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CHAPTER I
Indefinite Integral

Introduction

When our study calculus, focused our attention on the operation differentiation, i.e. defined and use of derivatives function this side of the topic called "Differential Calculus".

Now we will study the second part of the calculus, which is called "Integral Calculus" .

Example

We saw in the calculus that if the s(t) the distance traveled by a moving object in a time t, the instantaneous velocity v(t) be equal to the derivative distance s (t) any that v (t) = s ′(t), which he credited to velocity v, we simply discover which distance s (t), but may happen to be known for is our function velocity v(t) and calculate the velocity s, in such a case be known to us is derived s′(t) is required to find the function s(t), and this inverse operation of the differentiation.

In other when we studied the differentiation derivation functions in certain intervals given the known range of the function, will study here reverse operation to process differentiation, in the sense that if we gave a function f (x) on the open (a, b) is required to find another function F (x) on the same period so that:

$$F'(x) = f(x)$$

We will review here the most important basic rules that we studied in the calculus, and will symbolized the symbol D derivative function, in the form:

$$Df(x) = d/dx\, f(x) = f'(x)$$

If the functions f (x), g (x), h (x) the differential functions then the following are true:

1- Differential of summation functions

$$D[f(x) \pm g(x)] = Df(x) \pm Dg(x) = f'(x) \pm g'(x)$$

2- Multiplying the differential function in a fixed amount

$D[cf(x)] = c\,Df(x) = c\,f'(x)$ c is an arbitrary constant

3- Differential of multiplication functions

$$D[f(x)\,g(x)] = f(x)\,Dg(x) + g(x)\,Df(x) = f(x)\,g'(x) + g(x)\,f'(x)$$

4- Differential of quotients of two functions

$$D\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)Df(x) - f(x)Dg(x)}{[g(x)]^2} \quad ; \quad g(x) \neq 0 .$$

The chain rule

If y=f(u) since u=g(x) and y=f(g(x)) then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = f'(u)\frac{du}{dx} = f'[g(x)]g'(x)$$

Result:

If y=f(u) since u=g(v) and v=h(x) then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}$$

Differential of inverse functions

If y=f(x) then $x = f^{-1}(y)$ and say the inverse function for $\frac{dy}{dx}$, $\frac{dx}{dy}$ br the relations : $\frac{dy}{dx} = \frac{1}{dx/dy}$.

For example, differential inverse trigonometric functions and inverse hyperbolic trigonometric functions.

Differential of parametric functions

If y=f(t) and y=g(t) then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g(t)}{f(t)}$$

Differential of some special functions

(1) $D(c) = c' = 0$

(2) $Dx^n = nx^{n-1} \Rightarrow Du^n = nu^{n-1}\frac{du}{dx}$

(3) $D\sin x = \cos x \Rightarrow D\sin u = \cos u\frac{du}{dx}$

(4) $D\cos x = -\sin x \Rightarrow D\cos u = -\sin u\frac{du}{dx}$

(5) $D\tan x = \sec^2 x \Rightarrow D\tan u = \sec^2 u\frac{du}{dx}$

(6) $D\cot x = \operatorname{cos}ec^2 x \Rightarrow D\tan u = -\operatorname{cos}ec^2 u\frac{du}{dx}$

(7) $D\sec x = \sec x \tan x \Rightarrow D\sec u = \sec u \tan u\frac{du}{dx}$

(8) $D\operatorname{cos}ecx = -\operatorname{cos}ecx\cot x \Rightarrow D\operatorname{cos}ecu = -\operatorname{cos}ecu\cot u\frac{du}{dx}$

(9) $D\log_a x = \frac{1}{x}\log_a e = \frac{1}{x\ln a} \Rightarrow D\log_a u = \frac{1}{u\ln a}.\frac{du}{dx}$; $a \succ 0,$ $a \neq 1$

(10) $D\ln u = \frac{1}{u}\frac{du}{dx}$ $\log_a e = \ln e = 1;$ $a = e$

(11) $Da^x = a^x \ln a \Rightarrow Da^u = a^u \ln a \frac{du}{dx}$

(12) $De^x = e^u \Rightarrow De^u = e^u \frac{du}{dx}$

(13) $D\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \Rightarrow D\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

(14) $D\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}} \Rightarrow D\cos^{-1} u = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$

(15) $D\tan^{-1} x = \frac{1}{1+x^2} \Rightarrow D\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx}$

(16) $D\cot^{-1} x = \frac{-1}{1+x^2} \Rightarrow D\cot^{-1} u = \frac{-1}{1+u^2}\frac{du}{dx}$

(17) $D\sec^{-1} x = \pm\frac{1}{x\sqrt{x^2-1}}$ $\begin{cases} +if & x>1 \\ -if & x<-1 \end{cases}$

$\Rightarrow D\sec^{-1} u = \pm\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$ $\begin{cases} +if & x>1 \\ -if & x<-1 \end{cases}$

(18) $D\mathrm{cos}ec^{-1} x = \mp\frac{-1}{x\sqrt{x^2-1}}$ $\begin{cases} -if & x>1 \\ +if & x<-1 \end{cases}$

$\Rightarrow D\sec^{-1} u = \mp\frac{-1}{u\sqrt{u^2-1}}\frac{du}{dx}$ $\begin{cases} -if & x>1 \\ +if & x<-1 \end{cases}$

Properties of Indefinite Integral

Suppose that $f(x)$; $g(x)$ two integrals function then, we have

1- $\int cf(x)dx = c\int f(x)dx$
2- $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

Some properties for the indefinite integrals:

Power rule

$$\int x^r dx = \frac{1}{r+1}x^{r+1} + c; \qquad r \neq -1$$

Example:

Find the following integral

$$\int (6x^4 - 10x^2 + 7 + \frac{1}{2\sqrt{x}})dx$$

Solution :

$$\int (6x^4 - 10x^2 + 7 + \frac{1}{\sqrt{x}})dx = 6\int x^4 dx - 10\int x^2 dx + 7\int dx + \frac{1}{2}\int x^{-\frac{1}{2}} dx$$

then we have

$$6\int x^4 dx - 10\int x^2 dx + 7\int dx + \frac{1}{2}\int x^{-\frac{1}{2}} dx = \frac{6}{5}x^5 - \frac{10}{3}x^3 + 7x + x^{\frac{1}{2}}$$

and

$$\int (6x^4 - 10x^2 + 7 + \frac{1}{2\sqrt{x}})dx = \frac{6}{5}x^5 - \frac{10}{3}x^3 + 7x + x^{\frac{1}{2}} + c$$

Example:

Find the following integral $\int 6x(5-x)dx$

Solution:

$$\int 6x(5-x)dx = 6\int (5x - x^2)dx = 6(\frac{5}{2}x^2 - \frac{1}{3}x^3) + c = x^2(15-2x) + c$$

Example:

Find the following integral $\int [4t-3)/t^3]dt$

Solution:

We see that

$$\int \frac{4t-3}{t^3}dt = \int (\frac{4}{t^2} - \frac{3}{t^3})dt = 4\int t^{-2}dt - 3\int t^{-3}dt = -4t^{-1} - \frac{3}{2}t^{-2} + c$$
$$= (\frac{-4}{t} - \frac{3}{2t^2}) + c = \frac{-8t+3}{2t^2} + c$$

Generalized power rule

$$\int g^r(x).g'(x)dx = \frac{1}{r+1}g^{r+1}(x) + c; \qquad r \neq -1$$

Example:

Find the following integral $\int 4x^2\sqrt{5-x^3}dx$

Solution:

Put the $g(x) = 5 - x^3$ we see that

$g^{\frac{1}{2}}(x).g'(x)$ i.e. $r = \frac{1}{2}$ we get

$$\int 4x^2\sqrt{5-x^3}dx = 4\int (5-x^3)^{\frac{1}{2}}(-3x^2)(-\frac{1}{3})dx = -\frac{4}{3}\int (5-x^3)^{\frac{1}{2}}(-3x^2)dx$$
$$= -\frac{4}{3}(5-x^3)^{\frac{1}{2}+1}(\frac{2}{3}) + c = -\frac{8}{9}(5-x^3)^{\frac{3}{2}} + c$$

Some different rules for integral functions which can be written as follows:

(1) $\int x^{r}dx=\frac{1}{r+1}x^{r+1}+c\quad ;\quad r\neq -1$

(2) $\int g^{r}(x).g'(x)dx=\frac{1}{r}g^{r+1}(x)+c\quad ;\quad r\neq -1$

(3) $\int \frac{dx}{x}=\ln|x|+c$

(4) $\int \frac{g'(x)}{g(x)}=\ln|g(x)|+c$

(5) $\int e^{x}dx=e^{x}+c$

(6) $\int e^{g(x)}g'(x)dx=e^{g(x)}+c$

(7) $\int \sin\alpha x dx=\frac{-1}{\alpha}\cos\alpha x+c$

(8) $\int \cos\alpha x dx=\frac{1}{\alpha}\sin\alpha x+c$

(9) $\int \sec^{2}\alpha x dx=\frac{1}{\alpha}\tan\alpha x+c$

(10) $\int \operatorname{cosec}^{2}\alpha x dx=-\frac{1}{\alpha}\cot\alpha x+c$

(11) $\int \sec\alpha x\tan\alpha x dx=\frac{1}{\alpha}\sec\alpha x+c$

(12) $\int \operatorname{cosec}\alpha x\cot\alpha x dx=-\frac{1}{\alpha}\operatorname{cosec}\alpha x+c$

(13) $\int \frac{dx}{\sqrt{a^{2}-x^{2}}}=\sin^{-1}\frac{x}{a}+c;\quad a\succ 0$

(14) $\int \frac{-dx}{\sqrt{a^{2}-x^{2}}}=\cos^{-1}\frac{x}{a}+c;\quad a\succ 0$

(15) $\int \frac{dx}{a^{2}+x^{2}}=\frac{1}{a}\tan^{-1}\frac{x}{a}+c;\quad a\neq 0$

(16) $\int \frac{-dx}{a^{2}+x^{2}}=\frac{1}{a}\cot^{-1}\frac{x}{a}+c;\quad a\neq 0$

(17) $\int \frac{dx}{x\sqrt{x^{2}-a^{2}}}=\frac{1}{a}\sec^{-1}\frac{x}{a}+c;\quad a\prec x$

(18) $\int \frac{-dx}{x\sqrt{x^{2}-a^{2}}}=\frac{1}{a}\operatorname{cosec}^{-1}\frac{x}{a}+c;\quad a\prec x$

Examples :

Example:

Find the following integrals

(i) $\int dt$

(ii) $\int (x+5)dx$

(iii) $\int (ax+b)dx$

(iv) $\int (3x^2-8x)dx$

(v) $\int b(x-a)dx$

(vi) $\int \frac{5}{\sqrt{3}}dx$

Solution:

Now, we find the following integrals

(i) $\int dt = t+c$

(ii) $\int (x+5)dx = \frac{1}{2}x^2+5x+c$

(iii) $\int (ax+b)dx = \int axdx + b\int dx = \frac{a}{2}x^2+bx+c$

(iv) $\int (3x^2-8x)dx = 3\int x^2dx - 8\int xdx = 3.\frac{1}{3}x^3 - 8.\frac{1}{2}x^2 + c$

$= x^3-4x^2+c$

(v) $\int b(x-a)dx = b(\int xdx - a\int dx) = b(\frac{1}{2}x^2-ax)+c = \frac{1}{2}bx(x-2a)+c$

(vi) $\int \frac{5}{\sqrt{3}}dx = \frac{5}{\sqrt{3}}\int dx = \frac{5}{\sqrt{3}}x+c$

Example:

Find the following integrals

(i) $\int \pi r^2 dr$

(ii) $\int (t^3 + t + 5)/7dt$

(iii) $\int 4x^2(x+1)dx$

(iv) $\int (y+5)^2 dy$

(v) $\int [x(x^3+1) - \frac{1}{x^4}]dx$

(vi) (vi) $\int \frac{4}{\sqrt[3]{t}} dt$

Solution:

Now, we find the following integrals

(i) $\int \pi r^2 dr = \pi \int r^2 dr = \pi r^3 + c$

(ii) $\int (t^3 + t + 5)/7dt = \frac{1}{7}(\int t^3 dt + \int tdt + 5\int dt) = \frac{1}{7}(\frac{t^4}{4} + \frac{t^2}{2} dt + 5t) + c$

(iii) $\int 4x^2(x+1)dx = 4\int (x^3 + x^2)dx = 4[\int (x^3 dx + \int x^2 dx]$

$$= 4[\frac{x^4}{4} + \frac{x^3}{3}] + c = x^4 + \frac{4}{3}x^3 + c$$

(iv) $\int (y+5)^2 dy = \frac{1}{3}(y+5)^3 + c$

or

$$\int (y+5)^2 dy = \int (y^2 + 10y + 25)dy = \frac{1}{3}y^3 + 5y^2 + 25y + c$$

(v) $\int [x(x^3+1) - \frac{1}{x^4}]dx = \int x^4 dx + \int xdx - \int x^{-4} dx = \frac{1}{5}x^5 + \frac{1}{2}x^2 + \frac{1}{3}x^{-3} + c$

(vi) $\int \frac{4}{\sqrt{t}} dt = 4\int t^{-\frac{1}{2}} dt = 4\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + c = 8t^{\frac{1}{2}} + c = 8\sqrt{t} + c$

Example:

Find the following integrals

(i) $\frac{1}{2}\int \sqrt{4-9x}dx$

(ii) $\int \frac{5x^2-12x-10}{x^4}dx$

(iii) $\int \sqrt{y}(3y+1)^2 dy$

(iv) $\int (3-4x)^{12}dx$

(v) $\int (3x+4)^{-6}dx$

(vi) $\int (8x-6)(4x^2-6x)^3 dx$

(vii) $\int [x+\frac{1}{(4x+3)^2}]dx$

(viii) $\int y\sqrt{2y^2-1}dy$

Solution:

Now, we find the following integrals

(i) $$\frac{1}{2}\int \sqrt{4-9x}dx=\frac{1}{2}\int (4-9x)^{\frac{1}{2}}(-9)(-\frac{1}{9})dx$$

$$=-\frac{1}{18}\int (4-9x)^{\frac{1}{2}}(-9)dx$$

$$=-\frac{1}{18}(4-9x)^{3/2}\frac{2}{3}+c$$

$$=-\frac{1}{27}(4-9x)^{3/2}+c$$

(ii) $$\int \frac{5x^2-12x-10}{x^4}dx=5\int x^{-2}dx-12\int x^{-3}dx-10\int x^{-4}dx$$

$$=\frac{5}{-1}x^{-1}-\frac{12}{-2}x^{-2}-\frac{10}{-3}x^{-3}+c$$

$$=\frac{5}{x}+\frac{6}{x^2}+\frac{10}{3x^3}+c$$

$$=\frac{-15x^2+18x+10}{3x^3}+c$$

(iii) $\int \sqrt{y}(3y+1)^2 dy = \int y^{\frac{1}{2}}(9y^2+6y+1)dy$

$= 9\int y^{\frac{5}{2}}dy + 6\int y^{\frac{3}{2}}dy + \int y^{\frac{1}{2}}dy$

$= \frac{18}{7}y^{\frac{7}{2}} + \frac{6(2)}{5}y^{\frac{5}{2}} + \frac{2}{3}y^{\frac{3}{2}} + c$

$= \sqrt{y}(\frac{18}{7}y^3 + \frac{12}{5}y^2 + \frac{2}{3}y) + c$

(iv) $\int (3-4x)^{12}dx = \int (3-4x)^{12}(-4)(-\frac{1}{4})dx$

$= (-\frac{1}{4})\int (3-4x)^{12}(-4)dx$

$= (-\frac{1}{4(13)})(3-4x)^{13} + c$

$= -\frac{1}{52}(3-4x)^{13} + c$

(v) $\int (3x+4)^{-6}dx = \frac{1}{3}\int (3x+4)^{-6}(3)dx$

$= \frac{1}{3(-5)}(3x+4)^{-5} + c = -\frac{1}{15}(3x+4)^{-5} + c$

(vi) $\int (8x-6)(4x^2-6x)^3 dx = \int (4x^2-6x)^3(8x-6)dx$

$= \frac{1}{4}(4x^2-6x)^4 + c$

(vii) $\int [x + \frac{1}{(4x+3)^2}]dx = \int xdx + \int \frac{1}{(4x+3)^2}]dx$

$= \frac{1}{2}x^2 + \frac{1}{4}\int (4x+3)^{-2}(4)dx$

$= \frac{1}{2}x^2 + \frac{1}{-4}(4x+3)^{-1} + c = \frac{1}{2}x^2 - \frac{1}{4(4x+3)} + c$

(viii) $\int y\sqrt{2y^2-1}dy = \int (2y^2-1)^{\frac{1}{2}}ydy$

$= \frac{1}{4}\int (2y^2-1)^{\frac{1}{2}}(4y)dy$

$= \frac{1}{4}(2y^2-1)^{\frac{3}{2}}(\frac{2}{3}) + c$

$= \frac{1}{6}(2y^2-1)^{\frac{3}{2}} + c$

Integrals of Logarithmic & Exponential Functions

Example:

Find the following integrals

(i) $\int e^{x}dx$

(ii) $\int e^{ax}dx$

(iii) $\int 2e^{3x+2}dx$

(iv) $\int (e^{x}+e^{-x})^{2}dx$

(v) $\int \left(\frac{(e^{x}+e^{-x})}{e^{x}}\right)dx$

(vi) $\int e^{s}\cos e^{s}ds$

(vii) $\int e^{\sin x}\cos x dx$

Solution:

Now, we find the following integrals by the above rules

(i) $\int e^{x}dx = e^{x}+c$

(ii) $\int e^{ax}dx = \frac{1}{a}e^{ax}+c$

Thus

$$\int e^{ax}dx = \frac{1}{a}\int e^{ax}a dx = \frac{1}{a}e^{ax}+c$$

(iii) $\int 2e^{3x+2}dx = 2\int e^{3x+2}3(\frac{1}{3})dx$

$$= \frac{2}{3}\int e^{3x+2}3dx = \frac{2}{3}e^{3x+2}3+c$$

(iv) $\int (e^{x}+e^{-x})^{2}dx = \int e^{2x}dx + 2\int dx + \int e^{-2x}dx$

$$= \frac{1}{2}e^{2x}+2x-\frac{1}{2}e^{-2x}+c = \frac{1}{2}(e^{2x}+4x-e^{-2x})+c$$

(v) $\int \left(\frac{(e^x + e^{-x})}{e^x}\right) dx = \int (1 + e^{-2x}) dx$

$$= \int dx + 2\int e^{-2x} dx = x + \frac{e^{-2x}}{-2} + c = \frac{2xe^x - e^{-x}}{2e^x} + c$$

(vi) $\int e^s \cos e^s ds = \sin e^s + c$

(vii) $\int e^{\sin x} \cos x dx = e^{\sin x} + c$

Example:

Find the following integrals

(i) $\int \frac{2x}{x^2 + 3} dx$

(ii) $\int \frac{4}{3t - 11} dt$

(iii) $\int \frac{4x}{x^2 + 3} dx$

(iv) $\int \frac{u + 1}{u^2 + 2u + 11} du$

(v) $\int \frac{\sec^2 \theta}{a + b \tan \theta} d\theta$

Solution:

Now, we find the following integrals by the above rules

(i) $\int \frac{2x}{x^2 + 3} dx = \ln|x^2 + 3| + c$

(ii) $\int \frac{4}{3t - 11} dt = \frac{4}{3}\int \frac{3}{3t - 11} dt = \frac{4}{3}\ln|3t - 11| + c$

(iii) $\int \frac{4x}{x^2 + 3} dx = 2\int \frac{2x}{x^2 + 3} dx = 2\ln|x^2 + 3| + c$

(iv) $\int \frac{u + 1}{u^2 + 2u + 11} du = \frac{1}{2}\int \frac{2u + 2}{u^2 + 2u + 11} du = \frac{1}{2}\ln|u^2 + 2u + 11| + c$

(v) $\int \frac{\sec^2 \theta}{a + b \tan \theta} d\theta = \frac{1}{b}\int \frac{\sec^2 \theta}{a + b \tan \theta} d\theta = \frac{1}{b}\ln|a + b \tan \theta| + c$

Integrals containing trigonometric functions and inverse trigonometric

Example:

Find the following integrals

(i) $\int \sin 2x dx$

(ii) $\int 3\cos 3\theta d\theta$

(iii) $\int \sec^2(-2y)dy$

(iv) $\int b\,\mathrm{cosec}^2 ax dx, \qquad a \neq 0$

(v) $\int \sqrt{\cos s}\,\sin x dx$

(vi) $\int x \sin x^2 dx$

(vii) $\int \sin^3 t dt$

Solution:

Now, we find the following integrals

(i) $\int \sin 2x dx = \frac{-1}{2}\cos 2x + c$

(ii) $\int 3\cos 3\theta d\theta = \sin 3\theta + c$

(iii) $\int \sec^2(-2y)dy = -\frac{1}{2}\int (-2)\sec^2(-2y)dy = -\frac{1}{2}\tan(-2y) + c$

(iv) $\int b\,\mathrm{cosec}^2 ax dx = b\int \mathrm{cosec}^2 ax dx = \frac{-b}{a}\cot ax + c$

(v) $\int \sqrt{\cos x}\,\sin x dx = \int (\cos x)^{\frac{1}{2}}\sin x dx = \frac{2}{3}(\cos x)^{\frac{3}{2}} + c$

(vi) $\int x\sin x^2 dx = \frac{1}{2}\int \sin x^2(2x)dx = -\frac{1}{2}\cos x^2 + c$

(vii) $\int \sin^3 t dt = \int \sin^2 t \sin t dt = \int (1-\cos^2 t)\sin t dt$

$$= \int \sin t dt - \int \cos^2 t \sin t dt = \cos t + \frac{1}{3}\cos^3 t + c$$

Example:

Find the following integrals

(i) $\int \frac{6du}{9+u^2}$

(ii) $\int \frac{6dy}{1+9y^2}$

(iii) $\int \frac{dx}{\sqrt{8-x^2}}$

(iv) $\int \frac{\sin 2x}{\sqrt{3-\sin^2 x}}dx$

Solution:

Now, we find the following integrals

(i) $\int \frac{6du}{9+u^2} = 6\int \frac{du}{3^2+u^2} = 6.\frac{1}{3}\tan^{-1}\frac{u}{3}+c$

(ii) $\int \frac{6dy}{1+9y^2} = \frac{1}{9}\int \frac{6dy}{\frac{1}{9}+y^2} = \frac{1}{9}\int \frac{6dy}{(\frac{1}{3})^2+y^2} = \frac{6}{9}(\frac{1}{1/3}\tan^{-1}\frac{y}{1/3})+c$

$= \frac{2}{3}\tan^{-1}3y + c$

(iii) $\int \frac{dx}{\sqrt{8-x^2}} = \sin^{-1}\frac{x}{2\sqrt{2}}+c$

(iv) $\int \frac{\sin 2x}{\sqrt{3-\sin^2 x}}dx = -\int (3-\sin^2 x)^{-\frac{1}{2}}(-2\sin x\cos x)dx$

$= -(3-\sin^2 x)^{\frac{1}{2}}.2+c$

$= -2\sqrt{3-\sin^2 x}+c$

Since

$$\sin 2x = 2\sin x\cos x \; .$$

Integrals of Trigonometric Functions

(i) $\int \sin x dx = -\cos x + c$

(ii) $\int \cos x dx = \sin x + c$

(iii) $\int \sec x \tan x dx = \sec x + c$

(iv) $\int \cos ec x \cot an x dx = -\cos ec x + c$

(v) $\int \sec^2 x dx = \tan x + c$

(vi) $\int \cos ec^2 x dx = -\cot x + c$

(vii) $\int \tan x dx = \int \frac{\sin x}{\cos x} dx = \int \frac{d \cos x}{\cos x}$

$$= -\ln|\cos x| + c = \ln|\sec x| + c$$

(viii) $\int \cot x dx = \int \frac{\cos x}{\sin x} dx = \int \frac{d \sin x}{\sin x} = \ln|\sin x| + c$

(ix) $\int \sec x dx = \int \frac{\sec x + \tan x}{\sec x + \tan x} \sec x dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$

$$= \int \frac{d(\sec x + \tan x)}{\sec x + \tan x} = \ln|\sec x + \tan x| + c$$

(x) $\int \cos ec x dx = \int \frac{\cos ec x - \cot x}{\cos ec x - \cot x} \cos ec x dx = \int \frac{\cos ec^2 x - \cos ec x \cot x}{\cos ec x - \cot x} dx$

$$= \int \frac{d(\cos ec x - \cot x)}{\cos ec x - \cot x} = \ln|\cos ec x + \cot x| + c$$

Generalized Integrals of Trigonometric Functions

$\int \sin f(x) f'(x) dx = -\cos f(x) + c$

$\int \cos f(x) f'(x) dx = \sin f(x) + c$

$\int \sec f(x) \tan f(x) f'(x) dx = \sec f(x) + c$

$\int \cos ec f(x) \cot an f(x) f'(x) dx = -\cos ec f(x) + c$

$\int \sec^2 f(x) f'(x) dx = \tan f(x) + c$

$\int \cos ec^2 f(x) f'(x) dx = -\cot f(x) + c$

Also,

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = \int \frac{d \cos x}{\cos x} = -\ln|\sin x| + c = \ln|\sec x| + c$$

$$\int \cot x dx = \int \frac{\cos x}{\sin x} dx = \int \frac{d \sin x}{\sin x} = \ln|\sin x| + c$$

$$\int \sec x dx = \int \frac{\sec x + \tan x}{\sec x + \tan x} \sec x dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

$$= \int \frac{d(\sec x + \tan x)}{\sec x + \tan x} = \int \ln|\sec x + \tan x| + c$$

$$\int \mathrm{cos}ecx dx = \int \frac{\mathrm{cos}ecx - \cot x}{\mathrm{cos}ecx - \cot x} \mathrm{cos}ecx dx = \int \frac{\mathrm{cos}ec^2 x - \mathrm{cos}ecx \cot x}{\mathrm{cos}ecx - \cot x} dx$$

$$= \int \frac{d(\mathrm{cos}ecx - \cot x)}{\mathrm{cos}ecx - \cot x} = \int \ln|\mathrm{cos}ecx - \cot x| + c$$

Integrals of Hyperbolic Functions

(i) $\int \sinh x dx = \cosh x + c$

(ii) $\int \cosh x dx = \sinh x + c$

(iii) $\int \mathrm{sec}hx \tanh x dx = -\mathrm{sec}hx + c$

(iv) $\int \mathrm{cos}echx \coth anxdx = -\mathrm{cos}echx + c$

(v) $\int \mathrm{sec}h^2 x dx = \tanh x + c$

(vi) $\int \mathrm{cos}ech^2 x dx = -\coth x + c$

(vii) $\int \tanh x dx = \int \frac{\sinh x}{\cosh x} dx = \int \frac{d \cosh x}{\cosh x}$

$$= \ln|\cos hx| + c$$

(viii) $\int \coth x dx = \int \frac{\cosh x}{\sinh x} dx = \int \frac{d \sinh x}{\sinh x} = \ln|\sinh x| + c$

(ix)

$$\int \mathrm{sec}hx dx = \int \frac{\mathrm{sec}\,hx + \tanh x}{\mathrm{sec}\,hx + \tanh x} \mathrm{sec}\,h\, x dx = \int \frac{\mathrm{sec}h^2 x + \mathrm{sec}hx \tanh x}{\mathrm{sec}hx + \tanh x} dx$$

$$= \int \frac{d(\mathrm{sec}hx + \tanh x)}{\mathrm{sec}hx + \tanh x} = \ln|\mathrm{sec}\,hx + \tanh x| + c$$

(x) $\int \operatorname{cosech}x\,dx = \int \frac{\operatorname{cosech}x - \coth x}{\operatorname{cosech}x - \coth x}\operatorname{cosech}x\,dx$

$$= \int \frac{\operatorname{cosech}^2 x - \operatorname{cosech}x\coth x}{\operatorname{cosech}x - \coth x}dx$$

$$= \int \frac{d(\operatorname{cosech}x - \coth x)}{\operatorname{cosech}x - \coth x} = \ln|\operatorname{cosech}x + \coth x| + c$$

Also,

$$\int \sinh f(x) f'(x)dx = -\cosh f(x) + c$$
$$\int \cosh f(x) f'(x)dx = \sinh x + c$$
$$\int \operatorname{sech}x \tanh x\ f'(x)dx = \operatorname{sech}x + c$$
$$\int \operatorname{cosech}x \coth x\ f'(x)dx = -\operatorname{cosech}x + c$$
$$\int \operatorname{sech}^2 x\ f'(x)dx = \tanh x + c$$
$$\int \operatorname{cosech}^2 x\ f'(x)dx = -\coth x + c$$

Also,

$$\int \tanh x\,dx = \int \frac{\sinh x}{\cosh x}dx = \int \frac{d\cosh x}{\cosh x} = -\ln|\sinh x| + c = \ln|\operatorname{sech}x| + c$$

$$\int \coth x\,dx = \int \frac{\cosh x}{\sinh x}dx = \int \frac{d\sinh x}{\sinh x} = \ln|\sinh x| + c$$

$$\int \operatorname{sech}x\,dx = \int \frac{\operatorname{sech}x + \tanh x}{\operatorname{sech}x + \tanh x}\sec x\,dx = \int \frac{\operatorname{sech}^2 x + \operatorname{sech}x\tanh x}{\operatorname{sech}x + \tanh x}dx$$

$$= \int \frac{d(\operatorname{sech}x + \tanh x)}{\operatorname{sech}x + \tanh x} = \int \ln|\operatorname{sech}x + \tanh x| + c$$

$$\int \operatorname{cosech}x\,dx = \int \frac{\operatorname{cosech}x - \coth x}{\operatorname{cosech}x - \coth x}\operatorname{cosech}x\,dx = \int \frac{\operatorname{cosech}^2 x - \operatorname{cosech}x\coth x}{\operatorname{cosech}x - \coth x}dx$$

$$= \int \frac{d(\operatorname{cosech}x - \coth x)}{\operatorname{cosech}x - \coth x} = \int \ln|\operatorname{cosech}x - \coth x| + c$$

Integrals squares of Trigonometric and Hyperbolic Functions

(i) $\int \sin^2 x dx = \int \frac{1}{2}(1-\cos 2x)dx = \frac{1}{2}(\int dx - \int \cos 2x dx)$

$= \frac{1}{2}(x - \frac{1}{2}\sin 2x) + c$

(ii) $\int \cos^2 x dx = \int \frac{1}{2}(1+\cos 2x)dx = \frac{1}{2}(\int dx + \int \cos 2x dx)$

$= \frac{1}{2}(x + \frac{1}{2}\sin 2x) + c$

(iii) $\int \tan^2 x dx = \int (\sec^2 x - 1)dx = \tan x - x + c$

(iv) $\int \cot^2 x dx = \int (\operatorname{cosec}^2 x - 1)dx = \cot x - x + c$

(iv) $\int \sinh^2 x dx = \int \frac{1}{2}(\cosh 2x - 1)dx = \frac{1}{2}(\frac{1}{2}\sinh 2x - x) + c$

(v) $\int \cosh^2 x dx = \int \frac{1}{2}(\cosh 2x + 1)dx = \frac{1}{2}(\frac{1}{2}\sinh 2x + x) + c$

(vi) $\int \tanh^2 x dx = \int (1 - \operatorname{sech}^2 x)dx = x - \tanh x + c$

(vii) $\int \coth^2 x dx = \int (\operatorname{cosech}^2 x + 1)dx = -\coth x + x + c$

Integrals Multiplying of Trigonometric and Hyperbolic Functions

(i) $\int \sin ax \cos bx dx = \frac{1}{2}\int \{\sin(a+b)x + \sin(a-b)x\}dx$

$= \frac{1}{2}\left\{\frac{\cos(a+b)x}{a+b} + \frac{\cos(a-b)x}{a-b}\right\} + c, \qquad a \neq b$

(ii) $\int \cos ax \cos bx dx = \frac{1}{2}\int \{\cos(a+b)x + \cos(a-b)x\}dx$

$= \frac{1}{2}\left\{\frac{\sin(a+b)x}{a+b} + \frac{\sin(a-b)x}{a-b}\right\} + c, \qquad a \neq b$

(iii) $\int \sin ax \sin bx dx = \frac{1}{2}\int \{\cos(a+b)x - \cos(a-b)x\}dx$

$= \frac{1}{2}\left\{\frac{\sin(a+b)x}{a-b} + \frac{\sin(a-b)x}{a+b}\right\} + c, \qquad a \neq b$

(iv) $\int \sinh ax \cosh bx dx = \frac{1}{2}\int \{\sinh(a+b)x + \sinh(a-b)x\}dx$

$$= \frac{1}{2}\left\{\frac{\cosh(a+b)x}{a+b} + co\frac{\cosh(a-b)x}{a-b}\right\} + c, \qquad a \neq b$$

(v) $\int \cosh ax \cosh bx dx = \frac{1}{2}\int \{\cosh(a+b)x + \cosh(a-b)x\}dx$

$$= \frac{1}{2}\left\{\frac{\sinh(a+b)x}{a+b} + \frac{\sinh(a-b)x}{a-b}\right\} + c, \qquad a \neq b$$

(vi) $\int \sinh ax \sinh bx dx = \frac{1}{2}\int \{\cosh(a+b)x - \cosh(a-b)x\}dx$

$$= \frac{1}{2}\left\{\frac{\sinh(a+b)x}{a-b} + \frac{\sinh(a-b)x}{a+b}\right\} + c, \qquad a \neq b$$

Integrals of Inverse Trigonometric Functions & Inverse Hyperbolic Functions .

(i) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c, \qquad |x| < a$

(ii) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c, \qquad |x| \neq a$

(iii) $\int \frac{dx}{x\sqrt{a^2 - x^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c, \qquad |x| > a$

(iv) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\frac{x}{a} + c$

(v) $\int \frac{dx}{a^2 - x^2} = \frac{1}{a}\tanh^{-1}\frac{x}{a} + c$

Some rules of the integrals

Suppose that the following relations :

$$\int \frac{dx}{ax^2+bx+c} \qquad , \qquad \int \frac{dx}{\sqrt{ax^2+bx+c}}$$

Example:

Calculate the following integrals

(i) $\int \frac{dx}{x^2+2x+10}$

(ii) $\int \frac{dx}{\sqrt{5+4x-x^2}}$

(iii) $\int \frac{dx}{x^2+4x+13}$

Solution :

(i) $\int \frac{dx}{x^2+2x+10} = \int \frac{d(x+1)}{(x+1)^2+9}$

$= \frac{1}{3}\tan^{-1}\frac{x+1}{3} + c$

(ii) $\int \frac{dx}{\sqrt{5+4x-x^2}} = \int \frac{dx}{\sqrt{5-(x^2-4x)}}$

$= \int \frac{dx}{\sqrt{9-(x-2)^2}}$

$= \sin^{-1}\frac{x-2}{3} + c$

(iii) $\int \frac{xdx}{x^2+4x+13} = \frac{1}{2}\int \frac{2xdx}{x^2+4x+13} = \frac{1}{2}\int \frac{(2x+4)-4}{x^2+4x+13}dx$

$= \frac{1}{2}\int \frac{(2x+4)}{x^2+4x+13}dx - \int \frac{2}{(x+2)^2+9}dx$

$= \frac{1}{2}\ln(x^2+4x+13) - \frac{2}{3}\tan^{-1}\frac{x+2}{3} + c$.

Exercises :

(1) Calculate the following integrals

(1) $\int (x^4 + 5x^3 - 1)dx$

(2) $\int (3 - 2x - 5x^4)dx$

(3) $\int (x+2)(4x+3)dx$

(4) $\int (x^2 - 1)^2 dx$

(5) $\int (\sqrt{x} - x + \frac{2}{\sqrt{x}})dx$

(6) $\int \frac{dx}{(1+x)^{\frac{1}{2}}}$

(7) $\int \frac{(1+\sqrt{x})dx}{\sqrt{x}}$

(8) $\int (\sqrt{x}(3-x)dx$

(9) $\int \frac{dx}{\sqrt{x+3}}$

(10) $\int \sqrt{2x-1}dx$

(11) $\int \frac{dx}{2x+3}$

(12) $\int \frac{(x-1)dx}{x+1}$

(13) $\int x(x^2+1)^{\frac{3}{2}}dx$

(14) $\int x\sqrt{x^2+1}dx$

(15) $\int \frac{e^{\frac{1}{x^2}}}{x^3}dx$

(16) $\int \frac{e^{2x}}{e^{2x}+3}dx$

(17) $\int \frac{(x+1)}{\sqrt{x^3+2x-4}}dx$

(2) Calculate the following integrals

(1) $\int \frac{\sin x}{1+\cos x}dx$

(2) $\int \sin 2x dx$

(3) $\int e^{\sin 2x}\cos 2x dx$

(4) $\int \sin x\sqrt{\cos x}dx$

(5) $\int (1+\sin 5x)^5\cos 5x dx$

(6) $\int \frac{\sin 2x}{1+\sin^2 x}dx$

(7) $\int \frac{\operatorname{cosec}^2 x}{1+\cot x}dx$

(8) $\int \frac{(\sin\ln x)(\cos\ln x)}{x}dx$

(9) $\int \tan^3 x\sec^2 x dx$

(10) $\int \sec 5x dx$

(11) $\int \operatorname{cosec}^2 x\cot^2 x dx$

(3) Calculate the following integrals

(1) $\int \operatorname{sech}(2x-1)dx$

(2) $\int \operatorname{cosech}2x\coth 5x dx$

(3) $\int \cosh^3\frac{x}{2}dx$

(4) $\int \frac{(4x+1)\operatorname{sech}^2(2x^2+1)}{\sqrt{3+\tanh(2x^2+x)}}dx$

(5) $\int \frac{\sinh x}{\cosh^4 x}dx$

(6) $\int \coth^6 2x\operatorname{cosech}^2 2x dx$

(4) Calculate the following integrals

(1) $\int \dfrac{dx}{\sqrt{4-x^2}}dx$

(2) $\int \dfrac{dx}{\sqrt{x^2-4}}dx$

(3) $\int \dfrac{dx}{4+x^2}dx$

(4) $\int \dfrac{dx}{\sqrt{9x^2-16}}$

(5) $\int \dfrac{dx}{\sqrt{9x^2+16}}$

(6) $\int \dfrac{dx}{x\sqrt{9x^2-16}}$

(7) $\int \dfrac{(x+1)dx}{x\sqrt{x^2-9}}$

(8) $\int \dfrac{\sin 8x dx}{9+\sin^2 4x}$

(9) $\int \dfrac{\sec^2 x dx}{\sqrt{1-4\tan^2 x}}$

(10) $\int \dfrac{dx}{\sqrt{2x^2-7x+5}}$

(11) $\int \dfrac{(3x-4)dx}{\sqrt{2x^2-7x+5}}$

(12) $\int \sqrt{\dfrac{1-x}{3+x}}dx$

(13) $\int \dfrac{dx}{40+6x-x^2}$

(14) $\int \dfrac{(x+4)dx}{84+8x-x^2}$

(15) $\int \dfrac{(2x+3)dx}{\sqrt{24+6x-x^2}}$

(5) Calculate the following integrals

(1) $\int \sin^3 x dx$ (2) $\int \cos^5 x dx$

(3) $\int \sin^2 x \cos^3 x dx$ (4) $\int \tan^4 x dx$

(5) $\int \sec^4 2x dx$ (6) $\int \operatorname{sech}^4 2x dx$

(7) $\int \cot^4 2x dx$ (8) $\int \sinh^3 x dx$

(9) $\int \cosh^3 \frac{3}{2} x dx$ (10) $\int \tan^5 x \sec^2 x dx$.

(6) Calculate the following integrals

(1) $\int \frac{\sqrt{x^2-9}}{x} dx$

(2) $\int \frac{1}{(1-x^2)^{\frac{5}{2}}} dx$

(3) $\int \frac{x^3}{(9+x^2)^{\frac{5}{2}}} dx$

(4) $\int \frac{x^2}{\sqrt{9-x^2}} dx$

(5) $\int \frac{1}{x^4\sqrt{x^2+4}} dx$

(6) $\sqrt{3-2x-x^2} dx$

(7) $\int \frac{\sin 2x}{(2-\sin^2 x)^{3/2}} dx$

(8) $\int (x-3)\sqrt{10+6x-x^2} dx$

(9) $\int \frac{x^2}{\sqrt{1+x^2}} dx$

(10) $\int \frac{1}{\left(4-9x^2\right)^2} dx$

(11) $\int \sqrt{4\cosh x - \cosh^2 x} \sinh x dx$

(12) $\int \frac{x^2}{\sqrt{x^2-2x}} dx$

(7) Calculate the following integrals

(1) $\int \; 3(x+\frac{5}{2}x^3)dx$

(2) $\int \; x(2x-3)dx$

(3) $\int \; (3x^3-8x)dx$

(4) $\int \; (x\sqrt{x}-2\sqrt{x}-3)dx$

(5) $\int \; 3x(1+\sqrt{x})dx$

(6) $\int \; (z^3(4-2z)+6)dz$

(7) $\int \; cr(a+r^2)dr$

(8) $\int \; 3(x+5+\frac{1}{x^2})dx$

(9) $\int \; (3x^{-2}+\frac{11}{3}x^{-5})dx$

(10) $\int \; (\sqrt{w}-\frac{1}{\sqrt{w}})dw$

(11) $\int \; (\sqrt{7z}dz$

(12) $\int \; \frac{u-1}{u^3}du$

(13) $\int \; (x+\frac{1}{x})^2dx$

(14) $\int \; u^2(u^3+3)^{10}du$

(15) $\int \; 3x^2(x^3-10)dx$

(16) $\int \; 3x(x^2+6)^3dx$

(17) $\int \; (x^2+6)^3dx$

(18) $\int \; (x+\sqrt{1-x})dx$

(19) $\int \; \frac{dx}{\sqrt{2x+3}}$

(20) $\int \; \frac{dt}{(t-1)^2}$

(21) $\int \; (6z^2+5)(6z^3+5z-1)^4dz$

(8) Calculate the following integrals

(1) $\int e^{ax+b}dx$

(2) $\int e^{\sin x}\cos x dx$

(3) $\int \dfrac{3dx}{e^{4x+1}}$

(4) $\int e^{-t/2}dt$

(5) $\int \dfrac{e^{x}dx}{1+e^{2x}}$

(6) $\int \dfrac{8x+10}{2x^{2}+5x}dx$

(7) $\int \dfrac{1-\sin x}{x+\cos x}dx$

CHAPTER II
Methods of Integration

Integration by Parts

The integration by parts is important in the field of integration and can be written rule derivative the product as follows:

$$D[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

This means that $f(x)g(x)$ is antiderivative for right hand side and

$$f(x)g(x) = \int f(x)g'(x)dx + \int g(x)f'(x)dx$$

Also, we can write:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx \qquad (1)$$

If we put $u = f(x),\qquad v = g(x)$ then

$$du = f'(x)dx, \qquad dv = g'(x)dx$$

Thus the formula (1) can we written as follows

$$\int udv = uv - \int vdu \qquad (2)$$

Thus we get the formula (1) and (2) are true for $f(x), \quad g(x)$ which have two derivatives are continuous , which is the main methods for integration by parts.

Example:

Find the following integral

$$\int x\cos x dx$$

Solution :

By integration by parts we get

$u = x, \qquad dv = \cos x dx$

$du = dx, \qquad v = \sin x + c_1$

From the relation (2) we get

$$\int x\cos x dx = x(\sin x + c_1) - \int (\sin x + c_1)dx = x\sin x + \cos x + c$$

Note :

It should be noted that the constant integration c_1 in the previous example, which shows of the end of integral .

Example:

Find the following integral

$$\int x^2 e^x dx$$

Solution :

By integration by parts we get

$u = x^2, \qquad dv = e^x dx$

$du = 2x dx, \qquad v = e^x$

By substituting in integration by parts we get

$$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx \qquad (*)$$

again we used the integration by parts we have

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + c$$

By substituting in equation (*) we get

$$\int x^2 e^x dx = x^2 e^x - 2(xe^x - e^x) + c = (x^2 - 2x + 2)e^x + c$$

When we used the integration by parts there exist several method to chose u and dv as follows :

(1) $u = x, \qquad dv = xe^x dx$
(2) $u = 1, \qquad dv = x^2 e^x dx$
(3) $u = xe^x, \qquad dv = xdx$
(4) $u = e^x, \qquad dv = x^2 dx$

Each of these choices, we find

$$udv = x^2 e^x dx$$

From which we have

$$u = e^x, \qquad dv = x^2 dx$$

$$du = e^x dx, \qquad dv = \frac{1}{3}x^3$$

Thus

$$\int x^2 e^x dx = \frac{1}{3}x^3 e^x - \frac{1}{3}\int x^3 e^x dx$$

Example:

Find the following integral

$$\int \tan^{-1} x dx$$

Solution :

By integration by parts we get

$$u = \tan^{-1} x, \qquad dv = dx$$

$$du = \frac{dx}{1+x^2}, \qquad v = x$$

Thus we have

$$\int \tan^{-1} x dx = x\tan^{-1} x - \int \frac{xdx}{1+x^2} dx = x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + c$$

Example:

Find the following integral

$$\int e^x \sin x dx$$

Solution :

By integration by parts we get

$$u = \sin x, \qquad dv = e^x dx$$
$$du = \cos x dx, \qquad v = e^x$$

Thus we have

$$\int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx$$

Another time by integration by parts we get

$$u = \cos x, \qquad dv = e^x dx$$

$$du = -\sin x dx, \qquad v = e^x$$

Thus we have

$$\int e^x \cos x dx = e^x \cos x + \int e^x \sin x dx$$

and

$$\int e^x \sin x dx = e^x \sin x - (e^x \cos x + \int e^x \sin x dx)$$
$$= e^x (\sin x - \cos x) - \int e^x \sin x dx$$
$$2\int e^x \sin x dx = e^x (\sin x - \cos x)$$
$$\int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x) + c$$

Integration by Trigonometric Substitution

The integration method for compensation one of the important ways to resolve many of the issues of integration and through an appropriate compensation and now write

$$\sin^2 x + \cos^2 x = 1$$
$$\tan^2 x + 1 = \sec^2 x$$
$$\tan^2 x = \sec^2 x - 1$$
$$\cot^2 x + 1 = \cos ec^2 x$$
$$\cot^2 x = \cos ec^2 x - 1$$

Example:

Find the following integral

$$\int \sec^4 2x \tan 2x dx$$

Solution :

To solve this integration we write

$$I = \int \sec^4 2x \tan 2x dx = \int \sec^3 2x (\sec 2x \tan 2x) dx$$

and using

$$\int g^n(x) g(x) dx = \frac{g^{n+1}(x)}{n+1} + c \quad ; \quad n \neq -1$$

we get

$$I = \frac{1}{2} \int \sec^3 2x (2 \sec 2x \tan 2x) dx$$
$$= \frac{1}{2} \cdot \frac{1}{4} (\sec 2x)^4 + c = \frac{1}{8} (\sec 2x)^4 + c$$

Example:

Find the following integral

$$\int \tan^3 x dx$$

Solution :

To solve this integration we write

$$\int \tan^3 x dx = \int \tan^2 x \tan x dx$$
$$= \int (\sec^2 x - 1) \tan x dx$$
$$= \int \sec^2 x \tan x dx - \int \tan x dx$$
$$= \frac{1}{2}\tan^2 x - \int \frac{\sin x}{\cos x} dx = \frac{1}{2}\tan^2 x - \ln|\cos x| + c$$

Now, we introduce the following some important trigonometric identities

$$\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$$
$$\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$
$$\cos x \sin y = \frac{1}{2}[\sin(x+y) - \sin(x-y)]$$
$$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$
$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$
$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$
$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$
$$\sin 2x = 2 \sin x \cos x.$$

Example:

Find the following integral

$$\int \cos 2x \cos 5x dx$$

Solution :

To solve this integration we write

$$\int \cos 2x \cos 5x dx = \frac{1}{2}\int (\cos 7x + \cos 3x) dx$$
$$= \frac{1}{14}\sin 7x + \frac{1}{6}\sin 3x + c$$

Using

$\sqrt{x^2 + a^2}$, $\sqrt{a^2 \pm x^2}$

we get

$\int \sqrt{a^2+x^2}dx = \sin^{-1}\frac{x}{a}+c$

Now using the following relations

$x = a\sin x,\qquad x = a\sec x,\qquad x = a\tan x$

and

$1-\sin^2 x = \cos^2 x;\qquad \tan^2 x+1 = \sec^2 x;\qquad \tan^2 x = \sec^2 x - 1$

Example:

Find the following integral

$$\int \frac{\sqrt{a^2-x^2}}{x^2}dx; \qquad a>0$$

Solution :

To solve this integration we write

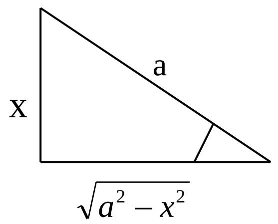
put $x = a\sin\theta$ where $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

$\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta} = a\sqrt{1-\sin^2\theta} = a\cos\theta$

Then

$$\int \frac{\sqrt{a^2-x^2}}{x^2}dx = \int \frac{a\cos\theta}{a^2\sin^2\theta}a\cos\theta d\theta$$
$$= \int \cot^2\theta d\theta = \int (\mathrm{cosec}^2\theta-1)d\theta$$
$$= \int \mathrm{cosec}^2\theta d\theta - \int d\theta = \cot\theta-\theta+c$$

Therefore $x = a\sin\theta$ and $\sin\theta = \frac{x}{a}$ we have



$$\int \frac{\sqrt{a^2-x^2}}{x^2}dx = -\frac{\sqrt{a^2-x^2}}{x} - \sin^{-1}\frac{x}{a}+c$$

Example:

Find the following integral

$$\int \frac{1}{(x^2-1)^{\frac{3}{2}}}dx$$

Solution :

To solve this integration we write

put $x = \sec\theta$ where $dx = \sec\theta\tan\theta d\theta$

$$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \sqrt{\sec^2\theta - 1} = \tan\theta$$

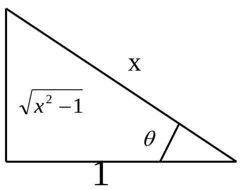
Therefore

$$\int \frac{1}{(x^2-1)^{\frac{3}{2}}}dx = \int \frac{\sec\theta\tan\theta}{\tan^3\theta}d\theta$$

$$= \int \frac{\cos\theta}{\sin^2\theta}d\theta$$

$$= -\frac{1}{\sin\theta} + c$$

$$= -\frac{x}{\sqrt{x^2-1}} + c$$



Integration by Partial Functions

Example:

Find the following integral

$$\int \frac{1}{(x-1)(x+2)}dx$$

Solution :

To solve this integration we write

$$\frac{1}{(x-1)} - \frac{1}{(x+2)} = \frac{(x+2)-(x-1)}{(x-1)(x+2)} = \frac{3}{(x-1)(x+2)}$$

i.e.

$$\frac{(x+1)}{(x-1)(x+2)} = \frac{2}{3(x-1)} + \frac{1}{3(x+2)}$$

Integration by Substitution

Example:

Find the following integral

$$\int \frac{dx}{x - x^{4/3}}$$

Solution :

To solve this integration we write

Put $x = u^3$

$$\int \frac{dx}{x - x^{4/3}} = \int \frac{3u^2 du}{u^3 - u^4} = \int \frac{3du}{u(1-u)}$$

and

$$3\int \frac{du}{u(1-u)} = 3\int \left(\frac{1}{u} + \frac{1}{(1-u)}\right) du = 3(\ln|u| + \ln|1-u|) + c$$

$$= 3\ln\left|\frac{u}{u-1}\right| + c = 3\ln\left|\frac{x^{1/3}}{1 - x^{1/\frac{1}{3}}}\right| + c$$

Example:

Find the following integral

$$\int \frac{dx}{1 + \cos x}$$

Solution :

To solve this integration we write

$$\int \frac{(1-\cos x)dx}{(1-\cos x)(1+\cos x)} = \int \frac{(1-\cos x)dx}{1-\cos^2 x}$$

$$= \int \frac{(1-\cos x)dx}{\sin^2 x}$$

$$= \int (coc^2 x - \cos ecx \cot x) dx$$

$$= -\cot x + \cos ecx + c$$

Example:

Find the following integrals

(1) $\int \frac{x+3}{\sqrt{1-x^2}} dx$

(2) $\int \frac{dx}{x^2 + 15x + 30}$

(3) $\int \frac{x+1}{x^2 - 4x + 8} dx$

(4) $\int \frac{x+2}{\sqrt{4x-x^2}}dx$

Solution :

To solve this integrations we write

(1) $\int \frac{x+3}{\sqrt{1-x^2}}dx = \int \frac{x}{\sqrt{1-x^2}}dx + \int \frac{3}{\sqrt{1-x^2}}dx$

$$= -\sqrt{1-x^2} + 3\sin^{-1} x + c$$

(2) $\int \frac{dx}{x^2+15x+30} = \int \frac{dx}{(x+5)^2+5} = \int \frac{dx}{(x+5)^2+(\sqrt{5})^2}$

$$= \frac{1}{(\sqrt{5})^2}\tan^{-1}(\frac{x+5}{\sqrt{5}}) + c$$

(3) $\int \frac{x+1}{x^2-4x+8}dx = \frac{1}{2}\int \frac{(2x-4)+6}{x^2-4x+8}dx$

$$= \frac{1}{2}\int \frac{(2x-4)}{x^2-4x+8}dx + 3\int \frac{1}{x^2-4x+8}dx$$

$$= \frac{1}{2}\ln(x^2-4x+8) + 3\int \frac{1}{(x-2)^2+4}dx$$

$$= \ln\sqrt{(x^2-4x+8)} + \frac{3}{2}\tan^{-1}(\frac{x-2}{2}) + c$$

(4) $\int \frac{x+2}{\sqrt{4x-x^2}}dx = -\frac{1}{2}\int \frac{-2x-4}{\sqrt{4x-x^2}}dx$

$$= -\frac{1}{2}\int \frac{(-2x+4)-8}{\sqrt{4x-x^2}}dx$$

$$= -\frac{1}{2}\int \frac{4-2x}{\sqrt{4x-x^2}}dx + 4\int \frac{dx}{\sqrt{4x-x^2}}$$

$$= -\sqrt{4x-x^2} + 4\sin^{-1}(\frac{x-2}{2}) + c$$

Example:

Find the following integrals

(1) $\int \sin 3x \sin 2x dx$

(2) $\int \sin 3x \sin 5x dx$

(3) $\int \cos 4x \cos 2x dx$

Solution :

To solve this integrations we write

(1) $\int \sin 3x \sin 2x dx = \frac{1}{2}\int [\cos(3x-2x)-\cos(3x+2x)]dx$

$$= \frac{1}{2}\int [\cos x - \cos 5x]dx$$

$$= \frac{1}{2}\sin x - \frac{1}{10}\sin 5x + c$$

Likewise, the integral with respect to other integrals can be calculated by using different laws for calculating trigonometry.

Example:

Find the following integrals

(1) $\int \sqrt{21-4x-x^2}dx$

(2) $\int \sqrt{x^2+2x+10}dx$

(3) $\int \sqrt{x^2+6x+5}dx$

Solution:

(1) $\int \sqrt{21-4x-x^2}dx = \int \sqrt{21-(x^2+4x)}dx$

$$= \int \sqrt{21-(x+2)^2-4}dx$$

$$= \int \sqrt{25-(x+2)^2}d(x+2)$$

then

$$(x+2) = 5\sin\theta, \qquad dx = 5\cos\theta d\theta$$

$$25-(x+2)^2 = 25-25\sin^2\theta = 25(1-\sin^2\theta) = 25\cos^2\theta$$

$$\int \sqrt{21-4x-x^2}dx = \int \sqrt{25\cos^2\theta}.5\cos\theta d\theta = \int 25\cos^2\theta d\theta$$

$$= \frac{25}{2}\int (1+\cos 2\theta)d\theta = \frac{25}{2}(\theta + \frac{\sin 2\theta}{2}) + c$$

$$= \frac{25}{2}(\theta + \frac{2\sin\theta\cos\theta}{2}) + c$$

$$= \frac{25}{2}\left\{\sin^{-1}\frac{x+2}{5} + \frac{x+2}{5}\sqrt{1-\left(\frac{x+2}{5}\right)^2}\right\} + c$$

$$= \frac{25}{2}\left\{\sin^{-1}\frac{x+2}{5} + \frac{x+2}{5}\sqrt{25-(x+2)^2}\right\} + c$$

$$= \frac{25}{2}\left\{\sin^{-1}\frac{x+2}{5} + \frac{x+2}{5}\sqrt{21--4x-x^2}\right\} + c$$

(2) $\int \sqrt{x^2+2x+10}dx = \int \sqrt{(x+1)^2+9}dx$

put

$$(x+1) = 3\sinh\theta, \qquad dx = 3\cosh\theta d\theta$$

$$(x+1)^2+9 = 9\sinh^2\theta + 9 = 9\cosh^2\theta$$

$$\int \sqrt{21-4x-x^2}dx = \int 3\cosh\theta.3\cosh\theta d\theta = \int 9\cosh^2\theta d\theta$$

$$= \frac{9}{2}\int (1+\cosh 2\theta)d\theta = \frac{9}{2}(\theta + \frac{\sinh 2\theta}{2}) + c$$

$$= \frac{1}{2}\left\{\sinh^{-1}\frac{x+1}{3} + \frac{x+1}{3}\sqrt{\left(\frac{x+1}{3}\right)^2+1}\right\} + c$$

$$= \frac{1}{2}\left\{\sinh^{-1}\frac{x+1}{3} + \frac{x+1}{3}\sqrt{x^2+2x+10}\right\} + c$$

$$= \frac{1}{2}\left\{9\sinh^{-1}\frac{x+1}{3} + (x+1)\sqrt{x^2+2x+10}\right\} + c$$

(3) $\int \sqrt{x^2+6x+5}dx = \int \sqrt{(x+3)^2-4}dx$

put

$$(x+3) = 2\cosh\theta, \qquad dx = 2\sinh\theta d\theta$$

$$(x+3)^2-4 = 4\cosh^2\theta - 4 = 4\sinh^2\theta$$

$$\int \sqrt{x^2+6x+5}dx = 2\int \sinh\theta.3\cosh\theta d\theta$$

$$= 2\int (\cosh 2\theta - 1)d\theta = 2\left(\frac{\sinh 2\theta}{2} - \theta\right) + c$$

$$= 2(\sinh\theta\cosh\theta - \theta) + c$$

$$= 2\left\{\frac{x+3}{2}\sqrt{\left(\frac{x+3}{2}\right)^2 - 1} - \cosh^{-1}\frac{x+3}{2}\right\} + c$$

$$= \frac{1}{2}\left\{(x+3)\sqrt{x^2+6x+5} - 2\cosh^{-1}\frac{x+3}{2}\right\} + c$$

Exercise

(1) Find the following integrals (using the fractional integration method):

$\int x\ln x dx$, $\int x^3 \ln x dx$, $\int x^n \ln x dx$, $\int \ln\left(x^2\right)dx$, $\int xe^{-x}dx$

$\int \cos^{-1} x dx$, $\int x\sin 2x dx$, $\int x^2 \cos 2x dx$, $\int x^3 e^x dx$

$\int \ln(x^x)dx$, $\int \frac{x}{e^{2x}}dx$

(2) Find the value of the following integrals (using partial fractional integration or any other method):

(1) $\int \frac{x-1}{(x-2)(x-3)}dx$

(2) $\int \frac{dx}{x^2-2x-8}$

(3) $\int \frac{1}{x(x+1)}dx$

(4) $\int \frac{dx}{x^2-8x+15}$

(5) $\int \frac{1}{t^2-a^2}dt$

(6) $\int \frac{4}{x(x^2-1)}dx$

(3) Find the value of the following integrals:

(1) $\int \frac{\sin 3x}{\sqrt{2\cos 3x+5}}dx$

(2) $\int \frac{\sec x}{1+\tan x}dx$

(3) $\int \frac{1}{e^x(3+e^{-x})}dx$

(4) $\int \frac{2+\log x}{x}dx$

(5) $\int \frac{\log^3 x}{x}dx$

(6) $\int \frac{\sin x}{e^x}dx$

(7) $\int (\sec x e^{\tan x})^2 dx$

(8) $\int \frac{\tan x}{\log \cos x}dx$

(9) $\int \sec 5x dx$

(10) $\int \frac{\cot x}{\log \sin x}dx$

Section III
Integral Calculus

Chapter **1**

INDEFINITE INTEGRAL

In differential calculus, we studied about the methods to finds the derivatives of different functions and their applications in Engineering and simple problems of mathematics. For example to find the slopes of tangents/normals, rate of change of quantities, maxima/minima of functions etc.

Now, in the following section, we shall be given the derivative of a function, and we are to find the function, whose derivative is given. This process of finding the function, whose derivative is given is called Integration.

Thus integration is the inverse process of differentiation. Application of integrations is very useful in finding the areas plane regions, lengths of arcs, volume of solid of revolution etc.

Antiderivative or Primitive of a function

Let f(x) and g(x) be two functions such that $\frac{d}{dx}(g(x)) = f(x)$; then g(x) is called antiderivative or primitive of f(x).

For example, $\frac{d}{dx}(x^3) = 3x^2$, thus x^3 is the antiderivative or primitive of $3x^2$.

Indefinite Integral

Let g(x) be the primitive or antiderivative of function f(x).

Thus $\frac{d}{dx}(g(x)) = f(x)$

Also $\frac{d}{dx}[g(x)+c] = f(x)$, for any constant c.

Thus if g(x) is the antiderivative of f(x) then for every value of constant g(x)+c is also an antiderivative of f(x). Thus the derivative of every function is unique but its antiderivative is not unique but they are infinitely many in numbers. Thus the antiderivative of a function is not definite, but is indefinite. Due to this fact the antiderivative of a function f(x) is called indefinite integral and symbolically written as $\int f(x)dx = g(x) + c$; c is called Constant of Integration and 'dx' indicates that the integration is carried out w.r.t. x. The function f(x) whose integral is to be found is called Integrand.

Some Standard Elementary Integrals

Derivatives	Integrals
1. $\because \frac{d}{dx}(c) = 0$; where c is any constant	$\therefore \int o\, dx = c$
2. $\because \frac{d}{dx}\left[\frac{x^{n+1}}{n+1}\right] = x^n;\ n \neq -1$	$\therefore \int x^n dx = \frac{x^{n+1}}{n+1}$+c for $n \neq -1$
3. $\because \frac{d}{dx}(\log x) = \frac{1}{x};\ x>0$	$\therefore \int \frac{1}{x} dx = \log\lvert x\rvert + c$
4. $\because \frac{d}{dx}(e^x) = e^x$	$\therefore \int e^x dx = e^x + c$
5. $\because \frac{d}{dx}(a^x) = a^x \log a,\ a > 0,\ a \neq 1$	$\therefore \int a^x dx = \frac{a^x}{\log a} + c;\ a>0,\ a \neq 1$
6. $\because \frac{d}{dx}(x) = 1$	$\therefore \int 1 dx = x + c$
7. $\because \frac{d}{dx}[ax+b]^{n+1} = (n+1)a(ax+b)^n;\ n \neq -1$	$\therefore \int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c,\ n \neq -1$
8. $\because \frac{d}{dx}(\sin x) = \cos x$	$\therefore \int \cos x\, dx = \sin x + c$
9. $\because \frac{d}{dx}(\cos x) = -\sin x$	$\therefore \int \sin x\, dx = -\cos x + c$
10. $\because \frac{d}{dx}(\tan x) = \sec^2 x$	$\therefore \int \sec^2 x\, dx = \tan x + c$
11. $\because \frac{d}{dx}(\cot x) = -\text{cosec}^2 x$	$\therefore \int \text{cosec}^2 x\, dx = -\cot x + c$
12. $\because \frac{d}{dx}(\sec x) = \sec x.\ \tan x$	$\therefore \int \sec x \tan x = \sec x + c$
13. $\because \frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x$	$\therefore \int \text{cosec}\, x \cot x\, dx = -\text{cosec}\, x + c$
14. $\because \frac{d}{dx}(\log \sin x) = \cot x$	$\therefore \int \cot x dx = \log\ \lvert\sin x\rvert + c$ $= -\log\lvert\text{cosec}\, x\rvert + c$

15. $\because \frac{d}{dx}(\log \sec x) = \tan x$	$\therefore \int \tan x dx = \log\lvert\sec x\rvert + c$ $= -\log\lvert\cos x\rvert + c$
16. $\because \frac{d}{dx}[\log(\sec x + \tan x)] = \sec x$	$\therefore \int \sec x\, dx = \log\lvert\sec x + \tan x\rvert + c$ $= \log\left\lvert\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\rvert + c$
17. $\because \frac{d}{dx}[\log(\text{cosec } x - \cot x) = \text{cosec } x$	$\therefore \int \text{cosec}\, dx = \log\lvert\text{cosec } x - \cot x\rvert + c$ $= \log\left\lvert\tan\frac{x}{2}\right\rvert + c$

Some Important Results

1) $\int cf(x)dx = c\int f(x)dx$; where c is any constant
2) $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

Remarks :

1. $\int f(x).g(x)\, dx \neq \int f(x)dx \,.\, \int g(x)dx$
2. $\int \frac{f(x)}{g(x)}dx \neq \frac{\int f(x)dx}{\int g(x)dx}$

Example 1 Evaluate the following Integrals

i) $\int x^6 dx$ ii) $\int \sqrt{x}\, dx$

iii) $\int x^{4/5}\, dx$ iv) $\int \sqrt[3]{x}\, dx$

v) $\int 5x^2 dx$ vi) $\int \frac{x^5 + x^2}{\sqrt{x}}dx$

Solution

i) $\int x^6 dx$

$= \frac{x^7}{7} + c$ $\left[\because \int x^n dx = \frac{x^{n+1}}{n+1} + c \text{ for } n \neq -1\right]$

ii) $\int \sqrt{x}\, dx$

$= \int x^{\frac{1}{2}} dx$

$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c \quad = \quad \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{2}{3}x^{\frac{3}{2}} + c$$

iii) $\int x^{4/5}\, dx$

$$= \frac{x^{\frac{4}{5}+1}}{\frac{4}{5}+1} + c \quad = \quad \frac{x^{\frac{9}{5}}}{\frac{9}{5}} + c$$

$$= \frac{5}{9}x^{\frac{9}{5}} + c$$

iv) $\int \sqrt[3]{x}\, dx$

$$= \int (x)^{\frac{1}{3}} dx$$

$$= \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} + c \quad = \quad \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + c$$

$$= \frac{3}{4}x^{\frac{4}{3}} + c$$

v) $\int 5x^2 dx$

$= 5 \int x^2\, dx$ $\left[\because \int cf(x)dx = c\int f(x)dx\right]$

$= 5\frac{x^3}{3} + c$ $\left[\because \int x^n dx = \frac{x^{n+1}}{n+1} + c \text{ for } n \neq -1\right]$

$$= \frac{5}{3}x^3 + c$$

vi) $\int \frac{x^5 + x^2}{\sqrt{x}} dx$

$= \int \left(\frac{x^5}{\sqrt{x}} + \frac{x^2}{\sqrt{x}} \right) dx$

$= \int \frac{x^5}{\sqrt{x}} dx + \int \frac{x^2}{\sqrt{x}}$ $\quad \left[\because \int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx \right]$

$= \int \frac{x^5}{x^{\frac{1}{2}}} dx + \int \frac{x^2}{x^{\frac{1}{2}}} dx$

$= \int x^{5-\frac{1}{2}} dx + \int x^{2-\frac{1}{2}} dx$

$= \int x^{\frac{9}{2}} dx + \int x^{\frac{3}{2}} dx$

$= \frac{x^{\frac{9}{2}+1}}{\frac{9}{2}+1} + \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + c$

$= \frac{2}{11} x^{\frac{11}{2}} + \frac{2}{5} x^{\frac{5}{2}} + c$

Example 2 Evaluate the following integrals :-

i) $\int (x^3+4x^2-5x+7)dx$ ii) $\int \left(x^3 - 2x^{\frac{2}{3}} + 7x^{\frac{5}{4}} + 9 \right) dx$

iii) $\int \frac{x^2 - x + 1}{\sqrt{x}} dx$ iv) $\int \left(\frac{5x + 3\sqrt{x} + 1}{x^2} \right) dx$

v) $\int \frac{1}{\sqrt{x^3}} dx$ vi) $\int (x^3 + \sqrt[3]{x^2} + x + 5) dx$

Solution

i) $\int (x^3+4x^2-5x+7)dx$

$= \frac{x^4}{4} + \frac{4x^3}{3} - \frac{5x^2}{2} + 7x + c$

ii) $\int\left(x^3 - 2x^{\frac{2}{3}} + 7x^{\frac{5}{4}} + 9\right)dx$

$$= \frac{x^{3+1}}{3+1} - 2\frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} + 7\frac{x^{\frac{5}{4}+1}}{\frac{5}{4}+1} + 9x + c$$

$$= \frac{x^4}{4} - \frac{6}{5}x^{\frac{5}{3}} + \frac{28}{9}x^{\frac{9}{4}} + 9x + c$$

iii) $\int\frac{x^2 - x + 1}{\sqrt{x}}dx$

$$= \int\left(\frac{x^2}{\sqrt{x}} - \frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right)dx$$

$$= \int\left(x^{\frac{3}{2}} - x^{\frac{1}{2}} + x^{\frac{-1}{2}}\right)dx$$

$$= \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} - \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$= \frac{2}{5}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + c$$

iv) $\int\left(\frac{5x + 3\sqrt{x} + 1}{x^2}\right)dx$

$$= \int\left(\frac{5x}{x^2} + \frac{3\sqrt{x}}{x^2} + \frac{1}{x^2}\right)dx$$

$$= \int\left(5.\frac{1}{x} + 3.x^{-\frac{3}{2}} + x^{-2}\right)dx$$

$$= 5\int\frac{1}{x}dx + 3\int x^{-\frac{3}{2}}dx + \int x^{-2}dx$$

$$= 5\log|x| + 3\frac{(x)^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} + \frac{x^{-2+1}}{-2+1} + c$$

$$= 5\log|x| - 6x^{-\frac{1}{2}} - x^{-1} + c$$

$$= 5\log|x| - \frac{6}{\sqrt{x}} - \frac{1}{x} + c$$

v) $\int \frac{1}{\sqrt{x^3}} dx$

$$= \int \frac{1}{(x^3)^{\frac{1}{2}}} dx$$

$$= \int \frac{1}{x^{\frac{3}{2}}} dx$$

$$= \int x^{-\frac{3}{2}} dx$$

$$= \frac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} + c$$

$$= \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + c$$

$$= -2x^{-\frac{1}{2}} + c$$

$$= -\frac{2}{\sqrt{x}} + c$$

vi) $\int (x^3 + \sqrt[3]{x^2} + x + 5) dx$

$$= \int x^3 dx + \int x^{\frac{2}{3}} dx + \int x dx + \int 5 dx$$

$$= \frac{x^4}{4} + \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} + \frac{x^2}{2} + 5x + c$$

$$= \frac{x^4}{4} + \frac{3}{5}x^{\frac{5}{3}} + \frac{x^2}{2} + 5x + c$$

Example 3 Evaluate the following integrals

i) $\int (e^x+a^x+x^a)\,dx$ ii) $\int (e^x.a^x+\cos\alpha)\,dx$

iii) $\int (1+x)\sqrt{x}\,dx$ iv) $\int \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 dx$

v) $\int \frac{(1-x)^2}{\sqrt{x}}dx$

Solution

i) $\int (e^x+a^x+x^a)\,dx$; a>0 and a ≠ 1,

$$= e^x + \frac{a^x}{\log a} + \frac{x^{a+1}}{a+1} + c$$

ii) $\int (e^x.a^x+\cos\alpha)\,dx$

$= \int ((ea)^x+\cos\alpha)\,dx$

$= \int (ea)^x\,dx + \int \cos\alpha\,dx$, [∵ cos α is constant]

$$= \frac{(ea)^x}{\log ea} + (\cos\alpha)x + c$$

iii) $\int (1+x)\sqrt{x}\,dx$

$$= \int \left(\sqrt{x} + x\sqrt{x}\right)dx$$

$$= \int \left(x^{\frac{1}{2}} + x^{\frac{3}{2}}\right)dx$$

$$= \int x^{\frac{1}{2}}dx + \int x^{\frac{3}{2}}dx$$

$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + c$$

$$= \frac{2}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}} + c$$

iv) $\int\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

$$= \int\left(x + \frac{1}{x} + 2\right)dx$$

$$= \int x dx + \int \frac{1}{x} dx + 2\int 1 dx$$

$$= \frac{x^2}{2} + \log x + 2x + c$$

v) $\int \frac{(1-x)^2}{\sqrt{x}} dx$

$$= \int \frac{1 + x^2 - 2x}{\sqrt{x}} dx$$

$$= \int \frac{1}{\sqrt{x}} dx + \int \frac{x^2}{\sqrt{x}} dx - 2\int \frac{x}{\sqrt{x}} dx$$

$$= \int x^{-\frac{1}{2}} dx + \int x^{\frac{3}{2}} dx - 2\int x^{\frac{1}{2}} dx$$

$$= \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} - 2\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$= 2x^{\frac{1}{2}} + \frac{2}{5}(x)^{\frac{5}{2}} - \frac{4}{3}x^{\frac{3}{2}} + c$$

Example 4 Evaluate the following integrals

i) $\int\left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right)dx$ ii) $\int\left(\frac{x}{a} + \frac{a}{x} - x^a + a^x + ax\right)dx$

iii) $\int x\sqrt{1+x}\, dx$ iv) $\int \frac{1}{\sqrt{x} + \sqrt{x+1}} dx$

Solution

i) $\int\left(x+\frac{1}{x}\right)\left(x^2+\frac{1}{x^2}\right)dx$

$= \int\left(x^3+\frac{1}{x}+x+\frac{1}{x^3}\right)dx$

$= \int x^3dx+\int\frac{1}{x}dx+\int xdx+\int x^{-3}dx$

$= \frac{x^4}{4}+\log|x|+\frac{x^2}{2}+\frac{x^{-3+1}}{-3+1}+c$

$= \frac{x^4}{4}+\log|x|+\frac{x^2}{2}-\frac{x^{-2}}{2}+c$

ii) $\int\left(\frac{x}{a}+\frac{a}{x}-x^a+a^x+ax\right)dx$ $\quad ; a \neq 0, \pm 1$

$= \int\frac{x}{a}dx+\int\frac{a}{x}dx+\int x^a dx+\int a^x dx+\int axdx$

$= \frac{1}{a}\int xdx+a\int\frac{1}{x}dx+\int x^a dx+\int a^x dx+a\int xdx$

$= \frac{1}{a}.\frac{x^2}{2}+a\log|x|+\frac{x^{a+1}}{a+1}+\frac{a^x}{\log a}+\frac{ax^2}{2}+c$

iii) $\int x\sqrt{1+x}dx$

$= \int(1+x-1)\sqrt{1+x}dx$

$= \int(1+x)\sqrt{1+x}dx-\int\sqrt{1+x}dx$

$= \int(1+x)^{\frac{3}{2}}dx-\int(1+x)^{\frac{1}{2}}dx$

$= \frac{(1+x)^{\frac{3}{2}+1}}{\left(\frac{3}{2}+1\right)}-\frac{(1+x)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)}+c$ $\quad \left[\because \int(ax+b)^n dx=\frac{(ax+b)^{n+1}}{a(n+1)} \text{ for } n\neq -1\right]$

$= \frac{2}{5}(1+x)^{\frac{5}{2}}-\frac{2}{3}(1+x)^{\frac{3}{2}}+c$

iv) $\int \frac{1}{\sqrt{x}+\sqrt{x+1}}dx$

$= \int \frac{1}{\sqrt{x}+\sqrt{x+1}} \times \frac{\sqrt{x}-\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}}dx$

$= \int \frac{\sqrt{x}-\sqrt{x+1}}{x-x-1}dx$

$= \int \left(\sqrt{x+1}-\sqrt{x}\right)dx$

$= \int (x+1)^{\frac{1}{2}}dx - \int (x)^{\frac{1}{2}}dx$

$= \frac{(x+1)^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{(x)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$

$= \frac{2}{3}(x+1)^{\frac{3}{2}} - \frac{2}{3}(x)^{\frac{3}{2}} + c$

Example 5 Evaluate the integrals of following functions w.r.t. x.

i) $2\sin x - 3\cos x$ ii) $\cos x \, . \tan x + e^x$

iii) $\sec^2 x + \tan^2 x$ iv) $\cot^2 x + a^x - 1; a \neq 1, a > 0$

v) $\operatorname{cosec} x \cot x$ vi) $\tan x (\sec x + 1)$

Solution

i) $\int (2\sin x - 3\cos x)\, dx$

$= 2\int \sin x\, dx - 3\int \cos x dx$

$= 2(-\cos x) - 3\sin x + c$

$= -2\cos x - 3\sin x + x$

ii) $\int (\cos x \tan x + e^x)\, dx$

$= \int \cos x \, . \frac{\sin x}{\cos x} dx + \int e^x dx$

$= \int \sin x\, dx + \int e^x dx$

$= -\cos x + e^x + c$

iii) $\int (\sec^2 x + \tan^2 x) dx$

$= \int \sec^2 x + (\sec^2 x - 1)\, dx$, $[\because \sec^2 x - \tan^2 x = 1]$

$= \int (2\sec^2 x - 1)\, dx$

$= 2\int \sec^2 x\, dx - \int 1 dx$

$= 2\tan x - x + c$

iv) $\int(\cot^2x+a^x-1)\,dx$; $a \neq 1, a>0$

$= \int \cot^2x\,dx + \int a^x\,dx - \int 1dx$

$= \int(\mathrm{cosec}^2x-1)dx + \int a^x dx - \int 1dx$, $[\because \mathrm{cosec}^2x - \cot^2 = 1]$

$= -\cot x - x + \dfrac{a^x}{\log a} - x + c$

v) $\int \mathrm{cosec}\,x \cot x\,dx$

$= -\mathrm{cosec}\,x + c$

vi) $\int \tan x(\sec x + 1)\,dx$

$= \int \sec x \tan x\,dx + \int \tan x\,dx$

$= \sec x + \log|\sec x| + c$

Three Important Results:

i) If $\int f(x)\,dx = g(x) + c$

then $\int f(ax+b)dx = \dfrac{g(ax+b)}{a} + c$

ii) $\int [f(x)]^n f'(x)dx = \dfrac{[f(x)]^{n+1}}{n+1} + c$, for $n \neq -1$

iii) $\int \dfrac{f'(x)}{f(x)}dx = \log|f(x)| + c$

Example 6 Evaluate the following Integrals.

i) $\int \sin(2x+3)\,dx$ ii) $\int \sec^2(3-4x)\,dx$

iii) $\int e^{4x}\,dx$ iv) $\int (\tan x)^5 \sec^2x\,dx$

v) $\int \dfrac{(\log x)^2}{x}dx$ vi) $\int \dfrac{x^3-4x^2+6}{x^4+3x^2-8x}dx$

Solution

i) $\int \sin(2x+3)\,dx$

$= -\dfrac{\cos(2x+3)}{2} + c$, $\left[\begin{array}{l} \because \int f(x)dx = \int \sin(x)dx = -\cos x + c \\ \because \int f(2x+3)dx = \int \sin(2x+3)dx = -\dfrac{\cos(2x+3)}{2} + c \end{array}\right]$

ii) $\int \sec^2(3-4x)\,dx$

$= \dfrac{\tan(3-4x)}{-4} + c$, $\left[\because \int \sec^2(ax+b)dx = \dfrac{1}{a}\tan(ax+b) + c\right]$

$= -\frac{1}{4}\tan(3-4x)+c$

iii) $\int e^{4x}\,dx$

$= \frac{e^{4x}}{4}+c$, $\left[\because \int e^{ax+b}dx = \frac{e^{ax+b}}{a}+c\right]$

iv) $\int (\tan x)^5 \sec^2 x\,dx$

$= \frac{(\tan x)^{5+1}}{5+1}+c$, $\left[\because \int [f(x)]^n f'(x) = \frac{[f(x)]^{n+1}}{n+1}+c\right]$

$= \frac{1}{6}(\tan x)^6 + c$

$= \frac{1}{6}\tan^6 x + c$

v) $\int \frac{(\log x)^2}{x}dx$

$= \int (\log x)^2 . \frac{1}{x}dx$, $\left[\because \frac{d}{dx}(\log x) = \frac{1}{x}\right]$

$= \frac{(\log x)^{2+1}}{2+1}+c$

$= \frac{1}{3}(\log x)^3 + c$

vi) $\int \frac{x^3-4x^2+6}{3x^2-8x}dx$, $\left[\because \frac{d}{dx}(x^3-4x^2+6) = 3x^2-8x \text{ and } \int \frac{f'(x)}{f(x)} = \log|f(x)|+c\right]$

$= \log|3x^2-8x|+c$

Example 7 Evaluate the integrals of the following functions w.r.t. x.

i) $\sin^2 x$ ii) $\cos^2 x$

iii) $\sec^2(3x-4)$ vi) $\cos 3x \cos 7x$

Solution

i) $\int \sin^2 x\,dx$

$= \int \frac{1-\cos 2x}{2}dx$, $\left[\because 1-\cos 2x = 2\sin^2 x \quad \therefore \frac{1-\cos 2x}{2} = \sin^2 x\right]$

$$= \int \frac{1}{2} dx - \int \frac{1}{2} \cos 2x dx$$

$$= \frac{1}{2}\int 1dx - \frac{1}{2}\int \cos 2xdx$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$$

ii) $\int \cos^2 x \, dx$

$$= \int \frac{1 + \cos 2x}{2} dx \qquad \left[\because 1 + \cos 2x = 2\cos^2 x \quad \therefore \frac{1 + \cos 2x}{2} = \cos^2 x\right]$$

$$= \int \frac{1}{2} dx + \int \frac{1}{2} \cos 2x dx$$

$$= \frac{1}{2}\int 1dx + \frac{1}{2}\int \cos 2xdx$$

$$= \frac{1}{2}x + \frac{1}{4}\sin 2x + c$$

iii) $\int \sec^2(3x-4) \, dx$

$$= \frac{\tan(3x-4)}{3} + c$$

iv) $\cos 3x \cos 7x$

$$= \frac{1}{2}\int 2\cos 7x \cos 3x dx$$ [multiplying and dividing by 2]

$$= \frac{1}{2}\int [\cos(7x + 3x) + \cos(7x - 3x)]dx,$$ [$\because$ 2cosA cosB = cos(A+B)+cos(A−B)]

$$= \frac{1}{2}\int (\cos 10x + \cos 4x)dx$$

$$= \frac{1}{2}\frac{\sin 10x}{10} + \frac{1}{2}\frac{\sin 4x}{4} + c$$

$$= \frac{1}{20}\sin 10x + \frac{1}{8}\sin 4x + c$$

Example 8 Evaluate the Integrals of following functions w.r.t. x.

i) $\cos^3 x$ ii) $\dfrac{\sin^3 x - \cos^3 x}{\sin^2 x.\cos^2 x}$

iii) $\dfrac{1}{1-\cos x}$ iv) $\dfrac{1+\cos 2x}{1-\cos 2x}$

Solution

i) $\int \cos^3 x \, dx$

$$= \int\left(\frac{1}{4}\cos 3x + \frac{3}{4}\cos x\right)dx \quad , \quad \left[\begin{array}{l} \because \cos 3x = 4\cos^3 x - 3\cos x \\ \therefore 4\cos^3 x = \cos 3x + 3\cos x \\ \cos^3 x = \frac{1}{4}\cos 3x + \frac{3}{4}\cos x \end{array}\right.$$

$$= \frac{1}{4}\int \cos 3x dx + \frac{3}{4}\int \cos x dx$$

$$= \frac{1}{4}\frac{\sin 3x}{3} + \frac{3}{4}\sin x + c$$

$$= \frac{1}{12}\sin 3x + \frac{3}{4}\sin x + c$$

ii) $\int \dfrac{\sin^3 x - \cos^3 x}{\sin^2 x.\cos^2 x}$

$$= \int \frac{\sin^3 x}{\sin^2 x \cos^2 x}\, dx - \int \frac{\cos^3 x}{\sin^2 x \cos^2 x}\, dx$$

$$= \int \frac{\sin x}{\cos^2 x}\, dx - \int \frac{\cos x}{\sin^2 x}\, dx$$

$= \int \sec x \tan x \, dx - \int \text{cosec}\, x \cot x \, dx$

$= \sec x + \text{cosec}\, x + c$

iii) $\int \dfrac{1}{1-\cos x}$

$$= \int \frac{1}{1-\cos x} \times \frac{1+\cos x}{1+\cos x} dx$$

$$= \int \frac{1+\cos x}{1-\cos^2 x}\, dx$$

$$= \int \frac{1+\cos x}{\sin^2 x}\, dx$$

$$= \int \left(\frac{1}{\sin^2 x} + \frac{\cos x}{\sin^2 x}\right) dx$$

$= \int (\text{cosec}^2 x + \text{cosec } x \cot x)\, dx$

$= \int \text{cosec}^2 x\, dx + \int \text{cosec } x \cot x\, dx$

$= -\cot x - \text{cosec } x + c$

iv) $\int \frac{1+\cos 2x}{1-\cos 2x}\, dx$

$= \int \frac{2\cos^2 x}{2\sin^2 x} dx$, $\left[\begin{matrix} \because 1+\cos 2\theta = 2\cos^2\theta \\ 1-\cos 2\theta = 2\sin^2\theta \end{matrix}\right]$

$= \int \cot^2 x\, dx$

$= \int (\text{cosec}^2 x - 1) dx$, $[\because \text{cosec}^2 x - \cot^2 x = 1]$

$= -\cot x - x + c$

Example 9 Evaluate the following Integrals $\int \frac{\sin x}{\sin(x-a)} dx$

Solution $\int \frac{\sin x}{\sin(x-a)} dx$

$$= \int \frac{\sin(x-a+a)}{\sin(x-a)} dx$$

$$= \int \frac{\sin(x-a)\cos a + \cos(x-a)\sin a}{\sin(x-a)} dx$$

$[\because \sin(A+B) = \sin A \cos B + \cos A \sin B]$

$= \int (\cos a)\, dx + \int \sin a \cot(x-a)\, dx$

$= (\cos a) \int 1 dx + \sin a \int \cot(x-a)\, dx$

$= x \cos a + \sin a \log |\sin(x-a)| + c$

Example 10 Evaluate $\int \frac{dx}{\cos(x-a)\cos(x-b)}$

Solution $\int \frac{dx}{\cos(x-a)\cos(x-b)}$

$$= \frac{1}{\sin(b-a)} \int \frac{\sin(b-a)dx}{\cos(x-a)\cos(x-b)}$$

$$= \frac{1}{\sin(b-a)} \int \frac{\sin[(x-a)-(x-b)]dx}{\cos(x-a)\cos(x-b)}$$

$$= \frac{1}{\sin(b-a)} \int \frac{\sin(x-a)\cos(x-b) - \cos(x-a)\sin(x-b)}{\cos(x-a)\cos(x-b)} dx$$

[∵ sin (A−B) = sinA cosB − cosA sinB]

$$= \frac{1}{\sin(b-a)} \int [\tan(x-a) - \tan(x-b)]\, dx$$

$$= \frac{1}{\sin(b-a)} \int \tan(x-a)\, dx \quad \frac{1}{\sin(b-a)} \int \tan(x-b)\, dx$$

$$= \frac{1}{\sin(b-a)} \log |\sec(x-a)| - \frac{1}{\sin(b-a)} \log |\sec(x-b)| + c$$

$$= \frac{1}{\sin(b-a)} \log \left| \frac{\sec(x-a)}{\sec(x-b)} \right| + c$$

$$= \frac{1}{\sin(b-a)} \log \left| \frac{\cos(x-b)}{\cos(x-a)} \right| + c$$

Integration By Substitution :

So, far we were dealing with the functions whose integration can be found by using the standard elementary integrals. e.g. $\int \sin(ax+b)dx = \frac{-\cos(ax+b)}{a}$, $\int \frac{1}{x} dx = \log x$, $\int (\sec^2 x) + \tan x)\, dx = \tan x + \log|\sec x| + c$ etc.

Now, further we are to integrate the functions which can not be integrated by simply using the standard elementary integrals and important results. Sometimes these type of integrals can be calculated by transforming the given integrand into one of the standard elementary integrands by taking a suitable substitution of the independent variable of given integrand e.g. $I = \int \frac{\sin(\log x)}{x} dx$

To solve it, we put log x = t

Differentiating both sides w.r.t. x. $\frac{1}{x} = \frac{dt}{dx}$

or $\quad \frac{1}{x}\, dx = dt$

$\therefore$ $I = \int \sin t\, dt = -\cos t + c$

$= -\cos(\log x) + c$

In the above example, we put $\log x = t$ or $x = e^t$

i.e. the independent variable x is substituted by e^t.

Example 11 Integrate the following functions

i) $\dfrac{\sec^2(\log x)}{x}$ ii) $x^2 \sin(x^3+5)$

iii) $(2x-7)\sqrt{x^2-7x+5}$ iv) $e^x \sin(e^x)$

v) $\dfrac{x-2}{\sqrt{x^2-4x+5}}$

Solution

i) $\int \dfrac{\sec^2(\log x)}{x}$

put $\log x = t$

$\therefore$ $\dfrac{1}{x} = \dfrac{dt}{dx}$

or $\dfrac{1}{x}\, dx = dt$

$\therefore$ $\int \dfrac{\sec^2(\log x)}{x} dx = \int \sec^2 t\, dt$

$= \tan t + c$

$= \tan(\log x) + c$

ii) $x^2 \sin(x^3+5)$

put $x^3 + 5 = t$

$\therefore$ $3x^2 = \dfrac{dt}{dx}$

or $3x^2\, dx = dt$

or $x^2 dx = \dfrac{dt}{3}$

$\therefore$ $\int x^2 \sin(x^3+5) dx = \int \sin t \dfrac{dt}{3}$

$= \dfrac{1}{3} \int \sin t\, dt$

$$= -\frac{1}{3}\cos t + c$$

$$= -\frac{1}{3}\cos(x^3 + 5) + c$$

iii) $\int (2x - 7)\sqrt{x^2 - 7x + 5}$

put $x^2 - 7x + 5 = t$

$\therefore \quad 2x - 7 = \frac{dt}{dx}$

or $(2x - 7)dx = dt$

$\therefore \quad \int (2x - 7)\sqrt{x^2 - 7x + 5}\, dx$

$$= \int \sqrt{t}\, dt$$

$$= \int (t)^{\frac{1}{2}}\, dt$$

$$= \frac{2}{3}(t)^{\frac{3}{2}} + c$$

$$= \frac{2}{3}(x^2 - 7x + 5)\sqrt{x^2 - 7x + 5} + c$$

iv) $\int e^x \sin e^x dx$

put $e^x = t$

$\therefore \quad e^x = \frac{dt}{dx}$

or $e^x\, dx = dt$

$\therefore \quad \int e^x \sin e^x dx = \int \sin t\, dt$

$$= -\cos t + c$$

$$= -\cos e^x + c$$

v) $\int \frac{x - 2}{\sqrt{x^2 - 4x + 5}} dx$

put $x^2 - 4x + 5 = t$

$\therefore \quad 2x - 4 = \frac{dt}{dx}$

$(x - 2) = \frac{dt}{2dx}$

$$(x-2)\,dx = \frac{dt}{2}$$

$$\therefore \int \frac{(x-2)}{\sqrt{x^2-4x+5}} = \int \frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{2}\int t^{-\frac{1}{2}} dt$$

$$= \frac{1}{2}\frac{(t)^{-\frac{1}{2}+1}}{\left(-\frac{1}{2}+1\right)} + c$$

$$= \left(\frac{1}{2}\right)\frac{(t)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + c$$

$$= \sqrt{t} + c$$

$$= \sqrt{x^2-4x+5} + c$$

Example 12 Evaluate

i) $\int \frac{1}{a^2-x^2}$ ii) $\int \frac{1}{x^2-a^2}dx$ iii) $\int \frac{1}{a^2+x^2}dx$

Solution

i) $\int \frac{1}{a^2-x^2}\,dx$

Put $x = a\sin\theta$

$$\frac{dx}{d\theta} = a\cos\theta$$

$$\therefore dx = a\cos\theta\, d\theta$$

$$\therefore \int \frac{1}{a^2-x^2}dx = \int \frac{1}{a^2-a^2\sin^2\theta}.a\cos d\theta$$

$$= \frac{1}{a^2}\int \frac{a\cos\theta}{\cos^2\theta}d\theta \quad , \quad \because \left[1-\sin^2\theta = \cos^2\theta\right]$$

$$= \frac{1}{a}\int \frac{1}{\cos\theta}d\theta$$

$$= \frac{1}{a}\int \sec\theta d\theta$$

$$= \frac{1}{a}\log|\sec\theta + \tan\theta| + c \qquad = \frac{1}{a}\log\left|\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right| + c$$

$$= \frac{1}{a}\log\left|\frac{1+\sin\theta}{\cos\theta}\right| + c$$

$$= \frac{1}{a}\log\left|\frac{1+\frac{x}{a}}{\sqrt{1-\frac{x^2}{a^2}}}\right| + c \qquad = \frac{1}{a}\log\left|\frac{a+x}{\sqrt{(a+x)(a-x)}}\right| + c$$

$$= \frac{1}{a}\log\left|\sqrt{\frac{a+x}{a-x}}\right| + c \qquad = \frac{1}{a}\log\left|\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}\right| + c$$

$$= \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$$

ii) $\int \frac{1}{x^2 - a^2}dx$

Put $x = a\sec\theta$

$dx = a\sec\theta\tan\theta\, d\theta$

$$\therefore \int \frac{1}{x^2 - a^2}dx = \int \frac{1}{a^2\sec^2\theta - a^2} a\sec\theta\tan\theta d\theta$$

$$= \int \frac{\sec\theta.\tan\theta d\theta}{a\tan^2\theta}, \qquad \because [\sec^2\theta - \tan^2\theta = 1]$$

$$= \frac{1}{a}\int \frac{\sec\theta}{\tan\theta}d\theta$$

$$= \frac{1}{a}\int \frac{1}{\cos\theta}.\frac{\cos\theta}{\sin\theta}d\theta$$

$$= \frac{1}{a}\int \operatorname{cosec}\theta d\theta$$

$$= \frac{1}{a}\log|\operatorname{cosec}\theta - \cot\theta| + c$$

$$= \frac{1}{a}\log\left|\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right| + c$$

$$= \frac{1}{a}\log\left|\frac{1-\frac{a}{x}}{\sqrt{1-\left(\frac{a^2}{x^2}\right)}}\right| + c \quad = \frac{1}{a}\log\left|\frac{x-a}{\sqrt{(x-a)(x+a)}}\right| + c$$

$$= \frac{1}{a}\log\left|\sqrt{\frac{x-a}{x+a}}\right| + c$$

$$= \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$$

iii) $\int \frac{1}{a^2+x^2}dx$

Put $x = a\tan\theta$

$dx = a\sec^2\theta\, d\theta$

$$\therefore \quad \int \frac{1}{a^2\tan^2\theta + a^2} a\sec^2\theta\, d\theta$$

$$= \frac{1}{a}\int \frac{\sec^2\theta}{(\tan^2\theta+1)}d\theta \quad = \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\theta + c$$

$$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

Remark : the above three integrals should be remembered and can be directly used as standard integrals.

Thus $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

$$\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

Example 13 Evaluate

i) $\int \frac{1}{\sqrt{a^2 - x^2}} dx$ ii) $\int \frac{1}{\sqrt{x^2 - a^2}} dx$ iii) $\int \frac{1}{\sqrt{a^2 + x^2}} dx$

Solution

i) $\int \frac{1}{\sqrt{a^2 - x^2}} dx$

Put $x = a \sin\theta$

$\frac{dx}{d\theta} = a \cos\theta$

$\therefore dx = a \cos\theta \, d\theta$

$$\therefore \int \frac{1}{\sqrt{a^2 - x^2}} dx = \int \frac{1}{\sqrt{a^2 - a^2 \sin^2\theta}} . a \cos d\theta$$

$$= \int \frac{a \cos\theta}{\cos\theta} d\theta$$

$$= \int 1 d\theta$$

$$= \theta + c$$

$$= \sin^{-1}\left(\frac{x}{a}\right) + c \qquad \left[\begin{array}{l} \because x = a \sin\theta \\ \therefore \frac{x}{a} = \sin\theta \text{ or } \sin^{-1}\left(\frac{x}{a}\right) = \theta \end{array}\right]$$

ii) $\int \frac{1}{\sqrt{x^2 - a^2}} dx$

Put $x = a \sec\theta$

$\therefore dx = a \sec\theta \tan\theta \, d\theta$

$$\therefore \int \frac{1}{\sqrt{x^2 - a^2}} dx = \int \frac{1}{\sqrt{a^2 \sec^2\theta - a^2}} a \sec\theta \tan\theta d\theta = \int \frac{\sec\theta \tan\theta d\theta}{\tan\theta}$$

$$= \int \sec\theta d\theta$$

$$= \log|\sec\theta + \tan\theta| + c_1$$

$$= \log\left|\frac{x}{a} + \sqrt{\sec^2\theta - 1}\right| + c_1$$

$$= \log\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right| + c_1$$

$$= \log\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + c_1$$

$$= \log\left|x+\sqrt{x^2-a^2}\right| - \log a + c_1 \;, \quad \because \left[\log\frac{m}{n} = \log m - \log n\right]$$

$$= \log\left|x+\sqrt{x^2-a^2}\right| + c \qquad , \text{ [where } c = -\log a + c_1]$$

iii) $\int \frac{1}{\sqrt{a^2+x^2}} dx$

Put $x = a\tan\theta$

$\therefore \quad dx = a\sec^2\theta \, d\theta$

$$\therefore \quad \int \frac{1}{\sqrt{a^2+x^2}} dx = \int \frac{1}{\sqrt{a^2\tan^2\theta + a^2}} a\sec^2\theta d\theta = \int \frac{\sec^2\theta d\theta}{\sqrt{\sec^2\theta}}$$

$$= \int \sec\theta d\theta$$

$$= \log|\sec\theta + \tan\theta| + c_1$$

$$= \log\left|\tan\theta + \sqrt{1+\tan^2\theta}\right| + c_1 \;, \qquad \because \left[\sec\theta\sqrt{1+\tan^2\theta}\right]$$

$$= \log\left|\frac{x}{a} + \sqrt{1+\frac{x^2}{a^2}}\right| + c_1$$

$$= \log\left|\frac{x+\sqrt{a^2+x^2}}{a}\right| + c_1$$

$$= \log\left|x+\sqrt{a^2+x^2}\right| - \log a + c_1 \;, \quad \because \left[\log\frac{m}{n} = \log m - \log n\right]$$

$$= \log\left|x+\sqrt{a^2+x^2}\right| + c \qquad , \text{ [where } c = -\log a + c_1]$$

Remark : The above three integrals should be remembered and can be used directly as standard integrals.

Hence $\quad \int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \log\left|x+\sqrt{x^2-a^2}\right|+c$$

and $$\int \frac{1}{\sqrt{a^2+x^2}}dx = \log\left|x+\sqrt{a^2+x^2}\right|+c$$

Example 14 Evaluate the following integrals

i) $\int \frac{1}{4x^2-9}$ ii) $\int \frac{1}{16-25x^2}dx$ iii) $\int \frac{1}{16+9x^2}dx$

Solution

i) $\int \frac{1}{4x^2-9}$

$$= \frac{1}{4}\int \frac{1}{x^2-\left(\frac{3}{2}\right)^2}dx$$

$$= \frac{1}{4}\times\frac{1}{2\left(\frac{3}{2}\right)}\log\left|\frac{x-\frac{3}{2}}{x+\frac{3}{2}}\right|+c \quad , \quad \because \int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$$

$$= \frac{1}{12}\log\left|\frac{2x-3}{2x+3}\right|+c$$

ii) $\int \frac{1}{16-25x^2}dx$

$$= \frac{1}{25}\int \frac{1}{\frac{16}{25}-x^2}dx$$

$$= \frac{1}{25}\int \frac{1}{\left(\frac{4}{5}\right)^2-x^2}dx$$

$$= \frac{1}{25}\times\frac{1}{2\left(\frac{4}{5}\right)}\log\left|\frac{\frac{4}{5}+x}{\frac{4}{5}-x}\right|+c \quad , \quad \because \int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c$$

$$= \frac{1}{40}\log\left|\frac{4+5x}{4-5x}\right| + c$$

iii) $\int \frac{1}{16+9x^2}dx$

$$= \frac{1}{9}\int \frac{1}{\frac{16}{9}+x^2}dx$$

$$= \frac{1}{9}\int \frac{1}{\left(\frac{4}{3}\right)^2+x^2}dx$$

$$= \frac{1}{9}\times\frac{1}{\left(\frac{4}{3}\right)}\tan^{-1}\left(\frac{x}{\frac{4}{3}}\right)+c \qquad \because \int\frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$$

$$= \frac{1}{12}\tan^{-1}\left(\frac{3x}{4}\right)+c$$

Example 15 Evaluate the following integrals

i) $\int \frac{1}{\sqrt{4-9x^2}}dx$ ii) $\int \frac{1}{\sqrt{16x^2-25}}dx$ iii) $\int \frac{1}{\sqrt{9+25x^2}}dx$

Solution i) $\int \frac{1}{\sqrt{4-9x^2}}dx$

$$= \frac{1}{\sqrt{9}}\int \frac{1}{\sqrt{\left(\frac{4}{9}\right)-x^2}}dx$$

$$= \frac{1}{\sqrt{9}}\int \frac{1}{\sqrt{\left(\frac{2}{3}\right)^2-x^2}}dx$$

$$= \frac{1}{3}\sin^{-1}\left(\frac{x}{\frac{2}{3}}\right)+c \qquad \because \int\frac{dx}{\sqrt{a^2-x^2}} = \frac{1}{a}\sin^{-1}\left(\frac{x}{a}\right)+c$$

$$= \frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right) + c$$

ii) $\int \frac{1}{\sqrt{16x^2 - 25}} dx$

$$= \frac{1}{\sqrt{16}} \int \frac{1}{\sqrt{x^2 - \left(\frac{25}{16}\right)}} dx$$

$$= \frac{1}{4} \int \frac{1}{\sqrt{x^2 - \left(\frac{5}{4}\right)^2}} dx$$

$$= \frac{1}{4} \log\left|x + \sqrt{x^2 - \frac{25}{16}}\right| + c \quad , \quad \because \int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + c$$

$$= \frac{1}{4} \log\left|x + \sqrt{\frac{16x^2 - 25}{4}}\right| + c$$

$$= \frac{1}{4} \log\left|4x + \sqrt{16x^2 - 25}\right| + c'$$ *Ans*

iii) $\int \frac{1}{\sqrt{9 + 25x^2}} dx$

$$= \frac{1}{\sqrt{25}} \int \frac{1}{\sqrt{\left(\frac{9}{25}\right) + x^2}} dx$$

$$= \frac{1}{5} \int \frac{1}{\sqrt{\left(\frac{3}{5}\right)^2 + x^2}} dx$$

$$= \frac{1}{5} \log\left|x + \sqrt{\left(\frac{3}{5}\right)^2 + x^2}\right| + c \quad , \quad \because \int \frac{1}{\sqrt{a^2 - x^2}} dx = \log\left|x + \sqrt{a^2 + x^2}\right| + c$$

$$= \frac{1}{5} \log\left|5x + \sqrt{9 + 25x^2}\right| + c'$$

Example 16 Evaluate the following integrals $\int \frac{1}{x\sqrt{x^2 - a^2}} dx$

Solution $\int \frac{1}{x\sqrt{x^2 - a^2}} dx$

Put $x = a \sec\theta$

$\therefore\ dx = a \sec\theta \tan\theta \, d\theta$

$\therefore \quad \int \frac{1}{x\sqrt{x^2 - a^2}} dx$

$$= \int \frac{1}{a\sec\theta\sqrt{a^2\sec^2\theta - a^2}} a\sec\theta\tan\theta d\theta$$

$$= \int \frac{\tan\theta}{\sqrt{a^2\tan^2\theta}} d\theta$$

$$= \frac{1}{a}\int d\theta = \frac{1}{a}\theta + c$$

$$= \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c \quad , \quad \left[\begin{array}{l} \because x = a\sec\theta \\ \therefore \frac{x}{a} = \sec\theta \text{ or } \sec^{-1}\left(\frac{x}{a}\right) = \theta \end{array}\right]$$

Remark : The above result can be remembered and used directly as standard integral.

Example 17 Evaluate $\int \frac{1}{x\sqrt{9x^2 - 16}} dx$

Solution $\int \frac{1}{x\sqrt{9x^2 - 16}} dx$

$$= \frac{1}{\sqrt{9}}\int \frac{1}{x\sqrt{x^2 - \frac{16}{9}}} dx$$

$$= \frac{1}{3}\int \frac{1}{x\sqrt{x^2 - \left(\frac{4}{3}\right)^2}} dx$$

$$= \frac{1}{3} \times \frac{1}{\left(\frac{4}{3}\right)} \sec^{-1}\left(\frac{x}{\frac{4}{3}}\right) + c$$

$$= \frac{1}{4}\sec^{-1}\left(\frac{3x}{4}\right) + c$$

Example 18 Evaluate the following integrals

i) $\int \frac{\cos x - \sin x}{\sin x + \cos x} dx$ ii) $\int \frac{1}{x} \log x \, dx$

iii) $\int \frac{\cos x}{(1 + \sin x)^3} dx$ iv) $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$

v) $\int \frac{1}{x + \sqrt{x}} dx$ vi) $\int \frac{e^{2x} + 1}{e^{2x} - 1} dx$

Solution

i) Let $I = \int \frac{\cos x - \sin x}{\sin x + \cos x} dx$

Put $\sin x + \cos x = t$

$\therefore (\cos x - \sin x)\, dx = dt$

$$\therefore \quad I = \int \frac{dt}{t}$$

$= \log |t| + c$

$= \log |\sin x + \cos x| + c$

ii) $\int \frac{1}{x} \log x \, dx$

Put $\log x = t$

$\therefore \quad \frac{1}{x} dx = dt$

$$\therefore \quad \int \frac{1}{x} \log x \, dx = \int t \, dt$$

$$= \frac{t^2}{2} + c$$

$$= \frac{1}{2}(\log x)^2 + c$$

iii) $\int \frac{\cos x}{(1+\sin x)^3} dx$

Put $(1+\sin x) = t$

$\therefore \cos x \, dx = dt$

$$\therefore \int \frac{\cos x}{(1+\sin x)^3} dx = \int \frac{dt}{t^3}$$

$$= \int t^{-3} dt$$

$$= \frac{t^{-3+1}}{-3+1} + c$$

$$= \frac{t^{-2}}{-2} + c$$

$$= -\frac{1}{2t^2} + c$$

$$= \frac{-1}{2(1+\sin x)^2} + c$$

iv) $\int \frac{\sin\sqrt{x}}{\sqrt{x}} dx$

Put $\sqrt{x} = t$

$\therefore \frac{1}{2\sqrt{x}} dx = dt$

or $\frac{dx}{\sqrt{x}} = 2dt$

$$\therefore \int \frac{\sin\sqrt{x}}{\sqrt{x}} dx = \int \sin t \,(2dt)$$

$$= \int 2\sin t \, dt$$

$$= -2\cos t + c$$

$$= -2\cos\sqrt{x} + c$$

v) $\int \frac{1}{x+\sqrt{x}} dx$

$$= \int \frac{1}{\sqrt{x}.\sqrt{x}+\sqrt{x}} dx$$

$$= \int \frac{1}{\sqrt{x}(\sqrt{x}+1)}\, dx$$

Put $(\sqrt{x} + 1) = t$

$\therefore \quad \frac{1}{2\sqrt{x}}\, dx = dt$

or $\frac{dx}{\sqrt{x}} = 2dt$

$$\therefore \quad \int \frac{1}{\sqrt{x}(\sqrt{x}+1)}\, dx = \int \frac{2dt}{t}$$

$$= 2\int \frac{dt}{t}$$

$$= 2\log|t| + c$$

$$= 2\log|\sqrt{x} + 1| + c$$

vi) Let $I = \int \frac{e^{2x}+1}{e^{2x}-1}\, dx$

$$= \int \frac{(e^{2x}+1)/e^{x}}{(e^{2x}-1)/e^{x}}\, dx$$ [Dividing Numerator and denominator by e^x]

$$= \int \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}\, dx$$

Put $e^x - e^{-x} = t$

$\therefore \quad [e^x - (-e^{-x})]dx = dt$

or $(e^x + e^{-x})dx = dt$

$$\therefore \quad I = \int \frac{dt}{t}$$

$$= \log|t| + c$$

$$= \log|e^x - e^{-x}| + c$$

$$= \log\left|\frac{e^{2x}-1}{e^{x}}\right| + c$$

$$= \log|e^{2x}-1| - \log e^x + c$$

$$= \log|e^{2x}-1| - x + c$$

Example 19 Evaluate the following integrals :

i) $\int \frac{1}{(1+x^2)}(\tan^{-1}x)^3 dx$ ii) $\int \frac{\left(e^{\sin^{-1}x}\right)^4}{\sqrt{1-x^2}} dx$

iii) $\int \frac{1}{(1+x^2)\tan^{-1}x} dx$ iv) $\int \frac{\sec^{-1}x}{x\sqrt{x^2-1}} dx$

v) $\int \frac{\operatorname{cosec} x}{\log|\operatorname{cosec} x - \cot x|} dx$ vi) $\int \sin^4 x \cos x \, dx$

Solution

i) $\int \frac{1}{(1+x^2)}(\tan^{-1}x)^3 dx$

Put $\tan^{-1}x = t$

$\therefore \frac{1}{1+x^2} dx = dt$, $\left[\because \frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}\right]$

$$\therefore \int \frac{1}{(1+x^2)}(\tan^{-1}x)^3 dx = \int t^3 dt$$

$$= \frac{t^4}{4} + c$$

$$= \frac{1}{4}(\tan^{-1}x)^4 + c$$

ii) $\int \frac{\left(e^{\sin^{-1}x}\right)^4}{\sqrt{1-x^2}} dx$

Put $e^{\sin^{-1}x} = t$

$\therefore e^{\sin^{-1}x} \cdot \frac{1}{\sqrt{1-x^2}} dx = dt$, $\left[\because \frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{\sqrt{1-x^2}}\right]$

$$\therefore \int \frac{\left(e^{\sin^{-1}x}\right)^4}{\sqrt{1-x^2}} dx = \int \left(e^{\sin^{-1}x}\right)^3 \cdot \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}} dx$$

$$= \int (t)^3 dt$$

$$= \frac{t^4}{4} + c$$

$$= \frac{\left(e^{\sin^{-1} x}\right)^4}{4} + c$$

iii) $\int \frac{1}{(1+x^2)\tan^{-1} x} dx$

Put $\tan^{-1}x = t$

$\therefore \quad \frac{1}{1+x^2} dx = dt \quad , \quad \left[\because \frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}\right]$

$\therefore \quad \int \frac{1}{(1+x^2)\tan^{-1} x} = \int \frac{dt}{t}$

$= \log | \ t \ | + c$

$= \log | \tan^{-1}x | + c$

iv) $\int \frac{\sec^{-1} x}{x\sqrt{x^2 - 1}} dx$

Put $\sec^{-1}x = t$

$\therefore \quad \frac{1}{x\sqrt{x^2 - 1}} dx = dt \quad , \quad \left[\because \frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2 - 1}}\right]$

$\therefore \quad \int \frac{\sec^{-1} x}{x\sqrt{x^2 - 1}} dx = \int t \, dt$

$$= \frac{t^2}{2} + c$$

$$= \frac{1}{2}(\sec^{-1} x)^2 + c$$

v) $\int \frac{\operatorname{cosec} x}{\log |\operatorname{cosec} x - \cot x|} dx$

Put $\log |\operatorname{cosec} x - \cot x| = t$

$\therefore \operatorname{cosec} x \, dx = dt$

$$[\because \frac{d}{dx}[\log|\text{cosec } x - \cot x|] = \text{cosec } x]$$

$$\therefore \quad \int \frac{\text{cosec } x}{\log|\text{cosec } x - \cot x|} dx = \int \frac{dt}{t}$$

$$= \log|t| + c$$

$$= \log|\log|\text{cosec } x - \cot x|| + c$$

vi) $\int \sin^4 x \cos x \, dx$

Put $\sin x = t$

$\therefore \cos x \, dx = dt$

$$\therefore \quad \int \sin^4 x \cos x \, dx = \int (t)^4 \, dt$$

$$= \frac{t^5}{5} + c$$

$$= \frac{1}{5}\sin^5 x + c$$

Example 20 Evaluate the following integrals

i) $\int \frac{\sin 2x dx}{(a^2 - b^2\cos^2 x)}$ ii) $\int x^3 e^{x^4} \cos(e^{x^4}) dx$

iii) $\int \frac{(1 + \log x)}{x^x} dx$ iv) $\int \frac{(x+1)(x+\log x)^3}{x} dx$

Solution

i) $\int \frac{\sin 2x dx}{(a^2 - b^2\cos^2 x)}$

Put $a^2 - b^2\cos^2 x = t$

$-b^2(2\cos x(-\sin x))dx = dt$

$b^2 (\sin 2x)dx = dt$

$\therefore \sin 2x \, dx = \frac{dt}{b^2}$

$$\therefore \quad \int \frac{\sin 2x dx}{(a^2 - b^2\cos^2 x)} = \frac{1}{b^2}\int \frac{dt}{t}$$

$$= \frac{1}{b^2}\log|t| + c$$

$$= \frac{1}{b^2}\log|a^2 - b^2\cos^2 x| + c$$

ii) $\int x^3 e^{x^4} \cos(e^{x^4})dx$

Put $e^{x^4} = t$

$\therefore e^{x^4} . 4x^3 dx = dt$

$\therefore x^3 e^{x^4} dx = \frac{1}{4} dt$

$\therefore \int x^3 e^{x^4} \cos(e^{x^4})dx = \int \frac{1}{4} \cos t\, dt$

$= \frac{1}{4} \sin t + c$

$= \frac{1}{4} \sin\left(e^{x^4}\right) + c$

iii) $\int \frac{(1+\log x)}{x^x} dx$

Put $x^x = t$

$\therefore x^x (1+\log x)\, dx = dt$

$(1+\log x)dx = \frac{dt}{x^x}$

$= \frac{dt}{t}$

$\therefore \int \frac{(1+\log x)}{x^x} dx = \int \frac{1}{t} \frac{dt}{t}$

$= \int \frac{dt}{t^2} = \int t^{-2} dt$

$= \frac{t^{-2+1}}{-2+1} + c$

$= -\frac{1}{t} + c$

$= -\frac{1}{x^x} + c$

$= -x^{-x} + c$

iv) $\int \frac{(x+1)(x+\log x)^3}{x} dx$

$= \int \left(1+\frac{1}{x}\right)(x+\log x)^3 dx$

Put (x+logx) = t

$\therefore \left(1+\frac{1}{x}\right)dx = dt$

$= \int t^3 dt$

$= \int \frac{t^4}{4} + c$

$= \frac{1}{4}(x+\log x)^4 + c$

INTEGRATION OF THE FORM

$\int \frac{1}{ax^2+bx+c} dx \; ; a \neq 0$

Steps of solving

i) Make the co-efficient of x^2 unity i.e. equal to 1.

ii) Make the Denominator - (a complete square) $\pm (K)^2$

iii) Use one of the standard integrals $\int \frac{1}{x^2-a^2}dx$, $\int \frac{1}{a^2-x^2}dx$, $\int \frac{1}{a^2+x^2}dx$, to find the integral.

Example 21 Evaluate $\int \frac{1}{4x^2+6x+9}dx$,

Solution $\int \frac{1}{4x^2+6x+9}dx$,

$= \frac{1}{4}\int \frac{1}{x^2+\frac{3}{2}x+\frac{9}{4}}dx$ [Making the coefficient of x^2 equal to 1]

$= \frac{1}{4}\int \frac{1}{x^2+\frac{3}{2}x+\frac{9}{16}-\frac{9}{16}+\frac{9}{4}}dx$

[Adding and subtracting (½ coefficient of x)^2

$$= \frac{1}{4}\int \frac{1}{\left(x+\frac{3}{2}\right)^2 + \frac{27}{16}}dx$$

$$= \frac{1}{4}\int \frac{1}{\left(x+\frac{3}{2}\right)^2 + \left(\frac{\sqrt{27}}{4}\right)^2}dx \quad , \qquad \left[\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$= \frac{1}{4}\times \frac{1}{\left(\frac{\sqrt{27}}{4}\right)^2}\tan^{-1}\left[\frac{x+\frac{3}{4}}{\frac{\sqrt{27}}{4}}\right]+c$$

$$= \frac{1}{\sqrt{27}}\tan^{-1}\left[\frac{4x+3}{\sqrt{27}}\right]+c$$

$$= \frac{1}{3\sqrt{3}}\tan^{-1}\left[\frac{4x+3}{\sqrt{27}}\right]+c$$

INTEGRATION OF THE FORM

$$\int \frac{1}{\sqrt{ax^2+bx+c}}dx;\ a \neq o$$

Steps of solving

(a) First make the Quadratic Polynomial inside the square root in the standard form by previous knowledge.

(b) Now use one of the standard Integrals $\int \frac{1}{\sqrt{a^2-x^2}}dx$, $\int \frac{1}{\sqrt{x^2-a^2}}dx$, $\int \frac{1}{\sqrt{a^2+x^2}}dx$.

Example 22 $\int \frac{1}{\sqrt{3x^2-4x-6}}dx$

Solution $\int \frac{1}{\sqrt{3x^2-4x-6}}dx$

$$= \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{x^2 - \frac{4}{3}x - 2}}\,dx$$

$$= \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{x^2 - \frac{4}{3}x + \frac{4}{9} - \frac{4}{9} - 2}}\,dx$$

$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\left(x - \frac{2}{3}\right)^2 - \frac{22}{9}}}\,dx$$

$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\left(x - \frac{2}{3}\right)^2 - \left(\frac{\sqrt{22}}{3}\right)^2}}\,dx$$

$$= \frac{1}{\sqrt{2}}\log\left|\left(x - \frac{2}{3}\right) + \sqrt{\left(x - \frac{2}{3}\right)^2 - \left(\frac{\sqrt{22}}{3}\right)^2}\right| + c$$

$$\left[\because \int \frac{1}{\sqrt{x^2 - a^2}}dx = \log|x + \sqrt{x^2 - a^2}| + c\right]$$

$$= \frac{1}{\sqrt{2}}\log\left|\left(\frac{3x - 2}{3}\right) + \sqrt{\left(x - \frac{2}{3}\right)^2 - \frac{22}{9}}\right| + c$$

Example 23 Evalute $\int \frac{1}{\sqrt{8 + 4x - 9x^2}}dx$

Solution $\int \frac{1}{\sqrt{8 + 4x - 9x^2}}dx$

$$= \frac{1}{\sqrt{9}}\int \frac{1}{\sqrt{\frac{8}{9} + \frac{4}{9}x - x^2}}\,dx$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{\frac{8}{9}-\left(x^2-\frac{4}{9}x\right)}}dx$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{\frac{8}{9}+\frac{4}{81}-\left(x^2-\frac{4}{9}x+\frac{4}{81}\right)}}dx$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{\left(\frac{72+4}{81}\right)-\left(x-\frac{2}{9}\right)^2}}dx$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{\left(\frac{\sqrt{76}}{9}\right)^2-\left(x-\frac{2}{9}\right)^2}}dx$$

$$= \frac{1}{3}\sin^{-1}\left[\frac{\left(x-\frac{2}{9}\right)}{\left(\frac{\sqrt{76}}{9}\right)}\right]+c \qquad \left[\because \int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right)+c\right]$$

$$= \frac{1}{3}\sin^{-1}\left(\frac{9x-2}{\sqrt{76}}\right)+c$$

$$= \frac{1}{3}\sin^{-1}\left(\frac{9x-2}{2\sqrt{19}}\right)+c$$

INTEGRATION OF THE FORM

$$\int \frac{px+q}{ax^2+bx+c}dx \ ; a \neq 0$$

Example 24 Evaluate $\int \frac{3x-4}{2x^2+4x-5}dx$

Solution:- Let $I = \int \frac{3x-4}{2x^2+4x-5}dx$

Put $(3x - 4) = \lambda + \mu \frac{d}{dx}(2x^2 + 4x - 5)$

$3x - 4 = \lambda + \mu(4x + 4)$

$\therefore \quad 3x - 4 = 4\mu x + (\lambda + 4\mu)$

Compairing the coefficients of x and constant terms we get,

$4\mu = 3 \quad$ and $\lambda + 4\mu = -4$

$\therefore \quad \mu = \dfrac{3}{4} \quad$ and $\lambda = -4 - 4\mu$

$$= -4 - 4\left(\frac{3}{4}\right) = -7$$

$$\therefore\ I = \int \frac{-7 + \frac{3}{4}(4x+4)}{2x^2 + 4x - 5}\,dx$$

$$= -7\int \frac{1}{2x^2+4x-5}\,dx + \frac{3}{4}\int \frac{4x+4}{2x^2+4x-5}\,dx$$

$$= -7I_1 + \frac{3}{4}\log|2x^2 + 4x - 5| + c_1 \quad \text{--- (1)}, \quad \left[\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c\right]$$

where $I_1 = \int \dfrac{1}{2x^2+4x-5}\,dx$

$$= \frac{1}{2}\int \frac{1}{x^2 + 2x - \frac{5}{2}}\,dx$$

$$= \frac{1}{2}\int \frac{1}{(x^2+2x+1) - 1 - \frac{5}{2}}\,dx$$

$$= \frac{1}{2}\int \frac{1}{(x+1)^2 - \left(\frac{\sqrt{7}}{\sqrt{2}}\right)^2}\,dx$$

$$= \frac{1}{2} \times \frac{1}{2\left(\frac{\sqrt{7}}{\sqrt{2}}\right)} \log\left|\frac{(x+1) - \frac{\sqrt{7}}{\sqrt{2}}}{(x+1) + \frac{\sqrt{7}}{\sqrt{2}}}\right| + c_2$$

$$\therefore\ I_1 = \frac{1}{2\sqrt{14}} \log\left|\frac{\sqrt{2}(x+1) - \sqrt{7}}{\sqrt{2}(x+1) + \sqrt{7}}\right| + c_2, \quad \left[\because \int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$$

using the value of I_1 in (1) we get,

$$I = \frac{-7}{2\sqrt{14}}\log\left|\frac{\sqrt{2}(x+1)-\sqrt{7}}{\sqrt{2}(x+1)+\sqrt{7}}\right| + \frac{3}{4}\log|2x^2+4x-5| + c_1 - 7c_2$$

$$\text{or} \quad I = \frac{-7}{2\sqrt{14}}\log\left|\frac{\sqrt{2}(x+1)-\sqrt{7}}{\sqrt{2}(x+1)+\sqrt{7}}\right| + \frac{3}{4}\log|2x^2+4x-5| + c$$

INTEGRATION OF THE FORM

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx \quad ; \; a \neq 0$$

Example 25 Evaluate $\int \frac{x+3}{\sqrt{4x^2+5x+2}}\,dx$

Solution : Let $I = \int \frac{x+3}{\sqrt{4x^2+5x+2}}\,dx$

Put $(x+3) = \lambda + \mu \frac{d}{dx}(4x^2+5x+2)$

$= \lambda + \mu\,[8x+5]$

$x + 3 = 8\mu x + (5\mu + \lambda)$, compairing the coefficient of x and constant forms we get

$\therefore \; 8\mu = 1$ and $5\mu + \lambda = 3$

$\mu = \frac{1}{8}$ and $5\left(\frac{1}{8}\right) + \lambda = 3$

$\lambda = 3 - \frac{5}{8}$

$\lambda = \frac{19}{8}$

$$\therefore \; I = \int \frac{\frac{19}{8} + \frac{1}{8}(8x+5)}{\sqrt{4x^2+5x+2}}\,dx$$

$$= \frac{19}{8}\int \frac{1}{\sqrt{4x^2+5x+2}}\,dx + \frac{1}{8}\int \frac{8x+5}{4x^2+5x+2}\,dx$$

$$= \frac{19}{8} I_1 + \frac{1}{8} I_2 \qquad ---(1)$$

Now $I_1 = \int \frac{1}{\sqrt{4x^2+5x+2}}\,dx$

$$= \frac{1}{\sqrt{4}} \int \frac{1}{\sqrt{x^2 + \frac{5}{4}x + \frac{1}{2}}}\,dx$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{x^2 + \frac{5}{4}x + \frac{25}{64} + \frac{1}{2} - \frac{25}{64}}}\,dx$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{\left(x+\frac{5}{8}\right)^2 + \left(\frac{\sqrt{7}}{8}\right)^2}} \quad , \quad \left[\text{use } \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \log\left|x+\sqrt{x^2+a^2}\right| + c\right]$$

$$= \frac{1}{2}\log\left|\left(x+\frac{5}{8}\right) + \sqrt{\left(x+\frac{5}{8}\right)^2 + \left(\frac{\sqrt{7}}{8}\right)^2}\right| + c_1 \qquad ---(2)$$

and $I_2 = \int \frac{8x+5}{\sqrt{4x^2+5x+2}}\,dx$

Put $4x^2 + 5x + 2 = t$

$\therefore$ $(8x + 5)\,dx = dt$

$\therefore$ $I_2 = \int \frac{dt}{\sqrt{t}}$

$$= \int t^{-\frac{1}{2}}\,dt$$

$$= \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + c_2$$

$$= 2\sqrt{t} + c_2$$

$$= 2\sqrt{4x^2+5x+2} + c_2 \qquad ---(3)$$

using he value of I_1 and I_2 from (2) and (3) in (1) we get,

$$I = \frac{19}{16}\log\left|\left(x+\frac{5}{8}\right)+\sqrt{\left(x+\frac{5}{8}\right)^2+\left(\frac{\sqrt{7}}{8}\right)^2}\right| + \frac{1}{4}\sqrt{4x^2+5x+2} + \left(\frac{19}{8}c_1 + \frac{1}{8}c_2\right)$$

or $$I = \frac{19}{16}\log\left|\left(x+\frac{5}{8}\right)+\sqrt{\left(x+\frac{5}{8}\right)^2+\left(\frac{\sqrt{7}}{8}\right)^2}\right| + \frac{1}{4}\sqrt{4x^2+5x+2} + c$$

INTEGRATION BY PARTS

So far, we were dealing with the Integration of the functions which were single functions or if they were the product of two or more functions, then the corresponding Integrals can be converted to simple standard elementary Integrals.

e.g $I = \int e^x \sin e^x dx$

if we Put $e^x dx = dt$

$\therefore \quad I = \int \sin t \, dt = -\cos t + c$

$= -\cos e^x + c.$

However, some times the Integrals is the Product of two functions, and it is not possible to convert the corresponding Integrals into standard elementary Integrals. e. g. $\int x \sin x \, dx$, $\int e^x \sin x \, dx$ etc.

This problem leads to the Concept of "INTEGRATION BY PARTS".

If $f(x) = g(x) . h(x)$

then, $\int f(x) \, dx = \int g(x) . h(x) \, dx$

$$= g(x) \int h(x) \, dx - \int \left(\frac{d}{dx}(g(x)) . \int h(x) dx \right) dx$$

i.e. $\int$ (Ist function) x (IInd function) dx

$$= (\text{Ist}) \int (\text{IInd}) dx - \int \left[\frac{d}{dx}(\text{Ist}) . \int (\text{IInd}) dx \right] dx$$

Remarks-

1. The above method can be used to integrate any Integrand which is the Product of two functions.
2. The order of Preference for the function to be taken as first function can be made by a word ' ILATE', where I stands for Inverse function, L stands for Logarithmic function, A stands for Algebraic function, T stands for Trigonometric functions, and E stands for Exponential functions.

Example 26 Integrate x sin x w.r.t. x

Solution $\int \underset{I}{x} \underset{II}{\sin x}\, dx$

$\therefore$ x is an algebraic function and Sin x is trigonometric, So by the word of preference 'ILATE' , 'x' should be treated as Ist function and (sin x) as second.

$$\therefore \int \underset{I}{x} \underset{II}{\sin x}\, dx = x\int \sin x dx - \int\left[\frac{d}{dx}(x)\int \sin x dx\right]dx$$

$= x(-\cos x) - \int 1\,(-\cos x)dx$

$= -x \cos x + \int \cos x\, dx$

$= -x\cos x + \sin x + c$

Example 27 Evaluate $\int x \log x\, dx$

Solution $\int \underset{II}{x} \underset{I}{\log x}\, dx$

$$= (\log x)\int x\, dx - \int\left(\frac{d}{dx}(\log x).\int x dx\right)dx$$

$$= (\log x)\left(\frac{x^2}{2}\right) - \int \frac{1}{x}.\frac{x^2}{2}dx$$

$$= \frac{1}{2}x^2 \log x - \frac{1}{2}\int x dx$$

$$= \frac{1}{2}x^2 \log x - \frac{1}{4}x^2 + c$$

Example 28 Evalute $\int x^2 \sin^2 x\, dx$

Solution $\int x^2 \sin^2 x\, dx$

$$= \int x^2 \frac{(1-\cos 2x)}{2}\, dx \qquad [\because 1-\cos 2x = 2\sin^2 x]$$

$$= \frac{1}{2}\int x^2 dx - \frac{1}{2}\int \underset{I}{x^2} \underset{II}{\cos 2x} dx$$

$$= \frac{1}{2}\left(\frac{x^3}{3}\right) - \frac{1}{2}\left[x^2\int \cos 2x dx - \int\left[\frac{d}{dx}(x^2)\int \cos 2x dx\right]dx\right]$$

$$= \frac{1}{6}x^3 - \frac{1}{2}\left[\frac{x^2 \sin 2x}{2} - \int \frac{2x \sin 2x}{2}dx\right]$$

$$= \frac{1}{6}x^3 - \frac{1}{4}x^2 \sin 2x + \frac{1}{2}\int \underset{I}{x} \underset{II}{\sin 2x}\, dx$$

$$= \frac{1}{6}x^3 - \frac{1}{4}x^2 \sin 2x + \frac{1}{2}\left[x\int \sin 2x dx - \int \frac{d}{dx}(x)\int \sin 2x dx\right]dx$$

[Integrating again by parts]

$$= \frac{1}{6}x^3 - \frac{1}{4}x^2 \sin 2x + \frac{1}{2}x\frac{(-\cos 2x)}{2} - \frac{1}{2}\int (1)\frac{(-\cos 2x)}{2}dx$$

$$= \frac{1}{6}x^3 - \frac{1}{4}x^2 \sin 2x - \frac{1}{4}x\cos 2x + \frac{1}{4}\int \cos 2x\, dx$$

$$= \frac{1}{6}x^3 - \frac{1}{4}x^2 \sin 2x - \frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x + c$$

Example 29 Evalute $\int (2x - 7) \sin (3x + 4)\, dx$

Solution Let $I = \int \underset{I}{(2x - 7)} \underset{II}{\sin (3x+ 4)}\, dx$

$$= (2x - 7) \int \sin (3x + 4)dx - \int\left[\frac{d}{dx}(2x-7)\int \sin(3x+4)dx\right]dx$$

$$= \frac{(2x-7)(-\cos(3x+4))}{3} - \int \frac{2[-\cos(3x+4)]}{3}\, dx$$

$$= -\frac{1}{3}(2x - 7) \cos (3x + 4) + \frac{2}{3}\int \cos (3x + 4)\, dx$$

$$= -\frac{1}{3}(2x - 7) \cos (3x + 4) + \frac{2}{9} \sin (3x+ 4) + c$$

Example 30 Evaluate $\int \text{Sec}^3\theta\, d\theta$

Solution Let $I = \int \sec^3\theta\, d\theta$

$$= \int \underset{II}{\text{Sec}^2\theta} . \underset{I}{\text{Sec}\, \theta}\, d\theta$$

$$= \sec\theta \int \sec^2\theta\, d\theta - \int [\frac{d}{d\theta}(\sec\theta). \int \sec^2 \theta\, d\theta]d\theta$$

$$= \sec\theta . \tan \theta - \int \sec \theta \tan\theta . \tan\theta\, d\theta$$

$$= \text{Sec}\theta . \tan \theta - \int \sec \theta . \tan^2 \theta\, d\theta$$

$$= \sec\theta . \tan \theta - \int \sec\theta . (\sec^2\theta - 1)d\theta$$

$$I = \sec \theta . \tan \theta - \int \sec^3\theta\, d\theta + \int \sec \theta\, d\theta$$

$$I = \sec\theta \tan\theta - I + \log|\sec\theta + \tan\theta|$$

or $$I + I = \sec\theta . \tan\theta + \log|\sec\theta + \tan\theta|$$

or $$2I = \sec\theta . \tan\theta + \log|\sec\theta + \tan\theta|$$

or $$I = \frac{1}{2}\sec\theta . \tan\theta + \frac{1}{2}\log|\sec\theta + \tan\theta| + c$$

Example 31 Evaluate $\int x^2 e^x dx$

Solution $\int \underset{I}{x^2} \underset{II}{e^x} dx$

$$= x^2 \int e^x dx - \int [\frac{d}{dx}(x^2) . \int e^x dx] dx$$

$$= x^2 e^x - \int 2x . e^x dx$$

$$= x^2 e^x - 2\int \underset{I}{x} . \underset{II}{e^x} dx$$

$$= x^2 e^x - 2\left[x\int e^x dx - \int\left[\frac{d}{dx}(x)\int e^x dx\right]dx\right]$$

[Integrating again by parts]

$$= x^2 e^x - 2[x e^x - \int 1.e^x dx]$$

$$= x^2 e^x - 2x e^x + 2e^x + c$$

Example 32 Evaluate $\int \log x dx$

Solution $\int \log x$

$$= \int \underset{II}{I}.\underset{I}{(\log x)}$$

$$= \log x \int 1 dx - \int\left(\frac{d}{dx}(\log x).\int 1 dx\right)dx$$

$$= (\log x)(x) - \int \frac{1}{x} .x.dx$$

$$= x \log x - \int 1 dx$$

$$= x \log x - x + c$$

An Important Result to remember

$$\int e^x [f(x) + f'(x) dx = e^x f(x) + c$$

Example 33 Integrate the following functions w.r.t x.

(i) $\int e^x\left(\frac{x-1}{x^2}\right)dx$

(ii) $\int e^x [\tan x + \sec^2 x]dx$

(iii) $\int (\sin x + \cos x)\, dx$

(iv) $\int e^x \sec x\, (1 + \tan x)\, dx$

Solution

(i) $\int e^x\left(\frac{x-1}{x^2}\right)dx$

$= \int e^x\left[\frac{x}{x^2}-\frac{1}{x^2}\right]dx$

$= \int e^x\left[\frac{1}{x}-\frac{1}{x^2}\right]dx$

$= e^x.\frac{1}{x}+c$, (using the above result)

(ii) $\int e^x[\tan x + \sec^2 x]\, dx$

$= e^x \tan x + c$, (using the above result)

(iii) $\int e^x (\sin x + \cos x)\, dx$

$= e^x \sin x+c$, (using the above result)

(iv) $\int e^x \sec x\, (1+\tan x)\, dx$

$= \int e^x (\sec x + \sec x \tan x)\, dx$

$= e^x \sec x + c$, (using the above result)

Example 34 Evaluate the following Integrals:-

(i) $\int \sqrt{a^2 - x^2}dx$ (ii) $\int \sqrt{a^2 + x^2}dx$

(iii) $\int \sqrt{x^2 - a^2}dx$

Solution :-

(i) $\int \sqrt{a^2 - x^2}dx$ put $x= a \sin\theta$

$\therefore\ dx = a \cos\theta\, d\theta$

$\therefore\ \int \sqrt{a^2 - x^2}dx = \int \sqrt{a^2 - a^2 \sin^2\theta}.a\cos\theta d\theta$

$$= a^2\int \frac{1+\cos 2\theta}{2} d\theta$$

$$= \frac{1}{2}a^2\int 1 d\theta + \frac{1}{2}a^2\int \cos 2\theta d\theta$$

$$= \frac{1}{2}a^2\theta + \frac{1}{2}a^2\frac{\sin 2\theta}{2} + c$$

$$= \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{4}a^2(2\sin\theta \,.\, \cos\theta) + c$$

$$= \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right) + \frac{a^2}{2}\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}} + c$$

$$= \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right) + \frac{a}{2}x\sqrt{\frac{a^2-x^2}{a}} + c$$

$$= \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2-x^2}}{2} + c$$

$$\therefore \int\sqrt{a^2-x^2}dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$$

ii) $\int\sqrt{a^2+x^2}dx$

Put $x = a\tan\theta$

$\therefore dx = a\sec^2\theta \, d\theta$

$$\therefore \int\sqrt{a^2+x^2}dx = \int\sqrt{a^2+a^2\tan^2\theta}\,.a\sec^2\theta\, d\theta$$

$$= \int a\sec\theta \,.\, a\sec^2\theta\, d\theta$$

$$= a^2\int \sec^3\theta\, d\theta$$

$$= a^2\left[\frac{1}{2}\sec\theta.\tan\theta + \frac{1}{2}\log|\sec\theta+\tan\theta|\right] + c$$

[see example 30]

$$= \frac{a^2}{2}\sec\theta\tan\theta + \frac{a^2}{2}\log|\sec\theta + \tan\theta| + c$$

$$= \frac{a^2}{2}\sqrt{1+\tan^2\theta}.\tan\theta + \frac{a^2}{2}\log\left|\sqrt{1+\tan^2\theta}+\tan\theta\right| + c$$

$$= \frac{a^2}{2}\sqrt{1+\tan^2\theta}.\tan\theta + \frac{a^2}{2}\log\left|\sqrt{1+\tan^2\theta}+\tan\theta\right| + c$$

$$= \frac{a^2}{2}\sqrt{1+\frac{x^2}{a^2}}.\left(\frac{x}{a}\right) + \frac{a^2}{2}\log\left|\sqrt{1+\frac{x^2}{a^2}}+\frac{x}{a}\right| + c$$

$$= \frac{a^2}{2}\sqrt{\frac{a^2+x^2}{a^2}} + \frac{x}{a} + \frac{a^2}{2}\log\left|\frac{\sqrt{a^2+x^2}}{a}+\frac{x}{a}\right| + c$$

$$= \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\log\left|\frac{x+\sqrt{a^2+x^2}}{a}\right| + c$$

$$= \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\log\left|x+\sqrt{a^2+x^2}\right| - \frac{a^2}{2}\log a + c$$

$$= \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\log\left|x+\sqrt{a^2+x^2}\right| + c'$$

Let I $= \int \underset{II}{1}.\underset{I}{\sqrt{a^2+x^2}}\,dx$

$$= \sqrt{a^2+x^2}\int 1dx - \int\left[\frac{d}{dx}\left(\sqrt{a^2+x^2}\right)\int 1dx\right]dx$$

$$= x\sqrt{a^2+x^2} - \int \frac{1.2x}{2\sqrt{a^2+x^2}}.x\,dx$$

$$= x\sqrt{a^2+x^2} - \int \frac{x^2}{\sqrt{a^2+x^2}}\,dx$$

$$= x\sqrt{a^2+x^2} - \int \frac{\left(a^2+x^2-a^2\right)}{\sqrt{a^2+x^2}}\,dx$$

$$= x\sqrt{a^2+x^2} - \int\sqrt{a^2+x^2}dx + \int \frac{a^2}{\sqrt{a^2+x^2}}\,dx$$

$$= x\sqrt{a^2+x^2} - I + a^2\int \frac{dx}{\sqrt{a^2+x^2}}$$

or $\quad I = \dfrac{x\sqrt{a^2+x^2}}{2} + \dfrac{a^2}{2}\log\left|x+\sqrt{a^2+x^2}\right| + c$

(iii) $\int \sqrt{x^2-a^2}\,dx$

put $x = a\sec\theta$

$dx = a\sec\theta\tan\theta\,d\theta$

$$\therefore \int\sqrt{x^2-a^2}\,dx = \int\sqrt{a^2\sec^2\theta-a^2}\,(a\sec\theta\tan\theta\,d\theta)$$

$$= \int a\tan\theta \,.\, a\sec\theta\tan\theta\,d\theta$$

$$= a^2\int\sec\theta\,.\,\tan^2\theta\;d\theta$$

$$= a^2\int\sec\theta\,[\sec^2\theta-1]d\theta$$

$$= a^2\int\sec^3\theta\,d\theta - a^2\int\sec\theta\,d\theta$$

$$= a^2\left[\frac{1}{2}\sec\theta\,.\,\tan\theta + \frac{1}{2}\log|\sec\theta+\tan\theta|\right]$$
$$-a^2\log|\sec\theta+\tan\theta| + c$$

$$= \frac{a^2}{2}\sec\theta\tan\theta + \left(\frac{a^2}{2}-a^2\right)\log|\sec\theta+\tan\theta| + c$$

$$= \frac{a^2}{2}\sec\theta\tan\theta - \frac{a^2}{2}\log|\sec\theta+\tan\theta| + c$$

$$= \frac{a^2}{2}\left(\frac{x}{a}\right)\sqrt{\sec^2\theta-1} - \frac{a^2}{2}\log\left|\frac{x}{a}+\sqrt{\sec^2\theta-1}\right| + c$$

$$= \frac{ax}{2}\sqrt{\frac{x^2}{a^2}-1} - \frac{a^2}{2}\log\left|\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right| + c$$

$$= \frac{ax}{2}\sqrt{\frac{x^2-a^2}{a^2}} - \frac{a^2}{2}\log\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + c$$

$$= \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}| + c$$

Note : the above three result can be remembered and used as standard Integrals

(1) $\int\sqrt{a^2-x^2}\,dx = \dfrac{x\sqrt{a^2-x^2}}{2} + \dfrac{a^2}{2}\sin^{-1}\left(\dfrac{x}{a}\right) + c$

(1) $\int\sqrt{a^2-x^2}\,dx=\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$

(2) $\int\sqrt{a^2+x^2}\,dx=\frac{x\sqrt{a^2+x^2}}{2}+\frac{a^2}{2}\log|x+\sqrt{a^2+x^2}|+c$

(3) $\int\sqrt{x^2-a^2}\,dx=\frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\log|x+\sqrt{a^2-x^2}|+c$

Integration by Partial Fraction :-

In Previous semester we studied , how to make partial fractions of a given fraction, now we are to integrate a given fraction, after making its partial fractions.

Example 35 Evaluate $\int\frac{5}{(x+2)(x-1)}dx$

Solution Let $\frac{5}{(x+2)(x-1)}=\frac{A}{(x+2)}+\frac{B}{(x-1)}$ ----(a)

$\therefore \quad \frac{5}{(x+2)(x-1)}=\frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$

$\therefore \quad 5 = A(x-1) + B(x+2)$ ----(1)

put $(x+2) = 0$ i.e. $x = -2$ in (1)

$\therefore \quad 5 = A(-2-1) + B(0)$

or $A = -\frac{5}{3}$

Now , put $(x-1) = 0$ i.e. $x = 1$ in (1)

$\therefore \quad 5 = A(0) + B(1+2)$

or $B = \frac{5}{3}$

using the values of A and B in Eq. (a) we get

$$\frac{5}{(x+2)(x-1)}=\frac{-\frac{5}{3}}{(x+2)}+\frac{\frac{5}{3}}{(x-1)}$$

Integrating both sides we gave

$\therefore \quad \int\frac{5}{(x+2)(x-1)}dx=-\frac{5}{3}\int\frac{1}{(x+2)}dx+\frac{5}{3}\int\frac{1}{(x-1)}dx$

$$= -\frac{5}{3}\log|x+2| + \frac{5}{3}\log|x-1| + c$$

$$\left[\because \int \frac{1}{ax+b}dx = \frac{\log|ax+b|}{a}\right]$$

$$= \frac{5}{3}\log\left|\frac{x-1}{x+2}\right| + c$$

Example 36 Evaluate $\int \frac{\sin x \cos x}{(3\sin x - 2)(2\sin x - 1)}dx$

Solution $\int \frac{\sin x \cos x}{(3\sin x - 2)(2\sin x - 1)}dx$

Put $\sin x = t$
$\therefore \cos x\, dx = dt$

$$= \int \frac{\sin x \cos x}{(3\sin x - 2)(2\sin x - 1)}dx$$

$$= \int \frac{t\, dt}{(3t-2)(2t-1)}$$

Let $\frac{t}{(3t-2)(2t-1)} = \frac{A}{(3t-2)} + \frac{B}{(2t-1)}$ ----(1)

$\therefore t = A(2t - 1) + B(3t - 2)$ ----(2)

put $(2t - 1) = 0$ i.e. $t = \frac{1}{2}$ in (2)

$$\frac{1}{2} = A(0) + B\left[3\left(\frac{1}{2}\right) - 2\right]$$

or $B = \frac{1}{2} \times \left(-\frac{2}{1}\right) = -1$

$\therefore B = -1$

Now, put $3t - 2 = 0$ i.e. $t = \frac{2}{3}$ in (2)

$$\frac{2}{3} = A\left[2\left(\frac{2}{3}\right) - 1\right] + B(0)$$

or $A = \frac{2}{3} \times \frac{3}{1}$ $\therefore A = 2$

using the values of A and B in equation (1) we get

$$\frac{t}{(3t-2)(2t-1)} = \frac{2}{3t-2} + \frac{(-1)}{(2t-1)}$$

$$\therefore \int \frac{t}{(3t-2)(2t-1)} = \int \frac{2}{3t-2}\,dt + \int \frac{(-1)}{(2t-1)}\,dt$$

$$= \frac{2\log|3t-2|}{3} - \frac{\log|2t-1|}{2} + c$$

$$= \log|3t-2|^{\frac{2}{3}} - \log|2t-1|^{\frac{1}{2}} + c$$

$$= \log\left(\frac{|3t-2|^{\frac{2}{3}}}{|2t-1|^{\frac{1}{2}}}\right) + c$$

$$= \log\left(\frac{|3\sin x-2|^{\frac{2}{3}}}{|2\sin x-1|^{\frac{1}{2}}}\right) + c$$

Example 37 Evaluate $\int \frac{\sin 2x\,dx}{(4+\sin^2 x)(9+\sin^2 x)}$

Solution $\int \frac{\sin 2x\,dx}{(4+\sin^2 x)(9+\sin^2 x)}$

Put $\sin^2 x = t$

$\therefore 2\sin x \cos x\, dx = dt$

$\therefore \sin 2x\, dx = dt$

$$\therefore \int \frac{\sin 2x\,dx}{(4+\sin^2 x)(9+\sin^2 x)}$$

$$= \int \frac{dt}{(4+t)(9+t)}$$

$$= \int \left[\frac{1}{5(4+t)} - \frac{1}{5(9+t)}\right] dt$$

[By making Partial tractions]

$$= \frac{1}{5}\int \frac{1}{4+t}\,dt - \frac{1}{5}\int \frac{1}{9+t}\,dt$$

$$= \frac{1}{5}\log\ |4+t| - \frac{1}{5}\log\ |9+t| + c$$

$$= \frac{1}{5}\left|\frac{4+t}{9+t}\right| + c \quad = \frac{1}{5}\log\left|\frac{4+\sin^2 x}{9+\sin^2 x}\right| + c$$

Example 38 Evaluate $\int \frac{\tan x}{3+2\tan^2 x}dx$

Solution $\int \frac{\tan x}{3+2\tan^2 x}dx$

Put $\tan^2 x = t$

or $2\tan x \cdot \sec^2 x\, dx = dt$

$$\therefore \quad \tan x\, dx = \frac{1}{2\sec^2 x}dt$$

$$\text{or} \quad \tan x\, dx = \frac{1}{2(1+\tan^2 x)}dt$$

$$= \frac{1}{2(1+t)}dt$$

$$\therefore \quad \int \frac{\tan x}{3+2\tan^2 x}dx = \int \frac{1}{2(1+t)(3+2t)}dt$$

$$= \frac{1}{2}\int\left[\frac{1}{(1+t)} - \frac{2}{(3+2t)}\right]dt$$

(By making Partial fractions)

$$= \frac{1}{2}\int \frac{1}{1+t}dt - \int \frac{1}{3+2t}dt$$

$$= \frac{1}{2}\log|1+t| - \frac{\log|3+2t|}{2} + c$$

$$= \frac{1}{2}\log\left|\frac{1+t}{3+2t}\right| + c$$

$$= \frac{1}{2}\log\left|\frac{1+\tan^2 x}{3+2\tan^2 x}\right| + c$$

$$= \log\sqrt{\frac{1+\tan^2 x}{3+2\tan^2 x}} + c$$

SOME SPECIAL TYPES OF INTEGRATIONS

TYPE I **INTEGRALS OF THE TYPE**

$$\int \frac{1}{a+b\cos^2 x}dx,\ \int \frac{1}{a+b\sin^2 x}dx \text{ and } \int \frac{1}{a\sin^2 x+b\cos^2 x+c}\, dx$$

Example 39 Evaluate $\int \frac{1}{3\sin^2 x+4\cos^2 x+2}\, dx$

Solution Let $I = \int \frac{1}{3\sin^2 x+4\cos^2 x+2}\, dx$

Dividing Numerator and Denominator by $\cos^2 x$, we get

$$I = \int \frac{\frac{1}{\cos^2 x}dx}{3\tan^2 x+4+2\sec^2 x}$$

$$= \int \frac{\sec^2 dx}{3\tan^2 x+4+2(1+\tan^2 x)}$$

$$= \int \frac{\sec^2 dx}{6+5\tan^2 x}$$

put $\tan x = t$

$\therefore \sec^2 dx = dt$

$$= \int \frac{dt}{6+5t^2}$$

$$= \frac{1}{5}\int \frac{dt}{\frac{6}{5}+t^2}$$

$$= \frac{1}{5}\int \frac{dt}{\left(\sqrt{\frac{6}{5}}\right)^2+t^2}$$

$$= \frac{1}{5}\times\frac{1}{\left(\sqrt{\frac{6}{5}}\right)}\tan^{-1}\left[\frac{t}{\sqrt{\frac{6}{5}}}\right]+c \quad , \quad \left[\because \int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$$

$$= \frac{1}{\sqrt{30}} \tan^{-1}\left(\frac{\sqrt{5}}{\sqrt{6}} t\right) + c$$

$$= \frac{1}{\sqrt{30}} \tan^{-1}\left(\sqrt{\frac{5}{6}} \tan x\right) + c$$

Type II **Integrals of the Type**

$$\int \frac{1}{a + b\sin x} dx, \int \frac{1}{a + b\cos x} dx \text{ and } \int \frac{1}{a\sin x + b\cos x + c} dx$$

Example 40 Evaluate i) $\int \frac{1}{2 + 4\sin x} dx$ ii) $\int \frac{1}{4\sin x - 2\cos x + 5} dx$

Solution i) $\int \frac{1}{2 + 4\sin x} dx$

$$\text{Let } I = \int \frac{1}{2 + 4\left[\frac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}}\right]} \qquad \left[\because \sin 2\theta = \frac{2\tan\theta}{1 + \tan^2\theta}\right]$$

$$= \int \frac{\left[1 + \tan^2\frac{x}{2}\right] dx}{2\left[1 + \tan^2\frac{x}{2}\right] + 8\tan\frac{x}{2}},$$

Put $\tan\frac{x}{2} = t$

$\therefore \frac{1}{2}\sec^2\frac{x}{2} dx = dt$

or $\sec^2\frac{x}{2} dx = 2dt$

$$= \int \frac{\sec^2\frac{x}{2}}{2\tan\frac{x}{2} + 8\tan\frac{x}{2} + 2}$$

$$\therefore I = \int \frac{2dt}{2t^2 + 8t + 2}$$

$$= \int \frac{dt}{t^2 + 4t + 2}$$

$$= \int \frac{dt}{t^2 + 4t + 4 - 4 + 1}$$

$$= \int \frac{dt}{(t+2)^2 - 3}$$

$$= \int \frac{dt}{(t+2)^2 - \left(\sqrt{3}\right)^2}$$

$$= \frac{1}{2\sqrt{3}} \log\left|\frac{t+2-\sqrt{3}}{t+2+\sqrt{3}}\right| + c \qquad \left[\because \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log\left[\frac{x-a}{x+a}\right]\right]$$

$$= \frac{1}{2\sqrt{3}} \log\left|\frac{\tan\frac{x}{2} + 2 - \sqrt{3}}{\tan\frac{x}{2} + 2 + \sqrt{3}}\right| + c$$

ii) Let $I = \int \frac{1}{4\sin x - 2\cos x + 5} dx$

$$= \int \frac{1}{4\left[\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right] - 2\left[\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right] + 5} dx$$

$$= \int \frac{\left(1+\tan^2\frac{x}{2}\right)dx}{8\tan\frac{x}{2} - 2 + 2\tan^2\frac{x}{2} + 5 + 5\tan^2\frac{x}{2}}$$

$$= \int \frac{\left(\sec^2\frac{x}{2}\right)dx}{7\tan^2\frac{x}{2} + 8\tan\frac{x}{2} + 3}$$

Put $\tan\frac{x}{2} = t$

$\therefore \left(\sec^2\frac{x}{2}\right)\left(\frac{1}{2}\right)dx = dt$

$\therefore \sec^2\frac{x}{2} dx = 2dt$

$$\therefore I = \int \frac{2dt}{7t^2 + 8t + 3}$$

$$= \frac{2}{7}\int \frac{dt}{t^2 + \frac{8}{7}t + \frac{3}{7}}$$ [Adding and sibstracting $(\frac{1}{2}\text{coefficient of } t)^2$]

$$= \frac{2}{7}\int \frac{dt}{t^2 + \frac{8}{7}t + \frac{16}{49} + \frac{3}{7} - \frac{16}{49}}$$

$$= \frac{2}{7}\int \frac{dt}{\left(t + \frac{4}{7}\right)^2 + \frac{5}{49}}$$

$$= \frac{2}{7}\int \frac{dt}{\left(t + \frac{4}{7}\right)^2 + \left(\frac{\sqrt{5}}{7}\right)^2}$$

$$= \frac{2}{7} \times \frac{1}{\left(\frac{\sqrt{5}}{7}\right)} \tan^{-1}\left[\frac{t + \frac{4}{7}}{\frac{\sqrt{5}}{7}}\right] + c ,$$ $\left[\because \int \frac{1}{a^2 + x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$

$$= \frac{2}{\sqrt{5}} \tan^{-1}\left[\frac{7t + 4}{\sqrt{5}}\right] + c$$

$$= \frac{2}{\sqrt{5}} \tan^{-1}\left[\frac{7\tan\frac{x}{2} + 4}{\sqrt{5}}\right] + c$$

Type III **Integrals of the Type**

$$\int \sqrt{ax^2 + bx + c}\ dx$$

Example 41 Evaluate $\int \sqrt{(4x^2 + 9x + 4)}\ dx$

Solution $\int \sqrt{(4x^2 + 9x + 4)}\ dx$

$$= \int \sqrt{4}\sqrt{\left(x^2 + \frac{9}{4}x + 1\right)}\, dx$$

$$= 2\int\sqrt{\left(x^2+\frac{9}{4}x+1\right)}\,dx$$

$$= 2\int\sqrt{\left(x^2+\frac{9}{4}x+\frac{81}{64}-\frac{81}{64}+1\right)}\,dx$$

$$= 2\int\sqrt{\left(x+\frac{9}{8}\right)^2-\left(\frac{17}{64}\right)}\,dx$$

$$= 2\int\sqrt{\left(x+\frac{9}{8}\right)^2-\left(\frac{\sqrt{17}}{8}\right)^2}\,dx$$

$$= 2\frac{\left(x+\frac{9}{8}\right)\sqrt{\left(x+\frac{9}{8}\right)^2-\left(\frac{\sqrt{17}}{8}\right)^2}}{2}$$

$$-\left(\frac{\sqrt{17}}{8}\right)^2\log\left|\left(x+\frac{9}{8}\right)+\sqrt{\left(x+\frac{9}{8}\right)^2-\left(\frac{\sqrt{17}}{8}\right)^2}\right|+c$$

$$\left[\because\int\sqrt{x^2-a^2}\,dx=\frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|\right]$$

$$= \left(\frac{8x+9}{8}\right)\sqrt{x^2+\frac{9}{4}x+1}-\frac{17}{64}\log\left|\frac{8x+9}{8}+\sqrt{x^2+\frac{9}{4}x+1}\right|+c$$

Type IV **Integral of the Type**

$$\int(px+q)\sqrt{ax^2+bx+c}\,dx$$

Example 42 Evaluate $\int(2x-3)\sqrt{x^2+4x+2}\,dx$

Solution Let $I = \int(2x-3)\sqrt{x^2+4x+2}\,dx$

$$=\int(2x+4-7)\sqrt{x^2+4x+2}\,dx$$

$$=\int[-7+1(2x+4)]\sqrt{x^2+4x+2}\,dx$$

$$= -7\int \sqrt{x^2+4x+2}\ dx + \int (2x+4)\sqrt{x^2+4x+2}\ dx$$

$$= -7\int \sqrt{x^2+4x+4-4+2}\ dx + \int (2x+4)\left(x^2+4x+2\right)^{\frac{1}{2}} dx$$

$$= -7\int \sqrt{(x+2)^2-2}\ dx + \frac{(x^2+4x+2)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c_1,$$

$$\left[\because \int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{(n+1)} + c\right]$$

$$= -7\int \sqrt{(x+2)^2-\left(\sqrt{2}\right)^2}\ dx + \frac{2}{3}(x^2+4x+2)^{\frac{3}{2}} + c_1$$

$$= -7\left[\frac{(x+2)\sqrt{(x+2)^2-\left(\sqrt{2}\right)^2}}{2} - \frac{\left(\sqrt{2}\right)^2}{2}\log\left|(x+2)+\sqrt{(x+2)^2-\left(\sqrt{2}\right)^2}\right|\right] + c_2$$

$$+\frac{2}{3}(x^2+4x+2)^{\frac{3}{2}} + c_1$$

$$= -7\frac{(x+2)\sqrt{x^2+4x+2}}{2} + 7\log\left|(x+2)+\sqrt{x^2+4x+2}\right| + \frac{2}{3}\left(x^2+4x+2\right)^{\frac{3}{2}} + c$$

Example 43 Evaluate $\int \sqrt{9+4x^2}\ dx$

Solution Let $I = \int \sqrt{9+4x^2}\ dx$

$$= \int \sqrt{4}\sqrt{\frac{9}{4}+x^2}\ dx$$

$$= 2\int \sqrt{\left(\frac{3}{2}\right)^2 + x^2}\ dx$$

$$= 2\left[\frac{x\sqrt{\left(\frac{3}{2}\right)^2+x^2}}{2} + \frac{\left(\frac{3}{2}\right)^2}{2}\log\left|x+\sqrt{\left(\frac{3}{2}\right)^2+x^2}\right|\right] + c$$

$$\left[\because \int \sqrt{a^2+x^2}\,dx = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\log\left|x+\sqrt{a^2+x^2}\right|+c\right]$$

$$= x\sqrt{\frac{9}{4}+x^2} + \frac{9}{4}\log\left|x+\sqrt{\frac{9}{4}+x^2}\right|+c$$

Example 44 Evaluate $\int \sqrt{9x^2-16}\,dx$

Solution Let I $= \int \sqrt{9x^2-16}\,dx$

$$= \int \sqrt{9}\sqrt{x^2-\left(\frac{16}{9}\right)}\;dx$$

$$= 3\int \sqrt{x^2-\left(\frac{4}{3}\right)^2}\;dx$$

$$= \frac{3x\sqrt{x^2-\left(\frac{4}{3}\right)^2}}{2} - \frac{3\left(\frac{4}{3}\right)^2}{2}\log\left|x+\sqrt{x^2-\left(\frac{4}{3}\right)^2}\right|+c$$

$$\left[\because \int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+c\right]$$

$$= \frac{3x}{2}\sqrt{x^2-\frac{16}{9}} - \frac{8}{3}\log\left|x+\sqrt{x^2-\frac{16}{9}}\right|+c$$

Example 45 Evaluate $\int \sqrt{25-16x^2}\;dx$

Solution $\int \sqrt{25-16x^2}\;dx$

$$= \int \sqrt{16}\sqrt{\frac{25}{16}-x^2}\;dx$$

$$= \int 4\sqrt{\left(\frac{5}{4}\right)^2-x^2}\;dx$$

$$= 4\left[\frac{x\sqrt{\left(\frac{5}{4}\right)^2 - x^2}}{2} + \frac{\left(\frac{5}{4}\right)^2}{2}\sin^{-1}\left(\frac{x}{\frac{5}{4}}\right)\right] + c$$

$$= 2x\sqrt{\frac{25}{16} - x^2} + \frac{25}{8}\sin^{-1}\left(\frac{4x}{5}\right) + c$$

$$= \frac{x\sqrt{25 - 16x^2}}{2} + \frac{25}{8}\sin^{-1}\left(\frac{4x}{5}\right) + c$$

Type V **Integrals of The Type**

$\int \sin^p x \cos^q x \, dx$; (p and q are integers and either p is odd or q is odd)

Example 46 Evaluate $\int \sin^4 x \cos^3 x \, dx$

Solution Because index of cos x is odd, Put sin x = t

$\therefore$ cos x dx = dt

$\therefore$ $\int \sin^4 x \cos^3 x \, dx = \int (t)^4 (\cos^2 x)(\cos x \, dx)$

$= \int t^4 (1-\sin^2 x) dt$

$= \int t^4 (1-t^2) dt$

$= \int (t^4 - t^6) dt$

$$= \frac{t^5}{5} - \frac{t^7}{7} + c$$

$$= \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + c$$

Example 47 Evaluate $\int \sin^5 x \cos^4 x \, dx$

Solution $\int \sin^5 x \cos^4 x \, dx$

$= \int \cos^4 x \sin^4 x \, . \sin x \, dx$

Put cos x = t

$\therefore$ - sin x dx = dt

or sin x dx = -dt

$\therefore$ $\int \sin^5 x \cos^4 x \, dx = \int (\cos x)^4 (\sin^2 x)^2 (\sin x \, dx)$

$= \int (\cos x)^4 (1-\cos^2 x)^2 (\sin x dx)$

$= \int (t)^4 (1-t^2)^2 (-dt)$

$= -\int t^4 (1+t^4-2t^2) dt$

$= -\int t^4 dt - \int t^8 dt + 2\int t^6 dt$

$= -\frac{t^5}{5} - \frac{t^9}{9} + \frac{2t^7}{7} + c$

$= -\frac{1}{5}\cos^5 x - \frac{1}{9}\cos^9 x + \frac{2}{7}\cos^7 x + c$

Example 48 Evaluate $\int \sin^3 x \cos^5 x \, dx$

Solution Put $\sin x = t$

$\therefore$ $\cos x \, dx = dt$

$\therefore$ $\int \sin^3 x \cos^5 x \, dx = \int \sin^3 x \cos^4 x \cos x \, dx$

$= \int \sin^3 x (1-\sin^2 x)^2 . \cos x \, dx$

$= \int (t)^3 (1-t^2)^2 \, dt$

$= \int t^3 (1-2t^2+t^4) dt$

$= \int (t^3-2t^5+t^7) dt$

$= \frac{t^4}{4} - \frac{2t}{6} + \frac{t^8}{8} + c$

$= \frac{1}{4}\sin^4 x - \frac{1}{3}\sin^6 x + \frac{1}{8}\sin^8 x + c$

$= \frac{1}{4}\sin^8 x - \frac{1}{3}\sin^6 x + \frac{1}{4}\sin^4 x + c$

Type VI **Integral of the Type**

$\int \frac{p\sin x + q\cos x}{a\sin x + b\cos x} dx$; $\int \frac{p\sin x}{a\sin x + b\cos x} dx$

$\int \frac{q\cos x}{a\sin x + b\cos x} dx$

Example 49 Evaluate $\int \frac{2\sin x - 3\cos x}{4\sin x + 5\cos x} dx$

Solution $\int \frac{2\sin x - 3\cos x}{4\sin x + 5\cos x} dx$

Put $(2\sin x - 3\cos x) = A(4\sin x + 5\cos x) + B\frac{d}{dx}(4\sin x + 5\cos x)$ -- (1)

$\therefore$ $(2\sin x - 3\cos x) = A(4\sin x + 5\cos x) + B[4\cos x - 5\sin x]$

or $[2\sin x - 3\cos x] = (4A-5B)\sin x + (5A+4B)\cos x$

Comp airing the coefficients of sin x and cos x, we get

$4A - 5B = -2$

and $5A + 4B = -3$

Solving for A and B, we get

$$A = -\frac{7}{41} \text{ and } B = -\frac{22}{41}$$

Putting the values of A and B in Equation (1) we get,

$$\therefore \quad (2\sin x - 3\cos x) = -\frac{7}{41}(4\sin x + 5\cos x) - \frac{22}{41}\frac{d}{dx}(4\sin x + 5\cos x)$$

$$\therefore \quad \int \frac{2\sin x - 3\cos x}{4\sin x + 5\cos x}\,dx$$

$$= \int \frac{-\frac{7}{41}(4\sin x + 5\cos x) - \frac{22}{41}(4\cos x - 5\sin x)}{(4\sin x + 5\cos x)}\,dx$$

$$= -\frac{7}{41}\int \frac{4\sin x + 5\cos x}{4\sin x + 5\cos x}\,dx - \frac{22}{41}\int \frac{4\cos x - 5\sin x}{4\sin x + 5\cos x}\,dx$$

$$= -\frac{7}{41}\int 1\,dx - \frac{22}{41}\int \frac{(4\cos x - 5\sin x)}{4\sin x + 5\cos x}\,dx$$

$$= -\frac{7}{41}x - \frac{22}{41}\log|4\sin x + 5\cos x| + c$$

FEW TYPICAL EXAMPLES

Example 50 Integrals the following functions w.r.t. x.

i) $\tan^{-1}x$ ii) $x\cos^3 x$

Solution

i) $\int \tan^{-1}x\,dx$

Put $x = \tan t$

$\therefore \quad dx = \sec^2 t\,dt$

$$\therefore \quad \int \tan^{-1}x\,dx = \int \underset{I}{t}\,\underset{II}{\sec^2 t}\,dt$$

$$= t\int \sec^2 t\,dt - \int\left(\frac{d}{dt}(t)\int \sec^2 t\,dt\right)dt$$

(on integrating by parts)

$= t\tan t - \int 1.\tan t\,dt$

$= t\tan t - \log\sec t + c$

$= t\tan t - \log\sqrt{1+\tan^2 t} + c$

$$= (\tan^{-1}x)\, x - \log\sqrt{1+x^2} + c$$

$$= x\tan^{-1}x - \log\sqrt{1+x^2} + c$$

$$= x\tan^{-1}x - \frac{1}{2}\log(1+x^2) + c$$

ii) $\int x\cos^3 x\,dx$

$$= \int x\left[\frac{1}{4}\cos 3x + \frac{3}{4}\cos x\right]dx \qquad [\because \cos 3x = 4\cos^3 x - 3\cos x]$$

$$= \frac{1}{4}\int x\cos 3x\,dx + \frac{3}{4}\int x\cos x\,dx$$

$$= \frac{1}{4}\left[\frac{x\sin 3x}{3} - \int\frac{(1)\sin 3x}{3}dx\right] + \frac{3}{4}\left[x\sin x - \int 1.\sin x\,dx\right]$$

[Integrating by Parts]

$$= \frac{1}{12}x\sin x - \frac{1}{12}\frac{(-\cos 3x)}{3} + \frac{3}{4}x\sin x - \frac{3}{4}(-\cos x) + c$$

$$= \frac{1}{12}x\sin 3x - \frac{1}{36}\cos 3x + \frac{3}{4}x\sin x + \frac{3}{4}\cos x + c$$

Example 51 Integrate the following functions w.r.t. x.

i) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ ii) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ iii) $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

Solution

i) Let $I = \int\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

put $x = \tan\theta$

$dx = \sec^2\theta\,d\theta$

$$\therefore\ I = \int\sin^{-1}\left(\frac{2\tan\theta}{1+\tan\theta}\right)\sec^2\theta\,d\theta$$

$$= \int\sin^{-1}(\sin 2\theta)\sec^2\theta\,d\theta$$

$$= \int \underset{I}{2\theta}\,\underset{II}{\sec^2\theta}\,d\theta$$

$$= 2\left[\theta\int\sec^2\theta d\theta - \int\left(\frac{d}{d\theta}(\theta).\int\sec^2\theta d\theta\right)d\theta\right]$$

$= 2[\theta \tan\theta - \int 1.\tan\theta\, d\theta]$

$= 2\theta \ \tan\theta - 2\log\sec\theta + c$

$= 2\theta \tan\theta - 2\log\sqrt{1+\tan^2\theta} + c$

$= 2x \tan^{-1}x - 2\log\sqrt{1+x^2} + c$

(ii) Let $I = \int \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Put $x = \tan\theta$

$\therefore dx = sex^2\theta\, d\theta$

$\therefore I = \int \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$= \int \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)\sec^2\theta\, d\theta$

$= \int \cos^{-1}(\cos 2\theta)\,.\,\sec^2\theta\, d\theta$

$= \int \underset{I}{2\theta}\, \underset{II}{\sec^2\theta}\, d\theta$

$= 2\left[\theta\int \sec^2\theta d\theta - \int\left(\frac{d}{d\theta}(\theta).\int \sec^2\theta d\theta\right)d\theta\right]$

$= 2[\theta.\tan\theta - \int(1)\tan\theta\, d\theta]$

$= 2[\theta\tan\theta - \log\sec\theta] + c$

$= 2\theta\tan\theta - 2\log\sec\theta + c$

$= 2(\tan^{-1}x)\, x - 2\log\sqrt{1+\tan^2\theta} + c$

$= 2x \tan^{x} - 2\log\sqrt{1+x^2} + c$

$\therefore I = 2x\tan^{x} - \log(1+x^2) + c$

(iii) Let $I = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

Put $x = \tan\theta$

$dx = \sec^2 d\theta$

$\therefore I = \int \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right)\sec^2\theta\, d\theta$

$= \int \tan^{-1}(\tan 3\theta)\sec^2\theta\, d\theta$

$= \int 3\theta \sec^2 d\theta$

$= 3 \int \underset{I}{\theta} \underset{II}{\sec^2\theta}\, d\theta$

$= 3\left[\theta \int \sec^2\theta d\theta - \int\left(\frac{d}{d\theta}(\theta).\int \sec^2\theta d\theta\right)d\theta\right]$

$= 3[\theta \tan\theta - \int 1.\tan\theta\, d\theta]$

$= 3[\theta \tan\theta - \log \sec\theta + c$

$= 3(\tan^{-1}x)(x) - 3 \log\sqrt{1+\tan^2\theta} + c$

$= 3x \tan^{-1} x - 3 \log\sqrt{1+x^2} + c$

$= 3x \tan^{-1} x - \frac{3}{2}\log(1+x^2) + c$

MULTIPLE CHOICE QUESTIONS

Q 1. $\int 5dx$ is equal to

(a) 0 (b) $5x+c$ (c) $\frac{5x^2}{2}+c$ (d) 5

Q2. $\int dx$ is equal to

(a) 1 (b) $\frac{x^2}{2}+c$ (c) $x+c$ (d) 0

Q3. $\int \frac{1}{x} dx$ is equal to

(a) $x^{-1}+c$ (b) $-\frac{1}{x^2}+c$ (c) $\log|x| + c$ (d) 0

Q4. $\int e^x dx$ is equal to

(a) $e^x + c$ (b) e^x (c) $\frac{e^x}{2}+c$ (d) $e^{-x}+c$

Q 5. $\int a^x dx$ is equal to ; $(a>0)$

(a) $a^x \log a$ (b) $\frac{a^x}{\log a}+c$ (c) $a^x + c$ (d) None of these.

Q6. $\int e^{2x} dx$ is equal to

(a) $e^{2x}+c$ (b) $\frac{e^{2x}}{2}+c$ (c) $xe^{2x}+c$ (d) xe^{2x}

Q7. $\int a^{3x}\,dx$ is equal to ; (a>0)

(a) $\frac{a^{3x}}{3\log a}+c$ (b) $a^{3x}+c$ (c) $[a^{3x}\log a]+c$ (d) $\frac{a^{3x}}{\log a}+c$

Q8. $\int x^n\,dx = \frac{x^{n+1}}{n+1}+c$ for

(a) $n \geq 0$ (b) $n \leq 0$ (c) $n \neq -1$ (d) $n \neq 0$

Q9. $\int \sqrt{x}\,dx =$

(a) $\sqrt{x}+c$ (b) $\frac{\sqrt{x}}{2}+c$ (c) $\frac{1}{\sqrt{2}}x+c$ (d) $\frac{2}{3}x\sqrt{x}+c$

Q10. $\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)dx =$

(a) $\frac{2}{3}x\sqrt{x}+2\sqrt{x}+c$ (b) $\frac{2}{3}x\sqrt{x}-2\sqrt{x}+c$

(c) $\frac{2}{3}x\sqrt{x}+\sqrt{x}+c$ (d) $2\sqrt{x}-\frac{1}{\sqrt{x}}+c$

Q11. $\int(2x^2+3x-5)dx =$

(a) $2x^3+\frac{3x^2}{2}-5x+c$ (b) $\frac{2x^3}{3}+\frac{3x^2}{2}-5x+c$

(c) $\frac{2x^3}{3}-\frac{3x^2}{2}+5x+c$ (d) None of these.

Q12. $\int x^{\frac{12}{5}}dx$

(a) $\frac{5}{17}x^{\frac{17}{5}}+c$ (b) $\frac{12}{5}x^{\frac{5}{12}}+c$ (c) $\frac{17}{5}x^{\frac{17}{5}}+c$ (d) None of these

Q13. $\int\frac{x^2-x+2}{\sqrt{x}}dx$

(a) $\frac{2}{3}x^{\frac{5}{2}}-\frac{2}{3}x^{\frac{3}{2}}+4x^{\frac{1}{2}}+c$ (b) $\frac{2}{5}x^{\frac{5}{2}}-\frac{2}{3}x^{\frac{3}{2}}+4x^{\frac{1}{2}}+c$

(c) $\frac{2}{5}x^{\frac{5}{2}}-\frac{2}{3}x^{\frac{3}{2}}-4x^{\frac{1}{2}}+c$ (d) None of these

Q14. $\int K\, f(x)\, dx$, (where K is a constant) is equal to

(a) $Kf(x) + c$ (b) $K + \int f(x)\, dx + c$ (c) $K\int f(x)\, dx + c$ (d) None of these

Q15. $\int |x|\, dx\ (x < 0)$ is equal to

(a) $\frac{x^2}{2} + c$ (b) $2x^2 + c$ (c) $-\frac{x^2}{2} + c$ (d) $\frac{|x|^2}{2} + c$

Q16. $\int |x|\, dx\ (x \geq 0)$ is equal to

(a) $\frac{x^2}{2} + c$ (b) $2x^2 + c$ (c) $-\frac{x^2}{2} + c$ (d) $\frac{|x|^2}{2} + c$

Q17. $\int (x^a + a^x + a^a)\, dx$ (for a > 0) is equal to

(a) $\frac{x^{a+1}}{a+1} + a^x + a^a + c$

(b) $\frac{x^{a+1}}{a+1} + a^x \log a + c$

(c) $\frac{x^{a+1}}{a+1} + \frac{a^x}{\log a} + a^a + c$

(d) $\frac{x^{a+1}}{a+1} + \frac{a^x}{\log a} + a^a .\, x + c$

Q18. $\int (2x - 3)^5 dx =$

(a) $\frac{(2x-3)^6}{6} + c$ (b) $\frac{(2x-3)^6}{12} + c$ (c) $\frac{(2x-3)^6}{-18} + c$ (d) $2(2x - 3)^6 + c$

Q 19. $\int \sqrt{3x-5}\, dx =$

(a) $\frac{2}{3}(3x - 5)^{3/2} + c$

(b) $\frac{2}{9}(3x - 5)^{3/2} + c$

(c) $\frac{2}{9}(3x - 5)^{1/2} + c$

(d) None of these.

Q20. $\int \sqrt[3]{5-2x}\, dx =$

(a) $\frac{3}{8}(5 - 2x)^{4/3} + c$

(b) $-\frac{3}{8}(5-2x)^{1/3} + c$

(c) $\frac{3}{4}(5 - 2x)^{4/3} + c$

(d) $-\frac{3}{8}(5-2x)^{4/3} + c$

Q21. $\int \underset{I}{f} . \underset{II}{g}\, dx$ where f and g are functions of 'x' is equal to

(a) $f \int g\, dx + \int \left(\frac{df}{dx}\int g dx\right) dx$

(b) $g \int f\, dx - \int \left(\frac{df}{dx}\int g dx\right) dx$

(c) $f\int g\,dx - \int\left(\frac{df}{dx}\int g dx\right)dx$ (c) None of these

Q22. $\int xe^x\,dx =$

(a) $xe^x + c$ (b) $(x-1)\,e^x + c$

(c) $(x+1)e^x + c$ (d) $xe^x + x + c$

Q23. $\int \log x\,dx =$

(a) $x \log x + x + c$ (b) $x \log x + x^2 + c$

(c) $x \log x - x + c$ (d) $\log x - x + c$

Q24. $\int \frac{1}{x}(\log x)^5\,dx =$

(a) $\frac{1}{x} + (\log x)^5 + c$ (b) $x(\log)^6 + c$ (c) $\frac{(\log x)^6}{6} + c$ (d) None of these

Q25. $\int \frac{1}{x \log x}\,dx =$

(a) $x + \log x + c$ (b) $\log|\log x| + c$

(c) $x - \log x + c$ (d) $x \log x + x + c$

Q26 $\int \frac{1}{x \log x^n}\,dx =$

(a) $x + \log x^n + c$ (b) $\frac{1}{n}\log|\log x| + c$

(c) $x - n\log x + c$ (d) $n\,x\log x + x^n + c$

Q27. $\int \log a\,\,dx =$

(a) $a \log a + a + c$ (b) $x \log a + ax^2 + c$

(c) $x(\log a) + c$ (d) $\log a - x + c$

Q 28. $\int \frac{2^x}{3^x}\,dx = ?$

(a) $\frac{2^x}{3^x}\log\frac{2}{3}$ (b) $\frac{\left(\frac{2}{3}\right)^x}{\log\frac{2}{3}} + c$ (c) $\frac{\left(\frac{3}{2}\right)^x}{\log\frac{3}{2}} + c$ (d) None of these

Q29. $\int 2^x.\,3^x\,dx = ?$

(a) $6^x \log 6 + c$ (b) $\frac{6^x}{\log 6} + c$ (c) $\frac{2^x \log 2}{3^x \log 3} + c$ (d) None of these

Q30. $\int e^x.e^{e^x} dx = ?$

(a) $e^x + c$ (b) $e^x + e^{e^x} + c$ (c) $e^x - e^{e^x} + c$ (d) $e^{e^x} + c$

Q31. $\int x \log x \, dx = ?$

(a) $\frac{x^2}{2}\log x - \frac{x^2}{4} + c$ (b) $x^2 \log x + \frac{x^2}{4} + c$

(c) $\frac{x^2}{2}\log x + \frac{x^2}{4} + c$ (d) $x \log x + x + c$

Q32 $\int x \log x^n \, dx = ?$

(a) $n\frac{x^2}{2}\log x - n\frac{x^2}{4} + c$ (b) $nx^2 \log x + n\frac{x^2}{4} + c$

(c) $n\frac{x^2}{2}\log x + c$ (d) $n x \log x + x^n + c$

Q33. $\int x \sin x \, dx$ is

(a) $x \cos x + \sin x + c$ (b) $\cos x + x \sin x + x$

(c) $-x \cos x + \sin x + c$ (d) $x \cos x - \sin x + c$

Q34 $\int x^2 \cos x \, dx$ is

(a) $(x^2 + 2) \sin x + 2x \cos + c$ (b) $(x^2 - 2)\sin x + 2x \cos x + c$

(c) $x^2 \sin x + 2x \cos x + c$ (d) $x^2 \cos x + 2x \sin x + c$

Q35 $\int \sec^2 x \, dx$ is

(a) $\tan x + c$ (b) $\cot x + c$ (c) $-\tan x + c$ (d) $-\cot x + c$

Q36. $\int \mathrm{cosec}^2 x \, dx$ is

(a) $\tan x + c$ (b) $\cot x + c$ (c) $-\tan x + c$ (d) $-\cot x + c$

Q37 $\int \cos(3x - 7) dx$ is

(a) $\sin(3x-7) + c$ (b) $\frac{1}{3}\sin(3x - 7) + c$ (c) $\frac{1}{3}\cos(3x-7)$ (d) None of these

Q38 $\int \sec x \tan x \, dx$ is

(a) $\sec^2 x + c$ (b) $\sec x \cot x$ (c) $\mathrm{cosec}x \tan x$ (d) $\sec x + c$

Q39 $\int \frac{\cot(8x-2)}{\sin(8x-2)} dx$ is

(a) $\frac{1}{8}\cot(3x - 2) + c$ (b) $\frac{1}{8}\mathrm{cosec}(8x - 2)$

(c) $-\frac{1}{8}\mathrm{cosec}(8x-2) + c$ (d) None of these

Q40 $\int \tan x\, dx$ is

(a) $\log|\sec x|+c$
(b) $-\log|\cos x|+c$
(c) $\log|\sin x|+c$
(d) Both (a) and (b)

Q41. $\int \cot(5x-2)dx$ is

(a) $\frac{1}{5}\log|\sin(5x-2)|+c$
(b) $-\frac{1}{5}\log|\text{cosec}(5x-2)|+c$
(c) $\frac{1}{5}\log|\cos(5x-2)|+c$
(d) Both (a) and (b)

Q42 $\int \sec(4x-3)\, dx$ is

(a) $\frac{1}{4}\log|\sec(4x-3)+\tan(4x-3)|+c$
(b) $\frac{1}{4}\log\left|\tan\left(\frac{\pi}{4}+\frac{4x-3}{2}\right)\right|$
(c) Both (a) and (b)
(d) None of these.

Q43. $\int \text{cosec}\,.\,3x\, dx$ is

(a) $\frac{1}{3}\log|\text{cosec}\,3x-\cot 3x|$
(b) $\frac{1}{3}\log\left|\tan\left(\frac{3x}{2}\right)\right|+c$
(c) Both (a) and (b)
(d) None of these.

Q44. $\int (\tan x)^4 \sec^2 x\, dx = ?$

(a) $\frac{1}{5}\tan^5 x+c$
(b) $\frac{1}{3}\sec^3 x+c$
(c) $\sec x\,.\,\tan x$
(d) $\sec^2 x+\tan^4 x+c$

Q45. $\int (\cot x)^3 \text{cosec}^2 x\, dx = ?$

(a) $\frac{1}{3}\cot^3 x+c$
(b) $\frac{1}{4}\cot^4 x+c$
(c) $-\frac{1}{4}\cot^4 x+c$
(d) None of these

Q46. $\int \frac{\sec^2 x}{\tan x}dx = ?$

(a) $\frac{1}{2}\tan^2 x+c$
(b) $\log|\tan x|+c$
(c) $\log|\sec x|+c$
(d) $\log|\sec x+\tan x|+c$

Q47. $\int \tan x\, \text{cosec}^2 x\, dx = ?$

(a) $\log|\cot x|+c$
(b) $-\log|\cot x|+c$
(c) $\log|\text{cosec}\, x|+c$
(d) $\log|\text{cosec}\, x-\cot x|+c$

Q48. $\int \sin x \cos x\, dx = ?$

(a) $\frac{1}{4}\sin 2x+c$
(b) $-\frac{1}{4}\sin 2x+c$
(c) $\frac{1}{4}\cos 2x+c$
(d) $-\frac{1}{4}\cos 2x+c$

Q49. $\int (\cos^2 x - \sin^2 x)\, dx = ?$

(a) $-\frac{1}{2}\sin 2x + c$ (b) $-\frac{1}{2}\cos 2x + c$ (c) $\frac{1}{2}\cos 2x + c$ (d) $\frac{1}{2}\sin 2x + c$

Q50. $\int (3 \sin x - 4 \sin^3 x)\, dx = ?$

(a) $-\frac{1}{3}\cos 3x + c$ (b) $\frac{1}{3}\cos 3x + c$ (c) $\frac{1}{3}\sin 3x + c$ (d) $-\frac{1}{3}\sin 3x + c$

Q51 $\int (4\cos^2 x - 3)\cos x\, dx = ?$

(a) $-\frac{1}{3}\cos 3x + c$ (b) $\frac{1}{3}\cos 3x + c$ (c) $\frac{1}{3}\sin 3x + c$ (d) $-\frac{1}{3}\sin 3x + c$

Q52 $\int \left(\frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}\right) dx = ?$

(a) $\frac{1}{3}\log|\operatorname{cosec} 3x| + c$ (b) $\frac{1}{3}\log|\sec 3x| + c$

(c) $-\frac{1}{3}\log|\sec 3x| + c$ (d) None of these.

Q53. $\int \sin^2 x\, dx = ?$

(a) $\frac{1}{3}\sin^3 x + c$ (b) $\frac{1}{3}\cos^3 x + c$

(c) $\frac{1}{2}x - \frac{1}{4}\sin 2x + c$ (d) None of these.

Q54 $\int \cos^2 (2x + 5)dx + ?$

(a) $\frac{1}{2}x + \frac{1}{8}\sin(4x+10) + c$ (b) $\frac{1}{2}x + \frac{1}{8}\cos(4x+10) + c$

(c) $\frac{1}{2}x - \frac{1}{8}\cos(4x+10) + c$ (d) $\frac{1}{2}x - \frac{1}{8}\sin(4x+10) + c$

Q55. $\int \frac{1 - \cos 2x}{1 + \cos 2x}\, dx = ?$

(a) $\tan x + x + c$ (b) $\tan x - x + c$ (c) $\cot x + x + c$ (d) $\cot x - x + c$

Q56. $\int \sin x^\circ\, dx = ?$

(a) $\cos x^\circ$ (b) $-\frac{180}{\pi}\cos x^\circ$ (c) $\frac{180}{\pi}\cos x^\circ$ (d) $-\frac{\pi}{180}\cos x^\circ$

Q57. $\int \frac{\sec 2x - 1}{\sec 2x + 1}\, dx = ?$

(a) $\tan x + x + c$ (b) $\cot x + x + c$ (c) $\tan x - x + c$ (d) $\cot x - x + c$

Q58. $\int e^x[f(x) + f'(x)]\,dx = ?$

(a) $e^x f(x) + c$ (b) $e^{-x} f(x) + c$ (c) $f(x) + c$ (d) $e^x f'(x) + c$

Q59. $\int e^x [\tan x + \sec^2 x]\,dx = ?$

(a) $e^x \sec x + c$ (b) $e^x \sec^2 x + c$ (c) $e^x \tan x + c$ (d) $e^x \tan^2 x + c$

Q60. $\int e^x\left[\frac{1}{x} - \frac{1}{x^2}\right]dx = ?$

(a) $xe^x + c$ (b) $x^2 e^x + c$ (c) $e^x \frac{1}{x} + c$ (d) $e^x \frac{1}{x^2} + c$

Q61. $\int e^x \sec x [1 + \tan x]dx = ?$

(a) $e^x \tan x + c$ (b) $e^x \tan^2 x + c$ (c) $e^x \sec^2 x + c$ (d) $e^x \sec x + c$

Q62. $\int e^x [\log|\sec x| + \tan x]\,dx = ?$

(a) $e^x \log|\sec x| + c$ (b) $e^x \log \tan x + c$ (c) $\log|\sec x| + c$ (d) $e^x \log|\tan x| + c$

Q63. $\int e^x (\sin x + \cos x)\,dx$

(a) $e^x \sin x + c$ (b) $e^x \cos x + c$ (c) $-e^x \sin x + c$ (d) $-e^x \cos x + c$

Q64. $\int e^x (\cos x - \sin x)\,dx$

(a) $-e^x \cos x + c$ (b) $e^x \cos x + c$ (c) $e^x \sin x + c$ (d) $-e^x \sin x + c$

Q 65. $\int e^{ax} \sin bx\,dx = ?$

(a) $\frac{e^{ax}}{a^2+b^2}[a \sin bx + b \cos bx] + c$ (b) $\frac{-e^{ax}}{a^2+b^2}[a \sin bx + b \cos bx]$

(c) $\frac{e^{ax}}{a^2+b^2}[a \sin bx - b \cos bx] + c$ (d) None of these

Q66 $\int e^{5x} \sin 3x\,dx = ?$

(a) $\frac{e^{5x}}{34}[5 \sin 3x + 3 \cos x\, 3x] + c$ (b) $\frac{-e^{5x}}{34}[5 \sin 3x] + c$

(c) $\frac{e^{5x}}{34}[5 \sin 3x - 3 \cos 3x] + c$ (d) None of these.

Q67 $\int e^{ax} \cos bx\,dx = ?$

(a) $\frac{e^{ax}}{a^2+b^2}[a \cos bx - b \sin bx] + c$ (b) $\frac{e^{ax}}{a^2+b^2}[a \cos bx + b \sin bx] + c$

(c) $e^{ax}[a \cos bx + b \sin bx] + c$ (d) None of these

Q68 $\int e^{2x} \cos (3x + 7)\,dx + ?$

(a) $\frac{e^{2x}}{13}$ [2cos (3x+7)+3 sin(3x + 7)] +c (b) $\frac{e^{2x}}{13}$ [3sin (3x + 7)] +c

(c) $\frac{e^{2x}}{13}$ [2 cos (3x+7)–3sin (3x + 7)] +c (d) None of these

Q69 $\int (2x-3)\sqrt{x^2-3x+5}\,dx = ?$

(a) $\frac{2}{3}(x^2 -3x + 5)^{1/2} +c$ (b) $\frac{2}{3}(x^2 -3x + 5)^{5/2} +c$

(c) $(x^2 -3x + 5)^{3/2} +c$ (d) $\frac{2}{3}(x^2 -3x + 5)^{3/2} +c$

Q70 $\int \frac{2x-3}{(x^2-3x+5)}dx = ?$

(a) $x^2 -3x + 5)^2 +c$ (b) $\log | x^2 -3x + 5 | +c$

(c) $\frac{1}{2}(x^2 -3x + 5)^{2} +c$ (d) None of these.

Q71 $\int \frac{(x+1)dx}{(x^2+2x+7)}$ is qual to

(a) $\frac{1}{2}(x^2 +2x + 7)+c$ (b) $\log|x^2 + 2x + 7|+c$

(c) $\frac{1}{2}\log|x^2 + 2x +7|+c$ (d) None of these

Q72 $\int \frac{e^{\tan^{-1}x}}{1+x^2}dx = ?$

(a) $e^{\tan^{-1}x}+c$ (b) $\tan^{-1} +c$ (c) $e^{-\tan^{-1}x}+c$ (d) $e^{\cot^{-1}x}+c$

Q 73 $\int \frac{e^{\sin^{-1}x}}{\sqrt{1+x^2}}dx = ?$

(a) $e^{\sin^{-1}x}+c$ (b) $\cos^{-1} x +c$ (c) $e^{-\sin^{-1}x}+c$ (d) $e^{\cos^{-1}x}+c$

Q 74. $\int \frac{1}{x^2+9}dx = ?$

(a) $\frac{1}{3}\cot^{-1}\left(\frac{x}{3}\right)+c$ (b) $\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+c$ (c) $\frac{1}{3}\sec^{-1}\left(\frac{x}{3}\right)+c$ (d) $\frac{1}{3}\text{cosec}^{-1}\left(\frac{x}{3}\right)+c$

Q75 $\int \frac{1}{x\sqrt{x^2-9}}\,dx = ?$

(a) $-\frac{1}{3}\sec^{-1}\left(\frac{x}{3}\right)+c$ (b) $\frac{1}{3}\sec^{-1}\left(\frac{x}{3}\right)+c$ (c) $\frac{1}{3}\sin^{-1}\left(\frac{x}{3}\right)+c$ (d) $\sin^{-1}\left(\frac{x}{3}\right)+c$

Q76 $\int \frac{1}{\sqrt{4-x^2}}\,dx = ?$

(a) $-\frac{1}{2}\cos^{-1}\left(\frac{x}{2}\right)+c$ (b) $-\cos^{-1}\left(\frac{x}{2}\right)+c$ (c) $\sin^{-1}\left(\frac{x}{2}\right)+c$ (d) Both (b) and (c)

Q77 $\int \frac{1}{\sqrt{x^2-4}}\,dx + ?$

(a) $\sin^{-1}\left(\frac{x}{2}\right)+c$

(b) $\cos^{-1}\left(\frac{x}{2}\right)+c$

(c) $\frac{1}{2}\sec^{-1}\left(\frac{x}{2}\right)+c$

(d) $\log|x+\sqrt{x^2-4}|+c$

Q78 $\int \frac{1}{\sqrt{4x^2+9}}\,dx = ?$

(a) $\frac{3}{2}\tan^{-1}\left(\frac{2x}{3}\right)+c$

(b) $\log|2x+\sqrt{4x^2+9}|+c$

(c) $\frac{1}{2}|2x+\sqrt{4x^2+9}|+c$

(d) None of these.

Q79 $\int \frac{1}{4x^2-25}\,dx = ?$

(a) $\frac{1}{10}\log\left|\frac{2x-5}{2x+5}\right|+c$

(b) $\frac{1}{20}\log\left|\frac{2x-5}{2x+5}\right|+c$

(c) $\frac{1}{10}\log\left|\frac{2x+5}{2x-5}\right|+c$

(d) None of these.

Q80. $\int \frac{1}{25-16x^2}\,dx = ?$

(a) $\frac{1}{40}\log\left|\frac{5+4x}{5-4x}\right|+c$

(b) $\frac{1}{32}\log\left|\frac{5+4x}{5-4x}\right|+c$

(c) $\frac{1}{10}\log\left|\frac{5+4x}{5-4x}\right|+c$

(d) None of thses

Q81. $\int\sqrt{9-x^2}\,dx = ?$

(a) $\frac{x\sqrt{9-x^2}}{2}+\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)+c$

(b) $\frac{x\sqrt{9-x^2}}{2}+\frac{9}{2}\cos^{-1}\left(\frac{x}{3}\right)+c$

(c) $\log|x+\sqrt{9-x^2}|+c$

(d) None of these.

Q82. $\sqrt{9^2+x^2}\,dx = ?$

(a) $\frac{x\sqrt{9-x^2}}{2}+\frac{9}{2}\tan^{-1}\left(\frac{x}{3}\right)+c$

(b) $\frac{x\sqrt{9-x^2}}{2}+\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)+c$

(c) $\frac{x\sqrt{9+x^2}}{2}+\frac{9}{2}\log|x+\sqrt{9+x^2}|+c$

(d) None of these.

Q83. $\int\sqrt{4x^2-9}\,dx = ?$

(a) $\frac{x\sqrt{4x^2-9}}{2}-\frac{9}{4}\log|2x+\sqrt{4x^2-9}|+c$

(b) $\frac{x\sqrt{4x^2-9}}{2}+\frac{9}{2}\log|2x+\sqrt{4x^2-9}|+c$

(c) $\frac{x\sqrt{4x^2-9}}{2}+\frac{9}{2}\sin^{-1}\left(\frac{2x}{3}\right)+c$

(d) None of these.

Q84. $\int x^x(1+\log x)\,dx = ?$

(a) $x^x \log x + c$

(b) $x^x + \log x + c$

(c) $x^x + c$

(d) None of these.

Q85. $\int x^{\sin x}\left(\frac{1}{x}\sin x+\cos x\log x\right)dx = ?$

(a) $x^{\sin x}+c$

(b) $(\sin x)^x+c$

(c) $x+\sin x+c$

(d) None of these.

Q86 $\int\frac{1}{(x-a)(x-b)}\,dx = ?$

(a) $\log|(x-a)(x-b)|+c$

(b) $\log\left|\frac{x-b}{x-a}\right|+c$

(c) $\log\left|\frac{x-a}{x-b}\right|+c$

(d) $\frac{1}{(b-a)}\log\left|\frac{x-b}{x-a}\right|+c$

Chapter **2**

DEFINITE INTEGRAL

If f(x) is a continuous function on a closed interval [a, b], then $\int_a^b f(x)dx$ is called Definite Integral of a function f(x) from x = a to x = b.

If F(x) is the primitive or anti derivative of f(x) , then,

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

x = a is called Lower limit and

x = b is called Upper limit of Integration.

The interval [a, b] is called the range of integration.

The value of Definite Integral is Unique.

Let $\int f(x)\,dx = F(x) + c$

Then $\int_a^b f(x)dx = \left[\int f(x)dx\right]_{x=b} - \left[\int f(x)dx\right]_{x=a}$

$= [F(x) + c]_{x=b} - [F(x) + c]_{x=a}$

$= [F(b) + c] - [F(a) + c]$

$= F(b) - F(a)$,

which is unique being independent of constant of integration c.

Example 1 **Evaluate.**

(i) $\int_1^2 (x^3 + x^2 - 1)\,dx$ (ii) $\int_2^3 \sqrt{x}\,dx$

(iii) $\int_0^1 (2x+3)^4\,dx$ (iv) $\int_2^3 e^{2x}\,dx$

(v) $\int_2^3 \frac{1}{(3x-5)}\,dx$ (vi) $\int_0^1 \frac{3x^3 - 4x^2 + 1}{\sqrt{x}}\,dx$

Solution

(i) $\int_1^2 (x^3 + x^2 - 1)\,dx$

$$= \left[\frac{x^4}{4} + \frac{x^3}{3} - x\right]_1^2$$

$$= \left[\frac{x^4}{4} + \frac{x^3}{3} - x\right]_{x=2} - \left[\frac{x^4}{4} + \frac{x^3}{3} - x\right]_{x=1}$$

$$= \left[\frac{(2)^4}{4} + \frac{(2)^3}{3} - 2\right] - \left[\frac{(1)^4}{4} + \frac{(1)^3}{3} - 1\right]$$

$$= \left[4 + \frac{8}{3} - 2\right] - \left[\frac{1}{4} + \frac{1}{3} - 1\right]$$

$$= \left(2 + \frac{8}{3}\right) - \left[\frac{3 + 4 - 12}{12}\right] = \frac{61}{12}$$

(ii) $\int_2^3 \sqrt{x}\, dx$

$$= \int_2^3 (x)^{\frac{1}{2}}\, dx \quad , \qquad \left[\because \int x^n dx = \frac{x^{n+1}}{n+1} \text{ for } n \neq -1\right]$$

$$= \left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_2^3$$

$$= \left[\frac{2}{3} x^{\frac{3}{2}}\right]_2^3$$

$$= \frac{2}{3}\left[(3)^{\frac{3}{2}} - (2)^{\frac{3}{2}}\right]$$

$$= \frac{2}{3}\left[3\sqrt{3} - 2\sqrt{2}\right]$$

(iii) $\int_0^1 (2x + 3)^4\, dx$

$= \left[\frac{(2x+3)^5}{5(2)}\right]_0^1$, $\left[\because \int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{(n+1)a} \text{ for } n \neq -1\right]$

$= \frac{1}{10}[(2(1)+3)^5 - (2(0)+3)^5]$

$= \frac{1}{10}[(5)^5 - (3)^5]$

$= \frac{1}{10}[3125 - 243]$

$= \frac{1}{10}[2882] =$ 288.2

(iv) $\int_2^3 e^{2x}\,dx$

$= \left[\frac{e^{2x}}{2}\right]_2^3$, $\left[\because \int e^{ax}dx = \frac{e^{ax}}{a} + c\right]$

$= \frac{1}{2}\left[e^6 - e^4\right]$

$= \frac{1}{2}e^4(e^2 - 1)$

(v) $\int_2^3 \frac{1}{(3x-5)}\,dx$

$= \left[\frac{\log|3x-5|}{3}\right]_2^3$, $\left[\because \int \frac{1}{ax+b}\,dx = \frac{\log|ax+b|}{a} + c\right]$

$= \frac{1}{3}\log|3(3)\text{–}5| - \frac{1}{3}\log|3(2)\text{–}5|$

$= \frac{1}{3}\log 4 - \frac{1}{3}\log 1$

$= \frac{1}{3}\left[\log\left(\frac{4}{1}\right)\right]$, $\left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$

$= \frac{1}{3}\log 4$

$= \frac{1}{3}\log(2)^2 \qquad , \qquad [\because \log m^n = n\log m]$

$= \frac{2}{3}\log 2$

(vi) $\int_0^1 \frac{3x^3 - 4x^2 + 1}{\sqrt{x}}\,dx$

$$= \int_0^1 \left(\frac{3x^3}{x^{\frac{1}{2}}} - \frac{4x^2}{x^{\frac{1}{2}}} + \frac{1}{x^{\frac{1}{2}}} \right) dx$$

$$= \int_0^1 \left(3x^{\frac{5}{2}} - 4x^{\frac{3}{2}} + x^{-\frac{1}{2}} \right) dx$$

$$= \left[3\frac{x^{\frac{7}{2}}}{\frac{7}{2}} - 4\frac{x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} \right]_0^1$$

$$= \left[\frac{6}{7}x^{\frac{7}{2}} - \frac{8}{5}x^{\frac{5}{2}} + 2x^{\frac{1}{2}} \right]_0^1$$

$$= \left[\frac{6}{7} - \frac{8}{5} + 2 \right] - [0 - 0 + 0]$$

$$= \left[\frac{30 - 56 + 70}{35} \right] - 0 = \frac{44}{35}$$

Example 2 **Evaluate**

(i) $\int_0^{\pi/4} (6\sec^2 x + \sin 4x)\,dx$ (ii) $\int_0^{\pi/4} (\tan^2 x + \cos x)\,dx$

(iii) $\int_0^{\pi/3} \cos 2x \sin x\,dx$ (iv) $\int_0^1 \frac{5dx}{1 + x^2}$

(v) $\int_0^{\pi/2}\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^2 d\theta$ (vi) $\int_{\pi/4}^{\pi}\frac{dx}{1+\cos 2x}$

Solution

(i) $\int_0^{\pi/4}(6\sec^2 x + \sin 4x)\,dx$

$$= 6\int_0^{\pi/4}\sec^2 x\,dx + \int_0^{\pi/4}\sin 4x\,dx$$

$$= 6[\tan x]_0^{\pi/4} + \left[\frac{-\cos 4x}{4}\right]_0^{\pi/4}$$

$$= 6\left[\tan\frac{\pi}{4} - \tan 0\right] - \frac{1}{4}\left[\cos 4.\frac{\pi}{4} - \cos 0\right]$$

$$= 6[1 - 0] - \frac{1}{4}[\cos\pi - \cos 0]$$

$$= 6 - \frac{1}{4}[-1-1] \quad , \quad \begin{bmatrix}\because \cos\pi = -1 \\ \cos 0 = 1\end{bmatrix}$$

$$= 6 + \frac{1}{2} = \frac{13}{2}$$

(ii) $\int_0^{\pi/4}(\tan^2 x + \cos x)\,dx$

$$= \int_0^{\pi/4}[(\sec^2 x - 1) + \cos x]\,dx$$

$$= \int_0^{\pi/4}(\sec^2 x - 1 + \cos x)\,dx$$

$$= [\tan x - x + \sin x]_0^{\pi/4}$$

$$= \left[\tan\frac{\pi}{4} - \frac{\pi}{4} + \sin\frac{\pi}{4}\right] - [\tan 0 - 0 + \sin 0]$$

$$= \left[1 - \frac{\pi}{4} + \frac{1}{\sqrt{2}}\right] - [0] \quad = 1 + \frac{1}{\sqrt{2}} - \frac{\pi}{4}$$

(iii) $\int_0^{\pi/3} \cos 2x \sin x \, dx$

$$= \frac{1}{2}\int_0^{\pi/3} 2\cos 2x \sin x \, dx$$

$$= \frac{1}{2}\int_0^{\pi/3} [\sin(2x + x) - \sin(2x - x)]dx$$

$[\because 2\cos A \sin B = \sin(A + B) - \sin(A-B)]$

$$= \frac{1}{2}\int_0^{\pi/3} [\sin 3x - \sin x]\, dx$$

$$= \frac{1}{2}\left[\frac{-\cos 3x}{3} + \cos x\right]_0^{\pi/3}$$

$$= \frac{1}{2}\left[\left(\frac{-\cos 3.\frac{\pi}{3}}{3} + \cos\frac{\pi}{3}\right) - \left(\frac{-\cos 0}{3} + \cos 0\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{-\cos \pi}{3} + \cos\frac{\pi}{3}\right) - \left(-\frac{1}{3} + 1\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{3} + \frac{1}{2} + \frac{1}{3} - 1\right]$$

$$= \frac{1}{2}\left[\frac{2+3+2-6}{6}\right]$$

$$= \frac{1}{2}\left(\frac{1}{6}\right) = \frac{1}{12}$$

(iv) $\int_0^1 \frac{5dx}{1+x^2}$

$$= 5\left[\tan^{-1} x\right]_0^1 \qquad \left[\because \int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$= 5[\tan^{-1}1 - \tan^{-1}0]$$

$$= 5\left[\frac{\pi}{4} - 0\right] = \frac{5\pi}{4}$$

(v) $\int_0^{\pi/2}\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^2 d\theta$

$$= \int_0^{\pi/2}\left(\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)d\theta$$

$$= \int_0^{\pi/2}(1+\sin\theta)d\theta \qquad [\because 2\sin A\cos A = \sin 2A]$$

$$= [\theta-\cos\theta]_0^{\pi/2}$$

$$= \left[\frac{\pi}{2}-\cos\frac{\pi}{2}\right]-[0-\cos 0]$$

$$= \left(\frac{\pi}{2}-0\right)-(0-1) \qquad = \frac{\pi}{2}+1$$

(vi) $\int_{\pi/4}^{\pi}\frac{dx}{1+\cos 2x}$

$$= \int_{\pi/4}^{\pi}\frac{dx}{2\cos^2 x}$$

$$= \frac{1}{2}\int_{\pi/4}^{\pi}\sec^2 x\, dx$$

$$= \frac{1}{2}[\tan x]_{\pi/4}^{\pi}$$

$$= \frac{1}{2}\left[\tan\pi-\tan\frac{\pi}{4}\right]$$

$$= \frac{1}{2}[0-1] \quad = \quad -\frac{1}{2}$$

Example 3 **Evaluate**

(i) $\int_0^{\sqrt{2}}\sqrt{2-x^2}\,dx$ (ii) $\int_0^{\infty}\frac{dx}{(x^2+a^2)(x^2+b^2)}$; $a \neq b$

(iii) $\int_{\pi/6}^{\pi/2}\frac{\sin 2x}{\sin x}\,dx$

Solution

(i) $\int_0^{\sqrt{2}} \sqrt{2-x^2}\, dx$

$$= \int_0^{\sqrt{2}} \sqrt{(\sqrt{2})^2 - x^2}\, dx$$

$$= \left[\frac{x\sqrt{(\sqrt{2})^2 - x^2}}{2} + \frac{(\sqrt{2})^2}{2}\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^{\sqrt{2}}$$

$$= \left[\frac{\sqrt{2}\sqrt{(\sqrt{2})^2 - (\sqrt{2})^2}}{2} + \sin^{-1}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)\right] - \left[0 + \sin^{-1} 0\right]$$

$$= [0 + \sin^{-1}1] - [0]$$

$$= \frac{\pi}{2}$$

(ii) $\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$; $a \neq b$

$$= \frac{1}{(a^2-b^2)} \int_0^{\infty} \frac{(a^2-b^2)dx}{(x^2+a^2)(x^2+b^2)}$$

$$= \frac{1}{a^2-b^2} \int_0^{\infty} \frac{(x^2+a^2)-(x^2+b^2)}{(x^2+a^2)(x^2+b^2)} dx$$

$$= \frac{1}{a^2-b^2} \int_0^{\infty} \left[\frac{1}{(x^2+b^2)} - \frac{1}{(x^2+a^2)}\right] dx$$

$$= \frac{1}{a^2-b^2} \left[\int_0^{\infty} \frac{1}{x^2+b^2} dx - \int_0^{\infty} \frac{1}{x^2+a^2} dx\right]$$

$$= \frac{1}{a^2-b^2} \left[\frac{1}{b}\tan^{-1}\frac{x}{b} - \frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^{\infty}$$

$$= \frac{1}{a^2 - b^2}\left[\left(\frac{1}{b}\tan^{-1}\infty - \frac{1}{b}\tan^{-1}0\right) - \left(\frac{1}{a}\tan^{-1}\infty - \frac{1}{a}\tan^{-1}0\right)\right]$$

$$= \frac{1}{a^2 - b^2}\left[\left(\frac{1}{b}.\frac{\pi}{2} - 0\right) - \left(\frac{1}{a}.\frac{\pi}{2} - 0\right)\right]$$

$$= \frac{1}{a^2 - b^2}\left[\frac{\pi}{2b} - \frac{\pi}{2a}\right]$$

$$= \frac{\pi}{a^2 - b^2}\left[\frac{a-b}{2ab}\right]$$

$$= \frac{1}{(a-b)(a+b)}.\frac{(a-b)}{2ab}\pi$$

$$= \frac{\pi}{2ab(a+b)}$$

(iii) $\displaystyle\int_{\pi/6}^{\pi/2} \frac{\sin 2x}{\sin x}\,dx$

$$= \int_{\pi/6}^{\pi/2} \frac{2\sin x \cos x}{\sin x}\,dx$$

$$= \int_{\pi/6}^{\pi/2} 2\cos x\,dx$$

$$= 2[\sin x]_{\pi/6}^{\pi/2}$$

$$= 2\left[\sin\frac{\pi}{2} - \sin\frac{\pi}{6}\right]$$

$$= 2\left[1 - \frac{1}{2}\right] = 2\left(\frac{1}{2}\right) = 1$$

Example 4 **Evaluate**

(i) $\displaystyle\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}dx$ (ii) $\displaystyle\int_1^2 2x\sqrt{5-x^2}\,dx$

(iii) $\displaystyle\int_0^1 \frac{(\tan^{-1}x)^2}{1+x^2}\,dx$ (iv) $\displaystyle\int_1^3 \frac{\sqrt{\log x}}{x}\,dx$

Soluion

(i) $\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}dx$

Put $\sqrt{1-x^2} = t$

$\therefore \frac{1}{2\sqrt{1-x^2}}(-2x)dx = dt$

$\therefore \frac{x}{\sqrt{1-x^2}}dx = -dt$

when x = 0, $t = \sqrt{1-(0)^2} = 1$

when x = $\frac{1}{2}$, $t = \sqrt{1-\left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

$\therefore \int_0^{1/2} \frac{x}{\sqrt{1+x^2}}dx = \int_1^{\sqrt{3}/2} -dt$

$= -[t]_1^{\sqrt{3}/2} = -\left[\frac{\sqrt{3}}{2} - 1\right]$

$= -\frac{\sqrt{3}}{2} + 1 = \frac{2-\sqrt{3}}{2}$

(ii) $\int_1^2 2x\sqrt{5-x^2}\,dx$

Put 5- x^2 = t
$\therefore$ -2x dx = dt
$\therefore$ 2x dx = -dt
when x = 1, t = 5 - $(1)^2$ = 4
when x = 2, t = 5 - $(2)^2$ = 1

$\therefore \int_1^2 2x\sqrt{5-x^2}\,dx = \int_4^1 (t)^{\frac{1}{2}}(-dt)$

$= -\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_4^1 = -\frac{2}{3}\left[(1)^{\frac{3}{2}} - (4)^{\frac{3}{2}}\right]$

$$= -\frac{2}{3}[1-8]$$

$$= -\frac{2}{3}(-7) = \frac{14}{3}$$

(iii) $\int_0^1 \frac{(\tan^{-1}x)^2}{1+x^2}\,dx$

Put $\tan^{-1}x = t$

$\therefore \frac{1}{1+x^2}\,dx = dt$

when $x = 0$, $t = \tan^{-1} 0 = 0$

when $x = 1$, $t = \tan^{-1}(1) = \frac{\pi}{4}$

$$\therefore \int_0^1 \frac{(\tan^{-1}x)^2}{1+x^2}\,dx = \int_0^{\pi/4} (t)^2.dt$$

$$= \left[\frac{t^3}{3}\right]_0^{\pi/4}$$

$$= \frac{1}{3}\left[\left(\frac{\pi}{4}\right)^3 - (0)^3\right] = \frac{\pi^3}{192}$$

(iv) $\int_1^3 \frac{\sqrt{\log x}}{x}\,dx$

put $\log x = t$

$\frac{1}{x}\,dx = dt$

when $x = 1$, $t = \log 1 = 0$
when $x = 3$, $t = \log 3$

$$\therefore \int_1^3 \frac{\sqrt{\log x}}{x}\,dx = \int_0^{\log 3} \sqrt{t}\,dt$$

$$= \int_0^{\log 3} (t)^{\frac{1}{2}}\,dt$$

$$= \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^{\log 3}$$

$$= \frac{2}{3}\left[(\log 3)^{\frac{3}{2}} - (0)^{\frac{3}{2}} \right]$$

$$= \frac{2}{3}(\log 3)^{\frac{3}{2}}$$

Example 5 **Evaluate**

(i) $\int_0^{\pi/4} \frac{\sin x}{\cos 3x + 3\cos x}\, dx$ (ii) $\int_0^{\pi/2} \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x}$

(iii) $\int_0^{\pi} \frac{dx}{5 + 3\cos x}$ (iv) $\int_1^{2} \frac{dx}{(x+1)\sqrt{x^2 - 1}}$

(v) $\int_0^{1} \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

Solution

(i) $\int_0^{\pi/4} \frac{\sin x}{\cos 3x + 3\cos x}\, dx$

$$= \int_0^{\pi/4} \frac{\sin x}{(4\cos^3 x - 3\cos x + 3\cos x)}\, dx \quad , \quad [\because \cos 3x = 4\cos^3 x - 3\cos x]$$

$$= \int_0^{\pi/4} \frac{\sin x}{4\cos^3 x}\, dx$$

$$= \frac{1}{4}\int_0^{\pi/4} \tan x \sec^2 x \, dx$$

Put $\tan x = t$

$\therefore$ $\sec^2 x \, dx = dt$

when $x = 0$, $t = \tan o = 0$

when $x = \frac{\pi}{4}$, $t = \tan \frac{\pi}{4} = 1$

$$\therefore \quad \int_0^{\pi/4} \frac{\sin x}{\cos 3x + 3\cos x}\, dx = \frac{1}{4}\int_0^1 t.\, dt$$

$$= \frac{1}{4}\left[\frac{t^2}{2}\right]_0^1 = \frac{1}{8}\left[(1)^2 - (0)^2\right]$$

$$= \frac{1}{8}$$

(ii) $\displaystyle\int_0^{\pi/2} \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x}$

Dividing Numerator and Denominator by $\cos^2 x$, we get

$$\int_0^{\pi/2} \frac{\frac{1}{\cos^2 x} dx}{a^2 \frac{\sin^2 x}{\cos^2 x} + b^2 \frac{\cos^2 x}{\cos^2 x}}$$

$$= \int_0^{\pi/2} \frac{\sec^2 x\, dx}{a^2 \tan^2 x + b^2}$$

Put $\tan x = t$

$\quad Sec^2 x\, dx = dt$

when $x = 0, t = 0$

when $x = \frac{\pi}{2}, t = \infty$

$$= \int_0^{\infty} \frac{dt}{a^2 t^2 + b^2}$$

$$= \frac{1}{a^2} \int_0^{\infty} \frac{dt}{t^2 + \left(\frac{b^2}{a^2}\right)}$$

$$= \frac{1}{a^2} \left| \frac{1}{\left(\frac{b}{a}\right)} \tan^{-1} \frac{t}{\left(\frac{b}{a}\right)} \right|_0^{\infty}$$

$$= \frac{1}{ab}\left|\tan^{-1}\left(\frac{at}{b}\right)\right|_0^{\infty}$$

$$= \frac{1}{ab}\left[\tan^{-1}\infty - \tan^{-1}0\right]$$

$$= \frac{1}{ab}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{2ab}$$

(iii) $\int_0^{\pi}\frac{dx}{5+3\cos x}$

$$= \int_0^{\pi}\frac{dx}{5+3\left[\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right]}$$

$$= \int_0^{\pi}\frac{\left(1+\tan^2\frac{x}{2}\right)dx}{5+5\tan^2\frac{x}{2}+3-3\tan^2\frac{x}{2}}$$

$$= \int_0^{\pi}\frac{\sec^2\frac{x}{2}\,dx}{2\tan^2\frac{x}{2}+8}$$

$$= \frac{1}{2}\int_0^{\pi}\frac{\sec^2\frac{x}{2}\,dx}{\left(\tan^2\frac{x}{2}+4\right)}$$

Put $\tan\frac{x}{2} = t$

$\therefore \quad \frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$

when $x = 0$, $t = 0$,
when $x = \pi$, $t = \infty$

$$\therefore \int_0^{\pi} \frac{dx}{5+3\cos x} = \int_0^{\infty} \frac{dt}{t^2+4}$$

$$= \int_0^{\infty} \frac{dt}{t^2+(2)^2}$$

$$= \left|\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right|_0^{\infty}$$

$$= \frac{1}{2}\left[\tan^{-1}\infty - \tan^{-1}0\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2}-0\right] = \frac{\pi}{4}$$

iv) $\displaystyle\int_1^2 \frac{dx}{(x+1)\sqrt{x^2-1}}$

Put $\quad x+1 = \dfrac{1}{t}$

$\therefore \quad dx = -\dfrac{1}{t^2}dt$

when $x = 1, t = \dfrac{1}{2}$, when $x = 2, t = \dfrac{1}{3}$

$$= \int_{1/2}^{1/3} \frac{\left(-\frac{1}{t^2}dt\right)}{\frac{1}{t}\sqrt{\left(\frac{1}{t}-1\right)^2-1}}$$

$$= \int_{1/2}^{1/3} \frac{-1dt}{t\sqrt{\frac{1}{t^2}+1-\frac{2}{t}-1}}$$

$$= \int_{1/2}^{1/3} \frac{-1}{t\sqrt{\frac{1-2t}{t^2}}}dt$$

$$= \int_{1/2}^{1/3} \frac{-1}{\sqrt{1-2t}} dt$$

$$= -\int_{1/2}^{1/3} (1-2t)^{-1/2} dt$$

$$= -\left[\frac{(1-2t)^{\frac{1}{2}}}{(-2)\left(\frac{1}{2}\right)}\right]_{\frac{1}{2}}^{\frac{1}{3}}$$

$$= \left[\left(1-\frac{2}{3}\right)^{\frac{1}{2}} - (1-1)^{\frac{1}{2}}\right]$$

$$= \left[\left(\frac{1}{3}\right)^{\frac{1}{2}}\right] = \frac{1}{\sqrt{3}}$$

(v) $\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

Put $\sqrt{x} = t$

$\therefore \quad \frac{1}{2\sqrt{x}} dx = dt$

$\frac{1}{\sqrt{x}} dx = 2dt$

when $x = 0, t = 0$, when $x=1, \ t = 1$

$$= \int_0^1 e^t (2dt)$$

$$= 2\left[e^t\right]_0^1 \qquad = 2[e^1 - e^0]$$

$$= 2(e-1)$$

$$= 8 \times \frac{\pi}{2} = 4\pi$$

Properties of Definite Integrals

Property 1

The change of variable does not change the value of definite Integral.

i.e. $\int_a^b f(x)dx = \int_a^b f(z)dz$

Proof : Let $\int f(x)dx = F(x)$ and $\int f(z)dx = F(z)$

$\therefore \int_a^b f(x)dx = F(b) - F(a)$ ------ (1)

and $\int_a^b f(z)dz = F(b) - F(a)$ ------ (2)

$\therefore$ From (1) and (2) $\int_a^b f(x)dx = \int_a^b f(z)dz$

Property 2

If the limits of a definite Integral are interchanged, then the value of definite integral thus obtained remains unchanged but with opposite sign.

i.e. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Proof : Let $\int f(x)\, dx = F(x)$

$\therefore \int_a^b f(x)dx = F(b) - F(a)$ ------- (1)

and $\int_b^a f(x)dx = F(a) - F(b)$ ------- (2)

clearly from (1) & (2)

$\int_a^b f(x)dx = \int_b^a f(x)dx$

Property 3

If [a, b] is the range of integreation and a<c<b, then

$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Proof : Let $\int f(x)\,dx = F(x)$

then $\int_a^b f(x)dx = F(b) - F(a)$ ------ (1)

and $\int_a^c f(x)dx + \int_c^b f(x)dx$

$= [F(c) - F(a)] + [F(b) - F(c)]$

$= F(b) - F(a)$ ------ (2)

$\therefore$ from (1) and (2)

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Property 4

$$\int_{-a}^a f(x)dx = \begin{cases} 0, \text{ if } f(x) \text{ is an odd function i.e. } f(-x) = -f(x) \\ 2\int_0^a f(x)dx, \text{ if } f(x) \text{ is an even function i. e. } f(-x) = f(x) \end{cases}$$

Proof: From Property (3)

$$\int_{-a}^a f(x)dx = \int_{-a}^0 f(x)dx + \int_0^a f(x)dx \text{ ----- (1) }, [\because -a < o < a]$$

Let f(x) be an odd function

Consider, $\int_{-a}^0 f(x)dx$

Put $x = -z$

$\therefore$ $dx = -dz$

when $x = -a$, $z = a$, when $x = 0$, $z = 0$

$$\therefore \int_{-a}^0 f(x)dx = \int_a^0 f(-z)(-dz)$$

$= \int_0^a f(-z)dz$, [by property (2)]

$= -\int_0^a f(z)dz$, [$\because$ f is an odd function $\therefore f(-z)=-f(z)$]

$= -\int_0^a f(x)dx$, [By property (1)

$\therefore$ (1) becomes,

$$\int_{-a}^{a} f(x)dx = -\int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx = 0$$

Now , let f(x) be an even function

Now , consider $\int_{-a}^{0} f(x)dx$

Put $x = -z$

$\therefore$ $dx = -dz$

when $x = -a$, $z = a$ and when $x = 0$, $z = 0$

$$\therefore \int_{-a}^{0} f(x)dx = \int_{a}^{0} f(-z)(-dz)$$

$$= \int_{0}^{a} f(-z)dz \quad , \quad \text{[By Property (2)]}$$

$$= \int_{0}^{a} f(z)dz \quad , \quad [\because \text{ f is an even function } \therefore f(-z) = f(z)]$$

$$= \int_{0}^{a} f(x)dx \quad , \quad \text{[By Property 1]}$$

$\therefore$ (1) be comes

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx$$

$$= 2\int_{0}^{a} f(x)dx$$

Thus.

$$\int_{-a}^{a} f(x)dx = \begin{cases} 0, \text{ if f(x) is an odd function} \\ 2\int_{0}^{a} f(x)dx \text{ , if f(x) is an even function} \end{cases}$$

Property 5 $\int_{0}^{a} f(x)dx = \int_{0}^{a} f(a-x)dx$

Proof put $x = (a - z)$ in $\int_{0}^{a} f(x)dx$

$\therefore \quad dx = -dz$

when $x = 0$, $z = a$, when $x = a$, $z = 0$

$$\therefore \int_0^a f(x)dx = \int_a^0 f(a-z)(-dz)$$

$$= \int_0^a f(a-z)dz, \qquad \text{[By Property (2)]}$$

$$= \int_0^a f(a-x)dx, \qquad \text{[By Property (1)]}$$

$$\therefore \iint_0^a f(x)dx = \int_0^a f(a-x)dx$$

Property 6 $\displaystyle\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

Proof $\displaystyle\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx$ ------ (1) [By property 3]

Consider the integral, $\displaystyle\int_a^{2a} f(x)dx$

put $x = 2a - z$

$\therefore \quad dx = -dz$

when $x = a$, $z = a$, when $x = 2a$, $z = 0$

$$\therefore \int_a^{2a} f(x)dx = \int_a^0 f(2a-z)(-dz)$$

$$= \int_0^a f(2a-z)dz, \qquad \text{[By Property (2)]}$$

$$= \int_0^a f(2a-x)dx, \qquad \text{[By Property (1)]} \quad \text{---- (2)}$$

From (1) and (2), we have

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$$

Property 7 $\int_0^{2a} f(x)dx = \begin{cases} 0, \text{if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)dx, \text{ if } f(2a-x) = f(x) \end{cases}$

Proof From property (6)

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx \qquad \text{------ (1)}$$

If f(2a–x) = –f(x), then from (1)

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a -f(x)dx$$

$$= \int_0^a f(x)dx - \int_0^a -f(x)dx = 0$$

If f(2a–x) = f(x), then from (1)

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx$$

$$= 2\int_0^a f(x)dx$$

Thus, $\int_0^{2a} f(x)dx = \begin{cases} 0, \text{if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)dx, \text{ if } f(2a-x) = f(x) \end{cases}$

Example 9 **Evaluate the following Definite Integrals :-**

(i) $\int_1^4 f(x)dx$ Where $f(x) = \begin{cases} 4x+5, \text{ if } 1 \le x \le 2 \\ 3x-5, \text{ if } 2 \le x \le 4 \end{cases}$

(ii) $\int_{-\pi/2}^{\pi/2} |\sin x| dx$ (iii) $\int_0^{\pi} |\cos x| dx$

Solution

(i) $\int_1^4 f(x)dx$; $f(x) = \begin{cases} 4x+5, \text{ if } 1 \le x \le 2 \\ 3x-5, \text{ if } 2 \le x \le 4 \end{cases}$

Because the function is defined in two ways in the interval [1,4] first in the subinterval $1 \le x \le 2$ and the other in the subinterval $2 \le x \le 4$.

Thus we shall use property (3) as follows:

$$\int_1^4 f(x)dx = \int_1^2 f(x)dx + \int_2^4 f(x)dx$$

$$= \int_1^2 (4x+5)dx + \int_2^4 (3x-5)dx$$

$$= \left[\frac{4x^2}{2}+5x\right]_1^2 + \left[\frac{3x^2}{2}-5x\right]_2^4$$

$$= \left[\{2(2)^2+5(2)\}-\{2(1)^2+5(1)\}\right] + \left[\left\{\frac{3}{2}(4)^2-5(4)\right\}-\left\{\frac{3}{2}(2)^2-5(2)\right\}\right]$$

$= [18-7]+[(24-20)-(6-10)]$

$= 11+4+4 \quad = 19$

(ii) $\int_{-\pi/2}^{\pi/2} |\sin x|dx$

$|\sin x| = -\sin x$ for $\frac{-\pi}{2} \le x \le 0$, $[\because$ x is in IVth quad.]

and $|\sin x| = \sin x$ for $0 \le x \le \frac{\pi}{2}$, $[\because$ x is in Ist quad.]

By property (3)

$$\int_{-\pi/2}^{\pi/2} |\sin x|dx = \int_{-\pi/2}^{0} |\sin x|dx + \int_0^{\pi/2} |\sin x|dx$$

$$= \int_{-\pi/2}^{0} (-\sin x)dx + \int_0^{\pi/2} \sin x\, dx$$

$$= -[-\cos x]_{-\pi/2}^{0} + [-\cos x]_0^{\pi/2}$$

$$= \left[\cos 0 - \cos\left(-\frac{\pi}{2}\right)\right] - \left[\cos\frac{\pi}{2} - \cos 0\right]$$

$= [1-0]-[0-1] \quad = 1+1 \quad = 2$

(iii) $\int_0^{\pi} |\cos x| dx$

$|\cos x| = \cos x$ for $0 \le x \le \frac{\pi}{2}$, [$\because$ x is in Ist quad.]

and $|\cos x| = -\cos x$ for $\frac{\pi}{2} \le x \le \pi$, [$\because$ x is in IInd quad.]

$\therefore$ By property (3)

$$\int_0^{\pi} |\cos x| dx = \int_0^{\pi/2} |\cos x| dx + \int_{\pi/2}^{\pi} |\cos x| dx$$

$$= \int_0^{\pi/2} \cos x \, dx + \int_{\pi/2}^{\pi} (-\cos x) dx$$

$$= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi}$$

$$= \left[\sin\frac{\pi}{2} - \sin 0\right] - \left[\sin \pi - \sin\frac{\pi}{2}\right]$$

$$= [1-0]-[0-1] = 1+1 = 2$$

Example 10 Evaluate the following Definite Integrals :

(i) $\int_{-1}^{1} x^3 dx$ (ii) $\int_{-\pi}^{\pi} \sin x \; dx$ (iii) $\int_{-2}^{2} x^4 dx$

(iv) $\int_{-\pi}^{\pi} \cos 2x \; dx$ (v) $\int_{-\pi/2}^{\pi/2} x \sin x \; dx$ (vi) $\int_{-\pi/2}^{\pi/2} x \cos x \; dx$

(vii) $\int_{-1}^{1} |x| dx$

Solution

(i) $\int_{-1}^{1} x^3 dx$

$f(x) = x^3$

and $f(-x) = (-x)^3 = -x^3$

$\therefore$ $f(-x) = -f(x)$

$\therefore$ f(x) is an odd function

$\therefore \quad \int_{-1}^{1} x^3 dx = 0$ [By property (4)]

[Remark : $\int_{-a}^{a} x^n dx = 0$ for n tobe odd integer]

(ii) $\int_{-\pi}^{\pi} \sin x \; dx$

$f(x) = \sin x$

and $f(-x) = \sin(-x) = -\sin x$

$\therefore \quad f(-x) = -f(x)$

$\therefore$ f(x) is an odd function

$\therefore \quad \int_{-\pi}^{\pi} \sin x \; dx = 0$ [Remark : $\int_{-a}^{a} (\sin x)^n dx = 0$ for n to be odd integer]

(iii) $\int_{-2}^{2} x^4 dx$

$f(x) = x^4$

and $f(-x) = (-x)^4 = x^4$

$\therefore \quad f(-x) = f(x)$

$\therefore$ f(x) is an even function

$\therefore \quad \int_{-2}^{2} x^4 dx = 2\int_{0}^{2} x^4 dx$, [By Property (4)]

$= 2\left[\frac{x^5}{5}\right]_0^2 = \frac{2}{5}\left[(2)^5 - (0)^5\right]$

$= \frac{2}{5}[32 - 0] = \frac{64}{5}$

[Remark : $\int_{-a}^{a} x^n dx = 2\int_{0}^{a} x^n dx$; for n to be even integer]

(iv) $\int_{-\pi}^{\pi} \cos 2x \; dx$

$f(x) = \cos 2x$

and $f(-x) = \cos 2(-x) = \cos 2x$

$\therefore \quad f(-x) = f(x)$

$\therefore$ f(x) is an even function

$$\therefore \int_{-\pi}^{\pi} \cos 2x \, dx = 2\int_{0}^{\pi} \cos 2x \, dx, \qquad \text{[By Property (4)]}$$

$$= 2\left[\frac{\sin 2x}{2}\right]_0^{\pi}$$

$$= [\sin 2\pi - \sin 0] = [0 - 0] = 0$$

(v) $\int_{-\pi/2}^{\pi/2} x \sin x \, dx$

$f(x) = x \sin x$

and $f(-x) = (-x) \sin(-x)$

$= (-x)(-\sin x)$

$= x \sin x$

$\therefore$ $f(-x) = f(x)$

or f(x) is an even function

$$\therefore \int_{-\pi/2}^{\pi/2} x \sin x \, dx = 2\int_{0}^{\pi/2} \underset{I}{x} \underset{II}{\sin x} \, dx, \qquad \text{[By Property (4)]}$$

$$= 2\left[\left|x\int \sin x \, dx\right|_0^{\pi/2} - \int_0^{\pi/2}\left(\frac{d}{dx}(x)\int \sin x \, dx\right)dx\right]$$

$$= 2\left[-|x \cos x|_0^{\pi/2} - \int_0^{\pi/2} 1(-\cos x)dx\right]$$

$$= -2\left[\frac{\pi}{2}\cos\frac{\pi}{2} - 0\right] + 2\int_0^{\pi/2} \cos x \, dx$$

$$= 0 + 2[\sin x]_0^{\pi/2} = 2\left[\sin\frac{\pi}{2} - 0\right]$$

$$= 2(1) = 2$$

(vi) $\int_{-\pi/2}^{\pi/2} x \cos x \, dx$

$f(x) = x \cos x$

and $f(-x) = (-x) \cos(-x)$

$= (-x) \cos x$

$= -x \cos x$

$\therefore \quad f(-x) = -f(x)$

or $\quad$ f(x) is an odd function

$$\therefore \quad \int_{-\pi/2}^{\pi/2} x\cos x \; dx = 0 \qquad \text{[By property (4)]}$$

Remarks :

i) The product of two odd functions is an even function, similar to (-)(-) = +

ii) The productof an odd and an even function is an off function, similar to (-)(+) = - ; considering '-' for odd and '+' for even.

(vii) $\int_{-1}^{1} |x| dx$

$f(x) = |x|$

and $f(-x) = |-x| = |x|$

$\therefore \quad f(-x) = f(x)$

or $\quad$ f(x) is an even function

$$\therefore \quad \int_{-1}^{1} |x| dx = 2\int_{0}^{1} |x| dx \quad \text{, [By Property (4)]}$$

$$= 2\int_{0}^{1} x \; dx \quad , [\because \; |x| = x \text{ for } 0 \le x \le 1]$$

$$= 2\left[\frac{x^2}{2}\right]_0^1 = \left[(1)^2 - (0)^2\right] = 1$$

Example 11 Evaluate the following Definite Integral

(i) $\int_{0}^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$ $\qquad$ (ii) $\int_{0}^{\pi/2} \frac{dx}{1 + \tan x}$

(iii) $\int_{0}^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}} dx$ $\qquad$ (iv) $\int_{0}^{\pi/2} \frac{\cos^5 x}{\sin^5 x + \cos^5 x} dx$

(v) $\int_{0}^{\pi/2} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} dx$ $\qquad$ (vi) $\int_{0}^{\pi/2} \log(\tan x) \, dx$

Solution

$$\text{Let} \quad I = \int_{0}^{\pi/2} \frac{\sin x}{\sin x + \cos x} \qquad \text{-----} (1)$$

Also $\quad I \quad = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)} dx \quad , \quad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\cos x \, dx}{\cos x + \sin x} \qquad \text{-----} (2)$$

Adding (1) and (2), we have

$$2I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx + \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx$$

$$= \int_0^{\pi/2} \left(\frac{\sin x + \cos x}{\cos x + \sin x}\right) dx$$

$$= \int_0^{\pi/2} 1 \, dx$$

$$= [x]_0^{\pi/2}$$

$$= \frac{\pi}{2} - 0$$

$$\therefore \quad 2I = \frac{\pi}{2}$$

$$\text{or} \quad I = \frac{\pi}{4}$$

(ii) Let $I = \int_0^{\pi/2} \frac{dx}{1+\tan x}$

$$= \int_0^{\pi/2} \frac{dx}{1+\frac{\sin x}{\cos x}} = \int_0^{\pi/2} \frac{\cos x \, dx}{\cos x + \sin x} \qquad \text{-----} (1)$$

Also $\quad I \quad = \int_0^{\pi/2} \frac{\cos\left(\frac{\pi}{2}-x\right)}{\cos\left(\frac{\pi}{2}-x\right)+\sin\left(\frac{\pi}{2}-x\right)} dx \quad , \quad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx \qquad \text{-----} (2)$$

Adding (1) and (2), we have

$$2I = \int_0^{\pi/2} \left(\frac{\cos x + \sin x}{\sin x + \cos x} \right) dx$$

$$= \int_0^{\pi/2} 1\, dx$$

$$= [x]_0^{\pi/2}$$

$$= \frac{\pi}{2} - 0$$

$$\therefore \ 2I = \frac{\pi}{2}$$

$$\text{or } \ I = \frac{\pi}{4}$$

(iii) Let $I = \int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}}\, dx$ ----- (1)

Also $I = \int_0^{\pi/2} \frac{\sqrt{\cot x\left(\frac{\pi}{2} - x\right)}}{\sqrt{\cot\left(\frac{\pi}{2} - x\right)} + \sqrt{\tan\left(\frac{\pi}{2} - x\right)}}\, dx$, $\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx \right]$

$$I = \int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x} + \sqrt{\cot x}}\, dx \qquad \text{-----} (2)$$

Adding (1) and (2), we have

$$2I = \int_0^{\pi/2} \frac{\sqrt{\cot x} + \sqrt{\tan x}}{\sqrt{\tan x} + \sqrt{\cot x}}\, dx$$

$$= \int_0^{\pi/2} 1\, dx$$

$$= [x]_0^{\pi/2}$$

$$= \frac{\pi}{2} - 0$$

$$\therefore \quad 2I = \frac{\pi}{2}$$

$$\text{or} \quad I = \frac{\pi}{4}$$

(iv) Let $I = \int_0^{\pi/2} \frac{\cos^5 x \, dx}{\sin^5 x + \cos^5 x}$ ----- (1)

Also $I = \int_0^{\pi/2} \frac{\cos^5\left(\frac{\pi}{2} - x\right)}{\sin^5\left(\frac{\pi}{2} - x\right) + \cos^5\left(\frac{\pi}{2} - x\right)} dx$, $\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$= \int_0^{\pi/2} \frac{\sin^5 x}{\cos^5 x + \sin^5 x} dx$ ----- (2)

Adding (1) and (2), we have

$$2I = \int_0^{\pi/2} \left(\frac{\cos^5 x + \sin^5 x}{\cos^5 x + \sin^5 x}\right) dx$$

$$= \int_0^{\pi/2} 1 \, dx$$

$$= [x]_0^{\pi/2}$$

$$= \frac{\pi}{2} - 0$$

$$\therefore \quad 2I = \frac{\pi}{2}$$

$$\text{or} \quad I = \frac{\pi}{4}$$

(v) Let $I = \int_0^{\pi/2} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} dx$ ----- (1)

Also I $= \int_0^{\pi/2} \frac{\sin^{3/2}\left(\frac{\pi}{2}-x\right)}{\sin^{3/2}\left(\frac{\pi}{2}-x\right)+\cos^{3/2}\left(\frac{\pi}{2}-x\right)}dx$, $\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$= \int_0^{\pi/2} \frac{\cos^{3/2} x}{\cos^{3/2} x + \sin^{3/2} x} dx$ ----- (2)

Adding (1) and (2), we get

2I $= \int_0^{\pi/2} \left(\frac{\sin^{3/2} x + \cos^{3/2} x}{\cos^{3/2} x + \sin^{3/2} x}\right) dx$

$= \int_0^{\pi/2} 1\, dx$

$= [x]_0^{\pi/2}$

$= \frac{\pi}{2} - 0$

$\therefore$ 2I $= \frac{\pi}{2}$

or I $= \frac{\pi}{4}$

(vi) Let I $= \int_0^{\pi/2} \log(\tan x)\, dx$ ----- (1)

Also I $= \int_0^{\pi/2} \log\left[\tan\left(\frac{\pi}{2}-x\right)\right] dx$, $\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$= \int_0^{\pi/2} \log(\cot x)\, dx$ ----- (2)

Adding (1) and (2), we get

2I $= \int_0^{\pi/2} [\log(\tan x) + \log(\cot x)]\, dx$

$$= \int_0^{\pi/2} \log(\tan x . \cot x)\, dx \quad , \qquad [\because \log m + \log n = \log mn]$$

$$= \int_0^{\pi/2} \log(1)\, dx$$

$$= \int_0^{\pi/2} 0\, dx \quad = 0$$

$\therefore\ 2I = 0$

or $I = 0$

Example 12 Evaluate $\int_0^{\pi} \frac{x \sin x}{1+\sin x} dx$

Solution $\int_0^{\pi} \frac{x \sin x}{1+\sin x} dx$

$$\text{Let} \quad I = \int_0^{\pi} \frac{x \sin x}{1+\sin x} dx \qquad \text{-----} (1)$$

$$\text{Also} \quad I = \int_0^{\pi} \frac{(\pi - x)\sin x(\pi - x)}{1+\sin x(\pi - x)} dx \quad , \qquad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\sin x} dx$$

$$= \int_0^{\pi} \frac{\pi \sin x}{1+\sin x} dx - \int_0^{\pi} \frac{x \sin x}{1+\sin x} dx$$

$$\therefore \quad I = \pi \int_0^{\pi} \frac{\sin x}{1+\sin x} dx - I$$

$$\text{or} \quad I + I = \pi \int_0^{\pi} \frac{\sin x}{1+\sin x} dx$$

$$2I = \pi \int_0^{\pi} \frac{\sin x}{1+\sin x} dx$$

$$= \pi \int_0^{\pi} \frac{1 + \sin x - 1}{1+\sin x} dx$$

$$= \pi\int_0^{\pi} 1\,dx - \pi\int_0^{\pi}\frac{1}{1+\sin x}dx$$

$$= \pi[x]_0^{\pi} - \pi\int_0^{\pi}\frac{1-\sin x}{1-\sin^2 x}dx$$

$$= \pi^2 - \pi\int_0^{\pi}\frac{1-\sin x}{\cos^2 x}dx$$

$$= \pi^2 - \pi\int_0^{\pi}(\sec^2 x - \sec x.\tan x)\,dx$$

$$= \pi^2 - \pi[\tan x - \sec x]_0^{\pi}$$

$$= \pi^2 - \pi\,[\{\tan\pi - \sec\pi\} - \{\tan 0 - \sec 0\}]$$

$$= \pi^2 - \pi\,[\{0-(-1)\} - \{0-1\}]$$

$$= \pi^2 - \pi\,[1+1]$$

$$2I = \pi^2 - 2\pi$$

or $$I = \frac{\pi^2}{2} - \pi$$

Two Standard Formula (Walli's Formulae)

1) $$\int_0^{\pi/2}\sin^n x\,dx = \int_0^{\pi/2}\text{con}^n x\,dx$$

$$= \frac{(n-1)\times \text{go on diminishing by 2}}{n\times \text{go on diminishing by 2}}\times \boxed{\frac{\pi}{2}}$$

Here $\boxed{\frac{\pi}{2}}$ indicates that, multiplication by $\frac{\pi}{2}$ only when n is even and no multiplication by $\frac{\pi}{2}$ if n is odd.

2) $$\int_0^{\pi/2}\sin^m x\cos^n dx$$

$$= \frac{[(m-1)\times \text{go on diminishing by 2}][(n-1)\times \text{go on diminishing by 2}]}{(m+n)\times \text{go on diminishing by 2}}\times \boxed{\frac{\pi}{2}}$$

Here $\boxed{\frac{\pi}{2}}$ indicates that; multiplication by $\frac{\pi}{2}$ only if both m and n are even and no multiplication by $\frac{\pi}{2}$ if atleast one of m and n is odd.

Remark : In above formulae, (n–1) and n are diminishes by 2 so long as the factors are positive (i.e. the last factor should be either 1 or 2)

Example 13 **Evaluate**

(i) $\int_0^{\pi/2} \sin^8 x \, dx$ (ii) $\int_0^{\pi/2} \cos^9 x \, dx$

(iii) $\int_0^{\pi/2} \sin^5 x \cos^7 x \, dx$ (iv) $\int_0^{\pi/2} \sin^4 x \cos^6 x \, dx$

Solution

(i) $\int_0^{\pi/2} \sin^8 x \, dx$

$$= \frac{(n-1) \times \text{ go on diminishing by 2}}{n \times \text{ go on diminishing by 2}} \times \frac{\pi}{2}$$

$$= \frac{7x5x3x1}{8x6x4x2} x \frac{\pi}{2} = \frac{35\pi}{256}$$

(ii) $\int_0^{\pi/2} \cos^9 x \, dx$

Here n = 9 = odd no., n–1 = 8

$$\therefore \int_0^{\pi/2} \cos^9 x \, dx = \frac{(n-1) \times \text{go on diminishing by 2 [upto + ve factor]}}{n \times \text{go on diminishing by 2 [upto + ve factor]}}$$

$$= \frac{8x6x4x2}{9x7x5x3x1} = \frac{128}{315}$$

(iii) $\int_0^{\pi/2} \sin^5 x \cos^7 x \, dx$

Here m = 5, n = 7, both are odd:
m–1 = 4, n–1 = 6, m+n = 12

$$\therefore \int_0^{\pi/2} \sin^5 x \cos^7 x \, dx$$

$$= \frac{[(m-1)\times \text{go on diminishing by 2}]\times[(n-1)\times \text{go on diminishing by 2}]}{(m+n)\times \text{go on diminishing by 2}}$$

$$= \frac{[4x2]\times[6x4x2]}{[12x10x8x6x4x2]} = \frac{1}{120}$$

(iv) $\int_0^{\pi/2} \sin^4 x \cos^6 x \, dx$

Here m = 4, n = 6, both are even
m-1 = 3, n-1 = 5, m+n = 10

$\therefore \int_0^{\pi/2} \sin^4 x \cos^6 x \, dx$

$$= \frac{[(m-1)\times \text{go on diminishing by 2}]\times[(n-1)\times \text{go on diminishing by 2}]}{(m+n)\times \text{go on diminishing by 2}} \times \frac{\pi}{2}$$

$$= \frac{(3x1)(5x3x1)}{(10x8x6x4x2)} \times \frac{\pi}{2} = \frac{3\pi}{512}$$

Example 14 Evaluate

(i) $\int_0^{\pi/6} \sin^7 3x \, dx$ (ii) $\int_0^{\pi/4} \sin^7 2x \cos^8 2x \, dx$

(iii) $\int_0^{a} x^4 \sqrt{a^2 - x^2} \, dx$ (iv) $\int_0^{\infty} \frac{x^4}{(1+x^2)^4} dx$

Solution

(i) $\int_0^{\pi/6} \sin^7 3x \, dx$

Put $3x = \theta$
$\therefore$ $3dx = d\theta$
or $dx = \frac{1}{3} d\theta$

When $x = 0, \theta = 0$, when $x = \frac{\pi}{6}, \theta = \frac{\pi}{2}$

$$\therefore \int_0^{\pi/6} \sin^7 3x \, dx = \int_0^{\pi/2} \sin^7 \theta \frac{d\theta}{3}$$

$$= \frac{1}{3} \cdot \frac{(7-1) \times \text{go on diminishing by 2}}{7 \times \text{go on diminishing by 2}}$$

$$= \frac{1}{3} \cdot \frac{6x4x2}{7x5x3}$$

$$= \frac{1}{3} \cdot \frac{16}{35} = \frac{16}{105}$$

(ii) $\int_0^{\pi/4} \sin^7 2x \cos^8 2x \, dx$

Put $2x = \theta$

$\therefore$ $2dx = d\theta$

or $dx = \frac{d\theta}{2}$

When $x = 0, \theta = 0$, when $x = \frac{\pi}{4}, \theta = \frac{\pi}{2}$

$$\therefore \int_0^{\pi/4} \sin^7 2x \cos^8 2x \, dx = \int_0^{\pi/2} \sin^7 \theta \cos^8 \theta . \frac{d\theta}{2}$$

$$= \frac{1}{2} \cdot \frac{[(7-1)x4x2] \times [(8-1)x5x3]}{(7+8)x13x11x9x7x5x3x1}$$

$$= \frac{1}{2} \times \frac{6x4x2x7x5x3}{15x13x11x9x7x5x3x1} = \frac{8}{6435}$$

(iii) $\int_0^{a} x^4 \sqrt{a^2 - x^2} \, dx$

Put $x = a\sin\theta$

$\therefore$ $dx = a\cos\theta \, d\theta$

When $x = 0, \theta = 0$, when $x = a, \theta = \frac{\pi}{2}$

$$\therefore \int_0^{a} x^4 \sqrt{a^2 - x^2} \, dx = \int_0^{\pi/2} a^4 \sin^4 \theta \sqrt{a^2 - a^2 \sin^2 \theta} . a\cos\theta \, d\theta$$

$$= a^4 \int_0^{\pi/2} \sin^4 \theta . a^2 \cos^2 \theta \, d\theta$$

$$= a^6 \int_0^{\pi/2} \sin^4\theta \cos^2\theta \, d\theta$$

$$= a^6 . \frac{[3x1] \times [1]}{6x4x2} \times \frac{\pi}{2} \text{ , Here } m = 4, n = 2 \text{ by Walli's Formula}$$

$$= \frac{\pi a^6}{32}$$

(iv) $\int_0^{\infty} \frac{x^4}{(1+x^2)^4} dx$

Put $x = \tan\theta$

$\therefore$ $dx = \sec^2\theta \, d\theta$

When $x = 0, \theta = 0$, when $x = \infty$, $\theta = \frac{\pi}{2}$

$$\therefore \int_0^{\infty} \frac{x^4}{(1+x^2)^4} dx = \int_0^{\pi/2} \frac{\tan^4\theta}{(1+\tan^2\theta)^4} . \sec^2\theta \, d\theta$$

$$= \int_0^{\pi/2} \frac{\tan^4\theta}{(\sec^2\theta)^4} . \sec^2\theta \, d\theta$$

$$= \int_0^{\pi/2} \frac{\tan^4\theta}{\sec^6\theta} d\theta$$

$$= \int_0^{\pi/2} \cos^6\theta \frac{\sin^4\theta}{\cos^4\theta} d\theta$$

$$= \int_0^{\pi/2} \sin^4\theta \cos^2\theta \, d\theta$$

$$= \frac{[3x1] \times [1]}{6x4x2} \times \frac{\pi}{2} \text{ , Here } m = 4, n = 2 \text{ by Walli's Formula}$$

$$= \frac{\pi}{32}$$

MULTIPLE CHOICE QUESTIONS

Q 1. $\int_a^b k\,dx$; where K is constant is equal to

(a) k (b) k(b–a) (c) 0 (d) k(a–b)

Q 2. $\int_a^b dx = ?$

(a) b–a (b) a–b (c) x(a–b) (d) x(b–a)

Q 3. $\int_a^a f(x)\,dx = ?$

(a) f(a) (b) a (c) 1 (d) 0

Q 4. $\int_a^b \sqrt{x}\,dx = ?$

(a) $\frac{3}{2}\left[a^{\frac{3}{2}} - b^{\frac{3}{2}}\right]$ (b) $\frac{2}{3}\left[(b)^{\frac{3}{2}} - (a)^{\frac{3}{2}}\right]$ (c) $\frac{2}{3}(b-a)^{\frac{3}{2}}$ (d) $\frac{2}{3}\left[a^{\frac{3}{2}} - b^{\frac{3}{2}}\right]$

Q 5. $\int_1^3 (x^2 + 7x - 5)dx = ?$

(a) $\frac{40}{3}$ (b) $\frac{20}{3}$ (c) $\frac{80}{3}$ (d) $\frac{60}{3}$

Q 6. $\int_a^b \frac{1}{x} dx = ?$

(a) (b–a) (b) $\log\frac{a}{b}$ (c) $\log\frac{b}{a}$ (d) log(b–a)

Q 7. $\int_0^4 x\sqrt{x}\,dx = ?$

(a) $\frac{32}{5}$ (b) $\frac{64}{5}$ (c) $\frac{128}{5}$ (d) None of these

Q 8. $\int_1^2 \frac{x^2}{1+x^3}\,dx$

(a) $\frac{1}{3}\log\left(\frac{9}{2}\right)$ (b) $3\log\left(\frac{9}{2}\right)$ (c) $\frac{1}{3}\log\left(\frac{2}{9}\right)$ (d) $3\log\left(\frac{2}{9}\right)$

Q 9. $\int_a^b e^x \, dx$

(a) $e^{(b-a)}$ (b) $e^b - e^a$ (c) $e^a - e^b$ (d) e^{a-b}

Q 10. $\int_0^{\sqrt{3}} \frac{e^{m\tan^{-1}x}}{1+x^2} \, dx = ?$

(a) $\left(e^{m\pi/3} - 1\right)$ (b) $\frac{1}{m}\left(e^{m\pi/6} - 1\right)$ (c) $\frac{1}{m}\left(e^{m\pi/3} - 1\right)$ (d) $\left(e^{m\pi/6} - 1\right)$

Q 11. $\int_0^1 \frac{\tan^{-1}x}{1+x^2} \, dx$

(a) $\frac{\pi}{32}$ (b) $\frac{\pi^2}{16}$ (c) $\frac{\pi^2}{32}$ (d) $\frac{\pi}{16}$

Q 12. $\int_0^{1/\sqrt{2}} \frac{\sin^{-1}x}{\sqrt{1+x^2}} \, dx = ?$

(a) $\frac{\pi}{32}$ (b) $\frac{\pi^2}{32}$ (c) $\frac{\pi^2}{16}$ (d) $\frac{\pi}{16}$

Q 13. $\int_0^{\infty} \frac{\sin(\tan^{-1}x)}{1+x^2} \, dx$

(a) 1 (b) -1 (c) 0 (d) $\frac{1}{\sqrt{2}}$

Q 14. $\int_0^{\pi/3} \sec^2 x \, dx = ?$

(a) $\frac{1}{\sqrt{3}}$ (b) 1 (c) 0 (d) $\sqrt{3}$

Q 15. $\int_{\pi/6}^{\pi/4} \operatorname{cosec}^2 x \, dx = ?$

(a) $\sqrt{3}$ (b) $\sqrt{3}+1$ (c) $\sqrt{3}-1$ (d) $-\sqrt{3}+1$

Q 16. $\int_0^{\pi/4} \tan^{2x} \sec^2 x \, dx = ?$

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) 0 (d) 1

Q 17. $\int_0^{\pi/4} \tan^n \sec^2 x \, dx$, $(n \neq -1)$ is equal to

(a) n (b) $\frac{1}{n}$ (c) $\frac{1}{n+1}$ (d) None of these

Q 18. $\int_0^{\pi/2} \sin^2 x \, dx = ?$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{8}$

Q 19. $\int_0^{\pi} \cos^2 x \, dx = ?$

(a) $\frac{\pi}{8}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

Q 20. $\int_0^{\pi} (\cos^2 x - \sin^2 x) \, dx$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) 0 (d) 1

Q 21. $\int_0^{\pi/2} \frac{\sin 3x}{\sin x} \, dx = ?$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) 0 (d) 1

Q 22. $\int_a^b \frac{1}{x} (\log x)^n \, dx = ?$

(a) $\frac{1}{n}\left(b^n - a^n\right)$ (b) $\frac{1}{n+1}\left[(b)^{n+1} - (a)^{n+1}\right]$

(c) $\frac{1}{n+1} \log\left(\frac{b}{a}\right)^{n+1}$ (d) None of these

Q 23. $\int_0^{\pi/2} e^{3x} \sin 2x \, dx$

(a) $\frac{2}{13}\left[e^{3\pi/2}+1\right]$ (b) $\frac{2}{13}\left[e^{3\pi/2}-1\right]$ (c) $\frac{1}{13}\left[e^{3\pi/2}-1\right]$ (d) $\frac{1}{13}\left[e^{3\pi/2}+1\right]$

Q 24. $\int_0^2 \frac{x^3-1}{(x^2+x+1)}\,dx = ?$

(a) 1 (b) 45 (c) 0 (d) 42

Q 25. $\int_{\pi/6}^{\pi/4} \frac{\sec^2 x}{\tan x}\,dx = ?$

(a) $-\log\sqrt{3}$ (b) $-\log 3$ (c) $\log\frac{1}{3}$ (d) $\frac{1}{2}\log 3$

Q 26. $\int_1^2 \frac{3x^2-4}{x^3-4x+5}\,dx$

(a) $\log\frac{2}{5}$ (b) $\log\frac{3}{2}$ (c) $\log\frac{5}{2}$ (d) $\log\frac{1}{2}$

Q 27. If $\int_a^b f(x)\,dx = k$, then $\int_b^a f(x)\,dx = ?$

(a) -k (b) k (c) 0 (d) k(a-b)

Q 28. $\int_0^a f(x)\,dx = ?$

(a) $\int_0^{2a} f(x)dx$ (b) $\int_0^a f(a-x)dx$ (c) $\int_0^a f(x-a)dx$ (d) $\int_0^a -f(x)dx$

Q 29. $\int_0^{\pi/2} (\sin x + \tan x)\,dx$

(a) $\int_0^{\pi/2} (\cos x - \cot x)dx$ (b) $\int_0^{\pi/2} (\cos x + \cot x)dx$

(c) $\int_0^{\pi/2} (\cos x + \tan x)dx$ (d) None of these

Q 30. $\int_0^{\pi/2} \frac{dx}{1+\cot x} = ?$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{8}$

Q 31. $\int_0^{\pi/2} \frac{\sin^4 x}{\cos^4 x + \sin^4 x} dx = ?$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{8}$ (d) $\frac{\pi}{4}$

Q 32. $\int_0^{\pi/4} \frac{\sqrt{\cot x}}{\sqrt{\tan x} + \sqrt{\cot x}} dx = ?$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{8}$

Q 33. $\int_0^{\pi/2} \frac{dx}{1+\sqrt{\cot x}} dx = ?$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{8}$

Q 34. $\int_0^{\pi/2} \cos^7 x \, dx = ?$

(a) $\frac{16}{35}$ (b) $\frac{8}{35}$ (c) $\frac{4}{35}$ (d) $\frac{12}{35}$

Q 35. $\int_0^{\pi/2} \cos^6 x \, dx = ?$

(a) $\frac{5}{16}$ (b) $\frac{5}{32}\pi$ (c) $\frac{5}{32}$ (d) $\frac{7}{32}\pi$

Q 36. $\int_0^{\pi/2} \sin^5 x \, dx = ?$

(a) $\frac{2}{15}$ (b) $\frac{4}{15}$ (c) $\frac{1}{3}$ (d) $\frac{8}{15}$

Q 37. $\int_0^{\pi/2} \sin^4 x \, dx = ?$

(a) $\frac{2}{15}\pi$ (b) $\frac{3}{16}\pi$ (c) $\frac{5}{16}\pi$ (d) $\frac{7}{16}\pi$

Q 38. $\int_0^{\pi/2} \sin^5 x \cos^7 x \, dx = ?$

(a) $\frac{1}{100}$ (b) $\frac{1}{35}$ (c) $\frac{1}{120}$ (d) $\frac{1}{12}$

Q 39. $\int_0^{\pi/2} \sin^4 x \cos^7 x \, dx = ?$

(a) $\frac{16}{1155}$ (b) $\frac{8}{1155}$ (c) $\frac{6}{1155}$ (d) $\frac{7}{1155}$

Q 40. $\int_0^{\pi/2} \sin^4 x \cos^6 x \, dx = ?$

(a) $\frac{5}{128}\pi$ (b) $\frac{3}{128}\pi$ (c) $\frac{3}{256}\pi$ (d) $\frac{3}{512}\pi$

Q 41. $\int_0^{\pi/2} \sin^4 x \cos^3 x \, dx = ?$

(a) $\frac{1}{35}$ (b) $\frac{2}{35}$ (c) $\frac{3}{35}$ (d) $\frac{4}{35}$

Q 42. $\int_0^{\pi/6} \sin^4 3x \cos^6 3x \, dx = ?$

(a) $\frac{\pi}{1024}$ (b) $\frac{\pi}{256}$ (c) $\frac{\pi}{512}$ (d) $\frac{\pi}{128}$

Q 43. $\int_0^{\pi/4} \sin^5 2x \, dx$

(a) $\frac{1}{15}$ (b) $\frac{2}{15}$ (c) $\frac{3}{15}$ (d) $\frac{4}{15}$

Q 44. $\int_{-a}^{a} f(x) \, dx = ?$; where $f(-x) = -f(x)$

(a) 0 (b) $2\int_0^a f(x)dx$ (c) $\int_0^a f(x)dx$ (d) None of these

Chapter **3**

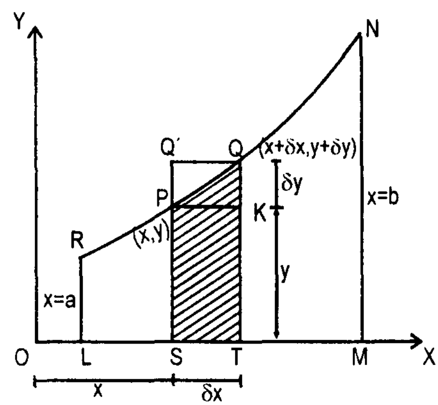
APPLICATION OF INTEGRATION AND NUMERICAL INTEGRATION

Definite Integral as area under the curve

We know that integration is the inverse process of differentiation. Now we are going to prove that, the definite integral may be regarded as the area under the curve.

Let $y = f(x)$ be a continuous function in $a \le x \le b$, then the area under the curve $y = f(x)$, bounded by x–axis, ordinates $x = a$ and $x = b$ is given by $\int_a^b y\,dx$ or $\int_a^b f(x)\,dx$

Proof : Let the adjacent figure shows the graph of function $y = f(x)$ in $[a,b]$.



Let the ordinate be moving from initial position RL to final position NM.

Let during this movement, the area swept out by the ordinate be A

i.e. area LMNQPR = A

Let while moving from RL to NM, PS be an intermediate position of ordinate such that P has co-ordinates (x,y). Let Q be any point on the curve adjacent to point P and having co-ordinates

$(x+\delta x,\ y+\delta y)$.

Let δA = Area (STQP)

Now area of Rectangle (STKP) = $\delta x.y$ and area of rectangle (STQQ′) = $(y+\delta y).\ \delta x$

Clearly

$$\text{Area of (STKP)} \le \text{Area of (STQP)} \le \text{Area of (STQQ}')$$

i.e. $$\delta x.y \le \delta A \le (y+\delta y).\ \delta x$$

or $$y \le \frac{\delta A}{\delta x} \le (y+\delta y)$$

Taking limit as $\delta x \to 0$, $[\delta y \to 0 \text{ as } \delta x \to 0]$

$$\lim_{\delta x \to 0} y \le \lim_{\delta x \to 0} \frac{\delta A}{\delta x} \le \lim_{\delta x \to 0} (y+\delta y)$$

$$\therefore \quad y \leq \frac{dA}{dx} \leq \psi$$

$$\text{or} \quad \frac{dA}{dx} = y$$

$$\therefore \quad A = \int_{x=a}^{x=b} y \, dx$$

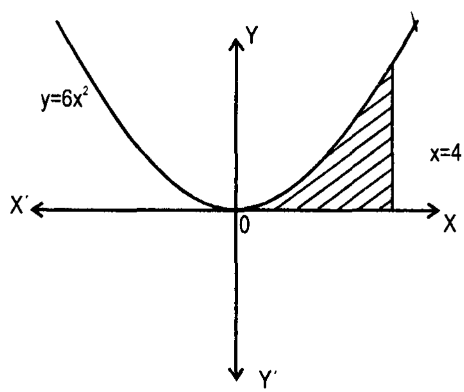
$$\text{or} \quad A = \int_a^b f(x) \, dx$$

Thus, $\int_a^b f(x)\,dx$ = Area under the curve y = f(x) and bounded by x-axis, ordinates x = a and x = b.

Similarly, $\int_c^d x\,dy$ = Area bounded between the curve x = f(y), y = axis and the lines y = c and y = d

Example 1 Find the area under the curve $y = 6x^2$ from x = 0 to x = 4.

Solution



$$\text{Area} = \int_0^4 y\,dx$$

$$= \int_0^4 6x^2\,dx$$

$$= 6\left[\frac{x^3}{3}\right]_0^4$$

$= 2[(4)^3 - (0)^3]$

$= 2\times64 = 128$ Square units.

Example 2 Find the area under the curve y = x–2, above x–axis and bounded by the line x = 5.

Solution The line y = x–2 cuts x–axis, where y = 0

i.e. x – 2 = 0, x = 2

Thus we are to find the area under the curve y = x – 2, above x–axis and between the ordinates x = 2 and x = 5

$\therefore$ Area $= \int_2^5 y\, dx$

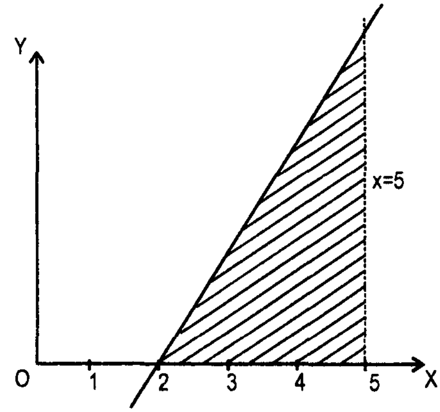
$= \int_2^5 (x-2)\, dx$

$= \left[\frac{x^2}{2} - 2x\right]_2^5$

$= \left[\frac{(5)^2}{2} - 2(5)\right] - \left[\frac{(2)^2}{2} - 2(2)\right]$

$= \frac{25}{2} - 10 - \frac{4}{2} + 4$

$= \frac{25 - 20 - 4 + 8}{2} = \frac{9}{2}$ sq. units



Example 3 Find the area bounded by the curve $x = 4y^2$ from $y = 2$ to $y = 4$ and by y - axis.

Solution Required area

$= \int_2^4 x\, dy$

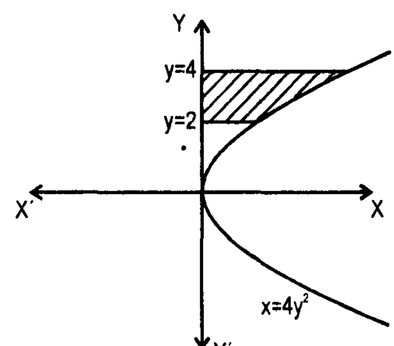
$= \int_2^4 4y^2\, dy$

$= 4\left[\frac{y^3}{3}\right]_2^4$

$= \frac{4}{3}\left[(4)^2 - (2)^3\right]$

$= \frac{4}{3}[64 - 8] = \frac{4}{3} \times 56$

$= \frac{224}{3} = 74\frac{2}{3}$



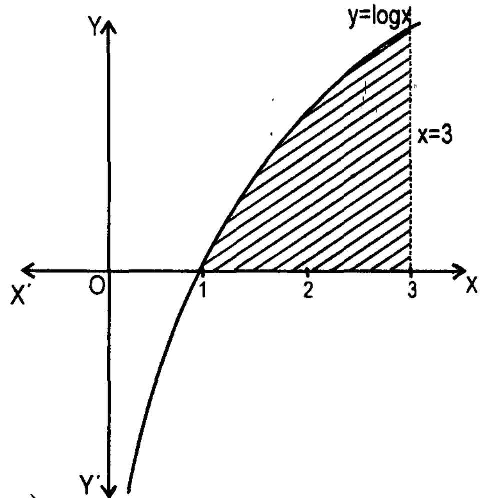
Example 4 Find the area bounded by the curve $y = \log x$, above x–axis and ordinate $x = 3$.

Solution The curve $y = \log x$ cuts x–axis where y = 0

i.e. $\log x = 0$

=> $x = 1$

∴ Required area



$$= \int_1^3 y\, dx$$

$$= \int_1^3 \log x\, dx$$

$$= \int_1^3 \underset{II}{(1)}\underset{I}{(\log x)}\, dx$$

$$= \left|\log x \int 1\, dx\right|_1^3 - \int_1^3 \left(\frac{d}{dx}(\log x)\int 1 dx\right) dx$$

$$= [(\log x)x]_1^3 - \int_1^3 \frac{1}{x}.x\, dx$$

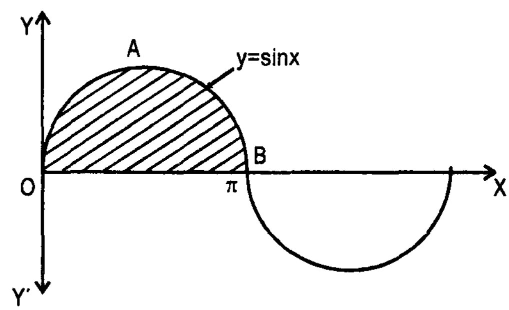
$$= [(\log x)x]_1^3 - [x]_1^3$$

$= [3 \log 3 - \log 1]-[3-1]$

$= 3 \log 3-2$, $[\because \log 1 = 0]$

Example 5 Find the bounded by the curve y = sin x and x-axis from x = 0 to x = π.

Solution Required Area



$$= \int_0^\pi y\, dx$$

$$= \int_0^\pi \sin x\, dx$$

$$= [-\cos x]_0^\pi$$

$= -[\cos\pi - \cos 0]$

$= -[-1-1]$

= 2 sq. units *Ans.*

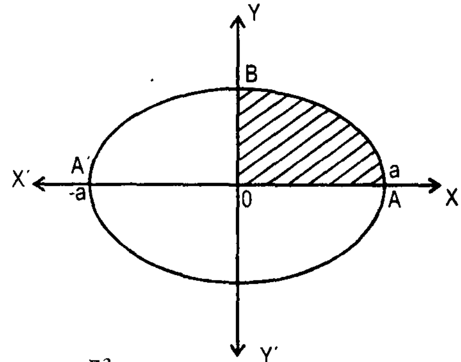
Example 6 Find the area of Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (a>b)

Solution The Ellipse cuts x-axis where y = 0 i.e. $\frac{x^2}{a^2} = 1$, or $x^2 = a^2$ or $x = \pm a$

i.e. at (–a,0) and (a,0)

$\because$ Ellipse is a symmetric curve about both x–axis as well as y–axis.

$\therefore$ Area of Ellipse = 4(Area OAB)



$$= 4\int_0^a y\, dx$$

$$= 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx$$

$$= 4\frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\, dx$$

$$= \frac{4b}{a}\left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{4b}{a}\left[a \times 0 + \frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right) - 0 - 0\right]$$

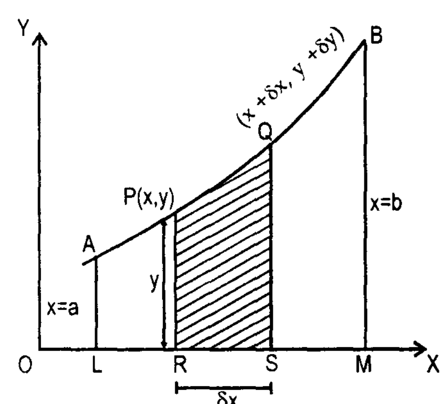
$$= \frac{4b}{a}\left[\frac{a^2}{2}\sin^{-1} 1\right]$$

$$= \frac{4b}{a}\left[\frac{a^2}{2}.\frac{\pi}{2}\right]$$

$= \pi ab$ sq. units

Volume of solid of Revolution

When a plane area is rotated about any line (as its axis), then we get a solid (imaginary) in the space. The volume of this imaginary solid is called **Volume of solid of Revolution.**



Let $y = f(x)$ be a continuous curve from $x = a$ to $x = b$.

Let $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be two neighbouring points on the curve $y = f(x)$.

Now, if we rotate the curve AB about x–axis, then the area PQSR will also rotate about x–axis.

Due to rotation of ordinate PR = y, about x-axis, a circle of area πy^2 is generated and due to rotation of plane area PRSQ a cylinder of volume $\pi y^2 \delta x$ is generated.

i.e. $\delta v = \pi y^2 \delta x$

$\therefore$ The volume of solid generated by the revolution by the complete plane area ABML will be given by

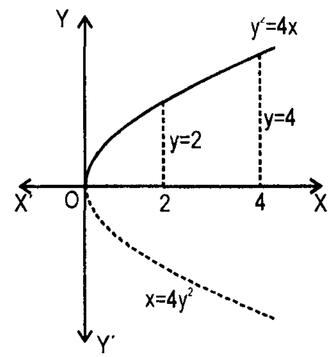
$$V = \int_a^b \pi y^2 \, dx \quad , \quad [\text{Here } y = f(x)]$$

Similarly, if the same curve is rotated about y-axis, then the volume of solid of revolution will be given by

$$V = \int_c^d \pi x^2 \, dy \quad , \quad [\text{Here } x = g(y) = f^{-1}(y),\ c = f(a) \text{ and } d = f(b)]$$

Example 7 Find the volume of solid of revolution generated by revolving the curve $y^2 = 4x$ $(y>0)$ about x-axis and between the ordinates x = 2 and x = 4.

Solution :



$$\text{Volume} = \int_2^4 \pi y^2 \, dx$$

$$= \pi \int_2^4 4x \, dx$$

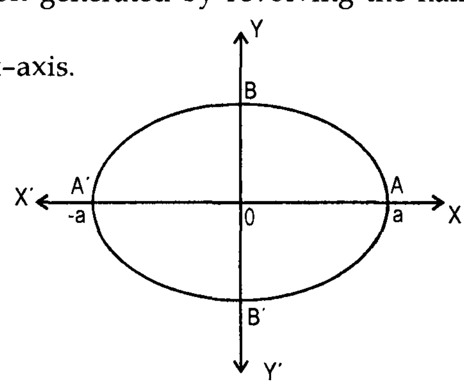
$$= 4\pi \left[\frac{x^2}{2} \right]_2^4$$

$$= 2\pi \, [(4)^2 - (2)^2]$$

$$= 2\pi \, [16-4]$$

$$= 2\pi \, (12) \qquad = 24\pi \text{ cubic units}$$

Example 8 Find the volume of solid of revolution generated by revolving the half branch of Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about x-axis.



Solution : The Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Intersects x-axis at the points where y=0

i.e. $\frac{x^2}{a^2} = 1$ or $x = \pm a$

i.e the points A′(-a,0) and A(a,0)

$\therefore$ Volume $= \int_{-a}^{a} \pi y^2 \, dx$

$= \int_{-a}^{a} \frac{b^2}{a^2}\left(a^2 - x^2\right) dx$

$= \frac{\pi b^2}{a^2} \int_{-a}^{a} \left(a^2 - x^2\right) dx$

$= \frac{2\pi b^2}{a^2} \int_{-a}^{a} \left(a^2 - x^2\right) dx$, $\left[\because \int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx, \text{ if } f(-x) = f(x)\right]$

$= \frac{2\pi b^2}{a^2}\left[a^2 x - \frac{x^3}{3}\right]_0^a$

$= \frac{2\pi b^2}{a^2}\left[\left(a^3 - \frac{a^3}{3}\right) - (0 - 0\right]$

$= \frac{2\pi b^2}{a^2}\left[\frac{2}{3}a^3\right]$

$= \frac{4}{3}\pi a b^2$ cubic units

Example 9 Find the volume of sphere of radius 'a'.

Solution A sphere of radius 'a' can be generated by revolving a semi-circle of radius 'a' about x-axis. Let it be upper half of circle of radius 'a'.

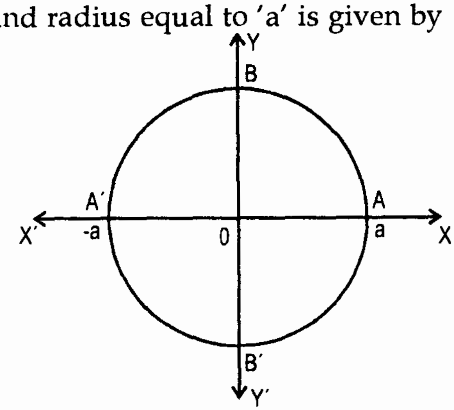
Equation of circle having center (0,0) and radius equal to 'a' is given by $x^2+y^2 = a^2$.

It intersects x-axis, where y = 0

i.e. $x^2 = a^2$

$\therefore$ $x = \pm a$

i.e. At the point A′ (-a,0) and A(a,0)



$\therefore$ Reqd. volume $= \int_{-a}^{a} \pi y^2 \, dx$

$= \pi \int_{-a}^{a} \left(a^2 - x^2\right) dx$

$$= 2\pi\int_0^a \left(a^2 - x^2\right) dx \qquad \left[\because \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx \text{ where } f(-x) = f(x)\right]$$

$$= 2\pi\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= 2\pi\left[\left(a^2x - \frac{a^3}{3}\right) - (0-0)\right]$$

$$= 2\pi\left[a^3 - \frac{a^3}{3}\right]$$

$$= 2\pi\left[\frac{2}{3}a^3\right] = \frac{4}{3}\pi a^3$$

Average or Mean Value

The average or mean value of a function f(x) between x = a and x = b is defined as

$$\text{A.V. or M.V.} = \int_a^b \frac{f(x)dx}{b-a}$$

Root Mean Square value (R.M.S.V.) - Root mean square value of a function f(x) between x = a and x = b is defined by,

$$\text{R.M.S.V.} = \sqrt{\frac{\int_a^b \left[f(x)^1\right]^2 dx}{(b-a)}}$$

Example 10 Find the Average Value of $y = x^2+5x+2$ from x = 1 to x = 4

Solution

$$\text{A.V. or M.V.} = \int_1^4 \frac{f(x)dx}{(4-1)}$$

$$= \int_1^4 \frac{x^2+5x+2}{3} dx$$

$$= \frac{1}{3}\left[\frac{x^3}{3} + \frac{5x^2}{2} + 2x\right]_1^4$$

$$= \frac{1}{3}\left[\left\{\frac{(4)^3}{3}+\frac{5(4)^2}{2}+2(4)\right\}-\left\{\frac{(1)^3}{3}+\frac{5(1)^2}{2}+2(1)\right\}\right]$$

$$= \frac{1}{3}\left[\left\{\frac{64}{3}+\frac{80}{2}+8\right\}-\left\{\frac{1}{3}+\frac{5}{2}+2\right\}\right]$$

$$= \frac{1}{3}\left[\frac{128+240+48}{6}\right]-\frac{1}{3}\left[\frac{2+15+12}{6}\right]$$

$$= \frac{43}{2} \quad = \quad 21.5$$

Example 11 Find the Average Value of $\cos^2\omega t$ from $t = 0$ to $t = \frac{2\pi}{\omega}$

Solution A.V. $= \int_a^b \frac{f(t)}{b-a}\, dt$

$$= \int_0^{2\pi/\omega} \frac{\cos^2 \omega t}{\left(\frac{2\pi}{\omega}-0\right)}\, dt$$

$$= \int_0^{2\pi/\omega} \frac{(1+\cos 2\omega)}{2\left(\frac{2\pi}{\omega}\right)}\, dt \quad , \quad [\because 1+\cos 2\theta = 2\cos^2\theta]$$

$$= \frac{\omega}{4\pi}\int_0^{2\pi/\omega}(1+\cos 2\omega t)\ dt$$

$$= \frac{\omega}{4\pi}\left[t+\frac{\sin 2\omega t}{2\omega}\right]_0^{2\pi/\omega}$$

$$= \frac{\omega}{4\pi}\left[\left(\frac{2\pi}{\omega}+\frac{1}{2\omega}\sin 4\pi\right)-\left(0+\frac{\sin 0}{2\omega}\right)\right]$$

$$= \frac{\omega}{4\pi}\left[\frac{2\pi}{\omega}+0-0\right]$$

$$= \frac{1}{2}$$

MULTIPLR CHOICE QUESTIONS

Q1. The area under the curve y = x–3, bounded by x–axis and ordinate x = 5
(a) 1 sq. unit (b) 2 sq. units (c) 3 sq. units (d) 5 sq. units

Q.2 The area under the curve $y = 8x^2$ from x = 1 to x = 4 and above x–axis is ?
(a) 63 sq. units (b) 126 sq. units (c) 168 sq. units (d) 21 sq. units

Q.3 The area bounded by the curve y = log x between the x– axis and the ordinates x = 3 and x = 4 is
(a) $\log\left(\frac{256}{3e}\right)$ (b) $\log\left(\frac{256}{3}\right)$ (c) $\log\left(\frac{256}{e}\right)$ (d) log(256e)

Q.4 The area bounded by the parabola $y^2 = 4ax$ above x–axis and latus rectum is
(a) $\frac{4a^2}{3}$ (b) $\frac{8a^2}{3}$ (c) $\frac{6a^2}{3}$ (d) $\frac{a^2}{3}$

Q.5 The area of Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is given by
(a) πab (b) $\pi\sqrt{ab}$ (c) $\pi a^2 b$ (d) πab^2

Q.6 The area bounded by the curve y = sin2x from x = 0 to $\frac{\pi}{2}$ and x–axis is
(a) 0 (b) 1 (c) 2 (d) $\frac{3}{2}$

Q.7 The area bounded by the curve $y = e^{2x}$ from x = 0 to 2 is
(a) (e^4-1) (b) (e^2-1) (c) $\frac{1}{2}(e^4-1)$ (d) $\frac{1}{4}(e^4-1)$

Q.8 The area of circle of radius 5 in 1st quadrant is
(a) $\frac{25}{2}\pi$ (b) $\frac{25}{3}\pi$ (c) $\frac{25}{8}\pi$ (d) $\frac{25}{4}\pi$

Q.9 The area bounded by the curve y = cosx from x = 0 to $\frac{\pi}{2}$, and x–axis is
(a) 0 (b) –1 (c) 2 (d) 1

Q.10 The area bounded by the curve y = cosx from x = 0 to π and x–axis will be
(a) 0 (b) –1 (c) 2 (d) 1

Q.11 The area under the curve $y=6x^2$ from x=0 to x=4 is
(a) 128 (b) 124 (c) 136 (d) None of these

Q.12 The area bounded by the curve $y = 3x^2+2x-1$ from x=1 to x=4 is
(a) 76 (b) 75 (c) 72 (d) 78

Q.13 $\int_a^b f(x)\,dx$ represents

(a) Volume (b) Area (c) Length (d) None of these

Q.14 $\int_a^b \pi y^2\,dx$ represents

(a) Volume (b) Area (c) Length (d) None of these

Q.15 The volume of sphere of radius R is

(a) $\frac{1}{3}\pi R^3$ (b) $\frac{4}{3}\pi R^3$ (c) $\frac{2}{3}\pi R^3$ (d) $4\pi R^2$

Q.16 The volume of solid of revolution obtained by revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about x-axis is given by

(a) $\frac{2}{3}\pi ab^2$ (b) $\frac{1}{3}\pi ab^2$ (c) πab^2 (d) $\frac{4}{3}\pi ab^2$

Q.17 The area under the curve $y = \sqrt{3x+4}$ between x = 0 and x=4 is

(a) $\frac{118}{9}$ (b) $\frac{112}{9}$ (c) $\frac{121}{9}$ (d) $\frac{110}{9}$

Q.18 The area bounded by the curve y=3cosx, x-axis and y-axis is

(a) 3 (b) 2 (c) 1 (d) 4

Q.19 The area bounded by y=log x, y=0 and x=2 is

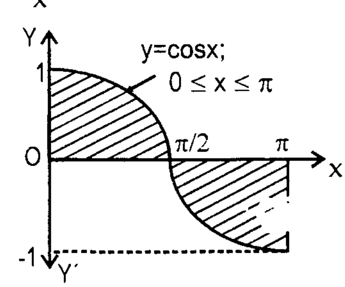
(a) log 3–1 (b) log 5–1 (c) log 4–1 (d) log 2–1

Q.20 The area of Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in Ist quadrant is

(a) $\frac{\pi ab}{4}$ (b) $\frac{\pi ab}{2}$ (c) $\frac{3\pi ab}{4}$ (d) $\frac{3\pi ab}{2}$

Q.21 The area enclosed between the curve of function $y = \frac{4}{x^2}$, the x-axis and the ordinates x=1 and x=3 is

(a) $\frac{2}{3}$ (b) $\frac{4}{3}$ (c) $\frac{1}{2}$ (d) $\frac{8}{3}$



Q.22 The following curve shows the graph of y = cosx [0 ≤ x ≤ π]. The area of shaded region will be

(a) 2 (b) 0 (c) 3 (d) 2.5