1,0)$.

Answers.

1. (i) $(0,1,0)$ (ii) $(0,-1,0)$ (iii) $\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{2}\right)$ (iv) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$ (v) $\left(\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}\right)$
2. $\frac{(1 + i)}{\sqrt{2}}$; 1; i.

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2. Analytic Functions

2.0. Introduction

In this chapter we study in detail the concepts of limit and continuity for functions of a complex variable. We also introduce the notion of differentiability for functions of a complex variable and see how the derivative of a complex function of one complex variable sometimes behaves like the derivative of a real function of one real variable and othertimes is comparable to the partial derivatives of a real function of two variables.

2.1. Functions of a Complex Variable

We use the letters z and w to denote complex variables. Thus to denote a complex valued function of a complex variable we use the notation $w = f(z)$. Throughout this chapter we shall consider functions whose domain of definition is a region of the complex plane.

The function $w = iz + 3$ is defined in the entire complex plane.

The function $w = \dfrac{1}{z^2+1}$ is defined at all points of the complex plane except at $z = \pm i$.

The function $w = |z|$ is defined in the entire complex plane and this is a real valued function of the complex variable z.

If $a_0, a_1, \cdots\cdots, a_n$ are complex constants the function $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is defined in the entire complex plane and is called a **polynomial** in z.

If $P(z)$ and $Q(z)$ are polynomials the quotient $\dfrac{P(z)}{Q(z)}$ is called a **rational function** and it is defined for all z with $Q(z) \neq 0$.

The function $f(z) = x^4 + y^4 + i(x^2 + y^2)$ is defined over the entire complex plane.

In general if $u(x, y)$ and $v(x, y)$ are real valued functions of two variables both defined on a region S of the complex plane then $f(z) = u(x, y) + iv(x, y)$ is a **complex valued function** defined on S.

Conversely each complex function $w = f(z)$ can be put in the form

$$w = f(z) = u(x, y) + iv(x, y)$$

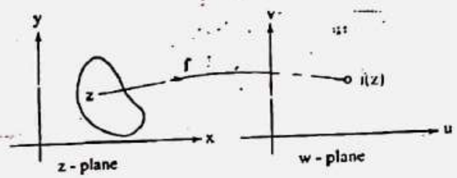
where u and v are real valued functions of the real variables x and y.

$u(x, y)$ is called the **real part** and $v(x, y)$ is called the **imaginary part of the function** $f(z)$.

For example, $f(z) = z^2 = (x + iy)^2 = (x^2 - y^2) + i(2xy)$ so that $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$.

Thus a complex function $w = f(z)$ can be viewed as a function of the complex variable z or as a function of two real variables x and y.

To have a geometric representation of the function $w = f(z)$ it is convenient to draw separate complex planes for the variables z and w so that corresponding to each point $z = x + iy$ of the z-plane there is a point $w = u + iv$ in the w-plane.



Exercises.

1. Express each of the following functions in the form $u(x, y) + iv(x, y)$

(i) $w = z^3$ (ii) $w = 2\bar{z}^2 + 1$ (iii) $w = \dfrac{1}{z}$

(iv) $w = \dfrac{z}{1+z}$ (v) $w = z + \dfrac{1}{z}$ (vi) $w = z\bar{z}$

Answers

(i) $(x^3 - 3xy^2) + i(3x^2y - y^3)$ (ii) $(2x^2 - 2y^2 + 1) - i(4xy)$

(iii) $\dfrac{x}{x^2+y^2} - \dfrac{iy}{x^2+y^2}$ (iv) $\dfrac{x^2+x+y^2}{(x+1)^2+y^2} + \dfrac{iy}{(x+1)^2+y^2}$

(v) $\dfrac{x(x^2+y^2+1)}{x^2+y^2} + i\dfrac{y(x^2+y^2-1)}{x^2+y^2}$ (vi) $x^2 + y^2$

2.2. Limits

Let $w = f(z)$ be a function defined in some region containing a point z_0 except perhaps at the point z_0. It may happen that as z approaches z_0 the value $f(z)$ of the function is arbitrarily close to a complex number l. Then we say that the limit of the function $f(z)$ as z *approaches* z_0 is l. This idea is expressed in a precise form in the following definition.

Definition. A function $w = f(z)$ is said to have the *limit* l as z tends to z_0 if given $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |z - z_0| < \delta \Rightarrow |f(z) - l| < \varepsilon$. In this case we write $\lim_{z \to z_0} f(z) = l$.

Geometrically the definition states that given any open disc with centre l and radius ε there exists an open disc with centre z_0 and radius δ such that for every point $z(\neq z_0)$ in the disc $|z - z_0| < \delta$ the image $w = f(z)$ lies in the disc $|w - l| < \varepsilon$.

Remark 1. The above definition does not give any method of determining the limit l and it only provides a means of testing whether l is the limit of $f(z)$ as $z \to z_0$.

Remark 2. The condition $0 < |z-z_0| < \delta$ excludes the point $z = z_0$ from consideration. Hence the definition of limit employs only values of z in some disc $|z - z_0| < \delta$ other than z_0. Therefore the value of $f(z)$ at z_0 is immaterial and in fact to consider the limit of $f(z)$ as $z \to z_0$, $f(z)$ need not even be defined at z_0. Even if $f(z_0)$ is defined it is not necessary that $\lim_{z\to z_0} f(z) = f(z_0)$.

Lemma. When the limit of a function $f(z)$ exists as z tends to z_0 then the limit has a unique value.

Proof. Suppose that $\lim_{z\to z_0} f(z)$ has two values l_1 and l_2.

Then given $\varepsilon > 0$ there exists δ_1 and $\delta_2 > 0$ such that

$$0 < |z - z_0| < \delta_1 \Rightarrow |f(z) - l_1| < \varepsilon/2 \quad \text{and}$$
$$0 < |z - z_0| < \delta_2 \Rightarrow |f(z) - l_2| < \varepsilon/2.$$

Now let $\delta = \min\{\delta_1, \delta_2\}$.

Then if $0 < |z - z_0| < \delta$ we have

$$\begin{aligned} |l_1 - l_2| &= |l_1 - f(z) + f(z) - l_2| \\ &\leq |f(z) - l_1| + |f(z) - l_2| \quad \text{(using triangle inequality)} \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary $|l_1 - l_2| = 0$ so that $l_1 = l_2$.

Example 1. Let $f(z) = \begin{cases} z^2 & \text{if } z \neq i \\ 0 & \text{if } z = i \end{cases}$

As z approaches i, $f(z)$ approaches $i^2 = -1$. Hence we expect that $\lim_{z\to i} f(z) = -1$.

To prove that we must show that given $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |z - i| < \delta \Rightarrow |z^2 + 1| < \varepsilon$.

$$\begin{aligned} \text{Now, } |z^2 + 1| &= |(z + i)(z - i)| \\ &= |z + i||z - i| \quad \ldots(1) \end{aligned}$$

Note that if we can find a $\delta > 0$ satisfying the requirements of the definition then we can choose another $\delta \leq 1$ satisfying the requirements of the definition.

$$\begin{aligned} \text{Now } 0 < |z - i| < 1 \Rightarrow |z + i| &= |z - i + 2i| \\ &\leq |z - i| + |2i| \\ &< 1 + 2 = 3. \end{aligned}$$

$\therefore\ |z + i| < 3.$

Using this in (1) we obtain $0 < |z - i| < 1$.

$$\Rightarrow |z^2 + 1| < 3|z - i|$$

Hence if we choose $\delta = \min\left\{1, \frac{\varepsilon}{3}\right\}$ we get

$$0 < |z - i| < \delta \Rightarrow |z^2 + 1| < \varepsilon.$$

$\therefore\ \lim_{z\to i} f(z) = -1.$

Example 2. $\lim_{z\to 2} \dfrac{z^2 - 4}{z - 2} = 4.$

Let $f(z) = \dfrac{z^2 - 4}{z - 2}$. Hence $f(z)$ is not defined at $z = 2$ and when $z \neq 2$ we have

$$f(z) = \frac{(z+2)(z-2)}{z-2} = z + 2.$$

$\therefore\ |f(z) - 4| = |z + 2 - 4| = |z - 2|$ when $z \neq 2$

Now given $\varepsilon > 0$, we choose $\delta = \varepsilon$.

Then $0 < |z - 2| < \delta \Rightarrow |f(z) - 4| < \varepsilon$.

$\therefore\ \lim_{z\to 2} f(z) = 4$

Remark. For a function of one real variable there are only two directions along which a point x can travel and tend to a point x_0. Hence we distinguish between left limit and right limit at a point x_0 and the limit at x_0 exists if and only if the left and right limit at x_0 exist and are equal.

For functions of a complex variable infinitely many modes of approaches are possible for a point z to tend to a point z_0. If a function $f(z)$ approaches to two different values as z tends to z_0 along two different paths then $\lim_{z\to z_0} f(z)$ does not exist.

Example 3. The function $f(z) = \dfrac{\bar{z}}{z}$ does not have a limit as $z \to 0$.

$$f(z) = \frac{\bar{z}}{z} = \frac{x - iy}{x + iy}$$

Suppose $z \to 0$ along the path $y = mx$.

Along this path $f(z) = \frac{x - imx}{x + imx} = \frac{1 - im}{1 + im}$ as $x \neq 0$.

Hence if $z \to 0$ along the path $y = mx$, $f(z)$ tends to $\frac{1 - im}{1 + im}$ which is different for different values of m.

Hence $f(z)$ does not have a limit as $z \to 0$.

Example 4. Let $f(z) = \frac{x^2 y^2}{(x + y^2)^3}$, $z \neq 0$. Then $f(z)$ does not have a limit as $z \to 0$.

Along the parabola $y^2 = mx$ we have $f(z) = \frac{mx^3}{(x + mx)^3} = \frac{m}{(1 + m)^3}$.

Hence if $z \to 0$ along the parabola $y^2 = mx$, $f(z)$ tends to $\frac{m}{(1 + m)^3}$ which depends on m.

Hence $f(z)$ does not have a limit as $z \to 0$.

Exercises.

1. Use the definition of limit to prove the following

   (a) $\lim_{z \to 0} c = c$ where c is any constant.

   (b) $\lim_{z \to z_0} (az + b) = az_0 + b$

   (c) $\lim_{z \to z_0} z^2 = z_0^2$

   (d) $\lim_{z \to z_0} \bar{z} = \bar{z}_0$

   (e) $\lim_{z \to z_0} \text{Re } z = \text{Re } z_0$.

2. Prove that each of the following functions $f(z)$ does not have a limit as $z \to 0$.

   (a) $f(z) = \frac{\sqrt{|xy|}}{x + iy}$, $z = x + iy \neq 0$.

   (b) $f(z) = \frac{-ix^3 y}{x^6 + y^2}$, $z \neq 0$

   (c) $f(z) = \frac{xy}{x^2 + y^2}$, $z \neq 0$.

we now formulate the definition of $\lim_{z \to z_0} f(z) = l$ where z_0 or l is infinite.

Definition. We say $\lim_{z \to \infty} f(z) = l$ if given $\varepsilon > 0$ there exists a number $m > 0$ such that $|z| > m \Rightarrow |f(z) - l| < \varepsilon$.

We say that $\lim_{z \to z_0} f(z) = \infty$ if for given $n > 0$ there exists $\delta > 0$ such that $0 < |z - z_0| < \delta \Rightarrow |f(z)| > n$.

We say that $\lim_{z \to \infty} f(z) = \infty$ if for given $n > 0$ there exists $m > 0$ such that $|z| > m \Rightarrow |f(z)| > n$.

2.3. Theorems on Limit

We state without proof the following theorem on the limits of sum, product and quotient of two functions. The proof is analogus to that of real functions.

Theorem 2.1. Let f and g be two functions whose limits at z_0 exist.

Let $\lim_{z \to z_0} f(z) = l$ and $\lim_{z \to z_0} g(z) = m$.

Then (i) $\lim_{z \to z_0} [f(z) + g(z)] = l + m$

(ii) $\lim_{z \to z_0} f(z) g(z) = lm$

(iii) $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{l}{m}$ provided $m \neq 0$.

Theorem 2.2.

(i) If $\lim_{z \to z_0} f(z) = l$ then $\lim_{z \to z_0} \overline{f(z)} = \bar{l}$.

(ii) If $\lim_{z \to z_0} f(z) = l$, then $\lim_{z \to z_0} |f(z)| = |l|$.

(iii) $\lim_{z \to z_0} f(z) = l$ iff $\lim_{z \to z_0} \text{Re } f(z) = \text{Re } l$ and $\lim_{z \to z_0} \text{Im } f(z) = \text{Im}(l)$

Proof. (i) Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that $0 < |z - z_0| < \delta \Rightarrow |f(z) - l| < \varepsilon$.

Now $|\overline{f(z)} - \bar{l}| = |\overline{f(z) - l}| = |f(z) - l|$

Hence $0 < |z - z_0| < \delta \Rightarrow |\overline{f(z)} - \bar{l}| < \varepsilon$ so that $\lim_{z \to z_0} \overline{f(z)} = \bar{l}$.

(ii) $||f(z)| - |l|| \leq |f(z) - l|$ and hence

$$0 < |z - z_0| < \delta \Rightarrow ||f(z)| - |l|| < \varepsilon.$$

$\therefore \lim_{z \to z_0} |f(z)| = |l|$.

(iii) Let $\lim_{z \to z_0} f(z)| = l$.

Since $\text{Re } f(z) = \frac{1}{2}[f(z) + \overline{f(z)}]$ we have

$$\lim_{z \to z_0} \text{Re } f(z) = \frac{1}{2}\left[\lim_{z \to z_0} f(z) + \lim_{z \to z_0} \overline{f(z)}\right]$$
$$= \frac{1}{2}(l + \bar{l})$$
$$= \text{Re } l.$$

Similarly $\lim_{z \to z_0} \text{Im } f(z) = \text{Im } l$.

Conversely, let $\lim_{z \to z_0} \text{Re } f(z) = \text{Re } l$ and let $\lim_{z \to z_0} \text{Im } f(z) = \text{Im } l$. Since $f(z) = \text{Re } f(z) + i \text{ Im } f(z)$ it follows that $\lim_{z \to z_0} f(z) = \text{Re } l + i \text{ Im } l = l$.

Remark. It follows immediately from the definition of limit that $\lim_{z \to z_0} z = z_0$ and $\lim_{z \to z_0} c = c$ where c is any constant.

Hence using (i) and (ii) of theorem 2.1 we see that for any polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, $\lim_{z \to z_0} P(z) = P(z_0)$.

Exercises.

1. Evaluate the following limits using theorems 2.1 and 2.2

(i) $\lim_{z \to 2i} (2x + iy^2)^2$ (ii) $\lim_{z \to 1-i} [x + i(2x + y)]$

(iii) $\lim_{z \to 1+i} (z^2 - 5z + 10)$ (iv) $\lim_{z \to -2i} \frac{(z+3)(z-4)}{z^2 + 5z + 9}$

(v) $\lim_{z \to -i} \frac{\bar{z} + z^2}{1 - \bar{z}}$ (vi) $\lim_{z \to z_0} \frac{1}{z^n}$, $z_0 \neq 0$.

Answers.

(i) -16 (ii) $1 + i$ (iii) $5 - 3i$

(iv) $-\frac{2(3 - 2i)(4 + 2i)}{5(1 - 2i)}$ (v) -1 (vi) $\frac{1}{z_0^n}$.

2.4. Continuous Functions

Definition. Let f be a complex valued function defined on a region D of the complex plane. Let $z_0 \in D$. Then f is said to be **continuous** at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$.

Thus f is **continuous** at z_0 if given $\varepsilon > 0$ there exists a $\delta > 0$ such that $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$.

f is said to be continuous in D if it is continuous at each point of D.

The following are immediate consequences of the corresponding results on limits given in theorems 2.1 and 2.2.

Theorem 2.3. (i) If f and g are continuous at z_0 then $f + g$, fg and $\bar{f}$ are continuous at z_0 and f/g is continuous at z_0 if $g(z_0) \neq 0$.

(ii) If f is continuous at z_0 then $|f|$ is also continuous at z_0.

(iii) If f is continuous at z_0 iff Re f and Im f are continuous at z_0.

(iv) Any polynomial $P(z)$ is continuous at each point of the complex plane and any rational function $\frac{P(z)}{Q(z)}$ is continuous at all points where $Q(z) \neq 0$.

2.5. Differentiability

Definition. Let f be a complex function defined in a region D and let $z \in D$. Then f is said to be **differentiable** at z if $\lim_{h \to 0} \frac{f(z + h) - f(z)}{h}$ exists and is finite. This limit is denoted by $f'(z)$ or $\frac{df}{dz}$ and is called the **derivative** of $f(z)$ at z.

The function is said to be differentiable in D if it is **differentiable** at all points of D.

Example 1. The function $f(z) = z^2$ is differentiable at every point and $f'(z) = 2z$.

Proof.
$$\frac{f(z + h) - f(z)}{h} = \frac{(z + h)^2 - z^2}{h}$$
$$= 2z + h$$

Hence
$$\lim_{h \to 0} \frac{f(z + h) - f(z)}{h} = \lim_{h \to 0} (2z + h)$$
$$= 2z$$
$$\therefore f'(z) = 2z.$$

Example 2. The function $f(z) = \bar{z}$ is nowhere differentiable.

Proof.
$$\frac{f(z + h) - f(z)}{h} = \frac{\overline{(z + h)} - \bar{z}}{h}$$
$$= \frac{\bar{z} + \bar{h} - \bar{z}}{h} = \frac{\bar{h}}{h}$$

$\lim_{h \to 0} \frac{\bar{h}}{h}$ does not exist (refer example 3 of 2.2).

$\therefore$ $f(z) = \bar{z}$ is nowhere differentiable.

Remark 1. If $f(z)$ is differentiative at a point z then it is continuous at that point.

Proof. $\lim_{h\to 0}[f(z+h)-f(z)] = \lim_{h\to 0}\left[\frac{f(z+h)-f(z)}{h}\right] \times \lim_{h\to 0} h$

$$= f'(z) \times 0$$

$$= 0$$

$\therefore \lim_{h\to 0} f(z+h) = f(z)$ so that f is continuous at z.

The converse of the above result is not true.

For example, $f(z) = \bar{z}$ is continuous everywhere but it is nowhere differentiable (refer example 2).

The definition of derivative for complex functions is identical to the definition for real functions and the following formal rules of differentiation are true for complex functions also and the proof is left as an exercise.

Theorem 2.4. Let $f(z)$ and $g(z)$ be differentiable at a point z. Then

(i) $(f+g)'(z) = f'(z) + g'(z)$.

(ii) $(fg)'(z) = f(z)g'(z) + f'(z)g(z)$.

(iii) $\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{[g(z)]^2}$ provided $g(z) \neq 0$.

(iv) Suppose g is differentiable at z and f is differentiable at $g(z)$. Let $F(z) = f(g(z))$. Then $F'(z) = f'(g(z))g'(z)$. (This is the usual chain rule for the derivative of composite functions).

(v) Let n be any positive integer. The function $f(z) = z^n$ is differentiable at every point and $f'(z) = nz^{n-1}$.

(vi) The polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ is differentiable at every point and $P'(z) = a_1 + 2a_2 z + \cdots + na_n z^{n-1}$.

(vii) If n is a negative integer $f(z) = z^n$ is differentiable at every point $z \neq 0$ and $f'(z) = nz^{n-1}$.

Exercises.

1. Find the derivative of the following functions.

   (i) $z^2 + 3z + 1$ (ii) $\frac{z+1}{2z+3}\left(z \neq -\frac{3}{2}\right)$ (iii) $\frac{(2+z^3)^4}{z-1}; z \neq 1$.

2. Prove that $f(z) = \frac{z-1}{z+1}$ is differentiable at every point $z \neq -1$ and find $f'(z)$.
3. Prove that $f(z) = \text{Re}\, z$ is not differentiable at any point.
4. Prove that $f(z) = \text{Im}\, z$ is not differentiable at any point.

Answers.

1. (i) $2z+3$ (ii) $\frac{1}{(2z+3)^2}$ (iii) $\frac{(2+z^3)^3(11z^3-12z^2-2)}{(z-1)^2}$

2. $\frac{2}{(z+1)^2}$.

2.6. The Cauchy-Riemann Equations

The existence of the derivative of a complex function of a complex variable $f(z)$ requires $\frac{f(z+h)-f(z)}{h}$ to approach to the same limit as $h \to 0$ along any path. This has some far reaching consequences. In this section we derive some important properties of the real and imaginary parts of the differentiable function $f(z) = u(x,y) + iv(x,y)$.

Theorem 2.5. Let $f(z) = u(x,y) + iv(x,y)$ be differentiable at a point $z_0 = x_0 + iy_0$. Then $u(x,y)$ and $v(x,y)$ have first order partial derivatives $u_x(x_0,y_0)$, $u_y(x_0,y_0)$, $v_x(x_0,y_0)$ and $v_y(x_0,y_0)$ at (x_0,y_0) and these partial derivatives satisfy the **Cauchy-Riemann equations** (C.R equations) given by

$$u_x(x_0,y_0) = v_y(x_0,y_0) \text{ and } u_y(x_0,y_0) = -v_x(x_0,y_0).$$

$$\text{Also, } f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0)$$

$$= v_y(x_0,y_0) - iu_y(x_0,y_0).$$

Proof. Since $f(z) = u(x,y) + iv(x,y)$ is differentiable at $z_0 = x_0 + iy_0$ $\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}$ exists and hence the limit is independent of the path in which h approuhces zero.

Let $h = h_1 + ih_2$.

$$\text{Now } \frac{f(z_0+h)-f(z_0)}{h}$$

$$= \frac{u(x_0+h_1, y_0+h_2) + iv(x_0+h_1, y_0+h_2) - u(x_0,y_0) - iv(x_0,y_0)}{h_1+ih_2}$$

$$= \left[\frac{u(x_0+h_1, y_0+h_2) - u(x_0,y_0)}{h_1+ih_2}\right] + i\left[\frac{v(x_0+h_1, y_0+h_2) - v(x_0,y_0)}{h_1+ih_2}\right]$$

Suppose $h \to 0$ along the real axis so that $h = h_1$.

$$\text{Then } f'(z_0) = \lim_{h_1\to 0}\left[\frac{f(z_0+h_1)-f(z_0)}{h_1}\right]$$

$$= \lim_{h_1\to 0}\left[\frac{u(x_0+h_1,y_0)-u(x_0,y_0)}{h_1}\right] + i\lim_{h_1\to 0}\left[\frac{v(x_0+h_1,y_0)-v(x_0,y_0)}{h_1}\right]$$

$$= u_x(x_0,y_0)+iv_x(x_0,y_0) \qquad \dots (1)$$

Now, suppose $h \to 0$ along the imaginary axis so that $h = ih_2$.

$$\therefore f'(z_0) = \lim_{ih_2\to 0}\left[\frac{f(z_0+h_2)-f(z_0)}{ih_2}\right]$$

$$= \lim_{h_2\to 0}\left[\frac{u(x_0,y_0+h_2)-u(x_0,y_0)}{ih_2}\right] + i\lim_{h_2\to 0}\left[\frac{v(x_0,y_0+h_2)-v(x_0,y_0)}{ih_2}\right]$$

$$= \left[\frac{u_y(x_0,y_0)}{i}\right]+i\left[\frac{v_y(x_0,y_0)}{i}\right]$$

$$= \frac{1}{i}u_y(x_0,y_0)+v_y(x_0,y_0)$$

$$= -iu_y(x_0,y_0)+v_y(x_0,y_0) \qquad \dots (2)$$

From (1) and (2) we get

$$f'(z_0) = u_x(x_0,y_0)+iv_x(x_0,y_0) = v_y(x_0,y_0)-iu_y(x_0,y_0)$$

Equating real and imaginary parts we get

$$u_x(x_0,y_0) = v_y(x_0,y_0)$$
$$u_y(x_0,y_0) = -v_x(x_0,y_0)$$

Remark 1. Since $f'(z) = u_x+iv_x = v_y-iu_y$ we have

$$|f'(z)|^2 = u_x^2+v_x^2 = u_y^2+v_y^2$$

$$\text{Also } |f'(z)|^2 = u_x^2+u_y^2 = v_x^2+v_y^2$$

$$\text{Further } |f'(z)|^2 = u_xv_y-u_yv_x = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)}$$

Remark 2. The Cauchy-Riemann equations provide a necessary condition for differentiability at a point. Hence if the C.R equations are not satisfied for a complex function at any point then we can conclude that the function is not differentiable.

For example, consider the function

$$f(z) = \bar{z} = x-iy.$$

Here $u(x,y) = x$ and $v(x,y) = -y$.

$\therefore u_x(x,y) = 1$ and $v_y(x,y) = -1$.

$\therefore u_x \neq v_y$ so that C.R equations are not satisfied at any point z.

Hence the function $f(z) = \bar{z}$ is nowhere differentiable.

Remark 3. The C.R equations are not sufficient for differentiability at a point as shown in the following examples.

Example 1. Let $f(z) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

Here $u(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

and $v(x,y) = 0$.

$$\text{Now, } u_x(0,0) = \lim_{h\to 0}\left[\frac{u(h,0)-u(0,0)}{h}\right] = \lim_{h\to 0}\left[\frac{0-0}{h}\right] = 0.$$

Similarly $u_y(0,0) = 0$.

Also $v_x(0,0) = 0$ and $v_y(0,0) = 0$.

Hence the C.R equations are satisfied at $z = 0$.

Now, along the path $y = mx$

$$f(z) = \frac{xmx}{x^2+m^2x^2} = \frac{m}{1+m^2} \text{ if } x \neq 0.$$

Hence if $z \to 0$ along the path $y = mx$, $f(z) \to \dfrac{m}{1+m^2}$, which is different for different values of m.

Hence $f(z)$ does not have a limit as $z \to 0$ so that $f(z)$ is not even continuous at $z = 0$.

Thus $f(z)$ is not differentiable at $z = 0$.

Example 2. Let $f(z) = \sqrt{|xy|}$

Here $u(x, y) = \sqrt{|xy|}$ and $v(x, y) = 0$.

$$u_x(0,0) = \lim_{h\to 0}\left[\frac{u(h,0) - u(0,0)}{h}\right]$$
$$= 0$$

Similarly $u_y(0, 0) = 0$.

Also $v_x(0, 0) = 0$ and $v_y(0, 0) = 0$.

Hence the C.R equations are satisfied at $z = 0$.

We claim that $f(z)$ is not differentiable at $(0, 0)$.

Along the path $y = mx$,

$$\frac{f(z) - f(0)}{z} = \frac{\sqrt{|xmx|}}{x + imx} = \frac{\sqrt{|m|}}{1 + im} \text{ if } x \neq 0.$$

Hence as $z \to 0$ along the path $y = mx$, $\frac{f(z) - f(0)}{z}$ tends to $\frac{\sqrt{|m|}}{1 + im}$ which depends on the path along which $z \to 0$ so that f is not differentiable at $z = 0$.

Note. In this example the function $f(z)$ is continuous and has partial derivatives which satisfy Cauchy-Riemann equations at 0 but is not differentiable at 0.

In the following theorem we prove that C.R equations together with the continuity of partial derivatives give a sufficient condition for differentiability of complex functions.

Theorem 2.6. Let $f(z) = u(x, y) + iv(x, y)$ be a function defined in a region D such that u, v and their first order partial derivatives are continuous in D. If the first order partial derivatives of u, v satisfy the Cauchy-Riemann equations at a point $(x, y) \in D$ then f is differentiable at $z = x + iy$.

Proof. Since $u(x, y)$ and its first order partial derivatives are continuous at (x, y) we have by the mean value theorem for functions of two variables

$$u(x + h_1, y + h_2) - u(x, y) = h_1 u_x(x, y) + h_2 u_y(x, y) + h_1\varepsilon_1 + h_2\varepsilon_2 \quad \ldots (1)$$

where ε_1 and $\varepsilon_2 \to 0$ as h_1 and $h_2 \to 0$.

Similarly

$$v(x + h_1, y + h_2) - v(x, y) = h_1 v_x(x, y) + h_2 v_y(x, y) + h_1\varepsilon_3 + h_2\varepsilon_4 \quad \ldots (2)$$

where $\varepsilon_3, \varepsilon_4 \to 0$ as h_1 and $h_2 \to 0$.

Let $h = h_1 + ih_2$.

Then
$$\frac{f(z+h) - f(z)}{h} = \frac{1}{h}[u(x + h_1, y + h_2) - u(x, y)$$
$$+ iv(x + h_1, y + h_2) - v(x, y)]$$
$$= \frac{1}{h}[\{h_1 u_x(x, y) + h_2 u_y(x, y) + h_1\varepsilon_1 + h_2\varepsilon_2\}$$
$$+ i\{h_1 v_x(x, y) + h_2 v_y(x, y) + h_1\varepsilon_3 + h_2\varepsilon_4\}]$$

using (1) and (2).

$$= \frac{1}{h}[h_1\{u_x(x, y) + iv_x(x, y)\} + h_2\{u_y(x, y) + iv_y(x, y)\}$$
$$+ h_1(\varepsilon_1 + i\varepsilon_3) + h_2(\varepsilon_2 + i\varepsilon_4)]$$
$$= \frac{1}{h}[(h_1 + ih_2)u_x(x, y) - i(h_1 + ih_2)u_y(x, y)$$
$$+ h_1(\varepsilon_1 + i\varepsilon_3) + h_2(\varepsilon_2 + i\varepsilon_4)]$$

(using C.R equations).

$$= \frac{1}{h}[hu_x(x, y) - ihu_y(x, y) + h_1(\varepsilon_1 + i\varepsilon_3) + h_2(\varepsilon_2 + i\varepsilon_4)]$$
$$= u_x(x, y) - iu_y(x, y) + \frac{h_1}{h}(\varepsilon_1 + i\varepsilon_3) + \frac{h_2}{h}(\varepsilon_2 + i\varepsilon_4).$$

Now, since $\left|\frac{h_1}{h}\right| \leq 1$, $\frac{h_1}{h}(\varepsilon_1 + i\varepsilon_3) \to 0$ as $h \to 0$.

Similarly $\frac{h_2}{h}(\varepsilon_2 + i\varepsilon_4) \to 0$ as $h \to 0$.

$\therefore \lim_{h\to 0}\frac{f(z+h) - f(z)}{h} = u_x(x, y) - iu_y(x, y)$. Hence f is differentiable.

Example 1. Let $f(z) = e^x(\cos y + i\sin y)$.

$\therefore\ u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$.

Then $u_x(x, y) = e^x \cos y = v_y(x, y)$

and $u_y(x, y) = -e^x \sin y = -v_x(x, y)$

Thus the first order partial derivatives of u and v satisfy the Cauchy-Riemann equations at every point.

Further $u(x, y)$ and $v(x, y)$ and their first order partial derivatives are continuous at every point. Hence f is differentiable at every point of the complex plane.

Example 2. Let $f(z) = |z|^2$.

$\therefore\ f(z) = u(x, y) + iv(x, y) = x^2 + y^2$.

$\therefore\ u(x, y) = x^2 + y^2$ and $v(x, y) = 0$.

Hence $u_x(x, y) = 2x$; $u_y(x, y) = 2y$

$v_x(x, y) = 2x = v_y(x, y)$.

Clearly the Cauchy-Riemann equations are satisfied at $z = 0$.

Further u and v and their first order partial derivatives are continuous and hence f is differentiable at $z = 0$.

Also we notice that the C.R equations are not satisfied at any point $z \neq 0$ and hence f is not differentiable at $z \neq 0$. Thus f is differentiable only at $z = 0$.

Alternate forms of Cauchy - Riemann equations

In the following theorem we express the Cauchy-Riemann equations in complex form.

Theorem 2.7. (Complex form of C-R equations)

Let $f(z) = u(x, y) + iv(x, y)$ be differentiable. Then the C.R equations can be put in the complex form as $f_x = -if_y$.

Proof. Let $f(z) = u(x, y) + iv(x, y)$.

$$\text{Then } f_x = u_x + iv_x$$
$$\text{and } f_y = u_y + iv_y.$$
$$\text{Hence } f_x = -if_y$$
$$\Leftrightarrow u_x + iv_x = -i(u_y + iv_y)$$
$$\Leftrightarrow u_x + iv_x = v_y - iu_y$$
$$\Leftrightarrow u_x = v_y \text{ and } v_x = -u_y.$$

Thus the two C.R equations are equivalent to the equation $f_x = -if_y$.

In the following theorem we express the Cauchy Riemann equations and the derivative of a complex function in terms of it polar coordinates.

Theorem 2.8. (C.R equations in polar coordinates)

Let $f(z) = u(r, \theta) + iv(r, \theta)$ be differentiable at $z = re^{i\theta} \neq 0$.

Then $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$ and $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$.

Further $f'(z) = \dfrac{r}{z}\left(\dfrac{\partial u}{\partial r} + i\dfrac{\partial v}{\partial r}\right)$.

Proof. We have $x = r\cos\theta$ and $y = r\sin\theta$.

$$\text{Hence } \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$
$$= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \qquad \ldots (1)$$

$$\text{Also } \frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta}$$
$$= \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta)$$
$$\therefore \frac{1}{r}\frac{\partial v}{\partial \theta} = -\frac{\partial v}{\partial x}\sin\theta + \frac{\partial v}{\partial y}\cos\theta$$
$$= \frac{\partial u}{\partial y}\sin\theta + \frac{\partial u}{\partial x}\cos\theta \quad \text{(using C.R equations)}$$
$$= \frac{\partial u}{\partial r} \quad \text{(using 1)}$$
$$\text{Thus } \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}.$$

Similarly we can prove that $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$.

Now,

$$r\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = r\left[\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\right) + i\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r}\right)\right]$$
$$= r\left[\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right) + i\left(\frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta\right)\right]$$
$$= r\cos\theta\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + r\sin\theta\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$
$$= x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + iy\left(\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\right)$$
$$= xf'(z) + iyf'(z)$$
$$= (x + iy)f'(z)$$
$$= zf'(z).$$
$$\therefore f'(z) = \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right).$$

We now proceed to express C.R equations in yet another form.

Let $f(z) = u(x, y) + iv(x, y)$.

Since $x = \dfrac{z + \bar{z}}{2}$ and $y = \dfrac{z - \bar{z}}{2i}$ we have

$$f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right).$$

Thus f can be thought of as a function of z and $\bar{z}$. Though z and $\bar{z}$ are not independent variables we form the partial derivatives $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$ as if z and $\bar{z}$ are independent variables. With this convention we have the following theorem.

Theorem 2.9. If $f(z)$ is a differentiable function, the C.R equations can be put in the form $\frac{\partial f}{\partial \bar{z}} = 0$.

Proof.
$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}}$$
$$= \frac{\partial f}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial f}{\partial y}\left(-\frac{1}{2i}\right)$$
$$= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

Thus $\frac{\partial f}{\partial \bar{z}} = 0 \Leftrightarrow \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$ which is the complex form of the C-R equations. (refer theorem 2.7)

Thus the C.R equations can be put in the form $\frac{\partial f}{\partial \bar{z}} = 0$.

Solved Problems

Problem 1. Verify Cauchy-Riemann equations for the function $f(z) = z^3$.

Solution. $f(z) = z^3 = (x+iy)^3$
$$= (x^3 - 3xy^2) + i(3x^2y - y^3)$$
$\therefore$ $u(x, y) = x^3 - 3xy^2$ and $v(x, y) = 3x^2y - y^3$

$\therefore$ $u_x = 3x^2 - 3y^2$ and $v_x = 6xy$

$u_y = -6xy$ and $v_y = 3x^2 - 3y^2$.

Here $u_x = v_y$ and $u_y = -v_x$

Hence the Cauchy-Riemann equations are statisfied.

Problem 2. Prove that the following functions are nowhere differentiable.

(i) $f(z) = \operatorname{Re} z$ (ii) $f(z) = e^x(\cos y - i\sin y)$

Solution.

(i) $f(z) = \operatorname{Re} z$
$= x$
$\therefore$ $u(x, y) = x$ and $v(x, y) = 0$.
$\therefore$ $u_x = 1$ and $v_x = 0$.
$u_y = 0$ and $v_y = 0$.
Since $u_x \neq v_y$ the C.R equations are not satisfied at any point.
Hence $f(z)$ is nowhere differentiable.

(ii) $f(z) = e^x(\cos y - i\sin y)$
$= e^x \cos y - ie^x \sin y$.
$\therefore$ $u(x, y) = e^x \cos y$ and $v(x, y) = -e^x \sin y$.
$\therefore$ $u_x = e^x \cos y$ and $v_x = -e^x \sin y$.
$u_y = -e^x \sin y$ and $v_y = -e^x \cos y$.
Clearly C.R equations are not satisfied at any point and hence $f(z)$ is nowhere differentiable.

Problem 3. Prove that $f(z) = \begin{cases} \frac{z \operatorname{Re} z}{|z|} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$ is continuous at $z = 0$ but not differentiable at $z = 0$.

Solution. First we shall prove that $\lim_{z\to 0} f(z) = 0$.

Now $|f(z) - 0| = \left|\frac{z \operatorname{Re} z}{|z|}\right| = |\operatorname{Re} z|$.

Further $|\operatorname{Re} z| \leq |z|$.

$\therefore$ For any given $\varepsilon > 0$ if we choose $\delta = \varepsilon$ we get
$$|z| = |z - 0| < \delta \Rightarrow |f(z) - 0| < \varepsilon.$$

Hence f is continuous at $z = 0$.

Now, we prove that $f(z)$ is not differentiable at $z = 0$.
$$\frac{f(z) - f(0)}{z - 0} = \frac{z \operatorname{Re} z}{z|z|} = \frac{\operatorname{Re} z}{|z|}$$
$$= \frac{x}{\sqrt{x^2+y^2}} \quad \text{where} \quad z = x + iy.$$

Along the path $y = mx$,
$$\frac{f(z) - f(0)}{z - 0} = \frac{x}{\sqrt{x^2 + m^2x^2}} = \frac{1}{\sqrt{1+m^2}}.$$

The value of the limit depends on m and hence on the path along which $z \to 0$.

$\therefore \lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$ does not exist.

$\therefore f(z)$ is not differentiable at $z=0$.

Problem 4. Prove that $f(z) = z \operatorname{Im} z$ is differentiable only at $z=0$ and find $f'(0)$.

Solution. $f(z) = z \operatorname{Im} z$

$= (x+iy)y$

$\therefore u(x,y) = xy$ and $v(x,y) = y^2$.

$\therefore u_x = y;\ v_x = 0;\ u_y = x$ and $v_y = 2y$.

Clearly the C.R equation are satisfied only at $z=0$.

Further all the first order partial derivatives are continuous.

Hence $f(z)$ is differentiable at $z=0$.

Also $f'(0) = u_x(0,0) + i v_x(0,0) = 0$.

Problem 5. Show that $f(z) = \begin{cases} \frac{xy^2(x+iy)}{x^2+y^4} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$ is not differentiable at $z=0$.

Solution. $\frac{f(z)-f(0)}{z-0} = \frac{xy^2(x+iy)}{x^2+y^4}\cdot\left(\frac{1}{x+iy}\right)$

$= \frac{xy^2}{x^2+y^4}$

$\therefore$ Along the path $x = my^2$

$$\frac{f(z)-f(0)}{z-0} = \frac{my^4}{m^2y^4+y^4} = \frac{m}{m^2+1}.$$

The value of the limit depends on m and hence depends on the path along which $z \to 0$.

$\therefore \lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$ does not exist.

$\therefore f(z)$ is not differentiable at $z=0$.

Problem 6. Prove that the function $f(z) = \begin{cases} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$ satisfies C-R equations at the origin but $f'(0)$ does not exist.

Solution. $f(z) = \begin{cases} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

Here $u(x,y) = \frac{x^3-y^3}{x^2+y^2}$ and $v(x,y) = \frac{x^3+y^3}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and $u(0,0) = v(0,0) = 0$.

Now, $u_x(0,0) = \lim_{h\to 0} \frac{u(h,0)-u(0,0)}{h}$

$= \lim_{h\to 0}\left(\frac{h^3/h^2 - 0}{h}\right) = 1.$

Similarly $u_y(0,0) = -1$; $v_x(0,0) = 1$ and $v_y(0,0) = 1$. (verify)

Thus $u_x(0,0) = v_y(0,0) = 1$ and

$u_y(0,0) = -v_x(0,0) = -1$, so that

C.R. equations are satisfied at $z=0$.

Now, $\frac{f(z)-f(0)}{z-0} = \frac{x^3-y^3}{(x^2+y^2)(x+iy)} + i\frac{x^3+y^3}{(x^2+y^2)(x+iy)}$

Along the path $y = mx$ we have

$$\frac{f(z)-f(0)}{z-0} = \frac{x^3-m^3x^3}{(x^2+m^2x^2)(x+imx)} + i\frac{x^3+m^3x^3}{(x^2+m^2x^2)(x+imx)}$$

$$= \frac{1-m^3}{(1+m^2)(1+im)} + i\frac{1+m^3}{(1+m^2)(1+im)}$$

Hence the value of the limit depends on the path along which $z \to 0$.

Thus $\lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$ does not exist.

Hence f is not differentiable at 0.

Problem 7. Prove that $f(z) = \sin x \cosh y + i \cos x \sinh y$ is differentiable at every point.

Solution. $f(z) = \sin x \cosh y + i \cos x \sinh y$

$\therefore u(x,y) = \sin x \cosh y$ and $v(x,y) = \cos x \sinh y$

$u_x = \cos x \cosh y$ and $v_x = -\sin x \sinh y$

$u_y = \sin x \sinh y$ and $v_y = \cos x \cosh y$

$\therefore$ $u_x = v_y$ and $u_y = -v_x$ for all x, y.

Hence C.R. equations are satisfied at every point.

Further all the first order partial derivatives are continuous.

Hence $f(z)$ is differentiable at every point.

[redacted] Find constants a and b so that the function $f(z) = a(x^2 - y^2) + ibxy + c$ is differentiable at every point.

Solution. Here $u(x, y) = a(x^2 - y^2) + c$ and $v(x, y) = bxy$.

$u_x = 2ax$; $v_x = by$.

$u_y = -2ay$ and $v_y = bx$.

Clearly $u_x = v_y$ and $u_y = -v_x$ iff $2a = b$.

$\therefore$ C-R equations are satisfied at all points iff $2a = b$.

$\therefore$ The function $f(z)$ is *differentiable for all values of* a, b with $2a = b$.

[redacted] Show that $f(z) = \sqrt{r}(\cos \theta/2 + i \sin \theta/2)$ where $r > 0$ and $0 < \theta < 2\pi$ is differentiable and find $f'(z)$.

Solution. $f(z) = \sqrt{r}\,(\cos \theta/2 + i \sin \theta/2)$.

$u = \sqrt{r} \cos(\theta/2)$ and $v = \sqrt{r} \sin(\theta/2)$.

$$\therefore \frac{\partial u}{\partial r} = \frac{1}{2\sqrt{r}} \cos(\theta/2) \text{ and } \frac{\partial v}{\partial r} = \frac{1}{2\sqrt{r}} \sin(\theta/2)$$

$$\frac{\partial u}{\partial \theta} = \frac{-\sqrt{r}}{2} \sin(\theta/2) \text{ and } \frac{\partial v}{\partial \theta} = \frac{\sqrt{r}}{2} \cos(\theta/2)$$

$$\text{Now, } \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r}\left(\frac{\sqrt{r}}{2} \cos(\theta/2)\right)$$

$$= \frac{1}{2\sqrt{r}} \cos(\theta/2)$$

$$= \frac{\partial u}{\partial r}$$

$$\text{Thus } \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}.$$

$$\text{Similarly } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

$$= \frac{1}{2\sqrt{r}} \sin(\theta/2).$$

Hence the C-R equations (in polar form) are satisfied.

Further all the first order partial derivatives are continuous.

Hence $f'(z)$ exists.

$$\text{Also } f'(z) = \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) \quad \text{(refer theorem 2.8)}$$

$$= \frac{r}{z}\left(\frac{1}{2\sqrt{r}} \cos(\theta/2) + \frac{i}{2\sqrt{r}} \sin(\theta/2)\right)$$

$$= \frac{r}{2\sqrt{r}z}(\cos \theta/2 + i \sin \theta/2)$$

$$= \frac{1}{2z}\left[\sqrt{r}(\cos \theta/2 + i \sin \theta/2)\right]$$

$$= \frac{1}{2z}[\sqrt{z}] = \frac{1}{2\sqrt{z}}$$

$$\text{Hence } f'(z) = \frac{1}{2\sqrt{z}}.$$

Exercises.

1. Verify C.R. equations for the following functions
   (i) $f(z) = az + b$
   (ii) $f(z) = e^z$
   (iii) $f(z) = (1/z), z \neq 0$
   (iv) $f(z) = iz + 2$
   (v) $f(z) = e^{-x}(\cos y - i \sin y)$
   (vi) $f(z) = \cos x \cosh y - i \sin x \sinh y$
   (vii) $f(z) = \sin z$
   (viii) $f(z) = ze^{-z}$

2. Prove that the following are nowhere differentiable.
   (i) $f(z) = |z|$ (ii) $f(z) = \text{Im } z$
   (iii) $f(z) = xy + iy$ (iv) $f(z) = z - \bar{z}$
   (v) $f(z) = 2x + ixy^2$

3. Prove that for the following functions the C.R. equations are satisfied at $z = 0$, but the function is not differentiable at $z = 0$.

$$\text{(i) } f(z) = \begin{cases} \dfrac{xy^2}{x^2 + y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

$$\text{(ii) } f(z) = \begin{cases} \dfrac{x^2y^5(x + iy)}{x^4 + y^{10}} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

(iii) $f(z) = \begin{cases} \dfrac{x^3y(y-ix)}{x^6+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

(iv) $f(z) = \begin{cases} \dfrac{z^5}{|z^4|} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

4. Prove that the following functions are differentiable at every point.

(i) $f(z) = (x^3 - 3xy^2) + i(3x^2y - y^3)$
(ii) $f(z) = iz + 2$
(iii) $f(z) = x^2 - y^2 - 2xy + i(x^2 - y^2 + 2xy)$
(iv) $f(z) = 2x - 3y + i(3x + 2y)$.

5. Find constants a, b and c so that the following functions are differentiable at every point.

(i) $f(z) = x + ay - i(bx + cy)$
(ii) $f(z) = x + ay + i(bx + cy)$
(iii) $f(z) = ax^2 - by^2 + icxy$
(iv) $f(z) = e^x \cos ay + ie^x \sin (y + b) + c$
(v) $f(z) = \cos x(\cosh y + a \sinh y) + i \sin x(\cosh y + b \sinh y)$

Answers.

5. (i) $a = b; c = -1$ (ii) $a = -b; c = 1$ (iii) $a = c/2 = b$ (v) $a = -1 = b$.

2.7. Analytic [redacted]

Definition. A function f defined in a region D of the complex plane is said to be **analytic at a point** $a \in D$ if f is differentiable at every point of some neighbourhood of a.

Thus f is analytic at a if there exists $\varepsilon > 0$ such that f is differentiable at every point of the disc $S(a, \varepsilon) = \{z / |z - a| < \varepsilon\}$.

If f is analytic at every point of a region D then f is said to be analytic in D.

A function which is analytic at every point of the complex plane is called an entire function or integral function.

For example any polynomial is an entire function.

[redacted] If f is analytic at a point a then f is differentiable at a. [redacted]

For example, $f(z) = |z|^2$ is differentiable only at $z = 0$. (refer example 2 in theorem 2.7). Hence f is differentiable at $z = 0$ but not analytic at $z = 0$.

Remark 2. If $f(z)$ is analytic at a then there exists $\varepsilon > 0$ such that $f(z)$ is differentiable at each point of $S(a, \varepsilon)$. Now, let $z \in S(a, \varepsilon)$. Then we can find $\delta > 0$ such that $S(z, \delta) \subseteq S(a, \varepsilon)$. Hence f is differentiable at every point of $S(z, \delta)$ so that f is analytic at z.

Thus f is analytic at every point of $S(a, \varepsilon)$. Hence f is analytic at a and if only if f is analytic at each point of some neighbourhood of a. Hence the set of all points for which a given function is analytic forms an open set.

In particular, if a function is analytic in an arbitrary subset A of the complex plane then there exists an open set containing A in which the function is analytic.

[redacted] We shall later prove that if $f(z)$ is analytic at a point then $f(z)$ has derivatives of all orders at that point. In particular

$$f'(z) = u_x(x, y) + iv_x(x, y) = v_y(x, y) - iu_y(x, y)$$

is further differentiable and hence $f'(z)$ is continuous.

Hence u_x, v_x, u_y, v_y are all continuous.

This together with Theorem 2.6 gives the following result.

$f(z)$ is analytic in a region D if and only if the real and imaginary parts of $f(z)$ have continuous first order partial derivatives that satisfy the Cauchy-Riemanns equations for all points in D.

Further it follows that if $f(z)$ is analytic in D then u and v have continuous partial derivatives of all orders.

[redacted]0. An analytic function in a region D with its derivative zero at every point of the domain is a constant.

Proof. Let $f(z) = u(x, y) + iv(x, y)$ be analytic in D and $f'(z) = 0$ for all $z \in D$.

Since $f'(z) = u_x + iv_x = v_y - iu_y$ we have $u_x = u_y = v_x = v_y = 0$.

$\therefore$ $u(x, y)$ and $v(x, y)$ are constant functions and hence $f(z)$ is constant.

Remark 4. The above theorem is not true if the domain of $f(z)$ is not a region.

For example let $D = \{z / |z| < 1\} \cup \{z / |z| > 2\}$

D is not a connected subset of C so that D is not a region.

Let $f : D \to C$ be defined by

$$f(z) = \begin{cases} 1 & \text{if } |z| < 1 \\ 2 & \text{if } |z| > 2 \end{cases}$$

Clearly $f'(z) = 0$ for all points $z \in D$ and f is not a constant function in D.

Solved problems

Problem 1. An analytic function in a region with constant modulus is constant.

Solution. Let $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain D.

Since $|f(z)|$ is constant, we have $u^2+v^2=c$ where c is a constant.

Differentiating partially with respect to x we get $2uu_x+2vv_x=0$.

$$\text{(i.e)}\ uu_x+vv_x=0 \quad \dots (1)$$

Similarly differentiating partially with respect to y we get

$$uu_y+vv_y=0 \quad \dots (2)$$

Using C.R equations in (1) and (2) we get

$$uu_x-vu_y=0 \quad \dots (3)$$

$$uu_y+vu_x=0 \quad \dots (4)$$

Eliminating u_y from (3) and (4) we get $(u^2+v^2)u_x=0$.

Since $u^2+v^2=c$ we get $u_x=0$.

Similarly we can prove that $v_x=0$ so that $f'(z)=u_x+iv_x=0$.

Hence f is constant.

[redacted] Any analytic function $f(z)=u+iv$ with $\arg f(z)$ constant is itself a constant function.

Solution. $\arg f(z)=\tan^{-1}(v/u)=c$, where c is a constant.

$\therefore\ \frac{v}{u}=k$ where k is a constant.

$\therefore\ v=ku$.

Hence $v_x=ku_x$ and $v_y=ku_y$.

Eliminating k from the above equations we get $u_xv_y=v_xu_y$

$\therefore\ u_xv_y-u_yv_x=0$.

$\therefore\ u_x^2+u_y^2=0$ (using C.R. equations)

$\therefore\ u_x=0$ and $u_y=0$ and hence u is constant.

Similarly we can prove that v is constant.

$\therefore\ f=u+iv$ is constant.

Problem 3. If $f(z)$ and $\overline{f(z)}$ are analytic in a region D show that $f(z)$ is constant in that region.

Solution. Let $f(z)=u(x,y)+iv(x,y)$.

$$\therefore\ \overline{f(z)}=u(x,y)-iv(x,y)$$

$$=u(x,y)+i[-v(x,y)]$$

Since $f(z)$ is analytic in D we have $u_x=v_y$ and $u_y=-v_x$.

Since $\overline{f(z)}$ is analytic in D we have $u_x=-v_y$ and $u_y=v_x$.

Adding we get $u_x=0$ and $u_y=0$.

Hence $u_x=0=v_x$

$\therefore\ f'(z)=u_x+iv_x=0$

$\therefore\ f(z)$ is constant in D.

Problem 4. Prove that the functions $f(z)$ and $\overline{f(\bar z)}$ are simultaneously analytic.

Solution. Suppose $f(z)=u(x,y)+iv(x,y)$ is analytic in a region D.

Then the first order partial derivatives of u and v are continuous and satisfy the C-R equations

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad \dots (1)$$

$$\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y} \quad \dots (2)$$

Now, $\overline{f(\bar z)}=u(x,-y)-iv(x,-y)$.

$=u_1(x,y)+iv_1(x,y)$ where $u_1(x,y)=u(x,-y)$

and $v_1(x,y)=-v(x,-y)$.

$$\text{Hence } \frac{\partial u_1}{\partial x}=\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=\frac{\partial v_1}{\partial y} \text{ (using 1)}$$

$$\text{and } \frac{\partial u_1}{\partial y}=-\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}=-\frac{\partial v_1}{\partial x}$$

$\therefore$ The first order partial derivatives of u_1 and v_1 are continuous and satisfy the Cauchy-Riemann equations in D.

Hence $\overline{f(\bar z)}$ is analytic in D.

Similarly if $\overline{f(\bar z)}$ is analytic in D then $f(z)$ is also analytic in D.

Problem 5. If $\frac{\partial^2}{\partial x\partial y}=\frac{\partial^2}{\partial y\partial x}$ prove that $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=4\frac{\partial^2}{\partial z\partial \bar z}$

Solution. Let $z=x+iy$

$$\therefore\ x=\frac{1}{2}(z+\bar z) \text{ and } y=\frac{1}{2i}(z-\bar z)$$

$$\text{Hence } \frac{\partial}{\partial \bar z}=\frac{\partial}{\partial x}\frac{\partial x}{\partial \bar z}+\frac{\partial}{\partial y}\frac{\partial y}{\partial \bar z}$$

$$=\frac{1}{2}\frac{\partial}{\partial x}-\frac{1}{2i}\frac{\partial}{\partial y}$$

$$=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$$

$$\therefore \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{2}\left[\left(\frac{\partial^2}{\partial x^2} + i\frac{\partial^2}{\partial x \partial y}\right)\left(\frac{1}{2}\right) + \left(\frac{\partial^2}{\partial y \partial x} + i\frac{\partial^2}{\partial y^2}\right)\left(\frac{1}{2i}\right)\right]$$

$$= \frac{1}{4}\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + i\cdot\frac{\partial^2}{\partial x \partial y} + \frac{1}{i}\frac{\partial^2}{\partial y \partial x}\right]$$

$$= \frac{1}{4}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x \partial y}\left(i + \frac{1}{i}\right)\right]$$

$$= \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

$$\therefore \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}.$$

Exercises.

1. Prove that an analytic function whose real part is constant is itself a constant.
2. Prove that an analytic function whose imaginary part is constant is itself a constant.
3. If $f = u + iv$ is analytic in a region D and uv is constant in D then prove that f reduces to a constant.
4. If $f = u + iv$ is analytic in a region D and $v = u^2$ in D then prove that f reduces to a constant.
5. Determine the constants a and b in order that the function $f(z) = (x^2 + ay^2 - 2xy) + i(bx^2 - y^2 + 2xy)$ should be analytic. Find $f'(z)$.
6. Test whether the following functions are analytic
   (i) $z^3 + z$ (ii) $e^x(\cos y + i \sin y)$
   (iii) $e^x(\cos y - i \sin y)$ (iv) $e^{-x}(\cos y - i \sin y)$

Answers.

5. $a = -1$; $b = 1$; $f'(z) = (1+i)z^2$ 6. (i) yes (ii) yes (iii) no (iv) yes.

2.8. Harmonic Functions

Definition. Let $u(x, y)$ be a function of two real variables x and y defined in a region D. $u(x, y)$ is said to be a **harmonic function** if $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and this equation is called **Laplace's equation**.

Theorem 2.11. The real and imaginary parts of an analytic function are harmonic functions.

Proof. Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function.

Then u and v have continuous partial derivatives of first order which satisfy the C.R equations given by $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Further $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ and $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$.

$$\text{Now } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)$$

$$= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x}$$

$$= 0.$$

Thus u is a harmonic function.

Similarly we can prove that v is a harmonic function.

Remark 1. Laplace's equation provides a necessary condition for a function to be the real or imaginary part of an analytic function.

For example if $u(x, y) = x^2 + y$ we have

$$\frac{\partial^2 u}{\partial x^2} = 2; \ \frac{\partial^2 u}{\partial y^2} = 0 \text{ and } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2.$$

Thus $u(x, y)$ is not harmonic function and hence it cannot be the real part of any analytic function.

Definition. Let $f = u + iv$ be an analytic function in a region D. Then v is said to be a **conjugate harmonic** function of u.

Theorem 2.12. Let $f = u + iv$ be an analytic function in a region D. Then v is a harmonic conjugate of u if and only if u is a harmonic conjugate of $-v$.

Proof. Let v be a harmonic conjugate of u. Then $f = u + iv$ is analytic.

$\therefore$ $if = iu - v$ is also analytic.

Hence u is a harmonic conjugate of $-v$.

The proof for the converse is similar.

Theorem 2.13. Any two harmonic conjugates of a given harmonic function u in a region D differ by a real constant.

Proof. Let u be a harmonic function.

Let v and v^* be two harmonic conjugates of u.

$u + iv$ and $u + iv^*$ are analytic in D.

Hence by the Cauchy-Riemann equation we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial v^*}{\partial y}$$

$$\text{and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -\frac{\partial v^*}{\partial x}$$

$$\therefore \frac{\partial v}{\partial y} = \frac{\partial v^*}{\partial y} \text{ and } \frac{\partial v}{\partial x} = \frac{\partial v^*}{\partial x}$$

Hence $\frac{\partial}{\partial y}(v - v^*) = 0$ and $\frac{\partial}{\partial x}(v - v^*) = 0$.

$\therefore$ $v = v^* + c$ where c is a real constant.

Remark. The Cauchy-Riemann equations can be used to obtain a harmonic conjugate of a given harmonic function.

For example, let $u(x, y) = x^2 - y^2$.

Then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$ so that u is harmonic in the whole complex plane C.

Now let $v(x, y)$ be a harmonic conjugate of u.

$$\text{Then } \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x \quad \ldots (1)$$

$$\text{and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -2y \quad \ldots (2)$$

On integration of (1) with respect to y we get $v = 2xy + \varphi(x)$ where $\varphi(x)$ is a function of x alone.

Now from (2) $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ gives $2y + \varphi'(x) = 2y$

$\therefore$ $\varphi'(x) = 0$ so that $\varphi(x) = c$ (a constant).

$\therefore$ $v = 2xy + c$.

Thus the harmonic conjugate of $u(x, y) = x^2 - y^2$ is given by $v(x, y) = 2xy + c$ and the corresponding entire function is given by

$$f(z) = (x^2 - y^2) + i(2xy + c)$$
$$= z^2 + ic$$

Let $u(x, y)$ and $v(x, y)$ be given harmonic functions. We now describe a method, due to Milne-Thompson, of constructing an analytic function whose real part is $u(x, y)$ or imaginary part is $v(x, y)$.

Milne-Thompson method

Let $u(x, y)$ be a given harmonic function.

Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function

$$\text{Then } f'(z) = u_x(x, y) + iv_x(x, y)$$
$$= u_x(x, y) - iu_y(x, y).$$

Let $\varphi_1(x, y) = u_x(x, y)$ and $\varphi_2(x, y) = u_y(x, y)$.

We have $x = \frac{z + \bar{z}}{2}$ and $y = \frac{z - \bar{z}}{2i}$

Hence $f'(z) = \varphi_1\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) - i\varphi_2\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right)$.

Putting $z = \bar{z}$ we obtain $f'(z) = \varphi_1(z, 0) - i\varphi_2(z, 0)$.

Hence $f(z) = \int [\varphi_1(z, 0) - i\varphi_2(z, 0)]dz + c$.

Note. It can be proved in a similar way that the analytic function $f(z)$ with a given harmonic function $v(x, y)$ as imaginary part is given by

$$f(z) = \int [\psi_1(z, 0) + i\psi_2(z, 0)]dz + c$$

where $\psi_1(x, y) = v_y$ and $\psi_2(x, y) = v_x$.

Solved Problems

Problem 1. Prove that $u = 2x - x^3 + 3xy^2$ is harmonic and find its harmonic conjugate. Also find the corresponding analytic function.

Solution. $u = 2x - x^3 + 3xy^2$.

$\therefore$ $u_x = 2 - 3x^2 + 3y^2$; $u_{xx} = -6x$; $u_y = 6xy$; $u_{yy} = 6x$.

$\therefore$ $u_{xx} + u_{yy} = 0$. Hence u is harmonic.

Let v be a harmonic conjugate of u.

$\therefore$ $f(z) = u + iv$ is analytic.

By Cauchy-Riemann equations we have

$$v_y = u_x = 2 - 3x^2 + 3y^2.$$

$\therefore$ Integrating with respect to y we get

$$v = 2y - 3x^2y + y^3 + \lambda(x) \quad \ldots (1)$$

where $\lambda(x)$ is an arbitrary function of x.

$\therefore$ $v_x = -6xy + \lambda'(x)$.

Now $v_x = -u_y$ gives $-6xy + \lambda'(x) = -6xy$.

Hence $\lambda'(x) = 0$ so that $\lambda(x) = c$ where c is a constant.

Thus $v = 2y - 3x^2y + y^3 + c$ [from (1)].

$$\text{Now } f(z) = (2x - x^3 + 3xy^2) + i(2y - 3x^2y + y^3) + ic$$
$$= 2(x+iy) - [(x^3 - 3xy^2) + i(3x^2y - y^3)] + ic$$
$$= 2z - z^3 + ic.$$

$\therefore f(z) = 2z - z^2 + ic$ is the required analytic function.

Problem 2. Show that $u = \log\sqrt{x^2+y^2}$ is harmonic and determine its conjugate and hence find the corresponding analytic function $f(z)$.

Solution. $u = \log\sqrt{x^2+y^2} = \frac{1}{2}\log(x^2+y^2)$.

$$\therefore u_x = \frac{x}{x^2+y^2};\ u_{xx} = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

Similarly $u_{yy} = \dfrac{x^2-y^2}{(x^2+y^2)^2}$.

Obviously $u_{xx} + u_{yy} = 0$ and hence u is harmonic.

Let v be a harmonic conjugate of u.

$\therefore f(z) = u + iv$ is an analytic function.

By C.R. equations we have

$$v_y = u_x = \frac{x}{x^2+y^2}.$$

Integrating w.r.t y we get $v = \tan^{-1}\left(\frac{y}{x}\right) + \phi(x)$ where $\phi(x)$ is an arbitrary function of x.

$$\text{Now } v_x = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) + \phi'(x).$$

$$\text{Also } v_x = -u_y \Rightarrow \frac{-y}{x^2+y^2} + \phi'(x) = \frac{-y}{x^2+y^2} \text{ so that } \phi'(x) = 0.$$

Hence $\phi(x) = c$.

$$\therefore v = \tan^{-1}\left(\frac{y}{x}\right) + c$$

$$\therefore f(z) = u + iv = \log\sqrt{x^2+y^2} + i\left[\tan^{-1}\left(\frac{y}{x}\right) + c\right].$$

Problem 3. Show that

$$u(x,y) = \sin x\cosh y + 2\cos x\sinh y + x^2 - y^2 + 4xy$$

is harmonic. Find an analytic function $f(z)$ in terms of z with the given u for its real part.

Solution.
$$u_x = \cos x\cosh y - 2\sin x\sinh y + 2x + 4y.$$
$$u_{xx} = -\sin x\cosh y - 2\cos x\sinh y + 2.$$
$$u_y = \sin x\sinh y + 2\cos x\cosh y - 2y + 4x.$$
$$u_{yy} = \sin x\cosh y + 2\cos x\sinh y - 2.$$

$\therefore u_{xx} + u_{yy} = 0$. Hence u is harmonic.

Now let $\phi_1(x,y) = u_x$ and $\phi_2(x,y) = u_y$.

$$\therefore \phi_1(z,0) = \cos z\cosh 0 - 2\sin z\sinh 0 + 2z$$
$$= \cos z + 2z.$$

Similarly $\phi_2(z,0) = 2\cos z + 4z$.

$$\therefore f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]dz \text{ (by Milne Thompson method)}$$
$$= \int [\cos z + 2z - i(2\cos z + 4z)]dz$$
$$= \sin z + z^2 - 2i\sin z - 2iz^2 + c.$$

Problem 4. If $f(z) = u(x,y) + iv(x,y)$ is an analytic function and $u(x,y) = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$, find $f(z)$.

Solution. It can be verified that $u(x,y)$ is harmonic.

$$\text{Now, } u_x = \frac{(\cosh 2y + \cos 2x)2\cos 2x + 2\sin^2 2x}{(\cosh 2y + \cos 2x)^2}$$
$$= \frac{2\cosh 2y\cos 2x + 2}{(\cosh 2y + \cos 2x)^2}.$$

$$\text{Also, } u_y = \frac{-2\sin 2x\sinh 2y}{(\cosh 2y + \cos 2x)^2}.$$

Let $\phi_1(x,y) = u_x$ and $\phi_2(x,y) = u_y$.

$$\therefore \phi_1(z,0) = \frac{2\cos 2z\cosh 0 + 2}{(\cosh 0 + \cos 2z)^2} = \frac{2}{1+\cos 2z} = \sec^2 z$$

and $\phi_2(z,0) = 0$.

$$\text{Now, } f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]dz$$
$$= \int \sec^2 z\, dz,$$
$$= \tan z + c.$$
$$\therefore\ f(z) = \tan z + c.$$

Problem 5. Find the analytic function $f(z) = u + iv$ if $u + v = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$.

Solution.

$$u + v = \frac{\sin 2x}{\cosh 2y - \cos 2x} \quad \ldots (1)$$

$$\therefore\ u_x + v_x = \frac{2(\cosh 2y - \cos 2x)\cos 2x - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2} \quad \ldots (2)$$

$$\text{and } u_y + v_y = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}. \quad \ldots (3)$$

Since the required function $f(z) = u + iv$ is to be analytic, u and v satisfy the C.R. equations $u_x = v_y$ and $u_y = -v_x$.

Using these equations in (2) we get

$$u_x - u_y = \frac{2(\cosh 2y - \cos 2x)\cos 2x - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2}.$$

$$\therefore\ u_x(z,0) - u_y(z,0) = \frac{2(1 - \cos 2z)\cos 2z - 2\sin^2 2z}{(1 - \cos 2z)^2}$$

$$= \frac{2\cos 2z - 2(\cos^2 2z + \sin^2 2z)}{(1 - \cos 2z)^2}$$

$$= \frac{-2(1 - \cos 2z)}{(1 - \cos 2z)^2}$$

$$= \frac{-2}{2\sin^2 z} = -\operatorname{cosec}^2 z. \quad \ldots (4)$$

Using C.R. equations in (3) we get

$$u_y + u_x = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}.$$

$$\therefore\ u_y(z,0) + u_x(z,0) = 0 \quad \ldots (5)$$

Now adding (4) and (5) we get $2u_x(z,0) = -\operatorname{cosec}^2 z$.

$$\therefore\ u_x(z,0) = -\frac{1}{2}\operatorname{cosec}^2 z \quad \ldots (6)$$

Subtracting (4) from (5) we get $2u_y(z,0) = \operatorname{cosec}^2 z$.

$$\therefore\ u_y(z,0) = \frac{1}{2}\operatorname{cosec}^2 z \quad \ldots (7)$$

$$\text{Now } f(z) = u(z,0) + iv(z,0)$$
$$\Rightarrow f'(z) = u_x(z,0) + iv_x(z,0)$$
$$= u_x(z,0) - iu_y(z,0)$$
$$= -\frac{1}{2}(1+i)\operatorname{cosec}^2 z \text{ [using (6) and (7)]}$$

Integrating w.r.t z we have

$$f(z) = \left(\frac{1+i}{2}\right)\cot z + c.$$

Problem 6. Given $v(x,y) = x^4 - 6x^2y^2 + y^4$ find $f(z) = u(x,y) + iv(x,y)$ such that $f(z)$ is analytic.

Solution. It can be easily verified that $v(x,y)$ is harmonic.

Now, $v_x = 4x^3 - 12xy^2$ and $v_y = -12x^2y + 4y^3$.

Let $f(z) = u + iv$ be the required analytic function.

By Cauchy-Riemann equations $u_x = v_y$.

$\therefore\ u_x = -12x^2y + 4y^3$.

$\therefore$ Integrating with respect to x we get $u = -4x^3y + 4xy^3 + \lambda(y)$ where $\lambda(y)$ is an arbitrary function of y.

$$\therefore\ u_y = -4x^3 + 12xy^2 + \lambda'(y) = -v_x.$$
$$\therefore\ -(4x^3 - 12xy^2) = -4x^3 + 12xy^2 + \lambda'(y).$$
$$\therefore\ \lambda'(y) = 0 \text{ so that } \lambda(y) = c \text{ where } c \text{ is a constant.}$$

Thus $u = -4x^3y + 4xy^3 + c$.

$$\therefore\ f(z) = (-4x^3y + 4xy^3 + c) + i(x^4 - 6x^2y^2 + y^4)$$
$$= i[(x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3)] + c$$
$$= i(x + iy)^4 + c$$
$$= iz^4 + c.$$

Aliter (*Milne Thompson Method*).

Let $\psi_1(x, y) = v_y$ and $\psi_2(x, y) = v_x$.

$\therefore\ \psi_1(x, y) = -12x^2y + 4y^3$ and $\psi_2(x, y) = 4x^3 - 12xy^2$.

$\therefore\ \psi_1(z, 0) = 0$ and $\psi_2(z, 0) = 4z^3$.

$$\therefore\ f(z) = \int [\psi_1(z, 0) + i\psi_2(z, 0)]dz$$
$$= i\int 4z^3\, dz$$
$$= iz^4 + c.$$

Problem 7. Find the analytic function $f(z) = u + iv$ given that $u - v = e^x(\cos y - \sin y)$.

Solution. $u - v = e^x(\cos y - \sin y)$. ... (1)

$\therefore\ u_x - v_x = e^x(\cos y - \sin y)$ ... (2)

and $u_y - v_y = -e^x(\sin y + \cos y)$. ... (3)

Since the required function is to be analytic it has to satisfy the C.R. equations.

$\therefore$ Using C.R. equations in (3) we get

$$-v_x - u_x = -e^x(\sin y + \cos y). \quad \text{... (4)}$$

Solving (2) and (4) we get

$$u_x = e^x \cos y \quad \text{... (5)}$$
$$\text{and } v_x = e^x \sin y \quad \text{... (6)}$$

Integrating (6) with respect to x we get

$$v = e^x \sin y + f(y)$$
$$\therefore\ v_y = e^x \cos y + f'(y) \quad \text{... (7)}$$

Using C.R. equations in (5) and (7) we get $f'(y) = 0$.

Hence $f(y) = c_1$ where c_1 is a constant.

$\therefore\ v = e^x \sin y + c_1$.

From (1) $u = e^x \cos y + c_2$.

$$\text{Now, } f(z) = u + iv$$
$$= e^x(\cos y + ie^x \sin y + c_1 + ic_2$$
$$= e^x(\cos y + i\sin y) + (c_1 + ic_2)$$
$$= e^x e^{iy} + \alpha \text{ (where } \alpha \text{ is a complex constant).}$$
$$= e^{x+iy} + \alpha$$
$$= e^z + \alpha.$$

Problem 8. If $u + v = (x - y)(x^2 + 4xy + y^2)$ and $f(z) = u + iv$ find the analytic function $f(z)$ in terms of z.

Solution. $u + v = (x - y)(x^2 + 4xy + y^2)$ ... (1)

Differentiating (1) partially w.r.t x we get

$$u_x + v_x = (x^2 + 4xy + y^2) + (x - y)(2x + 4y). \quad \text{... (2)}$$

Differentiating (1) partially w.r.t y we get

$$u_y + v_y = -(x^2 + 4xy + y^2) + (x - y)(4x + 2y). \quad \text{... (3)}$$

Since $f = u + iv$ is analytic, u and v satisfy the C.R. equations

$$u_x = v_y \text{ and } u_y = -v_x.$$

$\therefore$ Using C.R. equations in (3) we get

$$-v_x + u_x = -(x^2 + 4xy + y^2) + (x - y)(4x + 2y). \quad \text{... (4)}$$

Adding (2) and (4) we get $2u_x = (x - y)(6x + 6y)$.

$$\therefore\ u_x = 3(x^2 - y^2). \quad \text{... (5)}$$

Substracting (4) from (2) we get $v_x = 6xy$. ... (6)

Using C.R. equations in (6) we get $u_y = -6xy$. ... (7)

Let $\phi_1(x, y) = u_x$ and $\phi_2(x, y) = u_y$.

$\therefore\ \phi_1(z, 0) = 3z^2$ and $\phi_2(z, 0) = 0$.

By Milne-Thompson method

$$f(z) = \int [\phi_1(z, 0) - i\phi_2(z, 0)]dz = \int 3z^2 dz = z^3 + c.$$

Problem 9. Find the real part of the analytic function whose imaginary part is $e^{-x}[2xy\cos y + (y^2 - x^2)\sin y]$. Construct the analytic function.

Solution. Let $v = e^{-x}[2xy\cos y + (y^2 - x^2)\sin y]$ and $f(z) = u + iv$ be the required analytic function.

We can prove that v is harmonic. We use Milne Thompson method to find the harmonic conjugate u of v.

Let $\psi_1(x, y) = v_y = e^{-x}(2x\cos y - 2xy\sin y + 2y\sin y + (y^2 - x^2)\cos y)$

and $\psi_2(x, y) = v_x = e^{-x}(-2xy\cos y - (y^2 - x^2)\sin y + 2y\cos y - 2x\sin y)$.

$\therefore\ \psi_1(z, 0) = e^{-z}(2z - z^2)$ and $\psi_2(z, 0) = 0$.

By Milne Thompson method

$$f(z) = \int [\psi_1(z,0) + i\psi_2(z,0)]dz$$
$$= \int e^{-z}(2z - z^2)dz$$
$$= \int 2ze^{-z}dz - \left[-z^2e^{-z} + \int e^{-z}2z\,dz\right]$$
$$= z^2e^{-z}$$
$$= (x+iy)^2e^{-(x+iy)}$$
$$= \left[(x^2-y^2) + 2ixy\right]e^{-x}(\cos y - i\sin y).$$
$$= e^{-x}\left[(x^2-y^2) + 2ixy\right](\cos y - i\sin y).$$

Real part of $f(z) = e^{-x}\left[(x^2-y^2)\cos y + 2xy\sin y\right]$.

(i.e.) $u = e^{-x}\left[(x^2-y^2)\cos y + 2xy\sin y\right]$.

Problem 10. Find the constant a so that $u(x, y) = ax^2 - y^2 + xy$ is harmonic. Find an analytic function $f(z)$ for which u is the real part. Also find its harmonic conjugate.

Solution. $u = ax^2 - y^2 + xy$.

Given that u is harmonic. Hence it satisfies Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

$$\text{Now, } \frac{\partial u}{\partial x} = 2ax + y \text{ and } \frac{\partial^2 u}{\partial x^2} = 2a;$$
$$\frac{\partial u}{\partial y} = -2y + x \text{ and } \frac{\partial^2 u}{\partial y^2} = -2.$$
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow 2a - 2 = 0.$$

Hence $a = 1$.

$$\therefore\ u = x^2 - y^2 + xy.$$

Hence $u_x = 2x + y$ and $u_y = -2y + x$.

Let $\phi_1(x, y) = u_x = 2x + y$ and $\phi_2(x, y) = u_y = -2y + x$.

$$\therefore\ \phi_1(z,0) = 2z \text{ and } \phi_2(z,0) = z.$$

$$\therefore\ f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\,dz$$
$$= \int (2z - iz)dz$$
$$= z^2 - \frac{iz^2}{2} + c$$
$$= (x+iy)^2 - i\frac{(x+iy)^2}{2} + c$$
$$= (x^2 - y^2 + 2ixy) - \frac{i}{2}(x^2 - y^2 + 2ixy) + c$$
$$= (x^2 - y^2 + xy) + i\left(2xy + \frac{y^2 - x^2}{2}\right) + c.$$

$\therefore\ v(x, y) = 2xy + \left(\frac{y^2 - x^2}{2}\right)$ is the harmonic conjugate of $u(x, y)$.

Problem 11. If $u(x, y)$ is a harmonic function in a region D prove that $f(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ is analytic in D.

Solution. Let $U = \frac{\partial u}{\partial x}$ and $V = -\frac{\partial u}{\partial y}$.

$\therefore\ f(z) = U + iV$. Since u is harmonic U and V have continuous first order partial derivatives and $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$ (1)

$$\text{Also } \frac{\partial U}{\partial x} = \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \quad \text{[using (1)]}$$
$$= \frac{\partial V}{\partial y}.$$
$$\text{Hence } \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}.$$
$$\text{Now, } \frac{\partial U}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = -\frac{\partial V}{\partial x}.$$
$$\text{Hence } \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}.$$

Thus the partial derivatives of U and V satisfy the Cauchy-Riemann equations. Hence f is analytic in D.

Problem 12. If u and v are harmonic functions satisfying the Cauchy-Riemann equations in a region D then $f = u + iv$ is analytic in D.

Solution. Since u and v are harmonic the first order partial derivatives of u and v are continuous. Also u and v satisfy the C.R. equations in D.
Hence $f = u + iv$ is analytic in D.

Problem 13. Prove that the real (imaginary) part of an analytic function when expressed in polar form satisfies the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

(This equation is the Laplace equation in polar form.)

Solution. We know that Cauchy-Riemann equations in polar form are given by

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad (1)$$

$$\text{and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}. \qquad (2)$$

We eliminate v from (1) and (2).
Differentiating (1) partially with respect to r and (2) partially with respect to θ we have

$$\frac{\partial^2 v}{\partial r \partial \theta} = r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} \qquad (3)$$

$$\frac{\partial^2 v}{\partial \theta \partial r} = -\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2}. \qquad (4)$$

Since $\frac{\partial^2 v}{\partial r \partial \theta} = \frac{\partial^2 v}{\partial \theta \partial r}$ we have $r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} = -\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2}$.

$\therefore \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$. Similarly $\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$.

Problem 14. ϕ and ψ are functions of x and y satisfying Laplace's equation. If $u = \phi_y - \psi_x$ and $v = \phi_x + \psi_y$ prove that $u + iv$ is analytic.

Solution. Given that ϕ and ψ satisfy Laplace's equation.

Hence $\phi_{xx} + \phi_{yy} = 0 \qquad (1)$

and $\psi_{xx} + \psi_{yy} = 0. \qquad (2)$

Also $u = \phi_y - \psi_x$ and $v = \phi_x + \psi_y$.

$$\text{Hence } u_x = \phi_{xy} - \psi_{xx}$$
$$u_y = \phi_{yy} - \psi_{yx}$$

$$v_x = \phi_{xx} + \psi_{xy}$$
$$= -\phi_{yy} + \psi_{xy} \text{ [by (1)]}$$
$$\text{and } v_y = \phi_{yx} + \psi_{yy}$$
$$= \phi_{yx} - \psi_{xx} \text{ [by (2)]}$$

Thus $u_x = v_y$ and $u_y = -v_x$.

Since ϕ and ψ are harmonic, all the partial derivatives are continuous.

Hence $u + iv$ is analytic.

Problem 15. Show that if u and v are conjugate harmonic functions the product uv is a harmonic function.

Solution. Since u and v are conjugate harmonic functions we have

$$u_{xx} + u_{yy} = 0 \qquad \ldots (1)$$
$$v_{xx} + v_{yy} = 0 \qquad \ldots (2)$$
$$u_x = v_y \qquad \ldots (3)$$
$$u_y = -v_x. \qquad \ldots (4)$$

Now let $\phi = uv$.

$$\phi_x = uv_x + vu_x.$$
$$\phi_{xx} = uv_{xx} + 2u_x v_x + vu_{xx}$$
$$\text{Similarly } \phi_{yy} = uv_{yy} + 2u_y v_y + vu_{yy}$$
$$= uv_{yy} - 2v_x u_x + vu_{yy} \text{ [using (3) and (4)].}$$
$$\text{Now } \phi_{xx} + \phi_{yy} = u(v_{xx} + v_{yy}) + v(u_{xx} + u_{yy})$$
$$= 0 \text{ [using (1) and (2)].}$$

$\therefore$ $\phi = uv$ is a harmonic function.

Problem 16. If $f(z)$ is analytic prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2.$$

Solution. Let $f(z) = u + iv$.

$\therefore$ $|f(z)|^2 = u^2 + v^2 = \phi$ (say) and $f'(z) = u_x + iv_x$.

$$\therefore \frac{\partial \phi}{\partial x} = 2uu_x + 2vv_x.$$

$$\therefore \frac{\partial^2 \phi}{\partial x^2} = 2[u_x^2 + uu_{xx} + v_x^2 + vv_{xx}] \qquad \ldots (1)$$

Similarly $\dfrac{\partial^2\phi}{\partial y^2} = 2[u_y^2 + uu_{yy} + v_y^2 + vv_{yy}] = 2[v_x^2 + uu_{yy} + u_x^2 + vv_{yy}]$ ... (2)

Since u and v are harmonic

$$u_{xx} + u_{yy} = 0 \text{ and } v_{xx} + v_{yy} = 0 \qquad \text{... (3)}$$

Adding (1) and (2) using (3) we get

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 4\left(u_x^2 + v_x^2\right)$$
$$= 4\,|u_x + iv_x|^2$$
$$= 4\,|f'(z)|^2$$

Hence the result.

Problem 17. If $f(z) = u + iv$ is analytic and $f(z) \neq 0$, prove that

(i) $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) \log|f(z)| = 0.$

(ii) $\nabla^2 \text{ amp } f(z) = 0.$

Solution. $\log f(z) = \log|f(z)| + i \text{ amp } f(z).$

Since $f(z) \neq 0$, $\log|f(z)|$ exists.

Further since $f(z)$ is analytic and $f(z) \neq 0$, $\log f(z)$ is also analytic.

$\therefore$ $\log|f(z)|$ and amp $f(z)$ are the real and imaginary parts of the analytic function $\log f(z)$.

Hence both $\log|f(z)|$ and amp $f(z)$ satisfy the Laplace equation.

(i) $\dfrac{\partial^2}{\partial x^2}(\log|f(z)|) + \dfrac{\partial^2}{\partial y^2}(\log|f(z)|) = 0$

(i.e.) $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) \log|f(z)| = 0.$

(ii) Also, $\dfrac{\partial^2}{\partial x^2}(\text{amp } f(z)) + \dfrac{\partial^2}{\partial y^2}(\text{amp } f(z)) = 0$

(i.e.) $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) \text{amp } f(z) = 0$

(i.e.) $\nabla^2 \text{ amp } f(z) = 0.$

Problem 18. Given the function $w = z^3$ where $w = u + iv$. Show that u and v satisfy the Cauchy-Riemann equations. Prove that the families of curves $u = c_1$ and $v = c_2$ (c_1 and c_2 are constants) are orthogonal to each other.

Solution. $w = z^3 = (x + iy)^3$
$$= (x^3 - 3xy^2) + i(3x^2y - y^3).$$

$\therefore$ $u = x^3 - 3xy^2$ and $v = 3x^2y - y^3$

$u_x = 3x^2 - 3y^2$ and $u_y = -6xy$;

$v_x = 6xy$ and $v_y = 3x^2 - 3y^2$.

We note that $u_x = v_y$ and $u_y = -v_x$.

Hence u and v satisfy the Cauchy-Riemann equations.

Now $u_{xx} = 6x$ and $u_{yy} = -6x$.

$\therefore$ $u_{xx} + u_{yy} = 6x - 6x = 0.$

Hence u satisfies the Laplace equations.

Similarly $v_{xx} + v_{yy} = 6y - 6y = 0.$

Hence v satisfies the Laplace equations.

$$u = c_1 \Rightarrow x^3 - 3xy^2 = c_1$$

Differentiating w.r.t x we get $3x^2 - 3\left(2xy\dfrac{dy}{dx} + y^2\right) = 0.$

$$\therefore \frac{dy}{dx} = \frac{3(x^2 - y^2)}{6xy} = \frac{x^2 - y^2}{2xy}.$$

$\therefore$ Slope of the tangent at (x_0, y_0) for the curve $u = c_1$ is given by $m_1 = \dfrac{x_0^2 - y_0^2}{2x_0y_0}$.

Now $v = c_2 \Rightarrow 3x^2y - y^3 = c_2$.

Differentiating w.r.t x we get

$$3\left(2xy + x^2\frac{dy}{dx}\right) - 3y^2\frac{dy}{dx} = 0. \text{ Hence } \frac{dy}{dx}(3x^2 - 3y^2) = -6xy.$$

$$\therefore \frac{dy}{dx} = \frac{-2xy}{x^2 - y^2}.$$

Slope of the tangent to the curve $u = c_2$ at (x_0, y_0) is given by $m_2 = \dfrac{-2x_0y_0}{x_0^2 - y_0^2}$.

Clearly, $m_1m_2 = -1$.

$\therefore$ The two families of curves are orthogonal.

Exercises.

1. Prove that the following functions are harmonic. Also find a harmonic conjugate.

 (i) $u = \sinh x \sin y$

 (ii) $u = 3x^2y + 2x^2 - y^3 - 2y^2$

 (iii) $u = e^x \cos y$.

2. Prove that the following functions are harmonic. Also find a function v such that $f(z) = u + iv$ is analytic and express $f(z)$ in terms of z.

 (i) $u = 2x(1-y)$ (ii) $u = e^x(x\cos y - y \sin y)$

 (iii) $u = 2xy + 3y$ (iv) $u = \dfrac{y}{x^2+y^2}$.

3. Find the function $f(z) = u + iv$ such that $f(z)$ is analytic given that

 (i) $u = x$ (ii) $u = e^x \cos y$

 (iii) $u = x^3 - 3xy^2$ (iv) $u = e^x \sin y$

 (v) $u = \cos x \cosh y$ (vi) $u = e^x(x\cos y - y\sin y)$

 (vii) $u = \dfrac{2\cos x \cosh y}{\cos 2x + \cosh 2y}$ (viii) $v = 3x^2y - y^3$

 (ix) $v = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

 (x) $u = \dfrac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$.

4. Find the analytic function $f(z) = u + iv$ if

 (i) $u - v = (x-y)(x^2 + 4xy + y^2)$

 (ii) $u + v = \dfrac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$

 (iii) $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - e^y - e^{-y}}$ given that $f(\pi/2)$

 (iv) $u + v = \dfrac{x}{x^2+y^2}$ given that $f(1) = 1$.

5. If $f(z) = u + iv$ is an analytic function prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2|f(z)|^{p-2}|f'(z)|^2.$$

6. If $f(z)$ is an analytic function of z show that

$$\left(\frac{\partial}{\partial x}|f(z)|\right)^2 + \left(\frac{\partial}{\partial y}|f(z)|\right)^2 = |f'(z)|^2.$$

7. Prove that the function $u(z)$ and $u(\bar{z})$ are simultaneously harmonic.

8. Prove that the function $u(x, y)$ and $u(x^2-y^2, 2xy)$ are simultaneously harmonic.

9. Prove that $u(x, y) = x^2 - y^2$ and $v(x, y) = -\dfrac{y}{x^2+y^2}$ are both harmonic but $u + iv$ is not analytic.

10. From the Laplace's equation for $u(x, y)$ prove that $\dfrac{\partial^2 u}{\partial z \partial \bar{z}} = 0$.

Answers.

1. (i) $v = -\cosh x \cos y$ (ii) $v = 4xy - x^3 + 3xy^2$ (iii) $v = e^x \sin y$
2. (i) $2z + iz^2$ (ii) ze^z (iii) $-i(z^2 + 3z)$ (iv) $1/z$
3. (i) z (ii) e^z (iii) z^3 (iv) $-ie^z$ (v) $\cos z$ (vi) ze^z (vii) $\sec z$ (viii) z^3 (ix) $i(z^3 + 3z)$ (x) $\cot z$
4. (ii) $\cot z$ (iii) $\frac{1}{2}(1 - \cot(z/2))$ (iv) $\dfrac{1+i}{2z} + \dfrac{1-i}{2}$

2.9. Conformal Mapping

In this section we study the geometric consequences of the existence of the derivatives of complex functions. In particular we prove that if an analytic function f has a non zero derivative at a point z_0 lying in a region then f preserves the angle between any two curves at z_0 both in magnitude and direction. We start with the necessary definitions.

Definition. A curve C in the complex plane is given by a continuous function $\gamma : [a, b] \to C$.

If $\gamma(t) = x(t) + iy(t)$ then the curve C is determined by the two continuous real valued functions of the real parameter t given by $x = x(t)$ and $y = y(t)$ where $a \le t \le b$. We also write $z = z(t) = x(t) + iy(t)$ where $a \le t \le b$. The point $z(a)$ is called the **origin** of the curve and $z(b)$ is called the **terminus** of the curve.

The curve C is said to be **simple** if $t_1 \neq t_2 \Rightarrow z(t_1) \neq z(t_2)$.

Equivalently C is simple if the function γ is 1-1.

The curve C is called a **closed curve** if $z(a) = z(b)$ and C is called a **simple closed curve** if (i) $z(a) = z(b)$ (ii) $z(t_1) \neq z(t_2)$ for any other pair of distinct real numbers $t_1, t_2 \in [a, b]$.

A simple closed curve is also called a **Jordan curve.**

A curve C is said to be ***differentiable*** if $z'(t)$ exists and is continuous. If further $z'(t) \neq 0$ then the curve is said to be **regular (smooth).**

Geometrically the regular curve has a tangent whose direction is determined by the argument of $z'(t)$.

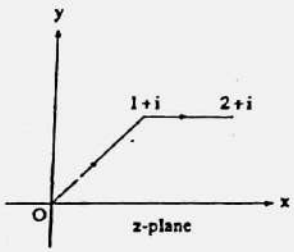
If C is a curve determined by the equation $z = z(t)$ where $a \le t \le b$ then the opposite curve of C denoted by $-C$ is given by the equation $z(t) = z(b+a-t)$ where $a \le t \le b$.

Example 1. The polygonal line given by

$$z(t) = \begin{cases} t + it & \text{if } 0 \le t \le 1 \\ t + i & \text{if } 1 \le t \le 2 \end{cases}$$

consisting of a line segment from 0 to $1+i$ followed by another line segment from $1+i$ to $2+i$ is a simple curve.

The equation of the curve automatically determines an orientation for the curve as shown in the figure.

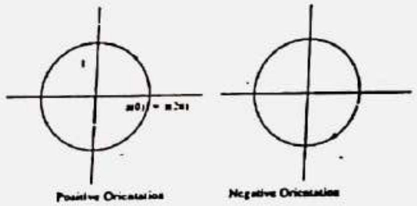


We notice that the above curve is differentiable except at $1+i$. Such a curve is called **a piecewise differentiable curve.**

Definition. A curve C given by $z = z(t)$ is said to be piecewise differentiable if it is differentiable except at a finite number of points and at any point where $z(t)$ is not differentiable it has a left derivative and right derivative.

Example 2.

The equation given by $z(t) = \cos t + i \sin t$ where $0 \le t \le 2\pi$ represents the unit circle C with centre O and radius 1 described in the anticlockwise direction. The origin and terminus of the curve are $z(0) = 1 = z(2\pi)$. The orientation of the circle as described in the figure is taken as the positive orientation.



The same circle with negative orientation $-C$ is given by the equation $z(t) = \cos(2\pi - t) + i \sin(2\pi - t)$. This is a simple closed curve.

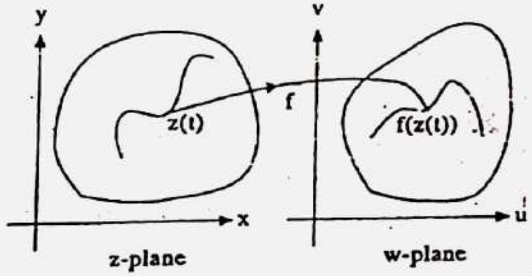
Example 3. In general the equation $z(t) = a + r(\cos t + i \sin t)$ where $0 \le t \le 2\pi$ represents a positively oriented circle with centre a and radius r. This is also a simple closed curve.

Example 4. The curve represented by $z(t) = \cos t + i \sin t$ where $0 \le t \le 4\pi$ is a closed curve. However it is not a simple closed curve, since $z(\pi/2) = z(5\pi/2)$. Actually the equation represents a unit circle traversed twice.

Example 5. The curve represented by $z(t) = \cos t + i \sin t$ where $0 \le t \le \pi$ represents a semi circular curve of unit radius above the real axis with the origin 1 and terminus -1. This is not a closed curve since $z(0) \ne z(\pi)$.

Definition. Let f be an analytic function in a region D. Let C be a curve given by the equation $z = z(t)$ where $a \le t \le b$ and lying in D.

Then the equation $w = w(t) = f(z(t))$ defines another curve C' in the w-plane and is called the image of the curve C under f.



Definition. Let f be a continuous function defined in the region D. Let $z_0 \in D$. Let C_1 and C_2 be two regular curves passing through z_0 and lying in D. Let C_1' and C_2' be the images of C_1 and C_2 respectively under f. If the angle between C_1 and C_2 is equal to the angle between C_1' and C_2' both in magnitude and direction then f is said to be conformal at z_0.

Thus a conformal mapping preserves angle both in magnitude and direction.

If the angle is preserved only in magnitude and direction is reversed then the map is said to be isogonal or indirectly conformal.

Theorem 2.14. Let f be an analytic function defined in a region D. Let $z_0 \in D$. If $f'(z_0) \ne 0$ then f is conformal at z_0.

Proof. Let C be a regular curve lying in the region D and passing thorugh $z_0 \in D$.

Suppose the equation C is given by $z = z(t)$ where $a \le t \le b$.

Let $z_0 = z(t_0)$ for $t_0 \in [a, b]$.

The equation of the image curve C' of C under f is given by $w = w(t) = f(z(t))$.

$$\therefore \; w'(t) = f'(z(t))z'(t).$$

$$\therefore \; w'(t_0) = f'(z(t_0))z'(t_0)$$
$$= f'(z_0)z'(t_0).$$

By hypothesis $f'(z_0) \ne 0$.

Also since C is regular $z'(t_0) \ne 0$.

Hence $w'(t_0) \ne 0$ and $\arg w'(t_0) = \arg f'(z(t_0)) + \arg z'(t_0)$ ... (1)

Hence $\varphi = \psi + \theta$ where $\arg w'(t_0) = \varphi$

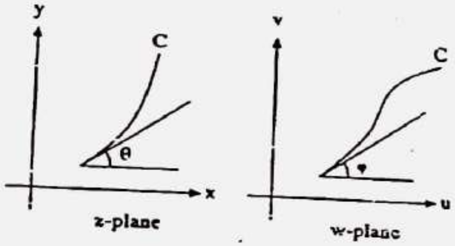
$\arg f'(z_0) = \psi$

and $\arg z'(t_0) = \theta$.

Clearly θ represents the angle made by the tangent to the curve C at z_0 with the positive direction of the x-axis in the z-plane.

Similarly φ represents the angle made by the tangent to the curve C' at $f(z_0)$ with the positive direction of the u-axis in the w-plane.

Hence it follows from (1) that the tangent to the regular curve C at z_0 is rotated through the angle ψ by the transformation $w = f(z)$.



Now, let C_1 and C_2 be two regular curves passing through $z_0 \in D$ and lying in D and C_1' and C_2' be their image curves under the map $w = f(z)$.

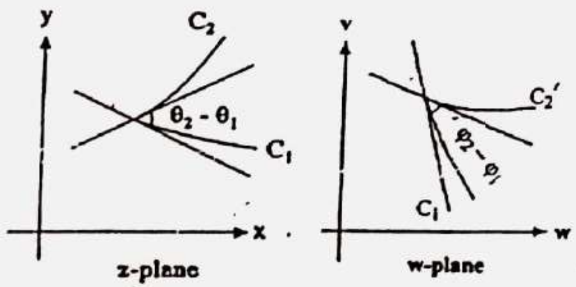
Let θ_1 and θ_2 be the angles made by the tangents to the curves C_1 and C_2 respectively at z_0 with the positive direction of the x-axis in the z-plane.

Let φ_1 and φ_2 be the angles made by the tangents to the curves C_1' and C_2' at $f(z_0)$ in the w-plane.

$$\therefore \varphi_1 = \psi + \theta_1 \text{ and } \varphi_2 = \psi + \theta_2.$$

$$\therefore \varphi_2 - \varphi_1 = \theta_2 - \theta_1.$$

$\therefore$ The angle $\varphi_2 - \varphi_1$ from C_1' to C_2' is the same in magnitude and sense as the angle $\theta_2 - \theta_1$ from C_1 to C_2 (refer figure).



Hence the function f preserves angle between the curves C_1 and C_2 at z_0 both in magnitude and direction.

Hence f is conformal at z_0.

Note 1. Under the conformal mapping $w = f(z)$ *angle of rotation* ψ at z_0 is $\arg w'(z_0)$ and the *scale factor* is $|f'(z_0)|$

Note 2. The condition for conformality at a point z_0, can also be written as $\dfrac{\partial(u, v)}{\partial(x, y)} \neq 0$.

Since $f'(z_0) \neq 0$ we have $|f'(z_0)| \neq 0$.

$\therefore \dfrac{\partial(u, v)}{\partial(x, y)} \neq 0$ (refer remark 1 in theorem 2.5)

Example 1. Consider the mapping $w = \bar{z}$. Geometrically it represents reflection about the real axis and its preserves the angle in magnitude but reverses the direction. Hence it is an isogonal mapping.

Example 2. Consider the mapping $w = z^2$. Angle between any two curves passing through the origin is doubled by this mapping.

Hence the mapping is not conformal at $z = 0$. We notice that $f'(0) = 0$.

Definition. Let $f(z)$ be an analytic function defined in D and let $z_0 \in D$. z_0 is called a critical point of $f(z)$ if $f'(z_0) = 0$.

Solved Problems

Problem 1. Determine the angle of rotation and scale factor at the point $z = 1 + i$ under the mapping $w = z^2$.

Solution. We know that the angle of rotation at $1 + i$ under $w = z^2$ is given by $\arg w'(1 + i)$ and the scale factor $|w'(1 + i)|$.

Now $w = z^2 \Rightarrow w'(z) = 2z$.

$\therefore w'(1 + i) = 2(1 + i)$.

$\therefore \arg w'(1 + i) = \arg[2(1 + i)] = \tan^{-1}(1) = \pi/4$.

$\therefore$ The angle of rotation at $1 + i$ is $\pi/4$.

Now the scale factor at $z = 1 + i$ is given by

$$|w'(1 + i)| = |2(1 + i)| = 2\sqrt{2}.$$

Problem 2. Find the points where the following mappings are conformal. Also find the critical points if any.

(i) $w = z^n$ (n positive integer) (ii) $w = \dfrac{1}{z}$

(iii) $w = z + \dfrac{1}{z}$ (iv) $w = e^z$

(v) $w = \sin z$ (vi) $w = \cosh z$

(vii) $w = az + b$ and $a \neq 0$.

Solution. We know $w = f(z)$ is conformal at a point z_0 if (a) w is analytic at z_0 and $f'(z_0) \neq 0$.

Also we know z_0 is a critical point of $w = f(z)$ if
(a) w is analytic at z_0 and (b) $f'(z_0) = 0$.

(i) $w = z^n$ (n is a positive integer)

$f(z) = z^n$ is analytic at all points.

Now $f'(z) = nz^{n-1}$ and $f'(z) = 0$ if and only if $z = 0$.

Hence the mapping is conformal at all points $z \neq 0$ and 0 is the only critical point of $f(z)$.

(ii) $w = \frac{1}{z}$. The mapping $f(z) = \frac{1}{z}$ is analytic at all points except $z = 0$.

Now $f'(z) = -\frac{1}{z^2}$ and $f'(z) \neq 0$ for all $z \neq 0$.

Hence the mapping is conformal at all points except $z = 0$ and there is no critical point for the mapping.

(iii) $w = z + \frac{1}{z}$.

$f(z) = z + \frac{1}{z}$ is not analytic at $z = 0$. Hence it is not conformal at $z = 0$.

Now $f'(z) = 1 - \frac{1}{z^2}$.

$$f'(z) = 0 \Rightarrow z^2 = 1 \Rightarrow z = \pm 1.$$

The only critical points are 1 and -1.

(iv) $w = e^z$.

$f(z) = e^z$ is analytic at all points in the complex plane.

Now $f'(z) = e^z$ and $e^z \neq 0$ for all z.

$\therefore$ $f(z)$ is conformal in the entire complex plane and there is no critical point.

(v) $w = \sin z$.

Then $f'(z) = \cos z$. Clearly $f'(z) = \cos z = 0$ if $z = \pi/2 + n\pi$ where $n \in Z$. These are the critical points and $f(z)$ is conformal at all other points.

(vi) $w = \cosh z$.

Hence $f'(z) = \sinh z$. The mapping is not conformal at points where $\sinh z = 0$ and the points are $z = 0, \pm\pi i, \pm 2\pi i, \ldots$ and these points are the critical points.

(vii) $w = az + b$ ($a \neq 0$)

$f(z) = az + b$ is analytic everywhere. Now $f'(z) = a \neq 0$.

$\therefore$ $f'(z) \neq 0$ for all z. Hence this mapping is conformal everywhere and there are no critical points.

Exercises

1. Find the angle of rotation and scale factor for the transformation $w = \frac{1}{z}$ at (i) $z = 1$ and (ii) $z = i$.
2. Show that under the mapping $w = z^2$ the angle of rotation at $z = 2 + i$ is $\tan^{-1}(1/2)$ and the scale factor is $2\sqrt{5}$.
3. Find the angle of rotation and scale factor for the mapping $w = z^3 + 8iz$ at $z = 1 - i$.
4. Find the coefficent of magnification and angle of rotation for (i) $w = z^3$ at $1 + i$ and (ii) $w = (1 - i)z$ at any point z.
5. Show that $w = iz$ represents a rotation through an angle $\pi/2$ and $w = -z$ a rotation through π.
6. Find where the following mappings are conformal and also find the critical points if any.
(i) $w = z^3$ (ii) $w = \cos z$ (iii) $w = \sinh z$

Answers.

1. (i) π (ii) 0
3. $\tan^{-1}\left(\frac{1}{4}\right)$; $\sqrt{68}$
4. (i) 6; $\frac{\pi}{2}$ (ii) $\sqrt{2}$; $-\frac{\pi}{4}$
6. (i) Conformal at all points except $z = 0$. Origin is a critical point.
(ii) Conformal except $z = 0, \pm\pi, \pm 2\pi, \ldots\ldots$. These are the critical points.
(iii) Conformal except at $z = \pm\left(\frac{\pi i}{2}\right), \frac{3\pi i}{2}, \ldots$ These are the critical points.

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6. Complex Integration

6.0. Introduction

In this chapter we develop the theory of integration for complex functions. We assume that the reader is familiar with the Riemann integral of a function defined on $[a, b]$. Using this we define the integeral of a complex valued function defined on $[a, b]$ and the integral of a function $f : D \to C$ where D is a region in C, along a curve C lying in D. We prove Cauchy's fundamental theorem and study the various consequences of this theorem.

6.1. Definite integral

We start with the definition of definite integral for a continuous complex valued function of a real variable.

Definition. Let $f(t) = u(t) + iv(t)$ be a continuous complex valued function defined on $[a, b]$.

$$\text{We define } \int_a^b f(t)\, dt = \int_a^b u(t)\, dt + i \int_a^b v(t)\, dt.$$

Remark. The following properties of the definite integral can be easily verified

1. $\operatorname{Re} \int_a^b f(t)\, dt = \int_a^b \operatorname{Re}[f(t)]\, dt$
2. $\operatorname{Im} \int_a^b f(t)\, dt = \int_a^b \operatorname{Im}[f(t)]\, dt$
3. $\int_a^b [f(t) + g(t)]\, dt = \int_a^b f(t)\, dt + \int_a^b g(t)\, dt$
4. $\int_a^b cf(t)\, dt = c \int_a^b f(t)\, dt$ where c is any complex constant.

Lemma. $\left|\int_a^b f(t)\, dt\right| \le \int_a^b |f(t)|\, dt$

Proof. Let $\int_a^b f(t)\, dt = re^{i\theta}$.

$$\therefore \left|\int_a^b f(t)\, dt\right| = r = e^{-i\theta} \int_a^b f(t)\, dt$$

$$= \operatorname{Re}\left(e^{-i\theta} \int_a^b f(t)\, dt\right) \quad \text{(since } r \text{ is real)}$$

$$= \operatorname{Re}\left(\int_a^b e^{-i\theta} f(t)\, dt\right) \quad \text{(using 4)}$$

$$= \int_a^b \operatorname{Re}\left(e^{-i\theta} f(t)\, dt\right) \quad \text{(using 1)}$$

$$\le \int_a^b \left|e^{-i\theta} f(t)\right| dt$$

$$= \int_a^b \left|e^{-i\theta}\right| |f(t)|\, dt$$

$$= \int_a^b |f(t)|\, dt$$

$$\text{Thus } \left|\int_a^b f(t)\, dt\right| \le \int_a^b |f(t)|\, dt$$

Definition. Let C be a piecewise differentiable curve given by the equation $z = z(t)$ where $a \le t \le b$. Let $f(z)$ be a continuous complex valued function defined in a region containing the curve C. We define

$$\int_C f(z)\, dz = \int_a^b f(z(t))z'(t)\, dt$$

Example 1. Consider $\int_C f(z)\, dz$ where $f(z) = \frac{1}{z}$ and C is the circle $|z| = r$ described in the positive sense. The parametric equation of the circle $|z| = r$ is given by $z = re^{it}$ where $0 \le t \le 2\pi$ and $z'(t) = ire^{it}$

$$\therefore \int_C f(z)dz = \int_C \frac{dz}{z} = \int_0^{2\pi} \frac{ire^{it}}{re^{it}}dt$$

$$= i\int_0^{2\pi} dt$$

$$= 2\pi i.$$

Example 2. In general $\int_C \frac{dz}{z-a} = 2\pi i$ where C is the circle with centre a radius r given by the equation $z = a + re^{it}, 0 \le t \le 2\pi$. For $\int_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{rie^{it}}{re^{it}}dt = i\int_0^{2\pi} dt = 2\pi i$

Theorem 6.1. $\int_{-C} f(z)dz = -\int_C f(z)dz.$

Proof. Suppose the equation of C is given by $z = z(t)$ where $a \le t \le b$. We know that the equation of $-C$ is given by

$$z(t) = z(b + a - t) \text{ where } a \le t \le b.$$

Now, $\int_{-C} f(z)\,dz = \int_a^b f(z(b+a-t))z'(b+a-t)(-dt).$

Put $b + a - t = u$. Then $-dt = du$.

Also $t = a \Rightarrow u = b$ and $t = b \Rightarrow u = a$

$$\therefore \int_{-C} f(z)\,dz = \int_b^a f(z(u))z'(u)du$$

$$= -\int_a^b f(z(u))z'(u)du$$

$$= -\int_C f(z)dz.$$

Remark. The following results are immediate consequences of the definition.

1. Let α be a complex constant. Then $\int_C \alpha f(z)dz = \alpha\int_C f(z)dz.$
2. $\int_C [f(z) + g(z)]dz = \int_C f(z)dz + \int_C g(z)dz$

Definition. Let C_1 be a differentiable curve with origin z_1 and terminus z_2. Let C_2 be another differentiable curve with origin z_2 and terminus z_3. Then the curve C which consists of C_1 followed by C_2 is a piecewise differentiable curve with origin z_1 and terminus z_3. This curve is denoted by $C_1 + C_2$.

Remark. If $C = C_1 + C_2$ then $\int_C f(z)dz = \int_{C_1} f(z) + \int_{C_2} f(z)dz$

In general if $C = C_1 + C_2 + ... + C_n$ then

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + ... + \int_{C_n} f(z)dz$$

Definition. Let C be a piecewise differentiable curve given by the equation $z = z(t)$ where $a \le t \le b$. Then the **length** l of C is defined by

$$l = \int_a^b |z'(t)|dt.$$

Example. Consider the circle C with centre a and radius . The parametric equation of C is given by $z = a + re^{it}$ where $0 \le t \le 2\pi$.

$$\therefore z'(t) = ire^{it}$$

$$\therefore l = \int_0^{2\pi} |z'(t)|dt = \int_0^{2\pi} |ire^{it}|dt = \int_0^{2\pi} r\,dt$$

$$= 2\pi r.$$

Theorem 6.2. $\left|\int_C f(z)dz\right| \le Ml$ where $M = max\{|f(z)|/z \in C\}$ and l is the length of C.

Proof. Supose C is given by the equation $z = z(t)$ where $a \le t \le b$.

By definition of M we have

$$|f(z(t))| \le M \text{ for all } t; a \le t \le b \qquad \ldots (1)$$

$$\text{Now, } \left|\int_C f(z)dz\right| = \left|\int_a^b f(z(t))z'(t)dt\right|$$

$$\leq \int_a^b |f(z(t))z'(t)|dt$$

$$= \int_a^b |f(z(t))||z'(t))|dt$$

$$\leq \int_a^b M|z'(t)|dt.(\text{using}(1))$$

$$= M\int_a^b |z'(t)|dt$$

$$= Ml$$

$$\therefore \left|\int_C f(z)dz\right| \leq Ml$$

Solved Problems

Problem 1. Evaluate $\int_C f(z)dz$ where $f(z) = y - x - i3x^2$ and C is the line segment from $z = 0$ to $z = 1 + i$.

Solution. The equation of the line segment C joining $z = 0$ and $z = 1 + i$ is given by $y = x$.

$\therefore$ The prametric equation of C can be taken as $x = t$ and $y = t$ where $0 \leq t \leq 1$. Hence $z(t) = x(t) + iy(t) = t + it$ so that $z'(t) = (1 + i)$.

$$\text{Now, } f(z(t)) = t - t - i3t^2 = -i3t^2.$$

$$\therefore \int_C f(z)dz = \int_0^1 f(z(t))z'(t)dt$$

$$= \int_0^1 -i3t^2(1+i)dt$$

$$= -3i(1+i)\left[\frac{t^3}{3}\right]_0^1$$

$$= 1 - i$$

Problem 2. Prove that $\int_C \frac{dz}{(z-a)^n} = \begin{cases} 0 \text{ if } n \neq 1 \\ 2\pi i \text{ if } n = 1 \end{cases}$ where C is the circle with centre a and radius r and $n \in \mathbb{Z}$.

Solution. The parameters equation of the circle C is given by $z - a = re^{it}, 0 \leq t \leq 2\pi$.

$$\therefore z'(t) = ire^{it}$$

$$\text{Now, } \int_C \frac{dz}{(z-a)^n} = \int_0^{2\pi} \frac{ire^{it}}{(re^{it})^n}dt$$

$$= \frac{i}{r^{n-1}}\int_0^{2\pi} e^{i(1-n)t}dt$$

$$= \frac{i}{r^{n-1}}\left[\frac{e^{i(1-n)t}}{i(1-n)}\right]_0^{2\pi} \text{ provided } n \neq 1$$

$$= \frac{1}{(1-n)r^{n-1}}\left[e^{i(1-n)2\pi} - 1\right]$$

$$= \frac{1}{(1-n)r^{n-1}}[1-1]$$

$$= 0.$$

If $n = 1$, $\int_C \frac{dz}{z-a} = 2\pi i$. (Refer example 2 of 6.1)

Hence the result.

Problem 3. Evaluate $\int_c \bar{z}\, dz$ from $z = 0$ to $z = 4 + 2i$ along the curve C consisting of the line segment from $z = 0$ to $z = 2i$ followed by the line segment from $z = 2i$ to $z = 4 + 2i$.

Solution. Let C_1 denote the line segment joining 0 to $2i$ and C_2 denote the line segment joining $2i$ to $4 + 2i$. Then $C = C_1 + C_2$.

Now the parametric equation of C_1 is given by $x(t) = 0$ and $y(t) = t$ where $0 \le t \le 2$.

Hence $z(t) = x(t) + iy(t) = it$ so that $z'(t) = i$.

$$\text{Hence } \int_{C_1} \bar{z}\, dz = \int_0^2 (-it)i\,dt = \int_0^2 t\,dt = 2.$$

Now the parametric equation of C_2 is given by $x(t) = t$ and $y(t) = 2$ where $0 \le t \le 4$.

Hence $z(t) = t + 2i$ and $z'(t) = 1$.

$$\therefore \int_{C_2} \bar{z}dz = \int_0^4 (t - 2i)dt$$

$$= \left[\frac{t^2}{2} - 2it\right]_0^4$$

$$= 8 - 8i.$$

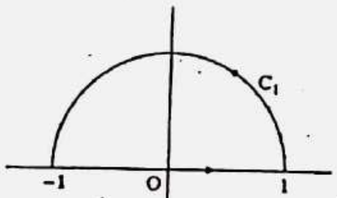
$$\therefore \int_C \bar{z}\, dz = \int_{C_1} \bar{z}\, dz + \int_{C_2} \bar{z}\, dz = 2 + 8 - 8i = 10 - 8i.$$

Problem 4. Evaluate $\int_c |z|\, \bar{z}\, dz$ where C is the closed curve consisting of the upper semicircle $|z| = 1$ and the segment $-1 \le x \le 1$.

Solution. Let $f(z) = |z|\,\bar{z}$

$$\therefore \int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$

where C_1 is the upper semicircle $|z| = 1$ and C_2 is the line segment $-1 \le x \le 1$.



The parametric equation of C_1 is given by $z = e^{it}, 0 \le t \le \pi$.

$$\text{Hence } z'(t) = ie^{it}$$

$$\therefore \int_{C_1} f(z)dz = \int_0^{\pi} e^{-it} ie^{it} dt$$

$$= \pi i.$$

The parametric equation of C_2 is given by $y = 0, x = t$ where $-1 < t \le 1$.

Hence $z(t) = t$ and $z'(t) = 1$.

$$\text{Also } |z(t)| = \begin{cases} -t \text{ if } -1 \le t < 0 \\ t \text{ if } 0 < t \le 1 \end{cases}$$

$$\text{Hence } \int_{C_2} |z|\,\bar{z}dz = \int_{-1}^{0} -t \cdot t dt + \int_0^1 t \cdot t dt$$

$$= \left[\frac{-t^3}{3}\right]_{-1}^0 + \left[\frac{t^3}{3}\right]_0^1$$

$$= -\frac{1}{3} + \frac{1}{3} = 0.$$

$$\text{Hence } \int_C |z|\ \bar{z}\, dz = \int_{C_1} |z|\ \bar{z}\, dz + \int_{C_2} |z|\ \bar{z}\, dz$$

$$= \pi i.$$

Problem 5. Prove $\int_C \bar{z}^2 dz = \begin{cases} 0 \text{ if } C \text{ is the unit circle } |z| = 1 \\ 4\pi i \text{ if } C \text{ is the circle } |z - 1| = 1 \end{cases}$

Solution. Let C be the unit circle $|z| = 1$

The parametric equation of C is given by $z(t) = e^{it}$ where $0 \le t \le 2\pi$.

Hence $z'(t) = ie^{it}$ and $[\bar{z}(t)]^2 = e^{-2it}$

$$\therefore \int_C \bar{z}^2 dz = \int_0^{2\pi} [\overline{z}(t)]^2 z'(t)dt$$

$$= i\int_0^{2\pi} e^{-it}dt$$

$$= -\left[e^{-it}\right]_0^{2\pi}$$

$$= 0.$$

Now, let C be the circle $|z-1| = 1$.
The parametric equation of C is given by $z(t) = 1 + e^{it}$ where $0 \le t \le 2\pi$.
Hence $z'(t) = ie^{it}$.

$$\int_C \bar{z}^2 dz = \int_0^{2\pi} (1+e^{-it})^2 ie^{it}dt$$

$$= i\int_0^{2\pi} (e^{it} + e^{-it} + 2)dt$$

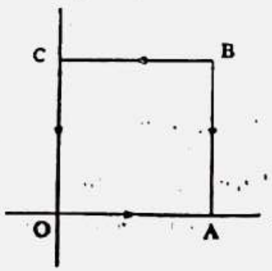
$$= i\left[\frac{e^{it}}{i} - \frac{e^{-it}}{i} + 2t\right]_0^{2\pi}$$

$$= \left[e^{it} - e^{-it} + 2it\right]_0^{2\pi}$$

$$= 4\pi i.$$

Problem 6. Show that $\int_C |z|^2 dz = -1 + i$ where C is the square with vertices $O(0,0)$, $A(1,0)$, $B(1,1)$ and $C(0,1)$.

Solution. $C = C_1 + C_2 + C_3 + C_4$ where C_1, C_2, C_3 and C_4 are the line segments OA, AB, BC and CO as shown in the figure. The parametric equation of C_1 is given by $x = t$ and $y = 0$ where $0 \le t \le 1$.



Hence $z(t) = t$ and $z'(t) = 1$.

$$\therefore \int_{C_1} |z|^2 dz = \int_0^1 t^2 dt = \frac{1}{3}.$$

The parametric equation of C_2 is given by $y = t$ and $x = 1$ where $0 \le t \le 1$. Hence $z(t) = 1 + it$ and $z'(t) = i$.

$$\therefore \int_{C_2} |z|^2 dz = \int_0^1 |1+it|^2 i dt$$

$$= i\int_0^1 (1+t^2)dt$$

$$= i\left[t + \frac{t^3}{3}\right]_0^1 = \frac{4i}{3}.$$

The parametric equation of C_3 is given by $y = 1$ and $x = 1 - t, 0 \le t \le 1$.

Hence $z(t) = (1-t) + i$ and $z'(t) = -1$.

$$\therefore \int_{C_3} |z|^2 dz = \int_0^1 [(1-t)^2 + 1](-1)dt$$

$$= -\int_0^1 (t^2 - 2t + 2)dt = -\frac{4}{3}.$$

The parametric equation of C_4 is given by $x = 0, y = 1 - t, 0 \le t \le 1$. Hence $z(t) = i(1-t)$ and $z'(t) = -i$.

$$\int_{C_4} |z|^2 dz = \int_0^1 (1-t)^2(-i)dt = i\left[\frac{(1-t)^3}{3}\right]_0^1 = -\frac{i}{3}.$$

Hence $\int_C f(z)dz = \frac{1}{3} + \frac{4i}{3} - \frac{4}{3} - \frac{i}{3} = -1 + i.$

Problem 7. Evaluate the integral $\int_C (x^2 - iy^2)dz$ where C is the parabola $y = 2x^2$ from $(1,2)$ to $(2,8)$.

Solution. Let $f(z) = x^2 - iy^2$. The parametric equation of C is given by $x = t$ and $y = 2t^2$ where $1 \le t \le 2$.

$\therefore z(t) = x(t) + iy(t) = t + i2t^2$ and $z'(t) = 1 + 4it.$

$$\therefore \int_C (x^2 - iy^2)dz = \int_1^2 (t^2 - 4it^4)(1 + 4it)dt,$$

$$= \int_1^2 \left[(t^2 + 16t^5) + i(4t^3 - 4t^4)\right] dt$$

$$= \left[\left(\frac{t^3}{3} + \frac{16t^6}{6}\right) + i\left(t^4 - \frac{4t^5}{5}\right)\right]_1^2$$

$$= \frac{511}{3} - \frac{49}{5}i.$$

Problem 8. Evaluate $\int_C \frac{z+2}{z} dz$ where C is the semi circle $z = 2e^{i\theta}$ where $0 \le \theta \le \pi$.

Solution. Here $z'(\theta) = 2ie^{i\theta}$ so that $dz = 2ie^{i\theta}d\theta$.

$$\int_C \frac{z+2}{z} dz = \int_0^\pi \left(\frac{2e^{i\theta} + 2}{2e^{i\theta}}\right)(2ie^{i\theta}d\theta)$$

$$= 2i\int_0^\pi (1 + e^{i\theta})d\theta$$

$$= 2i\left[\theta + \frac{e^{i\theta}}{i}\right]_0^\pi$$

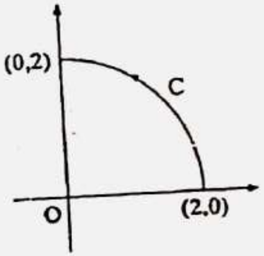
$$= 2i\left[\left(\pi - \frac{1}{i}\right) - \left(\frac{1}{i}\right)\right]$$

$$= 2i\left[\frac{\pi i - 2}{i}\right]$$

$$= -4 + 2\pi i.$$

Problem 9. Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant. Without actually evaluting the integral show that $\left|\int_C \frac{dz}{z^2+1}\right| \le \frac{\pi}{3}$.

Solution Let $f(z) = \frac{1}{z^2+1}$.



Since C is the circular arc of radius 2 lying in the first quadrant the length of C is given by $l = \frac{1}{4}(2\pi \times 2) = \pi$

Also on C, $|z^2 + 1| = |z^2 - (-1)| \ge |z^2| - |-1| = |z|^2 - 1 = 3$

Thus $|z^2 + 1| \ge 3$

$$\therefore \left|\frac{1}{z^2+1}\right| \le \frac{1}{3}.$$

Hence by theorem 6.2 $\left|\int_C \frac{dz}{z^2+1}\right| \le \frac{\pi}{3}$.

Problem 10. Without evaluating the integral show that $\left|\int_C \frac{dz}{z^4}\right| \le 4\sqrt{2}$ where C is the line segment from $z = i$ to $z = 1$.

Solution. C is the line segment joining (0, 1) to (1, 0) and its length is obviously $\sqrt{2}$.

As z varies on C, the minimum value of $|z|$ is the perpendicular distance from the origin to the line segment C.

Thus on C, $|z| \ge \frac{1}{\sqrt{2}}$ so that $|z|^4 \ge \frac{1}{4}$.

$$\therefore \left|\frac{1}{z^4}\right| \le 4.$$

$\therefore$ By theorem 6.2 $\left|\int_C \frac{dz}{z^4}\right| \le 4\sqrt{2}$.

Exercises

1. Evaluate $\int_C x\,dz$ where C is the directed line segment from 0 to $1+i$

2. Evaluate $\int_C x\,dz$ where C is the circle $|z| = r$.

3. Show that $\int_C x\,dz = \frac{i\pi}{2}$ and $\int_C y\,dz = -\frac{\pi}{2}$ where C is the semi circle $|z| = 1$ and $0 \le \arg z \le \pi$ with inital point $z = 1$.

4. Find the value of the integral $\int_0^{1+i} (x - y + ix^2)dz$

   (i) along the straight line from $z = 0$ to $z = 1+i$.

   (ii) along the imaginary axis from $z = 0$ to $z = i$ and then along a line parallel to the real axis from $z = i$ to $z = 1+i$.

5. Show that $\int_{C_1} \bar{z}\,dz = \pi i$ and $\int_{C_2} \bar{z}\,dz = -\pi i$ if C_1 in the upper half of the circle $|z| = 1$ with $z = 1$ as the initial point and C_2 is the same semicircle with $z = -1$ as the inital point.

6. Evaluate the integrals $I_1 = \int_C x\,dz$ and $I_2 = \int_C y\,dz$ along the following paths.

   (i) along the radius vector of the point $z = 2+i$.

   (ii) along the semicircle $|z| = 1$, $0 \le \arg z \le \pi$ with starting point 1.

   (iii) around the circle $|z - a| = r$.

7. Evaluate $\int_C |z|\,dz$ along the following paths.

   (i) around the semicircle $|z| = 1$, $0 \le z \le \pi$ with starting point 1.

   (ii) around the semicircle $|z| = 1$, $-\frac{\pi}{2} \le \arg z \le \frac{\pi}{2}$ with starting point $-i$

   (iii) around the circle $|z| = r$.

8. Evlauate $\int_C |z|\bar{z}\,dz$ where C is the closed curve consisting of the upper semicircle $|z| = 1$ and the segment $-1 \le x \le 1$, $y = 0$.

9. Compute $\int_{|z|=1} |z - 1|\,|dz|$

10. Evaluate $\int_{(0,3)}^{(2,4)} (2y + x^2)dx + (3x - y)\,dy$ along

    (i) the parabola $x = 2t$, $y = t^2 + 3$

    (ii) the straight line segments from (0, 3) to (2, 3) and then from (2, 3) to (2, 4).

    (iii) a straight line from (0, 3) to (2, 4).

11. Evaluate $\int_C (x + 2y)dx + (y - 2x)dy$ where C is the ellipse defined by $x = 4\cos\theta$, $y = 3\sin\theta$ and C is described in the anticlockwise direction.

12. Evlauate $\int_{(0,1)}^{(2,5)} (3x + y)dx + (2y - x)dy$ along

    (i) the curve $y = x^2 + 1$

    (ii) the straight line joining (0, 1) and (2, 5)

    (iii) the straight lines from (0, 1) to (0, 5) and then from (0, 5) to (2, 5)

    (iv) the straight lines from (0, 1) to (2, 1) and then from (2, 1) to (2, 5)

13. Compute $\int_C \frac{dz}{z}$ where C denotes

    (i) the square described in the positive sense with sides parallel to the axes and of length $2a$ and having its centre at the origin.

    (ii) the circle $|z| = r$ described in the positive sense.

14. Evaluate $\int_C |z|\,dz$ where C is the circle $|z - 1| = 1$ in the positive sense.

15. If C is the curve $y = x^3 - 3x^2 + 4x - 1$ joining the points (1,1) and (2,3) evaluate $\int_C (12z^2 - 4iz)dz$.

16. Find $\int z^n dz$ along the line joining the points $z = 0$ and $z = 2+i$. Show that the integral is independent of the path.

17. Show that (i) $\int_{(2,1)}^{(3,2)} (2xy^3 - 2y^2 - 6y)dx + (3x^2y^2 - 4xy - 6x)dy$ is independent of the path joining points (2, 1) and (3, 2). (ii) Evaluate the integral in (i)

18. Show that the value of $\int \frac{dz}{z-2}$ is $2\pi i$ if it is along the circle $|z - 2| = 1$ or along the circle $|z - 2| = 4$.

19. Evaluate $\int_C z^2 dz$ along C where

    (i) C is the segment joining the points (1, 1) and (2, 4).

    (ii) C is the curve $x = t$, $y = t^2$ joining (1, 1) and (2, 4).

20. Evaluate $\int_C \frac{z+2}{z}dz$ where C is

(i) the semicircle $z = 2e^{i\theta}$ where $\pi \le \theta \le 2\pi$

(ii) the circle $z = 2e^{i\theta}$ where $-\pi \le \theta \le \pi$

21. If C is the boundary of the square with vertices at the points $z = 0, z = 1, z = 1+i, z = i$ and the orientation of C is counter clockwise then show that $\int_C (3z+1)dz = 0$.

22. Show that $\left|\int_C (x^2 - iy^2)dz\right| \le \frac{5}{2}$

(i) where C is the interval $[-i, i]$ on the y- axis.

(ii) where C is the semi circle $z = \cos\theta + i\sin\theta$; $-\pi/2 \le \theta \le \pi/2$.

23. Show that if C is the boundary of the triangle with vertices at the points $z = 0$, $z = 3i$, $z = -4$ and the orientation of C is anticlockwise then $\left|\int_C (e^z - \bar{z})dz\right| \le 60$.

24. Evaluate $\int_C f(z)dz$ where $f(z) = e^z$ and C is the arc from $z = \pi i$ to $z = 1$ consisting of the line segment joining these points.

25. Evaluate $\int_0^{1+i} (z^2 + z)dz$ by choosing two different paths of integration and show that the results are same.

Answers

1. $(1+i)/2$ 2. $i\pi r^2$ 4. (i) $(-1+i)/3$ (ii) $-(3+i)/6$ 6. (i) $2+i$; $(2+i)/2$ (ii) $i\pi/2; -\pi/2$ (iii) $i\pi r^2; -\pi r^2$ 7. (i) -2 (ii) $2i$ (iii) 0 8. πi 9. 8 10. (i) $\frac{33}{2}$ (ii) $\frac{103}{6}$ (iii) $\frac{97}{6}$ 11. -48π 12. (i) $\frac{88}{3}$ (ii) 32 (iii) 40 (iv) 24 13. (i) $2\pi i$ (ii) $2\pi i$ 14. $8i/3$ 15. $-156+38i$ 17. (ii) 24 19. (i) $-\frac{86}{3}-6i$ (ii) $-\frac{86}{3}-6i$ 20. (i) $4+2\pi i$ (ii) $4\pi i$ 24. $1+e$ 25. $\frac{5i}{3}-\frac{2}{3}$.

6.2. Cauchy's Theorem

In this section we prove the fundamental theorem of integration known as Cauchy's theorem which forms the basis for the theory of complex integration

Definition. Let $p(x, y)$ and $q(x, y)$ be two real valued functions. Then the differential equation $p(x, y)dx + q(x, y)dy = 0$ is said to be exact if there exists a function $u(x, y)$ such that $\frac{\partial u}{\partial x} = p$ and $\frac{\partial u}{\partial y} = q$.

We assume the following theorem without proof.

Theorem 6.3. $\int_C pdx + qdy$ depends only on the end points of C if and only if the integrand is exact.

Remark. The above theorem is true if p and q are complex valued functions as well.

We now apply the above theorem for complex functions to get a charecterisation for $\int_C f(z)dz$ to depend only on end points of C.

Theorem 6.4. Let $f(z)$ be a continuous complex valued function defined on a region D. Then $\int_C f(z)dz$ depends only on the end points of C if and only if there exists an analytic function $F(z)$ such that $F'(z) = f(z)$ in D.

Proof.
$$\int_C f(z)dz = \int_C f(z)(dx + idy) \text{ (since } z = x+iy)$$
$$= \int_C f(z)dx + if(z)dy$$

$\int_C f(z)dz$ depends only on the end points of C if and only if there exists a function $F(z)$ defined on D such that $\frac{\partial F}{\partial x} = f(z)$ and $\frac{\partial F}{\partial y} = if(z)$.

$\therefore$ $\frac{\partial F}{\partial x} = \frac{1}{i}\frac{\partial F}{\partial y}$ so that $\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}$ which is the complex form of the Cauchy-Riemann equation for $F(z)$ (refer theorem 2.7)

Since $f(z)$ is continuous the partial derivatives of $F(z)$ are also continuous and hence $F(z)$ is analytic in D and $F'(z) = f(z)$. Hence the theorem.

Corollary 1. Let $f(z)$ be a continuous complex valued function defined on a region D then $\int_C f(z)dz = 0$ for every closed curve C lying in D iff there exists an analytic function $F(z)$ such that $F'(z) = f(z)$ in D.

Corollary 2. $\int_C (z-a)^n dz = 0$ for every closed curve C provided $n \ge 0$.

Proof. Let $F(z) = \dfrac{(z-a)^{n+1}}{n+1}$.

Clearly $F'(z) = (z-a)^n$

$= f(z)$.

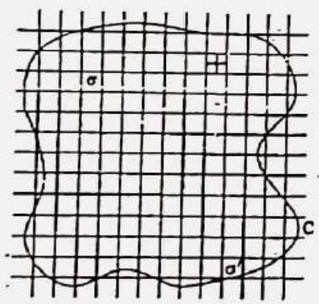
$\therefore$ By corollary (1), $\int_C f(z)dz = 0$. Hence $\int_C (z-a)^n dz = 0$ for all $n \geq 0$.

Lemma. Let C be a simple closed curve. Let D denote the closed region consisting of all points interior to C together with the points on C. Let f be a function analytic in D. Then given $\varepsilon > 0$ it is possible to cover D with a finite number of squares and partial squares whose boundaries are denoted by C_j such that there exists points z_j lying inside or on each C_j satisfying

$$\left|\frac{f(z)-f(z_j)}{z-z_j} - f'(z_j)\right| < \varepsilon \ (j = 1, 2, \ldots\ldots, n) \qquad \ldots (1)$$

for all points z distinct from each z_j and lying inside or on C_j.

Proof We subdivide the region D into squares and partial squares by drawing equally spaced lines parallel to the coordinate axes (refer figure). (A square is a closed region consisting of all points on and interior to it. If a particular square contains points which are not in D we remove those points and call what remains a partial square. In this figure σ is a square and σ' is a partial square). This gives a finite number of squares and partial squares which cover the region D.



Suppose the Lemma is false. Then in the covering constructed as above there exists a subregion with boundary C_j such that no point z_j exists satisfying (1).

Let σ_0 denote that subregion if it is a square. If it is a partial square let σ_0 denote the entire square of which it is a part.

We now subdivide σ_0 into four smaller squares by drawing line segments joining the mid points of the opposite sides. At least one of the four smaller squares say σ_1 is such that σ_1 contains points of D and no point z_j satisfying (1) exists.

Continuing this process we obtain a nested infinite sequence of squares $\sigma_1, \sigma_2, \ldots\ldots, \sigma_n \ldots\ldots$ such that for each σ_n no z_j satisfying (1) exists.

Now there exists a point z_0 common to each σ_n such that for any $\delta > 0$ the neighbourhood $|z - z_0| < \delta$ contains all the squares σ_n for all sufficiently large values of n.

Hence every neighbourhood of z_0 contains points of D distinct from z_0. Hence z_0 is a limit point of D. Since D is closed $z_0 \in D$.

Since $f(z)$ is analytic at z_0 there exists $\delta > 0$ such that

$$|z - z_0| < \delta \Rightarrow \left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| < \varepsilon \qquad \ldots (2)$$

Choose N such that the square σ_N is contained in the neighbourhood $|z - z_0| < \delta$. Then for every point z in σ_N (2) holds.

$\therefore$ z_0 serves as the point z_j stated in the lemma. This is a contradiction since there is no z_j in σ_N satisfying (1).

This contradiction proves the lemma.

Theorem 6.5. (Cauchy's theorem) Let f be a function which is analytic at all points inside and on a simple closed curve C. Then $\int_C f(z)dz = 0$.

Proof. Let D be the closed region consisting of all points interior to C together with the points on C.

Let $\varepsilon > 0$ be given.

Let $C_j (j = 1, 2, \ldots\ldots, n)$ denote the boundaries of the squares and partial squares covering D such that there exists a point z_j lying inside or on C_j satisfying

$$\left|\frac{f(z)-f(z_j)}{z-z_j} - f'(z_j)\right| < \varepsilon \qquad \ldots (1)$$

for all z distinct from z_j and lying within or on C_j.

$$\text{Let } \delta_j(z) = \begin{cases} \dfrac{f(z)-f(z_j)}{z-z_j} - f'(z_j) & \text{if } z \neq z_j \\ 0 & \text{if } z = z_j \end{cases}$$

Clearly $\delta_j(z)$ is a continuous function and

$$f(z) = f(z_j) - z_j f'(z_j) + z f'(z_j) + (z - z_j)\delta_j(z).$$

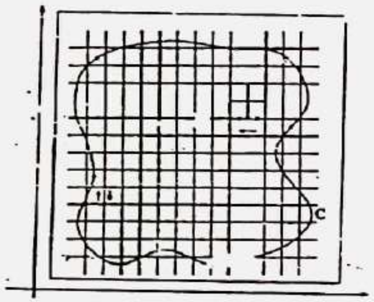
$$\therefore \int_{C_j} f(z)dz = \int_{C_j} f(z_j)dz - \int_{C_j} z_j f'(z_j)dz + \int_{C_j} z f'(z_j)dz + \int_{C_j} (z - z_j)\delta_j(z)dz$$

$$= f(z_j)\int_{C_j} dz - z_j f'(z_j)\int_{C_j} dz + f'(z_j)\int_{C_j} z dz + \int_{C_j} (z - z_j)\delta_j(z)dz$$

$$= \int_{C_j} (z - z_j)\delta_j(z)dz \ \left(\text{since} \int_{C_j} dz = 0 \text{ and } \int_{C_j} z dz = 0\right)$$

$$\therefore \sum_{j=1}^{n} \int_{C_j} f(z)dz = \sum_{j=1}^{n} \int_{C_j} (z - z_j)\delta_j(z)dz \qquad \ldots (2)$$

Now, in the sum $\sum_{j=1}^{n} \int_{C_j} f(z)dz$ the integrals along the common boundary of every pair of adjacent subregions cancel each other. (since the integral is taken in one direction along that line segment in one subregion and in the opposite direction in the other) (refer figure)



Hence only the integrals along the arcs which are the parts of C remain.

$$\therefore \sum_{j=1}^{n} \int_{C_j} f(z)dz = \int_{C} f(z)dz.$$

From (2) $\int_C f(z)dz = \sum_{j=1}^{n} \int_{C_j} (z - z_j)\delta_j(z)dz.$

$$\therefore \left| \int_C f(z)dz \right| = \left| \sum_{j=1}^{n} \int_{C_j} (z - z_j)\delta_j(z)dz \right|$$

$$\leq \sum_{j=1}^{n} \int_{C_j} |(z - z_j)\delta_j(z)|dz$$

$$= \sum_{j=1}^{n} \int_{C_j} |z - z_j||\delta_j(z)|dz$$

$$\therefore \left| \int_C f(z)dz \right| \leq \sum_{j=1}^{n} \int_{C_j} |z - z_j||\delta_j(z)|dz \qquad \ldots (3)$$

Now if C_j is a square and s_j is the length of its side then $|z - z_j| < \sqrt{2}s_j$ for all z on C_j.

Also from (1) we have $|\delta_j(z)| < \varepsilon$ and hence

$$\int_{C_j} |z - z_j||\delta_j(z)|dz < (\sqrt{2}s_j\varepsilon)(4s_j) \text{(by theorem 6.2)}$$

$$= 4\sqrt{2}A_j\varepsilon. \qquad \ldots (4)$$

where A_j is the area of the square C_j

Similarly for a partial square with boundary C_j if l_j is the length of the arc of C which forms a part of C_j. We have

$$\int_{C_j} |z - z_j||\delta_j(z)|dz < \sqrt{2}s_j\varepsilon(4s_j + l_j)$$

$$< 4\sqrt{2}A_j\varepsilon + \sqrt{2}Sl_j\varepsilon \qquad \ldots (5)$$

where S is the length of a side of some square containing the entire region D as well as all the squares orignially used in covering D.

We observe that the sum of all A_j's that occur in the right hand side of (4) and (5) do not exceed S^2 and the sum of all the l_j's is equal to L (the length of C).

Using (4) and (5) in (3) we obtain

$$\left| \int_C f(z)dz \right| < (4\sqrt{2}S^2 + \sqrt{2}SL)\,\varepsilon$$

$$= k\varepsilon \text{ where } k = 4\sqrt{2}S^2 + \sqrt{2}SL \text{ is a constant.}$$

Thus $\left| \int_C f(z)dz \right| < k\varepsilon.$

Since ε is arbitary we have $\int_C f(z)dz = 0.$

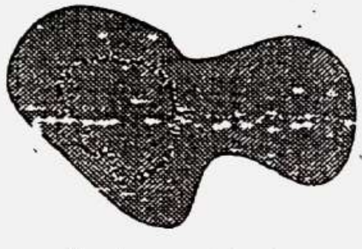
Note. Cauchy's theorem was first proved by using Green's theorem with the additional hypothesis that $f'(z)$ is continuous. Later Goursat proved the theorem without the hypothesis that $f'(z)$ is continuous. For this reason the theorem is sometimes known as Cauchy-Goursat theorem.

Definition. A region D is said to be **simply connected** if every simple closed curve lying in D encloses only points of D.

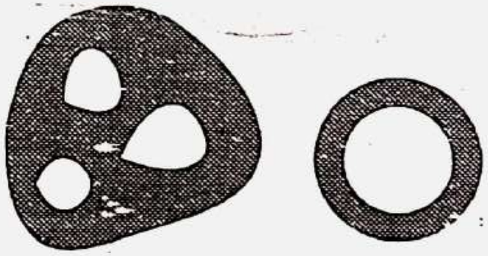
For example the interior of a simple closed curve is a simply conected region. The annular region enclosed by two concentric circles is not simply connected.

A region which is not a simply connected is said to be a **multiply connected region.**

Intutively a simply connected region is one which does not have any holes in it.



Simply connected region



Multiply connected regions

We observe that Cauchy's theorem can be restated as follows

Theorem 6.6. (Cauchy's theorem for simply connected regions).

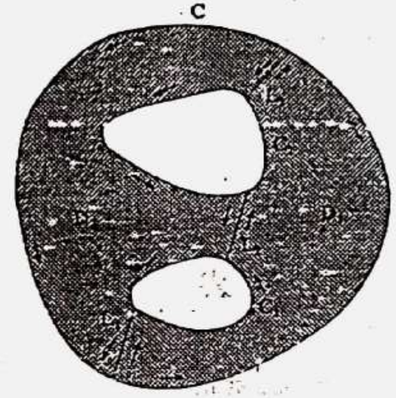
Let f be a function which is analytic in a simply connected region D. Let C be any simple closed curve lying within D. Then $\int_C f(z)dz = 0$.

We now extend Cauchy's theorem to certain types of multiply connected regions.

Theorem 6.7. (Cauchy's theorem for multiply connected regions)

Let C be a simple closed curve. Let $C_j (j = 1, 2, \ldots \ldots, n)$ be a finite number of simple closed curves lying in the interior of C such that the interiors of C_j's are disjoint. Let D be the closed region consisting of all points within and on C except the points interior to each C_j. Let B denote the entire oriented boundary of D consisting of C and all the C_j described in a direction such that the points of D are to the left of B. Let f be a function which is analytic in D. Then $\int_B f(z)dz = 0$.

Proof. Let L_1 be a polygonal path joining a point of C to a point of C_1; L_2 a polygonal path joining a point of C_1 to a point of C_2; ...... ; L_i a polygonal path joining a point of C_{i-1} to a point of C_i and L_{n+1} a polygonal path joining a point of C_n to a point of C such that no two L_i^s cross each other (refer figure).



This divides the region D into two simply connected regions D_1 and D_2. Let B_1 and B_2 denote the boundaries of D_1 and D_2 respectively.

By Cauchy's theorem for simply connected region

$$\int_{B_1} f(z)dz = 0 \text{ and } \int_{B_2} f(z)dz = 0.$$

Also $\int_{B_1} f(z)dz + \int_{B_2} f(z)dz = \int_B f(z)dz$ since the integrals along L_j are taken twice in the opposite directions and cancel each other.

$$\therefore \int_B f(z)dz = 0.$$

We observe that $B = C - C_1 - C_2 - \ldots - C_n$ and hence the above theorem can also be written in the form

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \ldots + \int_{C_n} f(z)dz.$$

In particular if C is a simple closed curve and C_0 is another simple closed curve lying in the interior of C and f is analytic in the region D consisting of all points inside and on C excluding the points interior to C_0 then $\int_C f(z)dz = \int_{C_0} f(z)dz$.

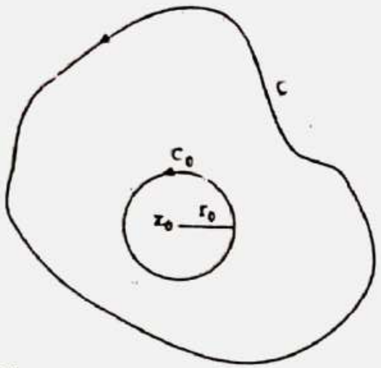
6.3. Cauchy's Integral Formula

In this section we establish another fundamental result known as Cauchy's integral formula using Cauchy's theorem.

Theorem 6.8. Let $f(z)$ be a function which is analytic inside and on a simple closed curve C. Let z_0 be any point in the interior of C.

$$\text{Then } f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz.$$

Proof. Choose a circle C_0 with centre z_0 and radius r_0 such that C_0 lies in the interior of C.



Now, z_0 is the only point inside C at which the function $\frac{f(z)}{z-z_0}$ is not analytic and hence is analytic in the region D consisting of all points inside and on C except the points interior to C_0.

$$\text{Hence } \int_C \frac{f(z)dz}{z-z_0} = \int_{C_0} \frac{f(z)dz}{z-z_0}$$

$$= \int_{C_0} \left(\frac{f(z)-f(z_0)+f(z_0)}{z-z_0}\right)dz$$

$$= \int_{C_0} \left(\frac{f(z)-f(z_0)}{z-z_0}\right)dz + \int_{C_0} \frac{f(z_0)}{z-z_0}dz$$

$$= \int_{C_0} \left(\frac{f(z)-f(z_0)}{z-z_0}\right)dz + f(z_0)\int_{C_0} \frac{dz}{z-z_0}$$

$$= \int_{C_0} \left(\frac{f(z)-f(z_0)}{z-z_0}\right)dz + f(z_0)(2\pi i).$$

$$\text{Thus } \int_C \frac{f(z)dz}{z-z_0} = \int_{C_0} \left(\frac{f(z)-f(z_0)}{z-z_0}\right)dz + 2\pi i f z_0 \quad \dots (1)$$

We now claim that $\int_{C_0} \left(\frac{f(z)-f(z_0)}{z-z_0}\right)dz = 0.$

Since $f(z)$ is analytic inside and on C it is continuous at z_0.

$\therefore$ Given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|z-z_0| < \delta \Rightarrow |f(z)-f(z_0)| < \varepsilon.$$

If we choose $r_0 < \delta$, then $|z-z_0| < r_0 \Rightarrow |f(z)-f(z_0)| < \varepsilon$.

$$\text{Hence } \left|\int_{C_0} \left(\frac{f(z)-f(z_0)}{z-z_0}\right)dz\right| < \left(\frac{\varepsilon}{r_0}\right)(2\pi r_0) \text{(by theorem 6.2)}$$

$$= 2\pi\varepsilon$$

$$\text{Thus } \left|\int_{C_0} \left(\frac{f(z)-f(z_0)}{z-z_0}\right)dz\right| < 2\pi\varepsilon.$$

Since ε is arbitrary we have $\int_{C_0} \left(\frac{f(z)-f(z_0)}{z-z_0}\right)dz = 0.$

$\therefore$ From (1) we get $\int_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0).$

$$\therefore f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz.$$

Theorem 6.9. Let $f(z)$ be analytic in a region D bounded by two concentric circles C_1 and C_2 and on the boundary. Let z_0 be any point in D. Then

$$f(z_0) = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-z_0} - \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z-z_0}dz.$$

Proof. Let L_1 and L_2 be two disjoint line segments not passing through z_0 both joining a point of C_1 to a point of C_2 as shown in the figure. This divides the region D into two simply connected regions D_1 and D_2. Let B_1 and B_2 denote the oriented boundary of D_1 and D_2 respectively. Then $B_1 + B_2 = C_1 - C_2$ ... (1)

We assume without loss of generality that $z_0 \in D_1$.

By Cauchy's integral formula,

$$\frac{1}{2\pi i}\int_{B_1} \frac{f(z)}{z-z_0}dz = f(z_0) \quad \dots (2)$$

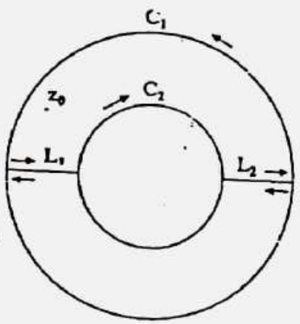
Also $\frac{f(z)}{z-z_0}$ is analytic in D_2 and hence by Cauchy's theorem

$$\frac{1}{2\pi i}\int_{B_2}\frac{f(z)}{z-z_0}dz = 0 \qquad \dots (3)$$

Adding (2) and (3) and using (1) we get:

$$f(z_0) = \frac{1}{2\pi i}\int_{C_1-C_2}\frac{f(z)}{z-z_0}dz$$

$$= \frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{z-z_0}dz - \frac{1}{2\pi i}\int_{C_2}\frac{f(z)}{z-z_0}dz$$



Example 1. Consider $\int_C \frac{dz}{z-3}$ where C is the circle $|z-2| = 5$.

Let $f(z) = 1$

The point $z = 3$ lies inside C.

Hence by Cauchy's integral formula $\int_C \frac{dz}{z-3} = 2\pi i f(3) = 2\pi i$.

Example 2. Let C denote the unit circle $|z| = 1$

Then $\int_C \frac{e^z}{z}dz = \int_C \frac{e^z}{z-0}dz = 2\pi i e^0 = 2\pi i$.

Theorem 6.10. Let $f(z)$ be analytic inside and on the circle C with centre a and radius r. Then $f(a) = \frac{\int_0^l f(z)ds}{l}$ where s is the arc length and l is the circumference of the circle.

(i.e) The value of the function at the centre is equal to the mean of the value of the function on the circumference.

Proof. By Cauchy's integral's formula we have $f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)}dz$.

Now the equation of the circle C is given by $z = a + re^{i\theta}$ where $0 \le \theta \le 2\pi$.

$$\therefore dz = ire^{i\theta}d\theta.$$

$$\therefore f(a) = \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(a+re^{i\theta})}{re^{i\theta}}\left(ire^{i\theta}\,d\theta\right)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})d\theta.$$

Also we have $s = r\theta$ and s varies from 0 to l.

$$\therefore d\theta = \frac{ds}{r}$$

$$\therefore f(a) = \frac{1}{2\pi r}\int_0^l f(a+re^{i\theta})\,ds$$

$$= \frac{1}{l}\int_0^l f(z)\,ds.$$

Hence the theorem.

Theorem 6.11. (Maximum Modulus Theorem) Let $f(z)$ be continuous in a closed and bounded region D and analytic and nonconstant in the interior of D. Then $|f(z)|$ attains its maximum value on the boundary of D and never in the interior of D.

Proof. Since f is continuous in a closed and bounded region D, $|f(z)|$ is bounded and attains its bound.

$\therefore$ There exists a positive real number M such that

$$|f(z)| \le M \text{ for all } z \in D \qquad \dots (1)$$

and equality holds for at least one point z in D. Suppose that there exists an interior point $z_0 \in D$ such that

$$|f(z_0)| = M \qquad \dots (2)$$

Choose a circle with centre z_0 and radius r such that the circular disc $|z - z_0| \le r$ is contained in D. Then we have

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})e^{i\theta}d\theta. \text{ (refer proof of theorem 6.10)}$$

$$\therefore |f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi}|f(z_0+re^{i\theta})|d\theta \qquad \dots (3)$$

Also from (1) and (2) we have $|f(z_0 + re^{i\theta})| \le |f(z_0)|$

$$\therefore \int_0^{2\pi} |f(z_0 + re^{i\theta})| d\theta \le 2\pi |f(z_0)|$$

$$\therefore |f(z_0)| \ge \frac{1}{2\pi} \int_C^{2\pi} |f(z_0 + re^{i\theta})| d\theta \qquad \ldots (4)$$

From (3) and (4) we get $|f(z_0)| = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{i\theta})| d\theta$

$$\therefore \quad 2\pi |f(z_0)| = \int_0^{2\pi} |f(z_0 + re^{i\theta})| d\theta.$$

$$\int_0^{2\pi} |f(z_0)| d\theta = \int_0^{2\pi} |f(z_0 + re^{i\theta})| d\theta$$

$$\therefore \int_0^{2\pi} \left[|f(z_0)| - |f(z_0 + re^{i\theta})| \right] d\theta = 0.$$

Since the integrand in the above expression is continuous and non- negative we have $|f(z_0)| - |f(z_0 + re^{i\theta})| = 0$

(ie) $|f(z_0)| = |f(z_0 + re^{i\theta})|$ for all z in the circular disc $|z - z_0| < r$.

(ie) $|f(z_0)| = |f(z)|$ for all z in the circular disc.

$\therefore$ $f(z)$ is constant in a neighbourhood of z_0.

Since $f(z)$ is continuous it follows that $f(z)$ is constant throughout D which is a contradiction.

$\therefore$ The maximum of $|f(z)|$ is not attained at any of the interior points of D. Hence the theorem.

Solved Problems

Problem 1. Evaluate using Cauchy's integral formula

$$\frac{1}{2\pi i} \int_C \frac{z^2+5}{z-3} dz \text{ where } C \text{ is } |z| = 4.$$

Solution. $f(z) = z^2 + 5$ is analytic inside and on $|z| = 4$ and $z = 3$ lies inside it.

$\therefore$ By Cauchy's integral formula $\frac{1}{2\pi i} \int_C \frac{z^2+5}{z-3} dz = f(3) = 3^2 + 5 = 14.$

Problem 2. Evaluate $\int_C \frac{z\,dz}{z^2-1}$ where C is the positively oriented circle $|z| = 2$.

Solution. $\frac{1}{z^2-1} = \frac{1}{(z+1)(z-1)} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right).$

$$\therefore \int_C \frac{z}{z^2-1} dz = \frac{1}{2} \int_C \frac{z}{z-1} dz - \frac{1}{2} \int_C \frac{z\,dz}{z+1}.$$

$f(z) = z$ is analytic and $1, -1$ lie in the interior of C.

$\therefore$ By Cauchy's integral formula $\int_C \frac{z\,dz}{z-1} = 2\pi i f(1) = 2\pi i.$

$$\text{Also } \int_C \frac{z dz}{z+1} = 2\pi i f(-1) = -2\pi i.$$

$$\therefore \int_C \frac{z\,dz}{z^2-1} = \frac{1}{2}(2\pi i) - \frac{1}{2}(-2\pi i) = 2\pi i.$$

Problem 3. Evaluate $\int_C \frac{e^z}{z^2+4} dz$ where C is positively oriented circle $|z - i| = 2$.

Solution. $\frac{1}{z^2+4} = \frac{1}{(z+2i)(z-2i)}$

$$= \frac{1}{4i}\left(\frac{1}{z-2i} - \frac{1}{z+2i}\right) \text{ (by partial fraction).}$$

Now, $2i$ lies inside C and by Cauchy's integral formula we have $\int_C \frac{e^z}{z-2i} dz = 2\pi i e^{2i}$.

Also $-2i$ lies outside C and hence $\frac{e^z}{z+2i}$ is analytic inside and on C.

Hence by Cauchy's theorem $\int_C \frac{e^z}{z+2i} dz = 0.$

$$\therefore \int_C \frac{e^z}{z^2+4} dz = \frac{1}{4i}(2\pi i e^{2i} - 0) = \frac{\pi}{2} e^{2i}.$$

Problem 4. Evaluate $\int_C \left(\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\right) dz$ where C is the circle $|z| = 3$.

Solution. By partial fractions $\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$.

Let $f(z) = \sin \pi z^2 + \cos \pi z^2$. Then $f(z)$ is analytic inside and on C and the points 1 and 2 lie inside C. Hence by Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-1} dz = 2\pi i f(1)$$

$$= 2\pi i(\sin \pi + \cos \pi)$$

$$= -2\pi i$$

Similarly $\int_C \frac{f(z)}{z-2} dz = 2\pi i f(2)$

$$= 2\pi i(\cos 4\pi + \sin 4\pi)$$

$$= 2\pi i.$$

Hence $\int_C \frac{f(z)}{(z-1)(z-2)} dz = 2\pi i - (-2\pi i) = 4\pi i$

Problem 5. Let C denote the boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$ where C is described in the positive sense.

Evaluate (i) $\int_C \frac{z\, dz}{2z+1}$ and (ii) $\int_C \frac{\cos z}{z(z^2+8)}$.

Solution.

(i) $\int_C \frac{z\, dz}{2z+1} = \frac{1}{2}\int_C \frac{z\, dz}{z+\frac{1}{2}}$

$$= \frac{1}{2}(2\pi i)\left(-\frac{1}{2}\right) \text{ (by Cauchy's integral formula)}$$

$$= \frac{-\pi i}{2}.$$

(ii) Let $f(z) = \frac{\cos z}{z^2+8}$. The points where $f(z)$ is not analytic are $\pm i2\sqrt{2}$ and these points lie outside C. Hence $f(z)$ is analytic inside and on C.

$\therefore$ By Cauchy's integral formula

$$\int_C \frac{\cos z}{z(z^2+8)} dz = \int_C \frac{f(z)}{z} dz = 2\pi i f(0) = 2\pi i\left(\frac{1}{8}\right) = \frac{\pi i}{4}.$$

Problem 6. Evaluate $\frac{z\, dz}{(9-z^2)(z+i)}$ where C is the circle $|z| = 2$ taken in the positive sense.

Solution. Let $f(z) = \frac{z}{9-z^2}$. Clearly $f(z)$, is analytic within and on C.

$\therefore$ By Cauchy's integral formula

$$\int_C \frac{z\, dz}{(9-z^2)(z+i)} = \int_C \frac{f(z)}{z+i} dz$$

$$= 2\pi i f(-i)$$

$$= 2\pi i\left(\frac{-i}{10}\right) = \frac{\pi}{5}.$$

Exercises

1. Prove that $\int_C \frac{z\, dz}{z^2-1} = 2\pi i$ where C is the positively oriented circle $|z| = 2$.
2. Evaluate $\int_C \frac{dz}{z^2+4}$ where C is $|z - i| = 2$ in the positive sense.
3. Evaluate $\int_C \frac{e^z\, dz}{z^2+1}$ where C is the circle of radius 1 with centre at
   (i) $z = i$ and
   (ii) $z = -i$.
4. Evaluate $\int_C \frac{\cos \pi z}{z^2-1} dz$ where C is a rectangle with vertices at (i) $2 \pm i, -2 \pm i$ and (ii) $-i, 2-i, 2+i, i$.
5. Show that $\frac{1}{2\pi i}\int_C \frac{e^{zt}\, dz}{z^2+1} = \sin t$ if $t > 0$ and C is the circle $|z| = 3$.

6. Evaluate $\int_C \frac{e^{3z}\,dz}{z-\pi i}$ where C is the circle $|z-1|=4$.

7. Evaluate $\int_C \frac{\sin 3z}{z+\pi/2}$ where C is the circle $|z|=5$.

8. Evaluate $\int_C \frac{dz}{z-3-i}$ where C is

(i) the square bounded by the real and imaginary axis and the lines $x=1$ and $y=1$;

(ii) the rectangle bounded by the real and imaginary axes and the lines $x=4$ and $y=3$ described in the anti-clockwise direction.

9. Evaluate $\int_C \frac{dz}{z^2(z-1)}$ where C is (i) $|z|=\frac{3}{4}$ (ii) $|z|=\frac{3}{2}$.

10. Evaluate $\int_C \frac{3z-1}{z^3-z}\,dz$ where C is (i) $|z|=\frac{1}{2}$ (ii) $|z|=2$.

11. Evaluate $\int_C \frac{dz}{z^3(z-1)}$ where C is (i) $|z|=2$ (ii) $|z-1|=\frac{1}{2}$.

12. Evaluate $\int_C \frac{dz}{z^2(z^2+4)}$ where C is

(i) $|z|=\frac{3}{2}$ (ii) $|z|=1$ (iii) $|z-2i|=3$

(iv) $|z+2i|=3$ (v) $|z|=3$.

13. Evaluate the integral $\int_C \frac{z\,dz}{z^4-1}$ where C is the circle $|z-2|=2$.

14. Evaluate $\int_C \frac{(z+4)\,dz}{z^2+2z+5}$ where C is

(i) $|z|=1$ (ii) $|z+1-i|=2$ (iii) $|z+1+i|=2$

15. Evaluate $\int_C \frac{z\,dz}{(9-z^2)(z+1)}$ where C is the circle $|z|=2$.

16. Prove $\int_C \frac{dz}{z^2+1}=0$ where C is the positively oriented circle $|z|=2$.

Answers.

2. $\frac{\pi}{2}$ 3. (i) $\pi(\cos 1+i\sin 1)$ (ii) $-\pi(\cos 1-i\sin 1)$ 4. (i) 0 (ii) $-\pi i$ 6. $-2\pi i$ 7. $2\pi i$ 8. (i) 0 (ii) $2\pi i$ 9. (i) $-2\pi i$ (ii) 0 10. (i) $2\pi i$ (ii) 0 11. (i) $-2\pi i$ (ii) $2\pi i$ 12. (i) 0 (ii) 0 (iii) 2π (iv) -2π (v) 0 13. $\frac{\pi i}{2}$ 14. (i) 0 (ii) $\pi\left(3+\frac{7}{2}\right)$ (iii) $\frac{\pi(2i-3)}{2}$ 15. $-\frac{\pi i}{4}$.

6.4. Higher Derivatives

In this section we shall prove that an analytic function has derivatives of all orders. It follows, in particular, that *the derivative of an analytic function is again an analytic function.*

Consider a function $f(z)$ which is analytic in a region D. Let $z\in D$. Let C be any circle with centre z such that the circle and its interior is contained in D. By Cauchy's integral formula we have

$$f(z)=\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta-z}\,d\zeta.$$

We now proceed to prove that $f'(z)=\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z)^2}\,d\zeta$

and in general $f^{(n)}(z)=\frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z)^{n+1}}\,d\zeta.$

Theorem 6.12. Let f be analytic inside and on a simple closed curve C. Let z be any point inside C. Then $f'(z)=\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z)^2}\,d\zeta$

Proof. By Cauchy's integral formula we have $f(z)=\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta-z}\,d\zeta$

$$\therefore \frac{f(z+h)-f(z)}{h}=\frac{1}{h(2\pi i)}\int_C\left(\frac{f(\zeta)}{\zeta-z-h}-\frac{f(\zeta)}{\zeta-z}\right)d\zeta$$

$$=\frac{1}{h\,2\pi i}\int_C\left[\frac{hf(\zeta)}{(\zeta-z-h)(\zeta-z)}\right]d\zeta$$

$$= \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z - h)(\zeta - z)} \qquad \ldots (1)$$

$$\text{Now} \int_C \frac{f(\zeta)d\zeta}{(\zeta - z - h)(\zeta - z)} - \int_C \frac{f(\zeta)d\zeta}{(\zeta - z)^2}$$

$$= \int_C \left[\frac{f(\zeta)}{(\zeta - z - h)(\zeta - z)} - \frac{f(\zeta)}{(\zeta - z)^2}\right] d\zeta$$

$$= \int_C \frac{f(\zeta)}{(\zeta - z)}\left(\frac{1}{\zeta - z - h} - \frac{1}{\zeta - z}\right) d\zeta$$

$$= \int_C \frac{f(\zeta)}{(\zeta - z)}\left[\frac{h}{(\zeta - z - h)(\zeta - z)}\right] d\zeta$$

$$= h\int_C \frac{f(\zeta)d\zeta}{(\zeta - z - h)(\zeta - z)^2}$$

$$\therefore \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z - h)(\zeta - h)} - \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z)^2}$$

$$= \frac{h}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z - h)(\zeta - z)^2}$$

$$\therefore \frac{f(z+h) - f(z)}{h} - \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z)^2}$$

$$= \frac{h}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z - h)(\zeta - z)^2} \text{(using(1))} \qquad \ldots (2)$$

Now, let M denote the maximum value of $|f(\zeta)|$ on C. Let L be the length of C and d be the shortest distance from z to any point on the curve C.

$\therefore$ For any point ζ on C we have

$$|\zeta - z| \geq d \text{ and } |\zeta - z - h| \geq |\zeta - z| - |h| \geq d - |h|$$

$$\text{Hence } \left|\frac{f(\zeta)}{(\zeta - z)^2(\zeta - z - h)}\right| \leq \frac{M}{d^2(d - |h|)}$$

From (2) we get

$$\left|\frac{f(z+h) - f(z)}{h} - \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z)^2}\right| \leq \frac{|h|}{2\pi}\left(\frac{M}{d^2(d - |h|)}\right)$$

$$\therefore \lim_{h \to 0}\left(\frac{f(z+h) - f(z)}{h} - \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z)^2}\right) = 0$$

$$\therefore \lim_{h \to 0}\frac{f(z+h) - f(z)}{h} = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z)^2}$$

$$\therefore f'(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z)^2}$$

Remark. By using induction on n we can prove that for any positive integer n we have $f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta.$

Note. Thus an analytic function has derivatives of all orders and the derivate of an analytic function is again analytic.

Example 1. $\int_C \frac{e^z}{z^n}\, dz = \frac{2\pi i}{(n-1)!}$ where C is the circle $|z| = 1$.

Let $f(z) = e^z$. Clearly $f(z)$ is analytic and $f^{(n)}(z) = e^z$ for all n.

By the formula for higher derivatives

$$\int_C \frac{e^z}{z^n}\, dz = \int_C \frac{e^z}{(z-0)^n}\, dz = \frac{2\pi i}{(n-1)!} e^0 = \frac{2\pi i}{(n-1)!}$$

Example 2. $\int_C \frac{\sin^2 z}{(z - \pi/6)^3}\, dz = \pi i$ where C is the circle $|z| = 1$.

Solution. Let $f(z) = \sin^2 z$. Then $f'(z) = 2 \sin z \cos z = \sin 2z$. $f''(z) = 2\cos 2z$. Also $\pi/6$ lies inside C.

$$\therefore \int_C \frac{\sin^2 z}{(z-\pi/6)^3}\,dz = \frac{2\pi i}{2!} f''(\pi/6)$$

$$= \pi i(2\cos \pi/3)$$

$$= \pi i$$

Theorem 6.13. (**Cauchy's inequality**)

Let $f(z)$ be analytic inside and on the circle C with centre z_0 and radius r. Let M denote the maximum of $|f(z)|$ on C. Then $\left|f^{(n)}(z_0)\right| \le \frac{n!M}{r^n}$

Proof. We have $f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}$

$$\therefore \left|f^{(n)}(z_0)\right| \le \frac{n!}{2\pi}\left(\frac{M}{r^{n+1}}\right)(2\pi r) = \frac{n!M}{r^n}$$

Hence $\left|f^{(n)}(z_0)\right| \le \frac{n!M}{r^n}$

Theorem 6.14. (**Liouville's theorem**)

A bounded entire function in the complex plane is constant.

Proof. Let $f(z)$ be a bounded entire function.

Since $f(z)$ is bounded there exists a real number M such that $|f(z)| \le M$ for all z. Let z_0 be any complex number and $r > 0$ be any real number.

By Cauchy's inequality we have $|f'(z_0)| \le \frac{M}{r}$.

Taking the limit as $r \to \infty$ we get $f'(z_0) = 0$.

Since z_0 is arbitrary $f'(z) = 0$ for all z in the complex plane.

$\therefore f(z)$ is a constant function.

Theorem 6.14. (**Fundamental theorem of algebra**)

Every polynomial of degree ≥ 1 has atleast one zero (root) in C.

Proof. Let $f(z)$ be a polynomial of degree ≥ 1.

Suppose $f(z)$ has no zero in C. Then $f(z) \ne 0$ for all z.

Further $f(z)$ is an entire function in the complex plane.

$\therefore \frac{1}{f(z)}$ is also an entire function. Also as $z \to \infty$, $f(z) \to \infty$.

$\therefore \frac{1}{f(z)} \to 0$ as $z \to \infty$.

$\therefore \frac{1}{f(z)}$ is a bounded function.

Hence by Liouville's theorem $\frac{1}{f(z)}$ is a constant function.

$\therefore f(z)$ is a constant function and hence it is a polynomial of degree zero which is a contradiction.

Hence $f(z)$ has at least one root in C.

Hence the theorem.

Theorem 6.15. (**Morera's theorem**)

If $f(z)$ is continuous in a simply connected domain D and if $\int_C f(z)\,dz = 0$ for every simple closed curve C lying in D then $f(z)$ is analytic in D.

(This theorem is the converse of Cauchy's theorem)

Proof. By corollary 1 of 6.2 there exists an analytic function $F(z)$ such that $F'(z) = f(z)$ in D.

Also we know the derivative of an analytic function is an analytic function.

Hence $F'(z)$ is analytic in D.

$\therefore f(z)$ is analytic in D.

Solved Problems

Problem 1. Evaluate $\int_C \frac{\sin z}{(z-\pi/2)^2}\,dz$ where C is the circle $|z| = 2$.

Solution. Let $f(z) = \sin z$. Hence $f'(z) = \cos z$. Also $\pi/2$ lies inside $|z| = 2$.

$$\text{Hence } \int_C \frac{\sin z\,dz}{(z-\pi/2)^2} = 2\pi i f'(\pi/2)$$

$$= 2\pi i(\cos \pi/2)$$

$$= 0.$$

Problem 2. Evaluate $\int_C \frac{z^3\,dz}{(2z+i)^3}$ where C is the unit circle.

Solution. $\int_C \frac{z^3\,dz}{(2z+i)^3} = \frac{1}{8}\int_C \frac{z^3\,dz}{(z+i/2)^3}$

Let $f(z) = z^3$. Then $f'(z) = 3z^2$ and $f''(z) = 6z$

Also $-\frac{i}{2}$ lies inside C.

$$\text{Hence} \int_C \frac{z^3\,dz}{(2z+i)^3} = \frac{1}{8}\left(\frac{2\pi i}{2!}\right) f''\left(-\frac{i}{2}\right)$$

$$= \frac{2\pi i}{16}(-3i)$$

$$= \frac{3\pi}{8}$$

Problem 3. Evaluate $\int_C \frac{(e^z + z\sinh z)\,dz}{(z-\pi i)^2}$ where C is the circle $|z| = 4$.

Solution. Let $f(z) = e^z + z\sinh z$

Therefore $f'(z) = e^z + z\cosh z + \sinh z$

Also πi lies inside C.

$$\text{Hence} \int_C \frac{f(z)}{(z-\pi i)^2}\,dz = 2\pi i f'(\pi i)$$

$$= 2\pi i[e^{\pi i} + \pi i\cosh \pi i + \sinh \pi i]$$

$$= 2\pi i(-1 - \pi i)$$

$$= -2\pi i(1 + \pi i).$$

Problem 4. Show that when f is analytic within and on a simple closed curve C and z_0 is not on C then $\int_C \frac{f'(z)\,dz}{z-z_0} = \int_C \frac{f(z)\,dz}{(z-z_0)^2}$

Solution. *Case i.* Suppose z_0 is in the exterior of C. Then both $\frac{f(z)}{(z-z_0)^2}$ and $\frac{f'(z)}{z-z_0}$ are analytic inside and on C.

$\therefore$ By Cauchy's theorem $\int_C \frac{f'(z)\,dz}{z-z_0} = \int_C \frac{f(z)\,dz}{(z-z_0)^2} = 0$

Case ii. z_0 lies in the interior of C.

Then by Cauchy's integral formula, $\int_C \frac{f'(z)}{(z-z_0)}\,dz = 2\pi i\ f'(z_0)$.

Also by the formula for higher derivatives $\int_C \frac{f(z)}{(z-z_0)^2}\,dz = 2\pi i f'(z_0)$.

$$\text{Hence} \int_C \frac{f'(z)}{z-z_0}\,dz = \int_C \frac{f(z)}{(z-z_0)^2}\,dz.$$

Problem 5. Let the function $f(z) = u(x, y) + iv\ (x, y)$ be continuous in a closed bounded region D and let it be analytic and not constant in the interior of D. Show that the function $u(x, y)$ reaches its maximum value on the boundary of D and never in the interior of D.

Solution. Consider the function $e^{f(z)}$. Since $f(z)$ is continuous in a closed bounded region D and nonconstant in the interior of D, $e^{f(z)}$ is also continuous in the closed bounded region D and analytic and nonconstant in the interior of D.

Now, the maximum value of $|e^{f(z)}|$ is attained only at a boundary point of D. (by theorem 6.11).

$\therefore$ But $|e^{f(z)}| = e^{u(x,y)}$

$\therefore$ Maximum value $e^{u(x,y)}$ is attained only at a boundary point of D.

$\therefore$ Maximum value of $u(x, y)$ is attained only at a boundary point of D.

Problem 6. Evaluate $\int_C \frac{\sin 2z\ dz}{(z-\pi i/4)^4}$ where C is $|z| = 1$.

Solution. Let $f(z) = \sin 2z$. Since $f(z)$ is analytic and $\pi i/4$ lies inside C.

$$\therefore \int_C \frac{\sin 2z}{(z-\pi i)^4} = \frac{2\pi i}{3!} f'''\left(\frac{\pi i}{4}\right).$$

Now $f'(z) = 2\cos 2z$, $f''(z) = -4\sin 2z$; $f'''(z) = -8\cos 2z$.

$$\text{Hence } f'''(\pi i/4) = -8\cos(\pi i/2)$$

$$= -8\cosh(\pi/2).$$

$$\therefore \int_C \frac{\sin z}{(z-\pi i)^4}\,dz = -\frac{8\pi i}{3}\cosh(\pi/2).$$

Problem 7. Evaluate $\int_C \frac{e^{2z}}{(z+1)^4}\,dz$ where C is the circle $|z| = 2$.

Solution. Let $f(z) = e^{2z}$. Clearly $f(z)$ is analytic and

$f'(z) = 2e^{2z}$; $f''(z) = 4e^{2z}$; $f'''(z) = 8e^{2z}$

By the formula for higher derivatives

$$\int_C \frac{e^{2z}}{(z+1)^4}\,dz = \left(\frac{2\pi i}{3!}\right) f'''(-1)$$

$$= \left(\frac{2\pi i}{6}\right)(8e^{-2})$$

$$= \frac{i8\pi e^{-2}}{3}.$$

Problem 8. Evaluate $\int_C \frac{e^z}{(z+2)(z+1)^2}\,dz$ where C is $|z| = 3$.

Solution. $\frac{1}{(z+2)(z+1)^2} = \frac{(z+2)-(z+1)}{(z+2)(z+1)^2}$

$$= \frac{1}{(z+1)^2} - \frac{1}{(z+2)(z+1)}$$

$$= \frac{1}{(z+1)^2} - \frac{1}{z+1} + \frac{1}{z+2}$$

$$\int_C \frac{e^z}{(z+2)(z+1)}\,dz = \int_C \frac{e^z}{z+2}\,dz - \int_C \frac{e^z}{z+1}\,dz + \int_C \frac{e^z}{(z+1)^2}\,dz \quad \ldots (1)$$

We note that $z = -2, \ -1$ lie in the interior of C.

Let $f(z) = e^z$. It is analytic in C. Also $f'(z) = e^z$.

By Cauchy's integral formula

$$\int_C \frac{e^z}{z+2}\,dz = 2\pi i\ f(-2) = 2\pi i e^{-2}.$$

$$\int_C \frac{e^z}{z+1}\,dz = 2\pi i\ f(-1) = 2\pi i e^{-1}.$$

$$\int_C \frac{e^z}{(z+1)^2}\,dz = \left(\frac{2\pi i}{1!}\right) f'(-1) = 2\pi i e^{-1}.$$

$\therefore$ From (1) $\int_C \frac{e^z}{(z+2)(z+1)^2}\,dz = 2\pi i\left[e^{-2} - e^{-1} + e^{-1}\right]$

$$= 2\pi i e^{-2}$$

Exercises.

Evaluate the following

1. $\int_C \frac{(z+2)\,dz}{z^2}$ where C is $|z| = 1$
2. $\int_C \frac{\cos z\,dz}{(z-\pi/2)^2}$ where C is $|z| = 2$
3. $\int_C \frac{e^{2z}\,dz}{(z-1)^4}$ where C is $|z| = \frac{3}{2}$

$\int_C \frac{e^{az}\,dz}{z^{n+1}}$ where C is $|z| = \frac{1}{2}$

5. If C is $|z| = 2$ prove the following.
   (i) $\int_C \frac{e^z\,dz}{z-1} = 2\pi i e$
   (ii) $\int_C \frac{e^z\,dz}{(z-1)^3} = \pi i e$
   (iii) $\int_C \frac{e^z\,dz}{(z-4)^4} = \frac{\pi i e}{3}$
6. Evaluate $\int_C \frac{e^{iz}}{z^3}\,dz$ where C is $|z| = 2$
7. Evaluate $\int_C \frac{dz}{z^2(z-3)}$ where C is $|z| = 2$
8. Evaluate $\int_C \frac{\tan z\,.dz}{(z-\pi/4)^2}$ where C is $|z| = 1$
9. Evaluate the integral $\frac{1}{2\pi i}\int_C \frac{ze^z\,dz}{(z-a)^3}$ if the point lies inside the simple closed curve C.
10. Evaluate the integral $\int_C \frac{e^z\,dz}{z(1-z)^3}$ when
    (i) the point 0 lies inside and the point 1 is outside C.
    (ii) the point 1 lies inside and the point 0 is outside C.
    (iii) the points 0 and 1 both lie inside C.

11. Prove that $\int_C \frac{dz}{(z^2+4)^2} = \frac{\pi}{6}$ where C is $|z-i| = 2$

12. Evaluate $\int_C \frac{z^2\, dz}{(2z-1)^2}$ where C is $|z| = 1$

13. Evaluate $\int_C \frac{(z^3+1))}{(3z+1)^3}$ where C is $|z| = 1$

14. Evaluate $\int_C \frac{z\, e^z\, dz}{(4z+\pi i)^2}$ where C is $|z| = 1$

15. Evaluate $\int_C \frac{\sin z\; dz}{z^2\, e^z}$ where C is $|z| = 1$

16. Evaluate $\int_C \frac{\cos z\; dz}{z^{2n+1}}$ where C is $|z| = 1$

17. Evaluate $\int_C \frac{\sin^6 z\; dz}{(z-\pi/6)^3}$ where C is $|z| = 1$

18. Prove that $\frac{1}{2\pi i}\int_C \frac{e^{zt}\, dz}{(z^2+1)^2} = \frac{t \sin t}{2}$ if $t > 0$ and C is the circle $|z| = 3$.

19. If C is a closed curve described in the positive sense and $\varphi(z_0) = \int_C \frac{(z^4+z)\, dz}{(z-z_0)^3}$ show that $\varphi(z_0) = 12\pi i z_0^2$ when z_0 is inside C and $\varphi(z_0) = 0$ when z_0 lies outside C.

Answers

1. $2\pi i$ 2. $-2\pi i$ 3. $\frac{2\pi i e^2}{3}$ 4. $\frac{a^n 2\pi i}{n!}$

6. $-\pi i$ 7. $\frac{-2\pi i}{9}$ 9. $e^a(1+a/2)$ 10. (i) $2\pi i$

(ii) $-\pi e i$ (iii) $(2-e)\pi i$

12. $\frac{\pi i}{2}$ 13. $\frac{-4\pi i}{27}$ 14. $\frac{\pi}{\sqrt{2}}\left[1+\frac{\pi}{4}+i\left(1-\frac{4}{\pi}\right)\right]$

15. $2\pi i$ 16. $\frac{(-1)^n 2\pi i}{(2n)!}$ 17. $\frac{21\pi i}{16}$

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7. Series Expansions

7.0. Introduction

In this chapter we consider the problem of representing a given function as a power series. We prove that if a function is analytic at a point z_0 then it can be expanded as a power series called *Taylor's series* consisting of non-negative powers of $z - z_0$ and the expansion is valid in some neighbourhood of z_0. We also prove that a function $f(z)$ which is analytic in an annular region $a < |z - z_0| < b$ can be expanded as a series called *Laurent's series* consisting of positive and negative powers of $z - z_0$. We also introduce the concept of *singular points* of a function and classify the singular points and discuss the behaviour of the function in the neighbourhood of a singularity.

7.1. Taylor's Series

Theorem 7.1. (Taylor's theorem)
Let $f(z)$ be analytic in a region D containing z_0. Then $f(z)$ can be represented as a power series in $z - z_0$ given by

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + \cdots\cdots$$

The expansion is valid in the largest open disc with centre z_0 contained in D.

Proof. Let $r > 0$ be such that the disc $|z - z_0| < r$ is contained in D.

Let $0 < r_1 < r$. Let C_1 be the circle $|z - z_0| = r_1$.

By Cauchy's integral formula we have

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{(\zeta - z)} d\zeta \quad \ldots (1)$$

Also by theorem on higher derivatives we have

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{C_1} \frac{f(\zeta)d\zeta}{(\zeta - z)^{n+1}} \quad \ldots (2)$$