Lectures in

Special and General Relativity

For

Third Year Students

Faculties of Science and Education

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Important topics in special relativity

- The failure of the Galilean transformations.
- Lorentz transformations.
- Time and Space in special relativity.
- Relativistic momentum and Energy.

Course Objectives

After completion of this course, student will be able to

- Demonstrate knowledge and broad understanding of Special Relativity.
- Explain the meaning and significance of the postulate of Special Relativity.
- Recall the setup and significance of Michelson-Morley experiment.
- Time dilation: moving clocks run (tick) slower. (Proper Time).
- Length contraction: moving rods contract. (Proper Length).
- Loss of simultaneity.
- What causes the Doppler shift for light? Time dilation, together with geometrical effects.
- Velocity transformation between inertial reference frames.
- Explain true nature of Lorentz transformation and Doppler effect.
- Explain relativistic momentum and Einstein field equations.

Chapter (1)

Relativity (I)

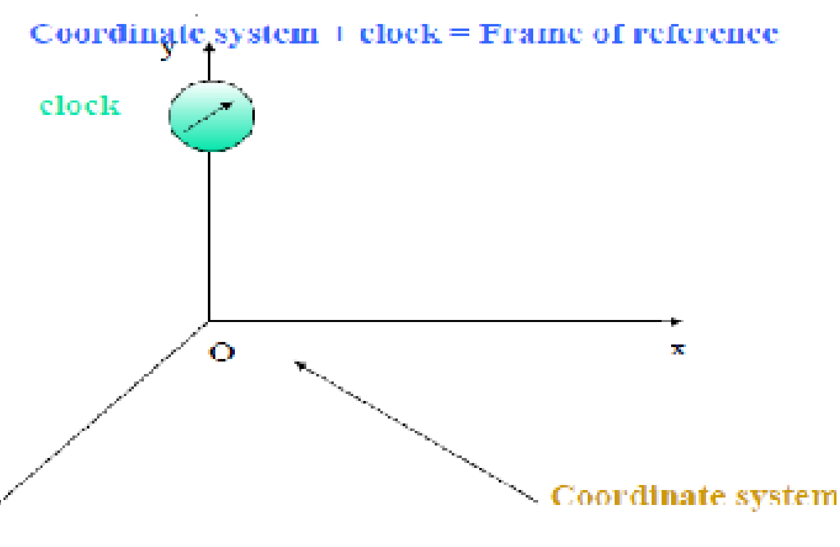
Chapter (1)
Pre-Relativistic Physics

1-1 : Galilean-Newton Relativity

In order to study General Relativity one starts discussing Special Relativity. To this end, it is important to briefy look at pre-relativistic Physics to see how Special Relativity arose.

The starting point of Special Relativity is the study of motion. For this one needs the following ingredients :

- Frames of reference. These consist of an origin in space, 3 orthogonal axes and a clock.



- Events. : This notion denotes a single point in space together with a single point in time. Thus, events are characterized by 4 real numbers: an ordered triple (x; y; z) giving the location in space relative to a _xed coordinate system and a real number giving the Newtonian time. One denotes the event by E = (t; x; y; z).

There are an infinite number of frames of reference. Motion relative to each frame looks, in principle, different. Hence, it is natural to ask: is there a subset of these frames which are in some sense simple, preferred or natural? The answer to this question is yes.

These are the so-called inertial frames. In an inertial frame an isolated, non-rotating, un-accelerated body moves on a straight line and uniformly.

Inertial frames are not unique. There are actually an infinite number of these. This raises the question: can one tell in which inertial frame are we in? It turns out that within the framework of Newtonian Mechanics this is not possible. More precisely, one has the following :

Galilean Principle of Relativity : Laws of mechanics cannot distinguish between inertial frames. This implies that there is no absolute rest. In other words, the laws of Mechanics retain the same form in different inertial frames. In this sense, Relativity predates Einstein.

1.2 : Laws of Newton

The three Laws of Newtonian Mechanics4 are :

(1) Any material body continues in its state of rest or uniform motion (in a straight line) unless it is made to change the state by forces acting on it. This principle is equivalent to the statement of existence of inertial frames.

(2) The rate of change of momentum is equal to the force.

(3) Action and reaction are equal and opposite.

These laws or principles, together with the following fundamental assumptions (some of which are implicitly assumed in Newton's laws) amount to the Newtonian framework:

1- Absolute time, motion and space as defined in Newton's "Philosophiae Principia Mathematica"
2- **Space and time are continuous** |i.e. not discrete. This is necessary to make use of the Calculus. There is no limit to the accuracy with which quantities such as time and space can be measured.

3- **There is a universal (absolute) time**. Different observers in different frames measure the same time. In fact, Newton also regarded space to be absolute as well.

However, the absoluteness of space is not necessary for the development of the Newtonian framework, as space intervals turn out to be invariant under Galilean transformations. Historically, Newton demanded this for subjective reasons.

4- Mass remains invariant (**absolute**) as viewed from different inertial frames.

5- The Geometry of space is Euclidean. For example, the sum of angles in any triangle equals 180 degrees.

6- lengths or distances are absolute no matter what is the velocity.

7- Infinitely many inertial frames of reference, each one in relative motion to absolute space.

8- Universal time throughout space where all inertial frames share this universal time.

9- All laws of physics are the same in all inertial frames of reference whether at rest or in motion relative to absolute space.

10- Speed of light is instantaneous (information transmission is instantaneous).

11- All possibilities of unbounded, relative motion.

12- Newton's Law of Universal Gravitation holds as derived from Newton's "Axioms, or Laws of Motion".

13- Inertial frames of reference are related by Galilean rules of transformation - i.e., simple vector addition and subtraction .

14- Action at a distance - corpuscular theory of the (luminiferous) aether.

15- Euclidean geometry of space - Cartesian or polar coordinates systems.

16- Conservation Laws of energy, mass and momentum.

17- Gravity attraction is directly related to mass and inversely to absolute distance.

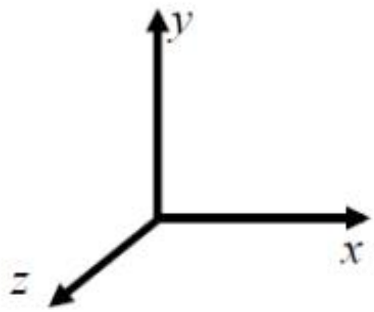
18- Bodies or objects are constituted of matter whose measure is the amount of (inertial) mass.

19- Energy and mass are distinct entities as is force

4- Galileo's relativity

Newton's laws of motion must be implemented relative to some reference frame.

A reference frame is called an **inertial frame** if Newton's laws are valid in that frame.



Such a frame is established when a body, not subjected to net external forces, moves in rectilinear motion **at constant velocity**.

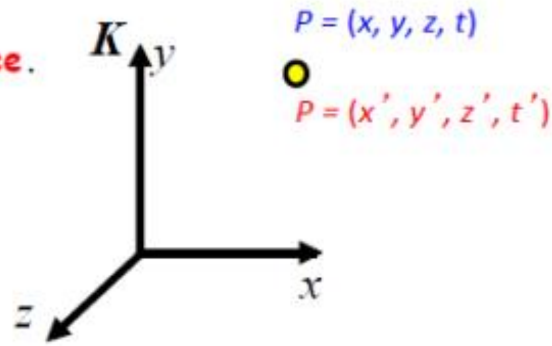
If Newton's laws are valid in one reference frame, then they are also valid in another reference frame moving at a uniform velocity relative to the first system.

This is referred to as the **Galilean invariance**.

Galilean transformation: for a point P

In one frame K: $P = (x, y, z, t)$

In another frame K: $P = (x', y', z', t')$



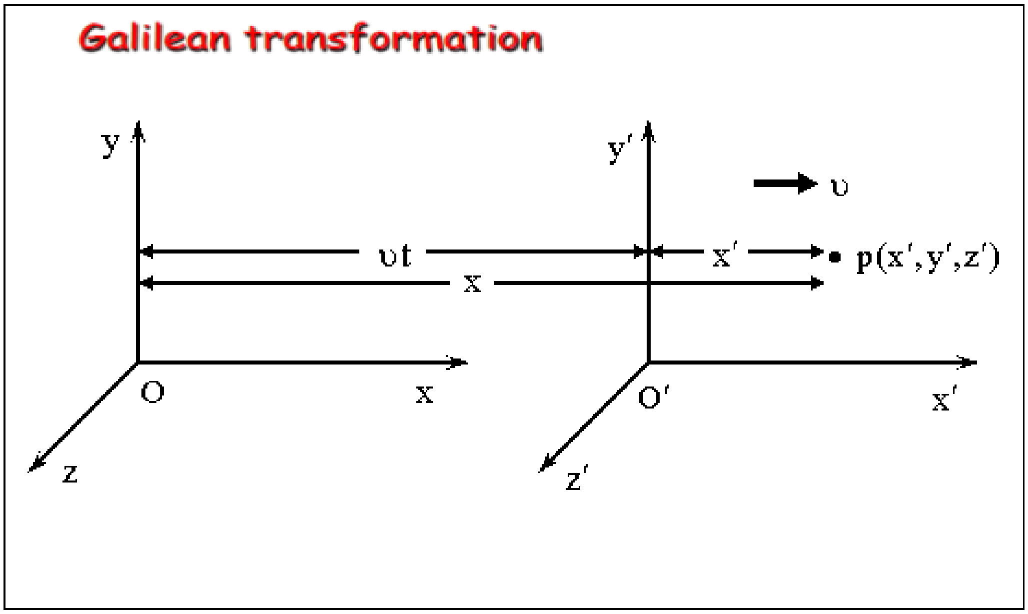
FRAME OF REFERENCE

- **Inertial frame**
  - Obeys Newton Ist law of motion or law of inertia
    - Rest frame
    - Moving with uniform speed
- **Non-inertial frame**
  - Does not obey Newton Ist law of motion or law of inertia
    - Accelerating frame
    - Rotating frame

Galilean Transformation :

Conditions of the Galilean Transformation

- Parallel axes
- O' has a constant relative velocity in the *x*-direction with respect to O
- **Time** (t) for all observers is a *Fundamental invariant*, i.e., the same for all inertial observers
- Time is universal, so therefore t = t'.



Galilean transformation

$$x = x' + v_x \Delta t'$$
$$y = y'$$
$$z = z'$$
$$\Delta t = \Delta t'$$

Inverse Galilean transformation

$$x' = x - v_x \Delta t'$$
$$y' = y$$
$$z' = z$$
$$\Delta t' = \Delta t$$

The corresponding velocity transformations are

$$u_x = \frac{dx}{dt} = \frac{dx'}{dt} + v_x = u_x' + v_x$$
$$u_y = \frac{dy}{dt} = \frac{dy'}{dt} = u_y'$$
$$u_z = \frac{dz}{dt} = \frac{dz'}{dt} = u_z'$$

Inverse $\Rightarrow$

$$u_x' = \frac{dx'}{dt} = \frac{dx}{dt} - v_x = u_x - v_x$$
$$u_y' = \frac{dy'}{dt} = \frac{dy}{dt} = u_y$$
$$u_z' = \frac{dz'}{dt} = \frac{dz}{dt} = u_z$$

For acceleration

$$a_x = \frac{du_x}{dt} = a_x' \qquad\qquad a_x' = \frac{du_x'}{dt} = a_x$$

$$a_y = \frac{du_y}{dt} = a_y' \qquad \text{Inverse} \Rightarrow \qquad a_y' = \frac{du_y'}{dt} = a_y$$

$$a_z = \frac{du_z}{dt} = a_z' \qquad\qquad a_z' = \frac{du_z'}{dt} = a_z$$

Accelerations are the same in both K and K' frames!

Note that for two inertial frames, the $a_x = a_x'$, $a_y = a_y'$, and $a_z = a_z'$.

Sucesse Or Advanteges of Galilean Transformation :

Galileo Transformation has proved several of physics' laws such as conservation of mass , energy, momentum, and second newton's law.

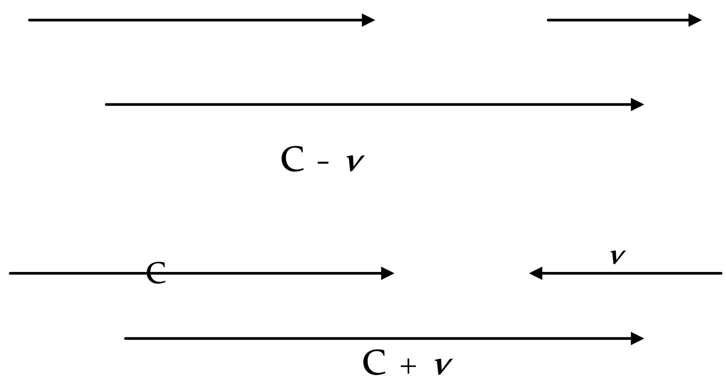
Frailer of Galilean Transformation :

Maxwell's equations, which summaries electricity and magnetism, cause the Galilean Transformation to fail on following :

1. Galilean Transformation failed to explain the actual result of Michelson-Morley experiment.
2. They are not invariant under the Galilean Transformation.
3. Galilean Transformation failed to prove unchanged of electromagnetism's laws , specially the rectilinear theory of light.
4. They predict the speed of light is independent of the inertial reference frames instead of ($c' = c + v$) as required by Galilean Relativity.

Example : The nature of rectilinear propagation of light

Assume there is a signal of light moves by a velocity (C) in Either medium , if there is an observer exist in this medium and measures the light speed as (C). If there is another observer exist in another inertial frame reference and moves with relative velocity (v) with respect to the first inertial reference frame. This observer will measure light velocity given by C − v , if he moves in the same direction of light propagation and he will measures it by C + v , if he was moves in opposite direction of the propagation of light as shown in the following Fig.



This means that, the speed of light not constant , it differ from reference frame to another, and it only has constant value at inertial reference frame , it is the Either medium.

The Transition to Modern Relativity

●Although Newton's laws of motion had the same form under the Galilean transformation, Maxwell's equations did not.

●In 1905, Albert Einstein proposed a fundamental connection between space and time and that Newton's laws are only an approximation.

1: The Need for Ether

●The wave nature of light suggested that there existed a propagation medium called the *luminiferous ether* or just ether.

- ❑ Ether had to have such a low density that the planets could move through it without loss of energy

- It also had to have an elasticity to support the high velocity of light waves

Maxwell's Equations

●In Maxwell's theory the speed of light, in terms of the permeability and permittivity of free space, was given by

$$v = c = 1/\sqrt{\mu_0 \varepsilon_0}$$

●Thus the velocity of light between moving systems must be a constant

An Absolute Reference System

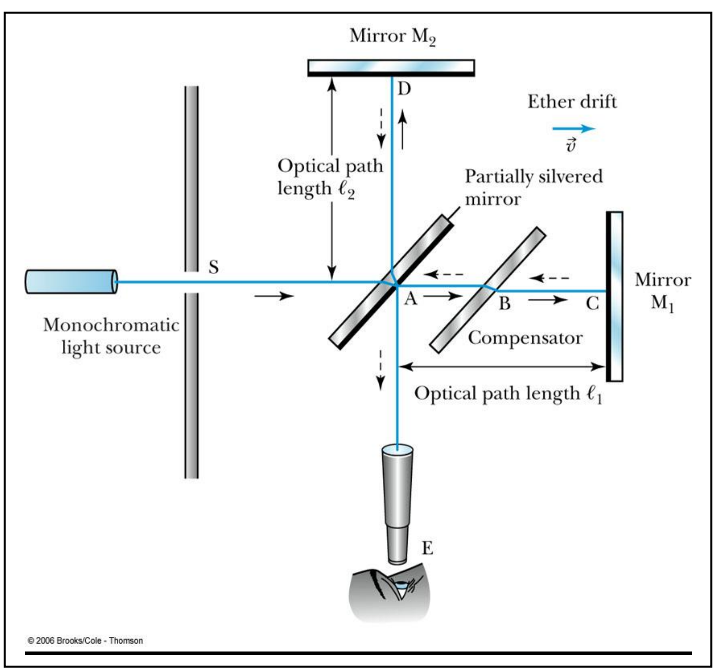
●Ether was proposed as an absolute reference system in which the speed of light was this constant and from which other measurements could be made.

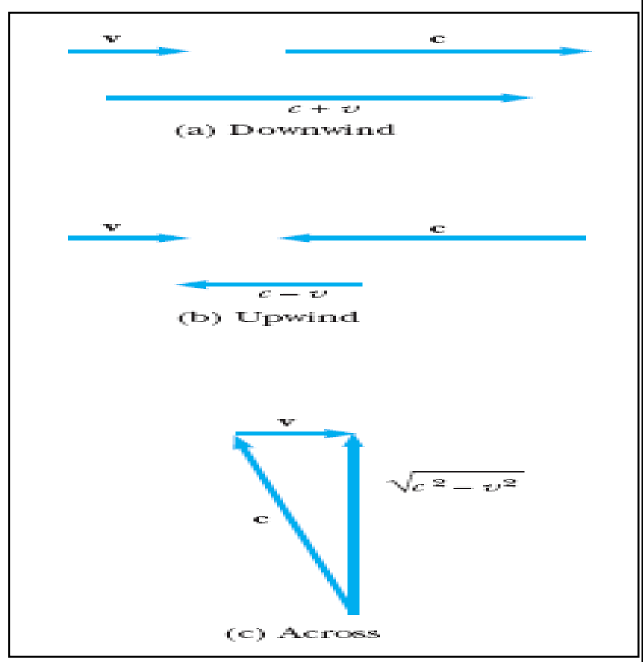
●The Michelson–Morley experiment was an attempt to show the existence of ether.

2.2: The Michelson-Morley Experiment

- Albert Michelson (1852–1931) was the first U.S. citizen to receive the Nobel Prize for Physics (1907), and built an extremely precise device called an *interferometer* to measure the minute phase difference between two light waves traveling in mutually orthogonal directions.

The Michelson Interferometer



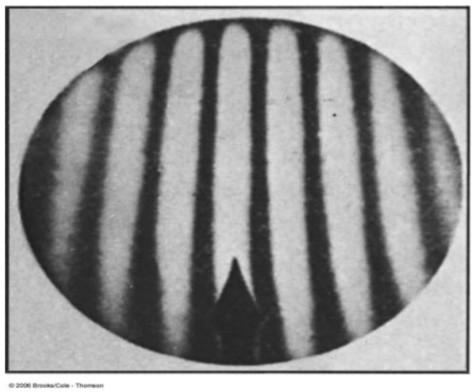


1-AC is parallel to the motion of the Earth inducing an “ether wind”

2. Light from source S is split by mirror A and travels to mirrors C and D in mutually perpendicular directions

3. After reflection the beams recombine at A slightly out of phase due to the “ether wind” as viewed by telescope E.

Typical interferometer fringe pattern expected when the system is rotated by 90°



The Analysis

Assuming the Galilean Transformation

Time t_1 from A to C and back:

$$t_1 = \frac{\ell_1}{c+v} + \frac{\ell_1}{c-v} = \frac{2c\ell_1}{c^2-v^2} = \frac{2\ell_1}{c}\left(\frac{1}{1-v^2/c^2}\right)$$

Time t_2 from A to D and back:

$$t_2 = \frac{2\ell_2}{\sqrt{c^2-v^2}} = \frac{2\ell_2}{c} = \frac{1}{\sqrt{1-v^2/c^2}}$$

So that the change in time is:

$$\Delta t = t_2 - t_1 = \frac{2}{c}\left(\frac{\ell_2}{\sqrt{1-v^2/c^2}} - \frac{\ell_1}{\sqrt{1-v^2/c^2}}\right)$$

Upon rotating the apparatus, the optical path lengths ℓ_1 and ℓ_2 are interchanged producing a different change in time: (note the change in denominators)

$$\Delta t' = t'_2 - t'_1 = \frac{2}{c}\left(\frac{\ell_2}{1-v^2/c^2} - \frac{\ell_1}{\sqrt{1-v^2/c^2}}\right)$$

Thus a time difference between rotations is given by:

$$\Delta t' - \Delta t = \frac{2}{c}\left(\frac{\ell_1+\ell_2}{1-v^2/c^2} - \frac{\ell_1+\ell_2}{1-v^2/c^2}\right)$$

and upon a binomial expansion, assuming $v/c << 1$, this reduces to

$$\Delta t' - \Delta t \approx v^2(\ell_1+\ell_2)/c^3$$

Because $v^2/c^2 \ll 1$, this expression can be simplified by using the following binomial expansion after dropping all terms higher than second order:

$$(1 - x)^n \approx 1 - nx \quad (\text{for } x \ll 1)$$

In our case, $x = \frac{v^2}{c^2}$, and we find

$$\Delta t = t_1 - t_2 \approx \frac{Lv^2}{c^3}$$

The time difference between the two light beams gives rise to a phase difference between the beams, producing the interference fringe pattern when they combine at the position of the telescope. A difference in the pattern (Fig. 1.6) should be detected by rotating the interferometer through 90^o in a horizontal plane, such that the two beams exchange roles. Then

$$\Delta d \equiv c\Delta t \sim 10^{-7}\ m \quad \text{for } L = 10 \text{ m and } (v/c)^2 \sim 10^{-8}.$$

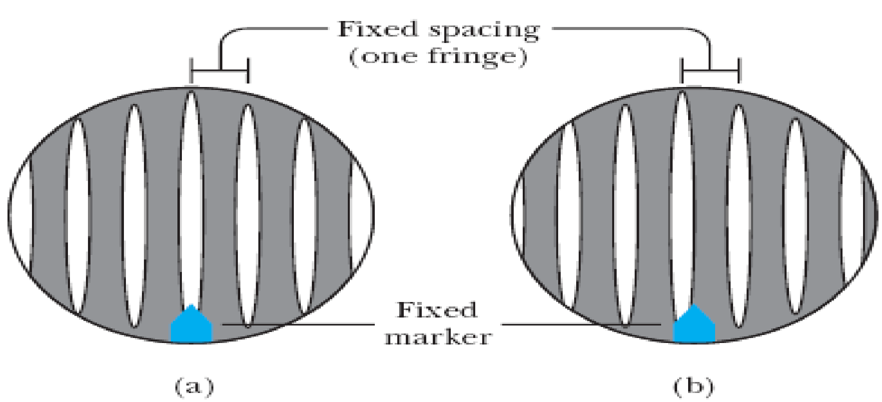


Figure 1.6: Interference fringe schematic showing (a) fringes before rotation and (b) expected fringe shift after a rotation of the interferometer by 90°.

The corresponding fringe shift is equal to this path difference divided by the wavelength of light ,λ , because a change in path of 1 wavelength corresponds to a shift of 1 fringe.

$$\text{Shift} = \frac{2Lv^2}{\lambda c^2}$$

Specifically, using light of wavelength 500 nm, and $\Delta d = 2.2 \times 10^{-7}\,\text{m}$ we find a fringe shift for rotation through 90^{o} of

$$\text{Shift} = \frac{\Delta d}{\lambda} = \frac{2.2 \times 10^{-7}\ \text{m}}{5.0 \times 10^{-7}\ \text{m}} \approx 0.40$$

Result: The precision instrument designed by Michelson and Morley had the capability of detecting a shift in the fringe pattern as small as 0.01 fringe. However, *they detected no shift in the fringe pattern*.

Results

Using the Earth's orbital speed as: $V = 3 \times 10^4$ m/s

●together with $\ell_1 \approx \ell_2 = 1.2$ m

●So that the time difference becomes $\Delta t' - \Delta t \approx v^2(\ell_1 + \ell_2)/c^3 = 8 \times 10^{-17}$ s

Although a very small number, it was within the experimental range of measurement for light waves.

Michelson's Conclusion

●Michelson noted that he should be able to detect a phase shift of light due to the time difference between path lengths but found none.

●He thus concluded that the hypothesis of the stationary ether must be incorrect.

●After several repeats and refinements with assistance from Edward Morley (1893-1923), again *a null result.*

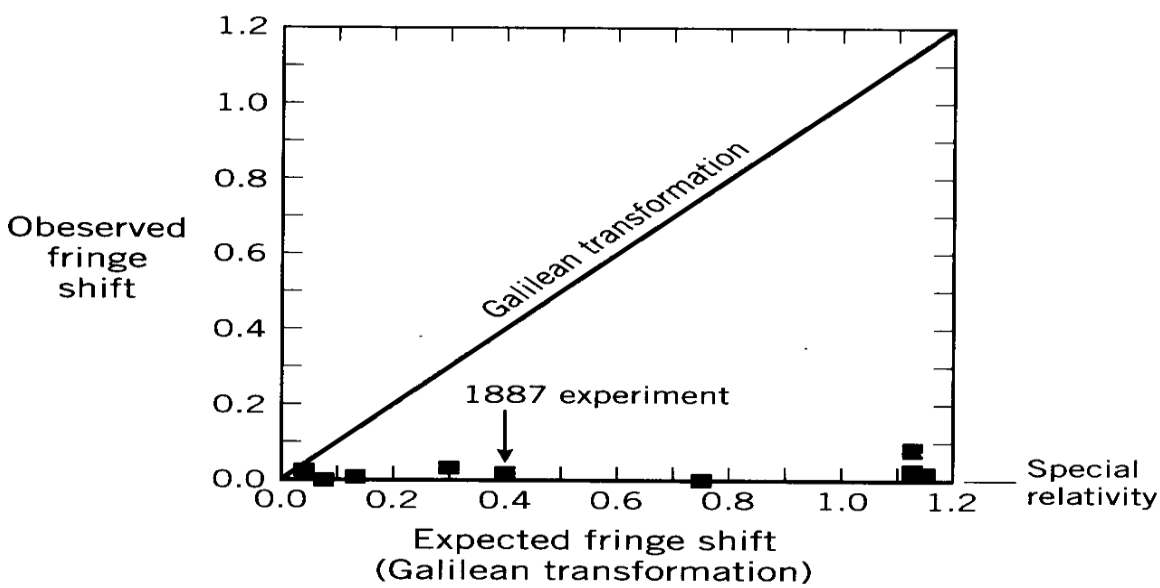
●***Thus, ether does not seem to exist!***

Possible Explanations

●Many explanations were proposed but the most popular was the *ether drag* hypothesis.

- This hypothesis suggested that the Earth somehow "dragged" the ether along as it rotates on its axis and revolves about the sun.

This was contradicted by *stellar abberation* wherein telescopes had to be tilted to observe starlight due to the Earth's motion. If ether was dragged along, this tilting would not exist. The Michelson-Morley experiment has been refined and repeated many times. Several of these results from the period 1881-1930 are summarized in figure (6). On the vertical axis we plot the observed fringe shift and on the horizontal axis we plot the expected fringe shift as calculated from the Galilean transformation. If the speed of light is constant, then zero fringe shift is expected. In practice a small fringe shift is observed due to the finite precision of the experimental apparatus.



The difficulties raised by this null result were tremendous, not only implying that light waves were a new kind of wave propagating without a medium but that the Galilean transformations were flawed for inertial frames moving at high relative speeds. The stage was set for Albert Einstein, who solved these problems in 1905 with his special theory of relativity.

Chapter (2)

Relativity (I)

Chapter (2) : Relativity (I)

Special Relativity

Difference between the special and general theories of

The special theory of relativity

• The special theory of relativity is based on two ideas: all observers measure the same speed of light in a vacuum, regardless of their speed relative to the light source; and there is no absolute frame of reference.

• To a stationary observer watching an object travelling at close to the speed of light: time appears to slow down; mass appears to increase, and length in the direction of travel appears to decrease.

• Because all motion is relative, an observer travelling at close to the speed of light (relative to a 'stationary' observer) will see that time, mass and length for a stationary observer have all changed.

• Objects with mass cannot reach or exceed the speed of light.

• Energy and mass are equivalent (E = mc2).

The General theory of relativity

•Space and time merge into four-dimensional spacetime.

• Large masses distort spacetime.

• Gravity results from the distortion of spacetime.

• Gravity causes light to bend towards large masses because of the distortion of spacetime.

• Time runs slower in areas where the gravitational field is stronger. Einstein versus Newton

in areas where the gravitational field is stronger. Einstein versus Newton

Postulates of special relativity:

1. **The Principle of Relativity:** All the laws of physics have the same form in all inertial reference frames.

   In other words, covariance applies to electromagnetism (there is no ether) as well as to mechanics.

2. **The Constancy of the Speed of Light:** The speed of light in vacuum has the same value, $c = 3.00 \times 10^8$ m/s, in all inertial frames, regardless of the velocity of the observer or the velocity of the source emitting the light.

   This postulate is in fact more or less required by the first postulate. If the speed of light was different in different frames, the Maxwell equations governing the propagation of light would have to be frame-dependent.

   In fact, Einstein said he was completely unaware of the MM experiment at the time he proposed his postulates. He was just thinking about the theory of light as being absolute and frame independent.

These two apparently simple postulates imply dramatic changes in how we must visualize length, time and simultaneity.

1. The distance between two points and the time interval between two events both depend on the frame of reference in which they are measured.

2. Events at different locations that occur simultaneously in one frame are not simultaneous in another frame moving uniformly with respect to the first.

To see what exactly is true, we need to first think about how an inertial reference frame is defined. We use a coordinate grid and a set of synchronized clocks throughout all space.

As an aside, we should note that we are already questioning that such a picture actually exists when looking at very tiny distance scales where effects of *quantum gravity* are expected to enter.

An inertial reference frame is probably only an effective description that is only valid up to the Planck mass scale.

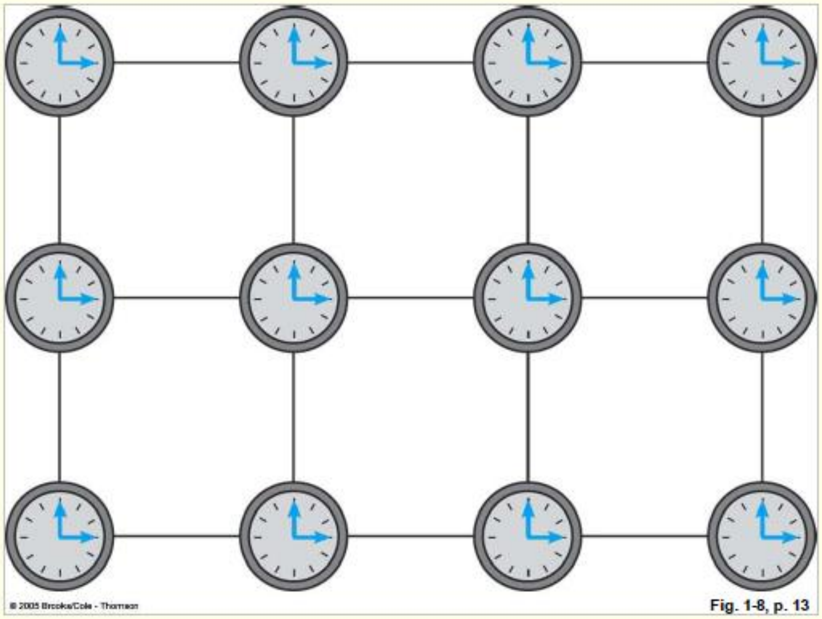


Figure 3: Picture of an inertial reference frame.

Success or Advantages of Special Relativity Postulates:

1. No absolute time, motion or space, rather (space - time) spacetime
2. Lengths contract at velocities approaching *c* relative to stationary observers.
3. Infinitely many inertial frames of reference, each one in relative motion to each other
4. Each inertial frame has its own time dilation
5. All laws of physics are the same in all inertial frames of reference which leads to the invariance of the speed of light, *c*
6. Speed of light is finite (information is finite transmitted)
7. All possibilities of unbounded, relative motion with upper limit of velocities of bodies and particles held at *c*, speed of light
8. Newton's Law of Universal Gravitation is only true as an approximation of physical reality for relative velocities $v << c$
9. Inertial frames of reference are related by (Fitzgerald -) Lorentz Transformation rules

10. Conservation Laws of energy, mass and momentum are maintained but in relativistic terms with finite c

11. Energy and mass are equivalent, force being a manifestation of energy

Let's return to the concept of time.

Example A

Suppose time were uniquely definable and the same in all frames.

Consider a (small) plane moving at speed v (and very close to ground) in the $+x$ direction relative to someone on the ground.

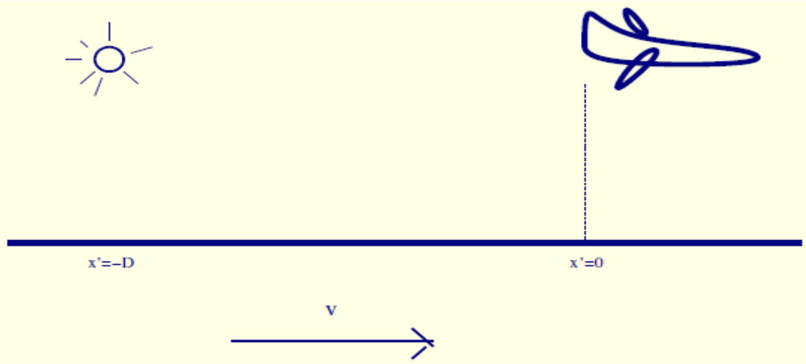


Figure 4: The frame for a plane moving relative to the earth.

When the plane is at $x' = 0$ someone at $x' = -D$ flashes a light (these are the plane's coordinates). If time is universal then both P (plane) and G (ground) agree that the light flashes at a certain time, say $t = t' = 0$.

The time at which P thinks the light arrives at $x' = 0$ (the plane never moves from $x' = 0$ – he is at rest in his coordinate system) is $t' = D/c$ (assuming light travels with velocity c).

The time at which G thinks the light arrives at the plane would also be $t = t' = D/c$ if time is universal. However, since the plane has moved

by an amount

$$extra\ distance = v\frac{D}{c} \tag{9}$$

according to G while the light has been traveling, the G observer concludes that the velocity of light is

$$\frac{distance}{time} = \frac{D + v\frac{D}{c}}{\frac{D}{c}} = v + c. \tag{10}$$

Well, this contradicts Einstein's postulates of relativity. It has to be that the clocks in the G and P frames are not synchronized in the manner we assumed or that distance scales are not the same in the two frames. In fact, both apply.

Example B

- Consider 2 observers A and B that pass one another, with, say, B moving with velocity v in the x direction relative to A who we envision is at rest in "our" frame.

  A burst of light is emitted as they pass one another. Each claims that the light travels outward in spherical waves with velocity c, with the spheres centered on themselves.

  A modern day application is that a terrorist dropping a bomb (that immediately detonates) from a fast moving car might hope to quickly leave behind the destruction and explosion. But, to the extent that electromagnetic radiation was the only consideration, he would always be at the center of the explosion no matter how fast he was moving in some other frame.

- This is completely different from what one would conclude if light traveled in a medium like water.

  Consider two boats, one (A) at rest in a pond, the other (B) moving rapidly (but without creating any wake) relative to the first boat.

B drops a rock in the pond as he passes A.

Because A is at rest in the pond, the ripples spread out in concentric circles from his position and B, looking back, agrees. Indeed, he could even go faster than the ripples, in which case they would never catch up to him.

Putting Einstein's visualization into mathematical language, we would say that the expanding spheres of electromagnetic radiation should obey

$$x^2 + y^2 + z^2 = c^2t^2 \text{ for } A; \quad x'^2 + y'^2 + z'^2 = c^2t'^2 \text{ for } B\,. \qquad (11)$$

These are the equations defining how each sees the light fronts (in his own frame) emanating from the initial flash.

Demanding that the transformation from the unprime to prime system be a linear transformation[1] with coefficients determined only by the fundamental constant c and by the relative velocity of the two frames v, *and* requiring that $x' = 0$ must correspond to $x = vt$ (see Fig. 1), there is only one solution, the so-called *Lorentz* **transformation:**

Transformation between systems

The constancy of the speed of light is not compatible with Galilean transformations. Consider a wave front starting at the origin of two frames whose origin coincide at $t = 0$. In terms of the coordinates of the two frames

$$x^2 + y^2 + z^2 = c^2 t^2$$
$$x'^2 + y'^2 + z'^2 = c^2 t'^2$$

According to the Galilean transformation

$$x' = x - vt$$
$$y' = y, \quad z' = z, \quad t' = t$$

$$\Rightarrow \quad x'^2 + y'^2 + z'^2 = (x^2 - 2xvt + v^2t^2) + y^2 + z^2 \neq c^2t^2 \tag{A.6}$$

Therefore the Galilean transformation is not compatible with the constancy of the speed of light

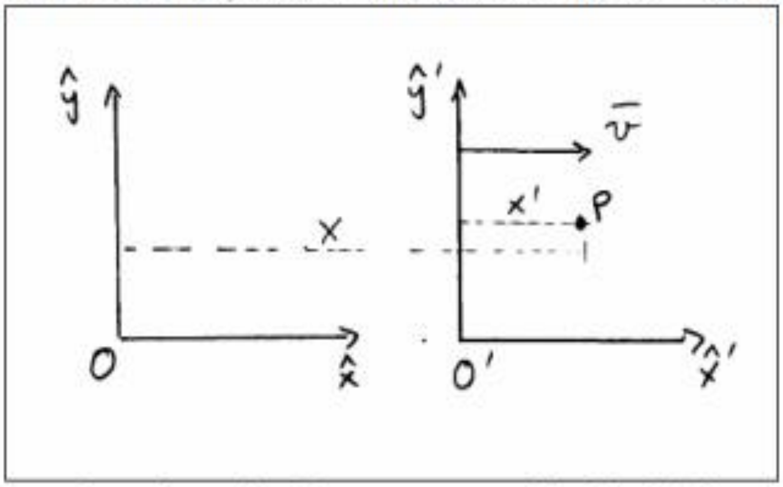
There are a couple of extra terms ($-2xvt + v^2t^2$) in the primed frame.

4. Lorentz Transformation

Now we wish to derive the transformation equations for the displacement and velocity of an object—the relativistic version of the Galilean transformation equations. In what follows, we'll be setting $\gamma = \dfrac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}$.

a. Two frames

Consider two inertial reference frames, S & S' and assume that O = O' at $t' = 0$.



What is the x-distance from O to the point P, as measured in the S' frame?

In effect, then, we'll have $\Delta t = t$ and $\Delta t' = t'$.

$$\ell' = x' + vt'$$

In the S frame, $\ell = x$, so $\ell' = \dfrac{x}{\gamma}$ also. Set 'em equal.

$$\frac{x}{\gamma} = x' + vt'$$

$$x = \gamma(x' + vt')$$

On the other hand, as measured in the S frame, $x = vt + \dfrac{x'}{\gamma}$. Set them equal.

Lorentz Transformation can be written as :

$$X^{`} = \frac{x' + Vt}{\sqrt{1 - V^2/C^2}}$$

$$y^{`} = y'$$

$$Z^{`} = Z'$$

$$t^{`} = \frac{t' + Vx/C^2}{\sqrt{1 - V^2/C^2}}$$

Inverse relations

$$x'^{`} = \frac{x - Vt}{\sqrt{1 - V^2/C^2}}$$

$$y'^{`} = y$$

$$Z'^{`} = Z$$

$$t'^{`} = \frac{t - Vx/C^2}{\sqrt{1 - V^2/C^2}}$$

Lorentz Transformation for velocity can be written as :

$$\because u_x = dx/dt \quad , \quad u_y = dy/dt \quad , \quad u_z = dz/dt$$

$$\therefore u_x = \frac{dx}{dt} = \frac{\gamma(dx' + Vdt')}{\gamma\left[dt' + (V^2/C^2)dx'\right]} = \frac{ux' + V}{1 + u'x\frac{V}{C^2}}$$

$$u_y = \frac{dy}{dt} = \frac{dy'}{\gamma\left[dt' + (V^2/C^2)dx'\right]} = \frac{uy'}{\gamma(1 + u'x\frac{V}{C^2})}$$

$$u_z = \frac{dz}{dt} = \frac{dz'}{\gamma\left[dt' + (V^2/C^2)dx'\right]} = \frac{uz'}{\gamma(1 + u'x\frac{V}{C^2})}$$

The Inverse Lorentz velocity transformation

$$\therefore u_x = \frac{dx'}{dt} = \frac{u_x - V}{1 - u_x V/C^2}$$

$$\therefore u_y = \frac{dy'}{dt} = \frac{u_y - V}{\gamma(1 - u_x V/C^2)}$$

$$\therefore u_z = \frac{dz'}{dt} = \frac{u_z - V}{\gamma(1 - u_x V/C^2)}$$

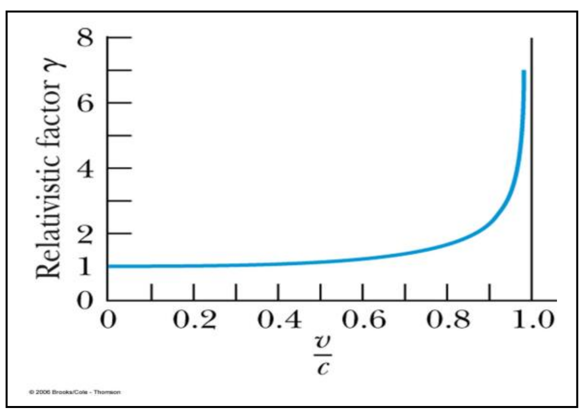
Note that if $v/c \ll 1$, then $\gamma \to 1$ and we get back the Galilean *approximation*.

But, if $v/c \to 1$, then $\gamma \gg 1$ and there are big changes.

properties of γ

Recall $\beta = v/c < 1$ for all observers.

1) $\gamma \geq 1$ equals 1 only when $v = 0$.
2) Graph of β: (note $v \neq c$)



An important consequence

Using the Lorentz transform equations, we can easily show that $x^2 - c^2t^2 = x'^2 - c^2t'^2$, for any choices of x, t and the corresponding values of x', t'.

$x_4 \cdot x_4 \equiv c^2t^2 - x^2 - y^2 - z^2$. In the prime frame, $x_4' = (ct', x', y', z')$ and $x_4' \cdot x_4' \equiv c^2t'^2 - x'^2 - y'^2 - z'^2$.

We see that a restatement of the Lorentz transformation equations, equivalently Einstein's frame-independent for the velocity of light, is to say that the square of a 4-vector is frame-independent.

c. 4-vectors

Suppose that when O = O', a flash of light is emitted from the origin O. In the S frame, the distance the light wave front travels in time t is $r^2 = x^2 + y^2 + z^2 = c^2t^2$. Measured in the S' frame, it's $r'^2 = x'^2 + y'^2 + z'^2 = c^2t'^2$. Subtract the second expression from the first and collect the S frame on one side of the equal sign, the S' frame on the other side.

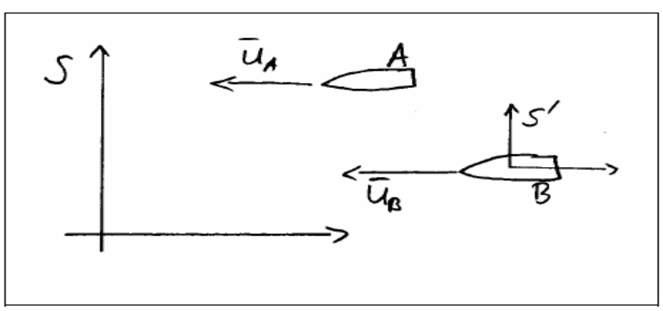
$$r^2 - r'^2 = c^2t^2 - c^2t'^2$$
$$r^2 - c^2t^2 = r'^2 - c^2t'^2$$

There is this quantity, a generalized displacement (call it s) which is the <u>same</u> in the two inertial reference frames.

$$s^2 = s'^2$$

We see that the quantity (ict) "acts like" a component of displacement along a fourth axis. The *interval* between any two events in space-time is $\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2\Delta t^2$. The interval is invariant under the Lorentz Transformation. That is, as measured in any two inertial frames, $\Delta s^2 = \Delta s'^2$. This is an extension of the invariance of lengths under a rotation of the coordinate axes.

example:



$u_A = -0.5c$ and $u_B = -0.8c$, both as measured in the S frame. The S' frame rides along with spaceship B. Therefore, $\vec{v} = \vec{u}_B$.

$$u'_A = \frac{u_A - v}{1 - \frac{v}{c^2}u_A} = \frac{-0.5c - (-0.8c)}{1 - \frac{0.8c}{c^2}0.5c} = \frac{c}{2}$$

Be careful with the directions of the velocities.

Note that when $u << c$ and $v << c$, then $\frac{vu}{c^2} \to 0$ and $u' = u - v$. On the other hand, if $u = c$,

then $u' = \frac{c - v}{1 - \frac{cv}{c^2}} = \frac{c\left(1 - \frac{v}{c}\right)}{1 - \frac{v}{c}} = c$.

Important sucsseses of the special relativity

1- Time dilation

The most important outcome of Lorentz transformation that ($t^{`} \neq t$) . isTh means that, if we put a clock inside different coordinates, the rate of its rotation will changes .

To explain the mean of time dilation using **Einstein train experiment** , if there is a light signal emits from a bulb light exist at train's car surface and incident on a mirror fixed at top roof of the car and reflected back the surface. The train moves with velocity (v) .

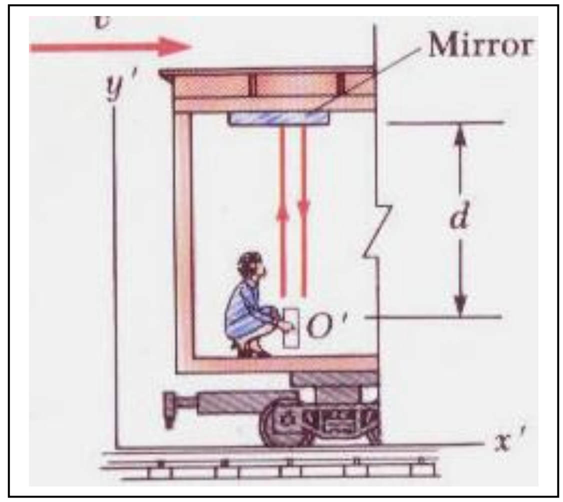
Event : Is the Light Signal , studding its propagation from the starting point of its emission until it reflection from the fixed mirror at roof of the train's car surface .

Observers O and O˜ : Suppose there are two observers, one exist at fixed inertial reference frame (O˜). and the other exist inside the train's car (O).

First: Measurements of the Observer at a fixed inertial reference frame (O˜).

The observer (O˜). Measures the travel of time by dividing the resultant distance by the speed of light .

The O˜ observer will measure the time of this event by dividing resultant distance by the speed of light .

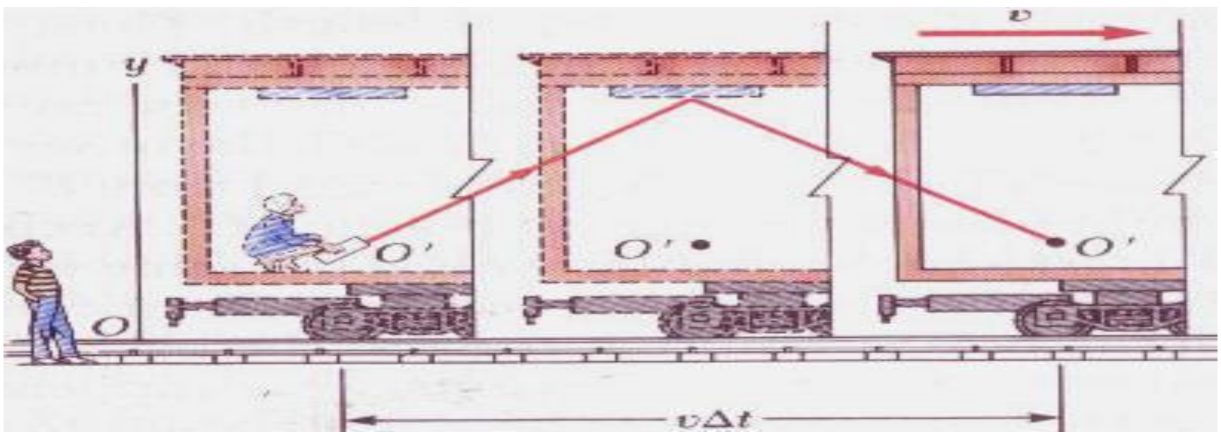


Notes : The O˜ observer is fixed with respect to the event , its position coordinates doesn't change between the starting and ending the event as shown in the drown fighter.

Travel distance equals twice height of roof of the train's car (2 L\ ) and the event time period with respect to O˜ observer calculated as :

$$t^{\backslash} = \frac{2L^{\backslash}}{C} - - - - - - - - -(1)$$

Second: Measurements of the moving observere (O).



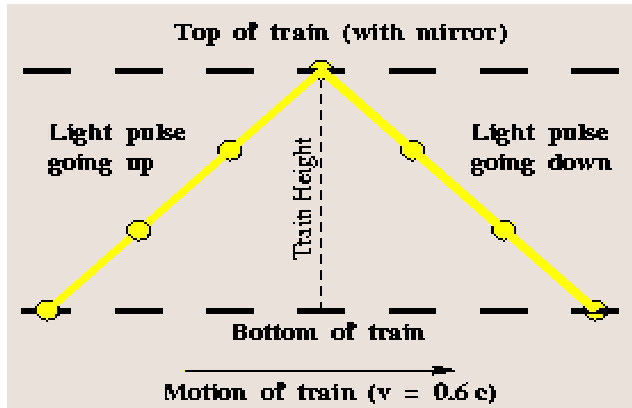
The O observer carried out his measurements, but he moves with respect to the event (or the event moves with respect to him) . Here, the initial and final point of event occurs at two different positions with respect to the O observer as shown in a given fig. During this time period of the event , the train travel forward to the right a distance (vt) , where t is event time measured by the O observer.

This Fig. illustrate the path of light signal with respect to the O observer , here the path length of the light signal is greater than its path with respect to the O˜ observer.

From the second postulate of special relativity, the velocity of the light signal is constant (not change) with respect to the tow observers and equals the speed of light (c) . Because the path of the light signal with respect to the O observer is longer than that of the $O^{\sim}$ observer, then the measured time by O observer is greater than that of $O^{\sim}$ observer.

Mathmatical Forms btween the measurements of the two observers O and $O^{\sim}$

Consider the bold line represents the path of signal light which measured by $O^{\sim}$ observer, and dashed line represents the path of light measured by O observer . In order to the light travel the distance from the surface of train's car to the mirror in the roof of the car , it spend a half (t/2) of total time. Using mathematical Fesajorse theory on the left triangle. Considering $L^{\backslash}$ is the height of the train's car as follow :



$$\left(\frac{ct}{2}\right)^2 = \left(\frac{vt}{2}\right)^2 + L^{\backslash 2}$$

$$\Rightarrow \frac{c^2t^2}{4} = \frac{v^2t^2}{4} + L^{\backslash 2}$$

$$\Rightarrow \frac{c^2t^2}{4} - \frac{v^2t^2}{4} = L^{\backslash 2} \Rightarrow \frac{t^2}{4}\left(c^2 - v^2\right) = L^{\backslash 2}$$

$$\Rightarrow t^2\left(c^2 - v^2\right) = 4L^{\backslash 2} \Rightarrow t^2 = \frac{4L^{\backslash 2}}{\left(c^2 - v^2\right)}$$

$$\Rightarrow t^2 = \frac{\left(2L^{\backslash}\right)^2}{c^2\left(1 - v^2/c^2\right)} \Rightarrow t^2 = \frac{\left(2L^{\backslash}\right)^2/c^2}{\left(1 - v^2/c^2\right)}$$

$$\Rightarrow t = \frac{2L^{\backslash}/c}{\sqrt{\left(1 - v^2/c^2\right)}} ----------------(2)$$

$$t^{\backslash} = \frac{2L^{\backslash}}{C} .... from...Eq(2)$$

We got the relation between time measurements of each observer

$$t = \frac{t^{\backslash}}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma . t^{\backslash} \qquad \text{------------------- (3)}$$

We put ; $\gamma = 1\Big/\sqrt{\left(1 - v^2/c^2\right)}$

Note : Equation (3) represents the measured time dilation according to special relativity. Always the parameter, $(\gamma) > 1$, This leads to $\Delta t' > \Delta t$. e.g : this mean that the time which measured by a fixed observer is greater, while the moving watch its measured time seems to be delay with respect to a fixed observer.

The studies of time dilation are important in the emission of radiation processes, nuclear diffusion and interaction of primary particles, because their velain is smallerocities are much greater , because it close to the speed of light. This cause the changes in (γ) parameter values of are greater.

Important Notes :

1- **Because the velocity of the train is smaller than the speed of light , this leads to $\gamma > 1$. For this reason, time measurements of O** $> O^{\sim}$ e.g : $t < t^{\backslash}$ **.**

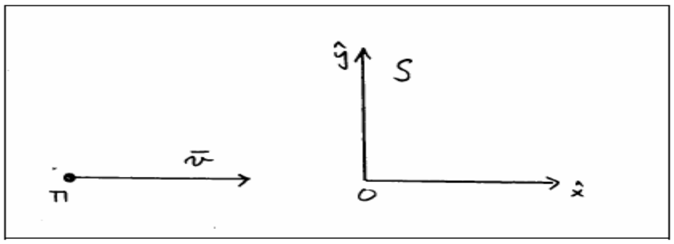
2- **For velocities** $v << c$**, such as velocities of cars, train, rock . These velocities are smaller compared to light speed. This mack the calculation of the dominator in Eq (3) = 1 . This mack the measurements are equal fro the two observers, This means that dilation of time can not measured except at high velocities compared to light speed .**

3- **At high velocities , the moving observer watch with respect to the event , it measure long time than that of the fixed observer with respect to the event.**

4- The proper Time, it is the measured time by the fixed observer with respect to the event.

Example

The lifetime of a *pion* in its own rest frame is $\Delta t' = 2.6x10^{-8}$ sec. Consider a pion moving with speed $v = 0.95c$ in a lab—what will be measured as its lifetime in the lab?



$$\Delta t = \frac{\Delta t'}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{2.6x10^{-8}\text{ sec}}{\sqrt{1-0.95^2}} = \frac{2.6x10^{-8}\text{ sec}}{0.312} = 8.33x10^{-8}\text{ sec}.$$

The lifetime of a fast-moving particle is measured by noting how far it travels before decaying. In this example $\ell = v\Delta t = 0.95c \cdot 8.33x10^{-8}$ sec $= 23.7$ m. In practice, we measure ℓ and compute Δt.

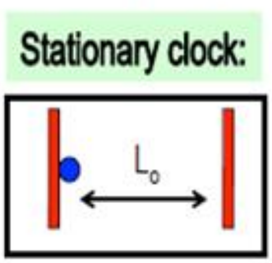
c. Proper time

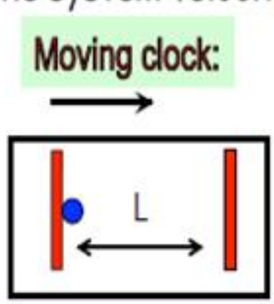
The *proper time* is the time interval measured by an observer for whom the two events occur at the same place, so that $\Delta x' = \Delta y' = \Delta z' = 0$.

Space contraction

Consider the time for a pulse parallel to the system velocity to do a round trip:

$$t_0 = 2\frac{L_0}{c}$$





$$t = \frac{L}{c-v} + \frac{L}{c+v}$$

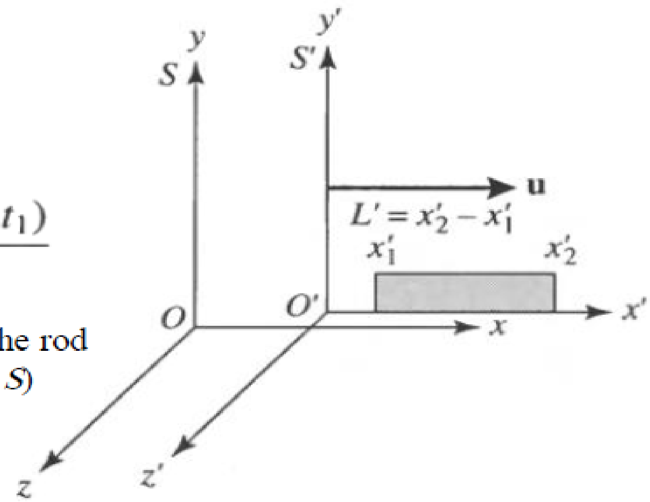
$$\Rightarrow L = L_0 \cdot \sqrt{1 - v^2/c^2}$$

An observer moving along an object will find it shorter than it would be if the observer was standing still.

Length Contraction (Lorentz Contraction)

- Suppose that a rod lies at rest along the x'-axis of frame S'. Let the left end of the rod be at x_1' and the right end at x_2', so that the length of the rod as measured in frame S' is $L' = x_2' - x_1'$. What is the length of the rod as measured in frame S?
- **From Lorntze transformation**

$$x' = \frac{x - ut}{\sqrt{1 - u^2/c^2}}$$



we find that

$$x_2' - x_1' = \frac{(x_2 - x_1) - u(t_2 - t_1)}{\sqrt{1 - u^2/c^2}}$$

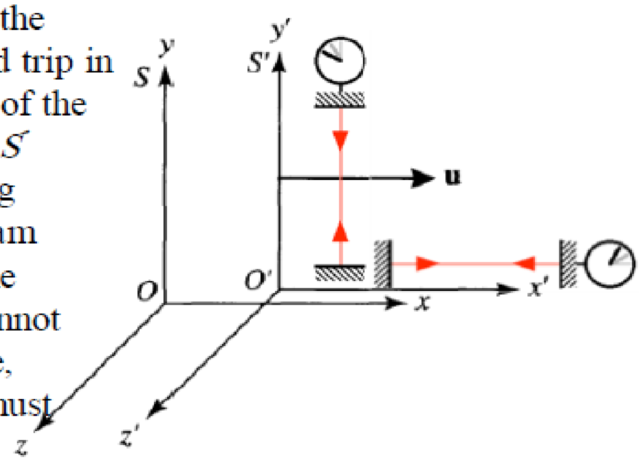
and with $t_1 = t_2$ (naturally both ends of the rod are measured at the same time in frame S)

$$L' = \frac{L}{\sqrt{1 - u^2/c^2}}$$

Which is shorter, L or L' ?

Length Contraction and Proper Length

- Once again, we are simply applying an additional postulate to Newtonian physics, that is Einstein's 2nd postulate.
- Consider the light clock described before, but now with one clock oriented orthogonal to and the other clock parallel to the direction of motion of the S' frame. They must both tick at the same rate as measured by an observer in the S frame (as time can only run at one rate in a given reference frame), although suffering from the effect of time dilation.
- According to the observer in the S frame, if there is no length contraction, the light pulse of the orthogonally-oriented clock makes a round trip in a shorter time interval than the light pulse of the parallel-oriented clock. (Imagine that the S' frame is stationary and the S frame is being carried to the left by a river: the cross-stream swimmer makes a round trip faster than the upstream-downstream swimmer.) This cannot happen as both clocks tick at the same rate, implying that the parallel-oriented clock must suffer length contraction.

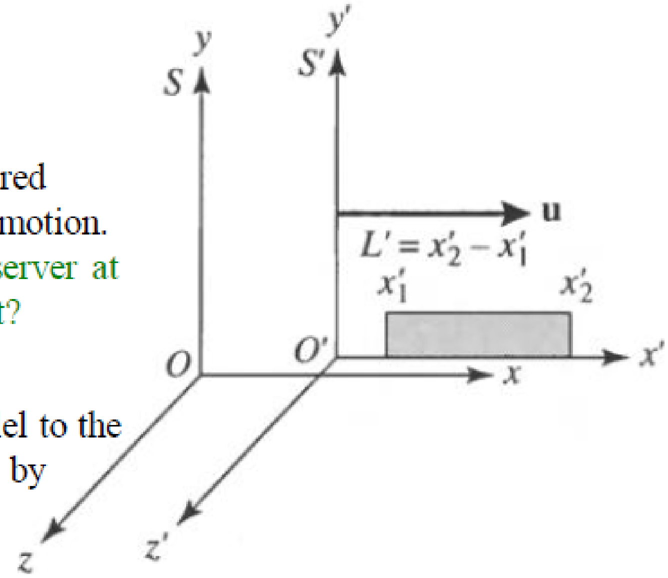


- Let us return to the derivation for the length of a rod in frame S' as measured by an observer in frame S where the rod aligned in the direction of motion:

$$L = L' \sqrt{1 - u^2/c^2}$$

- Let us call $L' = L_{rest}$ as the rod is at rest in frame S'. Let us cal $L = L_{moving}$ as the rod is moving in frame S. Then

$$L_{moving} = L_{rest}\sqrt{1 - u^2/c^2}.$$



- Lengths (distances) are therefore measured differently by two observers in relative motion. Who measures the longer length, an observer at rest or moving with respect to the object?

- Note that only lengths (distances) parallel to the direction of relative motion are affected by length contraction. Lengths (distances) perpendicular to the direction of relative motion remain unchanged

Simultaneity

a. Space-time

Each event has associated with it four numbers: x, y, z coordinates and a "value of time" which we read off a clock located at that spatial location. There is no central universal clock, rather there is a clock at every point in space.

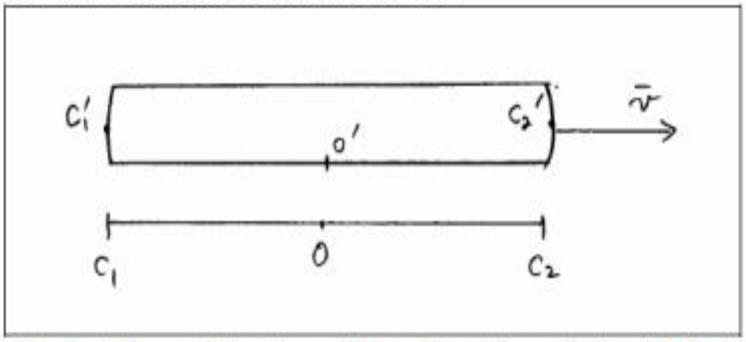
b. Synchronization

We would like all clocks in a reference frame to display exactly the same reading simultaneously, but can this be arranged? Only by the exchange of signals, which is another way of saying only in terms of intervals. However, as we have seen, intervals are not the same for observers in different inertial reference frames. Therefore, the concept of two events being simultaneous has no absolute meaning.

c. Non-simultaneity

Two events viewed as simultaneous in one frame will not be seen as occurring simultaneously in another frame.

example: a train moving with constant velocity on a straight, smooth track. One observer rides on the train, the other observer stands beside the track.



Flashes of light are emitted at the points C_1 and C_2 when the origins (O & O') of the two frames coincide. To the trackside observer at O, the flashes are simultaneous. To the observer on the train, however, the flash emitted at C'_2 is received before the flash emitted at C'_1. Yet both observers measure the same speed of light, c.

Twin Paradox

The Set–up

Twins Mary and Frank at age 30 decide on two career paths: Mary decides to become an astronaut and to leave on a trip 8 light years (ly) from the Earth at a great speed and to return; Frank decides to reside on the Earth.

The Problem : Upon Mary's return, Frank reasons that her clocks measuring her age must run slow. As such, she will return younger. However, Mary claims that it is Frank who is moving and consequently his clocks must run slow.

The Paradox : Who is younger upon Mary's return?

The Resolution

1) Frank's clock is in an **inertial system** during the entire trip; however, Mary's clock is not. As long as Mary is traveling at constant speed away from Frank, both of them can argue that the other twin is aging less rapidly.
2) When Mary slows down to turn around, she leaves her original inertial system and eventually returns in a completely different inertial system.

3) Mary's claim is no longer valid, because she does not remain in the same inertial system. There is also no doubt as to who is in the inertial system. Frank feels no acceleration during Mary's entire trip, but Mary does.

4) Frank's clock is in an **inertial system** during the entire trip; however, Mary's clock is not. As long as Mary is traveling at constant speed away from Frank, both of them can argue that the other twin is aging less rapidly.

5) When Mary slows down to turn around, she leaves her original inertial system and eventually returns in a completely different inertial system.

6) Mary's claim is no longer valid, because she does not remain in the same inertial system. There is also no doubt as to who is in the inertial system. Frank feels no acceleration during Mary's entire trip, but Mary does.

Spacetime

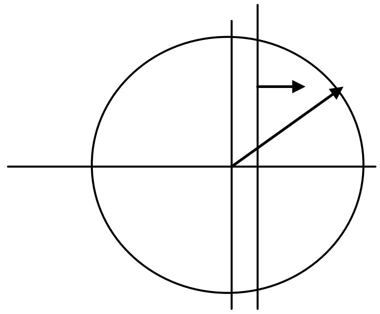
●When describing events in relativity, it is convenient to represent events on a spacetime diagram.

●In this diagram one spatial coordinate *x,* to specify position, is used and instead of time *t*, *ct* is used as the other coordinate so that both coordinates will have dimensions of length.

●Spacetime diagrams were first used by H. Minkowski in 1908 and are often called Minkowski diagrams. Paths in Minkowski spacetime are called worldlines.

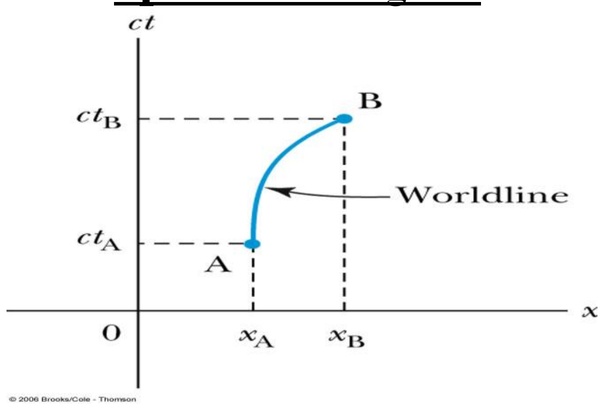
Spacetime Interval

Since all observers "see" the same speed of light, then all observers, regardless of their velocities, must see spherical wave fronts.

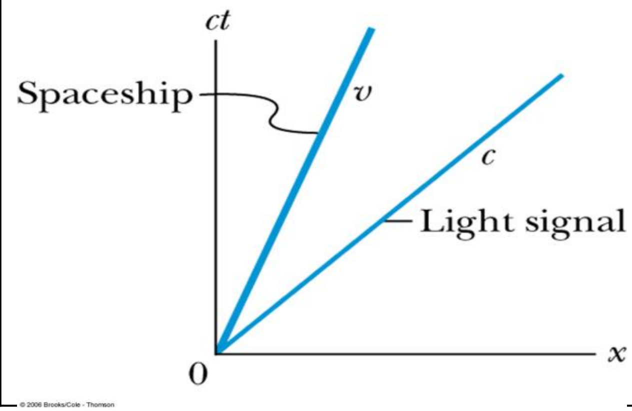


$$s^2 = x^2 - c^2t^2 = (x')^2 - c^2(t')^2 = (s')^2$$

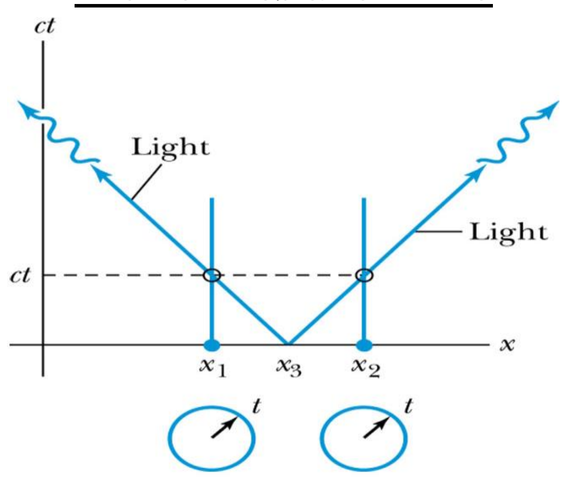
Spacetime Diagram



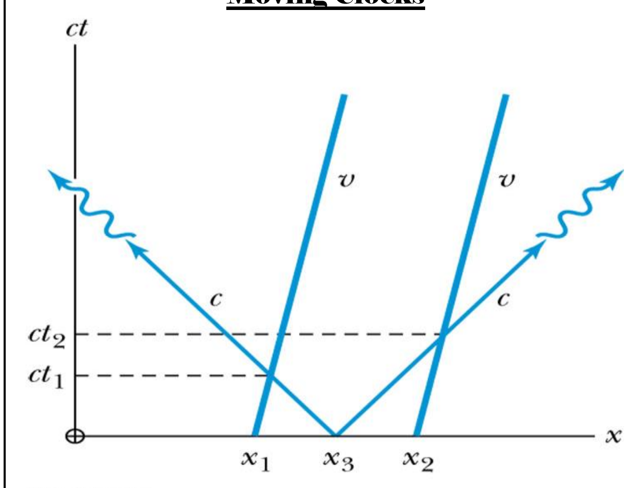
Particular Worldlines

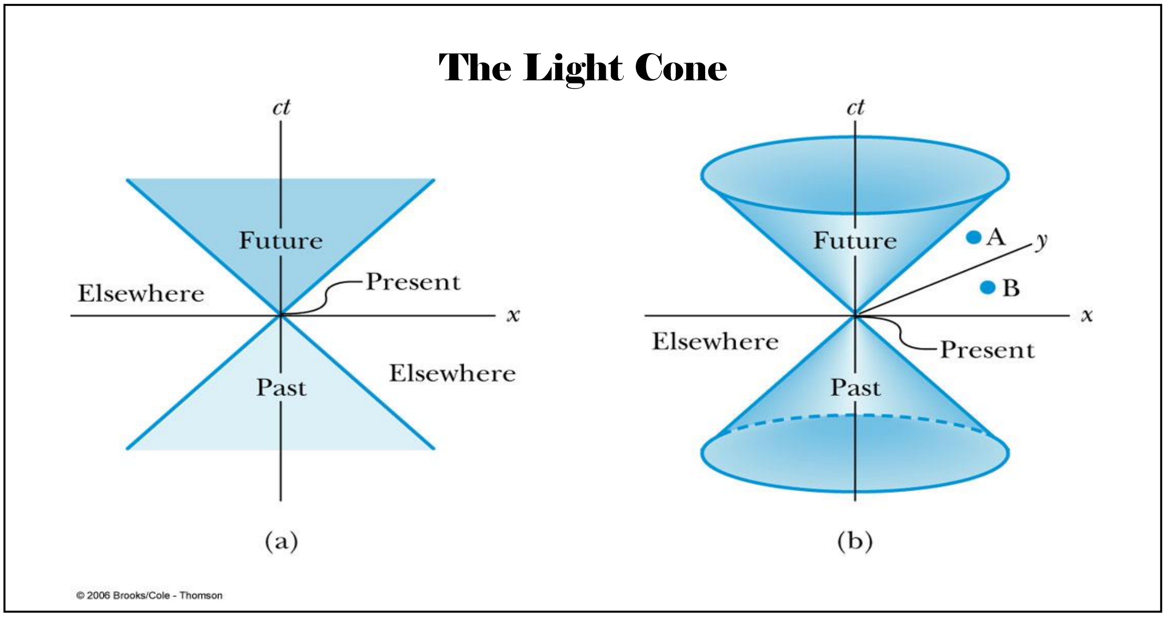


Worldlines and Time



Moving Clocks





●If we consider two events, we can determine the quantity Δs^2 between the two events, and we find that it is **invariant** in any inertial frame. The quantity Δs is known as **the spacetime interval** between two events.

There are three possibilities for the invariant quantity Δs^2:

1) $\Delta s^2 = 0$: $\Delta x^2 = c^2 \Delta t^2$, and the two events can be connected only by a light signal. The events are said to have a **lightlike** separation.
2) $\Delta s^2 > 0$: $\Delta x^2 > c^2 \Delta t^2$, and no signal can travel fast enough to connect the two events. The events are not causally connected and are said to have a **spacelike** separation.
3) $\Delta s^2 < 0$: $\Delta x^2 < c^2 \Delta t^2$, and the two events can be causally connected. The interval is said to be **timelike**.

relativistic Doppler shift.

This is a particularly important application as it is the Doppler "red"-shift that we use to tell us that the universe is expanding from something like an initial big bang.

You are all familiar with the usual Doppler shift in which the pitch of sound for a whistle on a train headed towards you has a higher pitch than the whistle sound when the train is moving away.

This is because successive waves emitted by a source moving towards you are closer together than normal because of the advance of the source — and since their separation is the wavelength of sound, the corresponding frequency is higher.

The formula in the case of sound is probably something you have derived in an earlier course.

$$f = f_0 \left(\frac{1 + \frac{v}{c}}{1 - \frac{V}{c}} \right) , \tag{32}$$

where f_0 is the frequency of the sound as measured by the source itself, f is the frequency as measured by the observer, c is the speed of sound, v is the speed of the observer ($+$ for motion toward source), and V is the speed of the source ($+$ for motion toward the observer).

This classical Doppler effect evidently varies depending upon whether the source, the observer, or both are moving.

- This does not violate relativity because sound does travel in a medium — unlike light.
- Since light does not travel in a medium, the light wave Doppler effect will be different.
  It can be derived using the concepts of time dilation. My derivation is an alternative approach to that given in the book. Note, in particular, that the book assumes that the source is moving towards the observer.

Imagine a light source as a clock that ticks f_0 times per second and emits a light wave peak at each tick. The proper time in the source rest frame between ticks is $t_0 = 1/f_0$.

Consider a source at rest and an observer moving away from it with velocity v. The interval between ticks as seen by this observer is given by time dilation: $t = \gamma t_0$

As viewed by the observer, he travels the distance vt away from the source between ticks.

Thus, each tick takes a time vt/c longer to reach him than the simple time t between ticks.

The total time between the arrival of successive peaks (successive ticks) is then

$$\begin{aligned} T &= t + \frac{vt}{c} = t(1 + \frac{v}{c}) = \gamma t_0(1 + \frac{v}{c}) \\ &= t_0 \frac{1 + \frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}} = t_0 \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}\,. \end{aligned} \quad (33)$$

The corresponding frequency of the ticks or wave peaks is just the inverse:

$$f(\text{receding case}) = \frac{1}{T} = \frac{1}{t_0}\sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} = f_0\sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} \quad (34)$$

- Now, although I derived this for an observer moving away from the source, the same result applies if the source moves away from the observer.
- You should also note that the frequency shift depends only on the relative velocity of the source and observer. One does not need to reference any medium in which light travels.

- Since wavelength and frequency are inversely related, $f\lambda = c$, the shift in λ obeys the inverse formula.

Example:

Determining the speed of recession of the Galaxy Hydra.

A certain absorption line that would be at $\lambda_0 = 394\ nm$ were Hydra at rest, is shifted to $\lambda = 475\ nm$ according to observations on earth.

We use

$$\lambda = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}\lambda_0 \quad (35)$$

to find that

$$\frac{v}{c} = \frac{\frac{\lambda^2}{\lambda_0^2} - 1}{\frac{\lambda^2}{\lambda_0^2} + 1} = 0.185\,. \quad (36)$$

Therefore, Hydra is receding from us with a velocity of $v = 0.185\ c = 5.54 \times 10^7\ m/s$.

Show tape #42 on space-time diagram starting at 20 min mark. *

Solved Problems

Problem (1) : The time age for nuclear particle befor transferring to another form equals 1.8 x 10^{-8} sec, when it at rest in laboratory research. What the time age of this particle when its velocity becomes 0.95 of light speed?

Solution

IN this case , $\upsilon = 0.95\ c$, Then :

$$\Delta t' = \frac{1.8\times 10^{-8}}{\sqrt{1-\left(\frac{0.95\,c}{c}\right)^2}} = 5.76\times 10^{-8}\ \text{Sec.}$$

The result shows that the time age of the particle equals three time its time age when it becomes at rest

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Problem (2) : what is the speed of the airplane, which has a clock rotates by one second delay with another clock exist on Earth frame ?.

Solution

Put : $\Delta t = 3600\ \text{Sec}$ & $\Delta t' = 3601\ \text{Sec}$

$$\therefore 3601 = \frac{3600}{\sqrt{1-\left(\frac{\upsilon}{c}\right)^2}}$$

$$\sqrt{1-\left(\frac{\upsilon}{c}\right)^2} = \frac{3600}{3601} \Rightarrow \left(\frac{\upsilon}{c}\right)^2 = 1-\left(\frac{3600}{3601}\right)^2$$

$$\upsilon = \sqrt{1-\left(\frac{3600}{3601}\right)^2}\times c \quad \Rightarrow \quad \upsilon = 7.1\times 10^6\ \text{m/Sec}$$

Problem (3) : If the average age time of the Meson particle (μ) which moves by a velocity $0.9\ c$ is $6\times 10^{-6}\ \text{Sec}$. Calculate its average age in a steady frame reference.

Solution

$\Delta t' = 6\times 10^{-6}\ \text{Sec}$ ، $\upsilon = 0.9\,c$

$$\therefore \Delta t = \sqrt{1-\left(\frac{0.9\,c}{c}\right)^2} \times 6\times10^{-6}$$
$$= \sqrt{0.19} \times 6\times10^{-6}$$
$$= 2.62\times10^{-6} \text{ Sec}$$

Problem (4) : Airplane moves with respect to the Erath's frame reference by a velocity 600 m/Sec , its orignal length equals 50 m . what is the decrease in length , which seems to be less with respect to an observer exist on the inerial Earth's surface frame .

Solution

$\upsilon = 600 \text{ m/Sec}$ ، $c = 3\times10^{6} \text{ m/Sec}$ ، $L = 50 \text{ m}$

$$L' = L\sqrt{1-\left(\frac{\upsilon}{c}\right)^2} = 50\sqrt{1-\left(\frac{600}{3\times10^{8}}\right)^2} = 50\sqrt{1-4\times10^{-12}} \qquad \therefore \Delta L \approx 10^{-10} \text{ m}$$

Problem (5) : **A particle has a mass of 5 Kgm , How much its mass when it moves by a velocity 0.6 c . ?**

solution

$m = 5$ Kg. ، $\upsilon = 0.6\,c$

$$\therefore m = \frac{m_0}{\sqrt{1-\left(\frac{\upsilon}{c}\right)^2}} = \frac{5}{\sqrt{1-(0.6)^2}} = \frac{5}{\sqrt{0.64}} = \frac{50}{8} = 6.25 \text{ Kg.}$$

Problem (6) : Particle has a mass 100 Kg, it moves by a velocity 0.8 c. How much its resident mass

Solution

$m = 100$ Kg. ، $\upsilon = 0.8c$:

$$\because m = \frac{m_0}{\sqrt{1-\left(\frac{\upsilon}{c}\right)^2}} \Rightarrow \therefore m_0 = m\sqrt{1-\left(\frac{\upsilon}{c}\right)^2} \Rightarrow \therefore m_0 = 100\sqrt{1-(0.8)^2}$$

$$= 0.6 \text{ Kg.} = 100 \times 0.6$$

Problem (7)

Time Dilation ("Moving Clocks Run Slow")

• At what speed does a clock move if it runs at a rate which is one-half the rate of a clock at rest?

Solution

We assume that the clock is at rest in S'. As observed by stationary observers in S, the clock moves in the positive x-direction with speed v. Text Eq. (1.30):

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right) \tag{1}$$

relates the time t measured in S with the time t' measured in S' where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{2}$$

and

$$\beta = v/c \tag{3}$$

Let $\Delta t'$ be a time interval measured by an observer at rest in S'. ($\Delta t'$ is a proper time. $\Delta t'$ is measured when $\Delta x' = 0$.)

Let Δt be the time interval measured by observers at rest in S.

Then

$$\Delta t = \gamma \Delta t' \tag{4}$$

and therefore

$$\beta = \sqrt{1 - \left(\frac{\Delta t'}{\Delta t}\right)^2} \tag{5}$$

It follows from Eq. (5) that $\beta = 0.866$ when $\Delta t = 2\Delta t'$.

Eq. (4) indicates that the time interval Δt measured by observers at rest in S is larger than the time interval $\Delta t'$ measured by an observer at rest with respect to the clock. That is, “moving clocks run slow”.

It is important to note that Eq. (4) relates clock readings on a single clock in S' with clock readings on two separate clocks in S.

"Moving clocks run slow" is illustrated by the LIght Pulse Clock. For this clock, a light pulse is directed along the positive y' axis and reflected back to its starting point. The traversal time is recorded as $\Delta t'$. In S, the clock moves along the positive x axis with speed v. A stationary observer records the time the light pulse starts and a second stationary observer, farther to the right along the x axis, records the time when the light pulse returns to its starting point. This traversal time is recorded as Δt. Because of the sideways motion, the light pulse travels farther in S than in S'. Since the speed of light is the same in both frames, it follows that the Δt is larger than $\Delta t'$. The moving clock runs slow.

Problem (8)

Length Contraction (“Moving Rods Contract”)

- At what speed does a meter stick move if its length is observed to shrink to 0.5 m?

Solution

We assume that the meter stick is at rest in S'. As observed by stationary observers in S, the meter stick moves in the positive x-direction with speed v. Text Eq. (1.25):

$$x' = \gamma(x - vt) \tag{6}$$

relates the position x' measured in S' with the position x measured in S.

Let $\Delta x'$ be the length of the meter stick measured by an observer at rest in S'. ($\Delta x'$ is the proper length of the meter stick.)

The meter stick is moving with speed v along the x axis in S. To determine its length in S, the positions of the front and back of the meter stick are observed by two stationary observers in S at the same time. The length of the meter stick as measured in S is the distance Δx between the two stationary observers at $\Delta t = 0$.

Then

$$\Delta x' = \gamma \Delta x \tag{7}$$

where γ is given by Eq. (2). It follows that

$$\beta = \sqrt{1 - \left(\frac{\Delta x}{\Delta x'}\right)^2} \tag{8}$$

It follows from Eq. (8) that $\beta = 0.866$ when $\Delta x = \Delta x'/2$.

Eq. (7) indicates that the length Δx of an object measured by observers at rest in S is smaller than the length $\Delta x'$ measured by an observer at rest with respect to the meter stick. That is, "moving rods contract".

It is important to note that Eq. (7) compares an actual length measurement in S' (a proper length) with a length measurement determined at equal times on two separate clocks in S.

Problem (9)

Time Dilation for a Slow Moving Object ("Moving Clocks Run Slow")

An atomic clock is placed in a jet airplane. The clock measures a time interval of 3600 s when the jet moves with speed 400 m/s.

•How much larger a time interval does an identical clock held by an observer at rest on the ground measure?

Solution

We take the S frame to be attached to the Earth and the S' frame to be the rest frame of the atomic clock.

It follows from Eq. (2) that

$$\gamma \simeq 1 + \beta^2/2 \tag{9}$$

and from Eq. (4) that

$$\delta t = \Delta t - \Delta t' \simeq \beta^2 \Delta t'/2 \tag{10}$$

It follows that $\delta t = 3.2$ ns when $v = 400$ m/s and $\Delta t' = 3600$ s.

Problem (10)

Muon Decay: Time Dilation ("Moving Clocks Run Slow")

The muon is an unstable particle that spontaneously decays into an electron and two neutrinos. If the number of muons at $t = 0$ is N_0, the number N at time t is

$$N = N_0 e^{-t/\tau} \tag{11}$$

where $\tau = 2.20\mu s$ is the mean lifetime of the muon. Suppose the muons move at speed $0.95c$.

- What is the observed lifetime of the muons?
- How many muons remain after traveling a distance of 3.0 km?

Solution

We take the S frame to be attached to the Earth and the S' frame to be the rest frame of the muon.

It follows from Eq. (4) that $\Delta t = 7.046\ \mu s$ when $\Delta t' = 2.2\ \mu s$ and $\beta = 0.95$.

A muon at this speed travels 3.0 km in 10.53 μs. After travelling this distance, N muons remain from an initial population of N_0 muons where

$$N = N_0 e^{-t/\tau} = N_0 e^{-10.53/7.046} = 0.225 N_0 \tag{12}$$

Problem (11)

Length Contraction and Rotation

A rod of length L_0 moves with speed v along the horizontal direction. The rod makes an angle θ_0 with respect to the x' axis.

- Determine the length of the rod as measured by a stationary observer.
- Determine the angle θ the rod makes with the x axis.

Solution

We take the S' frame to be the rest frame of the rod.

A rod of length L_0 in S' makes an angle θ_0 with the x' axis. Its projected lengths $\Delta x'$ and $\Delta y'$ are

$$\Delta x' = L_0 \cos\theta_0 \tag{13}$$

$$\Delta y' = L_0 \sin\theta_0 \tag{14}$$

In a frame S in which the rod moves at speed v along the x axis, the projected lengths Δx and Δy are given by Eq. (7) and

$$\Delta y' = \Delta y \tag{15}$$

which equation follows from text Eq. (1.26):

$$y' = y \tag{16}$$

The length L of the rod as measured by a stationary observer in S is

$$L = \sqrt{(\Delta x)^2 + (\Delta y)^2} = L_0 \left(1 - \beta^2 \cos^2 \theta_0\right)^{1/2}. \tag{17}$$

The rod makes an angle θ with the x axis in S where

$$\tan\theta = \Delta y/\Delta x = \gamma \tan\theta_0. \tag{18}$$

The rod in S appears contracted and rotated. See also text Fig. (1.14) on page 19 of the text.

Problem (12)

Relativistic Doppler Shift

- How fast and in what direction must galaxy A be moving if an absorption line found at wavelength 550 nm (green) for a stationary galaxy is shifted to 450 nm (blue) (a "blue-shift") for galaxy A?
- How fast and in what direction is galaxy B moving if it shows the same line shifted to 700 nm (red) (a "red shift")?

Solution

Galaxy A is approaching since an absorption line with wavelength 550 nm for a stationary galaxy is shifted to 450 nm. To find the speed v at which A is approaching, we use text Eq. (1.13):

$$f_{obs} = \sqrt{\frac{1+\beta}{1-\beta}} f_{source} \tag{19}$$

and $\lambda = c/f$ to write

$$\lambda_{obs} = \sqrt{\frac{1-\beta}{1+\beta}} \lambda_{source} \tag{20}$$

from which

$$\beta = \frac{\lambda_{source}^2 - \lambda_{obs}^2}{\lambda_{source}^2 + \lambda_{obs}^2} \tag{21}$$

It follows that $\beta = 0.198$ when $\lambda_{source} = 550$ nm and $\lambda_{obs} = 450$ nm.

Galaxy B is receding since the same absorption line is shifted to 700 nm. Proceeding as above,

$$\beta = \frac{\lambda_{obs}^2 - \lambda_{source}^2}{\lambda_{obs}^2 + \lambda_{source}^2} \tag{22}$$

It follows that $\beta = 0.237$ when $\lambda_{source} = 550$ nm and $\lambda_{obs} = 700$ nm.

Problem (13)

Lorentz Velocity Transformation

Two spaceships approach each other, each moving with the same speed as measured by a stationary observer on the Earth. Their relative speed is $0.70c$,

- Determine the velocities of each spaceship as measured by the stationary observer on Earth.

Solution

Text Eq. (1.32) gives the Lorentz velocity transformation:

$$u'_x = \frac{u_x - v}{1 - u_x v/c^2} \tag{23}$$

where u_x is the velocity of an object measured in the S frame, u'_x is the velocity of the object measured in the S' frame and v is the velocity of the S' frame along the x axis of S.

We take the S frame to be attached to the Earth and the S' frame to be attached to the spaceship moving to the right with velocity v. The other spaceship has velocity $u_x = -v$ in S and velocity $u'_x = -0.70c$ in S'.

It follows from Eq. (23) that

$$0.70 = \frac{2\beta}{1 + \beta^2} \tag{24}$$

solving which yields $\beta = 0.41$. As measured by the stationary observer on Earth, the spaceships are moving with velocities $\pm 0.41c$.

Problem (14)

Lorentz Velocity Transformation

A stationary observer on Earth observes spaceships A and B moving in the same direction toward the Earth. Spaceship A has speed $0.5c$ and spaceship B has speed $0.80c$.

- Determine the velocity of spaceship A as measured by an observer at rest in spaceship B.

Solution

We take the S frame to be attached to the Earth and the S' frame to be attached to spaceship B moving with velocity $v = -0.8c$ along the x axis. Spaceship A has velocity $u_x = -0.50c$ in S.

It follows from Eq. (23) that spaceship A has velocity $u'_x = 0.50c$ in S'. Spaceship A moves with velocity $0.5c$ as measured by an observer at rest in spaceship B.

Problem (15)

Speed of Light in a Moving Medium

The motion of a medium such as water influences the speed of light. This effect was first observed by Fizeau in 1851.

Consider a light beam passing through a horizontal column of water moving with velocity v.

• Determine the speed u of the light measured in the lab frame when the beam travels in the same direction as the flow of the water.

• Determine an approximation to this expression valid when v is small.

Solution

We assume that the light beam and the tube carrying water are oriented along the positive x-direction of the lab frame. The speed of light u in the lab frame is related to the speed of light u' in a frame moving with the water by text Eq. (1.34):

$$u = \frac{u' + v}{1 + u'v/c^2} \tag{25}$$

where v is the speed of the water in the lab frame.

Now $u' = c/n$, where n is the index of refraction of water, so

$$u = \frac{c}{n}\left(\frac{1 + n\beta}{1 + \beta/n}\right) \tag{26}$$

Using

$$(1 + \beta/n)^{-1} \simeq (1 - \beta/n) \tag{27}$$

it follows that

$$u \simeq \frac{c}{n} + v\left(1 - \frac{1}{n^2}\right). \tag{28}$$

This equation agrees with Fizeau's experimental result. The equation shows that the Lorentz velocity transformation and not the Galilean velocity is correct for light.

Problem (16)

Lorentz Velocity Transformations for Two Components

As seen from Earth, two spaceships A and B are approaching along perpendicular directions.

- If A is observed by a stationary Earth observer to have velocity $u_y = -0.90c$ and B to have velocity $u_x = +0.90c$, determine the speed of ship A as measured by the pilot of ship B.

Solution

We take the S frame to be attached to the Earth and the S' frame to be attached to spaceship B moving with $\beta = 0.90$ along the x axis. Spaceship A has velocity components $u_x = 0, u_y = -0.90c$ in S.

Eq. (23) and text Eq. (1.33):

$$u'_y = \frac{u_y}{\gamma(1 - u_x v/c^2)} \tag{29}$$

give the velocity components of spaceship A in S', from which

$$u'_x = -v = -0.90c \tag{30}$$

$$u'_y = u_y/\gamma = -0.39c \tag{31}$$

so

$$u' = \sqrt{(u_x{'})^2 + (u_y{'})^2} = 0.98c. \tag{32}$$

Problem (17)

Lorentz Velocity Transformation

A spaceship moves away from Earth with speed v and fires a shuttle craft in the forward direction at a speed v relative to the spaceship. The pilot of the shuttle craft launches a probe in the forward direction at speed v relative to the shuttle craft.

- Determine the speed of the shuttle craft relative to the Earth.
- Determine the speed of the probe relative to the Earth.

Solution

We take the S frame to be attached to the Earth and the S' frame to be attached to the spaceship moving with speed v along the x axis. The shuttle craft has speed $u'_x = v$ in S'. Text Eq. (1.34):

$$u_x = \frac{u'_x + v}{1 + u'_x v/c^2} \tag{112}$$

gives its speed u_x in S as

$$u_x = \frac{2v}{1+\beta^2}. \tag{113}$$

We now take the S' frame to be attached to the shuttle craft moving with speed

$$\tilde{v} = \frac{2v}{1+\beta^2} \tag{114}$$

along the x axis. The probe has speed $u'_x = v$ in S'. Its speed u_x in S is given by Eq. (112) with v replaced by $\tilde{v}$ from which

$$u_x = \left(\frac{3+\beta^2}{1+3\beta^2}\right)v \qquad (115)$$

It follows from Eq. (115) that $u_x \to 3v$ when $\beta \ll 1$ and $u_x \to c$ when $\beta \to 1$.

Problem (18)

Lorentz Velocity Transformation, Length Contraction, Time Dilation

Two powerless rockets are heading towards each other on a collision course. As measured by Liz, a stationary Earth observer, Rocket 1 has speed $0.800c$, Rocket 2 has speed $0.600c$, both rockets are 50.0 m in length, and they are initially 2.52 Tm apart.

- What are their respective proper lengths?
- What is the length of each rocket as observed by a stationary observer in the other rocket?
- According to Liz, how long before the rockets collide?
- According to Rocket 1, how long before they collide?
- According to Rocket 2, how long before they collide?
- If the crews are able to evacuate their rockets safely within 50 min (their own time), will they be able to do so before the collision?

Solution

Eq. (7) relates a rocket's proper length $\Delta x'$ with its length Δx when moving with speed v as measured a stationary observer. It follows that the proper length of Rocket 1 is 83.3 m and the proper length of Rocket 2 62.5 m.

We take the S frame to be attached to the Earth with the rockets moving along the x axis; the velocity of Rocket 1 is $0.800c$, the velocity of Rocket 2 is -$0.600c$. To determine the velocity of Rocket 1 as measured by a stationary observer in Rocket 2, we take the S' frame to be attached to Rocket 2. In S, the velocity u_x of Rocket 1 is $0.800c$, the velocity v of S' is -$0.600c$. It follows from Eq. (23) that the velocity u'_x of Rocket 1 in S' is $0.946c$. Similarly, the velocity of Rocket 2 as measured by a stationary observer in Rocket 1 is $-0.946c$. In both cases, $\gamma = 3.083$. It follows from Eq. (7) that the length of Rocket 1 as measured by a stationary observer in Rocket 2 is 27.0 m and the length of Rocket 2 as measured by a stationary observer in Rocket 1 is 20.3 m.

Liz observes that, in a time Δt, Rocket 1 travels a distance $0.800c\Delta t$ and Rocket 2 travels a distance $0.600c\Delta t$. The total distance traveled by the two rockets is 2.52 Tm. It follows that Δt=100 min.

Eq. (4) relates the proper time $\Delta t'$ in a rocket with the time Δt when moving with speed v as measured by a stationary observer. It follows that the time before collision as measured by a stationary observer in Rocket 1 is 60 min and the time before collision as measured by a stationary observer in Rocket 2 is 80 min. The crews are able to evacuate their rockets safely before collision.

Chapter (3)

Relativity (II)

Relativistic Kinematics

Chapter (3)
Relativistic Kinematic

Relativistic mass (changing mass with velocity)

One of the important outcomes of special relativity is the effect of velocity on mass of the moving object. From Newtonian mechanics, the velocity of any object has a mass (m) ,increases under a continuous affecting force according to the following equation :

$$\upsilon_t = \upsilon_0 + a\,t = \upsilon_0 + \frac{F}{m}\,t \qquad (1)$$

υ_0 = velocity at any moment (t)

a = acceleration of the object $a = \frac{F}{m}$

This tell us that , υ_0 increased to infinity after an infinite time . This result not true , if we consider from the postulate of the special relativity that there is a maximum velocity , it is the light speed. To solve this problem, consider the mass of the object increases by increasing the velocity of the object according the following equation :

$$m = \gamma\, m_0 \qquad (2)$$

m_0 = Rest mass of the object

m = Mass of the object when it moves by a velocity (v)

Mass of the object given by :

$$m = \frac{m_0}{\sqrt{1 - \left(\frac{\upsilon}{c}\right)^2}} \qquad (3)$$

This equation indicates as the velocity of the object close to the light speed , the mass of the object highly increases until it becomes infinite value, if v close to C.

The mass becomes infinity , it need big force to moves, and this force not exist in nature.

Note : According to Newtonian mechanics, the mass of the object is constant, and it doesn't depend on the its velocity. This not agree with special relatively, because the mass increases by increasing its velocity,

$$m = m_\circ / (1 - v^2 / c^2)^{\frac{1}{2}}$$

m_0 = Rest mass of the body with respect to the velocity observer

m = Effective mass of the moving body with velocity (v)

The above relativistic mas equation tell us , in the direction of the motion, the mass of a body in any moving inertial reference frame increases with increasing its velocity and reach infinity when (v) close to(C) . But at very low velocity (v),compared to (C), the term [$\beta = v^2/c^2$] can be neglected, in this case the mass of the body still as it is, as it was described in classical mechanics.

Relativity II: Mass, Energy and Momentum

Energy and Momentum

We require that all the "Laws" of Physics be the same in all inertial reference frames. We require further that when $v << c$, we recover the familiar Newtonian forms of the "Laws." The latter requirement is called a *Correspondence Principle*. What are those "Laws"?

1. Conservation of Momentum

We define a *relativistic momentum* so that the two conditions above are satisfied.

$$\vec{p} = \gamma m \vec{u}$$

This m is the *rest mass*—the mass measured by an observer at rest with respect to the object. This quantity should be the same in all inertial reference frames. With this definition, $\vec{p}_{initial} = \vec{p}_{final}$ in all inertial reference frames.

2. Relativistic Energy

a. Work-energy theorem (one dimensional)

The work done by a force on an object changes its kinetic energy, thus

$$\Delta K = W_{12} = \int_{x_1}^{x_2} F dx \,.$$

$$\Delta K = \int_{x_1}^{x_2} \frac{dp}{dt} dx$$

$$\Delta K = \int_{t_1}^{t_2} \frac{dp}{dt} \frac{dx}{dt} dt$$

$$\Delta K = \int u dp$$

Integrate by parts.

$$\Delta K = up\Big|_{u_1}^{u_2} - \int_{u_1}^{u_2} p du$$

$$\Delta K = up\Big|_{u_1}^{u_2} - \int_{u_1}^{u_2} \frac{mu}{\sqrt{1 - \frac{u^2}{c^2}}} du$$

Recall that $udu = \frac{du^2}{2}$.

$$\Delta K = up\Big|_{u_1}^{u_2} - \frac{m}{2}\int \frac{du^2}{\sqrt{1-\frac{u^2}{c^2}}}$$

Look up the form $\int \frac{dx}{\sqrt{a+bx}}$ in a math tables book.

$$\Delta K = up\Big|_{u_1}^{u_2} - \frac{m}{2}\left[\frac{2\sqrt{1-\frac{u^2}{c^2}}}{-\frac{1}{c^2}}\right]_{u_1}^{u_2}$$

$$\Delta K = \left[\frac{mu^2}{\sqrt{1-\frac{u^2}{c^2}}} + mc^2\sqrt{1-\frac{u^2}{c^2}}\right]_{u_1}^{u_2}$$

$$\Delta K = \left[\frac{mu^2 + mc^2 - mu^2}{\sqrt{1-\frac{u^2}{c^2}}}\right]_{u_1}^{u_2} = \Delta\left[\frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}}\right]$$

Now, if we started from rest, then $u_1 = 0$ and $u_2 = u$ and $\Delta K = \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}} - mc^2$. Therefore, we define the *relativistic kinetic energy* to be

$$K = \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}} - mc^2 .$$

The quantity mc^2 is called the *rest energy*, because it's independent of u. The *total relativistic energy* is $E = K + mc^2 + V$, where V is the potential energy, if any. If $V = 0$, then

$$E = K + mc^2 = \gamma mc^2 .$$

b. Energy-momentum relation

Take a look at the quantity ($V = 0$)

$$E^2 - m^2c^4 = \frac{m^2c^4}{1-\frac{u^2}{c^2}} - m^2c^4 = \frac{m^2c^2u^2}{1-\frac{u^2}{c^2}} = c^2\left(\frac{m^2u^2}{1-\frac{u^2}{c^2}}\right).$$

$$E^2 - m^2c^4 = c^2p^2$$

$$E^2 = c^2p^2 + m^2c^4$$

For photons, $m = 0$ and $E = pc$.

c. Units of mass-energy

It is convenient to express energy in units of electron-volts (eV). An *electron-volt* is the energy gained by an electron upon being accelerated through a one Volt potential difference. Thus 1 eV = $1.60\text{x}10^{-19}$Joules. The rest energy of an electron is

$$mc^2 = 9.11x10^{-31}kg\left(3x10^8 m/\sec\right)^2 = 8.20x10^{-14}J = 0.511x10^6 eV = 0.511MeV .$$

Often, mass is expressed in terms of MeV/c^2 so that the electron mass is $0.511MeV/c^2$. Sometimes, the c^2 is dropped, but it's understood to still be there. Similarly, momentum is expressed in terms of MeV/c, since pc = units of MeV.

3. Relativistic Mechanics

a. Force

We want the "Laws" of Mechanics to be invariant under the Lorentz Transformation. Also, we want to recover the classical result when $u \ll c$. So, we define the *relativistic force* component to be $F_x = \frac{dp_x}{dt}$, where $p_x = \frac{mu_x}{\sqrt{1-\frac{u^2}{c^2}}}$.

Let's say the motion and force are entirely along the x-direction.

$$F = \frac{d}{dt}\left[\frac{mu}{\sqrt{1-\frac{u^2}{c^2}}}\right] = \frac{m}{\sqrt{1-\frac{u^2}{c^2}}}\frac{du}{dt} + mu\frac{d}{dt}\left[\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\right]$$

$$F = \frac{m}{\sqrt{1-\frac{u^2}{c^2}}}\frac{du}{dt} + mu\left(-\frac{1}{2}\right)\left(1-\frac{u^2}{c^2}\right)^{-3/2}\left(-\frac{2u}{c^2}\right)\frac{du}{dt}$$

$$F = m\frac{du}{dt}\left[\left(1-\frac{u^2}{c^2}\right)^{-1/2} + \frac{u^2}{c^2}\left(1-\frac{u^2}{c^2}\right)^{-3/2}\right]$$

$$F = m\left(\frac{1}{1-\frac{u^2}{c^2}}\right)^{3/2}\frac{du}{dt}$$

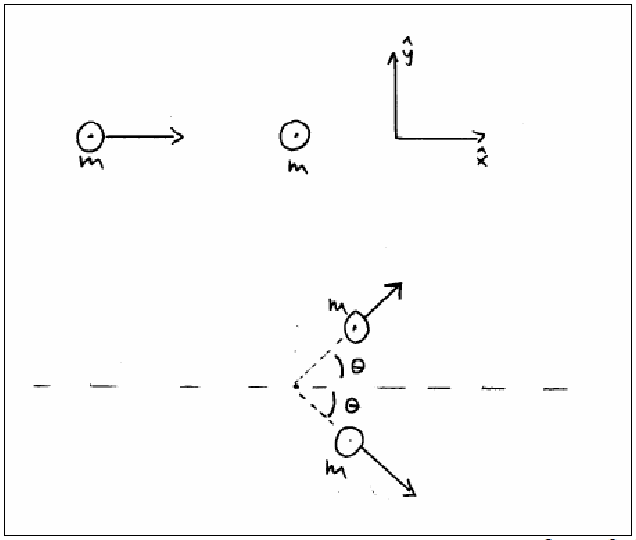
Solve for the acceleration.

$$\frac{du}{dt} = \frac{F}{m}\left(1-\frac{u^2}{c^2}\right)^{3/2}$$

The result is, that as $u \to c$, $\frac{du}{dt} \to 0$, no matter how large the applied force. At the other extreme, when $u \ll c$, $\frac{du}{dt} = \frac{F}{m}$.

b. Collisions—conservation of momentum

Consider the collision of two billiard balls. They have equal masses, m. Let's say that one ball is initially at rest while the second ball has momentum p_o and energy E_o before the collision. After the collision, both balls have the same energy, E, and mass, m. It's an elastic collision. Momentum and energy are conserved.



In the x direction, $p_o = 2p\cos\theta$. Substitute for p_o and p using $E^2 = p^2c^2 + m^2c^4$.

$$\frac{1}{c}\sqrt{E_o^2 - m^2c^4} = \frac{2}{c}\sqrt{E^2 - m^2c^4}\cos\theta$$

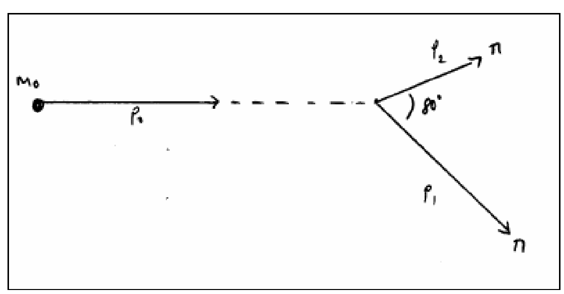
Conservation of energy allows us to eliminate E, since it was given that $E_o + mc^2 = 2E$. Keep in mind that E_o is the relativistic total energy of the second ball, while mc^2 is the rest energy of the first (target) ball. At the same time, we solve for $\cos\theta$, the cosine of the scattering angle.

$$\cos\theta = \frac{\sqrt{E_o^2 - m^2c^4}}{\sqrt{(E_o + mc^2)^2 - 4m^2c^4}} = \sqrt{\frac{(E_o + mc^2)(E_o - mc^2)}{(E_o + 3mc^2)(E_o - mc^2)}}$$

$$\cos\theta = \sqrt{\frac{E_o + mc^2}{E_o + 3mc^2}}$$

In the classical limit, $E_o \approx mc^2$ and therefore

$\cos\theta \approx \sqrt{\frac{2mc^2}{4mc^2}} = \sqrt{\frac{1}{2}} \Rightarrow \theta = 45^o$. But, as $E_o >> mc^2$, $\cos\theta \to 1 \Rightarrow \theta \to 0^o$!



c. Decay of a high-energy particle

An unidentified high-energy particle is observed to decay into two pions (π mesons), as shown.

Knowing the momenta and masses of the decay products, we determine the mass of the incident particle, hoping to identify it.

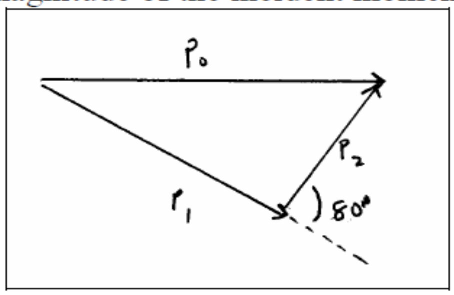
$$p_1 = 910\frac{MeV}{c},\ p_2 = 323\frac{MeV}{c},\ m_1c^2 = m_2c^2 = mc^2 = 139.6Mev\ .$$

The energy and momenta are conserved. The total energy is

$$E_o = E_1 + E_2 = \sqrt{p_1^2c^2 + m^2c^4} + \sqrt{p_2^2c^2 + m^2c^4}$$

$$E_o = 921MeV + 352MeV = 1273MeV$$

The quickest way to obtain the magnitude of the incident momentum is to use the law of cosines:



$$p_o^2c^2 = p_1^2c^2 + p_2^2c^2 - 2p_1p_2c^2\cos\theta = 1034229MeV^2$$

$$p_oc = 1017MeV$$

Now that we have the total energy and the kinetic energy, the mass is obtained from

$$E_o^2 = p_o^2c^2 + m_o^2c^4$$

$$m_oc^2 = \sqrt{E_o^2 - p_o^2c^2} = 765MeV$$

Evidently, the incident particle was a ρ meson. What was its speed before it decayed? Well, the total energy is also $E_o^2 = \dfrac{m_o^2c^4}{1-\dfrac{u^2}{c^2}}$, so solve that for u.

$$\frac{u}{c} = 1 - \sqrt{\frac{m_o^2c^4}{E_o^2}} = 0.8$$

d. Mass-energy equivalence

When we speak of the total energy being conserved that includes the total rest energy. For instance, consider the decay of a neutron that is initially at rest.

$$n \rightarrow p + e + \bar{\nu}$$

The neutron decays into a proton, an electron and an anti-neutrino. The three product particles are observed to have total kinetic energy of $K = 0.781\ MeV$. The initial energy is just the rest energy of the neutron, $E_i = 939.57\ MeV$. The total final energy is

$$E_f = m_pc^2 + m_ec^2 + K = 938.28MeV + 0.511MeV + 0.781MeV = 939.57MeV$$

Notes: i) The rest energy of the anti-neutrino is too small to bother with.

ii) Keep in mind the rounding of numbers and significant digits when substituting numerical values into the formulae.

iii) Notice that $m_n \neq m_p + m_e$. A portion of the neutron's rest energy has been converted into kinetic energy.

Solved Problems

Problem (1) : If a resident particle splits into two moving parts in opiate direction , their masses and velocities respectively are 3 Kg and 5.33 Kg and 0.6 c and 0.8 c. How much the mass of the original particle ?

Solution

$m_{o1} = 5.33 \text{ Kg}$, $m_{o2} = 3 \text{ Kg}$, $\upsilon_1 = 0.8c$, $\upsilon_2 = 0.6c$

Initial Energy = Final Energy

$$\because mc^2 = \frac{m_{o1}c^2}{\sqrt{1-\left(\frac{0.8c}{c}\right)^2}} + \frac{m_{o2}c^2}{\sqrt{1-\left(\frac{0.6c}{c}\right)^2}}$$

$$m_o = \frac{5.33}{\sqrt{1-0.64}} + \frac{3}{\sqrt{1-0.36}} = \frac{5.33}{0.6} + \frac{3}{0.8} = 12.63 \text{ Kg}$$

Problem (2) : Calculate the momentum of electron with kinetic energy equals one Million electron volt .

Solution

$$E = \sqrt{m_o^2 c^4 + P^2 c^2}$$

$$E^2 = m_o^2 c^4 + P^2 c^2$$

ergynergyE Kintic Energy + Potental = Total En

$$E = m_o c^4 + K$$

$$\therefore (m_o c^2 + 1\text{MeV})^2 = m_o^2 c^4 + P^2 c^2$$

$$(0.511+1)^2 = (0.511)^2 + P^2 c^2$$

$$1 + 2(0.511 + (0.511)^2 = (0.511)^2 + P^2 c^2$$

$$1 + 1.022 = P^2 c^2$$

$$P = \sqrt{\frac{2.022}{c^2}}$$

Problem (3) : An electron was accelerated from its rest position with velocity $0.5c$. Calculate the change in its kinetic energy.

Solution

$$E_0 = m_0 c^2 = 0.511 \text{ MeV}$$

At the speed $0.5\,c$:

$$E = mc^2 = \frac{m_0 c^2}{\sqrt{1-(\upsilon/c)^2}} = \frac{0.511 \text{ MeV}}{\sqrt{1-0.25}} = \frac{0.511}{\sqrt{0.95}}$$

$$E = 0.59 \text{ MeV}$$

$$K = 0.59 - 0.511 = 0.079 \text{ MeV}$$

Problem (4) : Calculate the Kinetic energy of the electron which its momentum equals $(2/c)$ Mev

Solution

$$E^2 = \left(m_0 c^2\right)^2 + (Pc)^2$$

$$\left(K + m_0 c^2\right)^2 = \left(m_0 c^2\right)^2 + (Pc)^2$$

$$(K + 0.511)^2 = \left(\frac{2 \text{ MeV}}{c} \times c\right)^2 + (0.511)^2$$

$$K = 1.55 \text{ MeV}$$

Problem (5) : Calculate the velocity of an electron its kinetic energy 2 MeV

Solution

$$K = (m - m_0)\, c^2$$

$$K = \frac{m_0 c^2}{\sqrt{1-(\upsilon/c)^2}} - m_0 c^2$$

$$2 \text{ MeV} = \frac{0.511 \text{ MeV}}{\gamma} - 0.511 \text{ MeV}$$

$$\gamma = \frac{0.511}{2.511} = 0.203$$

$$\gamma = \frac{\upsilon}{c} \quad \Rightarrow \quad \upsilon = \gamma\, c$$

$$\upsilon = 0.89\,c$$

Problem (6) :

a- Calculate the mass and the velocity, if its kinetic energy equals 1.5 MeV

b- What is the energy by the unit of electron volt which required for the electron to transfer from its rest position to a velocity equals (0.9 C)

Solution

$$K = (m - m_0)\,c^2 \qquad (1)$$

$$m = \frac{m_0}{\sqrt{1-(\upsilon/c)^2}} \qquad (2)$$

Frome Eq. (1) : $m = m_0 + K/c^2$

The electron mass : $m_0 = 9.1\times10^{-31}$ Kg

Light speed : $c = 3\times10^{8}$ m/Sec

$$K = 1.5\,\text{MeV} = 1.5\times10^{6}\ \text{eV} = 1.5\times10^{6}\,(1.6\times10^{-19})$$

$$K = 1.5\times1.6\times10^{-13}\ \text{Joule}$$

$$m = (9.11\times10^{-31}) + \frac{1.5\times1.6\times10^{-13}}{(9\times10^{16})}$$

$$m = (9.11\times10^{-31}) + (26.7\times10^{-31})$$

$$m = 35.8\times10^{-31}\ \text{Kg}$$

The value of moving mass equals four time it's a rest mass

The velocity of the electron from Eq (2)

$$m = \frac{m_0}{\sqrt{1-(\upsilon/c)^2}}$$

Squaring the both side of this equation :

$$1-(\upsilon/c)^2 = (m_0/m)^2$$

$$\frac{\upsilon^2}{c^2} = 1 - \left(\frac{m_0}{m}\right)^2 \quad \Rightarrow \quad \frac{\upsilon}{c} = \sqrt{1 - \left(\frac{m_0}{m}\right)^2}$$

$$\upsilon = c\sqrt{1 - \left(\frac{m_0}{m}\right)^2} = 3\times10^8 \times \sqrt{1 - \left(\frac{9.11}{35.8}\right)^2}$$

$$\upsilon = 3\times10^8 \times \sqrt{1 - 0.065} = 3\times10^8 \times \sqrt{0.935}$$

$$\upsilon = 3\times10^8 \times 0.968 = 2.9\times10^8 \text{ m/Sec}$$

(b) : Required energy for transferring the electron from rest to 0.9 c si :

$$K = (m - m_0)\, c^2$$

$$m = \frac{m_0}{\sqrt{1 - (\upsilon/c)^2}}$$

$$K = \left(\frac{m_0}{\sqrt{1 - (\upsilon/c)^2}} - m_0\right) c^2$$

$$K = \left(\frac{1}{\sqrt{1 - (\upsilon/c)^2}} - 1\right) m_0 c^2$$

$$\upsilon = 0.9\,c \qquad \therefore (\upsilon/c)^2 = 0.81$$

$$\frac{1}{\sqrt{1 - (\upsilon/c)^2}} = \frac{1}{\sqrt{1 - 0.81}} = \frac{1}{\sqrt{0.19}} = 2.29$$

$$K = (2.29 - 1)\, m_0 c^2 = 1.29\; m_0 c^2$$

$$K = 1.29\times9.11\times10^{-31} \times 9\times10^{16}$$

$$K = 10.6\times10^{-14} \text{ Joule} = \frac{10.6\times10^{-14}}{1.6\times10^{-19}} = 6.63\times10^5 \text{ eV}$$

Problem (7)

Relativistic Form of Newton's Second Law

Consider the relativistic form of Newton's Second Law.

- Show that when **F** is parallel to **v**

$$F = m\left(1 - \frac{v^2}{c^2}\right)^{-3/2}\frac{dv}{dt} \tag{33}$$

where m is the mass of the object and v is its speed.

Solution

The force **F** on a particle with rest mass m is the rate of change its momentum **p** as given by text Eq. (1.36):

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{34}$$

where as given by text Eq. (1.35):

$$\mathbf{p} = \gamma m\,\mathbf{v} \tag{35}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{36}$$

where **v** is the velocity of the particle. Eq. (34) with **p** given by Eq. (35) is the relativistic generalization of Newton's Second Law.

Now

$$\mathbf{v} = v\mathbf{u}_t \tag{37}$$

where $\mathbf{u}_t$ is a unit vector along the tangent of the particle's trajectory, and the acceleration **a** of the particle is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dv}{dt}\mathbf{u}_t + \frac{v^2}{\rho}\mathbf{u}_n \tag{38}$$

where $\mathbf{u}_n$ is a unit vector orthogonal to $\mathbf{u}_t$ directed towards the centre of curvature of the trajectory and ρ is the radius of curvature of the trajectory, so

$$\frac{d\mathbf{p}}{dt} = \gamma m\left(\gamma^2\frac{dv}{dt}\mathbf{u}_t + \frac{v^2}{\rho}\mathbf{u}_n\right) \tag{39}$$

When

$$\mathbf{F} = F\mathbf{u}_t \tag{40}$$

that is, when **F** is parallel to **v**, if follows that

$$\rho = \infty \tag{41}$$

That is, the particle moves in a straight line, and

$$a = \tilde{a}/\gamma^3 \tag{42}$$

where

$$a = \frac{dv}{dt} \tag{43}$$

and

$$\tilde{a} = \frac{F}{m} \tag{44}$$

Problem (8)

Relativistic Form of Newton's Second Law: Particle in an Electric Field

A charged particle moves along a straight line in a uniform electric field **E** with speed v.

• If the motion and the electric field are both in the x direction, show that the magnitude of the acceleration of the charge q is given by

$$a = \frac{dv}{dt} = \frac{qE}{m}\left(1 - \frac{v^2}{c^2}\right)^{3/2} \tag{45}$$

• Discuss the significance of the dependence of the acceleration on the speed.

• If the particle starts from rest $x = 0$ at $t = 0$, find the speed of the particle and its position after a time t has elapsed.

• Comment of the limiting values of v and x as $t \to \infty$.

Solution

When a particle of charge q moves in an electric field **E**, the force **F** on the particle is **F** = q**E**. If the particle moves in the direction of **E**, then **F** and **v** are parallel. Accordingly, Eq. (42) holds with $\tilde{a} = qE/m$.

It follows from Eq. (42) that $a \to 0$ as $v \to c$ and also that

$$F \simeq ma \quad \text{when } v \ll c \tag{46}$$

which is the nonrelativistic result.

When F is constant and the particle starts from rest at $t = 0$, its speed $v(t)$ is found by integrating Eq. (42):

$$\int_0^{v(t)} \frac{dv'}{(1 - v'^2/c^2)^{3/2}} = \tilde{a}t \tag{47}$$

to be

$$v(t) = \frac{\tilde{a}t}{\sqrt{1 + (\tilde{a}t/c)^2}}. \tag{48}$$

It follows that $v(t) \to c$ as $t \to \infty$ and also that

$$v(t) \simeq \tilde{a}t \quad \text{when } \tilde{a}t \ll c \tag{49}$$

which is the non-relativistic result.

The position $x(t)$ of the particle is found by integrating $v = dx/dt$:

$$x(t) = \int_0^t v(t')dt' = \left(\sqrt{1 + (\tilde{a}t/c)^2} - 1\right) c^2/\tilde{a}. \tag{50}$$

It follows that $x \to ct$ as $t \to \infty$ and also that

$$x(t) \simeq \frac{1}{2}\tilde{a}t^2 \quad \text{when } \tilde{a}t \ll c \tag{51}$$

which is the nonrelativistic result.

The position $x(v)$ is found by integrating $a(v) = dv/dt = vdv/dx$:

$$x(v) - x(0) = \int_{x(0)}^{v} \frac{v'dv'}{a(v')} = (\gamma - 1)c^2/\tilde{a}. \tag{52}$$

It follows that $x(v) - x(0) \to \infty$ as $v \to c$ and also that

$$v^2 \simeq 2\tilde{a}[x(v) - x(0)] \quad \text{when } v \ll c \tag{53}$$

which is the nonrelativistic result.

Problem (9)

Relativistic Form of Newton's Second Law: Particle in a Magnetic Field

The force $\mathbf{F}$ on a particle with rest mass m and charge q moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ is

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}. \tag{54}$$

• If the particle moves in a circular orbit with a fixed speed v in the presence of a constant magnetic field, use Newton's Second Law to show that the frequency of its orbital motion is

$$f = \frac{qB}{2\pi m}\left(1 - \frac{v^2}{c^2}\right)^{1/2} \tag{55}$$

Solution

When a particle moves with constant speed, that is, when

$$\frac{dv}{dt} = 0 \tag{56}$$

it follows from Eqs. (34) and (39) that

$$\tilde{\omega}\rho = \gamma v \tag{57}$$

where

$$\tilde{\omega} = \frac{qB\sin\theta}{m} \tag{58}$$

where θ is the angle that $\mathbf{v}$ makes with $\mathbf{B}$. For a proton moving perpendicular to a 1.00 T magnetic field, $\tilde{\omega} = 95.8$ MHz.

The right side of Eq. (57) is constant. Accordingly, the radius of curvature ρ of the particle's trajectory changes to accommodate changes in the magnetic field $\mathbf{B}$.

When $\mathbf{B}$ is constant (that is, time-independent and homogeneous), it follows from Eq. (57) that the particle moves in a circle with radius

$$r = \gamma v/\tilde{\omega} \tag{59}$$

and speed

$$v = \tilde{\omega}r/\gamma = \frac{\tilde{\omega}r}{\sqrt{1 + (\tilde{\omega}r/c)^2}}. \tag{60}$$

The angular frequency $\omega = v/r$ of the circular motion is

$$\omega = \tilde{\omega}/\gamma = \frac{\tilde{\omega}}{\sqrt{1 + (\tilde{\omega}r/c)^2}} \tag{61}$$

as above.

It follows that $r \to \infty$ as $v \to c$ and and also that

$$v \simeq \tilde{\omega} r \quad \text{when } v \ll c \tag{62}$$

which is the nonrelativistic result.

It follows also that

$$\omega \simeq \tilde{\omega} \quad \text{when } \tilde{\omega} r \ll c \tag{63}$$

For a proton moving perpendicular to a 1.00 T magnetic field, this requires that $r \ll 3.13\,\text{m}$.

The above results limit the range of speeds attainable in a conventional particle-accelerating cyclotron which relies, as with Eq. (63), on a constant-frequency accelerating potential to increase particle speeds and a time-independent homogeneous magnetic field to make particles move in circles.

This limitation is overcome at the TRIUMF cyclotron on the UBC campus which accelerates protons to 520 MeV ($0.75c$), and has a diameter of 17.1 m. This is accomplished by increasing the magnetic field with radius to accommodate the Lorentz factor γ. For more information on TRIUMF, see http://www.triumf.ca.

Problem (10)

Relativistic Form of Newton's Second Law: Particle in a Magnetic Field

• Show that the momentum of a particle having charge e moving in a circle of radius R is given by $p = 300BR$ where p is in MeV/c, B is in teslas and R is in meters.

Solution

It follows from Eqs. (35) and (59) that the momentum p of a particle of charge e moving perpendicular to a constant magnetic field B is

$$p = eBR \tag{64}$$

where R is the radius of the circular orbit. Using $e = 1.602 \times 10^{-19}$ C, MeV $= 1.602 \times 10^{-13}$ J and $c = 3.00 \times 10^{8}$ m/s, it follows that

$$p = 300BR \tag{65}$$

where p is in MeV/c, B is in teslas and R is in meters.

Problem (11)

Relativistic Kinematics: Energy-Momentum Relationship

• Show that the energy-momentum relationship $E^2 = p^2c^2 + m^2c^4$ follows from $E = \gamma mc^2$ and $p = \gamma mv$.

Solution

See also *Notes on a Few Topics in Special Relativity* by Malcolm McMillan.

It follows from $p = \gamma mv$ and Eq. (36) that

$$v = \frac{c}{\sqrt{1 + (mc/p)^2}} \tag{66}$$

and

$$\gamma = \sqrt{1 + (p/mc)^2} \tag{67}$$

which with $E = \gamma mc^2$ yields

$$E = \sqrt{p^2c^2 + m^2c^4} \tag{68}$$

It follows from Eq. (66) that a particle with rest mass $m = 0$ travels at the speed of light c.

It follows from Eq. (68) that the energy E and momentum p of a particle with rest mass $m = 0$ are related by

$$E = pc \tag{69}$$

Problem (12)

Relativistic Kinematics for an Electron

Electrons in projection television sets are accelerated through a potential difference of 50 kV.

- Calculate the speed of the electrons using the relativistic form of kinetic energy assuming the electrons start from rest.
- Calculate the speed of the electrons using the classical form of kinetic energy.
- Is the difference in speed significant in the design of this set?

Solution

A particle with rest mass m moving with speed v has kinetic energy K given by text Eq. (1.42):

$$K = (\gamma - 1)mc^2 \tag{70}$$

where γ is given by Eq. (36) from which it follows that

$$v = c\sqrt{1 - [1 + (K/mc^2)^{-2}]}. \tag{71}$$

It follows from Eq. (70) that

$$K \simeq \frac{1}{2}mv^2 \tag{72}$$

when $v \ll c$, which is the nonrelativistic result.

An electron (rest energy 511 keV) moving through a potential difference $V = 50$ kV acquires a kinetic energy $K = 50$ keV.

It follows from Eq. (71) that $v = 0.413c$. The nonrelativistic expression Eq. (72) yields $v = 0.442c$ which is 6% greater than the correct relativistic result.

Does it make a difference which number is used when building a TV set? If the distance between the filament off which electrons are boiled and the phosphor screen of a TV set is 50 cm, then the difference in travel times calculated relativistically and nonrelativistically is 26 ns. This time difference is too small to make a significant difference in the operation of a TV set.

Problem (13)

Relativistic Kinematics: Lorentz Invariant

The quantity $E^2 - p^2c^2$ is an invariant quantity in Special Relativity. This means that $E^2 - p^2c^2$ has the same value in all inertial frames even though E and p have different values in different frames.

• Show this explicitly by considering the following case: A particle of mass m is moving in the $+x$ direction with speed u and has momentum p and energy E in the frame S. If S' is moving at speed v in the standard way, determine the momentum p' and energy E' observed in S', and show that $E'^2 - p'^2c^2 = E^2 - p^2c^2$.

Solution

See also *Notes on a Few Topics in Special Relativity* by Malcolm McMillan.

In frame S, a particle of rest mass m has velocity $\mathbf{u}$. Its momentum $\mathbf{p}$ and energy E are given by text Eqs. (1.35) and (1.44):

$$\mathbf{p} = \frac{m\mathbf{u}}{\sqrt{1 - u^2/c^2}} \tag{73}$$

$$E = \frac{mc^2}{\sqrt{1 - u^2/c^2}}. \tag{74}$$

In frame S', which moves along the x-axis of S with speed v, the velocity of the particle is $\mathbf{u}'$ and its momentum $\mathbf{p}'$ and energy E' are

$$\mathbf{p}' = \frac{m\mathbf{u}'}{\sqrt{1 - u'^2/c^2}} \tag{75}$$

$$E' = \frac{mc^2}{\sqrt{1 - u'^2/c^2}}. \tag{76}$$

It follows from Eqs. (73) to (76) that

$$E'^2 - p'^2c^2 = E^2 - p^2c^2 = m^2c^4. \tag{77}$$

The relationship between the momentum and energy in S and S' follows from the Lorentz velocity transformations given by text Eqs. (1.32) to (1.34):

$$u'_x = \frac{u_x - v}{1 - u_x v/c^2} \tag{78}$$

$$u'_y = \frac{u_y}{\gamma(1 - u_x v/c^2)} \tag{79}$$

$$u'_z = \frac{u_z}{\gamma(1 - u_x v/c^2)} \tag{80}$$

where γ is given by Eq. (2). It follows that

$$\frac{1}{\sqrt{1 - u'^2/c^2}} = \frac{\gamma(1 - u_x v/c^2)}{\sqrt{1 - u^2/c^2}} \tag{81}$$

so

$$p'_x = \gamma(p_x - \beta p_0) \tag{82}$$

$$p'_y = p_y \tag{83}$$

$$p'_z = p_z \tag{84}$$

$$p'_0 = \gamma(p_0 - \beta p_x) \tag{85}$$

where

$$p_0 = E/c. \tag{86}$$

Eqs. (82) and (85) may be written as

$$p'_x = p_x \cosh u - p_0 \sinh u \tag{87}$$

$$p'_0 = p_0 \cosh u - p_x \sinh u \tag{88}$$

where u is the rapidity:

$$\tanh u = \beta \tag{89}$$

so $\cosh u = \gamma$ and $\sinh u = \beta\gamma$, from which

$$p_x = p'_x \cosh u + p'_0 \sinh u \tag{90}$$

$$p_0 = p'_0 \cosh u + p'_x \sinh u \tag{91}$$

We note that the first and fourth Lorentz spacetime transformations text Eqs. (1.25) to (1.28):

$$x' = \gamma(x - \beta x_0) \tag{92}$$

Problem (14)

Einstein Mass-Energy Relationship for the Decay of the Neutron

The free neutron is known to decay into a proton, an electron and an antineutrino (of zero rest mass) according to

$$n \to p + e^- + \bar{\nu}. \tag{102}$$

This is called *beta decay*. The decay products are measured to have a total kinetic energy of (0.781 ± 0.005) MeV.

- Show that this observation is consistent with the Einstein mass-energy relationship.

Solution

Using mass values given in text Appendix A (m_n=939.5656 MeV/c^2, m_p=938.2723 MeV/c^2, m_e=0.5110 MeV/c^2), it follows that

$$\Delta m = m_n - (m_p + m_{e^-} + m_{\bar{\nu}}) = 0.7823\,\text{MeV}/c^2. \tag{103}$$

By conservation of mass-energy, this decrease in rest mass energy is converted into total kinetic energy Q of the decay products as per text Eq. (1.50): $Q = \Delta mc^2 = 0.7823\,\text{MeV}$. This result is consistent with the observed value of (0.781 ± 0.005) MeV.

Problem (15)

Conservation of Energy and Momentum in Electron-Positron Annihilation

An electron e^- with kinetic energy 1.000 MeV makes a head-on collision with a positron e^+ at rest. (A positron is an antimatter particle that has the same mass as the electron but opposite charge.)

In the collision the two particles annihilate each other and are replaced by two photons of equal energy, each traveling at angles θ with the electron's direction of motion. (A photon γ is a massless particle of electromagnetic radiation having energy $E = pc$.) The reaction is

$$e^- + e^+ \to 2\gamma \tag{104}$$

- Determine the energy E , momentum p and angle of emission θ of each photon.

Solution

The incident electron, with rest mass $m = 0.511\text{MeV}/c^2$, has momentum p along the positive x-axis and kinetic energy K. It follows from Eqs. (67) and (70) that

$$p = \sqrt{K(K + 2m^2c^4)}/c \tag{105}$$

from which $p = 1.422\ \text{MeV}/c$ when $K = 1.000\ \text{MeV}$.

The total energy E of the electron and the stationary positron before the collision is

$$E = K + 2mc^2 = 2.022\ \text{MeV}. \tag{106}$$

The two photons emerge from the collision each with energy

$$E_\gamma = \frac{1}{2}E = 1.011\ \text{MeV} \tag{107}$$

as given by conservation of energy, and, using Eq. (69), each with magnitude of momentum

$$p_\gamma = E_\gamma/c = 1.011\ \text{MeV}/c. \tag{108}$$

The momentum vectors of the photons make angles $\pm\theta$ with the x-axis. Conservation of momentum in the x-direction is

$$p = 2p_\gamma \cos\theta \tag{109}$$

from which $\theta = 45.3°$.

Problem (16)

Conservation of Energy and Momentum in Neutral Kaon Decay

The K^0 meson decays into two charged pions according to

$$K^0 \to \pi^+ + \pi^- \tag{110}$$

The pions have equal and opposite charges as indicated and the same rest mass $m_\pi = 140\ \text{MeV}/c^2$.

Suppose that a K^0 at rest decays into two pions in a bubble chamber in which a magnetic field B=2.0 T is present.

- If the radius of curvature of the pions is 34.4 cm, determine the momenta and speeds of the pions and the rest mass of the K^0.

Solution

It follows from Eq. (65) that the momentum of each pion is $p = 206$ MeV/c.

It follows from Eq. (66) that the speed of each pion is $v = 0.827c$.

It follows from Eq. (68) that the energy of each pion is $E = 249$ MeV.

Conservation of energy:

$$m_K c^2 = 2E \tag{111}$$

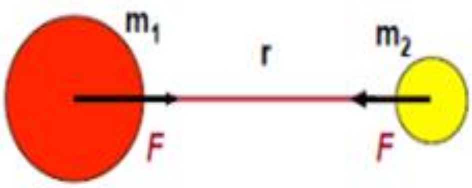
yields $m_K = 498$ MeV/c^2.

Chapter (4)

General Relativity

Chapter (4)
General Relativity

Newton's law of gravitation



$$F = G\frac{m_1 m_2}{r^2} \quad (A.25)$$

We know there are **4 forces of nature**:

- Gravity, Electromagnetism, Weak & Strong Nuclear forces
- Gravity is by far the weakest force, but it is also the most obvious: it's universal, acting the same on all forms of matter

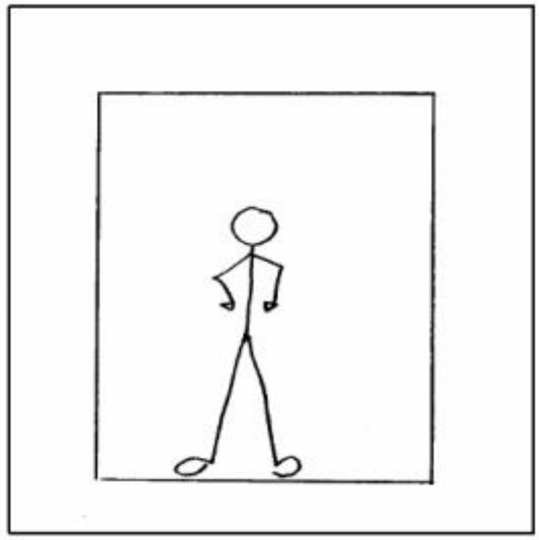
Einstein realized there that is an equivalence between gravity and acceleration: you are weightless in a plummeting elevator. This is the **equivalence principle.**

Another form of **Einstein's equivalence principle**: an observer inside an enclosed box cannot tell the difference between being at rest on Earth's surface (a) or being accelerated in outer space (b).

1. Equivalence

In Special Relativity it is asserted that all inertial reference frames are equivalent—the "laws" of physics are the same in all inertial reference frames. No experiment done in one frame can detect its uniform motion relative to another frame. Can the same be said for reference frames that have a relative acceleration?

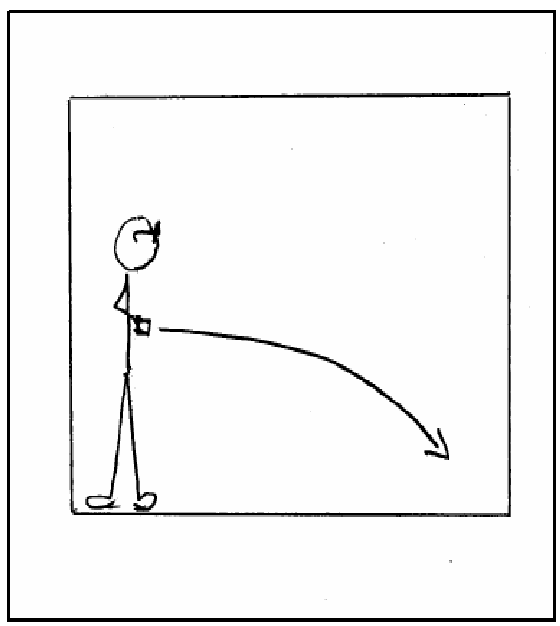
a. Elevator



Recall the past discussion of a person standing in an elevator. If the elevator moves perfectly smoothly and there are no floor indicator lights, then the person inside will have no perception of the elevator's motion, except for feeling perhaps the elevator floor pressing upward on his or her feet. [Keep in mind: the person gets no information from any source outside the reference frame of the elevator.] Contrast this situation with that of another person standing in a similar elevator, but this elevator is simply resting level on the Earth's surface. The person in this elevator also feels the floor pressing upward on his or her feet, also has no perception of the elevator's motion. We, as omniscient external observers, know that this second elevator is resting on the surface of a planet, and that what the person inside is experiencing is the gravitational force exerted by that planet. The point is that there is no experiment that either of the persons inside the elevators could perform that would distinguish between the two situations. Pendula would swing back and forth just the same; projectiles would follow the same kinds of arcs, etc.

b. Light and gravity

Imagine ourselves as observers far from any source of gravitational force. Nearby, we observe a closed "elevator" which is accelerating, relative to us, at a constant rate, $\vec{a}_o$. A person standing inside the "elevator" sends a series of light pulses toward one wall—he or she and we see the light pulses dropping toward the floor as they approach the wall. The light follows a curved path inside the elevator.



The Postulate of General Relativity asserts that the "laws" of physics have the same form for observers in any frame of reference, regardless of its acceleration relative to another frame. We have seen that an accelerated frame is equivalent to one in a gravitational field. It follows that the force of gravity must affect a beam of light just as it affects the motion of a massive projectile. Indeed, experiment has shown that it does. But, light has no mass.

2. Curvature

Classically, we would say that a mass, such as a planet, exerts a gravitational force on another mass, such as a moon or a person. However, a person in an "elevator" cannot determine whether his or her "elevator" is in the gravitational field of a planet or is being accelerated at a constant rate by, say rocket motors. If the "elevator" is in a gravitational field, we can nonetheless mathematically transform the "laws" of physics into versions of the same mathematical form that do not include gravity yet which make equally accurate predictions of the motions of particles and of light beams.

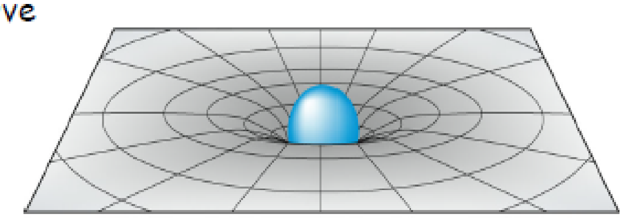
What Einstein did was to formulate such a version of the "laws" of motion. Objects and light beams move always in straight lines, but in a curved space-time. Empty space-time is flat, but the presence of mass at any location curves space-time to a degree proportional to the amount of mass that is present.

In 1916 Einstein published the final form of the **General Theory of Relativity**.

We can think of gravity as a feature of the background in which we live. This background is space and time: **spacetime**

What we experience as gravity is actually **the curvature of spacetime**

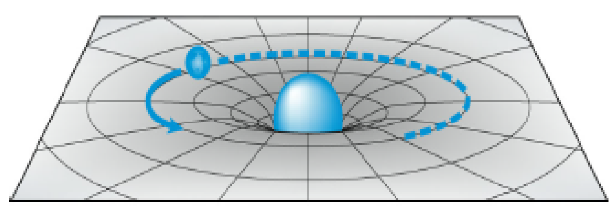
In general relativity (GR), matter warps space-time, so that the straightest and shortest path *(geodesic)* looks like a curve to us.



Mass tells space how to curve.

Space tells matter how to move.

The figure shows an analogy: weight on a tight rubber sheet depresses it (a) , so a ball is deflected around it (b). That is how GR describes the motion of a planet around the Sun, and not by means of a force, as implied by Newton's gravitational force, Eq. (A.25). However, Einstein showed us that Newton's law is a limit of GR for small masses.



We know how to describe motion of objects exactly (remember rocket science) using Newton's gravitational law. There must be a way describe exactly the motion without forces, according to GR.

Well, it is complicated.

I will give a very short tour of GR next.

The **space-time interval**, Δs defined as

$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2\Delta t^2 \qquad (A.26)$$

is Lorentz invariant. That is, if we use the Lorentz transformations with Eq. (A. 13), with $\Delta x = dx$, $\Delta t = dt$, etc., we get $\Delta s'^2 = \Delta s^2$.

This interval can be written in terms of the **space-time metric**

$$\Delta s^2 = \begin{bmatrix} c\Delta t & \Delta x & \Delta y & \Delta z \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \qquad (A.27)$$

The space-time metric

We can rewrite the expression for the space-time interval

$$\Delta s^2 = \sum_{\mu=0,\dots,3} \sum_{\nu=0,\dots,3} \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu \qquad (A.28)$$

where

$$x^0 = ct \quad x^1 = x \quad x^2 = y \quad x^3 = z \qquad (A.29)$$

and

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (A.30)$$

(rows and columns indexed 0, 1, 2, 3)

It is economical to use the **Summation Notation**: the summed indices occur once as **subscripts** and again as **superscripts**:

$$\Delta s^2 = \sum_{\mu=0,\dots,3} \sum_{\nu=0,\dots,3} \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu \qquad (A.31)$$

Use of Greek letters means v = 0, 1, 2, 3

$$\Delta s^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu \qquad (A.32)$$

When the same index appears as a superscript and a subscript, summation is assumed, and we can omit the summation symbols.

Tensors

In General Relativity, space is curved, and the space-time metric can be more complex. The more general metric coefficients of general relativity (which may not be -1's, 0's, and 1's) are denoted by $g_{\mu\nu}$:

$$\Delta s^2 = g_{\mu\nu}\,\Delta x^{\mu}\Delta x^{\nu} \qquad (A.33)$$

Example: An expanding (flat) universe

$$\Delta s^2 = a(t)^2\,[\Delta x^2 + \Delta y^2 + \Delta z^2] - c^2\Delta t^2 \quad \text{where } a(t) \sim t^{\,q} \qquad (A.34)$$

Values of q range from *1/2* (in a radiation-dominated universe) to *2/3* (in a matter-dominated universe). **(See Lecture 4).**

$g_{\mu\nu}$ is an tensor. A **tensor** is a function of one or more vectors that yields a real number. $g_{\mu\nu}$ takes two input vectors and yields a number: the interval Δs^2.

Because $g_{\mu\nu}$ operates on two vectors, we say it's a tensor of **rank 2**.

Example: a **vector** can undergo dot products with other vectors to yield a number, so **it's a tensor of rank 1**. **Scalars have rank zero**.

The rank is also the number of indices on the tensor and the dimension of the matrix necessary to write it down.

Geodesics

GR distinguishes between vectors and tensors that are **covariant** (with lower indices) and **contravariant** (with upper indices). To raise or lower an index, simply multiply by the metric:

$$x_{\mu} = g_{\mu\nu}x^{\nu} \qquad (A.35) \qquad\qquad \Gamma^{\alpha}_{\ \mu\kappa} = g_{\mu\nu}\Gamma^{\alpha\nu}_{\ \ \ \kappa} \qquad (A.36)$$

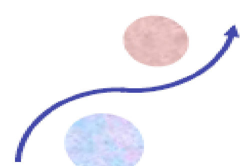
Ordinarily, we don't usually have to worry about this because our metric is simple, and covariant and contravariant tensors are essentially the same.

To raise the indices of the metric $g_{\mu\nu}$ itself, just take its inverse

$$g^{\mu\nu} = [g]^{-1}_{\ \ \mu\nu} \qquad (A.37)$$

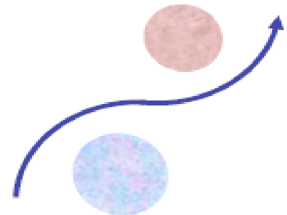
In Newtonian space, **geodesics** are straight lines, and one way of saying this is that acceleration is zero

$$\frac{d^2x^{\alpha}}{d\tau^2} = 0 \qquad (A.38)$$



where τ is **proper time** (i.e., the time measured in the frame of reference of the particle) , and x^{α} is the position vs. τ of the particle.

Curved Spaces

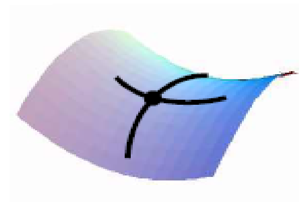


In curved space, this expression generalizes to

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} \qquad (A.39)$$

where $\Gamma^\alpha_{\beta\gamma}$ is called a **Christoffel symbol**, given by

$$g_{\alpha\delta}\Gamma^\delta_{\beta\gamma} = \tfrac{1}{2}\left(\frac{dg_{\alpha\beta}}{dx^\gamma} + \frac{dg_{\alpha\gamma}}{dx^\beta} - \frac{dg_{\beta\gamma}}{dx^\alpha}\right) \qquad (A.40)$$



The curvature of space-time is complicated because there are several dimensions, and the curvature at each point can be different in each dimension (including time). Think of a saddle in two dimensions for which the curvature depends on the direction.

The curvature of space-time is given by the **Ricci Tensor**

$$R_{\alpha\beta} = \frac{d\Gamma^\gamma_{\alpha\beta}}{dx^\gamma} - \frac{d\Gamma^\gamma_{\alpha\gamma}}{dx^\beta} + \Gamma^\gamma_{\alpha\beta}\Gamma^\delta_{\gamma\delta} - \Gamma^\gamma_{\alpha\delta}\Gamma^\delta_{\beta\gamma} \qquad (A.41)$$

Einstein Tensor

The **Einstein tensor** can be written in terms of the Ricci tensor as

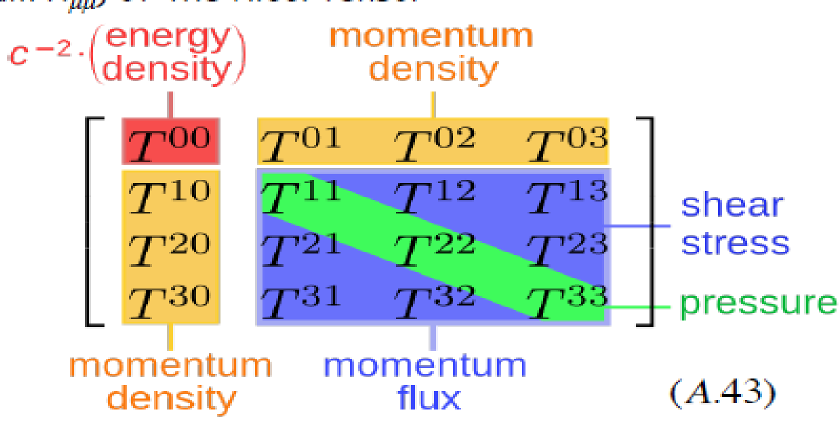
$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}R\,g_{\mu\nu} \qquad (A.42)$$

where R is the trace (i.e., the sum $R_{\mu\mu}$) of the Ricci tensor

Matter's effect on space-time occurs through the **stress-energy tensor**, T.

$T_{00} = T_{tt}$ is the **mass-energy density**

$T_{10} = T_{xt}$, $T_{20} = T_{yt}$ and $T_{30} = T_{zt}$ are how fast the matter is moving — its **momentum**



(A.43)

$T_{11} = T_{xx}$, $T_{22} = T_{yy}$ and $T_{33} = T_{zz}$ are the **pressures** in each of the three directions $T_{12} = T_{xy}$, $T_{13} = T_{xz}$ and $T_{23} = T_{yz}$ are the **stresses in the matter**.

Einstein Field Equations

The following set of coupled nonlinear partial differential equations (one for each element) relates the curvature of space, $G_{\mu\nu}$, to the energy-momentum tensor, $T_{\mu\nu}$:

$$G_{\mu\nu} = \frac{8\pi}{c^4} G\, T_{\mu\nu} \qquad (A.44)$$

Only six component equations are independent.

where G is the usual gravitational constant.

The goal is to solve for $g_{\mu\nu}$, for all values of μ and ν. In free space, where $T_{\mu\nu} = 0$, this reduces to

$$R_{\mu\nu} = 0 \qquad (A.45)$$

One can show that Einstein's Field Equations reduce to Newton's law of gravity in the weak-field and slow-motion limit.

As mentioned in Lecture 3, Einstein introduced the **Cosmological constant** by modifying his equation to

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = \frac{8\pi}{c^4} G\, T_{\mu\nu} \qquad (A.46)$$

Ex: The Schwarzschild Solution

Using spherical coordinates, ρ, θ, ϕ, and spherical symmetry, we can solve Einstein's Field Equations (with $\Lambda = 0$) for the metric to find

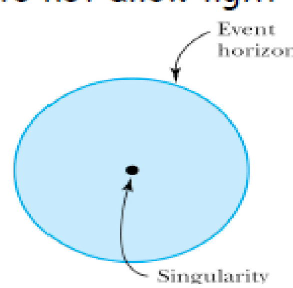
$$\Delta s^2 = \left(1 - \frac{2GM}{rc^2}\right)^{-1} \Delta r^2 + r^2 \Delta\Omega^2 - \left(1 - \frac{2GM}{rc^2}\right) \Delta t^2 \qquad (A.47)$$

The other elements of $g_{\mu\nu}$ are zero, and $\Delta\Omega^2 = \Delta\theta^2 + \sin^2\theta\, \Delta\phi^2$

Note that, when $R_s = 2GM/c^2$ (called the **Schwarzschild radius**), this becomes

$$\Delta s^2 = \infty\, \Delta r^2 + r^2 \Delta\Omega^2 - 0\, \Delta t^2 \qquad (A.48)$$

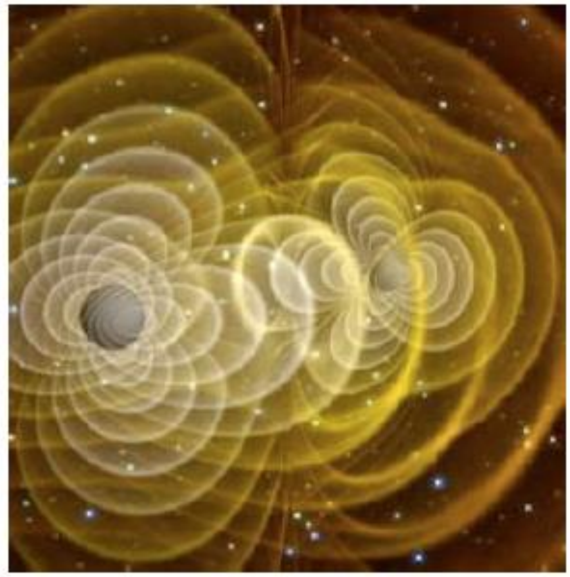
When a star's thermonuclear fuel is depleted, no heat is left to counteract the force of gravity, which becomes dominant. The star's mass collapses into an incredibly dense ball that could warp space-time enough to not allow light to escape. The point at the center is called a ***singularity***.



A collapsing star greater than 3 solar masses will distort space-time in this way to create a **black hole**.

Schwarzschild determined the radius of a black hole, known as the ***event horizon***. The Schwarzschild radius is given by Eq. (4.24) in Lecture 4.

Gravitational Waves



When a **charge accelerates**, the electric field surrounding the charge redistributes itself. This change in the electric field **produces an electromagnetic wave**, which is easily detected. Similarly, an **accelerated mass** should also **produce gravitational waves**.

Gravitational waves carry energy and momentum, travel at the speed of light, and are characterized by frequency and wavelength.

As gravitational waves pass through space-time, they cause small ripples. The stretching and shrinking is on the order of 1 part in 10^{21} even due to a strong gravitational wave source.

Due to their small magnitude, gravitational waves are difficult to detect. Large astronomical events could create measurable space-time waves such as the collapse of a neutron star, a black hole or the Big Bang.

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