APPLIED(2) COURSE

Statics & Mechanics

For the Student of

Faculty of Education Level 1

STATICS PART

Contents

Introduction

Education Is Not The Learning of Facts; I's Rather The Traininig Of The Mind To Think (Albert Einstein)

1 Review

2 Force

2.1 What is force

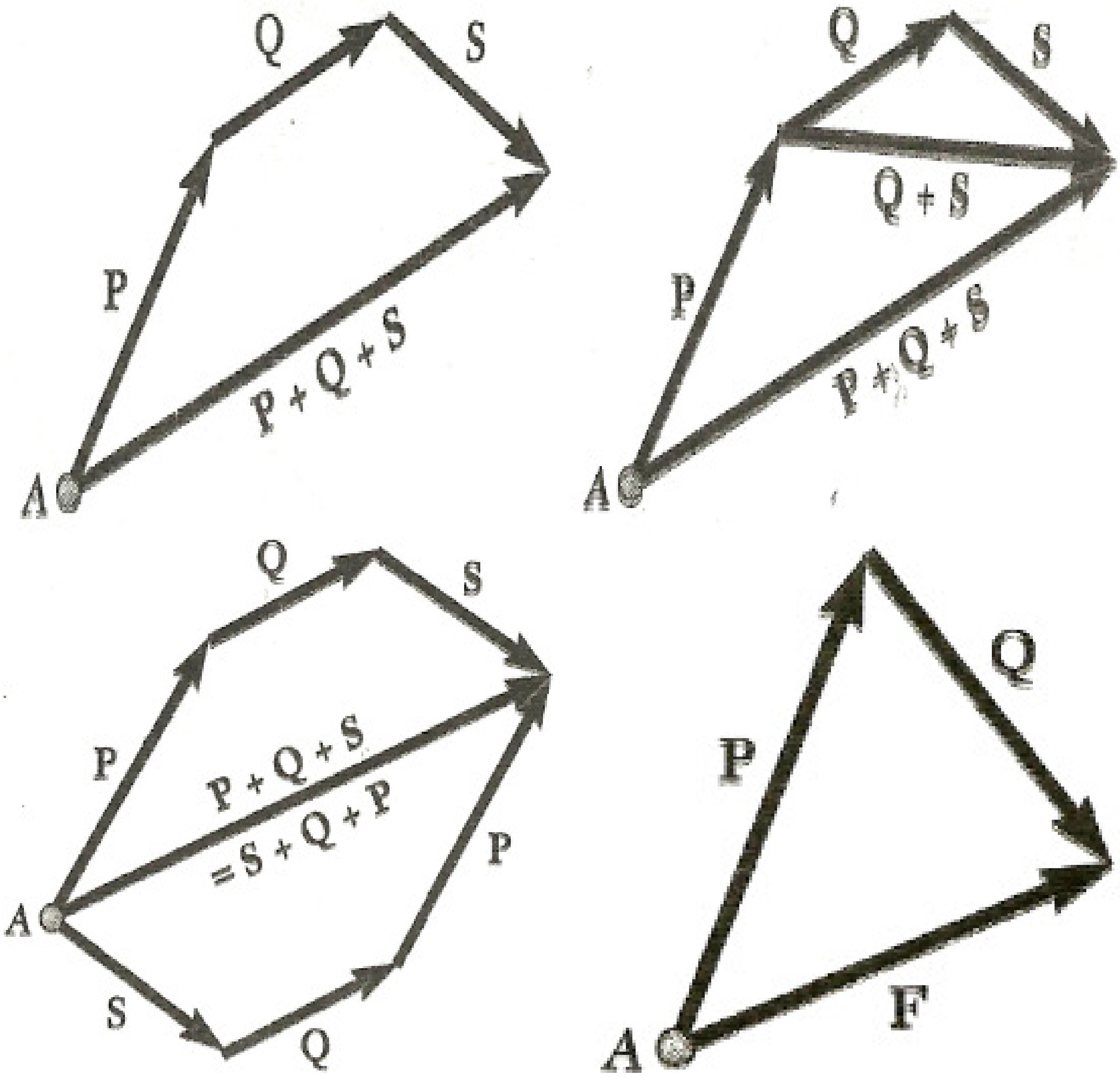
A force represents the action on one body into another and is generally charectraized by its point of application, its magnitude and its direction. Forces acting on a given particle, have the same point of application. The direction of the force is defined by the line of action and the sense of force. The sense of the force should be indicated by an arrwohead. It's unite is Newton (N)

2.2 Resolutant of several current forces

A concurrent forces acted upon a particle A contained in the same plane, all pass through A. The resultant force $\vec{R}$ of the given concurrent forces is obtained by addition using polygon rule. The resultant force $\vec{R}$ has the same effect on the particle A as the given forces.

2.3 Resolution of a forces into componenets

In the next section, we sayed that two or more forces acting on sa particle may be replaced by a single foce which has the same effect on the particle. Conversely, a single force $\vec{F}$ acting on a particle may be replaced by two or



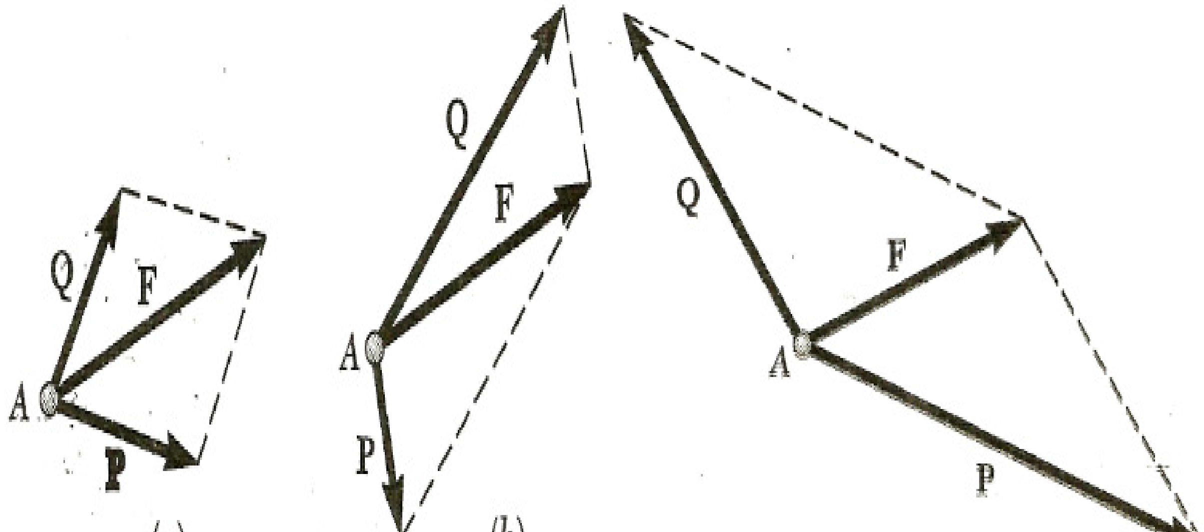
more forces which have together the same effect on a particle. These forces called the components of the original force $\vec{F}$. The process of substituting the components for $\vec{F}$ is called resolving the force $\vec{F}$ into components.

For

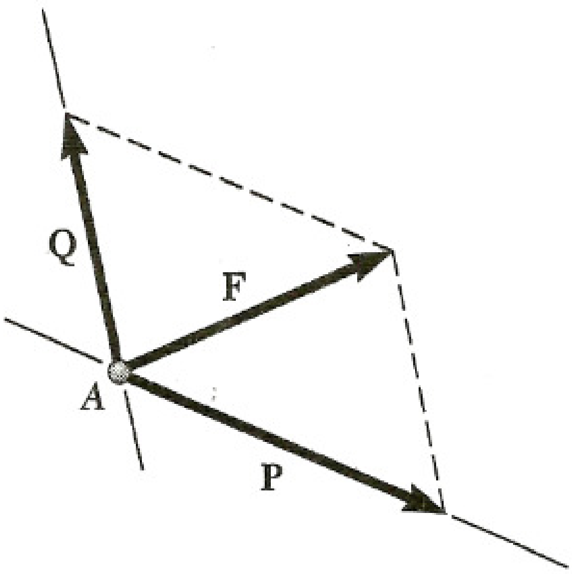
$$\vec{F} = F_x\vec{i} + F_y\vec{j}$$

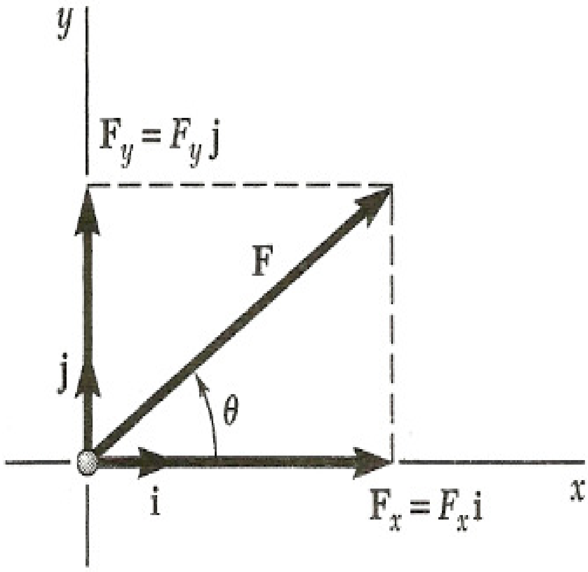
$F_x = |\vec{F}|\cos\theta$ represents the x-component of the force $\vec{F}$, while, $F_y = |\vec{F}|\sin\theta$ represents the y-component of the force $\vec{F}$.

Magnitude of force



(a) (b) (e)

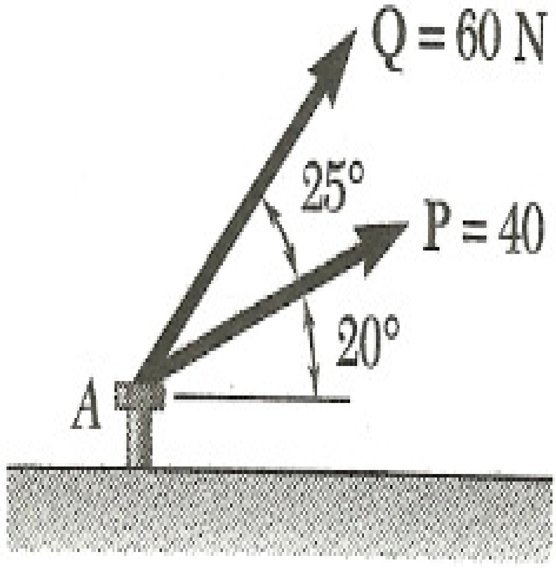


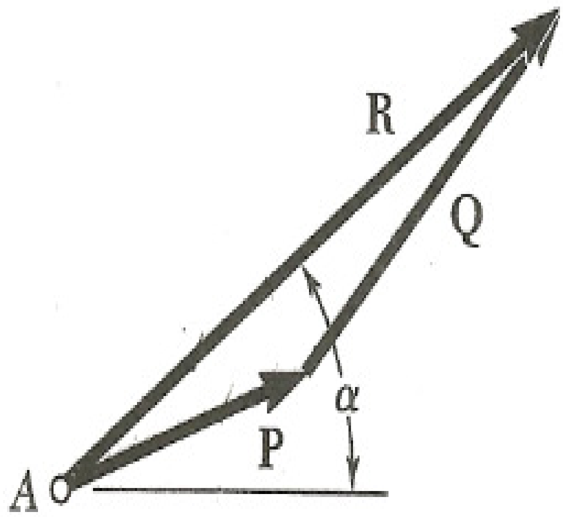


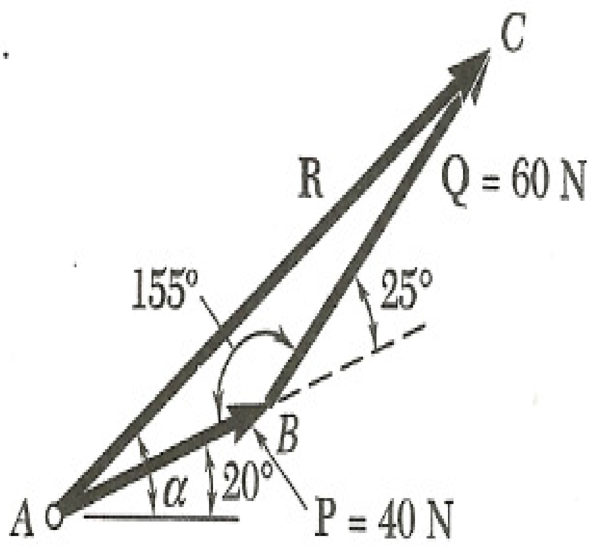
As $\vec{F}$ is a vector, so it follows the law computing the magnitude of vector, as $|\vec{F}| = \sqrt{F_x^2 + F_y^2}$

Example: A two forces $\vec{P}$ and $\vec{Q}$ acting on a bolt A. Determine their resultant.

SOLUTION







We use the traingle rule, two sides and the included angle are known. We apply the law of cosines:

$$R^2 = P^2 + Q^2 - 2PQ\cos\theta$$

$$R^2 = (40)^2 + (60)^2 - 2(40)(60)cos155^o = 97.73N$$

Now applying the law of sines, we write

$$\frac{\sin\phi}{Q} = \frac{\sin\theta}{R} \Longrightarrow \frac{\sin\phi}{60} = \frac{\sin 155^o}{97.73} \tag{2.1}$$

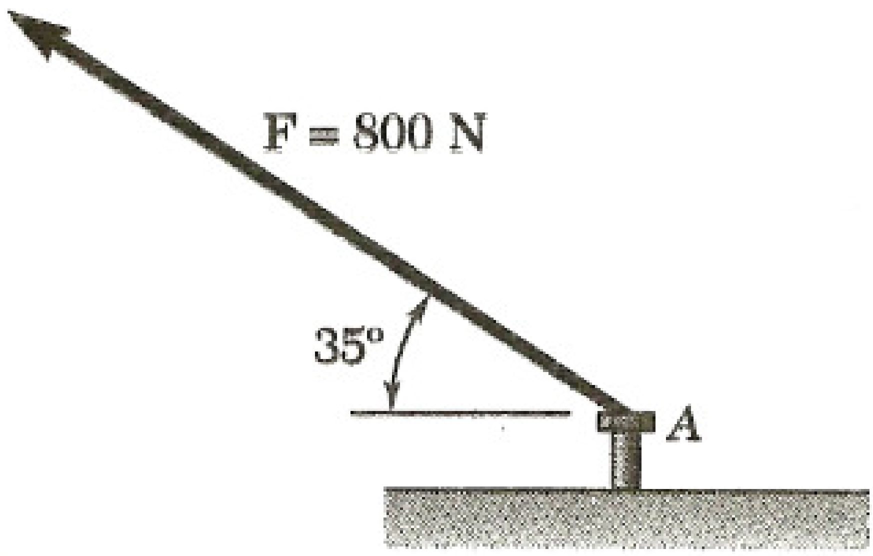
Solving (2.1) for $\sin\phi$, we have

$$\sin\phi = \frac{(60)\sin 155^o}{97.73} \Longrightarrow \phi = 15.04^o$$

This means

$$\alpha = 20^o + 15.04^o = 35.04^o$$

Example: A force of $800N$ is exerted on a bolt A as shown in the figure. Determine the horizontal and vertical componenets of the force.

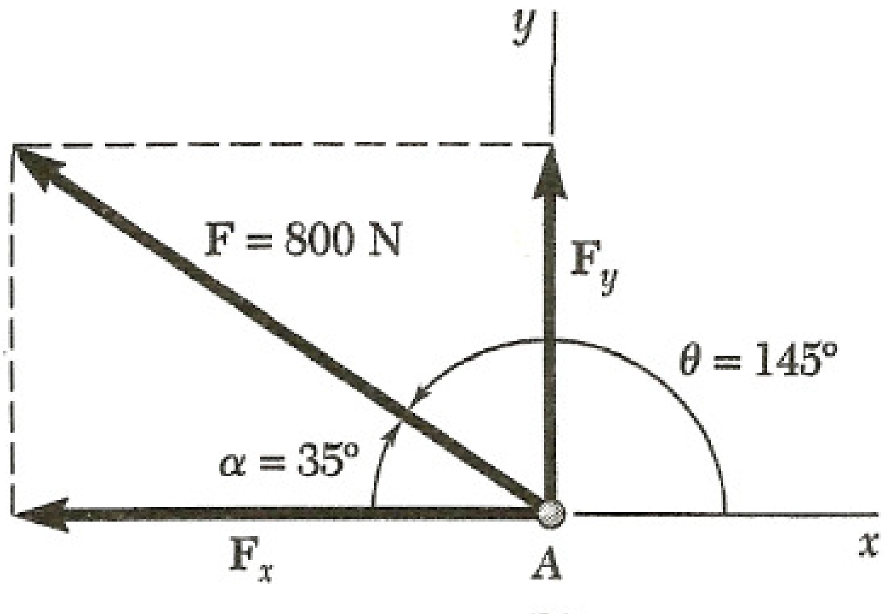


(a)

SOLUTION

Using the expressions of $\vec{F}$ components

$$F_x = F\cos\theta$$



(b)

$$F_y = F \sin\theta$$

Where $\alpha = 34^o$. From the figure, we note that $F - x$ will be in negative direction of x-axis, while F_y in positive direction of y-axis, so

$$F_x = -F\cos\alpha = -(800)\cos 35^o = -655N$$

$$= F_y = +F\sin\alpha = +(800)\sin 35^o = +459N$$

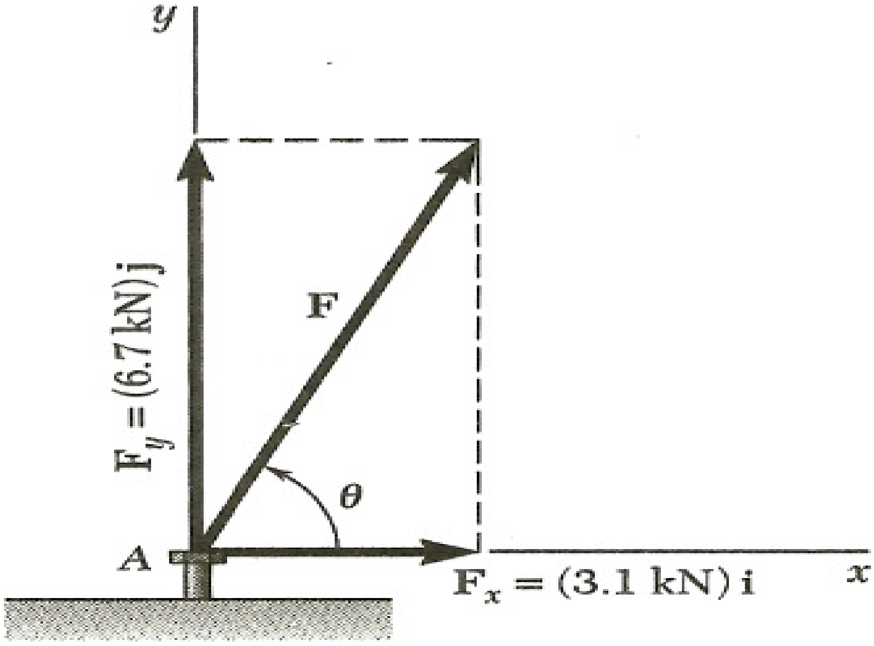
The vector components of $\vec{F}$ are

$$\vec{F}_x = -655\vec{i} \qquad \vec{F} - y = 459\vec{j}$$

and we may write $\vec{F}$ in the form

$$\vec{F} = -655\vec{i} + 459\vec{j}$$

Example: A force $\vec{F} = 3.1kN\vec{i} + 6.7kN\vec{j}$ is applied to a bolt A as shown in the figure. Determine the magnitude of the force and the angle θ it forms with the horizontal.



SOLUTION

From the figure we note that the components of the force $\vec{F}$ represented completely by the the two neighnouring arms of the rectangle, so, we apply the law of sines to obtain the angle θ as

$$\tan\theta = \frac{F_y}{F_x} = \frac{6.7\ kN}{3.1\ kN} \qquad \Rightarrow \qquad \theta = 65.17^o$$

To determine the the magnitude of the force $\vec{F}$, using the relation

$$F_y = F\sin\theta \qquad \Rightarrow \qquad F = \frac{F_y}{\sin\theta} = \frac{6.7}{\sin 65.17^o} = 7.4kN$$

2.4 Resultant force by adding X and Y components

Another method to obtain the resultant force is to resolving the forces into their x- and y-componenets, then adding the corresponding components to obtain the conponents of the resultant force. For example, the three forces $\vec{P}, \vec{Q}$ and $\vec{S}$ acting on a particle A. Their resultant is defined by the relation

$$R_x\vec{i} + R_y\vec{j} = P_x\vec{i} + P_y\vec{j} + Q_x\vec{i} + Q_y\vec{j} + S_x\vec{i} + S_y\vec{j}$$

$$= (P_x + Q_x + S_x)\vec{i} + (P_y + Q_y + S_y)\vec{j}$$

From which it follows that

$$R_x = P_x + Q_x + S_x = \sum F_x, \qquad R_y = P_y + Q_y + S_y = \sum F_y$$

Example: Four forces act on a particle A as shown. Determine the resultant of the forces on the particle.

SOLUTION

Resolving the forces into their x- and y-componenets as shown, then summing the corresponding forces, we obtain

$$\vec{R} = R_x\vec{i} + R_y\vec{j} = (F_1\cos 30^o + F_4\cos 15^o - F_2\sin 20^o)\vec{i} + (F_2\cos 20^o + F_1\sin 30^o - F_4\,\mathrm{si}$$

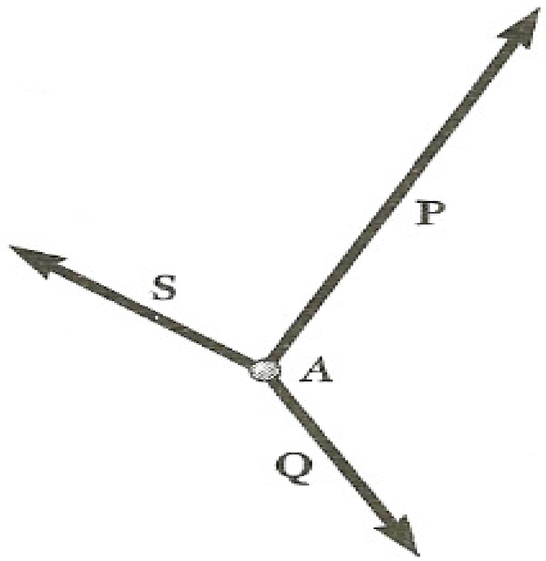
$$= 199.1\vec{i} + 14.3\vec{j}$$

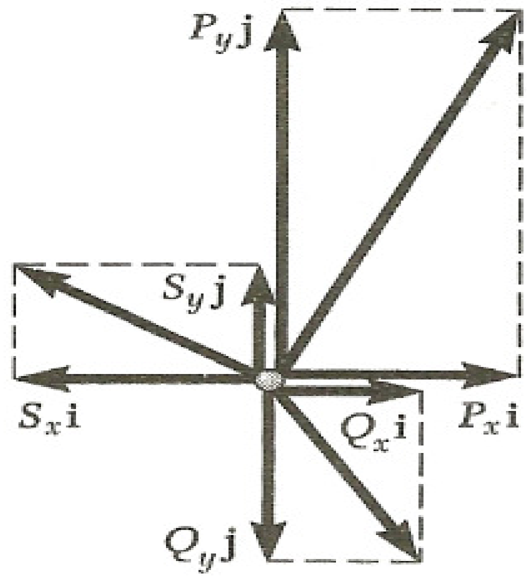
The magnitude and direction of the resultant force $\vec{R}$, can be determined from the traiangle rule as

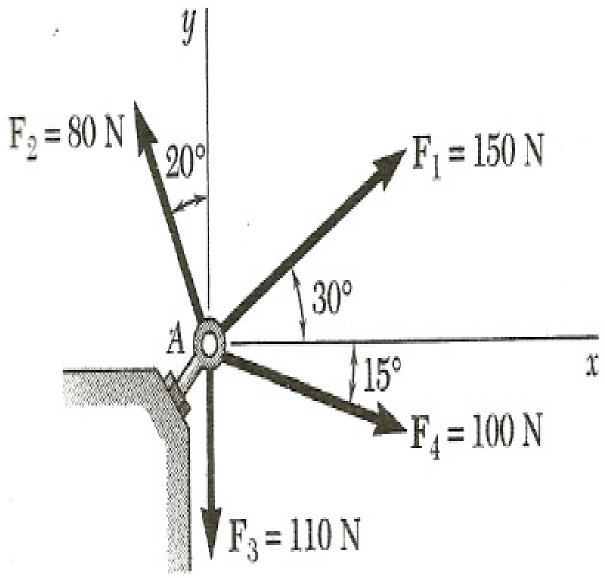
$$\tan\alpha = \frac{R_r}{R_x} = \frac{14.3}{199.1} \qquad \Rightarrow \qquad \alpha = 4.1^o$$

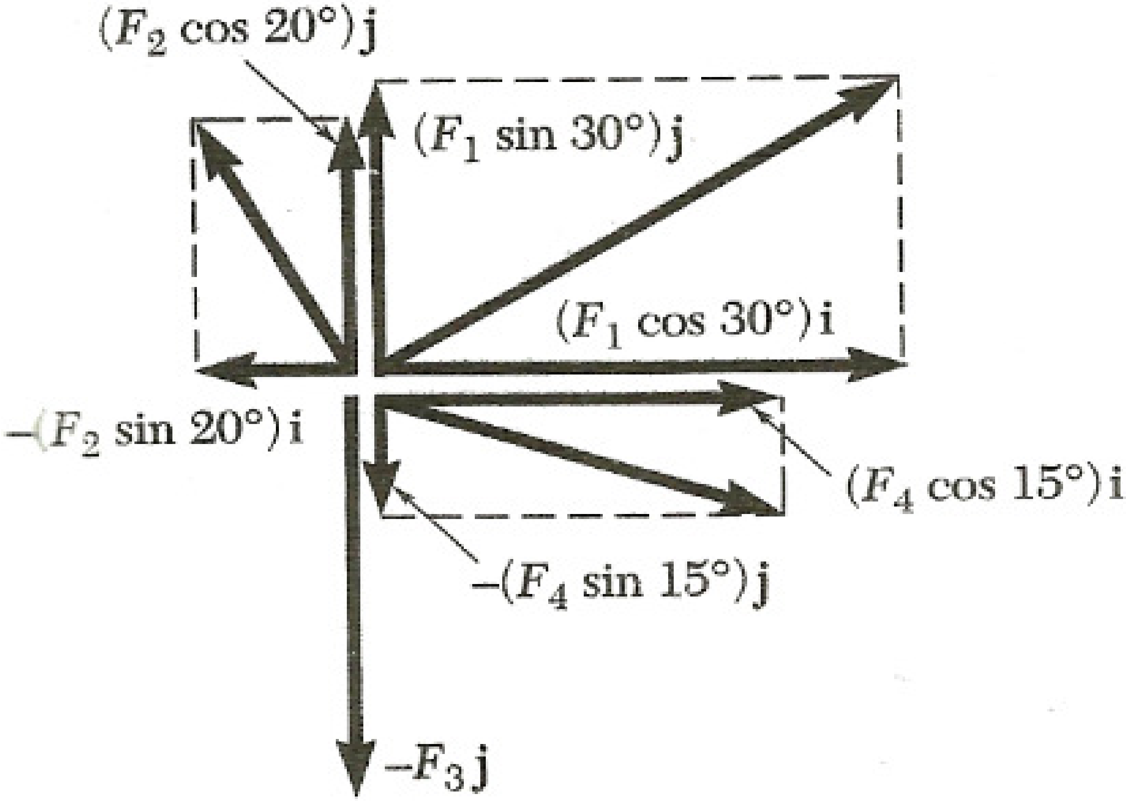
While the magnitude is obtained as

$$R = \frac{14.3}{\sin\alpha} = 199.6N$$


P
S
A
Q


$P_y\mathbf{j}$
$S_y\mathbf{j}$
$S_x\mathbf{i}$
$Q_x\mathbf{i}$
$P_x\mathbf{i}$
$Q_y\mathbf{j}$


y
$F_2 = 80$ N
20°
$F_1 = 150$ N
30°
A
x
15°
$F_4 = 100$ N
$F_3 = 110$ N



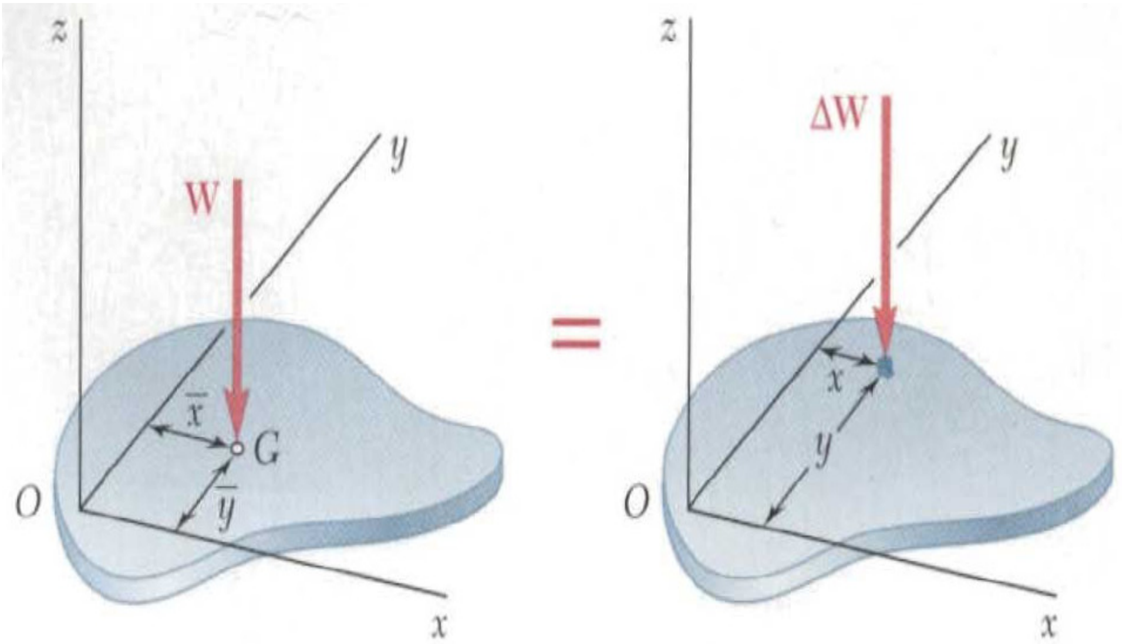
3 Centroids and center of gravity

3.1 Center of gravity of a two-dimentional bady (Areas and lines)

Let us consider a flat horizontal plate. We can divide the plate into n small elements. The coordinates of the first element denoted by x_1, y_1, of the second element by x_2, y_2, etc. The forces exerted by the earth on the elements of the plate are weights $\Delta W_1, \Delta W_2, \ldots, \Delta W_n$ and directed twowrds the center of the earth; however, for all practical purposes they can be assumed to be parallel. Their resultant is W, i.e.,

$$W = \Delta W - 1 + \Delta W_2 + \cdots + \Delta W_n$$

To obtain the coordinates $\bar{x}, \bar{y}$ of the point G where the resultant W is acting, we take the moments of W about the x and y axes which are equal to the sum



of the corresponding moments of the elemental weigts,

$$\bar{x}W = x_1\Delta W_1 + x_2\Delta W_2 + \cdots + x_n\Delta W_n = \sum x\Delta W$$

$$\bar{y}W = y_1\Delta W_1 + y_2\Delta W_2 + \cdots + y_n\Delta W_n = \sum y\Delta W$$

This means that

$$\bar{x} = \frac{\sum x\Delta W}{W}$$

If we increase the number of elements into which the plate is divided and simaltaneousely derease the size of each element, we obtain in the limit the following expressions

$$W = \int dW, \qquad \bar{x} = \frac{\int xdW}{W}, \qquad \bar{y} = \frac{\int ydW}{W}$$

We can write the previous equations for areas and lines, respectively as
for lines, with length l

$$\bar{x} = \frac{\int xdl}{l}, \qquad \bar{y} = \frac{\int ydl}{l}$$

for area, with erea A

$$\bar{x} = \frac{\int x dA}{A}, \qquad \bar{y} = \frac{\int y dA}{A}$$

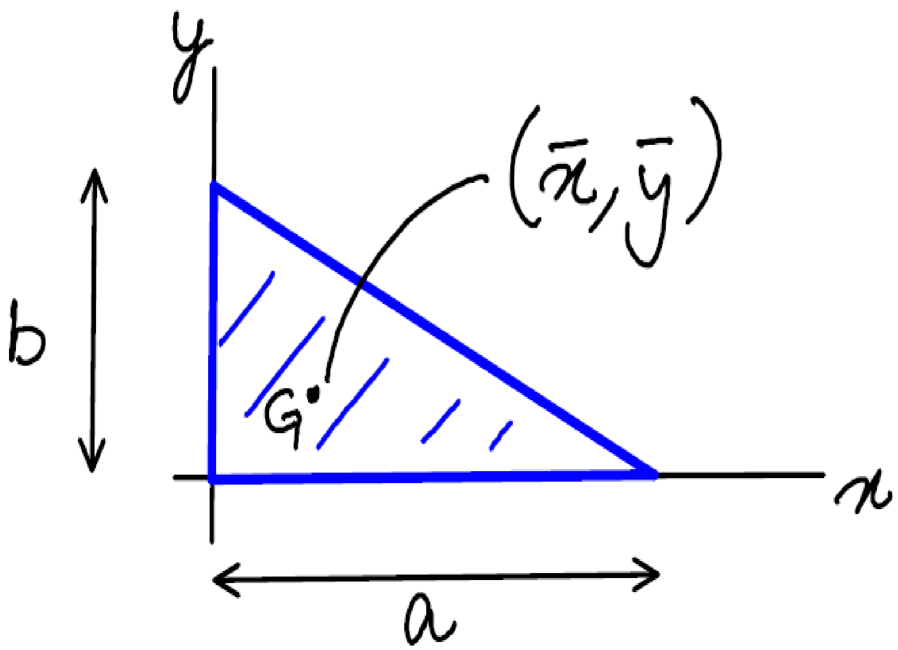
In three dimensions, for volums, with volum v,

$$\bar{x} = \frac{\int x dv}{v}, \qquad \bar{y} = \frac{\int y dv}{v}$$

3.2 Examples

Example: Find the center of gravity of an area of triangle of height of b and base of a.

SOLUTION

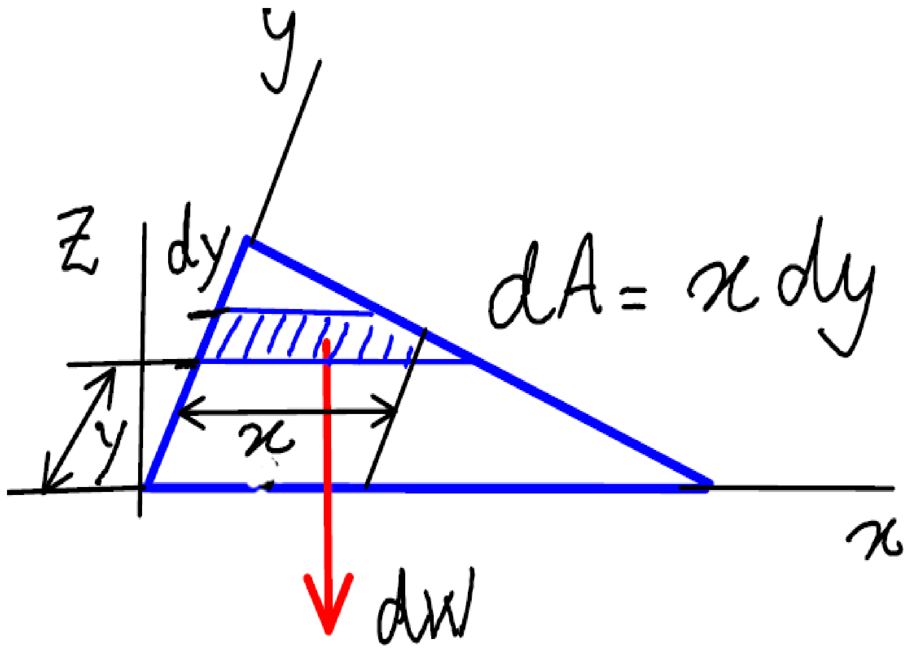


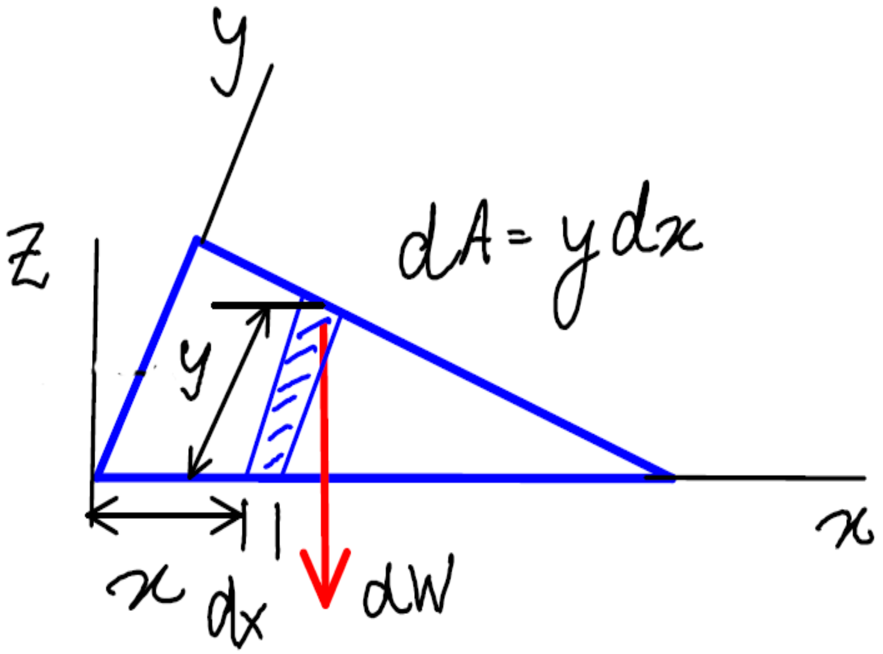
To solve, we notice from the figure that

$$x = -\frac{a}{b}y + a, \qquad\qquad x = -\frac{a}{b}y + b$$

Now

$$\bar{x} = \frac{\int_0^a x dW}{\int_0^a dW} = \frac{\int_0^a x \rho y dx}{\int_0^a \rho y dx}$$


y
Z
dy
dA = x dy
y
x
x
dW


y
Z
dA = y dx
y
x
x
dx
dW

$$= \frac{\int_0^a x[(-b/a)x + a]dx}{\int_0^a \rho[(-b/a)x + a]dx} = a/3$$

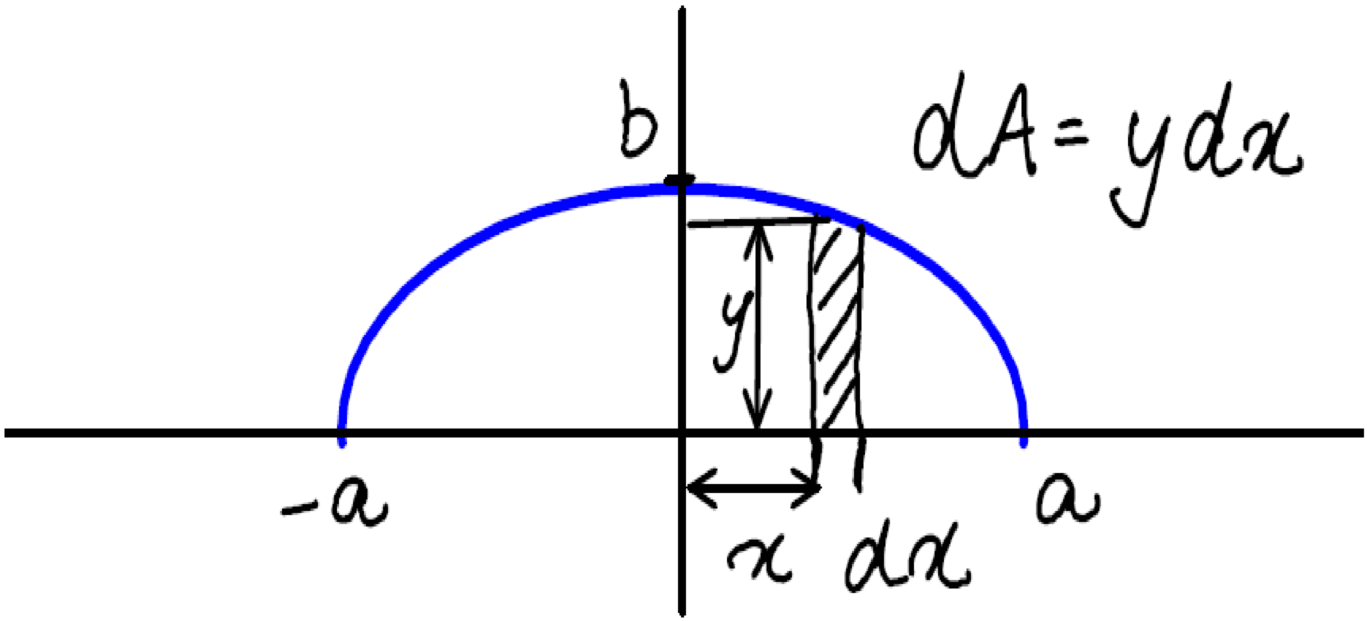
Also

$$\bar{y} = \frac{\int_0^a ydW}{\int_0^a dW} = \frac{\int_0^a y\rho xdy}{\int_0^a \rho xdy}$$

$$= \frac{\int_0^a y[(-b/a)y + a]dy}{\int_0^a \rho[(-b/a)y + a]dy} = b/3$$

<u>**Example:**</u> Find the center of gravity of an area of a semi ellipse.

SOLUTION



From ellipse equation we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \rightarrow \quad y = b\sqrt{1 - \frac{x^2}{a^2}} = \frac{b}{a}\sqrt{a^2 - x^2}$$

Then

$$\bar{x} = \frac{\int_{-a}^a xdW}{\int_{-a}^a dW} = \frac{\int_{-a}^a x\rho ydx}{\int_{-a}^a \rho ydx}$$

$$= \frac{\int_{-a}^a x[\frac{b}{a}\sqrt{a^2 - x^2}]dx}{\int_{-a}^a [\frac{b}{a}\sqrt{a^2 - x^2}]dx}$$

If we substitute

$$x = a\cos\theta, \qquad dx = -a\sin\theta d\theta, \quad \text{at} \quad x = -a \rightarrow \theta = \pi, \quad \text{at} \quad x = a \rightarrow \theta = 0$$

This means that

$$\bar{x} = \frac{-a\int_{\pi}^{0}\cos\theta\sin^2\theta d\theta}{\int_{\pi}^{0}\sin^2\theta d\theta} = \frac{-a\int_{\pi}^{0}\sin^2\theta d\sin\theta}{\int_{\pi}^{0}\frac{(1-\cos 2\theta)}{2}d\theta}$$

$$= \frac{\frac{-a}{3}[\sin^3\theta]_{\pi}^{0}}{[\theta/2]_{\pi}^{0} + [(1/4)\cos 2\theta]_{\pi}^{0}} = 0$$

Also

$$\bar{y} = \frac{\int_0^a ydW}{\int_0^a dW} = \frac{\int_0^b y\rho x dy}{\int_0^b \rho x dy}$$

$$= \frac{\int_0^b y[\frac{a}{b}\sqrt{b^2 - y^2}]dy}{\int_0^b \rho[\frac{a}{b}\sqrt{b^2 - y^2}]dy}$$

$$= \frac{\int_0^b y\sqrt{b^2 - y^2}dy}{\int_0^b \sqrt{b^2 - y^2}dy} =$$

Put $y = b\cos\theta \quad \rightarrow \quad dy = -b\sin\theta d\theta, y = 0 \rightarrow \theta = \pi/2, y = b \rightarrow \theta = 0$

Then we have

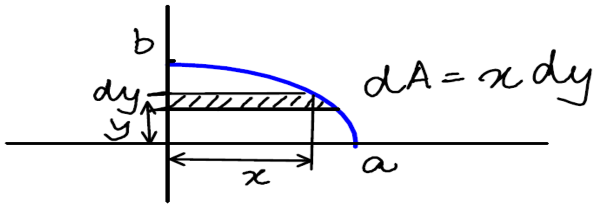
$$\bar{y} = \frac{b\int_{\pi/2}^{0}\cos\theta\sin\theta d\theta}{\int_{\pi/2}^{0}\sin^2\theta d\theta} =$$

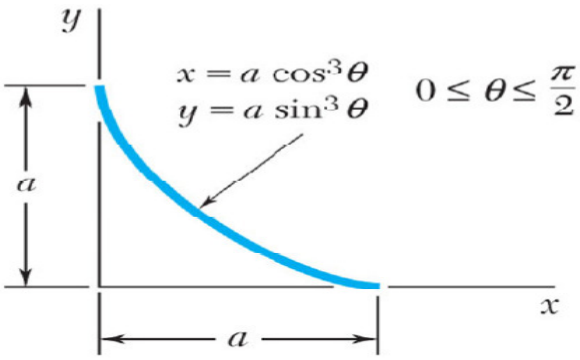
$$\bar{y} = \frac{\frac{b}{3}[\sin^3\theta]_{\pi/2}^{0}}{[\theta/2]_{\pi/2}^{0} + (1/4)[\cos 2\theta]_{\pi/2}^{0}} = \frac{-b/3}{[(-\pi/4) + (1/4)(1-1)]} = \frac{4b}{3\pi}$$

Example: Find the center of gravity of an area of a quarter ellipse.

SOLUTION

Example: Find the center of gravity of the wire as shown.





SOLUTION

We divide the wire into small elements of length dl where from the figure, we notice

$$dl = \sqrt{dx^2 + dy^2} = \sqrt{(-3a\cos^2\theta\sin\theta)^2 + (3a\sin^2\theta\cos\theta)^2}d\theta$$

$$= 3a\sin\theta\cos\theta d\theta,$$

From the defination, we known

$$\bar{x} = \frac{\int_0^{\pi/2} x dW}{\int_0^{\pi/2} dW} = \frac{\int_0^{\pi/2} x\rho g dl}{\int_0^{\pi/2} \rho g dl} = \frac{a\int_0^{\pi/2}\cos^4\theta\sin\theta d\theta}{\int_0^{\pi/2}\sin\theta\cos\theta d\theta}$$

$$= \frac{2a}{5}\frac{\cos^5\theta\Big|_0^{\pi/2}}{\cos^2\theta\Big|_0^{\pi/2}} = \frac{2a}{5}$$

From symmetry,

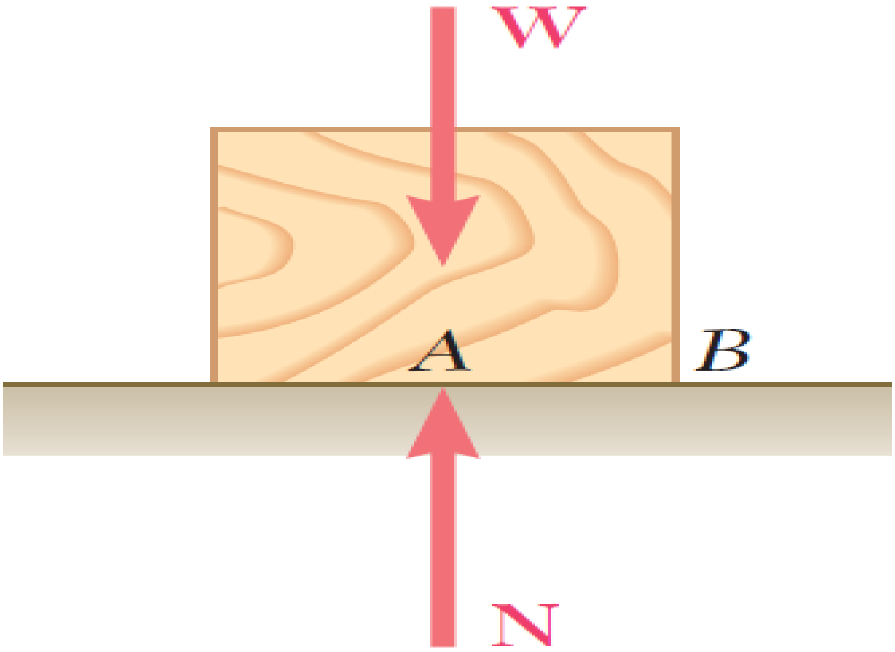
$$\bar{y} = \frac{2a}{5}$$

4 Friction

When two surfaces are in contact, tangential forces, called friction forces, will always develop if one attempts to move one surface with respect to the other.

4.1 The Laws of Dry Friction. Coefficients of Friction

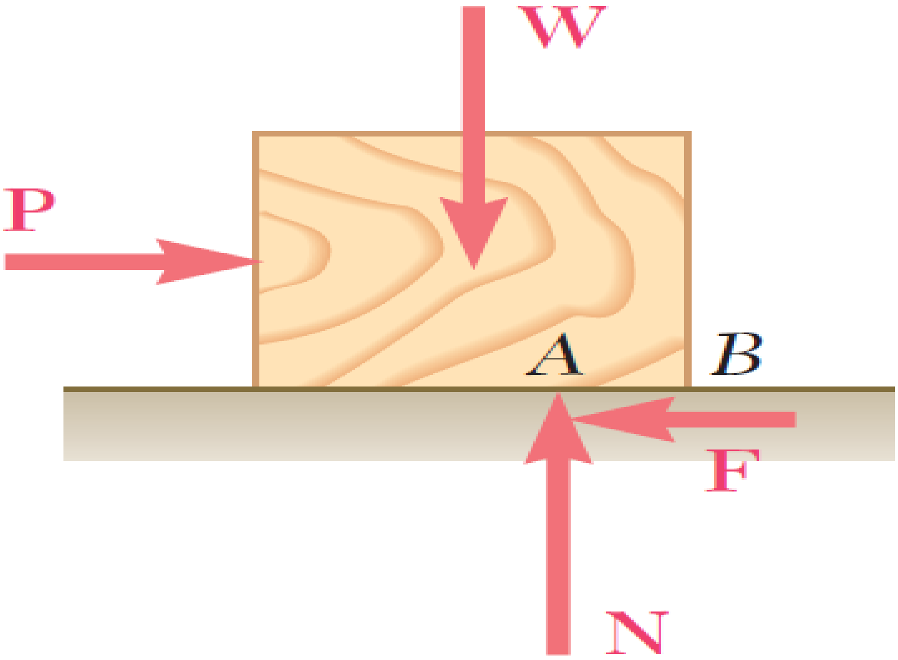
- Block of weight W placed on horizontal surface. Forces acting on block are its weight and reaction of surface N.

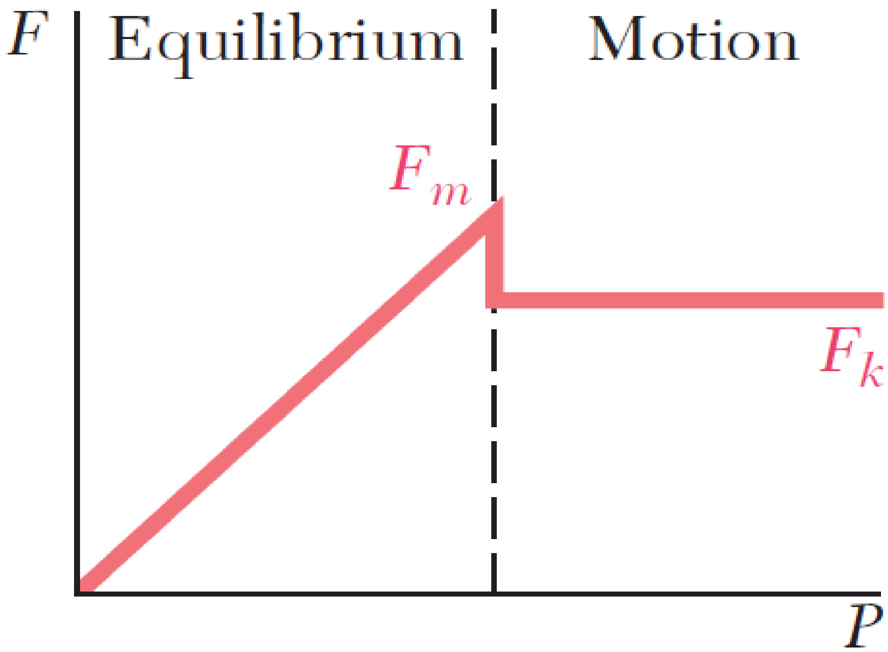


- Small horizontal force P applied to block. For block to remain stationary, in equilibrium, a horizontal component F of the surface reaction is required. F is a static-friction force.

- As P increases, the static-friction force F increases as well until it reaches a maximum value F_m.

$$\vec{F}_m = \mu_s N$$


W
P
A
B
F
N

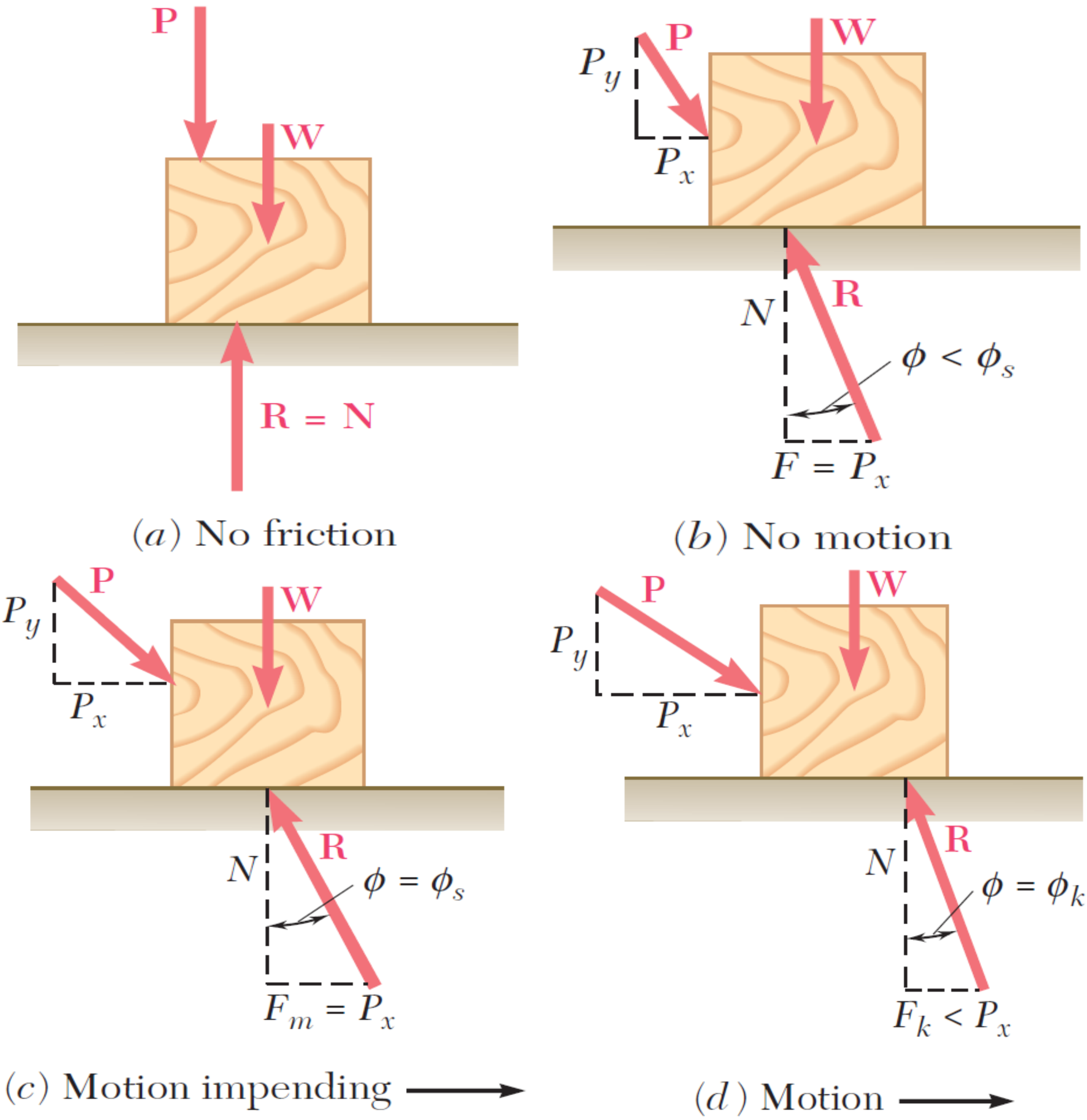

F
Equilibrium
Motion
F_m
F_k
P

- Further increase in P causes the block to begin to move as F drops to a smaller kinetic-friction force F_k.

$$\vec{F}_k = \mu_k N$$

4.2 Angles of Friction

It is sometimes convenient to replace normal force N and friction force F by their resultant R:



- If no horizontal force is applied to the block, the resultant R reduces to the normal force N

- If the applied force P has a horizontal component P_x which tends to move the block, the force R will have a horizontal component F and, thus, will form an angle ϕ with the normal to the surface.

- If P_x is increased until motion becomes impending, the angle between R and the vertical grows and reaches a maximum value. This value is called the angle of static friction and is denoted by ϕ_s, we note that

$$\tan \phi_s = \frac{F_m}{N} = \frac{\mu_s N}{N} = \mu_s$$

- If motion actually takes place, the magnitude of the friction force drops to F_k; similarly, the angle ϕ between R and N drops to a lower value ϕ_k, called the angle of kinetic friction, we write

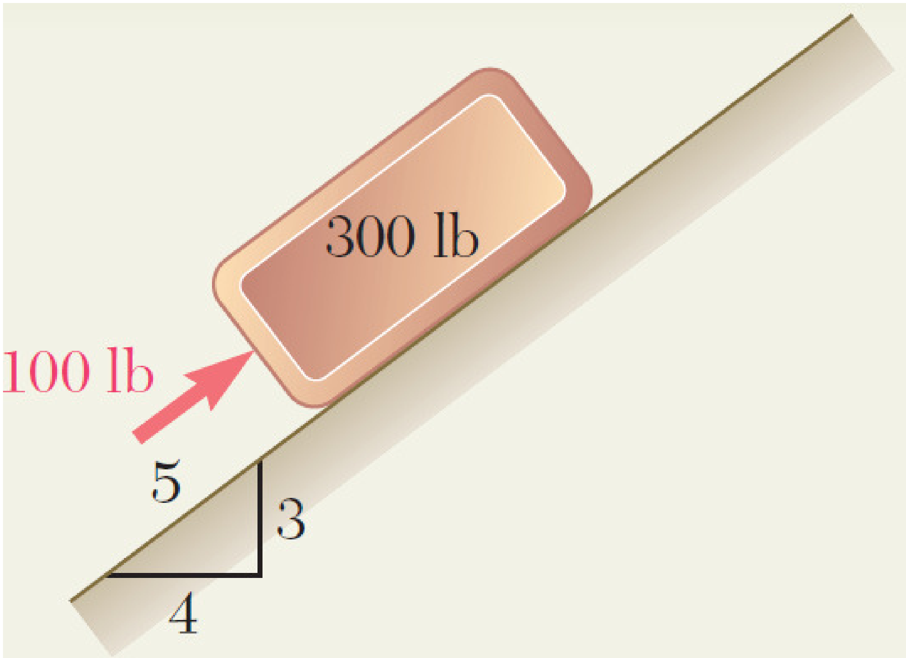
$$\tan \phi_k = \frac{F_k}{N} = \frac{\mu_k N}{N} = \mu_k$$

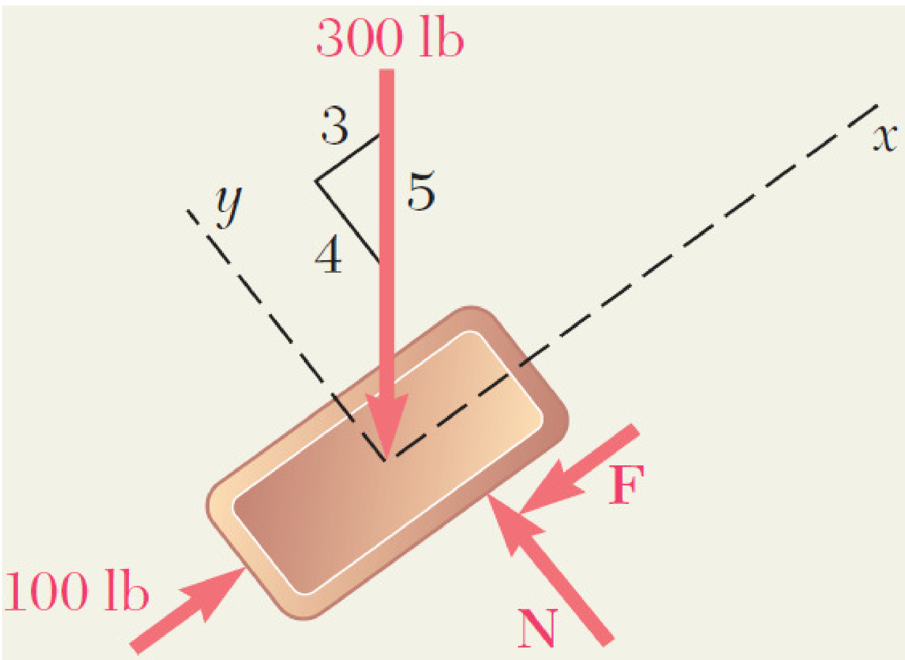
4.3 Examples

Example: A 100-lb force acts as shown on a 300-lb block placed on an inclined plane. The coefficients of friction between the block and the plane are $\mu_s = 0.25$ and $\mu_k = 0.20$. Determine whether the block is in equilibrium, and find the value of the friction force.

Solution

Force Required for Equilibrium: We first determine the value of the friction force required to maintain equilibrium. Assuming that F is directed


300 lb
100 lb
5
3
4


300 lb
3
5
4
y
x
F
N
100 lb

down and to the left, we draw the free-body diagram of the block and write

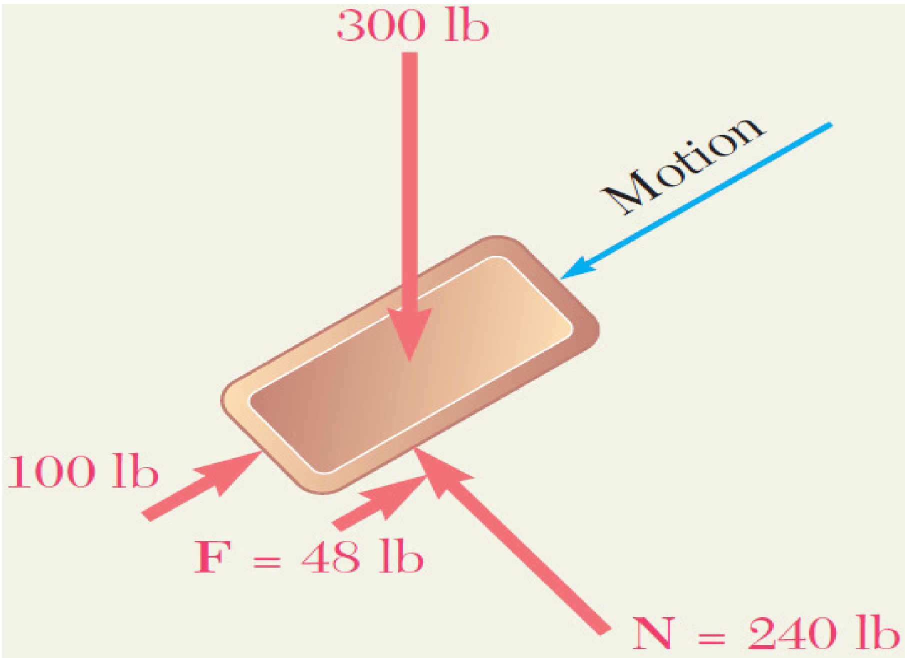
$$+\nearrow \sum F_x = 0: \qquad 100\text{ lb} - \frac{3}{5}(300\text{ lb}) - F = 0 \tag{4.1}$$

$$F = 280\text{ lb} \qquad F = 80\text{ lb} \nearrow \tag{4.2}$$

$$+\nwarrow \sum F_y = 0: \qquad N - \frac{4}{5}(300\text{ lb}) = 0 \tag{4.3}$$

$$N = 1240\text{ lb} \qquad N = 240\text{ lb} \nwarrow$$

The force F required to maintain equilibrium is an 80-lb force directed up and



to the right; the tendency of the block is thus to move down the plane.

Maximum Friction Force: The magnitude of the maximum friction force which may be developed is

$$F_m = \mu_s N \qquad F_m = 0.25(240\text{ lb}) = 60\text{ lb}$$

Since the value of the force required to maintain equilibrium (80 lb) is larger than the maximum value which may be obtained (60 lb), equilibrium will not be maintained and the block will slide down the plane.

Actual Value of Friction Force: The magnitude of the actual friction force is obtained as follows:

$$F_{actual} = F_k = \mu_k N = 0.20(240 \text{ lb}) = 48 \text{ lb}$$

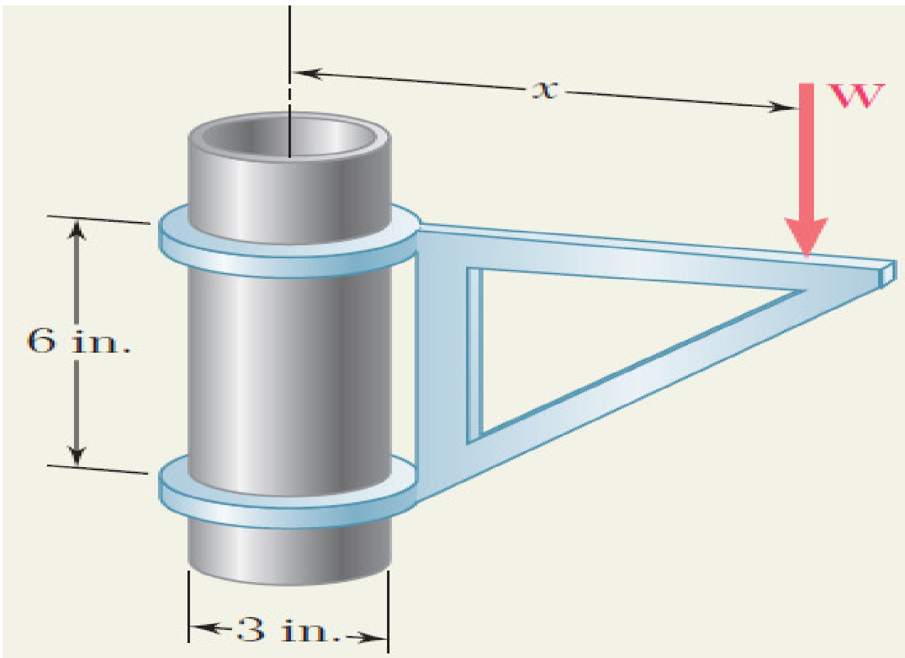
The sense of this force is opposite to the sense of motion; the force is thus directed up and to the right:

$$F_{actual} = 48 \text{ lb} \nearrow$$

It should be noted that the forces acting on the block are not balanced; the resultant is

$$\frac{3}{5}(300 \text{ lb}) - 100 \text{ lb} - 48 \text{ lb} = 32 \text{ lb} \swarrow$$

Example: The movable bracket shown may be placed at any height on the 3-in.-diameter pipe. If the coefficient of static friction between the pipe and bracket is 0.25, determine the minimum distance x at which the load W can be supported. Neglect the weight of the bracket.



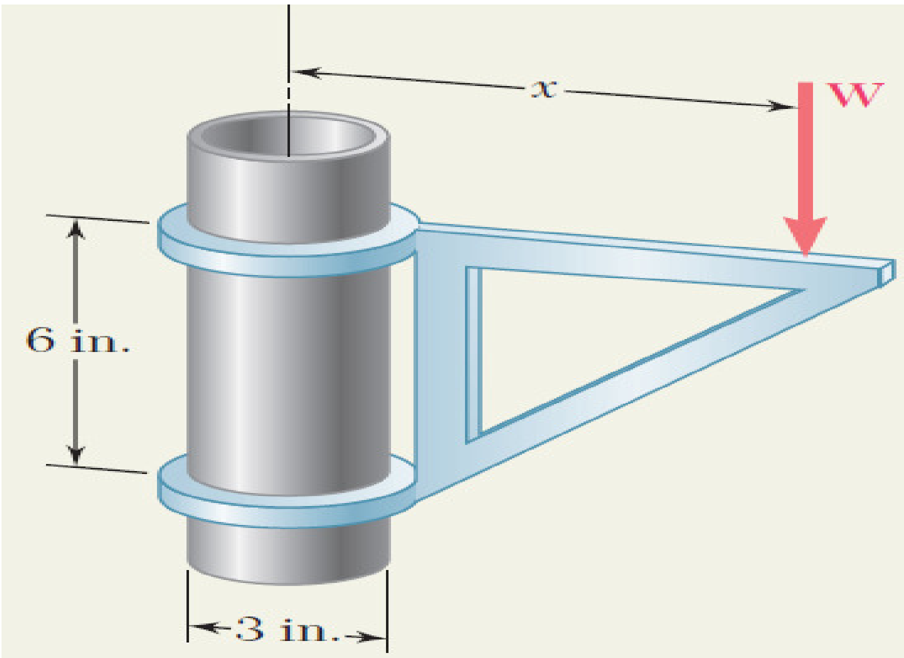
Solution

Free-Body Diagram. We draw the free-body diagram of the bracket. When W is placed at the minimum distance x from the axis of the pipe, the bracket is just about to slip, and the forces of friction at A and B have reached their maximum values:

$$F_A = \mu_s N_A = 0.25 N_A$$

$$F_B = \mu_s N_B = 0.25 N_B$$

Equilibrium Equations:



$$\overset{+}{\rightarrow} \sum F_x = 0 : \qquad N_B - N_A = 0$$

$$N_B = N_A$$

$$+\uparrow \sum F_y = 0 : \qquad F_A + F_B - W = 0$$

$$0.25 N_A + 0.25 N_B = W$$

And, since N_B has been found equal to N_A,

$$0.50N + A = W$$

$$N + A = 2W$$

$$+\sum M_B = 0: \qquad N_A(6in.) - F_A(3in.) - W(x - 1.5in.) = 0$$

$$6N_A - 3(0.25N_A) - Wx + 1.5W = 0$$

$$6(2W) - 0.75(2W) - Wx + 1.5W = 0$$

Dividing through by W and solving for x,

$$x = 12 \text{ in.}$$

5 Method of Virtual Work

Principle of virtual work is another method for solving certain types of equilibrium problems.

5.1 PRINCIPLE OF VIRTUAL WORK

Consider a particle acted upon by several forces $F_1, F_2, ..., F_n$. We can imagine that the particle undergoes a small displacement from A to A'. This displacement is possible, but it will not necessarily take place. The forces may be balanced and the particle at rest, or the particle may move under the action of the given forces in a direction different from that of AA'. Since the displacement considered does not actually occur, it is called a virtual displacement and is denoted by δr.

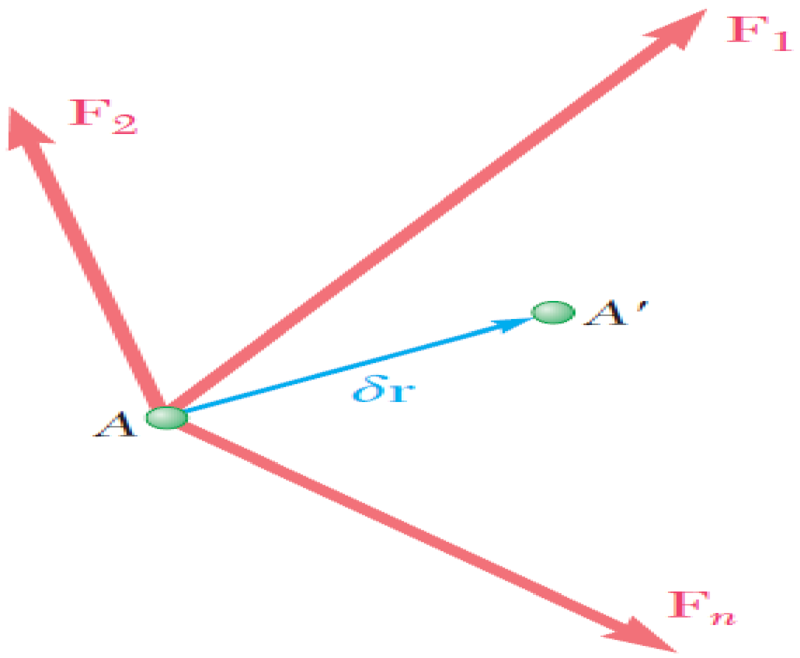
The virtual work of all the forces acting on the particle of is

$$\delta U = \vec{F}_1 \cdot \delta\vec{r} + \vec{F}_2 \cdot \delta\vec{r} + ... + \vec{F}_n \cdot \delta\vec{r} = (\vec{F}_1 + \vec{F}_2 + ... + \vec{F}_n) \cdot \delta\vec{r}$$

or

$$\delta U = \vec{R} \cdot \delta \vec{r}$$

where $\vec{R}$ is the resultant of the given forces. Thus, the total virtual work of



the forces $F_1, F_2, ..., F_n$ is equal to the virtual work of their resultant R.

5.2 Examples

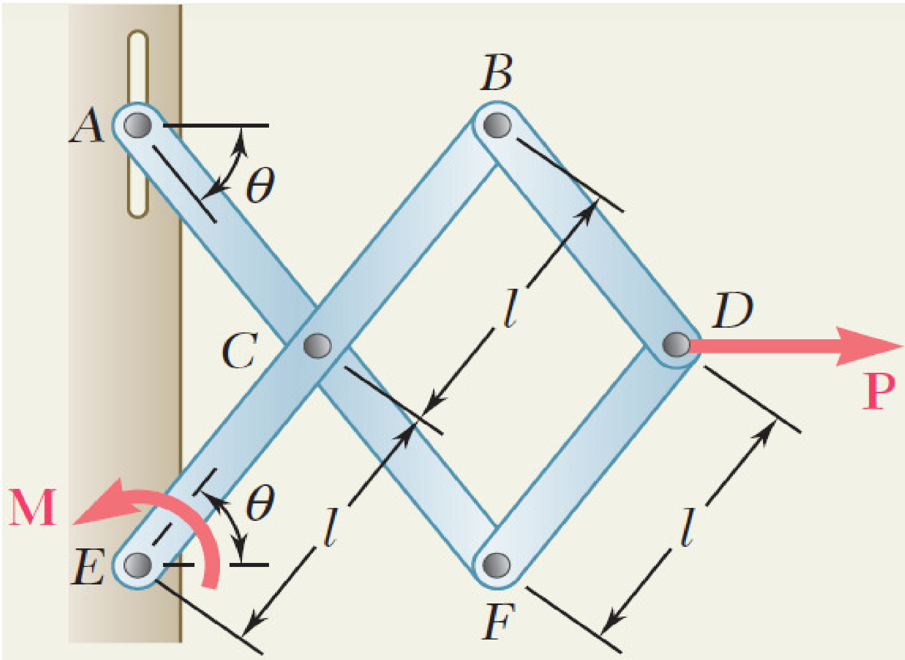
Example: Using the method of virtual work, determine the magnitude of the couple M required to maintain the equilibrium of the mechanism shown.

Solution

Choosing a coordinate system with origin at E, we write

$$x_D = 3l\cos\theta \qquad \delta x_D = -3l\sin\theta\delta\theta$$

Principle of Virtual Work. Since the reactions $\vec{A}$, E_x, and E_y will do no work during the virtual displacement, the total virtual work done by $\vec{M}$ and $\vec{P}$ must



be zero. Noting that $\vec{P}$ acts in the positive x direction and $\vec{M}$ acts in the positive θ direction, we write

$$\delta U = 0: \qquad +M\delta\theta + P\delta x_D = 0$$

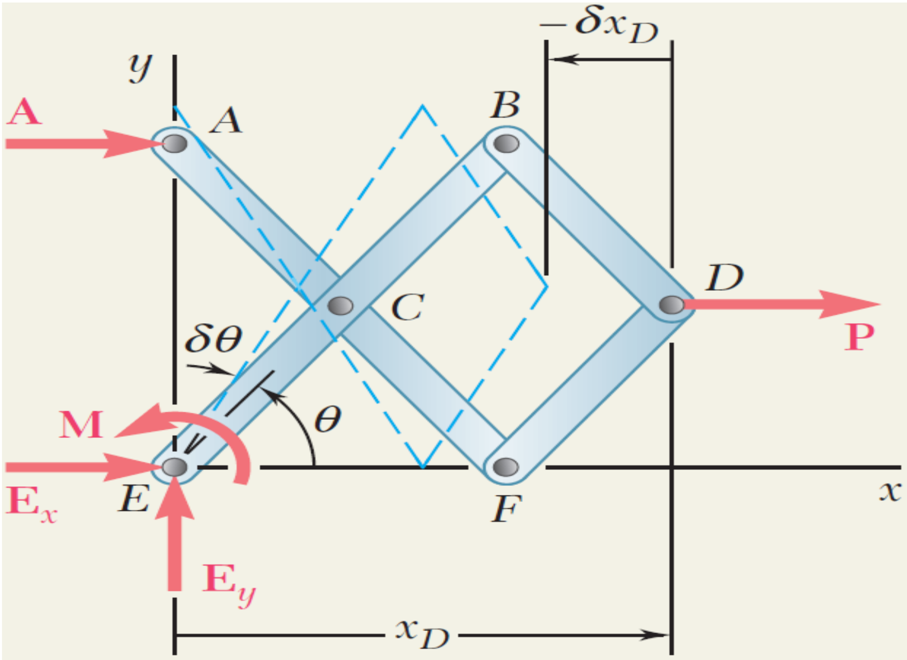
$$+M\delta\theta + P(-3l\sin\theta\delta\theta) = 0$$

$$M = 3Pl\sin\theta$$

5.3 Examples

Example: Determine the expressions for θ and for the tension in the spring which correspond to the equilibrium position of the mechanism. The unstretched length of the spring is h, and the constant of the spring is k. Neglect the weight of the mechanism.

Solution



With the coordinate system shown

$$y_B = l\sin\theta \qquad\qquad y_C = 2l\sin\theta$$

$$\delta y_B = l\cos\theta\delta\theta \qquad\qquad \delta y_C = 2l\cos\theta\delta\theta$$

The elongation of the spring is $s = y_C - h = 2l\sin\theta - h$

The magnitude of the force exerted at C by the spring is

$$F = ks = k(2l\sin\theta - h)$$

Principle of Virtual Work. Since the reactionsA_x, A_y, and C do no work, the total virtual work done by P and F must be zero.

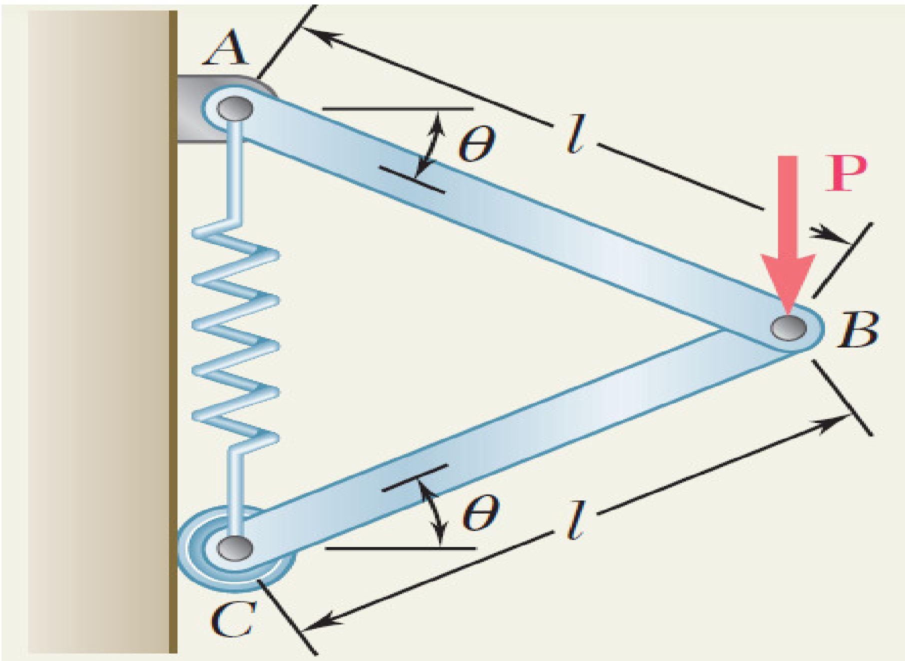
$$\delta U = 0: \qquad\qquad P\delta y_B - F\delta y_C = 0$$

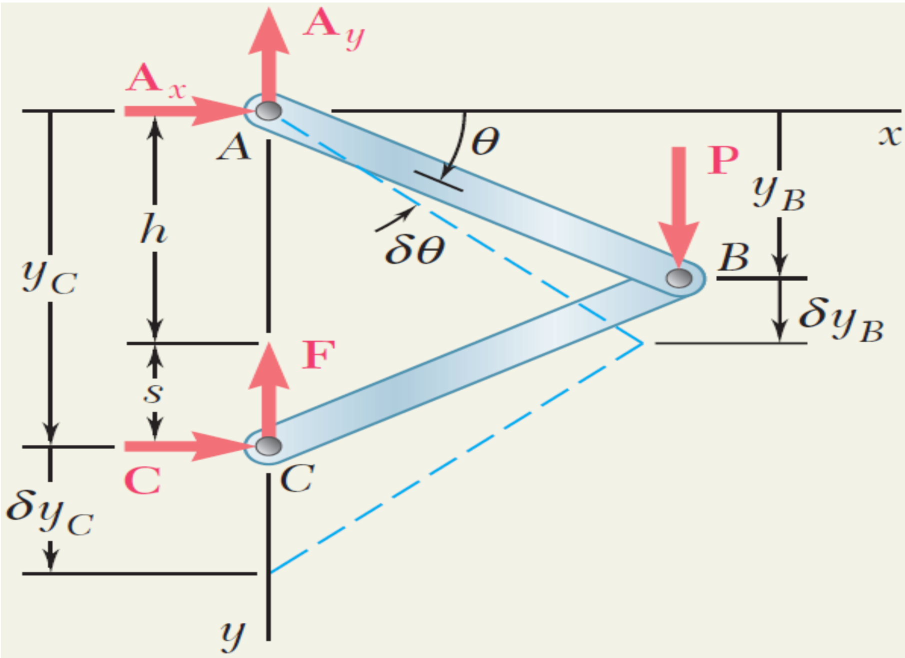
$$P(l\cos\theta\delta\theta) - k(2l\sin\theta - h)(2l\cos\theta\delta\theta) = 0$$

$$\sin\theta = \frac{P + 2kh}{4kl}$$

Substituting this expression, we obtain

$$F = \frac{1}{2}P$$

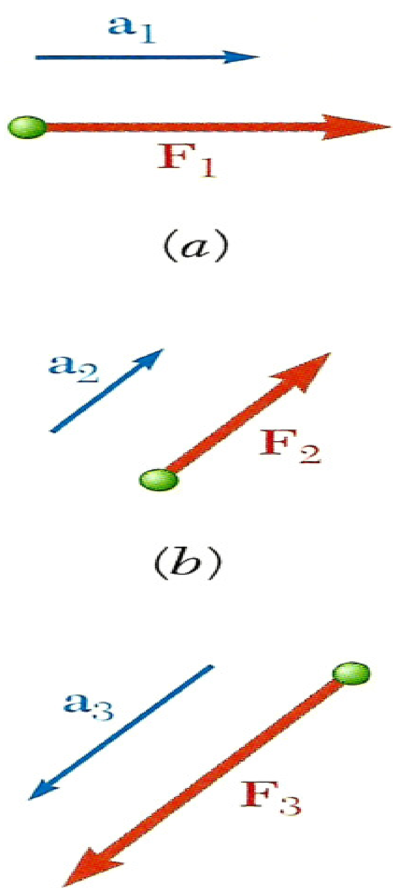

A
θ
l
P
B
θ
l
C


A_y
A_x
A
θ
x
P
y_B
h
$\delta\theta$
B
y_C
δy_B
F
s
C
C
δy_C
y

DYNAMICS PART

6 Newton's second law of motion

Newton's second law of motion can be stated as follows:
If the resultant force acting on a particle is not zero, the particle will have an acceleration proportional to the magnitude of the resultant force and in the direction of this resultant force.
To uderstand Newton's second law, imagin a particle subjected to a force $\vec{f_1}$ of constant direction and constant magnitude f_1. Under the action of that force, the particle is observed to move in a straight line in the direction of the force. By determining the position of the particle at various instants, we find that its acceleration has a constant magnitude. Repeating this action with forces $\vec{f_2}, \vec{f_3}\ldots$, of different magnitudes or directions, we find that the particle moves each time in the direction of the force acting on it and that the magnitudes $a_1, a_2, \ldots$ of the acceleration are proportional to the magnitudes $f_1, f_2, \ldots$ of the corresponding forces. $f - 1/a_1 = f_2/a_2 = \cdots = const.$
The constsnt value is the mass of the particle and is denoted by m. When a particle of mass m is acted upon by force $\vec{f}$, the force $\vec{F}$ and the acceleration $\vec{a}$ of the particle satisfy the relation:
$\vec{f} = m\vec{a}$
Note that the vectors $\vec{f}$ and $\vec{a}$ have the same direction.
When a particle is subjected simaltaneousely to several forces, we have
$\sum \vec{f} = m\vec{a}$
where $\sum \vec{f}$ represents the sum of the forces acting on the particle.



* Equation of motion

Consider a particle of mass m acted upon by several forces. Newton's second law can be expressed as

$\sum \vec{f} = m\vec{a}$

which relates the forces acting on the particle and the vector $m\vec{a}$. Using the components of both vectors $\vec{f} = (f_x, f_y, f_z)$ and $\vec{a} = (a_x, a_y, a_z)$, we have

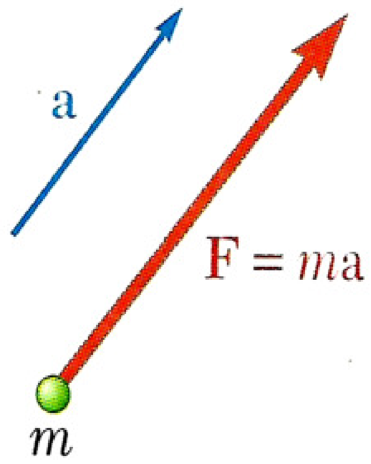
$\sum(f_x\vec{i} + f_y\vec{j} + f_z\vec{k}) = m(a_x\vec{i} + a_y\vec{j} + a_z\vec{k})$,

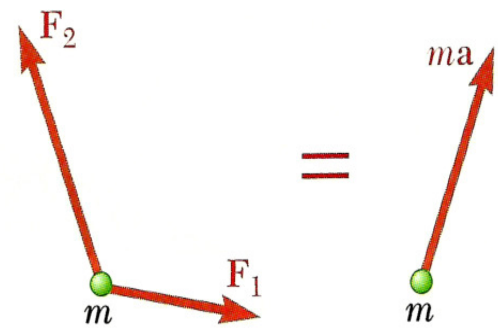
from which it follows that:

$\sum f_x = ma_x, \quad \sum f_y = ma_y, \qquad \sum f_z = ma_z,$

where

$a_x = d^2x/dt^2 = \ddot{x}, \qquad a_y = d^2y/dt^2 = \ddot{y}, \qquad a_z = d^2z/dt^2 = \ddot{z}.$



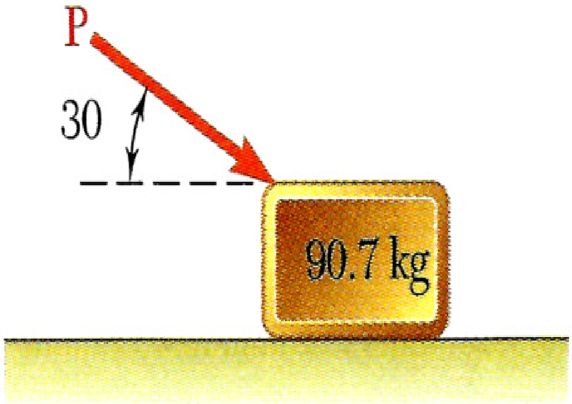


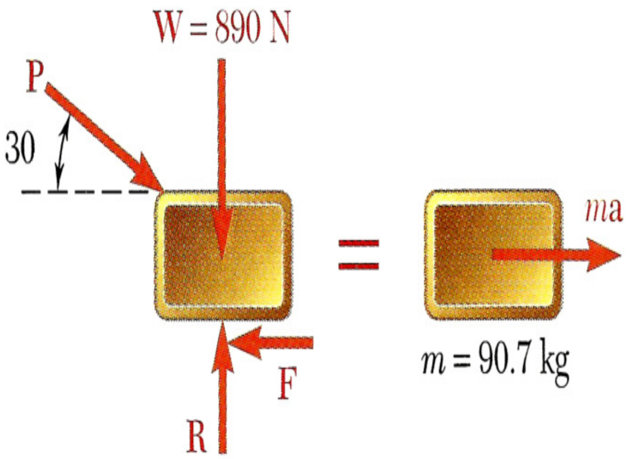
Example: A $90.7kg$ block rests on a horizontal plane. Find the magnitude of the force $\vec{P}$ required to give the block an acceleration of $3m/s^2$ to the right. The coefficient of kinetic fraction between the block and the plane is $\mu = 0.25$

Solution

The mass of the block is, weight of the block, $W = mg = 890N$, $m = 90.7g$
Note that:
$\vec{f} = muR = 0.25R$ and that $a = 3m/s^2$. expressing that the forces acting on the block are equivalent to the vector $m\vec{a}$, we write

$$(+) \to \sum f_x = ma \quad \to P\cos 30 - 0.25R = (90.7kg)(3m/s^2)$$





$$P\cos 30 - 0.25R = 272N(1)$$

$$(+)\uparrow \sum f_y = 0 \rightarrow R - P\sin 30 = 890N = 0(2)$$

Solving (2) for R and substituting the result into (1), we see

$$R = P\sin 30 + 890$$

then

$$P\cos 30 - 0.25(P\sin 30 + 890) = 272$$

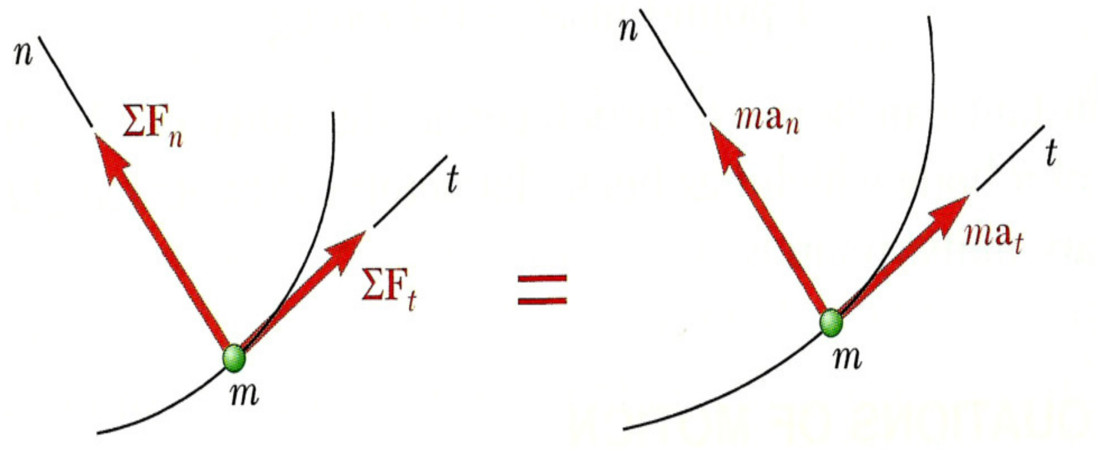
then

$$P = 667.3N$$

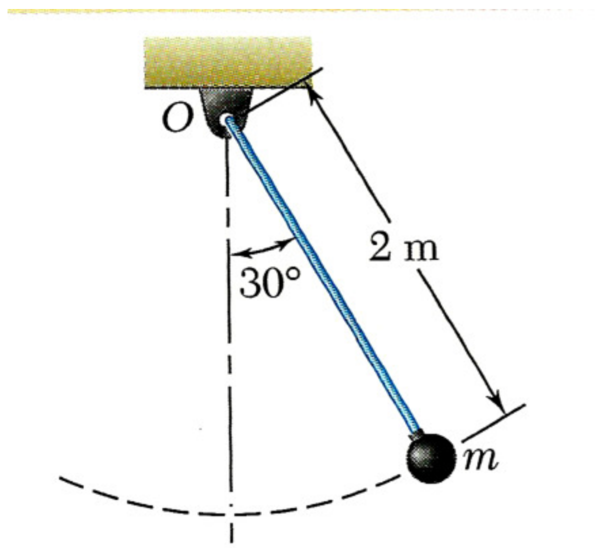
* Tangential and Normal components

Resolving the forces and acceleration of the particle into two components along the tangent to the path (in the direction of motion) and the normal (towards the inside of the path), we obtain

$\sum f_t = ma_t = mdv/dt$ $\sum f_n = ma_n = mv^2/\rho$. **Example 2** The

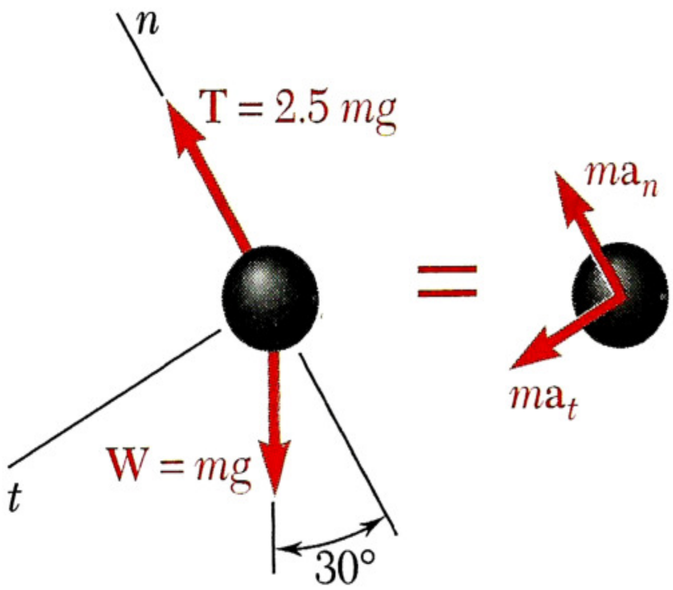


bob of a 2-m pendulum describes an arc of a circle in a vertical plane. If the tension in the cord is 2.5 times the weight of the bob for the position shown, find the velocity and acceleration of the bob in that position.



Solution

The weight of the bob is $W = mg$, the tension in the cord is thus $2.5mg$. Recalling that a_n, is directed toward o and assuming a_t as shown, we apply Newton's second law and obtain



$$(+)\sum f_t = ma_t \qquad mg\sin 30 = ma_t$$

$$(+)\sum f_n = ma_n \qquad 2.5mg - mg\cos 30 = ma_n$$

$$a_n = 1.634g = 16.01m/s^2$$

since $a_n = v^2/\rho$, we have $v^2 = \rho a_n = (2)(16.03m/s^2)$

then

$$v = \pm 5.66m/s$$

7 Curvilinear motion of particles

7.1 Position, velocity, and acceleration

When a particle moves along a curve other than straight line, we say that the particle curivlinear motion (motion in a plane). We can define the position of the particle by replacing the distance x in straight line by the vector $\vec{r}$, and follow the same procedure for defining the velocity v and acceleration $\vec{a}$ at any time t. So, we find

$$\vec{v} = \frac{d\vec{t}}{dt}, \qquad \vec{a} = \frac{d\vec{v}}{dt}$$

In this case both velocity $\vec{v}$ and acceleration $\vec{a}$ have two components, one in x-axis direction and other in y-axis direction.

$$\vec{v} = \frac{d\vec{t}}{dt} = v_x\vec{i} + v_y\vec{j} = \dot{x}\vec{i} + \dot{y}\vec{j}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = a_x\vec{i} + a_y\vec{j} = \ddot{x}\vec{i} + \ddot{y}\vec{j}$$

8 Motion of projectile

Applying the equation of curvilinear motion on a projectile case, we have

$$a_x = \ddot{x} = 0, \qquad a_y = \ddot{y} = -g$$

Denoting by, x_0, y_0 the coordinates of the projectiler, $(v_x)_0, (v_y)_0$ the components of initial velocity $\vec{v}_0$ of the projectile, on integrating twice , we obtain

$$v_x = \dot{x} = (v_x)_0, \qquad v_y = \dot{y} = (v_y)_0 - gt$$

$$x = x_0 + (v_x)_0 t, \qquad y = y_0 + (v_y)_0 - \frac{1}{2}gt^2$$

8.1 Examples

Example 1 A projectile is fired from the edge of a $150 - m$ cliff with an initial velocity of $180m/s$ at angle of 30 with the horizontal. Neglecting air resistance, find (a) the horizontal distance from the gun to the point where the projectile stricke the ground (b) the greatest elevation above the ground reached by the projectile.

Solution

The vertical motion:
Choosing the positive sense of the y axis upward and placing the origin at the gun, we have:

$$(v_y)_0 = (180m/s)\sin 30 = +90m/s$$

$$a = -9.81m/s^2$$

Substituting into the equation of uniformaly accelerated motion, we have:

$$v_y = (v - y)_0 + at \rightarrow v_y = 90 - 9.81t$$

$$y = (v_y)_0 t + (1/2)at^2 \rightarrow y = 90t - 4.90t^2 \qquad (2)$$

$$v_y^2 = (v_y)_0 + 2ay \rightarrow v_y^2 = 8100 - 19.62y$$

Horizontal motion:

Choosing the positive sense of the x axis to the right, we have

$$(v_x)_0 = (180m/s)\cos 30 = +155.9m/s$$

Substituting into the equation of uniform motion, we obtain

$$x = (v_x)_0 t \rightarrow x = 155.9t \qquad (4)$$

Horizontal distance:

When the projectile strickes the ground, we have

$$y = -150m$$

carrying this value into Eq (2) for the vertical motion, we write

$$-150 = 90t - 4.90t^2 \rightarrow t62 - 18.37t - 30.6 = 0$$

$$t = 19.91s$$

Carrying $t = 19.91s$ into Eq (4) for the horizontal motion, we obtain

$$x = 155.9(19.91) = 3100m$$

Greatest elevation: When the projectile reaches its greatest elevation, we have $v_y = 0$, carrying this value into Eq (3) for the vertical motion, we write

$$0 = 8100 - 19.62y \rightarrow y = 413m$$

The greatest elevation above ground $= 150m + 413m = 563m$ **Example 2** A projectile is fired with an initial velocity of $240m/s$ at a target B located $600m$ above the gun A and a horizontal distance of $3600m$. Neglecting air resistance, determine the value of the firing angle α.

Solution

The horizontal and vertical motion will be considered seoarately.

Horizontal motion: Placing the origin of the coordinate axes at the gun, we have

$$(v_x)_0 = 240\cos\alpha$$

Substituting into the equation of uniform horizontal motion, we obtain

$$x = (v_x)_0 t \qquad\qquad x = (240\cos\alpha)t$$

the time required for the projectile to move through a horizontal distance of $3600m$ is obtianed by setting x equal to $3600m$, then

$$3600 = (240\cos\alpha)t \qquad \Rightarrow \qquad t = \frac{2600}{240\cos\alpha} = \frac{15}{\cos\alpha}$$

Vertical motion:

$$(v_y)_0 = 240\sin\alpha; \qquad\qquad a = -9.80m/s^2$$

Substituting into the equation of uniformally accelerated vertical motion, we obtain

$$y = (v_y)_0 t + \frac{1}{2}at^2 \qquad\qquad y = (240\sin\alpha)t - 4.9t^2$$

Projectile hits target: When $x = 3600m$, we must have $y = 600m$. Substituting for y and setting t equal to the value found above, we have

$$600 = 240\sin\alpha\frac{15}{\cos\alpha} - 4.9(15/\cos\alpha)^2$$

Since $1/\cos^2\alpha = \sec^2\alpha = 1 + \tan^2\alpha$, we have

$$600 = 240(15)\tan\alpha - 4.9(15)^2(1 + \tan^2\alpha)$$

This means

$$1103 \tan^2 \alpha - 3600 \tan \alpha + 1703 = 0$$

Solving for $\tan \alpha$, we have

$$\tan \alpha = 0,565 \qquad \text{and} \qquad \tan \alpha = 2.75$$

Which means tha

$$\alpha = 29.5^o \qquad \text{and} \qquad \alpha = 70.0^o$$

Then the target will be hit if either of these two firing angles is used. **Example 3**

A motorist is traviling on a curved section of highway of radius $750m$ at the speed of $100km/h$. The motorist suddenly applies the brakes, causing the automobile to slow down at a constant rate. Knowing that after $8s$ the speed has been reduced to $75km/h$, determine the acceleration of the automobile immediatley after the brakes have been applied.

Solution

Tangential components of acceleration: First the speeds are expressed in m/s

$$100km/h = (100km/h)(1000m/1km)(1h/3600s)$$

=27.8 m/s Similarly $75km/h = 20.8m/s$

Since the automobile slows down at a constant rate, we have

$$a_t = \text{average} a_t = \frac{\Delta v}{\Delta t} = \frac{20.8 - 27.8}{8} = -0.875m/s^2$$

Normal components of acceleration: Immediately after the brakes have been applied, the speed is still $27.8m/s$, and we have

$$a_n = v^2/\rho = \frac{(27.8)^2}{750} = 1.03m/s^2$$

Magnitude and direction of acceleration: The magnitude and direction of acceleration of the resultant $\vec{a}$ of the components a_n and a_t are

$$\tan\alpha = \frac{a_n}{a_t} = \frac{1.03}{0.875} \qquad \Rightarrow \qquad \alpha = 49.7^o$$

$$a = |\vec{a}| = \frac{a_n}{\sin\alpha} = \frac{1.03}{\sin 49.7} \qquad \Rightarrow \qquad a = |\vec{a}| = 1.35 m/s^2$$

Example 4 Determine the rated speed of a highwaycurve of radius $\rho = 120m$ banked through an angle $\theta = 18^o$. The rated speed of a banked highway curve is the speed at which a car should travel if no leteral friction force is to be exerted on its wheels.

Solution

Suppose that the car mass $m = W/g$, where W is the car weight. Since there is no friction force is exerted on the car, the reaction R of the road is perpendicular to the road way. Using the normal and tangential components of the acceleration with Newton's second law, we obtain

$$+\uparrow \sum F_y = 0 \qquad \rightarrow \qquad R\cos\theta - mg = 0 \quad \rightarrow \quad R = \frac{mg}{\cos\theta}$$

$$\overset{+}{\leftarrow} \sum F_n = ma_n \qquad \rightarrow \quad R\sin\theta = ma_n$$

Substituting for R, using $a_n = v^2/\rho$, we obtain

$$\frac{mg}{\cos\theta}\sin\theta = m\frac{v^2}{\rho} \qquad \rightarrow \quad v^2 = g\rho\tan\theta$$

Substituting $\rho = 120$ and $\theta = 18^o$ into this equation, we obtain

$$v^2 = (9.8)(120)\tan 18^o$$

$$\rightarrow \qquad v = 19.6 m/s = 19.6(\frac{1}{1000})(3600) = 70.6 km/h$$

9 Work and energy

Consider a particle which moves from a point A to a point A'. If $\vec{r}$ denotes the position vector corresponding to point A, the small vector joining A and A' can be denoted by the differential $d\vec{r}$, the vector $d\vec{r}$ is called the displacement of the particle. Let us assume that a force $\vec{F}$ corresponding to the displacement $d\vec{r}$ is defined as

$$dU = \vec{F} \cdot d\vec{r} \qquad (1)$$

obtained by forming the scalar product of the foce $\vec{F}$ and the displacement $d\vec{r}$. Recalling the scalar product definition, we write

$$dU = F ds \cos\alpha$$

We can express the dU in terms of the rectangular components of the force and desplacement

$$dU = F_x dx + F_y dy + F_z dz$$

The work of $\vec{F}$ during a finite displacement of the particle from A_1 and A_2 is obtained by integrating eq (1) along the path. This work, denoted by $U_{1\to2}$ is

$$U_{1\to2} = \int_{A_1}^{A_2} \vec{F} \cdot d\vec{r}$$

obtaining that $F\cos\alpha$ represents the tangential component F_t of the force, we can express the work $U_{1\to2}$ as

$$U_{1\to2} = \int_{A_1}^{A_2} (F\cos\alpha) ds = \int_{s_1}^{s_2} F_t ds$$

In terms of the rectangular component, we have

$$U_{1\to2} = \int_{A_1}^{A_2} (F_x dx + F_y dy + F_z dz)$$

9.1 Kinetic energy of a particle: Priciple of work and energy

Consider a particle of mass m acted upon by a force $\vec{F}$ and moving alone a path which is either rectangular or curved. Expressing Newton's seconf law in terms of the tangential components of the force and of acceleration, we write

$$F_t = ma_t \qquad \text{or} \qquad F_t = mdv/dt$$

where v is the speed of the particle.
Then

$$F_t = mdv/dsds/dt = mvdv/ds$$

Then

$$F_t ds = mvdv$$

integrating from A_1 where $s = s_1$ and $v = v_1$ to A_2, wher $s = s_2$ and $v = v_2$

$$\int_{s_1}^{s_2} F - tds = m\int_{v_1}^{v_2} vdv = (1/2)mv_2^2 - (1/2)mv_1^2$$

The l.h.s. represnets the work $U_{1\to 2}$ of the force $\vec{F}$. The expression $(1/2)mv^2$ is the kinetic energy of the particle and denoted by T, i.e.,

$$T = (1/2)mv^2$$

this means

$$U_{1\to 2} = T_2 - T_1$$

which epresses that "when a particle moves from A_1 to A_2 under the action of a force $\vec{F}$, the work of the force $\vec{F}$ is equal to the change in the kinetic energy of the particle". This is known as the principle of work and energy.

9.1.1 Potential energy

Let us consider a body of weight w which moves along a curved path from point A_1 of elevation y_1 to a point A_2 of elevation y_2. We know the work of the force of gravity w during this displacement is

$$U_{1\to 2} = W_{y_1} - W_{y_2}$$

The function W is called the potential energy of the body with respect to the force of gravity w and is denoted by

$$U_{1\to 2} = (V_g)_1 - (V_g)_2 \text{with} V_g = W_y$$

Note, if $(V_g)_2 > (V_g)_1$, if the potential energy increases during the displacement, the work $U_{1\to 2}$ is negative. If on the other hand, the work of w is positive, the potential energy decreases. Therefore, the potential energy V_g of the body provides a measure of the work of the work which can be done its weight w.

9.1.2 Conservation force

As previous, a force $\vec{F}$ on a particle is said to be conservative if its work $U_{1\to 2}$ is independent of the path followed by the particle as it moves from point A to B.

We can write:

$$U_{1\to 2} = V(x_1, y_1, z_1) - V(x_2, y_2, z_2)$$

or, for short,

$$U_{1\to 2} = V_1 - V_2$$

As we know, the function $V(x, y, z)$ is the potential energy of potential function. If A is coincide with B, we have $V_1 = V_2$ and the work is zero. Thus for any conservation force $\vec{F}$ we have,

$$\oint \vec{F} \cdot d\vec{r} = 0$$

where the circle on the integral sign indicates that the path is closed.
For two neighboring points $A(x, y, z)$ and $B(x + dx, y + dy, z + dz)$, the elementary work dU corresponding to the displacement $d\vec{r}$ from A to B is

$$dU = V(x, y, z) - V(x + dx, y + dy, z + dz)$$

or

$$dU = -dV(x, y, z) \qquad (1)$$

we know that

$$dU = F_x dx + F_y dy + F_z dz \qquad (2)$$

Using (1) and (2), we obtain

$$F_x dx + F_y dy + F_z dz = -[(\partial V/\partial x)dx + (\partial V/\partial y)dy + (\partial V/\partial z)dz]$$

this means

$$F_x = -\partial V/\partial x, \qquad F_y = -\partial V/\partial y, \qquad F_z = -\partial V/\partial z$$

this means that the force $\vec{F}$ can be expressed as

$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k} = -[(\partial V/\partial x)\vec{i} + (\partial V/\partial y)\vec{j} + (\partial V/\partial z)\vec{k}]$$

The vector in parantheses is known as the gradient of the scalar function V and is denoted by $gradV$. For any conservative force

$$\vec{F} = -\vec{grad}V$$

9.2 Examples

Example 1 A 9 kg collar slides without friction along a vertical rod. The spring attached to the collar has an undeformed length of 10 cm and a constant of 525 N/m. If the collar is released from rest in position 1, determine its velocity after it has moved 15 cm to position 2. % position 1: Potential energy
The elnogation og the spring is

$$x_1 = 20cm - 10cm = 10cm$$

and we have

$$V_e = (1/2)kx_1^2 = (1/2)(525)(0.1)^2 = 2.625N.m$$

Choosing the datum we have $V_g = 0$, therefore,

$$V_1 = V_e + V_g = 2.625N.m$$

Kinetic energy:
Since the velocity in position 1 is zero, $T_1 = 0$ % position 2: Potential energy
The elongation of the spring is

$$x_2 = 25cm - 10cm = 15cm$$

and we have

$$V_e = (1/2)kx_2^2 = (1/2)(525)(0.15)^2 = 5.9N.m$$

Therefore,

$$V_2 = V_e + V_g = 5.9 - 13.25 = -7.35N.m$$

Kinetic energy:

$$T_2 = (1/2)mv_2^2 = (1/2)(9)v_2^2 = 4.5v_2^2$$

Conservation of energy: Applying the principle of conservation of energy between position 1 and position 2 we obtain

$$T_1 + V_1 = T_2 + V_2$$

this means

$$0 + 2.625N.m = 4.5v_2^2 - 7.35N.m$$

we obtain

$$v_2 = \pm 1.5m/s$$

9.2.1 Kinematics of rigid bodies

In the following we investigate the relation existing between the time, the positions, the velocities, and the accelerations of the variuos particles forming the rigid body:

1. Translation: A motion is said to be a translation if any straight line inside the body keeps the same direction during the motion. It can also be observed that along parallel paths. If these paths are straight lines, the motion is said to be a rectiline transition, if the paths are curved lines, the motion is a curvilinear translation.
2. Rotation about a fixed axis: In this motion, the particles forming the rigid body move in parallel planes along circles centered on the same fixed axis. If this axis, called the axis of rotation, intersects the rigid body, the particles located on the axis have zero velocity and zero acceleration.
3. Central plane motion: There are many other types of planes motion that is, motions in which all the particles of the body move in parallel

planes. Any plane motion which is neither a rotation nor a translation is referred to as a general plane motion.

4. Motion about a fixed point: The three-dimensional motion of a rigid body attached at a fixed point O, for example, the motion of a top on a rough floor, is known as motion about fixed point.

5. General motion: Any motion of a rigid body which does not fall in any of the categories above is referred to as a general motion.

(1) Translation: Consider a rigid body in translation (either rectilinear or curvilinear translation), and let A and B by any two of its particles. Denoting, respectively, by $\vec{r}_A$ and $\vec{r}_B$ the position vectors of A and B with respect to a fixed frame of reference and by $\vec{r}_{AB}$. the vector joining A and B, we write

$$\vec{r}_B = \vec{r}_A + \vec{r}_{AB}$$

Let us differentiate this relation with respect to t, bearing im mind that $\vec{r}_{AB}$ is constsnt in direction and magnitude, we have

$$\vec{v}_b = \vec{v}_b$$

Differentiating once more, we write

$$\vec{a}_B = \vec{a}_A$$

Thus, when a rigid body is in translation, all the points of the body have the same velocity and the same acceleration at any given instant.

(2) Rotation about a fixed axis: Consider a rigid body which rotates about a fixed axis AA'. The angle θ is known as the angular coordinate of the

body and is defined as positive when viewed as counter clockwise from A'. $\dot{\theta}$ denotes the time derivative of θ. The velocity v is perpendicular to the plane containinig AA' and $\vec{r}$. The vector $\vec{\omega} = \dot{\theta}\vec{k}$ which formed the vector product $\vec{\omega} \times \vec{r}$, we thus write

$$\vec{v} = d\vec{r}/dt = \vec{\omega} \times \vec{r}$$

The vector $\vec{\omega} = \omega\vec{k} = \dot{\theta}\vec{k}$, is called the angular velocity of the body. The acceleration $\vec{a}$ is obtained by

$$\vec{a} = d\vec{v}/dt = d/dt(\vec{\omega} \times \vec{r})$$

$$= d\vec{\omega}/dt \times \vec{r} + \vec{\omega} \times d\vec{r}/dt$$

$$= d\vec{\omega}/dt \times \vec{r} + \vec{\omega} \times \vec{v}$$

The vector $d\vec{\omega}/dt$ is denoted by $\vec{\alpha}$ and is called the angular acceleration of the body.
Substituting for $\vec{v} = \vec{\omega} \times \vec{r}$, we have

$$\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

also

$$\vec{\alpha} = \alpha\vec{k} = \dot{\omega}\vec{k} = \ddot{\theta}\vec{k}$$

(3) Equations defining the rotation of a rigid body about a fixed axis: The motion of rigid body rotating about a fixed AA' is said to be known when its angular coordinate θ can be expressed as a known that

$$\omega = d\theta/dt \qquad (1)$$

$$\alpha = d\omega/dt = d^2\theta/dt^2 \qquad (2)$$

or, solving (1) for dt and substituting into (2)

$$\alpha = \omega d\omega/dt$$

Two particular cases of rotation are

(i) uniformaly accelerated rotation: In this case, the angular acceleration is costant. It is appeared that

$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + (1/2)\alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

Example: Load B is connected to a double pulley by one of two inextensible cables. The motion of the pulley is controlled by cable C, which has a constant acceleration of $0.25 m/s^2$ and an initial velocity of 0.3 m/s, both directed to the right. Determine

(1) The number of revolutions executed by the pulley in 2 s.

(2) The velocity and change in position of the load B after 2 s, and

(3) The acceleration of point D on the rim of the inner pulley at $t = 0$

Solution

(1) Motion of the pulley: Since the cable is inextensible, the velocity of point D is equal to the velocity of D is equal to the acceleration of C.

$$(v_D)_0 - (V_c)_0 = 0.3 m/s$$

$$(a_D)_t = a_c = 0.25 m/s^2$$

since the distance from D to the center of the pulley is 3 in, we write

$$(v_D)_0 = r\omega_0 \rightarrow 0.3 = (0.1)\omega_0 \rightarrow \omega_0 = 3rad/s$$

$$(a_D)_t = r\alpha \rightarrow 0.25 = (0.1)\alpha \rightarrow \alpha = 2.5rad/s^2$$

Using the equations of uniformaly accelerated motion, we obtain, for $t = 2$ s

$$\omega = \omega_0 + \alpha t = 3 + (2.5)(2) = 8rad/s$$

$$\theta = \omega_0 t + (1/2)\alpha t^2 = (3)(0.25) + (1/2)(2.5)(2)^2 = 11rad$$

Number of revolution = 11(1 rev/2π rad) = 1.75 rev.

(2) Motion of load B: Using the following relation between linear and angular motion, with $r = 0.15$ in., we write

$$v_B = r\omega = (0.15 \ \text{m})(8 \ \text{rad/s}) = 12 \ \text{m/s} \rightarrow v_B = 12 \ \text{m/s}$$

$$\Delta y_B = r\theta = (0.15 \ \text{m})(11 \ \text{rad}) = 1.65cm$$

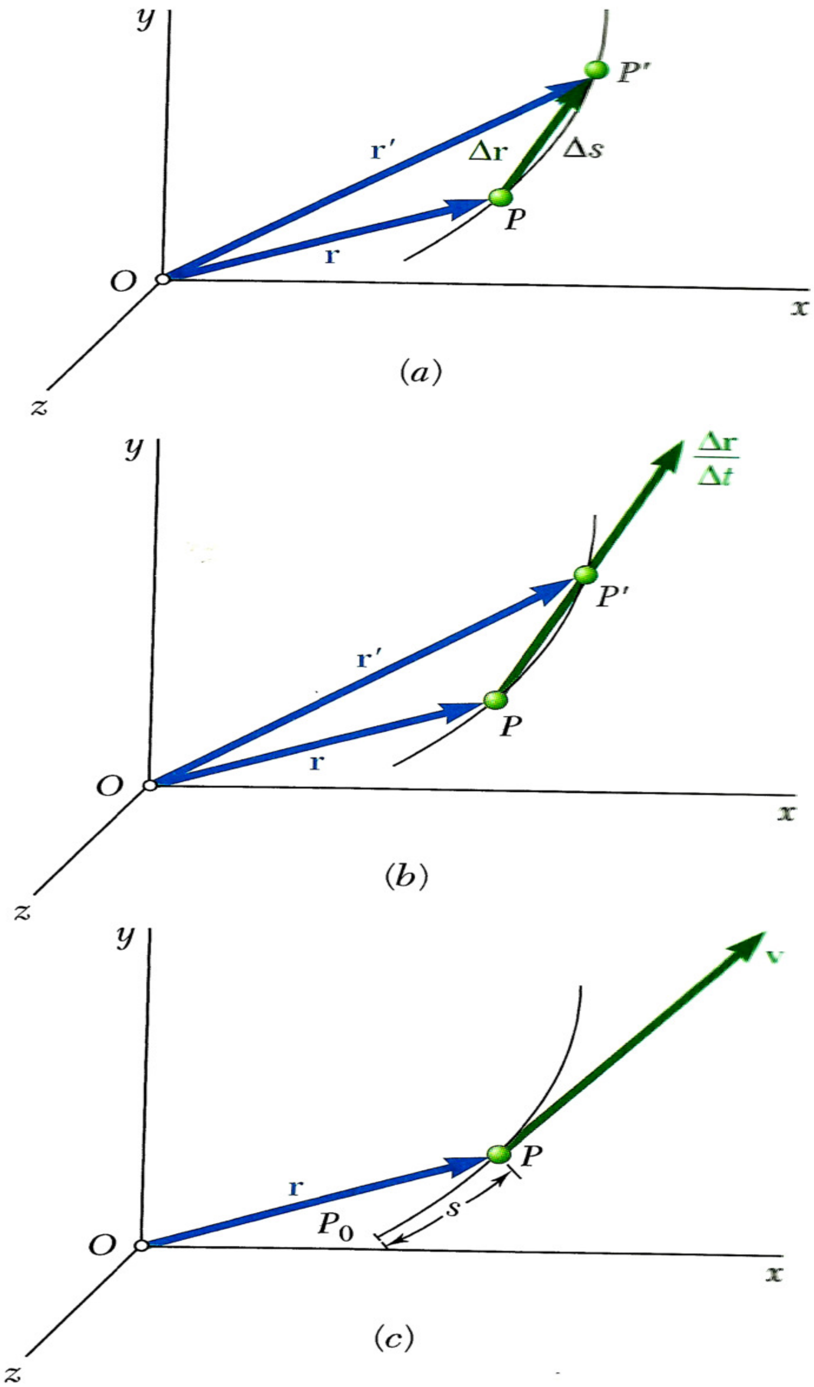
(3) Acceleration of point D at $t = 0$: The tangential component of the acceleration is $(a_D)_t = a_c = 0.25$ m$/s^2$
since, at $t = 0$, $\omega_0 = 4$ rad/s, the normal component of the acceleration is

$$(a_D)_n = r_D\omega_0^2 = (0.1 \ \text{m})(3 \ \text{rad}/s^2) = 0.9 \ \text{m}/s^2$$

The magnitude and directiion of the total acceleration can be obtained by writing

$$\tan\phi = (0.9)(0.25) \rightarrow \phi = 74.48^o$$

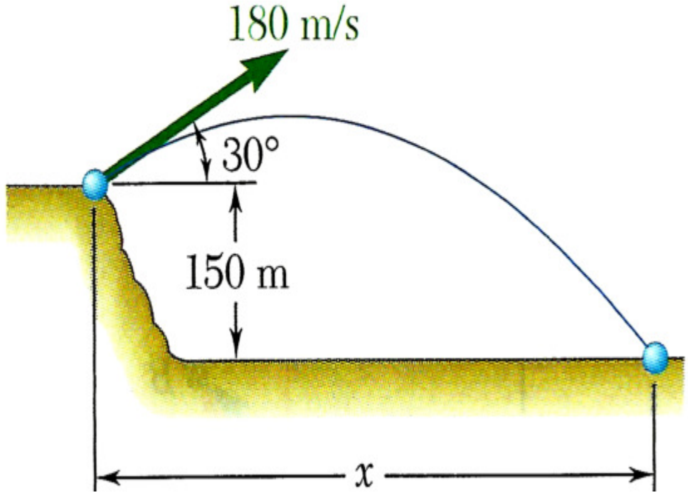
$$a_D \sin 74.48^o = 0.9m/s^2 \rightarrow a_D = 0.93m/s^2$$



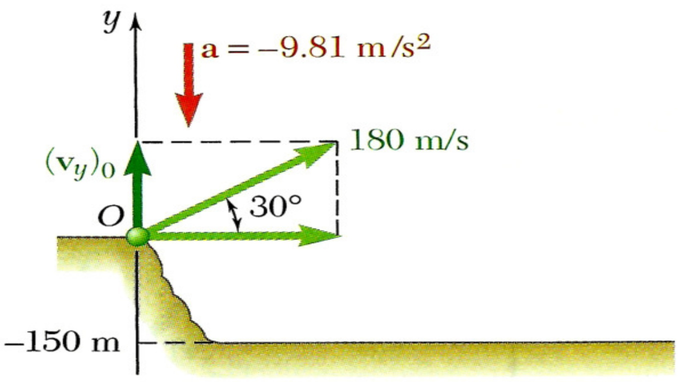
(a)

(b)

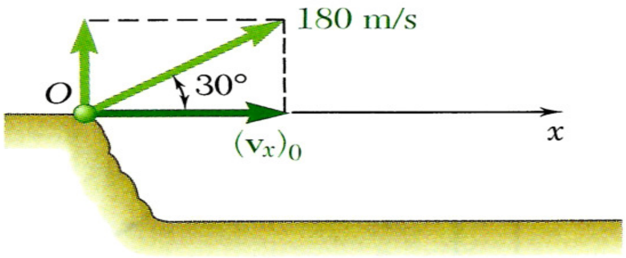
(c)



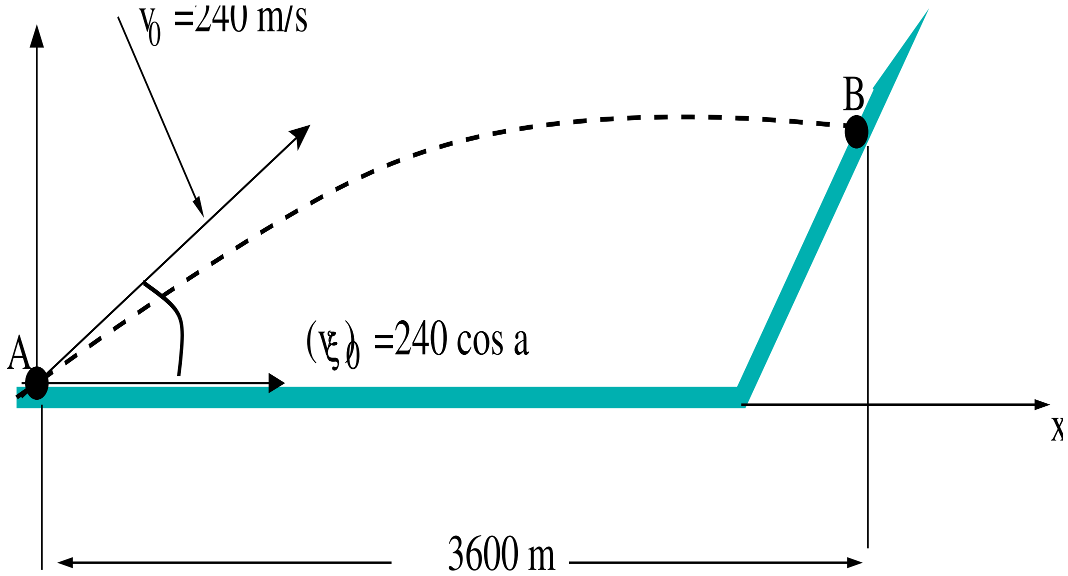
180 m/s
30°
150 m
x



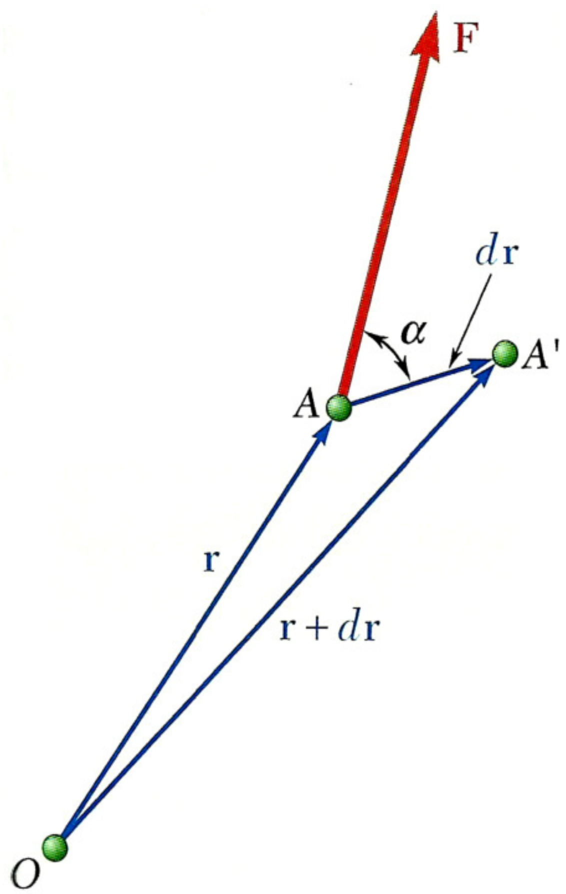
y
a = –9.81 m/s²
180 m/s
(v_y)_0
O
30°
–150 m



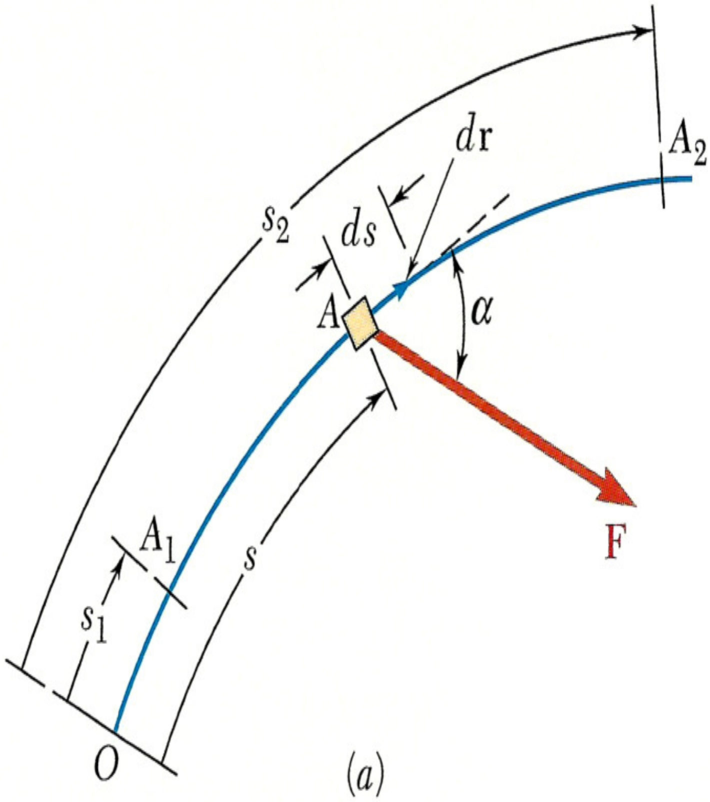
180 m/s
O
30°
(v_x)_0
x

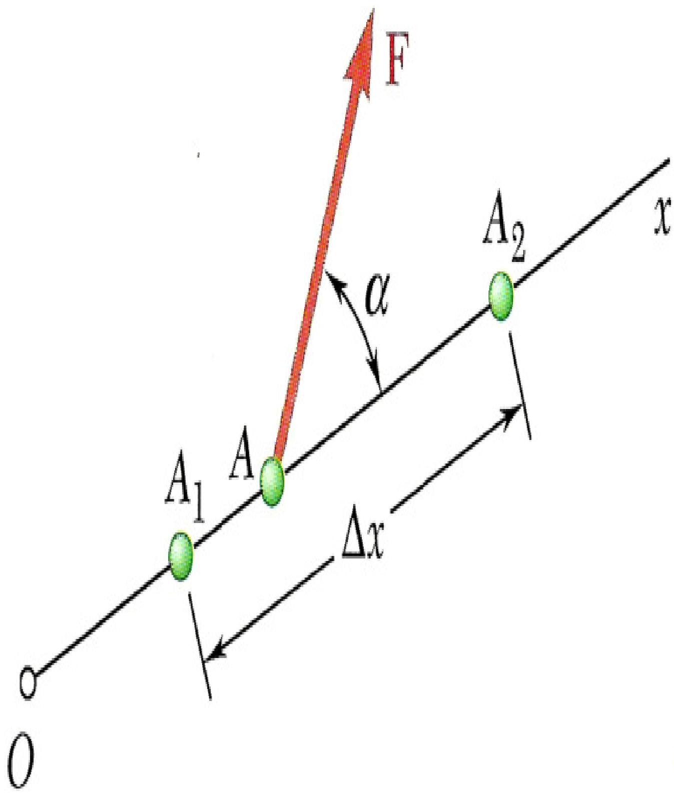


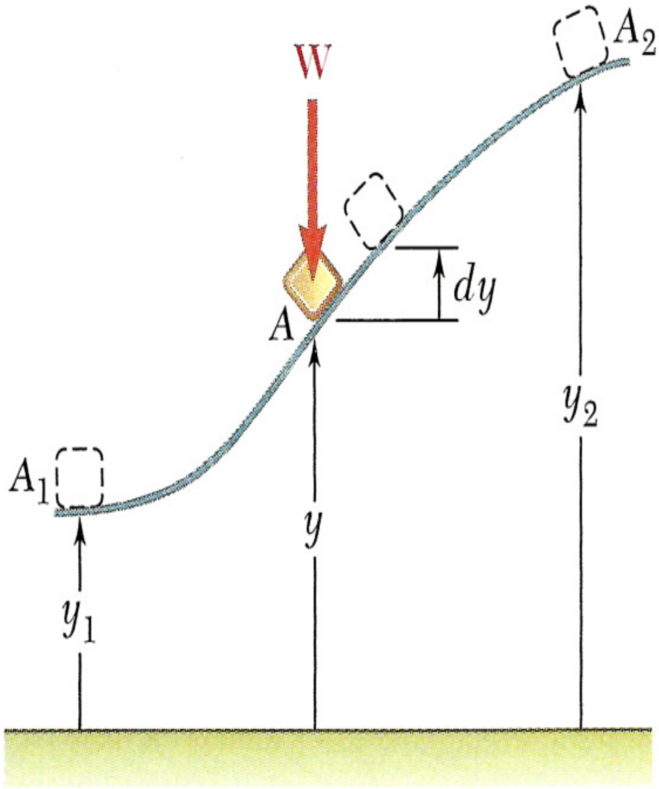
v_0 =240 m/s
B
A
$(v_x)_0$ =240 cos a
x
3600 m

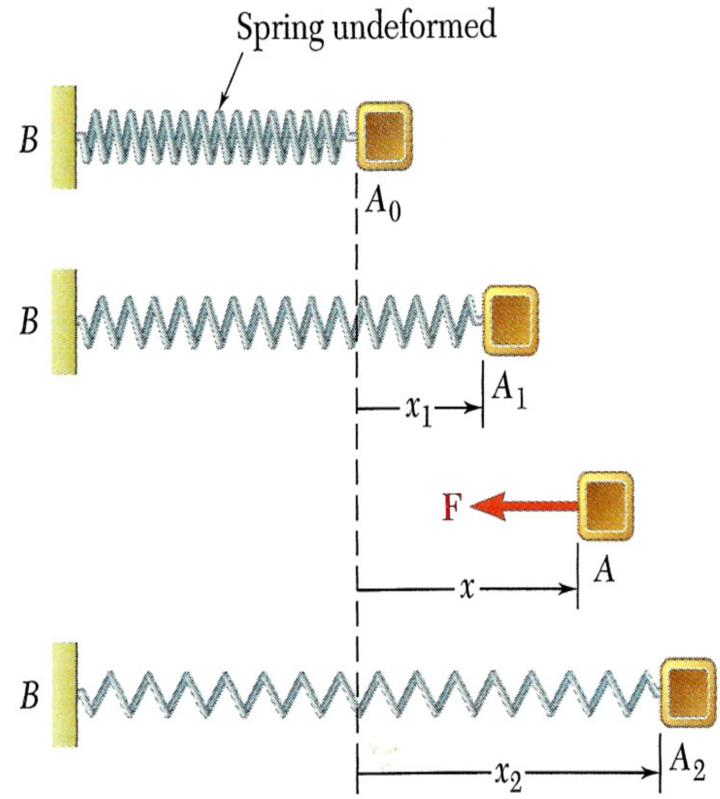


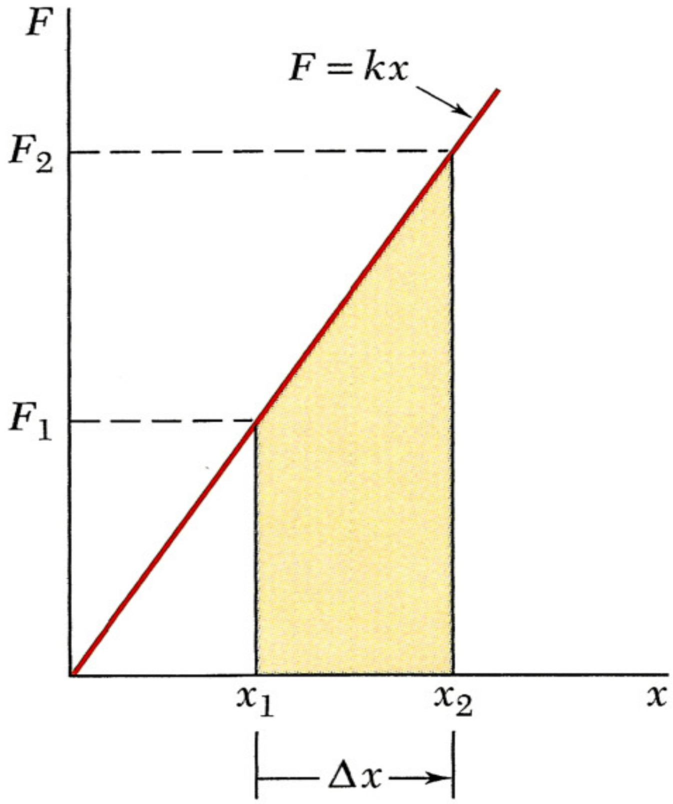
F
dr
α
A'
A
r
r + dr
O

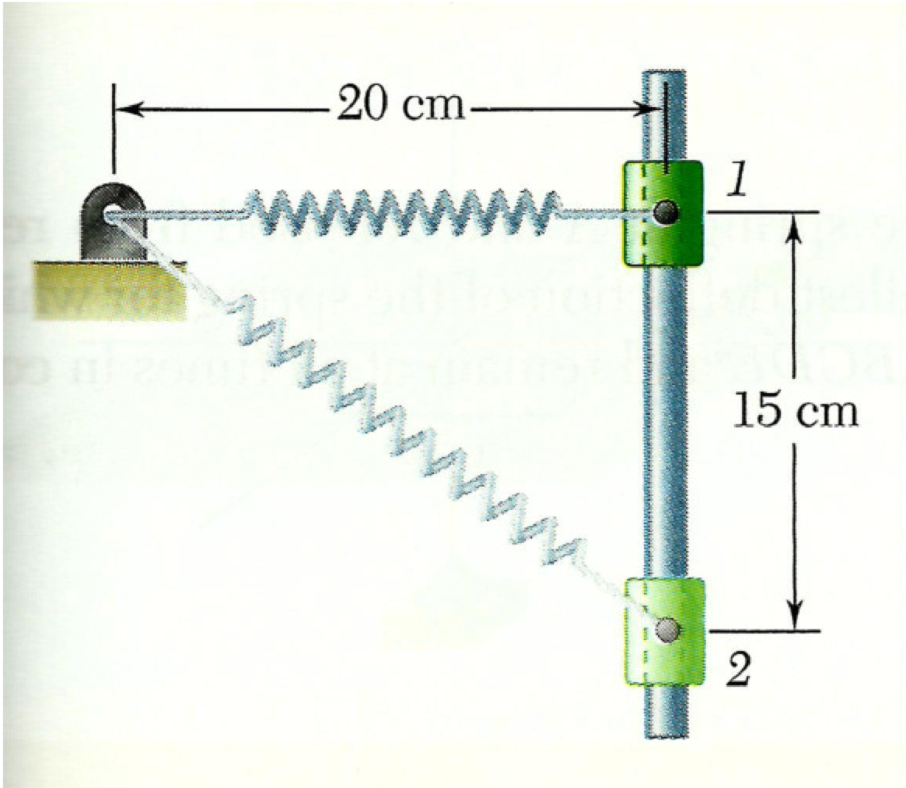

s2
dr
A2
ds
A
α
F
A1
s
s1
O
(a)


F
x
A2
α
A
A1
Δx
O

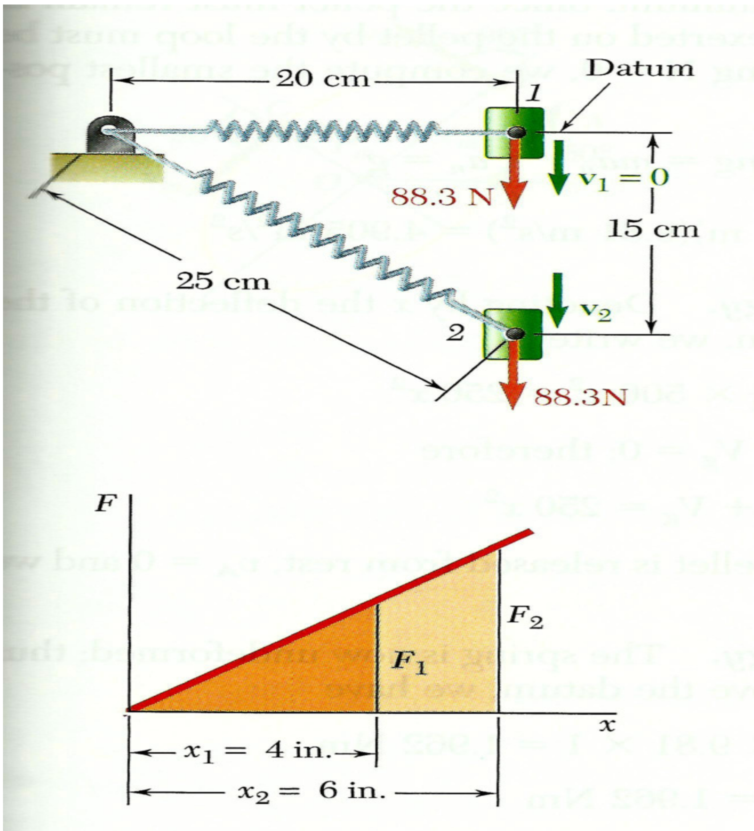

W
A_2
dy
A
y_2
A_1
y
y_1


Spring undeformed
B
A_0
B
x_1
A_1
F
A
x
B
x_2
A_2


F
F = kx
F2
F1
x1
x2
x
Δx



20 cm
1
15 cm
2



20 cm
Datum
1
v1 = 0
88.3 N
15 cm
25 cm
2
v2
88.3N
F
F2
F1
x
x1 = 4 in.
x2 = 6 in.

Chapter 23 Simple Harmonic Motion

Chapter 23 Simple Harmonic Motion

…Indeed it is not in the nature of a simple pendulum to provide equal and reliable measurements of time, since the wide lateral excursions often made may be observed to be slower than more narrow ones; however, we have been led in a different direction by geometry, from which we have found a means of suspending the pendulum, with which we were previously unacquainted, and by giving close attention to a line with a certain curvature, the time of the swing can be chosen equal to some calculated value and is seen clearly in practice to be in wonderful agreement with that ratio. As we have checked the lapses of time measured by these clocks after making repeated land and sea trials, the effects of motion are seen to have been avoided, so sure and reliable are the measurements; now it can be seen that both astronomical studies and the art of navigation will be greatly helped by them…[1]

Christian Huygens

23.1 Introduction: Periodic Motion

There are two basic ways to measure time: by duration or periodic motion. Early clocks measured duration by calibrating the burning of incense or wax, or the flow of water or sand from a container. Our calendar consists of years determined by the motion of the sun; months determined by the motion of the moon; days by the rotation of the earth; hours by the motion of cyclic motion of gear trains; and seconds by the oscillations of springs or pendulums. In modern times a second is defined by a specific number of vibrations of radiation, corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom.

Sundials calibrate the motion of the sun through the sky, including seasonal corrections. A clock escapement is a device that can transform continuous movement into discrete movements of a gear train. The early escapements used oscillatory motion to stop and start the turning of a weight-driven rotating drum. Soon, complicated escapements were regulated by pendulums, the theory of which was first developed by the physicist Christian Huygens in the mid 17^{th} century. The accuracy of clocks was increased and the size reduced by the discovery of the oscillatory properties of springs by Robert Hooke. By the middle of the 18^{th} century, the technology of timekeeping advanced to the point that William Harrison developed timekeeping devices that were accurate to one second in a century.

23.1.1 Simple Harmonic Motion: Quantitative

[1] Christian Huygens, *The Pendulum Clock or The Motion of Pendulums Adapted to Clocks By Geometrical Demonstrations,* tr. Ian Bruce, p. 1.

One of the most important examples of periodic motion is ***simple harmonic motion*** (SHM), in which some physical quantity varies sinusoidally. Suppose a function of time has the form of a sine wave function,

$$y(t) = A\sin(2\pi t / T) \tag{23.1.1}$$

where $A > 0$ is the ***amplitude*** (maximum value). The function $y(t)$ varies between A and $-A$, because a sine function varies between $+1$ and -1. A plot of $y(t)$ *vs.* time is shown in Figure 23.1.

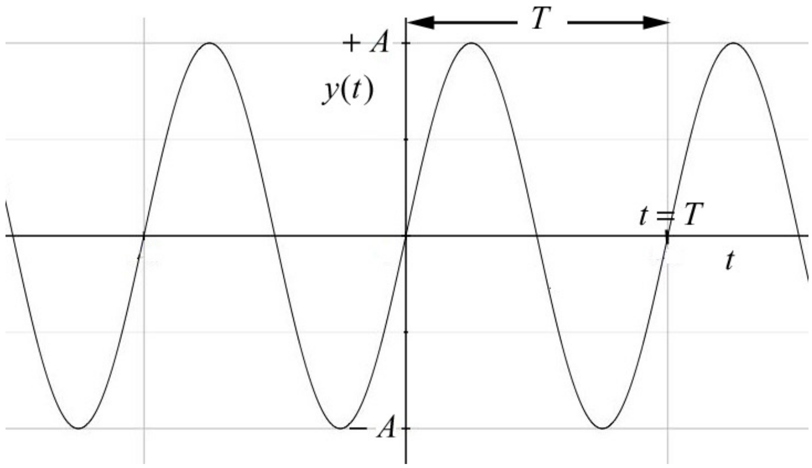


Figure 23.1 Sinusoidal function of time

The sine function is periodic in time. This means that the value of the function at time t will be exactly the same at a later time $t' = t + T$, where T is the ***period***. That the sine function satisfies the periodic condition can be seen from

$$y(t+T) = A\sin\left[\frac{2\pi}{T}(t+T)\right] = A\sin\left[\frac{2\pi}{T}t + 2\pi\right] = A\sin\left[\frac{2\pi}{T}t\right] = y(t)\,. \tag{23.1.2}$$

The ***frequency***, f, is defined to be

$$f \equiv 1/T\,. \tag{23.1.3}$$

The SI unit of frequency is inverse seconds, $\left[\mathrm{s}^{-1}\right]$, or hertz $[\mathrm{Hz}]$. The ***angular frequency*** of oscillation is defined to be

$$\omega_0 \equiv 2\pi / T = 2\pi f\,, \tag{23.1.4}$$

and is measured in radians per second. (The angular frequency of oscillation is denoted by ω_0 to distinguish from the angular speed $\omega = |d\theta / dt|$.) One oscillation per second, $1\,\mathrm{Hz}$, corresponds to an angular frequency of 2π $\mathrm{rad \cdot s^{-1}}$. (Unfortunately, the same symbol ω is used for angular speed in circular motion. For uniform circular motion the angular speed is equal to the angular frequency but for non-uniform motion the angular speed is not constant. The angular frequency for simple harmonic motion is a constant by definition.) We therefore have several different mathematical representations for sinusoidal motion

$$y(t) = A\sin(2\pi t / T) = A\sin(2\pi f\, t) = A\sin(\omega_0 t) \ . \tag{23.1.5}$$

23.2 Simple Harmonic Motion: Analytic

Our first example of a system that demonstrates simple harmonic motion is a spring-object system on a frictionless surface, shown in Figure 23.2

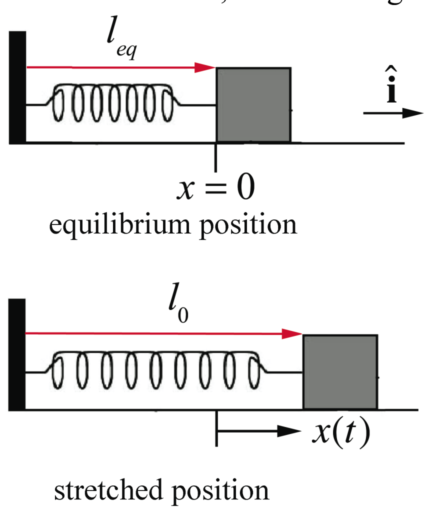


Figure 23.2 Spring-object system

The object is attached to one end of a spring. The other end of the spring is attached to a wall at the left in Figure 23.2. Assume that the object undergoes one-dimensional motion. The spring has a spring constant k and equilibrium length l_{eq}. Choose the origin at the equilibrium position and choose the positive x-direction to the right in the Figure 23.2. In the figure, $x > 0$ corresponds to an extended spring, and $x < 0$ to a compressed spring. Define $x(t)$ to be the position of the object with respect to the equilibrium position. The force acting on the spring is a linear restoring force, $F_x = -k\,x$ (Figure 23.3). The initial conditions are as follows. The spring is initially stretched a distance l_0 and given some initial speed v_0 to the right away from the equilibrium position. The initial position of the stretched spring from the equilibrium position (our choice of origin) is $x_0 = (l_0 - l_{eq}) > 0$ and its initial x-component of the velocity is $v_{x,0} = v_0 > 0$.

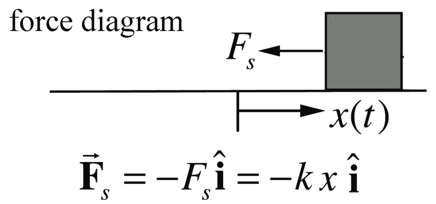


Figure 23.3 Free-body force diagram for spring-object system

Newton's Second law in the x-direction becomes

$$-k\,x = m\frac{d^2x}{dt^2}. \tag{23.2.1}$$

This equation of motion, Eq. (23.2.1), is called the ***simple harmonic oscillator equation*** (SHO). Because the spring force depends on the distance x, the acceleration is not constant. Eq. (23.2.1) is a second order linear differential equation, in which the second derivative of the dependent variable is proportional to the negative of the dependent variable,

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x. \tag{23.2.2}$$

In this case, the constant of proportionality is k/m,

Eq. (23.2.2) can be solved from energy considerations or other advanced techniques but instead we shall first guess the solution and then verify that the guess satisfies the SHO differential equation (see Appendix 22.3.A for a derivation of the solution).

We are looking for a position function $x(t)$ such that the second time derivative position function is proportional to the negative of the position function. Since the sine and cosine functions both satisfy this property, we make a preliminary *ansatz* (educated guess) that our position function is given by

$$x(t) = A\cos((2\pi/T)t) = A\cos(\omega_0\,t), \tag{23.2.3}$$

where ω_0 is the angular frequency (as of yet, undetermined).

We shall now find the condition that the angular frequency ω_0 must satisfy in order to insure that the function in Eq. (23.2.3) solves the simple harmonic oscillator equation, Eq. (23.2.1). The first and second derivatives of the position function are given by

$$\frac{dx}{dt} = -\omega_0 A \sin(\omega_0 t)$$
$$\frac{d^2x}{dt^2} = -\omega_0^2 A \cos(\omega_0 t) = -\omega_0^2 x. \tag{23.2.4}$$

Substitute the second derivative, the second expression in Eq. (23.2.4), and the position function, Equation (23.2.3), into the SHO Equation (23.2.1), yielding

$$-\omega_0^2 A \cos(\omega_0 t) = -\frac{k}{m} A \cos(\omega_0 t)\,. \tag{23.2.5}$$

Eq. (23.2.5) is valid for all times provided that

$$\omega_0 = \sqrt{\frac{k}{m}}\,. \tag{23.2.6}$$

The period of oscillation is then

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}\,. \tag{23.2.7}$$

One possible solution for the position of the block is

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}\,t\right), \tag{23.2.8}$$

and therefore by differentiation, the x-component of the velocity of the block is

$$v_x(t) = -\sqrt{\frac{k}{m}}\,A\sin\left(\sqrt{\frac{k}{m}}\,t\right). \tag{23.2.9}$$

Note that at $t = 0$, the position of the object is $x_0 \equiv x(t=0) = A$ since $\cos(0) = 1$ and the velocity is $v_{x,0} \equiv v_x(t=0) = 0$ since $\sin(0) = 0$. The solution in (23.2.8) describes an object that is released from rest at an initial position $A = x_0$ but does not satisfy the initial velocity condition, $v_x(t=0) = v_{x,0} \neq 0$. We can try a sine function as another possible solution,

$$x(t) = B\sin\left(\sqrt{\frac{k}{m}}\,t\right). \tag{23.2.10}$$

This function also satisfies the simple harmonic oscillator equation because

$$\frac{d^2x}{dt^2} = -\frac{k}{m}B\sin\left(\sqrt{\frac{k}{m}}\,t\right) = -\omega_0{}^2\,x\,, \qquad (23.2.11)$$

where $\omega_0 = \sqrt{k/m}$. The x-component of the velocity associated with Eq. (23.2.10) is

$$v_x(t) = \frac{dx}{dt} = \sqrt{\frac{k}{m}}B\cos\left(\sqrt{\frac{k}{m}}\,t\right). \qquad (23.2.12)$$

The proposed solution in Eq. (23.2.10) has initial conditions $x_0 \equiv x(t=0) = 0$ and $v_{x,0} \equiv v_x(t=0) = (\sqrt{k/m})B$, thus $B = v_{x,0} / \sqrt{k/m}$. This solution describes an object that is initially at the equilibrium position but has an initial non-zero x-component of the velocity, $v_{x,0} \neq 0$.

23.2.1 General Solution of Simple Harmonic Oscillator Equation

Suppose $x_1(t)$ and $x_2(t)$ are both solutions of the simple harmonic oscillator equation,

$$\begin{aligned}\frac{d^2}{dt^2}x_1(t) &= -\frac{k}{m}x_1(t)\\ \frac{d^2}{dt^2}x_2(t) &= -\frac{k}{m}x_2(t).\end{aligned} \qquad (23.2.13)$$

Then the sum $x(t) = x_1(t) + x_2(t)$ of the two solutions is also a solution. To see this, consider

$$\frac{d^2x(t)}{dt^2} = \frac{d^2}{dt^2}(x_1(t) + x_2(t)) = \frac{d^2x_1(t)}{dt^2} + \frac{d^2x_2(t)}{dt^2}. \qquad (23.2.14)$$

Using the fact that $x_1(t)$ and $x_2(t)$ both solve the simple harmonic oscillator equation (23.2.13), we see that

$$\begin{aligned}\frac{d^2}{dt^2}x(t) &= -\frac{k}{m}x_1(t) + -\frac{k}{m}x_2(t) = -\frac{k}{m}\big(x_1(t) + x_2(t)\big)\\ &= -\frac{k}{m}x(t).\end{aligned} \qquad (23.2.15)$$

Thus the *linear combination* $x(t) = x_1(t) + x_2(t)$ is also a solution of the SHO equation, Eq. (23.2.1). Therefore the sum of the sine and cosine solutions is the *general solution*,

$$x(t) = C\cos(\omega_0\,t) + D\sin(\omega_0\,t)\,, \qquad (23.2.16)$$

where the constant coefficients C and D depend on a given set of initial conditions $x_0 \equiv x(t=0)$ and $v_{x,0} \equiv v_x(t=0)$ where x_0 and $v_{x,0}$ are constants. For this general solution, the x-component of the velocity of the object at time t is then obtained by differentiating the position function,

$$v_x(t) = \frac{dx}{dt} = -\omega_0 C \sin(\omega_0 t) + \omega_0 D \cos(\omega_0 t). \tag{23.2.17}$$

To find the constants C and D, substitute $t=0$ into the Eqs. (23.2.16) and (23.2.17). Because $\cos(0)=1$ and $\sin(0)=0$, the initial position at time $t=0$ is

$$x_0 \equiv x(t=0) = C. \tag{23.2.18}$$

The x-component of the velocity at time $t=0$ is

$$v_{x,0} = v_x(t=0) = -\omega_0 C \sin(0) + \omega_0 D \cos(0) = \omega_0 D. \tag{23.2.19}$$

Thus

$$C = x_0 \quad \text{and} \quad D = \frac{v_{x,0}}{\omega_0}. \tag{23.2.20}$$

The position of the object-spring system is then given by

$$x(t) = x_0 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + \frac{v_{x,0}}{\sqrt{k/m}} \sin\left(\sqrt{\frac{k}{m}}\, t\right) \tag{23.2.21}$$

and the x-component of the velocity of the object-spring system is

$$v_x(t) = -\sqrt{\frac{k}{m}}\, x_0 \sin\left(\sqrt{\frac{k}{m}}\, t\right) + v_{x,0} \cos\left(\sqrt{\frac{k}{m}}\, t\right). \tag{23.2.22}$$

Although we had previously specified $x_0 > 0$ and $v_{x,0} > 0$, Eq. (23.2.21) is seen to be a valid solution of the SHO equation for any values of x_0 and $v_{x,0}$.

Example 23.1: Phase and Amplitude

Show that $x(t) = C\cos\omega_0 t + D\sin\omega_0 t = A\cos(\omega_0 t + \phi)$, where $A = (C^2 + D^2)^{1/2} > 0$, and $\phi = \tan^{-1}(-D/C)$.

Solution: Use the identity $A\cos(\omega_0 t+\phi) = A\cos(\omega_0 t)\cos(\phi) - A\sin(\omega_0 t)\sin(\phi)$. Thus $C\cos(\omega_0 t) + D\sin(\omega_0 t) = A\cos(\omega_0 t)\cos(\phi) - A\sin(\omega_0 t)\sin(\phi)$. Comparing coefficients we see that $C = A\cos\phi$ and $D = -A\sin\phi$. Therefore

$$(C^2 + D^2)^{1/2} = A^2(\cos^2\phi + \sin^2\phi) = A^2 \ .$$

We choose the positive square root to ensure that $A > 0$, and thus

$$A = (C^2 + D^2)^{1/2} \tag{23.2.23}$$

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{-D / A}{C / A} = -\frac{D}{C},$$

$$\phi = \tan^{-1}(-D / C). \tag{23.2.24}$$

Thus the position as a function of time can be written as

$$x(t) = A\cos(\omega_0 t + \phi). \tag{23.2.25}$$

In Eq. (23.2.25) the quantity $\omega_0 t + \phi$ is called the ***phase***, and ϕ is called the ***phase constant.*** Because $\cos(\omega_0 t + \phi)$ varies between $+1$ and -1, and $A > 0$, A is the amplitude defined earlier. We now substitute Eq. (23.2.20) into Eq. (23.2.23) and find that the amplitude of the motion described in Equation (23.2.21), that is, the maximum value of $x(t)$, and the phase are given by

$$A = \sqrt{x_0^2 + (v_{x,0} / \omega_0)^2}. \tag{23.2.26}$$

$$\phi = \tan^{-1}(-v_{x,0} / \omega_0 x_0). \tag{23.2.27}$$

A plot of $x(t)$ vs. t is shown in Figure 23.4a with the values $A = 3$, $T = \pi$, and $\phi = \pi / 4$. Note that $x(t) = A\cos(\omega_0 t + \phi)$ takes on its maximum value when $\cos(\omega_0 t + \phi) = 1$. This occurs when $\omega_0 t + \phi = 2\pi n$ where $n = 0, \pm 1, \pm 2, \cdots$. The maximum value associated with $n = 0$ occurs when $\omega_0 t + \phi = 0$ or $t = -\phi / \omega_0$. For the case shown in Figure 23.4a where $\phi = \pi / 4$, this maximum occurs at the instant $t = -T / 8$. Let's plot $x(t) = A\cos(\omega_0 t + \phi)$ vs. t for $\phi = 0$ (Figure 23.4b). For $\phi > 0$, Figure 23.4a shows the plot $x(t) = A\cos(\omega_0 t + \phi)$ vs. t. Notice that when $\phi > 0$, $x(t)$ is shifted to the left compared with the case $\phi = 0$ (compare Figures 23.4a with 23.4b). The function $x(t) = A\cos(\omega_0 t + \phi)$ with $\phi > 0$ reaches its maximum value at an earlier time than the function $x(t) = A\cos(\omega_0 t)$. The difference in phases for these two cases is $(\omega_0 t + \phi) - \omega_0 t = \phi$ and ϕ is sometimes referred to as the *phase shift*. When $\phi < 0$, the function $x(t) = A\cos(\omega_0 t + \phi)$ reaches its maximum value at a later time $t = T/8$ than the function $x(t) = A\cos(\omega_0 t)$ as shown in Figure 23.4c.

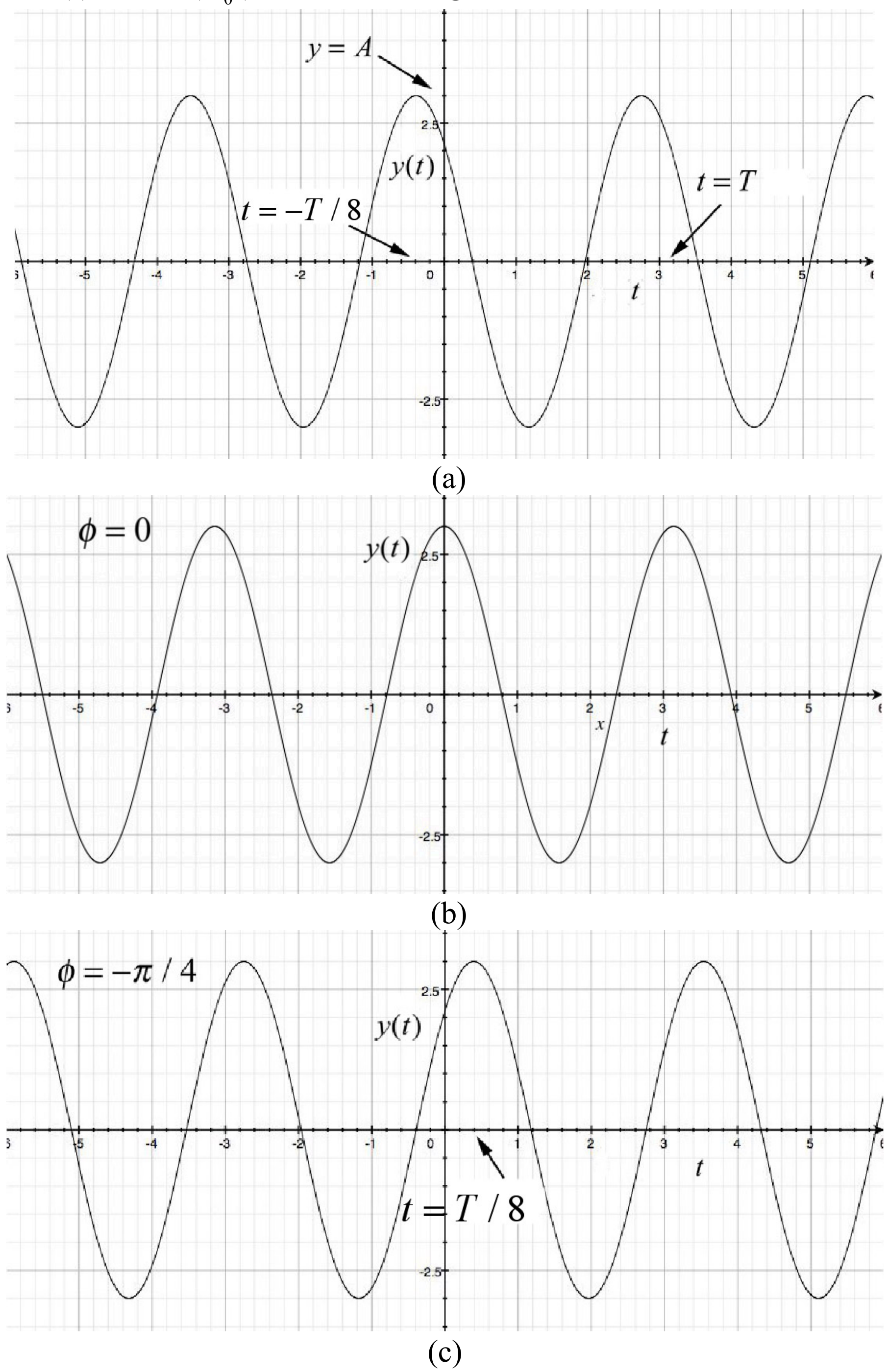


Figure 23.4 Phase shift of $x(t) = A\cos(\omega_0 t + \phi)$ (a) to the left by $\phi = \pi/4$, (b) no shift $\phi = 0$, (c) to the right $\phi = -\pi/4$

Example 23.2: Block-Spring System

A block of mass m is attached to a spring with spring constant k and is free to slide along a horizontal frictionless surface. At $t = 0$, the block-spring system is stretched an amount $x_0 > 0$ from the equilibrium position and is released from rest, $v_{x,0} = 0$. What is the period of oscillation of the block? What is the velocity of the block when it first comes back to the equilibrium position?

Solution: The position of the block can be determined from Eq. (23.2.21) by substituting the initial conditions $x_0 > 0$, and $v_{x,0} = 0$ yielding

$$x(t) = x_0 \cos\left(\sqrt{\frac{k}{m}}\, t\right), \tag{23.2.28}$$

and the x-component of its velocity is given by Eq. (23.2.22),

$$v_x(t) = -\sqrt{\frac{k}{m}}\, x_0 \sin\left(\sqrt{\frac{k}{m}}\, t\right). \tag{23.2.29}$$

The angular frequency of oscillation is $\omega_0 = \sqrt{k / m}$ and the period is given by Eq. (23.2.7),

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}\,. \tag{23.2.30}$$

The block first reaches equilibrium when the position function first reaches zero. This occurs at time t_1 satisfying

$$\sqrt{\frac{k}{m}}\, t_1 = \frac{\pi}{2}, \quad t_1 = \frac{\pi}{2}\sqrt{\frac{m}{k}} = \frac{T}{4}\,. \tag{23.2.31}$$

The x-component of the velocity at time t_1 is then

$$v_x(t_1) = -\sqrt{\frac{k}{m}} x_0 \sin\left(\sqrt{\frac{k}{m}}\, t_1\right) = -\sqrt{\frac{k}{m}} x_0 \sin(\pi / 2) = -\sqrt{\frac{k}{m}} x_0 = -\omega_0\, x_0 \tag{23.2.32}$$

Note that the block is moving in the negative x-direction at time t_1; the block has moved from a positive initial position to the equilibrium position (Figure 23.4(b)).

23.3 Energy and the Simple Harmonic Oscillator

Let's consider the block-spring system of Example 23.2 in which the block is initially stretched an amount $x_0 > 0$ from the equilibrium position and is released from rest, $v_{x,0} = 0$. We shall consider three states: state 1, the initial state; state 2, at an arbitrary time in which the position and velocity are non-zero; and state 3, when the object first comes back to the equilibrium position. We shall show that the mechanical energy has the same value for each of these states and is constant throughout the motion. Choose the equilibrium position for the zero point of the potential energy.

State 1: all the energy is stored in the object-spring potential energy, $U_1 = (1/2)k\,x_0^2$. The object is released from rest so the kinetic energy is zero, $K_1 = 0$. The total mechanical energy is then

$$E_1 = U_1 = \frac{1}{2}k\,x_0^2\,. \tag{23.3.1}$$

State 2: at some time t, the position and x-component of the velocity of the object are given by

$$\begin{aligned} x(t) &= x_0 \cos\left(\sqrt{\frac{k}{m}}t\right) \\ v_x(t) &= -\sqrt{\frac{k}{m}}x_0 \sin\left(\sqrt{\frac{k}{m}}t\right). \end{aligned} \tag{23.3.2}$$

The kinetic energy is

$$K_2 = \frac{1}{2}m v^2 = \frac{1}{2}k\,x_0^2 \sin^2\left(\sqrt{\frac{k}{m}}\,t\right), \tag{23.3.3}$$

and the potential energy is

$$U_2 = \frac{1}{2}k\,x^2 = \frac{1}{2}k\,x_0^2 \cos^2\left(\sqrt{\frac{k}{m}}\,t\right). \tag{23.3.4}$$

The mechanical energy is the sum of the kinetic and potential energies

$$\begin{aligned} E_2 &= K_2 + U_2 = \frac{1}{2}m {v_x}^2 + \frac{1}{2}k\,x^2 \\ &= \frac{1}{2}k\,x_0^2\left(\cos^2\left(\sqrt{\frac{k}{m}}\,t\right) + \sin^2\left(\sqrt{\frac{k}{m}}t\right)\right) \\ &= \frac{1}{2}k\,x_0^2, \end{aligned} \tag{23.3.5}$$

where we used the identity that $\cos^2\omega_0 t + \sin^2\omega_0 t = 1$, and that $\omega_0 = \sqrt{k/m}$ (Eq. (23.2.6)).

The mechanical energy in state 2 is equal to the initial potential energy in state 1, so the mechanical energy is constant. This should come as no surprise; we isolated the object-spring system so that there is no external work performed on the system and no internal non-conservative forces doing work.

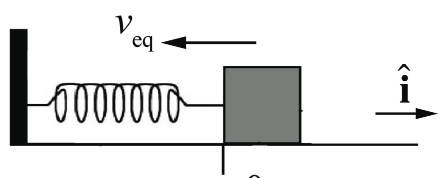


Figure 23.5 State 3 at equilibrium and in motion

State 3: now the object is at the equilibrium position so the potential energy is zero, $U_3 = 0$, and the mechanical energy is in the form of kinetic energy (Figure 23.5).

$$E_3 = K_3 = \frac{1}{2} m v_{eq}^2 . \tag{23.3.6}$$

Because the system is closed, mechanical energy is constant,

$$E_1 = E_3 . \tag{23.3.7}$$

Therefore the initial stored potential energy is released as kinetic energy,

$$\frac{1}{2} k x_0^2 = \frac{1}{2} m v_{eq}^2 , \tag{23.3.8}$$

and the x-component of velocity at the equilibrium position is given by

$$v_{x,eq} = \pm\sqrt{\frac{k}{m}}\, x_0 . \tag{23.3.9}$$

Note that the plus-minus sign indicates that when the block is at equilibrium, there are two possible motions: in the positive x-direction or the negative x-direction. If we take $x_0 > 0$, then the block starts moving towards the origin, and $v_{x,eq}$ will be negative the first time the block moves through the equilibrium position.

We can show more generally that the mechanical energy is constant at all times as follows. The mechanical energy at an arbitrary time is given by

$$E = K + U = \frac{1}{2} m v_x^{\,2} + \frac{1}{2} k x^2 \ . \tag{23.3.10}$$

Differentiate Eq. (23.3.10)

$$\frac{dE}{dt} = m v_x \frac{dv_x}{dt} + k x \frac{dx}{dt} = v_x \left(m \frac{d^2 x}{dt^2} + k x \right) \ . \tag{23.3.11}$$

Now substitute the simple harmonic oscillator equation of motion, (Eq. (23.2.1)) into Eq. (23.3.11) yielding

$$\frac{dE}{dt} = 0 \ , \tag{23.3.12}$$

demonstrating that the mechanical energy is a constant of the motion.

23.3.1 Simple Pendulum: Force Approach

A pendulum consists of an object hanging from the end of a string or rigid rod pivoted about the point P. The object is pulled to one side and allowed to oscillate. If the object has negligible size and the string or rod is massless, then the pendulum is called a ***simple pendulum***. Consider a simple pendulum consisting of a massless string of length l and a point-like object of mass m attached to one end, called the *bob*. Suppose the string is fixed at the other end and is initially pulled out at an angle θ_0 from the vertical and released from rest (Figure 23.6). Neglect any dissipation due to air resistance or frictional forces acting at the pivot.

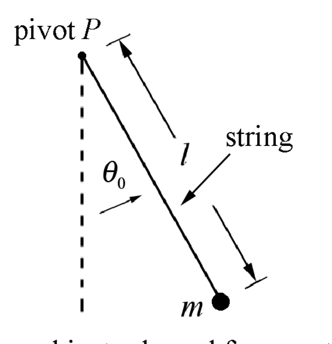


Figure 23.6 Simple pendulum

Let's choose polar coordinates for the pendulum as shown in Figure 23.7a along with the free-body force diagram for the suspended object (Figure 23.7b). The angle θ is defined with respect to the equilibrium position. When $\theta > 0$, the bob is has moved to the right, and when $\theta < 0$, the bob has moved to the left. The object will move in a circular arc centered at the pivot point. The forces on the object are the tension in the string

$\vec{\mathbf{T}} = -T\,\hat{\mathbf{r}}$ and gravity $m\vec{\mathbf{g}}$. The gravitation force on the object has $\hat{\mathbf{r}}$- and $\hat{\boldsymbol{\theta}}$-components given by

$$m\vec{\mathbf{g}} = mg(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}). \tag{23.3.13}$$

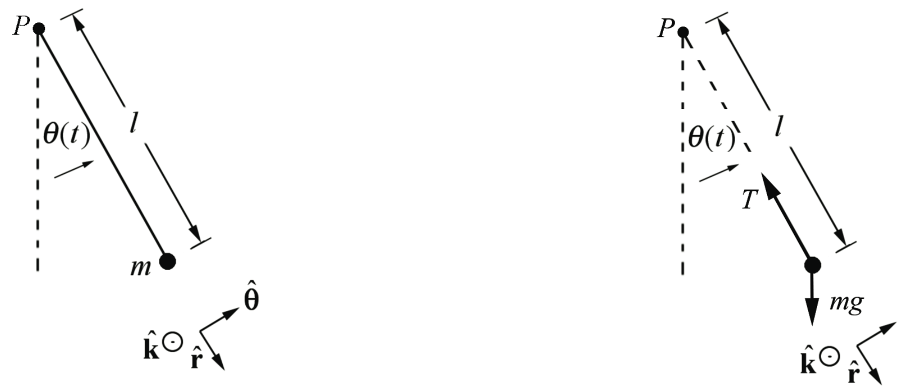


Figure 23.7 (a) Coordinate system **Figure 23.7 (b)** free-body force diagram

Our concern is with the tangential component of the gravitational force,

$$F_\theta = -mg\sin\theta. \tag{23.3.14}$$

The sign in Eq. (23.3.14) is crucial; the tangential force tends to restore the pendulum to the equilibrium value $\theta = 0$. If $\theta > 0$, $F_\theta < 0$ and if $\theta < 0$, $F_\theta > 0$, where we are that because the string is flexible, the angle θ is restricted to the range $-\pi/2 < \theta < \pi/2$. (For angles $|\theta| > \pi/2$, the string would go slack.) In both instances the tangential component of the force is directed towards the equilibrium position. The tangential component of acceleration is

$$a_\theta = l\alpha = l\frac{d^2\theta}{dt^2}. \tag{23.3.15}$$

Newton's Second Law, $F_\theta = m\,a_\theta$, yields

$$-mgl\sin\theta = ml^2\frac{d^2\theta}{dt^2}. \tag{23.3.16}$$

We can rewrite this equation is the form

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta. \tag{23.3.17}$$

This is not the simple harmonic oscillator equation although it still describes periodic motion. In the limit of small oscillations, $\sin\theta \cong \theta$, Eq. (23.3.17) becomes

$$\frac{d^2\theta}{dt^2} \cong -\frac{g}{l}\theta. \tag{23.3.18}$$

This equation is similar to the object-spring simple harmonic oscillator differential equation

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x. \tag{23.3.19}$$

By comparison with Eq. (23.2.6) the angular frequency of oscillation for the pendulum is approximately

$$\omega_0 \simeq \sqrt{\frac{g}{l}}, \tag{23.3.20}$$

with period

$$T = \frac{2\pi}{\omega_0} \simeq 2\pi\sqrt{\frac{l}{g}}. \tag{23.3.21}$$

The solutions to Eq. (23.3.18) can be modeled after Eq. (23.2.21). With the initial conditions that the pendulum is released from rest, $\frac{d\theta}{dt}(t=0)=0$, at a small angle $\theta(t=0)=\theta_0$, the angle the string makes with the vertical as a function of time is given by

$$\theta(t) = \theta_0\cos(\omega_0 t) = \theta_0\cos\left(\frac{2\pi}{T}t\right) = \theta_0\cos\left(\sqrt{\frac{g}{l}}\,t\right). \tag{23.3.22}$$

The z-component of the angular velocity of the bob is

$$\omega_z(t) = \frac{d\theta}{dt}(t) = -\sqrt{\frac{g}{l}}\theta_0\sin\left(\sqrt{\frac{g}{l}}\,t\right). \tag{23.3.23}$$

Keep in mind that the component of the angular velocity $\omega_z = d\theta/dt$ changes with time in an oscillatory manner (sinusoidally in the limit of small oscillations). The angular frequency ω_0 is a parameter that describes the system. The z-component of the angular velocity $\omega_z(t)$, besides being time-dependent, depends on the amplitude of oscillation θ_0. In the limit of small oscillations, ω_0 does not depend on the amplitude of oscillation.

The fact that the period is independent of the mass of the object follows algebraically from the fact that the mass appears on both sides of Newton's Second Law and hence cancels. Consider also the argument that is attributed to Galileo: if a pendulum, consisting of two identical masses joined together, were set to oscillate, the two halves would not exert forces on each other. So, if the pendulum were split into two pieces, the pieces would oscillate the same as if they were one piece. This argument can be extended to simple pendula of arbitrary masses.

23.3.2 Simple Pendulum: Energy Approach

We can use energy methods to find the differential equation describing the time evolution of the angle θ. When the string is at an angle θ with respect to the vertical, the gravitational potential energy (relative to a choice of zero potential energy at the bottom of the swing where $\theta = 0$ as shown in Figure 23.8) is given by

$$U = mgl(1-\cos\theta) \tag{23.3.24}$$

The θ-component of the velocity of the object is given by $v_\theta = l(d\theta / dt)$ so the kinetic energy is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\left(l\frac{d\theta}{dt}\right)^2. \tag{23.3.25}$$

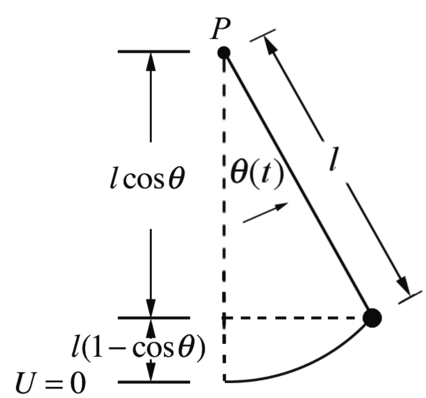


Figure 23.8 Energy diagram for simple pendulum

The mechanical energy of the system is then

$$E = K + U = \frac{1}{2}m\left(l\frac{d\theta}{dt}\right)^2 + mgl\left(1-\cos\theta\right). \tag{23.3.26}$$

Because we assumed that there is no non-conservative work (i.e. no air resistance or frictional forces acting at the pivot), the energy is constant, hence

$$\begin{aligned} 0 = \frac{dE}{dt} &= \frac{1}{2}m2l^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + mgl\sin\theta\frac{d\theta}{dt} \\ &= ml^2\frac{d\theta}{dt}\left(\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta\right). \end{aligned} \tag{23.3.27}$$

There are two solutions to this equation; the first one $d\theta / dt = 0$ is the equilibrium solution. That the z-component of the angular velocity is zero means the suspended object is not moving. The second solution is the one we are interested in

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0, \tag{23.3.28}$$

which is the same differential equation (Eq. (23.3.16)) that we found using the force method.

We can find the time t_1 that the object first reaches the bottom of the circular arc by setting $\theta(t_1) = 0$ in Eq. (23.3.22)

$$0 = \theta_0 \cos\left(\sqrt{\frac{g}{l}}\, t_1\right). \tag{23.3.29}$$

This zero occurs when the argument of the cosine satisfies

$$\sqrt{\frac{g}{l}}\, t_1 = \frac{\pi}{2}. \tag{23.3.30}$$

The z-component of the angular velocity at time t_1 is therefore

$$\frac{d\theta}{dt}(t_1) = -\sqrt{\frac{g}{l}}\,\theta_0 \sin\left(\sqrt{\frac{g}{l}}\, t_1\right) = -\sqrt{\frac{g}{l}}\,\theta_0 \sin\left(\frac{\pi}{2}\right) = -\sqrt{\frac{g}{l}}\,\theta_0 . \tag{23.3.31}$$

Note that the negative sign means that the bob is moving in the negative $\hat{\boldsymbol{\theta}}$-direction when it first reaches the bottom of the arc. The θ-component of the velocity at time t_1 is therefore

$$v_\theta(t_1) \equiv v_1 = l\frac{d\theta}{dt}(t_1) = -l\sqrt{\frac{g}{l}}\,\theta_0 \sin\left(\sqrt{\frac{g}{l}}\, t_1\right) = -\sqrt{lg}\,\theta_0 \sin\left(\frac{\pi}{2}\right) = -\sqrt{lg}\,\theta_0 . \tag{23.3.32}$$

We can also find the components of both the velocity and angular velocity using energy methods. When we release the bob from rest, the energy is only potential energy

$$E = U_0 = mgl\left(1 - \cos\theta_0\right) \cong mgl\frac{\theta_0^2}{2}, \tag{23.3.33}$$

where we used the approximation that $\cos\theta_0 \cong 1 - \theta_0^2 / 2$. When the bob is at the bottom of the arc, the only contribution to the mechanical energy is the kinetic energy given by

$$K_1 = \frac{1}{2} m v_1^{\,2} . \qquad (23.3.34)$$

Because the energy is constant, we have that $U_0 = K_1$ or

$$mgl\frac{\theta_0^2}{2} = \frac{1}{2} m v_1^{\,2} . \qquad (23.3.35)$$

We can solve for the θ-component of the velocity at the bottom of the arc

$$v_{\theta,1} = \pm\sqrt{gl}\ \theta_0 . \qquad (23.3.36)$$

The two possible solutions correspond to the different directions that the motion of the bob can have when at the bottom. The z-component of the angular velocity is then

$$\frac{d\theta}{dt}(t_1) = \frac{v_1}{l} = \pm\sqrt{\frac{g}{l}}\ \theta_0 , \qquad (23.3.37)$$

in agreement with our previous calculation.

If we do not make the small angle approximation, we can still use energy techniques to find the θ-component of the velocity at the bottom of the arc by equating the energies at the two positions

$$mgl\left(1-\cos\theta_0\right) = \frac{1}{2} m v_1^{\,2} , \qquad (23.3.38)$$

$$v_{\theta,1} = \pm\sqrt{2gl\left(1-\cos\theta_0\right)} . \qquad (23.3.39)$$

23.4 Worked Examples

Example 23.3: Rolling Without Slipping Oscillating Cylinder

Attach a solid cylinder of mass M and radius R to a horizontal massless spring with spring constant k so that it can roll without slipping along a horizontal surface. At time t, the center of mass of the cylinder is moving with speed V_{cm} and the spring is compressed a distance x from its equilibrium length. What is the period of simple harmonic motion for the center of mass of the cylinder?

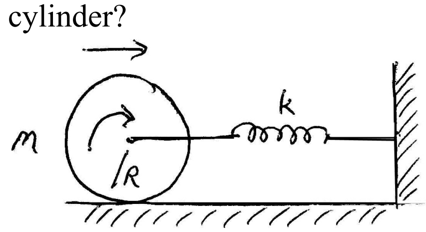


Figure 23.9 Example 23.3

Solution: At time t, the energy of the rolling cylinder and spring system is

$$E = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I_{cm} \left(\frac{d\theta}{dt} \right)^2 + \frac{1}{2} k x^2 \ . \tag{23.4.1}$$

where x is the amount the spring has compressed, $I_{cm} = (1/2) M R^2$, and because it is rolling without slipping

$$\frac{d\theta}{dt} = \frac{V_{cm}}{R} \ . \tag{23.4.2}$$

Therefore the energy is

$$E = \frac{1}{2} M V_{cm}^2 + \frac{1}{4} M R^2 \left(\frac{V_{cm}}{R} \right)^2 + \frac{1}{2} k x^2 = \frac{3}{4} M V_{cm}^2 + \frac{1}{2} k x^2 \ . \tag{23.4.3}$$

The energy is constant (no non-conservative force is doing work on the system) so

$$0 = \frac{dE}{dt} = \frac{3}{4} 2 M V_{cm} \frac{dV_{cm}}{dt} + \frac{1}{2} k 2 x \frac{dx}{dt} = V_{cm} (\frac{3}{2} M \frac{d^2 x}{dt^2} + kx) \tag{23.4.4}$$

Because V_{cm} is non-zero most of the time, the displacement of the spring satisfies a simple harmonic oscillator equation

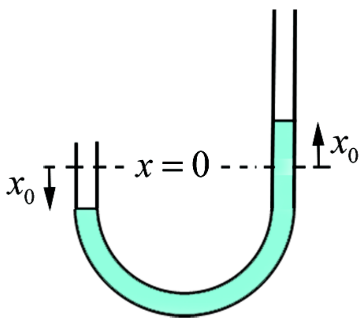
$$\frac{d^2 x}{dt^2} + \frac{2k}{3M} x = 0 \ . \tag{23.4.5}$$

Hence the period is

$$T = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{3M}{2k}} \ . \tag{23.4.6}$$

Example 23.4: U-Tube

A U-tube open at both ends is filled with an incompressible fluid of density ρ. The cross-sectional area A of the tube is uniform and the total length of the fluid in the tube is L. A piston is used to depress the height of the liquid column on one side by a distance x_0, (raising the other side by the same distance) and then is quickly removed (Figure 23.10). What is the angular frequency of the ensuing simple harmonic motion? Neglect any resistive forces and at the walls of the U-tube.



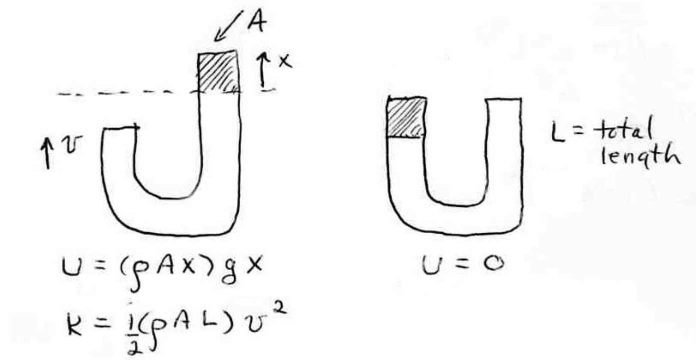


Figure 23.10 Example 23.4 **Figure 23.11** Energy diagram for water

Solution: We shall use conservation of energy. First choose as a zero for gravitational potential energy in the configuration where the water levels are equal on both sides of the tube. When the piston on one side depresses the fluid, it rises on the other. At a given instant in time when a portion of the fluid of mass $\Delta m = \rho A x$ is a height x above the equilibrium height (Figure 23.11), the potential energy of the fluid is given by

$$U = \Delta m g x = (\rho A x) g x = \rho A g x^2 \ . \tag{23.4.7}$$

At that same instant the entire fluid of length L and mass $m = \rho A L$ is moving with speed v, so the kinetic energy is

$$K = \frac{1}{2} m v^2 = \frac{1}{2} \rho A L v^2 \ . \tag{23.4.8}$$

Thus the total energy is

$$E = K + U = \frac{1}{2} \rho A L v^2 + \rho A g x^2 \ . \tag{23.4.9}$$

By neglecting resistive force, the mechanical energy of the fluid is constant. Therefore

$$0 = \frac{dE}{dt} = \rho A L v \frac{dv}{dt} + 2 \rho A g x \frac{dx}{dt} . \tag{23.4.10}$$

If we just consider the top of the fluid above the equilibrium position on the right arm in Figure 23.13, we rewrite Eq. (23.4.10) as

$$0 = \frac{dE}{dt} = \rho A L v_x \frac{dv_x}{dt} + 2 \rho A g x \frac{dx}{dt} \ , \tag{23.4.11}$$

where $v_x = dx / dt$. We now rewrite the energy condition using $dv_x / dt = d^2 x / dt^2$ as

$$0 = v_x \rho A \left(L\frac{d^2x}{dt^2} + 2gx \right) . \tag{23.4.12}$$

This condition is satisfied when $v_x = 0$, i.e. the equilibrium condition or when

$$0 = L\frac{d^2x}{dt^2} + 2gx \ . \tag{23.4.13}$$

This last condition can be written as

$$\frac{d^2x}{dt^2} = -\frac{2g}{L}x \ . \tag{23.4.14}$$

This last equation is the simple harmonic oscillator equation. Using the same mathematical techniques as we used for the spring-block system, the solution for the height of the fluid above the equilibrium position is given by

$$x(t) = B\cos(\omega_0 t) + C\sin(\omega_0 t) \ , \tag{23.4.15}$$

where

$$\omega_0 = \sqrt{\frac{2g}{L}} \tag{23.4.16}$$

is the angular frequency of oscillation. The x-component of the velocity of the fluid on the right-hand side of the U-tube is given by

$$v_x(t) = \frac{dx(t)}{dt} = -\omega_0 B\sin(\omega_0 t) + \omega_0 C\cos(\omega_0 t) \ . \tag{23.4.17}$$

The coefficients B and C are determined by the initial conditions. At $t = 0$, the height of the fluid is $x(t=0) = B = x_0$. At $t = 0$, the speed is zero so $v_x(t=0) = \omega_0 C = 0$, hence $C = 0$. The height of the fluid above the equilibrium position on the right hand-side of the U-tube as a function of time is thus

$$x(t) = x_0 \cos\left(\sqrt{\frac{2g}{L}} t \right) . \tag{23.4.18}$$

23.5 Damped Oscillatory Motion

Let's now consider our spring-block system moving on a horizontal frictionless surface but now the block is attached to a damper that resists the motion of the block due to viscous friction. This damper, commonly called a *dashpot,* is shown in Figure 23.13. The viscous force arises when objects move through fluids at speeds slow enough so that there is no turbulence. When the viscous force opposes the motion and is proportional to the velocity, so that

$$\vec{\mathbf{f}}_{\text{vis}} = -b\vec{\mathbf{v}}\,, \tag{23.5.1}$$

the dashpot is referred to as a *linear dashpot*. The constant of proportionality b depends on the properties of the dashpot.

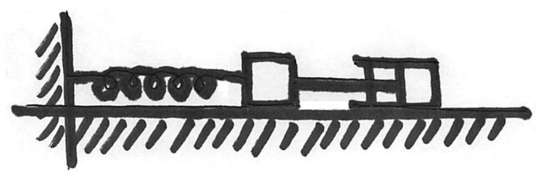


Figure 23.12 Spring-block system connected to a linear dashpot

Choose the origin at the equilibrium position and choose the positive x-direction to the right in the Figure 23.13. Define $x(t)$ to be the position of the object with respect to the equilibrium position. The x-component of the total force acting on the spring is the sum of the linear restoring spring force, and the viscous friction force (Figure 23.13),

$$F_x = -k\,x - b\frac{dx}{dt} \tag{23.5.2}$$

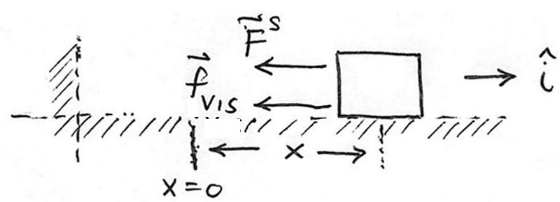


Figure 23.13 Free-body force diagram for spring-object system with linear dashpot

Newton's Second law in the x-direction becomes

$$-k\,x - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}\,. \tag{23.5.3}$$

We can rewrite Eq. (23.5.3) as

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0\,. \tag{23.5.4}$$

When $(b/m)^2 < 4k/m$, the oscillator is called *underdamped*, and the solution to Eq. (23.5.4) is given by

$$x(t) = x_{\text{m}} e^{-\alpha t}\cos(\gamma t + \phi) \tag{23.5.5}$$

where $\gamma = (k/m - (b/2m)^2)^{1/2}$ is the angular frequency of oscillation, $\alpha = b/2m$ is a parameter that measured the exponential decay of the oscillations, x_{m} is a constant and ϕ is the phase constant. Recall the undamped oscillator has angular frequency $\omega_0 = (k/m)^{1/2}$, so the angular frequency of the underdamped oscillator can be expressed as

$$\gamma = (\omega_0{}^2 - \alpha^2)^{1/2} . \tag{23.5.6}$$

In Appendix 23B: Complex Numbers, we introduce complex numbers and use them to solve Eq.(23.5.4) in Appendix 23C: Solution to the Underdamped Simple Harmonic Oscillator Equation.

The x-component of the velocity of the object is given by

$$v_x(t) = dx/dt = \left(-\gamma x_{\mathrm{m}} \sin(\gamma t + \phi) - \alpha x_{\mathrm{m}} \cos(\gamma t + \phi)\right) e^{-\alpha t} . \tag{23.5.7}$$

The position and the x-component of the velocity of the object oscillate but the amplitudes of the oscillations decay exponentially. In Figure 23.14, the position is plotted as a function of time for the underdamped system for the special case $\phi = 0$. For that case

$$x(t) = x_{\mathrm{m}} e^{-\alpha t} \cos(\gamma t) . \tag{23.5.8}$$

and

$$v_x(t) = dx/dt = \left(-\gamma x_{\mathrm{m}} \sin(\gamma t) - \alpha x_{\mathrm{m}} \cos(\gamma t)\right) e^{-\alpha t} . \tag{23.5.9}$$

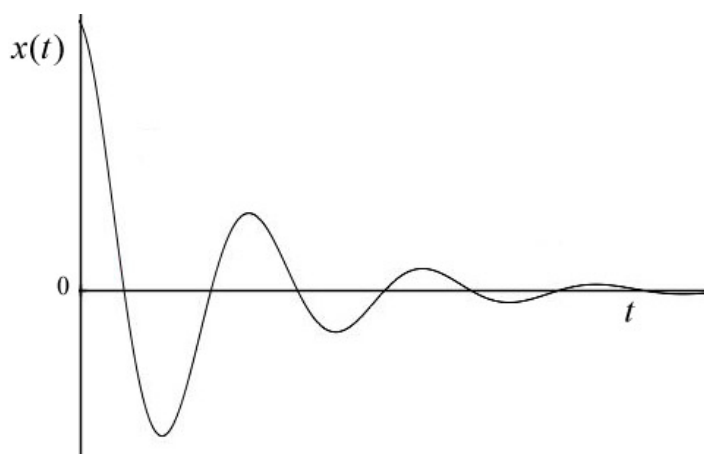


Figure 23.14 Plot of position $x(t)$ of object for underdamped oscillator with $\phi = 0$

Because the coefficient of exponential decay $\alpha = b/2m$ is proportional to the b, we see that the position will decay more rapidly if the viscous force increases. We can introduce a *time constant*

$$\tau = 1/\alpha = 2m/b . \tag{23.5.10}$$

When $t = \tau$, the position is

$$x(t=\tau) = x_{\mathrm{m}} \cos(\gamma\tau) e^{-1} . \tag{23.5.11}$$

The envelope of exponential decay has now decreases by a factor of e^{-1}, i.e. the amplitude can be at most $x_{\mathrm{m}} e^{-1}$. During this time interval $[0, \tau]$, the position has undergone a number of oscillations. The total number of radians associated with those oscillations is given by

$$\gamma\tau = (k/m - (b/2m)^2)^{1/2}(2m/b) . \tag{23.5.12}$$

The closest integral number of cycles is then

$$n = \left[\gamma\tau / 2\pi\right] = \left[(k/m - (b/2m)^2)^{1/2}(m/\pi b)\right] . \tag{23.5.13}$$

If the system is very weakly damped, such that $(b/m)^2 << 4k/m$, then we can approximate the number of cycles by

$$n = \left[\gamma\tau/2\pi\right] \simeq \left[(k/m)^{1/2}(m/\pi b)\right] = \left[\omega_0 (m/\pi b)\right], \tag{23.5.14}$$

where $\omega_0 = (k/m)^{1/2}$ is the angular frequency of the undamped oscillator.

We define the *quality*, Q, of this oscillating system to be proportional to the number of integral cycles it takes for the exponential envelope of the position function to fall off by a factor of e^{-1}. The constant of proportionality is chosen to be π. Thus

$$Q = n\pi . \tag{23.5.15}$$

For the weakly damped case, we have that

$$Q \simeq \omega_0 (m/b) . \tag{23.5.16}$$

23.5.1 Energy in the Underdamped Oscillator

For the underdamped oscillator, $(b/m)^2 < 4k/m$, $\gamma = (k/m - (b/2m)^2)^{1/2}$, and $\alpha = b/2m$. Let's choose $t = 0$ such that the phase shift is zero $\phi = 0$. The stored energy in the system will decay due to the energy loss due to dissipation. The mechanical energy stored in the potential and kinetic energies is then given by

$$E = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 . \tag{23.5.17}$$

where the position and the x-component of the velocity are given by Eqs. (23.5.8) and (23.5.9). The mechanical energy is then

$$E = \frac{1}{2}kx_{\text{m}}{}^2\cos^2(\gamma t)e^{-2\alpha t} + \frac{1}{2}m\left(-\gamma x_{\text{m}}\sin(\gamma t) - \alpha x_{\text{m}}\cos(\gamma t)\right)^2 e^{-2\alpha t}. \quad (23.5.18)$$

Expanding this expression yields

$$E = \frac{1}{2}(k + m\alpha^2)x_{\text{m}}{}^2\cos^2(\gamma t)e^{-2\alpha t} + m\gamma\alpha x_{\text{m}}{}^2\sin(\gamma t)\cos(\gamma t)e^{-2\alpha t} + \frac{1}{2}m\gamma^2 x_{\text{m}}{}^2\sin^2(\gamma t)e^{-2\alpha t} \quad (23.5.19)$$

The kinetic energy, potential energy, and mechanical energy are shown in Figure 23.15.

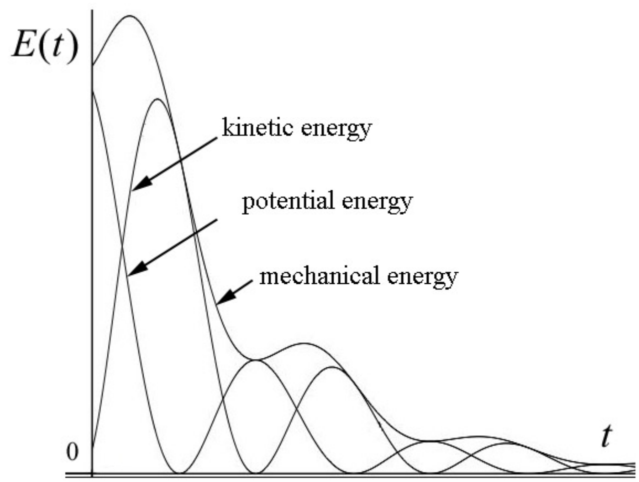


Figure 23.15 Kinetic, potential and mechanical energy for the underdamped oscillator

The stored energy at time $t = 0$ is

$$E(t=0) = \frac{1}{2}(k + m\alpha^2)x_{\text{m}}{}^2 \quad (23.5.20)$$

The mechanical energy at the conclusion of one cycle, with $\gamma T = 2\pi$, is

$$E(t=T) = \frac{1}{2}(k + m\alpha^2)x_{\text{m}}{}^2 e^{-2\alpha T} \quad (23.5.21)$$

The change in the mechanical energy for one cycle is then

$$E(t=T) - E(t=0) = -\frac{1}{2}(k + m\alpha^2)x_{\text{m}}{}^2(1 - e^{-2\alpha T}). \quad (23.5.22)$$

Recall that $\alpha^2 = b^2/4m^2$. Therefore

$$E(t=T) - E(t=0) = -\frac{1}{2}(k + b^2/4m)x_{\text{m}}^{\;2}(1 - e^{-2\alpha T}) . \qquad (23.5.23)$$

We can show (although the calculation is lengthy) that the energy dissipated by the viscous force over one cycle is given by the integral

$$E_{\text{dis}} = \int_0^T \vec{\mathbf{F}}_{\text{vis}} \cdot \vec{\mathbf{v}}\, dt = -\left(k + \frac{b^2}{4m}\right)\frac{x_{\text{m}}^{\;2}}{2}(1 - e^{-2\alpha t}) . \qquad (23.5.24)$$

By comparison with Eq. (23.5.23), the change in the mechanical energy in the underdamped oscillator during one cycle is equal to the energy dissipated due to the viscous force during one cycle.

23.6 Forced Damped Oscillator

Let's drive our damped spring-object system by a sinusoidal force. Suppose that the x-component of the driving force is given by

$$F_x(t) = F_0 \cos(\omega t) \;, \qquad (23.6.1)$$

where F_0 is called the *amplitude* (maximum value) and ω is the *driving angular frequency*. The force varies between F_0 and $-F_0$ because the cosine function varies between $+1$ and -1. Define $x(t)$ to be the position of the object with respect to the equilibrium position. The x-component of the force acting on the object is now the sum

$$F_x = F_0 \cos(\omega t) - kx - b\frac{dx}{dt} \;. \qquad (23.6.2)$$

Newton's Second law in the x-direction becomes

$$F_0 \cos(\omega t) - kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2} \;. \qquad (23.6.3)$$

We can rewrite Eq. (23.6.3) as

$$F_0 \cos(\omega t) = m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx \;. \qquad (23.6.4)$$

We derive the solution to Eq. (23.6.4) in Appendix 23E: Solution to the forced Damped Oscillator Equation. The solution to is given by the function

$$x(t) = x_0 \cos(\omega t + \phi) \ , \tag{23.6.5}$$

where the amplitude x_0 is a function of the driving angular frequency ω and is given by

$$x_0(\omega) = \frac{F_0 / m}{\left((b/m)^2 \omega^2 + (\omega_0^{\ 2} - \omega^2)^2\right)^{1/2}} \ . \tag{23.6.6}$$

The phase constant ϕ is also a function of the driving angular frequency ω and is given by

$$\phi(\omega) = \tan^{-1}\left(\frac{(b/m)\omega}{\omega^2 - \omega_0^{\ 2}} \right) \ . \tag{23.6.7}$$

In Eqs. (23.6.6) and (23.6.7)

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{23.6.8}$$

is the natural angular frequency associated with the undriven undamped oscillator. The x -component of the velocity can be found by differentiating Eq. (23.6.5),

$$v_x(t) = \frac{dx}{dt}(t) = -\omega x_0 \sin(\omega t + \phi) \ , \tag{23.6.9}$$

where the amplitude $x_0(\omega)$ is given by Eq. (23.6.6) and the phase constant $\phi(\omega)$ is given by Eq. (23.6.7).

23.6.1 Resonance

When $b/m << 2\omega_0$ we say that the oscillator is *lightly damped*. For a lightly-damped driven oscillator, after a transitory period, the position of the object will oscillate with the same angular frequency as the driving force. The plot of amplitude $x_0(\omega)$ vs. driving angular frequency ω for a lightly damped forced oscillator is shown in Figure 23.16. If the angular frequency is increased from zero, the amplitude of the $x_0(\omega)$ will increase until it reaches a maximum when the angular frequency of the driving force is the same as the natural angular frequency, ω_0, associated with the undamped oscillator. This is called **resonance.** When the driving angular frequency is increased above the natural angular frequency the amplitude of the position oscillations diminishes.

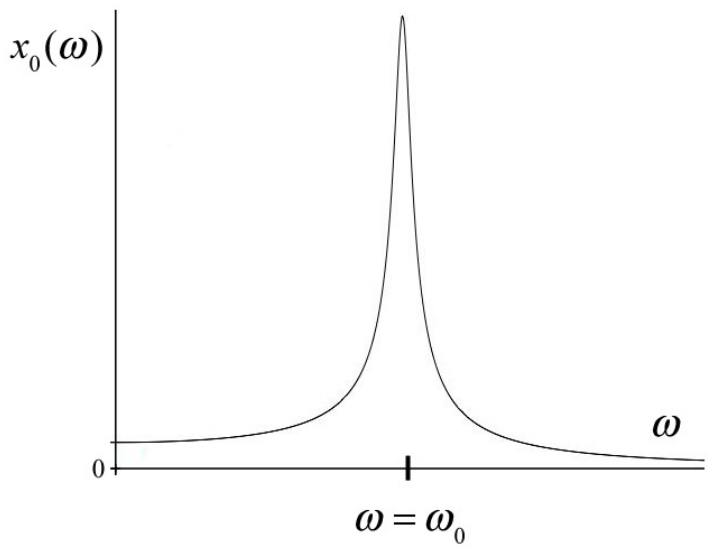


Figure 23.16 Plot of amplitude $x_0(\omega)$ vs. driving angular frequency ω for a lightly damped oscillator with $b/m << 2\omega_0$

We can find the angular frequency such that the amplitude $x_0(\omega)$ is at a maximum by setting the derivative of Eq. (23.6.6) equal to zero,

$$0 = \frac{d}{dt}x_0(\omega) = -\frac{F_0(2\omega)}{2m}\frac{((b/m)^2 - 2(\omega_0{}^2 - \omega^2))}{\left((b/m)^2\omega^2 + (\omega_0{}^2 - \omega^2)^2\right)^{3/2}} \ . \qquad (23.6.10)$$

This vanishes when

$$\omega = (\omega_0{}^2 - (b/m)^2/2)^{1/2} \ . \qquad (23.6.11)$$

For the lightly-damped oscillator, $\omega_0 >> (1/2)b/m$, and so the maximum value of the amplitude occurs when

$$\omega \simeq \omega_0 = (k/m)^{1/2} \ . \qquad (23.6.12)$$

The amplitude at resonance is then

$$x_0(\omega = \omega_0) = \frac{F_0}{b\omega_0} \ \text{(lightly damped)} \ . \qquad (23.6.13)$$

The plot of phase constant $\phi(\omega)$ vs. driving angular frequency ω for a lightly damped forced oscillator is shown in Figure 23.17.

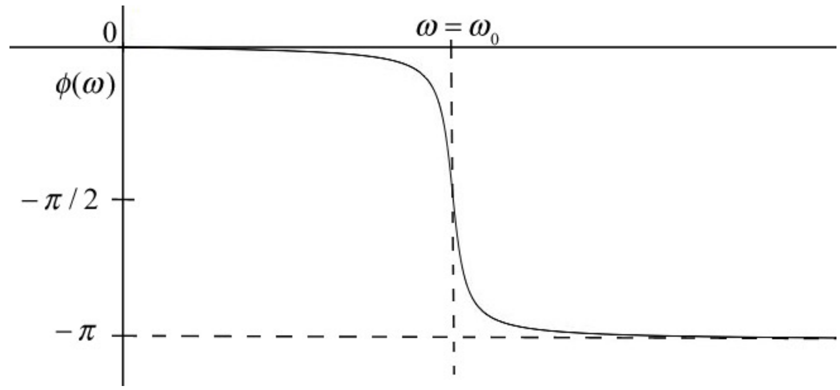


Figure 23.17 Plot of phase constant $\phi(\omega)$ vs. driving angular frequency ω for a lightly damped oscillator with $b/m << 2\omega_0$

The phase constant at resonance is zero,

$$\phi(\omega = \omega_0) = 0 \ . \quad (23.6.14)$$

At resonance, the x-component of the velocity is given by

$$v_x(t) = \frac{dx}{dt}(t) = -\frac{F_0}{b}\sin(\omega_0 t) \quad \text{(lightly damped)} \ . \quad (23.6.15)$$

When the oscillator is not lightly damped ($b/m \simeq \omega_0$), the resonance peak is shifted to the left of $\omega = \omega_0$ as shown in the plot of amplitude vs. angular frequency in Figure 23.18. The corresponding plot of phase constant vs. angular frequency for the non-lightly damped oscillator is shown in Figure 23.19.

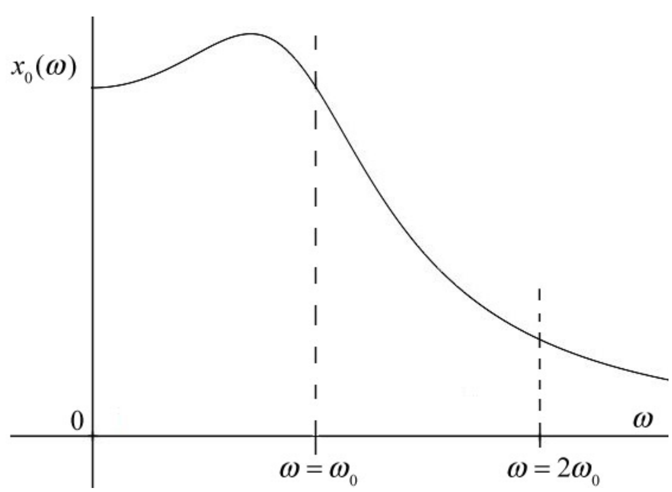


Figure 23.18 Plot of amplitude vs. angular frequency for lightly-damped driven oscillator where $b/m \simeq \omega_0$

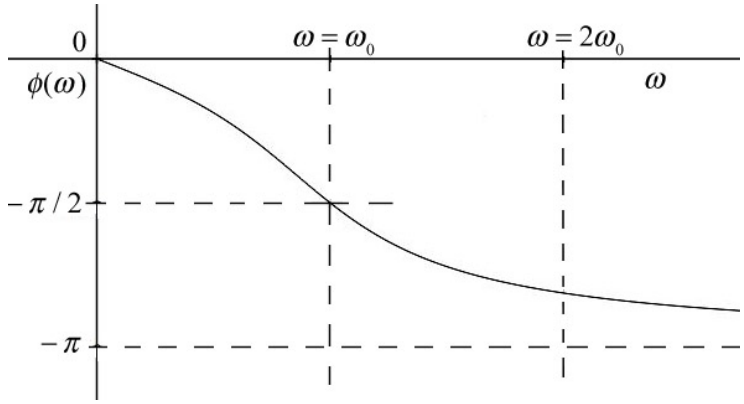


Figure 23.19 Plot of phase constant vs. angular frequency for lightly-damped driven oscillator where $b/m \simeq \omega_0$

23.6.2 Mechanical Energy

The kinetic energy for the driven damped oscillator is given by

$$K(t) = \frac{1}{2}mv^2(t) = \frac{1}{2}m\omega^2 x_0^{\ 2}\sin^2(\omega t + \phi) \ . \tag{23.6.16}$$

The potential energy is given by

$$U(t) = \frac{1}{2}kx^2(t) = \frac{1}{2}kx_0^{\ 2}\cos^2(\omega t + \phi) \ . \tag{23.6.17}$$

The mechanical energy is then

$$E(t) = \frac{1}{2}mv^2(t) + \frac{1}{2}kx^2(t) = \frac{1}{2}m\omega^2 x_0^{\ 2}\sin^2(\omega t + \phi) + \frac{1}{2}kx_0^{\ 2}\cos^2(\omega t + \phi) \ . \tag{23.6.18}$$

Example 23.5: Time-Averaged Mechanical Energy

The period of one cycle is given by $T = 2\pi/\omega$. Show that

(i) $$\frac{1}{T}\int_0^T \sin^2(\omega t + \phi)\,dt = \frac{1}{2} \ , \tag{23.6.19}$$

(ii) $$\frac{1}{T}\int_0^T \cos^2(\omega t + \phi)\,dt = \frac{1}{2} \ , \tag{23.6.20}$$

(iii) $$\frac{1}{T}\int_0^T \sin(\omega t)\cos(\omega t)\,dt = 0 \ . \tag{23.6.21}$$

Solution: (i) We use the trigonometric identity

$$\sin^2(\omega t+\phi)) = \frac{1}{2}(1-\cos(2(\omega t+\phi)) \tag{23.6.22}$$

to rewrite the integral in Eq. (23.6.19) as

$$\frac{1}{T}\int_0^T \sin^2(\omega t+\phi))\,dt = \frac{1}{2T}\int_0^T (1-\cos(2(\omega t+\phi))\,dt \tag{23.6.23}$$

Integration yields

$$\begin{aligned}&\frac{1}{2T}\int_0^T (1-\cos(2(\omega t+\phi))\,dt = \frac{1}{2} - \left(\frac{\sin(2(\omega t+\phi))}{2\omega}\right)\Bigg|_{T=0}^{T=2\pi/\omega}\\ &= \frac{1}{2} - \left(\frac{\sin(4\pi+2\phi)}{2\omega} - \frac{\sin(2\phi)}{2\omega}\right) = \frac{1}{2},\end{aligned} \tag{23.6.24}$$

where we used the trigonometric identity that

$$\sin(4\pi+2\phi) = \sin(4\pi)\cos(2\phi) + \sin(2\phi)\cos(4\pi) = \sin(2\phi)\ , \tag{23.6.25}$$

proving Eq. (23.6.19).

(ii) We use a similar argument starting with the trigonometric identity that

$$\cos^2(\omega t+\phi)) = \frac{1}{2}(1+\cos(2(\omega t+\phi))\ . \tag{23.6.26}$$

Then

$$\frac{1}{T}\int_0^T \cos^2(\omega t+\phi))\,dt = \frac{1}{2T}\int_0^T (1+\cos(2(\omega t+\phi))\,dt\ . \tag{23.6.27}$$

Integration yields

$$\begin{aligned}&\frac{1}{2T}\int_0^T (1+\cos(2(\omega t+\phi))\,dt = \frac{1}{2} + \left(\frac{\sin(2(\omega t+\phi))}{2\omega}\right)\Bigg|_{T=0}^{T=2\pi/\omega}\\ &= \frac{1}{2} + \left(\frac{\sin(4\pi+2\phi)}{2\omega} - \frac{\sin(2\phi)}{2\omega}\right) = \frac{1}{2}.\end{aligned} \tag{23.6.28}$$

(iii) We first use the trigonometric identity that

$$\sin(\omega t)\cos(\omega t) = \frac{1}{2}\sin(\omega t) \ . \tag{23.6.29}$$

Then

$$\begin{aligned}
&\frac{1}{T}\int_0^T \sin(\omega t)\cos(\omega t)\,dt = \frac{1}{T}\int_0^T \sin(\omega t)\,dt \\
&= -\frac{1}{T}\frac{\cos(\omega t)}{2\omega}\bigg|_0^T = -\frac{1}{2\omega T}(1-1) = 0.
\end{aligned} \tag{23.6.30}$$

The values of the integrals in Example 23.5 are called the **time-averaged values**. We denote the time-average value of a function $f(t)$ over one period by

$$\langle f \rangle \equiv \frac{1}{T}\int_0^T f(t)\,dt \ . \tag{23.6.31}$$

In particular, the time-average kinetic energy as a function of the angular frequency is given by

$$\langle K(\omega) \rangle = \frac{1}{4} m\omega^2 x_0^{\ 2} \ . \tag{23.6.32}$$

The time-averaged potential energy as a function of the angular frequency is given by

$$\langle U(\omega) \rangle = \frac{1}{4} k x_0^{\ 2} \ . \tag{23.6.33}$$

The time-averaged value of the mechanical energy as a function of the angular frequency is given by

$$\langle E(\omega) \rangle = \frac{1}{4} m\omega^2 x_0^{\ 2} + \frac{1}{4} k x_0^{\ 2} = \frac{1}{4}(m\omega^2 + k)x_0^{\ 2} \ . \tag{23.6.34}$$

We now substitute Eq. (23.6.6) for the amplitude into Eq. (23.6.34) yielding

$$\langle E(\omega) \rangle = \frac{F_0^{\ 2}}{4m}\frac{(\omega_0^{\ 2} + \omega^2)}{\left((b/m)^2\omega^2 + \left(\omega_0^{\ 2} - \omega^2\right)^2\right)} \ . \tag{23.6.35}$$

A plot of the time-averaged energy versus angular frequency for the lightly-damped case ($b/m << 2\omega_0$) is shown in Figure 23.20.

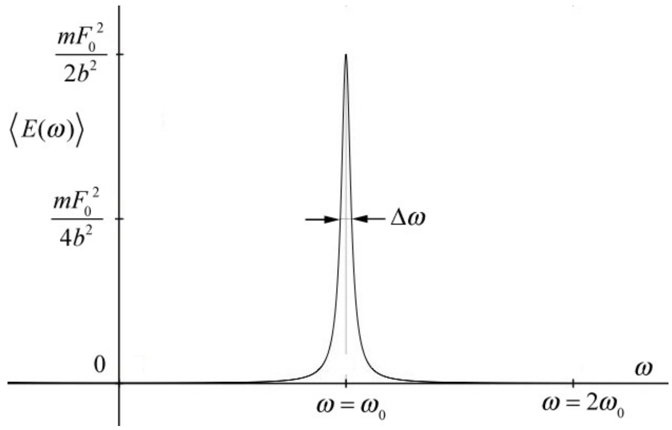


Figure 23.20 Plot of the time-averaged energy versus angular frequency for the lightly-damped case ($b/m << 2\omega_0$)

We can simplify the expression for the time-averaged energy for the lightly-damped case by observing that the time-averaged energy is nearly zero everywhere except where $\omega = \omega_0$, (see Figure 23.20). We first substitute $\omega = \omega_0$ everywhere in Eq. (23.6.35) except the term $\omega_0^{\ 2} - \omega^2$ that appears in the denominator, yielding

$$\langle E(\omega)\rangle = \frac{F_0^{\ 2}}{2m}\frac{(\omega_0^{\ 2})}{\left((b/m)^2\omega_0^{\ 2} + \left(\omega_0^{\ 2} - \omega^2\right)^2\right)}. \tag{23.6.36}$$

We can approximate the term

$$\omega_0^{\ 2} - \omega^2 = (\omega_0 - \omega)(\omega_0 + \omega) \simeq 2\omega_0(\omega_0 - \omega) \tag{23.6.37}$$

Then Eq. (23.6.36) becomes

$$\langle E(\omega)\rangle = \frac{F_0^{\ 2}}{2m}\frac{1}{\left((b/m)^2 + 4(\omega_0 - \omega)^2\right)} \quad \text{(lightly damped)}. \tag{23.6.38}$$

The right-hand expression of Eq. (23.6.38) takes on its maximum value when the denominator has its minimum value. By inspection, this occurs when $\omega = \omega_0$. Alternatively, to find the maximum value, we set the derivative of Eq. (23.6.35) equal to zero and solve for ω,

$$
\begin{aligned}
0 &= \frac{d}{d\omega}\left\langle E(\omega)\right\rangle = \frac{d}{d\omega}\frac{F_0^{\,2}}{2m}\frac{1}{\left((b/m)^2 + 4(\omega_0-\omega)^2\right)} \\
&= \frac{4F_0^{\,2}}{m}\frac{(\omega_0-\omega)}{\left((b/m)^2 + 4(\omega_0-\omega)^2\right)^2}
\end{aligned}
. \qquad (23.6.39)
$$

The maximum occurs when occurs at $\omega = \omega_0$ and has the value

$$
\left\langle E(\omega_0)\right\rangle = \frac{mF_0^{\,2}}{2b^2} \quad \text{(underdamped)}. \qquad (23.6.40)
$$

23.6.3 The Time-averaged Power

The time-averaged power delivered by the driving force is given by the expression

$$
\left\langle P(\omega)\right\rangle = \frac{1}{T}\int_0^T F_x v_x\, dt = -\frac{1}{T}\int_0^T \frac{F_0^{\,2}\omega\cos(\omega t)\sin(\omega t+\phi)}{m\left((b/m)^2\omega^2 + (\omega_0^{\,2}-\omega^2)^2\right)^{1/2}}dt\ , \qquad (23.6.41)
$$

where we used Eq. (23.6.1) for the driving force, and Eq. (23.6.9) for the x-component of the velocity of the object. We use the trigonometric identity

$$
\sin(\omega t+\phi) = \sin(\omega t)\cos(\phi) + \cos(\omega t)\sin(\phi) \qquad (23.6.42)
$$

to rewrite the integral in Eq. (23.6.41) as two integrals

$$
\begin{aligned}
\left\langle P(\omega)\right\rangle &= -\frac{1}{T}\int_0^T \frac{F_0^{\,2}\omega\cos(\omega t)\sin(\omega t)\cos(\phi)}{m\left((b/m)^2\omega^2 + (\omega_0^{\,2}-\omega^2)^2\right)^{1/2}}dt \\
&\quad -\frac{1}{T}\int_0^T \frac{F_0^{\,2}\omega\cos^2(\omega t)\sin(\phi)}{m\left((b/m)^2\omega^2 + (\omega_0^{\,2}-\omega^2)^2\right)^{1/2}}dt.
\end{aligned}
\qquad (23.6.43)
$$

Using the time-averaged results from Example 23.5, we see that the first term in Eq. (23.6.43) is zero and the second term becomes

$$
\left\langle P(\omega)\right\rangle = \frac{F_0^{\,2}\omega\sin(\phi)}{2m\left((b/m)^2\omega^2 + (\omega_0^{\,2}-\omega^2)^2\right)^{1/2}} \qquad (23.6.44)
$$

For the underdamped driven oscillator, we make the same approximations in Eq. (23.6.44) that we made for the time-averaged energy. In the term in the numerator and the term on the left in the denominator, we set $\omega \simeq \omega_0$, and we use Eq. (23.6.37) in the term on the right in the denominator yielding

$$\langle P(\omega)\rangle = \frac{F_0^{\,2}\sin(\phi)}{2m\left((b/m)^2 + 2(\omega_0 - \omega)\right)^{1/2}} \qquad \text{(underdamped)} . \qquad (23.6.45)$$

The time-averaged power dissipated by the resistive force is given by

$$\langle P_{\text{dis}}(\omega)\rangle = \frac{1}{T}\int_0^T (F_x)_{\text{dis}} v_x \, dt = -\frac{1}{T}\int_0^T b v_x^{\,2} \, dt = \frac{1}{T}\int_0^T \frac{F_0^{\,2}\omega^2 \sin^2(\omega t + \phi) dt}{m^2\left((b/m)^2\omega^2 + (\omega_0^{\,2} - \omega^2)^2\right)} \quad ,(23.6.46)$$
$$= \frac{F_0^{\,2}\omega^2 dt}{2m^2\left((b/m)^2\omega^2 + (\omega_0^{\,2} - \omega^2)^2\right)} .$$

where we used Eq. (23.5.1) for the dissipative force, Eq. (23.6.9) for the x-component of the velocity of the object, and Eq. (23.6.19) for the time-averaging.

23.6.4 Quality Factor

The plot of the time-averaged energy vs. the driving angular frequency for the underdamped oscullator has a width, $\Delta\omega$ (Figure 23.20). One way to characterize this width is to define $\Delta\omega = \omega_+ - \omega_-$, where $\omega_\pm$ are the values of the angular frequency such that time-averaged energy is equal to one half its maximum value

$$\langle E(\omega_\pm)\rangle = \frac{1}{2}\langle E(\omega_0)\rangle = \frac{mF_0^{\,2}}{4b^2} . \qquad (23.6.47)$$

The quantity $\Delta\omega$ is called the *line width at half energy maximum* also known as the *resonance width.* We can now solve for $\omega_\pm$ by setting

$$\langle E(\omega_\pm)\rangle = \frac{F_0^{\,2}}{2m}\frac{1}{\left((b/m)^2 + 4(\omega_0 - \omega_\pm)^2\right)} = \frac{mF_0^{\,2}}{4b^2} , \qquad (23.6.48)$$

yielding the condition that

$$(b/m)^2 = 4(\omega_0 - \omega_\pm)^2 . \qquad (23.6.49)$$

Taking square roots of Eq. (23.6.49) yields

$$\mp(b/2m) = \omega_0 - \omega_\pm . \qquad (23.6.50)$$

Therefore

$$\omega_{\pm} = \omega_0 \pm (b/2m). \tag{23.6.51}$$

The half-width is then

$$\Delta\omega = \omega_+ - \omega_- = (\omega_0 + (b/2m)) - (\omega_0 - (b/2m)) = b/m \;. \tag{23.6.52}$$

We define the quality Q of the resonance as the ratio of the resonant angular frequency to the line width,

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0}{b/m} \;. \tag{23.6.53}$$

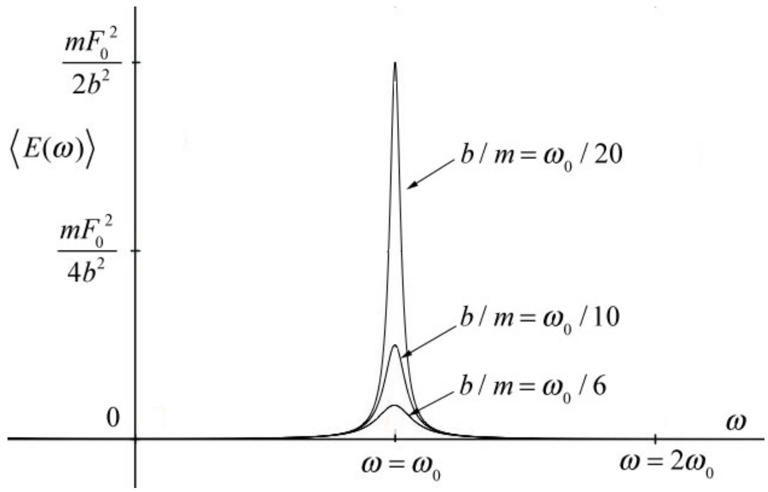


Figure 23.21 Plot of time-averaged energy vs. angular frequency for different values of b/m

In Figure 23.21 we plot the time-averaged energy vs. angular frequency for several different values of the quality factor $Q = 10, 5, \text{and } 3$. Recall that this was the same result that we had for the quality of the free oscillations of the damped oscillator, Eq. (23.5.16) (because we chose the factor π in Eq. (23.5.16)).

23.7 Small Oscillations

Any object moving subject to a force associated with a potential energy function that is quadratic will undergo simple harmonic motion,

$$U(x) = U_0 + \frac{1}{2}k(x - x_{eq})^2. \tag{23.7.1}$$

where k is a "spring constant", x_{eq} is the equilibrium position, and the constant U_0 just depends on the choice of reference point x_{ref} for zero potential energy, $U(x_{ref}) = 0$,

$$0 = U(x_{ref}) = U_0 + \frac{1}{2}k(x_{ref} - x_{eq})^2 . \tag{23.7.2}$$

Therefore the constant is

$$U_0 = -\frac{1}{2}k(x_{ref} - x_{eq})^2 . \tag{23.7.3}$$

The minimum of the potential x_0 corresponds to the point where the x-component of the force is zero,

$$\left.\frac{dU}{dx}\right|_{x=x_0} = 2k(x_0 - x_{eq}) = 0 \Rightarrow x_0 = x_{eq} , \tag{23.7.4}$$

corresponding to the equilibrium position. Therefore the constant is $U(x_0) = U_0$ and we rewrite our potential function as

$$U(x) = U(x_0) + \frac{1}{2}k(x - x_0)^2 . \tag{23.7.5}$$

Now suppose that a potential energy function is not quadratic but still has a minimum at x_0. For example, consider the potential energy function

$$U(x) = -U_1\left(\left(\frac{x}{x_1}\right)^3 - \left(\frac{x}{x_1}\right)^2\right), \tag{23.7.6}$$

(Figure 23.22), which has a stable minimum at x_0.

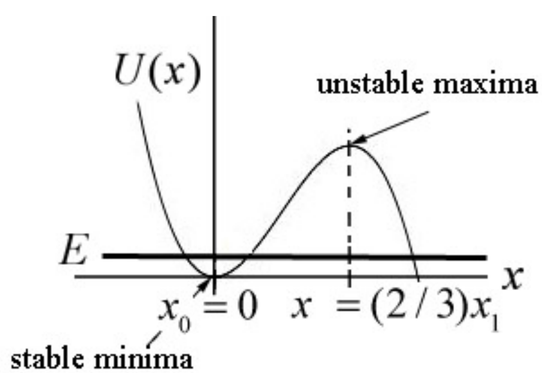


Figure 23.22 Potential energy function with stable minima and unstable maxima

When the energy of the system is very close to the value of the potential energy at the minimum $U(x_0)$, we shall show that the system will undergo small oscillations about the minimum value x_0. We shall use the Taylor formula to approximate the potential function as a polynomial. We shall show that near the minimum x_0, we can approximate the potential function by a quadratic function similar to Eq. (23.7.5) and show that the system undergoes simple harmonic motion for small oscillations about the minimum x_0.

We begin by expanding the potential energy function about the minimum point using the Taylor formula

$$U(x) = U(x_0) + \left.\frac{dU}{dx}\right|_{x=x_0}(x-x_0) + \frac{1}{2!}\left.\frac{d^2U}{dx^2}\right|_{x=x_0}(x-x_0)^2 + \frac{1}{3!}\left.\frac{d^3U}{dx^3}\right|_{x=x_0}(x-x_0)^3 + \cdots \quad (23.7.7)$$

where $\frac{1}{3!}\left.\frac{d^3U}{dx^3}\right|_{x=x_0}(x-x_0)^3$ is a third order term in that it is proportional to $(x-x_0)^3$, and $\left.\frac{d^3U}{dx^3}\right|_{x=x_0}$, $\left.\frac{d^2U}{dx^2}\right|_{x=x_0}$, and $\left.\frac{dU}{dx}\right|_{x=x_0}$ are constants. If x_0 is the minimum of the potential energy, then the linear term is zero, because

$$\left.\frac{dU}{dx}\right|_{x=x_0} = 0 \quad (23.7.8)$$

and so Eq. ((23.7.7)) becomes

$$U(x) \simeq U(x_0) + \frac{1}{2}\left.\frac{d^2U}{dx^2}\right|_{x=x_0}(x-x_0)^2 + \frac{1}{3!}\left.\frac{d^3U}{dx^3}\right|_{x=x_0}(x-x_0)^3 + \cdots \quad (23.7.9)$$

For small displacements from the equilibrium point such that $\left|x-x_0\right|$ is sufficiently small, the third order term and higher order terms are very small and can be ignored. Then the potential energy function is approximately a quadratic function,

$$U(x) \simeq U(x_0) + \frac{1}{2}\left.\frac{d^2U}{dx^2}\right|_{x=x_0}(x-x_0)^2 = U(x_0) + \frac{1}{2}k_{eff}(x-x_0)^2 \quad (23.7.10)$$

where we define k_{eff}, the ***effective spring constant***, by

$$k_{eff} \equiv \left.\frac{d^2U}{dx^2}\right|_{x=x_0}. \quad (23.7.11)$$

Because the potential energy function is now approximated by a quadratic function, the system will undergo simple harmonic motion for small displacements from the minimum with a force given by

$$F_x = -\frac{dU}{dx} = -k_{eff}(x - x_0)\,. \qquad (23.7.12)$$

At $x = x_0$, the force is zero

$$F_x(x_0) = \frac{dU}{dx}(x_0) = 0\,. \qquad (23.7.13)$$

We can determine the period of oscillation by substituting Eq. (23.7.12) into Newton's Second Law

$$-k_{eff}(x - x_0) = m_{eff}\frac{d^2x}{dt^2} \qquad (23.7.14)$$

where m_{eff} is the ***effective mass.*** For a two-particle system, the effective mass is the reduced mass of the system.

$$m_{eff} = \frac{m_1 m_2}{m_1 + m_2} \equiv \mu_{red}\,, \qquad (23.7.15)$$

Eq. (23.7.14) has the same form as the spring-object ideal oscillator. Therefore the angular frequency of small oscillations is given by

$$\omega_0 = \sqrt{\frac{k_{eff}}{m_{eff}}} = \sqrt{\frac{\left.\dfrac{d^2U}{dx^2}\right|_{x=x_0}}{m_{eff}}}\,. \qquad (23.7.16)$$

Example 23.6: Quartic Potential

A system with effective mass m has a potential energy given by

$$U(x) = U_0\left(-2\left(\frac{x}{x_0}\right)^2 + \left(\frac{x}{x_0}\right)^4\right), \qquad (23.7.17)$$

where U_0 and x_0 are positive constants and $U(0) = 0$. (a) Find the points where the force on the particle is zero. Classify these points as stable or unstable. Calculate the value of $U(x)/U_0$ at these equilibrium points. (b) If the particle is given a small displacement from an equilibrium point, find the angular frequency of small oscillation.

Solution: (a) A plot of $U(x)/U_0$ as a function of x/x_0 is shown in Figure 23.23.

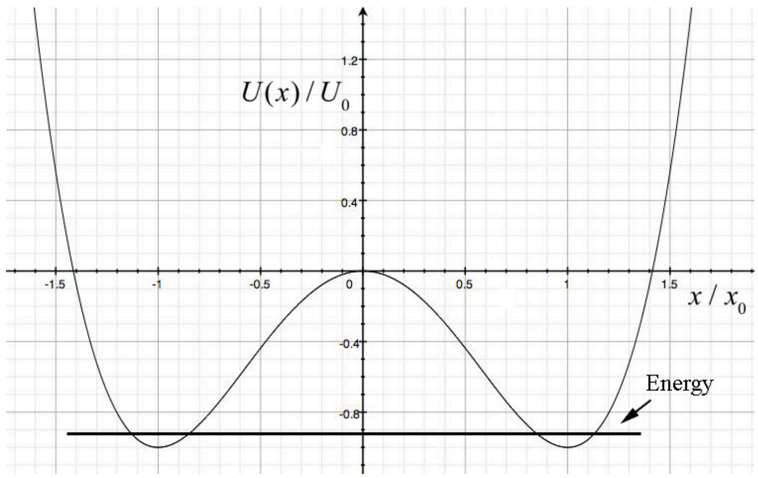


Figure 22.23 Plot of $U(x)/U_0$ as a function of x/x_0

The force on the particle is zero at the minimum of the potential energy,

$$\begin{aligned} 0 &= \frac{dU}{dx} = U_0\left(-4\left(\frac{1}{x_0}\right)^2 x + 4\left(\frac{1}{x_0}\right)^4 x^3\right) \\ &= -4U_0 x\left(\frac{1}{x_0}\right)^2\left(1 - \left(\frac{x}{x_0}\right)^2\right) \Rightarrow x^2 = x_0{}^2 \text{ and } x = 0. \end{aligned} \tag{23.7.18}$$

The equilibrium points are at $x = \pm x_0$ which are stable and $x = 0$ which is unstable. The second derivative of the potential energy is given by

$$\frac{d^2U}{dx^2} = U_0\left(-4\left(\frac{1}{x_0}\right)^2 + 12\left(\frac{1}{x_0}\right)^4 x^2\right). \tag{23.7.19}$$

If the particle is given a small displacement from $x = x_0$ then

$$\left.\frac{d^2U}{dx^2}\right|_{x=x_0} = U_0\left(-4\left(\frac{1}{x_0}\right)^2 + 12\left(\frac{1}{x_0}\right)^4 x_0^2\right) = U_0\frac{8}{x_0^2}. \tag{23.7.20}$$

(b) The angular frequency of small oscillations is given by

$$\omega_0 = \sqrt{\left.\frac{d^2U}{dx^2}\right|_{x=x_0} / m} = \sqrt{\frac{8U_0}{mx_0^2}}. \tag{23.7.21}$$

Example 23.7: Lennard-Jones 6-12 Potential

A commonly used potential energy function to describe the interaction between two atoms is the Lennard-Jones 6-12 potential

$$U(r) = U_0\left[(r_0 / r)^{12} - 2(r_0 / r)^6\right];\ r > 0\ , \qquad (23.7.22)$$

where r is the distance between the atoms. Find the angular frequency of small oscillations about the stable equilibrium position for two identical atoms bound to each other by the Lennard-Jones interaction. Let m denote the effective mass of the system of two atoms.

Solution: The equilibrium points are found by setting the first derivative of the potential energy equal to zero,

$$0 = \frac{dU}{dr} = U_0\left[-12r_0^{\,12}r^{-13} + 12r_0^{\,6}r^{-7}\right] = U_0 12r_0^{\,6}r^{-7}\left[-\left(\frac{r_0}{r}\right)^6 + 1\right]. \qquad (23.7.23)$$

The equilibrium point occurs when $r = r_0$. The second derivative of the potential energy function is

$$\frac{d^2U}{dr^2} = U_0\left[+(12)(13)r_0^{\,12}r^{-14} - (12)(7)r_0^{\,6}r^{-8}\right]\ . \qquad (23.7.24)$$

Evaluating this at $r = r_0$ yields

$$\left.\frac{d^2U}{dr^2}\right|_{r=r_0} = 72U_0r_0^{\,-2}\ . \qquad (23.7.25)$$

The angular frequency of small oscillation is therefore

$$\omega_0 = \sqrt{\left.\frac{d^2U}{dr^2}\right|_{r=r_0} / m} = \sqrt{72U_0 / mr_0^{\,2}}\ . \qquad (23.7.26)$$

Appendix 23A: Solution to Simple Harmonic Oscillator Equation

In our analysis of the solution of the simple harmonic oscillator equation of motion, Equation (23.2.1),

$$-k\,x = m\frac{d^2x}{dt^2}, \tag{23.A.1}$$

we assumed that the solution was a linear combination of sinusoidal functions,

$$x(t) = A\cos(\omega_0\,t) + B\sin(\omega_0\,t), \tag{23.A.2}$$

where $\omega_0 = \sqrt{k/m}$. We shall now derive Eq. (23.A.2).

Assume that the mechanical energy of the spring-object system is given by the constant E. Choose the reference point for potential energy to be the unstretched position of the spring. Let x denote the amount the spring has been compressed ($x<0$) or stretched ($x>0$) from equilibrium at time t and denote the amount the spring has been compressed or stretched from equilibrium at time $t=0$ by $x(t=0)\equiv x_0$. Let $v_x = dx/dt$ denote the x-component of the velocity at time t and denote the x-component of the velocity at time $t=0$ by $v_x(t=0)\equiv v_{x,0}$. The constancy of the mechanical energy is then expressed as

$$E = K + U = \frac{1}{2}k\,x^2 + \frac{1}{2}m\,v^2. \tag{23.A.3}$$

We can solve Eq. (23.A.3) for the square of the x-component of the velocity,

$$v_x^{\,2} = \frac{2E}{m} - \frac{k}{m}x^2 = \frac{2E}{m}\left(1 - \frac{k}{2E}x^2\right). \tag{23.A.4}$$

Taking square roots, we have

$$\frac{dx}{dt} = \sqrt{\frac{2E}{m}}\sqrt{1-\frac{k}{2E}x^2}\,. \tag{23.A.5}$$

(why we take the positive square root will be explained below).

Let $a_1 \equiv \sqrt{2E/m}$ and $a_2 \equiv k/2E$. It's worth noting that a_1 has dimensions of velocity and w has dimensions of [length]$^{-2}$. Eq. (23.A.5) is separable,

$$\frac{dx}{dt} = a_1\sqrt{1 - a_2 x^2}$$
$$\frac{dx}{\sqrt{1 - a_2 x^2}} = a_1\, dt. \tag{23.A.6}$$

We now integrate Eq. (23.A.6),

$$\int \frac{dx}{\sqrt{1 - a_1 x^2}} = \int a_1\, dt \ . \tag{23.A.7}$$

The integral on the left in Eq. (23.A.7) is well known, and a derivation is presented here. We make a change of variables $\cos\theta = \sqrt{a_2}\, x$ with the differentials $d\theta$ and dx related by $-\sin\theta\, d\theta = \sqrt{a_2}\, dx$. The integration variable is

$$\theta = \cos^{-1}\left(\sqrt{a_2}\, x\right). \tag{23.A.8}$$

Eq. (23.A.7) then becomes

$$\int \frac{-\sin\theta\, d\theta}{\sqrt{1 - \cos^2\theta}} = \int \sqrt{a_2}\, a_1\, dt\ . \tag{23.A.9}$$

This is a good point at which to check the dimensions. The term on the left in Eq. (23.A.9) is dimensionless, and the product $\sqrt{a_2}\, a_1$ on the right has dimensions of inverse time, $[\text{length}]^{-1}[\text{length}\cdot\text{time}^{-1}] = [\text{time}^{-1}]$, so $\sqrt{a_2}\, a_1\ dt$ is dimensionless. Using the trigonometric identity $\sqrt{1 - \cos^2\theta} = \sin\theta$, Eq. (23.A.9) reduces to

$$\int d\theta = -\int \sqrt{a_2}\, a_1\ \ dt\ . \tag{23.A.10}$$

Although at this point in the derivation we don't know that $\sqrt{a_2}\, a_1$, which has dimensions of frequency, is the angular frequency of oscillation, we'll use some foresight and make the identification

$$\omega_0 \equiv \sqrt{a_2}\, a_1 = \sqrt{\frac{k}{2E}}\sqrt{\frac{2E}{m}} = \sqrt{\frac{k}{m}}\,, \tag{23.A.11}$$

and Eq. (23.A.10) becomes

$$\int_{\theta=\theta_0}^{\theta} d\theta = -\int_{t=0}^{t} \omega_0\, dt\,. \tag{23.A.12}$$

After integration we have

$$\theta - \theta_0 = -\omega_0\, t\,, \tag{23.A.13}$$

where $\theta_0 \equiv -\phi$ is the constant of integration. Because $\theta = \cos^{-1}\left(\sqrt{a_2}\; x(t)\right)$, Eq. (23.A.13) becomes

$$\cos^{-1}\left(\sqrt{a_2}\; x(t)\right) = -(\omega_0\, t + \phi)\,. \tag{23.A.14}$$

Take the cosine of each side of Eq. (23.A.14), yielding

$$x(t) = \frac{1}{\sqrt{a_2}}\cos(-(\omega_0\, t + \phi)) = \sqrt{\frac{2E}{k}}\cos(\omega_0\, t + \phi)\,. \tag{23.A.15}$$

At $t = 0$,

$$x_0 \equiv x(t=0) = \sqrt{\frac{2E}{k}}\cos\phi\,. \tag{23.A.16}$$

The x-component of the velocity as a function of time is then

$$v_x(t) = \frac{dx(t)}{dt} = -\omega_0\sqrt{\frac{2E}{k}}\sin(\omega_0\, t + \phi)\,. \tag{23.A.17}$$

At $t = 0$,

$$v_{x,0} \equiv v_x(t=0) = -\omega_0\sqrt{\frac{2E}{k}}\sin\phi\,. \tag{23.A.18}$$

We can determine the constant ϕ by dividing the expressions in Eqs. (23.A.18) and (23.A.16),

$$-\frac{v_{x,0}}{\omega_0 x_0} = \tan\phi. \tag{23.A.19}$$

Thus the constant ϕ can be determined by the initial conditions and the angular frequency of oscillation,

$$\phi = \tan^{-1}\left(-\frac{v_{x,0}}{\omega_0 x_0}\right). \tag{23.A.20}$$

Use the identity

$$\cos(\omega_0 t + \phi) = \cos(\omega_0 t)\cos(\phi) - \sin(\omega_0 t)\sin(\phi) \tag{23.A.21}$$

to expand Eq. (23.A.15) yielding

$$x(t) = \sqrt{\frac{2E}{k}}\cos(\omega_0 t)\cos(\phi) - \sqrt{\frac{2E}{k}}\sin(\omega_0 t)\sin(\phi)\,, \qquad (23.A.22)$$

and substituting Eqs. (23.A.16) and (23.A.18) into Eq. (23.A.22) yields

$$x(t) = x_0 \cos\omega_0 t + \frac{v_{x,0}}{\omega_0}\sin\omega_0 t\,, \qquad (23.A.23)$$

agreeing with Eq. (23.2.21).

So, what about the missing $\pm$ that should have been in Eq. (23.A.5)? Strictly speaking, we would need to redo the derivation for the block moving in different directions. Mathematically, this would mean replacing ϕ by $\pi - \phi$ (or $\phi - \pi$) when the block's velocity changes direction. Changing from the positive square root to the negative *and* changing ϕ to $\pi - \phi$ have the collective action of reproducing Eq. (23.A.23).

Appendix 23B: Complex Numbers

A complex number z can be written as a sum of a real number x and a purely imaginary number iy where $i = \sqrt{-1}$,

$$z = x + iy\,. \qquad (23.B.1)$$

The complex number can be represented as a point in the x-y plane as show in Figure 23B.1.

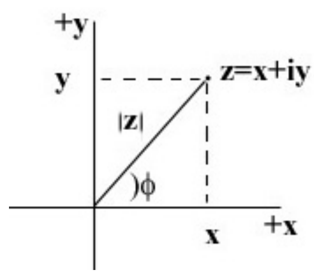


Figure 23B.1 Complex numbers

The complex conjugate $\bar{z}$ of a complex number z is defined to be

$$\bar{z} = x - iy\,. \qquad (23.B.2)$$

The modulus of a complex number is

$$|z| = (z\bar{z})^{1/2} = ((x+iy)(x-iy))^{1/2} = (x^2 + y^2)^{1/2}\,. \qquad (23.B.3)$$

where we used the fact that $i^2 = -1$. The modulus $|z|$ represents the length of the ray from the origin to the complex number z in Figure 23B.1. Let ϕ denote the angle that the ray with the positive x-axis in Figure 23B.1. Then

$$x = |z|\cos\phi, \tag{23.B.4}$$

$$y = |z|\sin\phi. \tag{23.B.5}$$

Hence the angle ϕ is given by

$$\phi = \tan^{-1}(y/x). \tag{23.B.6}$$

The inverse of a complex number is then

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{x - iy}{(x^2 + y^2)}. \tag{23.B.7}$$

The modulus of the inverse is the inverse of the modulus;

$$\left|\frac{1}{z}\right| = \frac{1}{(x^2 + y^2)^{1/2}} = \frac{1}{|z|}. \tag{23.B.8}$$

The sum of two complex numbers, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, is the complex number

$$z_3 = z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) = x_3 + iy_3, \tag{23.B.9}$$

where $x_3 = x_1 + x_2$, $y_3 = y_1 + y_2$. We can represent this by the vector sum in Figure 23B.2,

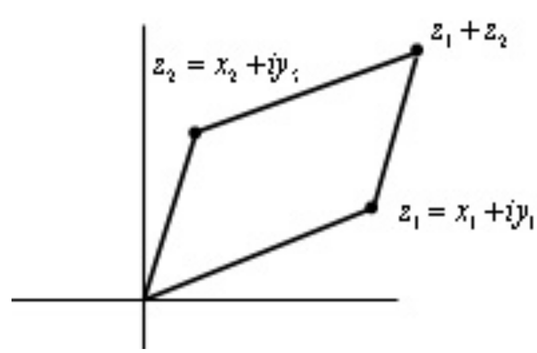


Figure 23B.2 Sum of two complex numbers

The product of two complex numbers is given by

$$z_3 = z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) = x_3 + iy_3, \quad (23.B.10)$$

where $x_3 = x_1 x_2 - y_1 y_2$, and $y_3 = x_1 y_2 + x_2 y_1$.

One of the most important identities in mathematics is the Euler formula,

$$e^{i\phi} = \cos\phi + i\sin\phi. \quad (23.B.11)$$

This identity follows from the power series representations for the exponential, sine, and cosine functions,

$$e^{i\phi} = \sum_{n=0}^{n=\infty} \frac{1}{n!}(i\phi)^n = 1 + i\phi - \frac{\phi^2}{2} - i\frac{\phi^3}{3!} + \frac{\phi^4}{4!} + i\frac{\phi^5}{5!} \ldots, \quad (23.B.12)$$

$$\cos\phi = 1 - \frac{\phi^2}{2} + \frac{\phi^4}{4!} - \ldots, \quad (23.B.13)$$

$$\sin\phi = \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \ldots. \quad (23.B.14)$$

We define two projection operators. The first one takes the complex number $e^{i\phi}$ and gives its real part,

$$\operatorname{Re} e^{i\phi} = \cos\phi. \quad (23.B.15)$$

The second operator takes the complex number $e^{i\phi}$ and gives its imaginary part, which is the real number

$$\operatorname{Im} e^{i\phi} = \sin\phi. \quad (23.B.16)$$

A complex number $z = x + iy$ can also be represented as the product of a modulus $|z|$ and a phase factor $e^{i\phi}$,

$$z = |z| e^{i\phi}. \quad (23.B.17)$$

The inverse of a complex number is then

$$\frac{1}{z} = \frac{1}{|z| e^{i\phi}} = \frac{1}{|z|} e^{-i\phi}, \quad (23.B.18)$$

where we used the fact that

$$\frac{1}{e^{i\phi}} = e^{-i\phi}. \quad (23.B.19)$$

In terms of modulus and phase, the sum of two complex numbers, $z_1 = |z_1| e^{i\phi_1}$ and $z_2 = |z_2| e^{i\phi_2}$, is

$$z_1 + z_2 = |z_1|e^{i\phi_1} + |z_2|e^{i\phi_2} . \qquad (23.B.20)$$

A special case of this result is when the phase angles are equal, $\phi_1 = \phi_2$, then the sum $z_1 + z_2$ has the same phase factor $e^{i\phi_1}$ as z_1 and z_2,

$$z_1 + z_2 = |z_1|e^{i\phi_1} + |z_2|e^{i\phi_1} = (|z_1| + |z_2|)e^{i\phi_1} . \qquad (23.B.21)$$

The product of two complex numbers, $z_1 = |z_1|e^{i\phi_1}$, and $z_2 = |z_2|e^{i\phi_2}$ is

$$z_1 z_2 = |z_1|e^{i\phi_1}|z_2|e^{i\phi_2} = |z_1||z_2|e^{i\phi_1+\phi_2} . \qquad (23.B.22)$$

When the phases are equal, the product does not have the same factor as z_1 and z_2,

$$z_1 z_2 = |z_1|e^{i\phi_1}|z_2|e^{i\phi_1} = |z_1||z_2|e^{i2\phi_1} . \qquad (23.B.23)$$

Appendix 23C: Solution to the Underdamped Simple Harmonic Oscillator

Consider the underdamped simple harmonic oscillator equation (Eq. (23.5.4)),

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0 . \qquad (23.C.1)$$

When $(b/m)^2 < 4k/m$, we show that the equation has a solution of the form

$$x(t) = x_{\mathrm{m}} e^{-\alpha t}\cos(\gamma t + \phi) . \qquad (23.C.2)$$

Solution: Let's suppose the function $x(t)$ has the form

$$x(t) = A\,\mathrm{Re}(e^{zt}) \qquad (23.C.3)$$

where z is a number (possibly complex) and A is a real number. Then

$$\frac{dx}{dt} = zAe^{zt} \qquad (23.C.4)$$

$$\frac{d^2x}{dt^2} = z^2 Ae^{zt} \qquad (23.C.5)$$

We now substitute Eqs. (23.C.3), (23.C.4), and (23.C.5), into Eq. (23.C.1) resulting in

$$z^2 Ae^{zt} + \frac{b}{m} zAe^{zt} + \frac{k}{m} Ae^{zt} = 0 . \tag{23.C.6}$$

Collecting terms in Eq. (23.C.6) yields

$$\left(z^2 + \frac{b}{m} z + \frac{k}{m} \right) Ae^{zt} = 0 \tag{23.C.7}$$

The condition for the solution is that

$$z^2 + \frac{b}{m} z + \frac{k}{m} = 0 \; . \tag{23.C.8}$$

This quadratic equation has solutions

$$z = \frac{-(b/m) \pm ((b/m)^2 - 4k/m)^{1/2}}{2} \; . \tag{23.C.9}$$

When $(b/m)^2 < 4k/m$, the oscillator is called *underdamped*, and we have two solutions for z, however the solutions are complex numbers. Let

$$\gamma = (k/m - (b/2m)^2)^{1/2} ; \tag{23.C.10}$$

and

$$\alpha = b/2m . \tag{23.C.11}$$

.

Recall that the imaginary number $i = \sqrt{-1}$. The two solutions are then $z_1 = -\alpha + i\gamma t$ and $z_2 = -\alpha - i\gamma t$. Because our system is linear, our general solution is a linear combination of these two solutions,

$$x(t) = A_1 e^{-\alpha + i\gamma t} + A_2 e^{-\alpha - i\gamma t} = (A_1 e^{i\gamma t} + A_2 e^{-i\gamma t}) e^{-\alpha t} , \tag{23.C.12}$$

where A_1 and A_2 are constants. We shall transform this expression into a more familiar equation involving sine and cosine functions with help from the Euler formula,

$$e^{\pm i\gamma t} = \cos(\gamma t) \pm i \sin(\gamma t) . \tag{23.C.13}$$

Therefore we can rewrite our solution as

$$x(t) = \left(A_1(\cos(\gamma t) + i\sin(\gamma t)) + A_2(\cos(\gamma t) - i\sin(\gamma t)) \right) e^{-\alpha t} . \tag{23.C.14}$$

A little rearrangement yields

$$x(t) = \big((A_1 + A_2)\cos(\gamma t) + i(A_1 - A_2)\sin(\gamma t)\big)e^{-\alpha t} . \tag{23.C.15}$$

Define two new constants $C = A_1 + A_2$ and $D = i(A_1 - A_2)$. Then our solution looks like

$$x(t) = (C\cos(\gamma t) + D\sin(\gamma t))e^{-\alpha t} . \tag{23.C.16}$$

Recall from Example 23.5 that we can rewrite

$$C\cos(\gamma t) + D\sin(\gamma t) = x_{\mathrm{m}}\cos(\gamma t + \phi) \tag{23.C.17}$$

,

where

$$x_{\mathrm{m}} = (C^2 + D^2)^{1/2} \text{, and } \phi = \tan^{-1}(D/C) .$$

Then our general solution for the underdamped case (Eq. (23.C.16)) can be written as

$$x(t) = x_{\mathrm{m}}e^{-\alpha t}\cos(\gamma t + \phi) . \tag{23.C.18}$$

There are two other possible cases which we shall not analyze: when $(b/m)^2 > 4k/m$, a case referred to as *overdamped*, and when $(b/m)^2 = 4k/m$, a case referred to as *critically damped*.

Appendix 23D: Solution to the Forced Damped Oscillator Equation

We shall now use complex numbers to solve the differential equation

$$F_0\cos(\omega t) = m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx \ . \tag{23.D.1}$$

We begin by assuming a solution of the form

$$x(t) = x_0\cos(\omega t + \phi) \ . \tag{23.D.2}$$

where the amplitude x_0 and the phase constant ϕ need to be determined. We begin by defining the complex function

$$z(t) = x_0 e^{i(\omega t + \phi)} \ . \tag{23.D.3}$$

Our desired solution can be found by taking the real projection

$$x(t) = \mathrm{Re}(z(t)) = x_0 \cos(\omega t + \phi) \ . \tag{23.D.4}$$

Our differential equation can now be written as

$$F_0 e^{i\omega t} = m\frac{d^2 z}{dt^2} + b\frac{dz}{dt} + kz \ . \tag{23.D.5}$$

We take the first and second derivatives of Eq. (23.D.3),

$$\frac{dz}{dt}(t) = i\omega x_0 e^{i(\omega t+\phi)} = i\omega z \ . \tag{23.D.6}$$

$$\frac{d^2 z}{dt^2}(t) = -\omega^2 x_0 e^{i(\omega t+\phi)} = -\omega^2 z \ . \tag{23.D.7}$$

We substitute Eqs. (23.D.3), (23.D.6), and (23.D.7) into Eq. (23.D.5) yielding

$$F_0 e^{i\omega t} = (-\omega^2 m + bi\omega + k)z = (-\omega^2 m + bi\omega + k)x_0 e^{i(\omega t+\phi)} \ . \tag{23.D.8}$$

We divide Eq. (23.D.8) through by $e^{i\omega t}$ and collect terms using yielding

$$x_0 e^{i\phi} = \frac{F_0 / m}{((\omega_0{}^2 - \omega^2) + i(b/m)\omega)} \ . \tag{23.D.9}$$

where we have used $\omega_0{}^2 = k/m$. Introduce the complex number

$$z_1 = (\omega_0{}^2 - \omega^2) + i(b/m)\omega \ . \tag{23.D.10}$$

Then Eq. (23.D.9) can be written as

$$x_0 e^{i\phi} = \frac{F_0}{my} \ . \tag{23.D.11}$$

Multiply the numerator and denominator of Eq. (23.D.11) by the complex conjugate $\overline{z}_1 = (\omega_0{}^2 - \omega^2) - i(b/m)\omega$ yielding

$$x_0 e^{i\phi} = \frac{F_0 \overline{z}_1}{m z_1 \overline{z}_1} = \frac{F_0}{m}\frac{((\omega_0{}^2 - \omega^2) - i(b/m)\omega)}{((\omega_0{}^2 - \omega^2)^2 + (b/m)^2\omega^2)} \equiv u + iv \ . \tag{23.D.12}$$

where

$$u = \frac{F_0}{m}\frac{(\omega_0{}^2 - \omega^2)}{((\omega_0{}^2 - \omega^2)^2 + (b/m)^2\omega^2)} \ , \tag{23.D.13}$$

$$v = -\frac{F_0}{m}\frac{(b/m)\omega}{((\omega_0^{\ 2}-\omega^2)^2+(b/m)^2\omega^2)} \ . \tag{23.D.14}$$

Therefore the modulus x_0 is given by

$$x_0 = (u^2+v^2)^{1/2} = \frac{F_0/m}{((\omega_0^{\ 2}-\omega^2)^2+(b/m)^2\omega^2)} \ , \tag{23.D.15}$$

and the phase is given by

$$\phi = \tan^{-1}(v/u) = \frac{-(b/m)\omega}{(\omega_0^{\ 2}-\omega^2)} \ . \tag{23.D.16}$$