Multiple Integrals

Introduction

In Calculus I we moved on to the subject of integrals once we had finished the discussion of derivatives. The same is true in this course. Now that we have finished our discussion of derivatives of functions of more than one variable we need to move on to integrals of functions of two or three variables.

Most of the derivatives topics extended somewhat naturally from their Calculus I counterparts and that will be the same here. However, because we are now involving functions of two or three variables there will be some differences as well. There will be new notation and some new issues that simply don't arise when dealing with functions of a single variable.

Here is a list of topics covered in this chapter.

Double Integrals – We will define the double integral in this section.

Iterated Integrals – In this section we will start looking at how we actually compute double integrals.

Double Integrals over General Regions – Here we will look at some general double integrals.

Double Integrals in Polar Coordinates – In this section we will take a look at evaluating double integrals using polar coordinates.

Triple Integrals – Here we will define the triple integral as well as how we evaluate them.

Triple Integrals in Cylindrical Coordinates – We will evaluate triple integrals using cylindrical coordinates in this section.

Triple Integrals in Spherical Coordinates – In this section we will evaluate triple integrals using spherical coordinates.

Change of Variables – In this section we will look at change of variables for double and triple integrals.

Surface Area – Here we look at the one real application of double integrals that we're going to look at in this material.

Area and Volume Revisited – We summarize the area and volume formulas from this chapter.

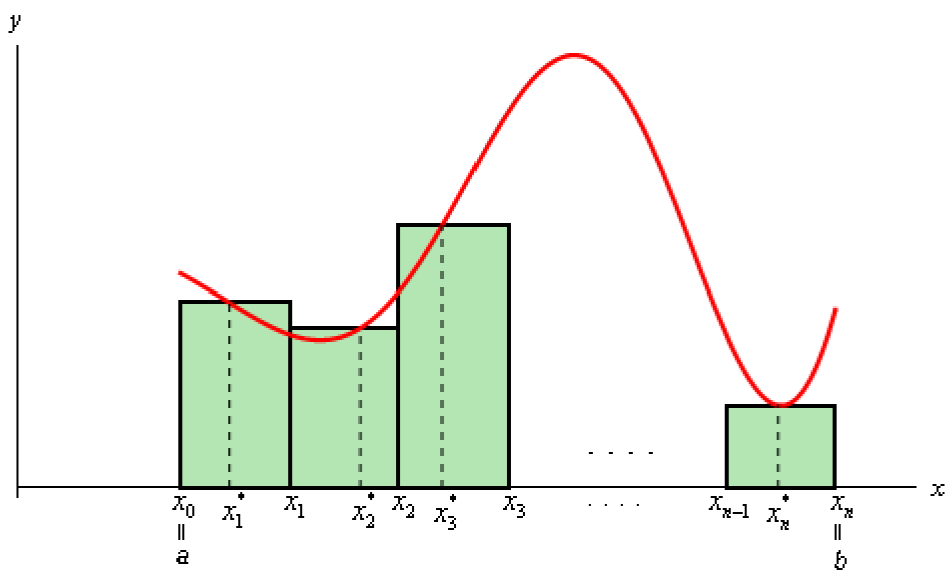
Double Integrals

Before starting on double integrals let's do a quick review of the definition of a definite integrals for functions of single variables. First, when working with the integral,

$$\int_a^b f(x)\,dx$$

we think of x's as coming from the interval $a \le x \le b$. For these integrals we can say that we are integrating over the interval $a \le x \le b$. Note that this does assume that $a < b$, however, if we have $b < a$ then we can just use the interval $b \le x \le a$.

Now, when we derived the definition of the definite integral we first thought of this as an area problem. We first asked what the area under the curve was and to do this we broke up the interval $a \le x \le b$ into n subintervals of width Δx and choose a point, x_i^*, from each interval as shown below,



Each of the rectangles has height of $f\left(x_i^*\right)$ and we could then use the area of each of these rectangles to approximate the area as follows.

$$A \approx f\left(x_1^*\right)\Delta x + f\left(x_2^*\right)\Delta x + \cdots + f\left(x_i^*\right)\Delta x + \cdots + f\left(x_n^*\right)\Delta x$$

To get the exact area we then took the limit as n goes to infinity and this was also the definition of the definite integral.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

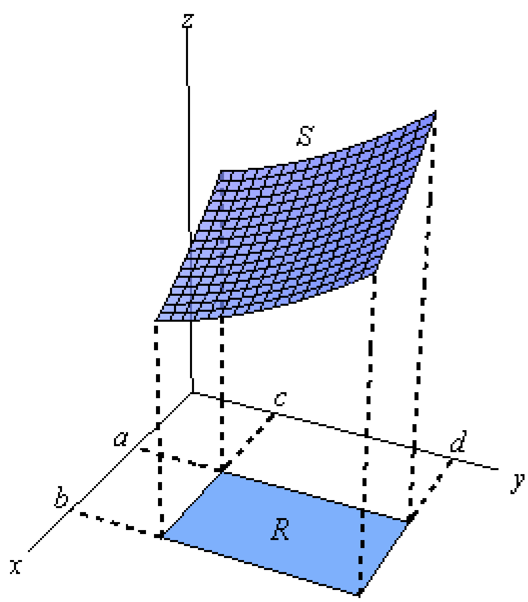
In this section we want to integrate a function of two variables, $f(x, y)$. With functions of one variable we integrated over an interval (*i.e.* a one-dimensional space) and so it makes some sense then that when integrating a function of two variables we will integrate over a region of $\mathbb{R}^2$ (two-dimensional space).

We will start out by assuming that the region in $\mathbb{R}^2$ is a rectangle which we will denote as follows,

$$R = [a,b] \times [c,d]$$

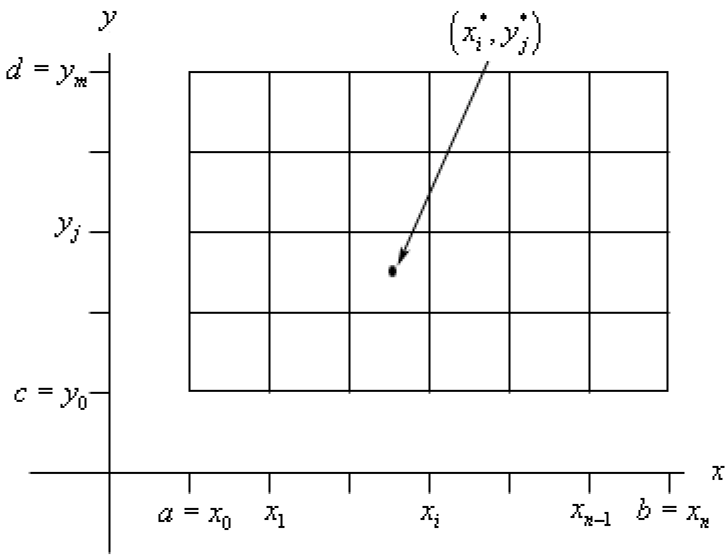
This means that the ranges for *x* and *y* are $a \le x \le b$ and $c \le y \le d$.

Also, we will initially assume that $f(x,y) \ge 0$ although this doesn't really have to be the case. Let's start out with the graph of the surface *S* given by graphing $f(x,y)$ over the rectangle *R*.

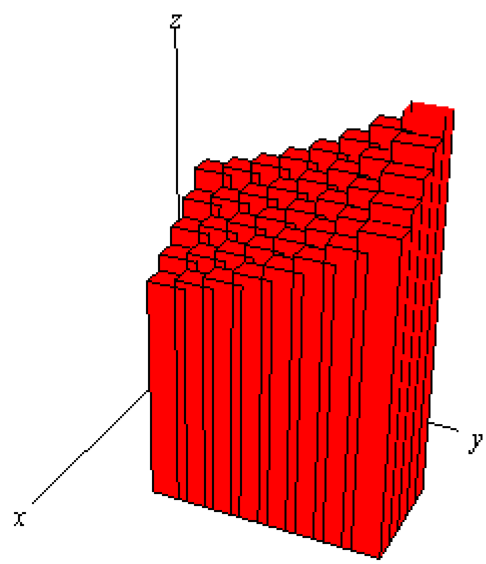


Now, just like with functions of one variable let's not worry about integrals quite yet. Let's first ask what the volume of the region under *S* (and above the *xy*-plane of course) is.

We will first approximate the volume much as we approximated the area above. We will first divide up $a \le x \le b$ into *n* subintervals and divide up $c \le y \le d$ into *m* subintervals. This will divide up *R* into a series of smaller rectangles and from each of these we will choose a point (x_i^*, y_j^*). Here is a sketch of this set up.



Now, over each of these smaller rectangles we will construct a box whose height is given by $f\left(x_i^*, y_j^*\right)$. Here is a sketch of that.



Each of the rectangles has a base area of ΔA and a height of $f\left(x_i^*, y_j^*\right)$ so the volume of each of these boxes is $f\left(x_i^*, y_j^*\right)\Delta A$. The volume under the surface *S* is then approximately,

$$V \approx \sum_{i=1}^{n}\sum_{j=1}^{m} f\left(x_i^*, y_j^*\right)\Delta A$$

We will have a *double* sum since we will need to add up volumes in both the *x* and *y* directions.

To get a better estimation of the volume we will take *n* and *m* larger and larger and to get the exact volume we will need to take the limit as both *n* and *m* go to infinity. In other words,

$$V = \lim_{n,\,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f\left(x_i^*, y_j^*\right)\Delta A$$

Now, this should look familiar. This looks a lot like the definition of the integral of a function of single variable. In fact this is also the definition of a double integral, or more exactly an integral of a function of two variables over a rectangle.

Here is the official definition of a double integral of a function of two variables over a rectangular region *R* as well as the notation that we'll use for it.

$$\iint\limits_R f\left(x, y\right) dA = \lim_{n,\,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f\left(x_i^*, y_j^*\right)\Delta A$$

Note the similarities and differences in the notation to single integrals. We have two integrals to denote the fact that we are dealing with a two dimensional region and we have a differential here as well. Note that the differential is *dA* instead of the *dx* and *dy* that we're used to seeing. Note as well that we don't have limits on the integrals in this notation. Instead we have the *R* written below the two integrals to denote the region that we are integrating over.

Note that one interpretation of the double integral of $f\left(x, y\right)$ over the rectangle *R* is the volume under the function $f\left(x, y\right)$ (and above the *xy*-plane). Or,

$$\text{Volume} = \iint\limits_R f\left(x, y\right) dA$$

We can use this double sum in the definition to estimate the value of a double integral if we need to. We can do this by choosing $\left(x_i^*, y_j^*\right)$ to be the midpoint of each rectangle. When we do this we usually denote the point as $\left(\overline{x}_i, \overline{y}_j\right)$. This leads to the **Midpoint Rule**,

$$\iint\limits_R f\left(x, y\right) dA \approx \sum_{i=1}^{n}\sum_{j=1}^{m} f\left(\overline{x}_i, \overline{y}_j\right)\Delta A$$

In the next section we start looking at how to actually compute double integrals.

Iterated Integrals

In the previous section we gave the definition of the double integral. However, just like with the definition of a single integral the definition is very difficult to use in practice and so we need to start looking into how we actually compute double integrals. We will continue to assume that we are integrating over the rectangle

$$R = [a,b] \times [c,d]$$

We will look at more general regions in the next section.

The following theorem tells us how to compute a double integral over a rectangle.

Fubini's Theorem

If $f(x,y)$ is continuous on $R = [a,b] \times [c,d]$ then,

$$\iint\limits_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

These integrals are called **iterated integrals**.

Note that there are in fact two ways of computing a double integral and also notice that the inner differential matches up with the limits on the inner integral and similarly for the outer differential and limits. In other words, if the inner differential is *dy* then the limits on the inner integral must be *y* limits of integration and if the outer differential is *dy* then the limits on the outer integral must be *y* limits of integration.

Now, on some level this is just notation and doesn't really tell us how to compute the double integral. Let's just take the first possibility above and change the notation a little.

$$\iint\limits_R f(x,y)\,dA = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

We will compute the double integral by first computing

$$\int_c^d f(x,y)\,dy$$

and we compute this by holding *x* constant and integrating with respect to *y* as if this were a single integral. This will give a function involving only *x*'s which we can in turn integrate.

We've done a similar process with partial derivatives. To take the derivative of a function with respect to *y* we treated the *x*'s as constants and differentiated with respect to *y* as if it was a function of a single variable.

Double integrals work in the same manner. We think of all the *x*'s as constants and integrate with respect to *y* or we think of all *y*'s as constants and integrate with respect to *x*.

Let's take a look at some examples.

Example 1 Compute each of the following double integrals over the indicated rectangles.

(a) $\iint\limits_R 6xy^2\,dA$, $R = [2,4]\times[1,2]$ [Solution]

(b) $\iint\limits_R 2x - 4y^3\,dA$, $R = [-5,4]\times[0,3]$ [Solution]

(c) $\iint\limits_R x^2y^2 + \cos(\pi x) + \sin(\pi y)\,dA$, $R = [-2,-1]\times[0,1]$ [Solution]

(d) $\iint\limits_R \frac{1}{(2x+3y)^2}\,dA$, $R = [0,1]\times[1,2]$ [Solution]

(e) $\iint\limits_R x\mathbf{e}^{xy}\,dA$, $R = [-1,2]\times[0,1]$ [Solution]

Solution

(a) $\iint\limits_R 6xy^2\,dA$, $R = [2,4]\times[1,2]$

It doesn't matter which variable we integrate with respect to first, we will get the same answer regardless of the order of integration. To prove that let's work this one with each order to make sure that we do get the same answer.

Solution 1

In this case we will integrate with respect to *y* first. So, the iterated integral that we need to compute is,

$$\iint\limits_R 6xy^2\,dA = \int_2^4\int_1^2 6xy^2\,dy\,dx$$

When setting these up make sure the limits match up to the differentials. Since the *dy* is the inner differential (*i.e.* we are integrating with respect to *y* first) the inner integral needs to have *y* limits.

To compute this we will do the inner integral first and we typically keep the outer integral around as follows,

$$\begin{aligned}\iint\limits_R 6xy^2\,dA &= \int_2^4 \left(2xy^3\right)\Big|_1^2\,dx\\ &= \int_2^4 16x - 2x\,dx\\ &= \int_2^4 14x\,dx\end{aligned}$$

Remember that we treat the *x* as a constant when doing the first integral and we don't do any integration with it yet. Now, we have a normal single integral so let's finish the integral by computing this.

$$\iint\limits_R 6xy^2\,dA = 7x^2\Big|_2^4 = 84$$

Solution 2
In this case we'll integrate with respect to x first and then y. Here is the work for this solution.

$$\begin{aligned}\iint_R 6xy^2\,dA &= \int_1^2\int_2^4 6xy^2\,dx\,dy\\ &= \int_1^2 \left(3x^2y^2\right)\Big|_2^4\,dy\\ &= \int_1^2 36y^2\,dy\\ &= 12y^3\Big|_1^2\\ &= 84\end{aligned}$$

Sure enough the same answer as the first solution.

So, remember that we can do the integration in any order.

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(b) $\displaystyle\iint_R 2x-4y^3\,dA$, $R=[-5,4]\times[0,3]$

For this integral we'll integrate with respect to y first.

$$\begin{aligned}\iint_R 2x-4y^3\,dA &= \int_{-5}^4\int_0^3 2x-4y^3\,dy\,dx\\ &= \int_{-5}^4 \left(2xy-y^4\right)\Big|_0^3\,dx\\ &= \int_{-5}^4 6x-81\,dx\\ &= \left(3x^2-81x\right)\Big|_{-5}^4\\ &= -756\end{aligned}$$

Remember that when integrating with respect to y all x's are treated as constants and so as far as the inner integral is concerned the $2x$ is a constant and we know that when we integrate constants with respect to y we just tack on a y and so we get $2xy$ from the first term.

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(c) $\displaystyle\iint_R x^2y^2+\cos(\pi x)+\sin(\pi y)\,dA$, $R=[-2,-1]\times[0,1]$

In this case we'll integrate with respect to x first.

$$\iint\limits_R x^2y^2+\cos(\pi x)+\sin(\pi y)\,dA = \int_0^1\int_{-2}^{-1} x^2y^2+\cos(\pi x)+\sin(\pi y)\,dx\,dy$$

$$= \int_0^1 \left(\frac{1}{3}x^3y^2+\frac{1}{\pi}\sin(\pi x)+x\sin(\pi y)\right)\Bigg|_{-2}^{-1} dy$$

$$= \int_0^1 \frac{7}{3}y^2+\sin(\pi y)\,dy$$

$$= \frac{7}{9}y^3-\frac{1}{\pi}\cos(\pi y)\Big|_0^1$$

$$= \frac{7}{9}+\frac{2}{\pi}$$

Don't forget your basic Calculus I substitutions!

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(d) $\displaystyle\iint\limits_R \frac{1}{(2x+3y)^2}\,dA\,,\quad R=[0,1]\times[1,2]$

In this case because the limits for x are kind of nice (*i.e.* they are zero and one which are often nice for evaluation) let's integrate with respect to x first. We'll also rewrite the integrand to help with the first integration.

$$\iint\limits_R (2x+3y)^{-2}\,dA = \int_1^2\int_0^1 (2x+3y)^{-2}\,dx\,dy$$

$$= \int_1^2 \left(-\frac{1}{2}(2x+3y)^{-1}\right)\Bigg|_0^1 dy$$

$$= -\frac{1}{2}\int_1^2 \frac{1}{2+3y}-\frac{1}{3y}\,dy$$

$$= -\frac{1}{2}\left(\frac{1}{3}\ln|2+3y|-\frac{1}{3}\ln|y|\right)\Bigg|_1^2$$

$$= -\frac{1}{6}(\ln 8-\ln 2-\ln 5)$$

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(e) $\displaystyle\iint\limits_R x\mathbf{e}^{xy}\,dA\,,\quad R=[-1,2]\times[0,1]$

Now, while we can technically integrate with respect to either variable first sometimes one way is significantly easier than the other way. In this case it will be significantly easier to integrate with respect to y first as we will see.

$$\iint_R x\mathbf{e}^{xy}\, dA = \int_{-1}^{2}\int_{0}^{1} x\mathbf{e}^{xy}\, dy\, dx$$

The y integration can be done with the quick substitution,

$$u = xy \qquad\qquad du = x\, dy$$

which gives

$$\begin{aligned}\iint_R x\mathbf{e}^{xy}\, dA &= \int_{-1}^{2} \mathbf{e}^{xy}\Big|_0^1\, dx \\ &= \int_{-1}^{2} \mathbf{e}^{x} - 1\, dx \\ &= \left(\mathbf{e}^{x} - x\right)\Big|_{-1}^{2} \\ &= \mathbf{e}^{2} - 2 - \left(\mathbf{e}^{-1} + 1\right) \\ &= \mathbf{e}^{2} - \mathbf{e}^{-1} - 3\end{aligned}$$

So, not too bad of an integral there provided you get the substitution. Now let's see what would happen if we had integrated with respect to x first.

$$\iint_R x\mathbf{e}^{xy}\, dA = \int_{0}^{1}\int_{-1}^{2} x\mathbf{e}^{xy}\, dx\, dy$$

In order to do this we would have to use integration by parts as follows,

$$u = x \qquad\qquad dv = \mathbf{e}^{xy}\, dx$$

$$du = dx \qquad\qquad v = \frac{1}{y}\mathbf{e}^{xy}$$

The integral is then,

$$\begin{aligned}\iint_R x\mathbf{e}^{xy}\, dA &= \int_0^1 \left(\frac{x}{y}\mathbf{e}^{xy} - \int \frac{1}{y}\mathbf{e}^{xy}\, dx\right)\Bigg|_{-1}^{2}\, dy \\ &= \int_0^1 \left(\frac{x}{y}\mathbf{e}^{xy} - \frac{1}{y^2}\mathbf{e}^{xy}\right)\Bigg|_{-1}^{2}\, dy \\ &= \int_0^1 \left(\frac{2}{y}\mathbf{e}^{2y} - \frac{1}{y^2}\mathbf{e}^{2y}\right) - \left(-\frac{1}{y}\mathbf{e}^{-y} - \frac{1}{y^2}\mathbf{e}^{-y}\right) dy\end{aligned}$$

We're not even going to continue here as these are very difficult integrals to do.

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As we saw in the previous set of examples we can do the integral in either direction. However, sometimes one direction of integration is significantly easier than the other so make sure that you think about which one you should do first before actually doing the integral.

The next topic of this section is a quick fact that can be used to make some iterated integrals somewhat easier to compute on occasion.

Fact

If $f(x,y) = g(x)h(y)$ and we are integrating over the rectangle $R = [a,b]\times[c,d]$ then,

$$\iint\limits_R f(x,y)\,dA = \iint\limits_R g(x)h(y)\,dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right)$$

So, if we can break up the function into a function only of *x* times a function of *y* then we can do the two integrals individually and multiply them together.

Let's do a quick example using this integral.

Example 2 Evaluate $\displaystyle\iint\limits_R x\cos^2(y)\,dA$, $R = [-2,3]\times\left[0,\frac{\pi}{2}\right]$.

Solution

Since the integrand is a function of *x* times a function of *y* we can use the fact.

$$\begin{aligned}\iint\limits_R x\cos^2(y)\,dA &= \left(\int_{-2}^{3} x\,dx\right)\left(\int_0^{\frac{\pi}{2}}\cos^2(y)\,dy\right)\\ &= \left(\frac{1}{2}x^2\right)\Bigg|_{-2}^{3}\left(\frac{1}{2}\int_0^{\frac{\pi}{2}}1+\cos(2y)\,dy\right)\\ &= \left(\frac{5}{2}\right)\left(\frac{1}{2}\left(y+\frac{1}{2}\sin(2y)\right)\Bigg|_0^{\frac{\pi}{2}}\right)\\ &= \frac{5\pi}{8}\end{aligned}$$

We have one more topic to discuss in this section. This topic really doesn't have anything to do with iterated integrals, but this is as good a place as any to put it and there are liable to be some questions about it at this point as well so this is as good a place as any.

What we want to do is discuss single indefinite integrals of a function of two variables. In other words we want to look at integrals like the following.

$$\int x\sec^2(2y) + 4xy\,dy$$

$$\int x^3 - \mathbf{e}^{-\frac{x}{y}}\,dx$$

From Calculus I we know that these integrals are asking what function that we differentiated to get the integrand. However, in this case we need to pay attention to the differential (*dy* or *dx*) in the integral, because that will change things a little.

In the case of the first integral we are asking what function we differentiated with respect to *y* to get the integrand while in the second integral we're asking what function differentiated with respect to x to get the integrand. For the most part answering these questions isn't that difficult. The important issue is how we deal with the constant of integration.

Here are the integrals.

$$\int x\sec^2(2y)+4xy\,dy=\frac{x}{2}\tan(2y)+2xy^2+g(x)$$

$$\int x^3-\mathbf{e}^{-\frac{x}{y}}\,dx=\frac{1}{4}x^4+y\,\mathbf{e}^{-\frac{x}{y}}+h(y)$$

Notice that the "constants" of integration are now functions of the opposite variable. In the first integral we are differentiating with respect to y and we know that any function involving only x's will differentiate to zero and so when integrating with respect to y we need to acknowledge that there may have been a function of only x's in the function and so the "constant" of integration is a function of x.

Likewise, in the second integral, the "constant" of integration must be a function of y since we are integrating with respect to x. Again, remember if we differentiate the answer with respect to x then any function of only y's will differentiate to zero.

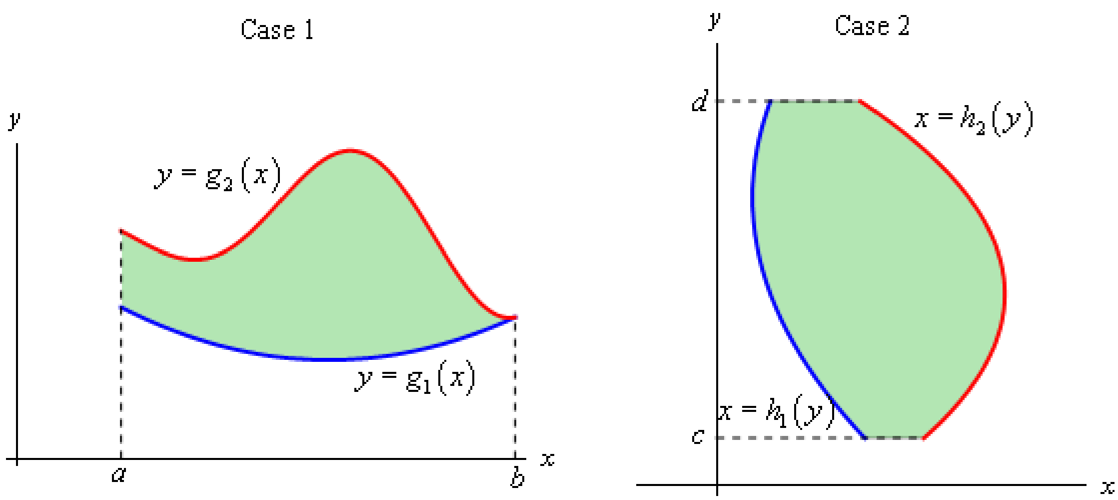
Double Integrals Over General Regions

In the previous section we looked at double integrals over rectangular regions. The problem with this is that most of the regions are not rectangular so we need to now look at the following double integral,

$$\iint\limits_D f(x,y)\,dA$$

where D is any region.

There are two types of regions that we need to look at. Here is a sketch of both of them.



We will often use *set builder notation* to describe these regions. Here is the definition for the region in Case 1

$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

and here is the definition for the region in Case 2.

$$D = \{(x,y) \mid h_1(y) \le x \le h_2(y), c \le y \le d\}$$

This notation is really just a fancy way of saying we are going to use all the points, (x,y), in which both of the coordinates satisfy the two given inequalities.

The double integral for both of these cases are defined in terms of iterated integrals as follows.

In Case 1 where $D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$ the integral is defined to be,

$$\iint\limits_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

In Case 2 where $D = \{(x,y) \mid h_1(y) \le x \le h_2(y), c \le y \le d\}$ the integral is defined to be,

$$\iint\limits_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Here are some properties of the double integral that we should go over before we actually do some examples. Note that all three of these properties are really just extensions of properties of single integrals that have been extended to double integrals.

Properties

1. $\displaystyle\iint\limits_D f(x,y)+g(x,y)\,dA = \iint\limits_D f(x,y)\,dA + \iint\limits_D g(x,y)\,dA$

2. $\displaystyle\iint\limits_D cf(x,y)\,dA = c\iint\limits_D f(x,y)\,dA$, where c is any constant.

3. If the region D can be split into two separate regions D_1 and D_2 then the integral can be written as

$$\iint\limits_D f(x,y)\,dA = \iint\limits_{D_1} f(x,y)\,dA + \iint\limits_{D_2} f(x,y)\,dA$$

Let's take a look at some examples of double integrals over general regions.

Example 1 Evaluate each of the following integrals over the given region D.

(a) $\displaystyle\iint\limits_D \mathbf{e}^{\frac{x}{y}}\,dA$, $D = \left\{(x,y)\,|\,1 \le y \le 2,\ y \le x \le y^3\right\}$ [Solution]

(b) $\displaystyle\iint\limits_D 4xy - y^3\,dA$, D is the region bounded by $y = \sqrt{x}$ and $y = x^3$. [Solution]

(c) $\displaystyle\iint\limits_D 6x^2 - 40y\,dA$, D is the triangle with vertices $(0,3)$, $(1,1)$, and $(5,3)$. [Solution]

Solution

(a) $\displaystyle\iint\limits_D \mathbf{e}^{\frac{x}{y}}\,dA$, $D = \left\{(x,y)\,|\,1 \le y \le 2,\ y \le x \le y^3\right\}$

Okay, this first one is set up to just use the formula above so let's do that.

$$\iint\limits_D \mathbf{e}^{\frac{x}{y}}\,dA = \int_1^2 \int_y^{y^3} \mathbf{e}^{\frac{x}{y}}\,dx\,dy = \int_1^2 \left. y\,\mathbf{e}^{\frac{x}{y}} \right|_y^{y^3} dy$$

$$= \int_1^2 y\,\mathbf{e}^{y^2} - y\mathbf{e}^1\,dy$$

$$= \left.\left(\frac{1}{2}\mathbf{e}^{y^2} - \frac{1}{2}y^2\mathbf{e}^1\right)\right|_1^2 = \frac{1}{2}\mathbf{e}^4 - 2\mathbf{e}^1$$

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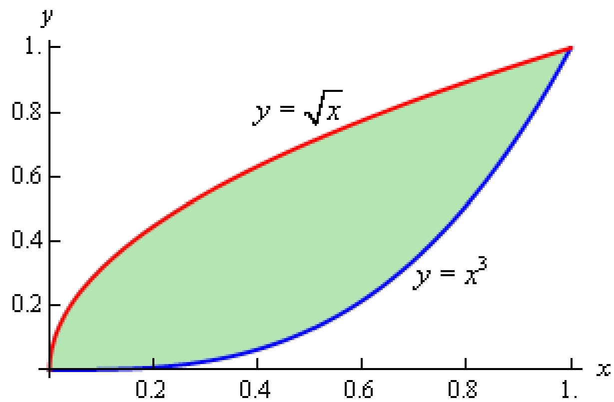
(b) $\iint\limits_D 4xy - y^3\, dA$**, *D* is the region bounded by** $y = \sqrt{x}$ **and** $y = x^3$**.**

In this case we need to determine the two inequalities for x and y that we need to do the integral. The best way to do this is the graph the two curves. Here is a sketch.



So, from the sketch we can see that that two inequalities are,

$$0 \le x \le 1 \qquad\qquad x^3 \le y \le \sqrt{x}$$

We can now do the integral,

$$\begin{aligned}\iint\limits_D 4xy - y^3\, dA &= \int_0^1 \int_{x^3}^{\sqrt{x}} 4xy - y^3\, dy\, dx \\ &= \int_0^1 \left(2xy^2 - \frac{1}{4}y^4 \right)\Bigg|_{x^3}^{\sqrt{x}} dx \\ &= \int_0^1 \frac{7}{4}x^2 - 2x^7 + \frac{1}{4}x^{12}\, dx \\ &= \left(\frac{7}{12}x^3 - \frac{1}{4}x^8 + \frac{1}{52}x^{13} \right)\Bigg|_0^1 = \frac{55}{156}\end{aligned}$$

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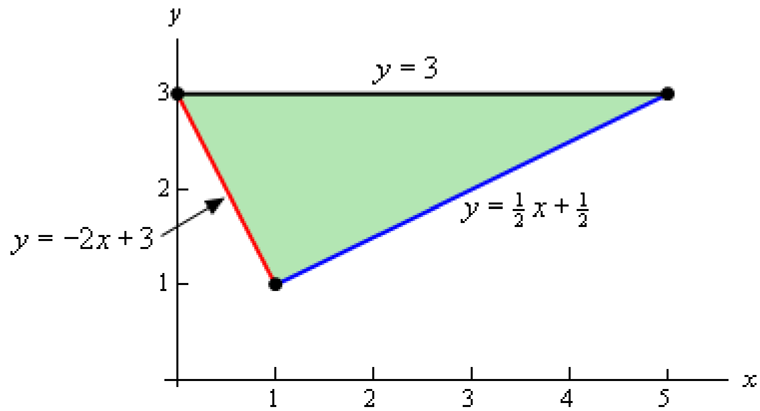
(c) $\iint\limits_D 6x^2 - 40y\, dA$**, *D* is the triangle with vertices** $(0,3)$**,** $(1,1)$**, and** $(5,3)$**.**

We got even less information about the region this time. Let's start this off by sketching the triangle.



Since we have two points on each edge it is easy to get the equations for each edge and so we'll leave it to you to verify the equations.

Now, there are two ways to describe this region. If we use functions of *x*, as shown in the image we will have to break the region up into two different pieces since the lower function is different depending upon the value of *x*. In this case the region would be given by $D = D_1 \cup D_2$ where,

$$D_1 = \left\{ (x,y) \mid 0 \le x \le 1, \ -2x+3 \le y \le 3 \right\}$$

$$D_2 = \left\{ (x,y) \mid 1 \le x \le 5, \ \frac{1}{2}x + \frac{1}{2} \le y \le 3 \right\}$$

Note the $\cup$ is the "union" symbol and just means that *D* is the region we get by combing the two regions. If we do this then we'll need to do two separate integrals, one for each of the regions.

To avoid this we could turn things around and solve the two equations for *x* to get,

$$y = -2x + 3 \qquad \Rightarrow \qquad x = -\frac{1}{2}y + \frac{3}{2}$$

$$y = \frac{1}{2}x + \frac{1}{2} \qquad \Rightarrow \qquad x = 2y - 1$$

If we do this we can notice that the same function is always on the right and the same function is always on the left and so the region is,

$$D = \left\{ (x,y) \mid -\frac{1}{2}y + \frac{3}{2} \le x \le 2y - 1, \ 1 \le y \le 3 \right\}$$

Writing the region in this form means doing a single integral instead of the two integrals we'd have to do otherwise.

Either way should give the same answer and so we can get an example in the notes of splitting a region up let's do both integrals.

Solution 1

$$\begin{aligned}\iint\limits_D 6x^2-40y\,dA &= \iint\limits_{D_1} 6x^2-40y\,dA+\iint\limits_{D_2} 6x^2-40y\,dA\\ &= \int_0^1\int_{-2x+3}^{3} 6x^2-40y\,dy\,dx+\int_1^5\int_{\frac{1}{2}x+\frac{1}{2}}^{3} 6x^2-40y\,dy\,dx\\ &= \int_0^1 \left(6x^2y-20y^2\right)\Big|_{-2x+3}^{3}\,dx+\int_1^5 \left(6x^2y-20y^2\right)\Big|_{\frac{1}{2}x+\frac{1}{2}}^{3}\,dx\\ &= \int_0^1 12x^3-180+20\left(3-2x\right)^2\,dx+\int_1^5 -3x^3+15x^2-180+20\left(\tfrac{1}{2}x+\tfrac{1}{2}\right)^2\,dx\\ &= \left(3x^4-180x-\tfrac{10}{3}\left(3-2x\right)^3\right)\Big|_0^1+\left(-\tfrac{3}{4}x^4+5x^3-180x+\tfrac{40}{3}\left(\tfrac{1}{2}x+\tfrac{1}{2}\right)^3\right)\Big|_1^5\\ &= -\frac{935}{3}\end{aligned}$$

That was a lot of work. Notice however, that after we did the first substitution that we didn't multiply everything out. The two quadratic terms can be easily integrated with a basic Calc I substitution and so we didn't bother to multiply them out. We'll do that on occasion to make some of these integrals a little easier.

Solution 2

This solution will be a lot less work since we are only going to do a single integral.

$$\begin{aligned}\iint\limits_D 6x^2-40y\,dA &= \int_1^3\int_{-\frac{1}{2}y+\frac{3}{2}}^{2y-1} 6x^2-40y\,dx\,dy\\ &= \int_1^3 \left(2x^3-40xy\right)\Big|_{-\frac{1}{2}y+\frac{3}{2}}^{2y-1}\,dy\\ &= \int_1^3 100y-100y^2+2\left(2y-1\right)^3-2\left(-\tfrac{1}{2}y+\tfrac{3}{2}\right)^3\,dy\\ &= \left(50y^2-\tfrac{100}{3}y^3+\tfrac{1}{4}\left(2y-1\right)^4+\left(-\tfrac{1}{2}y+\tfrac{3}{2}\right)^4\right)\Big|_1^3\\ &= -\frac{935}{3}\end{aligned}$$

So, the numbers were a little messier, but other than that there was much less work for the same result. Also notice that again we didn't cube out the two terms as they are easier to deal with using a Calc I substitution.

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As the last part of the previous example has shown us we can integrate these integrals in either order (*i.e.* x followed by y or y followed by x), although often one order will be easier than the other. In fact there will be times when it will not even be possible to do the integral in one order while it will be possible to do the integral in the other order.

Let's see a couple of examples of these kinds of integrals.

Example 2 Evaluate the following integrals by first reversing the order of integration.

(a) $\int_0^3 \int_{x^2}^9 x^3 \mathbf{e}^{y^3}\, dy\, dx$ [Solution]

(b) $\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}\, dx\, dy$ [Solution]

Solution

(a) $\int_0^3 \int_{x^2}^9 x^3 \mathbf{e}^{y^3}\, dy\, dx$

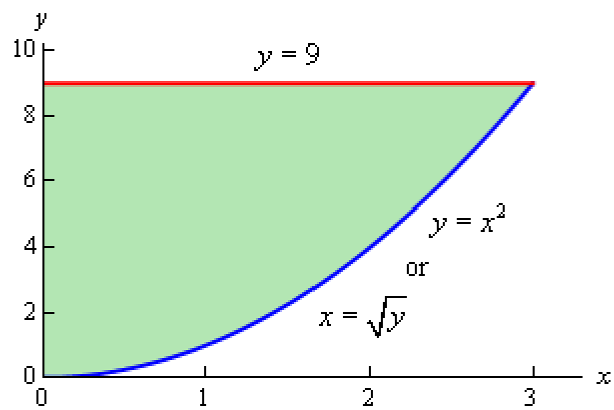
First, notice that if we try to integrate with respect to y we can't do the integral because we would need a y^2 in front of the exponential in order to do the y integration. We are going to hope that if we reverse the order of integration we will get an integral that we can do.

Now, when we say that we're going to reverse the order of integration this means that we want to integrate with respect to x first and then y. Note as well that we can't just interchange the integrals, keeping the original limits, and be done with it. This would not fix our original problem and in order to integrate with respect to x we can't have x's in the limits of the integrals. Even if we ignored that the answer would not be a constant as it should be.

So, let's see how we reverse the order of integration. The best way to reverse the order of integration is to first sketch the region given by the original limits of integration. From the integral we see that the inequalities that define this region are,

$$0 \le x \le 3$$
$$x^2 \le y \le 9$$

These inequalities tell us that we want the region with $y = x^2$ on the lower boundary and $y = 9$ on the upper boundary that lies between $x = 0$ and $x = 3$. Here is a sketch of that region.



Since we want to integrate with respect to x first we will need to determine limits of x (probably in terms of y) and then get the limits on the y's. Here they are for this region.

$$0 \le x \le \sqrt{y}$$
$$0 \le y \le 9$$

Any horizontal line drawn in this region will start at $x = 0$ and end at $x = \sqrt{y}$ and so these are the limits on the x's and the range of y's for the regions is 0 to 9.

The integral, with the order reversed, is now,

$$\int_0^3 \int_{x^2}^9 x^3 \mathbf{e}^{y^3}\, dy\, dx = \int_0^9 \int_0^{\sqrt{y}} x^3 \mathbf{e}^{y^3}\, dx\, dy$$

and notice that we can do the first integration with this order. We'll also hope that this will give us a second integral that we can do. Here is the work for this integral.

$$\begin{aligned}\int_0^3 \int_{x^2}^9 x^3 \mathbf{e}^{y^3}\, dy\, dx &= \int_0^9 \int_0^{\sqrt{y}} x^3 \mathbf{e}^{y^3}\, dx\, dy \\ &= \int_0^9 \left. \frac{1}{4} x^4 \mathbf{e}^{y^3} \right|_0^{\sqrt{y}} dy \\ &= \int_0^9 \frac{1}{4} y^2 \mathbf{e}^{y^3}\, dy \\ &= \left. \frac{1}{12} \mathbf{e}^{y^3} \right|_0^9 \\ &= \frac{1}{12}\left(\mathbf{e}^{729} - 1\right)\end{aligned}$$

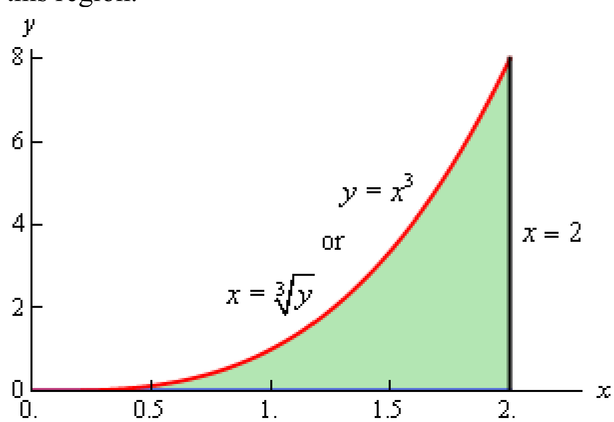
[Return to Problems]

(b) $\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4 + 1}\, dx\, dy$

As with the first integral we cannot do this integral by integrating with respect to x first so we'll hope that by reversing the order of integration we will get something that we can integrate. Here are the limits for the variables that we get from this integral.

$$\sqrt[3]{y} \le x \le 2$$
$$0 \le y \le 8$$

and here is a sketch of this region.



So, if we reverse the order of integration we get the following limits.

$$0 \le x \le 2$$
$$0 \le y \le x^3$$

The integral is then,

$$\begin{aligned}\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}\,dx\,dy &= \int_0^2 \int_0^{x^3} \sqrt{x^4+1}\,dy\,dx \\ &= \int_0^2 y\sqrt{x^4+1}\Big|_0^{x^3}\,dx \\ &= \int_0^2 x^3\sqrt{x^4+1}\,dx = \frac{1}{6}\left(17^{\frac{3}{2}}-1\right)\end{aligned}$$

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The final topic of this section is two geometric interpretations of a double integral. The first interpretation is an extension of the idea that we used to develop the idea of a double integral in the first section of this chapter. We did this by looking at the volume of the solid that was below the surface of the function $z = f(x,y)$ and over the rectangle R in the xy-plane. This idea can be extended to more general regions.

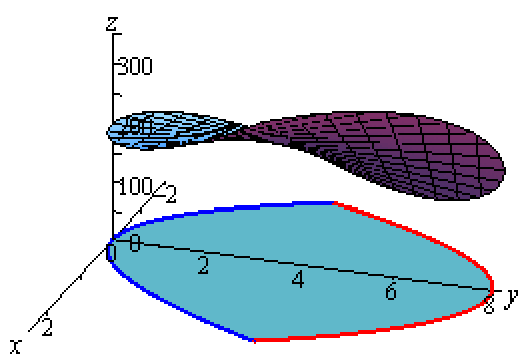
The volume of the solid that lies below the surface given by $z = f(x,y)$ and above the region D in the xy-plane is given by,

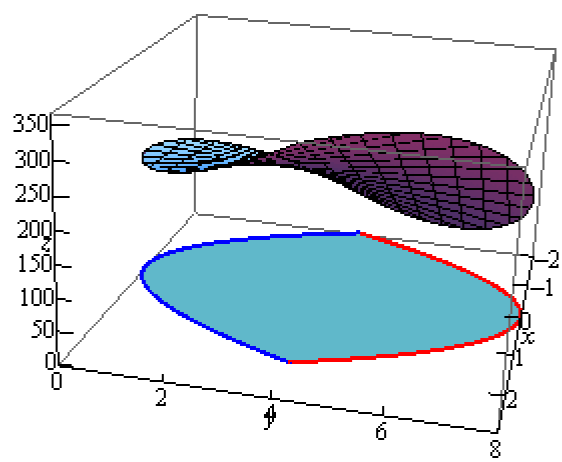
$$V = \iint_D f(x,y)\,dA$$

Example 3 Find the volume of the solid that lies below the surface given by $z = 16xy + 200$ and lies above the region in the xy-plane bounded by $y = x^2$ and $y = 8 - x^2$.

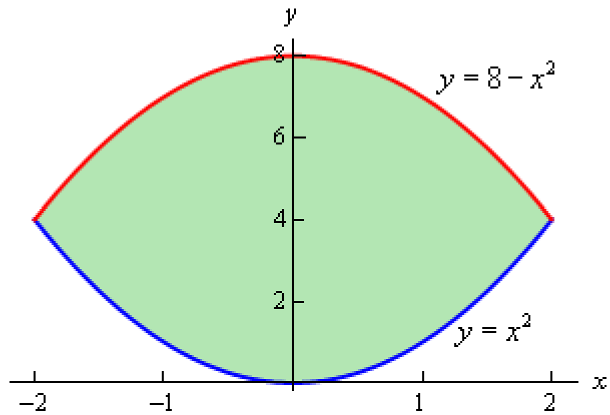
Solution

Here is the graph of the surface and we've tried to show the region in the xy-plane below the surface.





Here is a sketch of the region in the xy-plane by itself.



By setting the two bounding equations equal we can see that they will intersect at $x = 2$ and $x = -2$. So, the inequalities that will define the region D in the xy-plane are,

$$-2 \le x \le 2$$
$$x^2 \le y \le 8 - x^2$$

The volume is then given by,

$$\begin{aligned} V &= \iint\limits_D 16xy + 200\,dA \\ &= \int_{-2}^{2}\int_{x^2}^{8-x^2} 16xy + 200\,dy\,dx \\ &= \int_{-2}^{2}\left(8xy^2 + 200y\right)\Big|_{x^2}^{8-x^2}\,dx \\ &= \int_{-2}^{2} -128x^3 - 400x^2 + 512x + 1600\,dx \\ &= \left(-32x^4 - \frac{400}{3}x^3 + 256x^2 + 1600x\right)\Big|_{-2}^{2} = \frac{12800}{3} \end{aligned}$$

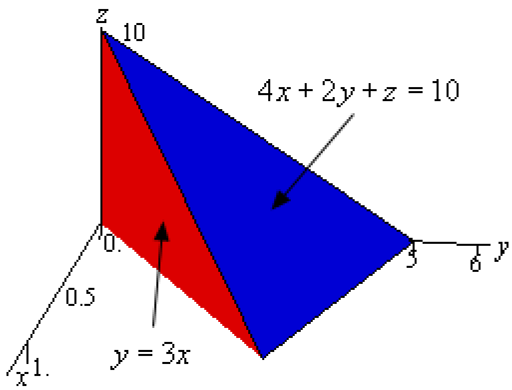
Example 4 Find the volume of the solid enclosed by the planes $4x + 2y + z = 10$, $y = 3x$, $z = 0$, $x = 0$.

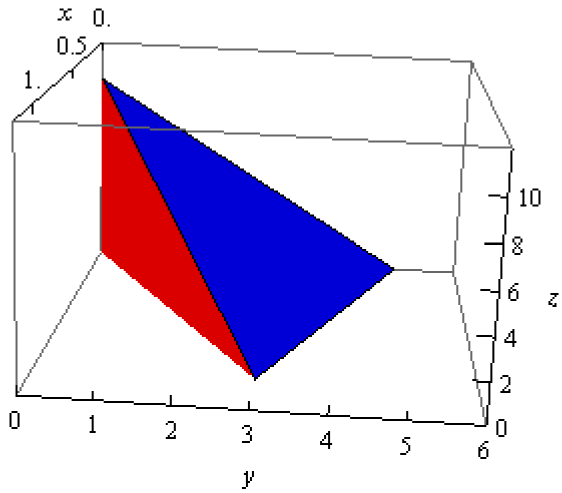
Solution This example is a little different from the previous one. Here the region D is not explicitly given so we're going to have to find it. First, notice that the last two planes are really telling us that we won't go past the xy-plane and the yz-plane when we reach them.

The first plane, $4x + 2y + z = 10$, is the top of the volume and so we are really looking for the volume under,

$$z = 10 - 4x - 2y$$

and above the region D in the xy-plane. The second plane, $y = 3x$ (yes that is a plane), gives one of the sides of the volume as shown below.

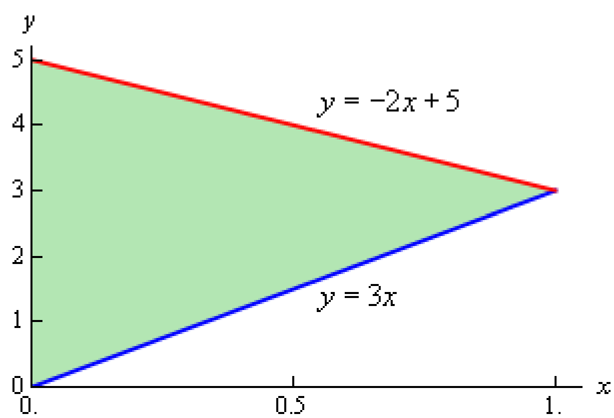




The region D will be the region in the *xy*-plane (*i.e.* $z=0$) that is bounded by $y=3x$, $x=0$, and the line where $z+4x+2y=10$ intersects the *xy*-plane. We can determine where $z+4x+2y=10$ intersects the *xy*-plane by plugging $z=0$ into it.

$$0+4x+2y=10 \qquad \Rightarrow \qquad 2x+y=5 \qquad \Rightarrow \qquad y=-2x+5$$

So, here is a sketch the region D.



The region D is really where this solid will sit on the *xy*-plane and here are the inequalities that define the region.

$$0 \le x \le 1$$
$$3x \le y \le -2x+5$$

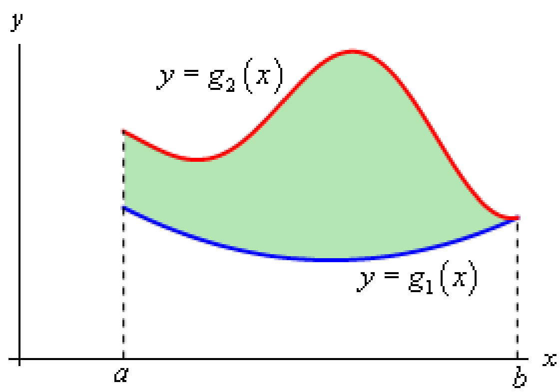
Here is the volume of this solid.

$$\begin{aligned}
V &= \iint_D 10 - 4x - 2y\,dA \\
&= \int_0^1 \int_{3x}^{-2x+5} 10 - 4x - 2y\,dy\,dx \\
&= \int_0^1 \left(10y - 4xy - y^2\right)\Big|_{3x}^{-2x+5}\,dx \\
&= \int_0^1 25x^2 - 50x + 25\,dx \\
&= \left(\frac{25}{3}x^3 - 25x^2 + 25x\right)\Big|_0^1 = \frac{25}{3}
\end{aligned}$$

The second geometric interpretation of a double integral is the following.

$$\text{Area of } D = \iint_D dA$$

This is easy to see why this is true in general. Let's suppose that we want to find the area of the region shown below.



From Calculus I we know that this area can be found by the integral,

$$A = \int_a^b g_2(x) - g_1(x)\,dx$$

Or in terms of a double integral we have,

$$\begin{aligned}
\text{Area of } D &= \iint_D dA \\
&= \int_a^b \int_{g_1(x)}^{g_2(x)} dy\,dx \\
&= \int_a^b y\Big|_{g_1(x)}^{g_2(x)}\,dx = \int_a^b g_2(x) - g_1(x)\,dx
\end{aligned}$$

This is exactly the same formula we had in Calculus I.

Double Integrals in Polar Coordinates

To this point we've seen quite a few double integrals. However, in every case we've seen to this point the region D could be easily described in terms of simple functions in Cartesian coordinates. In this section we want to look at some regions that are much easier to describe in terms of polar coordinates. For instance, we might have a region that is a disk, ring, or a portion of a disk or ring. In these cases, using Cartesian coordinates could be somewhat cumbersome. For instance, let's suppose we wanted to do the following integral,

$$\iint\limits_D f(x,y)\,dA, \quad D \text{ is the disk of radius 2}$$

To this we would have to determine a set of inequalities for x and y that describe this region. These would be,

$$-2 \le x \le 2$$
$$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$$

With these limits the integral would become,

$$\iint\limits_D f(x,y)\,dA = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\,dy\,dx$$

Due to the limits on the inner integral this is liable to be an unpleasant integral to compute.

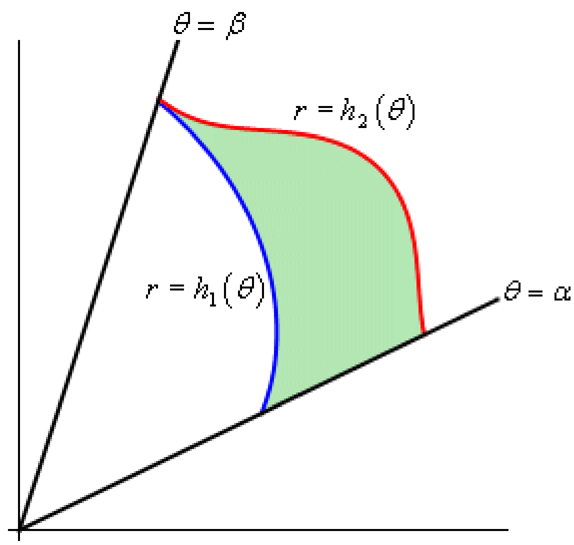
However, a disk of radius 2 can be defined in polar coordinates by the following inequalities,

$$0 \le \theta \le 2\pi$$
$$0 \le r \le 2$$

These are very simple limits and, in fact, are constant limits of integration which almost always makes integrals somewhat easier.

So, if we could convert our double integral formula into one involving polar coordinates we would be in pretty good shape. The problem is that we can't just convert the dx and the dy into a dr and a $d\theta$. In computing double integrals to this point we have been using the fact that $dA = dx\,dy$ and this really does require Cartesian coordinates to use. Once we've moved into polar coordinates $dA \ne dr\,d\theta$ and so we're going to need to determine just what dA is under polar coordinates.

So, let's step back a little bit and start off with a general region in terms of polar coordinates and see what we can do with that. Here is a sketch of some region using polar coordinates.

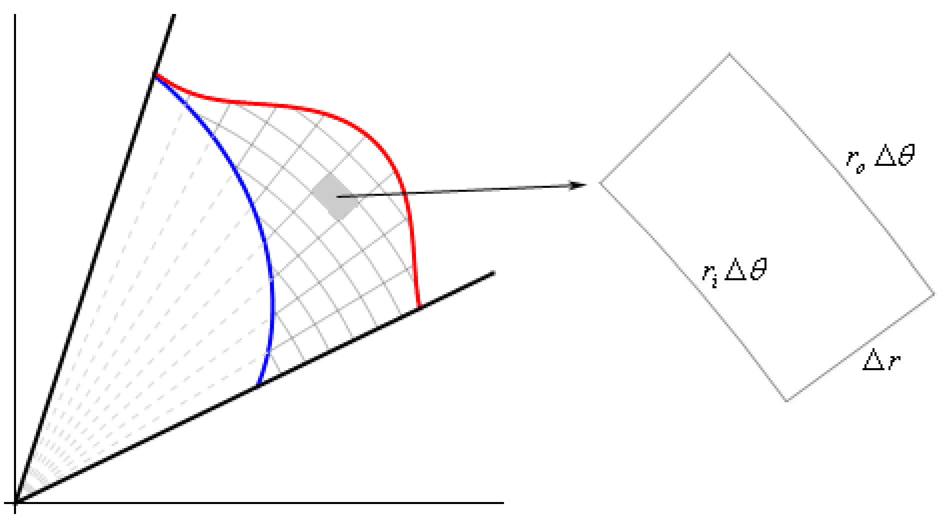


So, our general region will be defined by inequalities,

$$\alpha \le \theta \le \beta$$

$$h_1(\theta) \le r \le h_2(\theta)$$

Now, to find *dA* let's redo the figure above as follows,



As shown, we'll break up the region into a mesh of radial lines and arcs. Now, if we pull one of the pieces of the mesh out as shown we have something that is almost, but not quite a rectangle. The area of this piece is ΔA. The two sides of this piece both have length $\Delta r = r_o - r_i$ where r_o is the radius of the outer arc and r_i is the radius of the inner arc. Basic geometry then tells us that the length of the inner edge is $r_i\,\Delta\theta$ while the length of the out edge is $r_o\,\Delta\theta$ where $\Delta\theta$ is the angle between the two radial lines that form the sides of this piece.

Now, let's assume that we've taken the mesh so small that we can assume that $r_i \approx r_o = r$ and with this assumption we can also assume that our piece is close enough to a rectangle that we can also then assume that,

$$\Delta A \approx r\,\Delta\theta\,\Delta r$$

Also, if we assume that the mesh is small enough then we can also assume that,

$$dA \approx \Delta A \qquad\qquad d\theta \approx \Delta\theta \qquad\qquad dr \approx \Delta r$$

With these assumptions we then get $dA \approx r\,dr\,d\theta$.

In order to arrive at this we had to make the assumption that the mesh was very small. This is not an unreasonable assumption. Recall that the definition of a double integral is in terms of two limits and as limits go to infinity the mesh size of the region will get smaller and smaller. In fact, as the mesh size gets smaller and smaller the formula above becomes more and more accurate and so we can say that,

$$dA = r\,dr\,d\theta$$

We'll see another way of deriving this once we reach the Change of Variables section later in this chapter. This second way will not involve any assumptions either and so it maybe a little better way of deriving this.

Before moving on it is again important to note that $dA \neq dr\,d\theta$. The actual formula for dA has an r in it. It will be easy to forget this r on occasion, but as you'll see without it some integrals will not be possible to do.

Now, if we're going to be converting an integral in Cartesian coordinates into an integral in polar coordinates we are going to have to make sure that we've also converted all the x's and y's into polar coordinates as well. To do this we'll need to remember the following conversion formulas,

$$x = r\cos\theta \qquad\qquad y = r\sin\theta \qquad\qquad r^2 = x^2 + y^2$$

We are now ready to write down a formula for the double integral in terms of polar coordinates.

$$\iint\limits_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

It is important to not forget the added r and don't forget to convert the Cartesian coordinates in the function over to polar coordinates.

Let's look at a couple of examples of these kinds of integrals.

Example 1 Evaluate the following integrals by converting them into polar coordinates.

(a) $\iint\limits_D 2x\,y\,dA$, D is the portion of the region between the circles of radius 2 and radius 5 centered at the origin that lies in the first quadrant. [Solution]

(b) $\iint\limits_D \mathbf{e}^{x^2+y^2}\,dA$, D is the unit circle centered at the origin. [Solution]

Solution

(a) $\iint\limits_D 2x\,y\,dA$**, *D* is the portion of the region between the circles of radius 2 and radius 5 centered at the origin that lies in the first quadrant.**

First let's get D in terms of polar coordinates. The circle of radius 2 is given by $r = 2$ and the circle of radius 5 is given by $r = 5$. We want the region between them so we will have the following inequality for r.

$$2 \le r \le 5$$

Also, since we only want the portion that is in the first quadrant we get the following range of θ's.

$$0 \le \theta \le \frac{\pi}{2}$$

Now that we've got these we can do the integral.

$$\iint\limits_D 2x\,y\,dA = \int_0^{\frac{\pi}{2}} \int_2^5 2\left(r\cos\theta\right)\left(r\sin\theta\right) r\,dr\,d\theta$$

Don't forget to do the conversions and to add in the extra r. Now, let's simplify and make use of the double angle formula for sine to make the integral a little easier.

$$\begin{aligned}\iint\limits_D 2x\,y\,dA &= \int_0^{\frac{\pi}{2}} \int_2^5 r^3 \sin\left(2\theta\right) dr\,d\theta \\ &= \int_0^{\frac{\pi}{2}} \left.\frac{1}{4} r^4 \sin\left(2\theta\right)\right|_2^5 d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{609}{4} \sin\left(2\theta\right) d\theta \\ &= \left.-\frac{609}{8}\cos\left(2\theta\right)\right|_0^{\frac{\pi}{2}} \\ &= \frac{609}{4}\end{aligned}$$

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(b) $\iint\limits_D \mathbf{e}^{x^2+y^2}\, dA$**, *D* is the unit circle centered at the origin.**

In this case we can't do this integral in terms of Cartesian coordinates. We will however be able to do it in polar coordinates. First, the region *D* is defined by,

$$0 \le \theta \le 2\pi$$
$$0 \le r \le 1$$

In terms of polar coordinates the integral is then,

$$\iint\limits_D \mathbf{e}^{x^2+y^2}\, dA = \int_0^{2\pi} \int_0^1 r\,\mathbf{e}^{r^2}\, dr\, d\theta$$

Notice that the addition of the *r* gives us an integral that we can now do. Here is the work for this integral.

$$\begin{aligned}\iint\limits_D \mathbf{e}^{x^2+y^2}\, dA &= \int_0^{2\pi} \int_0^1 r\,\mathbf{e}^{r^2}\, dr\, d\theta \\ &= \int_0^{2\pi} \left.\frac{1}{2}\mathbf{e}^{r^2}\right|_0^1 d\theta \\ &= \int_0^{2\pi} \frac{1}{2}(\mathbf{e}-1)\, d\theta \\ &= \pi(\mathbf{e}-1)\end{aligned}$$

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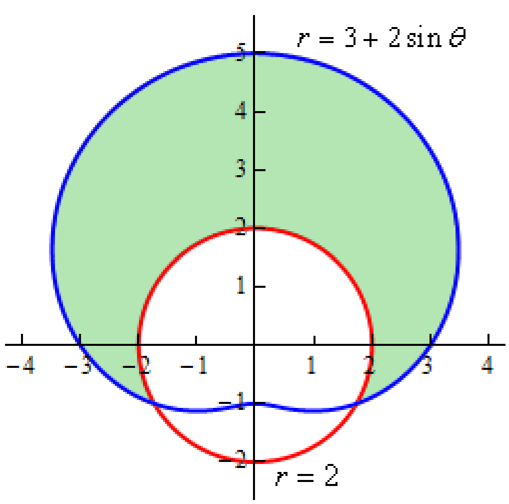
Let's not forget that we still have the two geometric interpretations for these integrals as well.

Example 2 Determine the area of the region that lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$.

Solution
Here is a sketch of the region, *D*, that we want to determine the area of.

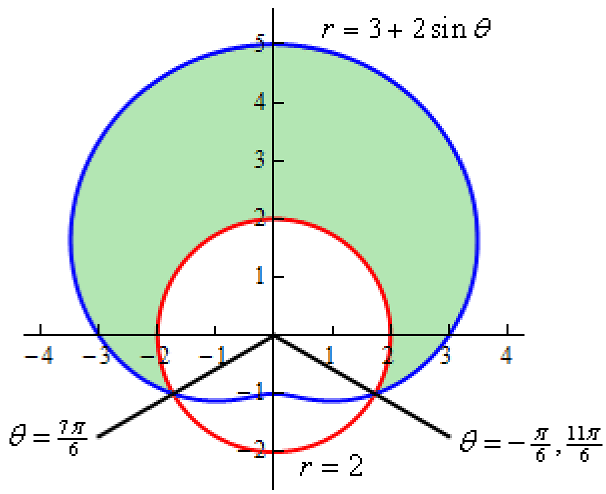


To determine this area we'll need to know that value of θ for which the two curves intersect. We can determine these points by setting the two equations equal and solving.

$$3+2\sin\theta = 2$$

$$\sin\theta = -\frac{1}{2} \qquad \Rightarrow \qquad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

Here is a sketch of the figure with these angles added.



Note as well that we've acknowledged that $-\frac{\pi}{6}$ is another representation for the angle $\frac{11\pi}{6}$. This is important since we need the range of θ to actually enclose the regions as we increase from the lower limit to the upper limit. If we'd chosen to use $\frac{11\pi}{6}$ then as we increase from $\frac{7\pi}{6}$ to $\frac{11\pi}{6}$ we would be tracing out the lower portion of the circle and that is not the region that we are after.

So, here are the ranges that will define the region.

$$-\frac{\pi}{6} \le \theta \le \frac{7\pi}{6}$$

$$2 \le r \le 3+2\sin\theta$$

To get the ranges for r the function that is closest to the origin is the lower bound and the function that is farthest from the origin is the upper bound.

The area of the region D is then,

$$
\begin{aligned}
A &= \iint\limits_D dA \\
&= \int_{-\pi/6}^{7\pi/6} \int_2^{3+2\sin\theta} r\,dr\,d\theta \\
&= \int_{-\pi/6}^{7\pi/6} \frac{1}{2} r^2 \Big|_2^{3+2\sin\theta} \,d\theta \\
&= \int_{-\pi/6}^{7\pi/6} \frac{5}{2} + 6\sin\theta + 2\sin^2\theta\,d\theta \\
&= \int_{-\pi/6}^{7\pi/6} \frac{7}{2} + 6\sin\theta - \cos(2\theta)\,d\theta \\
&= \left(\frac{7}{2}\theta - 6\cos\theta - \frac{1}{2}\sin(2\theta)\right)\Bigg|_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \\
&= \frac{11\sqrt{3}}{2} + \frac{14\pi}{3} = 24.187
\end{aligned}
$$

Example 3 Determine the volume of the region that lies under the sphere $x^2 + y^2 + z^2 = 9$, above the plane $z = 0$ and inside the cylinder $x^2 + y^2 = 5$.

Solution
We know that the formula for finding the volume of a region is,

$$V = \iint\limits_D f(x,y)\,dA$$

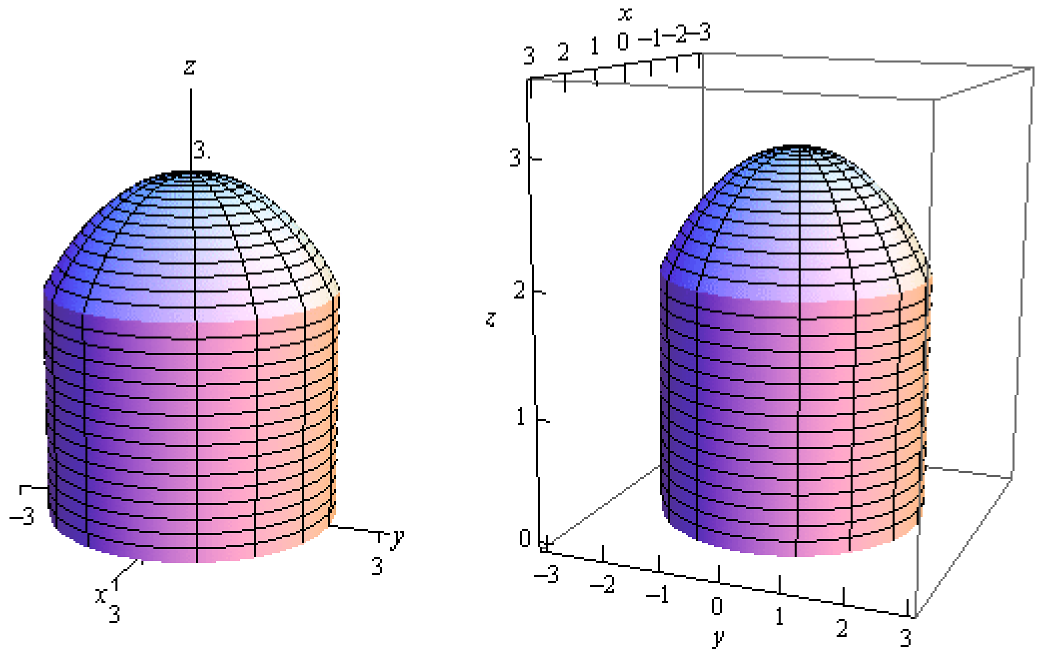
In order to make use of this formula we're going to need to determine the function that we should be integrating and the region *D* that we're going to be integrating over.

The function isn't too bad. It's just the sphere, however, we do need it to be in the form $z = f(x,y)$. We are looking at the region that lies under the sphere and above the plane $z = 0$ (just the *xy*-plane right?) and so all we need to do is solve the equation for *z* and when taking the square root we'll take the positive one since we are wanting the region above the *xy*-plane. Here is the function.

$$z = \sqrt{9 - x^2 - y^2}$$

The region *D* isn't too bad in this case either. As we take points, (x,y), from the region we need to completely graph the portion of the sphere that we are working with. Since we only want the portion of the sphere that actually lies inside the cylinder given by $x^2 + y^2 = 5$ this is also the region *D*. The region *D* is the disk $x^2 + y^2 \le 5$ in the *xy*-plane.

For reference purposes here is a sketch of the region that we are trying to find the volume of.



So, the region that we want the volume for is really a cylinder with a cap that comes from the sphere.

We are definitely going to want to do this integral in terms of polar coordinates so here are the limits (in polar coordinates) for the region,

$$0 \le \theta \le 2\pi$$

$$0 \le r \le \sqrt{5}$$

and we'll need to convert the function to polar coordinates as well.

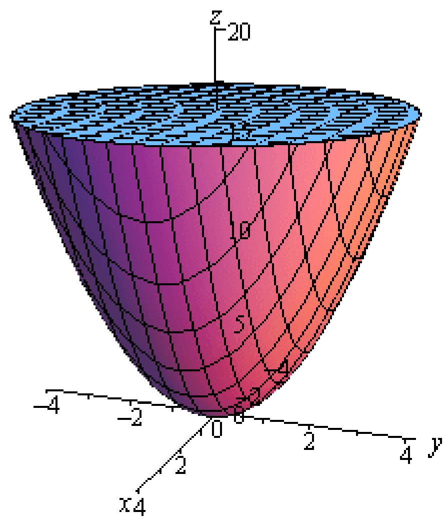
$$z = \sqrt{9 - \left(x^2 + y^2\right)} = \sqrt{9 - r^2}$$

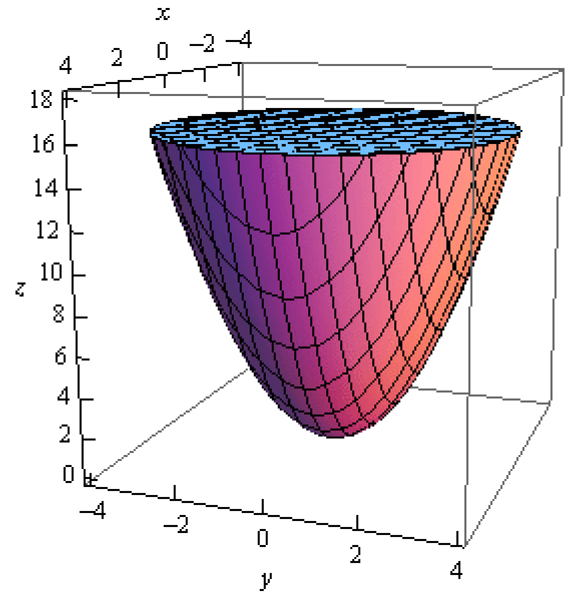
The volume is then,

$$\begin{aligned} V &= \iint\limits_D \sqrt{9 - x^2 - y^2}\, dA \\ &= \int_0^{2\pi} \int_0^{\sqrt{5}} r\sqrt{9 - r^2}\, dr\, d\theta \\ &= \int_0^{2\pi} -\frac{1}{3}\left(9 - r^2\right)^{\frac{3}{2}} \Big|_0^{\sqrt{5}}\, d\theta \\ &= \int_0^{2\pi} \frac{19}{3}\, d\theta \\ &= \frac{38\pi}{3} \end{aligned}$$

Example 4 Find the volume of the region that lies inside $z = x^2 + y^2$ and below the plane $z = 16$.

Solution
Let's start this example off with a quick sketch of the region.





Now, in this case the standard formula is not going to work. The formula

$$V = \iint_D f\left(x, y\right) dA$$

finds the volume under the function $f\left(x, y\right)$ and we're actually after the area that is above a function. This isn't the problem that it might appear to be however. First, notice that

$$V = \iint_D 16 \, dA$$

will be the volume under $z = 16$ (of course we'll need to determine *D* eventually) while

$$V = \iint_D x^2 + y^2 \, dA$$

is the volume under $z = x^2 + y^2$, using the same *D*.

The volume that we're after is really the difference between these two or,

$$V = \iint_D 16 \, dA - \iint_D x^2 + y^2 \, dA = \iint_D 16 - \left(x^2 + y^2\right) dA$$

Now all that we need to do is to determine the region *D* and then convert everything over to polar coordinates.

Determining the region *D* in this case is not too bad. If we were to look straight down the *z*-axis onto the region we would see a circle of radius 4 centered at the origin. This is because the top of the region, where the elliptic paraboloid intersects the plane, is the widest part of the region. We know the *z* coordinate at the intersection so, setting $z = 16$ in the equation of the paraboloid gives,

$$16 = x^2 + y^2$$

which is the equation of a circle of radius 4 centered at the origin.

Here are the inequalities for the region and the function we'll be integrating in terms of polar coordinates.

$$0 \le \theta \le 2\pi \qquad 0 \le r \le 4 \qquad z = 16 - r^2$$

The volume is then,

$$\begin{aligned} V &= \iint_D 16 - \left(x^2 + y^2\right) dA \\ &= \int_0^{2\pi} \int_0^4 r\left(16 - r^2\right) dr\, d\theta \\ &= \int_0^{2\pi} \left(8r^2 - \frac{1}{4} r^4\right)\Bigg|_0^4 d\theta \\ &= \int_0^{2\pi} 64\, d\theta \\ &= 128\pi \end{aligned}$$

In both of the previous volume problems we would have not been able to easily compute the volume without first converting to polar coordinates so, as these examples show, it is a good idea to always remember polar coordinates.

There is one more type of example that we need to look at before moving on to the next section. Sometimes we are given an iterated integral that is already in terms of x and y and we need to convert this over to polar so that we can actually do the integral. We need to see an example of how to do this kind of conversion.

Example 5 Evaluate the following integral by first converting to polar coordinates.

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \cos\left(x^2 + y^2\right) dx\, dy$$

Solution

First, notice that we cannot do this integral in Cartesian coordinates and so converting to polar coordinates may be the only option we have for actually doing the integral. Notice that the function will convert to polar coordinates nicely and so shouldn't be a problem.

Let's first determine the region that we're integrating over and see if it's a region that can be easily converted into polar coordinates. Here are the inequalities that define the region in terms of Cartesian coordinates.

$$0 \le y \le 1$$
$$0 \le x \le \sqrt{1 - y^2}$$

Now, the upper limit for the x's is,

$$x = \sqrt{1 - y^2}$$

and this looks like the right side of the circle of radius 1 centered at the origin. Since the lower limit for the x's is $x = 0$ it looks like we are going to have a portion (or all) of the right side of the disk of radius 1 centered at the origin.

The range for the y's however, tells us that we are only going to have positive y's. This means that we are only going to have the portion of the disk of radius 1 centered at the origin that is in the first quadrant.

So, we know that the inequalities that will define this region in terms of polar coordinates are then,

$$0 \le \theta \le \frac{\pi}{2}$$
$$0 \le r \le 1$$

Finally, we just need to remember that,

$$dx\,dy = dA = r\,dr\,d\theta$$

and so the integral becomes,

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \cos\left(x^2 + y^2\right) dx\,dy = \int_0^{\frac{\pi}{2}} \int_0^1 r\cos\left(r^2\right) dr\,d\theta$$

Note that this is an integral that we can do. So, here is the rest of the work for this integral.

$$\begin{aligned} \int_0^1 \int_0^{\sqrt{1-y^2}} \cos\left(x^2 + y^2\right) dx\,dy &= \int_0^{\frac{\pi}{2}} \left. \frac{1}{2}\sin\left(r^2\right)\right|_0^1 d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin\left(1\right) d\theta \\ &= \frac{\pi}{4}\sin\left(1\right) \end{aligned}$$

Triple Integrals

Now that we know how to integrate over a two-dimensional region we need to move on to integrating over a three-dimensional region. We used a double integral to integrate over a two-dimensional region and so it shouldn't be too surprising that we'll use a **triple integral** to integrate over a three dimensional region. The notation for the general triple integrals is,

$$\iiint\limits_E f(x,y,z)\,dV$$

Let's start simple by integrating over the box,

$$B = [a,b]\times[c,d]\times[r,s]$$

Note that when using this notation we list the x's first, the y's second and the z's third.

The triple integral in this case is,

$$\iiint\limits_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

Note that we integrated with respect to x first, then y, and finally z here, but in fact there is no reason to the integrals in this order. There are 6 different possible orders to do the integral in and which order you do the integral in will depend upon the function and the order that you feel will be the easiest. We will get the same answer regardless of the order however.

Let's do a quick example of this type of triple integral.

Example 1 Evaluate the following integral.

$$\iiint\limits_B 8xyz\,dV, \qquad B = [2,3]\times[1,2]\times[0,1]$$

Solution

Just to make the point that order doesn't matter let's use a different order from that listed above. We'll do the integral in the following order.

$$\begin{aligned}\iiint\limits_B 8xyz\,dV &= \int_1^2 \int_2^3 \int_0^1 8xyz\,dz\,dx\,dy \\ &= \int_1^2 \int_2^3 4xyz^2\Big|_0^1\,dx\,dy \\ &= \int_1^2 \int_2^3 4xy\,dx\,dy \\ &= \int_1^2 2x^2y\Big|_2^3\,dy \\ &= \int_1^2 10y\,dy = 15\end{aligned}$$

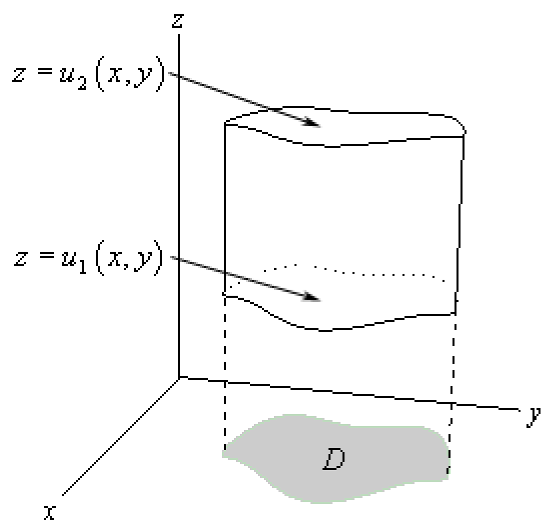
Before moving on to more general regions let's get a nice geometric interpretation about the triple integral out of the way so we can use it in some of the examples to follow.

Fact

The volume of the three-dimensional region E is given by the integral,

$$V = \iiint_E dV$$

Let's now move on the more general three-dimensional regions. We have three different possibilities for a general region. Here is a sketch of the first possibility.



In this case we define the region E as follows,

$$E = \left\{ (x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y) \right\}$$

where $(x,y) \in D$ is the notation that means that the point (x,y) lies in the region D from the xy-plane. In this case we will evaluate the triple integral as follows,

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA$$

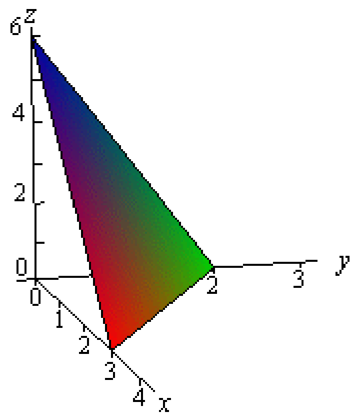
where the double integral can be evaluated in any of the methods that we saw in the previous couple of sections. In other words, we can integrate first with respect to x, we can integrate first with respect to y, or we can use polar coordinates as needed.

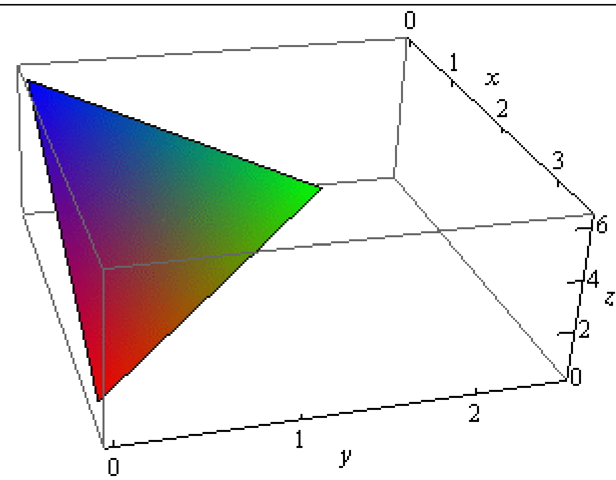
Example 2 Evaluate $\iiint_E 2x\,dV$ where E is the region under the plane $2x + 3y + z = 6$ that lies in the first octant.

Solution

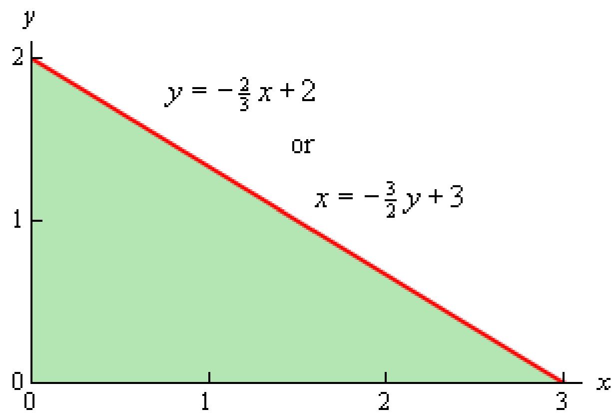
We should first define *octant*. Just as the two-dimensional coordinates system can be divided into four quadrants the three-dimensional coordinate system can be divided into eight octants. The first octant is the octant in which all three of the coordinates are positive.

Here is a sketch of the plane in the first octant.





We now need to determine the region *D* in the *xy*-plane. We can get a visualization of the region by pretending to look straight down on the object from above. What we see will be the region *D* in the *xy*-plane. So *D* will be the triangle with vertices at $\left(0,0\right)$, $\left(3,0\right)$, and $\left(0,2\right)$. Here is a sketch of *D*.



Now we need the limits of integration. Since we are under the plane and in the first octant (so we're above the plane $z = 0$) we have the following limits for *z*.

$$0 \le z \le 6 - 2x - 3y$$

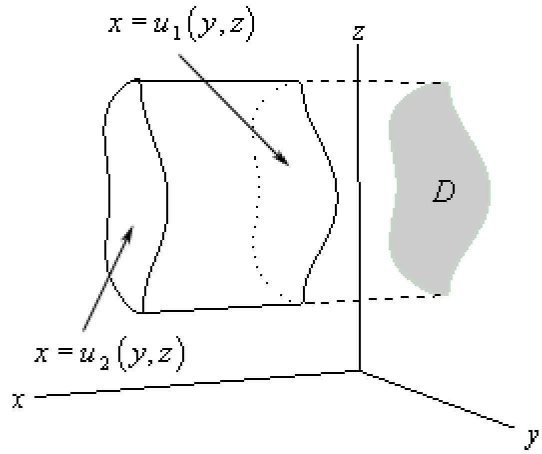
We can integrate the double integral over *D* using either of the following two sets of inequalities.

$0 \le x \le 3$	$0 \le x \le -\frac{3}{2}y + 3$
$0 \le y \le -\frac{2}{3}x + 2$	$0 \le y \le 2$

Since neither really holds an advantage over the other we'll use the first one. The integral is then,

$$
\begin{aligned}
\iiint\limits_E 2x\,dV &= \iint\limits_D \left[\int_0^{6-2x-3y} 2x\,dz\right] dA \\
&= \iint\limits_D 2xz\Big|_0^{6-2x-3y}\,dA \\
&= \int_0^3 \int_0^{-\frac{2}{3}x+2} 2x\left(6-2x-3y\right) dy\,dx \\
&= \int_0^3 \left(12xy-4x^2y-3xy^2\right)\Big|_0^{-\frac{2}{3}x+2} dx \\
&= \int_0^3 \frac{4}{3}x^3-8x^2+12x\,dx \\
&= \left(\frac{1}{3}x^4-\frac{8}{3}x^3+6x^2\right)\Bigg|_0^3 \\
&= 9
\end{aligned}
$$

Let's now move onto the second possible three-dimensional region we may run into for triple integrals. Here is a sketch of this region.



For this possibility we define the region E as follows,

$$E = \left\{ (x,y,z) \mid (y,z) \in D,\ u_1(y,z) \le x \le u_2(y,z) \right\}$$

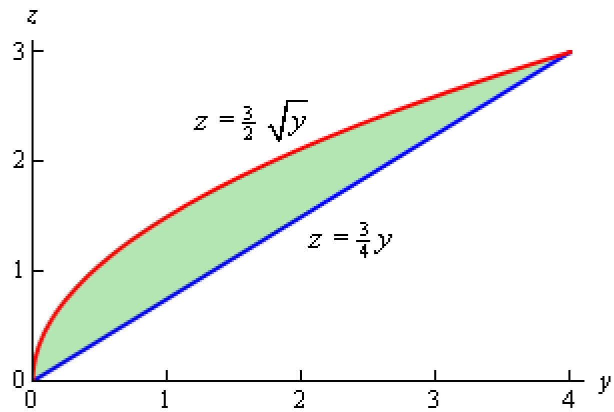
So, the region D will be a region in the yz-plane. Here is how we will evaluate these integrals.

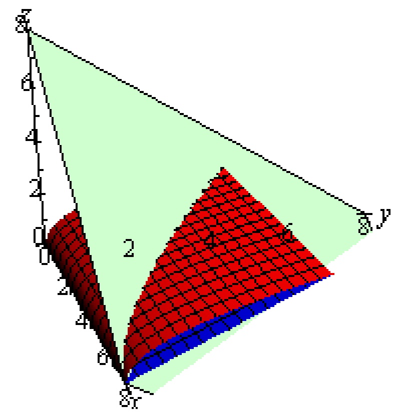
$$\iiint\limits_E f(x,y,z)\,dV = \iint\limits_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right] dA$$

As with the first possibility we will have two options for doing the double integral in the yz-plane as well as the option of using polar coordinates if needed.

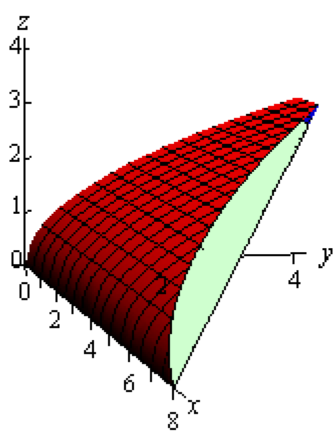
Example 3 Determine the volume of the region that lies behind the plane $x + y + z = 8$ and in front of the region in the *yz*-plane that is bounded by $z = \frac{3}{2}\sqrt{y}$ and $z = \frac{3}{4}y$.

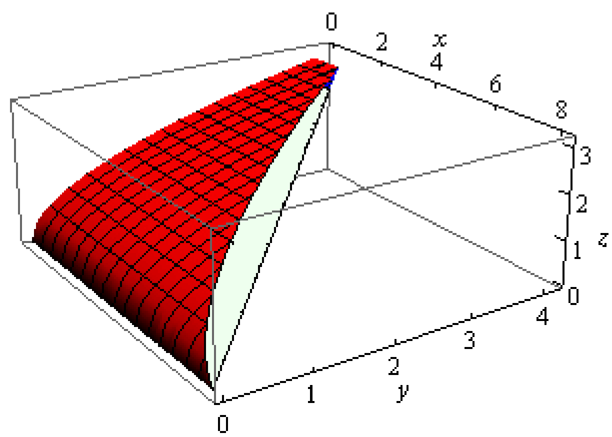
Solution
In this case we've been given *D* and so we won't have to really work to find that. Here is a sketch of the region *D* as well as a quick sketch of the plane and the curves defining *D* projected out past the plane so we can get an idea of what the region we're dealing with looks like.





Now, the graph of the region above is all okay, but it doesn't really show us what the region is. So, here is a sketch of the region itself.





Here are the limits for each of the variables.

$$0 \le y \le 4$$

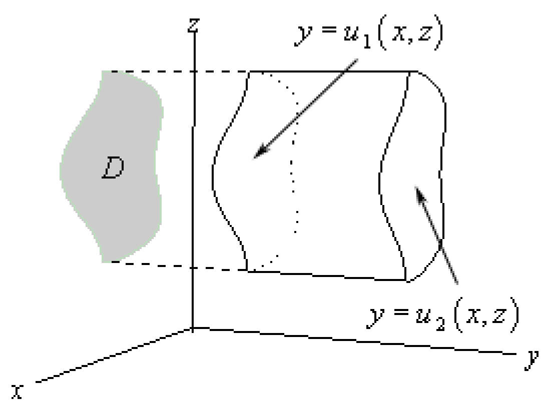
$$\frac{3}{4}y \le z \le \frac{3}{2}\sqrt{y}$$

$$0 \le x \le 8 - y - z$$

The volume is then,

$$V = \iiint\limits_E dV = \iint\limits_D \left[\int_0^{8-y-z} dx \right] dA$$
$$= \int_0^4 \int_{3y/4}^{3\sqrt{y}/2} 8 - y - z \, dz \, dy$$
$$= \int_0^4 \left(8z - yz - \frac{1}{2} z^2 \right) \Bigg|_{\frac{3y}{4}}^{\frac{3\sqrt{y}}{2}} dy$$
$$= \int_0^4 12 y^{\frac{1}{2}} - \frac{57}{8} y - \frac{3}{2} y^{\frac{3}{2}} + \frac{33}{32} y^2 \, dy$$
$$= \left(8 y^{\frac{3}{2}} - \frac{57}{16} y^2 - \frac{3}{5} y^{\frac{5}{2}} + \frac{11}{32} y^3 \right) \Bigg|_0^4 = \frac{49}{5}$$

We now need to look at the third (and final) possible three-dimensional region we may run into for triple integrals. Here is a sketch of this region.



In this final case E is defined as,

$$E = \left\{ (x, y, z) \mid (x, z) \in D, \ u_1(x, z) \le y \le u_2(x, z) \right\}$$

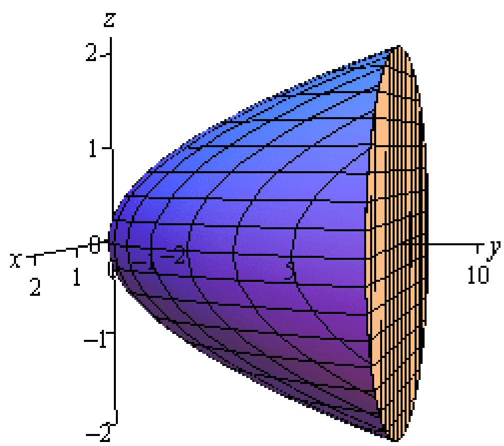
and here the region D will be a region in the xz-plane. Here is how we will evaluate these integrals.

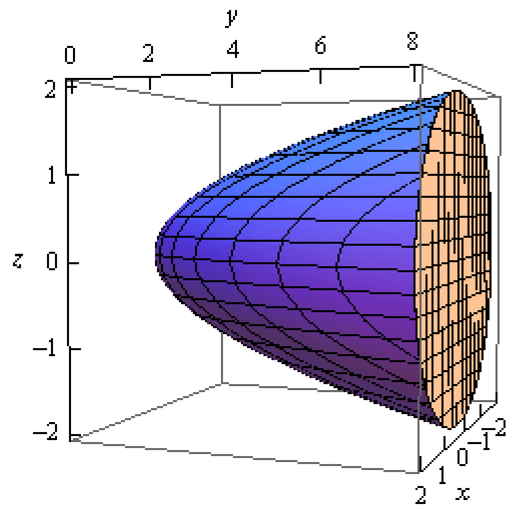
$$\iiint\limits_E f(x, y, z) \, dV = \iint\limits_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z) \, dy \right] dA$$

where we will can use either of the two possible orders for integrating D in the xz-plane or we can use polar coordinates if needed.

Example 4 Evaluate $\iiint\limits_E \sqrt{3x^2 + 3z^2}\, dV$ where E is the solid bounded by $y = 2x^2 + 2z^2$ and the plane $y = 8$.

Solution
Here is a sketch of the solid E.





The region D in the xz-plane can be found by "standing" in front of this solid and we can see that D will be a disk in the xz-plane. This disk will come from the front of the solid and we can determine the equation of the disk by setting the elliptic paraboloid and the plane equal.

$$2x^2 + 2z^2 = 8 \qquad \Rightarrow \qquad x^2 + z^2 = 4$$

This region, as well as the integrand, both seems to suggest that we should use something like polar coordinates. However we are in the xz-plane and we've only seen polar coordinates in the xy-plane. This is not a problem. We can always "translate" them over to the xz-plane with the following definition.

$$x = r\cos\theta \qquad\qquad z = r\sin\theta$$

Since the region doesn't have y's we will let z take the place of y in all the formulas. Note that these definitions also lead to the formula,

$$x^2 + z^2 = r^2$$

With this in hand we can arrive at the limits of the variables that we'll need for this integral.

$$2x^2 + 2z^2 \le y \le 8$$
$$0 \le r \le 2$$
$$0 \le \theta \le 2\pi$$

The integral is then,

$$\iiint_E \sqrt{3x^2+3z^2}\,dV = \iint_D \left[\int_{2x^2+2z^2}^{8} \sqrt{3x^2+3z^2}\,dy\right] dA$$
$$= \iint_D \left(y\sqrt{3x^2+3z^2} \right)\Big|_{2x^2+2z^2}^{8} dA$$
$$= \iint_D \sqrt{3\left(x^2+z^2\right)}\left(8-\left(2x^2+2z^2\right)\right) dA$$

Now, since we are going to do the double integral in polar coordinates let's get everything converted over to polar coordinates. The integrand is,

$$\sqrt{3\left(x^2+z^2\right)}\left(8-\left(2x^2+2z^2\right)\right) = \sqrt{3r^2}\left(8-2r^2\right)$$
$$= \sqrt{3}\; r\left(8-2r^2\right)$$
$$= \sqrt{3}\left(8r-2r^3\right)$$

The integral is then,

$$\iiint_E \sqrt{3x^2+3z^2}\,dV = \iint_D \sqrt{3}\,\left(8r-2r^3\right) dA$$
$$= \sqrt{3}\int_0^{2\pi}\int_0^2 \left(8r-2r^3\right) r\,dr\,d\theta$$
$$= \sqrt{3}\int_0^{2\pi} \left(\frac{8}{3}r^3-\frac{2}{5}r^5\right)\Big|_0^2 d\theta$$
$$= \sqrt{3}\int_0^{2\pi} \frac{128}{15}\,d\theta$$
$$= \frac{256\sqrt{3}\,\pi}{15}$$

Triple Integrals in Cylindrical Coordinates

In this section we want do take a look at triple integrals done completely in Cylindrical Coordinates. Recall that cylindrical coordinates are really nothing more than an extension of polar coordinates into three dimensions. The following are the conversion formulas for cylindrical coordinates.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

In order to do the integral in cylindrical coordinates we will need to know what dV will become in terms of cylindrical coordinates. We will be able to show in the Change of Variables section of this chapter that,

$$dV = r\,dz\,dr\,d\theta$$

The region, E, over which we are integrating becomes,

$$\begin{aligned} E &= \left\{ (x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y) \right\} \\ &= \left\{ (r,\theta,z) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta),\ u_1(r\cos\theta, r\sin\theta) \le z \le u_2(r\cos\theta, r\sin\theta) \right\} \end{aligned}$$

Note that we've only given this for E's in which D is in the xy-plane. We can modify this accordingly if D is in the yz-plane or the xz-plane as needed.

In terms of cylindrical coordinates a triple integral is,

$$\iiint\limits_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} r\, f(r\cos\theta, r\sin\theta, z)\,dz\,dr\,d\theta$$

Don't forget to add in the r and make sure that all the x's and y's also get converted over into cylindrical coordinates.

Let's see an example.

Example 1 Evaluate $\iiint\limits_E y\,dV$ where E is the region that lies below the plane $z = x + 2$ above the xy-plane and between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution

There really isn't too much to do with this one other than do the conversions and then evaluate the integral.

We'll start out by getting the range for z in terms of cylindrical coordinates.

$$0 \le z \le x + 2 \qquad \Rightarrow \qquad 0 \le z \le r\cos\theta + 2$$

Remember that we are above the xy-plane and so we are above the plane $z = 0$

Next, the region D is the region between the two circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ in the xy-plane and so the ranges for it are,

$$0 \le \theta \le 2\pi \qquad\qquad 1 \le r \le 2$$

Here is the integral.

$$\begin{aligned}\iiint\limits_E y\,dV &= \int_0^{2\pi}\int_1^2\int_0^{r\cos\theta+2}\left(r\sin\theta\right)r\,dz\,dr\,d\theta\\ &= \int_0^{2\pi}\int_1^2 r^2\sin\theta\left(r\cos\theta+2\right)dr\,d\theta\\ &= \int_0^{2\pi}\int_1^2 \frac{1}{2}r^3\sin\left(2\theta\right)+2r^2\sin\theta\,dr\,d\theta\\ &= \int_0^{2\pi}\left.\left(\frac{1}{8}r^4\sin\left(2\theta\right)+\frac{2}{3}r^3\sin\theta\right)\right|_1^2 d\theta\\ &= \int_0^{2\pi}\frac{15}{8}\sin\left(2\theta\right)+\frac{14}{3}\sin\theta\,d\theta\\ &= \left.\left(-\frac{15}{16}\cos\left(2\theta\right)-\frac{14}{3}\cos\theta\right)\right|_0^{2\pi}\\ &= 0\end{aligned}$$

Just as we did with double integral involving polar coordinates we can start with an iterated integral in terms of x, y, and z and convert it to cylindrical coordinates.

Example 2 Convert $\int_{-1}^{1}\int_0^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy$ into an integral in cylindrical coordinates.

Solution
Here are the ranges of the variables from this iterated integral.

$$-1 \le y \le 1$$
$$0 \le x \le \sqrt{1-y^2}$$
$$x^2+y^2 \le z \le \sqrt{x^2+y^2}$$

The first two inequalities define the region D and since the upper and lower bounds for the x's are $x=\sqrt{1-y^2}$ and $x=0$ we know that we've got at least part of the right half a circle of radius 1 centered at the origin. Since the range of y's is $-1 \le y \le 1$ we know that we have the complete right half of the disk of radius 1 centered at the origin. So, the ranges for D in cylindrical coordinates are,

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$
$$0 \le r \le 1$$

All that's left to do now is to convert the limits of the z range, but that's not too bad.

$$r^2 \le z \le r$$

On a side note notice that the lower bound here is an elliptic paraboloid and the upper bound is a cone. Therefore E is a portion of the region between these two surfaces.

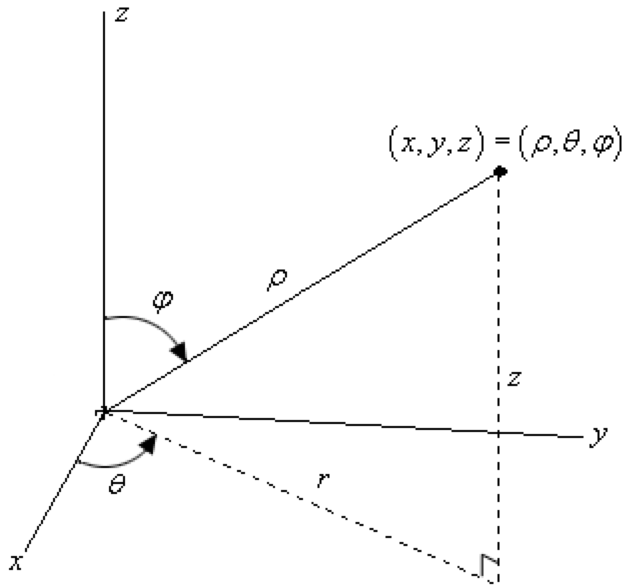
The integral is,

$$\begin{aligned}\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy &= \int_{-\pi/2}^{\pi/2}\int_{0}^{1}\int_{r^2}^{r} r\left(r\cos\theta\right)\left(r\sin\theta\right) z\,dz\,dr\,d\theta \\ &= \int_{-\pi/2}^{\pi/2}\int_{0}^{1}\int_{r^2}^{r} zr^3\cos\theta\sin\theta\,dz\,dr\,d\theta\end{aligned}$$

Triple Integrals in Spherical Coordinates

In the previous section we looked at doing integrals in terms of cylindrical coordinates and we now need to take a quick look at doing integrals in terms of spherical coordinates.

First, we need to recall just how spherical coordinates are defined. The following sketch shows the relationship between the Cartesian and spherical coordinate systems.



Here are the conversion formulas for spherical coordinates.

$$x = \rho\sin\varphi\cos\theta \qquad y = \rho\sin\varphi\sin\theta \qquad z = \rho\cos\varphi$$
$$x^2 + y^2 + z^2 = \rho^2$$

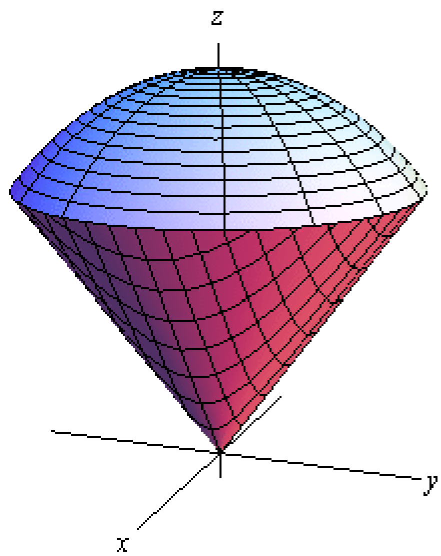
We also have the following restrictions on the coordinates.

$$\rho \geq 0 \qquad 0 \leq \varphi \leq \pi$$

For our integrals we are going to restrict *E* down to a spherical wedge. This will mean that we are going to take ranges for the variables as follows,

$$a \leq \rho \leq b$$
$$\alpha \leq \theta \leq \beta$$
$$\delta \leq \varphi \leq \gamma$$

Here is a quick sketch of a spherical wedge in which the lower limit for both ρ and φ are zero for reference purposes. Most of the wedges we'll be working with will fit into this pattern.



From this sketch we can see that E is really nothing more than the intersection of a sphere and a cone.

In the next section we will show that

$$dV = \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

Therefore the integral will become,

$$\iiint\limits_E f(x,y,z)\, dV = \int_\delta^\gamma \int_\alpha^\beta \int_a^b \rho^2 \sin\varphi \;\, f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\, d\rho\, d\theta\, d\varphi$$

This looks bad, but given that the limits are all constants the integrals here tend to not be too bad.

Example 1 Evaluate $\iiint\limits_E 16z\, dV$ where E is the upper half of the sphere $x^2 + y^2 + z^2 = 1$.

Solution
Since we are taking the upper half of the sphere the limits for the variables are,

$$0 \le \rho \le 1$$
$$0 \le \theta \le 2\pi$$
$$0 \le \varphi \le \frac{\pi}{2}$$

The integral is then,

$$\iiint_E 16z\,dV = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^1 \rho^2\sin\varphi\left(16\rho\cos\varphi\right)d\rho\,d\theta\,d\varphi$$
$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^1 8\rho^3\sin\left(2\varphi\right)d\rho\,d\theta\,d\varphi$$
$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi} 2\sin\left(2\varphi\right)d\theta\,d\varphi$$
$$= \int_0^{\frac{\pi}{2}} 4\pi\sin\left(2\varphi\right)d\varphi$$
$$= -2\pi\cos\left(2\varphi\right)\Big|_0^{\frac{\pi}{2}}$$
$$= 4\pi$$

Example 2 Convert $\int_0^3\int_0^{\sqrt{9-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} x^2+y^2+z^2\,dz\,dx\,dy$ into spherical coordinates.

Solution
Let's first write down the limits for the variables.

$$0 \le y \le 3$$
$$0 \le x \le \sqrt{9-y^2}$$
$$\sqrt{x^2+y^2} \le z \le \sqrt{18-x^2-y^2}$$

The range for x tells us that we have a portion of the right half of a disk of radius 3 centered at the origin. Since we are restricting y's to positive values it looks like we will have the quarter disk in the first quadrant. Therefore since D is in the first quadrant the region, E, must be in the first octant and this in turn tells us that we have the following range for θ (since this is the angle around the z-axis).

$$0 \le \theta \le \frac{\pi}{2}$$

Now, let's see what the range for z tells us. The lower bound, $z=\sqrt{x^2+y^2}$, is the upper half of a cone. At this point we don't need this quite yet, but we will later. The upper bound, $z=\sqrt{18-x^2-y^2}$, is the upper half of the sphere,

$$x^2+y^2+z^2=18$$

and so from this we now have the following range for ρ

$$0 \le \rho \le \sqrt{18} = 3\sqrt{2}$$

Now all that we need is the range for φ. There are two ways to get this. One is from where the cone and the sphere intersect. Plugging in the equation for the cone into the sphere gives,

$$\left(\sqrt{x^2+y^2}\right)^2 + z^2 = 18$$
$$z^2 + z^2 = 18$$
$$z^2 = 9$$
$$z = 3$$

Note that we can assume z is positive here since we know that we have the upper half of the cone and/or sphere. Finally, plug this into the conversion for z and take advantage of the fact that we know that $\rho = 3\sqrt{2}$ since we are intersecting on the sphere. This gives,

$$\rho\cos\varphi = 3$$
$$3\sqrt{2}\cos\varphi = 3$$
$$\cos\varphi = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \Rightarrow \qquad \varphi = \frac{\pi}{4}$$

So, it looks like we have the following range,

$$0 \le \varphi \le \frac{\pi}{4}$$

The other way to get this range is from the cone by itself. By first converting the equation into cylindrical coordinates and then into spherical coordinates we get the following,

$$z = r$$
$$\rho\cos\varphi = \rho\sin\varphi$$
$$1 = \tan\varphi \qquad \Rightarrow \qquad \varphi = \frac{\pi}{4}$$

So, recalling that $\rho^2 = x^2 + y^2 + z^2$, the integral is then,

$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} x^2 + y^2 + z^2 \, dz\, dx\, dy = \int_0^{\pi/4} \int_0^{\pi/2} \int_0^{3\sqrt{2}} \rho^4 \sin\varphi \, d\rho\, d\theta\, d\varphi$$

Change of Variables

Back in Calculus I we had the substitution rule that told us that,

$$\int_a^b f\left(g(x)\right)g'(x)\,dx = \int_c^d f(u)\,du \qquad \text{where} \quad u = g(x)$$

In essence this is taking an integral in terms of x's and changing it into terms of u's. We want to do something similar for double and triple integrals. In fact we've already done this to a certain extent when we converted double integrals to polar coordinates and when we converted triple integrals to cylindrical or spherical coordinates. The main difference is that we didn't actually go through the details of where the formulas came from. If you recall, in each of those cases we commented that we would justify the formulas for dA and dV eventually. Now is the time to do that justification.

While often the reason for changing variables is to get us an integral that we can do with the new variables, another reason for changing variables is to convert the region into a nicer region to work with. When we were converting the polar, cylindrical or spherical coordinates we didn't worry about this change since it was easy enough to determine the new limits based on the given region. That is not always the case however. So, before we move into changing variables with multiple integrals we first need to see how the region may change with a change of variables.

First we need a little notation out of the way. We call the equations that define the change of variables a **transformation**. Also we will typically start out with a region, R, in xy-coordinates and transform it into a region in uv-coordinates.

Example 1 Determine the new region that we get by applying the given transformation to the region R.

(a) R is the ellipse $x^2 + \frac{y^2}{36} = 1$ and the transformation is $x = \frac{u}{2}$, $y = 3v$. [Solution]

(b) R is the region bounded by $y = -x + 4$, $y = x + 1$, and $y = \frac{x}{3} - \frac{4}{3}$ and the transformation is $x = \frac{1}{2}(u + v)$, $y = \frac{1}{2}(u - v)$. [Solution]

Solution

(a) *R* is the ellipse $x^2 + \frac{y^2}{36} = 1$ and the transformation is $x = \frac{u}{2}$, $y = 3v$.

There really isn't too much to do with this one other than to plug the transformation into the equation for the ellipse and see what we get.

$$\left(\frac{u}{2}\right)^2 + \frac{(3v)^2}{36} = 1$$

$$\frac{u^2}{4} + \frac{9v^2}{36} = 1$$

$$u^2 + v^2 = 4$$

So, we started out with an ellipse and after the transformation we had a disk of radius 2.

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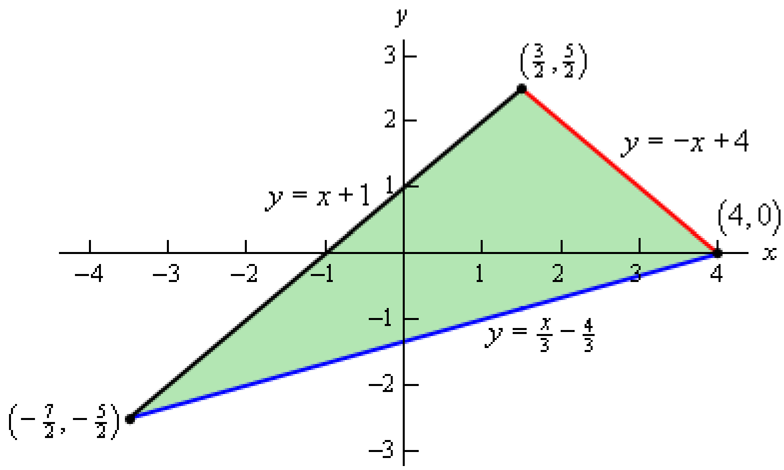
(b) *R* is the region bounded by $y = -x + 4$, $y = x + 1$, and $y = \frac{x}{3} - \frac{4}{3}$ and the transformation is $x = \frac{1}{2}(u + v)$, $y = \frac{1}{2}(u - v)$.

As with the first part we'll need to plug the transformation into the equation, however, in this case we will need to do it three times, once for each equation. Before we do that let's sketch the graph of the region and see what we've got.



So, we have a triangle. Now, let's go through the transformation. We will apply the transformation to each edge of the triangle and see where we get.

Let's do $y = -x + 4$ first. Plugging in the transformation gives,

$$\begin{aligned}\frac{1}{2}(u - v) &= -\frac{1}{2}(u + v) + 4 \\ u - v &= -u - v + 8 \\ 2u &= 8 \\ u &= 4\end{aligned}$$

The first boundary transforms very nicely into a much simpler equation.

Now let's take a look at $y = x + 1$,

$$\begin{aligned}\frac{1}{2}(u - v) &= \frac{1}{2}(u + v) + 1 \\ u - v &= u + v + 2 \\ -2v &= 2 \\ v &= -1\end{aligned}$$

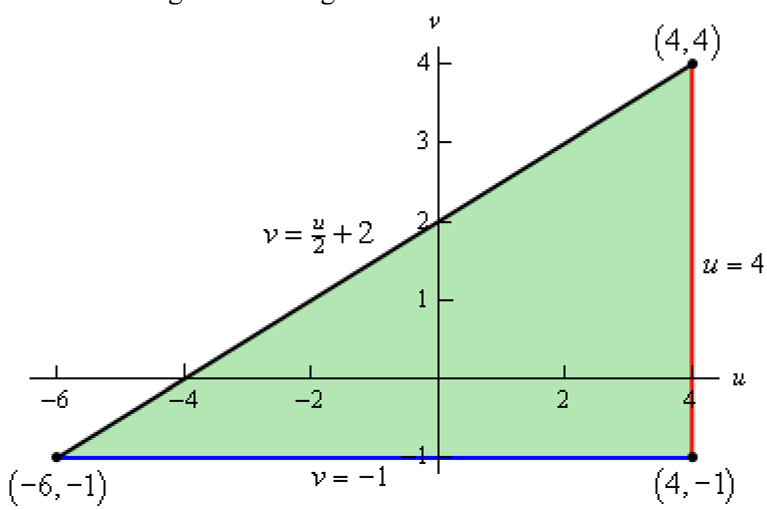
Again, a much nicer equation that what we started with.

Finally, let's transform $y = \frac{x}{3} - \frac{4}{3}$.

$$\frac{1}{2}(u - v) = \frac{1}{3}\left(\frac{1}{2}(u + v)\right) - \frac{4}{3}$$
$$3u - 3v = u + v - 8$$
$$4v = 2u + 8$$
$$v = \frac{u}{2} + 2$$

So, again, we got a somewhat simpler equation, although not quite as nice as the first two.

Let's take a look at the new region that we get under the transformation.



We still get a triangle, but a much nicer one.

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Note that we can't always expect to transform a specific type of region (a triangle for example) into the same kind of region. It is completely possible to have a triangle transform into a region in which each of the edges are curved and in no way resembles a triangle.

Notice that in each of the above examples we took a two dimensional region that would have been somewhat difficult to integrate over and converted it into a region that would be much nicer in integrate over. As we noted at the start of this set of examples, that is often one of the points behind the transformation. In addition to converting the integrand into something simpler it will often also transform the region into one that is much easier to deal with.

Now that we've seen a couple of examples of transforming regions we need to now talk about how we actually do change of variables in the integral. We will start with double integrals. In order to change variables in a double integral we will need the **Jacobian** of the transformation. Here is the definition of the Jacobian.

Definition

The **Jacobian** of the transformation $x = g(u,v)$, $y = h(u,v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

The Jacobian is defined as a determinant of a 2x2 matrix, if you are unfamiliar with this that is okay. Here is how to compute the determinant.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Therefore, another formula for the determinant is,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Now that we have the Jacobian out of the way we can give the formula for change of variables for a double integral.

Change of Variables for a Double Integral

Suppose that we want to integrate $f(x,y)$ over the region *R*. Under the transformation $x = g(u,v)$, $y = h(u,v)$ the region becomes *S* and the integral becomes,

$$\iint_R f(x,y)\,dA = \iint_S f\left(g(u,v),h(u,v)\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Note that we used *du dv* instead of *dA* in the integral to make it clear that we are now integrating with respect to *u* and *v*. Also note that we are taking the absolute value of the Jacobian.

If we look just at the differentials in the above formula we can also say that

$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Example 2 Show that when changing to polar coordinates we have $dA = r\,dr\,d\theta$

Solution

So, what we are doing here is justifying the formula that we used back when we were integrating with respect to polar coordinates. All that we need to do is use the formula above for *dA*.

The transformation here is the standard conversion formulas,

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

The Jacobian for this transformation is,

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta - \left(-r\sin^2\theta\right)$$

$$= r\left(\cos^2\theta + \sin^2\theta\right)$$

$$= r$$

We then get,

$$dA = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta = |r|\,dr\,d\theta = r\,dr\,d\theta$$

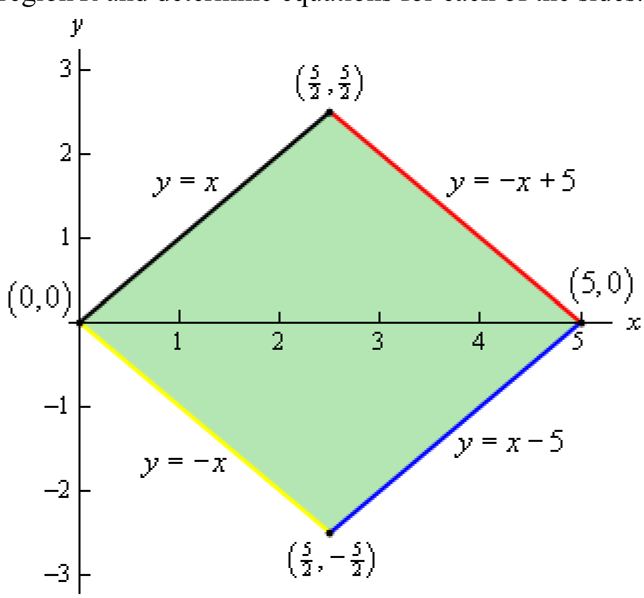
So, the formula we used in the section on polar integrals was correct.

Now, let's do a couple of integrals.

Example 3 Evaluate $\iint\limits_R x + y\,dA$ where R is the trapezoidal region with vertices given by $(0,0)$, $(5,0)$, $\left(\frac{5}{2},\frac{5}{2}\right)$ and $\left(\frac{5}{2},-\frac{5}{2}\right)$ using the transformation $x = 2u + 3v$ and $y = 2u - 3v$.

Solution

First, let's sketch the region R and determine equations for each of the sides.



Each of the equations was found by using the fact that we know two points on each line (*i.e.* the two vertices that form the edge).

While we could do this integral in terms of x and y it would involve two integrals and so would be some work.

Let's use the transformation and see what we get. We'll do this by plugging the transformation into each of the equations above.

Let's start the process off with $y = x$.

$$\begin{aligned} 2u - 3v &= 2u + 3v \\ 6v &= 0 \\ v &= 0 \end{aligned}$$

Transforming $y = -x$ is similar.

$$\begin{aligned} 2u - 3v &= -\left(2u + 3v\right) \\ 4u &= 0 \\ u &= 0 \end{aligned}$$

Next we'll transform $y = -x + 5$.

$$\begin{aligned} 2u - 3v &= -\left(2u + 3v\right) + 5 \\ 4u &= 5 \\ u &= \frac{5}{4} \end{aligned}$$

Finally, let's transform $y = x - 5$.

$$\begin{aligned} 2u - 3v &= 2u + 3v - 5 \\ -6v &= -5 \\ v &= \frac{5}{6} \end{aligned}$$

The region S is then a rectangle whose sides are given by $u = 0$, $v = 0$, $u = \frac{5}{4}$ and $v = \frac{5}{6}$ and so the ranges of u and v are,

$$0 \le u \le \frac{5}{4} \qquad\qquad 0 \le v \le \frac{5}{6}$$

Next, we need the Jacobian.

$$\frac{\partial\left(x, y\right)}{\partial\left(u, v\right)} = \begin{vmatrix} 2 & 3 \\ 2 & -3 \end{vmatrix} = -6 - 6 = -12$$

The integral is then,

$$\begin{aligned}\iint\limits_R x+y\,dA &= \int_0^{\frac{5}{6}}\int_0^{\frac{5}{4}}\left(\left(2u+3v\right)+\left(2u-3v\right)\right)\left|-12\right|du\,dv\\ &= \int_0^{\frac{5}{6}}\int_0^{\frac{5}{4}} 48u\,du\,dv\\ &= \int_0^{\frac{5}{6}} 24u^2\Big|_0^{\frac{5}{4}}\,dv\\ &= \int_0^{\frac{5}{6}}\frac{75}{2}\,dv\\ &= \frac{75}{2}v\Big|_0^{\frac{5}{6}}\\ &= \frac{125}{4}\end{aligned}$$

Example 4 Evaluate $\iint\limits_R x^2-xy+y^2\,dA$ where R is the ellipse given by $x^2-xy+y^2=2$ and using the transformation $x=\sqrt{2}\,u-\sqrt{\frac{2}{3}}\,v$, $y=\sqrt{2}\,u+\sqrt{\frac{2}{3}}\,v$.

Solution

The first thing to do is to plug the transformation into the equation for the ellipse to see what the region transforms into.

$$\begin{aligned}2 &= x^2-xy+y^2\\ &= \left(\sqrt{2}\,u-\sqrt{\frac{2}{3}}\,v\right)^2-\left(\sqrt{2}\,u-\sqrt{\frac{2}{3}}\,v\right)\left(\sqrt{2}\,u+\sqrt{\frac{2}{3}}\,v\right)+\left(\sqrt{2}\,u+\sqrt{\frac{2}{3}}\,v\right)^2\\ &= 2u^2-\frac{4}{\sqrt{3}}uv+\frac{2}{3}v^2-\left(2u^2-\frac{2}{3}v^2\right)+2u^2+\frac{4}{\sqrt{3}}uv+\frac{2}{3}v^2\\ &= 2u^2+2v^2\end{aligned}$$

Or, upon dividing by 2 we see that the equation describing R transforms into

$$u^2+v^2=1$$

or the unit circle. Again, this will be much easier to integrate over than the original region.

Note as well that we've shown that the function that we're integrating is

$$x^2-xy+y^2=2\left(u^2+v^2\right)$$

in terms of u and v so we won't have to redo that work when the time to do the integral comes around.

Finally, we need to find the Jacobian.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \sqrt{2} & -\sqrt{\frac{2}{3}} \\ \sqrt{2} & \sqrt{\frac{2}{3}} \end{vmatrix} = \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}} = \frac{4}{\sqrt{3}}$$

The integral is then,

$$\iint\limits_R x^2 - xy + y^2 \, dA = \iint\limits_S 2\left(u^2 + v^2\right)\left|\frac{4}{\sqrt{3}}\right| du\,dv$$

Before proceeding a word of caution is in order. Do not make the mistake of substituting $x^2 - xy + y^2 = 2$ or $u^2 + v^2 = 1$ in for the integrands. These equations are only valid on the boundary of the region and we are looking at all the points interior to the boundary as well and for those points neither of these equations will be true!

At this point we'll note that this integral will be much easier in terms of polar coordinates and so to finish the integral out will convert to polar coordinates.

$$\begin{aligned} \iint\limits_R x^2 - xy + y^2 \, dA &= \iint\limits_S 2\left(u^2 + v^2\right)\left|\frac{4}{\sqrt{3}}\right| du\,dv \\ &= \frac{8}{\sqrt{3}} \int_0^{2\pi} \int_0^1 \left(r^2\right) r\,dr\,d\theta \\ &= \frac{8}{\sqrt{3}} \int_0^{2\pi} \frac{1}{4} r^4 \Big|_0^1 \, d\theta \\ &= \frac{8}{\sqrt{3}} \int_0^{2\pi} \frac{1}{4} \, d\theta \\ &= \frac{4\pi}{\sqrt{3}} \end{aligned}$$

Let's now briefly look at triple integrals. In this case we will again start with a region *R* and use the transformation $x = g(u,v,w)$, $y = h(u,v,w)$, and $z = k(u,v,w)$ to transform the region into the new region *S*. To do the integral we will need a Jacobian, just as we did with double integrals. Here is the definition of the Jacobian for this kind of transformation.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

In this case the Jacobian is defined in terms of the determinant of a 3x3 matrix. We saw how to evaluate these when we looked at cross products back in Calculus II. If you need a refresher on how to compute them you should go back and review that section.

The integral under this transformation is,

$$\iiint_R f(x,y,z)\,dV = \iiint_S f\big(g(u,v,w),h(u,v,w),k(u,v,w)\big)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

As with double integrals we can look at just the differentials and note that we must have

$$dV = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

We're not going to do any integrals here, but let's verify the formula for dV for spherical coordinates.

Example 5 Verify that $dV = \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi$ when using spherical coordinates.

Solution
Here the transformation is just the standard conversion formulas.

$$x = \rho\sin\varphi\cos\theta \qquad y = \rho\sin\varphi\sin\theta \qquad z = \rho\cos\varphi$$

The Jacobian is,

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} &= \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\varphi & 0 & -\rho\sin\varphi \end{vmatrix} \\
&= -\rho^2\sin^3\varphi\cos^2\theta - \rho^2\sin\varphi\cos^2\varphi\sin^2\theta + 0 \\
&\qquad -\rho^2\sin^3\varphi\sin^2\theta - 0 - \rho^2\sin\varphi\cos^2\varphi\cos^2\theta \\
&= -\rho^2\sin^3\varphi\left(\cos^2\theta + \sin^2\theta\right) - \rho^2\sin\varphi\cos^2\varphi\left(\sin^2\theta + \cos^2\theta\right) \\
&= -\rho^2\sin^3\varphi - \rho^2\sin\varphi\cos^2\varphi \\
&= -\rho^2\sin\varphi\left(\sin^2\varphi + \cos^2\varphi\right) \\
&= -\rho^2\sin\varphi
\end{aligned}$$

Finally, dV becomes,

$$dV = \left|-\rho^2\sin\varphi\right| d\rho\, d\theta\, d\varphi = \rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi$$

Recall that we restricted φ to the range $0 \le \varphi \le \pi$ for spherical coordinates and so we know that $\sin\varphi \ge 0$ and so we don't need the absolute value bars on the sine.

We will leave it to you to check the formula for dV for cylindrical coordinates if you'd like to. It is a much easier formula to check.

Surface Area

In this section we will look at the lone application (aside from the area and volume interpretations) of multiple integrals in this material. This is not the first time that we've looked at surface area We first saw surface area in Calculus II, however, in that setting we were looking at the surface area of a solid of revolution. In other words we were looking at the surface area of a solid obtained by rotating a function about the x or y axis. In this section we want to look at a much more general setting although you will note that the formula here is very similar to the formula we saw back in Calculus II.

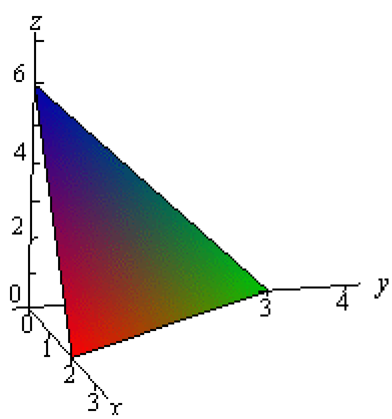
Here we want to find the surface area of the surface given by $z = f(x, y)$ where (x, y) is a point from the region D in the xy-plane. In this case the surface area is given by,

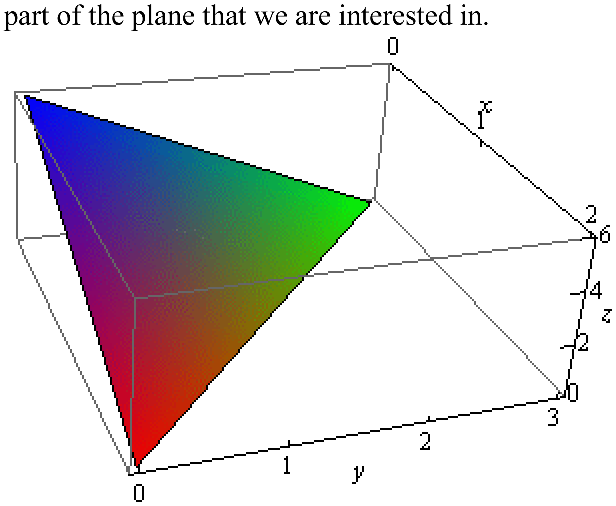
$$S = \iint\limits_D \sqrt{\left[f_x\right]^2 + \left[f_y\right]^2 + 1}\, dA$$

Let's take a look at a couple of examples.

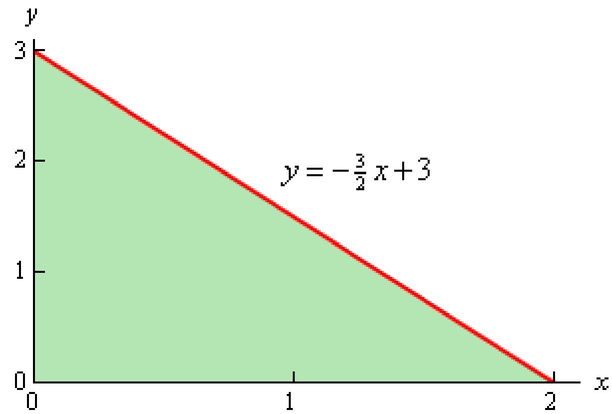
Example 1 Find the surface area of the part of the plane $3x + 2y + z = 6$ that lies in the first octant.

Solution
Remember that the first octant is the portion of the xyz-axis system in which all three variables are positive. Let's first get a sketch of the part of the plane that we are interested in.





We'll also need a sketch of the region D.



Remember that to get the region *D* we can pretend that we are standing directly over the plane and what we see is the region *D*. We can get the equation for the hypotenuse of the triangle by realizing that this is nothing more than the line where the plane intersects the *xy*-plane and we also know that $z = 0$ on the *xy*-plane. Plugging $z = 0$ into the equation of the plane will give us the equation for the hypotenuse.

Notice that in order to use the surface area formula we need to have the function in the form $z = f(x, y)$ and so solving for *z* and taking the partial derivatives gives,

$$z = 6 - 3x - 2y \qquad f_x = -3 \qquad f_y = -2$$

The limits defining *D* are,

$$0 \le x \le 2 \qquad 0 \le y \le -\frac{3}{2}x + 3$$

The surface area is then,

$$\begin{aligned} S &= \iint_D \sqrt{[-3]^2 + [-2]^2 + 1}\, dA \\ &= \int_0^2 \int_0^{-\frac{3}{2}x+3} \sqrt{14}\, dy\, dx \\ &= \sqrt{14} \int_0^2 -\frac{3}{2}x + 3\, dx \\ &= \sqrt{14}\left(-\frac{3}{4}x^2 + 3x\right)\Bigg|_0^2 \\ &= 3\sqrt{14} \end{aligned}$$

Example 2 Determine the surface area of the part of $z = xy$ that lies in the cylinder given by $x^2 + y^2 = 1$.

Solution

In this case we are looking for the surface area of the part of $z = xy$ where (x, y) comes from the disk of radius 1 centered at the origin since that is the region that will lie inside the given cylinder.

Here are the partial derivatives,

$$f_x = y \qquad\qquad f_y = x$$

The integral for the surface area is,

$$S = \iint\limits_D \sqrt{x^2 + y^2 + 1}\, dA$$

Given that D is a disk it makes sense to do this integral in polar coordinates.

$$\begin{aligned} S &= \iint\limits_D \sqrt{x^2 + y^2 + 1}\, dA \\ &= \int_0^{2\pi} \int_0^1 r\sqrt{1+r^2}\, dr\, d\theta \\ &= \int_0^{2\pi} \frac{1}{2}\left(\frac{2}{3}\right)\left(1+r^2\right)^{\frac{3}{2}}\Big|_0^1\, d\theta \\ &= \int_0^{2\pi} \frac{1}{3}\left(2^{\frac{3}{2}} - 1\right) d\theta \\ &= \frac{2\pi}{3}\left(2^{\frac{3}{2}} - 1\right) \end{aligned}$$

Area and Volume Revisited

This section is here only so we can summarize the geometric interpretations of the double and triple integrals that we saw in this chapter. Since the purpose of this section is to summarize these formulas we aren't going to be doing any examples in this section.

We'll first look at the area of a region. The area of the region D is given by,

$$\text{Area of } D = \iint_D dA$$

Now let's give the two volume formulas. First the volume of the region E is given by,

$$\text{Volume of } E = \iiint_E dV$$

Finally, if the region E can be defined as the region under the function $z = f(x,y)$ and above the region D in xy-plane then,

$$\text{Volume of } E = \iint_D f(x,y)\, dA$$

Note as well that there are similar formulas for the other planes. For instance, the volume of the region behind the function $y = f(x,z)$ and in front of the region D in the xz-plane is given by,

$$\text{Volume of } E = \iint_D f(x,z)\, dA$$

Likewise, the the volume of the region behind the function $x = f(y,z)$ and in front of the region D in the yz-plane is given by,

$$\text{Volume of } E = \iint_D f(y,z)\, dA$$

Line Integrals

Introduction

In this section we are going to start looking at Calculus with vector fields (which we'll define in the first section). In particular we will be looking at a new type of integral, the line integral and some of the interpretations of the line integral. We will also take a look at one of the more important theorems involving line integrals, Green's Theorem.

Here is a listing of the topics covered in this chapter.

Vector Fields – In this section we introduce the concept of a vector field.

Line Integrals – Part I – Here we will start looking at line integrals. In particular we will look at line integrals with respect to arc length.

Line Integrals – Part II – We will continue looking at line integrals in this section. Here we will be looking at line integrals with respect to x, y, and/or z.

Line Integrals of Vector Fields – Here we will look at a third type of line integrals, line integrals of vector fields.

Fundamental Theorem for Line Integrals – In this section we will look at a version of the fundamental theorem of calculus for line integrals of vector fields.

Conservative Vector Fields – Here we will take a somewhat detailed look at conservative vector fields and how to find potential functions.

Green's Theorem – We will give Green's Theorem in this section as well as an interesting application of Green's Theorem.

Curl and Divergence – In this section we will introduce the concepts of the curl and the divergence of a vector field. We will also give two vector forms of Green's Theorem.

Vector Fields

We need to start this chapter off with the definition of a vector field as they will be a major component of both this chapter and the next. Let's start off with the formal definition of a vector field.

Definition

A vector field on two (or three) dimensional space is a function $\vec{F}$ that assigns to each point (x, y) (or (x, y, z)) a two (or three dimensional) vector given by $\vec{F}(x, y)$ (or $\vec{F}(x, y, z)$).

That may not make a lot of sense, but most people do know what a vector field is, or at least they've seen a sketch of a vector field. If you've seen a current sketch giving the direction and magnitude of a flow of a fluid or the direction and magnitude of the winds then you've seen a sketch of a vector field.

The standard notation for the function $\vec{F}$ is,

$$\vec{F}(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j}$$

$$\vec{F}(x, y, z) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$$

depending on whether or not we're in two or three dimensions. The function *P*, *Q*, *R* (if it is present) are sometimes called **scalar functions**.

Let's take a quick look at a couple of examples.

Example 1 Sketch each of the following vector fields.

(a) $\vec{F}(x, y) = -y\vec{i} + x\vec{j}$ [Solution]

(b) $\vec{F}(x, y, z) = 2x\vec{i} - 2y\vec{j} - 2x\vec{k}$ [Solution]

Solution

(a) $\vec{F}(x, y) = -y\vec{i} + x\vec{j}$

Okay, to graph the vector field we need to get some "values" of the function. This means plugging in some points into the function. Here are a couple of evaluations.

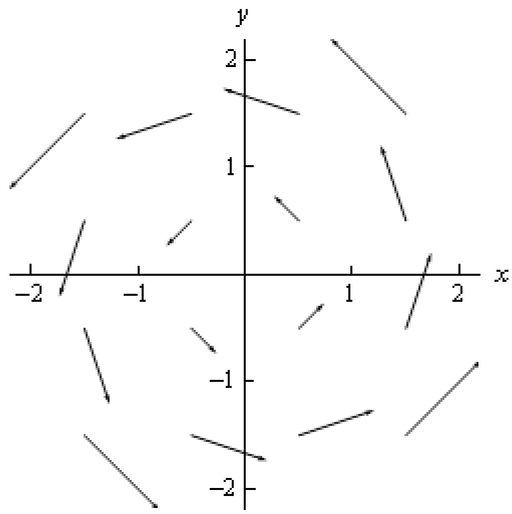
$$\vec{F}\left(\frac{1}{2}, \frac{1}{2}\right) = -\frac{1}{2}\vec{i} + \frac{1}{2}\vec{j}$$

$$\vec{F}\left(\frac{1}{2}, -\frac{1}{2}\right) = -\left(-\frac{1}{2}\right)\vec{i} + \frac{1}{2}\vec{j} = \frac{1}{2}\vec{i} + \frac{1}{2}\vec{j}$$

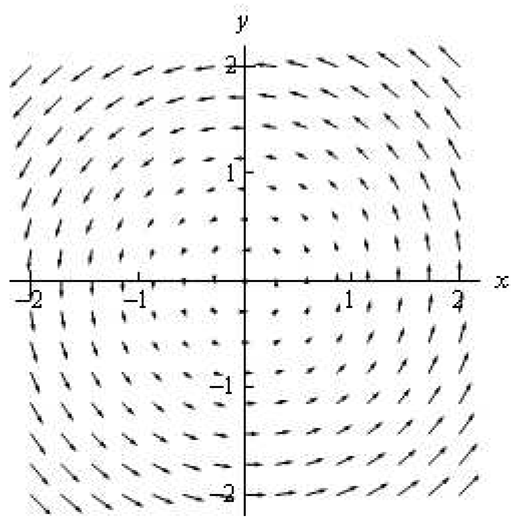
$$\vec{F}\left(\frac{3}{2}, \frac{1}{4}\right) = -\frac{1}{4}\vec{i} + \frac{3}{2}\vec{j}$$

So, just what do these evaluations tell us? Well the first one tells us that at the point $\left(\frac{1}{2}, \frac{1}{2}\right)$ we will plot the vector $-\frac{1}{2}\vec{i} + \frac{1}{2}\vec{j}$. Likewise, the third evaluation tells us that at the point $\left(\frac{3}{2}, \frac{1}{4}\right)$ we will plot the vector $-\frac{1}{4}\vec{i} + \frac{3}{2}\vec{j}$.

We can continue in this fashion plotting vectors for several points and we'll get the following sketch of the vector field.



If we want significantly more points plotted, then it is usually best to use a computer aided graphing system such as Maple or Mathematica. Here is a sketch with many more vectors included that was generated with Mathematica.



[Return to Problems]

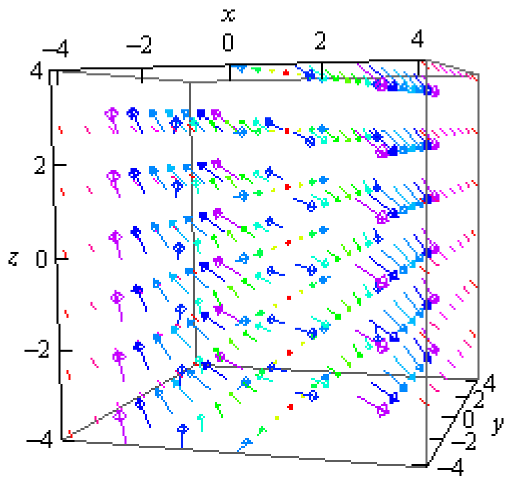
(b) $\vec{F}(x,y,z) = 2x\,\vec{i} - 2y\,\vec{j} - 2x\,\vec{k}$

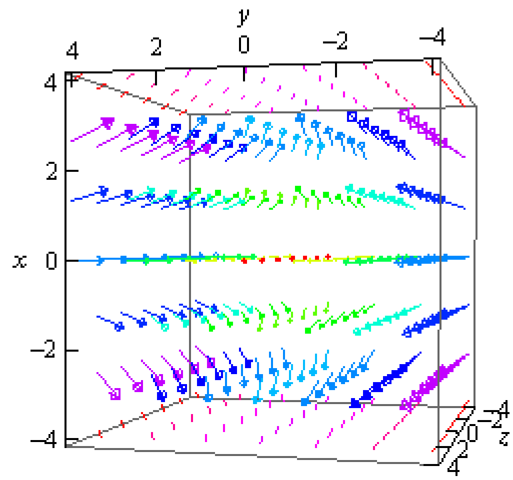
In the case of three dimensional vector fields it is almost always better to use Maple, Mathematica, or some other such tool. Despite that let's go ahead and do a couple of evaluations anyway.

$$\vec{F}(1,-3,2) = 2\,\vec{i} + 6\,\vec{j} - 2\,\vec{k}$$
$$\vec{F}(0,5,3) = -10\,\vec{j}$$

Notice that z only affect the placement of the vector in this case and does not affect the direction or the magnitude of the vector. Sometimes this will happen so don't get excited about it when it does.

Here is a couple of sketches generated by Mathematica. The sketch on the left is from the "front" and the sketch on the right is from "above".





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Now that we've seen a couple of vector fields let's notice that we've already seen a vector field function. In the second chapter we looked at the gradient vector. Recall that given a function $f(x,y,z)$ the gradient vector is defined by,

$$\nabla f = \left\langle f_x, f_y, f_z \right\rangle$$

This is a vector field and is often called a **gradient vector field**.

In these cases the function $f(x,y,z)$ is often called a scalar function to differentiate it from the vector field.

Example 2 Find the gradient vector field of the following functions.

(a) $f(x,y) = x^2 \sin(5y)$

(b) $f(x,y,z) = z\mathbf{e}^{-xy}$

Solution

(a) $f(x,y) = x^2 \sin(5y)$

Note that we only gave the gradient vector definition for a three dimensional function, but don't forget that there is also a two dimension definition. All that we need to drop off the third component of the vector.

Here is the gradient vector field for this function.

$$\nabla f = \left\langle 2x\sin(5y), 5x^2\cos(5y) \right\rangle$$

(b) $f(x, y, z) = z\mathbf{e}^{-xy}$

There isn't much to do here other than take the gradient.

$$\nabla f = \left\langle -yz\mathbf{e}^{-xy}, -xz\mathbf{e}^{-xy}, \mathbf{e}^{-xy} \right\rangle$$

Let's do another example that will illustrate the relationship between the gradient vector field of a function and its contours.

Example 3 Sketch the gradient vector field for $f(x, y) = x^2 + y^2$ as well as several contours for this function.

Solution

Recall that the contours for a function are nothing more than curves defined by,

$$f(x, y) = k$$

for various values of k. So, for our function the contours are defined by the equation,

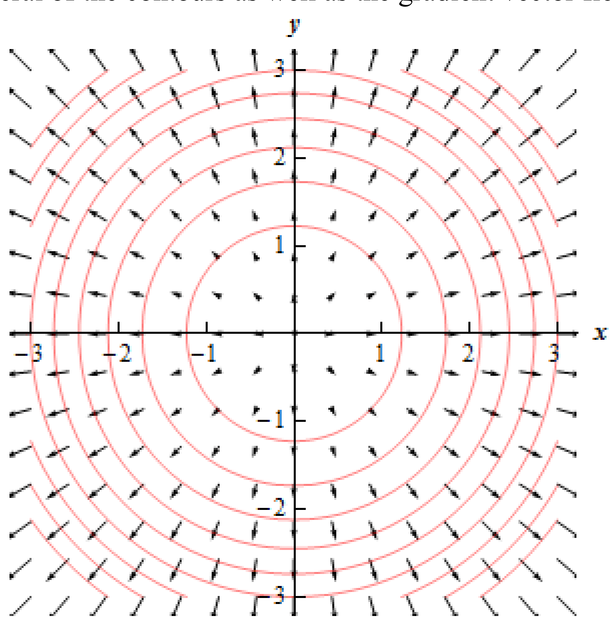
$$x^2 + y^2 = k$$

and so they are circles centered at the origin with radius $\sqrt{k}$.

Here is the gradient vector field for this function.

$$\nabla f(x, y) = 2x\,\vec{i} + 2y\,\vec{j}$$

Here is a sketch of several of the contours as well as the gradient vector field.



Notice that the vectors of the vector field are all perpendicular (or orthogonal) to the contours. This will always be the case when we are dealing with the contours of a function as well as its gradient vector field.

The k's we used for the graph above were 1.5, 3, 4.5, 6, 7.5, 9, 10.5, 12, and 13.5. Now notice that as we increased k by 1.5 the contour curves get closer together and that as the contour curves get closer together the larger the vectors become. In other words, the closer the contour curves are (as k is increased by a fixed amount) the faster the function is changing at that point. Also recall that the direction of fastest change for a function is given by the gradient vector at that point. Therefore, it should make sense that the two ideas should match up as they do here.

The final topic of this section is that of conservative vector fields. A vector field $\vec{F}$ is called a **conservative vector field** if there exists a function f such that $\vec{F} = \nabla f$. If $\vec{F}$ is a conservative vector field then the function, f, is called a **potential function** for $\vec{F}$.

All this definition is saying is that a vector field is conservative if it is also a gradient vector field for some function.

For instance the vector field $\vec{F} = y\,\vec{i} + x\,\vec{j}$ is a conservative vector field with a potential function of $f\left(x, y\right) = xy$ because $\nabla f = \left\langle y, x \right\rangle$.

On the other hand, $\vec{F} = -y\,\vec{i} + x\,\vec{j}$ is not a conservative vector field since there is no function f such that $\vec{F} = \nabla f$. If you're not sure that you believe this at this point be patient, we will be able to prove this in a couple of sections. In that section we will also show how to find the potential function for a conservative vector field.

Line Integrals – Part I

In this section we are now going to introduce a new kind of integral. However, before we do that it is important to note that you will need to remember how to parameterize equations, or put another way, you will need to be able to write down a set of parametric equations for a given curve. You should have seen some of this in your Calculus II course. If you need some review you should go back and review some of the basics of parametric equations and curves.

Here are some of the more basic curves that we'll need to know how to do as well as limits on the parameter if they are required.

Curve	**Parametric Equations**	
	Counter-Clockwise	Clockwise
$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ (Ellipse)	$x=a\cos(t)$ $y=b\sin(t)$ $0\le t\le 2\pi$	$x=a\cos(t)$ $y=-b\sin(t)$ $0\le t\le 2\pi$
	Counter-Clockwise	Clockwise
$x^2+y^2=r^2$ (Circle)	$x=r\cos(t)$ $y=r\sin(t)$ $0\le t\le 2\pi$	$x=r\cos(t)$ $y=-r\sin(t)$ $0\le t\le 2\pi$
$y=f(x)$	$x=t$ $y=f(t)$	
$x=g(y)$	$x=g(t)$ $y=t$	
Line Segment From (x_0,y_0,z_0) to (x_1,y_1,z_1)	$\vec r(t)=(1-t)\langle x_0,y_0,z_0\rangle+t\langle x_1,y_1,z_1\rangle \;,\; 0\le t\le 1$ or $x=(1-t)x_0+t\,x_1$ $y=(1-t)y_0+t\,y_1 \quad , \quad 0\le t\le 1$ $z=(1-t)z_0+t\,z_1$	

With the final one we gave both the vector form of the equation as well as the parametric form and if we need the two-dimensional version then we just drop the z components. In fact, we will be using the two-dimensional version of this in this section.

For the ellipse and the circle we've given two parameterizations, one tracing out the curve clockwise and the other counter-clockwise. As we'll eventually see the direction that the curve is traced out can, on occasion, change the answer. Also, both of these "start" on the positive x-axis at $t=0$.

Now let's move on to line integrals. In Calculus I we integrated $f(x)$, a function of a single variable, over an interval $[a,b]$. In this case we were thinking of x as taking all the values in this interval starting at a and ending at b. With line integrals we will start with integrating the function $f(x,y)$, a function of two variables, and the values of x and y that we're going to use will be the points, (x,y), that lie on a curve C. Note that this is different from the double integrals that we were working with in the previous chapter where the points came out of some two-dimensional region.

Let's start with the curve C that the points come from. We will assume that the curve is *smooth* (defined shortly) and is given by the parametric equations,

$$x = h(t) \qquad y = g(t) \qquad a \le t \le b$$

We will often want to write the parameterization of the curve as a vector function. In this case the curve is given by,

$$\vec{r}(t) = h(t)\vec{i} + g(t)\vec{j} \qquad a \le t \le b$$

The curve is called **smooth** if $\vec{r}'(t)$ is continuous and $\vec{r}'(t) \ne 0$ for all t.

The **line integral** of $f(x,y)$ along C is denoted by,

$$\int_C f(x,y)\,ds$$

We use a ds here to acknowledge the fact that we are moving along the curve, C, instead of the x-axis (denoted by dx) or the y-axis (denoted by dy). Because of the ds this is sometimes called the **line integral of *f* with respect to arc length**.

We've seen the notation ds before. If you recall from Calculus II when we looked at the arc length of a curve given by parametric equations we found it to be,

$$L = \int_a^b ds\,, \qquad \text{where} \quad ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

It is no coincidence that we use ds for both of these problems. The ds is the same for both the arc length integral and the notation for the line integral.

So, to compute a line integral we will convert everything over to the parametric equations. The line integral is then,

$$\int_C f(x,y)\,ds = \int_a^b f\big(h(t),g(t)\big)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Don't forget to plug the parametric equations into the function as well.

If we use the vector form of the parameterization we can simplify the notation up somewhat by noticing that,

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \left\| \vec{r}'(t) \right\|$$

where $\left\| \vec{r}'(t) \right\|$ is the magnitude or norm of $\vec{r}'(t)$. Using this notation the line integral becomes,

$$\int_C f(x,y)\,ds = \int_a^b f\left(h(t), g(t)\right) \left\| \vec{r}'(t) \right\| dt$$

Note that as long as the parameterization of the curve C is traced out exactly once as t increases from a to b the value of the line integral will be independent of the parameterization of the curve.

Let's take a look at an example of a line integral.

Example 1 Evaluate $\int_C xy^4\,ds$ where C is the right half of the circle, $x^2 + y^2 = 16$ rotated in the counter clockwise direction.

Solution
We first need a parameterization of the circle. This is given by,

$$x = 4\cos t \qquad y = 4\sin t$$

We now need a range of t's that will give the right half of the circle. The following range of t's will do this.

$$-\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

Now, we need the derivatives of the parametric equations and let's compute ds.

$$\frac{dx}{dt} = -4\sin t \qquad \frac{dy}{dt} = 4\cos t$$

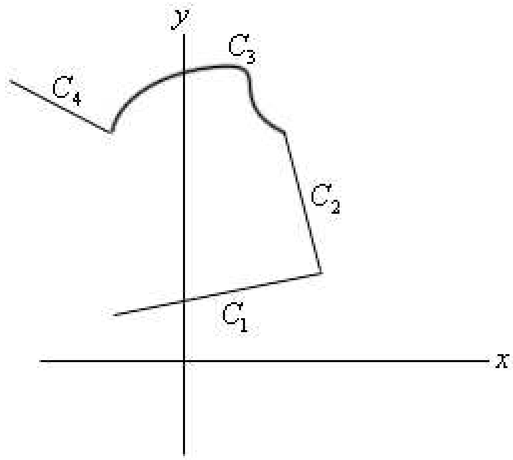
$$ds = \sqrt{16\sin^2 t + 16\cos^2 t}\,dt = 4\,dt$$

The line integral is then,

$$\begin{aligned}\int_C xy^4\,ds &= \int_{-\pi/2}^{\pi/2} 4\cos t\left(4\sin t\right)^4 (4)\,dt \\ &= 4096\int_{-\pi/2}^{\pi/2} \cos t\,\sin^4 t\,dt \\ &= \frac{4096}{5}\sin^5 t\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \frac{8192}{5}\end{aligned}$$

Next we need to talk about line integrals over **piecewise smooth curves**. A piecewise smooth curve is any curve that can be written as the union of a finite number of smooth curves, $C_1, \ldots, C_n$

where the end point of C_i is the starting point of C_{i+1}. Below is an illustration of a piecewise smooth curve.

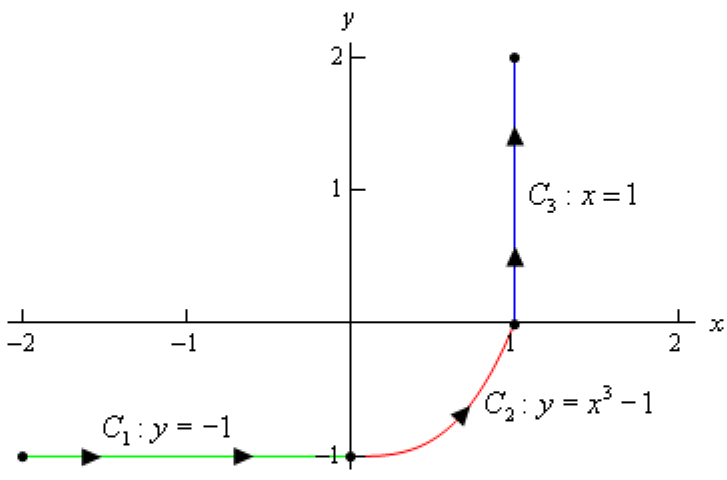


Evaluation of line integrals over piecewise smooth curves is a relatively simple thing to do. All we do is evaluate the line integral over each of the pieces and then add them up. The line integral for some function over the above piecewise curve would be,

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \int_{C_3} f(x,y)\,ds + \int_{C_4} f(x,y)\,ds$$

Let's see an example of this.

Example 2 Evaluate $\int\limits_C 4x^3\,ds$ where C is the curve shown below.



Solution

So, first we need to parameterize each of the curves.

$$\begin{aligned} C_1 &: \ x = t,\ y = -1, \qquad -2 \le t \le 0 \\ C_2 &: \ x = t,\ y = t^3 - 1, \qquad 0 \le t \le 1 \\ C_3 &: \ x = 1,\ y = t, \qquad 0 \le t \le 2 \end{aligned}$$

Now let's do the line integral over each of these curves.

$$\int_{C_1} 4x^3\,ds = \int_{-2}^{0} 4t^3\sqrt{(1)^2 + (0)^2}\,dt = \int_{-2}^{0} 4t^3\,dt = t^4\Big|_{-2}^{0} = -16$$

$$\begin{aligned} \int_{C_2} 4x^3\,ds &= \int_0^1 4t^3\sqrt{(1)^2 + \left(3t^2\right)^2}\,dt \\ &= \int_0^1 4t^3\sqrt{1+9t^4}\,dt \\ &= \frac{1}{9}\left(\frac{2}{3}\right)\left(1+9t^4\right)^{\frac{3}{2}}\Big|_0^1 = \frac{2}{27}\left(10^{\frac{3}{2}} - 1\right) = 2.268 \end{aligned}$$

$$\int_{C_3} 4x^3\,ds = \int_0^2 4(1)^3\sqrt{(0)^2 + (1)^2}\,dt = \int_0^2 4\,dt = 8$$

Finally, the line integral that we were asked to compute is,

$$\begin{aligned} \int_C 4x^3\,ds &= \int_{C_1} 4x^3\,ds + \int_{C_2} 4x^3\,ds + \int_{C_3} 4x^3\,ds \\ &= -16 + 2.268 + 8 \\ &= -5.732 \end{aligned}$$

Notice that we put direction arrows on the curve in the above example. The direction of motion along a curve *may* change the value of the line integral as we will see in the next section. Also note that the curve can be thought of a curve that takes us from the point $(-2,-1)$ to the point $(1,2)$. Let's first see what happens to the line integral if we change the path between these two points.

Example 3 Evaluate $\int_C 4x^3\,ds$ where C is the line segment from $(-2,-1)$ to $(1,2)$.

Solution

From the parameterization formulas at the start of this section we know that the line segment starting at $(-2,-1)$ and ending at $(1,2)$ is given by,

$$\begin{aligned} \vec r(t) &= (1-t)\langle -2,-1\rangle + t\langle 1,2\rangle \\ &= \langle -2+3t, -1+3t\rangle \end{aligned}$$

for $0 \le t \le 1$. This means that the individual parametric equations are,

$$x = -2 + 3t \qquad\qquad y = -1 + 3t$$

Using this path the line integral is,

$$\begin{aligned}\int_C 4x^3\,ds &= \int_0^1 4(-2+3t)^3\sqrt{9+9}\,dt \\ &= 12\sqrt{2}\left(\tfrac{1}{12}\right)(-2+3t)^4\Big|_0^1 \\ &= 12\sqrt{2}\left(-\frac{5}{4}\right) \\ &= -15\sqrt{2} = -21.213\end{aligned}$$

When doing these integrals don't forget simple Calc I substitutions to avoid having to do things like cubing out a term. Cubing it out is not that difficult, but it is more work than a simple substitution.

So, the previous two examples seem to suggest that if we change the path between two points then the value of the line integral (with respect to arc length) will change. While this will happen fairly regularly we can't assume that it will always happen. In a later section we will investigate this idea in more detail.

Next, let's see what happens if we change the direction of a path.

Example 4 Evaluate $\int_C 4x^3\,ds$ where C is the line segment from $(1,2)$ to $(-2,-1)$.

Solution

This one isn't much different, work wise, from the previous example. Here is the parameterization of the curve.

$$\begin{aligned}\vec{r}(t) &= (1-t)\langle 1,2\rangle + t\langle -2,-1\rangle \\ &= \langle 1-3t, 2-3t\rangle\end{aligned}$$

for $0 \le t \le 1$. Remember that we are switching the direction of the curve and this will also change the parameterization so we can make sure that we start/end at the proper point.

Here is the line integral.

$$\begin{aligned}\int_C 4x^3\,ds &= \int_0^1 4(1-3t)^3\sqrt{9+9}\,dt \\ &= 12\sqrt{2}\left(-\tfrac{1}{12}\right)(1-3t)^4\Big|_0^1 \\ &= 12\sqrt{2}\left(-\frac{5}{4}\right) \\ &= -15\sqrt{2} = -21.213\end{aligned}$$

So, it looks like when we switch the direction of the curve the line integral (with respect to arc length) will not change. This will always be true for these kinds of line integrals. However, there are other kinds of line integrals in which this won't be the case. We will see more examples of this in the next couple of sections so don't get it into your head that changing the direction will never change the value of the line integral.

Before working another example let's formalize this idea up somewhat. Let's suppose that the curve C has the parameterization $x = h(t)$, $y = g(t)$. Let's also suppose that the initial point on the curve is A and the final point on the curve is B. The parameterization $x = h(t)$, $y = g(t)$ will then determine an **orientation** for the curve where the positive direction is the direction that is traced out as t increases. Finally, let $-C$ be the curve with the same points as C, however in this case the curve has B as the initial point and A as the final point, again t is increasing as we traverse this curve. In other words, given a curve C, the curve $-C$ is the same curve as C except the direction has been reversed.

We then have the following fact about line integrals with respect to arc length.

Fact

$$\int_C f(x,y)\,ds = \int_{-C} f(x,y)\,ds$$

So, for a line integral with respect to arc length we can change the direction of the curve and not change the value of the integral. This is a useful fact to remember as some line integrals will be easier in one direction than the other.

Now, let's work another example

Example 5 Evaluate $\int_C x\,ds$ for each of the following curves.

(a) $C_1 : y = x^2$, $-1 \le x \le 1$ [Solution]

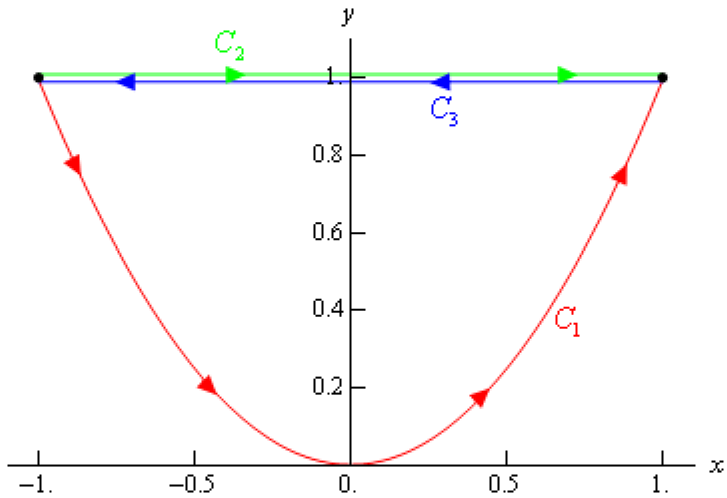
(b) C_2 : The line segment from $(-1,1)$ to $(1,1)$. [Solution]

(c) C_3 : The line segment from $(1,1)$ to $(-1,1)$. [Solution]

Solution

Before working any of these line integrals let's notice that all of these curves are paths that connect the points $(-1,1)$ and $(1,1)$. Also notice that $C_3 = -C_2$ and so by the fact above these two should give the same answer.

Here is a sketch of the three curves and note that the curves illustrating C_2 and C_3 have been separated a little to show that they are separate curves in some way even though they are the same line.



(a) $C_1 : y = x^2,\ -1 \le x \le 1$

Here is a parameterization for this curve.

$$C_1 : x = t,\ y = t^2,\ -1 \le t \le 1$$

Here is the line integral.

$$\int_{C_1} x\,ds = \int_{-1}^{1} t\sqrt{1+4t^2}\,dt = \frac{1}{12}\left(1+4t^2\right)^{\frac{3}{2}}\Big|_{-1}^{1} = 0$$

[Return to Problems]

(b) C_2 **: The line segment from** $(-1,1)$ **to** $(1,1)$**.**

There are two parameterizations that we could use here for this curve. The first is to use the formula we used in the previous couple of examples. That parameterization is,

$$\begin{aligned} C_2 : \vec{r}(t) &= (1-t)\langle -1,1\rangle + t\langle 1,1\rangle \\ &= \langle 2t-1,1\rangle \end{aligned}$$

for $0 \le t \le 1$.

Sometimes we have no choice but to use this parameterization. However, in this case there is a second (probably) easier parameterization. The second one uses the fact that we are really just graphing a portion of the line $y = 1$. Using this the parameterization is,

$$C_2 : x = t,\ y = 1,\ -1 \le t \le 1$$

This will be a much easier parameterization to use so we will use this. Here is the line integral for this curve.

$$\int_{C_2} x\,ds = \int_{-1}^{1} t\sqrt{1+0}\,dt = \frac{1}{2}t^2\Big|_{-1}^{1} = 0$$

Note that this time, unlike the line integral we worked with in Examples 2, 3, and 4 we got the same value for the integral despite the fact that the path is different. This will happen on occasion. We should also not expect this integral to be the same for all paths between these two points. At this point all we know is that for these two paths the line integral will have the same value. It is completely possible that there is another path between these two points that will give a different value for the line integral.

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(c) C_3 : The line segment from $(1,1)$ to $(-1,1)$.

Now, according to our fact above we really don't need to do anything here since we know that $C_3 = -C_2$. The fact tells us that this line integral should be the same as the second part (*i.e.* zero). However, let's verify that, plus there is a point we need to make here about the parameterization.

Here is the parameterization for this curve.

$$\begin{aligned} C_3 : \vec{r}(t) &= (1-t)\langle 1,1\rangle + t\langle -1,1\rangle \\ &= \langle 1-2t, 1\rangle \end{aligned}$$

for $0 \le t \le 1$.

Note that this time we can't use the second parameterization that we used in part (b) since we need to move from right to left as the parameter increases and the second parameterization used in the previous part will move in the opposite direction.

Here is the line integral for this curve.

$$\int\limits_{C_3} x\,ds = \int_0^1 (1-2t)\sqrt{4+0}\,dt = 2\left(t-t^2\right)\Big|_0^1 = 0$$

Sure enough we got the same answer as the second part.

[Return to Problems]

To this point in this section we've only looked at line integrals over a two-dimensional curve. However, there is no reason to restrict ourselves like that. We can do line integrals over three-dimensional curves as well.

Let's suppose that the three-dimensional curve C is given by the parameterization,

$$x = x(t), \qquad y = y(t) \qquad z = z(t) \qquad a \le t \le b$$

then the line integral is given by,

$$\int\limits_C f(x,y,z)\,ds = \int_a^b f\left(x(t),y(t),z(t)\right)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

Note that often when dealing with three-dimensional space the parameterization will be given as a vector function.

$$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$$

Notice that we changed up the notation for the parameterization a little. Since we rarely use the function names we simply kept the x, y, and z and added on the (t) part to denote that they may be functions of the parameter.

Also notice that, as with two-dimensional curves, we have,

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}=\left\|\vec{r}'(t)\right\|$$

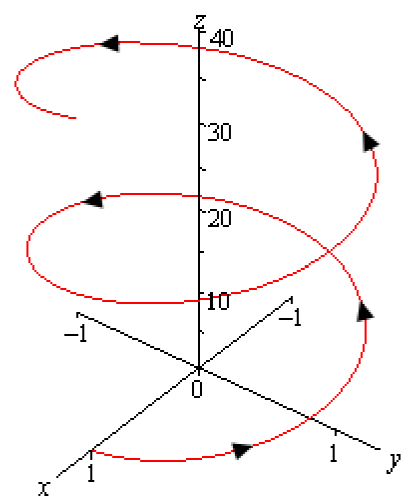
and the line integral can again be written as,

$$\int_C f(x,y,z)\,ds=\int_a^b f\left(x(t),y(t),z(t)\right)\left\|\vec{r}'(t)\right\|dt$$

So, outside of the addition of a third parametric equation line integrals in three-dimensional space work the same as those in two-dimensional space. Let's work a quick example.

Example 6 Evaluate $\int_C xyz\,ds$ where C is the helix given by, $\vec{r}(t)=\left\langle\cos(t),\sin(t),3t\right\rangle$, $0\le t\le 4\pi$.

Solution
Note that we first saw the vector equation for a helix back in the Vector Functions section. Here is a quick sketch of the helix.



Here is the line integral.

$$\int_C xyz\, ds = \int_0^{4\pi} 3t\cos(t)\sin(t)\sqrt{\sin^2 t + \cos^2 t + 9}\, dt$$
$$= \int_0^{4\pi} 3t\left(\frac{1}{2}\sin(2t)\right)\sqrt{1+9}\, dt$$
$$= \frac{3\sqrt{10}}{2}\int_0^{4\pi} t\sin(2t)\, dt$$
$$= \frac{3\sqrt{10}}{2}\left(\frac{1}{4}\sin(2t) - \frac{t}{2}\cos(2t)\right)\Bigg|_0^{4\pi}$$
$$= -3\sqrt{10}\,\pi$$

You were able to do that integral right? It required integration by parts.

So, as we can see there really isn't too much difference between two- and three-dimensional line integrals.

Line Integrals – Part II

In the previous section we looked at line integrals with respect to arc length. In this section we want to look at line integrals with respect to x and/or y.

As with the last section we will start with a two-dimensional curve C with parameterization,

$$x = x(t) \qquad y = y(t) \qquad a \le t \le b$$

The **line integral of *f* with respect to *x*** is,

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$$

The **line integral of *f* with respect to *y*** is,

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$$

Note that the only notational difference between these two and the line integral with respect to arc length (from the previous section) is the differential. These have a dx or dy while the line integral with respect to arc length has a ds. So when evaluating line integrals be careful to first note which differential you've got so you don't work the wrong kind of line integral.

These two integral often appear together and so we have the following shorthand notation for these cases.

$$\int_C P\,dx + Q\,dy = \int_C P(x,y)\,dx + \int_C Q(x,y)\,dy$$

Let's take a quick look at an example of this kind of line integral.

Example 1 Evaluate $\int_C \sin(\pi y)\,dy + yx^2\,dx$ where C is the line segment from $(0,2)$ to $(1,4)$.

Solution
Here is the parameterization of the curve.

$$\vec{r}(t) = (1-t)\langle 0,2\rangle + t\langle 1,4\rangle = \langle t, 2+2t\rangle \qquad 0 \le t \le 1$$

The line integral is,

$$\begin{aligned}\int_C \sin(\pi y)\,dy + yx^2\,dx &= \int_C \sin(\pi y)\,dy + \int_C yx^2\,dx \\ &= \int_0^1 \sin(\pi(2+2t))(2)\,dt + \int_0^1 (2+2t)(t)^2(1)\,dt \\ &= -\frac{1}{\pi}\cos(2\pi + 2\pi t)\Big|_0^1 + \left(\frac{2}{3}t^3 + \frac{1}{2}t^4\right)\Big|_0^1 \\ &= \frac{7}{6}\end{aligned}$$

In the previous section we saw that changing the direction of the curve for a line integral with respect to arc length doesn't change the value of the integral. Let's see what happens with line integrals with respect to x and/or y.

Example 2 Evaluate $\int\limits_C \sin(\pi y)\,dy + yx^2\,dx$ where C is the line segment from $(1,4)$ to $(0,2)$.

Solution
So, we simply changed the direction of the curve. Here is the new parameterization.

$$\vec{r}(t) = (1-t)\langle 1,4\rangle + t\langle 0,2\rangle = \langle 1-t, 4-2t\rangle \qquad 0 \le t \le 1$$

The line integral in this case is,

$$\begin{aligned}\int\limits_C \sin(\pi y)\,dy + yx^2\,dx &= \int\limits_C \sin(\pi y)\,dy + \int\limits_C yx^2\,dx\\ &= \int_0^1 \sin(\pi(4-2t))(-2)\,dt + \int_0^1 (4-2t)(1-t)^2(-1)\,dt\\ &= -\frac{1}{\pi}\cos(4\pi - 2\pi t)\Big|_0^1 - \left(-\frac{1}{2}t^4 + \frac{8}{3}t^3 - 5t^2 + 4t\right)\Bigg|_0^1\\ &= -\frac{7}{6}\end{aligned}$$

So, switching the direction of the curve got us a different value or at least the opposite sign of the value from the first example. In fact this will always happen with these kinds of line integrals.

Fact
If C is any curve then,

$$\int\limits_{-C} f(x,y)\,dx = -\int\limits_C f(x,y)\,dx \qquad \text{and} \qquad \int\limits_{-C} f(x,y)\,dy = -\int\limits_C f(x,y)\,dy$$

With the combined form of these two integrals we get,

$$\int\limits_{-C} P\,dx + Q\,dy = -\int\limits_C P\,dx + Q\,dy$$

We can also do these integrals over three-dimensional curves as well. In this case we will pick up a third integral (with respect to z) and the three integrals will be.

$$\int\limits_C f(x,y,z)\,dx = \int_a^b f(x(t),y(t),z(t))\,x'(t)\,dt$$

$$\int\limits_C f(x,y,z)\,dy = \int_a^b f(x(t),y(t),z(t))\,y'(t)\,dt$$

$$\int\limits_C f(x,y,z)\,dz = \int_a^b f(x(t),y(t),z(t))\,z'(t)\,dt$$

where the curve C is parameterized by

$$x = x(t) \qquad y = y(t) \qquad z = z(t) \qquad a \le t \le b$$

As with the two-dimensional version these three will often occur together so the shorthand we'll be using here is,

$$\int_C P\,dx + Q\,dy + R\,dz = \int_C P(x,y,z)\,dx + \int_C Q(x,y,z)\,dy + \int_C R(x,y,z)\,dz$$

Let's work an example.

Example 3 Evaluate $\int_C y\,dx + x\,dy + z\,dz$ where C is given by $x = \cos t$, $y = \sin t$, $z = t^2$, $0 \le t \le 2\pi$.

Solution

So, we already have the curve parameterized so there really isn't much to do other than evaluate the integral.

$$\begin{aligned}
\int_C y\,dx + x\,dy + z\,dz &= \int_C y\,dx + \int_C x\,dy + \int_C z\,dz \\
&= \int_0^{2\pi} \sin t(-\sin t)\,dt + \int_0^{2\pi} \cos t(\cos t)\,dt + \int_0^{2\pi} t^2(2t)\,dt \\
&= -\int_0^{2\pi} \sin^2 t\,dt + \int_0^{2\pi} \cos^2 t\,dt + \int_0^{2\pi} 2t^3\,dt \\
&= -\frac{1}{2}\int_0^{2\pi} \left(1-\cos(2t)\right)dt + \frac{1}{2}\int_0^{2\pi} \left(1+\cos(2t)\right)dt + \int_0^{2\pi} 2t^3\,dt \\
&= \left.\left(-\frac{1}{2}\left(t-\frac{1}{2}\sin(2t)\right) + \frac{1}{2}\left(t+\frac{1}{2}\sin(2t)\right) + \frac{1}{2}t^4\right)\right|_0^{2\pi} \\
&= 8\pi^4
\end{aligned}$$

Line Integrals of Vector Fields

In the previous two sections we looked at line integrals of functions. In this section we are going to evaluate line integrals of vector fields. We'll start with the vector field,

$$\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$$

and the three-dimensional, smooth curve given by

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} \qquad a \le t \le b$$

The **line integral of $\vec{F}$ along C** is

$$\int_C \vec{F} \bullet d\,\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \bullet \vec{r}\,'(t)\,dt$$

Note the notation in the left side. That really is a dot product of the vector field and the differential really is a vector. Also, $\vec{F}(\vec{r}(t))$ is a shorthand for,

$$\vec{F}(\vec{r}(t)) = \vec{F}(x(t), y(t), z(t))$$

We can also write line integrals of vector fields as a line integral with respect to arc length as follows,

$$\int_C \vec{F} \bullet d\,\vec{r} = \int_C \vec{F} \bullet \vec{T}\,ds$$

where $\vec{T}(t)$ is the unit tangent vector and is given by,

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\left\|\vec{r}\,'(t)\right\|}$$

If we use our knowledge on how to compute line integrals with respect to arc length we can see that this second form is equivalent to the first form given above.

$$\begin{aligned} \int_C \vec{F} \bullet d\,\vec{r} &= \int_C \vec{F} \bullet \vec{T}\,ds \\ &= \int_a^b \vec{F}(\vec{r}(t)) \bullet \frac{\vec{r}\,'(t)}{\left\|\vec{r}\,'(t)\right\|} \left\|\vec{r}\,'(t)\right\| dt \\ &= \int_a^b \vec{F}(\vec{r}(t)) \bullet \vec{r}\,'(t)\,dt \end{aligned}$$

In general we use the first form to compute these line integral as it is usually much easier to use. Let's take a look at a couple of examples.

Example 1 Evaluate $\int_C \vec{F} \bullet d\,\vec{r}$ where $\vec{F}(x,y,z) = 8x^2 yz\,\vec{i} + 5z\,\vec{j} - 4xy\,\vec{k}$ and C is the curve given by $\vec{r}(t) = t\,\vec{i} + t^2\,\vec{j} + t^3\,\vec{k}$, $0 \le t \le 1$.

Solution
Okay, we first need the vector field evaluated along the curve.

$$\vec{F}(\vec{r}(t)) = 8t^2(t^2)(t^3)\vec{i} + 5t^3\,\vec{j} - 4t(t^2)\vec{k} = 8t^7\,\vec{i} + 5t^3\,\vec{j} - 4t^3\,\vec{k}$$

Next we need the derivative of the parameterization.

$$\vec{r}'(t) = \vec{i} + 2t\,\vec{j} + 3t^2\,\vec{k}$$

Finally, let's get the dot product taken care of.

$$\vec{F}(\vec{r}(t)) \bullet \vec{r}'(t) = 8t^7 + 10t^4 - 12t^5$$

The line integral is then,

$$\begin{aligned} \int_C \vec{F} \bullet d\,\vec{r} &= \int_0^1 8t^7 + 10t^4 - 12t^5\,dt \\ &= \left(t^8 + 2t^5 - 2t^6\right)\Big|_0^1 \\ &= 1 \end{aligned}$$

Example 2 Evaluate $\int_C \vec{F} \bullet d\,\vec{r}$ where $\vec{F}(x,y,z) = xz\,\vec{i} - yz\,\vec{k}$ and C is the line segment from $(-1,2,0)$ and $(3,0,1)$.

Solution
We'll first need the parameterization of the line segment. We saw how to get the parameterization of line segments in the first section on line integrals. We've been using the two dimensional version of this over the last couple of sections. Here is the parameterization for the line.

$$\begin{aligned} \vec{r}(t) &= (1-t)\langle -1,2,0\rangle + t\langle 3,0,1\rangle \\ &= \langle 4t-1, 2-2t, t\rangle, \qquad 0 \le t \le 1 \end{aligned}$$

So, let's get the vector field evaluated along the curve.

$$\begin{aligned} \vec{F}(\vec{r}(t)) &= (4t-1)(t)\vec{i} - (2-2t)(t)\vec{k} \\ &= \left(4t^2 - t\right)\vec{i} - \left(2t - 2t^2\right)\vec{k} \end{aligned}$$

Now we need the derivative of the parameterization.

$$\vec{r}'(t) = \langle 4,-2,1\rangle$$

The dot product is then,

$$\vec{F}\left(\vec{r}\left(t\right)\right)\bullet\vec{r}'\left(t\right)=4\left(4t^2-t\right)-\left(2t-2t^2\right)=18t^2-6t$$

The line integral becomes,

$$\begin{aligned}\int\limits_C \vec{F}\bullet d\,\vec{r} &= \int_0^1 18t^2-6t\,dt\\ &=\left(6t^3-3t^2\right)\Big|_0^1\\ &=3\end{aligned}$$

Let's close this section out by doing one of these in general to get a nice relationship between line integrals of vector fields and line integrals with respect to x, y, and z.

Given the vector field $\vec{F}\left(x,y,z\right)=P\,\vec{i}+Q\,\vec{j}+R\,\vec{k}$ and the curve C parameterized by $\vec{r}\left(t\right)=x\left(t\right)\vec{i}+y\left(t\right)\vec{j}+z\left(t\right)\vec{k}$, $a\le t\le b$ the line integral is,

$$\begin{aligned}\int\limits_C \vec{F}\bullet d\,\vec{r} &= \int_a^b\left(P\,\vec{i}+Q\,\vec{j}+R\,\vec{k}\right)\bullet\left(x'\,\vec{i}+y'\,\vec{j}+z'\,\vec{k}\right)dt\\ &=\int_a^b Px'+Qy'+Rz'\,dt\\ &=\int_a^b Px'\,dt+\int_a^b Qy'\,dt+\int_a^b Rz'\,dt\\ &=\int\limits_C P\,dx+\int\limits_C Q\,dy+\int\limits_C R\,dz\\ &=\int\limits_C P\,dx+Q\,dy+R\,dz\end{aligned}$$

So, we see that,

$$\int\limits_C \vec{F}\bullet d\,\vec{r}=\int\limits_C P\,dx+Q\,dy+R\,dz$$

Note that this gives us another method for evaluating line integrals of vector fields.

This also allows us to say the following about reversing the direction of the path with line integrals of vector fields.

Fact

$$\int\limits_{-C} \vec{F}\bullet d\,\vec{r}=-\int\limits_C \vec{F}\bullet d\,\vec{r}$$

This should make some sense given that we know that this is true for line integrals with respect to x, y, and/or z and that line integrals of vector fields can be defined in terms of line integrals with respect to x, y, and z.

Fundamental Theorem for Line Integrals

In Calculus I we had the Fundamental Theorem of Calculus that told us how to evaluate definite integrals. This told us,

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

It turns out that there is a version of this for line integrals over certain kinds of vector fields. Here it is.

Theorem

Suppose that C is a smooth curve given by $\vec{r}(t)$, $a \le t \le b$. Also suppose that f is a function whose gradient vector, ∇f, is continuous on C. Then,

$$\int_C \nabla f \bullet d\,\vec{r} = f\left(\vec{r}(b)\right) - f\left(\vec{r}(a)\right)$$

Note that $\vec{r}(a)$ represents the initial point on C while $\vec{r}(b)$ represents the final point on C. Also, we did not specify the number of variables for the function since it is really immaterial to the theorem. The theorem will hold regardless of the number of variables in the function.

Proof

This is a fairly straight forward proof.

For the purposes of the proof we'll assume that we're working in three dimensions, but it can be done in any dimension.

Let's start by just computing the line integral.

$$\begin{aligned}\int_C \nabla f \bullet d\,\vec{r} &= \int_a^b \nabla f\left(\vec{r}(t)\right)\bullet\vec{r}\,'(t)\,dt \\ &= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right)dt\end{aligned}$$

Now, at this point we can use the Chain Rule to simplify the integrand as follows,

$$\begin{aligned}\int_C \nabla f \bullet d\,\vec{r} &= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right)dt \\ &= \int_a^b \frac{d}{dt}\left[f\left(\vec{r}(t)\right)\right]dt\end{aligned}$$

To finish this off we just need to use the Fundamental Theorem of Calculus for single integrals.

$$\int_C \nabla f \bullet d\,\vec{r} = f\left(\vec{r}(b)\right) - f\left(\vec{r}(a)\right)$$

■

Let's take a quick look at an example of using this theorem.

Example 1 Evaluate $\int_C \nabla f \bullet d\,\vec{r}$ where $f(x,y,z) = \cos(\pi x) + \sin(\pi y) - xyz$ and C is *any* path that starts at $\left(1, \frac{1}{2}, 2\right)$ and ends at $(2,1,-1)$.

Solution
First let's notice that we didn't specify the path for getting from the first point to the second point. The reason for this is simple. The theorem above tells us that all we need are the initial and final points on the curve in order to evaluate this kind of line integral.

So, let $\vec{r}(t)$, $a \le t \le b$ be any path that starts at $\left(1, \frac{1}{2}, 2\right)$ and ends at $(2,1,-1)$. Then,

$$\vec{r}(a) = \left\langle 1, \frac{1}{2}, 2 \right\rangle \qquad\qquad \vec{r}(b) = \langle 2,1,-1 \rangle$$

The integral is then,

$$\begin{aligned}\int_C \nabla f \bullet d\,\vec{r} &= f(2,1,-1) - f\left(1, \frac{1}{2}, 2\right) \\ &= \cos(2\pi) + \sin\pi - 2(1)(-1) - \left(\cos\pi + \sin\left(\frac{\pi}{2}\right) - 1\left(\frac{1}{2}\right)(2)\right) \\ &= 4\end{aligned}$$

Notice that we also didn't need the gradient vector to actually do this line integral. However, for the practice of finding gradient vectors here it is,

$$\nabla f = \left\langle -\pi\sin(\pi x) - yz, \pi\cos(\pi y) - xz, -xy \right\rangle$$

The most important idea to get from this example is not how to do the integral as that's pretty simple, all we do is plug the final point and initial point into the function and subtract the two results. The important idea from this example (and hence about the Fundamental Theorem of Calculus) is that, for these kinds of line integrals, we didn't really need to know the path to get the answer. In other words, we could use any path we want and we'll always get the same results.

In the first section on line integrals (even though we weren't looking at vector fields) we saw that often when we change the path we will change the value of the line integral. We now have a type of line integral for which we know that changing the path will NOT change the value of the line integral.

Let's formalize this idea up a little. Here are some definitions. The first one we've already seen before, but it's been a while and it's important in this section so we'll give it again. The remaining definitions are new.

Definitions

First suppose that $\vec{F}$ is a continuous vector field in some domain *D*.

1. $\vec{F}$ is a **conservative** vector field if there is a function *f* such that $\vec{F} = \nabla f$. The function *f* is called a **potential function** for the vector field. We first saw this definition in the first section of this chapter.

2. $\int_C \vec{F} \bullet d\,\vec{r}$ is **independent of path** if $\int_{C_1} \vec{F} \bullet d\,\vec{r} = \int_{C_2} \vec{F} \bullet d\,\vec{r}$ for any two paths C_1 and C_2 in D with the same initial and final points.

3. A path C is called **closed** if its initial and final points are the same point. For example a circle is a closed path.

4. A path C is **simple** if it doesn't cross itself. A circle is a simple curve while a figure 8 type curve is not simple.

5. A region D is **open** if it doesn't contain any of its boundary points.

6. A region D is **connected** if we can connect any two points in the region with a path that lies completely in D.

7. A region D is **simply-connected** if it is connected and it contains no holes. We won't need this one until the next section, but it fits in with all the other definitions given here so this was a natural place to put the definition.

With these definitions we can now give some nice facts.

Facts

1. $\int_C \nabla f \bullet d\,\vec{r}$ is independent of path.

   This is easy enough to prove since all we need to do is look at the theorem above. The theorem tells us that in order to evaluate this integral all we need are the initial and final points of the curve. This in turn tells us that the line integral must be independent of path.

2. If $\vec{F}$ is a conservative vector field then $\int_C \vec{F} \bullet d\,\vec{r}$ is independent of path.

   This fact is also easy enough to prove. If $\vec{F}$ is conservative then it has a potential function, f, and so the line integral becomes $\int_C \vec{F} \bullet d\,\vec{r} = \int_C \nabla f \bullet d\,\vec{r}$. Then using the first fact we know that this line integral must be independent of path.

3. If $\vec{F}$ is a continuous vector field on an open connected region D and if $\int_C \vec{F} \bullet d\,\vec{r}$ is independent of path (for any path in D) then $\vec{F}$ is a conservative vector field on D.

4. If $\int_C \vec{F} \bullet d\,\vec{r}$ is independent of path then $\int_C \vec{F} \bullet d\,\vec{r} = 0$ for every closed path C.

5. If $\int_C \vec{F} \bullet d\,\vec{r} = 0$ for every closed path C then $\int_C \vec{F} \bullet d\,\vec{r}$ is independent of path.

These are some nice facts to remember as we work with line integrals over vector fields. Also notice that 2 & 3 and 4 & 5 are converses of each other.

Conservative Vector Fields

In the previous section we saw that if we knew that the vector field $\vec{F}$ was conservative then $\int_C \vec{F} \bullet d\,\vec{r}$ was independent of path. This in turn means that we can easily evaluate this line integral provided we can find a potential function for $\vec{F}$.

In this section we want to look at two questions. First, given a vector field $\vec{F}$ is there any way of determining if it is a conservative vector field? Secondly, if we know that $\vec{F}$ is a conservative vector field how do we go about finding a potential function for the vector field?

The first question is easy to answer at this point if we have a two-dimensional vector field. For higher dimensional vector fields we'll need to wait until the final section in this chapter to answer this question. With that being said let's see how we do it for two-dimensional vector fields.

Theorem

Let $\vec{F} = P\,\vec{i} + Q\,\vec{j}$ be a vector field on an open and simply-connected region *D*. Then if *P* and *Q* have continuous first order partial derivatives in *D* and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

the vector field $\vec{F}$ is conservative.

Let's take a look at a couple of examples.

Example 1 Determine if the following vector fields are conservative or not.

(a) $\vec{F}(x,y) = \left(x^2 - yx\right)\vec{i} + \left(y^2 - xy\right)\vec{j}$ [Solution]

(b) $\vec{F}(x,y) = \left(2x\mathbf{e}^{xy} + x^2 y\mathbf{e}^{xy}\right)\vec{i} + \left(x^3\mathbf{e}^{xy} + 2y\right)\vec{j}$ [Solution]

Solution

Okay, there really isn't too much to these. All we do is identify *P* and *Q* then take a couple of derivatives and compare the results.

(a) $\vec{F}(x,y) = \left(x^2 - yx\right)\vec{i} + \left(y^2 - xy\right)\vec{j}$

In this case here is *P* and *Q* and the appropriate partial derivatives.

$$P = x^2 - yx \qquad \frac{\partial P}{\partial y} = -x$$

$$Q = y^2 - xy \qquad \frac{\partial Q}{\partial x} = -y$$

So, since the two partial derivatives are not the same this vector field is NOT conservative.

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(b) $\vec{F}(x,y) = \left(2x\mathbf{e}^{xy} + x^2y\mathbf{e}^{xy}\right)\vec{i} + \left(x^3\mathbf{e}^{xy} + 2y\right)\vec{j}$

Here is P and Q as well as the appropriate derivatives.

$$P = 2x\mathbf{e}^{xy} + x^2y\mathbf{e}^{xy} \qquad \frac{\partial P}{\partial y} = 2x^2\mathbf{e}^{xy} + x^2\mathbf{e}^{xy} + x^3y\mathbf{e}^{xy} = 3x^2\mathbf{e}^{xy} + x^3y\mathbf{e}^{xy}$$

$$Q = x^3\mathbf{e}^{xy} + 2y \qquad \frac{\partial Q}{\partial x} = 3x^2\mathbf{e}^{xy} + x^3y\mathbf{e}^{xy}$$

The two partial derivatives are equal and so this is a conservative vector field.

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Now that we know how to identify if a two-dimensional vector field is conservative we need to address how to find a potential function for the vector field. This is actually a fairly simple process. First, let's assume that the vector field is conservative and so we know that a potential function, $f(x,y)$ exists. We can then say that,

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = P\,\vec{i} + Q\,\vec{j} = \vec{F}$$

Or by setting components equal we have,

$$\frac{\partial f}{\partial x} = P \qquad \text{and} \qquad \frac{\partial f}{\partial y} = Q$$

By integrating each of these with respect to the appropriate variable we can arrive at the following two equations.

$$f(x,y) = \int P(x,y)\,dx \qquad \text{or} \qquad f(x,y) = \int Q(x,y)\,dy$$

We saw this kind of integral briefly at the end of the section on iterated integrals in the previous chapter.

It is usually best to see how we use these two facts to find a potential function in an example or two.

Example 2 Determine if the following vector fields are conservative and find a potential function for the vector field if it is conservative.

(a) $\vec{F} = \left(2x^3y^4 + x\right)\vec{i} + \left(2x^4y^3 + y\right)\vec{j}$ [Solution]

(b) $\vec{F}\left(x,y\right) = \left(2x\mathbf{e}^{xy} + x^2y\mathbf{e}^{xy}\right)\vec{i} + \left(x^3\mathbf{e}^{xy} + 2y\right)\vec{j}$ [Solution]

Solution

(a) $\vec{F} = \left(2x^3y^4 + x\right)\vec{i} + \left(2x^4y^3 + y\right)\vec{j}$

Let's first identify P and Q and then check that the vector field is conservative..

$$P = 2x^3y^4 + x \qquad \frac{\partial P}{\partial y} = 8x^3y^3$$

$$Q = 2x^4y^3 + y \qquad \frac{\partial Q}{\partial x} = 8x^3y^3$$

So, the vector field is conservative. Now let's find the potential function. From the first fact above we know that,

$$\frac{\partial f}{\partial x} = 2x^3y^4 + x \qquad \frac{\partial f}{\partial y} = 2x^4y^3 + y$$

From these we can see that

$$f\left(x,y\right) = \int 2x^3y^4 + x\,dx \qquad \text{or} \qquad f\left(x,y\right) = \int 2x^4y^3 + y\,dy$$

We can use either of these to get the process started. Recall that we are going to have to be careful with the "constant of integration" which ever integral we choose to use. For this example let's work with the first integral and so that means that we are asking what function did we differentiate with respect to x to get the integrand. This means that the "constant of integration" is going to have to be a function of y since any function consisting only of y and/or constants will differentiate to zero when taking the partial derivative with respect to x.

Here is the first integral.

$$\begin{aligned} f\left(x,y\right) &= \int 2x^3y^4 + x\,dx \\ &= \frac{1}{2}x^4y^4 + \frac{1}{2}x^2 + h\left(y\right) \end{aligned}$$

where $h\left(y\right)$ is the "constant of integration".

We now need to determine $h\left(y\right)$. This is easier that it might at first appear to be. To get to this point we've used the fact that we knew P, but we will also need to use the fact that we know Q to complete the problem. Recall that Q is really the derivative of f with respect to y. So, if we differentiate our function with respect to y we know what it should be.

So, let's differentiate f (including the $h\left(y\right)$) with respect to y and set it equal to Q since that is what the derivative is supposed to be.

$$\frac{\partial f}{\partial y} = 2x^4y^3 + h'(y) = 2x^4y^3 + y = Q$$

From this we can see that,

$$h'(y) = y$$

Notice that since $h'(y)$ is a function only of y so if there are any x's in the equation at this point we will know that we've made a mistake. At this point finding $h(y)$ is simple.

$$h(y) = \int h'(y)\,dy = \int y\,dy = \frac{1}{2}y^2 + c$$

So, putting this all together we can see that a potential function for the vector field is,

$$f(x,y) = \frac{1}{2}x^4y^4 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + c$$

Note that we can always check our work by verifying that $\nabla f = \vec{F}$. Also note that because the c can be anything there are an infinite number of possible potential functions, although they will only vary by an additive constant.

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(b) $\vec{F}(x,y) = \left(2x\mathbf{e}^{xy} + x^2y\mathbf{e}^{xy}\right)\vec{i} + \left(x^3\mathbf{e}^{xy} + 2y\right)\vec{j}$

Okay, this one will go a lot faster since we don't need to go through as much explanation. We've already verified that this vector field is conservative in the first set of examples so we won't bother redoing that.

Let's start with the following,

$$\frac{\partial f}{\partial x} = 2x\mathbf{e}^{xy} + x^2y\mathbf{e}^{xy} \qquad\qquad \frac{\partial f}{\partial y} = x^3\mathbf{e}^{xy} + 2y$$

This means that we can do either of the following integrals,

$$f(x,y) = \int 2x\mathbf{e}^{xy} + x^2y\mathbf{e}^{xy}\,dx \qquad \text{or} \qquad f(x,y) = \int x^3\mathbf{e}^{xy} + 2y\,dy$$

While we can do either of these the first integral would be somewhat unpleasant as we would need to do integration by parts on each portion. On the other hand the second integral is fairly simple since the second term only involves y's and the first term can be done with the substitution $u = xy$. So, from the second integral we get,

$$f(x,y) = x^2\mathbf{e}^{xy} + y^2 + h(x)$$

Notice that this time the "constant of integration" will be a function of x. If we differentiate this with respect to x and set equal to P we get,

$$\frac{\partial f}{\partial x} = 2x\mathbf{e}^{xy} + x^2y\mathbf{e}^{xy} + h'(x) = 2x\mathbf{e}^{xy} + x^2y\mathbf{e}^{xy} = P$$

So, in this case it looks like,

$$h'(x) = 0 \qquad \Rightarrow \qquad h(x) = c$$

So, in this case the "constant of integration" really was a constant. Sometimes this will happen and sometimes it won't.

Here is the potential function for this vector field.

$$f(x,y) = x^2\mathbf{e}^{xy} + y^2 + c$$

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Now, as noted above we don't have a way (yet) of determining if a three-dimensional vector field is conservative or not. However, if we are given that a three-dimensional vector field is conservative finding a potential function is similar to the above process, although the work will be a little more involved.

In this case we will use the fact that,

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} = P\,\vec{i} + Q\,\vec{j} + R\,\vec{k} = \vec{F}$$

Let's take a quick look at an example.

Example 3 Find a potential function for the vector field,

$$\vec{F} = 2xy^3z^4\,\vec{i} + 3x^2y^2z^4\,\vec{j} + 4x^2y^3z^3\,\vec{k}$$

Solution

Okay, we'll start off with the following equalities.

$$\frac{\partial f}{\partial x} = 2xy^3z^4 \qquad \frac{\partial f}{\partial y} = 3x^2y^2z^4 \qquad \frac{\partial f}{\partial z} = 4x^2y^3z^3$$

To get started we can integrate the first one with respect to *x*, the second one with respect to *y*, or the third one with respect to *z*. Let's integrate the first one with respect to *x*.

$$f(x,y,z) = \int 2xy^3z^4\,dx = x^2y^3z^4 + g(y,z)$$

Note that this time the "constant of integration" will be a function of both *y* and *z* since differentiating anything of that form with respect to *x* will differentiate to zero.

Now, we can differentiate this with respect to *y* and set it equal to *Q*. Doing this gives,

$$\frac{\partial f}{\partial y} = 3x^2y^2z^4 + g_y(y,z) = 3x^2y^2z^4 = Q$$

Of course we'll need to take the partial derivative of the constant of integration since it is a function of two variables. It looks like we've now got the following,

$$g_y(y,z) = 0 \qquad \Rightarrow \qquad g(y,z) = h(z)$$

Since differentiating $g(y,z)$ with respect to *y* gives zero then $g(y,z)$ could at most be a function of *z*. This means that we now know the potential function must be in the following form.

$$f(x,y,z) = x^2y^3z^4 + h(z)$$

To finish this out all we need to do is differentiate with respect to z and set the result equal to R.

$$\frac{\partial f}{\partial z} = 4x^2y^3z^3 + h'(z) = 4x^2y^3z^3 = R$$

So,

$$h'(z) = 0 \qquad \Rightarrow \qquad h(z) = c$$

The potential function for this vector field is then,

$$f(x,y,z) = x^2y^3z^4 + c$$

Note that to keep the work to a minimum we used a fairly simple potential function for this example. It might have been possible to guess what the potential function was based simply on the vector field. However, we should be careful to remember that this usually won't be the case and often this process is required.

Also, there were several other paths that we could have taken to find the potential function. Each would have gotten us the same result.

Let's work one more slightly (and only slightly) more complicated example.

Example 4 Find a potential function for the vector field,

$$\vec{F} = \left(2x\cos(y) - 2z^3\right)\vec{i} + \left(3 + 2y\mathbf{e}^z - x^2\sin(y)\right)\vec{j} + \left(y^2\mathbf{e}^z - 6xz^2\right)\vec{k}$$

Solution

Here are the equalities for this vector field.

$$\frac{\partial f}{\partial x} = 2x\cos(y) - 2z^3 \qquad \frac{\partial f}{\partial y} = 3 + 2y\mathbf{e}^z - x^2\sin(y) \qquad \frac{\partial f}{\partial z} = y^2\mathbf{e}^z - 6xz^2$$

For this example let's integrate the third one with respect to z.

$$f(x,y,z) = \int y^2\mathbf{e}^z - 6xz^2\,dz = y^2\mathbf{e}^z - 2xz^3 + g(x,y)$$

The "constant of integration" for this integration will be a function of both x and y.

Now, we can differentiate this with respect to x and set it equal to P. Doing this gives,

$$\frac{\partial f}{\partial x} = -2z^3 + g_x(x,y) = 2x\cos(y) - 2z^3 = P$$

So, it looks like we've now got the following,

$$g_x(x,y) = 2x\cos(y) \qquad \Rightarrow \qquad g(x,y) = x^2\cos(y) + h(y)$$

The potential function for this problem is then,

$$f(x,y,z) = y^2\mathbf{e}^z - 2xz^3 + x^2\cos(y) + h(y)$$

To finish this out all we need to do is differentiate with respect to y and set the result equal to Q.

$$\frac{\partial f}{\partial y} = 2y\mathbf{e}^{z} - x^2\sin(y) + h'(y) = 3 + 2y\mathbf{e}^{z} - x^2\sin(y) = Q$$

So,

$$h'(y) = 3 \qquad \Rightarrow \qquad h(y) = 3y + c$$

The potential function for this vector field is then,

$$f(x,y,z) = y^2\mathbf{e}^{z} - 2xz^3 + x^2\cos(y) + 3y + c$$

So, a little more complicated than the others and there are again many different paths that we could have taken to get the answer.

We need to work one final example in this section.

Example 5 Evaluate $\int_C \vec{F} \bullet d\vec{r}$ where $\vec{F} = \left(2x^3y^4 + x\right)\vec{i} + \left(2x^4y^3 + y\right)\vec{j}$ and C is given by $\vec{r}(t) = \left(t\cos(\pi t) - 1\right)\vec{i} + \sin\left(\frac{\pi t}{2}\right)\vec{j}$, $0 \le t \le 1$.

Solution

Now, we could use the techniques we discussed when we first looked at line integrals of vector fields however that would be particularly unpleasant solution.

Instead, let's take advantage of the fact that we know from Example 2a above this vector field is conservative and that a potential function for the vector field is,

$$f(x,y) = \frac{1}{2}x^4y^4 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + c$$

Using this we know that integral must be independent of path and so all we need to do is use the theorem from the previous section to do the evaluation.

$$\int_C \vec{F} \bullet d\vec{r} = \int_C \nabla f \bullet d\vec{r} = f\left(\vec{r}(1)\right) - f\left(\vec{r}(0)\right)$$

where,

$$\vec{r}(1) = \langle -2, 1 \rangle \qquad \vec{r}(0) = \langle -1, 0 \rangle$$

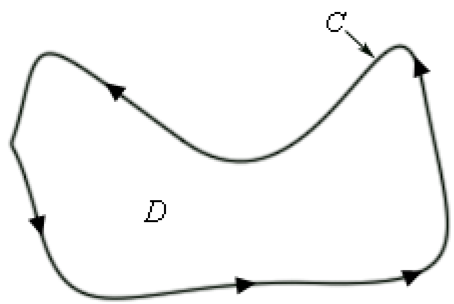
So, the integral is,

$$\begin{aligned}\int_C \vec{F} \bullet d\vec{r} &= f(-2,1) - f(-1,0) \\ &= \left(\frac{21}{2} + c\right) - \left(\frac{1}{2} + c\right) \\ &= 10\end{aligned}$$

Green's Theorem

In this section we are going to investigate the relationship between certain kinds of line integrals (on closed paths) and double integrals.

Let's start off with a simple (recall that this means that it doesn't cross itself) closed curve *C* and let *D* be the region enclosed by the curve. Here is a sketch of such a curve and region.



First, notice that because the curve is simple and closed there are no holes in the region *D*. Also notice that a direction has been put on the curve. We will use the convention here that the curve *C* has a **positive orientation** if it is traced out in a counter-clockwise direction. Another way to think of a positive orientation (that will cover much more general curves as well see later) is that as we traverse the path following the positive orientation the region *D* must always be on the left.

Given curves/regions such as this we have the following theorem.

Green's Theorem

Let *C* be a positively oriented, piecewise smooth, simple, closed curve and let *D* be the region enclosed by the curve. If *P* and *Q* have continuous first order partial derivatives on *D* then,

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Before working some examples there are some alternate notations that we need to acknowledge. When working with a line integral in which the path satisfies the condition of Green's Theorem we will often denote the line integral as,

$$\oint_C P\,dx + Q\,dy \qquad \text{or} \qquad \circlearrowleft\!\!\!\!\!\!\int_C P\,dx + Q\,dy$$

Both of these notations do assume that *C* satisfies the conditions of Green's Theorem so be careful in using them.

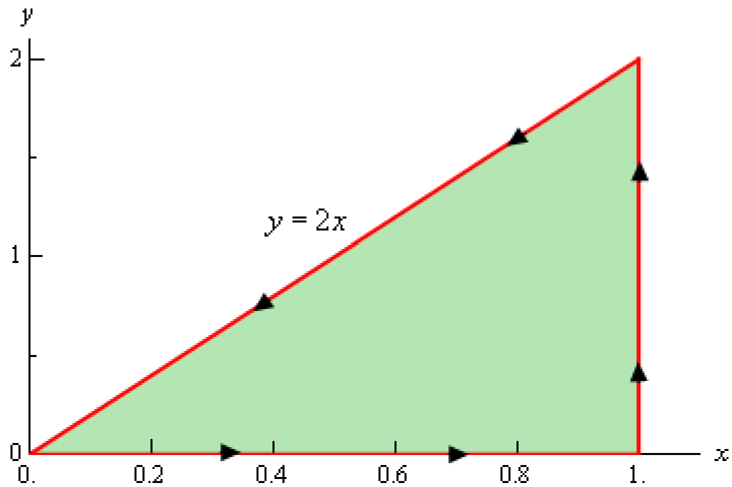
Also, sometimes the curve *C* is not thought of as a separate curve but instead as the boundary of some region *D* and in these cases you may see *C* denoted as ∂D.

Let's work a couple of examples.

Example 1 Use Green's Theorem to evaluate $\oint_C xy\,dx + x^2y^3\,dy$ where C is the triangle with vertices $(0,0)$, $(1,0)$, $(1,2)$ with positive orientation.

Solution

Let's first sketch C and D for this case to make sure that the conditions of Green's Theorem are met for C and will need the sketch of D to evaluate the double integral.



So, the curve does satisfy the conditions of Green's Theorem and we can see that the following inequalities will define the region enclosed.

$$0 \le x \le 1 \qquad\qquad 0 \le y \le 2x$$

We can identify P and Q from the line integral. Here they are.

$$P = xy \qquad\qquad Q = x^2y^3$$

So, using Green's Theorem the line integral becomes,

$$\begin{aligned}\oint_C xy\,dx + x^2y^3\,dy &= \iint_D 2xy^3 - x\,dA \\ &= \int_0^1\int_0^{2x} 2xy^3 - x\,dy\,dx \\ &= \int_0^1 \left.\left(\frac{1}{2}xy^4 - xy\right)\right|_0^{2x} dx \\ &= \int_0^1 8x^5 - 2x^2\,dx \\ &= \left.\left(\frac{4}{3}x^6 - \frac{2}{3}x^3\right)\right|_0^1 \\ &= \frac{2}{3}\end{aligned}$$

Example 2 Evaluate $\oint_C y^3\,dx - x^3\,dy$ where C is the positively oriented circle of radius 2 centered at the origin.

Solution
Okay, a circle will satisfy the conditions of Green's Theorem since it is closed and simple and so there really isn't a reason to sketch it.

Let's first identify P and Q from the line integral.

$$P = y^3 \qquad\qquad Q = -x^3$$

Be careful with the minus sign on Q!

Now, using Green's theorem on the line integral gives,

$$\oint_C y^3\,dx - x^3\,dy = \iint_D -3x^2 - 3y^2\,dA$$

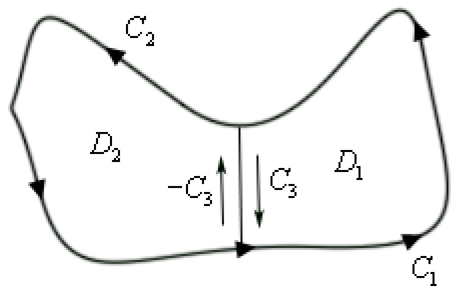
where D is a disk of radius 2 centered at the origin.

Since D is a disk it seems like the best way to do this integral is to use polar coordinates. Here is the evaluation of the integral.

$$\begin{aligned}\oint_C y^3\,dx - x^3\,dy &= -3\iint_D \left(x^2 + y^2\right) dA \\ &= -3\int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta \\ &= -3\int_0^{2\pi} \left.\frac{1}{4}r^4\right|_0^2 d\theta \\ &= -3\int_0^{2\pi} 4\,d\theta \\ &= -24\pi\end{aligned}$$

So, Green's theorem, as stated, will not work on regions that have holes in them. However, many regions do have holes in them. So, let's see how we can deal with those kinds of regions.

Let's start with the following region. Even though this region doesn't have any holes in it the arguments that we're going to go through will be similar to those that we'd need for regions with holes in them, except it will be a little easier to deal with and write down.



The region *D* will be $D_1 \cup D_2$ and recall that the symbol $\cup$ is called the union and means that *D* consists of both D_1 and D_2. The boundary of D_1 is $C_1 \cup C_3$ while the boundary of D_2 is $C_2 \cup \left(-C_3\right)$ and notice that both of these boundaries are positively oriented. As we traverse each boundary the corresponding region is always on the left. Finally, also note that we can think of the whole boundary, *C*, as,

$$C = \left(C_1 \cup C_3\right) \cup \left(C_2 \cup \left(-C_3\right)\right) = C_1 \cup C_2$$

since both C_3 and $-C_3$ will "cancel" each other out.

Now, let's start with the following double integral and use a basic property of double integrals to break it up.

$$\iint\limits_D \left(Q_x - P_y\right) dA = \iint\limits_{D_1 \cup D_2} \left(Q_x - P_y\right) dA = \iint\limits_{D_1} \left(Q_x - P_y\right) dA + \iint\limits_{D_2} \left(Q_x - P_y\right) dA$$

Next, use Green's theorem on each of these and again use the fact that we can break up line integrals into separate line integrals for each portion of the boundary.

$$\begin{aligned}\iint\limits_D \left(Q_x - P_y\right) dA &= \iint\limits_{D_1} \left(Q_x - P_y\right) dA + \iint\limits_{D_2} \left(Q_x - P_y\right) dA \\ &= \oint\limits_{C_1 \cup C_3} P\,dx + Q\,dy + \oint\limits_{C_2 \cup \left(-C_3\right)} P\,dx + Q\,dy \\ &= \oint\limits_{C_1} P\,dx + Q\,dy + \oint\limits_{C_3} P\,dx + Q\,dy + \oint\limits_{C_2} P\,dx + Q\,dy + \oint\limits_{-C_3} P\,dx + Q\,dy\end{aligned}$$

Next, we'll use the fact that,

$$\oint\limits_{-C_3} P\,dx + Q\,dy = -\oint\limits_{C_3} P\,dx + Q\,dy$$

Recall that changing the orientation of a curve with line integrals with respect to *x* and/or *y* will simply change the sign on the integral. Using this fact we get,

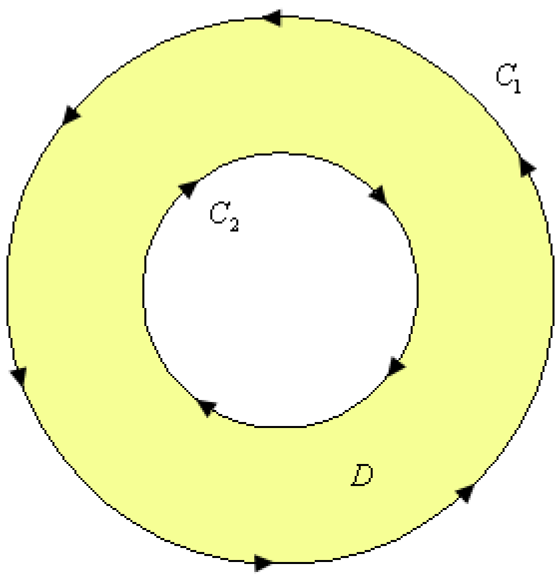
$$\begin{aligned}\iint\limits_D \left(Q_x - P_y\right) dA &= \oint\limits_{C_1} P\,dx + Q\,dy + \oint\limits_{C_3} P\,dx + Q\,dy + \oint\limits_{C_2} P\,dx + Q\,dy - \oint\limits_{C_3} P\,dx + Q\,dy \\ &= \oint\limits_{C_1} P\,dx + Q\,dy + \oint\limits_{C_2} P\,dx + Q\,dy\end{aligned}$$

Finally, put the line integrals back together and we get,

$$\begin{aligned}\iint\limits_D \left(Q_x - P_y\right) dA &= \oint\limits_{C_1} P\,dx + Q\,dy + \oint\limits_{C_2} P\,dx + Q\,dy \\ &= \oint\limits_{C_1 \cup C_2} P\,dx + Q\,dy \\ &= \oint\limits_{C} P\,dx + Q\,dy\end{aligned}$$

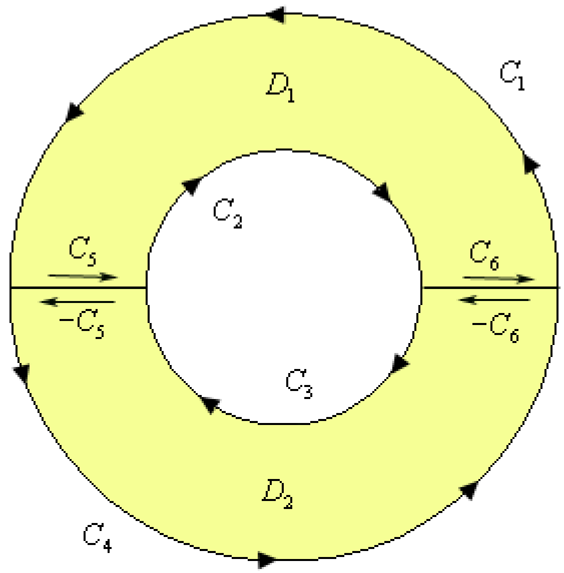
So, what did we learn from this? If you think about it this was just a lot of work and all we got out of it was the result from Green's Theorem which we already knew to be true. What this exercise has shown us is that if we break a region up as we did above then the portion of the line integral on the pieces of the curve that are in the middle of the region (each of which are in the opposite direction) will cancel out. This idea will help us in dealing with regions that have holes in them.

To see this let's look at a ring.



Notice that both of the curves are oriented positively since the region D is on the left side as we traverse the curve in the indicated direction. Note as well that the curve C_2 seems to violate the original definition of positive orientation. We originally said that a curve had a positive orientation if it was traversed in a counter-clockwise direction. However, this was only for regions that do not have holes. For the boundary of the hole this definition won't work and we need to resort to the second definition that we gave above.

Now, since this region has a hole in it we will apparently not be able to use Green's Theorem on any line integral with the curve $C = C_1 \cup C_2$. However, if we cut the disk in half and rename all the various portions of the curves we get the following sketch.



The boundary of the upper portion (D_1)of the disk is $C_1 \cup C_2 \cup C_5 \cup C_6$ and the boundary on the lower portion (D_2)of the disk is $C_3 \cup C_4 \cup \left(-C_5\right) \cup \left(-C_6\right)$. Also notice that we can use Green's Theorem on each of these new regions since they don't have any holes in them. This means that we can do the following,

$$\begin{aligned}\iint\limits_{D} \left(Q_x - P_y\right) dA &= \iint\limits_{D_1} \left(Q_x - P_y\right) dA + \iint\limits_{D_2} \left(Q_x - P_y\right) dA \\ &= \oint\limits_{C_1 \cup C_2 \cup C_5 \cup C_6} P\,dx + Q\,dy + \oint\limits_{C_3 \cup C_4 \cup (-C_5) \cup (-C_6)} P\,dx + Q\,dy\end{aligned}$$

Now, we can break up the line integrals into line integrals on each piece of the boundary. Also recall from the work above that boundaries that have the same curve, but opposite direction will cancel. Doing this gives,

$$\begin{aligned}\iint\limits_{D} \left(Q_x - P_y\right) dA &= \iint\limits_{D_1} \left(Q_x - P_y\right) dA + \iint\limits_{D_2} \left(Q_x - P_y\right) dA \\ &= \oint\limits_{C_1} P\,dx + Q\,dy + \oint\limits_{C_2} P\,dx + Q\,dy + \oint\limits_{C_3} P\,dx + Q\,dy + \oint\limits_{C_4} P\,dx + Q\,dy\end{aligned}$$

But at this point we can add the line integrals back up as follows,

$$\begin{aligned}\iint\limits_{D} \left(Q_x - P_y\right) dA &= \oint\limits_{C_1 \cup C_2 \cup C_3 \cup C_4} P\,dx + Q\,dy \\ &= \oint\limits_{C} P\,dx + Q\,dy\end{aligned}$$

The end result of all of this is that we could have just used Green's Theorem on the disk from the start even though there is a hole in it. This will be true in general for regions that have holes in them.

Let's take a look at an example.

Example 3 Evaluate $\oint_C y^3\,dx - x^3\,dy$ where C are the two circles of radius 2 and radius 1 centered at the origin with positive orientation.

Solution
Notice that this is the same line integral as we looked at in the second example and only the curve has changed. In this case the region D will now be the region between these two circles and that will only change the limits in the double integral so we'll not put in some of the details here.

Here is the work for this integral.

$$\begin{aligned}\oint_C y^3\,dx - x^3\,dy &= -3\iint_D \left(x^2+y^2\right)dA \\ &= -3\int_0^{2\pi}\int_1^2 r^3\,dr\,d\theta \\ &= -3\int_0^{2\pi} \left.\frac{1}{4}r^4\right|_1^2 d\theta \\ &= -3\int_0^{2\pi}\frac{15}{4}\,d\theta \\ &= -\frac{45\pi}{2}\end{aligned}$$

We will close out this section with an interesting application of Green's Theorem. Recall that we can determine the area of a region D with the following double integral.

$$A = \iint_D dA$$

Let's think of this double integral as the result of using Green's Theorem. In other words, let's assume that

$$Q_x - P_y = 1$$

and see if we can get some functions P and Q that will satisfy this.

There are many functions that will satisfy this. Here are some of the more common functions.

$P = 0$	$P = -y$	$P = -\frac{y}{2}$
$Q = x$	$Q = 0$	$Q = \frac{x}{2}$

Then, if we use Green's Theorem in reverse we see that the area of the region D can also be computed by evaluating any of the following line integrals.

$$A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx$$

where C is the boundary of the region D.

Let's take a quick look at an example of this.

Example 4 Use Green's Theorem to find the area of a disk of radius a.

Solution
We can use either of the integrals above, but the third one is probably the easiest. So,

$$A = \frac{1}{2}\oint_C x\,dy - y\,dx$$

where C is the circle of radius a. So, to do this we'll need a parameterization of C. This is,

$$x = a\cos t \qquad y = a\sin t \qquad 0 \le t \le 2\pi$$

The area is then,

$$\begin{aligned} A &= \frac{1}{2}\oint_C x\,dy - y\,dx \\ &= \frac{1}{2}\left(\int_0^{2\pi} a\cos t\left(a\cos t\right)dt - \int_0^{2\pi} a\sin t\left(-a\sin t\right)dt\right) \\ &= \frac{1}{2}\int_0^{2\pi} a^2\cos^2 t + a^2\sin^2 t\,dt \\ &= \frac{1}{2}\int_0^{2\pi} a^2 dt \\ &= \pi a^2 \end{aligned}$$

Curl and Divergence

In this section we are going to introduce a couple of new concepts, the curl and the divergence of a vector.

Let's start with the curl. Given the vector field $\vec{F} = P\,\vec{i} + Q\,\vec{j} + R\,\vec{k}$ the curl is defined to be,

$$\operatorname{curl}\vec{F} = \left(R_y - Q_z\right)\vec{i} + \left(P_z - R_x\right)\vec{j} + \left(Q_x - P_y\right)\vec{k}$$

There is another (potentially) easier definition of the curl of a vector field. To use it we will first need to define the ∇ *operator*. This is defined to be,

$$\nabla = \frac{\partial}{\partial x}\,\vec{i} + \frac{\partial}{\partial y}\,\vec{j} + \frac{\partial}{\partial z}\,\vec{k}$$

We use this as if it's a function in the following manner.

$$\nabla f = \frac{\partial f}{\partial x}\,\vec{i} + \frac{\partial f}{\partial y}\,\vec{j} + \frac{\partial f}{\partial z}\,\vec{k}$$

So, whatever function is listed after the ∇ is substituted into the partial derivatives. Note as well that when we look at it in this light we simply get the gradient vector.

Using the ∇ we can define the curl as the following cross product,

$$\operatorname{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

We have a couple of nice facts that use the curl of a vector field.

Facts

1. If $f\left(x,y,z\right)$ has continuous second order partial derivatives then $\operatorname{curl}\left(\nabla f\right) = \vec{0}$. This is easy enough to check by plugging into the definition of the derivative so we'll leave it to you to check.

2. If $\vec{F}$ is a conservative vector field then $\operatorname{curl}\vec{F} = \vec{0}$. This is a direct result of what it means to be a conservative vector field and the previous fact.

3. If $\vec{F}$ is defined on all of $\mathbb{R}^3$ whose components have continuous first order partial derivative and $\operatorname{curl}\vec{F} = \vec{0}$ then $\vec{F}$ is a conservative vector field. This is not so easy to verify and so we won't try.

Example 1 Determine if $\vec{F} = x^2 y\,\vec{i} + xyz\,\vec{j} - x^2 y^2\,\vec{k}$ is a conservative vector field.

Solution
So all that we need to do is compute the curl and see if we get the zero vector or not.

$$\begin{aligned}
\operatorname{curl}\vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 y & xyz & -x^2 y^2 \end{vmatrix} \\
&= -2x^2 y\,\vec{i} + yz\,\vec{k} - \left(-2xy^2\,\vec{j}\right) - xy\,\vec{i} - x^2\vec{k} \\
&= -\left(2x^2 y + xy\right)\vec{i} + 2xy^2\,\vec{j} + \left(yz - x^2\right)\vec{k} \\
&\neq \vec{0}
\end{aligned}$$

So, the curl isn't the zero vector and so this vector field is not conservative.

Next we should talk about a physical interpretation of the curl. Suppose that $\vec{F}$ is the velocity field of a flowing fluid. Then $\operatorname{curl}\vec{F}$ represents the tendency of particles at the point $\left(x, y, z\right)$ to rotate about the axis that points in the direction of $\operatorname{curl}\vec{F}$. If $\operatorname{curl}\vec{F} = \vec{0}$ then the fluid is called irrotational.

Let's now talk about the second new concept in this section. Given the vector field $\vec{F} = P\,\vec{i} + Q\,\vec{j} + R\,\vec{k}$ the divergence is defined to be,

$$\operatorname{div}\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

There is also a definition of the divergence in terms of the ∇ operator. The divergence can be defined in terms of the following dot product.

$$\operatorname{div}\vec{F} = \nabla \bullet \vec{F}$$

Example 2 Compute $\operatorname{div}\vec{F}$ for $\vec{F} = x^2 y\,\vec{i} + xyz\,\vec{j} - x^2 y^2\,\vec{k}$

Solution
There really isn't much to do here other than compute the divergence.

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}\left(x^2 y\right) + \frac{\partial}{\partial y}\left(xyz\right) + \frac{\partial}{\partial z}\left(-x^2 y^2\right) = 2xy + xz$$

We also have the following fact about the relationship between the curl and the divergence.

$$\operatorname{div}\left(\operatorname{curl}\vec{F}\right) = 0$$

Example 3 Verify the above fact for the vector field $\vec{F} = yz^2\,\vec{i} + xy\,\vec{j} + yz\,\vec{k}$.

Solution
Let's first compute the curl.

$$\begin{aligned}\operatorname{curl}\vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz^2 & xy & yz \end{vmatrix} \\ &= z\,\vec{i} + 2yz\,\vec{j} + y\,\vec{k} - z^2\vec{k} \\ &= z\vec{i} + 2yz\,\vec{j} + \left(y - z^2\right)\vec{k}\end{aligned}$$

Now compute the divergence of this.

$$\operatorname{div}\left(\operatorname{curl}\vec{F}\right) = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}\left(y - z^2\right) = 2z - 2z = 0$$

We also have a physical interpretation of the divergence. If we again think of $\vec{F}$ as the velocity field of a flowing fluid then $\operatorname{div}\vec{F}$ represents the net rate of change of the mass of the fluid flowing from the point (x, y, z) per unit volume. This can also be thought of as the tendency of a fluid to diverge from a point. If $\operatorname{div}\vec{F} = 0$ then the $\vec{F}$ is called incompressible.

The next topic that we want to briefly mention is the **Laplace** operator. Let's first take a look at,

$$\operatorname{div}\left(\nabla f\right) = \nabla \bullet \nabla f = f_{xx} + f_{yy} + f_{zz}$$

The Laplace operator is then defined as,

$$\nabla^2 = \nabla \bullet \nabla$$

The Laplace operator arises naturally in many fields including heat transfer and fluid flow.

The final topic in this section is to give two vector forms of Green's Theorem. The first form uses the curl of the vector field and is,

$$\oint_C \vec{F} \bullet d\,\vec{r} = \iint_D \left(\operatorname{curl}\vec{F}\right) \bullet \vec{k}\;dA$$

where $\vec{k}$ is the standard unit vector in the positive z direction.

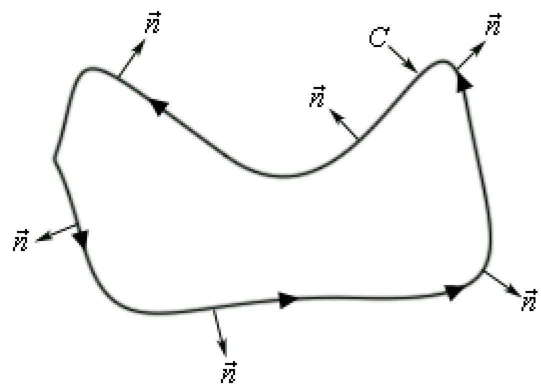
The second form uses the divergence. In this case we also need the outward unit normal to the curve C. If the curve is parameterized by

$$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j}$$

then the outward unit normal is given by,

$$\vec{n} = \frac{y'(t)}{\left\|\vec{r}'(t)\right\|}\vec{i} - \frac{x'(t)}{\left\|\vec{r}'(t)\right\|}\vec{j}$$

Here is a sketch illustrating the outward unit normal for some curve C at various points.



The vector form of Green's Theorem that uses the divergence is given by,

$$\boxed{\oint_C \vec{F} \bullet \vec{n}\, ds = \iint_D \operatorname{div} \vec{F}\, dA}$$