LINEAR ALGEBRA

Pure Mathematics 8

كلية التربية بالغردقة

Linear Algebra:

Systems of Linear Equations:

Definition: The most general linear equation is $a_1x_1 + a_2x_2 + ... + a_nx_n = b$ where there are n unknowns $x_1, x_2, ..., x_n$ and $a_1, a_2, ..., a_n, b$ are all known numbers.

A system of linear equations is nothing more than a collection of two or more linear equations. Here is a general system of m equations and n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\ ..... \quad ..... \quad ..... \quad & ..... \quad ..... \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &= b_m \end{aligned}$$

In this system the unknowns are $x_1, x_2, ..., x_n$ and the a_{ij} and b_i are known numbers.

The two matrices:

$$\begin{pmatrix} a_{11} & a_{12} & ..... & a_{1n} \\ a_{21} & a_{22} & ..... & a_{2n} \\ ..... & ..... & ..... & ..... \\ a_{m1} & a_{m2} & ..... & a_{mn} \end{pmatrix}, \begin{pmatrix} a_{11} & a_{12} & ..... & a_{1n} & b_1 \\ a_{21} & a_{22} & ..... & a_{2n} & b_2 \\ ..... & ..... & ..... & ..... & ..... \\ a_{m1} & a_{m2} & ..... & a_{mn} & b_m \end{pmatrix}$$

are the *coefficient matrix* , the *augmented matrix* for the system respectively.

The solution set to a system with n unknowns $x_1, x_2, ..., x_n$ is a set of numbers $t_1, t_2, ..., t_n$ so that we set $x_1 = t_1, x_2 = t_2, ..., x_n = t_n$ then all of the equations in the system will be satisfied.

Theorem: Given a system of m equations and n unknowns there will be one of three possibilities for solutions of the system:

1. There will be no solution.
2. There will be exactly one solution.
3. There will be infinitely many solutions.

If there is no solution to the system we call the system *inconsistent* , and if there is at least one solution to the system we call the system *consistent*.

Examples:

(1) The system of linear equations: $\begin{matrix} x + y = 1 \\ x + 8y = 1 \end{matrix}$ consistent; $x = 1$, $y = 0$.

Also, the system of linear equations: $\begin{matrix} x + y + 2z = 9 \\ 2x + 4y - 3z = 1 \\ 3x + 6y - 5z = 0 \end{matrix}$ consistent; $x = 1$, $y = 2$, $z = 3$.

(2) The system of linear equations: $\begin{matrix} 3x - 6y = 1 \\ 2x - 4y = 5 \end{matrix}$ inconsistent.

Also, the system of linear equations: $\begin{matrix} x - 2y + 3z = 2 \\ 2x + 3y - 2z = 5 \\ 4x - y + 4z = 1 \end{matrix}$ inconsistent.

The Row Reduction Algorithm:

- Interchange of any two rows.
- Multiplication of a row by a scalar.
- Addition of a multiple of one row to another row.

A matrix (any matrix) is said to be in *reduced row-echelon form* if it satisfies all four of the following conditions:

(1) If there are any rows of all zeros then they are at the bottom of the matrix.

(2) If the row does not consist of all zeros then its first non-zero entry is a 1 . This 1 is called a *leading 1*.

(3) In any two successive rows, neither of which consists of all zeros, the leading 1 of the lower row is to the right of the leading 1 of the higher row.

(4) If a column contains a leading 1 then all the other entries of that column are zero.

Examples:

1- The following matrices are all in reduced row-echelon form:

$$\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 5 \end{pmatrix}, \begin{pmatrix} 1 & -7 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

And the following matrices are not in reduced row-echelon form:

$$\begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 4 & 3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

2- Put the matrix $\begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 3 \\ 3 & 2 & -5 & 5 \end{pmatrix}$ in reduced row-echelon form.

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 3 \\ 3 & 2 & -5 & 5 \end{pmatrix} \xrightarrow[-3r_1+r_3]{-2r_1+r_2} \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -8 & -1 \end{pmatrix} \xrightarrow{-r_2} \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -8 & -1 \end{pmatrix} \xrightarrow{r_2+r_3} \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -7 & 0 \end{pmatrix} \xrightarrow{(-1/7)r_3}$$

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \xrightarrow{-r_2+r_1} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \xrightarrow{-r_3+r_2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Remark: The ***rank*** of a matrix is the number of Ones in the principal diagonal of the reduced matrix.

✓ <u>To find the solution set to a system of linear equations</u>:
we put the augmented matrix for the system in reduced row-echelon form, and write the final reduced matrix as a system of linear equations, then we can get the values of the unknowns (if the system consistent), that is called the *Gauss-Jordan Elimination*. As it shown in the following examples:

Examples:

Use the Row Reduction Algorithm to put the augmented matrix in reduced row-echelon form, then find the solution set for each of the following systems of linear equations:

(i) $\begin{aligned} x+y+2z&=9\\ 2x+4y-3z&=1\\ 3x+6y-5z&=0 \end{aligned}$ (ii) $\begin{aligned} x+y-z&=0\\ x-4y+2z&=0\\ 2x-3y+z&=0 \end{aligned}$ (iii) $\begin{aligned} x_1+5x_2+4x_3-13x_4&=3\\ 3x_1-x_2+2x_3+5x_4&=2\\ 2x_1+2x_2+3x_3-4x_4&=1 \end{aligned}$

(i) We reduce the augmented matrix as follows:

$$\begin{pmatrix} 1 & 1 & 2 & 9\\ 2 & 4 & -3 & 1\\ 3 & 6 & -5 & 0 \end{pmatrix} \xrightarrow[-3r_1+r_3]{-2r_1+r_2} \begin{pmatrix} 1 & 1 & 2 & 9\\ 0 & 2 & -7 & -17\\ 0 & 3 & -11 & -27 \end{pmatrix} \xrightarrow{(1/2)r_2} \begin{pmatrix} 1 & 1 & 2 & 9\\ 0 & 1 & -7/2 & -17/2\\ 0 & 3 & -11 & -27 \end{pmatrix}$$

$$\xrightarrow{-3r_2+r_3} \begin{pmatrix} 1 & 1 & 2 & 9\\ 0 & 1 & -7/2 & -17/2\\ 0 & 0 & -1/2 & -3/2 \end{pmatrix} \xrightarrow{-2r_3} \begin{pmatrix} 1 & 1 & 2 & 9\\ 0 & 1 & -7/2 & -17/2\\ 0 & 0 & 1 & 3 \end{pmatrix}$$

$$\xrightarrow{-r_2+r_1} \begin{pmatrix} 1 & 0 & 11/2 & 35/2\\ 0 & 1 & -7/2 & -17/2\\ 0 & 0 & 1 & 3 \end{pmatrix} \xrightarrow[(-11/2)r_3+r_1]{(7/2)r_3+r_2} \begin{pmatrix} 1 & 0 & 0 & 1\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & 3 \end{pmatrix}.$$

$$\therefore \begin{aligned} x&=1\\ y&=2\\ z&=3 \end{aligned}$$ (the system has unique solution).

(ii) We reduce the augmented matrix as follows:

$$\begin{pmatrix} 1 & 1 & -1 & 0\\ 1 & -4 & 2 & 0\\ 2 & -3 & 1 & 0 \end{pmatrix} \xrightarrow[-2r_1+r_3]{-r_1+r_2} \begin{pmatrix} 1 & 1 & -1 & 0\\ 0 & -5 & 3 & 0\\ 0 & -5 & 3 & 0 \end{pmatrix} \xrightarrow{-r_2+r_3} \begin{pmatrix} 1 & 1 & -1 & 0\\ 0 & -5 & 3 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{(-1/5)r_2}$$

$$\begin{pmatrix} 1 & 1 & -1 & 0\\ 0 & 1 & -\frac{3}{5} & 0\\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{-r_2+r_1} \begin{pmatrix} 1 & 0 & -\frac{2}{5} & 0\\ 0 & 1 & -\frac{3}{5} & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad \therefore x=\frac{2}{5}z\ ,\ y=\frac{3}{5}z$$

$z=0 \Rightarrow x=y=0\ ,\ \ z=5 \Rightarrow x=2, y=3\ ,\ \ldots\ ,$

Hence the system has more than one solution.

(iii) We reduce the augmented matrix as follows:

$$\begin{pmatrix} 1 & 5 & 4 & -13 & 3 \\ 3 & -1 & 2 & 5 & 2 \\ 2 & 2 & 3 & -4 & 1 \end{pmatrix} \xrightarrow[-2r_1+r_3]{-3r_1+r_2} \begin{pmatrix} 1 & 5 & 4 & -13 & 3 \\ 0 & -16 & -10 & 44 & -7 \\ 0 & -8 & -5 & 22 & -5 \end{pmatrix} \xrightarrow{-2r_3+r_2} \begin{pmatrix} 1 & 5 & 4 & -13 & 3 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & -8 & -5 & 22 & -5 \end{pmatrix}.$$

The system has no solution ; $0? = 3$.

Exercises:

Use the Row Reduction Algorithm to put the augmented matrix in reduced row-echelon form, then find the solution set for each of the following systems of linear equations:

(i) $\begin{aligned} -2x_1 + x_2 - x_3 &= 4 \\ x_1 + 2x_2 + 3x_3 &= 13 \\ 3x_1 + x_3 &= -1 \end{aligned}$ (ii) $\begin{aligned} x + 3y - 2z &= 0 \\ x - 8y + 8z &= 0 \\ 3x - 2y + 4z &= 0 \end{aligned}$ (iii) $\begin{aligned} x_1 + 2x_2 + 6x_3 &= 4 \\ 2x_1 + 4x_2 + 4x_3 &= -1 \\ -x_1 - 2x_2 + 2x_3 &= 8 \end{aligned}$

Vector(Linear) Space:

Def.: A nonempty set V (on which addition and scalar multiplication are defined) is called *vector space* or *linear space* over a field K if for $u,v,w \in V$, $\lambda,\mu \in K$ the following two conditions are satisfied: $(1) u+v \in V$ $(2) \lambda u \in V$
beside the following properties:

(VA1) $u+v=v+u$

(VA2) $u+(v+w)=(u+v)+w$

(VA3) $\exists 0 \in V$; $0+u=u+0=u$

(VA4) $\forall u \in V \exists -u \in V$; $u+(-u)=(-u)+u=0$

(VM1) $\lambda(\mu u)=(\lambda\mu)u$

(VM2) $1u=u$; $1 \in K$

(VD1) $\lambda(u+v)=\lambda\mu+\lambda v$

(VD2) $(\lambda+\mu)u=\lambda u+\mu u$

Solved Exercises:

Determine whether or not V is a vector space? Give reasons for your assertion.

(i) $V=\{(x,y):x,y \in R\}$, $(x_1,y_1)+(x_2,y_2)=(x_1+x_2,y_1+y_2)$, $k(x,y)=(2kx,2ky)$.

✓ $u=(x,y)$, $1u=1(x,y)=(2x,2y)\neq u$

$\therefore V$ is not vector space.

(ii) $V=\{(x,y):x,y \in R\}$, $(x_1,y_1)+(x_2,y_2)=(x_1+y_2,y_1+x_2)$, $k(x,y)=(kx,ky)$.

✓ $(1,0)+(0,1)=(2,0)$, $(0,1)+(1,0)=(0,2)$ *i.e.* $u+v\neq v+u$.

$\therefore V$ is not vector space.

(iii) $V=\{(x,y)\in R^2 : x=2\}$, $(x_1,y_1)+(x_2,y_2)=(x_1+x_2,y_1+y_2)$, $k(x,y)=(kx,ky)$.

✓ V is not vector space ; $(2,0)+(2,1)=(4,1)\notin V$.

(iv) $V=\{(x,y,z):x,y,z \in R\}$,

$(x_1,y_1,z_1)+(x_2,y_2,z_2)=(x_1+x_2,y_1+y_2,z_1+z_2)$, $k(x,y,z)=(kx,y,z)$.

$u=(x,y,z)$,$(\lambda+\mu)u=(\lambda+\mu)(x,y,z)=((\lambda+\mu)x,y,z)$,

✓ $\lambda u+\mu u=(\lambda x,y,z)+(\mu x,y,z)=((\lambda+\mu)x,2y,2z)$

i.e. $(\lambda+\mu)u\neq\lambda u+\mu u$.

$\therefore V$ is not vector space.

(v) $V=\{(x,y,z):x,y,z \in R\}$,

$(x_1,y_1,z_1)+(x_2,y_2,z_2)=(x_1+x_2,y_1+y_2,z_1+z_2)$, $k(x,y,z)=(kx,1,kz)$.

✓ $u=(x,y,z)$, $1u=1(x,y,z)=(x,1,z)\neq u$

$\therefore V$ is not vector space.

(vi) $V=\{(x,y,z):x,y,z \in R\}$,

$(x_1,y_1,z_1)+(x_2,y_2,z_2)=(x_2,y_1+y_2,z_2)$, $k(x,y,z)=(kx,ky,kz)$.

✓ $(1,2,3)+(4,5,6)=(4,7,6)$, $(4,5,6)+(1,2,3)=(1,7,3)$ *i.e.* $u+v\neq v+u$.

$\therefore V$ is not vector space. [Also (VA1),(VA3),(VA4),(VD2) × (verify that?)]

(vii) $V=\{(0,0,z):z \in R\}$,

$(0,0,z_1)+(0,0,z_2)=(0,0,z_1+z_2)$, $k(0,0,z)=(0,0,kz)$.

✓ V is a vector space ; all conditions and properties holds.

Subspaces:

Def.: Suppose that V is a vector space and W is a subset of V. If, under the addition and scalar multiplication that is defined on V, W is also a vector space then we call W is a *subspace* of V.

Theorem: Prove that a nonempty subset W of a vector space V is a subspace of V if and only if the following two conditions: $(i)\ u+v\in W$, $(ii)\ ku\in W\ \forall u,v\in W, k\ scalar$ are satisfied.

The proof:

(1) Suppose W be a subspace of V, the two conditions $(i),(ii)$ are satisfied from the definition of a subspace.

(2) Suppose the two conditions: $(i)\ u+v\in W$, $(ii)\ ku\in W\ \forall u,v\in W, k\ scalar$ are satisfied,

we prove that W is a subspace of V as follow:

the properites (VA1),(VA2),(VM1),(VM2),(VD1),(VD2) are true simply based on the fact that W is a subset of V, we only need to verify (VA3),(VA4):

From the condition $(ii)\ ku\in W\ \forall u\in W, k\ scalar$ put $k=0\Rightarrow ou=0\in W\ \forall u\in W$,and put $k=-1\Rightarrow(-1)u=-u\in W\ \forall u\in W$, therefore W is a subspace of V.

Examples:

(1) Let $W=\{(a,b,c)\in R^3 : b=2a\}$. Is W a subspace of a vector space R^3?

✓ It is shown that $W\subset R^3$,

(i) let $(a_1,b_1,c_1),(a_2,b_2,c_2)\in W \Rightarrow b_1=2a_1, b_2=2a_2$

$\therefore (a_1,b_1,c_1)+(a_2,b_2,c_2)=(a_1+a_2,b_1+b_2,c_1+c_2)\in W\ ;\ b_1+b_2=2(a_1+a_2).$

(ii) let $(a,b,c)\in W$, k scalar $\Rightarrow b=2a$

$\therefore k(a,b,c)=(ka,kb,kc)\in W\ ;\ kb=k(2a).$

$\therefore W$ is a subspace of a vector space R^3.

(2) Let $W=\{(a,b,c)\in R^3 : ab=0\}$. Is W a subspace of a vector space R^3?

✓ W is not subspace of a vector space R^3;

$(1,0,1),(0,1,1)\in W, but\ (1,0,1)+(0,1,1)=(1,1,2)\notin W$.

Exercises:

1- Let $W=\{(a,b,c)\in R^3 : b=a^2\}$. Is W a subspace of a vector space R^3?

2- Let $W=\{\begin{pmatrix}0 & a\\ b & 0\end{pmatrix} : a,b\in R\}$. Is W a subspace of a vector space $M_{2\times2}(R)$?

3- Let $W=\{A\in M_{2\times2}(R) : |A|=0\}$. Is W a subspace of a vector space $M_{2\times2}(R)$?

(Give reasons for your assertion).

Linear Combinations:

Def.: We say the vector u from the vector space V is a *linear combination* of the vectors $v_1, v_2, ..., v_n$, all from V , if there are scalars $c_1, c_2, ..., c_n$ so that u can be written $u = c_1 v_1 + c_2 v_2 + ... + c_n v_n$.

Example: Verify that the vector $u = (9,2,7)$ is a linear combination of the vectors $v_1 = (1,2,-1), v_2 = (6,4,2)$, but the vector $w = (4,-1,8)$ is not a linear combination of them.

✓ Let $u = c_1 v_1 + c_2 v_2$; c_1, c_2 *scalars*.

$\therefore (9,2,7) = c_1(1,2,-1) + c_2(6,4,2)$

$9 = c_1 + 6c_2$,

$\therefore 2 = 2c_1 + 4c_2$,

$7 = -c_1 + 2c_2$

We reduce the augmented matrix $\begin{pmatrix} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{pmatrix}$ as follows:

$$\begin{pmatrix} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{pmatrix} \xrightarrow[r_1+r_3]{-2r_1+r_2} \begin{pmatrix} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 8 & 16 \end{pmatrix} \xrightarrow{(-1/8)r_2} \begin{pmatrix} 1 & 6 & 9 \\ 0 & 1 & 2 \\ 0 & 8 & 16 \end{pmatrix} \xrightarrow[-6r_2+r_1]{-8r_2+r_3} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

$\therefore$ $c_1 = -3$, $c_2 = 2$, $\therefore$ $u = -3v_1 + 2v_2$.

Similarly, let $w = c_1 v_1 + c_2 v_2$; c_1, c_2 *scalars*.

$\therefore (4,-1,8) = c_1(1,2,-1) + c_2(6,4,2)$

$4 = c_1 + 6c_2$,

$\therefore -1 = 2c_1 + 4c_2$,

$8 = -c_1 + 2c_2$

We reduce the augmented matrix $\begin{pmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 8 \end{pmatrix}$ as follows:

$$\begin{pmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 8 \end{pmatrix} \xrightarrow[r_1+r_3]{-2r_1+r_2} \begin{pmatrix} 1 & 6 & 4 \\ 0 & -8 & -9 \\ 0 & 8 & 12 \end{pmatrix} \xrightarrow{-1r_2} \begin{pmatrix} 1 & 6 & 4 \\ 0 & 8 & 9 \\ 0 & 8 & 12 \end{pmatrix},$$

i.e. $c_1 + 6c_2 = 4$, $8c_2 = 9$, $8c_2 = 12$ $(c_2 = ???)$.

Hence the system of these equations has no solution.

$\therefore w = (4,-1,8)$ is not a linear combination of the vectors $v_1 = (1,2,-1), v_2 = (6,4,2)$.

Exercise: Verify that the vector $(0,-3,1)$ is a linear combination of the rows vectors of a matrix $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 3 & 3 & 1 \end{pmatrix}$, but is not a linear combination of its columns vectors.

Linear Dependence & Linear Independence:

Def.: Let V be a vector space over a field K. The vectors $v_1, v_2, ..., v_n \in V$ are said to be *linearly dependent* over K if there exist scalars $c_1, c_2, ..., c_n \in K$ not all of them 0, such that $c_1v_1 + c_2v_2 + ... + c_nv_n = 0$. Otherwise, the vectors are said to be *linearly independent* over K (i.e. $c_1v_1 + c_2v_2 + ... + c_nv_n = 0 \Rightarrow c_1 = 0,\ c_2 = 0,\ ...\ ,\ c_n = 0$).

A set $S = \{v_1, v_2, ..., v_n\}$ of vectors is linearly dependent if the vectors $v_1, v_2, ..., v_n$ are linearly dependent, otherwise S is linearly independent.

Example: The set $S_1 = \{v_1, v_2, v_3\}$; $v_1 = (2,-1,0,3), v_2 = (1,2,5,-1), v_3 = (7,-1,5,8)$ is linearly dependent in R^4,

and the set $S_2 = \{v_1, v_2, v_3\}$; $v_1 = (1,0,1,2), v_2 = (0,1,1,2), v_3 = (1,1,1,3)$ is linearly independent in R^4.

✓ Let $c_1v_1 + c_2v_2 + c_3v_3 = 0$,

$\therefore c_1(2,-1,0,3) + c_2(1,2,5,-1) + c_3(7,-1,5,8) = 0$

$$\therefore \begin{aligned} 2c_1 + c_2 + 7c_3 &= 0, \\ -c_1 + 2c_2 - c_3 &= 0, \\ 5c_2 + 5c_3 &= 0, \\ 3c_1 - c_2 + 8c_3 &= 0. \end{aligned}$$

We reduce the augmented matrix:

$$\begin{pmatrix} 2 & 1 & 7 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 5 & 5 & 0 \\ 3 & -1 & 8 & 0 \end{pmatrix} \xrightarrow{?} \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ (verify that?)}$$

$\therefore\ c_1 = -3c_3\ ,\ c_2 = -c_3$.

Hence the system of these equations has more than one solution.

$c_3 = 1 \Rightarrow c_1 = -3, c_2 = -1$, So S_1 is linearly dependent.

Similarly, $c_1(1,0,1,2) + c_2(0,1,1,2) + c_3(1,1,1,3) = 0$

$$\therefore \begin{aligned} c_1 + c_3 &= 0, \\ c_2 + c_3 &= 0, \\ c_1 + c_2 + c_3 &= 0, \\ 2c_1 + 2c_2 + 3c_3 &= 0. \end{aligned}$$

We reduce the augmented matrix:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 3 & 0 \end{pmatrix} \xrightarrow{?} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ (verify that?)}$$

$\therefore\ c_1 = c_2 = c_3 = 0$

So S_2 is linearly independent.

The Null Space & The Column Space of a Matrix:

Def.1: Suppose A is an $m \times n$ matrix.The *null space* of A is the set of all x in R^n such that $AX = 0$. (we denote the null space of a matrix A by $N(A)$)

Examples:

(1) Determine the null space of a matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

✓ We reduce the augmented matrix $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$ as follows:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \xrightarrow{-r_1+r_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \xrightarrow{-r_2+r_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \xrightarrow{-r_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \xrightarrow[-r_3+r_1]{-r_3+r_2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

$$\begin{aligned} & x_1 = 0, \\ \therefore\ & x_2 = 0, \\ & x_3 = 0 \end{aligned}$$

$\therefore\ N(A) = \{0\}$.

(2) Determine the null space of a matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix}$.

✓ We reduce the augmented matrix $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \end{pmatrix}$ as follows:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \end{pmatrix} \xrightarrow[-3r_1+r_3]{-2r_1+r_2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \end{pmatrix} \xrightarrow{-r_2+r_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\therefore \begin{matrix} x_1 + x_3 = 0, \\ x_2 - 2x_3 = 0 \end{matrix} \Rightarrow \begin{matrix} x_1 = -x_3, \\ x_2 = 2x_3 \end{matrix}$$

$\therefore\ N(A) = \{(x_1, x_2, x_3) = x_3(-1,2,1) : x_3 \in R\}$.

(3) For a matrix $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 3 & 3 & 1 \end{pmatrix}$:

✓ $N(A) = \{(x_1, x_2, x_3) = x_2(-2,1,3) : x_2 \in R\}$ (verify that?).

Def.2: Suppose A is an $m\times n$ matrix. The *column space* of A is the set $B=\{(b_1,b_2,...,b_m)\}\subset R^m$ such that $AX=B$.

(we denote the column space of a matrix A by $C(A)$)

Examples:

(1) Describe the column space of a matrix $A=\begin{pmatrix}1&1&2\\2&1&3\\3&2&5\end{pmatrix}$.

✓ We reduce the augmented matrix $\begin{pmatrix}1&1&2&b_1\\2&1&3&b_2\\3&2&5&b_3\end{pmatrix}$ as follows:

$$\begin{pmatrix}1&1&2&b_1\\2&1&3&b_2\\3&2&5&b_3\end{pmatrix}\xrightarrow[-3r_1+r_3]{-2r_1+r_2}\begin{pmatrix}1&1&2&b_1\\0&-1&-1&b_2-2b_1\\0&-1&-1&b_3-3b_1\end{pmatrix}\xrightarrow{-r_2+r_3}\begin{pmatrix}1&1&2&b_1\\0&-1&-1&b_2-2b_1\\0&0&0&b_3-b_2-b_1\end{pmatrix}$$

$\therefore C(A)=\{(b_1,b_2,b_3)\in R^3 : b_3=b_1+b_2\}$.

(2) For a matrix $A=\begin{pmatrix}1&2&-1\\2&4&-2\\-4&-8&4\end{pmatrix}$:

✓ $C(A)=\{(b_1,b_2,b_3)\in R^3 : b_2=2b_1 , b_3=-4b_1\}$ (verify that?).

Exercises: For a matrix $A=\begin{pmatrix}1&-2&0&0\\0&1&3&2\\2&-5&-3&-2\end{pmatrix}$.

(i) Determine the null space $N(A)$.

(ii) Describe the column space $C(A)$.

Exercises:

1- Verify that $\{(1,2,1),(2,9,0),(3,3,4)\}$ is linearly independent in R^3.

2- Verify that $\{(1,3,-1),(2,0,1),(1,-1,1)\}$ is linearly dependent in R^3.

3- True or False (explain): If $\{v_1,v_2,v_3\}$ is linearly independent, then so is $\{v_1,v_1+v_2,v_1+v_2+v_3\}$.

4- Let $S=\{v_1,v_2,v_3,...,v_n\}$ be a set of nonzero vectors such that $v_i \cdot v_j = 0 \ \forall\ i \neq j$. Verify that S is linearly independent.

✓ $c_1v_1+c_2v_2+c_3v_3+...+c_nv_n=0$ we need to prove that $c_1=c_2=c_3=...=c_n=0$:

$v_1.(c_1v_1+c_2v_2+c_3v_3+...+c_nv_n)=c_1v_1^{\,2}=0 \Rightarrow c_1=0\,;v_1^{\,2}\neq 0$,

$v_2.(c_1v_1+c_2v_2+c_3v_3+...+c_nv_n)=c_2v_2^{\,2}=0 \Rightarrow c_2=0\,;v_2^{\,2}\neq 0$,

$v_3.(c_1v_1+c_2v_2+c_3v_3+...+c_nv_n)=c_3v_3^{\,2}=0 \Rightarrow c_3=0\,;v_3^{\,2}\neq 0$, and, so on …

$v_n.(c_1v_1+c_2v_2+c_3v_3+...+c_nv_n)=c_nv_n^{\,2}=0 \Rightarrow c_n=0\,;v_n^{\,2}\neq 0$.

$\therefore S$ is linearly independent.

5- Show that the vectors $v_1,v_2,...,v_n$ are linearly dependent iff one of them is a linear combination of the others.

✓<u>The proof</u>:

For suppose, say, v_i is a linear combination of the others:

$v_i=c_1v_1+c_2v_2+...+c_{i-1}v_{i-1}+c_{i+1}v_{i+1}+...+c_nv_n$

$\therefore c_1v_1+c_2v_2+...+c_{i-1}v_{i-1}+(-1)v_i+c_{i+1}v_{i+1}+...+c_nv_n=0$,

where the coefficient of v_i is not 0; hence the vectors are linearly dependent.

Conversly, suppose the vectors $v_1,v_2,...,v_n$ are linearly dependent, then:

$c_1v_1+c_2v_2+...+c_{j-1}v_{j-1}+c_jv_j+c_{j+1}v_{j+1}+...+c_nv_n=0$

$\therefore -c_jv_j=c_1v_1+c_2v_2+...+c_{j-1}v_{j-1}+c_{j+1}v_{j+1}+...+c_nv_n$, where $c_j\neq 0$

$$\therefore v_j=(\frac{c_1}{-c_j})v_1+(\frac{c_2}{-c_j})v_2+...+(\frac{c_{j-1}}{-c_j})v_{j-1}+(\frac{c_{j+1}}{-c_j})v_{j+1}+...+(\frac{c_n}{-c_j})v_n$$

i.e. one of the vectors is a linear combination of the other vectors.

✓(**see**: Theorem 5.1 page # 87_ Lipschutz-LinearAlgebra.pdf)[1].

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References:

[1] S.Lipschutz:"Theory and Problems of Linear Algebra",Schaum's outline Series_ McGraw-Hill Book Company_(1974).

[2] Paul Dawkins:"Linear Algebra" (2007)_http://tutorial.math.lamar.edu/terms.aspx.

[3] Golden Maths Series:"Linear Algebra" Urheberrechtlich Geschautztes Material.

[4] L.Hogben:"Hand Book of Linear Algebra"_Discrete Mathematics and its Applications Series Editor Kenneth H.Rosen_(2007).

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