Lecture Notes in Integration for Pure 3

First Year

Mathematics

Contents

Introduction

Recall that there are two main parts of Calculus

1. **Derivatives:** Measures instantaneous change
2. **Integrals:** Measures cumulative amounts

We are now ready to begin part 2. It begins with the study of the reverse operation of taking the derivative.

Definition (Antiderivative)

A primitive or antiderivative of a function $f(x)$ is function $F(x)$ such that $F'(x) = f(x)$.

Example: Find an antiderivative of x^3, by trial and error.

Solution: Initial guess: x^4 (since derivation decreases the degree of a power function by 1):

$$\frac{d}{dx} x^4 = 4x^3.$$

Thus: $\frac{d}{dx}\left(\frac{1}{4}x^4\right) = \frac{1}{4}(4x^3) = x^3.$

Note: $\frac{d}{dx}\left(\frac{1}{4}x^4 - 7\right) = x^3$

All functions $F(x) = \frac{1}{4}x^4 + C$, C any constant, are antiderivatives.

Did we find all antiderivatives?

Theorem

Let $F(x)$ be an antiderivative of the function $f(x)$ defined on (a, b). Then any antiderivative on (a, b) of $f(x)$ is of the form $F(x) + C$ for some constant C.

Proof: Let $G(x)$ be another antiderivative of $F(x)$. Set $H(x) = G(x) - F(x)$. Then

$$H'(x) = G'(x) - F'(x) = f(x) - f(x) = 0.$$

We claim that $H(x)$ must be a constant function. For, if it would be not, there exist (at least) two points $x = u$ and $x = v$ in (a, b) with $H(u) \neq H(v)$. By the mean value theorem there exists then a point $x = c$ in (u, v) such that

$$\frac{H(u) - H(v)}{u - v} = H'(c).$$

But since $H(u) \neq H(v)$ this would mean $H'(c) \neq 0$, a contradiction. Thus $H(x) = C$ for some constant C. This implies $G(x) = F(x) + C$. q.e.d.

Definition (Indefinite Integral)

The indefinite integral or general antiderivative $\int f(x)\mathrm{d}x$ *of a function* $f(x)$ *stands for all possible antiderivatives of* $f(x)$ *defined on an interval, i.e.*

$$\int f(x)\,\mathrm{d}x = F(x) + C, \text{ where } C \text{ is a constant}$$

and $F(x)$ *is an arbitrary antiderivative of* $f(x)$.

Notation: In the expression $\displaystyle\int f(x)\mathrm{d}x$, the function $f(x)$ is called the **integrand** and $\mathrm{d}x$ is a differential (in its symbolic meaning). The constant C as above is called the **constant of integration**.

The indefinite integral should not be confused with the **definite integral** $\int_a^b f(x)\,\mathrm{d}x$ which we will consider next week and is defined as a limit of a sum. The symbol $\int$ is a stretched **S** and reminds about the **S**um. We will also explain the relation between the indefinite and the definite integral.

Power Rule: The indefinite integral of a power function $f(x) = x^n$, where $n \neq -1$ is

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C.$$

Raise the exponent by 1 and divide by the raised exponent.

Example: Find the indefinite integral of the following functions:

a) $f(x) = x^{13}$ $\quad \int f(x)\, dx = \frac{x^{14}}{14} + C$

b) $f(x) = \sqrt{x} = x^{1/2}$ $\quad \int f(x)\, dx = \frac{x^{3/2}}{3/2} + C = \frac{2x^{3/2}}{3} + C$

c) $f(x) = \frac{1}{x^3} = x^{-3}$ $\quad \int f(x)\, dx = \frac{x^{-2}}{-2} + C = -\frac{1}{2x^2} + C$

d) $f(x) = 1 = x^0$ $\quad \int f(x)\, dx = x + C$

Integration Rules (1)

$$(1) \int x^n \, dx = \frac{x^{n+1}}{n+1} + c \quad , n \neq -1$$

$$(2) \int a \, dx = ax + c \quad , \quad a \in R$$

$$(3) \int dx = x + c$$

Exercises

$$(1)\int x^{2}\,dx = \frac{1}{3}x^{3} + c$$

$$(2)\int x^{-\frac{1}{2}}\,dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c = 2x^{\frac{1}{2}} + c = 2\sqrt{x} + c$$

$$(3)\int \frac{3}{x^{4}}\,dx = \int 3x^{-4}\,dx = -x^{-3} + c = -\frac{1}{x^{3}} + c$$

$$(4)\int \sqrt{y}\,dy = \int y^{\frac{1}{2}}\,dy = \frac{y^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}\sqrt{y^{3}} + c$$

$$(5)\int \frac{5x}{\sqrt[3]{x}}\,dx = \int 5x\,x^{-\frac{1}{3}}\,dx = \int 5x^{\frac{2}{3}}\,dx = 3x^{\frac{2}{3}} + c$$
$$= 3\sqrt[3]{x^{2}} + c$$

$$(6)\int 4t\sqrt{t^{3}}\,dt = \int 4t\,t^{\frac{3}{2}}\,dt = \int 4t^{\frac{5}{2}}\,dt = \frac{8}{7}t^{\frac{7}{2}} + c$$
$$= \frac{8}{7}\sqrt{t^{7}} + c$$

$$(7)\int \sqrt{3t^{3}}\,dt = \int \sqrt{3}\,t^{\frac{3}{2}}\,dt = \frac{2\sqrt{3}}{5}t^{\frac{5}{2}} + c = \frac{2\sqrt{3}}{5}\sqrt{t^{5}}$$

$$(8)\int 8\,dx = 8x + c$$

$$(9)\int \sqrt{3}\,dx = \sqrt{3}\,x + c$$

$$(10)\int 3^{2}\,dx = 3^{2}x + c = 9x + c$$

$$(11)\int \ln 3\,dx = (\ln 3)x + c$$

$$(12)\int e^3\,dx = e^3 x + c$$

$$(13)\int \frac{1}{e^2}\,dx = \frac{1}{e^2}x + c$$

$$(14)\int 0\,dx = c$$

$$(15)\int \sin 30^\circ\,dx = x\sin 30^\circ + c$$

$$(16)\int 2^e\,dx = 2^e x + c$$

$$(17)\int \log 22\,dx = (\log 22)x + c$$

$$(18)\int \sin^{-1} 3\pi\,dx = (\sin^{-1} 3\pi)\,x + c$$

$$(19)\int (r + 3x)\,dx = rx + \frac{3}{2}x^2 + c$$

$$(20)\int (2x^2 + 5x)\,dx = \frac{2}{3}x^3 + \frac{5}{2}x^2 + c$$

$$(21)\int (x^3 - 5x + 9)\,dx = \frac{1}{4}x^4 - \frac{5}{2}x^2 + 9x + c$$

(22) $\int(\frac{2}{x^3}-\frac{5}{x^2}+x)dx=\int(2x^{-3}-5x^{-2}+x)dx$

$=-x^{-2}+5x^{-1}+\frac{1}{2}x^2+c=-\frac{1}{x^2}+\frac{5}{x}+\frac{1}{2}x^2+c$

(23) $\int(\frac{x^4+2x^3-7}{x^3})dx=\int(\frac{x^4}{x^3}+\frac{2x^3}{x^3}-\frac{7}{x^3})dx$

$\int(x+2-7x^{-3})dx=\frac{1}{2}x^2+2x+\frac{7}{2}x^{-2}+c=\frac{1}{2}x^2+2x+\frac{7}{2x^2}+c$

(24) $\int(\frac{x^2-25}{x-5})dx=\int\frac{(x-5)(x+5)}{x-5}dx=\int(x+5)dx$

$=\frac{1}{2}x^2+5x+c$

(25) $\int(\frac{4x^3-12x^2}{x-3})dx=\int\frac{4x^2(x-3)}{x-3}dx=\int 4x^2dx$

$=\frac{4}{3}x^3+c$

(26) $\int(\frac{x\ln 3+2\ln 3}{x+2})dx=\int\frac{(x+2)\ln 3}{x+2}dx=\int\ln 3dx=x\ln 3+c$

(27) $\int(x-2)(x+3)\,dx=\int(x^2+x-6)dx=\frac{1}{3}x^3+\frac{1}{2}x^2-6x+c$

Integration Rules (2)

$$(4)\ \int e^{F(x)}\, F'(x)\, dx = e^{F(x)} + c$$

$$(5)\ \int a^{F(x)}\, F'(x)\, dx = \frac{a^{F(x)}}{\ln a} + c$$

Power and Logarithmic Laws

$$(1)\ A^{n} A^{m} = A^{n+m}$$

$$(2)\ \frac{A^{n}}{A^{m}} = A^{n-m}$$

$$(3)\ (A^{n})^{m} = A^{nm}$$

$$(4)\ (A \cdot B)^{m} = A^{m} B^{m}$$

$$(5)\ \left(\frac{A}{B}\right)^{n} = \frac{A^{n}}{B^{n}}$$

$$(6)\ \sqrt[n]{A^{m}} = A^{\frac{m}{n}}$$

$$(1)\ \log_e A = \ln A$$

$$(2)\ \log_{10} A = \log A$$

$$(3)\ \log_a AB = \log_a A + \log_a B$$

$$(4)\ \log_a \frac{A}{B} = \log_a A - \log_a B$$

$$(5)\ e^{\ln F(x)} = F(x)$$

$$(6)\ a^{\log_a F(x)} = F(x)$$

$$(7)\ \ln e = 1$$

$$(8)\ \log_a a = 1$$

$$(9)\ \log_a B^{n} = n \log_a B$$

Exercises

$$(1)\ \int e^{3x}\,dx=\frac{1}{3}\int 3e^{3x}dx=\frac{1}{3}e^{3x}+c$$

$$(2)\ \int e^{-2x}\,dx=-\frac{1}{2}\int -2e^{-2x}dx=-\frac{1}{2}e^{-2x}+c$$

$$(3)\ \int e^{-7x+2}\,dx=-\frac{1}{7}\int -7e^{-7x+2}dx=-\frac{1}{7}e^{-7x+2}+c$$

$$(4)\ \int e^{\frac{3}{2}x}\,dx=\frac{2}{3}\int \frac{3}{2}e^{\frac{3}{2}x}dx=\frac{2}{3}e^{\frac{3}{2}x}+c$$

$$(5)\ \int xe^{4x^2}\,dx=\frac{1}{8}\int 8xe^{4x^2}\,dx=\frac{1}{8}e^{4x^2}+c$$

$$(6)\ \int x^2e^{-x^3}\,dx=-\frac{1}{3}\int -3x^2e^{-x^3}\,dx=-\frac{1}{3}e^{-x^3}+c$$

$$(7)\ \int x^{-2}e^{\frac{1}{x}}\,dx=-\int -x^{-2}e^{\frac{1}{x}}dx=-e^{\frac{1}{x}}+c$$

$$(8)\ \int \frac{e^{x^{-2}}}{x^3}\,dx=\int x^{-3}e^{x^{-2}}\,dx=-\frac{1}{2}\int -2x^{-3}e^{x^{-2}}\,dx=-\frac{1}{2}e^{x^{-2}}+c$$

(9) $\int (e^{2x}+e^{-8x})dx=\frac{1}{2}\int 2e^{2x}dx-\frac{1}{8}\int -8e^{-8x}dx=\frac{1}{2}e^{2x}-\frac{1}{8}e^{-8x}+c$

(10) $\int e\,dx=ex+c$

(11) $\int \frac{3}{e^{4x}}\,dx=3\int e^{-4x}\,dx=-\frac{3}{4}\int -4e^{-4x}\,dx=-\frac{3}{4}e^{-4x}+c$

(12) $\int \sqrt{e^{3x}}\,dx=\int (e^{3x})^{\frac{1}{2}}\,dx=\frac{2}{3}\int \frac{3}{2}e^{\frac{3}{2}x}\,dx=\frac{2}{3}e^{\frac{3}{2}x}+c$

(13) $\int e^{-1}\,dx=e^{-1}x+c$

(14) $\int \ln e\,dx=x\ln e+c=x+c$

(15) $\int \frac{e^{5x}}{e^{2x}}\,dx=\int e^{5x-2x}\,dx=\frac{1}{3}\int 3e^{3x}\,dx=\frac{1}{3}e^{3x}+c$

(16) $\int \frac{e^{5x}}{e}\,dx=\int e^{5x-1}\,dx=\frac{1}{5}\int 5e^{5x-1}\,dx=\frac{1}{5}e^{5x-1}+c$

(17) $\int e^{-e}\,dx=e^{-e}x+c$

$$(18)\ \int\frac{e^{x}+e^{2x}}{e^{3x}}\,dx=\int\frac{e^{x}}{e^{3x}}\,dx+\int\frac{e^{2x}}{e^{3x}}\,dx=-\frac{1}{2}\int-2e^{-2x}\,dx-\int-e^{-x}\,dx$$

$$=-\frac{1}{2}e^{-2x}-e^{-x}+c$$

$$(19)\ \int(e^{x}+e^{2x})(e^{x}-e^{2x})\,dx=\int(e^{2x}-e^{4x})dx=\frac{1}{2}\int 2e^{2x}dx-\frac{1}{4}\int 4e^{4x}dx$$

$$=\frac{1}{2}e^{2x}-\frac{1}{4}e^{4x}+c$$

$$(20)\ \int\ln\left(\frac{e^{x}}{e^{3x}}\right)dx=\int\ln(e^{x}e^{-3x})\,dx=\int\ln e^{-2x}\,dx=\int-2x\ln e\,dx$$

$$=\int-2x\,dx=-x^{2}+c$$

$$(21)\ \int\frac{e^{\frac{1}{x}}}{x^{2}}\,dx=\int e^{\frac{1}{x}}x^{-2}dx=-\int-e^{\frac{1}{x}}x^{-2}\,dx=-e^{\frac{1}{x}}+c$$

$$(22)\ \int\frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx=2\int e^{\sqrt{x}}\frac{1}{2\sqrt{x}}dx=2e^{\sqrt{x}}+c$$

$$(23)\ \int\frac{e^{\ln x}}{x}\,dx=\int e^{\ln x}\,\frac{1}{x}dx=e^{\ln x}+c=x+c$$

(24) $\int \frac{e^{\log x}}{x}\,dx = \ln 10 \int e^{\log x}\,\frac{1}{x\ln 10}\,dx = (\ln 10)e^{\log x} + c$

(25) $\int 3^{2x}\,dx = \frac{1}{2}\int 2\,(3^{2x})dx = \frac{3^{2x}}{2\ln 3} + c$

(26) $\int 5^{-7x+2}\,dx = -\frac{1}{7}\int -7(5^{-7x+2})dx = -\frac{5^{-7x+2}}{7\ln 5} + c$

(27) $\int (5^{2x} + 7^{-8x})dx = \frac{1}{2}\int 2(5^{2x})dx - \frac{1}{8}\int -8(7^{-8x})dx$

$= \frac{5^{2x}}{2\ln 5} - \frac{7^{-8x}}{8\ln 7} + c$

(28) $\int \frac{3}{5^{4x}}\,dx = 3\int 5^{-4x}\,dx = -\frac{3}{4}\int -4(5^{-4x})\,dx = -\frac{3(5^{-4x})}{4\ln 5} + c$

(29) $\int \sqrt{7^{3x}}\,dx = \int (7^{3x})^{\frac{1}{2}}\,dx = \frac{2}{3}\int \frac{3}{2}(7^{\frac{3}{2}x})\,dx = \frac{2(7^{\frac{3}{2}x})}{3\ln 7} + c$

(30) $\int 4^{2}\,dx = 4^{2}x + c$

(31) $\int \frac{5^{5x}}{5^{2x}}\,dx = \int 5^{5x-2x}\,dx = \frac{1}{3}\int 3(5^{3x})\,dx = \frac{5^{3x}}{3\ln 5} + c$

(32) $\int x^2 5^{-x^3}\,dx = -\frac{1}{3}\int -3x^2 5^{-x^3}\,dx = -\frac{5^{-x^3}}{3\ln 5}+c$

(33) $\int \frac{3^{x^{-2}}}{x^3}\,dx = \int x^{-3}3^{x^{-2}}\,dx = -\frac{1}{2}\int -2x^{-3}3^{x^{-2}}\,dx = -\frac{3^{x^{-2}}}{2\ln 3}+c$

(34) $\int \frac{7^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int 7^{\sqrt{x}}\frac{1}{2\sqrt{x}}\,dx = \frac{2(7^{\sqrt{x}})}{\ln 7}+c$

(35) $\int \log\left(\frac{10^x}{10^{3x}}\right)dx = \int \log(10^{x-3x})\,dx = \int \log 10^{-2x}\,dx = \int -2x\log 10\,dx$

$= \int -2x\,dx = -x^2+c$

(36) $\int \frac{4^x+6^x}{2^x}\,dx = \int \frac{4^x}{2^x}\,dx + \int \frac{6^x}{2^x}\,dx = \int \left(\frac{4}{2}\right)^x dx + \int \left(\frac{6}{2}\right)^x dx$

$\int 2^x\,dx + \int 3^x\,dx = \frac{2^x}{\ln 2}+\frac{3^x}{\ln 3}+c$

(37) $\int 3\left(\frac{5}{3}\right)^{2x} dx = \frac{3}{2}\int 2\left(\frac{5}{3}\right)^{2x} dx = \frac{3\left(\frac{5}{3}\right)^{2x}}{2\ln\frac{5}{3}}+c$

Integration Rules (3)

$$(6)\int \frac{F'(x)}{F(x)}\,dx = \ln|F(x)| + c$$

$$(7)\int [F(x)]^n F'(x)\,dx = \frac{[F(x)]^{n+1}}{n+1} + c \quad , n \neq -1$$

Logarithmic Laws

$$(1)\ \log_e A = \ln A$$

$$(2)\ \log_{10} A = \log A$$

$$(3)\ \log_a A\,B = \log_a A + \log_a B$$

$$(4)\ \log_a \frac{A}{B} = \log_a A - \log_a B$$

$$(5)\ e^{\ln F(x)} = F(x)$$

$$(6)\ a^{\log_a F(x)} = F(x)$$

$$(7)\ \ln e = 1$$

$$(8)\ \log_a a = 1$$

$$(9)\ \log_a B^n = n \log_a B$$

Exercises

(1) $\int \frac{dx}{2+x} = \ln|2+x|+c$

(2) $\int \frac{3dx}{5x+3} = \frac{3}{5}\int \frac{5dx}{5x+3} = \frac{3}{5}\ln|5x+3|+c$

(3) $\int \frac{2dx}{4-3x} = -\frac{2}{3}\int \frac{-3dx}{4-3x} = -\frac{2}{3}\ln|4-3x|+c$

(4) $\int \frac{x+5}{x^2+10x+5}dx = \frac{1}{2}\int \frac{2x+10}{x^2+10x+5}dx = \frac{1}{2}\ln|x^2+10x+5|+c$

(5) $\int \frac{x^2+4x^5}{x^3+2x^6}dx = \frac{1}{3}\int \frac{3x^2+12x^5}{x^3+2x^6}dx = \frac{1}{3}\ln|x^3+2x^6|+c$

(6) $\int \frac{2e^x}{2+e^x}dx = 2\int \frac{e^x}{2+e^x}dx = 2\ln|2+e^x|+c$

(7) $\int \frac{e^x+1}{e^x+x}dx = \ln|e^x+x|+c$

(8) $\int \frac{e^x-e^{-x}}{e^x+e^{-x}}dx = \ln|e^x+e^{-x}|+c$

(9) $\int \frac{3x^2+2x}{\sqrt{x^3+x^2}}dx = \int (3x^2+2x)(x^3+x^2)^{-\frac{1}{2}}dx = 2(x^3+x^2)^{\frac{1}{2}}+c$

$= 2\sqrt{x^3+x^2}+c$

$$(10)\ \int \frac{4x+1}{\sqrt{2x^2+x+3}}dx = \int (4x+1)(2x^2+x+3)^{-\frac{1}{2}}dx = 2(2x^2+x+3)^{\frac{1}{2}}+c$$

$$=2\sqrt{2x^2+x+3}+c$$

$$(11)\ \int \frac{e^x-x}{\sqrt[3]{2e^x-x^2}}dx = \frac{1}{2}\int (2e^x-2x)(2e^x-x^2)^{-\frac{1}{3}}dx = \frac{3}{4}(2e^x-x^2)^{\frac{2}{3}}+c$$

$$=\frac{3}{4}\sqrt[3]{(2e^x-x^2)^2}+c$$

$$(12)\ \int \frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}dx = \frac{1}{2}\int \frac{2e^{2x}+2e^{-2x}}{e^{2x}-e^{-2x}}dx = \frac{1}{2}\ln\left|e^{2x}-e^{-2x}\right|+c$$

$$(13)\ \int \frac{2^x+2^{-x}}{2^x-2^{-x}}dx = \frac{1}{\ln 2}\int \frac{2^x\ln 2+2^{-x}\ln 2}{2^x-2^{-x}}dx = \frac{1}{\ln 2}\ln\left|2^x-2^{-x}\right|+c$$

$$(14)\ \int \frac{dx}{e+x} = \ln|e+x|+c$$

$$(15)\ \int \frac{dx}{\pi+x} = \ln|\pi+x|+c$$

$$(16)\ \int \frac{(2+\ln x)^{10}dx}{x} = \int (2+\ln x)^{10}\frac{1}{x}dx = \frac{1}{11}(2+\ln x)^{11}+c$$

Definite Integration

We **define** the definite integral of the function $f(x)$ with respect to x from a to b to be

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)\,,$$

where $F(x)$ is the anti-derivative of $f(x)$. We call a and b the lower and upper limits of integration respectively. The function being integrated, $f(x)$, is called the integrand. Note the minus sign!

Note integration constants are not written in definite integrals since they always cancel in them:

$$\begin{aligned}\int_a^b f(x)dx &= F(x)\Big|_a^b \\ &= (F(b)+C) - (F(a)+C) \\ &= F(b) + C - F(a) - C \\ &= F(b) - F(a)\,.\end{aligned}$$

Example 1 Calculate the definite integral $\int_1^2 x^3 dx$.

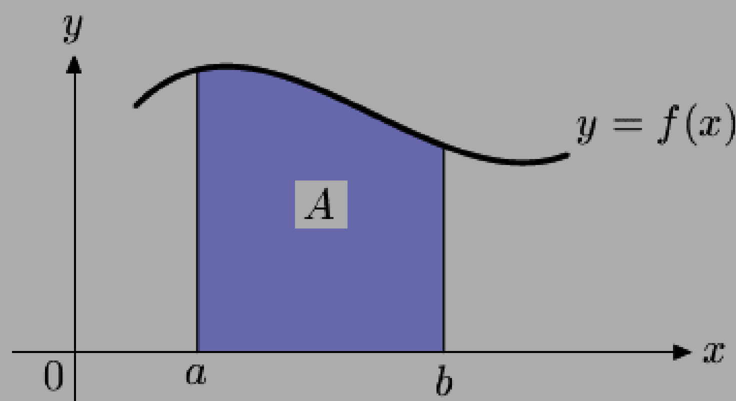
From the rule $\int ax^n dx = \frac{a}{n+1} x^{n+1}$ we have

$$\begin{aligned}\int_1^2 x^3 dx &= \frac{1}{3+1} x^{3+1}\Big|_1^2 \\ &= \frac{1}{4} x^4\Big|_1^2 = \frac{1}{4} \times 2^4 - \frac{1}{4} \times 1^1 \\ &= \frac{1}{4} \times 16 - \frac{1}{4} = 4 - \frac{1}{4} = \frac{15}{4}.\end{aligned}$$

The Area Under a Curve

The definite integral of a function $f(x)$ which lies above the x axis can be interpreted as the area under the curve of $f(x)$.

Thus the area shaded blue below



is given by the definite integral

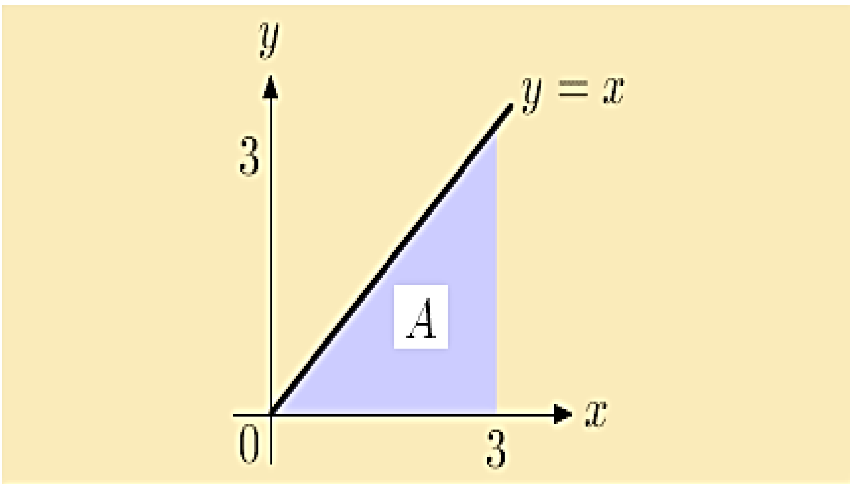
$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a).$$

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a) \quad , f(x) = F'(x)$$

$$EX: \int_1^2 2x\,dx = x^2 \Big|_1^2 = (2)^2 - (1)^2 = 4 - 1 = 3$$

Example 2 Consider the integral $\int_0^3 x\,dx$. The integrand $y = x$ (a straight line) is sketched below. The area underneath the line is the blue shaded triangle. The area of any triangle is half its base times the height. For the blue shaded triangle, this is

$$A = \frac{1}{2} \times 3 \times 3 = \frac{9}{2}.$$



As expected, the integral yields the same result:

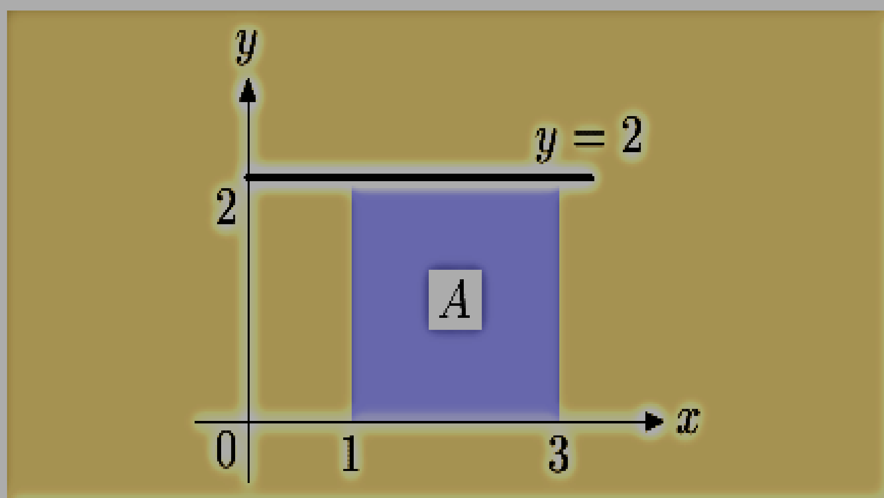
$$\int_0^3 x\,dx = \left.\frac{x^2}{2}\right|_0^3 = \frac{3^2}{2} - \frac{0^2}{2} = \frac{9}{2} - 0 = \frac{9}{2}.$$

Here is a quiz on this relation between definite integrals and the area under a curve.

Quiz Select the value of the definite integral

$$\int_1^3 2dx\,,$$

which is sketched in the following diagram:

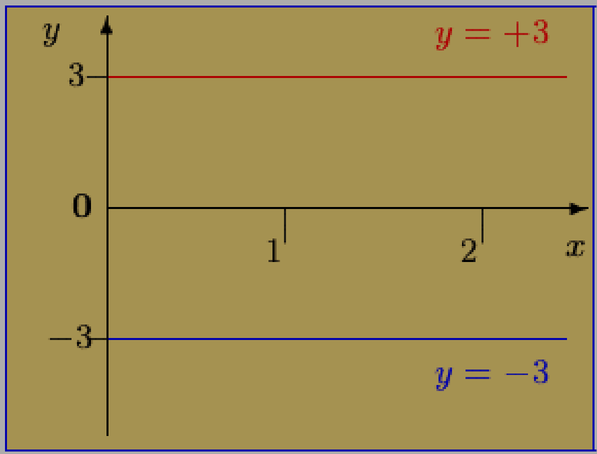


(a) 6 ,   (b) 2 ,   (c) 4 ,   (d) 8 .

Hint: 2 may be written as $2x^0$, since $x^0 = 1$.

Example 3 Consider the two lines: $y = 3$ and $y = -3$.

Let us integrate these functions in turn from $x = 0$ to $x = 2$.



a) For $y = 3$:

$$\int_0^2 (+3)dx = 3x\Big|_0^2 = 3\times2 - 3\times0 = 6\,.$$

and 6 is indeed the area of the rectangle of height 3 and length 2.

b) However, for $y = -3$:

$$\int_0^2 (-3)dx = -3x\Big|_0^2 = -3\times2 - (-3\times0) = -6\,.$$

Although both rectangles have the same area, the *sign* of this result is negative because the curve, $y = -3$, lies below the x axis. This indicates the **sign convention:**

> If a function lies below the x axis, its integral is negative.
> If a function lies above the x axis, its integral is positive.

Example 4 To calculate $\int_{-4}^{-2} 6x^2 dx$, use $\int ax^n dx = \frac{a}{n+1} x^{n+1}$. Thus

$$\begin{aligned} \int_{-4}^{-2} 6x^2\,dx &= \left.\frac{6}{2+1} x^{2+1}\right|_{-4}^{-2} \\ &= \left.\frac{6}{3} x^3\right|_{-4}^{-2} = \left.2x^3\right|_{-4}^{-2} \\ &= 2 \times (-2)^3 - 2 \times (-4)^3 = -16 + 128 = 112\,. \end{aligned}$$

Note that even though the integration range is for negative x (from -4 to -2), the integrand, $f(x) = 6x^2$, is a positive function. The definite integral of a positive function is positive. (Similarly it is negative for a negative function.)

Quiz Select the definite integral of $y = 5x^4$ with respect to x if the lower limit of the integral is $x = -2$ and the upper limit is $x = -1$

(a) -31 ,   (b) 31 ,   (c) 29 ,   (d) -27 .

Exercise 3. Use the integrals listed below to calculate the following definite integrals. (Click on the **green** letters for the solutions)

$f(x)$	x^n for $n \neq -1$	$\sin(ax)$	$\cos(ax)$	e^{ax}	$\frac{1}{x}$
$\int f(x)dx$	$\frac{1}{n+1}x^{n+1}$	$-\frac{1}{a}\cos(ax)$	$\frac{1}{a}\sin(ax)$	$\frac{1}{a}e^{ax}$	$\ln(x)$

(a) $\int_4^9 3\sqrt{t}dt$,

(b) $\int_{-1}^1 (x^2 - 2x + 4)dx$,

(c) $\int_0^\pi \sin(x)dx$,

(d) $\int_0^3 4e^{2x}dx$,

(e) $\int_1^2 \frac{3}{t}dt$,

(f) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\cos(4w)dw$.

Quiz Find the correct result for the definite integral

$$\int_a^{2b} x^2 dx \, .$$

(a) $\frac{8}{3}b^3 - \frac{1}{3}a^3$, (b) $4b - 2a$,

(c) $\frac{8}{3}b^3 + \frac{1}{3}a^3$, (d) $\frac{1}{3}b^3 - \frac{1}{3}a^3$.

Quiz Select the correct result for the definite integral

$$\int_2^3 \frac{1}{x^2} dx \, ,$$

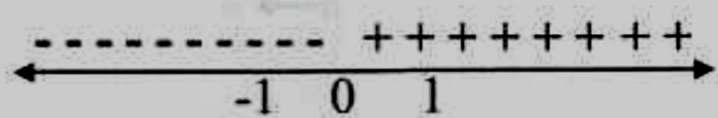
from the answers offered below

(a) -1 , (b) $\frac{1}{5}$, (c) $\frac{1}{36}$, (d) $\frac{1}{6}$.

Definite Integration of Absolute Value

(1) $\int_{-1}^{1} |x|\, dx$

$$F(x) = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$$



$$\int_{-1}^{1} |x|\, dx = \int_{-1}^{0} -x\, dx + \int_{0}^{1} x\, dx$$

$$= -\frac{1}{2}x^2 \Big|_{-1}^{0} + \frac{1}{2}x^2 \Big|_{0}^{1} = [-\frac{1}{2}(0)^2 + \frac{1}{2}(-1)^2] + [\frac{1}{2}(1)^2 - \frac{1}{2}(0)^2]$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

(2) $\int_{-2}^{0} |x+1|\, dx$

$$F(x) = \begin{cases} x+1 & , x \geq -1 \\ -(x+1) & , x < -1 \end{cases}$$

```
 - - - - - - - - - - -   + + + + + + + + +
<------------------------------------------->
                 -2   -1    0
```

$$\int_{-2}^{0} |x+1|\, dx = \int_{-2}^{-1} (-x-1)\, dx + \int_{-1}^{0} (x+1)\, dx$$

$$= (-\frac{1}{2}x^2 - x) \Big|_{-2}^{-1} + (\frac{1}{2}x^2 + x) \Big|_{-1}^{0}$$

$$= [-\frac{1}{2}(-1)^2 - (-1)] - [-\frac{1}{2}(-2)^2 + 2] + [\frac{1}{2}(0)^2 + (0)] - [\frac{1}{2}(-1)^2 - 1$$

$$= [-\frac{1}{2} + 1] - [-2 + 2] - [\frac{1}{2} - 1] = 1$$

(3) $\int_{0}^{1} |x+2|\, dx$

$$F(x) = \begin{cases} x+2 & , x \geq -2 \\ -(x+2) & , x < -2 \end{cases}$$

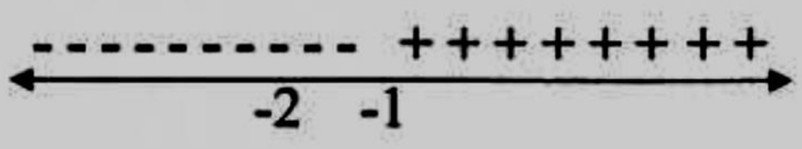
```
 - - - - - - - - - - -   + + + + + + + + +
<------------------------------------------->
                     -2     0     1
```

$$\int_{0}^{1} |x+2|\, dx = \int_{0}^{1} (x+2)\, dx$$

$$=\left(\frac{1}{2}x^2+2x\right)\Big|_{0}^{1} = \left[\frac{1}{2}(1)^2+2(1)\right]-\left[\frac{1}{2}(0)^2+2(0)\right]=\frac{5}{2}$$

(4) $\int_{-2}^{-1} |x+1|\, dx$

$$F(x)=\begin{cases} x+1 & , x \geq -1 \\ -(x+1) & , x < -1 \end{cases}$$



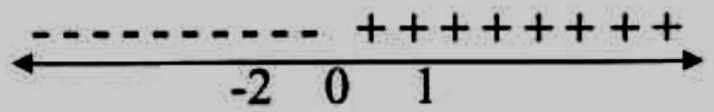
$$\int_{-2}^{-1} |x+1|\, dx = \int_{-2}^{-1} (-x-1)\, dx$$

$$=\left(-\frac{1}{2}x^2-x\right)\Big|_{-2}^{-1} = \left[-\frac{1}{2}(-1)^2-(-1)\right]-\left[-\frac{1}{2}(-2)^2+2\right]$$

$$=\left[-\frac{1}{2}+1\right]-\left[-2+2\right]=\frac{1}{2}$$

(5) $\int_{-2}^{1}(\,|x|+2\,)\,dx$

$$F(x)=\begin{cases} x & , x\geq 0 \\ -x & , x<0 \end{cases}$$



$$\int_{-2}^{1}(|x|+2)\,dx=\int_{-2}^{1}|x|dx+\int_{-2}^{1}2\,dx=\int_{-2}^{0}-x\,dx+\int_{0}^{1}x\,dx+\int_{-2}^{1}2\,dx$$

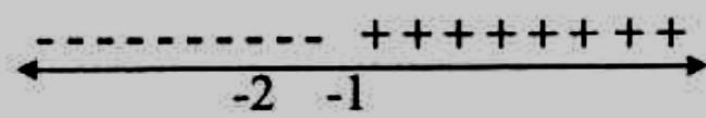
$$=-\frac{1}{2}x^2\Big|_{-2}^{0}+\frac{1}{2}x^2\Big|_{0}^{1}+2x\Big|_{-2}^{1}$$

$$=[-\frac{1}{2}(0)^2+\frac{1}{2}(-2)^2]+[\frac{1}{2}(1)^2-\frac{1}{2}(0)^2]+[2(1)-2(-2)]$$

$$=2+\frac{1}{2}+6=\frac{17}{2}$$

(6) $\int_{-2}^{-1}(|x+1|+2x)\,dx$

$$F(x)=\begin{cases} x+1 & , x\geq -1 \\ -(x+1) & , x<-1 \end{cases}$$



$$\int_{-2}^{-1}(|x+1|+2x)\,dx=\int_{-2}^{-1}|x+1|\,dx+\int_{-2}^{-1}2x\,dx$$

$$=\int_{-2}^{-1}(-x-1)\,dx+\int_{-2}^{-1}2x\,dx$$

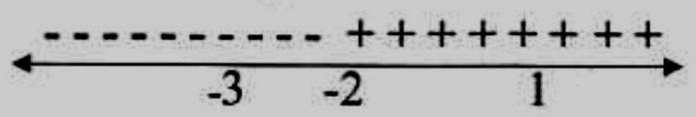
$$=(-\frac{1}{2}x^2-x)\Big|_{-2}^{-1}+x^2\Big|_{-2}^{-1}$$

$$=[-\frac{1}{2}(-1)^2-(-1)]-[-\frac{1}{2}(-2)^2+2]+[(-1)^2-(-2)^2]$$

$$=[-\frac{1}{2}+1]-[-2+2]-3=\frac{1}{2}-3=-\frac{5}{2}$$

(7) $\int_{-3}^{1}x|x+2|\,dx$

$$F(x)=\begin{cases} x+2 & , x\geq -2 \\ -(x+2) & , x<-2 \end{cases}$$

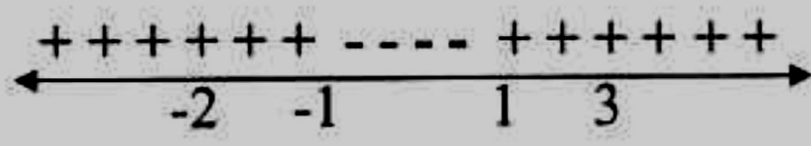


$$\int_{-3}^{1}x|x+2|\,dx=\int_{-3}^{-2}-x(x+2)\,dx+\int_{-2}^{1}x(x+2)\,dx$$

$$=\int_{-3}^{-2}(-x^2-2x)\,dx+\int_{-2}^{1}(x^2+2x)\,dx=(-\frac{1}{3}x^3-x^2)\Big|_{-3}^{-2}+(\frac{1}{3}x^3+x^2)\Big|_{-2}^{1}=$$

(8) $\int_{-2}^{3} |x^2 - 1| \, dx$

$$F(x) = \begin{cases} x^2 - 1 & , x \geq 1 \text{ or } x < -1 \\ -(x^2 - 1) & , -1 \leq x < 1 \end{cases}$$



$$\int_{-2}^{3} |x^2 - 1| \, dx = \int_{-2}^{-1} (x^2 - 1) \, dx + \int_{-1}^{1} -(x^2 - 1) \, dx + \int_{1}^{3} (x^2 - 1) \, dx$$

$$= \left(\frac{1}{3}x^3 - x\right)\Big|_{-2}^{-1} + \left(-\frac{1}{3}x^3 + x\right)\Big|_{-1}^{1} + \left(\frac{1}{3}x^3 - x\right)\Big|_{1}^{3} \approx 9.33$$

Middle Value Theory

If F(X) is a continous function over a closed interval [a,b], there is a number called C in the open interval which satisfies the following condition

$$F(c) = \frac{1}{b-a} \int_{a}^{b} F(x) \, dx$$

Exercises (Find the value of C which satisfies the given integration)

(1) $F(x) = x^2 \quad , x \in [1,2]$

$a = 1 \;,\; b = 2$

$F(c) = c^2$

$$c^2 = \frac{1}{2-1}\int_1^2 x^2 \, dx$$

$$c^2 = \frac{1}{3}x^3 \Big|_1^2$$

$$c^2 = \frac{1}{3}(2)^3 - \frac{1}{3}(1)^3$$

$$c^2 = \frac{8}{3} - \frac{1}{3} \Rightarrow c^2 = \frac{7}{3}$$

$$c = \sqrt{\frac{7}{3}} \in (1,2)$$

$$c = -\sqrt{\frac{7}{3}} \notin (1,2)$$

2) $F(x) = 2x - 3 \quad , x \in [1,4]$

$a = 1 \; , \; b = 4$

$F(c) = 2c - 3$

$$2c - 3 = \frac{1}{4-1}\int_{1}^{4}(2x-3)\;dx$$

$$2c - 3 = \left(\frac{1}{3}x^2 - x\right)\Big|_{1}^{4}$$

$$2c - 3 = \left[\frac{1}{3}(4)^2 - 4\right] - \left[\frac{1}{3}(1)^2 - 1\right]$$

$$2c - 3 = 2 \quad \therefore c = \frac{5}{2}$$

$$c = \frac{5}{2} \in (1,4)$$

3) $F(x) = x(3-x) \quad , x \in [0,3]$

$a = 0 \; , \; b = 3$

$F(c) = c(3-c) = 3c - c^2$

$$3c - c^2 = \frac{1}{3-0}\int_{0}^{3}(3x - x^2)\;dx$$

$$3c - c^2 = \frac{1}{3}\left(\frac{3}{2}x^2 - \frac{1}{3}x^3\right)\Big|_{0}^{3}$$

$$3c - c^2 = \frac{1}{3}\left[\frac{3}{2}(3)^2 - \frac{1}{3}(3)^3\right] - \frac{1}{3}\left[\frac{3}{2}(0)^2\right.$$

$$3c - c^2 = \frac{3}{2} \rightarrow 2c^2 - 6c + 3 = 0$$

$a=2\ ,b=-6\ ,c=3$

$$c=\frac{6\pm\sqrt{36-24}}{4}=\frac{6\pm\sqrt{12}}{4}$$

$$c=\frac{6\pm2\sqrt{3}}{4}=\frac{3}{2}\pm\frac{\sqrt{3}}{2}$$

$$c=\frac{3}{2}+\frac{\sqrt{3}}{2}\in[0,3]$$

$$c=\frac{3}{2}-\frac{\sqrt{3}}{2}\in(0,3)$$

(4) $F(x)=\frac{1}{x}\quad ,x\in[1,e]$

$a=1\ ,\ b=e$

$$F(c)=\frac{1}{c}$$

$$\frac{1}{c}=\frac{1}{e-1}\int_1^e\frac{1}{x}\,dx$$

$$\frac{1}{c}=\frac{1}{e-1}(\ln|x|)\Big]_1^e$$

$$\frac{1}{c}=\frac{1}{e-1}[\ln e-\ln 1]\Rightarrow\frac{1}{c}=\frac{1}{e-1}[1-0]$$

$$\therefore\ c=e-1$$

$$c=e-1\in(1,e)$$

Trigonometric Integration Rules (1)

(1) $\int \sin F(x) \; F'(x)\,dx = -\cos F(x) + c$

(2) $\int \cos F(x) \; F'(x)\,dx = \sin F(x) + c$

(3) $\int \tan F(x) \; F'(x)\,dx = \ln\left|\sec F(x)\right| + c$

(4) $\int \cot F(x) \; F'(x)\,dx = \ln\left|\sin F(x)\right| + c$

(5) $\int \sec F(x) \; F'(x)\,dx = \ln\left|\sec F(x) + \tan F(x)\right| + c$

(6) $\int \csc F(x) \; F'(x)\,dx = \ln\left|\csc F(x) - \cot F(x)\right| + c$

or $= -\ln\left|\csc F(x) + \cot F(x)\right| + c$

(7) $\int \sec F(x) \tan F(x) \; F'(x)\,dx = \sec F(x) + c$

(8) $\int \csc F(x) \; \cot F(x) \; F'(x)\,dx = -\csc F(x) + c$

(9) $\int \csc^2 F(x) \; F'(x)\,dx = -\cot F(x) + c$

(10) $\int \sec^2 F(x) \; F'(x)\,dx = \tan F(x) + c$

Trigonometric General Rules (1)

(1) $\sin^2 ax + \cos^2 ax = 1$

(2) $\sec^2 ax = 1 + \tan^2 ax$

(3) $\csc^2 ax = 1 + \cot^2 ax$

(4) $\sin^2 ax = \frac{1}{2}(1 - \cos 2ax)$

(5) $\cos^2 ax = \frac{1}{2}(1 + \cos 2ax)$

(6) $\sin 2ax = 2 \sin ax \ \cos ax$

(7) $\cos 2ax = \cos^2 ax - \sin^2 ax$

(8) $\sin ax = \frac{1}{\csc ax}$

(9) $\cos ax = \frac{1}{\sec ax}$

(10) $\tan ax = \frac{1}{\cot ax}$

(11) $\cot ax = \frac{\cos ax}{\sin ax}$

(12) $\tan ax = \frac{\sin ax}{\cos ax}$

(13) $\sin ax \ \cos bx = \frac{1}{2}[\sin(ax + bx) + \sin(ax - bx)]$

(14) $\cos ax \ \cos bx = \frac{1}{2}[\cos(ax + bx) + \cos(ax - bx)]$

(15) $\sin ax \ \sin bx = \frac{1}{2}[\cos(ax - bx) - \cos(ax + bx)]$

(16) $\sin(A \pm B) = \sin A \ \cos B \pm \cos A \sin B$

(17) $\cos(A \pm B) = \cos A \ \cos B \mp \sin A \sin B$

Exercises

1) $\int \sin 2x\,dx = \frac{1}{2}\int 2\sin 2x\,dx = -\frac{1}{2}\cos 2x + c$

2) $\int \cos\frac{2}{3}x\,dx = \frac{3}{2}\int \frac{2}{3}\cos\frac{2}{3}x\,dx = \frac{3}{2}\sin\frac{2}{3}x + c$

3) $\int \tan\frac{1}{3}x\,dx = 3\int \frac{1}{3}\tan\frac{1}{3}x\,dx = 3\ln\left|\sec\frac{1}{3}x\right| + c$

4) $\int x\tan x^2 \sec x^2\,dx = \frac{1}{2}\int 2x\tan x^2 \sec x^2\,dx = \frac{1}{2}\sec x^2 + c$

5) $\int 3^{2x}\cot 3^{2x}\csc 3^{2x}\,dx = \frac{1}{2\ln 3}\int 3^{2x}\,2(\ln 3)\cot 3^{2x}\csc 3^{2x}\,dx$

$= -\frac{1}{2\ln 3}\csc 3^{2x} + c$

6) $\int \frac{\sec\sqrt{x}}{\sqrt{x}}\,dx = 2\int \sec(\sqrt{x})\,\frac{1}{2\sqrt{x}}\,dx = 2\ln\left|\sec\sqrt{x} + \tan\sqrt{x}\right| + c$

7) $\int e^{3x}\csc^2 e^{3x}\,dx = \frac{1}{3}\int 3e^{3x}\csc^2 e^{3x}\,dx = -\frac{1}{3}\cot e^{3x} + c$

8) $\int e^{x}\sec^2 e^{x}\,dx = \tan e^{x} + c$

(9) $\int \frac{\sec 5x \tan 5x}{\sqrt{3+\sec 5x}}\,dx = \frac{1}{5}\int 5(\sec 5x \tan 5x)(3+\sec 5x$

$= \frac{2}{5}(3+\sec 5x)^{\frac{1}{2}} + c = \frac{2}{5}\sqrt{3+\sec 5x} + c$

(10) $\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx = \ln|\sec x + \tan x| + c$

(11) $\int \frac{\cos 2x}{\sqrt{1+\sin 2x}}\,dx = \frac{1}{2}\int 2(\cos 2x)(1+\sin 2x)^{-\frac{1}{2}}\,dx$

$= (1+\sin 2x)^{\frac{1}{2}} + c = \sqrt{1+\sin 2x} + c$

(12) $\int \sin x \cos x\,dx = \frac{1}{2}\sin^2 x + c$

(13) $\int \sin 2x \cos^3 2x\,dx = -\frac{1}{2}\int -2\sin 2x(\cos 2x)^3\,dx$

$= -\frac{1}{8}\cos^4 2x + c$

(14) $\int \frac{\sin t}{\sqrt{\cos t}}\,dt = -\int -\sin t\,(\cos t)^{-\frac{1}{2}}\,dt = -2(\cos t)^{\frac{1}{2}} + c$

$= -2\sqrt{\cos t} + c$

5) $\int \sqrt{x}\sin x^{\frac{3}{2}}\,dx = \frac{2}{3}\int \frac{3}{2}\sqrt{x}\sin x^{\frac{3}{2}}\,dx = -\frac{2}{3}\cos x^{\frac{3}{2}} + c$

16) $\int \frac{1}{t^{2}}\sin\frac{1}{t}\,dt = -\int -t^{-2}\sin t^{-1}\,dt = \cos t^{-1} + c$

17) $\int e^{\cos t}\sin t\,dt = -\int -e^{\cos t}\sin t\,dt = -e^{\cos t} + c$

18) $\int \cot 2x\csc^{2} 2x\,dx = -\frac{1}{2}\int -2\cot 2x\csc^{2} 2x\,dx$

$= -\frac{1}{4}\cot^{2} 2x + c$

19) $\int \cot^{4} 3x\csc^{2} 3x\,dx = -\frac{1}{3}\int -3\cot^{4} 3x\csc^{2} 3x\,dx$

$= -\frac{1}{15}\cot^{5} 3x + c$

(20) $\int \tan 5x\sec^{2} 5x\,dx = \frac{1}{5}\int 5\tan 5x\sec^{2} 5x\,dx$

$= \frac{1}{10}\tan^{2} 5x + c$

(21) $\int \tan^{4} 3x\sec^{2} 3x\,dx = \frac{1}{3}\int 3\tan^{4} 3x\sec^{2} 3x\,dx$

$= \frac{1}{15}\tan^{5} 3x + c$

$$(22)\ \int \frac{e^x}{\cos^2 e^x}\,dx = \int e^x \sec^2 e^x\,dx = \tan e^x + c$$

$$(23)\ \int \frac{x}{\sin^2 x^2}\,dx = \frac{1}{2}\int 2x \csc^2 x^2\,dx = -\frac{1}{2}\cot x^2 + c$$

$$(24)\ \int \frac{x\cos x + 2\cos x}{x+2}\,dx = \int \frac{\cos x\,(x+2)}{x+2}\,dx$$

$$= \int \cos x\,dx = \sin x + c$$

$$(25)\ \int \cos x \sec x\,dx = \int \cos x \frac{1}{\cos x}\,dx$$

$$= \int dx = x + c$$

$$(26)\ \int \sin x \sec x\,dx = \int \sin x \frac{1}{\cos x}\,dx = \int \frac{\sin x}{\cos x}\,dx$$

$$= \int \tan x\,dx = \ln|\sec x| + c$$

$$(27)\ \int \frac{\cot x \csc x}{1+\csc x}\,dx = -\int \frac{-\cot x \csc x}{1+\csc x}\,dx = -\ln|1+\csc x| + c$$

$$(28)\ \int \frac{1}{x}\sin(\ln x)\,dx = -\cos(\ln x) + c$$

$$(29)\ \int \frac{\sin 2x}{2+\cos 2x}\,dx = -\frac{1}{2}\int \frac{-2\sin 2x}{2+\cos 2x}\,dx = -\frac{1}{2}\ln|2+\cos 2x| + c$$

(30) $\int \sec^2 2x \tan^6 2x \, dx = \frac{1}{2} \int 2\sec^2 2x \tan^6 2x \, dx$

$= \frac{1}{14} \tan^7 2x + c$

(31) $\int (1+\cos 8x)^3 \sin 8x \, dx = -\frac{1}{8} \int -8(1+\cos 8x)^3 \sin 8x \, dx$

$= -\frac{1}{32}(1+\cos 8x)^4 + c$

(32) $\int \sqrt{1+\sin 2x} \cos 2x \, dx = \frac{1}{2} \int (1+\sin 2x)^{\frac{1}{2}} 2\cos 2x \, dx$

$= \frac{1}{3}(1+\sin 2x)^{\frac{3}{2}} + c$

(33) $\int \frac{\tan x}{\ln(\cos x)} dx = -\int \frac{-\tan x}{\ln(\cos x)} dx = -\ln\left|\ln(\cos x)\right| + c$

(34) $\int \frac{\cot x}{\ln(\sin x)} dx = \ln\left|\ln(\sin x)\right| + c$

(35) $\int \frac{\tan x}{\cos^2 x} dx = \int \tan x \frac{1}{\cos^2 x} dx = \int \tan x \sec^2 x \, dx =$

$= \frac{1}{2} \tan^2 x + c$

(36) $\int \frac{\cot x}{\sin^2 x} dx = \int \cot x \frac{1}{\sin^2 x} dx = -\int -\cot x \csc^2 x \, dx$

$= -\frac{1}{2} \cot^2 x + c$

(37) $\int \frac{\sec x}{\tan x}\,dx = \int \sec x \frac{1}{\tan x}\,dx = \int \frac{1}{\cos x}\ \frac{\cos x}{\sin x}\,dx$

$= \int \csc x\,dx = \ln|\csc x - \cot x| + c$

(38) $\int \frac{\csc x}{\cot x}\,dx = \int \csc x \frac{1}{\cot x}\,dx = \int \frac{1}{\sin x}\ \frac{\sin x}{\cos x}\,dx$

$= \int \sec x\,dx = \ln|\sec x + \tan x| + c$

(39) $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx = \int \sin^{-1} x \frac{1}{\sqrt{1-x^2}}\,dx = \frac{1}{2}(\sin^{-1} x)^2 + c$

(40) $\int \frac{\cos^{-1} 3x}{\sqrt{1-9x^2}}\,dx = -\frac{1}{3}\int \cos^{-1} 3x \frac{-3}{\sqrt{1-9x^2}}\,dx$

$= -\frac{1}{6}(\cos^{-1} 3x)^2 + c$

(41) $\int \frac{(\tan^{-1} 2x)^3}{1+4x^2}\,dx = \frac{1}{2}\int (\tan^{-1} 2x)^3 \frac{2}{1+4x^2}\,dx = \frac{1}{8}(\tan^{-1} 2x)^4 + c$

(42) $\int \frac{\sin 2x}{\cos^5 2x}\,dx = \int \sin 2x\ (\cos 2x)^{-5}\,dx$

$= -\frac{1}{2}\int -2\sin 2x\ (\cos 2x)^{-5}\,dx = \frac{1}{8}(\cos 2x)^{-4} + c$

$= \frac{1}{8\cos^4 2x} + c$

Trigonometric Integration Rules (2)

1) $\int \sinh F(x)\ F'(x)\,dx = \cosh F(x) + c$

2) $\int \cosh F(x)\ F'(x)\,dx = \sinh F(x) + c$

3) $\int \tanh F(x)\ F'(x)\,dx = \ln\left|\cosh F(x)\right| + c$

4) $\int \coth F(x)\ F'(x)\,dx = \ln\left|\sinh F(x)\right| + c$

5) $\int \operatorname{sech} F(x)\tanh F(x)\ F'(x)\,dx = -\operatorname{sech} F(x) + c$

6) $\int \operatorname{csch} F(x)\ \coth F(x)\ F'(x)\,dx = -\operatorname{csch} F(x) + c$

7) $\int \operatorname{csch}^2 F(x)\ F'(x)\,dx = -\coth F(x) + c$

8) $\int \operatorname{sech}^2 F(x)\ F'(x)\,dx = \tanh F(x) + c$

Trigonometric General Rules (2)

(1) $\cosh^2 ax - \sinh^2 ax = 1$

(2) $\mathrm{sech}^2 ax = 1 - \tanh^2 ax$

(3) $\mathrm{csch}^2 ax = \coth^2 ax - 1$

(4) $\sinh^2 ax = \frac{1}{2}(\cosh 2ax - 1)$

(5) $\cosh^2 ax = \frac{1}{2}(\cosh 2ax + 1)$

(6) $\sinh 2ax = 2 \sinh ax \cosh ax$

(7) $\cosh 2ax = \cosh^2 ax + \sinh^2 ax$

(8) $\sinh ax = \frac{1}{\mathrm{csch}\, ax}$

(9) $\cosh ax = \frac{1}{\mathrm{sech}\, ax}$

(10) $\tanh ax = \frac{1}{\coth ax}$

(11) $\coth ax = \frac{\cosh ax}{\sinh ax}$

(12) $\tanh ax = \frac{\sinh ax}{\cosh ax}$

(13) $\sinh ax \cosh bx = \frac{1}{2}[\sinh(ax+bx) + \sinh(ax-bx)]$

(14) $\cosh ax \cosh bx = \frac{1}{2}[\cosh(ax+bx) + \cosh(ax-bx)]$

(15) $\sinh ax \sinh bx = \frac{1}{2}[\cosh(ax+bx) - \cosh(ax-bx)]$

Exercises

(1) $\int \sinh 2x \, dx = \frac{1}{2}\int 2\sinh 2x \, dx = \frac{1}{2}\cosh 2x + c$

(2) $\int \cosh \frac{2}{3}x \, dx = \frac{3}{2}\int \frac{2}{3}\cosh \frac{2}{3}x \, dx = \frac{3}{2}\sinh \frac{2}{3}x + c$

(3) $\int \tanh \frac{1}{3}x \, dx = 3\int \frac{1}{3}\tanh \frac{1}{3}x \, dx = 3\ln\left|\cosh \frac{1}{3}x\right| + c$

(4) $\int \frac{\cosh 2x}{\sqrt{1+\sinh 2x}} \, dx = \frac{1}{2}\int 2(\cosh 2x)(1+\sinh 2x)^{-\frac{1}{2}} \, dx$

$= (1+\sinh 2x)^{\frac{1}{2}} + c = \sqrt{1+\sinh 2x} + c$

(5) $\int \frac{\cosh^{-1} 3x}{\sqrt{9x^2-1}} \, dx = \frac{1}{3}\int \cosh^{-1} 3x \frac{3}{\sqrt{9x^2-1}} \, dx$

$= \frac{1}{6}(\cosh^{-1} 3x)^2 + c$

(6) $\int \cosh^2 2x \, dx = \frac{1}{2}\int (1+\cosh 4x) \, dx = \int \frac{1}{2} dx + \frac{1}{8}\int 4\cosh 4x \, dx$

$= \frac{1}{2}x + \frac{1}{8}\sinh 4x + c$

(7) $\int \tanh^2 x \, dx = \int (1-\operatorname{sech}^2 x) \, dx = \int dx - \int \operatorname{sech}^2 x \, dx$

$= x - \tanh x + c$

(8) $\int \coth^2 x\, dx = \int (\operatorname{csch}^2 x + 1)\, dx = \int \operatorname{csch}^2 x\, dx + \int dx$

$= -\coth x + x + c$

(9) $\int \frac{dx}{(e^x - e^{-x})^2} = \frac{1}{4}\int \left(\frac{2}{e^x - e^{-x}}\right)^2 dx = \frac{1}{4}\int \operatorname{csch}^2 x\, dx = -\frac{1}{4}\coth x + c$

(10) $\int \frac{dx}{(e^x + e^{-x})^2} = \frac{1}{4}\int \left(\frac{2}{e^x + e^{-x}}\right)^2 dx = \frac{1}{4}\int \operatorname{sech}^2 x\, dx = \frac{1}{4}\tanh x + c$

(11) $\int \operatorname{sech} x\, dx = \int \frac{2}{e^x + e^{-x}}\, dx = \int \frac{2e^x}{e^x e^x + e^x e^{-x}}\, dx$

$= \int \frac{2e^x}{e^{2x} + 1}\, dx = 2\int \frac{e^x}{e^{2x} + 1}\, dx = 2\tan^{-1}(e^x) + c$

(12) $\int \operatorname{csch} x\, dx = \int \operatorname{csch} x \frac{\operatorname{csch} x - \coth x}{\operatorname{csch} x - \coth x}\, dx$

$\int \frac{\operatorname{csc}^2 hx - \operatorname{csc} hx \coth x}{\operatorname{csc} hx - \coth x}\, dx = \ln\left|\operatorname{csc} hx - \coth x\right| + c$

(13) $\int \tanh x\, dx = \int \frac{\sinh x}{\cosh x}\, dx = \ln\left|\cosh x\right| + c$

(14) $\int \coth x\, dx = \int \frac{\cosh x}{\sinh x}\, dx = \ln\left|\sinh x\right| + c$

(15) $\int \sinh 2x \sinh 3x \, dx$

$$= \int \frac{1}{2}[\cosh(2x+3x) - \cosh(2x-3x)] \, dx$$

$$= \int \frac{1}{2}[\cosh(5x) - \cosh(-x)] \, dx$$

$$= \frac{1}{10} \int 5\cosh(5x) \, dx - \frac{1}{2} \int \cosh(x) \, dx$$

$$= \frac{1}{10} \sinh 5x - \frac{1}{2} \sinh x + c$$

(16) $\int \frac{(1+\sinh^2 x)^{\frac{1}{2}}}{\sec hx \csc hx} \, dx = \int (1+\sinh^2 x)^{\frac{1}{2}} \frac{dx}{\sec hx \csc hx}$

$$= \frac{1}{2} \int (1+\sinh^2 x)^{\frac{1}{2}} 2\cosh x \sinh x \, dx = \frac{1}{3}(1+\sinh^2 x)^{\frac{3}{2}} + c$$

(17) $\int \frac{1+\cosh 2x}{\sinh^2 2x} \, dx = \int \frac{1}{\sinh^2 2x} dx + \int \frac{\cosh 2x}{\sinh^2 2x} \, dx$

$$= \int \csc h^2 2x \, dx + \int \frac{\cosh 2x}{\sinh 2x} \cdot \frac{1}{\sinh 2x} \, dx$$

$$= \frac{1}{2} \int 2 \csc h^2 2x \, dx + \frac{1}{2} \int 2 \coth 2x \csc h 2x dx$$

$$= -\frac{1}{2} \coth 2x - \frac{1}{2} \csc h 2x + c$$

(18) $\int \frac{dx}{\sqrt{9x^2-1}(\cosh^{-1} 3x)} = \frac{1}{3} \int \frac{\frac{3}{\sqrt{9x^2-1}}}{(\cosh^{-1} 3x)} \, dx$

$$= \frac{1}{3} \ln(\cosh^{-1} 3x) + c$$

Integration by substituting

We introduce the technique through some simple examples for which a linear substitution is appropriate.

Example

Suppose we want to find the integral

$$\int (x+4)^5 \, dx \tag{1}$$

You will be familiar already with finding a similar integral $\int u^5 \, du$ and know that this integral is equal to $\frac{u^6}{6} + c$, where c is a constant of integration. This is because you know that the rule for integrating powers of a variable tells you to increase the power by 1 and then divide by the new power.

In the integral given by Equation (1) there is still a power 5, but the integrand is more complicated due to the presence of the term $x + 4$. To tackle this problem we make a **substitution**. We let $u = x + 4$. The point of doing this is to change the integrand into the much simpler u^5. However, we must take care to substitute appropriately for the term dx too.

In terms of differentials we have

$$du = \left(\frac{du}{dx}\right) dx$$

Now, in this example, because $u = x + 4$ it follows immediately that $\frac{du}{dx} = 1$ and so $du = dx$. So, substituting both for $x + 4$ and for dx in Equation (1) we have

$$\int (x+4)^5 \, dx = \int u^5 du$$

The resulting integral can be evaluated immediately to give $\frac{u^6}{6} + c$. We can revert to an expression involving the original variable x by recalling that $u = x + 4$, giving

$$\int (x+4)^5 \, dx = \frac{(x+4)^6}{6} + c$$

We have completed the integration by substitution.

Example

Suppose now we wish to find the integral

$$\int \cos(3x+4)\,\mathrm{d}x \tag{2}$$

Observe that if we make a substitution $u = 3x + 4$, the integrand will then contain the much simpler form $\cos u$ which we will be able to integrate.

As before,

$$\mathrm{d}u = \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x$$

and so

$$\text{with } u = 3x+4 \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 3$$

It follows that

$$\mathrm{d}u = \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x = 3\,\mathrm{d}x$$

So, substituting u for $3x+4$, and with $\mathrm{d}x = \frac{1}{3}\mathrm{d}u$ in Equation (2) we have

$$\begin{aligned}\int \cos(3x+4)\,\mathrm{d}x &= \int \frac{1}{3}\cos u\,\mathrm{d}u \\ &= \frac{1}{3}\sin u + c\end{aligned}$$

We can revert to an expression involving the original variable x by recalling that $u = 3x + 4$, giving

$$\int \cos(3x+4)\,\mathrm{d}x = \frac{1}{3}\sin(3x+4) + c$$

We have completed the integration by substitution.

It is very easy to generalise the result of the previous example. If we want to find $\int \cos(ax+b)\mathrm{d}x$, the substitution $u = ax+b$ leads to $\frac{1}{a}\int \cos u\,\mathrm{d}u$ which equals $\frac{1}{a}\sin u + c$, that is $\frac{1}{a}\sin(ax+b)+c$.

A similar argument, which you should try, shows that $\int \sin(ax+b)\mathrm{d}x = -\frac{1}{a}\cos(ax+b) + c$.

Suppose we wish to find $\int \frac{1}{1-2x} \mathrm{d}x$.

We make the substitution $u = 1 - 2x$ in order to simplify the integrand to $\frac{1}{u}$. Recall that the integral of $\frac{1}{u}$ with respect to u is the natural logarithm of u, $\ln|u|$. As before,

$$\mathrm{d}u = \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right) \mathrm{d}x$$

and so

$$\text{with} \quad u = 1 - 2x \quad \text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = -2$$

It follows that

$$\mathrm{d}u = \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right) \mathrm{d}x = -2\,\mathrm{d}x$$

The integral becomes

$$\begin{aligned} \int \frac{1}{u}\left(-\frac{1}{2}\mathrm{d}u\right) &= -\frac{1}{2}\int \frac{1}{u}\,\mathrm{d}u \\ &= -\frac{1}{2}\ln|u| + c \\ &= -\frac{1}{2}\ln|1-2x| + c \end{aligned}$$

The result of the previous example can be generalised: if we want to find $\int \frac{1}{ax+b}\mathrm{d}x$, the substitution $u = ax + b$ leads to $\frac{1}{a}\int \frac{1}{u}\,\mathrm{d}u$ which equals $\frac{1}{a}\ln|ax+b| + c$.

This means, for example, that when faced with an integral such as $\int \frac{1}{3x+7}\,\mathrm{d}x$ we can immediately write down the answer as $\frac{1}{3}\ln|3x+7| + c$.

Key Point

$$\int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln|ax+b| + c$$

Example

Suppose we wish to find

$$\int_1^3 (9+x)^2\,dx$$

We make the substitution $u = 9 + x$. As before,

$$du = \left(\frac{du}{dx}\right) dx$$

and so

$$\text{with} \quad u = 9 + x \quad \text{and} \quad \frac{du}{dx} = 1$$

It follows that

$$du = \left(\frac{du}{dx}\right) dx = dx$$

The integral becomes

$$\int_{x=1}^{x=3} u^2\,du$$

where we have explicitly written the variable in the limits of integration to emphasise that those limits were on the variable x and not u. We can write these as limits on u using the substitution $u = 9 + x$. Clearly, when $x = 1$, $u = 10$, and when $x = 3$, $u = 12$. So we require

$$\begin{aligned}\int_{u=10}^{u=12} u^2\,du &= \left[\frac{1}{3}u^3\right]_{10}^{12} \\ &= \frac{1}{3}\left(12^3 - 10^3\right) \\ &= \frac{728}{3}\end{aligned}$$

Note that in this example there is no need to convert the answer given in terms of u back into one in terms of x because we had already converted the limits on x into limits on u.

3. Finding $\int f(g(x))g'(x)\,\mathrm{d}x$ by substituting $u = g(x)$

Example

Suppose now we wish to find the integral

$$\int 2x\sqrt{1+x^2}\,\mathrm{d}x \tag{3}$$

In this example we make the substitution $u = 1 + x^2$, in order to simplify the square-root term. We shall see that the rest of the integrand, $2x\,\mathrm{d}x$, will be taken care of automatically in the substitution process, and that this is because $2x$ is the derivative of that part of the integrand used in the substitution, i.e. $1 + x^2$.

As before,

$$\mathrm{d}u = \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x$$

and so

$$\text{with } u = 1 + x^2 \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$$

It follows that

$$\mathrm{d}u = \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x = 2x\,\mathrm{d}x$$

So, substituting u for $1 + x^2$, and with $2x\,\mathrm{d}x = \mathrm{d}u$ in Equation (3) we have

$$\begin{aligned}\int 2x\sqrt{1+x^2}\,\mathrm{d}x &= \int \sqrt{u}\,\mathrm{d}u \\ &= \int u^{1/2}\,\mathrm{d}u \\ &= \frac{2}{3}u^{3/2} + c\end{aligned}$$

We can revert to an expression involving the original variable x by recalling that $u = 1 + x^2$, giving

$$\int 2x\sqrt{1+x^2}\,\mathrm{d}x = \frac{2}{3}(1+x^2)^{3/2} + c$$

We have completed the integration by substitution.

Example

Suppose we wish to evaluate

$$\int \frac{4x}{\sqrt{2x^2+1}}\,\mathrm{d}x$$

By writing the integrand as $\dfrac{1}{\sqrt{2x^2+1}} \cdot 4x$ we note that it takes the form $\int f(g(x))g'(x)\mathrm{d}x$ where $f(u) = \dfrac{1}{\sqrt{u}}$, $g(x) = 2x^2+1$ and $g'(x) = 4x$.

The substitution $u = g(x) = 2x^2+1$ transforms the integral to

$$\int f(u)\,\mathrm{d}u = \int \frac{1}{\sqrt{u}}\,\mathrm{d}u$$

This is evaluated to give

$$\begin{aligned} \int \frac{1}{\sqrt{u}}\,\mathrm{d}u &= \int u^{-1/2}\,\mathrm{d}u \\ &= 2u^{1/2}+c \end{aligned}$$

Finally, using $u = 2x^2+1$ to revert to the original variable gives

$$\int \frac{4x}{\sqrt{2x^2+1}}\,\mathrm{d}x = 2(2x^2+1)^{1/2}+c$$

or equivalently

$$2\sqrt{2x^2+1}+c$$

Example

Suppose we wish to find $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,\mathrm{d}x$.

Consider the substitution $u = \sqrt{x}$. Then

$$\begin{aligned}\mathrm{d}u &= \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x \\ &= \frac{1}{2}x^{-1/2}\mathrm{d}x \\ &= \frac{1}{2x^{1/2}}\mathrm{d}x \\ &= \frac{1}{2\sqrt{x}}\mathrm{d}x\end{aligned}$$

so that

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,\mathrm{d}x = 2\int \sin u\,\mathrm{d}u$$

from which

$$\begin{aligned}2\int \sin u\,\mathrm{d}u &= -2\cos u + c \\ &= -2\cos\sqrt{x} + c\end{aligned}$$

We can also make the following observations:
the integrand can be written in the form $\sin\sqrt{x}\cdot\dfrac{1}{\sqrt{x}}$.

Writing $f(u) = \sin u$ and $g(x) = \sqrt{x}$ then $g'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$.

Further, $f(g(x)) = \sin\sqrt{x}$.

Hence we write the given integral as

$$2\int \sin\sqrt{x}\frac{1}{2\sqrt{x}}\,\mathrm{d}x$$

which is of the form

$$2\int f(g(x))g'(x)\,\mathrm{d}x$$

with f and g as given above.

As before the substitution $u = g(x) = \sqrt{x}$ produces the integral

$$2\int f(u)\,\mathrm{d}u = 2\int \sin u\,\mathrm{d}u$$

from which

$$\begin{aligned} 2\int \sin u\,\mathrm{d}u &= -2\cos u + c \\ &= -2\cos\sqrt{x} + c \end{aligned}$$

1. $\int (5x+4)^5 \, dx$

(a) Let $u = 5x + 4$

(b) Then $du = 5 \, dx$ or $\frac{1}{5} \, du = dx$.

(c) Now substitute

$$\begin{aligned} \int (5x+4)^5 \, dx &= \int u^5 \cdot \frac{1}{5} \, du \\ &= \int \frac{1}{5} u^5 \, du \\ &= \frac{1}{30} u^6 + C \\ &= \frac{1}{30}(5x+4)^6 + C \end{aligned}$$

2. $\int 3t^2(t^3+4)^5 \, dt$

(a) Let $u = t^3 + 4$

(b) Then $du = 3t^2 \, dt$

(c) Now substitute

$$\begin{aligned} \int 3t^2(t^3+4)^5 \, dt &= \int (t^3+4)^5 \cdot 3t^2 \, dt \\ &= \int u^5 \cdot du \\ &= \frac{1}{6} u^6 + C \\ &= \frac{1}{6}(t^3+4)^6 + C \end{aligned}$$

3. $\int \sqrt{4x-5} \, dx$

(a) Let $u = 4x - 5$

(b) Then $du = 4\,dx$ or $\frac{1}{4}\,du = dx$

(c) Now substitute

$$\begin{aligned} \int \sqrt{4x-5}\,dx &= \int \sqrt{u} \cdot \frac{1}{4}\,du \\ &= \int \frac{1}{4} u^{1/2}\,du \\ &= \frac{1}{4} u^{3/2} \cdot \frac{2}{3} + C \\ &= \frac{1}{6}(4x-5)^{3/2} + C \end{aligned}$$

4. $\int t^2(t^3+4)^{-1/2}\,dt$

(a) Let $u = t^3 + 4$

(b) Then $du = 3t^2\,dt$ or $\frac{1}{3}\,du = t^2\,dt$

(c) Now substitute

$$\begin{aligned} \int t^2(t^3+4)^{-1/2}\,dt &= \int (t^3+4)^{-1/2} \cdot t^2\,dt \\ &= \int u^{-1/2} \cdot \frac{1}{3}\,du \\ &= \int \frac{1}{3} u^{-1/2}\,du \\ &= \frac{1}{3} u^{1/2} \cdot \frac{2}{1} + C \\ &= \frac{2}{3} u^{1/2} + C \\ &= \frac{2}{3}(t^3+4)^{1/2} + C \end{aligned}$$

5. $\int \cos(2x+1)\,dx$

(c) Now substitute

$$\begin{aligned} \int \cos(2x+1)\,dx &= \int \cos(u)\cdot\frac{1}{2}\,du \\ &= \int \frac{1}{2}\cos(u)\,du \\ &= \frac{1}{2}\sin(u)+C \\ &= \frac{1}{2}\sin(2x+1)+C \end{aligned}$$

6. $\int \sin^{10}(x)\cos(x)\,dx$

(a) Let $u = \sin(x)\,dx$

(b) Then $du = \cos(x)\,dx$

(c) Now substitute

$$\begin{aligned} \int \sin^{10}(x)\cos(x)\,dx &= \int u^{10}\cdot du \\ &= \frac{1}{11}u^{11}+C \\ &= \frac{1}{11}\sin^{11}(x)+C \end{aligned}$$

7. $\int \frac{\sin(x)}{(\cos(x))^5}\,dx$

(a) Let $u = \cos(x)$

(b) Then $du = -\sin(x)\,dx$ or $-du = \sin(x)\,dx$

(c) Now substitute

$$\begin{aligned}
\int \frac{\sin(x)}{(\cos(x))^5}\,dx &= \int \frac{1}{(\cos(x)^5} \cdot \sin(x)\,dx \\
&= \int \frac{1}{u^5}\,(-du) \\
&= \int -u^{-5} + C \\
&= -\frac{u^{-4}}{-4} + C \\
&= u^{-4} + C \\
&= \frac{1}{(\cos(x))^4} + C
\end{aligned}$$

8. $\displaystyle\int \frac{(\sqrt{x}-1)^2}{\sqrt{x}}\,dx$

(a) Let $u = \sqrt{x} - 1$

(b) Then $du = \dfrac{1}{2\sqrt{x}}\,dx$ or $2\,du = \dfrac{1}{\sqrt{x}}\,dx$

(c) Now substitute

$$\begin{aligned}
\int \frac{(\sqrt{x}-1)^2}{\sqrt{x}}\,dx &= \int (\sqrt{x}-1)^2 \cdot \frac{1}{\sqrt{x}}\,dx \\
&= \int u^2\,(2\,du) \\
&= \int 2u^2\,du \\
&= \frac{2}{3}u^3 + C \\
&= \frac{2}{3}(\sqrt{x}-1)^3 + C
\end{aligned}$$

9. $\displaystyle\int \sqrt{x^3+x^2}\,(3x^2+2x)\,dx$

(a) Let $u = x^3 + x^2$

(b) Then $du = (3x^2 + 2x)\,dx$

(c) Now substitute

$$
\begin{aligned}
\int \sqrt{x^3 + x^2} \cdot (3x^2 + 2x)\, dx &= \int \sqrt{u}\, du \\
&= \int u^{1/2}\, du \\
&= \frac{2}{3} u^{3/2} + C \\
&= \frac{2}{3}(x^3 + x^2)^{3/2} + C
\end{aligned}
$$

10. $\displaystyle\int_{-1}^{1} \frac{x+1}{(x^2+2x+2)^3}\, dx$

(a) Let $u = x^2 + 2x + 2$

(b) Then $du = (2x+2)\, dx \rightarrow du = 2(x+1)\, dx$ or $\frac{1}{2}\, du = (x+1)\, dx$

(c) If $x = -1$, then $u = (-1)^2 + 2(-1) + 2 = 1$

(d) If $x = 1$, then $u = (1)^2 + 2(1) + 2 = 5$

(e) Now substitute

$$
\begin{aligned}
\int_{-1}^{1} \frac{(x+1)}{(x^2+2x+2)^3}\, dx &= \int_{-1}^{1} \frac{1}{(x^2+2x+2)^3} \cdot (x+1)\, dx \\
&= \int_{1}^{5} \frac{1}{u^3}\, \frac{1}{2}\, du \\
&= \int_{1}^{5} \frac{1}{2} u^{-3}\, du \\
&= \frac{1}{2} \frac{u^{-2}}{-2} \Big|_{1}^{5} \\
&= -\frac{1}{4u^2} \Big|_{1}^{5} \\
&= \left[-\frac{1}{4(5)^2}\right] - \left[-\frac{1}{4(1)^2}\right] \\
&= -\frac{1}{100} + \frac{1}{4} \\
&= \frac{24}{100} \\
&= \frac{6}{25}
\end{aligned}
$$

11. $\int_0^\pi \cos(x)\sqrt{\sin(x)}\,dx$

(a) Let $u = \sin(x)$

(b) Then $du = \cos(x)\,dx$

(c) If $x = 0$, then $u = \sin(0) = 0$.

(d) If $x = \pi$, then $u = \sin(\pi) = 0$

(e) Now substitute

$$\begin{aligned}
\int_0^\pi \cos(x)\sqrt{\sin(x)}\,dx &= \int_0^\pi \sqrt{\sin(x)}\cdot\cos(x)\,dx \\
&= \int_0^0 \sqrt{u}\,du \\
&= \int_0^0 u^{1/2}\,du \\
&= \frac{2}{3}u^{3/2}\Big|_0^0 \\
&= \left[\frac{2}{3}(0)^{3/2}\right] - \left[\frac{2}{3}(0)^{3/2}\right] \\
&= 0
\end{aligned}$$

Note, $\int_a^a f(x)\,dx = 0$. So we didn't actually need to go through the last 5 lines.

12. $\int (x+1)\sin(x^2+2x+3)\,dx$

(a) Let $u = x^2 + 2x + 3$

(b) Then $du = (2x+2)\,dx \to du = 2(x+1)\,dx$ or we can write $\frac{1}{2}\,du = (x+1)\,dx$

(c) Now substitute

$$\begin{aligned}
\int (x+1)\sin(x^2+2x+3)\,dx &= \int \sin(x^2+2x+3)\cdot(x+1)\,dx \\
&= \int \sin(u)\cdot\frac{1}{2}\,du \\
&= \int \frac{1}{2}\sin(u)\,du \\
&= -\frac{1}{2}\cos(u) + C \\
&= -\frac{1}{2}\cos(x^2+2x+3) + C
\end{aligned}$$

13. $\int \left(1+\frac{1}{t}\right)^3 \frac{1}{t^2}\,dt$

(a) Let $u = 1 + \frac{1}{t}$

(b) Then $du = -\frac{1}{t^2}\,dt$ or $-du = \frac{1}{t^2}\,dt$

(c) Now substitute

$$\begin{aligned}\int \left(1+\frac{1}{t}\right)^3 \cdot \frac{1}{t^2}\,dt &= \int u^3\,(-du)\\ &= -\frac{1}{4}u^4 + C\\ &= -\frac{1}{4}\left(1+\frac{1}{t}\right)^4 + C\end{aligned}$$

14. $\int_{-1}^{1} x^2\sqrt{x^3+1}\,dx$

(a) Let $u = x^3 + 1$

(b) Then $du = 3x^2\,dx$ or $\frac{1}{3}\,du = x^2\,dx$

(c) If $x = -1$, then $u = (-1)^3 + 1 = 0$.

(d) If $x = 1$, then $u = (1)^3 + 1 = 2$

(e) Now substitute

$$\begin{aligned}\int_{-1}^{1} x^2\sqrt{x^3+1}\,dx &= \int_{-1}^{1} \sqrt{x^3+1}\cdot x^2\,dx\\ &= \int_0^2 \sqrt{u}\cdot\frac{1}{3}\,du\\ &= \int_0^2 \frac{1}{3}u^{1/2}\,du\\ &= \frac{1}{3}u^{3/2}\cdot\frac{2}{3}\Big|_0^2\\ &= \frac{2}{9}u^{3/2}\Big|_0^2\\ &= \left[\frac{2}{9}(2)^{3/2}\right] - \left[\frac{2}{9}(0)^{3/2}\right]\\ &= \frac{2}{9}\sqrt{8} = \frac{4\sqrt{2}}{9}\end{aligned}$$

15. $\int \frac{2}{\sqrt{3x-7}}\,dx$

(a) Let $u = 3x - 7$

(b) Then $du = 3\,dx$ or $\frac{1}{3}\,du = dx$

(c) Now substitute

$$\begin{aligned} \int \frac{2}{\sqrt{3x-7}}\,dx &= \int \frac{2}{\sqrt{u}} \cdot \frac{1}{3}\,du \\ &= \int \frac{2}{3} u^{-1/2}\,du \\ &= \frac{2}{3} u^{1/2} \cdot \frac{2}{1} + C \\ &= \frac{4}{3} u^{1/2} + C \\ &= \frac{4}{3}\sqrt{3x-7} + C \end{aligned}$$

16. $\int_1^4 \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}\,dx$

(a) Let $u = \sqrt{x} + 1$

(b) Then $du = \frac{1}{2\sqrt{x}}\,dx$ or $2\,du = \frac{1}{\sqrt{x}}\,dx$

(c) If $x = 1$, then $u = \sqrt{1} + 1 = 2$.

(d) If $x = 4$, then $u = \sqrt{4} + 1 = 3$.

(e) Now substitute

$$
\begin{aligned}
\int_1^4 \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}\,dx &= \int_1^4 \frac{1}{(\sqrt{x}+1)^2}\cdot\frac{1}{\sqrt{x}}\,dx \\
&= \int_2^3 \frac{1}{u^2}\cdot(2\,du) \\
&= \int_2^3 2u^{-2}\,du \\
&= 2\frac{u^{-1}}{-1}\bigg|_2^3 \\
&= -\frac{2}{u}\bigg|_2^3 \\
&= \left[-\frac{2}{3}\right]-\left[-\frac{2}{2}\right] \\
&= -\frac{2}{3}+1 \\
&= \frac{1}{3}
\end{aligned}
$$

17. $\int_0^1 \frac{x}{\sqrt{x+1}}\,dx$

(a) Let $u = x + 1$, then $x = u - 1$ (need this for later)

(b) Then $du = dx$

(c) If $x = 0$, then $u = 1$.

(d) If $x = 1$, then $u = 2$

(c) Now substitute

$$\begin{aligned}
\int_0^1 \frac{x}{\sqrt{x+1}}\,dx &= \int_1^2 \frac{u-1}{\sqrt{u}}\,du \\
&= \int_1^2 \frac{u}{u^{1/2}} - \frac{1}{u^{1/2}}\,du \\
&= \int_1^2 u^{1/2} - u^{-1/2}\,du \\
&= \frac{2}{3}u^{3/2} - 2u^{1/2}\Big|_1^2 \\
&= \left[\frac{2}{3}(2)^{3/2} - 2(2)^{1/2}\right] - \left[\frac{2}{3}(1)^{3/2} - 2(1)^{1/2}\right] \\
&= \frac{4\sqrt{2}}{3} - 2\sqrt{2} - \frac{2}{3} + 2 \\
&= -\frac{2\sqrt{2}}{3} + \frac{4}{3} \\
&= \frac{4-2\sqrt{2}}{3}
\end{aligned}$$

18. $\int x\sqrt{2x+1}\,dx$

(a) Let $u = 2x+1$. Then $x = \frac{1}{2}(u-1)$ (need this for later)

(b) Then $du = 2\,dx$ or $\frac{1}{2}\,du = dx$

(c) Now substitute

$$\begin{aligned}
\int x\sqrt{2x+1}\,dx &= \int \frac{1}{2}(u-1)\sqrt{u}\cdot\frac{1}{2}\,du \\
&= \int \frac{1}{4}(u-1)\cdot u^{1/2}\,du \\
&= \int \frac{1}{4}u^{3/2} - \frac{1}{4}u^{1/2}\,du \\
&= \frac{1}{4}u^{5/2}\cdot\frac{2}{5} - \frac{1}{4}u^{3/2}\cdot\frac{2}{3} + C \\
&= \frac{1}{10}u^{5/2} - \frac{1}{6}u^{3/2} + C \\
&= \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C
\end{aligned}$$

19. $\int \sqrt{x}\sqrt{x\sqrt{x}+1}\,dx$

You should rewrite the integral as

$$\int x^{1/2}\sqrt{x^{3/2}+1}\,dx$$

to help identify u.

(a) Let $u = x^{3/2}+1$

(b) Then $du = \frac{3}{2}x^{1/2}\,dx$ or $\frac{2}{3}\,du = x^{1/2}\,dx$

(c) Now substitute

$$\begin{aligned}
\int x^{1/2}\sqrt{x^{3/2}+1}\,dx &= \int \sqrt{x^{3/2}+1}\cdot x^{1/2}\,dx \\
&= \int \sqrt{u}\cdot\frac{2}{3}\,du \\
&= \int \frac{2}{3}u^{1/2}\,du \\
&= \frac{2}{3}u^{3/2}\cdot\frac{2}{3}+C \\
&= \frac{4}{9}u^{3/2}+C \\
&= \frac{4}{9}(x^{3/2}+1)^{3/2}+C
\end{aligned}$$

20. $\int x^3\sqrt{x^2+1}\,dx$

(a) Let $u = x^2-1$. Then $x^2 = u-1$ (need this for later)

(b) Then $du = 2x\,dx$ or $\frac{1}{2}\,du = x\,dx$

(c) Now substitute

$$\begin{aligned}
\int x^3\sqrt{x^2+1}\,dx &= \int x^2\sqrt{x^2+1}\cdot x\,dx \\
&= \int (u-1)\sqrt{u}\,\frac{1}{2}\,du \\
&= \int \frac{1}{2}u^{3/2} - \frac{1}{2}u^{1/2}\,du \\
&= \frac{1}{2}u^{5/2}\cdot\frac{2}{5} - \frac{1}{2}u^{3/2}\cdot\frac{2}{3} + C \\
&= \frac{1}{5}u^{5/2} - \frac{1}{3}u^{3/2} + C \\
&= \frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C
\end{aligned}$$

21. $\int (x^2+1)\sqrt{x-2}\,dx$

(a) Let $u = x-2$. Then $u+2 = x$ and $(u+2)^2 = x^2$

$$x^2 = u^2 + 4u + 4$$

(b) Then $du = dx$

(c) Now substitute

$$\begin{aligned}
\int (x^2+1)\sqrt{x-2}\,dx &= \int (u^2+4u+4+1)\sqrt{u}\,du \\
&= \int (u^2+4u+5)\sqrt{u}\,du \\
&= \int u^{5/2} + 4u^{3/2} + 5u^{1/2}\,du \\
&= \frac{2}{7}u^{7/2} + \frac{8}{5}u^{5/2} + \frac{10}{3}u^{3/2} + C \\
&= \frac{2}{7}(x-2)^{7/2} + \frac{8}{5}(x-2)^{5/2} + \frac{10}{3}(x-2)^{3/2} + C
\end{aligned}$$