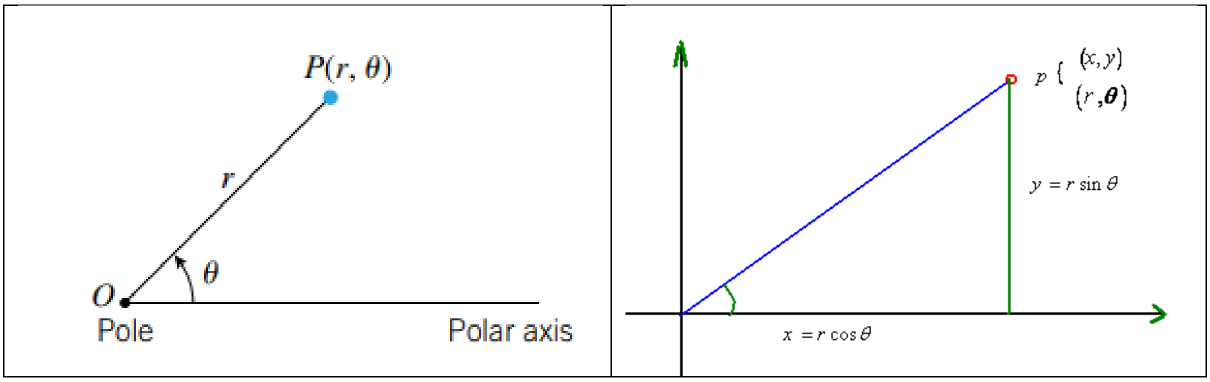
Analytic geometry

1 polar coordinates

A ***polar coordinate system*** in a plane consists of a fixed point O, called the ***pole*** (or ***origin***), and a ray emanating from the pole, called the ***polar axis***. In such a coordinate system we can associate with each point P in the plane a pair of ***polar coordinates*** (r, θ), where r is the distance from P to the pole and θ is an angle from the polar axis to the ray OP (Figure 11.1.1). The number r is called the ***radial coordinate*** of P and the number θ the ***angular coordinate*** (or ***polar angle***) of P. In Figure 11.1.2, the points $(6, 45^\circ)$, $(5, 120^\circ)$, $(3, 225^\circ)$, and $(4, 330^\circ)$ are plotted in polar coordinate systems. If P is the pole, then $r = 0$, but there is no clearly defined polar angle. We will agree that an arbitrary angle can be used in this case; that is, $(0, \theta)$ are polar coordinates of the pole for all choices of θ.



Relationship between polar and rectangular coordinates

Frequently, it will be useful to superimpose a rectangular xy-coordinate system on top of a polar coordinate system, making the positive x-axis coincide with the polar axis. If this is done, then every point P will have both rectangular coordinates (x, y) and polar coordinates (r, θ). As suggested by Figure 11.1.5, these coordinates are related by the equations

$$x = r\cos\theta, \quad y = r\sin\theta \tag{1}$$

These equations are well suited for finding x and y when r and θ are known. However, to find r and θ when x and y are known, it is preferable to use the identities $\sin^2\theta + \cos^2\theta = 1$ and $\tan\theta = \sin\theta / \cos\theta$ to rewrite (1) as

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x} \tag{2}$$

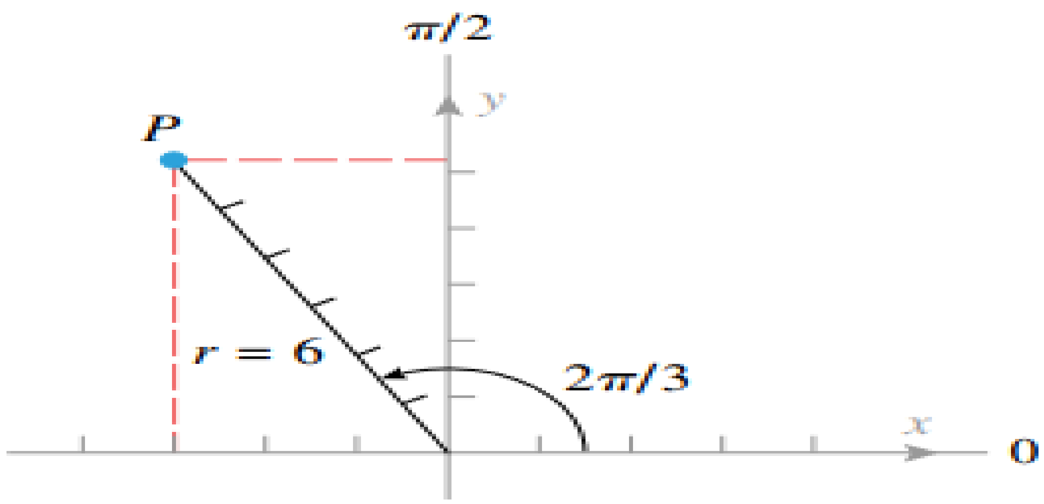
Example 1 Find the rectangular coordinates of the point P whose polar coordinates are $(6, 2\pi/3)$.

Solution. Substituting the polar coordinates $r = 6$ and $\theta = 2\pi/3$ in (1) yields

$$x = 6\cos\frac{2\pi}{3} = 6\left(-\frac{1}{2}\right) = -3$$

$$y = 6\sin\frac{2\pi}{3} = 6\left(\frac{\sqrt{3}}{2}\right) = 3\sqrt{3}$$

Thus, the rectangular coordinates of P are $(-3, 3\sqrt{3})$ (Figure 11.1.6). ◀



Example 2 Find polar coordinates of the point P whose rectangular coordinates are $(-2, 2\sqrt{3})$.

Solution. We will find the polar coordinates (r, θ) of P that satisfy the conditions $r > 0$ and $0 \le \theta < 2\pi$. From the first equation in (2),

$$r^2 = x^2 + y^2 = (-2)^2 + (2\sqrt{3})^2 = 4 + 12 = 16$$

so $r = 4$. From the second equation in (2),

$$\tan\theta = \frac{y}{x} = \frac{2\sqrt{3}}{-2} = -\sqrt{3}$$

From this and the fact that $(-2, 2\sqrt{3})$ lies in the second quadrant, it follows that the angle satisfying the requirement $0 \le \theta < 2\pi$ is $\theta = 2\pi/3$. Thus, $(4, 2\pi/3)$ are polar coordinates of P. All other polar coordinates of P are expressible in the form

$$\left(4, \frac{2\pi}{3} + 2n\pi\right) \quad \text{or} \quad \left(-4, \frac{5\pi}{3} + 2n\pi\right)$$

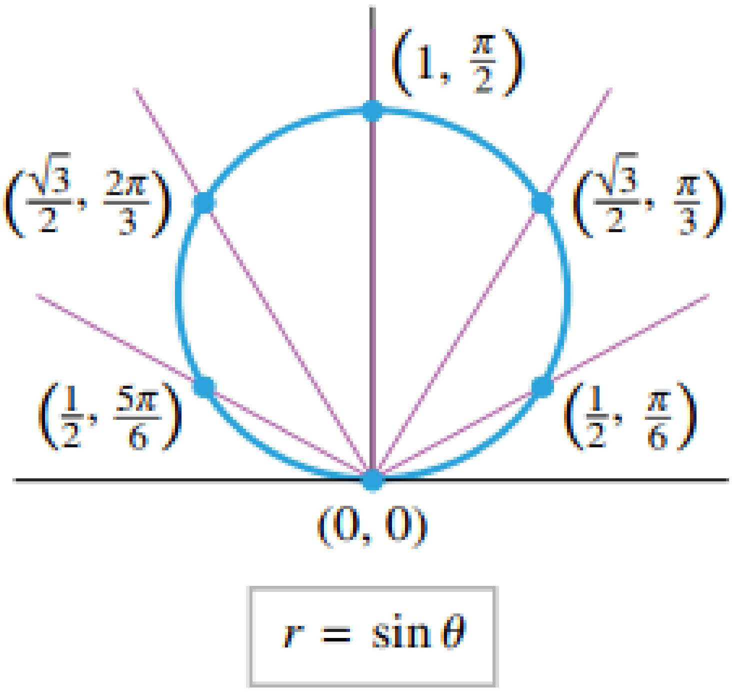
where n is an integer. ◀

Graphs in polar coordinates

We will now consider the problem of graphing equations of the form $r = f(\theta)$ in polar coordinates, where θ is assumed to be measured in radians.

Example 3 Sketch the graph of the equation $r = \sin\theta$ in polar coordinates by plotting points.

θ (RADIANS)	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$r = \sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0
(r, θ)	$(0, 0)$	$\left(\frac{1}{2}, \frac{\pi}{6}\right)$	$\left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$	$\left(1, \frac{\pi}{2}\right)$	$\left(\frac{\sqrt{3}}{2}, \frac{2\pi}{3}\right)$	$\left(\frac{1}{2}, \frac{5\pi}{6}\right)$	$(0, \pi)$	$\left(-\frac{1}{2}, \frac{7\pi}{6}\right)$	$\left(-\frac{\sqrt{3}}{2}, \frac{4\pi}{3}\right)$	$\left(-1, \frac{3\pi}{2}\right)$	$\left(-\frac{\sqrt{3}}{2}, \frac{5\pi}{3}\right)$	$\left(-\frac{1}{2}, \frac{11\pi}{6}\right)$	$(0, 2\pi)$



Observe that the points in Figure appear to lie on a circle. We can confirm that this is so by expressing the polar equation $r = \sin\theta$ in terms of x and y. To do this, we multiply the equation through by r to obtain

$$r^2 = r\sin\theta$$

which now allows us to apply Formulas (1) and (2) to rewrite the equation as

$$x^2 + y^2 = y$$

Rewriting this equation as $x^2 + y^2 - y = 0$ and then completing the square yields

$$x^2 + \left(y - \tfrac{1}{2}\right)^2 = \tfrac{1}{4}$$

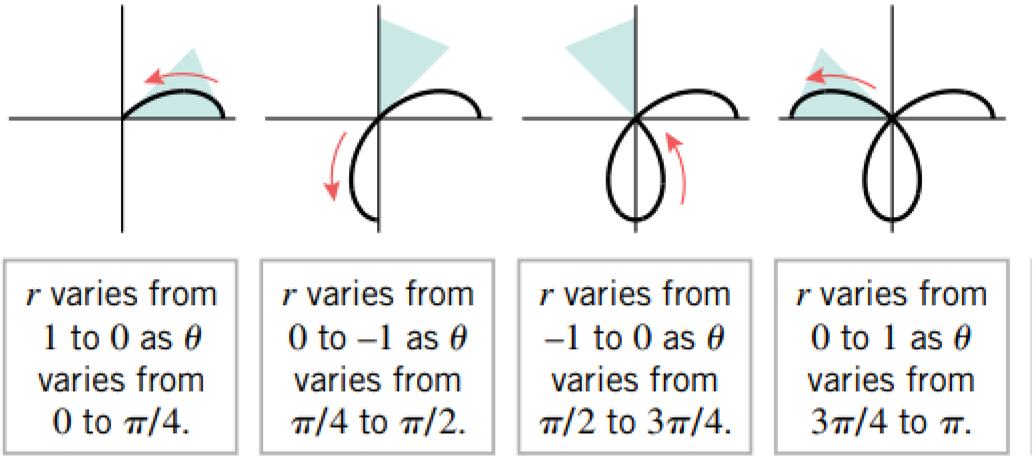
which is a circle of radius $\frac{1}{2}$ centered at the point $\left(0, \frac{1}{2}\right)$ in the xy-plane.

[Q] Transform the polar equation $r = sin\theta$ into Cartesian coordinates.

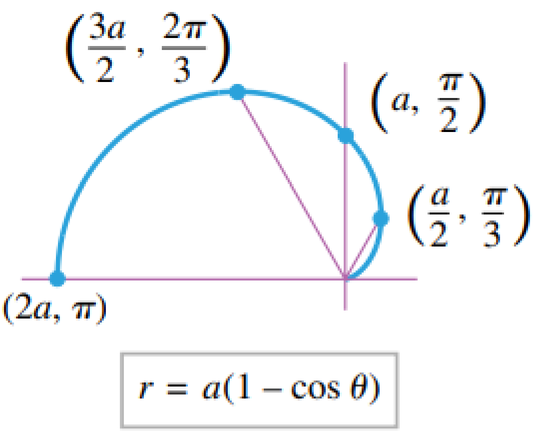
==

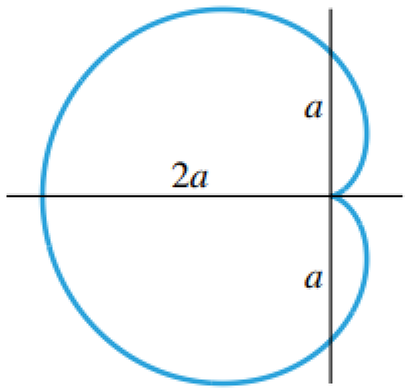
Exercises

1-Sketch the graph of r = cos 2θ in polar coordinates, θ∈ $[0, \pi]$.

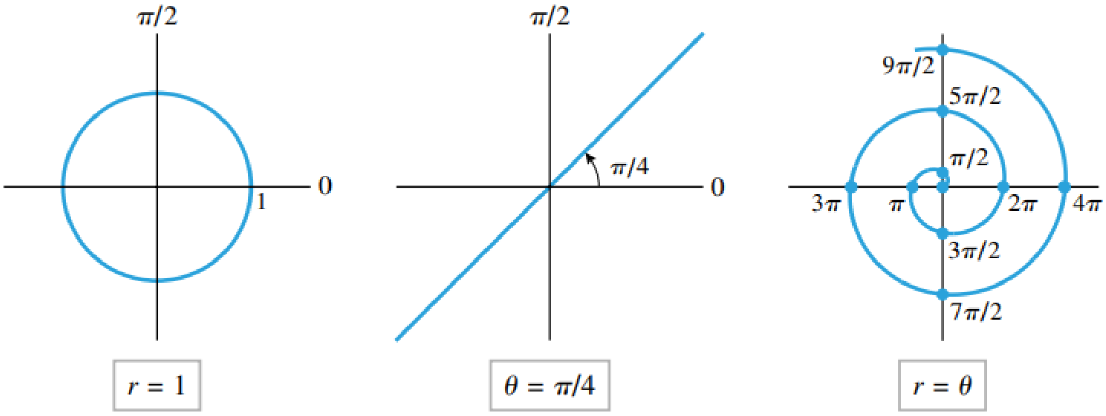


2-Sketch the graph of r = a(1 − cos θ) in polar coordinates, assuming a to be a positive constant, θ∈ $[0, \pi]$.





3-Sketch the curves (a) r = 1 (b) θ = π/4 (c) r = θ (θ ≥ 0) in polar coordinates.



2 tangent lines and arc length

for parametric and polar curves

Tangent lines to parametric curves

We will be concerned in this section with curves that are given by parametric equations

$$x = f(t), \quad y = g(t)$$

in which $f(t)$ and $g(t)$ have continuous first derivatives with respect to t. It can be proved that if $dx/dt \neq 0$, then y is a differentiable function of x, in which case the chain rule

implies that

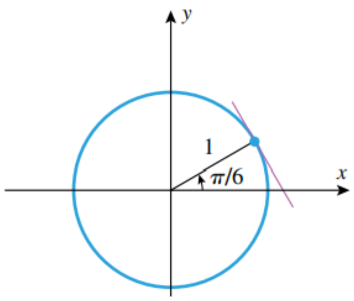
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \tag{1}$$

This formula makes it possible to find dy/dx directly from the parametric equations without eliminating the parameter.

Example 1 Find the slope of the tangent line to the unit circle

$$x = \cos t, \quad y = \sin t \qquad (0 \leq t \leq 2\pi)$$

at the point where $t = \pi/6$



Solution. From (1), the slope at a general point on the circle is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{-\sin t} = -\cot t \tag{2}$$

Thus, the slope at $t = \pi/6$ is

$$\left.\frac{dy}{dx}\right|_{t=\pi/6} = -\cot\frac{\pi}{6} = -\sqrt{3}$$ ◀

Example 2 In a disastrous first flight, an experimental paper airplane follows the trajectory

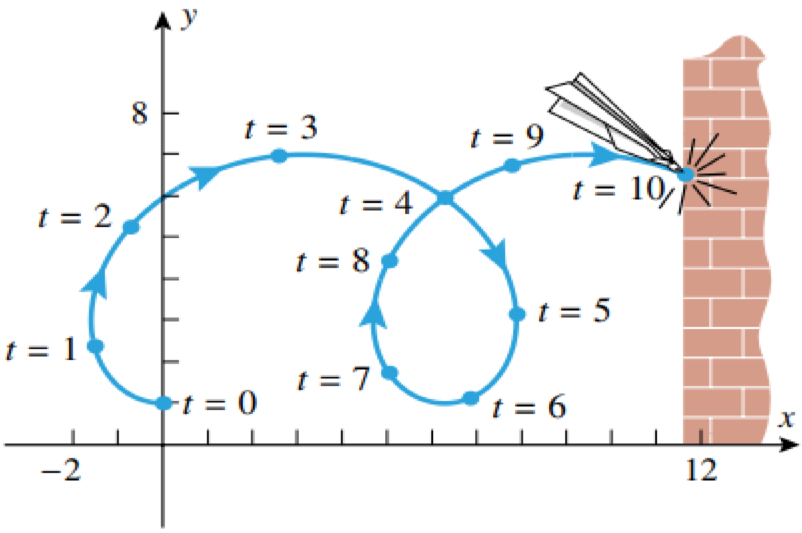
$$x = t - 3\sin t, \quad y = 4 - 3\cos t \qquad (t \geq 0)$$

but crashes into a wall at time $t = 10$

(a) At what times was the airplane flying horizontally?

(b) At what times was it flying vertically?

[Q]A butterfly flies between branches, following the path described by the two parametric equations



Solution (a). The airplane was flying horizontally at those times when $dy/dt = 0$ and $dx/dt \neq 0$. From the given trajectory we have

$$\frac{dy}{dt} = 3\sin t \quad \text{and} \quad \frac{dx}{dt} = 1 - 3\cos t \tag{3}$$

Setting $dy/dt = 0$ yields the equation $3\sin t = 0$, or, more simply, $\sin t = 0$. This equation has four solutions in the time interval $0 \leq t \leq 10$:

$$t = 0, \quad t = \pi, \quad t = 2\pi, \quad t = 3\pi$$

Since $dx/dt = 1 - 3\cos t \neq 0$ for these values of t (verify), the airplane was flying horizontally at times

$$t = 0, \quad t = \pi \approx 3.14, \quad t = 2\pi \approx 6.28, \quad \text{and} \quad t = 3\pi \approx 9.42$$

Solution (b). The airplane was flying vertically at those times when $dx/dt = 0$ and $dy/dt \neq 0$. Setting $dx/dt = 0$ in (3) yields the equation

$$1 - 3\cos t = 0 \quad \text{or} \quad \cos t = \tfrac{1}{3}$$

This equation has three solutions in the time interval $0 \leq t \leq 10$ (Figure 11.2.4):

$$t = \cos^{-1}\tfrac{1}{3}, \quad t = 2\pi - \cos^{-1}\tfrac{1}{3}, \quad t = 2\pi + \cos^{-1}\tfrac{1}{3}$$

Since $dy/dt = 3\sin t$ is not zero at these points (why?), it follows that the airplane was flying vertically at times

$$t = \cos^{-1}\tfrac{1}{3} \approx 1.23, \quad t \approx 2\pi - 1.23 \approx 5.05, \quad t \approx 2\pi + 1.23 \approx 7.51$$

Tangent lines to polar curves

Our next objective is to find a method for obtaining slopes of tangent lines to polar curves of the form $r = f(\theta)$ in which r is a differentiable function of θ. We showed in the last section that a curve of this form can be expressed parametrically in terms of the parameter θ by substituting $f(\theta)$ for r in the equations $x = r\cos\theta$ and $y = r\sin\theta$. This yields

$$x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta$$

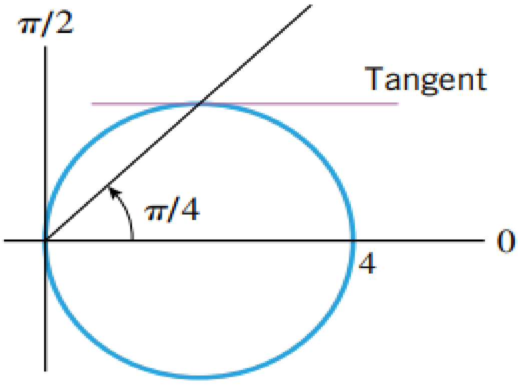
from which we obtain

$$\begin{aligned} \frac{dx}{d\theta} &= -f(\theta)\sin\theta + f'(\theta)\cos\theta = -r\sin\theta + \frac{dr}{d\theta}\cos\theta \\ \frac{dy}{d\theta} &= f(\theta)\cos\theta + f'(\theta)\sin\theta = r\cos\theta + \frac{dr}{d\theta}\sin\theta \end{aligned} \tag{6}$$

Thus, if $dx/d\theta$ and $dy/d\theta$ are continuous and if $dx/d\theta \neq 0$, then y is a differentiable function of x, and Formula (1) with θ in place of t yields

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\theta + \sin\theta\dfrac{dr}{d\theta}}{-r\sin\theta + \cos\theta\dfrac{dr}{d\theta}} \tag{7}$$

Example 5 Find the slope of the tangent line to the circle $r = 4\cos\theta$ at the point where $\theta = \pi/4$.



$r = 4\cos\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\theta + \sin\theta\dfrac{dr}{d\theta}}{-r\sin\theta + \cos\theta\dfrac{dr}{d\theta}}$$

Solution. From (7) with $r = 4\cos\theta$ we obtain (verify)

$$\frac{dy}{dx} = \frac{4\cos^2\theta - 4\sin^2\theta}{-8\sin\theta\cos\theta} = \frac{4\cos 2\theta}{-4\sin 2\theta} = -\cot 2\theta$$

Thus, at the point where $\theta = \pi/4$ the slope of the tangent line is

$$m = \left.\frac{dy}{dx}\right|_{\theta=\pi/4} = -\cot\frac{\pi}{2} = 0$$

which implies that the circle has a horizontal tangent line at the point where $\theta = \pi/4$

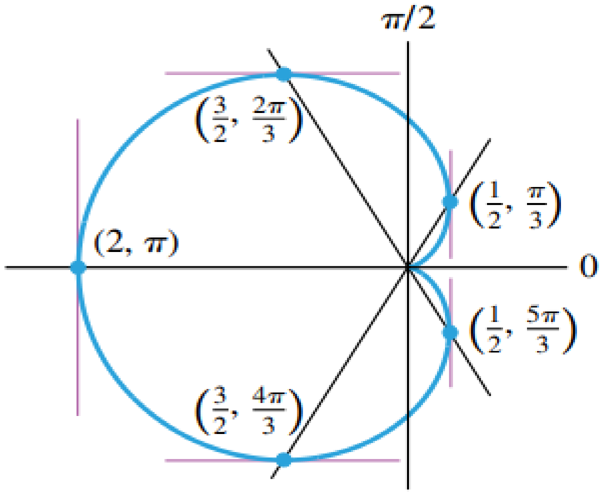
Example 6 Find the points on the cardioid $r = 1 - \cos\theta$ at which there is a horizontal tangent line, a vertical tangent line, or a singular point.

Solution. A horizontal tangent line will occur where $dy/d\theta = 0$ and $dx/d\theta \neq 0$, a vertical tangent line where $dy/d\theta \neq 0$ and $dx/d\theta = 0$, and a singular point where $dy/d\theta = 0$ and $dx/d\theta = 0$. We could find these derivatives from the formulas in (6). However, an alternative approach is to go back to basic principles and express the cardioid parametrically by substituting $r = 1 - \cos\theta$ in the conversion formulas $x = r\cos\theta$ and $y = r\sin\theta$. This yields

$$x = (1 - \cos\theta)\cos\theta, \quad y = (1 - \cos\theta)\sin\theta \qquad (0 \le \theta \le 2\pi)$$

Differentiating these equations with respect to θ and then simplifying yields (verify)

$$\frac{dx}{d\theta} = \sin\theta(2\cos\theta - 1), \quad \frac{dy}{d\theta} = (1 - \cos\theta)(1 + 2\cos\theta)$$



$r = 1 - \cos\theta$

Thus, $dx/d\theta = 0$ if $\sin\theta = 0$ or $\cos\theta = \frac{1}{2}$, and $dy/d\theta = 0$ if $\cos\theta = 1$ or $\cos\theta = -\frac{1}{2}$. We leave it for you to solve these equations and show that the solutions of $dx/d\theta = 0$ on the interval $0 \le \theta \le 2\pi$ are

$$\frac{dx}{d\theta} = 0: \quad \theta = 0, \ \frac{\pi}{3}, \ \pi, \ \frac{5\pi}{3}, \ 2\pi$$

and the solutions of $dy/d\theta = 0$ on the interval $0 \le \theta \le 2\pi$ are

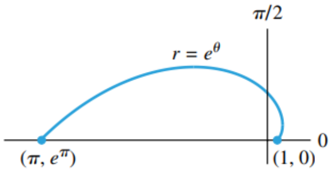
$$\frac{dy}{d\theta} = 0: \quad \theta = 0, \ \frac{2\pi}{3}, \ \frac{4\pi}{3}, \ 2\pi$$

Thus, horizontal tangent lines occur at $\theta = 2\pi/3$ and $\theta = 4\pi/3$; vertical tangent lines occur at $\theta = \pi/3$, π, and $5\pi/3$; and singular points occur at $\theta = 0$ and $\theta = 2\pi$
Note, however, that $r = 0$ at both singular points, so there is really only one singular point on the cardioid—the pole. ◀

Arc length of a polar curve

If no segment of the polar curve $r = f(\theta)$ is traced more than once as θ increases from α to β, and if $dr/d\theta$ is continuous for $\alpha \le \theta \le \beta$, then the arc length L from $\theta = \alpha$ to $\theta = \beta$ is

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad (8)$$

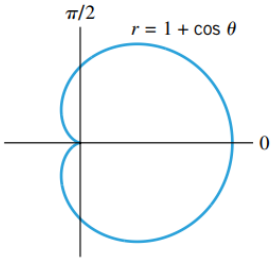


Example 8 Find the arc length of the spiral $r = e^{\theta}$ in between $\theta = 0$ and $\theta = \pi$.

Solution.

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{\pi} \sqrt{(e^{\theta})^2 + (e^{\theta})^2}\, d\theta$$

$$= \int_0^{\pi} \sqrt{2}\, e^{\theta}\, d\theta = \sqrt{2}\, e^{\theta}\Big]_0^{\pi} = \sqrt{2}(e^{\pi} - 1) \approx 31.3$$



Example 9 Find the total arc length of the cardioid $r = 1 + \cos\theta$.

Solution. The cardioid is traced out once as θ varies from $\theta = 0$ to $\theta = 2\pi$. Thus,

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{(1 + \cos\theta)^2 + (-\sin\theta)^2}\, d\theta$$

$$= \sqrt{2} \int_0^{2\pi} \sqrt{1 + \cos\theta}\, d\theta$$

$$= 2 \int_0^{2\pi} \sqrt{\cos^2 \frac{1}{2}\theta}\, d\theta$$ Identity (45) of Appendix E

$$= 2 \int_0^{2\pi} \left|\cos \frac{1}{2}\theta\right| d\theta$$

Since $\cos \frac{1}{2}\theta$ changes sign at π, we must split the last integral into the sum of two integrals: the integral from 0 to π plus the integral from π to 2π. However, the integral from π to 2π is equal to the integral from 0 to π, since the cardioid is symmetric about the polar axis Thus,

$$L = 2 \int_0^{2\pi} \left|\cos \frac{1}{2}\theta\right| d\theta = 4 \int_0^{\pi} \cos \frac{1}{2}\theta\, d\theta = 8 \sin \frac{1}{2}\theta\Big]_0^{\pi} = 8$$ ◀

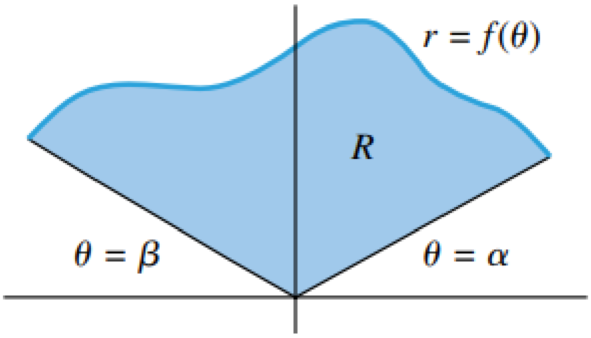
3 area in polar coordinates

Area in polar coordinates

Suppose that α and β are angles that satisfy the condition

$$\alpha < \beta \leq \alpha + 2\pi$$

and suppose that $f(\theta)$ is continuous for $\alpha \leq \theta \leq \beta$. Find the area of the region R enclosed by the polar curve $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$

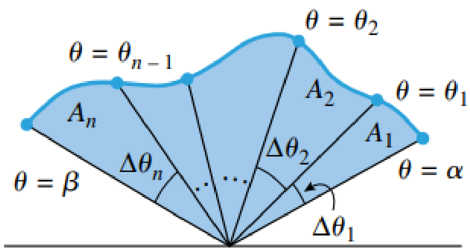


In polar coordinates it is better to divide the region into ***wedges*** by using rays

$$\theta = \theta_1,\ \theta = \theta_2, \ldots,\ \theta = \theta_{n-1}$$

such that

$$\alpha < \theta_1 < \theta_2 < \cdots < \theta_{n-1} < \beta$$

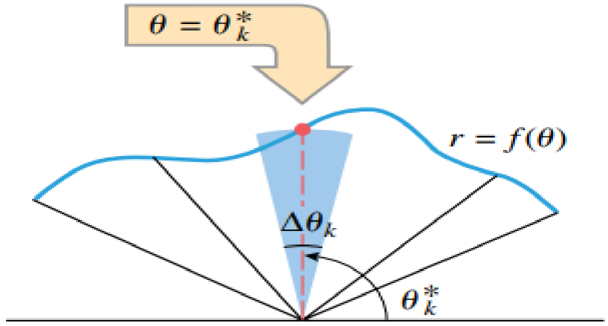


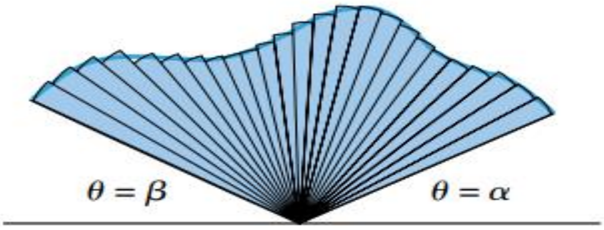
As shown in that figure, the rays divide the region R into n wedges with areas $A_1, A_2, \ldots, A_n$ and central angles $\Delta\theta_1, \Delta\theta_2, \ldots, \Delta\theta_n$. The area of the entire region can be written as

$$A = A_1 + A_2 + \cdots + A_n = \sum_{k=1}^{n} A_k \tag{1}$$

If $\Delta\theta_k$ is small, and if we assume for simplicity that $f(\theta)$ is nonnegative, then we can approximate the area A_k of the kth wedge by the area of a sector with central angle $\Delta\theta_k$ and radius $f(\theta_k^*)$, where $\theta = \theta_k^*$ is any ray that lies in the kth wedge Thus, from (1) and Formula (5) of Appendix E for the area of a sector, we obtain

$$A = \sum_{k=1}^{n} A_k \approx \sum_{k=1}^{n} \frac{1}{2}[f(\theta_k^*)]^2 \Delta\theta_k \tag{2}$$





If we now increase n in such a way that max $\Delta\theta_k \to 0$, then the sectors will become better and better approximations of the wedges and it is reasonable to expect that (2) will approach the exact value of the area A ; that is,

$$A = \lim_{\max \Delta\theta_k \to 0} \sum_{k=1}^{n} \frac{1}{2}[f(\theta_k^*)]^2 \Delta\theta_k = \int_{\alpha}^{\beta} \frac{1}{2}[f(\theta)]^2 \, d\theta$$

If α and β are angles that satisfy the condition

$$\alpha < \beta \leq \alpha + 2\pi$$

and if $f(\theta)$ is continuous for $\alpha \leq \theta \leq \beta$, then the area A of the region R enclosed by the polar curve $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$ is

$$A = \int_{\alpha}^{\beta} \frac{1}{2}[f(\theta)]^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta \qquad (3)$$

The hardest part of applying (3) is determining the limits of integration. This can be done as follows:

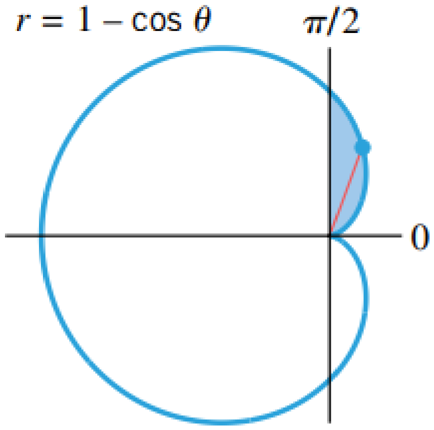
Step 1. Sketch the region R whose area is to be determined.

Step 2. Draw an arbitrary "radial line" from the pole to the boundary curve $r = f(\theta)$.

Step 3. Ask, "Over what interval of values must θ vary in order for the radial line to sweep out the region R?"

Step 4. Your answer in Step 3 will determine the lower and upper limits of integration.

Example 1 Find the area of the region in the first quadrant that is within the cardioid $r = 1 - \cos\theta$.



The shaded region is swept out by the radial line as θ varies from 0 to $\pi/2$.

Solution. The region and a typical radial line are shown in Figure 11.3.5. For the radial line to sweep out the region, θ must vary from 0 to $\pi/2$. Thus, from (3) with $\alpha = 0$ and $\beta = \pi/2$, we obtain

$$A = \int_0^{\pi/2} \frac{1}{2}r^2\,d\theta = \frac{1}{2}\int_0^{\pi/2}(1-\cos\theta)^2\,d\theta = \frac{1}{2}\int_0^{\pi/2}(1 - 2\cos\theta + \cos^2\theta)\,d\theta$$

With the help of the identity $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$, this can be rewritten as

$$A = \frac{1}{2}\int_0^{\pi/2}\left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta = \frac{1}{2}\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} = \frac{3}{8}\pi - 1 \quad ◀$$

Example 2 Find the entire area within the cardioid of Example 1.

Solution. For the radial line to sweep out the entire cardioid, θ must vary from 0 to 2π. Thus, from (3) with $\alpha = 0$ and $\beta = 2\pi$,

$$A = \int_0^{2\pi} \frac{1}{2}r^2\,d\theta = \frac{1}{2}\int_0^{2\pi}(1-\cos\theta)^2\,d\theta$$

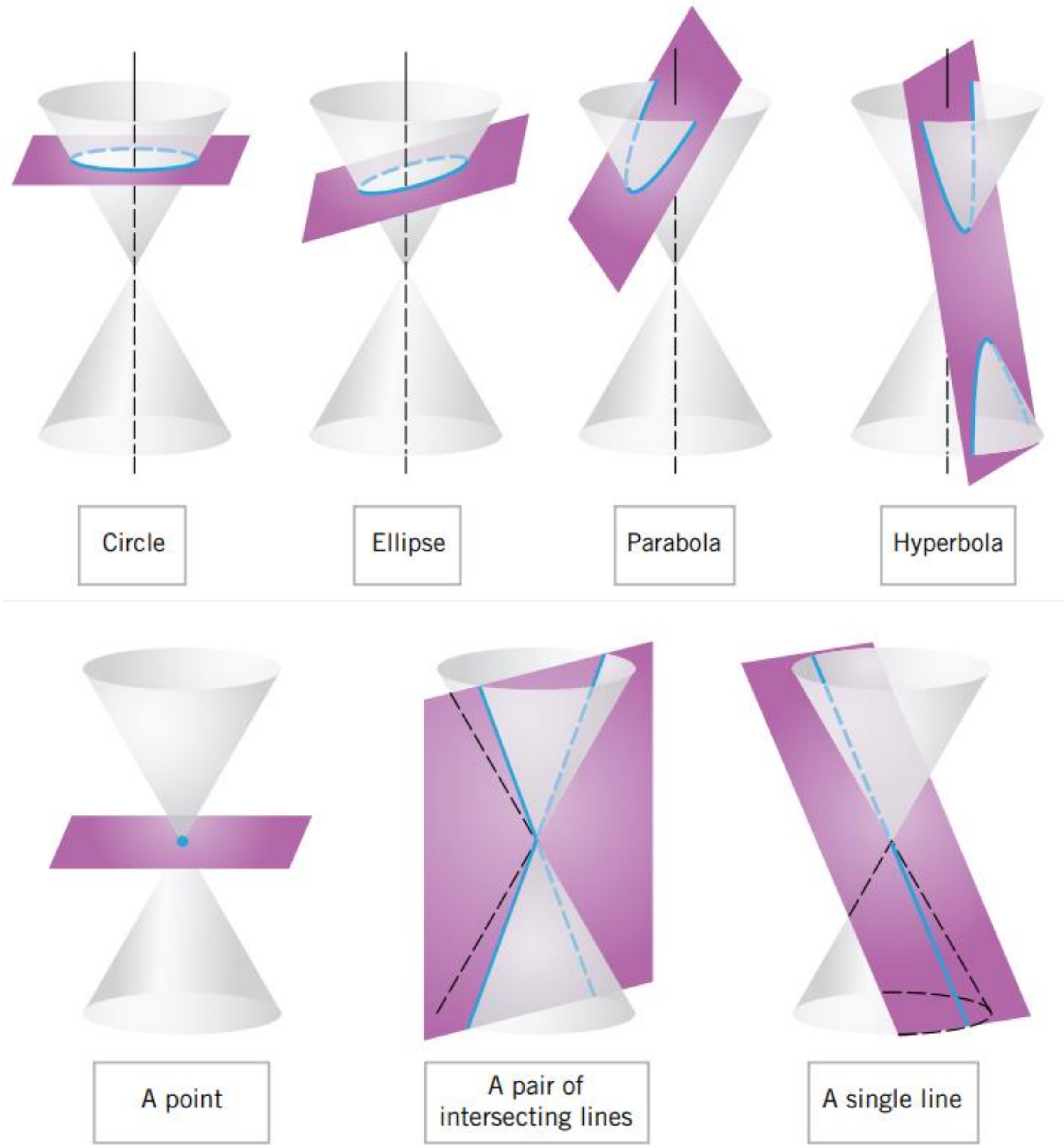
If we proceed as in Example 1, this reduces to

$$A = \frac{1}{2}\int_0^{2\pi}\left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta = \frac{3\pi}{2}$$

4 conic sections in calculus

Conic sections

Circles, ellipses, parabolas, and hyperbolas are called ***conic sections*** or ***conics*** because they can be obtained as intersections of a plane with a double-napped circular cone. If the plane passes through the vertex of the double-napped cone, then the intersection is a point, a pair of intersecting lines, or a single line. These are called ***degenerate conic sections***.

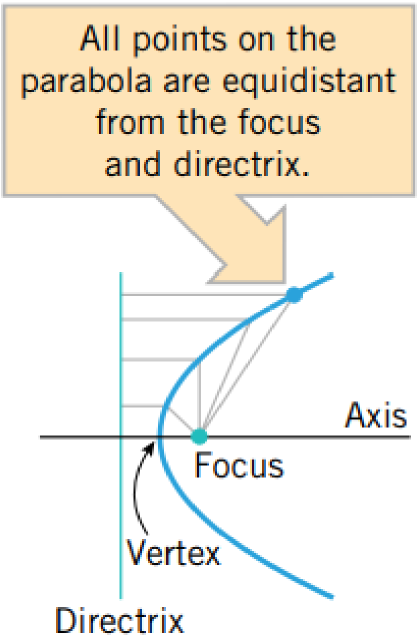


Definitions of the conic sections

DEFINITION. A ***parabola*** is the set of all points in the plane that are equidistant from a fixed line and a fixed point not on the line.

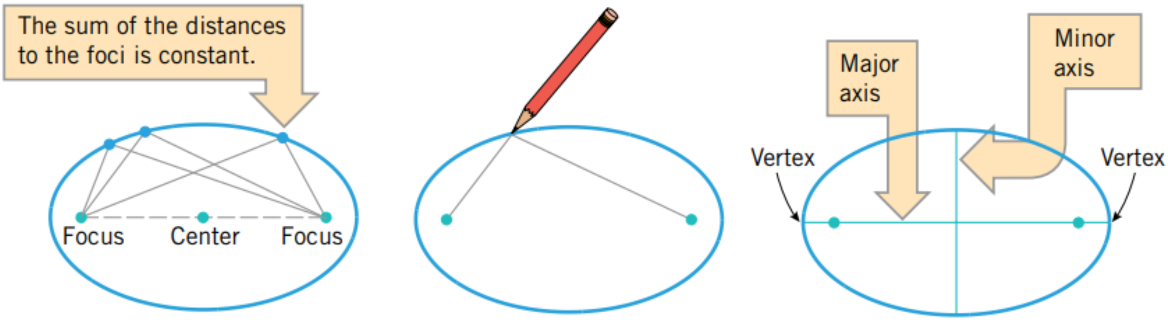
The line is called the ***directrix*** of the parabola, and the point is called the ***focus***

A parabola is symmetric about the line that passes through the focus at right angles to the directrix. This line, called the ***axis*** or the ***axis of symmetry*** of the parabola, intersects the parabola at a point called the ***vertex***.



DEFINITION. An ***ellipse*** is the set of all points in the plane, the sum of whose distances from two fixed points is a given positive constant that is greater than the distance between the fixed points.

The two fixed points are called the ***foci*** (plural of "focus") of the ellipse, and the midpoint of the line segment joining the foci is called the ***center*** (Figure *a*). To help visualize Definition 2, imagine that two ends of a string are tacked to the foci and a pencil traces a curve as it is held tight against the string (Figure *b*). The resulting curve will be an ellipse since the sum of the distances to the foci is a constant, namely the total length of the string. Note that if the foci coincide, the ellipse reduces to a circle. For ellipses other than circles, the line segment through the foci and across the ellipse is called the ***major axis*** (Figure *c*), and the line segment across the ellipse, through the center, and perpendicular to the major axis is called the ***minor axis***. The endpoints of the major axis are called ***vertices***.



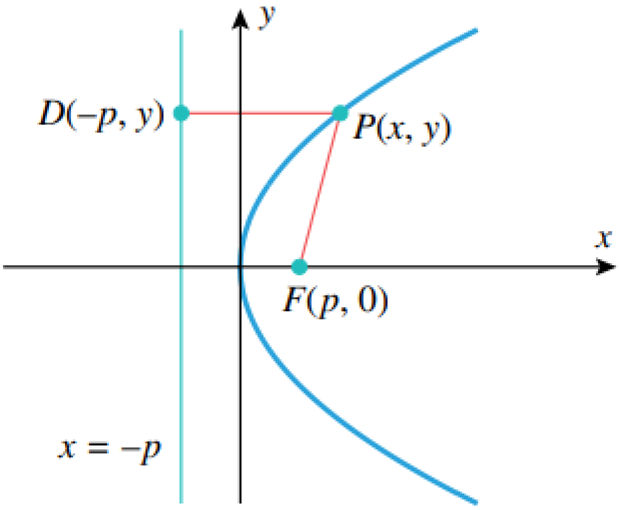
(a) (b) (c)

DEFINITION. A ***hyperbola*** is the set of all points in the plane, the difference of whose distances from two fixed distinct points is a given positive constant that is less than the distance between the fixed points.

Equations of parabolas in standard position

To illustrate how the equations are obtained, we will derive the equation for the parabola with focus $(p, 0)$ and directrix $x = -p$. Let $P(x, y)$ be any point on the parabola. Since P is equidistant from the focus and directrix, the distances PF and PD in Figure are equal; that is,

$$PF = PD \tag{1}$$



where $D(-p, y)$ is the foot of the perpendicular from P to the directrix. From the distance formula, the distances PF and PD are

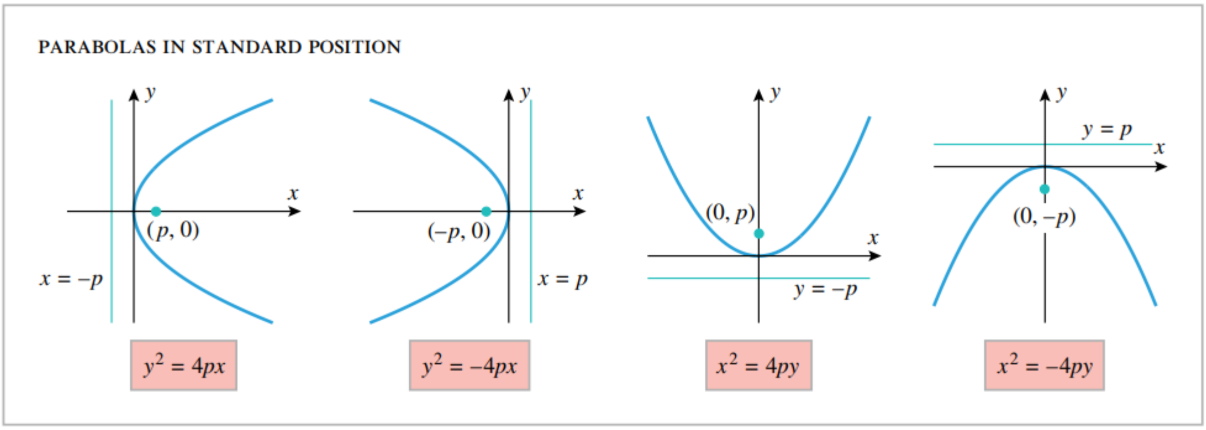
$$PF = \sqrt{(x-p)^2 + y^2} \quad \text{and} \quad PD = \sqrt{(x+p)^2} \tag{2}$$

Substituting in (1) and squaring yields

$$(x-p)^2 + y^2 = (x+p)^2 \tag{3}$$

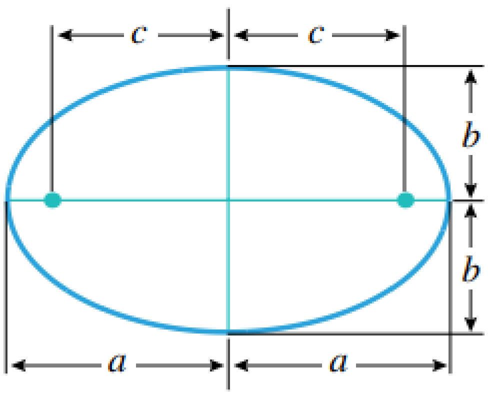
and after simplifying

$$y^2 = 4px$$



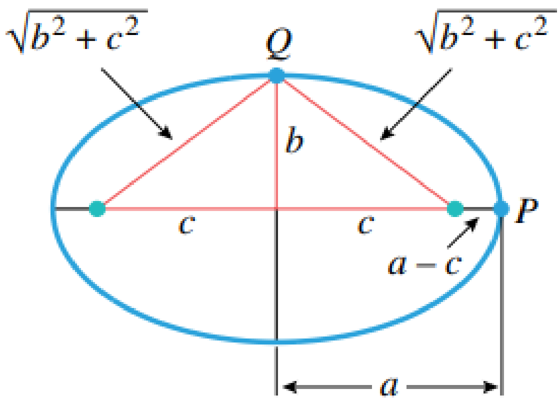
Equations of ellipses in standard position

It is traditional in the study of ellipses to denote the length of the major axis by $2a$, the length of the minor axis by $2b$, and the distance between the foci by $2c$ (Figure
The number a is called the ***semimajor axis*** and the number b the ***semiminor axis*** (standard but odd terminology, since a and b are numbers, not geometric axes).



There is a basic relationship between the numbers a, b, and c that can be obtained by examining the sum of the distances to the foci from a point P at the end of the major axis and from a point Q at the end of the minor axis (Figure From Definition 2, these sums must be equal, so we obtain

$$2\sqrt{b^2+c^2} = (a-c)+(a+c)$$



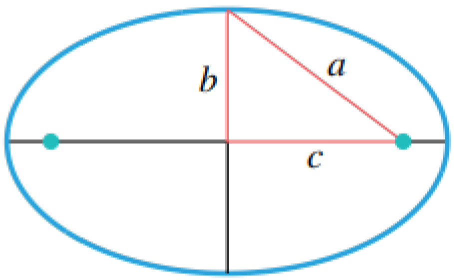
from which it follows that

$$a = \sqrt{b^2+c^2} \tag{6}$$

or, equivalently,

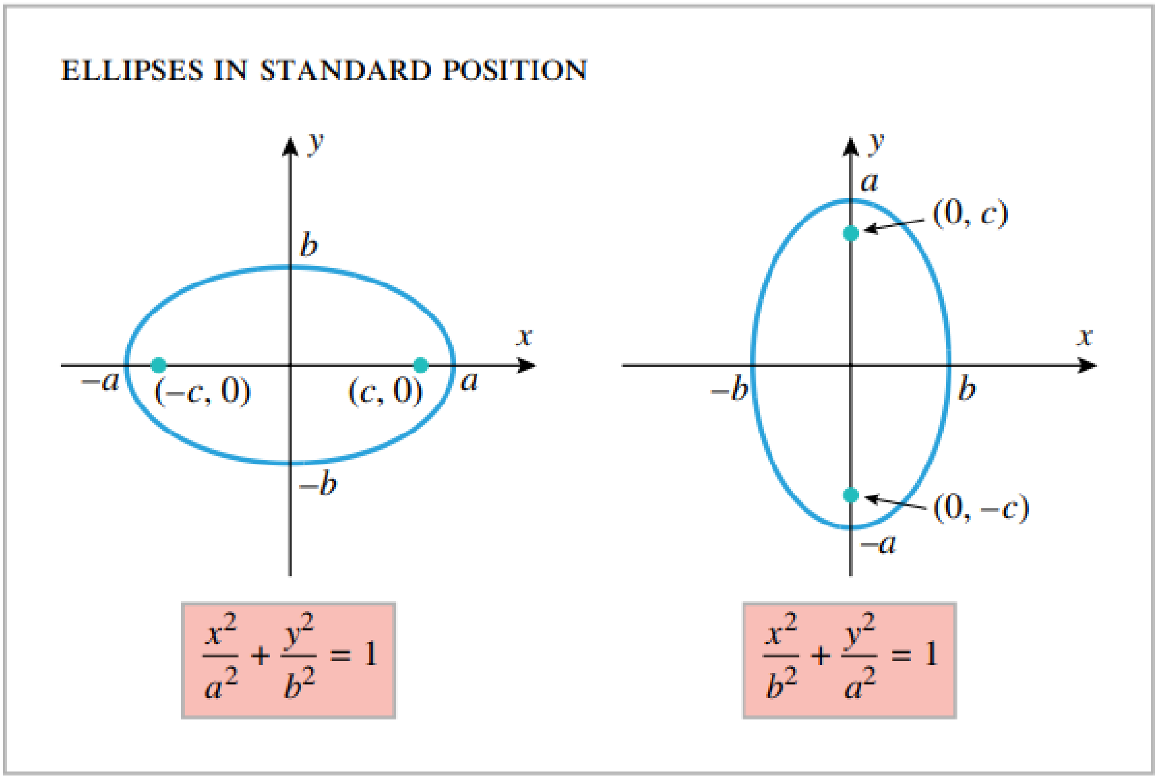
$$c = \sqrt{a^2-b^2} \tag{7}$$

From (6), the distance from a focus to an end of the minor axis is a which implies that for *all* points on the ellipse the sum of the distances to the foci is $2a$.



It also follows from (6) that $a \geq b$ with the equality holding only when $c = 0$. Geometrically, this means that the major axis of an ellipse is at least as large as the minor axis and that the two axes have equal length only when the foci coincide, in which case the ellipse is a circle.

The equation of an ellipse is simplest if the center of the ellipse is at the origin and the foci are on the x-axis or y-axis. The two possible such orientations are shown in Figure These are called the ***standard positions*** of an ellipse, and the resulting equations are called the ***standard equations*** of an ellipse.

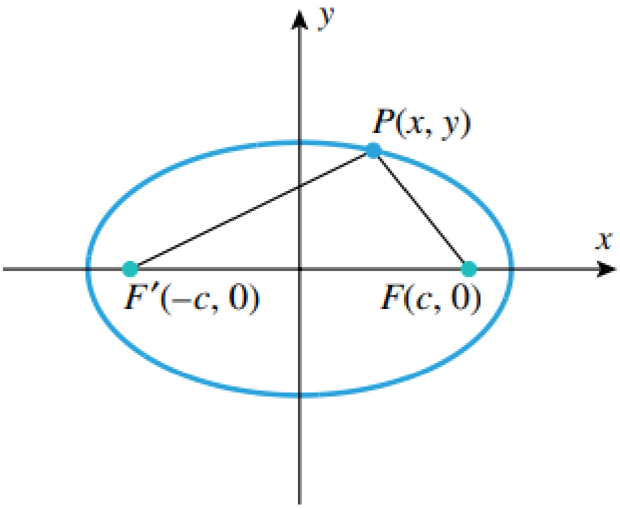


we will derive the equation for the ellipse with foci on the x-axis. Let $P(x, y)$ be any point on that ellipse. Since the sum of the distances from P to the foci is $2a$, it follows that

$$PF' + PF = 2a$$

so

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$



Transposing the second radical to the right side of the equation and squaring yields

$$(x + c)^2 + y^2 = 4a^2 - 4a\sqrt{(x - c)^2 + y^2} + (x - c)^2 + y^2$$

and, on simplifying,

$$\sqrt{(x - c)^2 + y^2} = a - \frac{c}{a}x \tag{8}$$

Squaring again and simplifying yields

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

which, by virtue of (6), can be written as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{9}$$

Conversely, it can be shown that any point whose coordinates satisfy (9) has $2a$ as the sum of its distances from the foci, so that such a point is on the ellipse.

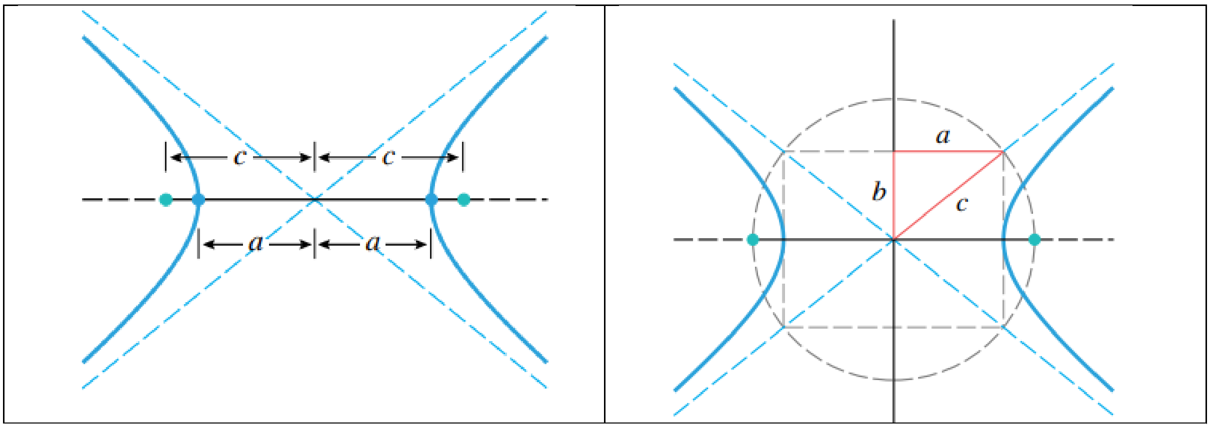
Equations of hyperbolas in standard position

It is traditional in the study of hyperbolas to denote the distance between the vertices by $2a$, the distance between the foci by $2c$ (, and to define the quantity b as

$$b = \sqrt{c^2 - a^2} \tag{10}$$

This relationship, which can also be expressed as

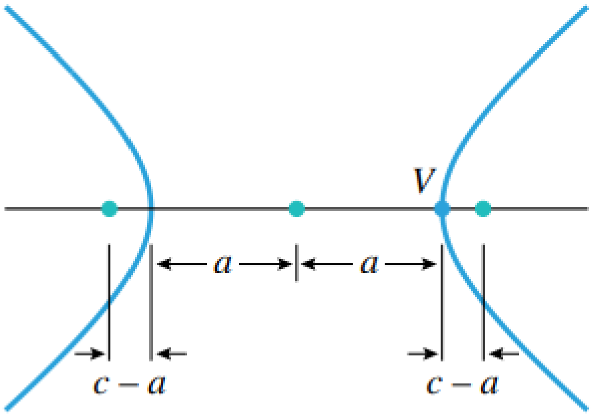
$$c = \sqrt{a^2 + b^2} \tag{11}$$



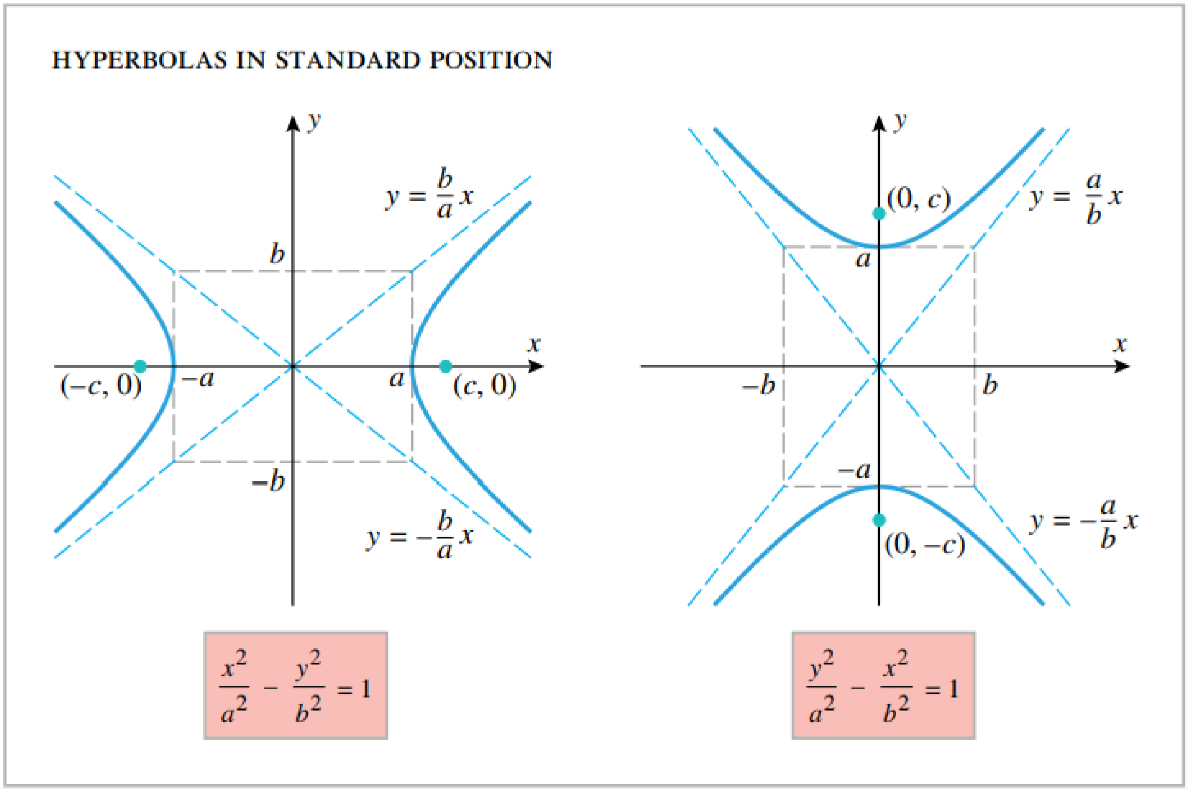
the asymptotes pass through the corners of a box extending b units on each side of the center along the conjugate axis and a units on each side of the center along the focal axis. The number a is called the ***semifocal axis*** of the hyperbola and the number b the ***semiconjugate axis***. (As with the semimajor and semiminor axes of an ellipse, these are numbers, not geometric axes).

If V is one vertex of a hyperbola, then, the distance from V to the farther focus minus the distance from V to the closer focus is

$$[(c - a) + 2a] - (c - a) = 2a$$



The equation of a hyperbola is simplest if the center of the hyperbola is at the origin and the foci are on the x-axis or y-axis. The two possible such orientations are shown in Figure These are called the ***standard positions*** of a hyperbola, and the resulting equations are called the ***standard equations*** of a hyperbola.



The derivations of these equations are similar to those already given for parabolas and ellipses, so we will leave them as exercises. However, to illustrate how the equations of the asymptotes are derived, we will derive those equations for the hyperbola

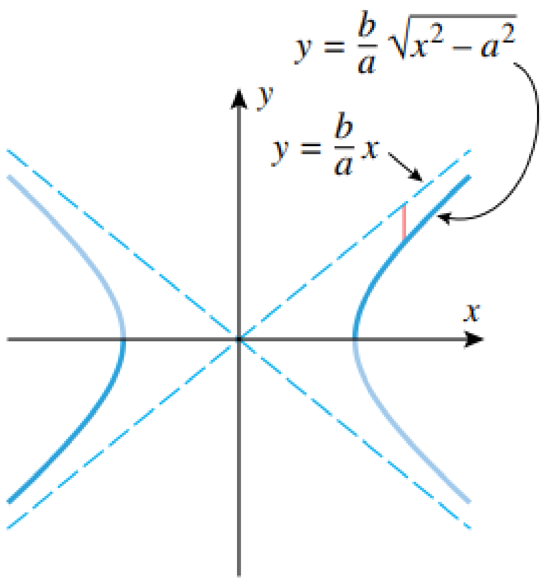
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

We can rewrite this equation as

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2)$$

which is equivalent to the pair of equations

$$y = \frac{b}{a}\sqrt{x^2 - a^2} \quad \text{and} \quad y = -\frac{b}{a}\sqrt{x^2 - a^2}$$



Thus, in the first quadrant, the vertical distance between the line $y = (b/a)x$ and the hyperbola can be written as

$$\frac{b}{a}x - \frac{b}{a}\sqrt{x^2 - a^2}$$

But this distance tends to zero as $x \to +\infty$ since

$$\lim_{x \to +\infty} \left(\frac{b}{a}x - \frac{b}{a}\sqrt{x^2 - a^2}\right) = \lim_{x \to +\infty} \frac{b}{a}(x - \sqrt{x^2 - a^2})$$

$$= \lim_{x \to +\infty} \frac{b}{a} \frac{(x - \sqrt{x^2 - a^2})(x + \sqrt{x^2 - a^2})}{x + \sqrt{x^2 - a^2}}$$

$$= \lim_{x \to +\infty} \frac{ab}{x + \sqrt{x^2 - a^2}} = 0$$

The analysis in the remaining quadrants is similar.

A quick way to find asymptotes

There is a trick that can be used to avoid memorizing the equations of the asymptotes of a hyperbola. They can be obtained, when needed, by substituting 0 for the 1 on the right side of the hyperbola equation, and then solving for y in terms of x. For example, for the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

we would write

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \quad \text{or} \quad y^2 = \frac{b^2}{a^2}x^2 \quad \text{or} \quad y = \pm\frac{b}{a}x$$

which are the equations for the asymptotes.

Translated conics

Equations of conics that are translated from their standard positions can be obtained by replacing x by $x - h$ and y by $y - k$ in their standard equations. For a parabola, this translates the vertex from the origin to the point (h, k); and for ellipses and hyperbolas, this translates the center from the origin to the point (h, k).

Parabolas with vertex* (h, k) *and axis parallel to x-axis

$$(y-k)^2 = 4p(x-h) \qquad \text{[Opens right]} \tag{12}$$

$$(y-k)^2 = -4p(x-h) \qquad \text{[Opens left]} \tag{13}$$

Parabolas with vertex* (h, k) *and axis parallel to y-axis

$$(x-h)^2 = 4p(y-k) \qquad \text{[Opens up]} \tag{14}$$

$$(x-h)^2 = -4p(y-k) \qquad \text{[Opens down]} \tag{15}$$

Ellipse with center* (h, k) *and major axis parallel to x-axis

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \quad [b \le a] \tag{16}$$

Ellipse with center* (h, k) *and major axis parallel to y-axis

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1 \quad [b \le a] \tag{17}$$

Hyperbola with center* (h, k) *and focal axis parallel to x-axis

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \tag{18}$$

Hyperbola with center* (h, k) *and focal axis parallel to y-axis

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1 \tag{19}$$

Example 7 Find an equation for the parabola that has its vertex at $(1, 2)$ and its focus at $(4, 2)$.

Solution. Since the focus and vertex are on a horizontal line, and since the focus is to the right of the vertex, the parabola opens to the right and its equation has the form

$$(y-k)^2 = 4p(x-h)$$

Since the vertex and focus are 3 units apart, we have $p = 3$, and since the vertex is at $(h, k) = (1, 2)$, we obtain

$$(y-2)^2 = 12(x-1)$$ ◀

================================

Sometimes the equations of translated conics occur in expanded form, in which case we are faced with the problem of identifying the graph of a quadratic equation in x and y:

$$Ax^2 + Cy^2 + Dx + Ey + F = 0 \qquad (20)$$

The basic procedure for determining the nature of such a graph is to complete the squares of the quadratic terms and then try to match up the resulting equation with one of the forms of a translated conic.

Example 8 Describe the graph of the equation

$$y^2 - 8x - 6y - 23 = 0$$

Solution. The equation involves quadratic terms in y but none in x, so we first take all of the y-terms to one side:

$$y^2 - 6y = 8x + 23$$

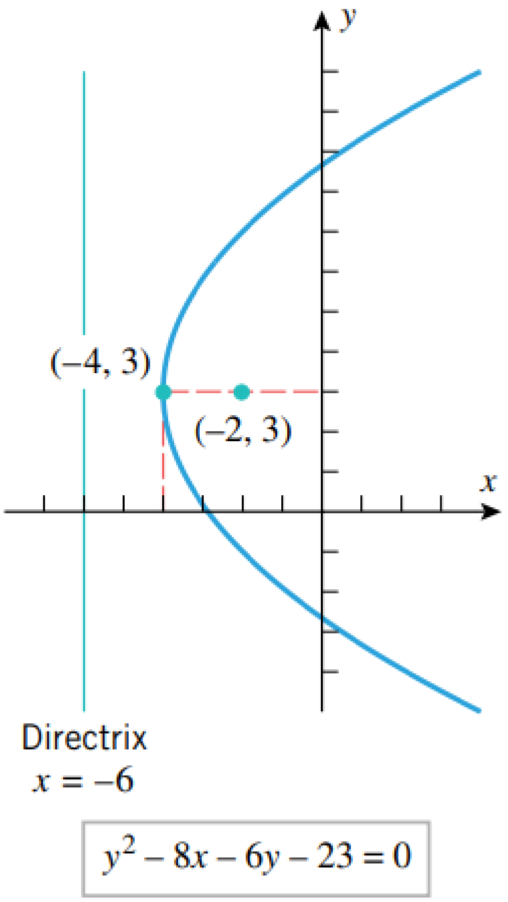
Next, we complete the square on the y-terms by adding 9 to both sides:

$$(y-3)^2 = 8x + 32$$

Finally, we factor out the coefficient of the x-term to obtain

$$(y-3)^2 = 8(x+4)$$

This equation is of form (12) with $h = -4$, $k = 3$, and $p = 2$, so the graph is a parabola with vertex $(-4, 3)$ opening to the right. Since $p = 2$, the focus is 2 units to the right of the vertex, which places it at the point $(-2, 3)$; and the directrix is 2 units to the left of the vertex, which means that its equation is $x = -6$. The parabola is shown in Figure 11.4.27. ◀



Example 9 Describe the graph of the equation

$$16x^2 + 9y^2 - 64x - 54y + 1 = 0$$

Solution. This equation involves quadratic terms in both x and y, so we will group the x-terms and the y-terms on one side and put the constant on the other:

$$(16x^2 - 64x) + (9y^2 - 54y) = -1$$

Next, factor out the coefficients of x^2 and y^2 and complete the squares:

$$16(x^2 - 4x + 4) + 9(y^2 - 6y + 9) = -1 + 64 + 81$$

or

$$16(x - 2)^2 + 9(y - 3)^2 = 144$$

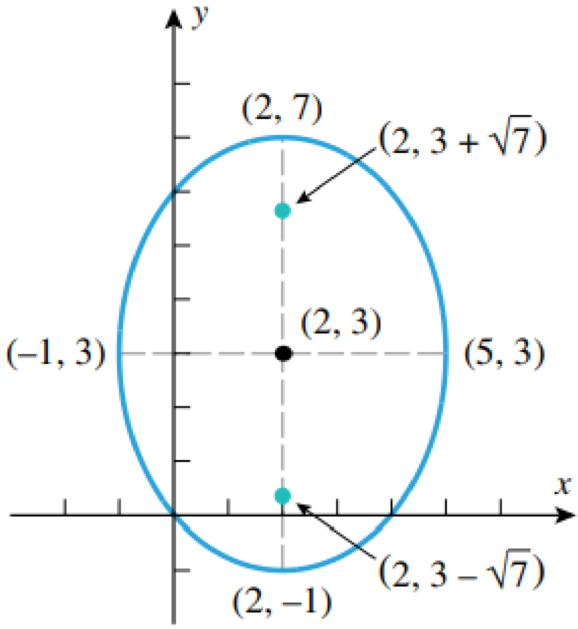
Finally, divide through by 144 to introduce a 1 on the right side:

$$\frac{(x - 2)^2}{9} + \frac{(y - 3)^2}{16} = 1$$

This is an equation of form (17), with $h = 2$, $k = 3$, $a^2 = 16$, and $b^2 = 9$. Thus, the graph of the equation is an ellipse with center (2, 3) and major axis parallel to the y-axis. Since $a = 4$, the major axis extends 4 units above and 4 units below the center, so its endpoints are (2, 7) and (2, −1) . Since $b = 3$, the minor axis extends 3 units to the left and 3 units to the right of the center, so its endpoints are (−1, 3) and (5, 3). Since

$$c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7}$$

the foci lie $\sqrt{7}$ units above and below the center, placing them at the points $(2, 3 + \sqrt{7})$ and $(2, 3 - \sqrt{7})$. ◀



$16x^2 + 9y^2 - 64x - 54y + 1 = 0$

Example 10 Describe the graph of the equation

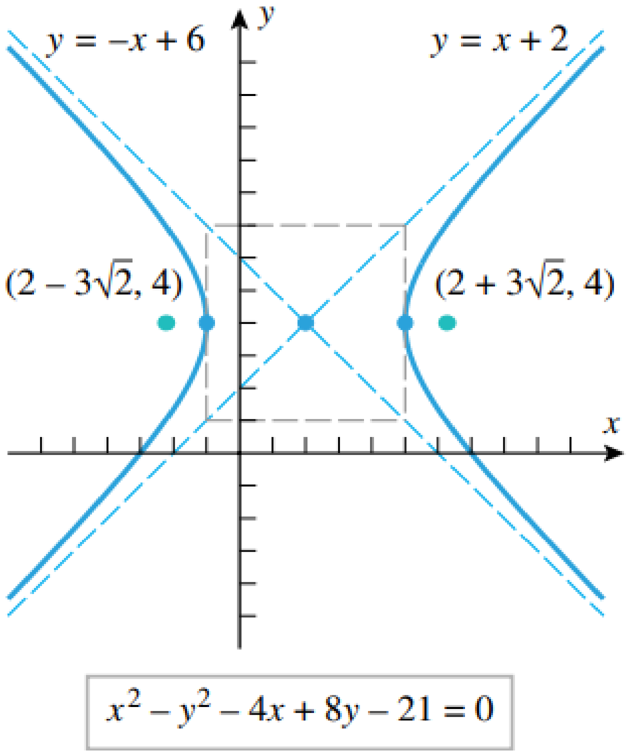
$$x^2 - y^2 - 4x + 8y - 21 = 0$$

Solution. This equation involves quadratic terms in both x and y, so we will group the x-terms and the y-terms on one side and put the constant on the other:

$$(x^2 - 4x) - (y^2 - 8y) = 21$$

We leave it for you to verify by completing the squares that this equation can be written as

$$\frac{(x-2)^2}{9} - \frac{(y-4)^2}{9} = 1 \tag{21}$$



This is an equation of form (18) with $h = 2$, $k = 4$, $a^2 = 9$, and $b^2 = 9$. Thus, the equation represents a hyperbola with center (2, 4) and focal axis parallel to the x-axis. Since $a = 3$, the vertices are located 3 units to the left and 3 units to the right of the center, or at the points $(-1, 4)$ and $(5, 4)$. From (11), $c = \sqrt{a^2 + b^2} = \sqrt{9 + 9} = 3\sqrt{2}$, so the foci are located $3\sqrt{2}$ units to the left and right of the center, or at the points $(2 - 3\sqrt{2}, 4)$ and $(2 + 3\sqrt{2}, 4)$.

The equations of the asymptotes may be found using the trick of substituting 0 for 1 in (21) to obtain

$$\frac{(x-2)^2}{9} - \frac{(y-4)^2}{9} = 0$$

This can be written as $y - 4 = \pm(x - 2)$, which yields the asymptotes

$$y = x + 2 \quad \text{and} \quad y = -x + 6$$

With the aid of a box extending $a = 3$ units left and right of the center and $b = 3$ units above and below the center, we obtain the sketch in Figure 11.4.29. ◀

5 rotation of axes; second-degree equations

Quadratic equations in x and y

We saw in Examples 8–10 of the preceding section that equations of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0 \tag{1}$$

can represent conic sections. Equation (1) is a special case of the more general equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{2}$$

which, if A, B, and C are not all zero, is called a ***second-degree equation*** or ***quadratic equation*** in x and y. We will show later in this section that the graph of any second-degree equation is a conic section (possibly a degenerate conic section). If $B = 0$, then (2) reduces to (1) and the conic section has its axis or axes parallel to the coordinate axes. However, if $B \neq 0$, then (2) contains a "cross-product" term Bxy, and the graph of the conic section represented by the equation has its axis or axes "tilted" relative to the coordinate axes. As an illustration, consider the ellipse with foci $F_1(1, 2)$ and $F_2(-1, -2)$ and such that the sum of the distances from each point $P(x, y)$ on the ellipse to the foci is 6 units. Expressing this condition as an equation, we obtain

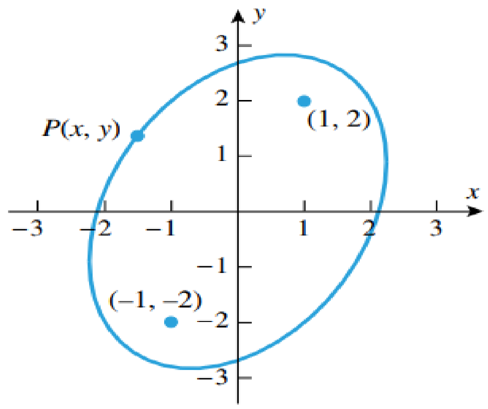
$$\sqrt{(x-1)^2 + (y-2)^2} + \sqrt{(x+1)^2 + (y+2)^2} = 6$$

Squaring both sides, then isolating the remaining radical, then squaring again ultimately yields

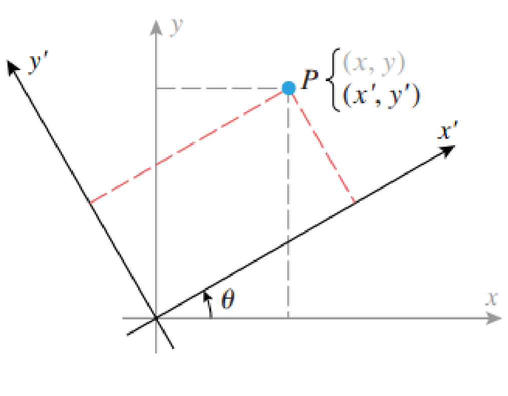
$$8x^2 - 4xy + 5y^2 = 36$$

as the equation of the ellipse.

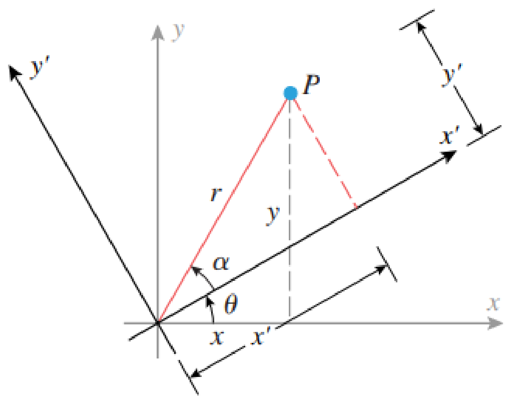
This is of form (2) with $A = 8$, $B = -4$, $C = 5$, $D = 0$, $E = 0$, $F = -36$.



Rotation of axes



(a)



(b)

In Figure a the axes of an xy-coordinate system have been rotated about the origin through an angle θ to produce a new $x'y'$-coordinate system. As shown in the figure, each point P in the plane has coordinates (x', y') as well as coordinates (x, y). To see how the two are related, let r be the distance from the common origin to the point P, and let α be the angle shown in Figure b. It follows that

$$x = r\cos(\theta + \alpha), \quad y = r\sin(\theta + \alpha) \tag{3}$$

and

$$x' = r\cos\alpha, \quad y' = r\sin\alpha \tag{4}$$

Using familiar trigonometric identities, the relationships in (3) can be written as

$$x = r\cos\theta\cos\alpha - r\sin\theta\sin\alpha$$
$$y = r\sin\theta\cos\alpha + r\cos\theta\sin\alpha$$

and on substituting (4) in these equations we obtain the following relationships called the ***rotation equations***:

$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned} \tag{5}$$

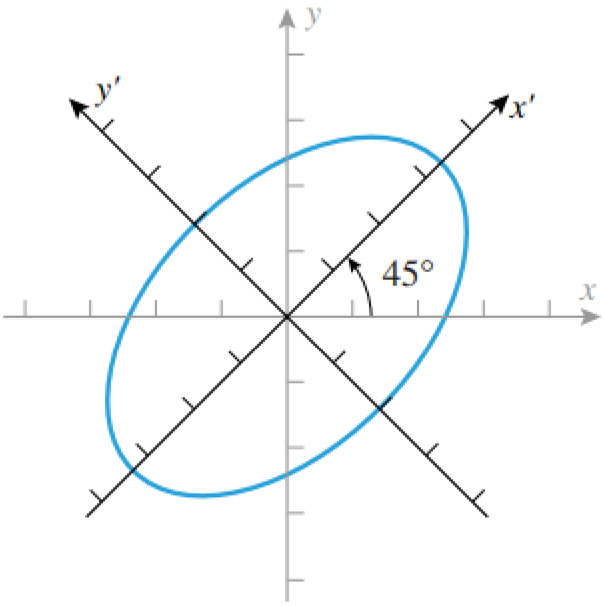
Example 1 Suppose that the axes of an xy-coordinate system are rotated through an angle of $\theta = 45^\circ$ to obtain an $x'y'$-coordinate system. Find the equation of the curve

$$x^2 - xy + y^2 - 6 = 0$$

in $x'y'$-coordinates.

Solution. Substituting $\sin\theta = \sin 45^\circ = 1/\sqrt{2}$ and $\cos\theta = \cos 45^\circ = 1/\sqrt{2}$ in (5) yields the rotation equations

$$x = \frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}} \quad \text{and} \quad y = \frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}$$



$x^2 - xy + y^2 - 6 = 0$

Substituting these into the given equation yields

$$\left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right)^2 - \left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right)\left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right) + \left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right)^2 - 6 = 0$$

or

$$\frac{x'^2 - 2x'y' + y'^2 - x'^2 + y'^2 + x'^2 + 2x'y' + y'^2}{2} = 6$$

or

$$\frac{x'^2}{12} + \frac{y'^2}{4} = 1$$

which is the equation of an ellipse ◀

====================================

If the rotation equations (5) are solved for x' and y' in terms of x and y, one obtains (Exercise 14):

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \end{aligned} \tag{6}$$

Example 2 Find the new coordinates of the point (2, 4) if the coordinate axes are rotated through an angle of $\theta = 30^\circ$.

Solution. Using the rotation equations in (6) with $x = 2$, $y = 4$, $\cos\theta = \cos 30^\circ = \sqrt{3}/2$, and $\sin\theta = \sin 30^\circ = 1/2$, we obtain

$$x' = 2(\sqrt{3}/2) + 4(1/2) = \sqrt{3} + 2$$
$$y' = -2(1/2) + 4(\sqrt{3}/2) = -1 + 2\sqrt{3}$$

Thus, the new coordinates are $(\sqrt{3} + 2, -1 + 2\sqrt{3})$. ◀

Eliminating the cross-product term

THEOREM. *If the equation*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{7}$$

is such that $B \neq 0$, *and if an* $x'y'$*-coordinate system is obtained by rotating the xy-axes through an angle* θ *satisfying*

$$\cot 2\theta = \frac{A - C}{B} \tag{8}$$

then, in $x'y'$*-coordinates, Equation* (7) *will have the form*

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$$

Proof. Substituting (5) into (7) and simplifying yields

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0$$

where

$$A' = A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta$$
$$B' = B(\cos^2\theta - \sin^2\theta) + 2(C - A)\sin\theta\cos\theta$$
$$C' = A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta$$
$$D' = D\cos\theta + E\sin\theta$$
$$E' = -D\sin\theta + E\cos\theta$$
$$F' = F$$

To complete the proof we must show that $B' = 0$ if

$$\cot 2\theta = \frac{A - C}{B}$$

or equivalently,

$$\frac{\cos 2\theta}{\sin 2\theta} = \frac{A - C}{B}$$

However, by using the trigonometric double-angle formulas, we can rewrite B' in the form

$$B' = B\cos 2\theta - (A - C)\sin 2\theta$$

Thus, $B' = 0$ if θ satisfies (10). ∎

REMARK. It is always possible to satisfy (8) with an angle θ in the range $0 < \theta < \pi/2$. We will always use such a value of θ.

Example 3 Identify and sketch the curve $xy = 1$.

Solution. As a first step, we will rotate the coordinate axes to eliminate the cross-product term. Comparing the given equation to (7), we have

$$A = 0, \quad B = 1, \quad C = 0$$

Thus, the desired angle of rotation must satisfy

$$\cot 2\theta = \frac{A - C}{B} = \frac{0 - 0}{1} = 0$$

This condition can be met by taking $2\theta = \pi/2$ or $\theta = \pi/4 = 45^\circ$. Substituting $\cos\theta = \cos 45^\circ = 1/\sqrt{2}$ and $\sin\theta = \sin 45^\circ = 1/\sqrt{2}$ in (5) yields

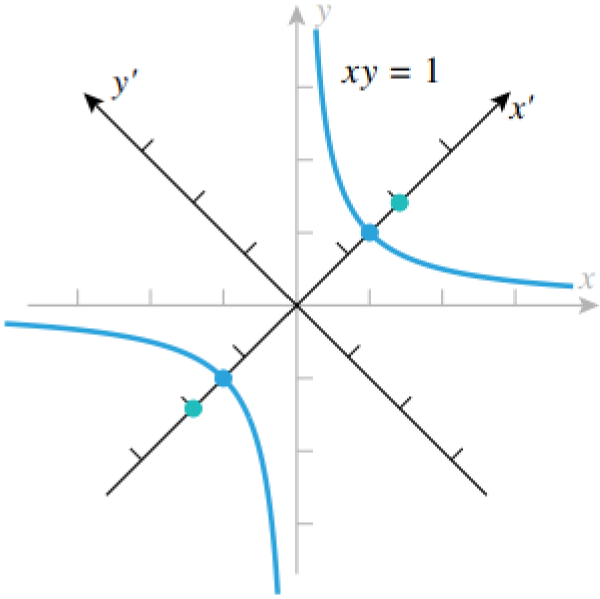
$$x = \frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}} \quad \text{and} \quad y = \frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}$$

Substituting these in the equation $xy = 1$ yields

Substituting these in the equation $xy = 1$ yields

$$\left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right)\left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right) = 1 \quad \text{and} \quad \frac{x'^2}{2} - \frac{y'^2}{2} = 1$$

which is the equation in the $x'y'$-coordinate system of an equilateral hyperbola with vertices at $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$ in that coordinate system ◀



In problems where it is inconvenient to solve

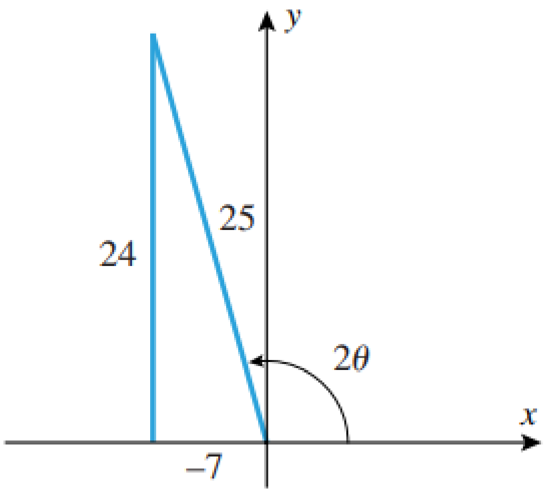
$$\cot 2\theta = \frac{A - C}{B}$$

for θ, the values of $\sin\theta$ and $\cos\theta$ needed for the rotation equations can be obtained by first calculating $\cos 2\theta$ and then computing $\sin\theta$ and $\cos\theta$ from the identities

$$\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}} \quad \text{and} \quad \cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}}$$

Example 4 Identify and sketch the curve

$$153x^2 - 192xy + 97y^2 - 30x - 40y - 200 = 0$$



Solution. We have $A = 153$, $B = -192$, and $C = 97$, so

$$\cot 2\theta = \frac{A - C}{B} = -\frac{56}{192} = -\frac{7}{24}$$

Since θ is to be chosen in the range $0 < \theta < \pi/2$, this relationship is represented by the triangle in Figure 11.5.5. From that triangle we obtain $\cos 2\theta = -\frac{7}{25}$, which implies that

$$\cos\theta = \sqrt{\frac{1 + \cos 2\theta}{2}} = \sqrt{\frac{1 - \frac{7}{25}}{2}} = \frac{3}{5}$$

$$\sin\theta = \sqrt{\frac{1 - \cos 2\theta}{2}} = \sqrt{\frac{1 + \frac{7}{25}}{2}} = \frac{4}{5}$$

Substituting these values in (5) yields the rotation equations

$$x = \tfrac{3}{5}x' - \tfrac{4}{5}y' \quad \text{and} \quad y = \tfrac{4}{5}x' + \tfrac{3}{5}y'$$

and substituting these in turn in the given equation yields

$$\tfrac{153}{25}(3x' - 4y')^2 - \tfrac{192}{25}(3x' - 4y')(4x' + 3y') + \tfrac{97}{25}(4x' + 3y')^2$$
$$- \tfrac{30}{5}(3x' - 4y') - \tfrac{40}{5}(4x' + 3y') - 200 = 0$$

which simplifies to

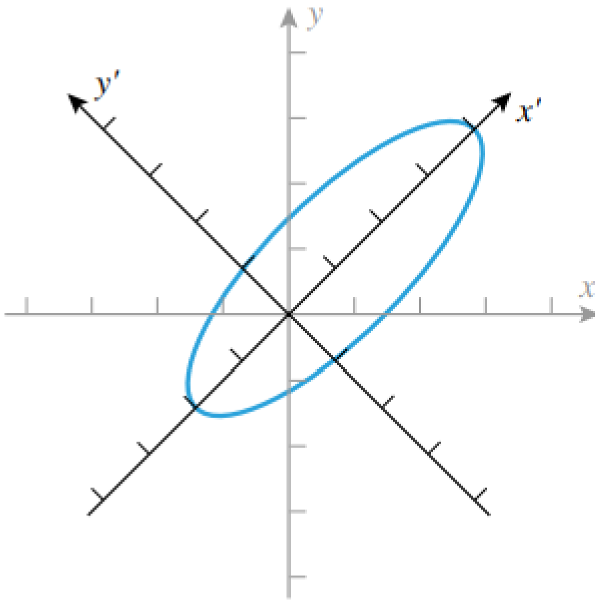
$$25x'^2 + 225y'^2 - 50x' - 200 = 0$$

or

$$x'^2 + 9y'^2 - 2x' - 8 = 0$$

Completing the square yields

$$\frac{(x' - 1)^2}{9} + y'^2 = 1$$



$$\frac{(x'-1)^2}{9} + y'^2 = 1$$

which is the equation in the $x'y'$-coordinate system of an ellipse with center $(1, 0)$ in that coordinate system and semiaxes $a = 3$ and $b = 1$ ◀

The discriminant

It is possible to describe the graph of a second-degree equation without rotating coordinate axes.

THEOREM. *Consider a second-degree equation*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{11}$$

(a) *If $B^2 - 4AC < 0$, the equation represents an ellipse, a circle, a point, or else has no graph.*

(b) *If $B^2 - 4AC > 0$, the equation represents a hyperbola or a pair of intersecting lines.*

(c) *If $B^2 - 4AC = 0$, the equation represents a parabola, a line, a pair of parallel lines, or else has no graph.*

The quantity $B^2 - 4AC$ in this theorem is called the ***discriminant*** of the quadratic equation. To see why Theorem is true, we need a fact about the discriminant. It can be shown (Exercise 19) that if the coordinate axes are rotated through any angle θ, and if

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0 \tag{12}$$

is the equation resulting from (11) after rotation, then

$$B^2 - 4AC = B'^2 - 4A'C' \tag{13}$$

In other words, the discriminant of a quadratic equation is not altered by rotating the coordinate axes. For this reason the discriminant is said to be ***invariant*** under a rotation of coordinate axes. In particular, if we choose the angle of rotation to eliminate the cross-product term, then (12) becomes

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0 \tag{14}$$

and since $B' = 0$, (13) tells us that

$$B^2 - 4AC = -4A'C' \tag{15}$$

Example 5 Use the discriminant to identify the graph of

$$8x^2 - 3xy + 5y^2 - 7x + 6 = 0$$

Solution. We have

$$B^2 - 4AC = (-3)^2 - 4(8)(5) = -151$$

Since the discriminant is negative, the equation represents an ellipse, a point, or else has no graph. (Why can't the graph be a circle?) ◀

In cases where a quadratic equation represents a point, a line, a pair of parallel lines, a pair of intersecting lines, or has no graph, we say that equation represents a ***degenerate conic section***. Thus, if we allow for possible degeneracy, it follows from Theorem that *every quadratic equation has a conic section as its graph.*