CHAPTER ONE
INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS

1.1 Introduction

Most natural phenomena, whether in the domain of fluid dynamics, electricity, magnetism, mechanics, optics, heat flow, economy, biology can be described in general by partial differential equations (PDEs). For example, the natural laws of physics, such as Maxwell's equations, Newton's law of cooling, the Navier-Stokes equations, Newton's equations of motion, and Schrodinger's equation of quantum mechanics, are stated (or can be) in terms of PDEs, that is, these laws describe physical phenomena by relating space and time derivatives. Derivatives occur in these equations because the derivatives represent natural things (like velocity, acceleration, force, friction, flux, current).

1.2 Basic Concepts and Definitions

Definition 1.1

A partial differential equation (usually denoted by PDE) is an equation that contains in addition to the dependent variable and independent variables, one or more partial derivatives of the dependent variable with respect to one or more independent variables. In general, it may be written in the form:

$$F\left(x, y, \ldots, u, u_x, u_y, \ldots, u_{xx}, u_{xy}, \ldots\right) = 0, \tag{1.1}$$

involving several independent variables $x, y, \ldots$, an unknown function $u(x, y, \ldots)$ of these variables, and the partial derivatives $u_x, u_y, \ldots$, $u_{xx}, u_{xy}, \ldots$, of the unknown function.

Definition 1.2

The general solution of a partial differential equation constitute of arbitrary functions of independent variables involved in (PDE) rather than on arbitrary constants. These arbitrary functions are defined on some domain $D \subset \mathbb{R}^n$ which is continuously differentiable such that all its partial derivatives involved in equation (1.1) exist and satisfy (1.1) identically.

We recall that in the case of ordinary differential equations, the first task is to find the general solution, and then a particular solution is determined by finding the values of arbitrary constants from the prescribed conditions. But, for partial differential equations, selecting a particular solution satisfying the additional conditions from the general solution of a partial differential equation may be as difficult as, or even more difficult than, the problem of finding the general solution itself. This is so because the general solution of a partial differential equation involves arbitrary functions; the specialization of such a solution to the particular form which satisfies supplementary conditions requires the determination of these arbitrary functions, rather than merely the determination of constants.

As indicated above, the general solution of a linear partial differential equation contains arbitrary functions. This means that there are infinitely many solutions and only by specifying the initial and/or boundary conditions can we determine a specific solution of interest.

Usually, both initial and boundary conditions arise from the physical problems.

In the case of partial differential equations in which one of the independent variables is the time t , an initial condition(s) specifies the physical state of the dependent variable $u(x,t)$ at a particular time $t = t_0$ or $t = 0$. Often $u(x,0)$ and/or $u_t(x,0)$ are specified to determine the function $u(x,t)$ at later times. Such conditions are called the **Cauchy** (or initial) conditions. It can be shown that these conditions are necessary and sufficient for the existence of a unique solution. The problem of finding the solution of the initial-value problem with prescribed Cauchy data on the line $t = 0$ is called the **Cauchy problem** or the **initial-value problem**.

In each physical problem, the governing equation is to be solved within a given domain D of space with prescribed values of the dependent variable $u(\mathrm{x},t)$ given on the boundary ∂D of D. Often, the boundary need not enclose a finite volume in which case, part of the boundary is at infinity. For problems with a boundary at infinity, boundedness conditions on the behavior of the solution at infinity must be specified. This kind of problem is typically known as a **boundary-value problem**, and it is one of the most fundamental problems in applied mathematics and mathematical physics.

There are three important types of boundary conditions which arise frequently in formulating physical problems.

(i) Dirichlet conditions

In this case the solution u is prescribed at each point of a boundary ∂D of a domain D. The problem of finding the solution of a given equation partial differential equation inside D with prescribed values of u on ∂D is called the **Dirichlet boundary-value problem**

(ii) Neumann conditions

In this case the values of normal derivative $\dfrac{\partial u}{\partial n}$ of the solution on the boundary ∂D are specified. Here, the problem is called the **Neumann boundary-value problem**

(iii) Robin conditions

where $\left(\dfrac{\partial u}{\partial n} + au\right)$ is specified on ∂D. The corresponding problem is called the Robin boundary-value problem.

Definition 1.3

The order of a partial differential equation is the order of the highest ordered partial derivative appearing in the equation. For example

$$u_{xx} + 2xu_{xy} + u_{yy} = e^{y},$$

is a second-order partial differential equation, and

$$u_{xxy} + xu_{yy} + 8u = 7y,$$

is a third-order partial differential equation, and

$$\frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} = 0,$$

is a fourth-order partial differential equation.

Definition 1.4

A partial differential equation is said to be linear if the function F is linear function in the dependent variable and all its derivatives with coefficients depending only on the independent variables, for example

$$x^2 y \frac{\partial u}{\partial x} + (x - y^2)\frac{\partial u}{\partial y} + yu = \sin(x + y),$$

is linear equation. While the equations

$$x^2 y \frac{\partial u}{\partial x} + (x - y^2)\frac{\partial u}{\partial y} + yu^2 = 0,$$

$$u\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = x\cos u,$$

are nonlinear equations.

In general, the linear partial differential equation of order n in two independent variables has the form

$$\sum_{i,j=0}^{i+j\le n} A_{ij}(x,y)\frac{\partial^{i+j} u}{\partial x^i \partial y^j} = G(x,y), \tag{1.2}$$

where $A_{ij}(x,y)$, $G(x,y)$ are functions of the independent variables x, y.

Definition 1.5

A partial differential equation it is said to be quasi-linear if it is linear in the highest-ordered derivative of the dependent variable. That is the coefficients of terms involve functions of only lower order derivatives of the dependent variables. However, terms with lower order derivatives can occur in any manner. For example, the equation

$$uu_x + u_t = u,$$

is first-order quasi-linear partial differential equation., while the equation

$$u_x u_{xx} + xuu_y = \sin y,$$

is a second-order quasi-linear partial differential equation.

Definition 1.6

A quasi-linear partial differential equation it is said to be semi-linear if the coefficients of highest derivatives are functions of the independent variables alone, for example

$$u_{xx} + u_{yy} = u^2,$$

Definition 1.7

The equation (1.2) is called homogeneous if the right hand side $G(x, y)$ is identically zero for all x and y. If $G(x, y)$ is not identically zero, then the equation is called nonhomogeneous.

Definition 1.8

The linear partial differential equation is called of homogeneous terms if all the terms of the linear partial differential equation have the same order, for example

$$u_{xxx} + xy^2 u_{xyy} - \sin x\, u_{yyy} = e^{x+y}.$$

1.3 A Few well-known PDEs

1.3.1 Heat equation

It is a partial differential equation gives the distribution of temperature in a specific region as a function of space and time when the temperature at the boundaries, the initial distribution of temperature, and the physical properties of the medium are given.

$$u_t = u_{xx}, \qquad \text{(heat equation in one dimension)}$$

$$u_t = u_{xx} + u_{yy}. \qquad \text{(heat equation in two dimensions)}$$

1.3.2 Laplace's equation

It is satisfied by the potential fields in source-free domains. For example, the Laplace equation is satisfied by the gravitational potential of the gravity force in domains free from attracting masses, the potential of an electrostatic field in a domain free from charges, etc.

$$u_{xx} + u_{yy} + u_{zz} = 0, \qquad \text{(Laplace's equation in Cartesian coordinates)}$$

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0. \quad \text{(Laplace's equation in polar coordinates)}$$

1.3.3 Wave equation

It is a partial differential equation describes various oscillatory processes and processes of wave propagation:

$$u_{tt} = u_{xx} + u_{yy} + u_{zz}. \qquad \text{(wave equation in three dimensions)}$$

1.3.4 Telegraph equation

It is a partial differential equation describes the voltage on an electrical transmission line with distance and time:

$$u_{tt} = u_{xx} + \alpha u_t + \beta u. \qquad \text{(telegraph equation)},$$

1.3.5 Schrödinger equation

It is a partial differential equation that governs the wave function of a quantum-mechanical system

$$i\hbar\frac{\partial u}{\partial t} = -\frac{\hbar^2}{2m}\Delta u + Vu.$$

1.4 Construction of Partial Differential Equations

There are two methods to form the partial differential equations:

- **Elimination of arbitrary constants.**
- **Elimination of arbitrary functions.**

1.4.1 Elimination of arbitrary constants

Consider a system of geometrical surfaces described by the equation

$$\phi(x,y,z,a,b)=0, \qquad (1.3)$$

where a and b are arbitrary parameters. We differentiate (1.3) with respect to x and y to obtain

$$\phi_x + p\phi_z = 0 \ , \quad \phi_y + q\phi_z = 0, \qquad (1.4)$$

where $p=\dfrac{\partial z}{\partial x}$, $q=\dfrac{\partial z}{\partial y}$.

The set of two equations (1.3) and (1.4) involves two arbitrary parameters a and b. In general, these two parameters can be eliminated from this set to obtain a first-order equation of the form

$$\psi(x,y,z,p,q)=0. \qquad (1.5)$$

Thus the system of surfaces (1.4) gives rise to a first-order partial differential equation (1.5).

In general, if the number of arbitrary constants to be eliminated is equal to the number of independent variables, then only one first-order partial differential equation arises. If the number of arbitrary constants to be eliminated is less than the number of independent variables, then more than one first-order partial differential equation is obtained. If the number of arbitrary constants to be eliminated is more than the number of independent variables, the partial differential equations obtained are of second or higher order.

Example (1.1)

Find the PDE corresponding to the family of spheres

$$x^2+y^2+(z-c)^2=r^2. \qquad (1.6)$$

Solution

Differentiating the equation (1.6) with respect to x and y gives

$$x+p(z-c)=0 \quad \text{and} \quad y+q(z-c)=0.$$

Eliminating the arbitrary constant c from these equations, we obtain the first-order, partial differential equation

$$yp-xq=0.$$

Example (1.2)

Find the PDE corresponding to the family of spheres

$$(x-a)^2+(y-b)^2+z^2=r^2.$$

Solution

We differentiate the equation of the family of spheres with respect to x and y to obtain

$$(x-a)+zp=0, \quad (y-b)+zq=0.$$

Eliminating the two arbitrary constants a and b, we find the nonlinear partial differential equation

$$z^2\left(p^2+q^2+1\right)=r^2.$$

Example (1.3)

Form the partial differential equation by eliminating the constants from

$$z = ax + by + ab. \qquad (1.7)$$

Solution

Differentiating Eq. (1.8) partially with respect to x and y, we obtain

$$\frac{\partial z}{\partial x}=a=p, \quad \frac{\partial z}{\partial y}=b=q$$

Substituting p and q for a and b in Eq. (1.7), we get the required PDE as

$$z = px + qy + pq.$$

Example (1.4)

Find the partial differential equation of the family of planes, the sum of whose x, y, z intercepts is equal to unity.

Solution

Let

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,$$

be the equation of the plane in intercept form, so that $a+b+c=1$. Thus, we have

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{1-a-b}=1, \qquad (1.8)$$

Differentiating Eq. (1.8) with respect to x and y, we have

$$\frac{p}{1-a-b}=-\frac{1}{a} \quad \text{and} \quad \frac{q}{1-a-b}=-\frac{1}{b}, \qquad (1.9)$$

From Eq. (1.9), we get

$$\frac{p}{q}=\frac{b}{a}. \qquad (1.10)$$

Also, from Eqs. (1.9) and (1.10), we get

$$pa=a+b-1=a+\frac{p}{q}a-1 \text{ or } a\left(1+\frac{p}{q}-p\right)=1.$$

Therefore,

$$a=\frac{q}{(p+q-pq)}. \qquad (1.11)$$

Similarly, from Eqs. (1.9) and (1.10), we find

$$b=\frac{p}{(p+q-pq)}\ , \tag{1.12}$$

Substituting the values of a and b from Eqs. (1.11) and (1.12) respectively to Eq. (1.8), we have

$$\frac{p+q-pq}{q}x+\frac{p+q-pq}{p}y+\frac{p+q-pq}{-pq}z=1,$$

or

$$\frac{x}{q}+\frac{y}{p}-\frac{z}{pq}=\frac{1}{p+q-pq},$$

That is,

$$px+qy-z=\frac{pq}{p+q-pq},$$

which is the required PDE.

Example (1.5)

Find the differential equation of all spheres of radius λ, having center in the xy -plane

Solution

Let

$$(x-a)^2+(y-b)^2+z^2=\lambda^2, \tag{1.13}$$

be the equation of the spheres having center at $(a,b,0)$ in the xy -plane.

Differentiating Eq. (1.13) with respect to x and y , we have

$$2(x-a)+2pz=0\ ,\ 2(y-b)+2qz=0. \tag{1.14}$$

Substituting of $(x-a)$ and $(y-b)$ from Eq. (1.14) to Eq. (1.13), we have

$$z^2\left(p^2+q^2+1\right)=\lambda^2,$$

which is the required PDE.

1.4.2 Elimination of arbitrary functions

Suppose u and v are any two given functions of x , y and z . Let F be an arbitrary function of u and v of the form

$$F(u,v)=0. \tag{1.15}$$

We can form a differential equation by eliminating the arbitrary function F . For this, we differentiate Eq. (1.15) partially with respect to x and y to get

$$\frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right]+\frac{\partial F}{\partial v}\left[\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right]=0, \tag{1.16}$$

and

$$\frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right]+\frac{\partial F}{\partial v}\left[\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right]=0, \tag{1.17}$$

where $p = \dfrac{\partial z}{\partial x}$, $q = \dfrac{\partial z}{\partial y}$.

Now, eliminating $\partial F / \partial u$ and $\partial F / \partial v$ from Eqs. (1.16) and (1.17), we obtain

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial z} p & \dfrac{\partial v}{\partial x} + \dfrac{\partial v}{\partial z} p \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} q & \dfrac{\partial v}{\partial y} + \dfrac{\partial v}{\partial z} q \end{vmatrix} = 0,$$

which simplifies to

$$p \frac{\partial(u,v)}{\partial(y,z)} + q \frac{\partial(u,v)}{\partial(z,x)} = \frac{\partial(u,v)}{\partial(x,y)}, \qquad (1.18)$$

where,

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix},$$

Eq. (1.18) is a linear PDE of the type

$$Pp + Qq = R,$$

where

$$P = \frac{\partial(u,v)}{\partial(y,z)}, \quad Q = \frac{\partial(u,v)}{\partial(z,x)}, \quad R = \frac{\partial(u,v)}{\partial(x,y)}.$$

Eq. (1.18) is called Lagrange's PDE of first order.

If z is given in the form

$$z = \phi(u) + \psi(v) \qquad (*)$$

where ϕ, ψ are arbitrary functions of u, v respectively, and u, v are functions of x, y. Differentiating Eq. (*) with respect to x and y, we have

$$\begin{cases} z_x = \phi'(u)u_x + \psi'(v)v_x \\ z_y = \phi'(u)u_y + \psi'(v)v_y \\ z_{xx} = \phi''(u)u_x^2 + \psi''(v)v_x^2 + \phi'(u)u_{xx} + \psi'(v)v_{xx} \\ z_{xy} = \phi''(u)u_x u_y + \psi''(v)v_x v_y + \phi'(u)u_{xy} + \psi'(v)v_{xy} \\ z_{yy} = \phi''(u)u_y^2 + \psi''(v)v_y^2 + \phi'(u)u_{yy} + \psi'(v)v_{yy} \end{cases} \qquad (**)$$

Now, eliminating $\phi'(u), \phi''(u), \psi'(v), \psi''(v)$ from Eq. (**), we obtain

$$\begin{vmatrix} z_x & u_x & v_x & 0 & 0 \\ z_y & u_y & v_y & 0 & 0 \\ z_{xx} & u_{xx} & v_{xx} & u_x^2 & v_x^2 \\ z_{xy} & u_{xy} & v_{xy} & u_x u_y & v_x v_y \\ z_{yy} & u_{yy} & v_{yy} & u_y^2 & v_y^2 \end{vmatrix} = 0 \qquad (***)$$

Equation (***) is a second-order linear partial differential equation, can be written in the form

$$Pz_{xx} + Qz_{xy} + Rz_{yy} + Sz_x + Tz_y = W \;,$$

where P, Q, R, S, T, W are certain functions of x, y. In general, elimination of arbitrary functions of a relation of the form

$$z = \sum_{k=1}^{n} f_k(u_k)$$

where $f_1, f_2, \cdots, f_n$ are the arbitrary function, and $u_1, u_2, \cdots, u_n$ are certain functions of x, y, implies a n^{th}-order linear partial differential equation.

The following examples illustrate the idea of formation of PDEs.

Example (1.6)

Form the PDE by eliminating the arbitrary function from

(i) $z = f(x+it) + g(x-it),$ where $i = \sqrt{-1}.$

(ii) $f\left(x + y + z, x^2 + y^2 + z^2\right) = 0.$

Solution

(i) Given

$$z = f(x+it) + g(x-it).$$

Differentiating it twice partially with respect to x and t, we get

$$\frac{\partial z}{\partial x} = f'(x+it) + g'(x-it),$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+it) + g''(x-it). \qquad (1.19)$$

Here, f' indicates derivative of f with respect to $(x+it)$ and g' indicates derivative of g with respect to $(x-it)$. Also, we have

$$\frac{\partial z}{\partial t} = if'(x+it) - ig'(x-it),$$

$$\frac{\partial^2 z}{\partial t^2} = -f''(x+it) - g''(x-it). \qquad (1.20)$$

From Eqs. (1.19) and (1.20), we at once, find that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0,$$

which is the required PDE.
(ii) The given relation can be written in the form

$$\phi(u,v)=0,$$

where $u=x+y+z$, $v=x^2+y^2+z^2$. Hence, the required PDE is of the form

$$Pp+Qq=R, \text{ (Lagrange equation)}$$

where

$$P=\frac{\partial(u,v)}{\partial(y,z)}=\begin{vmatrix}\frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z}\end{vmatrix}=\begin{vmatrix}1 & 2y\\ 1 & 2z\end{vmatrix}=2(z-y),$$

$$Q=\frac{\partial(u,v)}{\partial(z,x)}=\begin{vmatrix}\frac{\partial u}{\partial z} & \frac{\partial v}{\partial z}\\ \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x}\end{vmatrix}=\begin{vmatrix}1 & 2z\\ 1 & 2x\end{vmatrix}=2(x-z),$$

and

$$R=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x}\\ \frac{\partial u}{\partial v} & \frac{\partial v}{\partial v}\end{vmatrix}=\begin{vmatrix}1 & 2x\\ 1 & 2y\end{vmatrix}=2(y-x).$$

Hence, the required PDE is

$$2(z-y)p+2(x-z)q=2(y-x),$$

or

$$(z-y)p+(x-z)q=y-x.$$

Example (1.7)
Eliminate the arbitrary function from the following and hence, obtain the corresponding partial differential equation:

(i) $z=xy+f\left(x^2+y^2\right)$

(ii) $z=f\left(xy/z\right)$

Solution
(i) Given $z=xy+f\left(x^2+y^2\right)$. Differentiating it partially with respect to x and y , we obtain

$$\frac{\partial z}{\partial x}=y+2xf'\left(x^2+y^2\right)=p, \qquad (1.21)$$

$$\frac{\partial z}{\partial y}=x+2yf'\left(x^2+y^2\right)=q. \qquad (1.22)$$

Eliminating f' from Eqs. (1.21) and (1.22), we get

$$yp - xq = y^2 - x^2,$$

which is the required PDE.

(ii) Given $z = f(xy/z)$. Differentiating it partially with respect to x and y, we get

$$\frac{\partial z}{\partial x} = \frac{y(z - xp)}{z^2} f'(xy/z) = p, \qquad (1.23)$$

$$\frac{\partial z}{\partial y} = \frac{x(z - yq)}{z^2} f'(xy/z) = q. \qquad (1.24)$$

Eliminating f' from Eqs. (1.23) and (1.24), we find

$$xp - yq = 0$$

or

$$px = qy$$

Which is the required PDE

CHAPTER TWO
PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

2.1 Introduction

Many problems in mathematical, physical, and engineering sciences deal with the formulation and the solution of first-order partial differential equations. From a mathematical point of view, first-order equations have the advantage of providing a conceptual basis that can be utilized for second-, third-, and higher-order equations.

This chapter is concerned with first-order, quasi-linear, linear and nonlinear partial differential equations and their solution by using the Lagrange method of characteristics and its generalizations.

A first-order partial differential equation with n independent variables has the general form

$$F(x_1, x_2, \ldots, x_n, z, z_{x_1}, z_{x_2}, \ldots, z_{x_n}) = 0, \quad (x_1, x_2, \ldots, x_n) \subset \mathbb{R}^n \qquad (2.1)$$

where $z(x_1, x_2, \ldots, x_n)$ is the unknown function and F is a given function.

Equation (2.1) is called a quasi-linear partial differential equation if it is linear in first-partial derivatives of the unknown function $z(x_1, x_2, \ldots, x_n)$. So, the general quasi-linear equation must be of the form:

$$\sum_{i=1}^{n} a_i(x_1, x_2, \ldots, x_n, z) \frac{\partial z}{\partial x_i} = f(x_1, x_2, \ldots, x_n, z), \qquad (2.2)$$

where its coefficients a_i are functions of $x_1, x_2, \ldots, x_n$ and z. The following are examples of quasi-linear equations:

$$x\left(y^2 + z\right) z_x - y\left(x^2 + z\right) z_y = \left(x^2 - y^2\right) z,$$

$$z z_x + z_y + n z^2 = 0,$$

$$\left(y^2 - z^2\right) z_x - xyz_y = xz.$$

Equation (2.1) is called a semi-linear partial differential equation if its coefficients a_i are independent of z, and hence, the semilinear equation can be expressed in the form

$$\sum_{i=1}^{n} a_i(x_1, x_2, \ldots, x_n) \frac{\partial z}{\partial x_i} = f(x_1, x_2, \ldots, x_n, z). \qquad (2.3)$$

Examples of semilinear equations are

$$x z_x + y z_y = z^2 + x^2,$$

$$(x+1)^2 z_x + (y-1)^2 z_y = (x+y) z^2,$$

$$z_t + a z_x + z^2 = 0,$$

where a is a constant.

Equation (2.1) is said to be linear if F is linear in each of the variables z and $\frac{\partial z}{\partial x_i}$, and the coefficients of these variables are functions only of the independent variables $x_1, x_2, \ldots, x_n$. The general, first-order, linear partial differential equation has the form

$$\sum_{i=1}^{n} a_i(x_1, x_2, \ldots, x_n)\frac{\partial z}{\partial x_i} + b(x_1, x_2, \ldots, x_n)\, z = f(x_1, x_2, \ldots, x_n). \qquad (2.4)$$

Unless stated otherwise, the functions $a_i(x_1, x_2, \ldots, x_n)$, $b(x_1, x_2, \ldots, x_n)$, $f(x_1, x_2, \ldots, x_n)$ are assumed to be continuously differentiable. Equations of the form (2.4) are called homogeneous if $f(x_1, x_2, \ldots, x_n) = 0$ or non-homogeneous if $f(x_1, x_2, \ldots, x_n) \neq 0$.

Obviously, linear equations are a special kind of the quasi-linear equation (2.2) if a_i, are independent of z and f is a linear function in z. Similarly, semilinear equation (2.3) reduces to a linear equation if f is linear in z.

Examples of linear equations are

$$xz_x + yz_y - nz = 0,$$

$$nz_x + (x+y)z_y - z = e^x,$$

$$yz_x + xz_y = xy,$$

$$(y-z)u_x + (z-x)u_y + (x-y)u_z = 0.$$

An equation which is not linear is often called a nonlinear equation. So, first-order equations are often classified as linear and nonlinear.

2.2 Geometrical Interpretation of First-order Partial Differential Equations

Consider a first order quasi-linear PDE of the form

$$P(x,y,z)\frac{\partial z}{\partial x} + Q(x,y,z)\frac{\partial z}{\partial y} = R(x,y,z), \qquad (2.5)$$

or simply

$$Pp + Qq = R, \qquad (2.6)$$

where x and y are independent variables. The solution of Eq. (2.5) is a surface S lying in the $(x, y, z)-$space, called an integral surface. If we are given that $z = f(x,y)$ is an integral surface of the PDE (2.6). Then, the normal to this surface will have direction cosines proportional $(\partial z/\partial x, \partial z/\partial y, -1)$ or $(p, q, -1)$. Therefore, the direction of the normal is given by $\vec{n} = \{p, q, -1\}$.

Clearly, Eq. (2.6) can be written as the dot product of two vectors

$$(P,Q,R).(p,q,-1)=0, \tag{2.7}$$

From the PDE (2.7), we observe that the normal $\vec{n}$ is perpendicular to the direction defined by the vector $\vec{t}=\{P,Q,R\}$. This clearly shows that the vector $\vec{t}=\{P,Q,R\}$ must be a tangent vector of the integral surface $z=f(x,y)$ at the point (x,y,z), and hence, it determines a direction field called the the characteristic direction or Monge axis. This direction is of fundamental importance in determining a solution of equation (2.5), see Fig. 1.

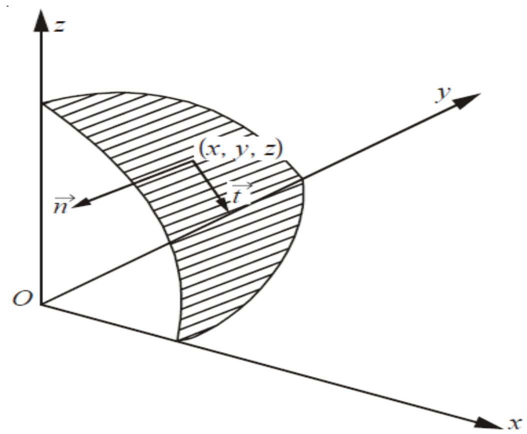

Fig. 1

A curve in (x,y,z)-space, whose tangent at every point coincides with the characteristic direction field (P,Q,R), is called a characteristic curve. If the parametric equations of this characteristic curve are

$$x=x(t), y=y(t), z=z(t).$$

Then the tangent vector to this curve is $\left(\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right)$ which must be equal to (P,Q,R). Therefore, the system of ordinary differential equations of the characteristic curve is given by

$$\frac{dx}{dt}=P(x,y,z),\ \frac{dy}{dt}=Q(x,y,z),\ \frac{dz}{dt}=R(x,y,z). \tag{2.8}$$

These are called the characteristic equations of the quasi-linear equation (2.5). Equivalently, the characteristic equations (2.8) in the nonparametric form are

$$\frac{dx}{P(x,y,z)}=\frac{dy}{Q(x,y,z)}=\frac{dz}{R(x,y,z)}. \tag{2.9}$$

2.3 Method of Characteristics and General Solutions

We can use the geometrical interpretation of first-order, partial differential equations and the properties of characteristic curves to develop a method for finding the general solution of quasi-linear equations. This is usually referred to as the method of characteristics due to Lagrange. This method of solution of quasi-linear equations can be described by the following result.

Theorem 2.1

The general solution of the linear PDE

$$Pp + Qq = R, \tag{2.10}$$

can be written in the form $F(u,v)=0$, where F is an arbitrary function, and $u(x,y,z)=C_1$, $v(x,y,z)=C_2$ are solution curves of the characteristic equations

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)} \tag{2.11}$$

Proof

Since $u(x,y,z)=C_1$ and $v(x,y,z)=C_2$ satisfy equations (2.11), then these equations must be compatible with the equations

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0,$$

and

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz = 0.$$

This is equivalent to the equations

$$\left.\begin{aligned} P\frac{\partial u}{\partial x}dx + Q\frac{\partial u}{\partial y}dy + R\frac{\partial u}{\partial z}dz = 0, \\ P\frac{\partial v}{\partial x}dx + Q\frac{\partial v}{\partial y}dy + R\frac{\partial v}{\partial z}dz = 0. \end{aligned}\right\} \tag{2.12}$$

We now solve (2.12), for P, Q, and R to obtain

$$\frac{P}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}} = \frac{Q}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}} = \frac{R}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}},$$

which can be rewritten as

$$\frac{P}{\frac{\partial(u,v)}{\partial(y,z)}} = \frac{Q}{\frac{\partial(u,v)}{\partial(z,x)}} = \frac{R}{\frac{\partial(u,v)}{\partial(x,y)}}. \tag{2.13}$$

Now, we may recall from Section 1.4 that the relation $F(u,\ v)=0,$ where F is an arbitrary function, leads to the partial differential equation

$$p\frac{\partial(u,v)}{\partial(y,z)} + q\frac{\partial(u,v)}{\partial(z,x)} = \frac{\partial(u,v)}{\partial(x,y)} \tag{2.14}$$

Substituting for $\frac{\partial(u,v)}{\partial(y,z)}, \frac{\partial(u,v)}{\partial(z,x)}$, and $\frac{\partial(u,v)}{\partial(x,y)}$ from (2.13) in (2.14), we find that $F(u, v) = 0$, is a solution of (2.10). This completes the proof.

We shall illustrate this method through following examples:

Example (2.1)

Find the general integral of the following linear partial differential equations:

(i) $y^2 p - xyq = x(z-2y)$,

(ii) $(y+zx)p - (x+yz)q = x^2 - y^2$.

Solution

(i) The integral surface of the given PDE is generated by the integral curves of the auxiliary equations

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}. \qquad (2.15)$$

The first two members of the above Eq. (2.15) give us

$$\frac{dx}{y} = \frac{dy}{-x} \quad \text{or} \quad xdx = -ydy,$$

which on integration results in

$$\frac{x^2}{2} = -\frac{y^2}{2} + C \quad \text{or} \quad x^2 + y^2 = C_1. \qquad (2.16)$$

The last two members of Eq. (2.15) give

$$\frac{dy}{-y} = \frac{dz}{z-2y} \quad \text{or} \quad zdy - 2ydy = -ydz,$$

that is,

$$2ydy = ydz + zdy,$$

which on integration yields

$$y^2 = yz + C_2 \quad \text{or} \quad y^2 - yz = C_2. \qquad (2.17)$$

Hence, the curves given by Eqs. (2.16) and (2.17) generate the required integral surface as

$$F\left(x^2 + y^2, y^2 - yz\right) = 0.$$

(ii) The integral surface of the given PDE is generated by the integral curves of the auxiliary equation

$$\frac{dx}{y+zx} = \frac{dy}{-(x+yz)} = \frac{dz}{x^2 - y^2}.$$

To get the first integral curve, let us consider the first combination as

$$\frac{xdx + ydy}{xy + zx^2 - xy - y^2 z} = \frac{dz}{x^2 - y^2},$$

or

$$\frac{xdx + ydy}{z\left(x^2 - y^2\right)} = \frac{dz}{x^2 - y^2}.$$

That is,

$$xdx + ydy = zdz.$$

On integration, we get

$$\frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{2} = C \quad \text{or} \quad x^2 + y^2 - z^2 = C_1. \tag{2.18}$$

Similarly, for getting the second integral curve, let us consider the combination such as

$$\frac{ydx + xdy}{y^2 + xyz - x^2 - xyz} = \frac{dz}{x^2 - y^2},$$

or

$$ydx + xdy + dz = 0,$$

which on integration results in

$$xy + z = C_2. \tag{2.19}$$

Thus, the curves given by Eqs. (2.18) and (2.19) generate the required integral surface as

$$F\left(x^2 + y^2 - z^2, xy + z\right) = 0.$$

Example (2.2)

Use Lagrange's method to solve the equation

$$\begin{vmatrix} x & y & z \\ \alpha & \beta & \gamma \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & -1 \end{vmatrix} = 0,$$

where $z = z(x, y)$.

Solution

The given PDE can be written as

$$x\left[-\beta - \gamma\frac{\partial z}{\partial y}\right] - y\left[-\alpha - \gamma\frac{\partial z}{\partial x}\right] + z\left[\alpha\frac{\partial z}{\partial y} - \beta\frac{\partial z}{\partial x}\right] = 0,$$

or

$$(\gamma y - \beta z)\frac{\partial z}{\partial x} + (\alpha z - \gamma x)\frac{\partial z}{\partial y} = \beta x - \alpha y.$$

The corresponding auxiliary equations are

$$\frac{dx}{(\gamma y - \beta z)} = \frac{dy}{(\alpha z - \gamma x)} = \frac{dz}{(\beta x - \alpha y)}. \tag{2.20}$$

Using multipliers x, y, and z we find that each fraction is

$$\frac{xdx + ydy + zdz}{0} = \text{one fraction.}$$

Therefore,

$$xdx + ydy + zdz = 0,$$

which on integration yields

$$x^2 + y^2 + z^2 = C_1.$$

Similarly, using multipliers α, β , and γ, we find from Eq. (2.20) that each fraction is equal to

$$\alpha dx + \beta dy + \gamma dz = 0,$$

which on integration gives

$$\alpha x + \beta y + \gamma z = C_2.$$

Thus, the general solution of the given equation is found to be

$$F\left(x^2 + y^2 + z^2, \alpha x + \beta y + \gamma z\right) = 0.$$

Example (2.3)

Find the general integrals of the following linear PDEs:

(i) $pz - qz = z^2 + (x+y)^2$,

(ii) $\left(x^2 - yz\right)p + \left(y^2 - zx\right)q = z^2 - xy$.

Solution

(i) The integral surface of the given PDE is generated by the integral curves of the auxiliary equation

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x+y)^2}. \qquad (2.21)$$

The first two members of Eq. (2.21) give

$$dx + dy = 0,$$

which on integration yields

$$x + y = C_1. \qquad (2.22)$$

Now, considering Eq. (2.22) and the first and last members of Eq. (2.21), we obtain

$$\frac{2zdz}{z^2 + C_1^2} = 2dx,$$

or

$$dx = \frac{zdz}{z^2 + C_1^2},$$

which on integration yields

$$\ln\left(z^2 + C_1^2\right) = 2x + C_2,$$

or

$$\ln\left[z^2 + (x+y)^2\right] - 2x = C_2. \qquad (2.23)$$

Thus, the curves given by Eqs. (2.22) and (2.23) generates the integral surface for the given PDE as

$$F\left(x+y,\ln\left\{x^2+y^2+z^2+2xy\right\}-2x\right)=0.$$

(ii) The integral surface of the given PDE is given by the integral curves of the auxiliary equation

$$\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy}, \tag{2.24}$$

Equation (2.24) can be rewritten as

$$\frac{dx-dy}{(x-y)(x+y+z)}=\frac{dy-dz}{(y-z)(x+y+z)}=\frac{dz-dx}{(z-x)(x+y+z)}. \tag{2.25}$$

Considering the first two terms of Eq. (2.25) and integrating, we get

$$\ln(x-y)=\ln(y-z)+\ln C_1,$$

$$\frac{x-y}{y-z}=C_1, \tag{2.26}$$

Similarly, considering the last two terms of Eq. (2.25) and integrating, we obtain

$$\frac{y-z}{z-x}=C_2, \tag{2.27}$$

Thus, the integral curves given by Eqs. (2.26) and (2.27) generate the integral surface

$$F\left(\frac{x-y}{y-z},\frac{y-z}{z-x}\right)=0.$$

2.4 Integral Surfaces Passing Through a Given Curve

In the previous section, we have seen how a general solution for a given linear PDE can be obtained. Now, we shall make use of this general solution to find an integral surface containing a given curve as explained below.

Suppose, we have obtained two integral curves described by

$$\left.\begin{aligned} u(x,y,z)&=C_1\\ v(x,y,z)&=C_2 \end{aligned}\right\}, \tag{2.28}$$

from the auxiliary equations of a given PDE. Then, the solution of the given PDE can be written in the form

$$F(u,v)=0. \tag{2.29}$$

Suppose, we wish to determine an integral surface, containing a given curve C described by the parametric equations of the form

$$x=x(t),\quad y=y(t),\quad z=z(t), \tag{2.30}$$

where t is a parameter. Then, the particular solution (2.28) must be like

$$\left.\begin{aligned} u\{x(t),y(t),z(t)\} &= C_1, \\ v\{x(t),y(t),z(t)\} &= C_2. \end{aligned}\right\} \qquad (2.31)$$

Thus, we have two relations, from which we can eliminate the parameter t to obtain a relation of the type

$$F\left(C_1, C_2\right) = 0, \qquad (2.32)$$

which leads to the solution given by Eq. (2.29). For illustration, let us consider the following couple of examples.

Example (2.4)

Find the integral surface of the linear PDE

$$x\left(y^2+z\right)p - y\left(x^2+z\right)q = \left(x^2-y^2\right)z,$$

containing the straight line $x+y=0,\ z=1$.

Solution

The auxiliary equations for the given PDE are

$$\frac{dx}{x\left(y^2+z\right)} = \frac{dy}{-y\left(x^2+z\right)} = \frac{dz}{\left(x^2-y^2\right)z}, \qquad (2.32)$$

Using the multiplier $x\,y\,z$, we have

$$yzdx + zxdy + xydz = 0.$$

On integration, we get

$$xyz = C_1, \qquad (2.33)$$

Using the multipliers x, y and z. Then we find that each fraction in Eq. (2.32) is equal to

$$xdx + ydy + zdz = 0,$$

which on integration yields

$$x^2 + y^2 + z^2 = C_2. \qquad (2.34)$$

For the initial curve in question, we have the equations in parametric form as

$$x = t, \quad y = -t, \quad z = 1.$$

Substituting these values in Eqs. (2.33) and (2.34), we obtain

$$\left.\begin{aligned} -t^2 &= C_1 \\ 2t^2+1 &= C_2 \end{aligned}\right\}.$$

Eliminating the parameter t, we find

$$1 - 2C_1 = C_2,$$

or

$$2C_1 + C_2 - 1 = 0.$$

Hence, the required integral surface is

$$x^2 + y^2 + z^2 + 2xyz - 1 = 0$$

Example (2.5)

Find the integral surface of the linear PDE

$$xp + yq = z,$$

which contains the circle defined by

$$x^2 + y^2 + z^2 = 4, \quad x + y + z = 2.$$

Solution

The integral surface of the given PDE is generated by the integral curves of the auxiliary equation

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}. \tag{2.35}$$

Integration of the first two members of Eq. (2.35) gives

$$\ln x = \ln y + \ln C,$$

or

$$\frac{x}{y} = C_1. \tag{2.36}$$

Similarly, integration of the last two members of Eq. (2.35) yields

$$\frac{y}{z} = C_2. \tag{2.37}$$

Hence, the integral surface of the given PDE is

$$F\left(\frac{x}{y}, \frac{y}{z}\right) = 0. \tag{2.38}$$

If this integral surface also contains the given circle, then we have to find a relation between x/y and y/z.

The equation of the circle is

$$x^2 + y^2 + z^2 = 4 \tag{2.39}$$

$$x + y + z = 2 \tag{2.40}$$

From Eqs. (2.36) and (2.37), we have

$$y = x/C_1, \quad z = y/C_2 = x/C_1C_2$$

Substituting these values of y and z in Eqs. (2.39) and (2.40), we find

$$x^2 + \frac{x^2}{C_1^2} + \frac{x^2}{C_1^2C_2^2} = 4, \quad \text{or} \quad x^2\left(1 + \frac{1}{C_1^2} + \frac{1}{C_1^2C_2^2}\right) = 4, \tag{2.41}$$

and

$$x + \frac{x}{C_1} + \frac{x}{C_1C_2} = 2, \quad \text{or} \quad x\left(1 + \frac{1}{C_1} + \frac{1}{C_1C_2}\right) = 2, \tag{2.42}$$

From Eqs. (2.41) and (2.42) we observe

$$1 + \frac{1}{C_1^2} + \frac{1}{C_1^2C_2^2} = \left(1 + \frac{1}{C_1} + \frac{1}{C_1C_2}\right)^2,$$

which on simplification gives us

$$\frac{2}{C_1}+\frac{2}{C_1C_2}+\frac{2}{C_1^2C_2}=0,$$

that is,

$$C_1C_2+C_1+1=0.$$

Now, replacing C_1 by x/y and C_2 by y/z , we get the required integral surface as

$$\frac{x}{y}\frac{y}{z}+\frac{x}{y}+1=0,$$

or

$$\frac{x}{z}+\frac{x}{y}+1=0,$$

or

$$xy+xz+yz=0.$$

2.5 The Cauchy Problem for First Order Equations

Many problems in applied mathematics, science, and engineering involve partial differential equations. We rarely try to find or discuss the properties of a solution to these equations in its most general form. In most cases of interest, we deal with those solutions of partial differential equations which satisfy certain supplementary conditions. In the case of a first-order partial differential equation, we determine the specific solution by formulating an initial-value problem or a Cauchy problem.

Theorem 2.2 (The Cauchy Problem for a First-Order Partial Differential Equation).

Suppose that C is a given curve in the (x, y)-plane with its parametric equations

$$x=x_0(t),\ y=y_0(t), \qquad (2.43)$$

where t belongs to an interval $I\subset\mathbb{R}$, and the derivatives $x_0'(t),\ y_0'(t),$ are piecewise continuous functions, such that $(x_0')^2+(y_0')^2\neq 0.$ Also, suppose that $z=z_0(t)$ is a given function on the curve C. Then, there exists a solution $z=z(x,y)$ of the equation

$$F(x,y,z,p,q)=0, \qquad (2.44)$$

in a domain $D\subset\mathbb{R}^2$ containing the curve C for all $t\in I$, and the solution $z=z(x,y)$ satisfies the given initial data, that is,

$$z\left[x_0(t),y_0(t)\right]=z_0(t), \qquad (2.45)$$

for all values of $t\in I$.

In short, the Cauchy problem is to determine a solution of equation (2.44) in a neighborhood of C, such that the solution $z=z(x,y)$ takes a prescribed value $z_0(t)$ on C. The curve C is called the initial curve of the problem, and $z_0(t)$ is called the initial data. Equation (2.45) is called the initial condition of the problem.

Theorem 2.3 (The Cauchy Problem for a Quasi-linear Equation).
Suppose that

(i) $x_0(t), y_0(t)$, and $z_0(t)$, are continuously differentiable functions of t in a closed interval, $0 \le t \le 1$.

(ii) $P(x,y,z), Q(x,y,z)$, and $R(x,y,z)$ are functions of x, y, and z with continuous first-order partial derivatives with respect to their arguments in some domain D of (x,y,z)-space containing the initial curve

$$C: \ x = x_0(t), \ y = y_0(t), \ z = z_0(t), \tag{2.46}$$

where $0 \le t \le 1$.

(iii)The functions P, Q, x_0 and y_0 satisfying the condition

$$P\left[x_0(t), y_0(t), z_0(t)\right] y_0'(t) - Q\left[x_0(t), y_0(t), z_0(t)\right] x_0'(t) \ne 0, \tag{2.47}$$

Then there exists a unique solution $z = z(x,y)$, of the quasi-linear equation

$$Pp + Qq = R, \tag{2.48}$$

in the neighborhood of $\Gamma: \ x = x_0(t), \ y = y_0(t)$, and this solution satisfies the initial condition

$$z\left[x_0(t), y_0(t)\right] = z_0(t), \quad \text{for} \ \ 0 \le t \le 1. \tag{2.49}$$

Note: The condition (2.47) excludes the possibility that Γ could be a characteristic.

Example 2.6

Show why there is no solution of the following partial differential equation

$$z_x + z_y = z, \tag{2.50}$$

that passes through the straight line $z(x,x) = 1$?

Solution

The initial curve in $\mathbb{R}^3$ can be given parametrically as

$$C: x_0(t) = t, \ y_0(t) = t, \ z_0(t) = 1 \ .$$

The characteristic equations in parametric form are

$$\frac{dx}{ds} = 1 = P, \ \ \frac{dy}{ds} = 1 = Q, \ \ \frac{dz}{ds} = z = R. \tag{2.51}$$

Using condition (2.47), we have

$$Py_0'(t) - Qx_0'(t) = (1)(1) - (1)(1) = 0$$

Hence, the problem has no solution.

Example 2.7

Discuss the following Cauchy problem

$$z_x + z_y = 0, \ \ z(\alpha x, x) = \mathrm{e}^{-y^2} \ , \ \alpha \in \mathbb{R} \tag{2.52}$$

Solution

The initial curve in $\mathbb{R}^3$ can be given parametrically as

$$C: x_0(t) = \alpha t \ , \ y_0(t) = t \ , \ z_0(t) = \mathrm{e}^{-t^2} . \qquad (2.53)$$

The characteristic equations in parametric form are

$$\frac{dx}{ds} = 1 = P, \ \frac{dy}{ds} = 1 = Q, \ \frac{dz}{ds} = 0 = R. \qquad (2.54)$$

Using condition (2.47), we have

$$Py_0'(t) - Qx_0'(t) = (1)(1) - (1)(\alpha) = 1 - \alpha$$

If $\alpha = 1$, then Cauchy problem (2.52) has no solution. Otherwise ($\alpha \neq 1$), the solutions of characteristic equations (2.54) are

$$x(s,t) = s + c_1, \ y(s,t) = s + c_2, \ z(s,t) = c_3 .$$

Using the initial conditions (2.53), we have

$$x(s,t) = s + \alpha t, \ y(s,t) = s + t, \ z(s,t) = \mathrm{e}^{-t^2} , \qquad (2.55)$$

which is the solution of (2.52) in the parametric form. To reach the solution of (2.52) in the Cartesian form, eliminating s and t form Eq. (2.55),

$$y - x = (1 - \alpha)t \quad \Rightarrow \ t = \frac{y - x}{1 - \alpha}$$

Thus, the solutions of Cauchy problem (2.52) is

$$z(x,y) = \exp\left[-\left(\frac{y - x}{1 - \alpha} \right)^2 \right]$$

Example 2.8

Find the solution of the equation

$$z(x+y)z_x + z(x-y)z_y = x^2 + y^2, \qquad (2.56)$$

with the Cauchy data $z = 0$ on $y = 2x$.

Solution

The characteristic equations are

$$\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2} = \frac{ydx + xdy - zdz}{0} = \frac{xdx - ydy - zdz}{0} .$$

Consequently,

$$d\left[\left(xy - \frac{1}{2}z^2 \right) \right] = 0 \quad \text{and} \quad d\left[\frac{1}{2}\left(x^2 - y^2 - z^2 \right) \right] = 0$$

These give two integrals

$$z^2 - x^2 + y^2 = C_1 \quad \text{and} \quad 2xy - z^2 = C_2,$$

where C_1 and C_2 are arbitrary constants. Hence, the general solution is

$$f(x^2 - y^2 - z^2, 2xy - z^2) = 0,$$

where f is an arbitrary function.

Using the Cauchy data in (2.56), we obtain $4C_1 = 3C_2$. Therefore

$$4\left(z^2 - x^2 + y^2 \right) = 3\left(2xy - z^2 \right).$$

Thus, the solution of equation (2.56) is given by

$$7z^2 = 6xy + 4\left(x^2 - y^2\right).$$

Example 2.9

Obtain the solution of the linear equation

$$z_x - z_y = 1, \tag{2.57}$$

with the Cauchy data

$$z(x,0) = x^2.$$

Solution

The characteristic equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{1}.$$

Obviously,

$$\frac{dy}{dx} = -1 \quad \text{and} \quad \frac{dz}{dx} = 1.$$

Clearly,

$$x + y = C_1 \quad \text{and} \quad z - x = C_2.$$

Thus, the general solution is given by

$$z = x + f(x+y),$$

where f is an arbitrary function.

We now use the Cauchy data to find $f(x) = x^2 - x$, and hence, the solution is

$$z(x,y) = (x+y)^2 - y.$$

Example 2.10

Obtain the solution of the equation

$$(y-z)z_x + (z-x)z_y = x - y, \tag{2.58}$$

with the condition $z = 0$ on $x\,y = 1$.

Solution

The characteristic equations for equation (2.58) are

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y}$$

The parametric forms of these equations are

$$\frac{dx}{dt} = y - z, \quad \frac{dy}{dt} = z - x, \quad \frac{dz}{dt} = x - y.$$

These lead to the following equations:

$$\dot{x} + \dot{y} + \dot{z} = 0 \quad \text{and} \quad x\dot{x} + y\dot{y} + z\dot{z} = 0, \tag{2.59}$$

where the dot denotes the derivative with respect to t.

Integrating (2.59), we obtain

$$x + y + z = C_1 \quad \text{and} \quad x^2 + y^2 + z^2 = C_2.$$

These equations represent circles. Using the Cauchy data, we find that

$$C_1^2 = (x+y)^2 = x^2 + y^2 + 2xy = C_2 + 2.$$

Thus, the integral surface is described by

$$(x+y+z)^2 = x^2 + y^2 + z^2 + 2.$$

Hence, the solution is given by

$$z(x,y) = \frac{1-xy}{x+y}.$$

Example 2.11

Solve the linear equation

$$yz_x + xz_y = z,$$

with the Cauchy data

$$z(x,0) = x^3 \quad \text{and} \quad z(0,y) = y^3.$$

Solution

The characteristic equations are

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{z}$$

or

$$\frac{dz}{z} = \frac{dx-dy}{y-x} = \frac{dx+dy}{y+x}$$

Solving these equations, we obtain

$$z = \frac{C_1}{x-y} = C_2(x+y),$$

or

$$z = C_2(x+y), \quad x^2 - y^2 = \frac{C_1}{C_2} = C.$$

Thus, the general solution is given by

$$f\left(\frac{z}{x+y}, x^2 - y^2\right) = 0,$$

or, equivalently,

$$z(x,y) = (x+y)g\left(x^2 - y^2\right).$$

Using the Cauchy data, we find that $g(x^2) = x^2$, that is $g(x) = x$. Consequently, the solution becomes

$$z(x,y) = (x+y)(x^2 - y^2).$$

Example 2.12

Determine the integral surfaces of the equation

$$x\left(y^2 + z\right)z_x - y\left(x^2 + z\right)z_y = \left(x^2 - y^2\right)z,$$

with the data

$$x + y = 0, \quad z = 1.$$

Solution

The characteristic equations are

$$\frac{dx}{x\left(y^2+z\right)}=\frac{dy}{-y\left(x^2+z\right)}=\frac{dz}{\left(x^2-y^2\right)z}, \qquad (2.60)$$

or

$$\frac{\frac{dx}{x}}{\left(y^2+z\right)}=\frac{\frac{dy}{y}}{-\left(x^2+z\right)}=\frac{\frac{dz}{z}}{\left(x^2-y^2\right)}=\frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0}.$$

Consequently,

$$\log(xyz)=\log C_1,$$

or

$$xyz=C_1.$$

From (2.60), we obtain

$$\frac{xdx}{x^2\left(y^2+z\right)}=\frac{ydy}{-y^2\left(x^2+z\right)}=\frac{du}{\left(x^2-y^2\right)u}=\frac{xdx+ydy-dz}{0},$$

whence we find that

$$x^2+y^2-2z=C_2.$$

Using the given data, we obtain

$$C_1=-x^2 \quad \text{and} \quad C_2=2x^2-2,$$

so that

$$C_2=-2\left(C_1+1\right).$$

Thus the integral surface is given by

$$x^2+y^2-2z=-2-2xyz,$$

or

$$2xyz+x^2+y^2-2z+2=0.$$

Example 2.13

Obtain the solution of the equation

$$xz_x+yz_y=x\exp(-z), \qquad (2.61)$$

with the data

$$z=0 \quad \text{on} \quad y=x^2$$

Solution

The characteristic equations are

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{x\exp(-z)}, \qquad (2.62)$$

or

$$\frac{y}{x}=C_1.$$

We also obtain from (2.62) that $dx = \mathrm{e}^z dz$ which can be integrated to find

$$\mathrm{e}^z = x + C_2.$$

Thus, the general solution is given by

$$f\left(\mathrm{e}^z - x, \frac{y}{x}\right) = 0,$$

or, equivalently,

$$\mathrm{e}^z = x + g\left(\frac{y}{x}\right).$$

Applying the Cauchy data, we obtain $g(x) = 1 - x$. Thus, the solution of (2.61) is given by

$$\mathrm{e}^z = x + 1 - \frac{y}{x},$$

or

$$z = \ln\left(x + 1 - \frac{y}{x}\right).$$

Example 2.14

Solve the initial-value problem

$$z_t + zz_x = x, \quad z(x,0) = f(x),$$

where (a) $f(x) = 1$ and (b) $f(x) = x$.

Solution

The characteristic equations are

$$\frac{dt}{1} = \frac{dx}{z} = \frac{dz}{x} = \frac{d(x+z)}{x+z}.$$

Integration gives

$$t = \ln(x+z) - \ln C_1,$$

or

$$(z+x)\mathrm{e}^{-t} = C_1.$$

Similarly, we get

$$z^2 - x^2 = C_2.$$

For case (a), we obtain

$$1 + x = C_1 \quad \text{and} \quad 1 - x^2 = C_2, \quad \text{and hence} \quad C_2 = 2C_1 - C_1^2.$$

Thus,

$$(z^2 - x^2) = 2(z+x)\mathrm{e}^{-t} - (z+x)^2 \mathrm{e}^{-2t}$$

or

$$z - x = 2\mathrm{e}^{-t} - (z+x)\mathrm{e}^{-2t}.$$

A simple manipulation gives the solution

$$z(x,t) = x \tanh t + \operatorname{sech} t.$$

Case (b) is left to the reader as an exercise.

Example 2.15

Find the integral surface of the equation

$$zz_x + z_y = 1, \tag{2.63}$$

so that the surface passes through an initial curve represented parametrically by

$$x = x_0(s), \quad y = y_0(s), \quad z = z_0(s),$$

where s is a parameter.

Solution

The characteristic equations for the given equations are

$$\frac{dx}{z} = \frac{dy}{1} = \frac{dz}{1},$$

which are, in the parametric form

$$\frac{dx}{d\tau} = z, \quad \frac{dy}{d\tau} = 1, \quad \frac{dz}{d\tau} = 1, \tag{2.64}$$

where τ is a parameter. Thus the solutions of this parametric system in general depend on two parameters s and τ. We solve this system (2.64) with the initial data

$$x(s,0) = x_0(s), \quad y(s,0) = y_0(s), \quad z(s,0) = z_0(s).$$

The solutions of (2.63) with the given initial data are

$$\left.\begin{aligned} x(s,\tau) &= \frac{\tau^2}{2} + \tau z_0(s) + x_0(s) \\ y(s,\tau) &= \tau + y_0(s) \\ u(s,\tau) &= \tau + z_0(s) \end{aligned}\right\}.$$

We choose a particular set of values for the initial data as

$$x(s,0) = 2s^2, \quad y(s,0) = 2s, \quad z(s,0) = 0, \quad s > 0.$$

Therefore, the solutions are given by

$$x = \frac{1}{2}\tau^2 + 2s^2, \quad y = \tau + 2s, \quad z = \tau. \tag{2.65}$$

Eliminating τ and s from (2.65) gives the integral surface

$$(z - y)^2 + z^2 = 2x$$

or

$$2z = y \pm \left(4x - y^2\right)^{\frac{1}{2}}.$$

The solution surface satisfying the data $z = 0$ on $y^2 = 2x$ is given by

$$2z = y - \left(4x - y^2\right)^{\frac{1}{2}}.$$

This represents the solution surface only when $y^2 < 4x$. Thus, the solution does not exist for $y^2 > 4x$ and is not differentiable when $y^2 = 4x$.

2.6 Canonical Forms of First-Order Linear Equations

It is often convenient to transform the more general first-order linear partial differential equation

$$P(x,y)z_x + Q(x,y)z_y + R(x,y)z = f(x,y), \qquad (2.66)$$

into a canonical (or standard) form which can be easily integrated to find the general solution of (2.66). We use the characteristics of equation (2.66) to introduce a new transformation by equations

$$\xi = \xi(x,y), \quad \eta = \eta(x,y), \qquad (2.67)$$

where ξ and η are once continuously differentiable and their Jacobian $J(x,y) \equiv \xi_x \eta_y - \xi_y \eta_x$ is nonzero in a domain of interest so that x and y can be determined uniquely from the equations (2.67). Thus, by chain rule,

$$z_x = z_\xi \xi_x + z_\eta \eta_x, \quad z_y = z_\xi \xi_y + z_\eta \eta_y, \qquad (2.68)$$

we substitute these partial derivatives (2.68) into (2.66) to obtain the equation

$$P^* z_\xi + Q^* z_\eta + R^* z = f^*, \qquad (2.69)$$

where

$$P^* = P\xi_x + Q\xi_y, \quad Q^* = P\eta_x + Q\eta_y, \; R^* = R, \; f^* = f. \qquad (2.70)$$

From (2.70) we see that $Q^* = 0$ if η is a solution of the first-order equation

$$P\eta_x + Q\eta_y = 0. \qquad (2.71)$$

This equation has infinitely many solutions. We can obtain one of them by assigning initial condition on a non-characteristic initial curve and solving the resulting initial-value problem according to the method described earlier. Since $\eta(x,y)$ satisfies equation (2.71), the curves $\eta(x,y) = C$ are always characteristic curves of equation (2.66). Thus, one set of the new transformations are the characteristic curves of (2.66). The second set, $\xi(x,y) == C$, can be chosen to be any one parameter family of smooth curves such that $J(x,y) \equiv \xi_x \eta_y - \xi_y \eta_x \neq 0$. Finally, since $Q^* = 0$ and $P^* \neq 0$ in D, we can divide (2.69) by P^* to obtain the canonical form

$$z_\xi + \alpha(\xi,\eta) z = \beta(\xi,\eta), \qquad (2.72)$$

where $\alpha(\xi,\eta) = \dfrac{R^*}{P^*}$ and $\beta(\xi,\eta) = \dfrac{f^*}{P^*}$.

Equation (2.72) represents an ordinary differential equation with ξ as the independent variable and η as a parameter which may be treated as constant. This equation (2.72) is called the canonical form of equation (2.66) in terms of the coordinates (ξ,η). Generally, the canonical equation

(2.72) can easily be integrated and the general solution of (2.66) can be obtained after replacing ξ and η by the original variables x and y. In practice, it is convenient to choose $\xi = \xi(x, y)$ and $\eta(x, y) = y$ or $\xi = x$ and $\eta = \eta(x, y)$ such that $J \neq 0$.

Example 2.16

Reduce each of the following equations:

$$z_x - z_y = z, \qquad (2.73)$$

$$yu_x + u_y = x. \qquad (2.74)$$

to canonical form, and obtain the general solution.

Solution

In (2.73), $a = 1, b = -1, c = -1$ and $d = 0$. The characteristic equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{z}.$$

The characteristic curves are $\xi = x + y = c_1$, and we choose $\eta = y = c_2$ where c_1 and c_2 are constants. Consequently, $z_x = z_\xi$ and $z_y = z_\xi + z_\eta$, and hence, equation (2.73) becomes

$$z_\eta = -z$$

Integrating this equation gives

$$\ln z(\xi, \eta) = -\eta + \ln f(\xi),$$

where $f(\xi)$ is an arbitrary function of ξ only. Equivalently,

$$z(\xi, \eta) = f(\xi) e^{-\eta}.$$

In terms of the original variables x and y , the general solution of equation (2.73) is

$$z(x, y) = f(x + y) e^{-y}$$

where f is an arbitrary function.

The characteristic equations of (2.74) are

$$\frac{dx}{y} = \frac{dy}{1} = \frac{du}{x}.$$

It follows from the first two equations that $\xi(x, y) = x - \frac{y^2}{2} = c_1$; we choose $\eta(x, y) = y = c_2$. Consequently, $u_x = u_\xi$ and $u_y = -yu_\xi + u_\eta$ and hence, equation (2.74) reduces to

$$u_\eta = \xi + \frac{1}{2}\eta^2.$$

Integrating this equation gives the general solution

$$u(\xi, \eta) = \xi\eta + \frac{1}{6}\eta^3 + f(\xi),$$

where f is an arbitrary function.

Thus, the general solution of (2.74) in terms of x and y is

$$u(x,y) = xy - \frac{1}{3}y^3 + f\left(x - \frac{y^2}{2}\right).$$

2.7 Method of Separation of Variables

During the last two centuries several methods have been developed for solving partial differential equations. Among these, a technique known as the method of separation of variables is perhaps the oldest systematic method for solving partial differential equations. Its essential feature is to transform the partial differential equations by a set of ordinary differential equations. The required solution of the partial differential equations is then exposed as a product $u(x,y) = X(x)Y(y) \neq 0$, or as a sum $u(x,y) = X(x) + Y(y)$, where $X(x)$ and $Y(y)$ are functions of x and y, respectively. Many significant problems in partial differential equations can be solved by the method of separation of variables. This method has been considerably refined and generalized over the last two centuries and is one of the classical techniques of applied mathematics, mathematical physics and engineering science.

Usually, the first-order partial differential equation can be solved by separation of variables without the need for Fourier series. The main purpose of this section is to illustrate the method by examples.

Example 2.17

Solve the initial-value problem

$$u_x + 2u_y = 0, \quad u(0,y) = 4e^{-2y}. \tag{2.75}$$

Solution

We seek a separable solution $u(x,y) = X(x)Y(y) \neq 0$ and substitute into the equation (2.75) to obtain

$$X'(x)Y(y) + 2X(x)Y'(y) = 0.$$

This can also be expressed in the form

$$\frac{X'(x)}{2X(x)} = -\frac{Y'(y)}{Y(y)}, \tag{2.76}$$

Since the left-hand side of this equation is a function of x only and the right-hand is a function of y only, it follows that (2.76) can be true if both sides are equal to the same constant value λ which is called an arbitrary separation constant. Consequently, (2.76) gives two ordinary differential equations

$$X'(x) - 2\lambda X(x) = 0, \quad Y'(y) + \lambda Y(y) = 0.$$

These equations have solutions given, respectively, by

$$X(x) = Ae^{2\lambda x} \quad \text{and} \quad Y(y) = Be^{-\lambda y},$$

Where A and B are arbitrary integrating constants. Consequently, the general solution is given by

$$u(x,y) = AB\exp(2\lambda x - \lambda y) = C\exp(2\lambda x - \lambda y),$$

where $C = AB$ is an arbitrary constant.
Using the initial condition in (2.75), we find

$$4e^{-2y} = u(0,y) = Ce^{-\lambda y},$$

and hence, we deduce that $C = 4$ and $\lambda = 2$. Therefore, the final solution is

$$u(x,y) = 4\exp(4x - 2y).$$

Example 2.18
Solve the equation

$$y^2u_x^2 + x^2u_y^2 = (xyu)^2. \tag{2.77}$$

Solution
We assume $u(x,y) = f(x)g(y) \neq 0$ is a separable solution of (2.77), and substitute into the equation. Consequently, we obtain

$$y^2\left\{f'(x)g(y)\right\}^2 + x^2\left\{f(x)g'(y)\right\}^2 = x^2y^2\{f(x)g(y)\}^2,$$

or, equivalently,

$$\frac{1}{x^2}\left\{\frac{f'(x)}{f(x)}\right\}^2 + \frac{1}{y^2}\left\{\frac{g'(y)}{g(y)}\right\}^2 = 1,$$

or

$$\frac{1}{x^2}\left\{\frac{f'(x)}{f(x)}\right\}^2 = 1 - \frac{1}{y^2}\left\{\frac{g'(y)}{g(y)}\right\}^2 = \lambda^2,$$

where λ^2 is a separation constant. Thus,

$$\frac{1}{x}\frac{f'(x)}{f(x)} = \lambda \quad \text{and} \quad \frac{g'(y)}{yg(y)} = \sqrt{1-\lambda^2}.$$

Solving these ordinary differential equations, we find

$$f(x) = A\exp\left(\frac{\lambda}{2}x^2\right) \quad \text{and} \quad g(y) = B\exp\left(\frac{1}{2}y\sqrt{1-\lambda^2}\right),$$

where A and B are arbitrary constant. Thus, the general solution is

$$u(x,y) = C\exp\left(\frac{\lambda}{2}x^2 + \frac{1}{2}y^2\sqrt{1-\lambda^2}\right). \tag{2.78}$$

Where $C = AB$ is an arbitrary constant. Using the initial condition $u(x,0) = 3\exp\left(x^2/4\right)$, we can determine both C and λ in (2.78). It turns out that $C = 3$ and $\lambda = (1/2)$, and the solution becomes

$$u(x,y) = 3\exp\left[\frac{1}{4}\left(x^2 + y^2\sqrt{3}\right)\right]$$

Example 2.19
Use the separation of variables $u(x,y) = f(x) + g(y)$ to solve the equation

$$u_x^2 + u_y^2 = 1. \qquad (2.79)$$

Solution

Obviously,

$\{f'(x)\}^2 = 1 - \{g'(y)\}^2 = \lambda^2$ where λ^2 is a separation constant. Thus, we obtain

$$f'(x) = \lambda \quad \text{and} \quad g'(y) = \sqrt{1-\lambda^2}.$$

Solving these ordinary differential equations, we find

$$f(x) = \lambda x + A \quad \text{and} \quad g(y) = y\sqrt{1-\lambda^2} + B,$$

where A and B are constants of integration. Finally, the solution of (2.79) is given by

$$u(x,y) = \lambda x + y\sqrt{1-\lambda^2} + C,$$

where $C = A + B$ is an arbitrary constant.

Example 2.20

Use $u(x,y) = f(x) + g(y)$ to solve the equation

$$u_x^2 + u_y + x^2 = 0. \qquad (2.80)$$

Solution

Obviously, equation (2.80) has the separable form

$$\{f'(x)\}^2 + x^2 = -g'(y) = \lambda^2,$$

where λ^2 is a separation constant. Consequently,

$$f'(x) = \sqrt{\lambda^2 - x^2} \text{ and } g'(y) = -\lambda^2.$$

These can be integrated to obtain

$$f(x) = \int\sqrt{\lambda^2 - x^2}\,dx + A = \lambda^2\int \cos^2\theta\, d\theta + A \qquad (x = \lambda\sin\theta)$$

$$= \frac{1}{2}\lambda^2\left[\sin^{-1}\left(\frac{x}{\lambda}\right) + \frac{x}{\lambda}\sqrt{1-\frac{x^2}{\lambda^2}}\right] + A,$$

and

$$g(y) = -\lambda^2 y + B$$

Finally, the general solution is given by

$$u(x,y) = \frac{1}{2}\lambda^2\sin^{-1}\left(\frac{x}{\lambda}\right) + \frac{x}{2}\sqrt{\lambda^2 - x^2} - \lambda^2 y + C,$$

where $C = A + B$ is an arbitrary constant.

Example 2.21

Use $v = \ln u$ and $v = f(x) + g(y)$ to solve the equation

$$x^2u_x^2 + y^2u_y^2 = u^2. \qquad (2.81)$$

Solution

In view of $v = \ln u$, $v_x = \frac{1}{u} u_x$ and $v_y = \frac{1}{u} u_y$, and hence, equation (2.81) becomes

$$x^2 v_x^2 + y^2 v_y^2 = 1. \qquad (2.82)$$

Substitute $v(x, y) = f(x) + g(y)$ into (2.82) to obtain

$$x^2 \{ f'(x) \}^2 + y^2 \{ g'(y) \}^2 = 1,$$

or

$$x^2 \{ f'(x) \}^2 = 1 - y^2 \{ g'(y) \}^2 = \lambda^2,$$

where λ^2 is a separation constant. Thus, we obtain

$$f'(x) = \frac{\lambda}{x} \quad \text{and} \quad g'(y) = \frac{1}{y}\sqrt{1-\lambda^2}.$$

Integrating these equations gives

$$f(x) = \lambda \ln x + A \quad \text{and} \quad g(y) = \sqrt{1-\lambda^2} \ln y + B,$$

where A and B are integrating constants. Therefore, the general solution of (2.82) is given by

$$v(x, y) = \lambda \ln x + \sqrt{1-\lambda^2} \ln y + \ln C = \ln\left(x^{\lambda} \cdot y^{\sqrt{1-\lambda^2}} \cdot C \right)$$

where $\ln C = A + B$. The final solution is

$$u(x, y) = \mathrm{e}^{v} = C x^{\lambda} \cdot y^{\sqrt{1-\lambda^2}},$$

where C is an integrating constant.

2.6 Surfaces Orthogonal to A Given System Of Surfaces

One of the useful applications of the theory of linear first order PDE is to find the system of surfaces orthogonal to a given system of surfaces.

Let a one-parameter family of surfaces is described by the equation

$$F(x, y, z) = C. \qquad (2.83)$$

Then, the task is to determine the system of surfaces which cut each of the given surfaces orthogonally. Let (x, y, z) be a point on the surface given by Eq. (2.83), where the normal to the surface will have direction ratios $(\partial F / \partial x, \partial F / \partial y, \partial F / \partial z)$ which may be denoted by P , Q, R .

Let

$$z = \phi(x, y)$$

be the surface which cuts each of the given system orthogonally (see Fig. 2). Then, its normal at the point (x, y, z) will have direction ratios $(\partial z / \partial x, \partial z / \partial y, -1)$ which, of course, will be perpendicular to the normal to the surfaces characterized by Eq. (2.83). As a consequence we have a relation

$$P \frac{\partial z}{\partial x} + Q \frac{\partial z}{\partial y} - R = 0,$$

or

$$Pp + Qq = R, \tag{2.84}$$

which is a linear PDE of Lagranges type, and can be recast into

$$\frac{\partial F}{\partial x}\frac{\partial z}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial z}{\partial y} = \frac{\partial F}{\partial z}. \tag{2.85}$$

Thus, any solution of the linear first order PDE of the type given by either Eq. (2.84) or (2.85) is orthogonal to every surface of the system described by Eq. (2.83). In other words, the surfaces orthogonal to the system (2.83) are the surfaces generated by the integral curves of the auxiliary equations

$$\frac{dx}{\partial F/\partial x} = \frac{dy}{\partial F/\partial y} = \frac{dz}{\partial F/\partial z}.$$

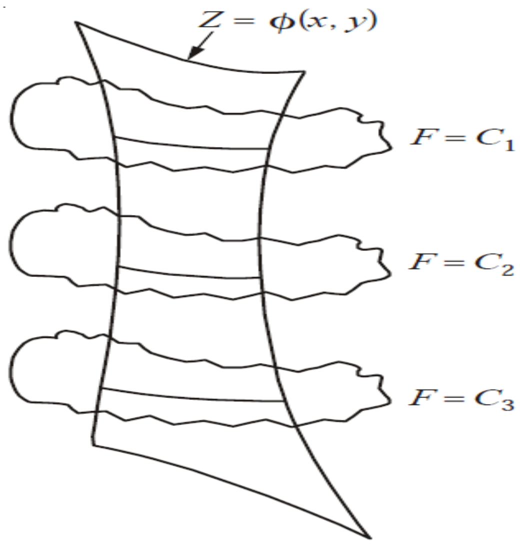

Fig. 2 Orthogonal surface to a given system of surfaces.

Example 2.22

Find the surface perpendicularly intersecting the family of surfaces with a parameter given by the equation $(x^2 + y^2)z = c$ and passing through the curve $y^2 = x$, $z = 0$: Here c is a parameter.

Solution

Let us write the given surface family as

$$f(x, y, z) = (x^2 + y^2)z - c = 0.$$

Using f_x , f_y and f_z

$$f_x = 2xz, \quad f_y = 2yz \ , \quad f_z = x^2 + y^2.$$

The Lagrange system, which corresponds to the partial differential equation of orthogonal surfaces is given by

$$\frac{dx}{2xz} = \frac{dy}{2yz} = \frac{dz}{x^2 + y^2}.$$

From this, two independent first integrals are as follows

$$\frac{y}{x}=c_1 \quad , \quad x^2+y^2-2z^2=c_2 .$$

General equation of surfaces perpendicular to the given family of surfaces are given by

$$F\left(\frac{y}{x}, x^2+y^2-2z^2\right)=0,$$

or

$$x^2+y^2-2z^2=g\left(\frac{y}{x}\right)$$

where F and g are arbitrary functions. To find the special surface that passes through the curve $y^2=x$, $z=0$, we write the parametric equation of curve as

$$x=t,\ y=t^2,\ z=0.$$

From this, we obtain

$$c_1=t,\ \ c_2=t^2+t^4\ \ \Rightarrow c_2=c_1^2+c_1^4 .$$

Thus, the desired surface has the equation

$$x^2+y^2-2z^2=\left(\frac{y}{x}\right)^4+\left(\frac{y}{x}\right)^2 .$$

Example 2.23

Find the surface perpendicularly intersecting the family of surfaces with a parameter given by the equation $z=cxy\left(x^2+y^2\right)$. Here c is a parameter.

Solution

Let's write the given surface family as

$$f(x,y,z)=\frac{xy\left(x^2+y^2\right)}{z}=\frac{1}{c}.$$

Using f_x, f_y and f_z ,

$$f_x=\frac{3x^2y+y^3}{z} \quad , \quad f_y=\frac{3y^2x+x^3}{z} \quad , \quad f_z=-\frac{xy\left(x^2+y^2\right)}{z^2}$$

The Lagrange system, which corresponds to the partial differential equation of orthogonal surfaces is given by

$$\frac{zdx}{3x^2y+y^3}=\frac{zdy}{3y^2x+x^3}=\frac{-z^2dz}{xy\left(x^2+y^2\right)}.$$

From this, we can write

$$\frac{xzdx+yzdy}{xy\left(3x^2+y^2\right)+xy\left(x^2+3y^2\right)}=\frac{-z^2dz}{xy\left(x^2+y^2\right)},$$

$$\frac{xzdx+yzdy}{xy\left(4x^2+4y^2\right)}=\frac{-z^2dz}{xy\left(x^2+y^2\right)},$$

$$xzdx+yzdy=-4z^2dz,$$

$$xdx+ydy=-4zdz,$$

$$x^2+y^2+4z^2=c_1.$$

The second solution is

$$\Rightarrow\frac{zdx+zdy}{(x+y)^3}=-\frac{zdx-zdy}{(x-y)^3}$$

$$\Rightarrow\frac{d(x+y)}{(x+y)^3}=-\frac{d(x-y)}{(x-y)^3}$$

$$\Rightarrow\frac{1}{(x+y)^2}+\frac{1}{(x-y)^2}=c_2.$$

The general equation of surfaces perpendicular to the given family of surfaces are given by

$$F\left(x^2+y^2+4z^2,\frac{1}{(x+y)^2}+\frac{1}{(x-y)^2}\right)=0,$$

$$\frac{1}{(x+y)^2}+\frac{1}{(x-y)^2}=g\left(x^2+y^2+4z^2\right),$$

where F and g are arbitrary functions.

Example 2.24

Find the equation of the system of surfaces which cut orthogonally the cones of the system $x^2+y^2+z^2=cxy$, c being a parameter.

Solution

The given system of surfaces is $F(x,y,z)=\dfrac{x^2+y^2+z^2}{xy}=c$. The auxiliary equations are

$$\frac{dx}{\partial F/\partial x}=\frac{dy}{\partial F/\partial y}=\frac{dz}{\partial F/\partial z}\Rightarrow\frac{dx}{\frac{1}{y}-\frac{y}{x^2}-\frac{z^2}{x^2y}}=\frac{dy}{-\frac{x}{y^2}+\frac{1}{x}-\frac{z^2}{xy^2}}=\frac{dz}{\frac{2z}{xy}},$$

i.e.

$$\frac{xdx}{x^2-y^2-z^2}=\frac{ydy}{-x^2+y^2-z^2}=\frac{dz}{2z}.$$

It follows that $xdx + ydy + zdz = 0$ and $\dfrac{xdx - ydy}{x^2 - y^2} = \dfrac{dz}{z}$ leading to the solutions

$$x^2 + y^2 + z^2 = c_1 \text{ and } \frac{x^2 - y^2}{z^2} = c_2,$$

respectively, where c_1 and c_2 are arbitrary constants. Thus the general solution of the given equation is $\phi\left(x^2 + y^2 + z^2, \dfrac{x^2 - y^2}{z^2}\right) = 0$ and the equation of the required system of surfaces is $x^2 + y^2 + z^2 = f\left(\dfrac{x^2 - y^2}{z^2}\right)$

Example 2.25

Find the surface which intersects the surfaces of the system $z(x + y) = c(3z + 1)$ orthogonally and passes through the circle $x^2 + y^2 = 1,\ z = 1$.

Solution

The given system of surfaces is $F(x, y, z) = \dfrac{z(x + y)}{3z + 1} = c$, c being parameter. The auxiliary equations are

$$\frac{dx}{\partial F / \partial x} = \frac{dy}{\partial F / \partial y} = \frac{dz}{\partial F / \partial z} \Rightarrow \frac{dx}{\frac{z}{3z + 1}} = \frac{dy}{\frac{z}{3z + 1}} = \frac{dz}{\frac{x + y}{(3z + 1)^2}},$$

i.e.

$$\frac{dx}{z} = \frac{dy}{z} = \frac{(3z + 1)dz}{x + y}. \ .$$

It follows that $dx - dy = 0$ and $(x + y)d(x + y) - \left(6z^2 + 2z\right)dz = 0$, whose solutions are

$$x - y = c_1 \text{ and } (x + y)^2 - 2z^2(2z + 1) = c_2,$$

where c_1 and c_2 are constants.

Now the given circle has parametric equations $x = \cos t$, $y = \sin t$, $z = 1$ so that $\cos t - \sin t = c_1$ and $(\cos t + \sin t)^2 - 6 = c_2$, i.e. $\sin 2t = 1 - c_1^2$ as well as $\sin 2t = 5 + c_2$. Eliminating t between these two relations, we get

$$c_1^2 + c_2 + 4 = 0 \Rightarrow (x - y)^2 + (x + y)^2 - 2z^2(2z + 1) + 4 = 0,$$

i.e.

$$x^2 + y^2 = 2z^3 + z^2 - 2 \ ,$$

which is the equation of the required surface.

2.7 First Order Non-Linear Equations

In this section, we will discuss the problem of finding the solution of first order non-linear partial differential equations (PDEs) in three variables of the form

$$F(x, y, z, p, q) = 0, \qquad (2.86)$$

where

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}.$$

We also assume that the function possesses continuous second order derivatives with respect to its arguments over a domain Ω of (x, y, z, p, q)-space, and either F_p or F_q is not zero at every point such that

$$F_p^2 + F_q^2 \neq 0.$$

The PDE (2.86) establishes the fact that at every point (x, y, z) of the region, there exists a relation between the numbers p and q such that $\phi(p,q) = 0$, which defines the direction of the normal $\vec{n} = \{p, q, -1\}$ to the desired integral surface $z = z(x, y)$ of Eq. (2.86). Thus, the direction of the normal to the desired integral surface at certain point (x, y, z) is not defined uniquely. However, a certain cone of admissible directions of the normals exist satisfying the relation $\varphi(p,q) = 0$ (see Fig.3).

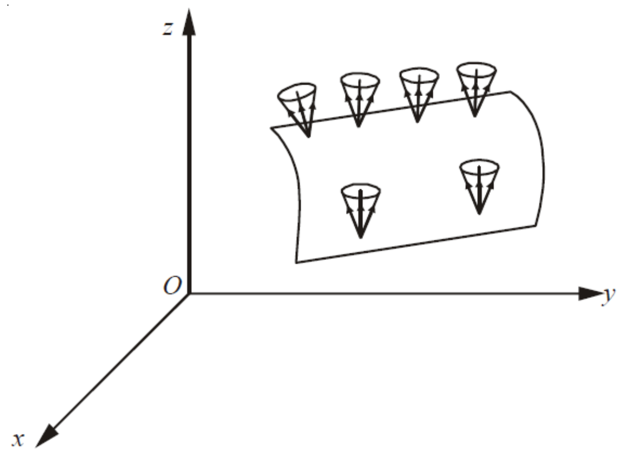

Fig. 3 Cones of normals to the integral surface.

Therefore, the problem of finding the solution of Eq. (2.86) reduces to finding an integral surface $z = z(x, y)$, which the normals at every point of it are directed along one of the permissible directions of the cone of normals at that point.

Thus, the integral or the solution of Eq. (2.86) essentially depends on two arbitrary constants in the form

$$f(x, y, z, a, b) = 0,$$

which is called a complete integral. Hence, we get a two-parameter family of integral surfaces through the same point.

2.7.1 Cauchy's Method of Characteristics

Here, we shall discuss the Cauchy's method for solving Eq. (2.86), which is based on geometrical considerations. Let $z = z(x, y)$ represents an integral surface S of Eq. (2.86) in (x, y, z)-space. Then, $\{p, q, -1\}$ are the direction ratios of the normal to S. Now, the differential equation (2.86) states that at a given point $P(x_0, y_0, z_0)$ on S, the relationship between p_0 and q_0, that is $F(x_0, y_0, z_0, p_0, q_0)$, need not be necessarily linear. Hence, all the tangent planes to possible integral surfaces through P form a family of planes enveloping a conical surface called Monge Cone with P as its vertex. In other words, the problem of solving the PDE (2.86) is to find surfaces which touch the Monge cone at each point along a generator.

Since an integral surface is touched by a Monge cone along its generator, we must have a method to determine the generator of the Monge cone of the PDE (2.86) which is explained below:

It may be noted that the equation of the tangent plane to the integral surface $z = z(x, y)$ at the point (x_0, y_0, z_0) is given by

$$p(x - x_0) + q(y - y_0) = (z - z_0).$$

Now, the given non-linear PDE (2.86) can be recasted into an equivalent form as

$$q = q(x_0, y_0, z_0, p),$$

indicating that p and q are not independent at (x_0, y_0, z_0). At each point of the surface S, there exists a Monge cone which touches the surface along the generator of the cone. The lines of contact between the tangent planes of the integral surface and the corresponding cones, that is the generators along which the surface is touched, define a direction field on the surface S. These directions are called the characteristic directions, or Monge directions on S and lie along the generators of the Monge cone. The integral curves of this field of directions on the integral surface S define a family of curves called characteristic curves as shown in Fig.5. The Monge cone can be obtained by eliminating p from the following equations:

$$p(x - x_0) + q(x_0, y_0, z_0, p)(y - y_0) = (z - z_0),$$

and

$$(x - x_0) + (y - y_0)\frac{dq}{dp} = 0. \qquad (2.87)$$

Observing that q is a function of p and differentiating Eq. (2.86) with respect to p, we get

$$\frac{dF}{dp} = \frac{\partial F}{\partial p} + \frac{\partial F}{\partial q}\frac{dq}{dp} = 0. \qquad (2.88)$$

Now, eliminating (dq/dp) from Eqs. (2.87) and (2.88), we obtain

$$\frac{\partial F}{\partial p}-\frac{\partial F}{\partial q}\frac{(x-x_0)}{(y-y_0)}=0,$$

or

$$\frac{x-x_0}{F_p}=\frac{y-y_0}{F_q}.$$

Therefore, the equations describing the Monge cone are given by

$$\left.\begin{aligned} &q=q(x_0,y_0,z_0,p),\\ &(x-x_0)p+(y-y_0)q=(z-z_0)\\ &\frac{x-x_0}{F_p}=\frac{y-y_0}{F_q}. \end{aligned}\right\} \qquad (2.89)$$

The second and third of Eqs. (2.89) define the generator of the Monge cone. Solving them for $(x-x_0),(y-y_0)$ and $(z-z_0)$, we get

$$\frac{x-x_0}{F_p}=\frac{y-y_0}{F_q}=\frac{z-z_0}{pF_p+qF_q}. \qquad (2.90)$$

Finally, replacing $(x-x_0),(y-y_0)$ and $(z-z_0)$ by dx, dy and dz respectively, which corresponds to infinitesimal movement from (x_0,y_0,z_0) along the generator, Eq. (2.90) becomes

$$\frac{dx}{F_p}=\frac{dy}{F_q}=\frac{dz}{pF_p+qF_q}. \qquad (2.91)$$

Denoting the ratios in Eq. (2.91) by dt, we observe that the characteristic curves on S can be obtained by solving the ordinary differential equations

$$\frac{dx}{dt}=F_p\{x,y,z(x,y),p(x,y),q(x,y)\}, \qquad (2.92)$$

and

$$\frac{dy}{dt}=F_q\{x,y,z(x,y),p(x,y),q(x,y)\}. \qquad (2.93)$$

Also, we note that

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}=p\frac{dx}{dt}+q\frac{dy}{dt}.$$

Therefore,

$$\frac{dz}{dt}=pF_p+qF_q.$$

Along the characteristic curve, p is a function of t, so that

$$\frac{dp}{dt} = \frac{\partial p}{\partial x}\frac{dx}{dt} + \frac{\partial p}{\partial y}\frac{dy}{dt}.$$

Now, using Eqs. (2.92) and (2.93), the above equation becomes

$$\frac{dp}{dt} = \frac{\partial p}{\partial x}\frac{\partial F}{\partial p} + \frac{\partial p}{\partial y}\frac{\partial F}{\partial q}.$$

Since $z_{xy} = z_{yx}$ or $p_y = q_x$, we have

$$\frac{dp}{dt} = \frac{\partial p}{\partial x}\frac{\partial F}{\partial p} + \frac{\partial q}{\partial x}\frac{\partial F}{\partial q}. \tag{2.94}$$

Also, differentiating Eq. (2.86) with respect to x, we find

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}p + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} = 0. \tag{2.95}$$

Using Eq. (2.95), Eq. (2.94) becomes

$$\frac{dp}{dt} = -\left(F_x + pF_z\right).$$

Similarly, we can show that

$$\frac{dq}{dt} = -\left(F_y + qF_z\right).$$

Thus, given an integral surface, we have shown that there exists a family of characteristic curves along which x, y, z, p and q vary according to the following equations

$$\left.\begin{aligned} \frac{dx}{dt} &= F_p \\ \frac{dy}{dt} &= F_q \\ \frac{dz}{dt} &= pF_p + qF_q \\ \frac{dp}{dt} &= -\left(F_x + pF_z\right) \\ \frac{dq}{dt} &= -\left(F_y + qF_z\right) \end{aligned}\right\} \tag{2.96}$$

These equations are known as characteristic equations of the given PDE (2.86). The last three equations of (2.96) are also called compatibility conditions. Without knowing the solution $z = z(x, y)$ of the PDE (2.86), it is possible to find the functions $x(t), y(t), z(t), p(t), q(t)$ from Eqs. (2.96). That is, we can find the curves $x = x(t), y = y(t), z = z(t)$ called characteristics. For illustration, we consider the following examples:

Example 2.26

Find the characteristics of the equation $pq = z$ and determine the integral surface which passes through the straight line $x = 1$, $z = y$.

Solution

The initial data curve is given in parametric form as

$$x_0(s) = 1, \quad y_0(s) = s, \quad z_0(s) = s,$$

then ordinarily the solution is sought in parametric form as

$$x = x(t,s), \quad y = y(t,s), \quad z = z(t,s).$$

Thus, using the given data, the differential equation becomes

$$p_0(s)q_0(s) - s = 0 = F,$$

and the strip condition gives

$$p_0 x_0'(s) + q_0 y_0'(s) = z_0'(s) \;\Rightarrow\; p_0(0) + q_0(1) = 1 \quad \Rightarrow \quad q_0 = 1.$$

Therefore,

$$q_0 = 1, \quad p_0 = s. \quad \text{(unique initial strip)}$$

Now, the characteristic equations for the given PDE are

$$\frac{dx}{dt} = q, \quad \frac{dy}{dt} = p, \quad \frac{dz}{dt} = 2pq, \quad \frac{dp}{dt} = p, \quad \frac{dq}{dt} = q.$$

On integration, we get

$$\left.\begin{aligned} &p = c_1 \exp(t), \quad q = c_2 \exp(t), \quad x = c_2 \exp(t) + c_3 \\ &y = c_1 \exp(t) + c_4, \quad z = c_1 c_2 \exp(2t) + c_5 \end{aligned}\right\}$$

Now, taking into account the initial conditions

$$x_0 = 1, \; y_0 = s, \; z_0 = s, \; p_0 = s, \; q_0 = 1,$$

we can determine the constants of integration and obtain (since $c_2 = 1, c_3 = 0$)

$$p = s\exp(t), \; q = \exp(t),$$

$$x = \exp(t), \; y = s\exp(t), \; z = s\exp(2t). \qquad (2.97)$$

Consequently, the required integral surface is obtained from Eq. (2.97) as $z = xy$

Example 2.27

Find the characteristics of the equation $pq = z$ and hence, determine the integral surface which passes through the parabola $x = 0$, $y^2 = z$.

Solution

The initial data curve is

$$x_0(s) = 0, \; y_0(s) = s, \; z_0(s) = s^2.$$

Using this data, the given PDE becomes

$$p_0(s)q_0(s) - s^2 = 0 = F.$$

The strip condition gives

$$p_0 x_0'(s) + q_0 y_0'(s) = z_0'(s) \;\Rightarrow\; p_0(0) + q_0(1) = 2s \quad \Rightarrow \quad q_0 = 2s.$$

Therefore,

$$q_0 = 2s \quad \text{and} \quad p_0 = z_0 / q_0 = s^2 / 2s = \frac{s}{2}.$$

Now, the characteristic equations of the given PDE are given by

$$\frac{dx}{dt} = q\,, \quad \frac{dy}{dt} = p\,, \quad \frac{dz}{dt} = 2pq\,, \quad \frac{dp}{dt} = p\,, \quad \frac{dq}{dt} = q.$$

On integration, we obtain

$$\left.\begin{array}{l} p = c_1 \exp(t)\,, \quad q = c_2 \exp(t)\,, \quad x = c_2 \exp(t) + c_3\,, \\ y = c_1 \exp(t) + c_4\,, \quad z = c_1 c_2 \exp(2t) + c_5\,. \end{array}\right\}$$

Taking into account the initial conditions

$$x_0 = 0,\ y_0 = s,\ z_0 = s^2,\ p_0 = s/2,\ q_0 = 2s,$$

we find

$$c_1 = s/2,\ c_2 = 2s,\ c_3 = -2s,\ c_4 = s/2, c_5 = 0.$$

Therefore, we have

$$\left.\begin{array}{l} p = \frac{s}{2}\exp(t),\ q = 2s\exp(t) \\ x = 2s[\exp(t) - 1], \quad y = \frac{s}{2}[\exp(t) + 1] \\ z = s^2 \exp(2t) \end{array}\right\} \qquad (2.98)$$

Eliminating s and t from the last three equations of (2.98), we get

$$16z = (4y + x)^2$$

This is the required integral surface.

Example 2.28

Find the characteristics of the PDE

$$p^2 + q^2 = 2,$$

and determine the integral surface which passes through $x = 0,\ z = y$.

Solution

The initial data curve is

$$x_0(s) = 0,\ y_0(s) = s,\ z_0(s) = s.$$

Using this data, the given PDE becomes

$$p_0^2 + q_0^2 - 2 = 0 = F\,,$$

and the strip condition gives

$$p_0 x_0'(s) + q_0 y_0'(s) = z_0'(s) \ \Rightarrow\ p_0(0) + q_0(1) = 1 \quad \Rightarrow \quad q_0 = 1.$$

Hence,

$$q_0 = 1, p_0 = \pm 1.$$

Now, the characteristic equations for the given PDE are given by

$$\left.\begin{aligned} &\frac{dx}{dt} = 2p\,,\ \frac{dy}{dt} = 2q\,,\ \frac{dz}{dt} = 2p^2 + 2q^2 = 4, \\ &\frac{dp}{dt} = 0\,,\ \frac{dq}{dt} = 0. \end{aligned}\right\}$$

On integration, we get

$$\left.\begin{aligned} &p = c_1\,,\ q = c_2\,,\ x = 2c_1 t + c_3\,, \\ &y = 2c_2 t + c_4\,,\ z = 4t + c_5\,. \end{aligned}\right\}$$

Taking into account the initial conditions

$$x_0 = 0,\ y_0 = s,\ z_0 = s,\ p_0 = \pm 1,\ q_0 = 1,$$

we find

$$\left.\begin{aligned} &p = \pm 1\,,\ q = 1,\ x = \pm 2t, \\ &y = 2t + s\,,\ \ z = 4t + s. \end{aligned}\right\} \qquad (2.99)$$

The last-three equations of (6) are parametric equations of the desired integral surface. Eliminating the parameters s and t, we get

$$z = y \pm x.$$

This is the required integral surface.

2.7.2 Compatible Systems of First Order Equations

Two first order PDEs are said to be compatible, if they have a common solution. We shall now derive the necessary and sufficient conditions for the two partial differential equations

$$f(x, y, z, p, q) = 0, \qquad (2.100)$$

and

$$g(x, y, z, p, q) = 0, \qquad (2.101)$$

to be compatible. Let

$$J = \frac{\partial(f, g)}{\partial(p, q)} \neq 0$$

Since Eqs. (2.100) and (2.101) have common solution, we can solve them and obtain explicit expressions for p and q in the form

$$p = \phi(x, y, z)\,,\ \ q = \psi(x, y, z), \qquad (2.102)$$

and then, the differential relation

$$pdx + qdy = dz,$$

or

$$\phi(x, y, z)dx + \psi(x, y, z)dy = dz,$$

should be integrable, for which the necessary condition (Pafaffian condition) is

$$\vec{n} \cdot (\nabla \vec{n}) = 0, \quad \text{(Pafaffian condition)}$$

where $n = \{\phi, \psi, -1\}$.That is,

$$(\phi\hat{i}+\psi\hat{j}-\hat{k})\cdot\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \partial l\partial x & \partial l\partial y & \partial/\partial z\\ \phi & \psi & -1\end{vmatrix}=0,$$

or

$$\phi(-\psi_z)+\psi(\phi_z)=\psi_x-\phi_y,$$

which can be rewritten as

$$\psi_x+\phi\psi_z=\phi_y+\psi\phi_z. \qquad (2.103)$$

Now, differentiating Eq. (2.100) with respect to x and z, we get

$$f_x+f_p\frac{\partial p}{\partial x}+f_q\frac{\partial q}{\partial x}=0,$$

and

$$f_z+f_p\frac{\partial p}{\partial z}+f_q\frac{\partial q}{\partial z}=0.$$

But, from Eq. (2.102), we have

$$\frac{\partial p}{\partial x}=\frac{\partial\phi}{\partial x},\quad \frac{\partial q}{\partial x}=\frac{\partial\psi}{\partial x}.$$

Using these results, the above equations can be recast into

$$f_x+f_p\phi_x+f_q\psi_x=0,$$

and

$$f_z+f_p\phi_z+f_q\psi_z=0.$$

Multiplying the second one of the above pair by ϕ and adding to the first one, we readily obtain

$$\left(f_x+\phi f_z\right)+f_p\left(\phi_x+\phi\phi_z\right)+f_q\left(\psi_x+\phi\psi_z\right)=0.$$

Similarly, from Eq. (2.101) we can deduce that

$$\left(g_x+\phi g_z\right)+g_p\left(\phi_x+\phi\phi_z\right)+g_q\left(\psi_x+\phi\psi_z\right)=0.$$

Solving the above pair of equations for $\left(\psi_x+\phi\psi_z\right)$, we have

$$\frac{\left(\psi_x+\phi\psi_z\right)}{f_p\left(g_x+\phi g_z\right)-g_p\left(f_x+\phi f_z\right)}=\frac{1}{f_q g_p-g_q f_p}=\frac{1}{J}$$

Or

$$\psi_x+\phi\psi_z=\frac{1}{J}\left[\left(f_p g_x-g_p f_x\right)+\phi\left(f_p g_z-g_p f_z\right)\right]$$

$$=\frac{1}{J}\left[\frac{\partial(f,g)}{\partial(x,p)}+\phi\frac{\partial(f,g)}{\partial(z,p)}\right], \qquad (2.104)$$

where $J=\dfrac{\partial(f,g)}{\partial(p,q)}\neq 0$. Similarly, differentiating Eq. (2.100) with respect to y and z and using Eq. (2.102), we can show that

$$\phi_y + \psi\phi_z = -\frac{1}{J}\left[\frac{\partial(f,g)}{\partial(y,q)} + \psi\frac{\partial(f,g)}{\partial(z,q)}\right]. \qquad (2.105)$$

Finally, substituting the values of $\psi_x + \phi\psi_z$ and $\phi_y + \psi_z$ from Eqs. (2.104) and (2.105) into Eq. (2.103), we obtain

$$\frac{\partial(f,g)}{\partial(x,p)} + \phi\frac{\partial(f,g)}{\partial(z,p)} = -\left[\frac{\partial(f,g)}{\partial(y,q)} + \psi\frac{\partial(f,g)}{\partial(z,q)}\right].$$

In view of Eqs. (2.102), we can replace ϕ and ψ by p and q, respectively to get

$$\frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = 0. \qquad (2.106)$$

This is the desired compatibility condition. For illustration, let us consider the following example:

Example 2.29

Show that the following PDEs

$$xp - yq = x \quad \text{and} \quad x^2 p + q = xz,$$

are compatible and hence, find their solution.

Solution

Suppose, we have

$$f = xp - yq - x = 0, \qquad (2.107)$$

and

$$g = x^2 p + q - xz = 0. \qquad (2.108)$$

Then,

$$\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} (p-1) & x \\ (2xp - z) & x^2 \end{vmatrix} = px^2 - x^2 - 2x^2 p + xz = xz - x^2 p - x^2,$$

$$\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} 0 & x \\ -x & x^2 \end{vmatrix} = x^2,$$

$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} -q & -y \\ 0 & 1 \end{vmatrix} = -q,$$

$$\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} 0 & -y \\ -x & 1 \end{vmatrix} = -xy,$$

and we find

$$\frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = xz - x^2 p - x^2 + px^2 - q - qxy$$

$$= xz - q - qxy - x^2$$

$$= xz - q - x(qy + x)$$

$$= xz - q - x^2 p = 0.$$

Hence, the given PDEs are compatible.
Now, solving Eqs. (2.107) and (2.108) for p and q, we obtain

$$p = \frac{x(1+yz)}{x(1+xy)} = \frac{1+yz}{1+xy},$$

and

$$q = \frac{x^2(z-x)}{x(1+xy)} = \frac{x(z-x)}{1+xy},$$

In order to get the solution of the given system, we have to integrate the equation

$$dz = pdx + qdy$$
$$= \frac{(1+yz)}{1+xy}dx + \frac{x(z-x)}{1+xy}dy,$$

or

$$dz - dx = \frac{y(z-x)}{1+xy}dx + \frac{x(z-x)}{1+xy}dy,$$

or

$$\frac{dz-dx}{z-x} = \frac{ydx+xdy}{1+xy}.$$

On integration, we get

$$\ln(z-x) = \ln(1+xy) + \ln c$$

That is,

$$z - x = c(1+xy)$$

Hence, the solution of the given system is found to be

$$z = x + c(1+xy),$$

which is of one-parameter family.

2.7.3 Charpit's Method

In this section, we will discuss a general method for finding the complete integral or complete solution of a nonlinear PDE of first order of the form

$$f(x,y,z,p,q) = 0. \qquad (2.109)$$

This method is known as Charpit's method. The basic idea in Charpit's method is the introduction of another PDE of first order of the form

$$g(x,y,z,p,q) = a, \qquad (2.110)$$

and then, solve Eqs. (2.109) and (2.110) for p and q and substitute in

$$dz = p(x,y,z,a)dx + q(x,y,z,a)dy. \qquad (2.111)$$

Now, the solution of Eq. (2.111) if it exists is the complete integral of Eq. (2.109). The main task is the determination of the second equation (2.110) which is already discussed in the previous section. Now, what is required, is to seek an equation of the form

$$g(x,y,z,p,q)=a,$$

compatible with the given equation

$$f(x,y,z,p,q)=0,$$

for which the necessary and sufficient condition is

$$\frac{\partial(f,g)}{\partial(x,p)}+p\frac{\partial(f,g)}{\partial(z,p)}+\frac{\partial(f,g)}{\partial(y,q)}+q\frac{\partial(f,g)}{\partial(z,q)}=0.$$

On expansion, we have

$$\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial p}-\frac{\partial f}{\partial p}\frac{\partial g}{\partial x}\right)+p\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial p}-\frac{\partial f}{\partial p}\frac{\partial g}{\partial z}\right)$$
$$+\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial q}-\frac{\partial f}{\partial q}\frac{\partial g}{\partial y}\right)+q\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial q}-\frac{\partial f}{\partial q}\frac{\partial g}{\partial z}\right)=0,$$

which can be recast into

$$f_p\frac{\partial g}{\partial x}+f_q\frac{\partial g}{\partial y}+\left(pf_p+qf_q\right)\frac{\partial g}{\partial z}-\left(f_x+pf_z\right)\frac{\partial g}{\partial p}-\left(f_y+qf_z\right)\frac{\partial g}{\partial q}=0. \quad (2.112)$$

This is a linear PDE, from which we can determine g. The auxiliary equations of (2.112) are

$$\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_p+qf_q}=\frac{dp}{-\left(f_x+pf_z\right)}=\frac{dq}{-\left(f_y+qf_z\right)}. \quad (2.113)$$

These equations are called Charpit's equations. Any integral of Eqs. (2.113) involving p or q or both can be taken as the second relation (2.110). Then, the integration of Eq. (2.111) gives the complete integral as desired. It may be noted that all Charpit's equations need not be used, but it is enough to choose the simplest of them. This method is illustrated through the following examples:

Example 2.30

Find the complete integral of

$$\left(p^2+q^2\right)y=qz. \quad (2.114)$$

Solution

Suppose

$$f=\left(p^2+q^2\right)y-qz=0,$$

then, we have

$$f_x=0,\quad f_y=p^2+q^2,\quad f_z=-q\,,\quad f_p=2py,\quad f_q=2qy-z.$$

Now, the Charpit's auxiliary equations are given by

$$\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_p+qf_q}=\frac{dp}{-\left(f_x+pf_z\right)}=\frac{dq}{-\left(f_y+qf_z\right)}.$$

That is,

$$\frac{dx}{2py}=\frac{dy}{2qy-z}=\frac{dz}{2p^2y+2q^2y-qz}=\frac{dp}{pq}=\frac{dq}{-\left[\left(p^2+q^2\right)-q^2\right]}. \quad (2.115)$$

From the last two members of Eq. (2.115), we have

$$\frac{dp}{pq}=\frac{dq}{-p^2},$$

or

$$pdp+qdq=0.$$

On integration, we get

$$p^2+q^2=a \text{ (constant)} . \quad (2.116)$$

From Eqs. (2.114) and (2.116), we obtain

$$ay-qz=0 \quad \text{or} \quad q=ay/z,$$

and

$$\left.p=\sqrt{a-\left(\frac{ay}{z}\right)^2}=\sqrt{\left(az^2-a^2y^2\right)/z^2}\right\}.$$

Substituting these values of p and q in

$$dz=pdx+qdy,$$

we get

$$dz=\frac{\sqrt{az^2-a^2y^2}}{z}dx+\frac{ay}{z}dy,$$

or

$$zdz-aydy=\sqrt{az^2-a^2y^2}dx,$$

which can be rewritten as

$$\frac{d\left(az^2-a^2y^2\right)^{1/2}}{a}=dx.$$

On integration, we find

$$\frac{\sqrt{az^2-a^2y^2}}{a}=x+b,$$

or

$$(x+b)^2=\left(z^2/a\right)-y^2.$$

Hence, the complete integral is

$$(x+b)^2+y^2=z^2/a.$$

Example 2.31

Find the complete integral of the PDE:

$$z^2=pqxy.$$

Solution

In this example, given

$$f = z^2 - pqxy.$$

Then, we have

$$f_x = -pqy, \quad f_y = -pqx, \quad f_z = 2z,$$
$$f_p = -qxy, \quad f_q = -pxy.$$

Now, the Charpit's auxiliary equations are given by

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}.$$

That is,

$$\frac{dx}{-qxy} = \frac{dy}{-pxy} = \frac{dz}{-2pqxy} = \frac{dp}{pqy - 2pz} = \frac{dq}{pqx - 2qz}. \qquad (2.117)$$

From Eq. (2.117), it follows that

$$\frac{dp/p}{qy - 2z} = \frac{dq/q}{px - 2z} = \frac{dx/x}{-qy} = \frac{dy/y}{-px},$$

which can be rewritten as

$$\frac{dp/p - dq/q}{qy - px} = \frac{-dx/x + dy/y}{qy - px},$$

or

$$\frac{dp}{p} - \frac{dq}{q} = \frac{dy}{y} - \frac{dx}{x}.$$

On integration, we find

$$\frac{p}{q}\frac{x}{y} = c, \text{ (constant)}$$

or

$$p = cqy / x.$$

From the given PDE, we have

$$z^2 = pqxy = cq^2y^2,$$

which gives

$$q^2 = z^2 / cy^2 \quad \text{or} \quad q = z / \sqrt{c}y = az / y,$$

where $a = 1/\sqrt{c}$. Hence,

$$p = z / ax.$$

Substituting these values of p and q in

$$dz = pdx + qdy,$$

we get

$$dz = \frac{z}{ax}dx + \frac{az}{y}dy,$$
$$\frac{dz}{z} = \frac{1}{a}\frac{dx}{x} + a\frac{dy}{y}.$$

On integration, we obtain

$$\ln z = \frac{1}{a}\ln x + a\ln y + \ln b,$$

or

$$z = bx^{1/a}y^{a},$$

which is the complete integral of the given PDE.

Example 2.32

Find the complete integral of

$$x^2p^2 + y^2q^2 - 4 = 0,$$

using Charpit's method.

Solution

The Charpit's equations for the given PDE can be written as

$$\frac{dx}{2x^2p} = \frac{dy}{2y^2q} = \frac{dz}{2\left(x^2p^2 + y^2q^2\right)} = \frac{dp}{-2xp^2} = \frac{dq}{-2yq^2}. \quad (2.118)$$

Considering the first and last but one of Eq. (2.118), we have

$$\frac{dx}{2x^2p} = \frac{dp}{-2xp^2} \quad \text{or} \quad \frac{dx}{x} + \frac{dp}{p} = 0.$$

On integration, we get

$$\ln(xp) = \ln a \quad \text{or} \quad xp = a. \quad (2.119)$$

From the given PDE and using the result (2.119), we get

$$y^2q^2 = 4 - a^2. \quad (2.120)$$

Substituting one set of p and q values from Eqs. (2.119) and (2.120) in

$$dz = pdx + qdy,$$

we find that

$$dz = a\frac{dx}{x} + \sqrt{4-a^2}\,\frac{dy}{y}.$$

On integration, the complete integral of the given PDE is found to be

$$z = a\ln x + \sqrt{4-a^2}\,\ln y + b.$$

2.8 Special Types of First Order Equations

Type I Equations Involving p and q only.

That is, equations of the type

$$f(p,q) = 0.$$

Let $z = ax + by + c = 0$ is a solution of the given PDE, described by

$$f(p,q) = 0,$$

then

$$p = \frac{\partial z}{\partial x} = a, \quad q = \frac{\partial z}{\partial y} = b.$$

Substituting these values of p and q in the given PDE, we get

$$f(a,b)=0.$$

Solving for b , we get, $b=\phi(a)$, say. Then,

$$z=ax+\phi(a)y+c$$

is the complete integral of the given PDE.

Example 2.33

Find a complete integral of the equation

$$\sqrt{p}+\sqrt{q}=1.$$

Solution

The given PDE is of the form $f(p,q)=0$. Therefore, let us assume the solution in the form

$$z=ax+by+c,$$

where

$$\sqrt{a}+\sqrt{b}=1 \quad \text{or} \quad b=(1-\sqrt{a})^2.$$

Hence, the complete integral is found to be

$$z=ax+(1-\sqrt{a})^2 y+c.$$

Example 2.34

Find the complete integral of the PDE

$$pq=1.$$

Solution

Since the given PDE is of the form $f(p,q)=0$, we assume the solution in the form $z=ax+by+c$, where $ab=1$ or $b=1/a$. Hence, the complete integral is

$$z=ax+\frac{1}{a}y+c.$$

Type II

Equations Not Involving the Independent Variables.

That is, equations of the type

$$f(z,p,q)=0.$$

As a trial solution, let us assume that z is a function of $u=x+ay$, where a is an arbitrary constant. Then,

$$z=f(u)=f(x+ay),$$

$$p=\frac{\partial z}{\partial x}=\frac{dz}{du}\cdot\frac{\partial u}{\partial x}=\frac{dz}{du},$$

$$q=\frac{\partial z}{\partial y}=\frac{dz}{du}\cdot\frac{\partial u}{\partial y}=a\frac{dz}{du}.$$

Substituting these values of p and q in the given PDE, we get

$$f\left(z,\frac{dz}{du},a\frac{dz}{du}\right)=0, \qquad (2.121)$$

which is an ordinary differential equation of first order.
Solving Eq. (2.121) for dz/du , we obtain

$$\frac{dz}{du} = \phi(z,a), \quad \text{(say)}$$

or

$$\frac{dz}{\phi(z,a)} = du.$$

On integration, we find

$$\int \frac{dz}{\phi(z,a)} = u + c.$$

That is,

$$F(z,a) = u + c = x + ay + c,$$

which is the complete integral of the given PDE.

Example 2.35

Find the complete integral of

$$p(1+q) = qz.$$

Solution

Let us assume the solution in the form

$$z = f(u) = x + ay.$$

Then,

$$p = \frac{dz}{du}, \quad q = a\frac{dz}{du}.$$

Substituting these values in the given PDE, we get

$$\frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = az\frac{dz}{du}.$$

That is,

$$a\frac{dz}{du} = az - 1 \quad \text{or} \quad a\frac{dz}{az-1} = du.$$

On integration, we find

$$\ln(az-1) = u + c = x + ay + c.$$

which is the required complete integral.

Example 2.36

Find the complete integral of the PDE:

$$p^2 z^2 + q^2 = 1.$$

Solution

Let us assume that $z = f(u) = x + ay$ is a solution of the given PDE. Then,

$$p = \frac{dz}{du}, \quad q = a\frac{dz}{du}.$$

Substituting these values of p and q in the given PDE, we obtain

$$\left(\frac{dz}{du}\right)^2 z^2 + a^2\left(\frac{dz}{du}\right)^2 = 1.$$

That is,

$$\left(\frac{dz}{du}\right)^2\left(z^2+a^2\right)=1 \quad \text{or} \quad \frac{dz}{du}=\frac{1}{\sqrt{z^2+a^2}},$$

or

$$\sqrt{z^2+a^2}\,dz = du.$$

On integration, we get

$$\frac{z\sqrt{z^2+a^2}}{2}+\frac{a^2}{2}\ln\left[\frac{z+\sqrt{z^2+a^2}}{a}\right]=x+ay+b,$$

which is the required complete integral of the given PDE.

Type III Separable Equations.

An equation in which z is absent and the terms containing x and p can be separated from those containing y and q is called a separable equation. That is, equations of the type

$$f(x,p)=F(y,q).$$

As a trial solution, let us assume that

$$f(x,p)=F(y,q)=a. \text{ (say)}$$

Now, solving them for p and q, we obtain

$$p=\phi(x,a), \quad q=\psi(y,a).$$

Since

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy = pdx+qdy,$$

or

$$dz=\phi(x,a)dx+\psi(y,a)dy.$$

On integration, we get the complete integral in the form

$$z=\int\phi(x,a)dx+\int\psi(y,a)dy+b.$$

Example 2.37

Find the complete integral of the PDE:

$$p^2y\left(1+x^2\right)=qx^2.$$

Solution

The given PDE is of separable type and can be rewritten as

$$\frac{p^2\left(1+x^2\right)}{x^2}=\frac{q}{y}=a. \quad \text{(say), an arbitrary constant.}$$

Then,

$$p = \frac{\sqrt{a}x}{\sqrt{1+x^2}}, \quad q = ay.$$

Substituting these values of p and q in

$$dz = pdx + qdy,$$

we get

$$dz = \frac{\sqrt{a}x}{\sqrt{1+x^2}}dx + aydy.$$

On integration, we obtain

$$z = \sqrt{a}\sqrt{1+x^2} + \frac{a}{2}y^2 + b,$$

which is the complete integral of the given PDE.

Example 2.38

Find the complete integral of

$$p^2 + q^2 = x + y.$$

Solution

The given PDE is of separable type and can be rewritten as

$$p^2 - x = y - q^2 = a, \quad \text{(say)}$$

Then,

$$p = \sqrt{x+a}, \quad q = \sqrt{y+a}.$$

Now, substituting these values of p and q in

$$dz = pdx + qdy,$$

we find

$$dz = \sqrt{x+a}dx + \sqrt{y+a}dy.$$

On integration, the complete integral is found to be

$$z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y+a)^{3/2} + b.$$

Type IV Clairaut's Form

A first order PDE is said to be of Clairaut's form if it can be written as

$$z = px + qy + f(p,q).$$

The corresponding Charpit's equations are

$$\frac{dx}{x+f_p} = \frac{dy}{y+f_q} = \frac{dz}{px+qy+pf_p+qf_q} = \frac{dp}{p-p} = \frac{dq}{q-q}. \qquad (2.122)$$

The integration of the last two equations of (2.122) gives us

$$p = a, \quad q = b.$$

Substituting these values of p and q in the given PDE, we get the required complete integral in the form

$$z = ax + by + f(a,b).$$

Example 2.39

Find the complete integral of the equation

$$z = px + qy + \sqrt{1 + p^2 + q^2}.$$

Solution

The given PDE is in the Clairaut's form. Hence, its complete integral is

$$z = ax + by + \sqrt{1 + a^2 + b^2}.$$

Example 2.40

Find the complete integral of

$$(p+q)(z - xp - yq) = 1.$$

Solution

The given PDE can be rewritten as

$$z = xp + yq + \frac{1}{p+q},$$

which is in the Clairaut's form,

$$z = px + qy + f(p,q).$$

Hence, the complete integral of the given PDE is

$$z = ax + by + \frac{1}{a+b}.$$

Example 2.41

In classical mechanics, the Hamilton-Jacobi equation for the problem of one-dimensional, Harmonic oscillator is given by the differential equation as

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}Kq^2 + \frac{\partial S}{\partial t} = 0, \qquad (2.123)$$

where $S = S(p,q,t)$, $p = \frac{\partial S}{\partial q}$ and K is a constant. Using Charpits method, find S.

Solution

Following the notation of Eq. (2.123) we rewrite

$$f\left(t,q,S,S_t,S_q\right) = \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}Kq^2 + \frac{\partial S}{\partial t},$$

which gives us

$$f_t = 0,\ f_q = Kq,\ f_S = 0,\ f_{S_t} = 1,\ f_{S_q} = \frac{S_q}{m}.$$

Then, the Charpits auxiliary equations assumes the following form:

$$\frac{dt}{1} = \frac{dq}{S_q / m} = \frac{dS}{S_t + S_q^2 / m} = \frac{dS_t}{0} = \frac{dS_q}{-Kq}.$$

Considering the second and last members, we have $\frac{dq}{S_q / m} = \frac{dS_q}{-Kq}$.

On integration, we get

$$\frac{S_q^2}{2m} + \frac{1}{2}Kq^2 = a \text{ (constant of integration).}$$

Equation (2.123) then becomes

$$S_t = -a \ , \ S_q^2 = Km\left(\frac{2a}{K} - q^2\right).$$

Substituting S_t and S_q into

$$dS = S_t dt + S_q dq,$$

and integrating, we arrive at

$$S = -at + \sqrt{Km}\int\left(\frac{2a}{K} - q^2\right)^{1/2} dq + C,$$

$$S = -at + \sqrt{Km}\int\left(\alpha^2 - q^2\right)^{1/2} dq + C.$$

where $\alpha^2 = \frac{2a}{K}$ and C is another constant of integration.

CHAPTER THREE
PARTIAL DIFFERENTIAL EQUATIONS OF SECOND ORDER

3.1 Introduction

An important ingredient of a systematic theory of partial differential equations is a classification scheme which identifies classes of equations with common properties. The “type” of an equation determines the nature of boundary and initial conditions which may be imposed, the nature of singularities which solutions may have and the nature of methods which can be used to approximate a solution. In this chapter, we present the classification of linear partial differential equations of the second order. Linear partial equations of the second order are divided into three main types. They are hyperbolic equations, parabolic equations and elliptic equations. Then we study the canonical forms of these main types. After that, we present methods for solving these linear equations with constant coefficients and generalize these methods to linear equations of higher orders than the second order.

3.2 Classification of second order PDEs

The most general linear second order PDE, with one dependent function u on a domain Ω of points $X=\left(x_1,x_2,\ldots,x_n\right),\, n>1$, is

$$\sum_{i,j=1}^{n} a_{ij}u_{xi}x_j+\sum_{i=1}^{n} b_i u_{xi}+Fu=G. \tag{3.1}$$

The classification of a PDE depends only on the highest order derivatives present. The classification of PDE is motivated by the classification of the quadratic equation of the form

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0, \tag{3.2}$$

which is elliptic, parabolic, or hyperbolic according as the discriminant B^2-4AC is negative, zero or positive. Thus, we have the following second order linear PDE in two variables x and y:

$$Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=G, \tag{3.3}$$

where the coefficients $A,B,C,\ldots$ are functions of x and y. Equation (3.3) is elliptic, parabolic or hyperbolic at a point $\left(x_0,y_0\right)$ according as the discriminant

$$B^2\left(x_0,y_0\right)-4A\left(x_0,y_0\right)C\left(x_0,y_0\right)$$

is negative, zero or positive. If this is true at all points in a domain Ω, then Eq. (3.3) is said to be elliptic, parabolic or hyperbolic in that domain. If the number of independent variables is two or three, a transformation can always be found to reduce the given PDE to a canonical form (also called normal form). In general, when the number of independent variables is greater than 3, it is not always possible to find such a transformation except in certain special cases. The idea of reducing the given

PDE to a canonical form is that the transformed equation assumes a simple form so that the subsequent analysis of solving the equation is made easy.

3.3 Canonical forms

Consider the most general transformation of the independent variables x and y of Eq. (3.3) to new variables ξ,η, where

$$\xi=\xi(x,y),\quad \eta=\eta(x,y), \tag{3.4}$$

such that the functions ξ and η are continuously differentiable and the Jacobian

$$J=\frac{\partial(\xi,\eta)}{\partial(x,y)}=\begin{vmatrix}\xi_x & \xi_y\\ \eta_x & \eta_y\end{vmatrix}=\left(\xi_x\eta_y-\xi_y\eta_x\right)\neq 0\,,$$

in the domain Ω where Eq. (3.3) holds. Using the chain rule of partial differentiation, the partial derivatives become

$$u_x=u_\xi\xi_x+u_\eta\eta_x\,,\ u_y=u_\xi\xi_y+u_\eta\eta_y\,,$$

$$u_{xx}=u_{\xi\xi}\xi_x^2+2u_{\xi\eta}\xi_x\eta_x+u_{\eta\eta}\eta_x^2+u_\xi\xi_{xx}+u_\eta\eta_{xx}\,,$$

$$u_{xy}=u_{\xi\xi}\xi_x\xi_y+u_{\xi\eta}\left(\xi_x\eta_y+\xi_y\eta_x\right)+u_{\eta\eta}\eta_x\eta_y+u_\xi\xi_{xy}+u_\eta\eta_{xyy}\,,$$

$$u_{yy}=u_{\xi\xi}\xi_y^2+2u_{\xi\eta}\xi_y\eta_y+u_{\eta\eta}\eta_y^2+u_\xi\xi_{yy}+u_\eta\eta_{yy}\,.$$

Substituting these expressions into the original differential equation (3.3), we get

$$\bar{A}u_{\xi\xi}+\bar{B}u_{\xi\eta}+\bar{C}u_{\eta\eta}+\bar{D}u_\xi+\bar{E}u_\eta+\bar{F}u=\bar{G}, \tag{3.5}$$

where

$$\left.\begin{aligned}
&\bar{A}=A\xi_x^2+B\xi_x\xi_y+C\xi_y^2\,,\\
&\bar{B}=2A\xi_x\eta_x+B\left(\xi_x\eta_y+\xi_y\eta_x\right)+2C\xi_y\eta_y\,,\\
&\bar{C}=A\eta_x^2+B\eta_x\eta_y+C\eta_y^2\,,\\
&\bar{D}=A\xi_{xx}+B\xi_{xy}+C\xi_{yy}+D\xi_x+E\xi_y\ ,\\
&\bar{E}=A\eta_{xx}+B\eta_{xy}+C\eta_{yy}+D\eta_x+E\eta_y\ ,\\
&\bar{F}=F,\quad \bar{G}=G.
\end{aligned}\right\} \tag{3.6}$$

It may be noted that the transformed equation (3.5) has the same form as that of the original equation (3.3) under the general transformation (3.4). Using Eq. (3.6) it can also be verified that

$$\bar{B}^2-4\bar{A}\bar{C}=\left(\xi_x\eta_y-\xi_y\eta_x\right)^2\left(B^2-4AC\right),$$

and therefore we conclude that the transformation of the independent variables does not modify the type of PDE.

Since the classification of Eq. (3.3) depends on the coefficients A,B and C we can also rewrite the equation in the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} = H\left(x, y, u, u_x, u_y\right).$$

It can be shown easily that under the transformation (3.4), Eq. (3.5) takes one of the following three canonical forms:

(i) In the hyperbolic case

$$u_{\xi\xi} - u_{\eta\eta} = \phi\left(\xi, \eta, u, u_\xi, u_\eta\right) \quad \text{or} \quad u_{\xi\eta} = \phi,\left(\xi, \eta, u, u_\xi, u_\eta\right). \quad (3.7)$$

(ii) In the elliptic case

$$u_{\xi\xi} + u_{\eta\eta} = \phi\left(\xi, \eta, u, u_\xi, u_\eta\right). \quad (3.8)$$

(iii) In the parabolic case

$$u_{\xi\xi} = \phi\left(\xi, \eta, u, u_\xi, u_\eta\right), \quad \text{or} \quad u_{\eta\eta} = \phi\left(\xi, \eta, u, u_\xi, u_\eta\right). \quad (3.9)$$

We shall discuss in detail each of these cases separately.

3.3.1 Canonical Form for Hyperbolic Equation

Since the discriminant $\bar{B}^2 - 4\bar{A}\bar{C} > 0$ for hyperbolic case, we set $\bar{A} = 0$ and $\bar{C} = 0$ in Eq. (3.6), which will give us the coordinates ξ and η that reduce the given PDE to a canonical form in which the coefficients of $u_{\xi\xi}$, $u_{\eta\eta}$ are zero. Thus we have

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0,$$

$$\bar{C} = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0,$$

which, on rewriting, become

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0,$$

$$A\left(\frac{\eta_x}{\eta_y}\right)^2 + B\left(\frac{\eta_x}{\eta_y}\right) + C = 0.$$

Solving these equations for $\left(\xi_x / \xi_y\right)$ and $\left(\eta_x / \eta_y\right)$, we get

$$\left.\begin{aligned} \frac{\xi_x}{\xi_y} &= \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \\ \frac{\eta_x}{\eta_y} &= \frac{-B - \sqrt{B^2 - 4AC}}{2A}. \end{aligned}\right\} \quad (3.10).$$

The condition $B^2 > 4AC$ implies that the slopes of the curves $\xi(x, y) = C_1, \eta(x, y) = C_2$ are real. Thus, if $B^2 > 4AC$, then at any point (x, y), there exists two real directions given by the two roots (3.10) along which the PDE (3.3) reduces to the canonical form. These are called characteristic equations. Though there are two solutions for each quadratic, we have considered only one solution for each. Otherwise we will end up with the same two coordinates. Along the curve $\xi(x,y)=c_1$, we have

$$d\xi = \xi_x dx + \xi_y dy = 0.$$

Hence,

$$\frac{dy}{dx} = -\left(\frac{\xi_x}{\xi_y}\right). \tag{3.11}$$

Similarly, along the curve $\eta(x,y)=c_2$, we have

$$\frac{dy}{dx} = -\left(\frac{\eta_x}{\eta_y}\right). \tag{3.12}$$

Integrating Eqs. (3.11) and (3.12), we obtain the equations of family of characteristics $\xi(x,y)=c_1$ and $\eta(x,y)=c_2$, which are called the characteristics of the PDE (3.3). Now to obtain the canonical form for the given PDE, we substitute the expressions of ξ and η into Eq. (3.3) which reduces to the second of (3.7). To make the ideas clearer, let us consider the following example:

Example 3.1

Classify and reduce the PDE

$$3u_{xx} + 10u_{xy} + 3u_{yy} = 0,$$

to a canonical form and solve it.

Solution

Comparing with the standard PDE (3.3), we have $A=3$, $B=10$, $C=3$, $B^2-4AC=64>0$. Hence the given equation is a hyperbolic PDE. The corresponding characteristics are:

$$\frac{dy}{dx} = -\left(\frac{\xi_x}{\xi_y}\right) = -\left(\frac{-B+\sqrt{B^2-4AC}}{2A}\right) = \frac{1}{3},$$

$$\frac{dy}{dx} = -\left(\frac{\eta_x}{\eta_y}\right) = -\left(\frac{-B-\sqrt{B^2-4AC}}{2A}\right) = 3.$$

To find ξ and η, we first solve for y by integrating the above equations. Thus, we get

$$y = 3x + c_1, \quad y = \frac{1}{3}x + c_2,$$

which give the constants as

$$c_1 = y - 3x, \quad c_2 = y - x/3.$$

Therefore,

$$\xi = y - 3x = c_1, \quad \eta = y - \frac{1}{3}x = c_2.$$

These are the characteristic lines for the given hyperbolic equation. In this example, the characteristics are found to be straight lines in the (x, y)-plane. To find the canonical equation, we substitute the expressions for ξ and η into Eq. (3.6) to get

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 3(-3)^2 + 10(-3)(1) + 3 = 0,$$

$$\bar{B} = 2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\,\eta_y$$

$$= 2(3)(-3)\left(-\frac{1}{3}\right) + 10\left[(-3)(1) + 1\left(-\frac{1}{3}\right)\right] + 2(3)(1)(1)$$

$$= 6 + 10\left(-\frac{10}{3}\right) + 6 = 12 - \frac{100}{3} = -\frac{64}{3},$$

$$\bar{C} = 0, \quad \bar{D} = 0, \quad \bar{E} = 0, \quad \bar{F} = 0.$$

Hence, the required canonical form is

$$\frac{64}{3}u_{\xi\eta} = 0 \quad \text{or} \quad u_{\xi\eta} = 0.$$

On integration, we obtain

$$u(\xi, \eta) = f(\xi) + g(\eta),$$

where f and g are arbitrary. Going back to the original variables, the general solution is

$$u(x, y) = f(y - 3x) + g(y - x/3).$$

3.3.2 Canonical Form for Parabolic Equation

For the parabolic equation, the discriminant $\bar{B}^2 - 4\bar{A}\bar{C} = 0$, which can be true if $\bar{B} = 0$ and $\bar{A}$ or $\bar{C}$ is equal to zero. Suppose we set first $\bar{A} = 0$ in Eq. (3.6). Then we obtain

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0,$$

or

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0,$$

which gives

$$\frac{\xi_x}{\xi_y} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

Using the condition for parabolic case, we get

$$\frac{\xi_x}{\xi_y} = -\frac{B}{2A}. \tag{3.13}$$

Hence, to find the function $\xi = \xi(x, y)$ which satisfies Eq. (3.13), we set

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = \frac{B}{2A},$$

and get the implicit solution

$$\xi(x,y) = C_1.$$

In fact, one can verify that $\bar{A} = 0$ implies $\bar{B} = 0$ as follows:

$$\bar{B} = 2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y.$$

Since $B^2 - 4AC = 0$, the above relation reduces to

$$\bar{B} = 2A\xi_x\eta_x + 2\sqrt{AC}\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y$$

$$= 2\left(\sqrt{A}\xi_x + \sqrt{C}\xi_y\right)\left(\sqrt{A}\eta_x + \sqrt{C}\eta_y\right).$$

However,

$$\frac{\xi_x}{\xi_y} = -\frac{B}{2A} = -\frac{2\sqrt{AC}}{2A} = -\sqrt{\frac{C}{A}}.$$

Hence,

$$\bar{B} = 2\left(\sqrt{A}\xi_x - \sqrt{A}\xi_x\right)\left(\sqrt{A}\eta_x + \sqrt{C}\eta_y\right) = 0.$$

We therefore choose ξ in such a way that both $\bar{A}$ and $\bar{B}$ are zero. Then η can be chosen in any way we like as long as it is not parallel to the ξ-coordinate. In other words, we choose η such that the Jacobian of the transformation is not zero. Thus we can write the canonical equation for parabolic case by simply substituting ξ and η into Eq. (3.3) which reduces to either of the forms (3.9).

To illustrate the procedure, we consider the following example:

Example 3.2

Classify and reduce the PDE

$$x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} = e^x,$$

to a canonical form.

Solution

The discriminant $B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0$, and hence the given PDE is parabolic everywhere. The characteristic equation is

$$\frac{dy}{dx} = -\frac{\xi_X}{\xi_y} = \frac{B}{2A} = -\frac{2xy}{2x^2} = -\frac{y}{x}.$$

On integration, we have

$$xy = c,$$

and hence $\xi = xy$ will satisfy the characteristic equation and we can choose $\eta = y$. To find the canonical equation, we substitute the expressions for ξ and η into Eq. (3.6) to get

$$\bar{A} = Ay^2 + Bxy + cx^2 = x^2y^2 - 2x^2y^2 + y^2x^2 = 0,$$
$$\bar{B} = 0, \quad \bar{C} = y^2, \quad \bar{D} = -2xy,$$
$$\bar{E} = 0, \quad \bar{F} = 0, \quad \bar{G} = e^x.$$

Hence, the transformed equation is

$$y^2 u_{\eta\eta} - 2xyu_\xi = e^x,$$

or

$$\eta^2 u_{\eta\eta} = 2\xi u_\xi + e^{\xi/\eta}.$$

The canonical form is, therefore,

$$u_{\eta\eta} = \frac{2\xi}{\eta^2} u_\xi + \frac{1}{\eta^2} e^{\xi/\eta}.$$

3.3.3 Canonical Form for Elliptic Equation

Since the discriminant $B^2 - 4AC < 0$, for elliptic case, the characteristic equations

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A},$$
$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A},$$

give us complex conjugate coordinates, say ξ and η. Now, we make another transformation from (ξ, η) to (α, β) so that

$$\alpha = \frac{\xi + \eta}{2}, \quad \beta = \frac{\xi - \eta}{2i},$$

which give us the required canonical equation in the form (3.9).

To illustrate the procedure, we consider the following example:

Example 3.3

Classify and reduce the PDE

$$u_{xx} + x^2 u_{yy} = 0,$$

to a canonical form.

Solution

The discriminant $B^2 - 4AC = -4x^2 < 0$. Hence, the given PDE is elliptic. The characteristic equations are

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = -\frac{\sqrt{-4x^2}}{2} = -ix,$$
$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = ix.$$

Integration of these equations yields

$$iy + \frac{x^2}{2} = c_1, \quad -iy + \frac{x^2}{2} = c_2.$$

Hence, we may assume that

$$\xi = \frac{1}{2}x^2 + iy, \quad \eta = \frac{1}{2}x^2 - iy.$$

Now, introducing the second transformation

$$\alpha = \frac{\xi + \eta}{2}, \quad \beta = \frac{\xi - \eta}{2i},$$

we obtain

$$\alpha = \frac{x^2}{2}, \quad \beta = y.$$

The canonical form can now be obtained by computing

$$\bar{A} = A\alpha_x^2 + \beta\alpha_x\alpha_y + c\alpha_y^2 = x^2,$$

$$\bar{B} = 2A\alpha_x\beta_x + B\left(\alpha_x\beta_y + \alpha_y\beta_x\right) + 2c\left(\alpha_y\beta_y\right) = 0,$$

$$\bar{C} = A\beta_x^2 + B\beta_x\beta_y + c\beta_y^2 = x^2,$$

$$\bar{D} = A\alpha_{xx} + B\alpha_{xy} + c\alpha_{yy} + D\alpha_x + E\alpha_y = 1,$$

$$\bar{E} = A\beta_{xx} + B\beta_{xy} + c\beta_{yy} + D\beta_x + E\beta_y = 0,$$

$$\bar{F} = 0, \quad \bar{G} = 0.$$

Thus the required canonical equation is

$$x^2u_{\alpha\alpha} + x^2u_{\beta\beta} + u_\alpha = 0,$$

or

$$u_{\alpha\alpha} + u_{\beta\beta} = -\frac{u_\alpha}{2\alpha}.$$

Example 3.3

Classify and reduce the relation

$$y^2u_{xx} - 2xyu_{xy} + x^2u_{yy} = \frac{y^2}{x}u_x + \frac{x^2}{y}u_y,$$

to a canonical form and solve it.

Solution

The discriminant of the given PDE is

$$B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0.$$

Hence the given equation is of a parabolic type. The characteristic equation is

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = \frac{B}{2A} = \frac{-2xy}{2y^2} = -\frac{x}{y}.$$

Integration gives $x^2 + y^2 = c_1$. Therefore, $\xi = x^2 + y^2$ satisfies the characteristic equation. The η-coordinate can be chosen arbitrarily so that it is not parallel to ξ, i.e. the Jacobian of the transformation is not zero. Thus we choose

$$\xi = x^2 + y^2, \quad \eta = y^2.$$

To find the canonical equation, we compute

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 4x^2y^2 - 8x^2y^2 + 4x^2y^2 = 0,$$

$$\bar{B} = 0, \quad \bar{C} = 4x^2y^2,$$

$$\bar{D} = \bar{E} = \bar{F} = \bar{G} = 0.$$

Hence, the required canonical equation is

$$4x^2y^2u_{\eta\eta} = 0 \quad \text{or} \quad u_{\eta\eta} = 0.$$

To solve this equation, we integrate it twice with respect to η to get

$$u_\eta = f(\xi), \quad u = f(\xi)\eta + g(\xi),$$

where $f(\xi)$ and $g(\xi)$ are arbitrary functions of ξ . Now, going back to the original independent variables, the required solution is

$$u = y^2 f\left(x^2 + y^2\right) + g\left(x^2 + y^2\right).$$

Example 3.4

Reduce the following equation to a canonical form:

$$\left(1 + x^2\right)u_{xx} + \left(1 + y^2\right)u_{yy} + xu_x + yu_y = 0.$$

Solution

The discriminant of the given PDE is

$$B^2 - 4AC = -4\left(1 + x^2\right)\left(1 + y^2\right) < 0.$$

Hence the given PDE is an elliptic type. The characteristic equations are

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = -\frac{\sqrt{-4\left(1 + x^2\right)\left(1 + y^2\right)}}{2\left(1 + x^2\right)} = -i\sqrt{\frac{1 + y^2}{1 + x^2}},$$

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = i\sqrt{\frac{1 + y^2}{1 + x^2}}.$$

On integration, we get

$$\xi = \ln\left(x + \sqrt{x^2 + 1}\right) - i\ln\left(y + \sqrt{y^2 + 1}\right) = c_1,$$

$$\eta = \ln\left(x + \sqrt{x^2 + 1}\right) + i\ln\left(y + \sqrt{y^2 + 1}\right) = c_2.$$

Introducing the second transformation

$$\alpha = \frac{\xi + \eta}{2}, \quad \beta = \frac{\eta - \xi}{2i},$$

we obtain

$$\alpha = \ln\left(x + \sqrt{x^2 + 1}\right),$$

$$\beta = \ln\left(y + \sqrt{y^2 + 1}\right).$$

Then the canonical form can be obtained by computing

$$\bar{A} = A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2 = 1, \quad \bar{B} = 0, \quad \bar{C} = 1, \quad \bar{D} = \bar{E} = \bar{F} = \bar{G} = 0.$$

Thus the canonical equation for the given PDE is

$$u_{\alpha\alpha} + u_{\beta\beta} = 0.$$

Example 3.5

Reduce the following equation to a canonical form and hence solve it:

$$u_{xx} - 2\sin x u_{xy} - \cos^2 x u_{yy} - \cos x u_y = 0.$$

Solution

Comparing with the general second order PDE (3.3), we have

$A = 1, \quad B = -2\sin x, \quad C = -\cos^2 x,$

$D = 0, \quad E = -\cos x, \quad F = 0, \quad G = 0$

The discriminant $B^2 - 4AC = 4\left(\sin^2 x + \cos^2 x\right) = 4 > 0$. Hence the given PDE is hyperbolic. The relevant characteristic equations are

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = -\sin x - 1,$$

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = 1 - \sin x.$$

On integration, we get

$$y = \cos x - x + c_1, \quad y = \cos x + x + c_2.$$

Thus, we choose the characteristic lines as

$$\xi = x + y - \cos x = c_1, \quad \eta = -x + y - \cos x = c_2.$$

In order to find the canonical equation, we compute

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0.$$

$$\bar{B} = 2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y.$$

$$= 2(\sin x + 1)(\sin x - 1) - 4\sin^2 x - 2\cos^2 x = -4.$$

$$\bar{C} = 0, \quad \bar{D} = 0, \quad \bar{E} = 0, \quad \bar{F} = 0, \quad \bar{G} = 0.$$

Thus, the required canonical equation is

$$u_{\xi\eta} = 0.$$

Integrating with respect to ξ, we obtain

$$u_\eta = f(\eta),$$

where f is arbitrary. Integrating once again with respect to η, we have

$$u = \int f(\eta)d\eta + g(\xi),$$

or

$$u = \psi(\eta) + g(\xi),$$

where $g(\xi)$ is another arbitrary function. Returning to the old variables x, y, the solution of the given PDE is

$$u(x,y) = \psi(y - x - \cos x) + g(y + x - \cos x).$$

Example 3.6

Reduce the Tricomi equation

$$u_{xx} + xu_{yy} = 0, \quad x \neq 0$$

for all x, y to a canonical form.

Solution

The discriminant $B^2 - 4AC = -4x$. Hence the given PDE is of mixed type: hyperbolic for $x < 0$ and elliptic for $x > 0$.

Case I

In the half-plane $x < 0$, the characteristic equations are

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = \frac{-2\sqrt{-x}}{2} = -\sqrt{-x},$$

$$\frac{dy}{dx} = -\frac{\eta_x}{\eta_y} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = \sqrt{-x}.$$

Integration yields

$$y = \frac{2}{3}(-x)^{3/2} + c_1,$$

$$y = -\frac{2}{3}(-x)^{3/2} + c_2.$$

Therefore, the new coordinates are

$$\xi(x,y) = \frac{3}{2}y - (\sqrt{-x})^3 = c_1,$$

$$\eta(x,y) = \frac{3}{2}y + (\sqrt{-x})^3 = c_2,$$

which are cubic parabolas. In order to find the canonical equation, we compute

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = -\frac{9}{4}x + 0 + \frac{9}{4}x = 0,$$

$$\bar{B} = 9x, \quad \bar{C} = 0, \quad \bar{D} = -\frac{3}{4}(-x)^{-1/2} = -\bar{E}, \quad \bar{F} = \bar{G} = 0.$$

Thus, the required canonical equation is

$$9xu_{\xi\eta} - \frac{3}{4}(-x)^{-1/2}u_\xi + \frac{3}{4}(-x)^{-1/2}u_\eta = 0,$$

or

$$u_{\xi\eta} = \frac{1}{6(\xi - \eta)}\left(u_\xi - u_\eta\right).$$

Case II

In the half-plane $x > 0$, the characteristic equations are given by

$$\frac{dy}{dx} = i\sqrt{x}\ , \quad \frac{dy}{dx} = -i\sqrt{x}.$$

On integration, we have

$$\xi(x, y) = \frac{3}{2}y - i(\sqrt{x})^3, \quad \eta(x, y) = \frac{3}{2}y + i(\sqrt{x})^3$$

Introducing the second transformation

$$\alpha = \frac{\xi + \eta}{2}, \quad \beta = \frac{\xi - \eta}{2i},$$

we obtain

$$\alpha = \frac{3}{2}y, \quad \beta = -(\sqrt{x})^3.$$

The corresponding normal or canonical form is

$$u_{\alpha\alpha} + u_{\beta\beta} + \frac{1}{3\beta}u_\beta = 0.$$

Example 3.7

Find the characteristics of the equation

$$u_{xx} + 2u_{xy} + \sin^2(x)u_{yy} + u_y = 0,$$

when it is of hyperbolic type.

Solution

The discriminant $B^2 - 4AC = 4 - 4\sin^2 x = 4\cos^2 x$. Hence for all $x \neq (2n-1)\pi/2$, the given PDE is of hyperbolic type. The characteristic equations are

$$\frac{dy}{dx} = \frac{B \mp \sqrt{B^2 - 4AC}}{2A} = 1 \mp \cos x.$$

On integration, we get

$$y = x - \sin x + c_1, \quad y = x + \sin x + c_2.$$

Thus, the characteristic equations are

$$\xi = y - x + \sin x, \quad \eta = y - x - \sin x.$$

Example 3.8

Reduce the following equation to a canonical form and hence solve it:

$$yu_{xx} + (x + y)u_{xy} + xu_{yy} = 0.$$

Solution

The discriminant $B^2 - 4AC = (x + y)^2 - 4xy = (x - y)^2 > 0$. Hence the given PDE is hyperbolic everywhere except along the line $y = x$; where-as on the line $y = x$, it is parabolic. When $y \neq x$, the characteristic equations are

$$\frac{dy}{dx} = \frac{B \mp \sqrt{B^2 - 4AC}}{2A} = \frac{(x+y) \mp (x-y)}{2y}.$$

Therefore,

$$\frac{dy}{dx} = 1, \quad \frac{dy}{dx} = \frac{x}{y}.$$

On integration, we obtain

$$y = x + c_1, \quad y^2 = x^2 + c_2.$$

Hence, the characteristic equations are

$$\xi = y - x, \quad \eta = y^2 - x^2.$$

These are straight lines and rectangular hyperbolas. The canonical form can be obtained by computing

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = y - x - y + x = 0, \quad \bar{B} = -2(x-y)^2,$$

$$\bar{C} = 0, \quad \bar{D} = 0, \quad \bar{E} = 2(x-y), \quad \bar{F} = \bar{G} = 0.$$

Thus, the canonical equation for the given PDE is

$$-2(x-y)^2 u_{\xi\eta} + 2(x-y)u_\eta = 0,$$

or

$$-2\xi^2 u_{\xi\eta} + 2(-\xi)u_\eta = 0,$$

or

$$\xi u_{\xi\eta} + u_\eta = \frac{\partial}{\partial \xi}\left(\xi \frac{\partial u}{\partial \eta}\right) = 0.$$

Integration yields

$$\xi \frac{\partial u}{\partial \eta} = f(\eta).$$

Again integrating with respect to η, we obtain

$$u = \frac{1}{\xi}\int f(\eta)d\eta + g(\xi).$$

Hence,

$$u = \frac{1}{y-x}\int f\left(y^2 - x^2\right) d\left(y^2 - x^2\right) + g(y-x),$$

is the general solution.

Example 3.9

Classify and transform the following equation to a canonical form:

$$\sin^2(x)u_{xx} + \sin(2x)u_{xy} + \cos^2(x)u_{yy} = x.$$

Solution

The discriminant of the given PDE is

$$B^2-4AC=\sin^2 2x-4\sin^2 x\cos^2 x=0.$$

Hence, the given equation is of parabolic type. The characteristic equation is

$$\frac{dy}{dx}=\frac{B}{2A}=\cot x.$$

Integration gives

$$y=\ln\sin x+c_1.$$

Hence, the characteristic equations are:

$$\xi=y-\ln\sin x,\quad \eta=y,$$

η is chosen in such a way that the Jacobian of the transformation is non-zero. Now the canonical form can be obtained by computing

$$\bar{A}=0,\quad \bar{B}=0,\quad \bar{C}=\cos^2 x,\quad \bar{D}=1,$$

$$\bar{E}=0,\quad \bar{F}=0,\quad \bar{G}=x.$$

Hence, the canonical equation is

$$\cos^2(x)u_{\eta\eta}+u_{\xi}=x,$$

or

$$\left[1-\mathrm{e}^{2(\eta-\xi)}\right]u_{\eta\eta}=\sin^{-1}\left(\mathrm{e}^{\eta-\xi}\right)-u_{\xi}.$$

Example 3.10

Show that the equation

$$u_{xx}+\frac{2N}{x}u_x=\frac{1}{a^2}u_{tt},$$

where N and a are constants, is hyperbolic and obtain its canonical form.

Solution

Comparing with the general PDE (3.3) and replacing y by t, we have $A=1, B=0, C=-1/a^2, D=2N/x$ and $E=F=G=0$. The discriminant $B^2-4AC=4/a^2>0$. Hence, the given PDE is hyperbolic. The characteristic equations are

$$\frac{dt}{dx}=\frac{B\mp\sqrt{B^2-4AC}}{2A}=\mp\frac{\sqrt{4/a^2}}{2}=\mp\frac{1}{a}.$$

Therefore,

$$\frac{dt}{dx}=-\frac{1}{a},\quad \frac{dt}{dx}=\frac{1}{a}.$$

On integration, we get

$$t=-\frac{x}{a}+c_1,\quad t=\frac{x}{a}+c_2.$$

Hence, the characteristic equations are

$$\xi=x+at,\quad \eta=x-at.$$

The canonical form can be obtained by computing

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_t + C\xi_t^2 = 0,$$

$$\bar{B} = 2A\xi_x\eta_x + B\left(\xi_x\eta_t + \xi_t\eta_x\right) + 2C\xi_t\eta_t = 4,$$

$$\bar{C} = 0, \quad \bar{D} = D\xi_x + E\xi_t = \frac{2N}{x}, \quad \bar{E} = D\eta_x + E\eta_t = \frac{2N}{x}.$$

Thus, the canonical equation for the given PDE is

$$4u_{\xi\eta} + \frac{2N}{x}\left(u_\xi + u_\eta\right) = 0.$$

Expressing x in terms of ξ and η, the required canonical equation is

$$u_{\xi\eta} + \frac{N}{\xi + \eta}\left(u_\xi + u_\eta\right) = 0.$$

Example 3.11

Transform the following differential equation to a canonical form:

$$u_{xx} + 2u_{xy} + 4u_{yy} + 2u_x + 3u_y = 0.$$

Solution

The discriminant $B^2 - 4AC = -12 < 0$. Hence, the given PDE is elliptic. The characteristic equations are

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = +1 - i\sqrt{3},$$

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = +1 + i\sqrt{3}.$$

Integration of these equations yields

$$y = +(1 - i\sqrt{3})x + c_1, \quad y = +(1 + i\sqrt{3})x + c_2.$$

Hence, we may take the characteristic equations in the form

$$\xi = y - (1 - i\sqrt{3})x, \quad \eta = y - (1 + i\sqrt{3})x.$$

In order to avoid calculations with complex variables, we introduce the second transformation

$$\alpha = \frac{\xi + \eta}{2}, \quad \beta = \frac{\xi - \eta}{2i}.$$

Therefore,

$$\alpha = y - x, \quad \beta = \sqrt{3}x.$$

The canonical form can now be obtained by computing

$$\bar{A} = A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2 = 3,$$
$$\bar{B} = 2A\alpha_x\beta_x + B\left(\alpha_x\beta_y + \alpha_y\beta_x\right) + 2C\alpha_y\beta_y = 0,$$
$$\bar{C} = A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2 = 3,$$
$$\bar{D} = A\alpha_{xx} + B\alpha_{xy} + C\alpha_{yy} + D\alpha_x + E\alpha_y = 1,$$
$$\bar{E} = A\beta_{xx} + B\beta_{xy} + C\beta_{yy} + D\beta_x + E\beta_y = 2\sqrt{3},$$
$$\bar{F} = 0, \quad \bar{G} = 0.$$

Thus the required canonical form is

$$3u_{\alpha\alpha} + 3u_{\beta\beta} + u_\alpha + 2\sqrt{3}u_\beta = 0,$$

or

$$u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{3}\left(u_\alpha + 2\sqrt{3}u_\beta\right).$$

CHAPTER FOUR
ELLIPTIC DIFFERENTIAL EQUATIONS

4.1 Introduction

In Chapter 3, we have seen the classification of second order partial differential equation into elliptic, parabolic and hyperbolic types. In this chapter we shall consider various properties and techniques for solving Laplace and Poisson equations which are elliptic in nature.

Various physical phenomena are governed by the well-known Laplace and Poisson equations. A few of them, frequently encountered in applications are: steady heat conduction, seepage through porous media, irrotational flow of an ideal fluid, distribution of electrical and magnetic potential, torsion of prismatic shaft, bending of prismatic beams, distribution of gravitational potential, etc.

4.2 Boundary Value Problems (BVPs)

The function u, satisfying the Laplace and Poisson equations in a bounded region $\mathbb{R}$ in R^3, should also satisfy certain boundary conditions on the boundary $\partial\mathbb{R}$ of this region. Such problems are referred to as boundary value problems (BVPs) for Laplace and Poisson equations. We shall denote the set of all boundary points of $\mathbb{R}$ by $\partial\mathbb{R}$. By the closure of $\mathbb{R}$, we mean the set of all interior points of $\mathbb{R}$ together with its boundary points and is denoted by $\bar{\mathbb{R}}$. Symbolically, $\bar{\mathbb{R}} = \mathbb{R} \cup \partial\mathbb{R}$.

If a function $f \in C^{(n)}$, then all its derivatives of order n are continuous. If $f \in C^{(0)}$, then we mean f is continuous.

There are mainly three types of boundary value problems for Laplace equation. If $f \in C^{(0)}$ and is specified on the boundary $\partial\mathbb{R}$ of some finite region $\mathbb{R}$, the problem of determining a function $u(x, y, z)$ such that $\nabla^2 u = 0$ within $\mathbb{R}$ and satisfying $u = f$ on $\partial\mathbb{R}$ is called the boundary value problem of first kind, or **interior Dirichlet problem**. For example, finding the steady state temperature within the region $\mathbb{R}$ when no heat sources or sinks are present and when the temperature is prescribed on the boundary $\partial\mathbb{R}$, is a Dirichlet problem. Another example would be to find the potential inside the region $\mathbb{R}$ when the potential is specified on the boundary $\partial\mathbb{R}$. These two examples correspond to the interior Dirichlet problem. Similarly, if $f \in C^{(0)}$ and is prescribed on the boundary $\partial\mathbb{R}$ of a finite simply connected region $\mathbb{R}$, determining a function $u(x, y, z)$ which satisfies $\nabla^2 u = 0$ outside $\mathbb{R}$ and is such that $u = f$ on $\partial\mathbb{R}$, is called an **exterior Dirichlet problem**. For example, determination of the distribution of the potential outside a body whose surface potential is prescribed, is an exterior Dirichlet problem. The second type of BVP is associated with von Neumann. The problem is to determine the function

$u(x, y, z)$ so that $\nabla^2 u = 0$ within $\mathbb{R}$ while $\dfrac{\partial u}{\partial n} = g$ at every point of $\partial\mathbb{R}$, where $\dfrac{\partial u}{\partial n}$ denotes the normal derivative of the field variable $u(x, y, z)$. This problem is called the **Neumann problem**. If $u(x, y, z)$ is the temperature, $\dfrac{\partial u}{\partial n}$ is the heat flux representing the amount of heat crossing per unit volume per unit time along the normal direction, which is zero when insulated. The third type of BVP is concerned with the determination of the function $u(x, y, z)$ such that $\nabla^2 u = 0$ within $\mathbb{R}$, while a boundary condition of the form $\dfrac{\partial u}{\partial n} + hu = f$, where $h \geq 0$ is specified at every point of $\partial\mathbb{R}$. This is called a mixed BVP or **Churchill's problem**. If we assume Newton's law of cooling, the heat lost is hu, where u is the temperature difference from the surrounding medium and $h > 0$ is a constant depending on the medium. The heat f supplied at a point of the boundary is partly conducted into the medium and partly lost by radiation to the surroundings. Equating these amounts, we get the third boundary condition.

4.3 Interior Dirichlet Problem for A Circle

The Dirichlet problem for the circle is defined as follows:

$$\begin{aligned} &\text{PDE}: \nabla^2 u = 0, && 0 \leq r \leq a,\ 0 \leq \theta \leq 2\pi \\ &\text{BC}: u(a, \theta) = f(\theta), && 0 \leq \theta \leq 2\pi \end{aligned} \qquad (4.1)$$

where $f(\theta)$ is a continuous function on $\partial\mathbb{R}$. The task is to find the value of u an point in the interior of the circle $\mathbb{R}$ in terms of its values on $\partial\mathbb{R}$ such that u is single valued and continuous on $\overline{\mathbb{R}}$.

In view of circular geometry, it is natural to choose polar coordinates to solve this problem and then use the variables separable method. The requirement of single-valuedness of u in $\bar{R}$ implies the periodicity condition, i.e.,

$$u(r, \theta + 2\pi) = u(r, \theta), \quad 0 \leq r \leq a$$

where $\nabla^2 u = 0$ in polar coordinates can be written as

$$\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0.$$

If $u(r, \theta) = R(r) H(\theta)$, the above equation reduces to

$$R'' H + \frac{1}{r} R' H + \frac{1}{r^2} R H'' = 0.$$

This equation can be rewritten as

$$\frac{r^2R''+rR'}{R}=-\frac{H''}{H}=k,$$

which means that a function of r is equal to a function of θ and, therefore, each must be equal to a constant k (a separation constant).

Case I

Let $k=\lambda^2$. Then

$$r^2R''+rR'-\lambda^2R=0,$$

which is a Euler type of equation and can be solved by setting $r=\mathrm{e}^z$. Its solution is

$$R=c_1\mathrm{e}^{\lambda z}+c_2\mathrm{e}^{-\lambda z}=c_1r^{\lambda}+c_2r^{-\lambda}.$$

Also,

$$H''+\lambda^2H=0,$$

whose solution is

$$H=c_3\cos\lambda\theta+c_4\sin\lambda\theta.$$

Therefore,

$$u(r,\theta)=\left(c_1r^{\lambda}+c_2r^{-\lambda}\right)\left(c_3\cos\lambda\theta+c_4\sin\lambda\theta\right). \qquad (4.2)$$

Case II

Let $k=-\lambda^2$. Then

$$r^2R''+rR'+\lambda^2R=0, \quad H''-\lambda^2H=0.$$

Their respective solutions are

$$R=c_1\cos(\lambda\ln r)+c_2\sin(\lambda\ln r),$$
$$H=c_3e^{\lambda\theta}+c_4e^{-\lambda\theta}.$$

Thus

$$u(r,\theta)=\left[c_1\cos(\lambda\ln r)+c_2\sin(\lambda\ln r)\right]\left(c_3e^{\lambda\theta}+c_4e^{-\lambda\theta}\right). \quad (4.3)$$

Case III

Let $k=0$. Then we have

$$rR''+R'=0.$$

Setting $R'(r)=V(r)$, we obtain

$$r\frac{dV}{dr}+V=0, \quad \text{i.e.,}\ \frac{dV}{V}+\frac{dr}{r}=0.$$

Integrating, we get $\ln Vr=\ln c_1$. Therefore,

$$V=\frac{c_1}{r}=\frac{dR}{dr}.$$

On integration,

$$R=c_1\ln r+c_2.$$

Also,

$$H''=0.$$

After integrating twice, we get

$$H = c_3\theta + c_4.$$

Thus,

$$u(r,\theta) = (c_1 \ln r + c_2)(c_3\theta + c_4). \tag{4.4}$$

Now, for the interior problem, $r = 0$ is a point in the domain $\mathbb{R}$ and since $\ln r$ is not defined at $r = 0$, the solutions (4.3) and (4.4) are not acceptable. Thus the required solution is obtained from Eq. (4.2). The periodicity condition in θ implies

$$c_3 \cos\lambda\theta + c_4 \sin\lambda\theta = c_3 \cos(\lambda(\theta + 2\pi)) + c_4 \sin(\lambda(\theta + 2\pi)),$$

i.e.

$$c_3[\cos\lambda\theta - \cos(\lambda\theta + 2\lambda\pi)] + c_4[\sin\lambda\theta - \sin(\lambda\theta + 2\lambda\pi)] = 0,$$

or

$$2\sin\lambda\pi\left[c_3 \sin(\lambda\theta + \lambda\pi) - c_4 \cos(\lambda\theta + \lambda\pi)\right] = 0.$$

Implying $\sin\lambda\pi = 0$, $\lambda\pi = n\pi$, $\lambda = n (n = 0,1,2,\ldots)$. Using the principle of superposition and renaming the constants, the acceptable general solution can be written as

$$u(r,\theta) = \sum_{n=0}^{\infty}\left(c_n r^n + d_n r^{-n}\right)\left(a_n \cos n\theta + b_n \sin n\theta\right).$$

At $r = 0$, the solution should be finite, which requires $d_n = 0$. Thus the appropriate solution assumes the form

$$u(r,\theta) = \sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n \left(A_n \cos n\theta + B_n \sin n\theta\right).$$

For $n = 0$, let the constant A_0 be $A_0 / 2$. Then the solution is

$$u(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \left(A_n \cos n\theta + B_n \sin n\theta\right), \tag{4.5}$$

which is a full-range Fourier series. Now we have to determine A_n and B_n so that the BC : $u(a,\theta) = f(\theta)$ is satisfied, i.e.,

$$f(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n \cos n\theta + B_n \sin n\theta\right).$$

Hence,

$$\begin{cases} A_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(\theta)\cos n\theta d\theta, \; n = 0,1,2,3,\ldots \\ B_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(\theta)\sin n\theta d\theta, \; n = 1,2,3,\ldots \end{cases} \tag{4.6}$$

In Eqs. (4.6) we replace the dummy variable θ by ϕ to distinguish this variable from the current variable θ in Eq. (4.5). Substituting Eq. (4.6) into Eq. (4.5), we obtain the relation

$$u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi} f(\phi)d\phi+\sum_{n=1}^{\infty}\left[\left(\frac{r}{a}\right)^n\frac{\cos n\theta}{\pi}\int_0^{2\pi}\cos(n\phi)f(\phi)d\phi\right.$$
$$\left.+\left(\frac{r}{a}\right)^n\frac{\sin n\theta}{\pi}\int_0^{2\pi}\sin(n\phi)f(\phi)d\phi\right].$$

Interchanging the order of summation and integration, we get

$$u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi} f(\phi)d\phi+\frac{1}{\pi}\int_0^{2\pi} f(\phi)\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\{\cos n\phi\cos n\theta+\sin n\phi\sin n\theta\}d\phi$$
$$=\frac{1}{2\pi}\int_0^{2\pi} f(\phi)\left[1+2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n(\phi-\theta)\right]d\phi. \qquad (4.7)$$

To obtain an alternative expression for $u(r,\theta)$ in closed integral form, we can proceed as follows:

Since

$$1+2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n(\phi-\theta)=\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\left(\mathrm{e}^{in(\phi-\theta)}+\mathrm{e}^{-in(\phi-\theta)}\right),$$
$$=1+\sum_{n=1}^{\infty}\rho^n\,\mathrm{e}^{in(\phi-\theta)}+\sum_{n=1}^{\infty}\rho^n\,\mathrm{e}^{-in(\phi-\theta)},\quad (\rho=\frac{r}{a})$$
$$=1+\frac{\rho\,\mathrm{e}^{i(\phi-\theta)}}{1-\rho\,\mathrm{e}^{i(\phi-\theta)}}+\frac{\rho\,\mathrm{e}^{-i(\phi-\theta)}}{1-\rho\,\mathrm{e}^{-i(\phi-\theta)}},$$

since $r<a$, $(r/a)=\rho<1$ and $\left|\mathrm{e}^{i(\phi-\theta)}\right|\le 1.$ So, we have

$$1+2\sum_{n=1}^{\infty}\rho^n\cos n(\phi-\theta)=\frac{1-\rho^2}{1-2\rho\cos(\phi-\theta)+\rho^2}.$$

Thus, the required solution takes the form

$$u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{\left(a^2-r^2\right)f(\phi)}{\left[a^2-2ar\cos(\phi-\theta)+r^2\right]}d\phi. \qquad (4.8)$$

This is known as Poisson's integral formula for a circle, which gives a unique solution for the Dirichlet problem. The solution (4.8) can be interpreted physically in many ways: It can be thought of as finding the potential $u(r,\theta)$ as a weighted average of the boundary potentials $f(\phi)$ weighted by the Poisson kernel P, given by

$$P=\frac{a^2-r^2}{\left[a^2-2ar\cos(\phi-\theta)+r^2\right]}.$$

It can also be thought of as a steady temperature distribution $u(r,\theta)$ in a circular disc, when the temperature u on its boundary $\partial\mathbb{R}$ is given by $u=f(\phi)$ which is independent of time.

4.4 Exterior Dirichlet Problem for A Circle

The exterior Dirichlet problem is described by

$$\begin{aligned} &\text{PDE: } \nabla^2 u = 0\,, \quad a \le r < \infty \\ &\text{BC}: u(a,\theta) = f(\theta), \end{aligned} \tag{4.9}$$

u must be bounded as $r \to \infty$.

By the method of separation of variables, the general solution (4.2) of $\nabla^2 u = 0$ in polar coordinates can be written as

$$u(r,\theta) = \sum_{n=0}^{\infty}\left(c_n r^n + d_n r^{-n}\right)\left(a_n \cos n\theta + b_n \sin n\theta\right).$$

Now as $r \to \infty$, we require u to be bounded, and, therefore, $c_n = 0$. After adjusting the constants, the general solution now reads

$$u(r,\theta) = \sum_{n=0}^{\infty} r^{-n}\left(A_n \cos n\theta + B_n \sin n\theta\right).$$

With no loss of generality, it can also be written as

$$u(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^{-n}\left(A_n \cos n\theta + B_n \sin n\theta\right). \tag{4.10}$$

Using the $\text{BC}: u(a,\theta) = f(\theta)$, we obtain

$$f(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n \cos n\theta + B_n \sin n\theta\right).$$

This is a full-range Fourier series in $f(\theta)$, where

$$\begin{cases} A_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(\theta)\cos n\theta d\theta, \ n = 0,1,2,3,\ldots \\ B_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(\theta)\sin n\theta d\theta, \ n = 1,2,3,\ldots \end{cases} \tag{4.11}$$

In Eq. (4.11) we replace the dummy variable θ by ϕ so as to distinguish it from the current variable θ. We then introduce the changed variable into solution (4.10) which becomes

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)d\phi + \sum_{n=1}^{\infty}\left[\frac{r^{-n}a^n}{\pi}\cos n\theta\int_0^{2\pi}\cos(n\phi)f(\phi)d\phi \right.$$
$$\left. + \frac{r^{-n}a^n}{\pi}\sin n\theta\int_0^{2\pi}\sin(n\phi)f(\phi)d\phi\right],$$

or

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\left[1 + 2\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n \cos n(\phi - \theta)\right]d\phi. \tag{4.12}$$

Thus the quantity in the square brackets on the right-hand side of Eq. (4.12) becomes

$$1+2\sum_{n=1}^{\infty}\rho^{-n}\cos n(\phi-\theta)=\frac{\rho^2-1}{\rho^2-2\rho\cos(\phi-\theta)+1}.$$

Therefore, the solution of the exterior Dirichlet problem reduces to that of an integral equation of the form

$$u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{\left(r^2-a^2\right)f(\phi)}{r^2-2ar\cos(\phi-\theta)+a^2}d\phi.$$

Example 4.1

Find the steady state temperature distribution in a semi-circular plate of radius a, insulated on both the faces with its curved boundary kept at a constant temperature U_0 and its bounding diameter kept at zero temperature as described in Fig. 4.1.

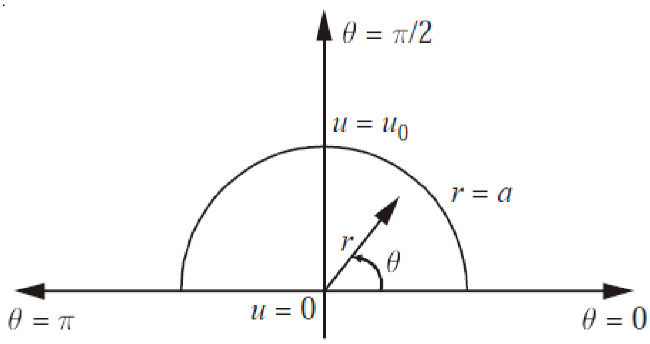

Fig. 4.1 Semi-circular plate.

Solution

The governing heat flow equation is

$$u_t=\nabla^2 u$$

In the steady state, the temperature is independent of time; hence $u_t=0$, and the temperature satisfies the Laplace equation. The problem can now be stated as follows: To solve

$$\text{PDE: }\nabla^2 u(r,\theta)=u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0,$$

$$\text{BCs}: u(a,\theta)=U_0,\quad u(r,0)=0,\quad u(r,\pi)=0.$$

The acceptable general solution is

$$u(r,\theta)=\left(cr^{\lambda}+dr^{-\lambda}\right)(A\cos\lambda\theta+B\sin\lambda\theta)$$

From the BC: $u(r,0)=0$, we get $A=0$; however, the BC: $u(r,\pi)=0$ also gives

$$B\sin\lambda\pi\left(cr^{\lambda}+dr^{-\lambda}\right)=0$$

implying either $B=0$ or $\sin\lambda\pi=0$. $B=0$ gives a trivial solution. For a non-trivial solution, we must have $\sin\lambda\pi=0$, implying

$\lambda\pi = n\pi, \quad n = 0,1,2,\dots$ meaning thereby $\lambda = n$. Hence, the possible solution is

$$u(r,\theta) = B\sin n\theta\left(cr^{\lambda} + dr^{-\lambda}\right). \tag{4.13}$$

In Eq. (4.13), we observe that as $r \to 0$, the term $r^{-\lambda} \to \infty$. But the solution should be finite at $r = 0$, and so $d = 0$. Then after adjusting the constants, it follows from the superposition principle that,

$$u(r,\theta) = \sum_{n=1}^{\infty} B_n \left(\frac{r}{a}\right)^n \sin n\theta.$$

Finally, using the first BC: $u(a,\theta) = U_0$, we get

$$u(a,\theta) = U_0 = \sum_{n=1}^{\infty} B_n \sin n\theta,$$

which is a half-range Fourier sine series. Therefore,

$$B_n = \frac{2}{\pi}\int_0^{\pi} U_0 \sin n\theta d\theta = \begin{cases} \dfrac{4U_0}{n\pi}, & \text{for } n = 1,3,\dots \\ 0, & \text{for } n = 2,4,\dots \end{cases}$$

Hence,

$$B_n = \frac{4U_0}{(2n+1)\pi}, \quad n = 0,1,2,\dots$$

With these values of B_n, the required solution is

$$u(r,\theta) = \frac{4U_0}{\pi}\sum_{n=0}^{\infty} \frac{1}{2n+1}\left(\frac{r}{a}\right)^{2n+1} \sin(2n+1)\theta.$$