South Valley University

Faculty of Science- Qena
Mathematics Department

Geometry I:Lecture

Notes Prepared by

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Cartesian & Polar Coordinates:

- The distance between two points $P_1(x_1, y_1), P_2(x_2, y_2)$ is:

$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Example: Find the lengths of the sides of the triangle whose vertices are $(5,1), (-3,7)$ and $(8,5)$,and prove that one of the angles is a right angle.

Solution: Let $P_1(5,1), P_2(-3,7), P_3(8,5)$

$$\therefore \overline{P_1P_2} = \sqrt{(-3-5)^2 + (7-1)^2} = \sqrt{64+36} = \sqrt{100} = 10 \ ,$$

$$\overline{P_1P_3} = \sqrt{(8-5)^2 + (5-1)^2} = \sqrt{9+16} = \sqrt{25} = 5 \ ,$$

$$\overline{P_2P_3} = \sqrt{(8-(-3))^2 + (5-7)^2} = \sqrt{121+4} = \sqrt{125} = 5\sqrt{5} \ ,$$

$$\therefore \overline{P_1P_2}^2 + \overline{P_1P_3}^2 = \overline{P_2P_3}^2$$

Hence the angle at P_1 is a right angle.

- The coordinates of a point (x, y) which divides the straight line joining two given points $P_1(x_1, y_1), P_2(x_2, y_2)$ internally$^+$ (externally$^-$) in the ratio $m_1 : m_2$ is: $(x = \dfrac{m_1x_2 \pm m_2x_1}{m_1 \pm m_2}, y = \dfrac{m_1y_2 \pm m_2y_1}{m_1 \pm m_2})$.

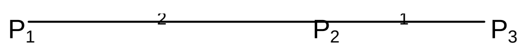
Example1: Find the coordinates of the point which divide the line joining the points $(2,-8)$ and $(-5,6)$ internally in the ratio $3:4$.

Solution:

$$(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}) = (\frac{3(-5) + 4(2)}{3+4}, \frac{3(6) + 4(-8)}{3+4}) = (-1,-2).$$

Example2: Find the coordinates of the point P_3 which divides the line joining the points $P_1(-3,-2), P_2(1,2)$ externally from the side of P_2 such that $\overline{P_1P_2} = 2\overline{P_2P_3}$.

Solution:



$$\frac{\overline{P_1P_3}}{\overline{P_2P_3}} = \frac{m_1}{m_2} = \frac{3}{1} \ ,$$

$$P_3\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}\right) = \left(\frac{3(1) - 1(-3)}{3-1}, \frac{3(2) - 1(-2)}{3-1}\right) = (3,4).$$

Example3: Find the coordinates of the two points P_3, P_4 which divides the line joining the points $P_1(2,-1), P_2(-1,5)$ into three equal parts.

Solution:

P$_1$ —1— P$_3$ —1— P$_4$ —1— P$_2$

$$\frac{\overline{P_1P_3}}{\overline{P_2P_3}} = \frac{m_1}{m_2} = \frac{1}{2} \quad ,$$

$$\therefore P_3\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right) = \left(\frac{1(-1) + 2(2)}{1+2}, \frac{1(5) + 2(-1)}{1+2}\right) = (1,1).$$

P_4 is the middle point between $P_3(1,1), P_2(-1,5)$,

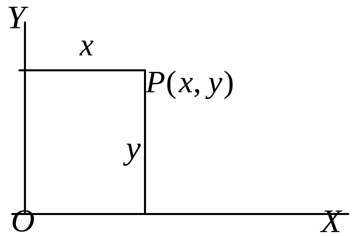
$$\therefore P_4\left(\frac{-1+1}{2}, \frac{5+1}{2}\right) = (0,3).$$

H.W: In what ratio does the point $(-1,-1)$ divide the join of $(-5,-3)$ and $(5,2)$?.

Coordinates System in a plane

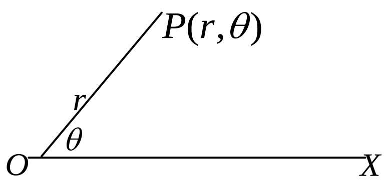
(1)- Cartesian Coordinates:

From a fixed point 0 at the plane is called the origin point we draw orthogonal straight lines OX,OY they are called axis coordinates If it is P at some point in the plane, P is completely determined by two number quantities (x, y) called point coordinates in the plane, where x represents the vertical dimension of the point P from the Y axis, and y represents the vertical dimension of the point P from the X (See figure):



(2)- Polar Coordinates:

Let O be a fixed point on the plane. From this fixed point we draw a straight horizontal constant that applies to the OX axis
(See figure):



if P is a point in the plane, then P must be completely defined if we know the distance OP (i.e. the the distance P from 0), and if we also know ,the angle that the rectal OP makes with the OX axis .

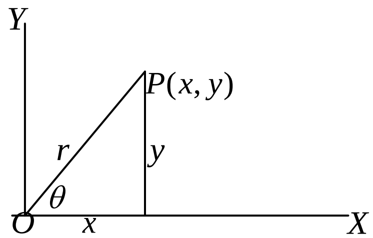
A fixed point 0 is called the starting line.

The OP dimension is called the polar dimension and symbolized by r, and the angle at which the OP straight from its original position applied to the OX axis to the OP position is called the polar angle of point P and is denoted by the symbol θ. The polar coordinates of point P in this case are the arranged two (r, θ).

The polar dimension OP is considered positive if measured from the O electrode in the straight direction that defines the polar angle θ, and is considered negative if measured in the opposite direction. The polar angle θ is considered positive if measured in an anti-clockwise direction, and is considered negative if measured in clockwise direction, and is: $(-\pi \le 0 \le \pi)$

(3)-<u>The relation between Cartesian and Polar Coordinates</u>:

Let P be a point in the plane of its polar coordinates (r, θ) and its Cartesian coordinates (x, y).as shone:



From the figure we see that: $x = r\cos\theta$ (1) , $y = r\sin\theta$ (2)

These two expressions x, y in terms of : (r, θ)

square the relations (1) and (2) and add them, we get:

$r^2 = x^2+y^2 \Rightarrow r = \sqrt{x^2+y^2}$ (3).

dividing (2) by (1), we get: $y/x = \tan\theta \Rightarrow \theta = \tan^{-1}(y/x)$ (4)

These two relations (3), (4) express (r, θ)in terms of(x, y)

<u>Example</u>:1 Find the Polar Coordinates of the point: $P(\sqrt{3},1)$

And determine the position of this point

<u>Solution</u>:

The point is given in Cartesian coordinates (x, y) = $(\sqrt{3},1)$, so:

$$r = \sqrt{x^2+y^2} = \sqrt{3+1} = \sqrt{4} = 2 \ , \ x = r\cos\theta, \ y = r\sin\theta$$

$$\sqrt{3} = 2\cos\theta, 1 = 2\sin\theta$$

$$\cos\theta=\frac{\sqrt{3}}{2},\ \sin\theta=\frac{1}{2},\ \theta=\frac{\pi}{6} \quad \text{then: } (r,\theta)=\left(2,\frac{\pi}{6}\right)$$

Then the angle θ is in the first quadrant of the plane.

Example:2

(i) Transform : $x^2+y^2-2x+2y=0$ into polar form.

(ii) Transform : $r=4a\cos\theta$ into Cartesian form.

Solution:

(i) put : $x=r\cos\theta,\ y=r\sin\theta$

$$\therefore (r\cos\theta)^2+(r\sin\theta)^2-2(r\cos\theta)+2(r\sin\theta)=0$$

$$\Rightarrow r^2(\cos^2\theta+\sin^2\theta)-2r(\cos\theta-\sin\theta)=0$$

$$\Rightarrow r=2(\cos\theta-\sin\theta).$$

(ii) $r=4a\cos\theta \Rightarrow r^2=4ar\cos\theta \Rightarrow x^2+y^2=4ax.$

Example:3

(i) Transform : $r^2=a^2\cos2\theta$ into Cartesian form.

(ii) Transform : $x^3=y^2(2-x)$ into polar form.

Solution:

$$(i)\quad r^2=a^2\cos2\theta=a^2\left(\cos^2\theta-\sin^2\theta\right) \Rightarrow r^4=a^2\left(r^2\cos^2\theta-r^2\sin^2\theta\right)$$

$$\Rightarrow\left(x^2+y^2\right)^2=a^2\left(x^2-y^2\right)$$

(ii) put $x=r\cos\theta,\ y=r\sin\theta$

$$\therefore (r\cos\theta)^3=(r\sin\theta)^2(2-r\cos\theta)$$

$$\Rightarrow r^3\cos^3\theta s=r^2\sin^2\theta(2-r\cos\theta)$$

$$\Rightarrow r^3\cos^3\theta+r^3\sin^2\theta\cos\theta=2r^2\sin^2\theta$$

$$\Rightarrow r^3\cos\theta\left(\cos^2\theta+\sin^2\theta\right)=2r^2\sin^2\theta$$

$$\Rightarrow r^3\cos\theta=2r^2\sin^2\theta \ \Rightarrow\ r5\cos\theta=2\sin^2\theta$$

$$r=2\tan\theta\sin\theta$$

Exercises:

1- Find the coordinates of the point P_3 which divides the line joining the points $P_1(0,-1), P_2(2,3)$ externally from the side of P_2 such that $\overline{P_1P_2}=2\overline{P_2P_3}$.

2- Find the coordinates of the two points P_3, P_4 which divides the line joining the points $P_1(1,1), P_2(-2,-5)$ into three equal parts.

3- Prove that the medians of a triangle with vertices

$P_1(x_1, y_1), P_2(x_2, y_2), P_3(x_3, y_3)$ is $M(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3})$.

4- Show that the distance between the two points $P_1(x_1, y_1), P_2(x_2, y_2)$ in polar coordinates is:

$\sqrt{r_1^{\,2} + r_2^{\,2} - 2r_1r_2\cos(\theta_2 - \theta_1)}$.

5- Find the Polar Coordinates for each of the following points::

$$P_1(-\sqrt{3},1), P_2(-1,\sqrt{3}), P_3(-1,1), P_4(-3,3\sqrt{3}), P_5(1,-\sqrt{3})$$

6- Find the Cartesian Coordinates for each of the following points:

$$P_1(2,-\frac{\pi}{2}), P_2(1,\frac{\pi}{3}), P_3(3,\frac{\pi}{4}), P_4(4,\frac{\pi}{3}), P_5(2,-\frac{\pi}{6})$$

7- Transform the following equations to the polar Coordinate:

(1) $(x^2 + y^2)^2 = 2a^2xy$ (2) $y^{2\,2} = x^3/(2a - x)$

(3) $x^4 + y^4 = a^2xy$ (4) $2x^2 - 2y^2 = 9$

8- Transform the following equations to the Cartesian Coordinate:

(1) $r = 1 - \cos\theta$ (2) $r^2 = 9\cos 2\theta$

(3) $r = 3/(2 + 3\sin\theta)$ (4) $r(2 - \cos\theta) = 2$

Change axes in the plane:

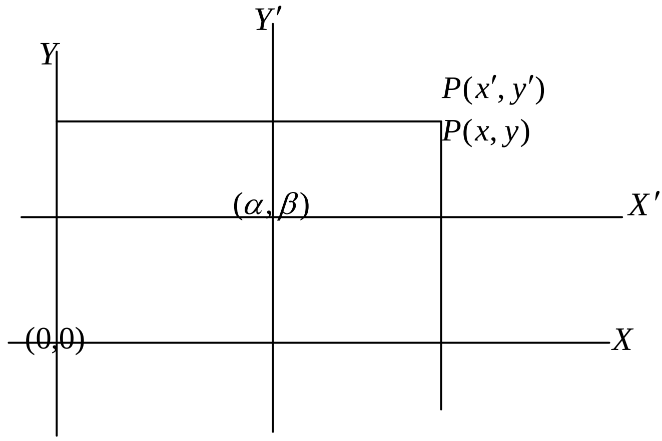
The purpose of changing coordinate axes is to place curve equations in the simplest form so that they can know their type and study their properties.

Below we will study three ways to change axes:

1-Transfer of the origin point (transfer of coordinate axes):

If : $y = f(x)$ it is the equation of a curve in a plane, and (x, y) it is the coordinates of a point P in the plane, and the origin point $(0,0)$ is moved to another point : (α, β)

While maintaining the direction of the axes, if the point P coordinates for the two new axes are: (x', y') , then it is of the form:



$$x = x' + \alpha \ , \ \ y = y' + \beta \ \ , \ \ x' = x - \alpha \ , \ y' = y - \beta$$

The relation between the new coordinates (x', y') and the old one (x, y) when the parallel axes are translated through the point (α, β) is: $x = x' + \alpha$, $y = y' + \beta$ and $x' = x - \alpha$, $y' = y - \beta$

Example1:Find the new coordinates for the point : when the parallel axes are translated through the point $(2,-5)$

<u>Solution</u>:

$$x = x' + \alpha \ , \quad \Rightarrow -3 = x' + 2 \Rightarrow x' = -5$$

$$y = y' + \beta \Rightarrow 4 = y' - 5 \Rightarrow y' = 9$$

<u>Example2</u> :Transform to parallel axes through the point $(1,0)$ the equation : $x^2 + xy + y^2 - 2x - y - 5 = 0$.

<u>Solution</u>: $x = x' + 1$, $y = y' + 0$

$$\therefore (x'+1)^2 + (x'+1)y' + y'^2 - 2(x'+1) - y' - 5 = 0$$

$$\Rightarrow (x'^2 + 2x' + 1) + (x'y' + y') + y'^2 - 2x' - 2 - y' - 5 = 0$$

$$\Rightarrow x'^2 + x'y' + y'^2 = 6.$$

<u>Example3:</u> Transform to parallel axes through the point $(2,-3)$ the equation $x^2 + y^2 - 4x + 6y = 36$.

Solution: $x = x' + 2\ ,\ \ y = y' - 3$

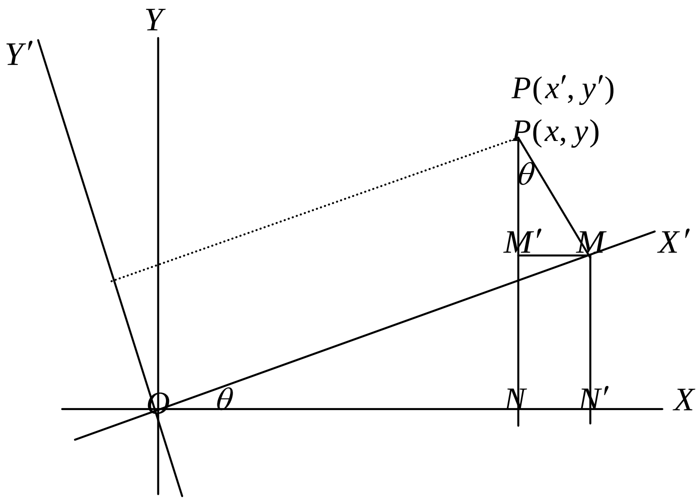
$$\therefore (x'+2)^2 + (y'-3)^2 - 4(x'+2) + 6(y'-3) = 36$$

$$\Rightarrow x'^2 + 4x' + 4 + + y'^2 6x' + 9 - 4x' - 8 + 6y' - 18 = 36$$

$$\Rightarrow x'^2 + y'^2 = 49.$$

2:Axes rotation

- The relation between the new coordinates (x', y') and the old one (x, y) when the parallel axes are rotated through an angle θ is:
  $x = x'\cos\theta - y'\sin\theta\ ,\ \ y = x'\sin\theta + y'\cos\theta$.



$x = ON = ON' - NN'$
$= ON' - M'M$
$= OM\cos\theta - PM\sin\theta$,
$= x'\cos\theta - y'\sin\theta.$

$y = PN = PM' + M'N$
$= PM' + MN'$
$= PM\cos\theta + OM\sin\theta$
$= y'\cos\theta + x'\sin\theta$
$= x'\sin\theta + y'\cos\theta.$

Then:

$x = x'\cos\theta - y'\sin\theta,$
$y = x'\sin\theta + y'\cos\theta.$

	x'	y'
x	$\cos\theta$	$-\sin\theta$
y	$\sin\theta$	$-\sin\theta$

Example1: What does the equation

$x^2 + 2xy + y^2 - 2\sqrt{2}\,x + 6\sqrt{2}\,y - 6 = 0$ become when the parallel axes are rotated through an angle of $\frac{\pi}{4}$.

Solution: $x = x'\cos\theta - y'\sin\theta$, $y = x'\sin\theta + y'\cos\theta$,

$$\theta = \frac{\pi}{4} \Rightarrow \sin\theta = \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow x = \frac{1}{\sqrt{2}}(x' - y') \ , \ y = \frac{1}{\sqrt{2}}(x' + y')$$

$$\therefore \frac{1}{2}(x'^2 - 2x'y' + y'^2) + (x'^2 - y'^2) + \frac{1}{2}(x'^2 + 2x'y' + y'^2) - 2(x' - y')$$
$$+ 6(x' + y') - 6 = 0$$
$$\therefore 2x'^2 + 4x' + 8y' - 6 = 0 \Rightarrow x'^2 + 2x' + 4y' - 3 = 0$$
$$\Rightarrow (x'+1)^2 - 1 + 4y' - 3 = 0$$
$$\Rightarrow (x'+1)^2 = -4(y'-1).$$

Example2: What does the equation: $2xy = 49$ become when the parallel axes are rotated through an angle of: $\frac{\pi}{4}$

Solution: $x = x'\cos\theta - y'\sin\theta$, $y = x'\sin\theta + y'\cos\theta$,

$$\theta = \frac{\pi}{4} \Rightarrow \sin\theta = \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow x = \frac{1}{\sqrt{2}}(x' - y') \ , \ y = \frac{1}{\sqrt{2}}(x' + y') \ ,$$
$$2\left\{\frac{1}{\sqrt{2}}(x' - y')\frac{1}{\sqrt{2}}(x' + y')\right\} = 49 \quad \Rightarrow (x'^2 - y'^2) = 49$$

Example3 - What does the equation $2x^2 - 3xy + y^2 = 0$ become when the parallel axes are rotated through an angle of $\theta = \tan^{-1}(1/2)$.

Solution: $x = x'\cos\theta - y'\sin\theta$, $y = x'\sin\theta + y'\cos\theta$,

$$\theta = \tan^{-1}(1/2) \Rightarrow \sin\theta = \frac{1}{\sqrt{5}} \ , \ \cos\theta = \frac{2}{\sqrt{5}} \Rightarrow x = \frac{1}{\sqrt{5}}(2x' - y') \ , \ y = \frac{1}{\sqrt{5}}(x' + 2y')$$
$$2\left\{\frac{1}{\sqrt{5}}(2x' - y')\right\}^2 - 3\left\{\frac{1}{\sqrt{5}}(x' + 2y')\frac{1}{\sqrt{5}}(2x' - y')\right\} + \left\{\frac{1}{\sqrt{5}}(x' + 2y')\right\} = 0$$
$$\frac{2}{5}(2x' - y')^2 - \frac{3}{5}(2x' - y')(x' + 2y') + \frac{1}{5}(x' + 2y')^2 = 0$$
$$\Rightarrow 2(4x'^2 - 4x'y' + y'^2) - 3(2x'^2 + 3x'y' - 2y'^2) + (x'^2 + 4x'y' + 4y'^2) = 0$$
$$\Rightarrow 3x'^2 - 13x'y' + 12y'^2 = 0.$$

Exercises:

1- Transform to parallel axes through the point $(3,5)$ the equation $x^2 + y^2 - 6x - 10y - 2 = 0$.

2- What does the equation $4x^2+2\sqrt{3}\,xy+2y^2=1$ become when the parallel axes are rotated through an angle of $\frac{\pi}{6}$

<u>3-Moving the axes and rotating them together:</u>

If the point of origin is moved to the point (α,β) and the axis coordinates OX, OY are rotated at an angle θ at the same time:

$$x = x'\cos\theta - y'\sin\theta + \alpha\ ,$$
$$y = x'\sin\theta + y'\cos\theta + \beta.$$

<u>Example (1):</u> If the point of origin is moved to the point : $(-1,2)$ and the axes rotate at an angle : $\tan^{-1}(1)$ Find the new coordinates for the point $(1,3)$ The new curve equation: $4x^2+y^2+8x-4y+7=0$

<u>Solution</u>: the relationship between the original coordinates x, y and new coordinates: x', y' When the point of origin is moved to the point: (α,β) and the axes rotate an angle θ is:

$$x = x'\cos\theta - y'\sin\theta + \alpha\ ,$$
$$y = x'\sin\theta + y'\cos\theta + \beta.$$

But from the data $(\alpha,\beta)=(-1,2)$, $\theta=\frac{\pi}{4} \Rightarrow \sin\theta=\cos\theta=\frac{1}{\sqrt{2}}$

$$\therefore x = \frac{1}{\sqrt{2}}(x'-y')-1. \qquad (*)$$

$$y = \frac{1}{\sqrt{2}}(x'+y')+2. \qquad (**)$$

To find the new coordinates of the point : $(1,3)$ we substituting by $x=1, y=3$ In the previous two relationships $(*),(**)$: we get:

$$1 = \frac{1}{\sqrt{2}}(x'-y')-1. \qquad (1)$$
$$3 = \frac{1}{\sqrt{2}}(x'+y')+2. \qquad (2)$$
$$\Rightarrow \quad (x'-y') = 2\sqrt{2}. \qquad (3)$$
$$(x'+y') = \sqrt{2}. \qquad (4)$$

Solve the two equations $(3),(4)$ together we get :

$x' = \frac{3}{\sqrt{2}}, y' = -\frac{1}{\sqrt{2}}$ so the new coordinates for the point : $(1,3)$

Is $:(\frac{3}{\sqrt{2}},-\frac{1}{\sqrt{2}})$.

To find the new equation for the curve $4x^2 + y^2 + 8x - 4y + 7 = 0$:
we substituting by x, y from $(*),(**)$ we get

$$4[\frac{1}{\sqrt{2}}(x' - y') - 1]^2 + [\frac{1}{\sqrt{2}}(x' + y') + 2]^2 + 8[\frac{1}{\sqrt{2}}(x' - y') - 1]$$
$$- 4[\frac{1}{\sqrt{2}}(x' + y') + 2] + 7 = 0.$$
$$\therefore 2(x'^2 - 2x'y' + y'^2) - \frac{8}{\sqrt{2}}(x' - y') + 4 + \frac{1}{2}(x'^2 + 2x'y' + y'^2)$$
$$+ \frac{4}{\sqrt{2}}(x' + y') + 4 + \frac{8}{\sqrt{2}}(x' - y') - 8 - \frac{4}{\sqrt{2}}(x' + y') - 8 + 7 = 0.$$
$$\therefore \frac{5}{2}x'^2 - 3x'y' + \frac{5}{2}y'^2 - 1 = 0.$$
$$\therefore 5x'^2 - 6x'y' + 5y'^2 = 2.$$

So this is the new curve equation required.

<u>Various examples:</u>

1- Find the new origin point that, if we move the axial coordinates, the curve equation : $12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0$
It becomes free from the absolute limit and first-class limits.

<u>Solution</u> : Let (α, β) It is the new point of origin substituting by

$x = x' + \alpha$, $y = y' + \beta$ In the curve equation $(^*)$

In order for the equation to be free of the absolute and first-degree limits, we equate the coefficients of the first-degree and absolute terms in the equation $(^*)$ Zero as follows:

$$24a - 10b + 11 = 0 \quad (1) \quad , \quad 10a + 4b - 5 = 0 \quad (2)$$
$$12a^2 - 10ab + 2b^2 + 11a - 5b + 2 = 0 \quad (3)$$

Solve the two equations $(1),(2)$ together we get:

$$\alpha = -\frac{3}{2} \ , \ \beta = -\frac{5}{2}$$

substituting in eq.$(^*)$ by $\alpha = -\frac{3}{2}$, $\beta = -\frac{5}{2}$

we git: $12x'^2 - 10x'y' + 2y'^2 = 0$

2- If the coordinate axis has a sharp angle : $\theta = \tan^{-1}\left(\frac{1}{2}\right)$ Find the new form to which the curve equation results :

$$2x^2 - 3xy + y^2 = 0$$

Solution: the relation between the original coordinates x, y and new coordinates x', y' when the axes rotate an angle θ is:

$$x = x'\cos\theta - y'\sin\theta \quad , \quad y = x'\sin\theta + y'\cos\theta.$$

then : $\theta = \tan^{-1}(\frac{1}{2}) \Rightarrow \sin\theta = \frac{1}{\sqrt{5}}$, $\cos\theta = \frac{2}{\sqrt{5}}$

$$\therefore x = \frac{1}{\sqrt{5}}(2x' - y') \quad , \quad y = \frac{1}{\sqrt{5}}(x' + 2y').$$

substituting by x, y in the curve equation we get:

$$\frac{2}{5}(2x' - y')^2 - \frac{3}{5}(2x' - y')(x' + 2y') + \frac{1}{5}(x' + 2y')^2 = 0$$
$$\Rightarrow 2(4x'^2 - 4x'y' + y'^2) - 3(2x'^2 + 3x'y' - 2y'^2) + (x'^2 + 4x'y' + 4y'^2) = 0$$
$$\Rightarrow 3x'^2 - 13x'y' + 12y'^2 = 0.$$

3- If the axes revolve around an angle $\frac{\pi}{4}$ Then the axes are then moved to the point (-2,-6) For the axis after rotation, check that the new form of the curve equation

$x^2 - y^2 - 4\sqrt{2}\,x - 8\sqrt{2}\,y + 4 = 0$ is $xy = 14$.

Solution: the relation between the original coordinates x, y and new coordinates x', y' when the axes rotate an angle θ is:

$$x = x'\cos\theta - y'\sin\theta \quad , \quad y = x'\sin\theta + y'\cos\theta.$$

Then from It is data : $\theta = \frac{\pi}{4} \Rightarrow \sin\theta = \cos\theta = \frac{1}{\sqrt{2}}$

$$\therefore x = \frac{1}{\sqrt{2}}(x' - y'), y = \frac{1}{\sqrt{2}}(x' + y').$$

substituting in the curve equation we get:

$$\frac{1}{2}(x' - y')^2 - \frac{1}{2}(x' + y')^2 - \frac{4\sqrt{2}}{\sqrt{2}}(x' - y') - \frac{8\sqrt{2}}{\sqrt{2}}(x' + y') + 4 = 0$$
$$\Rightarrow (x'^2 - 2x'y' + y'^2) - (x'^2 + 2x'y' + y'^2) - 8(x' - y') - 16(x' + y') + 8 = 0$$

$\Rightarrow -4x'y' - 24x' - 8y' + 8 = 0$
$\Rightarrow x'y' + 6x' + 2y' - 2 = 0.$

and substituting by $x' = x - 2$, $y' = y - 6$ we get

$(x-2)(y-6) + 6(x-2) + 2(y-6) - 2 = 0.$
$\Rightarrow xy - 14 = 0$
$\Rightarrow xy = 14.$

Exercises:

1-If the point of origin is moved to the point $(1,-2)$ Find the new coordinates for each of the following points:
$P_1(2,1)$, $P_2(5,-2)$, $P_3(0,1)$

2-Find the point where the origin point should be moved when:
a)- Until the point : $(-5,2)$ movies to the point : $(5,-2)$
b)- Until the equation : $(y-1)^2 = 4(x+2)$ becomes free from y and the absolute limit

3- If the point of origin is moved to the point : $(1,-3)$ and the axes rotate at an angle : $\tan^{-1}(\frac{3}{4})$ Find the new coordinates for the point $(2,-2)$ and the new curve of equation: $36x^2 + 24xy + +29y^2 + 150y + 45 = 0$

4-Find the coordinates of the point where the axes should be moved in order to:
a)- The equation $x^2 + 4xy + y^2 + 2x + 6y - 8 = 0$ turns to another free from the limits of the first degree.
b)- The equation $x^2 + y^2 + 2x + 6y + 10 = 0$ turns to another free from x and the absolute limit.

6- What does the equation: $x^2 + 2\sqrt{3}xy - y^2 = 5$ become when the parallel axes are rotated through an angle of $\frac{\pi}{6}$.

General 2nd degree Equ. & Pair of Lines:

1-The Condition for a general second degree equation :

$ax^2+2hxy+by^2+2gx+2fy+c=0$ to represent a pair of lines is:

$$\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=abc+2fgh-af^2-bg^2-ch^2=0$$ .And the angle θ

between these lines is given by:. $\tan\theta=\dfrac{2\sqrt{h^2-ab}}{a+b}$

From the previous relations, we can conclude that the two straight lines are:

1)- Real and different if : $h^2>ab$
2)- Imaginary if it was: $h^2<ab$
3)- Parallel (or applicable) if: $h^2=ab$
4)- Orthogonal if it is: $a+b=0$

Example1: Show that the equation $x^2+8xy+y^2+16x+4y+4=0$ represents a pair of lines, and calculate the angle between these lines.

Solution: $a=b=1, h=4, g=8, f=2, c=4$

$$\therefore\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=\begin{vmatrix} 1 & 4 & 8 \\ 4 & 1 & 2 \\ 8 & 2 & 4 \end{vmatrix}=1(4-4)-4(16-16)+8(8-8)=0,$$

$$\phi=\tan^{-1}[\frac{2\sqrt{h^2-ab}}{a+b}]=\tan^{-1}[\frac{2\sqrt{16-1}}{2}]=\tan^{-1}(\sqrt{15}).$$

Example2: Find the value of λ so that the equation

$x^2+2\lambda xy+y^2+6x+2y+9=0$ may represent a pair of lines.

Solution: $a=b=1, h=\lambda, g=3, f=1, c=9$

$$\therefore\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=abc+2fgh-af^2-bg^2-ch^2=0\Rightarrow 9+6\lambda-1-9-9\lambda^2=0$$

$$\Rightarrow -9\lambda^2+6\lambda-1=0\Rightarrow 9\lambda^2-6\lambda+1=0\Rightarrow(3\lambda-1)^2=0\Rightarrow\lambda=\frac{1}{3}.$$

Example2 :For what value of λ does the equation

$x^2-xy+\lambda y^2-3x-3y=0$ represent a pair of lines,

and what is then the angle between these lines

Solution: $a=1$, $b=\lambda, h=-1/2, g=-3/2, f=-3/2, c=0$

$$\therefore \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 1 & -1/2 & -3/2 \\ -3/2 & \lambda & -3/2 \\ -1/2 & -3/2 & 0 \end{vmatrix} = 1\left(\frac{-9}{4}\right) + \frac{1}{2}\left(\frac{-9}{4}\right) - \frac{1}{2}\left(-\frac{3}{4} + \frac{3\lambda}{2}\right) = 0$$

$$\frac{-9}{4} - \frac{9}{8} + \frac{9}{8} - \frac{3\lambda}{2} = 0 \Rightarrow \frac{-9-6\lambda}{4} = 0 \Rightarrow -9-6\lambda = 0 \Rightarrow \lambda = \frac{-3}{2}$$

$$\phi = \tan^{-1}[\frac{2\sqrt{h^2-ab}}{a+b}] = \tan^{-1}[\frac{2\sqrt{1+3/2}}{1-3/2}] = \tan^{-1}(\frac{2\sqrt{5/2}}{-1/\ 2}).$$

Example3: Show that the equation : $y^2 + xy - 2x^2 - 5x - y - 2 = 0$ represents a pair of lines, and find them, and calculate the angle between them, and find the point of their intersection.

Solution: $a = -2\ , b = 1, h = 1/2, g = -5/2, f = -1\ /2, c = -2$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} -2 & 1/2 & -5/2 \\ 1/2 & 1 & -1/2 \\ -5/2 & -1/2 & -2 \end{vmatrix} = -2\left(-2 - \frac{1}{4}\right) + \frac{-1}{2}\left(-1 - \frac{5}{4}\right) + \left(\frac{-5}{2}\right)\left(-\frac{1}{4} + \frac{5}{2}\right)$$

$$= \frac{36}{8} + \frac{9}{8} - \frac{45}{8} = 0$$

Then the equation represent a pair of lines.

To find a pair of lines we analysis the left side of a given equation:

$$y^2 + xy - 2x^2 - 5x - y - 2 = (y + 2x + \alpha)(y - x + \beta)$$

Compeering coefficient $x\ , y$ and absolute value we get:

$-\alpha + 2\beta = -5\ , \alpha + \beta = -1\ , \alpha\beta = -2$ so that $\alpha = 1\ , \beta = -2$

Then a pair of lines is: $y + 2x + 1 = 0\ , y + 2x + 1 = 0 \quad (*)$

$$\phi = \tan^{-1}[\frac{2\sqrt{h^2-ab}}{a+b}] = \tan^{-1}[\frac{2\sqrt{1/4+2}}{-2+1}] = \tan^{-1}(-3).$$

by solving the equations (*) we get the intersection point between these lines is : $(-1,1)$

Example4 :For what value of c does the equation

$12x^2 + 19xy + 4y^2 - 5x - 11y + c = 0$ represent a pair of lines,

Solution : $a = 12\ , b = 4, h - 19/2, g = -5/2, f = -11/2, c = c$

$$\therefore \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 12 & 19/2 & -5/2 \\ 19/2 & 4 & -11/2 \\ -5/2 & -11/2 & c \end{vmatrix} = 12\left(4c-\frac{121}{4}\right)-\frac{19}{2}\left(\frac{19c}{2}-\frac{55}{4}\right)-$$

$$\frac{5}{2}\left(-\frac{-309}{4}+\frac{20}{2}\right)=$$

$$12\left(\frac{16c-121}{4}\right)-\frac{19}{2}\left(\frac{38c-55}{4}\right)-\frac{5}{2}\left(\frac{-309+40}{4}\right)=0$$

$$=3(16c-121)-\frac{19}{8}(38c-55)-\frac{5}{8}(-269)=24(16c-121)-19(38c-55)+1345$$

$$=384c-2904-722c+1045+1345=-388c-514=0 \Rightarrow 388c=-514 \Rightarrow c=-3$$

Another Solution:

by analysis the left side of a given equation:

$$12x^2+19xy+4y^2-5x-11y+c=(4x+y+\alpha)(3x+4y+\beta)$$

Compeering coefficient $x\,,y$ and absolute value we get:

$3\alpha+4\beta=-5\ ,4\alpha+\beta=-11,\ \alpha\beta=c$ so that $\alpha=-3\ ,\beta=1$

Then : $c=\alpha\beta=(-3)(1)=-3$

2-The equation of any straight line passing by the intersection point of two known straight lines:

Let :

$a_1x + b_1y + c_1 = 0$ (1)

$a_2x + b_2y + c_2 = 0$ (2)

Be the equations of two straight lines and consider the equation: $(a_1x_1 + b_1y_1 + c_1) + k(a_2x_1 + b_2y_1 + c_2) = 0$ (3)

Where k is a constant.

It is clear that equation (3) is a first-degree equation in x, y and therefore it represents a straight line equation in the plane and if (x1, y1) is the point of intersection of the two lines (1), (2), it achieves both of them, and then it achieves the equation

(3), and on this equation (3) represents a straight line that passes by the point of intersection of the two lines (1), (2)
And by giving k different values, we get a group (bundle) of straight lines, all of which pass through the intersection point of the two straight lines (1), (2), which is called the head of the beam.

Example (1): Find the equation of a straight line that passes by the intersection point of the two lines:

$3x + 4y + 5 = 0$, $2x - 3y + 4 = 0$.

It passes the point of origin.

Solution: the equation of any straight line that passes the intersection point of the known straight lines is as follows: $(3x + 4y + 5) + k(2x - 3y + 4) = 0$ (*)

Since the straight line (*) passes the point of origin (0,0), it achieves its equation, so it is: $5+4k = 0 \Rightarrow k = -5/4$

Substituting k into the equation (*), we get:

$(3x + 4y + 5) + (-5/4)(2x - 3y + 4) = 0$

$\therefore 2x + 31y = 0$. and this is the line equation.

Example (2): Find the equation of the straight line that passes the intersection point of the two lines:

$2x - 4y + 1 = 0$, $3x + 5y - 6 = 0$

It is parallel to the line :$x + y + 2 = 0$.

Solution: the equation of any straight line that passes the intersection point of the known straight lines is as follows:

$(2x - 4y + 1) + k(3x + 5y - 6) = 0$ (*)

Since the straight (*) equals the straight $x + y + 2 = 0$

(Whose inclination is -1) is equal, so:

$-(2+3k)/(-4+5k) = -1 \Rightarrow k = 3$

This is the required line equation.

Example (3): Find the equation of the straight line that passes the intersection point of the two lines:

$y + 2x + 1 = 0$, $y - x - 2 = 0$

It is perpendicular to the straight line: $2x - y = 0$

Solution: the equation of any straight line that passes the intersection point of the known straight lines is: $(y + 2x + 1) + k(y - x - 2) = 0$

So the required line equation is : $2x - y = 0$

(Whose slope is equal to 2)The product of their slope is (-1),

and then: $[-(2-k)/(1+k)][2] = -1 \Rightarrow k = 1$

So the required line equation is:

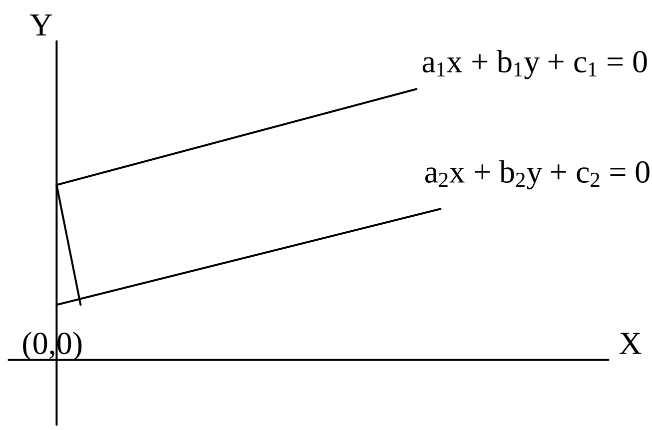
$(y + 2x + 1) + (y - x - 2) = 0$. So that: $x + 2y - 1 = 0$

3-The shortest distance between two straight lines is:

To find the shortest distance between the unbroken straight lines: $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$.

We find the intersection of one of them with one of the coordinate axes (Let OY hub beats.)

By putting x = 0), we find the value of y, and we get the intersection point (0, y), then we find the length of the column From this point it falls on the second straight line, so the length of this column is the shortest distance between the two straight lines given in the plane.



Example: Find the length of the shortest dimension between the two lines: $3x - 4y - 2 = 0$, $8y - 6x - 9 = 0$.

Solution: We find the point of intersection of the rectum

3x - 4y - 2 = 0 With the y-axis, we put x = 0, so y = -1/2, so the point of intersection is (0, -1/2).

The length of the shortest distance between the two straight lines equals the length of the column falling from this point on the straight line :8y – 6x – 9 = 0 is

$$h = \left| \frac{8(-\frac{1}{2}) - 6(0) - 9}{\sqrt{8^2 + 6^2}} \right| = \left| \frac{-13}{10} \right| = 1.3$$

4-<u>The angle between the two lines represented by the homogeneous Equation:</u>

Assume that the two straight lines represented by this equation $ax^2 + 2hxy + by^2 = 0.$ are: $y = m_1 x$, $y = m_2 x.$

Where m_1, m_2 they are inclined, so the common equation for them is as follows: $(y - m_1 x)(y - m_2 x) = ax^2 + 2hxy + by^2.$

$$\therefore y^2 - (m_1 + m_2)xy + (m_1 m_2)x^2 = (\frac{a}{b})x^2 + (\frac{2h}{b})xy + y^2.$$

Equal coefficients x^2, xy at both side we get:

$$m_1 m_2 = \frac{a}{b} \ , \ -(m_1 + m_2) = \frac{2h}{b}.$$

If it is the angle between the two straight lines, then it is:

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2} = \frac{\sqrt{(-\frac{2h}{b})^2 - 4(\frac{a}{b})}}{1 + \frac{a}{b}} = \frac{2\sqrt{h^2 - ab}}{a + b}.$$

$$\therefore \theta = \tan^{-1}(\frac{2\sqrt{h^2 - ab}}{a + b})$$

And if it is :$\theta = \frac{\pi}{2}$ Which :$\tan\theta = \infty$ and This is when $a + b = 0$

This is the condition of orthogonal linear straightness.

And if it is $\theta = 0$ Which $\tan\theta = 0$ This is when

$h^2 - ab = 0$ Which :$h^2 = ab$ and this is a condition of parallelism.

<u>5- Equation of the two straight lines which are fair to the two angles between the two lines represented by the equation:</u>

$ax^2 + 2hxy + by^2 = 0. \qquad (*)$

Assume that the two straight lines represented by this equation are: $y = m_1 x$, $y = m_2 x$. where m_1, m_2

Their inclination, so the equations of the two angles of these two angles are: $\frac{y - m_1 x}{\sqrt{1 + m_1^2}} = \pm \frac{y - m_2 x}{\sqrt{1 + m_2^2}}.$

The equation for them is:

$$\left(\frac{y - m_1 x}{\sqrt{1 + m_1^2}} - \frac{y - m_2 x}{\sqrt{1 + m_2^2}}\right)\left(\frac{y - m_1 x}{\sqrt{1 + m_1^2}} + \frac{y - m_2 x}{\sqrt{1 + m_2^2}}\right) = 0.$$

$$\therefore \frac{(y - m_1 x)^2}{1 + m_1^2} - \frac{(y - m_2 x)^2}{1 + m_2^2} = 0.$$

$$\therefore (m_1 + m_2)(y^2 - x^2) + 2xy - 2(m_1 m_2)xy = 0.$$

And substituting by: $m_1 + m_2 = -\frac{2h}{b}$, $m_1 m_2 = \frac{a}{b}$ we get

$$\left(-\frac{2h}{b}\right)(y^2 - x^2) + 2xy - 2\left(\frac{a}{b}\right)xy = 0.$$

$$\therefore h(x^2 - y^2) = (a - b)xy.$$

Thus, the equation shared by the two angles of the two straight lines represented by the homogeneous equation (*) in the form:

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

To find the common equation for the two angles of the two straight lines represented by the equation:

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \qquad (**)$

We transfer the axes of the coordinates to the intersection point of the two lines represented by the equation (**), so this equation turns into an image (*), and then we find the joint equation for the two angles of the two angles represented by the equation from the previous relationship. Then, back to the point of origin, we obtain the joint equation for the two halves of the two angles represented by the equation (**).

Example: Find the equation for the two equations between the two angles represented by the equation:

$$2x^2 + 3xy - 2y^2 - x + 3y - 1 = 0.$$

Solution: First we find the two straight lines represented by the given equation by analyzing the left side therein as follows:

$$2x^2 + 3xy - 2y^2 - x + 3y - 1 = (2x - y + \alpha)(x + 2y + \beta).$$

Comparing the coefficients of x, y and the absolute term at the two sides, we get :

$$\alpha + 2\beta = -1. \qquad (1)$$
$$2\alpha - \beta = 3. \qquad (2)$$
$$\alpha\beta = -1 \qquad (3)$$

From (1), (2) we get , $\alpha = 1$, $\beta = -1$. This achieves equation (3).

So the two lines are:

$$\mathbf{2x - y + 1 = 0 .} \qquad \mathbf{(4)}$$
$$x + 2y - 1 = 0. \qquad (5)$$

By solving equations (4) and (5), we get their intersection point (-1/5, 3/5).

By moving the axes to this point (i.e. placing $x = x` - 1/5$, $y = y` + 3/5$), the new image of the given equation is:

$$2x`^2 + 3x`y` - 2y`^2 = 0. \qquad (6)$$

The equation for the two straight lines between the two angles (6) is given by the relationship: $h(x`^2 - y`^2) = (a - b)(x`y`)$.

Substituting for a = 2, b = -2, h = 3/2 the equitable equation for the two lines (6) is: $3x`^2 - 8x`y` - 3y`^2 = 0$.

Returning to the original axes (i.e., x` = x +1/5, y` = y - 3/5), the common equation for the two straight lines of the two angles between the two lines represented by the given equation is:

$3(x+1/5)^2 - 8(x+1/5)(y-3/5) - 3(y-3/5)^2 = 0.$

That is: $3x^2 - 8xy - 3y^2 + 6x + 2y = 0.$

Various examples:

1-Verify that the equation: $x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0.$

They represent two parallel straight lines, and find the shortest length in between.

Solution: Requirement for representation of second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Two parallel straight lines are:

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 \ , \ h^2 = ab.$$

In substitution of the given equation it is:

$$\Delta = \begin{vmatrix} 1 & 3 & 2 \\ 3 & 9 & 6 \\ 2 & 6 & -5 \end{vmatrix} = 1(-45-36) - 3(-15-12) + 2(18-18) = -81 + 81 = 0,$$

$$h^2 = (3)^2 = 9 \ , \ ab = (1)(9) = 9. \therefore h^2 = ab$$

So the given equation represents two parallel straight lines.

Solve the equation given as a second degree equation in the following:

$$x = \frac{-(6y+4) \pm \sqrt{(6y+4)^2 - 4(1)(9y^2 + 12y - 5)}}{2(1)}$$

$$= \frac{-(6y+4) \pm \sqrt{36y^2 + 48y + 16 - 36y^2 - 48y + 20}}{2}$$

$$= \frac{-(6y+4) \pm \sqrt{36}}{2}.$$

So the two straight lines are: $x + 3y - 1 = 0$, $x + 3y + 5 = 0.$

And the intersection point of the first straight line: $x+3y-1=0$

With the y-axis are. $(0,\frac{1}{3})$

The length of the shortest distance h between the two lines Is equal to the length of the falling column from this point on the second straight line, which is: $x+3y+5=0$ where;

$$h=\left|\frac{1(0)+3(\frac{1}{3})+5}{\sqrt{1^2+3^2}}\right|=\frac{6}{\sqrt{10}}.$$

2 - Prove that equation: $3x^2-4xy-4y^2+14x+12y-5=0$

They represent two straight lines and find the common equation for the two halves between them.

Solution: The condition for the representation of the second degree equation : $ax^2+2hxy+by^2+2gx+2fy+c=0$

Two straight lines are: $\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$

And with compensation from the given equation be

$$\Delta=\begin{vmatrix} 3 & -2 & 7 \\ -2 & -4 & 6 \\ 7 & 6 & -5 \end{vmatrix}=3(20-36)-(-2)(10-42)+7(-12+28)=0.$$

So the given equation represents two straight lines.

To find them, we analyze the left side of the given equation as follows: $3x^2-4xy-4y^2+14x+12y-5=(3x+2y+\alpha)(x-2y+\beta)$. and comparing the coefficients of x, y and the absolute term of the two sides, we get:

$\alpha+3\beta=14.$ (1)

$-2\alpha+2\beta=12. \Rightarrow -\alpha+\beta=6.$ (2)

$\alpha\beta=-5.$ (3)

From (1), (2) we get : $\alpha = -1$, $\beta = 5$. This achieves equation (3). So the two lines are:

$3x + 2y - 1 = 0.$ (4)

$x - 2y + 5 = 0.$ (5)

and by solving equations (4) and (5), we get their intersection point (-1, 2).Also by moving the axes to this point (i.e., placing $x = x^{`}-1$, $y = y^{`} + 2$), the new form of the given equation is:

$3x^{`2} - 4x^{`}y^{`} - 4y^{`2} = 0.$ (6)

And the common equation for the two straight lines of the two angles between the two lines (6) is given by the relation:

$h(x^{`2} - y^{`2}) = (a - b)(x^{`}y^{`})$

Substituting for a = 3, b = -4, h = -2, the equitable equation for the straight lines (6) is: $2x^{`2} + 7x^{`}y^{`} - 2y^{`2} = 0$.

Returning to the original axes (i.e., put in $x^{`} = x + 1$, $y^{`} = y-2$), the common equation for the two straight lines of the two angles between the two lines represented by the given equation is:

$2(x + 1)^2 + 7(x + 1)(y - 2) - 2(y - 2)^2 = 0.$

$\therefore 2x^2 + 7xy - 2y^2 - 10x + 15y - 20 = 0.$

3 - Prove that the righteous: $a^2x^2 + 2h(a+b)xy + b^2y^2 = 0.$ (1)

they are equally inclined on the straight lines:

$ax^2 + 2hxy + by^2 = 0.$ (2)

Solution: If we prove that the joint equation of the two angles of the two straight lines (1) is the same as the common equation of the two halves of the two angles between the two straight lines (2) then the two straight lines (1) are equally inclined over the two straight lines (2). the common equation for the two halves of the two angles represented by equation (1) is:

$$\frac{x^2 - y^2}{a^2 - b^2} = \frac{xy}{h(a+b)} \quad , \text{ i.e. } \frac{x^2 - y^2}{a - b} = \frac{xy}{h} \quad (3)$$

And the equation for the two equations in the two angles represented by equation (2) is: $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$

That is, the common equation for the two equitable angles between the two straight lines (1) is the same for the two straight lines (2) and from that produces the required.

Exercises:

1- Verify that each of the following equations represents two straight lines:

(i) $x^2 - 4y^2 - 6x + 16y - 7 = 0$.

(ii) $6x^2 + 5xy - 6y^2 - 3x + 28y - 30 = 0$.

(iii) $15x^2 + 19xy - 10y^2 + 7x + 22y - 4 = 0$.

and find the point of their intersection, and the angle between them.

2-Find the value of k, which makes each of the following equations represent two straight lines:

(i) $12x^2 - 13xy - 14y^2 + 38x - 81y + k = 0$.

(ii) $x^2 - xy + ky^2 - 3x - 3y = 0$.

(iii) $x^2 + kxy + y^2 - 5x - 7y + 6 = 0$

3-Find the value of c that makes the equation:

$$6x^2 - 42xy + 60y^2 - 11x + 10y + c = 0$$

represents two straight lines. And prove that the angle between them is equal : $\tan^{-1}\left(\frac{3}{11}\right)$

4-Find the value of a, and c to represent the equation:

$$ax^2 + 3xy - 2y^2 - x + 3y + c = 0$$

Two straight orthogonal lines

5-Find the equation of a straight line that passes by the intersection point of the two lines:.

$4x - y + 1 = 0$, $2x + 5y - 6 = 0$.

It is perpendicular to the straight line $4x + 3y = 7$

6-Find the equation of a line that passes the intersection point of the two lines represented by the equation

$$10x^2 + 19xy + 6y^2 + 16x + 2y - 8 = 0$$

It is perpendicular to the straight line: $x - y = 0$

7-Find the longest distance between the two straight lines:

$2x + y - 3 = 0$, $4x + 2y + 1 = 0$.

8 - Prove that equation : $18x^2 - 48xy + 32y^2 + 9x - 12y - 54 = 0$

They represent two parallel straight lines, and find the shortest length in between.

9- Find the equation for the two straight lines between the two straight linesWith the equation :

$2x^2 - xy - y^2 + 4x + 5y - 6 = 0$

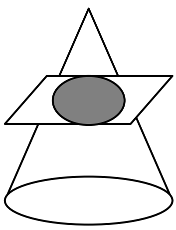
10 - Find the equation for the two equations that are fair to the two angles, between the two lines represented by the equation:

$x^2 + 5xy - 6y^2 - 7x - 4y + 2 = 0$

The **conical Sections**:

The **parabola**, **ellipse**, and **hyperbola** are cases of curves called **conic sections.** The name is derived from the fact that they may obtained as sections made by a plane with a double right circular cone.

The kind of curve produced is determined by the angle at which the plane intersects the surface.



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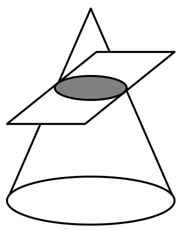
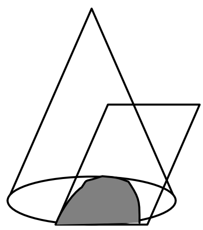
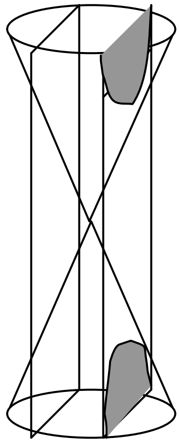


ellipse



parabola



hyperbola

Mathematical definition of conical Sections

Conic Section is the geometric place of a point moving in the plane so that the ratio between its distance from a fixed point in the plane and then from a straight line in the level is always a fixed amount. The fixed point is called the focus of the conic and a straight line is called directory and The fixed ratio is called eccentricity It is denoted by the symbol **e** and the straight line passing through the focus and perpendicular to the guide is called the cutoff axis, The straight line is called the focal length and perpendicular to the axis of the focal segment.

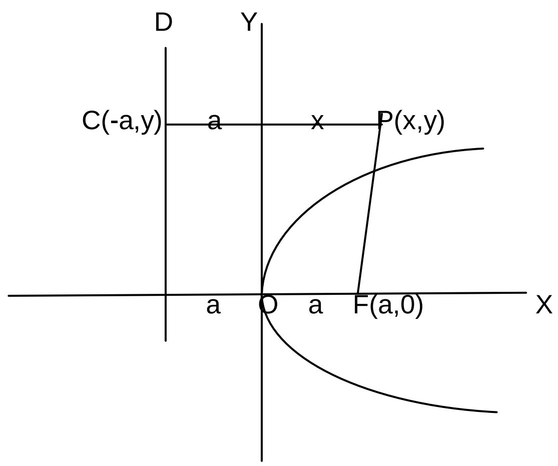
a)- If it is: $e \to \infty$ represents a pair of lines
b)- If it is: $e = 0$ represents a circle.
C)- If it is: $e = 1$ represents a parabola
d)- If it is: $e < 1$ represents ellipse
f)- If it is: $e > 1$ represents hyperbola

The Parabola

Definition: The locus of a point which moves such that it is equidistant from a fixed point and a fixed line is called a **parabola**.
The fixed point is called the **focus** and the fixed line the **directrix**.
The line passing through the focus and perpendicular to the directrix is called the **axis** of the parabola.
The middle point between the focus and the direcrix is called the **vertex** of the parabola.
The chord through focus paralled to the direcrix is called the **latus rectum**.
The distance of any point on the parabola from its focus is called the **focal distance** of the point.

Solved problem: Sketch, and then find the equation of the parabola whose vertex is $(0,0)$ and focus is $(a,0)$?.

Solution:
According to the definition of a parabola:



$$\overline{PF} = \overline{PC} \Rightarrow \sqrt{(x-a)^2 + (y-0)^2} = x + a.$$

$$\Rightarrow (x-a)^2 + (y-0)^2 = (x+a)^2$$

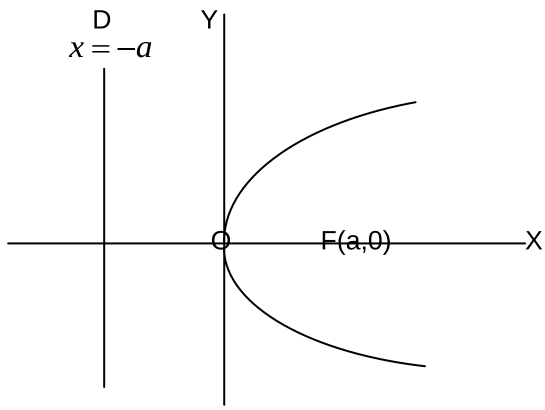
$$\Rightarrow x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$$

$$\therefore y^2 = 4ax.$$

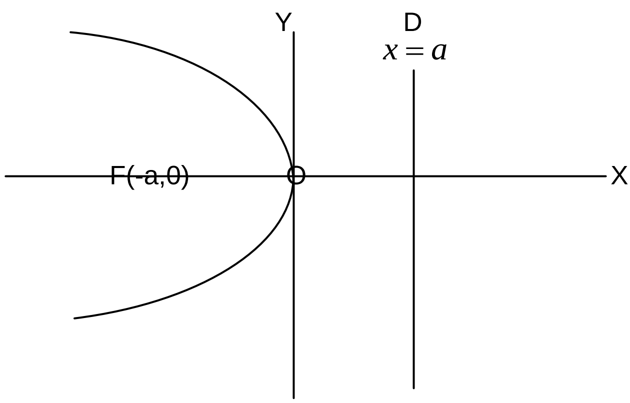
This is the equation of the parabola whose vertex is $(0,0)$, focus is $(a,0)$, direcrix $x = -a$, and the length of the latus rectum is $|4a|$.

- **Standard forms of parabola:**

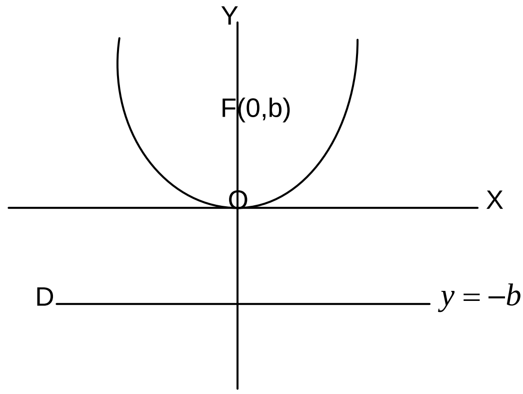
(1) The equation of the parabola with vertex at the origin $O(0,0)$ and focus at $(a,0)$ is: $y^2 = 4ax$



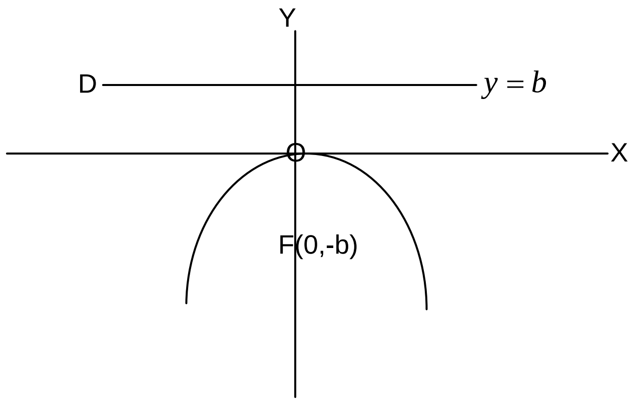
(2) The equation of the parabola with vertex at the origin $O(0,0)$ and focus at $(-a,0)$ is: $y^2 = -4ax$



(3) The equation of the parabola with vertex at the origin $O(0,0)$ and focus at $(0,b)$ is: $x^2 = 4by$



(4) The equation of the parabola with vertex at the origin $O(0,0)$ and focus at $(0,-b)$ is: $x^2 = -4by$



- The equation of parabola whose vertex (α,β), and the **axis** parallel to $X-axis$ is: $(y-\beta)^2 = 4a(x-\alpha)$
  (with focus $(a+\alpha,\beta)$, and direcrix $x = -a+\alpha$).
- The equation of parabola whose vertex (α,β), and the **axis** parallel to $Y-axis$ is: $(x-\alpha)^2 = 4b(y-\beta)$
  (with focus $(\alpha,-b+\beta)$, and direcrix $y = b+\beta$).

- **Remarks**:

1- The parabola $y^2 = 4ax$ ($(y-\beta)^2 = 4a(x-\alpha)$) opens to the right.

2- The parabola $y^2 = -4ax$ ($(y-\beta)^2 = -4a(x-\alpha)$) opens to the left

3- The parabola $x^2 = 4by$ ($(x-\alpha)^2 = 4b(y-\beta)$) opens to the upward.

4- The parabola $x^2 = -4by$ ($(x-\alpha)^2 = -4b(y-\beta)$) opens to the downward.

- **The General equation of parabola** whose focus (h,k) and direcrix $ax+by+c=0$ is: $(x-h)^2+(y-k)^2 = \dfrac{(ax+by+c)^2}{a^2+b^2}$.

Solved Problems:

1- Sketch, and then find the equation of the parabola whose:

(i) vertex is $(0,0)$ and focus is $(0,-2)$.

(ii) vertex is $(3,3)$ and focus is $(-3,3)$.

(iii) vertex is $(1,2)$ and focus is $(3,2)$.

(iv) vertex is $(1,2)$ and focus is $(1,0)$.

2- Find the vertex, focus, direcrix, and length of the latus rectum of the following parabolas:

(i) $y^2 = 12x$ (ii) $x^2 = -6y$

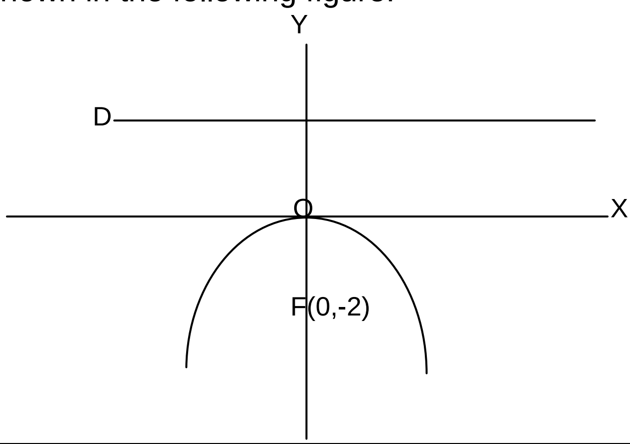
3- Find the equation of the parabola whose:

(i) focus is $(5,2)$ and direcrix is $x-1=0$.

(ii) focus is $(4,4)$ and direcrix is $y-5=0$.

The answer:

1- (i) As shown in the following figure:

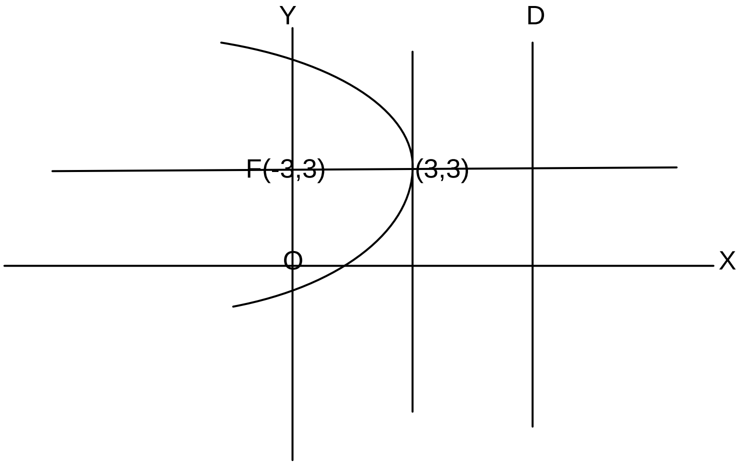


The form of the parabola is $x^2 = -4by$,

the focus $(0,-b) = (0,-2) \Rightarrow b = 2$

So, the equation of the parabola is $x^2 = -8y$.

(ii) As shown in the following figure:

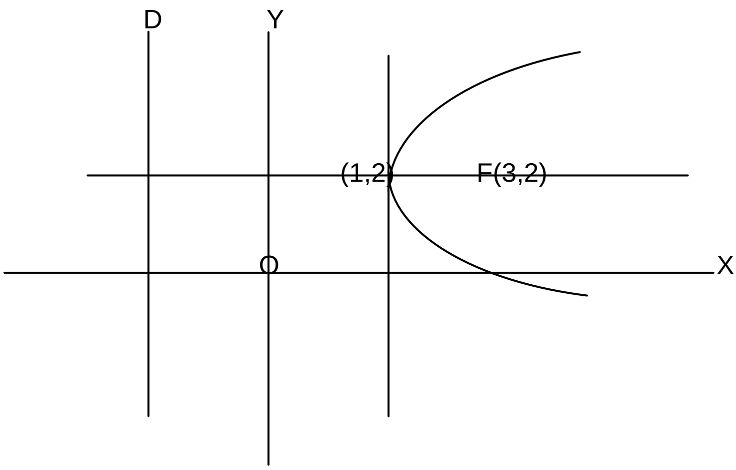


The form of the parabola is $(y-\beta)^2 = -4a(x-\alpha)$,

$V(\alpha,\beta) = (3,3)$, $F(a+\alpha,\beta) = (a+3,3) = (-3,3) \Rightarrow a = 6$

So, the equation of the parabola is $(y-3)^2 = -24(x-3)$.

(iii) As shown in the following figure:

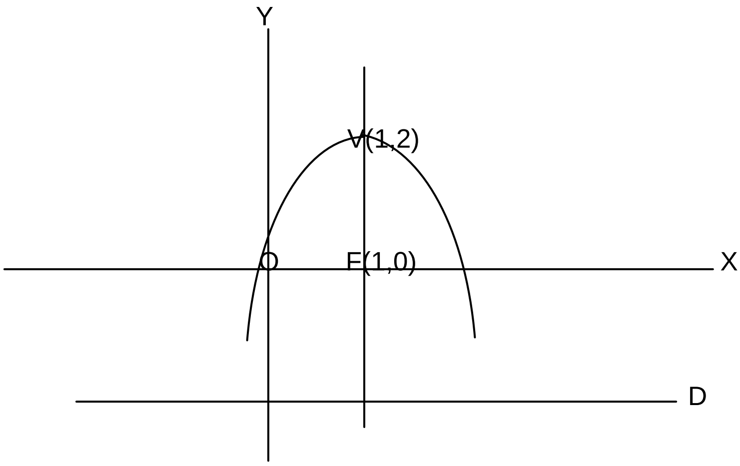


The form of the parabola is $(y-\beta)^2 = 4a(x-\alpha)$,

$V(\alpha,\beta) = (1,2)$, $F(a+\alpha,\beta) = (a+1,2) = (3,2) \Rightarrow a = 2$

So, the equation of the parabola is $(y-2)^2 = 8(x-1)$.

(iv) As shown in the following figure:



The form of the parabola is $(x-\alpha)^2 = -4b(y-\beta)$,
$V(\alpha,\beta) = (1,2)$, $F(\alpha,-b+\beta) = (1,0) = (1,-b+2) \Rightarrow b = 2$
So, the equation of the parabola is $(x-1)^2 = -8(y-2)$.

2- (i) $y^2 = 12x$ comparing with the form $y^2 = 4ax$:
$\therefore a = 3$ Hence $y^2 = 12x$ represents a parabola opens to the right, and whose vertex is $(0,0)$,
focus is $(a,0) = (3,0)$, and the length of latus rectum is $|4a| = 12$.

2- (ii) $x^2 = -6y$ comparing with the form $x^2 = -4by$:
$\therefore b = \dfrac{3}{2}$ Hence $x^2 = -6y$ represents a parabola opens to the downward, and whose vertex is $(0,0)$, focus is $(0,-b) = (0,-\dfrac{3}{2})$,
and the length of latus rectum is $|4b| = 6$.

4- The General equation of parabola whose focus (h,k), and direcrix $ax+by+c=0$ is: $(x-h)^2 + (y-k)^2 = \dfrac{(ax+by+c)^2}{a^2+b^2}$.

(i): $(x-5)^2 + (y-2)^2 = \dfrac{(x-1)^2}{1^2+0^2}$

$$\therefore (x^2-10x+25)+(y^2-4y+4)=x^2-2x+1$$
$$\Rightarrow y^2-4y=-8x+28$$
$$\Rightarrow (y-2)^2-4=-8x+28$$
$$\Rightarrow (y-2)^2=-8x+32$$
$$\Rightarrow (y-2)^2=-8(x-4).$$

(ii): $(x-4)^2+(y-4)^2=\dfrac{(y-5)^2}{0^2+1^2}$

$$\therefore (x^2-8x+16)+(y^2-8y+16)=y^2-10y+25$$
$$\Rightarrow x^2-8x=-2y-7$$
$$\Rightarrow (x-4)^2-16=-2y-7$$
$$\Rightarrow (x-4)^2=-2y+9$$
$$\Rightarrow (x-4)^2=-2(y-\frac{9}{4}).$$

Exercises:

1- Find the vertex, focus, direcrix, and length of the latus rectum of the following parabolas: (i) $y^2=8x$ (ii) $x^2=12y$

2- Find the equation of the parabola whose focus is $(1,2)$ and direcrix is $x+2=0$.

- References:

1- PK.Jain.Khalil Ahmed (2005). *Textbook of Analytic Geometry of Two Dimension. 2nd edition.* New Age International (P) Ltd.
2- J.H.Kindle. (1990). *Analytic Geometry*. Schaum's Outline.
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4- Yefimov, N.V. (1964). *A Brief course in Analytic Geometry.* Mir publishers.

Calculus of Integration

Prof. Dr. M. Abul-Ez

Indefinite Integration

If $F(x)$ is a function such that $F'(x) = f(x)$ on the interval $[a,b]$ Then $F(x)$ is called an anti-derivative or indefinite of $f(x)$. The indefinite integral of the given function is not unique for example x^2, x^2+3, x^2+5 are indefinite integral of $f(x)=2x$ since $\frac{d}{dx}(x^2)=\frac{d}{dx}(x^2+3)=\frac{d}{dx}(x^2+5)=2x$ All of indefinite integral of $f(x)=2x$ include in $f(x)=2x+c$ where c called the constant of integration, is an arbitrary constant.

1.1- Fundamental Integration Formula:

$$(1)\int \frac{d}{dx}f(x) = f(x)+c$$

$$(2)\int (u+v)dx = \int u\,dx + \int v\,dx$$

$$(3)\int \alpha u\,dx = \alpha\int u\,dx$$

$$(4)\int x^m\,dx = \frac{x^{m+1}}{m+1}+c \quad m \neq -1$$

$$(5)\int \frac{dx}{x} = \ln|x|+c$$

$$(6)\int a^x\,dx = \frac{a^x}{\ln a}+c$$

$$(7)\int e^x\,du = e^x+c$$

Integration of Trigonometric functions:

$$(8)\int \sin x\,dx = -\cos x+c$$

$$(9)\int \cos x\,dx = \sin x+c$$

$$(10)\int \tan x\,dx = \ln|\sec x|+c$$

$$(11)\int \cot x\,dx = \ln\cos x+c$$

(12) $\int \sec x\,dx = \ln|\sec x + \tan x| + c$

(13) $\int \text{cosec}\,dx = \ln|\csc x - \cot x| + c$

(14) $\int \sec^2 x\,dx = \tan x + c$

(15) $\int \text{cosec}^2\,dx = -\cot x + c$

(16) $\int \sec x\,\tan x\,dx = \sec x + c$

(17) $\int \text{cosec}\,x \cot x\,dx = -\text{cosec}\,x + c$

Integration tends to inverse of Trigonometric functions:

(18) $$\int \frac{dx}{\sqrt{a^2 - b^2x^2}} = \begin{cases} \frac{1}{b}\sin^{-1}\frac{bx}{a} + c \\ \frac{-1}{b}\cos^{-1}\frac{bx}{a} + c \end{cases}$$

(19) $$\int \frac{dx}{a^2 + b^2x^2} = \begin{cases} \frac{1}{ab}\tan^{-1}\frac{bx}{a} + c \\ \frac{-1}{ab}\cot^{-1}\frac{bx}{a} + c \end{cases}$$

(20) $$\int \frac{dx}{x\sqrt{b^2x^2 - a^2}}\,dx = \begin{cases} \frac{1}{a}\sec^{-1}\frac{bx}{a} + c \\ \frac{-1}{a}\text{cosec}^{-1}\frac{bx}{a} + c \end{cases}$$

(21) $$\int \frac{dx}{b^2x^2 - a^2}\,dx = \begin{cases} \frac{1}{ab}\coth^{-1}\frac{bx}{a} + c \\ \frac{1}{2ab}\ln\left|\frac{bx - a}{bx + a}\right| + c \end{cases}$$

(22) $$\int \frac{dx}{a^2 - b^2x^2} = \begin{cases} \frac{1}{ab}\tanh^{-1}\frac{bx}{a} + c \\ \frac{1}{2ab}\ln\left|\frac{a + bx}{a - bx}\right| + c \end{cases}$$

(23) $$\int \frac{dx}{\sqrt{x^2 + a^2}} = \begin{cases} \ln(x + \sqrt{x^2 + a^2}) + c \\ \sinh^{-1}\frac{x}{a} + c \end{cases}$$

$$(24)\int \frac{dx}{\sqrt{x^2-a^2}} = \begin{cases} \ln(x+\sqrt{x^2-a^2})+c \\ \cosh^{-1}\frac{x}{a}+c \end{cases}$$

$$(25)\int \sqrt{a^2-x^2} = \frac{1}{2}x\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c$$

$$(26)\int \sqrt{x^2+a^2} = \frac{1}{2}x\sqrt{x^2+a^2}+\frac{a^2}{2}\sinh^{-1}\frac{x}{a}+c$$

$$(27)\int \sqrt{x^2-a^2} = \frac{1}{2}x\sqrt{x^2-a^2}-\frac{a^2}{2}\cosh^{-1}\frac{x}{a}+c$$

In the following some lows witch we use to integrate the square of trigonometric functions

$$(1)\cos^2 x+\sin^2 x=1,$$

$$(2) 1+\tan^2 x=\sec^2 x,$$

$$(3)\cot^2 x+1=\csc^2 x$$

$$(4)\ \sin^2 x=\frac{1}{2}(1-\cos 2x)$$

$$(5)\cos^2 x=\frac{1}{2}(1+\cos 2x)$$

$$(6)\cos x\cos y=\frac{1}{2}\left[\cos(x+y)+\cos(x-y)\right]$$

$$(7)\sin x\sin y=\frac{1}{2}\left[\cos(x-y)-\cos(x+y)\right]$$

$$(8)\sin x\cos y=\frac{1}{2}\left[\sin(x+y)+\sin(x-y)\right]$$

Integration of square of trigonometric functions:

$$(1)\int \sin^2 x\,dx=\frac{1}{2}\int(1-\cos 2x)dx=\frac{1}{2}\left(x-\frac{1}{2}\cos 2x\right)+c$$

$$(2)\int \cos^2 x\,dx=\frac{1}{2}\int(1+\cos 2x)dx=\frac{1}{2}\left(x+\frac{1}{2}\cos 2x\right)+c$$

$$(3)\int \sec^2 x\,dx=\tan x+c,$$

$$(4)\int \operatorname{cosec}^2 x\,dx = -\cot x + c$$

$$(5)\int \tan^2 x\,dx = \int (\sec^2 - 1)dx = \tan x - x + c$$

$$(6)\int \cot^2 x\,dx = \int (\operatorname{cosec}^2 x - 1)dx = -\cot x - x + c$$

$$(7)\int \cos ax \cos bx\; dx = \frac{1}{2}\int \left[\cos(a+b)x + \cos(a-b)x\right]dx$$

$$= \frac{1}{2}\left[\frac{\sin(a+b)x}{a+b} + \frac{\sin(a-b)x}{(a-b)}\right] + c$$

$$(8)\ \int \sin ax \sin bx\,dx = \frac{1}{2}\int\left[\cos(a-b)x - \cos(a+b)x\right]dx$$

$$= \frac{1}{2}\left[\frac{\sin(a-b)x}{a-b} + \frac{\sin(a+b)x}{(a+b)}\right] + c$$

$$(9)\int \sin ax \cos bx\; dx = \frac{1}{2}\int\left[\sin(a+b)x + \sin(a-b)x\right]dx$$

$$= \frac{-1}{2}\left[\frac{\cos(a+b)x}{(a+b)} + \frac{\cos(a-b)x}{(a-b)}\right] + c$$

Solved Examples:

Exampl (1): $\int x^5\,dx = \frac{1}{6}x^6 + c$

Exampl (2): $\int \sqrt[3]{x}\; dx = \int x^{\frac{1}{3}}\; dx = \frac{3}{4}x^{\frac{4}{3}} + c$

Exampl (3): $\int (ax+b)^n\; dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{(n+1)} + c$

Exampl (4): $\int \frac{1}{(ax+b)}dx = \frac{1}{a}\ln\left|(ax+b)\right| + c$

Exampl (5): $\int \frac{1}{(ax+b)^n}dx = \int (ax+b)^{-n}\; dx = \frac{1}{a}\frac{(ax+b)^{-n+1}}{(-n+1)} + c,\;\; n \neq 1$

Exampl (6): $\int \frac{1}{\sqrt{ax+b}}dx = \frac{2}{a}\sqrt{ax+b}+c$

Exampl (7): $\int (1-x)\sqrt{x}\,dx = \int (x^{\frac{1}{2}} - x^{\frac{3}{2}})dx = \int x^{\frac{1}{2}}dx - \int x^{\frac{3}{2}}dx = \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} + c$

Exampl (8): Find $(i)\int (x^3+2)^5 x^2\,dx,\quad (ii)\int \frac{x^2dx}{\sqrt[4]{x^3+2}}dx \quad (iii)\int \sqrt{x^3+2}\;\; x^2\,dx$

$let\;\; u = x^3+2 \;\; \therefore du = 3x^2du$

substitute in the integral we have

$$(i)\int (x^3+2)^5 x^2\,dx = \frac{1}{3}\int u^5du = \frac{1}{18}u^6 + c = \frac{1}{18}(x^3+2)^6 + c$$

$$(ii)\int \frac{x^2dx}{\sqrt[4]{x^3+2}}dx = \frac{1}{3}\int \frac{du}{\sqrt[4]{u}}dx = \frac{1}{3}\int u^{\frac{-1}{4}}\,du = \frac{4}{9}u^{\frac{3}{4}} = \frac{4}{9}(x^3+2)^{\frac{3}{4}} + c$$

$$(iii)\int \sqrt{x^3+2}\;\;(x^2\,dx) = \frac{1}{9}\int \sqrt{u}\;\; du = \frac{2}{9}u^{\frac{3}{2}} + c = \frac{2}{9}(x^3+2)^{\frac{3}{2}} + c$$

Exampl (9):

$$\int x^3\sqrt{x^2+1}\,dx = \int (x^3+x-x)\sqrt{x^2+1}\,dx$$

$$= \int (x^3+x)\sqrt{x^2+1}\;\,dx - \int x\sqrt{x^2+1}\;dx$$

$$= \int x(x^2+1)\sqrt{x^2+1}\;dx - \int x\sqrt{x^2+1}\;dx$$

$$= \int x(x^2+1)^{\frac{3}{2}}dx - \int x\sqrt{x^2+1}\;dx = \frac{2}{10}(x^2+1)^{\frac{5}{2}} - \frac{2}{6}(x^2+1)^{\frac{3}{2}} + c$$

Exampl (10): $\int \frac{dx}{2x-3} = \frac{1}{2}\int \frac{2dx}{2x-3} = \frac{1}{2}\ln|2x-3| + c$

Exampl (11): $\int \frac{x^2dx}{1-2x^3} = -\frac{1}{6}\int \frac{-6x^2dx}{1-2x^3} = \frac{-1}{6}\ln(1-2x^3) + c$

Exampl (12): $\int \frac{x+5}{(x+1)}dx = \int \frac{(x+1)+4}{(x+1)}dx = \int \frac{(x+1)}{(x+1)}dx + \int \frac{4}{(x+1)}dx$

$$= \int \left(1 + \frac{4dx}{(x+1)}\right) = x + 4\ln|x+1| + c$$

Example(13): $\int \tan\frac{1}{2}x\,dx = 2\ln\left|\sec\frac{1}{2}x\right| + c$

Example(14): $\int x(\sec x^2)dx = \frac{1}{2}\int \sec x^2\,(d\,x^2) = \frac{1}{2}\ln\left|\sec x^2 + \tan x^2\right| + c$

Example(15): $\int \sin^2 x\,\cos x\,dx = \int \sin^2 x\,d(\sin x) = \frac{1}{3}\sin^3 x + c$

Example(16):

$$\int \frac{(1+x^2)^3}{\sqrt{x}}dx = \int \frac{(1+x^2)^3}{\sqrt{x}}dx = \int \frac{(1+3x^2+3x^4+x^6)}{\sqrt{x}}dx$$

$$= \int \left(\frac{1}{\sqrt{x}} + \frac{3x^2}{\sqrt{x}} + \frac{3x^4}{\sqrt{x}} + \frac{x^6}{\sqrt{x}}\right)dx$$

$$= \int \frac{dx}{\sqrt{x}} + \int 3x^{\frac{3}{2}}dx + \int 3x^{\frac{7}{2}}dx + \int x^{\frac{11}{2}}dx$$

$$= 2\sqrt{x} + \frac{6}{5}x^{\frac{5}{2}} + \frac{6}{9}x^{\frac{9}{2}} + \frac{2}{13}x^{\frac{13}{2}} + c$$

Example(17): $\int (e^x+1)^5 e^x\,dx = \int (e^x+1)^5\,d(e^x+1) = \frac{1}{6}(e^x+1)^6 + c$

Example(18): $\int a^{3x}\,dx = \frac{1}{3}\int a^{3x}\,(3dx) = \frac{1}{3\ln a}a^{3x} + c$

Example(19): $\int \frac{e^{2x}}{(e^{2x}-1)}dx = \frac{1}{2}\int \frac{d(e^{2x}-1)}{(e^{2x}-1)}dx = \frac{1}{2}\ln(e^{2x}-1) + c$

Example(20): $\int \frac{e^{2x}dx}{(e^{2x}-1)^5} = \frac{1}{2}\int \frac{d(e^{2x}-1)}{(e^{2x}-1)^5} = \frac{1}{2}\int (e^{2x}-1)^{-5}d(e^{2x}-1)$

$$= \frac{1}{2}\frac{(e^{2x}-1)^{-4}}{-4} + c$$

Example(21): $\int \sin 4x\,dx = \frac{-1}{4}\cos 4x + c,$ (19) $\int \cos 3x\,dx = \frac{1}{3}\sin 3x + c$

Example(22): $\int \cos^4 x\ \ \sin x\,dx = -\int \cos^4 x\,(-\sin x\,dx)$

Example(23): $\int \tan^3 x \sec^2 x\,dx = \int \tan^3 x \; d(\tan x) = \frac{1}{4}\tan^4 x + c$

Example(24): $\int \cot^5 x \operatorname{cosec}^2 x\,dx = -\int \cot^5 x(-\operatorname{cosec}^2 x)dx$

$$= -\int \cot^5 x \; d(\cot x) = -\frac{1}{6}\cot^6 x + c$$

Example(25): $\int \cos^5 x \sin^2 x\,dx = \int \cos^4 x \sin^2 x(\cos x dx)$

$$= \int (1-\sin^2 x)^2 \sin^2 x(\cos x dx)$$

$$= \int (1-2\sin^2 x + \sin^4 x)\sin^2 x(\cos x dx)$$

$$= \int (\sin^2 x - 2\sin^4 x + \sin^6 x)(\cos x dx)$$

let $y = \sin x \Rightarrow dy = \cos x dx$

$$\therefore I = \int (y^2 - 2y^4 + y^6)dy = \frac{1}{3}y^3 - \frac{2}{5}y^5 + \frac{1}{7}y^7 + c$$

$$= \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + c$$

Example(26): $\int \cos^4 x \sin^3 x\,dx = \int \cos^4 x \sin^2 x(\sin x\,dx)$

$$= \int \cos^4 x\,(1-\cos^2)(\sin x\; dx)$$

$$= \int (\cos^4 x - \cos^6 x)(\sin x dx)$$

let $y = \cos x \Rightarrow dy = -\sin x dx$

$$\therefore I = \int (y^4 - y^6)dy = \frac{1}{5}y^5 - \frac{1}{7}y^7 + c$$

$$= \frac{1}{5}\cos^5 x - \frac{1}{7}\cos^7 x + c$$

Try to solve $\int \sin^3 x \cos^5 x\,dx$

$\int \sin^4 x \cos^4 x\,dx$

Example(27): $\int \sec^4 x \tan x\, dx = \int \sec^3 x\, (\sec x \tan x)\, dx$

$$= \int \sec^3 x\, d(\sec x) = \frac{1}{4}\sec^4 + c$$

Example(28): $\int \frac{\sin x}{\cos^2 x} dx = \int \frac{\sin x}{\cos x}\frac{1}{\cos x} dx = \int \tan x \sec x\, dx = \sec x + c$

Example(29): $\int \frac{\cos x}{\sin^2 x} dx = \int \frac{\cos x}{\sin x}\, \frac{1}{\sin x} dx = \int \cot x\, \operatorname{cosec} x\, dx = -\cot x + c$

Example(30): $\int \frac{dx}{1+\cos x} = \int \frac{(1-\cos x)dx}{1-\cos^2 x} = \int \frac{(1-\cos x)dx}{\sin^2 x}$

$$= \int \frac{dx}{\sin^2 x} - \int \frac{\cos x\, dx}{\sin^2 x}$$

$$= \int \operatorname{cosec}^2 x dx - \int \cot x \operatorname{cosec} x\, dx$$

$$= -\cot x + \operatorname{cosec} x + c$$

Example(31): $\int \sec x\, dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$

$$= \ln\left|\sec x + \tan x\right| + c$$

Example(32): $\int \operatorname{cosec} x\, dx = \int \operatorname{cosec}\, x \frac{\operatorname{cosec}\, x - \cot x}{\operatorname{cosec}\, x - \cot x}\, dx$

$$= \int \frac{\operatorname{cosec}^2 x - \operatorname{cosec} x \cot x}{\operatorname{cosec}\, x - \cot x}\, dx = \ln\left|\operatorname{cosec} x - \cot x\right| + c$$

Example(33): *FindThefolowing* int *egrals*

$$(i) \quad I = \int \sin x \sqrt{(2+3\cos x)}\, dx,$$

$$(ii)\ J = \int \frac{\sin x\, dx}{\sqrt{(2+3\cos x)}}, \qquad (iii)\ K = \int \frac{\sin x\, dx}{(2+3\cos x)^4}$$

let $u = 2+3\cos x \Rightarrow du = -3\sin x\, dx$

$$\therefore (i) \quad I = \int \sin x \sqrt{(2+3\cos x)}\, dx = \frac{-1}{3}\int \sqrt{u}\, du = \left(\frac{-1}{3}\right).\left(\frac{2}{3}\right)u^{\frac{3}{2}}$$

$$= \left(\frac{-2}{9}\right)(2+3\cos x)^{\frac{3}{2}} + c$$

$$(ii)\ \ J=\int\frac{\sin x dx}{\sqrt{(2+3\cos x)}}=\frac{-1}{3}\int\frac{du}{\sqrt{u}}=\frac{-1}{3}.2\sqrt{u}+c=\frac{-2}{3}\sqrt{(2+3\cos x)}+c$$

$$(iii)\ \ K=\int\frac{\sin x dx}{(2+3\cos x)^4}=\frac{-1}{3}\int\frac{du}{u^4}=\frac{-1}{3}\int u^{-4}du=\frac{-1}{3}.\left(\frac{-1}{3}u^{-3}\right)$$

$$=\frac{1}{9}(2+3\cos x)^{-3}+c$$

Example(34):

$$\int(1+\tan x)^2\,dx=\int(1+2\tan x+\tan^2 x)\,dx$$

$$=\int\Big(1+2\tan x+(\sec^2 x-1)\Big)dx=\int\Big(2\tan x+\sec^2 x\Big)dx$$

$$=2\ln\left|\sec x\right|+\tan x+c$$

Example(35): *Find* $\int(1+\cot x)^2\,dx$

$$\int(1+\cot x)^2\,dx=\int(1+2\cot x+\cot^2 x)dx$$

$$=\int\Big(1+2\cot x+(\mathrm{cosec}^2 x-1)\Big)dx$$

$$=\int\Big(2\cot x+\mathrm{cosec}^2 x\Big)dx=2\ln\left|\sin x\right|-\mathrm{cosec}\,x+c$$

Example(36):

$$\int(\tan 3x+\sec 3x)^2\,dx=\int(\tan^2 3x+2\tan 3x\ \sec 3x+\sec^2 3x)\,dx$$

$$=\int\Big((\sec^2 3x-1)+2\tan 3x\ \sec 3x+\sec^2 3x\Big)dx$$

$$=\int\Big(2\sec^2 3x+2\tan 3x\ \sec 3x-1\Big)dx$$

$$=\frac{2}{3}\tan 3x+\frac{2}{3}\sec 3x-x+c$$

Example(37): $\int e^{\sin x}\cos x\,dx=\int e^{\sin x}\,d(\sin x)=e^{\sin x}+c$

Example(38): $\int a^{\tan x}\sec^2 x\,dx=\int a^{\tan x}\,d(\tan x)=\frac{1}{\ln a}\,a^{\tan x}+c$

Example(39): $\int e^{\cos x}\sin x\,dx=-\int e^{\cos x}\,d(\cos x)=-e^{\cos x}+c$

Example(40): $\int\ (\ln\sin x)\cot x\,dx\quad let\ \ y=\ln\sin x\Rightarrow dy=\cot x dx$

$$\therefore I=\int y dy=\frac{1}{2}y^2+c=\frac{1}{2}[\ln\sin x]^2+c$$

Example(41): $\int (1+\sec x)^3\, dx = \int \left(1+3\sec x+3\sec^2 x+\sec^3 x\right) dx$

$$=\int\left(1+3\sec x+3\sec^2 x+\sqrt{1+\tan^2 x}\,\sec^2 x\right) dx$$

$$=\int\left(dx+3\sec x\,dx+3\sec^2 x\,dx+\sqrt{1+\tan^2 x}\;\;d(\tan x)\right) dx$$

$$=x+3\ln\left|\sec x+\tan x\right|+\tan x+\frac{1}{2}\tan x\sqrt{1+\tan^2 x}$$

$$+\frac{1}{2}\ln(\tan x+\sqrt{1+\tan^2 x})+c$$

Example(42): $\int \frac{dx}{\operatorname{cosec}2x-\cot 2x}dx=\int\frac{dx}{\frac{1}{\sin 2x}-\frac{\cos 2x}{\sin 2x}}=\int\frac{\sin 2x dx}{1-\cos 2x}$

$$let\;\; u=1-\cos 2x \;\Rightarrow du=2\sin 2x dx$$

$$\therefore \int\frac{\sin 2x\;dx}{1-\cos 2x}=\frac{1}{2}\int\frac{du}{u}=\frac{1}{2}\ln|u|=\frac{1}{2}\ln|1-\cos 2x|+c$$

Example(43): find $\int(1+\tan x)^3 dx$

$$\int(1+\tan x)^3 dx=\int(1+3\tan x+3\tan^2 x+\tan^3 x)dx$$

$$=\int\left(1+3\tan x+3(\sec^2 x-1)+\tan x\tan^2 x\right) dx$$

$$=\int\left(1+3\tan x+3\sec^2 x-3+\tan x(\sec^2 x-1)\right) dx$$

$$=\int\left(1+3\tan x+3\sec^2 x-3+\tan x\sec^2 x-\tan x\right) dx$$

$$=\int\left(-2+2\tan x+3\sec^2 x+\tan x\sec^2 x\right) dx$$

$$=-\int 2dx+\int 2\tan x\,dx+\int 3\sec^2 x\,dx+\int\tan x\sec^2 x\,dx$$

$$=-2x+2\ln|\sec x|+3\tan x+\frac{1}{2}\tan^2 x+c$$

Example(44): $\int\frac{dx}{\sqrt{9-x^2}}dx=\sin^{-1}\frac{x}{3}+c$

Example(45): $\int\frac{dx}{5+x^2}=\frac{1}{\sqrt{5}}\tan^{-1}\frac{x}{\sqrt{5}}+c$

Example(46):

$$\int(1+\cos x)^3\,dx=\int(1+3\cos x+3\cos^2 x+\cos^3 x)dx$$

$$=\int\left(1+3\cos x+\frac{3}{2}(1+\cos 2x)+\cos x\cos^2 x\right)dx$$

$$=\int\left(1+3\cos x+\frac{3}{2}+\frac{3}{2}\cos 2x+\cos x(1-\sin^2 x)\right)dx$$

$$=\int\left(\frac{5}{2}+4\cos x+\frac{3}{2}\cos 2x-\sin^2 x\cos x\right)dx$$

$$=\frac{5}{2}x+4\sin x+\frac{3}{4}\sin 2x-\frac{1}{3}\sin^3 x+c$$

Example(47):

$$\int(1+\sin x)^3\,dx=\int(1+3\sin x+3\sin^2 x+\sin^3 x)\,dx$$

$$=\int\left(1+3\sin x+\frac{3}{2}(1-\cos 2x)+\sin x\sin^2 x\right)dx$$

$$=\int\left(1+3\sin x+\frac{3}{2}-\frac{3}{2}\cos 2x+\sin x(1-\cos^2 x)\right)dx$$

$$=\int\left(\frac{5}{2}+4\sin x+\frac{3}{2}\cos 2x-\sin x\cos^2 x\right)dx$$

$$=\frac{5}{2}x-4\cos x+\frac{3}{4}\sin 2x+\frac{1}{3}\cos^3 x+c$$

Example(48): *find* $\int\frac{dx}{4x^2+9}$

$$\int\frac{dx}{4x^2+9}=\frac{1}{4}\int\frac{dx}{x^2+\left(\frac{3}{2}\right)^2}=\frac{1}{4}.\frac{2}{3}\tan^{-1}\frac{2x}{3}+c=\frac{1}{6}\tan^{-1}\frac{2x}{3}+c$$

Another solution:

$$\int\frac{dx}{4x^2+9}=\frac{1}{9}\int\frac{dx}{\frac{4}{9}x^2+1}=\frac{1}{9}\int\frac{dx}{\left(\frac{2}{3}x\right)^2+1}=\frac{1}{9}.\frac{3}{2}\tan^{-1}\frac{2x}{3}=\frac{1}{6}\tan^{-1}\frac{2x}{3}+c$$

Another solution:

$$\int\frac{dx}{4x^2+9}=\frac{1}{2}\int\frac{2dx}{(2x)^2+3^2}=\frac{1}{2}.\frac{1}{3}\tan^{-1}\frac{2x}{3}+c$$

Example(49): $\int \frac{dx}{x\sqrt{4x^2-9}} = \int \frac{2dx}{2x\sqrt{(2x)^2-(3)^2}} = \frac{1}{3}\sec^{-1}\frac{2x}{3}+c$

According to $\int \frac{f'(x)}{f(x)\sqrt{\left(f(x)\right)^2-\left(a\right)^2}} = \frac{1}{a}\sec^{-1}\frac{f(x)}{a}+c$ where f(x) is linear

Example(50): $\int \frac{x^2dx}{\sqrt{1-x^6}} = \frac{1}{3}\int \frac{3x^2dx}{\sqrt{1-(x^3)^2}} = \frac{1}{3}\int \frac{d(x^3)}{\sqrt{1-(x^3)^2}} = \frac{1}{3}\sin^{-1}x^3+c$

Example(51): $\int \frac{\sin xdx}{\cos^2+1} = -\int \frac{d(\cos x)}{\cos^2+1} = -\tan^{-1}(\cos x)+c$

Example(52): $\int \frac{\cos x\sin xdx}{\cos^4+1} = \frac{1}{2}\int \frac{2\cos x\sin xdx}{(\cos^2 x)^2+1} = \frac{-1}{2}\int \frac{d(\cos^2 x)}{(\cos^2 x)^2+1}$

$$= \tan^{-1}(\cos^2 x)+c$$

Another solution: *let* $u=\cos^2 x \quad \Rightarrow du = 2\cos x\sin xdx$

$$\therefore \int \frac{\cos x\sin xdx}{\cos^4+1} = \frac{1}{2}\int \frac{du}{u^2+1} = \frac{-1}{2}\tan^{-1}u+c = \frac{-1}{2}\tan^{-1}(\cos^2 x)+c$$

Example(53): $\int \frac{dx}{\sqrt{25-16x^2}} = \frac{1}{4}\int \frac{4dx}{\sqrt{5^2-(4x)^2}} = \frac{1}{4}\sin^{-1}\frac{4x}{5}+c$

Example(54): $\int \frac{dx}{\sqrt{4-(x+2)^2}} = \int \frac{dx}{\sqrt{2^2-(x+2)^2}} = \sin^{-1}\frac{(x+2)}{2}+c$

Example(55): $\int \frac{3x^3-4x^2+3x}{x^2+1}dx = \int \left((3x-4)+\frac{4}{x^2+1}\right)dx$

$$= \frac{3x^2}{2}-4x+4\tan^{-1}x+c$$

Example(56): $\int \frac{dx}{x^2+4x+13} = \int \frac{dx}{(x^2+4x+4)+9} = \int \frac{dx}{(x+2)^2+3^2}$

$$= \frac{1}{3}\tan^{-1}\frac{(x+2)}{3}+c$$

Example(57):

$$\int \frac{dx}{\sqrt{20+8x-x^2}} = \int \frac{dx}{\sqrt{20-(x^2-8x)}} = \int \frac{dx}{\sqrt{36-(x^2-8x+16-16)}}$$

$$= \int \frac{dx}{\sqrt{6^2-(x-4)^2}} = \sin^{-1}\frac{(x-4)}{6} + c$$

Example(58):

$$\int \frac{(x+3)}{\sqrt{5-4x-x^2}} dx = \frac{-1}{2}\int \frac{-2x-6}{\sqrt{5-4x-x^2}} dx = \frac{-1}{2}\int \frac{(-2x-4)-2}{\sqrt{5-4x-x^2}} dx$$

$$= \frac{-1}{2}\int \frac{(-2x-2)}{\sqrt{5-4x-x^2}} dx + \frac{-1}{2}\int \frac{-2}{\sqrt{5-4x-x^2}} dx$$

$$= \frac{-1}{2}\int \frac{(-2x-2)}{\sqrt{5-4x-x^2}} dx + \int \frac{1}{\sqrt{9-(x+2)^2}} dx =$$

$$= \frac{-1}{2}.2\sqrt{5-4x-x^2} + \sin^{-1}\frac{(x+2)}{3} + c$$

Another solution $\frac{d}{dx}(5-4x-x^2) = -2x-4$

put $x+3 = A(\frac{d}{dx}(5-4x-x^2)) + B = A(-2x-4) + B$

$\therefore A = \frac{-1}{2}, B = 1, \quad x+3 = \frac{-1}{2}(-2x-4)+1,$

$$\int \frac{(x+3)}{\sqrt{5-4x-x^2}} dx = \int \frac{\frac{-1}{2}(-2x-4)+1}{\sqrt{5-4x-x^2}} dx$$

$$= \int \frac{\frac{-1}{2}(-2x-4)dx}{\sqrt{5-4x-x^2}} + \int \frac{dx}{\sqrt{5-4x-x^2}} = \frac{-1}{2}\int \frac{(-2x-2)dx}{\sqrt{5-4x-x^2}}$$

$$+ \int \frac{dx}{\sqrt{9-(x+2)^2}} = \frac{-1}{2}.2\sqrt{5-4x-x^2} + \sin^{-1}\frac{(x+2)}{3} + c$$

Example(59): $\int \frac{2x+3}{9x^2-12x+8} dx$

put $2x+3 = A(18x-12) + B \Rightarrow A = \frac{1}{9}, B = 3+12A = \frac{39}{9}$

$$\therefore \int \frac{2x+3}{9x^2-12x+8}dx = \frac{1}{9}\int \frac{(18x-12)+39}{9x^2-12x+8}dx$$

$$= \frac{1}{9}\int \frac{(18x-12)}{9x^2-12x+8}dx + \frac{1}{9}\int \frac{39}{9x^2-12x+8}dx$$

$$= \frac{1}{9}\int \frac{(18x-12)}{9x^2-12x+8}dx + \frac{1}{9}\int \frac{39}{(9x^2-12x+4)+4}dx$$

$$= \frac{1}{9}\int \frac{(18x-12)}{9x^2-12x+8}dx + \frac{1}{9}\int \frac{39}{(3x-2)^2+4}dx$$

$$= \frac{1}{9}\ln\left|9x^2-12x+8\right| + \frac{39}{9}.\frac{1}{3}.\frac{1}{2}\tan^{-1}\frac{(3x-2)}{2}+c$$

Example(60): $\int \frac{x+2}{\sqrt{4x-x^2}}dx$ $\quad$ *put* $x+2 = A(4-2x)+B$

$$\therefore A = \frac{-1}{2}, B = 4,\ x+2 = \frac{-1}{2}\left[(4-2x)+8\right]$$

$$\int \frac{x+2}{\sqrt{4x-x^2}}dx = \frac{-1}{2}\int \frac{\left[(4-2x)+8\right]}{\sqrt{4x-x^2}}dx = \frac{-1}{2}\int \frac{(4-2x)}{\sqrt{4x-x^2}}dx - \frac{1}{2}\int \frac{8}{\sqrt{4x-x^2}}dx$$

$$= \frac{-1}{2}\int \frac{(4-2x)dx}{\sqrt{4x-x^2}} - \frac{1}{2}\int \frac{8dx}{\sqrt{4-(4-4x+x^2)}}$$

$$= \frac{-1}{2}\int \frac{(4-2x)dx}{\sqrt{4x-x^2}} - \frac{1}{2}\int \frac{8dx}{\sqrt{4-(x-2)^2}} = -\sqrt{4x-x^2} - \frac{1}{2}\sin^{-1}\frac{(x-2)}{2}+$$

Exercise(1)

Integrate the following functions with respect to x:

(1) $(3-2x+4x^3)$ $\quad$ (2) $(x-3)(x+4)$ $\quad$ (3) $(\sqrt{x}+\frac{1}{2}x+\frac{2}{\sqrt{x}})$

(4) $(x+2)^4$ $\quad$ (5) $(3x-1)^3$ $\quad$ (6) $\frac{1}{\sqrt{4x-10)}}$

(7) $x(2x^2+3)$ $\quad$ (8) $x^2(3x^3-4)^7$ $\quad$ (9) $\frac{(x+1)}{\sqrt{x^2+2x+4}}$

(10) $\frac{(1+\sqrt{x})^2}{\sqrt{x}}$ (11) $\frac{x^2}{3x^2+1}$ (12) $\frac{x-1}{x+1}$

(13) $\frac{x^2+2x+2}{x+2}$ (14) $\frac{x+2}{x^2+2x+2}$ (15) $x^3\sqrt{(x^4+2)}$

(16) 10^{3x} (17) $a^{\sin x}\cos x$ (18) $5^{\tan x}\sec^2 x$

(19) $\sin x(3+2\cos x)^5$ (20) $\frac{\sec^2 x}{\sqrt{3-5\tan x}}$ (21) $\frac{e^{2x}-2}{e^{2x}+3}$

(22) $\frac{x\ln(x^2+1)}{(x^2+1)}$ (23) $\frac{(1+\ln x)^3}{x}$ (24) $\frac{(e^x+e^{-x})^2}{\sqrt{e^x}}$

(25) $(\cos x-\sin x)^2$ (26) $\sin^3 x\cos x$ (27) $\cos^4 x\sin x$

(28) $\tan^5 x\sec^2 x$ (29) $\cot^4 \operatorname{cosec}^2 x$ (30) $\frac{\sin 8x}{9+\sin^2 4x}$

(31) $\frac{\sin x}{\sqrt{(1+\cos x)}}$ (32) $\frac{1+\cos 2x}{2x+\sin 2x}$ (33) $\frac{\cot x((1+\operatorname{cosec} x)}{\operatorname{cosec} x}$

(34) $\frac{\sec^2 3x}{(1+\tan 3x)^2}$ (35) $\frac{1}{1-\sin 2x}$ (36) $\frac{1}{1-\cos 2x}$

(37) $\frac{(1+\cot x)^3}{1-\sin 2x}$ (38) $\frac{(1+\tan)^5}{1-\cos 2x}$ (39) $\frac{1}{5+x^2}$

(40) $\frac{\sec^2 x}{\sqrt{4-\tan^2 x}}$ (41) $\frac{e^{2x}}{\sqrt{1-e^{4x}}}$ (42) $\frac{\sin x\cos x}{1+\cos^2 2x}$

(43) $\frac{\tan x}{\sqrt{\cos^2 x-4}}$ (44) $\frac{\cot x}{\sqrt{\sin^2 x-4}}$ (45) $\frac{1}{x^2+2x-8}$

(46) $\frac{1}{\sqrt{x^2+2x-8}}$ (47) $\frac{2x-3}{x^2+6x+13}$ (48) $\frac{x-1}{3x^2-4x+3}$

(49) $\frac{x}{\sqrt{27+6x-x^2}}$ (50) $\frac{2x-1}{\sqrt{12+4x-x^2}}$

Integration of Hyperbolic Functions

For x any real number we define Hyperbolic functions as follows:

(1) $\sinh x = \frac{1}{2}(e^x - e^{-x})$	(4) $\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{(e^x - e^{-x})}$
(2) $\cosh x = \frac{1}{2}(e^x + e^{-x})$	(5) $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{(e^x + e^{-x})}$
(3) $\tanh x = \frac{\sinh x}{\cosh x} = \frac{(e^x - e^{-x})}{(e^x + e^{-x})}$,	(6) $\coth x = \frac{1}{\tanh x} = \frac{(e^x + e^{-x})}{(e^x - e^{-x})}$

and hyperbolic functions satisfy the following lows:

(1) $\cosh^2 x - \sinh^2 x = 1$

(2) $1 - \tanh^2 x = \operatorname{sech}^2 x$

(3) $\coth^2 x - 1 = \operatorname{cosech}^2 x$

(4) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

(5) $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$

(6) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

(7) $\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$

(8) $\sinh 2x = 2 \sinh x \cosh x$

(9) $\cosh 2x = \cosh^2 x + \sinh^2 x$

(10) $\sinh^2 x = \frac{1}{2}[\cosh 2x - 1]$

(11) $\cosh^2 x = \frac{1}{2}[\cosh 2x + 1]$

(12) $\cosh x + \sinh x = e^x$

(13) $\cosh x - \sinh x = e^{-x}$

(14) $\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$

we can proof this lows by using the definition
in the following we stat integration formula for hyperbolic functions

$$(1)\int \sinh x\,dx = \cosh x + c$$

$$(2)\int \cosh x\,dx = \sinh x + c$$

$$(3)\int \tanh x\,dx = \ln\cosh x + c$$

$$(4)\int \coth x\,dx = \ln\left|\sinh x\right| + c$$

$$(5)\int \operatorname{sech}^2 x\,dx = \tanh x + c$$

$$(6)\int \operatorname{cosech}^2 x\,dx = -\coth x + c$$

$$(7)\int \operatorname{sech} x \tanh x\ dx = -\operatorname{sech} x + c$$

$$(8)\int \operatorname{cosech} x\ \coth x\ dx = -\operatorname{cosech} x + c$$

$$(9)\int \frac{dx}{\sqrt{b^2x^2+a^2}} = \frac{1}{b}\sinh^{-1}\frac{bx}{a} + c$$

$$(10)\int \frac{dx}{\sqrt{b^2x^2-a^2}} = \frac{1}{b}\ \cosh^{-1}\frac{bx}{a} + c$$

$$(11)\int \frac{dx}{a^2-b^2x^2} = \begin{cases} \dfrac{1}{ab}\tanh^{-1}\dfrac{bx}{a} + c \\ \dfrac{1}{2ab}\ln\dfrac{a+bx}{a-bx} + c \end{cases}$$

$$(12)\int \frac{dx}{b^2x^2-a^2} = \begin{cases} \dfrac{1}{ab}\coth^{-1}\dfrac{bx}{a} + c \\ \dfrac{1}{2ab}\ln\dfrac{bx-a}{bx+a} + c \end{cases}$$

Solved Examples:

$$(1)\int \operatorname{sech} x\,dx = \int \frac{1}{\cosh x}dx = \int \frac{\cosh x}{\cosh^2 x}dx = \int \frac{\cosh x}{1+\sinh^2 x}dx = \tan^{-1}(\sinh x) + c$$

$$(2)\int \sinh^2 x\,dx = \frac{1}{2}\int(\cosh 2x - 1)dx = \frac{1}{4}\sinh 2x - \frac{1}{2}x + c$$

$$(3)\int \cosh^3 2x\,dx = \int(1+\sinh^2 2x)\cosh 2x\ dx = \int \cosh 2x\ dx + \int \sinh^2 2x\ \cosh 2x\ dx$$

$$= \frac{1}{2}\sinh 2x + \frac{1}{2}\frac{\sinh^3 2x}{3} + c$$

$$(4)\int e^{x}\cosh 2x\,dx=\int e^{x}.\frac{e^{2x}+e^{-2x}}{2}dx$$

$$=\frac{1}{2}\int(e^{3x}+e^{-x})dx=\frac{1}{2}(\frac{1}{3}e^{3x}-e^{-x})+c$$

$$(5)\int e^{3x}\sinh 5x\,dx=\int e^{3x}\frac{e^{5x}-e^{-5x}}{2}dx$$

$$=\frac{1}{2}\int(e^{8x}-e^{-2x})dx=\frac{1}{2}(\frac{1}{8}e^{8x}+\frac{1}{2}e^{-2x})+c$$

Solved Examples

Example(1): $\int\sinh 3x\,dx=\frac{1}{3}\cosh 3x+c$

Example(2): $\int\operatorname{sech} x\,dx=\int\frac{1}{\cosh x}dx=\int\frac{\cosh x}{\cosh^{2}x}dx=\int\frac{\cosh x}{1+\sinh^{2}x}dx$

$$=\int\frac{d(\sinh x)}{1+\sinh^{2}x}=\tan^{-1}(\sinh x)+c$$

Example(3): $\int\sinh^{2}x\,dx=\frac{1}{2}\int[\cosh 2x-1]\,dx=\frac{1}{2}\left[\frac{1}{2}\sinh 2x-x\right]+c$

Example(4): $\int\cosh^{2}3x\,dx=\frac{1}{2}\int[\cosh 6x+1]\,dx=\frac{1}{2}\left[\frac{1}{6}\sinh 6x+x\right]+c$

Example(5): $\int\tanh^{2}5x\,dx=\int\left[1-\operatorname{sech}^{2}5x\right]dx=\left[x-\frac{1}{5}\tanh 5x\right]+c$

Example(6): $\int\operatorname{sech}^{4}x\,dx=\int\left[1-\tanh^{2}x\right]\operatorname{sech}^{2}x\,dx$

$$=\int\operatorname{sech}^{2}x\,dx-\int\tanh^{2}x\operatorname{sech}^{2}x\,dx=\tanh x-\frac{1}{3}\tanh^{3}x+c$$

Example(7): $\int\sinh^{3}4x\,dx=\int\sinh^{2}4x(\sinh 4x)dx$

$$=\int(\cosh^{2}4x-1)(\sinh 4x)dx$$

$$=\int(\cosh^{2}4x(\sinh 4x)dx-\int(\sinh 4x)dx$$

$$=\frac{1}{12}\cosh^{3}4x-\frac{1}{4}\cosh 4x+c$$

Example(8): $\int e^{x}\cosh 3x\,dx=\int e^{x}\frac{(e^{3x}+e^{-3x})}{2}dx=\frac{1}{2}\int e^{x}(e^{3x}+e^{-3x})dx$

$$=\frac{1}{2}\int(e^{4x}+e^{-2x})dx=\frac{1}{2}\left(\frac{e^{4x}}{4}+\frac{e^{-2x}}{-2}\right)+c$$

Example(9): $\int e^{2x}\sinh 3x\,dx=\int e^{2x}\frac{(e^{3x}-e^{-3x})}{2}dx=\frac{1}{2}\int e^{2x}(e^{3x}-e^{-3x})dx$

$$=\frac{1}{2}\int(e^{5x}+e^{-x})dx=\frac{1}{2}\left(\frac{e^{5x}}{5}+e^{-x}\right)+c$$

Example(10): $\int x\sinh x\;dx=\frac{1}{2}\int x\left(e^{x}-e^{-x}\right)dx=\frac{1}{2}\int\left[xe^{x}-xe^{-x}\right]dx$

$$=\frac{1}{2}\int xe^{x}\,dx-\frac{1}{2}\int xe^{-x}dx$$

$$=\frac{1}{2}\left[xe^{x}-e^{x}\right]-\frac{1}{2}\left[-xe^{-x}-e^{-x}\right]+c$$

$$=\frac{1}{2}x\left[e^{x}-e^{-x}\right]-\frac{1}{2}\left[e^{x}-e^{-x}\right]+c$$

$$=x\cosh x-\sinh x+c$$

Try to solve $\int x\sinh x\;dx$ **by parts**

Example(11): $\int\frac{dx}{\sqrt{x^{2}+16}}=\sinh^{-1}\frac{x}{4}+c$

Example(12): $\int\frac{dx}{\sqrt{x^{2}-25}}=\cosh^{-1}\frac{x}{5}+c$

Example(13): $\int\sinh^{3}4x\,dx=\int\sinh^{2}4x(\sinh 4x)dx$

$$=\int(\cosh^{2}4x-1)(\sinh 4x)dx$$

$$=\int(\cosh^{2}4x(\sinh 4x)dx-\int(\sinh 4x)dx$$

$$=\frac{1}{12}\cosh^{3}4x-\frac{1}{4}\cosh 4x+c$$

Exercise(2)

Integrate the following functions:

(1) $\sinh 3x$ (2) $\operatorname{cosesh}^2\sqrt{x}$ (3) $x \quad \sinh x^2$

(4) $\sinh^2 x\cosh^2 x$ (5) $e^x \sinh 3x$ (6) $e^3 \cosh 3x$

(7) $\sinh x(3+2\cosh x)^5$ (8) $\dfrac{\operatorname{sech}^2 x}{\sqrt{3-5\tanh x}}$ (9) $(\cosh x+\sinh x)^2$

(10) $\sinh^3 x\cosh x$ (11) $\cosh^4 x \sinh x$ (12) $\tanh^5 x \operatorname{sech}^2 x$

(13) $\coth^4 \operatorname{cosech}^2 x$ (14) $\dfrac{\sinh 8x}{9+\sinh^2 4x}$ (15) $\dfrac{\sinh x\cosh x}{1+\cosh^2 2x}$

(16) $\dfrac{\sinh x}{\sqrt{(1+\cosh x)}}$ (17) $\dfrac{1+\cosh 2x}{2x+\sinh 2x}$ (18) $\dfrac{\coth x((1+\operatorname{cosech} x)}{\operatorname{cosech} x}$

(19) $\dfrac{\operatorname{sech}^2 3x}{(1+\tanh 3x)^2}$ (20) $\dfrac{1}{1-\sinh 2x}$ (21) $\dfrac{1}{1+\cosh 2x}$

(22) $\dfrac{(1+\coth x)^3}{1-\sinh 2x}$ (23) $\dfrac{(1+\tanh)^5}{1-\cosh 2x}$ (24) $\dfrac{\sec^2 x}{\sqrt{4-\tanh^2 x}}$

Methods of Integration:

(1) Integration by parts

When u and v are differentiable functions then

$$d(uv) = u\,dv + v\,du$$

$$u\,dv = d(uv) - v\,du$$

and by integrate $\int u\,dv = \int d(uv) - \int v\,du$ **(1)**

to apply this rule we refer to our problem by the integral $\int u\,dv$ and we must separate it into two parts one part being u and the other being dv and we find du and v by differentiation u and integrate dv .

Note that

It is very important how to chose the function to be integrated, and the function to be differentiated such that the integration on the right side in (1) is much easier to evaluate than the one on the left.

Solved Examples:

Example (1): **Find** $\int x e^x\,dx$

Solution:

If we chose $u = e^x$ to be differentiated and $dv = xdx$ to be integrated

$$\therefore \int xe^x dx = \frac{1}{2}x^2 e^x - \int \frac{1}{2}x^2 e^x dx$$

and its clear that the integration in the R.H.S is more difficult than the given integration then

we use the partation as follows

let $u = x \qquad dv = e^x dx$

then $du = dx \qquad v = e^x$

by substituting in the rule then $\int x e^x\,dx = x e^x - \int e^x dx$

Note that : The integral in the right side $\int e^x dx$ is simple than the integral $\int x e^x\,dx$ Finally $I = x e^x - e^x + c$.

Example (2): **Find:** $\int x^2 \ln x\, dx$

Solution:

Consider the partition $u = \ln x$ $dv = x^2 dx$

Then $du = \frac{1}{x} dx$ $v = \frac{x^3}{3}$

Substitute in the rule we have:

$$\therefore I = \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\frac{1}{x}dx = \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2 dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + c$$

Try to solve $I_n = \int x^n \ln x\, dx,\ n \in$

Example (3): **Find** $\int x\sqrt{1+x}\, dx$

Solution:

Let $u = x$ $dv = \sqrt{1+x}dx$

$\therefore du = dx$ $v = \frac{2}{3}(1+x)^{\frac{3}{2}}$

by using partition rule

$\int u\, dv = \int d(uv) - \int v\, du$

$$\therefore I = \frac{2x}{3}(1+x)^{\frac{3}{2}} - \frac{2}{3}\int (1+x)^{\frac{3}{2}}dx = \frac{2x}{3}(1+x)^{\frac{3}{2}} - (\frac{2}{3})(\frac{2}{5})(1+x)^{\frac{5}{2}} + c$$

(try with the new partation $u = \sqrt{1+x}$ $dv = xdx$ what hapen) try to solve $\int x\ (ax+b)^n\, dx,\ n \in$

Example (4): **Find** $\int x \sin x\, dx$

Solution:

Let $u = x$ $dv = \sin x dx$

$\therefore du = dx$ $v = -\cos x$

by using partition rule

$\int u\, dv = \int d(uv) - \int v\, du$

$$\therefore I = \int x \sin x\, dx = -x\cos x + \int \cos x dx = -x\cos x + \sin x + c$$

(try with the partation $u = \sin x \quad dv = x dx$ what hapen)
try to solve $I_n = \int x^n \sin x \, dx$ n is positive integer

Example (5): **Find** $\int x \cos x \, dx$

Solution:

let $u = x \qquad dv = \cos x \, dx$

$\therefore du = dx \qquad v = \sin x$

$\therefore I = x \sin x - \int \sin x \, dx = x \sin x - \cos x + c$

try to solve $I_n = \int x^n \cos x \, dx$ n is positive integer

Example (6): **Find** $\int x^2 \sin x \, dx$

Solution:

$let \quad u = x^2 \qquad dv = \sin x dx$

$\therefore du = 2x dx \qquad v = -\cos x$

by substituting in the rule

$\therefore I = -x^2 \cos x + 2\int x \cos x \, dx \qquad \textbf{(1)}$

we can solve $\int x \cos x dx$ *by parts as in example* (5)

$\int x \cos x \, dx \qquad let \quad u = x \qquad dv = \cos x \, dx$

$\therefore du = dx \qquad v = \sin x$

$\therefore I = x \sin x - \int \sin x \, dx = x \sin x - \cos x \qquad \textbf{(2)}$

from(2) in (1)

$\therefore I = -x^2 \cos x + 2\int x \cos x \, dx = -x^2 \cos x + 2(x \sin x - \cos x) + c$

$= -x^2 \cos x + 2x \sin x - 2\cos x + c$

Example (7): **Find** $\int x^2 \cos x \, dx$

Solution:

$u = x^2 \qquad dv = \cos x \, dx$

$\therefore du = 2x dx \qquad v = \sin x$

$$\therefore I = x^2 \sin x - 2\int x \sin x \, dx = x^2 \sin x - 2(-x\cos x + \sin x) + c$$
$$= x^2 \sin x + 2x\cos x - 2\sin x + c$$

<u>Example (8):</u> **Find** $\int x^2 e^x \, dx$

<u>Solution:</u>

$$u = x^2 \qquad dv = e^x dx$$
$$\therefore du = 2x dx \qquad v = e^x$$
$$\therefore I = x^2 e^x - \int 2x e^x \, dx = x^2 e^x + 2(xe^x - e^x) + c$$

try to solve $I_m = \int x^m e^x \, dx$ where m is a positive integer

<u>Example (9):</u> **Find** $\int \sin^{-1} x \, dx$

<u>Solution:</u>

$$u = \sin^{-1} x \qquad dv = dx$$
$$\therefore du = \frac{dx}{\sqrt{1-x^2}} \qquad let \quad v = x$$
$$I = \int u \, dv = uv - \int v \, du \; x\sin^{-1} x - \int \frac{x\,dx}{\sqrt{1-x^2}} = x\sin^{-1} x - \frac{1}{2}\int \frac{2x\,dx}{\sqrt{1-x^2}}$$
$$= x\sin^{-1} x + \frac{1}{2}\int \frac{-2x\,dx}{\sqrt{1-x^2}} = x\sin^{-1} x + \frac{1}{2}.2\sqrt{1-x^2} + c$$

try to solve $\int x \sin^{-1} x \, dx$ where m is a positive integer

<u>Example (10)</u>: **Find** $\int \tan^{-1} x \, dx$

<u>Solution</u>

$$u = \tan^{-1} x \qquad dv = dx$$
$$\therefore du = \frac{dx}{1+x^2} \qquad let \quad v = x$$
$$\therefore \int u \, dv = \int \tan^{-1} x \, dx = x\tan^{-1} x - \int \frac{x\,dx}{1+x^2} = x\tan^{-1} x - \frac{1}{2}\int \frac{2x\,dx}{1+x^2}$$

$$= x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + c$$

Example (11): **Find** $I = \int e^{ax}\sin bx\,dx, \qquad J = \int e^{ax}\cos bx\,dx$

Solution:

These integrals are of importance in the theory of electric currents, if each integral is evaluated by parts, the other one is obtained.

$$let \quad u = e^{ax} \qquad dv = \sin bx\,dx$$

$$\therefore du = a\,e^{ax}dx \qquad v = \frac{-1}{b}\cos bx$$

$$I = \int e^{ax}\sin bx\,dx = \frac{-1}{b}e^{ax}\cos bx - \int\left(\frac{-1}{b}\cos bx\right)\left(a\,e^{ax}dx\right)$$

$$= \frac{-e^{ax}}{b}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\,dx$$

$$\therefore I = \frac{-e^{ax}}{b}\cos bx + \frac{a}{b}J \qquad (1)$$

where $J = \int e^{ax}\cos bx\,dx$

Similarly taking the second integral J

let $u = e^{ax} \qquad dv = \cos bx\,dx$

$$\therefore du = ae^{ax}dx \qquad v = \frac{1}{b}\sin bx$$

$$\therefore J = \frac{1}{b}e^{ax}\sin bx - \int\frac{1}{b}\sin bx.a\,e^{ax}dx = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\int e^{ax}\sin bx\;dx$$

$$J = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}I \qquad (2)$$

from (1),(2) we get

$$\therefore I = \frac{-e^{ax}}{b}\cos bx + \frac{a}{b}\left(\frac{1}{b}e^{ax}\sin bx - \frac{a}{b}I\right) = \frac{-e^{ax}}{b}\cos bx + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}I$$

$$I + \frac{a^2}{b^2}I = \frac{-e^{ax}}{b}\cos bx + \frac{a}{b^2}e^{ax}\sin bx$$

$$I(1+\frac{a^2}{b^2}) = I(\frac{a^2+b^2}{b^2}) = \frac{-e^{ax}}{b}\cos bx + \frac{a}{b^2}e^{ax}\sin bx$$

$$\therefore I = \frac{b^2}{b^2+a^2}\left[\frac{-e^{ax}}{b}\cos bx + \frac{a}{b^2}e^{ax}\sin bx\right]$$

$$= \left[\frac{-b}{b^2+a^2}e^{ax}\cos bx + \frac{a}{b^2-a^2}e^{ax}\sin bx\right] + c$$

$$\boxed{I = \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{b^2+a^2}\left[-b\cos bx + a\sin bx\right] + c}$$

and from(2) $J = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}I = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\left[\frac{-e^{ax}}{b}\cos bx + \frac{a}{b}J\right]$

$$= \frac{1}{b}e^{ax}\sin bx + \frac{ae^{ax}}{b^2}\cos bx - \frac{a^2}{b^2}J$$

$$J + \frac{a^2}{b^2}J = J\left[1+\frac{a^2}{b^2}\right] = J\left[\frac{a^2+b^2}{b^2}\right] = \frac{1}{b}e^{ax}\sin bx + \frac{ae^{ax}}{b^2}\cos bx$$

$$\boxed{\therefore J = \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{b^2+a^2}\left[b\sin bx + a\cos bx\right] + c}$$

<u>in the integrals</u> $I = \int e^{ax}\sinh bx\,dx$, $J = \int e^{ax}\cosh bx\,dx$ we use the diffination of the hyperbolic functions $\sinh bx$, $\cosh bx$ as a functions of e^x then

$$Ih = \int e^{ax}\sinh bx\,dx = \int e^{ax}\left(\frac{e^{bx}-e^{-bx}}{2}\right)dx = \int\left(\frac{e^{(a+b)x}-e^{(a-b)x}}{2}\right)dx,$$

$$Jh = \int e^{ax}\cosh bx\,dx = \int e^{ax}\left(\frac{e^{bx}+e^{-bx}}{2}\right)dx = \int\left(\frac{e^{(a+b)x}+e^{(a-b)x}}{2}\right)dx,$$

<u>Example (12):</u> **Find** $I = \int\frac{dx}{(1+x^2)^{5/2}}$

Solution:

$$I=\int\frac{dx}{(1+x^2)^{5/2}}=\int(1+x^2)^{-5/2}dx$$

consider $J=\int\frac{dx}{(1+x^2)^{3/2}}=\int(1+x^2)^{-3/2}dx$

$$u=(1+x^2)^{-3/2} \qquad dv=dx$$

$$\therefore du=-3x(1+x^2)^{-5/2}dx \qquad v=x$$

$$J=x(1+x^2)^{-3/2}+3\int x^2(1+x^2)^{-5/2}dx$$

$$=x(1+x^2)^{-3/2}+3\int(1+x^2-1)(1+x^2)^{-5/2}dx$$

$$=x(1+x^2)^{-3/2}+3\int\left[(1+x^2)^{-3/2}-(1+x^2)^{-5/2}\right]dx$$

$$=x(1+x^2)^{-3/2}+3\int\left[(1+x^2)^{-3/2}-(1+x^2)^{-5/2}\right]dx=x(1+x^2)^{-3/2}+3J-3I$$

$$\therefore 3I=x(1+x^2)^{-3/2}+2J$$

$$I=\frac{x(1+x^2)^{-3/2}}{3}+\frac{2}{3}J \qquad (1)$$

To solve $J=\int\frac{dx}{(1+x^2)^{3/2}}$ consider $K=\int\frac{dx}{(1+x^2)^{1/2}}$ with the partation

$$u=(1+x^2)^{-1/2} \qquad dv=dx$$

$$du=-x(1+x^2)^{-3/2}dx \qquad v=x$$

$$K=x(1+x^2)^{-1/2}+\int x^2(1+x^2)^{-3/2}dx=x(1+x^2)^{-1/2}+\int(1+x^2-1)(1+x^2)^{-3/2}$$

$$=x(1+x^2)^{-1/2}+\int(1+x^2)^{-1/2}-(1+x^2)^{-3/2}dx=x(1+x^2)^{-1/2}+K-J$$

$$J=x(1+x^2)^{-1/2}=\frac{x}{(1+x^2)^{1/2}}+c$$

Substitute in(1)

$$\therefore \quad I=\frac{x(1+x^2)^{-3/2}}{3}+\frac{2}{3}\frac{x}{(1+x^2)^{1/2}}+c$$

Example (13): **Find** $\int \sec^3 x\, dx$

Solution:

$$\int \sec^3 x\, dx = \int \sec x \sec^2 x\, dx$$

$$u = \sec x \qquad dv = \sec^2 x\, dx$$

$$du = \sec x \tan x\, dx \qquad v = \tan x$$

$$\therefore \int \sec^3 x\, dx = \sec x \tan x - \int \sec x \tan^2 x\, dx = \sec x \tan x - \int \sec x(\sec^2 x - 1)\, dx$$

$$= \sec x \tan x - \int (\sec^3 x - \sec x)\, dx = \sec x \tan x - \int (\sec^3 x\, dx + \int \sec x\, dx$$

$$2\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx = \sec x \tan x + \ln|\sec x + \tan x| + c$$

$$\therefore \int \sec^3 x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + c$$

Example (14): **Find** $I = \int \sqrt{x^2 + a^2}\, dx$

Solution:

$$(14) I = \int \sqrt{x^2 + a^2}\, dx \qquad let \qquad u = \sqrt{x^2 + a^2} \qquad dv = dx$$

$$du = \frac{x\, dx}{\sqrt{x^2 + a^2}} \qquad v = x$$

$$I = \int \sqrt{x^2 + a^2}\, dx = x\sqrt{x^2 + a^2} - \int \frac{x^2\, dx}{\sqrt{x^2 + a^2}} = x\sqrt{x^2 + a^2} - \int \frac{(x^2 + a^2 - a^2)\, dx}{\sqrt{x^2 + a^2}}$$

$$= x\sqrt{x^2 + a^2} - \int \frac{(x^2 + a^2)\, dx}{\sqrt{x^2 + a^2}} + \int \frac{a^2\, dx}{\sqrt{x^2 + a^2}}$$

$$= x\sqrt{x^2 + a^2} - \int \sqrt{x^2 + a^2}\, dx + a^2 \sinh^{-1}\frac{x}{a}$$

$$\therefore 2\int \sqrt{x^2 + a^2}\, dx = x\sqrt{x^2 + a^2} + a^2 \sinh^{-1}\frac{x}{a}$$

$$\therefore \int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + c$$

Example (15): Find $I=\int\sqrt{a^2-x^2}\,dx$

Solution:

$$let \qquad u=\sqrt{a^2-x^2} \qquad\qquad dv=dx$$

$$du=\frac{-x\,dx}{\sqrt{a^2-x^2}} \qquad\qquad v=x$$

$$I=\int\sqrt{x^2+a^2}\,dx=x\sqrt{a^2-x^2}-\int\frac{-x^2\,dx}{\sqrt{a^2-x^2}}=x\sqrt{a^2-x^2}-\int\frac{(a^2-x^2-a^2)dx}{\sqrt{a^2-x^2}}$$

$$=x\sqrt{a^2-x^2}-\int\frac{(a^2-x^2)dx}{\sqrt{a^2-x^2}}+\int\frac{a^2\,dx}{\sqrt{a^2-x^2}}$$

$$=x\sqrt{a^2-x^2}-\int\sqrt{a^2-x^2}\,dx+a^2\sin^{-1}\frac{x}{a}$$

$$\therefore 2\int\sqrt{a^2-x^2}\,dx=x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}$$

$$\therefore \int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c$$

Exercise(3)

Integrate the following function with respect to x:

(1) (i) $x\sin x$	(ii) $x\sin 3x$	(iii) $x^2\sin x$	(iv) $x^3\cos x$
(2) (i) $x\ln x$	(ii) $x^2\ln x$	(iii) $x^3\ln x$	(iv) $x^n\ln x$
(3) (i) xe^{4x}	(ii) x^2e^{3x}	(iii) xe^{-2x}	(iv) $e^x\sin x$
(4) (i) $x\cos^{-1}$	(ii) $x\sin^{-1}$	(iii) $x\tan^{-1}$	(iv) $x\cot^{-1}$
(5) (i) $x\sin^2 x$	(ii) $x\sin x\cos x$	(iii) $x\sec^2 x$	(iv) $x\sinh x$
(6) (i) $x^2\sin^{-1}x$	(ii) $\sqrt{x^2+4}$	(iii) $\frac{\ln x}{x^3}$	(iv) $\sin x\sin 3x$
(7) (i) $x\cos^{-1}$	(ii) $x\sin^{-1}$	(iii) $x\tan^{-1}$	(iv) $\sin^3 x$
(8) (i) $\sin^3 x\cos^2 x$	(ii) $\cos^5 x$	(iii) $\sec^5 x$	(iv) $\operatorname{cosec}^3 x$

Reduction Formula

A Reduction Formula succeeds if ultimately it produces an integral which can be evaluated. We use the partition of integration to prove the following reduction formulas:

(1) If $I_m = \int \sin^m x\,dx$ **then show that** $I_m = \frac{-1}{m}\sin^{m-1} x \cos x + \frac{m-1}{m} I_{m-2}$

proof:

$$I_m = \int \sin^m x\,dx = \int \sin^{m-1} x\,(\sin x dx) = \int \sin^{m-1} x\; d(-\cos x)$$

let $u = \sin^{m-1} x$ $\qquad dv = d(-\cos x)$

$\therefore\quad du = (m-1)\sin^{m-2} x(\cos x)dx, \qquad v = -\cos x$

$$I_m = -\sin^{m-1} x\cos x + (m-1)\int \sin^{m-2} x(\cos^2 x)dx$$

$$= -\sin^{m-1} x\cos x + (m-1)\int \sin^{m-2} x(1-\sin^2 x)dx$$

$$= -\sin^{m-1} x\cos x + (m-1)\int \sin^{m-2} xdx - (m-1)\int \sin^m xdx$$

$$I_m = -\sin^{m-1} x\cos x + (m-1)I_{m-2} - (m-1)I_m$$

$$\therefore \big(1+(m-1)\big)I_m = -\sin^{m-1} x\cos x + (m-1)I_{m-2}$$

$$I_m = \frac{-1}{m}\sin^{m-1} x\cos x + \frac{(m-1)}{m} I_{m-2}$$

similarly we can prove that

(2) If $I_m = \int \cos^m x\,dx$ **then show that** $I_m = \frac{1}{m}\cos^{m-1} x\ \sin x + \frac{m-1}{m} I_{m-2}$

(3) If $I_m = \int x^m e^{ax}\,dx$ **then show that** $I_m = \frac{1}{a}x^m e^{ax} - \frac{m}{a} I_{m-1}$

(4) If $I_m = \int x^m a^x\,dx$ **then show that** $I_m = \frac{1}{\ln a}x^m a^x - \frac{m}{\ln a} I_{m-1}$

(5) If $I_m = \int [\log x]^m\,dx$ **then show that** $I_m = x[\log x]^m - mI_{m-1}$

(6)If $I_m = \int x^m \sin ax\,dx$

then show that $I_m = \frac{-x^m}{a}\cos ax + \frac{m}{a^2}x^{m-1}\sin ax - \frac{m(m-1)}{a} I_{m-2}$

(7)If $I_m = \int \tan^m dx$ **then show that** $I_m = \frac{\tan^{m-1} x}{m-1} - I_{m-2}$

(8)If $I_m = \int \sec^m dx$ **then show that** $I_m = \frac{\sec^{m-2} x \tan x}{m-1} + \frac{m-2}{m-1} I_{m-2}$

(9) If $I_m = \int \operatorname{cosec}^m x\,dx$ **then show that**

$I_m = \frac{-\operatorname{cosec}^{m-2} x \cot x}{m-1} + \frac{m-2}{m-1} I_{m-2}$

(10) If $I_n = \int x^n \cos bx\,dx$, $J_n = \int x^n \sin bx\,dx$

show that $I_n = x^n \sin bx - nJ_n.$ $J_n = x^n \cos bx - n I_n$

Exercise(4)

(1) if $I_n = \int x^n \cosh x\,dx$ and $J_n = \int x^n \sinh x\,dx$ prove that

$I_n = x^n \sinh x - n J_{n-1},$ $J_n = x^n \cosh x - n I_{n-1}$ find $I_4,\ J_4$

(2) Find the reduction formula connecting $I_{m,n} and\ I_{m-2,m+2}$

given that $I_{m,n} = \int \sin^m x \sin^n x\,dx$

(3) If $I_n = \int \cos^n x\,dx$, $J_n = \int \sin^n x\,dx$

Show that $n I_n = \sin x \cos^{n-1} x + (n-1) I_{n-2}.$

$nJ_n = -\cos x \sin^{n-1} x + (n-1) J_{n-2}.$

(4) prove that $\int \sin^{n-1} x \sin(n+1)x\,dx = \frac{1}{n}\sin^n x \sin nx$

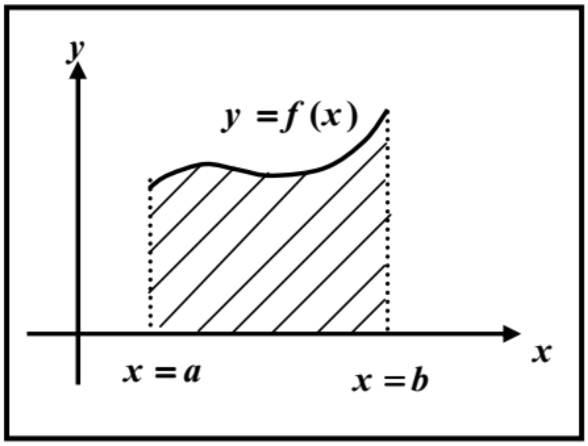
(5) Find the reducion formula for $\int x^m \left[1+x^2\right]^{\frac{n}{2}} dx,$

m, n are positive integer.

Definite integration

Area under the curve:

Given a continuous function $f(x)$ on the interval $[a,b]$ such that $f(x)>0$ we can approximate the area enclosed by the curve of $f(x)$, $x-$ axis and the two lines $x=a, x=b$ by dividing the interval $[a,b]$ into subintervals by the set of points $p=\{x_0, x_1, x_2, ..., x_n\}$ such that $a=x_0<x_1<x_2<...<x_n=b$ then the area given by:



$$S_n=(x_1-x_0)f(x_1)+(x_2-x_1)f(x_2)+...+(x_n-x_{n-1})f(x_n)$$

$$=\sum_{k=1}^{n}(x_k-x_{k-1})f(x_k)$$

if we divide the interval into n equal subintervals with length

$\Delta x_k=(x_k-x_{k-1}), k=1,2,...,n$

Then

$$S_n=\Delta x_1 f(x_1)+\Delta x_2 f(x_2)+...+\Delta x_n f(x_n)$$

$$=\sum_{k=1}^{n}\Delta x_k f(x_k)$$

Upper and lower sum:

To discuss the concept of integral of the function $f(x)$, we must first introduce some notation.

If $I=[a,b]$ is closed , bounded interval , then a partition of I is finite order d set $P:=\{x_0,x_1,....x_n\}$ of point of I such that

$a=x_0<x_1<x_2<...<x_n=b$

The points of the partition P can be used to divide I into non-overlapping subintervals $[x_0,x_1],[x_1,x_2],...,[x_{n-1},x_n]$.

Let $f(x)$ continuous function $f(x)$ defined on the interval $[a,b]$ and let P a partition of I we let

$m_k=\inf\{f(x):x\in[x_{k-1},x_k]\}$

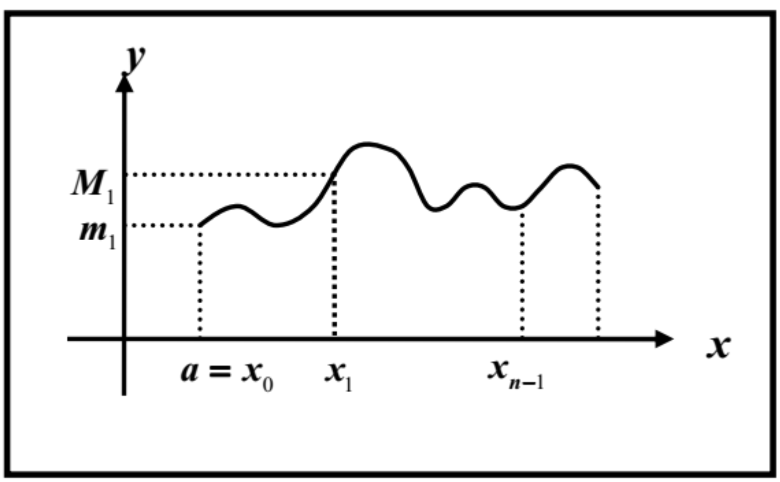
$M_k=\sup\{f(x):x\in[x_{k-1},x_k]\}$

The lower sum of $f(x)$ corresponding to the partition P is defined to be

$$L(P;f)=\sum_{k=1}^{n} m_k(x_k-x_{k-1})$$

The upper sum of $f(x)$ corresponding to the partition P is defined to be

$$U(P;f)=\sum_{k=1}^{n} M_k(x_k-x_{k-1})$$



if $f(x)$ is positive function then the lower sum $L(P;f)$ can be interpreted as the area of union of rectangles with base $[x_{k-1},x_k]$ and height m_k ,Similarly the Upper sum $U(P;f)$ can be interpreted as the area of union of rectangles with base $[x_{k-1}, x_k]$ and height M_k

Riemann's sum:

Let $f(x)$ is real valued function defined on the interval $[a,b]$, and let $p=\{x_0, x_1, x_2, ..., x_n\}$ is a partition on $[a, b]$. into n subintervals $[x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]$, choose a points $\xi_1, \xi_2, \xi_3, ..., \xi_n$ such that $\xi_k \in [x_{k-1}, x_k],\ k = 1, 2, 3, ..., n$

We define Riemann's sum in the form

$$S_n(P;f) = \sum_{k=1}^{n} f(\xi_k)\Delta x_k, \qquad \Delta x_k = x_k - x_{k-1}$$

$$= (x_1 - x_0)\ f(\xi_1) + (x_2 - x_1)\ f(\xi_2) + ... + (x_n - x_{n-1})\ f(\xi_n)$$

Since $m_k < f(x_k) < M_k$ then

$$\sum_{k=1}^{n} m_k(x_k - x_{k-1}) \le \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) \le \sum_{k=1}^{n} M_k(x_k - x_{k-1})$$

$$L(P;f) \le S_n(P;f) \le U(P;f)$$

Upper and lower Integrals:

The lower integral of $f(x)$ on I is the number

$$I_l = \lim_{n\to\infty} L(P;f) = \lim_{n\to\infty}\left[\sum_{k=1}^{n} m_k(x_k - x_{k-1})\right]$$

The Upper integral of $f(x)$ on I is the number

$$I_u = \lim_{n\to\infty} U(P;f) = \lim_{n\to\infty}\left[\sum_{k=1}^{n} M_k(x_k - x_{k-1})\right]$$

Riemann's Integral: **(Riemann's Criterion for integral)**

Let $f(x)$ is a continuous function on the interval $[a,b]$, and let p be a partition on $[a, b]$. $f(x)$ to be integrable on $[a,b]$ if the limit

$I_R = \lim\limits_{n\to\infty} S_n(P;f) = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} f(\xi_k)\Delta x_k$ exist and independent on choosing the points $\xi_1, \xi_2, \xi_3, ..., \xi_n$.

And we write

$$\int_a^b f(x)dx = I_R = \lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)\Delta x_k$$

Corollary:

If $f(x)$ is continuous function on $[a,b]$ the

$\lim\limits_{n\to\infty} S_n(P;f) = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} f(\xi_k)\Delta x_k$ is exist and independent on choosing

the points $\xi_1, \xi_2, \xi_3, ..., \xi_n$,and $f(x)$ is integrable on $[a,b]$.

Example(1):

Show that the function $f(x) = 4x - 1$ is integrable on the interval $[0,1]$ and find the value of the integral.

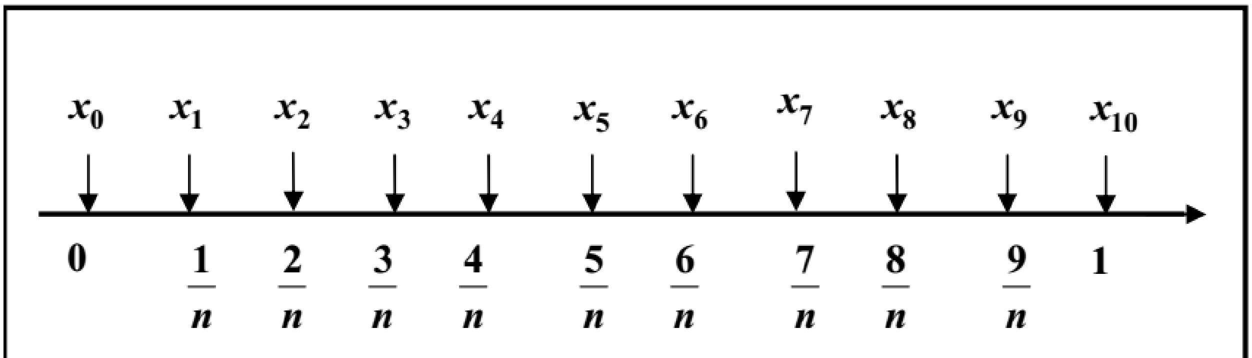
Solution:

Since the function is continuous function on the interval $[0,1]$ thus it
integrable on the interval $[0,1]$

Consider

$p = \{x_0 = 0,\ x_1,\ x_2, ...,\ x_{r-1},\ x_r, ...,\ x_n = 1\}$ be a partition to the interval

into n subinterval with length $\frac{b-a}{n} = \frac{1}{n}$ and



$$x_k = 0 + \frac{(b-a)k}{n} = \frac{k}{n},\ \ 1 \le k \le n \text{ and } \quad (x_k - x_{k-1}) = \frac{1}{n},\ \ k = 1, 2, ..., n$$

$$x_0 = 0,\ x_1 = \frac{1}{n},\ \ x_2 = \frac{2}{n}, ...,\ x_n = \frac{n}{n} = 1$$

Choose ξ_r such that

$$\xi_r = \frac{x_{k-1} + x_k}{2} = \frac{1}{2}\left[\frac{(k-1)}{n} + \frac{k}{n}\right] = \frac{2k-1}{2n}$$

$$f(\xi_k) = 4\xi_k - 1 = 4\left(\frac{2k-1}{2n}\right) - 1 = \frac{4k-2}{n} - 1$$

$$\therefore S_n(P; f) = \sum_{r=1}^{k} (x_k - x_{k-1})\ f(\xi_k) = \sum_{k=1}^{n} \frac{1}{n}\left(\frac{4k-2}{n} - 1\right)$$

$$= \sum_{k=1}^{n} \frac{1}{n^2}(4k-2) - \frac{1}{n} = \left(\frac{4}{n^2}\sum_{k=1}^{n} k - \frac{1}{n^2}\sum_{k=1}^{n} 2 - \frac{1}{n}\sum_{k=1}^{n} 1\right)$$

$$= \left(\frac{4}{n^2} . \frac{n(n+1)}{2} - \frac{1}{n^2} 2n - 1\right) = \left(\frac{2n(n+1)}{n^2} - \frac{2}{n} - 1\right)$$

$$\therefore \lim_{n\to\infty} S_n(P; f) = \lim_{n\to\infty}\left(\frac{2n(n+1)}{n^2} - \frac{2}{n} - 1\right) = 2 - 1 = 1$$

Example(2):

Show that the function $f(x) = 4x - 1$ is integrable on the interval $[1, 7]$ and find the value of the integral.

Solution:

Since the function is continuous function on the interval $[1, 7]$ thus it is integrable on the interval $[1, 7]$

Consider $p = \{x_0 = 1,\ x_1,\ x_2, ...,\ x_{r-1},\ x_r, ...,\ x_n = 7\}$ be a partition to the interval $[1, 7]$ Then

$$x_k = a + \frac{(b-a)k}{n} = 1 + \frac{6k}{n},\ \ 1 \le k \le n$$

$$(x_k - x_{k-1}) = \frac{6}{n},\ \ k = 1, 2, ..., n$$

$$x_0 = 1,\; x_1 = 1 + \frac{6}{n},\; x_2 = 1 + \frac{6}{n}\cdot 2, \ldots,\; x_n = 1 + \frac{6}{n}\cdot n = 7$$

Choose ξ_r such that

$$\xi_r = \frac{x_{k-1} + x_k}{2} = \frac{1}{2}\left[1 + \frac{6(k-1)}{n} + 1 + \frac{6k}{n}\right] = 1 + \frac{6k-3}{n}$$

$$f(\xi_k) = 4\xi_k - 1 = 4\left(1 + \frac{6k-3}{n}\right) - 1 = 3 + \frac{24k}{n} - \frac{12}{n}$$

$$\therefore S_n(P;f) = \sum_{r=1}^{k} (x_k - x_{k-1})\; f(\xi_k) = \sum_{k=1}^{n} \frac{6}{n}\left(3 + \frac{24k}{n} - \frac{12}{n}\right)$$

$$= \sum_{k=1}^{n}\left(\frac{18}{n} + \frac{144}{n^2}k - \frac{72}{n^2}\right)$$

$$= \frac{18}{n}\sum_{k=1}^{n} 1 + \frac{144}{n^2}\sum_{k=1}^{n} k - \frac{72}{n^2}\sum_{k=1}^{n} 1$$

$$= 18 + \frac{144}{n^2}\cdot\frac{n}{2}(n+1) - \frac{72}{n^2}\cdot n \;= 18 + 72\left(1 + \frac{1}{n}\right) - \frac{72}{n}$$

$$\therefore \lim_{n\to\infty} S_n(P;f) = \lim_{n\to\infty}\left[18 + 72\left(1 + \frac{1}{n}\right) - \frac{72}{n}\right] = 18 + 72 = 90$$

Example(3):

Show that the function $f(x) = 4x^2 + 3$ is integrable on the interval

$[2,10]$ and find the value of $\int_2^{10} (4x^2 + 3)\, dx$.

Solution:

Since the function is continuous function on the interval $[2,10]$ thus it is integrable on the interval $[2,10]$

Let

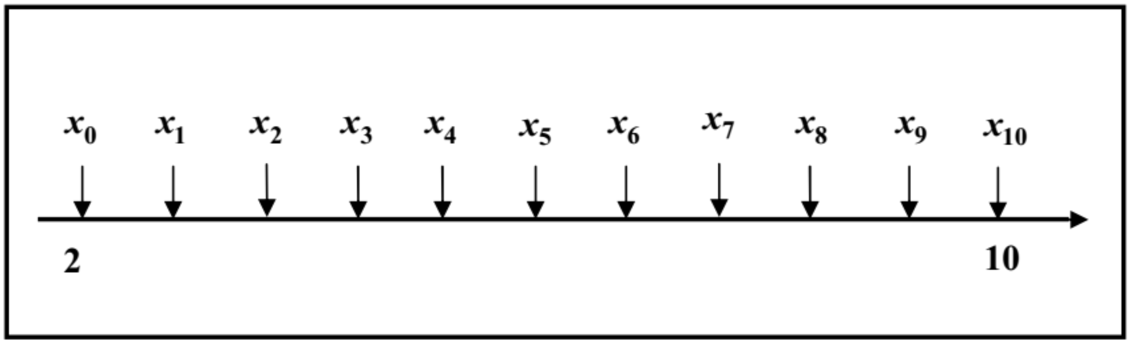
$p = \{x_0 = 2,\, x_1,\, x_2, \ldots,\, x_{r-1},\, x_r, \ldots,\, x_n = 20\}$ be a partition on the interval $[2,10]$ into n subintervals with length is

$$\Delta x_k = (x_k - x_{k-1}) = \frac{b-a}{n} = \frac{10-2}{n} = \frac{8}{n}$$

and

$$x_0 = 2,\ x_1 = 2 + \frac{8}{n},\ \ x_2 = 2 + \frac{16}{n}, \ldots,$$

$$x_{k-1} = 2 + \frac{8(k-1)}{n},\ \ x_k = 2 + \frac{8k}{n}, \ldots,\ \ x_n = 2 + \frac{8n}{n} = 10$$



let

$$\xi_r = \frac{x_{k-1} + x_k}{2} = 2 + \frac{8k}{n} - \frac{4}{n}$$

$$f(\xi_k) = 4\left(2 + \frac{8k}{n} - \frac{4}{n}\right)^2 + 3$$

$$= 4\left[4 + \frac{64k^2}{n^2} + \frac{32k}{n} + \frac{16}{n^2} - \frac{16}{n} - \frac{64k}{n^2}\right] + 3$$

$$= 19 + \frac{256}{n^2}k^2 + \frac{64}{n^2} + \frac{128}{n}k - \frac{64}{n} - \frac{256}{n^2}k$$

$$S_n(p;f) = \sum_{k=1}^{n}(x_k - x_{k-1})\ f(\xi_k) = \sum_{k=1}^{n}\ \frac{8}{n}\ f(\xi_k)$$

$$= \sum_{k=1}^{n}\left[\frac{152}{n} + \frac{2048}{n^3}k^2 + \frac{512}{n^3} + \frac{1024}{n^2}k - \frac{512}{n^2} - \frac{2048}{n^3}k\right]$$

$$= \frac{152}{n}n + \frac{2048}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) + \frac{512}{n^2}$$

$$+ \frac{1024}{2}\left(1 + \frac{1}{n}\right) - \frac{512}{n} - \frac{2048}{2n}\left(1 + \frac{1}{n}\right)$$

where

(1) $\sum_{1}^{n} r = \frac{1}{2} n(n+1)$,   (2) $\sum_{1}^{n} r^2 = \frac{1}{6} n(n+1)(2n+1)$

(3) $\sum_{1}^{n} r^3 = \left[\frac{1}{2} n(n+1)\right]^2$

since

$p = \{x_0, x_1, x_2, ..., x_n\}$

$\Delta x_k = \frac{8}{n}, \ \ \Delta x_k \to 0 \ \ as \ n \to \infty$

$$\lim_{\Delta x_k \to 0} S_n(p; f) = \lim_{n \to \infty} S_n(p; f) = 152 + \frac{1024}{3}(2) + 512 = \frac{4040}{3}$$

MATHEMATICS DYNAMICS

PROF. MAHDY

First year

FACULTY OF EDUCATION
PHYSICS AND CHEMISTRY DEPARTMENT

INTRODUCTION

The subject of Dynamics is generally divided into two branches: the first one, is called *Kinematics*, is concerned with the geometry of motion apart from all considerations of force, mass or energy; the second, is called *Kinetics*, is concerned with the effects of forces on the motion of bodies.

In order to describe the motion of a particle (or point) two things are needed,

(i) a frame of reference,
(ii) a time-keeper.

It is not possible to describe absolute motion, but only motion relative to surrounding objects; and a suitable frame of reference depends on the kind of motion that it is desired to describe. Thus if the motion is rectilinear the distance from a fixed point on the line is a sufficient description of the position of the moving point; and in more general cases systems of two or of three rectangular axes may be chosen as a frame of reference. For example, in the case of a body projected from the surface of the Earth a set of axes with the origin at the point of projection would be suitable for the description of motion relative to the Earth. But, for the description of the motion of the planets, it would be more convenient to take a frame of axes with an origin at the Sun's center (Polar co-ordinates).

■ Definitions

1. Mass: The mass of a body is the quantity of matter in the body. The unit of mass used in England is a pound and is defined to be the mass of a certain piece of platinum kept in the Exchequer Office.

2. A Particle (point): is a portion of matter which is indefinitely small in size, or which, for the purpose of our investigations, is so small that the distances between its different parts may be neglected.

3. A Body: may be regarded as an indefinitely large number of indefinitely small portions, or as a conglomeration of particles.

4. A Rigid Body is a body whose parts always preserve an invariable position with respect to one another.

5. Space is the boundless, three-dimensional extent in which objects and events occur and have relative position and direction. Two-dimensional space is described with two coordinates (x, y), while three-dimensional space (physical reality) is described in three coordinates (x, y, z).

6. Time is a part of the measuring system used to sequence events, to compare the durations of events and the intervals between them, and to quantify rates of change such as the motions of object (not related to analysis of statics problems).

7. Force is any influence that causes an object to undergo a change in speed, a change in direction, or in a change in shape. Force can also be described by intuitive concepts such as a push or pull that can cause an object with mass to change its velocity, i.e. accelerate. A force has both magnitude and direction, which is a vector quantity.

KINEMATICS IN ONE DIMENSION

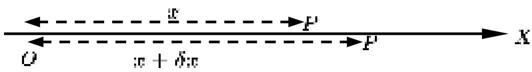
RECTILINEAR MOTION

Although motion in a straight line or rectilinear motion constitute the simplest of dynamical problems, yet it is very important because many physical problems reduce to this category, e.g., simple harmonic motion, motion under inverse square law, motion in a resisting medium and motion of a rocket. Therefore, in this chapter, we first proceed to determine the solution of the one dimensional equation of motion with subject to initial conditions. When a point (or particle) moves along a straight line, its motion is said to be a rectilinear motion. Here in this chapter we shall discuss the motion of a point (or particle) along a straight line which may be either horizontal or vertical. When a point (or particle) moves along a straight line, its motion is said to be a rectilinear motion. Here in this chapter we shall discuss the motion of a point (or particle) along a straight line which may be either horizontal or vertical.

■ Velocity and Acceleration

Suppose a particle moves along a straight line $\mathbf{OX}$ where $\mathbf{O}$ represents a fixed point on the line. Let $\mathbf{P}$ be the position of the particle at time t, where $\mathbf{OP} = x$ and $\mathbf{P'}$ be the position of the particle at time $t + \delta t$, with $\mathbf{OP'} = x + \delta x$. Therefore $\delta x / \delta t$ represents the average rate of displacement or the average velocity during the interval δt. If this ratio be independent of the interval δt, i.e. if it has the same value for all intervals of time, then the velocity is constant or uniform, and equal distances will be traversed in equal times. Whether the ratio $\delta x / \delta t$ be constant or not, its limiting value as δt tends to zero is defined to be the measure of the *velocity* (also known as instantaneous velocity) of the moving point at time t. But this limiting value is the differential coefficient of x with regard to t, so that if we denote the velocity by v, we have

$$v = \lim_{\delta t \to 0} \frac{\delta x}{\delta t} = \frac{dx}{dt} = \dot{x}$$



Again, *Acceleration* is similarly defined as the rate of change of velocity. Thus, if $v, v + \delta v$ denote the velocities of the moving point at times $t, t + \delta t$, then δv is the change of velocity in time δt and $\delta v / \delta t$ is the average rate of change of velocity during the interval δt. If this ratio is independent of the interval δt, then the acceleration is constant or uniform, or equal increments of velocity take place in equal intervals. Whether the ratio $\delta v / \delta t$ be constant or not, its limiting value as δt tends to zero is defined to be the measure of the acceleration of the moving point at time t. But this limiting value is the differential coefficient of v with regard to t, so that if we denote the acceleration by a, we have

$$\begin{aligned} a &= \lim_{\delta t \to 0} \frac{\delta v}{\delta t} = \frac{dv}{dt} \\ &= \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2 x}{dt^2} = \ddot{x} \end{aligned}$$

■ Other Expression for Acceleration

Let $v = \dfrac{dx}{dt}$. We can write (using chain rule in Differentiation)

$$\begin{aligned} a = \ddot{x} &= \frac{d^2 x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) \\ &= \frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt} = v\frac{dv}{dx} \end{aligned}$$

v

Therefore, $\frac{d^2x}{dt^2}$, $\frac{dv}{dt}$ and $v\frac{dv}{dx}$ are three expressions for representing the acceleration and any one of them can be used to suit the convenience in working out the problems.

■ Remember

The law of acceleration in a particular problem may be given by expressing the acceleration as a function of the time t, or the distance x, or the velocity v. The problem of further investigating the motion can then be solved as follows:

➤If acceleration is given as a function of the time t say $\varphi(t)$ so

$$a = \varphi(t) \qquad \Rightarrow \frac{dv}{dt} = \varphi(t)$$
$$\Rightarrow dv = \varphi(t)dt$$
$$\Rightarrow v = \int \varphi(t)dt + c_1$$

And then $\quad \because \quad v = \int \varphi(t)dt + c_1 \qquad \Rightarrow \frac{dx}{dt} = \int \varphi(t)dt + c_1$

$$\Rightarrow dx = \int \varphi(t)dt + c_1 \quad dt$$

$$\therefore x = \int \int \varphi(t)dt + c_1 \quad dt + c_2$$

➤If acceleration is given as a function of the distance x say $f(x)$ so

$$a = f(x) \qquad \Rightarrow v\frac{dv}{dx} = f(x) \qquad \Rightarrow vdv = f(x)dx$$
$$\Rightarrow v^2 = 2\int f(x)dx + c_3$$

Further, $\quad \because v^2 = 2\int f(x)dx + c_3$

$$\Rightarrow \frac{dx}{dt} = \mp\sqrt{2\int f(x)dx + c_3}$$

$$\Rightarrow \mp \frac{dx}{\sqrt{2\int f(x)dx + c_3}} = dt$$

$$\Rightarrow t + c_4 = \mp\int \frac{dx}{\sqrt{2\int f(x)dx + c_3}}$$

➤Again, Acceleration is given as a function of velocity v say $\varphi(v)$

$$a = \varphi(v) \qquad \Rightarrow \frac{dv}{dt} = \varphi(v)$$
$$\Rightarrow \frac{dv}{\varphi(v)} = dt \qquad \text{by integrating}$$
$$\Rightarrow t = \int \frac{dv}{\varphi(v)} + c_5$$

or we may connect velocity with distance by writing

$$v\frac{dv}{dx} = \varphi(v) \quad \Rightarrow \frac{vdv}{\varphi(v)} = dx \qquad \therefore x = \int \frac{vdv}{\varphi(v)} + c_6$$

where, $c_1 - c_6$ are constants of integration.

■ Illustrative Examples ■

Example

A car moves along a straight line such that its displacement x from a fixed point on the line (origin) at time t is given by $x = t^3 - 9t^2 + 24t + 6$. Determine the instant when the acceleration becomes zero, the position of the car at this instant and the velocity of the particle then.

Solution

Since, $x = t^3 - 9t^2 + 24t + 6$. Differentiating with respect to time (w.r.t),

the velocity $$v = \frac{dx}{dt} = 3t^2 - 18t + 24,$$

and the acceleration is $$a = \frac{dv}{dt} = 6t - 18$$

Now the acceleration vanishes i.e. $a = 0$ when $6t - 18 = 0 \Rightarrow t = 3$

When $t = 3$, the position is given by $x = 3^3 - 9(3^2) + 24(3) + 6 = 24$ units.

Again when $t = 3$ the velocity is given by $v = 3(3^2) - 18(3) + 24 = -3$, this means that at $t = 3$ the velocity of the particle equals 3 units and in the opposite direction of x.

Example

If at time t the displacement x of a particle moving away from the origin is given by $x = A\cos t + B\sin t$, where A, B are constants. Find the velocity and acceleration of the particle at in terms of time.

Solution

Given that $x = A\cos t + B\sin t$

Differentiating with respect to time (w.r.t), we obtain the velocity of the particle

$$v = \frac{dx}{dt} = B\cos t - A\sin t$$

Differentiating again, one get the acceleration at any time,

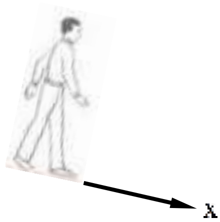
$$\begin{aligned} a = \frac{dv}{dt} &= -A\cos t - B\sin t \\ &= -(\underbrace{A\cos t + B\sin t}_{x}) \\ &= -x \end{aligned}$$

<u>Note that</u> the acceleration proportional to the displacement.

Example

A man moves along a straight line where its distance x from a fixed point on the line is given by $x = A\cos(\mu t + \epsilon)$. Prove that its acceleration varies as the distance measured from the origin and is directed towards the origin.

Solution



Since we have $x = A\cos(\mu t + \epsilon)$

Differentiating w.r.t $x = A\cos(\mu t + \epsilon)$, we get

$$\frac{dx}{dt} = -\mu A\sin(\mu t + \epsilon)$$

Differentiation again $\frac{d^2x}{dt^2} = -\mu^2 \underbrace{A\cos(\mu t + \epsilon)}_{x} = -\mu^2 x$

That is the acceleration varies as the distance x from the origin. The negative sign "-" indicates that it is in the negative sense of the x-axis, i.e., towards the origin.

Example

A truck moves along a straight line such that its distance x from a fixed point on it and the velocity v are related by $v^2 = \mu(b^2 - x^2)$. Prove that the acceleration varies as the distance from the origin and is directed towards the origin.

Solution

Since we have $v^2 = \mu(b^2 - x^2)$

Differentiating w.r.t x, we obtain

$$2v\frac{dv}{dx} = \mu(-2x) \qquad \therefore v\frac{dv}{dx} = a = -\mu x$$

Hence the acceleration varies as the distance x from the origin. The negative sign “-“ indicates that it is in the direction of x decreasing, i.e., towards the origin.

Example

A particle moves along a straight line such that its distance x from a fixed point on it and the time at any time t are related by $x = 2(1 - e^{-t})$. Find the velocity in terms of distance and the acceleration in terms of velocity.

Solution

In order to obtain the velocity with differentiating the function of position x with respect to time, we get

$$x = 2 \;\; 1 - e^{-t} \qquad \Rightarrow v = \frac{dx}{dt} = 2e^{-t} \qquad \text{Note} \quad \frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$$

$$\because x - 2 = -2e^{-t} \qquad \Rightarrow v = 2 - x$$

This equation illustrates the relation between velocity and distance.

Now to get the relation between acceleration and velocity

$$\because a = v\frac{dv}{dx} = v(-1) = -v \qquad \text{Note} \quad \frac{dv}{dx} = -1 \quad \therefore a = -v$$

Example

A car moves along a straight line such that its acceleration at any time t is given by $6t + 2$. Initially the mass at rest placed at the origin point. Determine the velocity and distance as a function of time. Determine the position of the car after 5 sec.

Solution

Since we have $a = 6t + 2,\qquad a = \dfrac{dv}{dt} \qquad \Rightarrow \dfrac{dv}{dt} = 6t + 2$

Thus, by separation of variables we get

$$dv = \; 6t + 2 \;\; dt \qquad \Rightarrow \int dv = \int \; 6t + 2 \;\; dt$$

$$\therefore \; v = 3t^2 + 2t + c_1$$

From initial conditions at $t = 0$, $v = 1$ then $c_1 = 0$

Again, $\therefore \; v = 3t^2 + 2t$ this equation gives the relation between velocity and time. Since $v = \dfrac{dx}{dt}$ that is

$\dfrac{dx}{dt} = 3t^2 + 2t \qquad \Rightarrow dx = \; 3t^2 + 2t \;\; dt$ (Separation variables)

$\int dx = \int \; 3t^2 + 2t \;\; dt$ or $x = t^3 + t^2 + c_2$

From initial conditions at $t = 0$, $x = 0$ then $c_2 = 0$, i.e.

$x = t^3 + t^2$

this equation gives the relation between distance and time.

The position at $t = 5$ is $x\big|_{t=5} = 5^3 + 5^2 = 150$

Example

A point moves along a straight line according to $v = u + bx$, where u, b are constants. Find the velocity and acceleration in terms of time and the acceleration in terms of distance and also as a function of velocity.

Solution

Velocity and acceleration can be obtained by differentiation the function of position and then velocity with respect to time, therefore

$$v = u + bx \quad \Rightarrow a = \frac{dv}{dt} = b\frac{dx}{dt} = bv = b(u + bx) \qquad \Rightarrow a = b(u + bx)$$

This equation gives the acceleration as a function of velocity $a = bv$ and as a function of distance $a = b(u + bx)$

Again to get the velocity and acceleration as functions of time

$$\because v = u + bx \quad \Rightarrow \frac{dx}{dt} = b(u + bx) \qquad \Rightarrow \frac{dx}{u + bx} = bdt$$

Multiply the previous relation by b and then integrate

$$\int \frac{bdx}{u + bx} = \int b^2 dt \quad \Rightarrow \ln(u + bx) = b^2 t + C$$

Where C is integration constant , the last relation can be rewritten as

$$\because \ln(u + bx) = b^2 t + C \qquad \Rightarrow \ln v = b^2 t + C \quad \text{Or}$$
$$\Rightarrow v = Ae^{b^2 t}, \quad A = e^C$$

This is the relation between velocity and time, also the acceleration given by

$$a = bv = bAe^{b^2 t}$$

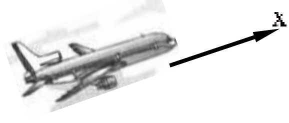
Example

A plane flies along a straight line with retardation $a = -2v^2$. Find the position at any instance if the point starts from origin with initial velocity equals unity.

Solution

The motion under retardation where $a = -2v^2$ but we know $a = \frac{dv}{dt}$, so

$$\because a = -2v^2 \qquad \Rightarrow \frac{dv}{dt} = -2v^2$$



By separation of variables and integrate, we obtain

$$-\int \frac{dv}{v^2} = \int 2dt + c_1 \qquad \Rightarrow \frac{1}{v} = 2t + c_1$$

The integration constant c_1 can be evaluated as $v = 1$ when $t = 0$, hence $1 = 2(0) + c_1 \quad \therefore c_1 = 1$ then the velocity can be obtained by

$$\frac{1}{v} = 2t + 1 \quad \text{but} \quad v = \frac{dx}{dt} \quad \therefore \frac{dt}{dx} = 2t + 1 \qquad \text{Or} \qquad \frac{dt}{2t+1} = dx$$

Again by integrating we get

$$\frac{2dt}{2t+1} = 2dx \quad \Rightarrow \ln(2t+1) = 2x + c_2$$

From initial condition $x = 0$ when $t = 0$ then $c_2 = 0$ and the relation between distance and time becomes

$$x = \frac{1}{2}\ln(2t+1)$$

Example

A particle starts from rest at a distance h from the origin $\mathbf{O}$ with retardation $-4x^{-3}$. Prove that the particle reach to distance ℓ from $\mathbf{O}$ in time $\frac{h}{2}\sqrt{h^2 - \ell^2}$ and then find its velocity at this position.

Solution

Since we have been given the retardation as $a = -16x^{-3}$ and $a = v\dfrac{dv}{dx}$ therefore,

$$\therefore v\frac{dv}{dx} = -4x^{-3} \qquad \Rightarrow vdv = -4x^{-3}\ dx$$

By integrating, we obtain

$$\therefore \int vdv = -\int 4x^{-3}\ dx + c_1 \qquad \text{Or} \qquad \frac{1}{2}v^2 = \frac{2}{x^2} + c_1 \quad \text{Or} \quad v^2 = \frac{4}{x^2} + c$$

The integration constant c can be evaluated as $v = 0$ when $x = h$, hence $0 = \dfrac{4}{h^2} + c$ i.e. $c_1 = -\dfrac{4}{h^2}$ and then we get

$$v^2 = \frac{4}{x^2} - \frac{4}{h^2} = \frac{4(h^2 - x^2)}{x^2 h^2} \quad \therefore v = \pm \frac{2}{h} \frac{\sqrt{h^2 - x^2}}{x}$$

We will consider the minus sign since the motion of the particle towards the origin –in decreasing x - and use $v = \frac{dx}{dt}$

$$\therefore \frac{dx}{dt} = -\frac{2}{h} \frac{\sqrt{h^2 - x^2}}{x} \quad \Rightarrow -\frac{xdx}{\sqrt{h^2 - x^2}} = \frac{2}{h} dt \quad \text{Or}$$

$$\Rightarrow -\int \frac{xdx}{\sqrt{h^2 - x^2}} = \int \frac{2}{h} dt + c_2$$

$$\Rightarrow \sqrt{h^2 - x^2} = \frac{2}{h} t + c_2$$

To obtain the constant c_2 when $x = h$ as $t = 0$ and then $c_2 = 0$ so

$$\therefore \sqrt{h^2 - x^2} = \frac{2}{h} t \quad \text{Or} \quad t = \frac{h}{2} \sqrt{h^2 - x^2}$$

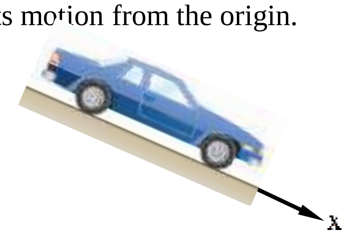
The spent time to reach to a distance ℓ from origin point is $t = \frac{h}{2} \sqrt{h^2 - \ell^2}$, to determine the velocity at this position, we put $x = \ell$ in velocity relation, that is

$$v\big|_{x=\ell} = \frac{2\sqrt{h^2 - \ell^2}}{h\ell}$$

Example

A car moves along a straight line according to the relation $v = (1 + x^2)t$. Find the distance as a function of time if the point starts its motion from the origin.

Solution



Since $v = (1 + x^2)t$ thus

$$\frac{dx}{dt} = (1 + x^2)t \quad \Rightarrow \frac{dx}{1 + x^2} = tdt$$

$$\therefore \int \frac{dx}{1+x^2} = \int t dt + c_1 \qquad \Rightarrow \tan^{-1} x = \frac{1}{2}t^2 + c_1$$

From initial condition where the point starts its motion at origin

$$\therefore \tan^{-1} 0 = \frac{1}{2}0^2 + c_1 \quad \Rightarrow 0 = 0 + c_1 \therefore c_1 = 0 \qquad \therefore x = \tan\left(\frac{1}{2}t^2\right)$$

Note that

$$\int \frac{f' dx}{1+f^2} = \tan^{-1} f$$

Example

If t be regarded as a function of velocity v, prove that the rate of decrease of acceleration is given by $a^3 \frac{d^2t}{dv^2}$, a being the acceleration.

Solution

Let a be the acceleration at time t. Then $a = \frac{dv}{dt}$. Now the rate of decrease of acceleration = $-\frac{da}{at}$

$$= -\frac{d}{at}\left(\frac{dv}{dt}\right) = -\frac{d}{at}\left(\frac{dt}{dv}\right)^{-1} \text{ regarded } t \text{ as a function of } v$$

$$= -\left(\frac{d}{av}\left(\frac{dt}{dv}\right)^{-1}\right)\frac{dv}{dt} = \left(\left(\frac{dt}{dv}\right)^{-2}\frac{d^2t}{dv^2}\right)\frac{dv}{dt}$$

$$= \left(\left(\frac{dv}{dt}\right)^{2}\frac{d^2t}{dv^2}\right)\frac{dv}{dt} = \underbrace{\left(\frac{dv}{dt}\right)^{3}}_{a}\frac{d^2t}{dv^2} = a^3\frac{d^2t}{dv^2}$$

Example

Prove that if a point moves with a velocity varying as any power (not less than unity) of its distance from a fixed point which it is approaching, it will never reach that point.

▌▌Solution ▶

If x is the distance of the particle from the fixed point $\mathbf{O}$ at any time t, then its speed v at this time is given by $v = kx^n$, where k is a constant and n is not less than 1. Since the particle is moving towards the fixed point i.e., in the direction decreasing, therefore

$$\frac{dx}{dt} = -v \qquad \text{or} \qquad \frac{dx}{dt} = -kx^n \qquad \text{....(1)}$$

Case 1. If $n = 1$, then from (1), we have

$$\frac{dx}{dt} = -kx \qquad \text{or} \qquad dt = -\frac{1}{k}\frac{dx}{x}$$

Integrating, $t = -\frac{1}{k}\ln x + A$ where A is a constant.

Putting $x = 0$ then the time t to reach the fixed point $\mathbf{O}$ is given by

$$t = -\frac{1}{k}\ln 0 + A = \infty$$

i.e., the particle will never reach the fixed point $\mathbf{O}$

Case 2. If $n > 1$, then from (1), we have

$$dt = -\frac{1}{k}x^{-n}dx$$

Integrating, $t = -\frac{1}{k}\frac{x^{1-n}}{1-n} + B$ where B is a constant.

Or

$$t = \frac{1}{k(n-1)x^{n-1}} + B$$

Putting $x = 0$ then the time t to reach the fixed point $\mathbf{O}$ is given by

$$t = \infty + B = \infty$$

i.e., the particle will never reach the fixed point $\mathbf{O}$

Hence if $n \geq 1$, the particle will never reach the fixed point, it is approaching.

PROBLEMS

❒ A particle moving in a straight line is subject to a resistance which produces the retardation kv^3, where v is the velocity and k is a constant. Show that v and t (the time) are given in terms of x (the distance) by the equations $v = \dfrac{u}{kux+1}$, $t = \dfrac{1}{2}kx^2 + \dfrac{x}{u}$, where u is the initial velocity.

❒ If the relation between x and t is of the form $t = bx^2 + kx$, find the velocity v as a function of x, and prove that the retardation of the particle is $2bv^3$.

❒ A particle is projected vertically upwards with speed u and moves in a vertical straight line under uniform gravity with no air resistance. Find the maximum height achieved by the particle and the time taken for it to return to its starting point.

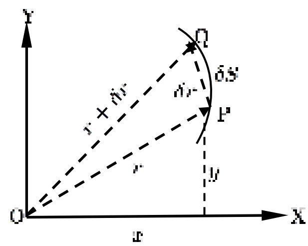
Kinematics in Two Dimensions

■ Velocity in Cartesian Coordinates

The velocity vector of a particle (or point) moving along a curve is the rate of change of its displacement with respect to time.

Let P and Q be the positions of a particle moving along a curve at times t and $t+\delta t$ respectively. With respect to $\mathbf{O}$ as the origin of vectors, let $\underline{\mathbf{O}P} = \underline{r}$ and $\underline{\mathbf{O}Q} = \underline{r} + \delta\underline{r}$ Then $\underline{\mathbf{P}Q} = \underline{\mathbf{O}Q} - \underline{\mathbf{OP}} = \delta\underline{r}$ represents the displacement of the particle in time δt and $\dfrac{\delta\underline{r}}{\delta t}$ indicates the average rate of displacement (or average velocity) during the interval δt. The limiting value of the average velocity $\dfrac{\delta\underline{r}}{\delta t}$ as δt tends to zero ($\delta t \to 0$) is the velocity. Therefore if the vector $\underline{v}$ represents the velocity of the particle at time t then

$$\underline{v} = \lim_{\delta t \to 0} \frac{\delta \underline{r}}{\delta t} = \frac{d\underline{r}}{dt} = \dot{\underline{r}}$$



Where $\underline{r}$ is the position vector of the particle.

Now, if $\qquad \underline{r} = x\hat{i} + y\hat{j}$

Then $\qquad \underline{v} = \dfrac{d\underline{r}}{dt} = \underbrace{\dfrac{dx}{dt}}_{v_x}\hat{i} + \underbrace{\dfrac{dy}{dt}}_{v_y}\hat{j} = \dot{x}\hat{i} + \dot{y}\hat{j}$

Note that $(\dot{x}, \dot{y})$ are called the components or resolved parts of the velocity $\underline{v}$ along the axes x and y respectively. The speed of the particle at P is given by

$$|\underline{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \frac{ds}{dt}$$

Also the angle θ which the direction of $\underline{v}$ makes with $\mathbf{OX}$ is

$$\tan\theta = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{dy}{dx}$$

■ Acceleration in Cartesian Coordinates

The acceleration vector of a particle moving along a curve is defined as the rate of change of its velocity vector.

if $\underline{v}$ and $\underline{v} + \delta\underline{v}$ are the velocities of a particle moving along a curve at times t and $t + \delta t$ respectively, then $\delta\underline{v}$ is the change in velocity vector in time δt and $\frac{\delta \underline{v}}{\delta t}$ is the average and then

$$\underline{a} = \lim_{\delta t \to 0} \frac{\delta \underline{v}}{\delta t} = \frac{d\underline{v}}{dt} = \frac{d}{dt}\left(\frac{d\underline{r}}{dt}\right) = \frac{d^2\underline{r}}{dt^2}$$

Substituting for $\underline{v} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$ we have,

$$\underline{a} = \frac{d}{dt}\left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}\right) = \underbrace{\frac{d^2x}{dt^2}}_{a_x}\hat{i} + \underbrace{\frac{d^2y}{dt^2}}_{a_y}\hat{j} = \ddot{x}\,\hat{i} + \ddot{y}\,\hat{j}$$

Here, $(\ddot{x}, \ddot{y})$ are called the components of the acceleration $\underline{a}$ along the axes x and y respectively. The magnitude of the acceleration is given by

$$|\underline{a}| = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}$$

Again, the angle φ which the direction of $\underline{a}$ makes with OX is

$$\tan\varphi = \frac{d^2y}{dt^2} \Big/ \frac{d^2x}{dt^2}$$

■ Illustrative Examples ■

Example

A point moves along the curve $x = t^3 + 1, \quad y = t^2$ where, t is the time. Determine the components of velocity and acceleration at $t = 1$

Solution

Let $\underline{r}$ be the position vector of the particle at time t , therefore

$$\underline{r} = x\,\hat{i} + y\,\hat{j} = (t^3 + 1)\,\hat{i} + t^2\,\hat{j}$$

Then the velocity vector is

$$\underline{v} = \frac{d\underline{r}}{dt} = 3t^2\,\hat{i} + 2t\,\hat{j} \quad \text{and} \qquad \underline{v}\Big|_{t=1} = 3(1)^2\,\hat{i} + 2(1)\,\hat{j} = 3\,\hat{i} + 2\,\hat{j}$$

Again the vector of acceleration is

$$\underline{a} = \frac{d\underline{v}}{dt} = 6t\,\hat{i} + 2\,\hat{j} \qquad \text{and} \qquad \underline{a}\Big|_{t=1} = 6(1)\,\hat{i} + 2\,\hat{j} = 6\,\hat{i} + 2\,\hat{j}$$

Example

The position of a moving point at time t is given by $x = 3\cos t, \quad y = 2\sin t$

Find its path velocity and acceleration vectors.

Solution

Since the parametric equations are $x = 3\cos t, \quad y = 2\sin t$ then

$$\left(\frac{x}{3}\right)^2 = \cos^2 t, \quad \left(\frac{y}{2}\right)^2 = \sin^2 t \qquad \Rightarrow \left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \qquad \text{or } 4x^2 + 9y^2 = 36$$

This is a the path equation which represents an Ellipse

Velocity vector is $\underline{v} = -3\sin t\ \hat{i} + 2\cos t\ \hat{j}$

While the acceleration vector is

$$\underline{a} = -3\cos t\ \hat{i} - 2\sin t\ \hat{j} = -\underbrace{(3\cos t\ \hat{i} + 2\sin t\ \hat{j})}_{\underline{r}} = -\underline{r}$$

Example

A particle moves along the curve $y = 2x^2$ such that its horizontal component of velocity is constant and equals 2. Calculate the components of acceleration and velocity when $y = 8$.

Solution

Since the horizontal component of velocity equals 2, i.e. $\dot{x} = 2$, therefore by differentiating w.r.t t we get

$\ddot{x} = 0$ and $y = 2x^2 \quad \Rightarrow \dot{y} = 4x\dot{x} = 8x \quad \therefore \ddot{y} = 8\dot{x} = 16$

That is the acceleration vector is given by

$\underline{a} = 16\,\hat{j}$

and the velocity components are $\dot{x} = 2$ and $\dot{y} = 8x$

Since as $y = 8$ gives $x = \pm 2$ thus, $\quad \underline{v} = 2\,\hat{i} + 8(\pm 2)\,\hat{j}, \quad |\underline{v}| = \sqrt{260}$

Example

A particle describes a plane curve such that its components of acceleration equal $(0, -\mu / y^2)$ with initial velocity $\sqrt{2\mu / b}$ parallel to **X-** axis and the initial position $(0, b)$. Find the path equation.

Solution

Here we are given that

$$\frac{d^2x}{dt^2} = 0, \qquad \frac{d^2y}{dt^2} = -\frac{\mu}{y^2}$$

Note that $\dfrac{d^2y}{dt^2} = \dfrac{d}{dt}\left(\dfrac{dy}{dt}\right) = \dfrac{d}{dy}\left(\dfrac{dy}{dt}\right) \times \dfrac{dy}{dt} = \dot{y}\dfrac{d\dot{y}}{dy}$ chain rule

Then $\dot{y}\dfrac{d\dot{y}}{dy}=-\dfrac{\mu}{y^2} \quad \Rightarrow \dot{y}d\dot{y}=-\dfrac{\mu}{y^2}dy \quad \Rightarrow \int \dot{y}d\dot{y} =-\int \dfrac{\mu}{y^2}dy$

$\dot{y}^2=\dfrac{2\mu}{y}+c_1 \qquad \left(\dot{y}=\dfrac{dy}{dt}\right)$

Initially $\dfrac{dy}{dt}=0$ when $y=b$, thus $\quad c_1=-\dfrac{2\mu}{b}$

$\dot{y}^2=\dfrac{2\mu}{y}-\dfrac{2\mu}{b} \qquad \Rightarrow \dot{y}^2=\dfrac{2\mu}{y}-\dfrac{2\mu}{b}=2\mu\left(\dfrac{1}{y}-\dfrac{1}{b}\right)=\dfrac{2\mu}{b}\dfrac{b-y}{y}$

Hence

$$\frac{dy}{dt}=-\sqrt{\frac{2\mu}{b}}\sqrt{\frac{b-y}{y}} \qquad (1)$$

(Negative sign has been taken because the particle is moving in the direction of y decreasing)

Again from $\dfrac{d^2x}{dt^2}=0 \qquad \Rightarrow \dfrac{dx}{dt}=c_3$

Initially when $t=0$, $\dfrac{dx}{dt}=\sqrt{\dfrac{2\mu}{b}}$ thus $\quad c_3=\sqrt{\dfrac{2\mu}{b}}$

$$\therefore \frac{dx}{dt}=\sqrt{\frac{2\mu}{b}} \qquad (2)$$

By dividing the two equations (1) and (2) we get

$\dfrac{dy}{dx}=-\sqrt{\dfrac{b-y}{y}} \qquad \Rightarrow \sqrt{\dfrac{y}{b-y}}dy=-dx$, then by integrating

$$b\left(\sin^{-1}\sqrt{\frac{y}{b}}-\sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}}\right)=-x+c_2$$

Hint to get the integration $\int\sqrt{\dfrac{y}{b-y}}dy$ let us use the transformation

$$y = b\sin^2\theta \qquad \Rightarrow dy = 2b\sin\theta\cos\theta\,d\theta$$

$$\therefore \int\sqrt{\frac{y}{b-y}}\,dy = \int\sqrt{\frac{b\sin^2\theta}{b-b\sin^2\theta}}\,2b\sin\theta\cos\theta\,d\theta$$

$$=\int\sqrt{\frac{b\sin^2\theta}{b\cos^2\theta}}\,2b\sin\theta\cos\theta\,d\theta$$

$$=\int\frac{\sin\theta}{\cancel{\cos\theta}}\,2b\sin\theta\,\cancel{\cos\theta}\,d\theta = 2b\int\sin^2\theta\,d\theta$$

$$\because \sin^2\theta = \frac{1}{2}\ \ 1-\cos 2\theta$$

$$\Rightarrow 2b\int\sin^2\theta\,d\theta = 2b\int\frac{1}{2}\ \ 1-\cos 2\theta\ \ d\theta$$

$$= b\int\ 1-\cos 2\theta\ \ d\theta = b\left(\theta-\frac{\sin 2\theta}{2}\right)$$

$$\therefore \int\sqrt{\frac{y}{b-y}}\,dy = b\left(\sin^{-1}\left(\sqrt{\frac{y}{b}}\right)-\sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}}\right)$$

The initial condition is $t = 0\ \ x = 0,\ \ y = b$ then from the equation

$$b\left(\sin^{-1}\sqrt{\frac{y}{b}}-\sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}}\right) = -x + c_2 \qquad \Rightarrow c_2 = b\frac{\pi}{2}$$

$$\therefore b\left(\sin^{-1}\left(\sqrt{\frac{y}{b}}\right)-\sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}}\right) = b\frac{\pi}{2}-x$$

$$\Rightarrow \sin^{-1}\sqrt{\frac{y}{b}} = \sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}}+\frac{\pi}{2}-\frac{x}{b}$$

$$\sqrt{\frac{y}{b}} = \sin\left(\sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}}+\frac{\pi}{2}-\sqrt{2\mu b}\,t\right)$$

$$= \cos\left(x-\sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}}\right)$$

$$y = b\cos^2\left(x-\sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}}\right)$$

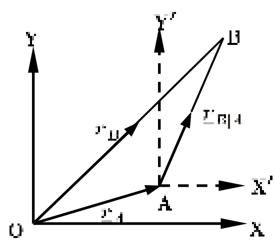
■ Relative motion of two particles

Motion does not happen in isolation. If you're riding in a train moving at 10 ms^{-1} east, this velocity is measured relative to the ground on which you're traveling. However, if another train passes you at 15 ms^{-1} east, your velocity relative to this other train is different from your velocity relative to the ground. Your velocity relative to the other train is 5 ms^{-1} west. To explore this idea further, we first need to establish some terminology.

♦ Reference Frames

To discuss relative motion in one or more dimensions, we first introduce the concept of reference frames. When we say an object has a certain velocity, we must state it has a velocity with respect to a given reference frame. In most examples we have examined so far, this reference frame has been Earth. If you say a person is sitting in a train moving at 10 m/s east, then you imply the person on the train is moving relative to the surface of Earth at this velocity, and Earth is the reference frame. We can expand our view of the motion of the person on the train and say Earth is spinning in its orbit around the Sun, in which case the motion becomes more complicated. In this case, the solar system is the reference frame. In summary, all discussion of relative motion must define the reference frames involved. We now develop a method to refer to reference frames in relative motion.

For two particles A and B moving in plane as shown, we considered the relative motion of B with respect to A, or more precisely, with respect to a moving frame attached to A and in translation with A. Denoting by $\underline{r}_{B|A}$ the relative position vector of B with respect to A, we had



$$\underline{r}_B = \underline{r}_A + \underline{r}_{B|A} \qquad \text{or} \qquad \underline{r}_{B|A} = \underline{r}_B - \underline{r}_A$$

Denoting by $\underline{v}_{B|A}$ and $\underline{a}_{B|A}$, respectively, the relative velocity and the relative acceleration of B with respect to A, we also showed that

Differentiating previous equation with respect to time

$$\frac{d\underline{r}_{B|A}}{dt} = \frac{d\underline{r}_B}{dt} - \frac{d\underline{r}_A}{dt} \qquad \textbf{or} \qquad \underline{v}_{B|A} = \underline{v}_B - \underline{v}_A$$

Differentiating previous equation with respect to time

$$\frac{d\underline{v}_{B|A}}{dt} = \frac{d\underline{v}_B}{dt} - \frac{d\underline{v}_A}{dt} \qquad \textbf{or} \qquad \underline{a}_{B|A} = \underline{a}_B - \underline{a}_A$$

■ Illustrative Examples ■

Example

Two points A and B are moving along a straight line such that $\underline{x}_A = t^3 - 2t$ and $\underline{x}_B = 2t^3 + t^2 - 5$. Find the relative velocity $\underline{v}_{BA}$ and acceleration $\underline{a}_{BA}$.

Solution

Since the relative position of point B with respect to point A, $\underline{x}_{B|A}$, is given by

$$\underline{x}_{B|A} = \underline{x}_B - \underline{x}_A$$

$$\Rightarrow \underline{x}_{B|A} = (2t^3 + t^2 - 5) - (t^3 - 2t) = t^3 + t^2 + 2t - 5$$

Hence the relative velocity $\underline{v}_{B|A}$ is obtained by

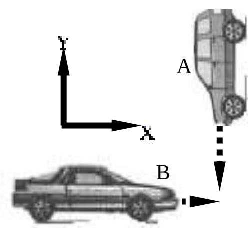
$$\underline{v}_{B|A} = \frac{d\underline{x}_{B|A}}{dt} = 3t^2 + 2t + 2$$

Again the relative acceleration $\underline{a}_{BA}$ is given by

$$\underline{a}_{B|A} = \frac{d\underline{v}_{B|A}}{dt} = 6t + 2$$

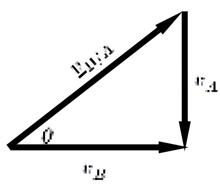
Example

A car A is traveling south at a speed of 70 km/h toward an intersection. A car B is traveling east toward the intersection at a speed of 80 km/h, as shown. Determine the velocity of the car B relative to the car A.



Solution

According to the given data the velocity of car A is $\underline{v}_A = -70\,\hat{j}$ and velocity of car B is $\underline{v}_B = 80\,\hat{i}$ then



$$\begin{aligned}\underline{v}_{B|A} &= \underline{v}_B - \underline{v}_A \\ &= 80\hat{i} - (-70\hat{j}) \\ &= 80\hat{i} + 70\hat{j}\end{aligned}$$

$$\Rightarrow \left|\underline{v}_{B|A}\right| = \sqrt{(80)^2 + (70)^2} = \sqrt{11300} \simeq 106.3 \text{ km h}^{-1}$$

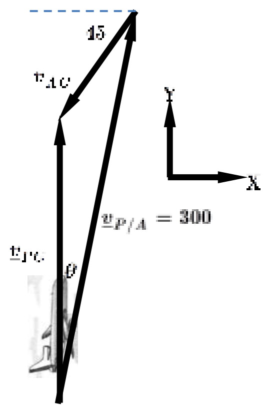
And make an angle θ with the velocity direction of car B obtained by

$$\tan\theta = \frac{70}{80} = \frac{7}{8} \qquad \Rightarrow \theta = \tan^{-1}\left(\frac{7}{8}\right)$$

Example

A pilot must fly his plane due north to reach his destination. The plane can fly at 300 km/h in still air. A wind is blowing out of the northeast at 90 km/h. Calculate the speed of the plane relative to the ground and in what direction must the pilot head her plane to fly due north.

Solution



The pilot must point her plane somewhat east of north to compensate for the wind velocity. We need to construct a vector equation that contains the velocity of the plane with respect to the ground, the velocity of the plane with respect to the air, and the velocity of the air with respect to the ground. Since these last two quantities are known, we can solve for the velocity of the plane with respect to the ground. We can graph the vectors and use this diagram to evaluate the magnitude of the plane's velocity with respect to the ground. The diagram will also tell us the angle the plane's velocity makes with north with respect to the air, which is the direction the pilot must head her plane.

From the given data the velocity of plane P is $\underline{v}_{P|A} = 300(\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$ and velocity of air A is $\underline{v}_{A|G} = -90\,(\cos 45^0\,\hat{i} + \sin 45^0\,\hat{j})$ and $\underline{v}_{P|G} = v_{P|G}\,\hat{j}$ then

$$\begin{aligned}\underline{v}_{P|G} &= \underline{v}_{P|A} + \underline{v}_{A|G} \\ &= 300(\sin\theta\,\hat{i} + \cos\theta\,\hat{j}) + -90\,(\cos 45^0\,\hat{i} + \sin 45^0\,\hat{j}) \\ &= (300\sin\theta - 90\cos 45^0)\hat{i} + (300\cos\theta - 90\sin 45^0)\hat{j}\end{aligned}$$

$$\Rightarrow 300\sin\theta - 90\cos 45^0 = 0$$

$$\sin\theta = \frac{45\sqrt{2}}{300} \qquad \text{And } v_{P|G} = 300\cos\theta - 90\sin 45^0 \simeq 230 \text{ km h}^{-1}$$

PROBLEMS

❒ The position of a moving point at time t is given by $x = at^2,\ y = 2at$

Find its velocity and acceleration

❒ A particle moves with constant velocity parallel to the axis of $\mathbf{Y}$ and a velocity proportional to y parallel to the axis of $\mathbf{X}$. Prove that it will describe a parabola

❒ A particle is acted on by a force parallel to the axis of $\mathbf{Y}$ whose acceleration is λy and is initially projected with a velocity $a\sqrt{\lambda}$ parallel to the axis of $\mathbf{X}$ at a point where $y = a$. Prove that it will describe the catenary $y = a\cosh(x / a)$

❒ A boat heads north in still water at 4.5 ms^{-1} directly across a river that is running east at 3.0 ms^{-1}. Find the velocity of the boat with respect to Earth.

PROJECTILE MOTION

Let us consider that u, v denote the resolved parts of the velocity of the particle parallel to the axes at time t and $u + \delta u$, $v + \delta v$ refer to the resolved parts at time $t + \delta t$ then the resolved parts of the acceleration are given as

$$a_x = \lim_{\delta t \to 0} \frac{\delta u}{\delta t} = \frac{du}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} = \ddot{x}$$

$$a_y = \lim_{\delta t \to 0} \frac{\delta v}{\delta t} = \frac{du}{dt} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2} = \ddot{y}$$

The consideration of component velocities and accelerations is of great importance when we have to deal with cases of motion where the path is not a straight line.

■ Equations of Motion of a Particle Moving in a Plane

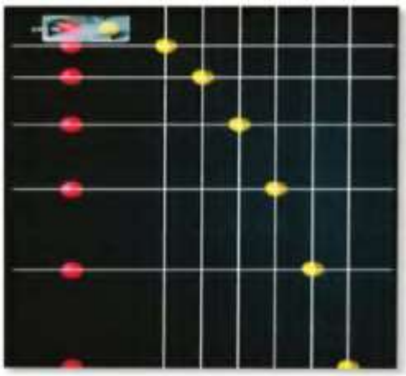
The position of a point in a straight line being determined by one co-ordinate, only one equation of motion is sufficient to determine the motion completely. In the case of a particle moving in a plane, two equations of motion are required in order to obtain the two co-ordinates which define the position of a point in a plane. The two equations of motion are obtained by resolving the forces in any two convenient directions at right angles to one another. If the two directions are taken parallel to the co-ordinate axes the equations of motion, as deduced from the second law of motion, will be of the form

$$m\frac{d^2x}{dt^2} = F_x \quad \text{and} \quad m\frac{d^2y}{dt^2} = F_y$$

where F_x, F_y are the sums of the resolved parts of the forces parallel to the axes of x and y:

■ Projectiles

As an example of motion in two dimension is the projectile motion. Recall that a particle has a mass but negligible size and shape. Therefore, we must limit application to those objects that have dimensions that are of no consequence in the analysis of the motion. In most problems, we will be focused in bodies of finite size, such as rockets, projectiles, or vehicles. Each of these objects can be considered as a particle, as long as the motion is characterized by the motion of its mass center and any rotation of the body is neglected. The free-flight motion of a projectile is often studied in terms of its rectangular components. The acceleration is of approximately 9.81 ms^{-2} or 32.2 ft s^{-2}.

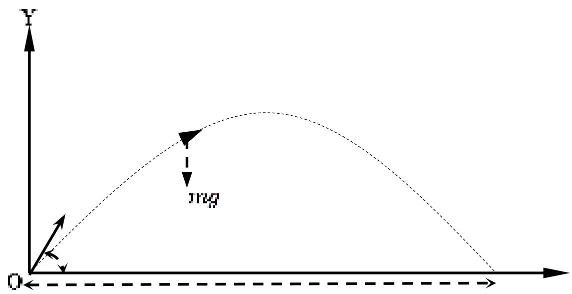


Each picture in this sequence is taken after the same time interval. The red ball falls from rest, whereas the yellow ball is given a horizontal velocity when released. Both balls accelerate downward at the same rate, and so they remain at the same elevation at any instant. This acceleration causes the difference in elevation between the balls to increase between successive photos. Also, note the horizontal distance between successive photos of the yellow ball is constant since the velocity in the horizontal direction remains constant.

We will discuss the motion of a particle projected in the field of gravity. We now consider the motion of a ***projectile***, that is, the motion of a body which is small enough to be regarded as a particle and which is projected in a direction oblique to the direction of gravity. A body that moves freely under uniform gravity, and possibly air resistance, is called a projectile. Projectile motion is very common. In ball games, the ball is a projectile, and controlling its trajectory is a large part of the skill of the game. On a larger scale, artillery shells are projectiles, but guided missiles, which have rocket propulsion, are not.

Note: Near the Earth's surface, we assume that the downward acceleration due to gravity is constant and the effect of air resistance is negligible.
We shall suppose the body to be projected in *vacuum* near the surface of the earth or, in other words, we shall suppose the resistance due to air and the slight variation in the force of gravity to be negligible. A particle of mass m is projected into the air with velocity u, in a direction making an angle α with the horizontal, to find its motion and the path described.

Let O, the point of projection, be taken as the origin and let the horizontal and the vertical lines through be taken as the axes of $\mathbf{X}$ and $\mathbf{Y}$. Again, let P be the position of the moving point, after time t. During the motion of the projectile, the only force acting on it is its weight acting downwards. The equations of motion, therefore, are



$$m\frac{d^2x}{dt^2} = 0 \qquad \text{and} \qquad m\frac{d^2y}{dt^2} = -mg$$

Or in other formula

$$\frac{d^2x}{dt^2} = 0 \qquad \text{and} \qquad \frac{d^2y}{dt^2} = -g$$

Integrating these equations, we get

$$\frac{dx}{dt} = C_1 \qquad \text{and} \qquad \frac{d^2y}{dt^2} = C_2 - gt \qquad (1)$$

where C_1, C_2 are integration constants

Initially at O when $t = 0$, $\frac{dx}{dt} = u\cos\alpha$ and $\frac{dy}{dt} = u\sin\alpha$ then

Equation (1) becomes

$$\frac{dx}{dt} = u\cos\alpha \quad \text{and} \quad \frac{dy}{dt} = u\sin\alpha - gt \qquad (2)$$

Integrating these equations again and applying initial conditions, *viz.*, when $t = 0,\ x = y = 0$, we obtain

$$x = u\cos\alpha\ t \quad \text{and} \quad y = u\sin\alpha\ t - \frac{1}{2}gt^2 \qquad (3)$$

Equation (2) gives the components of the velocity and (3) the displacements of the particle in the horizontal and vertical directions at any time t. These equations could also be written down at once by regarding the particle to be projected with a constant velocity $u\cos\alpha$ in the horizontal direction and with an initial velocity $u\sin\alpha$ under a retardation g in the vertical direction. Eliminating the time t the two parts of Equation (3) we have,

$$y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{u^2\cos^2\alpha} \qquad (4)$$

We now deduce the following facts from the five equations just obtained:

■ The Path Equation of Projectile

Equation (4) is of the second degree and the second degree term x^2 is a perfect square. It follow, therefore, that the path of the particle is a parabola. Equation (4) can be re-written in the form

$$y - \frac{u^2\sin^2\alpha}{2g} = \frac{-g}{2u^2\cos^2\alpha}\left(x - \frac{u^2\sin 2\alpha}{2g}\right)^2$$

It shows that the latus-rectum of the parabolic path $= 2u^2\cos^2\alpha\ /\ g$.

In the particular case when the particle is projected horizontally, $\alpha = 0$, and the Equation (4) of the path reduces to

$$y = -\frac{g}{2u^2}x^2$$

which is obviously a parabola the length of whose latus-rectum is $2u^2 / g$. The path of a projectile is called its ***trajectory***.

■ The Time of Flight

Let T, represents the time which the particle takes in reaching the horizontal plane through the point of projection.

Putting $y = 0$, in the second part of Equation (3) we get either $t = 0$ (at **O**)

And $t = \frac{2u\sin\alpha}{g} \qquad \Rightarrow T = \frac{2u\sin\alpha}{g}$

■ Greatest Height

This is also obtained either by finding by differentiation, the, maximum value of y from the second part of Equation (3) or by the fact that at the greatest height the vertical component of the velocity must vanish, i.e. from the second part of Equation (2)

$$\frac{dy}{dt} = u\sin\alpha - gt = 0 \qquad \Rightarrow t = \frac{u\sin\alpha}{g}$$

Substituting this in Equation (3) and simplifying we get

$$Y = \frac{u^2\sin^2\alpha}{2g}$$

■ Horizontal Range

The range $R = \mathrm{OB}$, on the horizontal plane through the point of projection the horizontal distance described by the particle in the time of flight T.

$$R = u\cos\alpha . T = u\cos\alpha\frac{2u\sin\alpha}{g} = \frac{u^2\sin 2\alpha}{g}$$

R can also be obtained by putting $y = 0$ in Equation (4).

Since, $R = \frac{u^2 \sin 2\alpha}{g}$ so R can be obtained by two values of projected angles

because $\sin 2\alpha = \sin(\pi - 2\alpha) = \sin 2\left(\frac{\pi}{2} - \alpha\right)$ $\quad (\alpha, \frac{\pi}{2} - \alpha)$

■ Maximum Horizontal Range

The range R is maximum when $\sin 2\alpha = 1$, i.e., when $\alpha = \frac{\pi}{4}$ Or $\alpha = 45^O$

therefore, the maximum range $R_{\max} = \frac{u^2}{g}$.

For a given velocity of projection, the horizontal range is the greatest when the angle of projection is 45°.

■ Range on an Inclined Plane

Let a particle be projected from a point O on a plane of inclination β, in the vertical plane through OP, the line of greatest slope of the inclined plane.

Let the velocity of projection be u at an elevation α to the horizontal. The equation to the path of the particle is

$$y = x \tan \alpha - \frac{1}{2} \frac{gx^2}{u^2 \cos^2 \alpha} \tag{10}$$

If the particle strikes the inclined Q plane at the point P, the distance, OP is called the range on the inclined plane. If OP$= R$ then the co-ordinates of P $(R \cos \beta, R \sin \beta)$ must satisfy Equation (10).

$$R \sin \beta = R \cos \beta \tan \alpha - \frac{1}{2} \frac{gr^2 \cos^2 \beta}{u^2 \cos^2 \alpha}$$

Then the range r

$$R = \frac{2u^2}{g} . \frac{(\cos \beta \tan \alpha - \sin \beta) \cos^2 \alpha}{\cos^2 \beta} = \frac{2u^2}{g} . \frac{\sin(\alpha - \beta) \cos \alpha}{\cos^2 \beta}$$
$$= \frac{u^2}{g \cos^2 \beta} \ \sin(2\alpha - \beta) - \sin \beta$$

The range down the plane may be obtained by putting $-\beta$ for β in this case the slope of the inclined plane is downwards.

■ Maximum Range on an Inclined Plane

u and β being known, the range varies with α, and it will be maximum when $\sin(2\alpha-\beta)$ is maximum. When $2\alpha-\beta=\frac{\pi}{2}$ **Or** $\alpha-\beta=\frac{\pi}{2}-\alpha$

Hence for maximum range, the direction of projection must bisect the angle between the vertical and the inclined plane. If OT be the direction of projection, then OT is tangent to the path at O, and the vertical through is perpendicular to the directrix. OT being equally inclined to OP and the vertical, the focus to the path must, therefore, lie on the line OP of the inclined plane, i.e., in the case of maximum range the focus lies in the range. The value of the maximum range is

$$R_{\max}=\frac{u^2(1-\sin\beta)}{g\cos^2\beta}$$

$$=\frac{u^2(1-\sin\beta)}{g(1-\sin^2\beta)}$$

$$=\frac{u^2\,\cancel{(1-\sin\beta)}}{g(1+\sin\beta)\cancel{(1-\sin\beta)}}$$

$$=\frac{u^2}{g(1+\sin\beta)}$$

■ Illustrative Examples ■

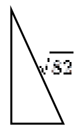
Example

If the maximum height for a projectile is 900 ft and the horizontal range is 400 ft. Find the velocity and its direction.

Solution

Since the maximum height and horizontal range are given by formulas

$$Y = \frac{u^2 \sin^2 \alpha}{2g}, \qquad R = \frac{u^2 \sin 2\alpha}{g}$$



Then using given values we get

$$900 = \frac{u^2 \sin^2 \alpha}{2g}, \qquad 400 = \frac{2u^2 \sin \alpha \cos \alpha}{g}$$

Then by dividing these two equations

$$\frac{9}{4} = \frac{u^2 \sin^2 \alpha}{2g} \Big/ \frac{2u^2 \sin \alpha \cos \alpha}{g} \Rightarrow \frac{9}{4} = \frac{\tan \alpha}{4} \quad \therefore \alpha = \tan^{-1} 9$$

which gives the angle of projection and the magnitude of the velocity of projection by using first equation

$$900 = \frac{u^2}{2g} \times \frac{81}{82} \Rightarrow u^2 = \frac{1800 \times 82 \times 32.2}{81} \quad \text{Or} \quad u \cong 242.23 \quad (g = 32.2 \text{ ft sec}^{-2})$$

Example

If the ratio between the magnitude of the velocity at maximum height and a height equals half of maximum height is $\sqrt{\frac{6}{7}}$. Show that the angle of projection is 30^0.

|| Example ▸

As it is obtained that $y=(u\sin\alpha)\,t-\frac{1}{2}gt^2$

Let the point A be the maximum height and hence $Y_A=\frac{u^2\sin^2\alpha}{2g}$

And **B** be the point where its height equals half of maximum height i.e.,

$$Y_B=\frac{1}{2}Y_A=\frac{u^2\sin^2\alpha}{4g}$$

The time spent from the projection of the particle reach point **B** is given by

$$\frac{u^2\sin^2\alpha}{4g}=(u\sin\alpha)\,t-\frac{1}{2}gt^2$$

Rewrite this equation again as (multiply by $4g$)

$$2(gt)^2-4(gt)u\sin\alpha+u^2\sin^2\alpha=0 \qquad \Rightarrow gt=\left(1-\frac{1}{\sqrt{2}}\right)u\sin\alpha$$

The components of velocity at point B are

$$\dot{x}_B=u\cos\alpha, \quad \dot{y}_B=u\sin\alpha-gt=u\sin\alpha-\left(1-\frac{1}{\sqrt{2}}\right)u\sin\alpha=\frac{1}{\sqrt{2}}u\sin\alpha$$

The resultant of the velocity at point B

$$v_B=\sqrt{\dot{x}_B^2+\dot{y}_B^2}=\sqrt{(u\cos\alpha)^2+\left(\frac{1}{\sqrt{2}}u\sin\alpha\right)^2}=\frac{u}{\sqrt{2}}\sqrt{1+\cos^2\alpha}$$

Since at the maximum height $\dot{x}_A=u\cos\alpha, \quad \dot{y}_A=0$ then

$$v_A=\sqrt{\dot{x}_A^2+\dot{y}_A^2}=u\cos\alpha$$

But as given $\frac{v_A}{v_B}=\sqrt{\frac{6}{7}}$ therefore,

$$\Rightarrow \frac{\sqrt{2}u\cos\alpha}{u\sqrt{1+\cos^2\alpha}}=\sqrt{\frac{6}{7}}$$
$$\Rightarrow \frac{\cos\alpha}{\sqrt{1+\cos^2\alpha}}=\sqrt{\frac{3}{7}}$$
$$\Rightarrow \frac{\cos^2\alpha}{1+\cos^2\alpha}=\frac{3}{7}$$

$7\cos^2\alpha = 3 + 3\cos^2\alpha \qquad \Rightarrow 4\cos^2\alpha = 3$

$$\Rightarrow \cos\alpha = \frac{\sqrt{3}}{2} \quad \text{Or} \quad \alpha = 30^0$$

Example

A particle is projected with a velocity of 24 ft $\sec^{-1}$ at an angle of elevation 60. Find (a) the equation to its path, (b) the greatest height attained, (c) the time for the range, (d) the length of the range.,

Solution

Since $u = 24$ and $\alpha = 60^o$, $g \simeq 32.2$ fts ec^{-2}

(a) the equation to the path is

$$y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{u^2\cos^2\alpha}, \qquad \text{therefore} \qquad y = \sqrt{3}\,x - \frac{1}{9}x^2$$

(b) The maximum height $= \dfrac{u^2\sin^2\alpha}{2g} = \dfrac{24\times 24}{2\times 32.2}\times\dfrac{3}{4} \simeq 6.71$ ft

(c) The time for the range $= \dfrac{2u\sin\alpha}{g} = \dfrac{2\times 24}{32.2}\times\dfrac{\sqrt{3}}{2} \simeq 1.29$ sec

(d) the length of the ran $= u\cos\alpha\,T = 24\times\dfrac{1}{2}\times\dfrac{3\sqrt{3}}{4} \simeq 15.49$ ft

Example

Find the maximum horizontal range of cricket ball projected with a velocity of 48 ft. per sec. If the ball is to have a range of $36\sqrt{3}$ ft., find the least angle of projection and the least time taken (let $g \simeq 32$ ft $\sec^{-2}$).

Solution

We have $u = 48$ and $\alpha = 45^o$, $g \simeq 32$ ft $\sec^{-2}$

$$R_{\max} = \frac{u^2}{g} \qquad \Rightarrow R_{\max} = \frac{48\times 48}{32} \simeq 72 \text{ ft}$$

If $R = \dfrac{u^2\sin 2\alpha}{g} = 36\sqrt{3} \qquad \Rightarrow \sin 2\alpha = \dfrac{36\sqrt{3}\times 32}{48\times 48} = \dfrac{\sqrt{3}}{2}$

Then $2\alpha = 60^o$ or 120^o that is $\alpha = 30^o$ or 60^o

Thus, the least angle of projection $\alpha = 30^o$

and the least time taken $= \dfrac{2u\sin\alpha}{g} = \dfrac{2\times 48}{32}\times\dfrac{1}{2} \simeq 1.5 \text{ sec}$

Example

A ball is projected from a point on the ground distant a from the foot of a vertical wall of height b, the angle of projection being α to the horizontal. If the ball just clears the wall prove that the greatest height reached is

$$\frac{a^2\tan^2\alpha}{4(a\tan\alpha - b)}$$

Solution

Let u be the velocity of projection, then since the ball passes through the top of the wall, a point (a,b), we have

$$b = a\tan\alpha - \frac{ga^2}{2u^2\cos^2\alpha} \qquad \text{Or} \qquad a\tan\alpha - b = \frac{ga^2}{2u^2\cos^2\alpha}$$

$$\therefore\ u^2 = \frac{ga^2}{2(a\tan\alpha - b)\cos^2\alpha}$$

Now the greatest height Y reached by the ball

$$\begin{aligned} Y &= \frac{u^2\sin^2\alpha}{2g} \\ &= \frac{\sin^2\alpha}{2g}\,\frac{ga^2}{2(a\tan\alpha - b)\cos^2\alpha} \\ &= \frac{a^2\tan^2\alpha}{4(a\tan\alpha - b)} \end{aligned}$$

Example

If T be the time taken to reach the other common point A of its path and T' the time to reach the horizontal plane through the point of projection. Find the height of the point A.

‖ Solution ▸

Since $x = u\cos\alpha\ t$ and the time of flight is $T + T'$ also $R = \dfrac{u^2 \sin 2\alpha}{g}$

Hence $u\cos\alpha\ (T + T') = \dfrac{2u^2 \cos\alpha \sin\alpha}{g} \quad \Rightarrow u\sin\alpha = \dfrac{1}{2}g(T + T')$

$$\because\ y\big|_A = u\sin\alpha\ T - \frac{1}{2}gT^2 \quad \Rightarrow \quad y\big|_A = \frac{1}{2}gT(T + T') - \frac{1}{2}gT^2 = \frac{1}{2}gTT'$$

‖ Example ▸

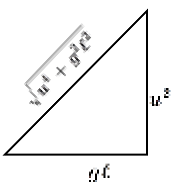
A particle is projected with a velocity u so as just to pass over the highest possible post at a horizontal distance ℓ from the point of projection $\mathbf{O}$. Prove that the greatest height above O attained by the particle in its flight is $\dfrac{u^6}{2g(u^4 + g^2\ell^2)}$.

‖ Solution ▸

Taking θ as the angle of projection and substituting ℓ for x the equation to the path, we have

$$y = \ell\tan\theta - \frac{g\ell^2}{2u^2\cos^2\theta} = \ell\tan\theta - \frac{g\ell^2}{2u^2}(1 + \tan^2\theta)$$

$$\therefore\ \frac{dy}{d\theta} = \ell\sec^2\theta - \frac{g\ell^2}{u^2}\tan\theta\sec^2\theta = \ell\sec^2\theta\left(1 - \frac{gd^2}{u^2}\tan\theta\right)$$



$$\frac{dy}{d\theta} = 0 \Rightarrow \tan\theta = \frac{u^2}{g\ell} \qquad \text{or} \qquad \sin^2\theta = \frac{u^4}{u^4 + g^2\ell^2}$$

y being positive and its minimum value being zero, the value of θ given in previous equation gives the maximum value of y. Now the greatest height attained by the particle

$$Y = \frac{u^2\sin^2\theta}{2g} = \frac{u^2}{2g}\left(\frac{u^4}{u^4 + g^2\ell^2}\right) = \frac{u^6}{2g\ \ u^4 + g^2\ell^2}$$

Example

Two particles are projected from the same point in the same vertical plane with equal velocities. If t, t' be the times taken to reach the common point of their paths and T, T' the times to the highest point, show that $tT + t'T'$ is independent of the directions of projection

Solution

Let α, β be the directions of projection

$$T = \frac{u\sin\alpha}{g}, \quad T' = \frac{u\sin\beta}{g}$$

If x is the horizontal distance of the common point, then

$$x = u\cos\alpha\ t, \ \ x = u\cos\beta\ t'$$

$$\therefore\ tT + t'T' = \frac{x}{u\cos\alpha}\frac{u\sin\alpha}{g} + \frac{x}{u\cos\beta}\frac{u\sin\beta}{g} = \frac{x}{g}(\tan\alpha + \tan\beta) \qquad (*)$$

Now the equations of the two- paths arc

$$y = x\tan\alpha - \frac{1}{2}\frac{gx^2\sec^2\alpha}{u^2}, \quad y = x\tan\beta - \frac{1}{2}\frac{gx^2\sec^2\beta}{u^2}$$

Subtracting we have,

$$x(\tan\alpha - \tan\beta) = \frac{1}{2}\frac{gx^2}{u^2}\ \sec^2\alpha - \sec^2\beta \ = \frac{1}{2}\frac{gx^2}{u^2}\ \tan^2\alpha - \tan^2\beta$$

$$\frac{x}{g}(\tan\alpha + \tan\beta) = \frac{2u^2}{g^2}$$

Hence from Equation (*)

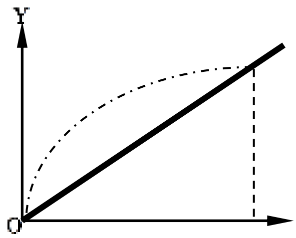
$$\therefore\ tT + t'T' = \frac{2u^2}{g^2}$$ which is independent of the directions of projection.

Example

A particle is projected with velocity u from a point on an inclined plate. If v_1 be its velocity on striking the plane when the range up the plane is maximum and v_2 the velocity on striking the plane when the range down the plane is maximum, prove that $u^2 = v_1 v_2$

Solution

Let R be the maximum range up the plane and α be the inclination of the plane, then



$$R = \frac{u^2}{g(1+\sin\alpha)}, \text{ and } v_1^2 = u^2 - 2gy = u^2 - 2gR\sin\alpha$$

$$\therefore v_1^2 = u^2 - 2g\sin\alpha \times \frac{u^2}{g(1+\sin\alpha)} = u^2 \times \frac{1-\sin\alpha}{1+\sin\alpha}$$

Similarly, by changing the sign of α, we have

$$\therefore v_2^2 = u^2 \times \frac{1+\sin\alpha}{1-\sin\alpha} \quad \text{Hence} \quad u^4 = v_1^2 v_2^2 \quad \text{Or} \quad u^2 = v_1 v_2$$

Example

A particle is projected and it paths through the two points $(12,12)$ and $(36,12)$ Find its velocity and the direction of projection.

Solution

The trajectory or path equation is $\qquad y = x\tan\alpha - \dfrac{gx^2}{2u^2\cos^2\alpha}$

The two points $(12,12)$ and $(36,12)$ lies on the path so that

With regard the point $(36,12)$ $\qquad 12 = 36\tan\alpha - \dfrac{g(36)^2}{2u^2\cos^2\alpha}$

With regard the point $(12,12)$ $\qquad 12 = 12\tan\alpha - \dfrac{g(12)^2}{2u^2\cos^2\alpha}$

By multiplying the second equation by 9 then subtracting, we have

$$96 = 72\tan\alpha \qquad \Rightarrow \tan\alpha = \frac{96}{72} = \frac{4}{3}$$

which gives the direction of velocity of projection, and to obtain the magnitude of the projection velocity, from first equation

$$\Rightarrow 12 = 36\left(\frac{4}{3}\right) - \frac{g(36)^2}{2u^2\left(\frac{3}{5}\right)^2} \qquad \Rightarrow \frac{g(36)^2}{2u^2\left(\frac{3}{5}\right)^2} = 36$$

$$\Rightarrow u^2 = 50\,g \qquad \text{Or} \qquad u = 5\sqrt{2g}$$

Example

A particle is projected and it paths through the two points (a, b) and (b, a) where (a, b) and (b, a) Prove that the range is given by $\dfrac{a^2 + ab + b^2}{a + b}$.

Solution

The trajectory or path equation is $\qquad y = x\tan\alpha - \dfrac{gx^2}{2u^2\cos^2\alpha}$

The two points (a, b) and (b, a) lies on the path so that

With regard the point (a, b) $\qquad a = b\tan\alpha - \dfrac{gb^2}{2u^2\cos^2\alpha}$

With regard the point (b, a) $\qquad b = a\tan\alpha - \dfrac{ga^2}{2u^2\cos^2\alpha}$

By multiplying the first equation by a and the second by b then subtracting, we have

$$\underbrace{a^2 - b^2}_{(a+b)\cancel{(a-b)}} = \frac{gab}{2u^2\cos^2\alpha}\cancel{(a-b)} \qquad \Rightarrow \quad a + b = \frac{gab}{2u^2\cos^2\alpha}$$

$$\text{Or} \qquad \frac{ab}{a+b} = \frac{2u^2\cos^2\alpha}{g}$$

Once again by multiplying the first equation by a^2 and the second by b^2 then subtracting, we have

$$\underbrace{a^3 - b^3}_{(a^2+ab+b^2)(a-b)} = ab(a-b)\tan\alpha \qquad \Rightarrow ab\tan\alpha = a^2 + ab + b^2$$

Since the range is given by $R = \dfrac{u^2 \sin 2\alpha}{g}$ therefore,

$$R = \frac{u^2 \sin 2\alpha}{g} = \frac{2u^2 \cos\alpha \sin\alpha}{g}$$

$$= \underbrace{\frac{2u^2 \cos^2\alpha}{g}}_{ab/(a+b)} \tan\alpha = \frac{ab}{a+b}\tan\alpha = \frac{a^2 + ab + b^2}{a+b}$$

$$\therefore R = \frac{a^2 + ab + b^2}{a+b}$$

Example

A particle is projected to reach a certain object located in the same horizontal plate of projection point, when it projected with angle α it falls down before the object by distance ℓ and when it projected with angle β it falls down after the object by distance ℓ. Find the exact angle to reach the object.

Solution

Let u be the velocity of projection and R is the exact range of the object then the range in first case is $R - \ell$ and the range in second case is $R + \ell$ therefore

$$R - \ell = \frac{u^2 \sin 2\alpha}{g} \qquad \text{and} \qquad R + \ell = \frac{u^2 \sin 2\beta}{g}$$

By addition the two equations, we get

$$\therefore 2R = \frac{u^2}{g}\ \sin 2\alpha + \sin 2\beta \qquad \Rightarrow R = \frac{u^2}{2g}\ \sin 2\alpha + \sin 2\beta$$

Now, let θ be the exact angle to reach the object so $R = \dfrac{u^2 \sin 2\theta}{g}$

By comparing (or dividing) the last two equations then

$$\Rightarrow \frac{u^2 \sin 2\theta}{g} = \frac{u^2}{2g}\ \sin 2\alpha + \sin 2\beta$$

$$\Rightarrow \sin 2\theta = \frac{\sin 2\alpha + \sin 2\beta}{2} \qquad \Rightarrow \theta = \frac{1}{2}\sin^{-1}\left(\frac{\sin 2\alpha + \sin 2\beta}{2}\right)$$

■ Projectiles with Resistance

We now proceed to include the effect of air resistance. From our earlier discussion of fluid drag, it is evident that in most practical instances of projectile motion through the Earth's atmosphere, it is the **quadratic law** of resistance that is appropriate. On the other hand, only the **linear law** of resistance gives rise to linear equations of motion and simple analytical solutions. This explains why mechanics textbooks contain extensive coverage of the linear case, even though this case is almost never appropriate in practice; the case that is appropriate cannot be solved! In the following example, we treat the linear resistance case.

Now suppose that the motion is opposed by a force proportional to the velocity. Thus if m denote the mass and v the velocity, let $m\gamma v$ denote the magnitude of the resistance. Therefore the components of the resistance parallel to horizontal and vertical axes **OX** , **OY** are

$$-m\gamma\dot{x}, \ -m\gamma\dot{y}$$

Let u denote the initial velocity in a direction making an angle α with the horizontal. The equations of motion give

$$\ddot{x} = -\gamma\dot{x} \qquad \text{and} \qquad \ddot{y} = -g - \gamma\dot{y}$$

By integrating we obtain

$$\ln\dot{x} = c_1 - \gamma t \qquad \text{and} \qquad \ln\left(\dot{y} + \frac{g}{\gamma}\right) = c_2 - \gamma t$$

since initially $x = y = 0$ and $\dot{x} = u\cos\alpha, \quad \dot{y} = u\sin\alpha$,then $c_1 = \ln u\cos\alpha$

and $c_2 = \ln\left(u\sin\alpha + \frac{g}{\gamma}\right)$, and hence

$$\dot{x} = u\cos\alpha \ e^{-\gamma t} \qquad \text{and} \qquad \dot{y} = \left(u\sin\alpha + \frac{g}{\gamma}\right)e^{-\gamma t} - \frac{g}{\gamma}$$

Once again integrate the previous formula

$$x = -\frac{u\cos\alpha}{\gamma}\, e^{-\gamma t} + c_3 \quad \text{and} \quad y = -\frac{1}{\gamma}\left(u\sin\alpha + \frac{g}{\gamma}\right)e^{-\gamma t} - \frac{g}{\gamma}t + c_4$$

Where, c_4, c_3 are constant, and $x = y = 0$ at $t = 0$ so that

$$c_3 = \frac{u\cos\alpha}{\gamma}, \qquad c_4 = \frac{1}{\gamma}\left(u\sin\alpha + \frac{g}{\gamma}\right)$$

So the last equation becomes

$$x = \frac{u\cos\alpha}{\gamma}\ 1 - e^{-\gamma t} \quad \text{and} \quad y = \frac{1}{\gamma}\left(u\sin\alpha + \frac{g}{\gamma}\right)\ 1 - e^{-\gamma t}\ - \frac{g}{\gamma}t$$

- The time spent to reach the maximum height is

$$T = \frac{1}{\gamma}\ln\left(\frac{\gamma u\sin\alpha}{g} + 1\right)$$

- The maximum height is

$$y = \frac{u\sin\alpha}{\gamma} - \frac{g}{\gamma^2}\ln\left(1 + \frac{\gamma u\sin\alpha}{g}\right)$$

- The time of flight is

$$T' = \frac{1}{\gamma}\left(\frac{\gamma u\sin\alpha}{g} + 1\right)\ 1 - e^{-\gamma T'}$$

- The path equation is

$$y = \frac{g}{\gamma u\cos\alpha}\left(\frac{\gamma u\sin\alpha}{g} + 1\right)x + \frac{g}{\gamma^2}\ln\left(1 - \frac{\gamma x}{u\cos\alpha}\right)$$

For instance to evaluate the spent time to reach the maximum height

Since $$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - .....$$

this is true for $|x| < 1$, and now let $\gamma \to 0$ in formula

$$T = \frac{1}{\gamma}\ln\left(\frac{\gamma u\sin\alpha}{g} + 1\right)$$

We get

$$T = \lim_{\gamma\to 0}\frac{1}{\gamma}\left(\frac{\gamma u\sin\alpha}{g} - \frac{\gamma^2 u^2 \sin^2\alpha}{2g^2} + \frac{\gamma^3 u^3 \sin^3\alpha}{3g} + ...\right)$$
$$= \lim_{\gamma\to 0}\left(\frac{u\sin\alpha}{g} - \frac{\gamma u^2 \sin^2\alpha}{2g^2} + \frac{\gamma u^3 \sin^3\alpha}{3g} + ...\right) = \frac{u\sin\alpha}{g}$$

This result obtained before when we neglected the resistance of air.

Example

A particle of mass m is projected with initial velocity u at an angle of elevation α through a resisting medium where its resistance proportional to v and the proportional constant is μm. Prove that the direction of the velocity makes an angle α with the horizontal $\frac{1}{\mu}\ln\left(1+\frac{\mu u}{g}(\sin\alpha+\cos\alpha)\right)$

Solution

By writing the equation of motion in **OX, OY** and then integrating and use the initial conditions as illustrated before we obtain the components of velocity of the particle at any instance

$$\dot{x} = u\cos\alpha \ e^{-\mu t} \qquad \text{and} \qquad \dot{y} = \left(u\sin\alpha + \frac{g}{\mu}\right)e^{-\mu t} - \frac{g}{\mu}$$

Since the angle of projection is α and the angle that the direction of velocity makes with the horizontal axis decreases until vanish at the highest position then it reverse to be α again downwards after time t which determines from

$$\tan \ -\alpha \ = \frac{\dot{y}}{\dot{x}} = \frac{\left(u\sin\alpha + \frac{g}{\mu}\right)e^{-\mu t} - \frac{g}{\mu}}{u\cos\alpha \ e^{-\mu t}} = -\tan \ \alpha$$

That is

$$\left(u\sin\alpha + \frac{g}{\mu}\right)e^{-\mu t} - \frac{g}{\mu} = -u\sin\alpha \ e^{-\mu t} \ \Rightarrow \left(2u\sin\alpha + \frac{g}{\mu}\right)e^{-\mu t} = \frac{g}{\mu}$$
$$\left(\frac{2\mu u\sin\alpha}{g} + 1\right) = e^{\mu t} \qquad \Rightarrow t = \frac{1}{\mu}\ln\left(\frac{2\mu u\sin\alpha}{g} + 1\right)$$

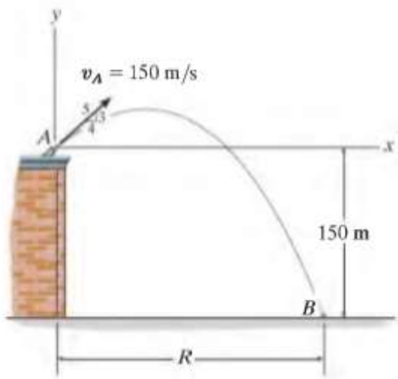
PROBLEMS

❒ A body, projected with a velocity of 120 ft sec^{-1} just clears a vertical wall 72 ft high and 360 ft. distant, find the two possible angles of projection and the corresponding horizontal ranges.

❒ A particle is projected so as just to clear a wall of height b at a horizontal distance a, and to have a range c from the point of projection, show that the velocity of projection V is given by

$$\frac{2V^2}{g} = \frac{a^2(c-a)^2 + b^2c^2}{ab(c-a)}.$$

❒ A projectile is fired with an initial velocity of V_A = 150 m/s off the roof of the building. Determine the range R where it strikes the ground at B.



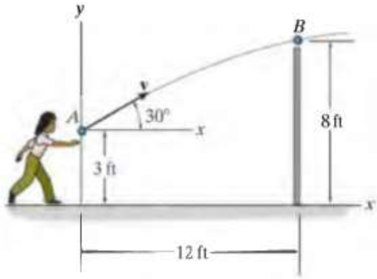
❒ A stone is projected with velocity V and elevation from a point O on level ground so as to hit a mark P on a wall whose distance from O is a, the height of P above the ground being 6. Prove that

$$2V^2(a\sin\theta\cos\theta - b\cos^2\theta) = ga^2 .$$

❒ A particle in projected with a velocity of 120 ft. per sec. at an angle of 60 with the horizontal from the foot of an inclined plane of inclination 30. Find the time of flight and the range on the inclined plane.

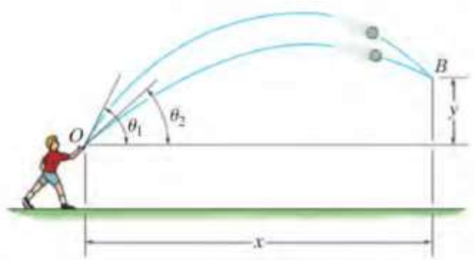
❒ A particle is projected from a point on a plane of inclination β with velocity u. Show that the maximum range down the plane is

$$\frac{u^2}{g}\left(\frac{1+\sin\beta}{\cos^2\beta}\right).$$



❒ A ball is thrown from A. If it is required to clear the wall at B, determine the minimum magnitude of its initial velocity V_A.

❒ A boy throws a ball at 0 in the air with a speed Vo at an angle θ_1. If he then throws another ball with the same speed v_0 at an angle $\theta_2 < \theta_1$ determine the time between the throws so that the balls collide in midair at B.



KINETICS OF A PARTICLE

This chapter is concerned with the foundations of dynamics and gravitation. Kinematics is concerned purely with geometry of motion, but dynamics seeks to answer the question as to what motion will actually occur when specified forces act on a body. The rules that allow one to make this connection are Newton's laws of motion. These are laws of physics that are founded upon experimental evidence and stand or fall according to the accuracy of their predictions. In fact, Newton's formulation of mechanics has been astonishingly successful in its accuracy and breadth of application, and has survived, essentially intact, for more than three centuries. The same is true for Newton's universal law of gravitation which specifies the forces that all masses exert upon each other.

Taken together, these laws represent virtually the entire foundation of classical mechanics and provide an accurate explanation for a vast range of motions from large molecules to entire galaxies.

■ Newton's Laws

Isaac Newton's∗ three famous laws of motion were laid down in Principia, written in Latin and published in 1687. These laws set out the founding principles of mechanics and have survived, essentially unchanged, to the present day. Even when translated into English, Newton's original words are hard to understand, mainly because the terminology of the seventeenth century is now archaic. Also, the laws are now formulated as applying to particles, a concept never used by Newton. A particle is an idealized body that occupies only a single point of space and has no internal structure. True particles do not exist in nature, but it is convenient to regard realistic bodies as being made up of particles. Using modern terminology, Newton's laws may be stated as follows:

➢**First Law:** When all external influences on a particle are removed, the particle moves with constant velocity. {This velocity may be zero in which case the particle remains at rest.}

➢**Second Law:** When a force $\boldsymbol{F}$ acts on a particle of mass m, the particle moves with instantaneous acceleration $\underline{a}$ given by the formula

$$\boldsymbol{F} = m\underline{a}$$

where the unit of force is implied by the units of mass and acceleration.

➢**Third Law:** When two particles exert forces upon each other, these forces are (i) equal in magnitude, (ii) opposite in direction, and (iii) parallel to the straight line joining the two particles.

■ The Law of Gravitation

Physicists recognize only four distinct kinds of interaction forces that exist in nature. These are gravitational forces, electromagnetic forces and weak/strong nuclear forces. The nuclear forces are important only within the atomic nucleus and will not concern us at all. The electromagnetic forces include electrostatic attraction and repulsion, but we will encounter them mainly as 'forces of contact' between material bodies. Since such forces are intermolecular, they are ultimately electromagnetic although we will make no use of this fact! The present section however is concerned with gravitation.

It is an observed fact that any object with mass attracts any other object with mass with a force called gravitation. When gravitational interaction occurs between particles, the Third Law implies that the interaction forces must be equal in magnitude, opposite in direction and parallel to the straight line joining the particles.

The gravitational forces that two particles exert upon each other each have magnitude

$$\underline{F} = \gamma \frac{Mm}{R^2} \hat{F} \qquad \text{....(1)}$$

where M, m are the particle masses, R is the distance between the particles, and γ, the constant of gravitation, is a universal constant. Since γ is not dimensionless, its numerical value depends on the units of mass, length and force.

This is the famous <u>inverse square law of gravitation</u> originally suggested by Robert Hooke, a scientific contemporary (and adversary) of Newton. In SI units, the constant of gravitation is given approximately by

$$\gamma = 6.67 \times 10^{-11} \ \text{Nm}^2\text{kg}^{-2}$$

this value being determined by observation and experiment. There is presently no theory (general relativity included) that is able to predict the value of γ. Indeed, the theory of general relativity does not exclude *repulsion* between masses!

To give some idea of the magnitudes of the forces involved, suppose we have two uniform spheres of lead, each with mass 5000 kg (five metric tons). Their common radius is about 47 cm which means that they can be placed with their centers 1 m apart. What gravitational force do they exert upon each other when they are in this position? We will show later that the gravitational force between uniform spheres of matter is exactly the same *as if* all the mass of each sphere were concentrated at its center. Given that this result is true, we can find the force that each sphere exerts on the other simply by substituting $M = m = 5000$ and $R = 1$ into equation (1). This gives $F = 0.00167\,\text{N}$ approximately, the weight of a few grains of salt! Such forces seem insignificant, but gravitation is the force that keeps the Moon in orbit around the Earth, and the Earth in orbit around the Sun. The reason for this disparity is that the masses involved are so much larger than those of the lead spheres in our example. For instance, the mass of the Sun is about 2×10^{30} kg.

■ Motion through a Resisting Medium

When a body moves in a medium like air or any other fluid, it experiences a resistance to its motion. The resistance which we have been neglecting so far, generally varies with the velocity. For small velocities the resistance is approximately proportional to the velocity, for greater velocities it varies as the square of the velocity and for still greater velocities, the resistance varies as the cube or even a higher power of the velocity. The forces of resistance being non-conservative, the principle of Conservation of Energy is not applicable to such cases.

■ Bodies Falling Vertically in a Resisting Medium

Suppose a particle with mass m is allowed to fall vertically subject to a resistance proportional to some power of the velocity v, e.g. a resistance force $\mu m v$, then we have the equation of motion

$$m\frac{dv}{dt} = mg - \mu m v \qquad \textbf{Or} \qquad \frac{dv}{dt} = g - \mu v$$

where μm is the constant of proportionality and g, the acceleration due to gravity, is supposed to remain constant. The equation shows that the acceleration of the particle decreases as its velocity increases and that it vanishes when g / μ . Separation of variables for the previous equation we get

$$\frac{dv}{g - \mu v} = dt \qquad \Rightarrow \frac{-\mu dv}{g - \mu v} = -\mu dt$$

Integration we have

$$\ln(g - \mu v) = -\mu t + c_1$$

If the initial velocity is u therefore, the constant c_1 may be obtained as $\ln(g-\mu u)=c_1$ then

$$\ln(g-\mu v)=-\mu t+\ln(g-\mu u) \qquad \Rightarrow \ln\frac{g-\mu v}{g-\mu u}=-\mu t$$

$$\Rightarrow g-\mu v=(g-\mu u)e^{-\mu t} \qquad Or \quad v=\frac{g}{\mu}-\frac{1}{\mu}(g-\mu u)e^{-\mu t}$$

The value $\frac{g}{\mu}$ is the greatest velocity attainable by the particle and is called the limiting or terminal velocity.

To get the height since $v=\frac{dy}{dt}$ then

$$\frac{dy}{dt}=\frac{g}{\mu}-\frac{1}{\mu}(g-\mu u)e^{-\mu t} \qquad \Rightarrow dy=\left(\frac{g}{\mu}-\frac{1}{\mu}(g-\mu u)e^{-\mu t}\right)dt$$

And integrate we get

$$y=\frac{g}{\mu}t+\frac{1}{\mu^2}(g-\mu u)e^{-\mu t}+c_2$$

Where $c_2=-\frac{1}{\mu^2}(g-\mu u)$ sine $y=0$ when $t=0$ that is

$$y=\frac{g}{\mu}t+\frac{1}{\mu^2}(g-\mu u)\ \ e^{-\mu t}\ -1$$

Subsequently the particle moves uniformly with this limiting velocity. The velocity for the rain drops at the surface of the earth cannot, therefore, give us any idea of the height from which they might have fallen, for after moving for some time they acquire the terminal velocity and continue to move uniformly with that velocity.

■ Illustrative Examples ■

Example

A particle with mass m moves horizontally through a resisting medium where its resistance proportional to v and the proportional constant is αm. If the particle starts its motion from the origin point with initial velocity u. Find the distance after time t.

Solution

The equation of motion of the particle is (horizontally)

$$m\frac{dv}{dt} = -\alpha m v \quad \Rightarrow \frac{dv}{v} = -\alpha\, dt$$

By integrating we have $\ln(v) = c_1 - \alpha t$ **(1)**

The constant c_1 can be determined from the initial conditions, $v = u$ at $t = 0$, therefore $c_1 = \ln u$ and equation (1) becomes

$$\ln(v) = \ln u - \alpha t \qquad \textbf{Or} \qquad v = ue^{-\alpha t} \qquad \textbf{(2)}$$

Equation (2) gives the velocity of the particle at any instance, and the position of the particle x can be obtained as follows

$$\frac{dx}{dt} = ue^{-\alpha t} \quad \Rightarrow \quad dx = ue^{-\alpha t}dt$$

v

$$\Rightarrow \int dx = \int ue^{-\alpha t}dt + c_2 \qquad \textbf{Or}$$

$$x = -\frac{u}{\alpha}e^{-\alpha t} + c_2 \qquad \textbf{(3)}$$

Where c_2 is integration constant that can be calculated from the initial conditions, $x = 0$ at $t = 0$, therefore $c_2 = \frac{u}{\alpha}$ and equation (3) turns into

$$x = \frac{u}{\alpha}\ 1 - e^{-\alpha t}$$

Example

A moving point with mass equals unity subject to a resistance $\lambda v+\mu v^2$ If the resisting force is the only force acting on the point. Find the distance where u is the initial velocity of the point.

Solution

Equation of motion is ($m=1$) – Note resisting force is the only acting force-

$$v\frac{dv}{dx}=-(\lambda v+\mu v^2) \qquad \Rightarrow \frac{\mu dv}{\lambda+\mu v}=-\mu dx$$

By integration we get

$$\ln(\lambda+\mu v)=c_1-\mu x \tag{1}$$

Where c_1 represents integration constant and can be obtained from the initial conditions, $v=u$ at $x=0$, therefore $c_1=\ln(\lambda+\mu u)$ and equation (1) turns into

$$\ln(\lambda+\mu v)=\ln(\lambda+\mu u)-\mu x \qquad \text{Or} \qquad \ln\left(\frac{\lambda+\mu u}{\lambda+\mu v}\right)=\mu x \tag{2}$$

Again from the last equation we can obtain the position of the point as the velocity vanishes

$$x\Big|_{v=0}=\frac{1}{\mu}\ln\left(\frac{\lambda+\mu u}{\lambda}\right)=\frac{1}{\mu}\ln\left(1+\frac{\mu}{\lambda}u\right)$$

Example

Two equal particles with mass m projected downwards from the same point and at the same instance with initial velocities u_1, u_2 subject to a resistance

μmv If u_1', u_2' are the velocities of the particles after time T. Prove that $u_1' - u_2' = (u_1 - u_2)e^{-\mu T}$.

Solution

With respect the first particle we suppose that its velocity at any time is v therefore, the equation of motion is

$$m\frac{dv}{dt} = mg - \mu mv \quad \Rightarrow \frac{dv}{g-\mu v} = dt \quad \text{Or} \quad \frac{-\mu dv}{g-\mu v} = -\mu dt$$

By integration we have

$$\ln(g - \mu v) = c - \mu t$$

look c indicates the integration constant which can be calculated from the initial conditions, $v = u_1$ when $t = 0$, therefore $c = \ln(g - \mu u_1)$ and the last equation become

$$\ln(g - \mu v) = \ln(g - \mu u_1) - \mu t \qquad \text{Or} \qquad g - \mu v = (g - \mu u_1)e^{-\mu t}$$

Now after time T, the velocity become u_1' that is

$$g - \mu u_1' = (g - \mu u_1)e^{-\mu T} \tag{1}$$

Now with respect the second particle we suppose that its velocity at any time is v' therefore, the equation of motion is

$$m\frac{dv'}{dt} = mg - \mu mv' \quad \Rightarrow \frac{dv'}{g-\mu v} = dt \quad \text{Or} \quad \frac{-\mu dv'}{g-\mu v'} = -\mu dt$$

By integration we have

$$\ln(g - \mu v') = c' - \mu t$$

where c refers to the integration constant which can be obtained from the initial conditions, $v' = u_2$ when $t = 0$, therefore $c' = \ln(g - \mu u_2)$ and the previous equation converted to

$$\ln(g - \mu v') = \ln(g - \mu u_2) - \mu t \qquad \text{Or} \qquad g - \mu v' = (g - \mu u_2)e^{-\mu t}$$

Again, Now after time T, the velocity become u_2' that is

$$g - \mu u_2' = (g - \mu u_2)e^{-\mu T} \qquad \textbf{(2)}$$

By subtracting Equations (1) and (2) we obtain

$$\mu \quad u_1' - u_2' \quad = \mu(u_1 - u_2)e^{-\mu T} \qquad \textbf{Or} \qquad u_1' - u_2' = (u_1 - u_2)e^{-\mu T}$$

Example

A point with mass m is projected vertically upwards with initial velocity $\sqrt{g\mu^{-1}}$ and the resistance of air produces retardation per unit mass μv^2 where v is the velocity and μ is constant. Find the highest position and the time spent to reach is $\dfrac{\pi}{4\sqrt{g\mu}}$.

Solution

The equation of motion – let the projection point be the origin-then

$$mv\frac{dv}{dy} = -mg - \mu mv^2 \quad \Rightarrow \frac{vdv}{g + \mu v^2} = -dy \quad \textbf{Or} \quad \frac{2\mu vdv}{g + \mu v^2} = -2\mu dy$$

By integration we get

$$\ln(g + \mu v^2) = c_1 - 2\mu y \qquad \textbf{(1)}$$

Note c_1 indicates the integration constant which can be obtained from the initial conditions, $v = \sqrt{g\mu^{-1}}$ when $y = 0$, therefore $c_1 = \ln 2g$ and equation (1) be

$$\ln(g + \mu v) = \ln 2g - 2\mu y \qquad \textbf{Or} \quad y = \frac{1}{2\mu}\ln\frac{2g}{g + \mu v} \qquad \textbf{(2)}$$

Equation (2) gives the position of the point at any instance t and at highest position the velocity is zero $v = 0$ and then

$$y = \frac{1}{2\mu}\ln\frac{2g}{g} \qquad \Rightarrow Y = \frac{1}{2\mu}\ln 2$$

And this is the highest position and to evaluate the spent time to reach since

$$m\frac{dv}{dt} = -mg - \mu m v^2 \quad \Rightarrow \frac{dv}{g + \mu v^2} = -dt \quad \text{Or} \quad \frac{\sqrt{\frac{\mu}{g}}dv}{1 + \left(\sqrt{\frac{\mu}{g}}\,v\right)^2} = -\sqrt{g\mu}\,dt$$

By integration we obtain

$$\tan^{-1}\left(\sqrt{\frac{\mu}{g}}\,v\right) = c_2 - \sqrt{g\mu}\,t \qquad (3)$$

Note c_2 is the integration constant which its value can be evaluated by the initial conditions, $v = \sqrt{g\mu^{-1}}$ when $t = 0$, therefore $c_2 = \frac{\pi}{4}$ and equation (3) turn into

$$\tan^{-1}\left(\sqrt{\frac{\mu}{g}}\,v\right) = \frac{\pi}{4} - \sqrt{g\mu}\,t \qquad \text{Or} \qquad v = \sqrt{\frac{g}{\mu}}\tan\left(\frac{\pi}{4} - \sqrt{g\mu}\,t\right)$$

This equation gives the velocity at any time t, and when $v = 0$ then t

$$\Rightarrow 0 = \sqrt{\frac{g}{\mu}}\tan\left(\frac{\pi}{4} - \sqrt{g\mu}\,t\right)$$

$$\Rightarrow \quad \left(\frac{\pi}{4} - \sqrt{g\mu}\,t\right) = 0 \qquad \text{Or} \quad t = \frac{\pi}{4\sqrt{g\mu}}$$

Example

A point with mass m is projected vertically upwards where the resistance of air produces a retardation $m\mu v$ where v is the velocity and μ is constant. If the velocity vanish at time T with a height ℓ from the point of projection Show that the initial velocity of the point is $\mu\ell + gT$.

Solution

The equation of motion –the point of projection is chosen to be the origin point-

$$m\frac{dv}{dt} = -mg - \mu mv \quad \Rightarrow \frac{\mu dv}{g+\mu v} = -\mu dt$$

By integrating we get

$$\ln(g+\gamma v) = c_1 - \mu t \tag{1}$$

here c_1 gives the integration constant which can be obtained from the initial conditions, $v = u$ when $t = 0$, -we suppose that the initial velocity is u which we need to obtain- therefore $c_1 = \ln(g+\mu u)$ and equation (1) takes the following formula

$$\ln(g+\mu v) = \ln(g+\mu u) - \mu t \quad \textbf{Or} \quad g+\mu v = \; g+\mu u \;\; e^{-\mu t}$$

$$\text{at } \; t = T, \;\; v = 0 \quad \Rightarrow \;\; g = \; g + \mu u \;\; e^{-\mu T} \tag{2}$$

In order to determine the height of the point we have

$$\because \;\; g+\mu v = (g+\mu u)e^{-\mu t} \qquad \textbf{Or} \qquad v = \frac{1}{\mu}\;\; (g+\mu u)e^{-\mu t} \;\; - g$$

But $v = \dfrac{dy}{dt}$ then

$$\frac{dy}{dt} = \frac{1}{\mu}\;\; (g+\mu u)e^{-\mu t} \;\; - g \qquad \Rightarrow dy = \frac{1}{\mu}\;\; (g+\mu u)e^{-\mu t} \;\; - g \;\; dt$$

By integration we get

$$y = -\frac{1}{\mu}\left(\frac{(g+\mu u)}{\mu}e^{-\mu t} \;\; + gt\right) + c_2 \tag{3}$$

here c_2 gives the integration constant which can be obtained from the initial conditions, $y = 0$ when $t = 0$, therefore $c_2 = \dfrac{(g+\mu u)}{\mu^2}$ and equation (2) become

$$y = \frac{g+\mu u}{\mu^2} - \frac{1}{\mu}\left(\frac{(g+\mu u)}{\mu}e^{-\mu t} \;\; + gt\right)$$

Now let $y = \ell$ when $t = T$

$$\ell = \frac{g+\mu u}{\mu^2} - \frac{(g+\mu u)}{\mu^2} e^{-\mu T} - \frac{gT}{\mu}$$

$$\Rightarrow \frac{(g+\mu u)}{\mu^2} e^{-\mu T} = \frac{g+\mu u}{\mu^2} - \frac{gT}{\mu} - \ell \quad \textbf{Or} \quad \frac{g}{\mu^2} = \frac{g+\mu u}{\mu^2} - \frac{gT}{\mu} - \ell$$

We use equation (2)

$$\frac{g}{\mu^2} = \frac{g+\mu u}{\mu^2} - \frac{gT}{\mu} - \ell \qquad \Rightarrow u = gT + \mu\ell$$

Example

A point with mass m is projected vertically upwards with initial velocity u and the resistance of air produces a retardation $m\gamma v^2$ where v is the velocity and γ is constant. Show that the velocity with which the point will return to the point of projection is $\frac{uu'}{\sqrt{u^2 + u'^2}}$ where $u' = \sqrt{g\gamma^{-1}}$.

Solution

To determine the velocity with which the point will return to the point of projection, we will consider the motion of the point upwards until it stop then it return.

The equation of motion of the point – consider $\mathbf{Y}$ axis to be vertically and the point of projection is chosen to be the origin point-

$$mv\frac{dv}{dy} = -mg - \gamma m v^2 \quad \Rightarrow \frac{2\gamma v dv}{g + \gamma v^2} = -2\gamma dy$$

By integration we get

$$\ln(g + \gamma v^2) = c_1 - 2\gamma y \qquad \textbf{(1)}$$

Where c_1 points out integration constant which can be obtained from the initial conditions, $v = u$ at $y = 0$, therefore $c_1 = \ln(g + \gamma u^2)$ and equation (1) takes the following formula

$$\ln(g + \gamma v^2) = \ln(g + \gamma u^2) - 2\gamma y \quad Or \quad y = \frac{1}{2\gamma}\ln\left(\frac{g + \gamma u^2}{g + \gamma v^2}\right)$$

The point will stop as $v = 0$, therefore

$$y\big|_{v=0} = Y = \frac{1}{2\gamma}\ln\left(\frac{g+\gamma u^2}{g}\right) = \frac{1}{2\gamma}\ln\left(1+\frac{u^2}{u'^2}\right), \quad (u'^2 = \frac{g}{\gamma})$$

Now by taking the motion where the point moves downwards, let the highest position represents the new origin point and the $\mathbf{Y}$ axis is chosen to be vertically downward. Moreover, the initial condition will be $v = 0$ when $y = 0$ where v is the velocity. The equation of motion

$$mv\frac{dv}{dy} = mg - \gamma mv^2 \quad \Rightarrow \frac{2\gamma v dv}{g-\gamma v^2} = -2\gamma dy$$

By integration we get

$$\ln(g-\gamma v^2) = c_2 - 2\gamma y \qquad \textbf{(2)}$$

Constant of integration c_2 can be obtained from the initial conditions, $v = 0$ at $y = 0$, therefore $c_2 = \ln g$ and equation (2) becomes

$$\ln(g-\gamma v^2) = \ln g - 2\gamma y \quad \textbf{Or} \quad y = \frac{1}{2\gamma}\ln\left(\frac{g}{g-\gamma v^2}\right)$$

And the velocity of the point with which the point will return to the point of projection is that is at $y = Y = \frac{1}{2\gamma}\ln\left(1+\frac{u^2}{u'^2}\right)$ hence

$$\frac{1}{2\gamma}\ln\left(1+\frac{u^2}{u'^2}\right) = \frac{1}{2\gamma}\ln\left(\frac{g}{g-\gamma v^2}\right) \quad \textbf{Or} \quad 1+\frac{u^2}{u'^2} = \frac{g}{g-\gamma v^2}$$

$$\Rightarrow \frac{u'^2+u^2}{u'^2} = \frac{g}{g-\gamma v^2}$$

$$\Rightarrow g-\gamma v^2 = \frac{gu'^2}{u'^2+u^2}$$

$$\Rightarrow \gamma v^2 = g - \frac{gu'^2}{u'^2+u^2}$$

$$
\begin{aligned}
v^2 &= u'^2 - \frac{u'^4}{u'^2 + u^2} \\
&= \frac{u'^2(u'^2 + u^2)}{u'^2 + u^2} - \frac{u'^4}{u'^2 + u^2} \\
&= \frac{u'^2 u^2}{u'^2 + u^2} \\
\therefore v &= \frac{uu'}{\sqrt{u^2 + u'^2}} \qquad \left(u'^2 = \frac{g}{\gamma}\right)
\end{aligned}
$$

PROBLEMS

❒ A particle of mass m is projected with velocity V along a smooth horizontal plane in a medium whose resistance per unit mass is γv, γ is a constant. Obtain the velocity v and the distance after a time t .

❐ A particle is projected vertically upwards with velocity u and the resistance of the air produces a retardation kv where v is the velocity. Determine the velocity with which the particle will return to the point of projection.

❒ A particle **P** moving along a horizontal straight line has retardation μv, where v is the velocity at time t. When $t = 0$, the particle is at **O** and has velocity u. Show that $u - v$ is proportional to **OP**.

❒ A particle subject to gravity describes a curved path in a resisting medium which causes retardation hv. Show that the resultant acceleration has a constant direction, and equals $a_0 e^{-ht}$ where a_0 is the acceleration when $t = 0$

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