South valley University

Faculty of science-Qena

Mathematics Department

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المحاضر: أ. د. جمال عبدالله أحمد حشودي

South Valley University

Faculty of Science

Mathematics Department

Mechanics of Rigid body: Lecture Notes

Prepared by

Department of Mathematics staff

Content

Chapter 1

Kinematics of Rigid Bodies

Deformable Body: Anybody that changes its shape and/or volume while being acted upon by any kind of external force.

Rigid body: A rigid body is a solid body in which deformation is zero or so small it can be neglected. The distance between any two given points on a rigid body remains constant in time regardless of external forces exerted on it

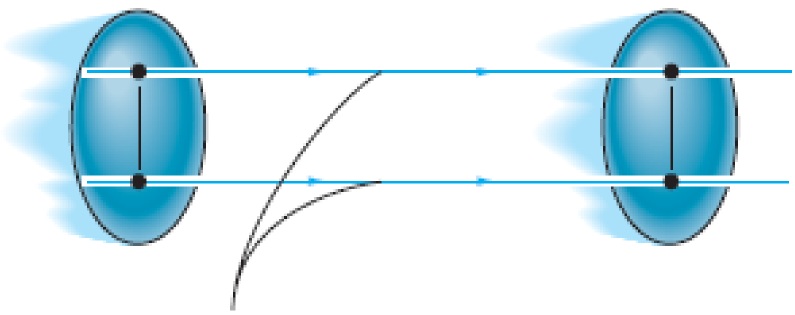
A deformable body is one that can distort. It would normally refer to a solid object so that as it deforms, it sort of deforms in a way that it could return to its starting shape if all the external forces were removed that caused it to deform.

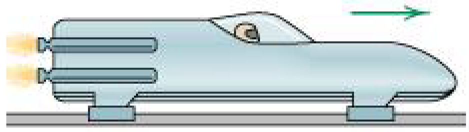
Types of Rigid Body Motion

Translation (Or Translation-al motion)

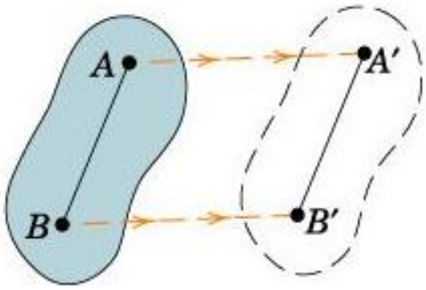
Translation. This type of motion occurs when a line in the body remains parallel to its original orientation throughout the motion.

Recti-linear translation: when the paths of motion for any two points on the body are parallel lines, the motion is called rectilinear translation



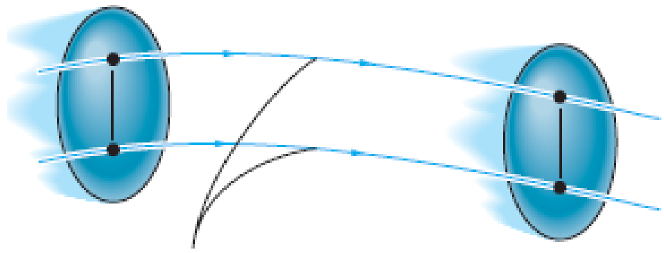


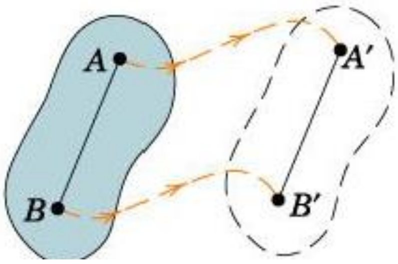
Rocket test sled

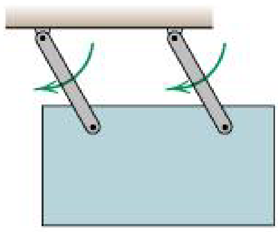


Curvi-linear Translation

If the paths of motion are along curved lines, the motion is called curvilinear translation



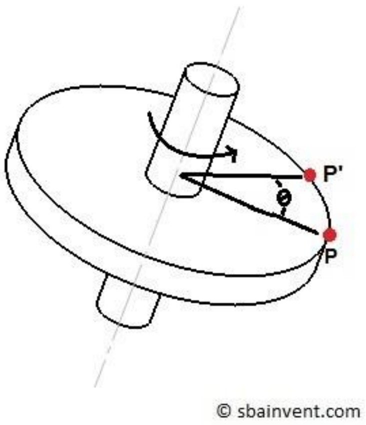


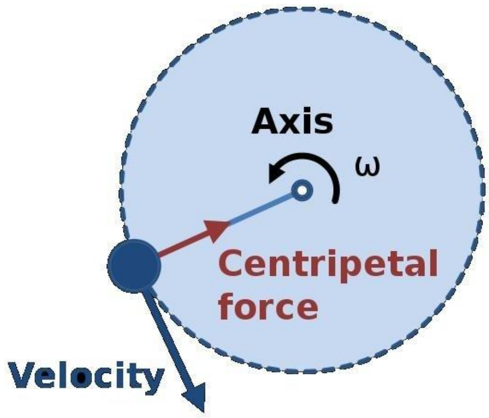


Parallel-link swinging plate

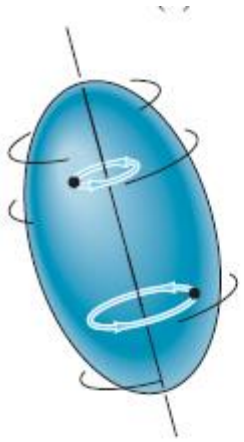
Rotation about a fixed axis

One straight line in the body is fixed. All other points in the body travel in circles around this line.



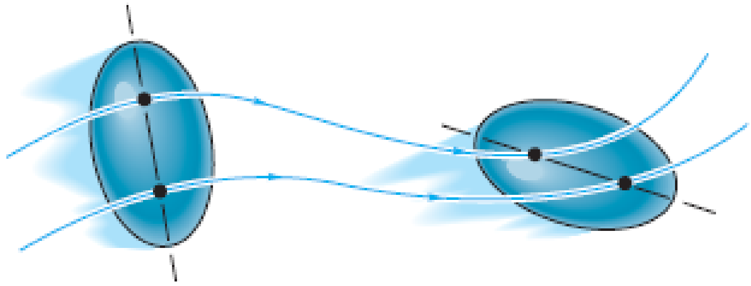


When a rigid body rotates about a fixed axis, all the particles of the body, except those which lies on the axis of rotation, move along circular paths except those which lies on the axis of rotation.



General plane motion

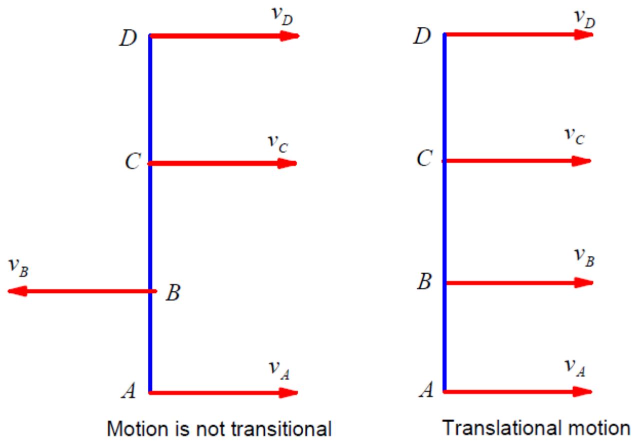
General plane motion. When a body is subjected to general plane motion, it undergoes a combination of translation and rotation, the translation occurs within a reference plane, and the rotation occurs about an axis perpendicular to the reference plane.



Velocity and Acceleration

In the Transitional motion , the velocity and acceleration of all points of the body at any moment are equal in magnitude and direction.

$$\vec{V}_A = \vec{V}_B = \vec{V}_C = \vec{V}_D = \vec{V}\,, \qquad \vec{f}_A = \vec{f}_B = \vec{f}_C = \vec{f}_D = \vec{f}$$



Rotational (Rotation) motion

Rotational motion is the motion of the body wrapping (Read: rapping) around its center

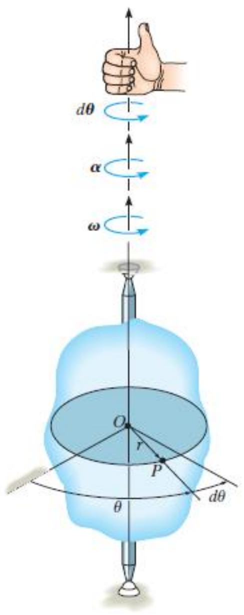


Fig. (a)

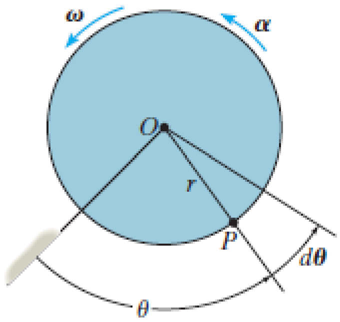


Fig. (b)

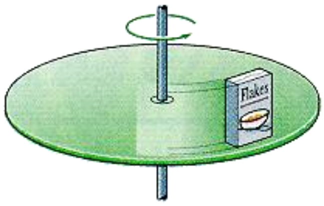


Fig. (c)

Note: One complete revolution is 360° = 2π radians.

Rotation about a Fixed Axis

When a body rotates about a fixed axis, any point P located in the body travels along a circular path. To study this motion it is first necessary to discuss the angular motion of the body about the axis.

Angular Motion. Since a point is without dimension, it cannot have angular motion. Only lines or bodies undergo angular motion. For example, consider the body shown in Figure and the angular motion of a radial line r located within the shaded plane.

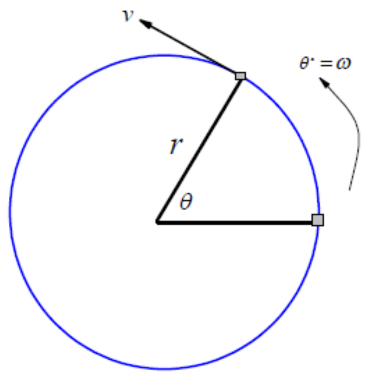
Angular Position. At the instant shown, the angular position of r is defined by the angle u, measured from a fixed reference line to r.

Angular Displacement. The change in the angular position, which can be measured as a differential dU, is called the angular displacement. This vector has a magnitude of dU, measured in degrees, radians, or revolutions, where 1 rev = 2p rad. Since motion is about a fixed axis, the direction of dU is always along this axis. Specifically, the direction is determined by the right-hand rule; that is, the fingers of the right hand are curled with the sense of rotation, so that in this case the thumb, or dU, points upward, Fig. 16–4a. In two dimensions, as shown by the top view of the shaded plane, Fig. 16–4b, both u and du are counterclockwise, and so the thumb points outward from the page.

Angular Velocity. The time rate of change in the angular position is called the *angular velocity* V (omega). Since *d*U occurs during an,

Remember that

When a body moves in a circular path, we can write both the velocity and acceleration in the form



$\vec{v}=\vec{\dot{r}}\ (\dot{r}\ ,\ r\dot{\theta}),\quad \vec{f}\ =\ (\ddot{r}-r\dot{\theta}^2,\ r\ddot{\theta}+2\dot{r}\dot{\theta})$

If $r=$ constant , then $r^{\bullet}=r^{\bullet\bullet}=0$. So, the velocity and acceleration becomes

$\vec{v}=\vec{\dot{r}}=(\ 0\ ,\ r\dot{\theta}),\quad \vec{f}\ =\ (0-r\dot{\theta}^2,\ r\ddot{\theta}+0)$, or $\vec{v}=\vec{\dot{r}}=(\ 0\ ,\ r\dot{\theta}),\quad \vec{f}\ =\ (-r\dot{\theta}^2,\ r\ddot{\theta}\ )$

The angular velocity in rotational (rotation) motion

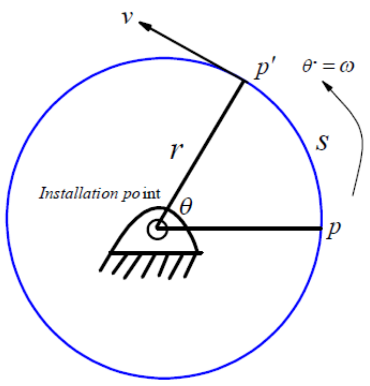
Angular velocity: The time rate of change in the angular position is called the angular velocity V (omega). Since dU occurs during an instant of time dt, then,

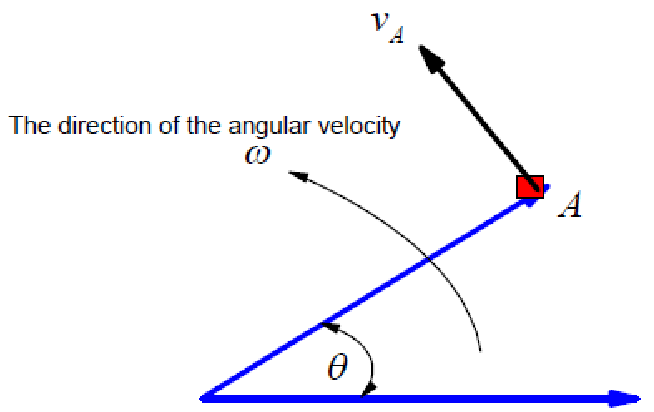
From the Figure $s=r\theta$, where θ is angler position and the angler velocity is $(\theta^{\bullet}=\omega)$.

It is clear that $\dfrac{d\theta}{dt}=\theta^{\bullet}=\omega$.

The relation between the angular velocity and transitional velocity is given from

$$\frac{ds}{dt}=s^{\bullet}=v=\frac{d(r\theta)}{dt}=r\frac{d\theta}{dt}=r\theta^{\bullet}=r\omega$$

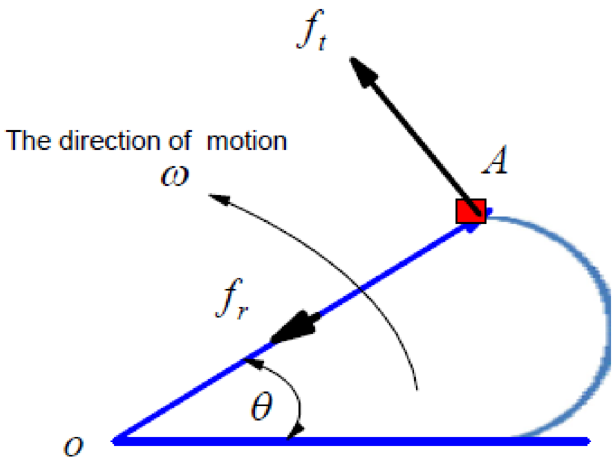




- The direction of translational velocity is in the same direction angular velocity

- The direction of the vector tells you the axis of the rotation, as well as whether the rotation is clockwise or counterclockwise.

- The relation between the angular acceleration and transitional acceleration is given from the transitional acceleration has two components, the first in Tangential directionand (f_t) the other in the normal direction (f_r)

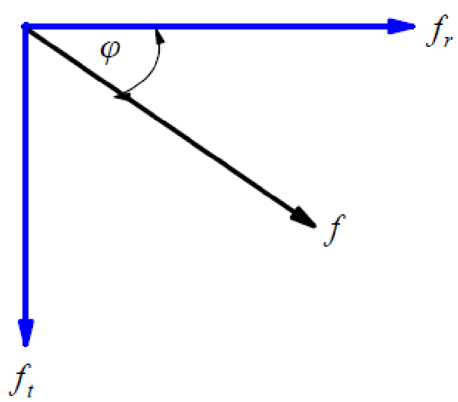


The components of the acceleration are given as

$f_t = r\dot{\omega}$, $f_r = r\omega^2$ Or $f_n = r\omega^2$

The Resultant of acceleration is given by $f = \sqrt{f_t^2 + f_r^2}$

While the direction is given by $\tan\varphi = \dfrac{f_t}{f_r}$



Special case of rotational motion

(i) In the case of constant angular velocity(i. e. the angular velocity does not $(\omega = \text{Constant})$ change with time $\frac{d\omega}{dt} = 0$) (ii) The pure rotational motion In the case of the angular acceleration is constant (i. e. $\omega^{\bullet} = \text{Constant}$). Then $\frac{d\omega}{dt} = C$ and $\omega = C\,t$

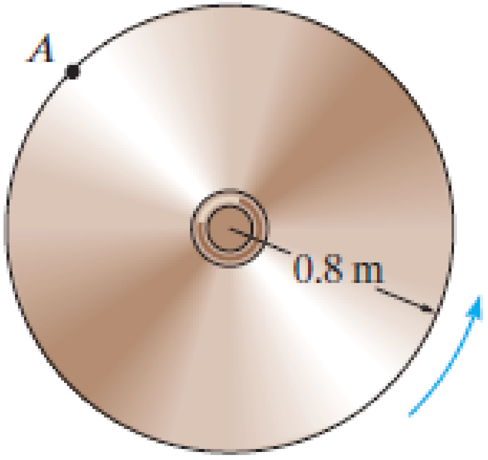
The relationship between the laws of motion in the case of linear motion with constant linear acceleration and rotational motion with constant angular acceleration

$\upsilon = \upsilon_o + a_c\, t,$	$\omega = \omega_o + \omega^{\bullet}\, t,$
$x = \upsilon_o\, t + \frac{1}{2} a_c\, t^2,$	$\theta = \omega_o\, t + \frac{1}{2}\omega^{\bullet} t^2,$
$\upsilon^2 = \upsilon_o^2 + 2\, a_c\, x,$	$\omega^2 = \omega_o^2 + 2\,\omega^{\bullet}\theta$

General Plane Motion (Translation + Rotation)

If a rigid body moves with both translational and rotational motion, it is said to be in general plane motion.

Example 1: The angular velocity of the disk is defined by $\omega = \left(5t^2 + 2\right) rad/\sec$ where t is in seconds. Determine the magnitudes of the velocity and acceleration of point A on the disk when $t = 0.5 \sec$?



Solution

$\omega = \left(5t^2 + 2\right) rad/\sec$

$\omega^{\bullet} = \frac{d\omega}{dt} = \left(10t\right) rad/\sec^2$

At $t = 0.5$ sec $\omega = \left(5(0.5)^2 + 2\right) = 3.25\ rad/\sec$, $\omega^{\bullet} = \left(10(0.5)\right) = 5\, rad/\sec^2$

But $v_A = (\omega)_{disk}(r)_{disk}$

Then $v_A = (3.25)(0.8) = 2.6\, m/\sec$,

$f_r = \omega^2 r, \quad f_t = \omega^{\bullet} r$

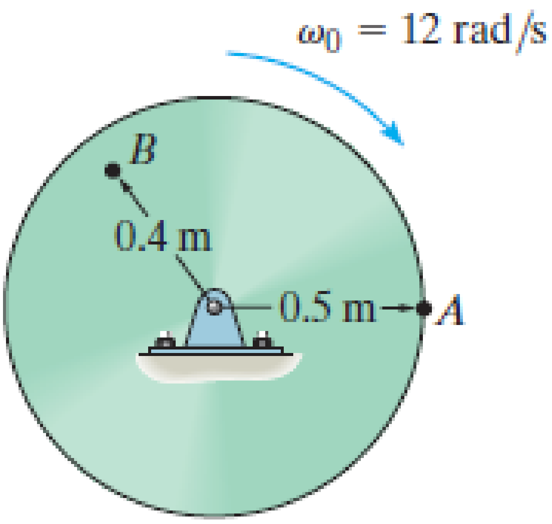
$f_{A_r} = (\omega^2)_{disk}(r)_{disk} = (3.25)^2(0.8m)) = (10.5625)(0.8) = 8.45\, m/\sec^2$

$f_{A_t} = (\omega^{\bullet})_{disk}(r\ )_{disk} = (5)(0.8m) = 4cm/\sec^2$ But $f_A = \sqrt{f_{A_t}^{\ 2} + f_{A_r}^{\ 2}}$

$f_A = \sqrt{(8.45)^2 + (4)^2} = \sqrt{71.4025 + 16} = \sqrt{87.4025} = 9.349\ m/\sec^2 = 9.35\ m/\sec^2$

$\tan\phi = \frac{f_t}{f_r} \rightarrow \tan\phi = \frac{f_{A_t}}{f_{A_r}} = \frac{4}{8.45} = 0.47337 \rightarrow \phi = \tan^{-1}(0.47337) \rightarrow \phi = 25^{O}.33'$

Example 2: The angular acceleration of the disk is defined by $\omega^{\bullet} = \left(3t^2 + 12\right) rad/\sec^2$ where t is in seconds. If the disk is originally rotating at $\omega_0 = 12\ rad/\sec$. Determine the magnitude of the velocity and two components of acceleration of point A and B on the disk when $t = 2\sec$.



Solution

The angular acceleration of the disk is given by $\omega^{\bullet} = \left(3t^2 + 12\right) rad/\sec^2$.

While the angular velocity is given by $\omega = \int \omega^{\bullet}\, dt = \int \left(3t^2 + 12\right) dt$

$$\omega = \left(\frac{3}{3}t^3 + 12t\right) + c_1$$

At the start rotating point (i. e. $t = 0$) $\omega_0 = 12\ rad/\sec$. So $c_1 = 12$. Then

The angular velocity is given as $\omega = \left\{t^3 + 12t + 12\right\} rad/\sec$ At

$t = 2\sec$, we have

$$\omega = \left\{(2)^3 + 12(2) + 12\right\} = 44 rad/\sec, \quad \omega^{\bullet} = \left(3(2)^2 + 12\right) = 24 rad/\sec^2$$

From the two relation $f_r = \omega^2 r, \quad f_t = \omega^{\bullet} r$,

<u>At the point A we have,</u> $(r)_{disk} = 0.5m$

$$f_{A_r} = (\omega^2)_{disk}(r)_{disk} = (44)^2(0.5m)) = (1936)(0.5) = 968\ m/\sec^2$$

$$f_{A_t} = (\omega^{\bullet})_{disk}(r\ )_{disk} = (24)(0.5m) = 12\ \ m/\sec^2$$

At the point A magnitude of the acceleration is given by $f_A = \sqrt{f_{A_t}^2 + f_{A_r}^2}$

$$f_A = \sqrt{(12)^2 + (968)^2} = \sqrt{144 + 937024} = \sqrt{937168} = 968.07\ m/\sec^2 = 968\ m/\sec^2$$

The direction of acceleration is given by $\tan\varphi_A = \dfrac{f_t}{f_r}$

$$\tan\varphi_A == \frac{f_{A_t}}{f_{A_r}} = \frac{12}{968} = 0.1239 \ \rightarrow\ \varphi_A = \tan^{-1}(0.1239) \ \rightarrow\ \varphi_A = 0.71024'$$

At the point B we have , $(r)_{disk} = 0.4m$

$$f_{A_r} = (\omega^2)_{disk}(r)_{disk} = (44)^2(0.4m)) = (1936)(0.4) = 774.4\,m/\sec^2$$

$$f_{A_t} = (\omega^{\bullet})_{disk}(r\ )_{disk} = (24)(0.4m) = 9.6\,m/\sec^2$$

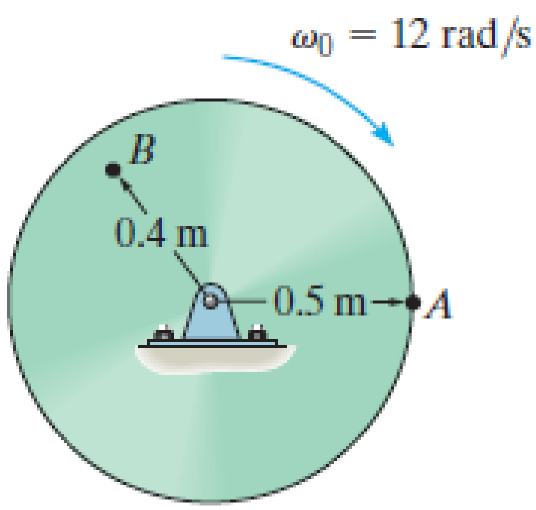
At the point B magnitude of the acceleration is given by $f_B = \sqrt{f_{B_t}^2 + f_{B_r}^2}$

$$f_B = \sqrt{(9.6)^2 + (774.4)^2} = \sqrt{92.16 + 599695} = \sqrt{599787} = 774.45\,m/\sec^2$$

The direction of the acceleration is given by $\tan\varphi_B = \dfrac{f_{B_t}}{f_{B_r}}$

$$\tan\varphi_B = \frac{f_{B_t}}{f_{B_r}} = \frac{9.6}{774.4} = 0.1239 \rightarrow \varphi_B = \tan^{-1}(0.1239) \rightarrow \varphi_B = 0.71024'$$

Example 3: The disk is originally rotating at $\omega_0 = 12\ rad/\sec$. If it is subjected to a constant angular acceleration of $\omega^{\bullet} = 20\ rad/\sec^2$. Determine the magnitudes of the velocity and the two components of acceleration of point A at the instant $t = 2\sec$?



Solution

Where the disk is subjected to a constant angular acceleration of $\omega^{\bullet} = 20\ rad/\sec^2$.

Then $\omega = \omega_o + \omega^{\bullet} t$, $\theta = \omega_o t + \frac{1}{2}\omega^{\bullet} t^2$, $\omega^2 = \omega_o^2 + 2\omega^{\bullet}\theta$

Where $\omega_0 = 12\ rad/\sec$, and after $t = 2\sec$ and form

Angular Motion: The angular velocity of the disk can be determined using from

$\omega = \omega_o + \omega^{\bullet} t$, we have $\omega = (12) + (20)(2) \rightarrow \omega = 52\ rad/\sec$

Motion of Point A. The magnitude of the velocity is given by

$$v_A = (\omega)_{disk}(r)_{disk} \rightarrow v_A = (52)_{disk}(0.5)_{disk} \rightarrow v_A = 26\ m/\sec$$

The tangential and normal component of acceleration are

$$f_{A_r} = (\omega^2)_{disk}(r)_{disk} = (52)^2(0.5m)) = (1936)(0.4) = 1352\ m/\sec^2$$

$$f_{A_t} = (\omega^{\bullet})_{disk}(r\ )_{disk} = (20)(0.5m) = 10\ m/\sec^2$$

At the point A magnitude of the acceleration is given by $f_A = \sqrt{f_{A_t}^2 + f_{A_r}^2}$

$$f_A = \sqrt{(10)^2 + (1352)^2} = 1352.04\ m/\sec^2$$

The direction of the acceleration is given by $\tan\phi_A = \frac{f_t}{f_r}$

$$\tan\phi_A = \frac{f_{A_t}}{f_{A_r}} = \frac{10}{1352.04} \rightarrow \phi_A = \tan^{-1}(0.00739) \rightarrow \phi_A = 0.423778'$$, then from

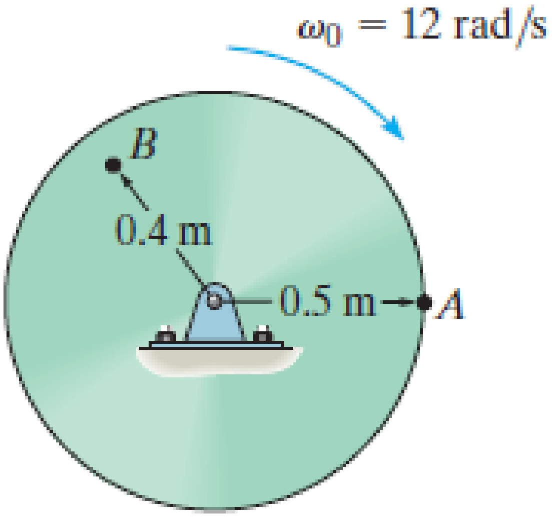
Eq. $\omega^2 = \omega_o^2 + 2\omega^{\bullet}\theta$, we have

$$(52)^2 = (12)^2 + 2(20)\theta \rightarrow \theta = \frac{2704-144}{40} = \frac{2560}{40} = \frac{256}{4} \rightarrow \theta = 64$$

The disk makes angle distance is given by ($\theta = 64$ rad)

The disk rotates laps $N = \frac{\theta}{2\pi} = \frac{64}{2\pi} = \frac{32}{\pi} \rightarrow \theta = 10.2\ rev$ (reflection-reversal)

Example 4: The disk is originally rotating at $\omega_0 = 12\ rad/\sec$. If it is subjected to a constant angular acceleration of $\omega^{\bullet} = 20\ rad/\sec^2$. Determine the magnitudes of the velocity and the two components of acceleration of point B when the disk undergoes 2 revolutions?



Solution

Where the disk is subjected to a constant angular acceleration of $\omega^{\bullet} = 20\ rad/\sec^2$ .Then

$$\omega = \omega_o + \omega^{\bullet}\, t, \qquad \theta = \omega_o\, t + \frac{1}{2}\omega^{\bullet} t^2, \qquad \omega^2 = \omega_o^2 + 2\,\omega^{\bullet}\theta$$

Where $\omega_0 = 12\ rad/\sec$.

When the disk undergoes 2 revolutions. Then $N = \dfrac{\theta}{2\pi} = 2 \rightarrow \theta = 4\pi\ \ rev$

Angular Motion: The angular velocity of the disk can be determined using from

$\omega^2 = \omega_o^2 + 2\,\omega^{\bullet}\theta$, we have $\omega^2 = (12)^2 + 2\,(20)\,(4\pi) = 144 + 160\pi = 646.6548$,

$\omega = 25.43\ rad/\sec$

Motion of Point B. The magnitude of the velocity is given by

$$v_A = (\omega)_{disk}(r)_B \rightarrow v_A = (25.43)_{disk}(0.4)_B \rightarrow v_A = 10.1717\ m/\sec \rightarrow v_A = 10.2\,m/\sec$$

The tangential and normal component of acceleration are

$$f_{B_r} = (\omega^2)_{disk} (r)_B = (25.43)^2 (0.4m)) = 258.674 m/\sec^2$$

$$f_{B_t} = (\omega^{\bullet})_{disk} (r\ )_B = (20)(0.4m) = 8\ m/\sec^2$$

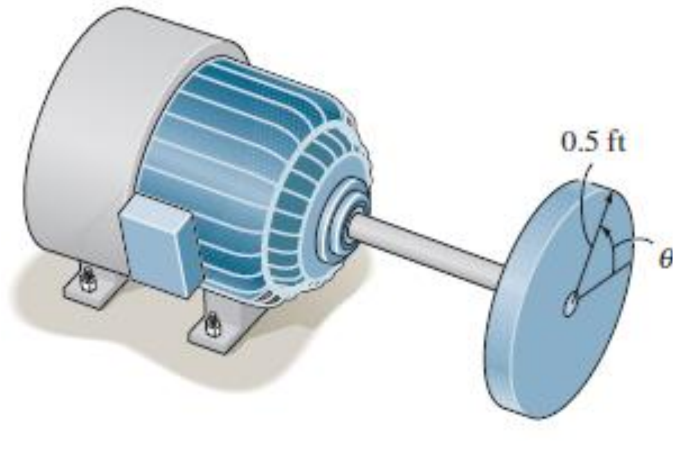
At the point A magnitude of the acceleration is given by $f_B = \sqrt{f_{B_t}^2 + f_{B_r}^2}$

$$f_B = \sqrt{(8)^2 + (\ 258.674)^2} = 258.798 m/\sec^2$$

The direction of the acceleration is given by $\tan\varphi_B = \dfrac{f_t}{f_r}$

$$\tan\varphi_B == \frac{f_{B_t}}{f_{B_r}} = \frac{8}{258.674} \rightarrow \varphi_B = \tan^{-1}(0.03092) \rightarrow \varphi_B = 1^O.4771\,4'$$

Example 5: The disk is driven by a motor such that the angular position of the disk is defined by $\theta = \left(20t + 4t^2\right) rad$ where t is in seconds. Determine the number of revolutions, the angular velocity, and angular acceleration of the disk when $t = 90\,s$?



Solution

At $t = 90\,\text{sec}$, we find that $\theta(t=90) = \left(20(90) + 4(90)^2\right) rad = \left(1800 + 4(8100)\right) rad$

$\theta(t=90) = (1800 + 32400)\, rad = 34200\, rad$

$\theta = 34200\, rad \dfrac{(1)\, rev}{2\pi\ rad} = \dfrac{34200}{2\pi} rev \quad \rightarrow \quad \theta = 5443 \quad rev$

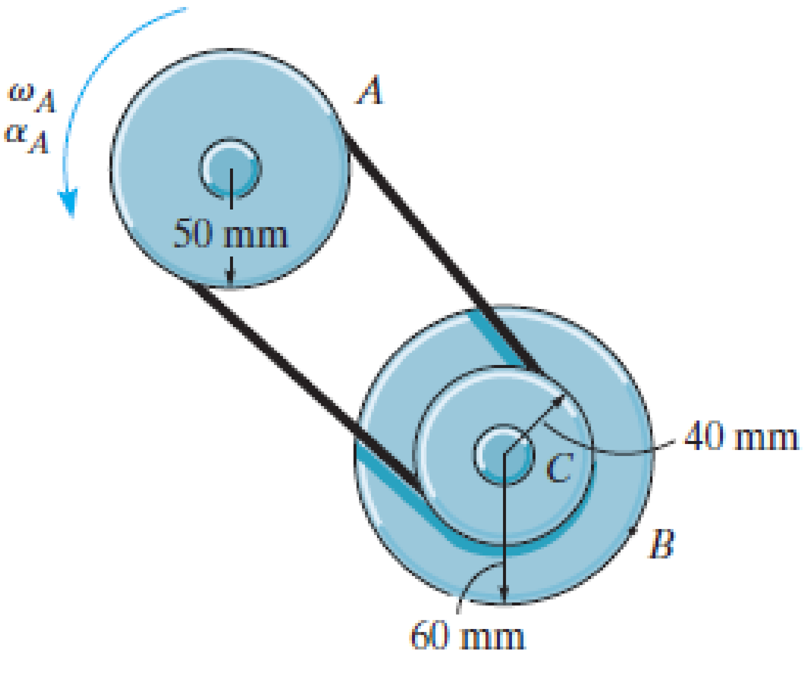
Angular Velocity: Applying Eq. $\omega = \dfrac{d\theta}{dt}$, we have

$\omega = \dfrac{d}{dt}\left(20t + 4t^2\right) = 20 + 8t$ and at $t = 90\,\text{sec}$, we have $\omega = 20 + 8(90) = 740\, rad/\sec$

Angular Acceleration: Applying Eq. A $\dot{\omega} = 8$, we have $\dot{\omega} = \dfrac{d\omega}{dt}$

$\dot{\omega} = 8\, rad/\sec^2 \qquad t = 90\,\text{sec}$

Example 6: At the instant $\omega_A = 5\ rad/\sec$ (it means initial the angular velocity), pulley A is given an angular acceleration $\dot{\omega}_A = 6\, rad/\sec^2$. Determine the magnitude of acceleration of point B on pulley C when A rotates 2 revolutions. Pulley C has an inner hub which is fixed to its outer one and turns with it?



Solution

Given $\left(\omega_A\right)_0 = 5\ rad/\sec$, $\dot{\omega}_A = 6\,rad/\sec^2$,

Where the angular acceleration of pulley A is constant. So we have

$$\omega = \omega_o + \dot{\omega}\, t, \qquad \theta = \omega_o\, t + \frac{1}{2}\dot{\omega} t^2, \qquad \omega^2 = \omega_o^2 + 2\,\dot{\omega}\theta$$

When the pulley A rotates 2 revolutions. Then $N_A = \dfrac{\theta_A}{2\pi} = 2 \rightarrow \theta_A = 4\pi\ \ rev$

Angular Motion: The angular velocity of the pulley A can be determined from

$\omega^2 = \omega_o^2 + 2\,\dot{\omega}\theta$, we have $\omega^2 = (5)^2 + 2\,(6)(4\pi) = 25 + 48\pi = 175.79644, \quad \omega = 13.2588\, rad/\sec$

Since pulleys A and C are connected by a non-slip belt. So, at any point on the pulleys A and C.

$v_A = v_C, \qquad f_{A_t} = f_{C_t}$. Then

$$v_A = v_C \rightarrow \omega_A r_A = \omega_C r_C \rightarrow (13.2588)\,(50) = \omega_C (40) \rightarrow \omega_C = 16.57\ rad/\sec$$

Also $f_{A_t} = f_{C_t} \rightarrow r_A \dot{\omega}_A = r_C \dot{\omega}_C \rightarrow (50)\,(6) = (40)\dot{\omega}_C \rightarrow \dot{\omega}_C = 7.5\,rad/\sec^2$

Motion of Point B. The tangential and normal component of acceleration of point B can be determined from,

$$f_{B_r} = (\omega^2)_C\,(r)_B = (16.57)^2\,(0.6\,m) = 164.739\,m/\sec^2$$

$$f_{B_t} = (\dot{\omega})_C\,(r)_B = (6)\,(0.6\,m) = 3.6\ m/\sec^2$$

$$f_B = \sqrt{(3.6)^2 + (164.77)^2} = 164.77\,m/\sec^2$$

Chapter 2

I. Mass Moment of Inertia

Definition of the Rigid body

In physics, a rigid body is a solid body in which deformation is zero or so small it can be neglected. The distance between any two given points on a rigid body remains constant in time regardless of external forces exerted on it. A rigid body is usually considered as a continuous distribution of mass.

Definition of moment of inertia

Physical; A measure of the resistance of a body to angular acceleration about a given axis

For an object rotating about an axis, the resistance of a body to accelerate is called inertia of mass

Mathematic; The Moment of Inertia is equal to the sum of the products of each element of mass in the body and the square of the element's distance from the axis.

It is defined as the sum of second moment of area of individual section about an axis

(1) The basic shapes

(2) Systems of particles

(3) Composite bodies (shapes)

(4) Uninform shapes

The Moment of Inertia of mass (Second moment of mass)

The mass moment of inertia about a fixed axis is the property of a body that measures the body's resilience to rotational acceleration. The greater its value, the greater the moment required to provide a given acceleration about a fixed pivot. The moment of inertia must be specified with respect to a chosen axis of rotation.

(1)- For a single mass, the moment of inertia can is expressed as

For the element dm that is located a distance a from the L-axis, the Moment of inertia referenced to L-axis is given as

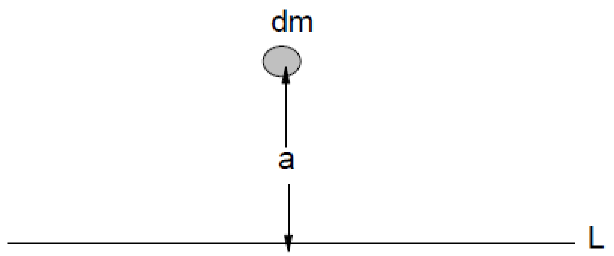


Fig. 1

$$I_{LL} = dm\, a^2$$

(2)- If a system consists of $n-$bodies, then the moment of inertia can be given as

For the $n-$elements, they have the mass $dm_1,\ dm_2,\ dm_3, \ldots\ldots, dm_n$ that is located a distance a from the L-axis, the moment of inertia referenced to L-axis is given as

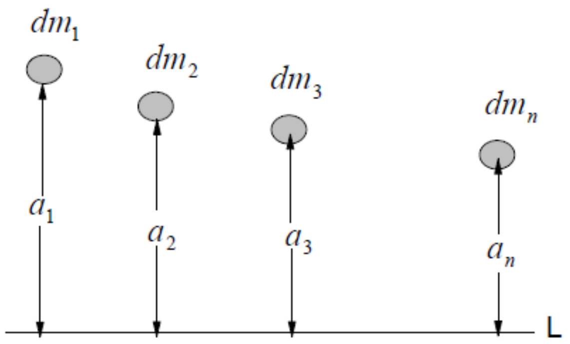


Fig. 2

$$I_{LL} = dm_1 a_1^2 + dm_2 a_2^2 + dm_3 a_3^2 + \dots\dots\dots + dm_n a_n^2 = \sum_{i=1}^{n} dm_i a_i^2$$

(3)- The Moment of Inertia in the plane

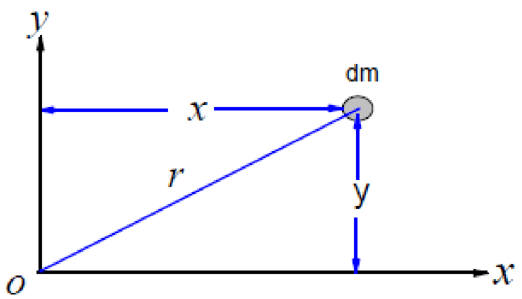


Fig. 3

Referenced to x-axis is given by $I_{xx} = dm\, y^2$,

Referenced to y-axis is given by $I_{yy} = dm x^2$,

Referenced to the original point (O) is given by

$$I_O = dmr^2 = m(x^2 + y^2) = I_{xx} + I_{yy}$$

I_O is called Polar moment inertial

(4)- The Moment of Inertia in the plane for number of elements

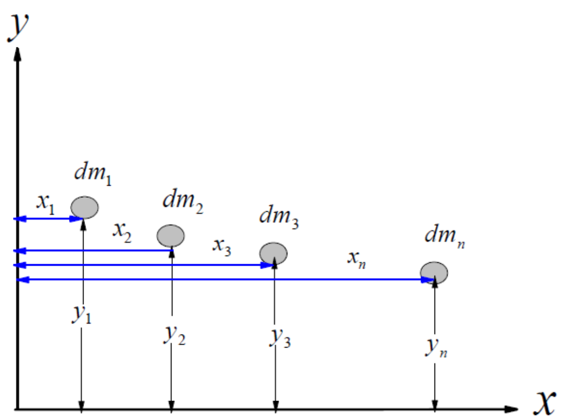


Fig. 4

Referenced to x-axis is given by $I_{xx} = \sum_{i=1}^{n} dm_i y_i^2$

Referenced to y-axis is given by $I_{yy} = \sum_{i=1}^{n} dm_i \, x_i^2$

(4)- The Moment of Inertia in space

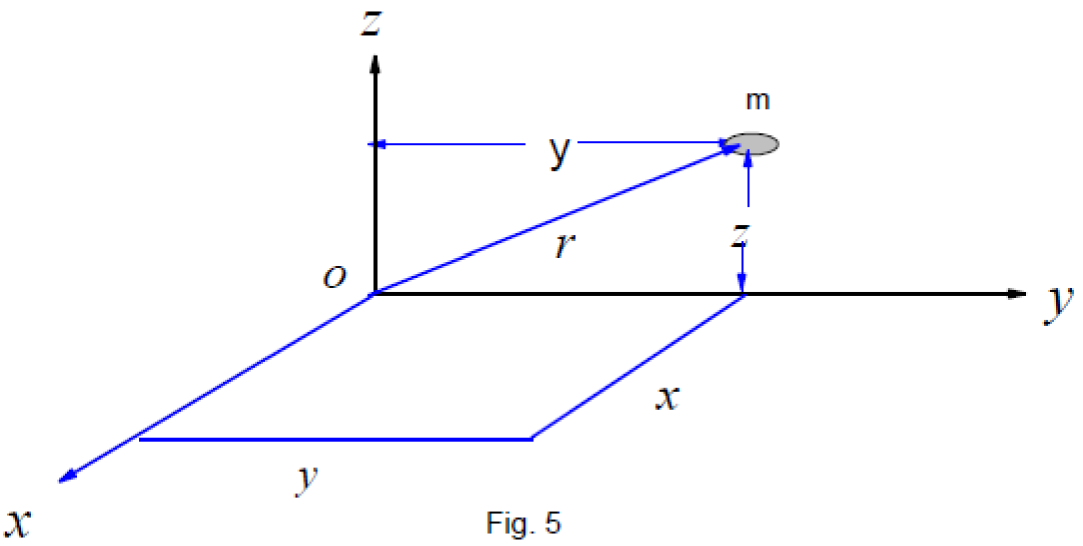


Fig. 5

Referenced to the original point (O) is given by

$$I_O = mr^2 = m(x^2 + y^2 + z^2) \quad (1)$$

Referenced to x-axis is given by $I_{xx} = m(y^2 + z^2)$,

Referenced to y-axis is given by $I_{yy} = m(x^2 + z^2)$,

Referenced to z-axis is given by $I_x = m(x^2 + y^2)$,

Referenced to the plane $-x = 0$ is given by $I_{xx} = m(y^2 + z^2)$,

Referenced to the plane $I_y = m(x^2 + z^2)$, is given by $-y = 0$

Referenced to the plane $z = 0$ is given by $I_z = m(x^2 + y^2)$,

From previous relation, we have

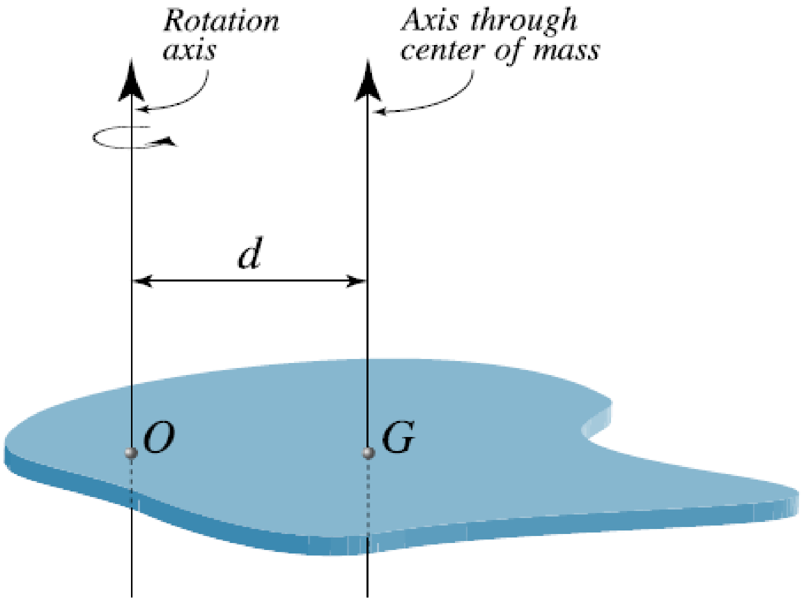
$$I_o = mr^2 = m(x^2 + y^2 + z^2) = I_{xoy} + I_{xoz} + I_{yoz}$$

$2I_o = I_{xx} + I_{yy} + I_{zz}$ or $I_o = mr^2 = m(x^2 + y^2 + z^2) = \frac{1}{2}(I_{xx} + I_{yy} + I_{zz})$

$$I_{xx} = m(y^2 + z^2) = I_{xoy} + I_{xoz}$$
$$I_{yy} = m(x^2 + z^2) = I_{xoy} + I_{yoz}$$
$$I_{zz} = m(x^2 + y^2) = I_{xoz} + I_{yoz}$$

Parallel axis theorem

Parallel axis theorem is applicable to bodies of any shape. The theorem of parallel axis states that the moment of inertia of a body about an axis parallel to an axis passing through the centre of mass is equal to the sum of the moment of inertia of body about an axis passing through centre of mass and product of mass and square of the distance between the two axes. The parallel axis theorem is much easier to understand in equation form than in words. Here it is:



In physics, the parallel axis theorem can be used to determine the moment of inertia of a rigid object about any axis, given the moment of inertia of the object about the parallel axis through the object's center of mass and the perpendicular distance between the axes.
We consider an element (m) and its center is (x_{cm}, y_{cm}) (see below Figure)

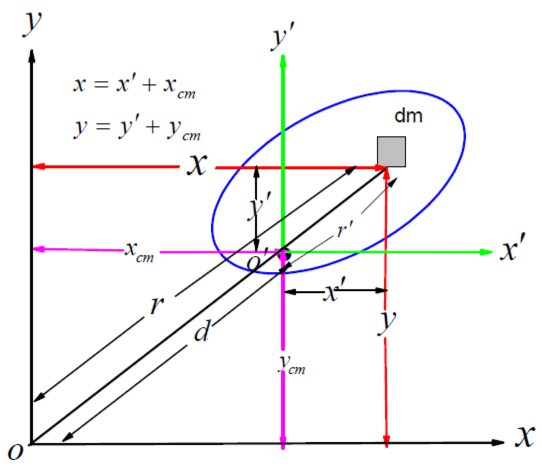


Fig. 7

$dI_{xx} = dm\, y^2$, the moment of inertial with respect to $x-$ axis

$dI_{yy} = dm\, x^2$, the moment of inertial with respect to $y-$ axis

$dI_O = dm\, r^2 = I_{xx} + I_{yy} = dm(x^2 + y^2)$, the moment of inertial with respect to the point(o)

$$I_O = \int r^2 dm = \int (x^2 + y^2) dm \qquad (1)$$

$$I_{cm} = \int r'^2 dm = \int (x'^2 + y'^2) dm \qquad (2)$$

$x = x' + x_{cm}, \quad y = y' + y_{cm}$

$$I_O = \int r^2 dm = \int \left\{ \left(x' + x_{cm} \right)^2 + \left(y' + y_{cm} \right)^2 \right\} dm$$

$$= \int \left\{ x'^2 + x_{cm}^2 + 2x' x_{cm} + y'^2 + y_{cm}^2 + 2y' y_{cm} \right\} dm$$

$$I_O = \underbrace{\int \left(x'^2 + y'^2 \right) dm}_{I_{cm}} + \int \underbrace{\left(x_{cm}^2 + y_{cm}^2 \right)}_{=d^2} dm + 2x_{cm} \int x' dm + 2y_{cm} \int y' dm$$

$$I_O = I_{cm} + \int d^2\, dm + 2x_{cm} \int x' dm + 2y_{cm} \int y' dm$$

$$I_O = I_{cm} + d^2 \int dm + 2x_{cm} \int x' dm + 2y_{cm} \int y' dm$$

$$I_O = I_{cm} + d^2 m + 2x_{cm} \int x' dm + 2y_{cm} \int y' dm \qquad (3)$$

$$\bar{x} = \frac{\int x' dm}{\int dm} \to \int x' dm = \bar{x} \int dm, \quad \bar{y} = \frac{\int y' dm}{\int dm} \to \int y' dm = \bar{y} \int dm \qquad (4)$$

$$I_O = I_{cm} + d^2 m + 2x_{cm} \left\{ \bar{x} \int dm \right\} + 2y_{cm} \left\{ \bar{y} \int dm \right\}$$

$$I_O = I_{cm} + d^2 m + 2x_{cm}\, \bar{x}\, m + 2y_{cm}\, \bar{y}\, m \qquad (5)$$

(6) $$I_O = I_{cm} + m\,d^2$$

Question: Let I_A and I_B be moments of inertia of a body about two axes A and B respectively. The axis A passes through the centre of mass of the body but B does not, So.

(A) $I_A < I_B$ (B) $I_A > I_B$ (C) If the axes are parallel $I_A < I_B$

(D) If the axes are parallel $I_A > I_B$ (E) If the axes are not parallel $I_A > I_B$

The moment of inertia is always less for an axis passing through the center of mass than any other parallel axis. We cannot say anything of the moment of inertia about a non parallel axis. Thus C is correct.

Perpendicular Axis Theorem

This theorem is applicable only to the planar bodies. Bodies which are flat with very less or negligible thickness. This theorem states that the moment of inertia of a planar body about an axis perpendicular to its plane is equal to the sum of its moments of inertia about two perpendicular axes concurrent with the perpendicular axis and lying in the plane of the body.

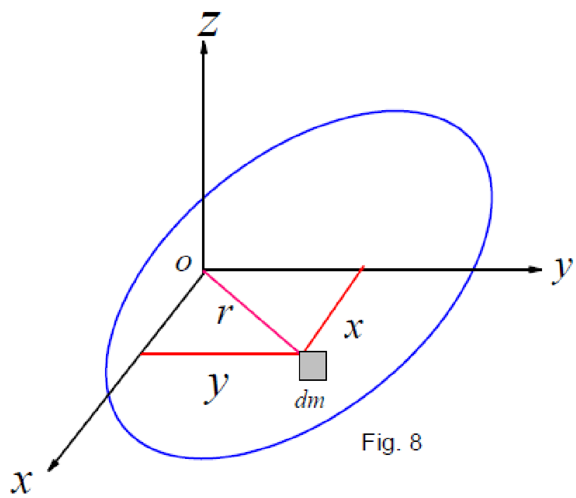


Fig. 8

$dI_{xx} = dm\,y^2$, the moment of inertial with respect to $x-$ axis

$dI_{yy} = dm\,x^2$, the moment of inertial with respect to $y-$ axis

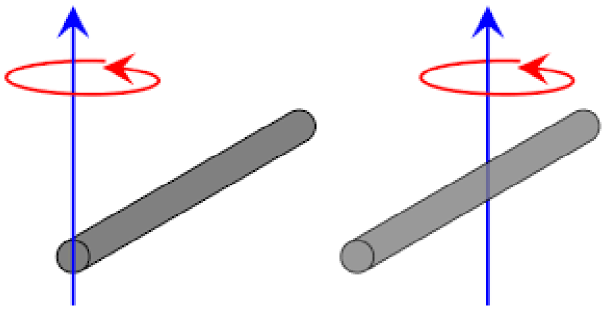
$dI_O = dm\,r^2 = I_{xx} + I_{yy} = dm(x^2 + y^2)$,, the moment of inertial with respect to the point (o)

$$I_O = \int (x^2 + y^2)\,dm = \int r^2 dm = r^2 \int dm = r^2 m \qquad (1)$$

$$I_{zz} = I_{xx} + I_{yy} \qquad (2)$$

Example:1 Find the Mass moment of inertia of a thin uniform rod?

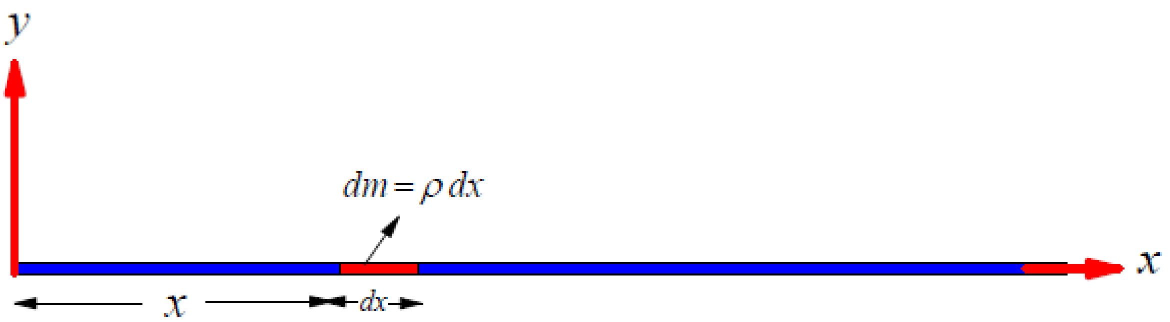
Solution

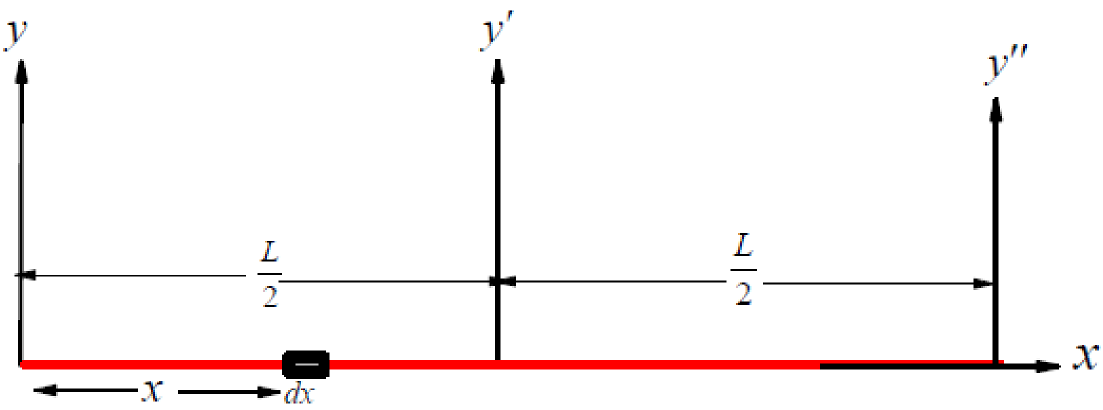


We consider L be the length of the Rod, M be the mass of the Rod and is the density ρ.

We divide the Rod into many small elements. We select one of them, that has length dx, mass dm and has the distance x from the left end of the Rod

For the small element $dm = \rho\, dx \to m = \int_0^L \rho\, dx = \rho \int_0^L dx = \rho\, x|_0^L \to m = \rho L$





The moment of inertia about its end is given by

$$I_{yy} = \int x^2 dm = \int_0^L x^2(\rho\, dx) = \frac{1}{3}\rho L^3 = \frac{1}{3}\rho L^3 \frac{m}{\rho L^3} = \frac{1}{3} m L^2 \qquad \therefore\ I_{yy} = \frac{1}{3} m L^2$$

This the moment of inertia of a thin uniform rod about an axis perpendicular to its length and passing through one of its ends.

The moment of inertia of a thin uniform rod about an axis perpendicular to its length and passing through its center. From the Parallel axis theorem

$$I_{yy} = I_{y'y'} + m\left(\frac{1}{2}L\right)^2 \to \frac{1}{3} m L^2 = I_{y'y'} + m\left(\frac{1}{2}L\right)^2 \to I_{y'y'} = \frac{1}{3} m L^2 - \frac{1}{4} mL^2 = \left(\frac{4-3}{12}\right) mL^2 = \frac{1}{12} mL^2$$

$$\therefore\ I_{y'y'} = \frac{1}{12} mL^2$$

The moment of inertia about its other end is given as

$$I_{y''y''} = I_{y'y'} + m\left(\frac{1}{2}L\right)^2 \to I_{y''y''} = \frac{1}{12} m L^2 + m\left(\frac{1}{2}L\right)^2 = \frac{1}{12} m L^2 + \frac{1}{4} mL^2 = \left(\frac{1+3}{12}\right) mL^2 = \frac{4}{12} mL^2$$

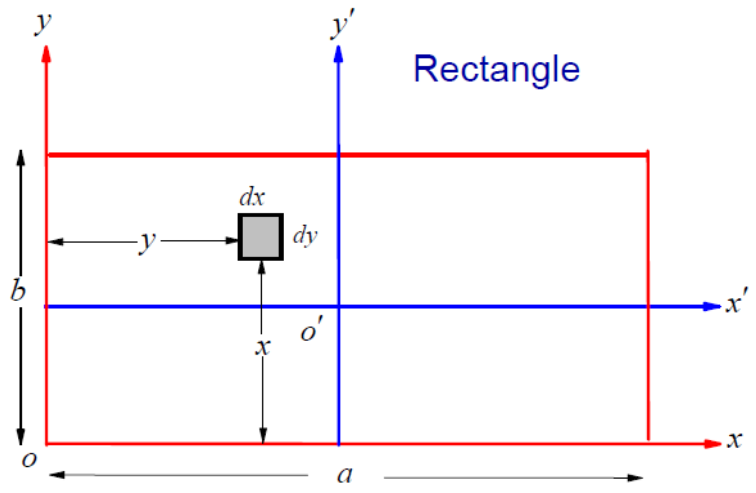
$$\therefore\ I_{y''y''} = \frac{1}{3} mL^2$$

Note: The moment of inertia for a thin uniform Rod that rotates about the axis perpendicular to the rod and passing through one end is $\frac{1}{3} mL^2$. If the axis of rotation passes through the center of the Rod, then the moment of inertia is $\frac{1}{12} mL^2$.

Example 2: Find the Mass moment of inertia of a thin uniform rectangular plate about its base and its one of edges axes?

Solution

We consider a uniform strip with the length (dx) and thickness (dy) as shown in below Figure, where the density is ρ .



$dm=\rho\,dx\,dy \rightarrow m=\rho\int_0^b\int_0^a dx\,dy \rightarrow \; m=\rho\,a\,b$. The moment of inertia about its corner is given by

$$dI_{yy}=x^2dm=\rho\,x^2\,dx\,dy \quad \rightarrow I_{yy}=\rho\int_0^b\int_0^a x^2\,dx\,dy=\rho\left[\frac{x^3}{3}\right]_0^b [y]_0^b=\frac{b\,a^3}{3}\rho=\frac{b\,a^3}{3}\rho\frac{m}{\rho\,a\,b} \quad \therefore\; I_{yy}=\frac{1}{3}m\,a^2$$

If we select a vertical strip (sector, section), we have

$$dI_{yy}=x^2dm=\rho\,x^2\,(b\,dx) \quad \rightarrow I_{yy}=\rho\,b\int_0^a x^2\;dx=\rho\,b\left[\frac{x^3}{3}\right]_0^a=\frac{b\,a^3}{3}\rho=\frac{b\,a^3}{3}\rho\frac{m}{\rho\,a\,b}$$

$$\therefore\; I_{yy}=\frac{1}{3}m\,a^2 \qquad\qquad I_{yy}=I_{y'y'}+m\left(\frac{1}{2}a\right)^2\rightarrow$$

$$\frac{1}{3}m\,a^2=I_{y'y'}+m\left(\frac{1}{2}a\right)^2\rightarrow I_{y'y'}=\frac{1}{3}m\,a^2-\frac{1}{4}ma^2=\left(\frac{4-3}{12}\right)ma^2=\frac{1}{12}ma^2$$

$\therefore\; I_{y'y'}=\frac{1}{12}ma^2$ Similarly, if we select a horizontal strip, we can prove that:

$$I_{xx}=\frac{1}{3}m\,b^2, \qquad I_{x'x'}=\frac{1}{12}m\,b^2$$

For axis is perpendicular ox,oy $I_{zz}=I_{xx}+I_{yy}=\frac{1}{3}m\,b^2+\frac{1}{3}m\,a^2=\frac{1}{3}m\,(a^2+b^2)$

For axis is perpendicular ox',oy' : $I_{z'z'}=I_{x'x'}+I_{y'y'}=\frac{1}{12}m\,b^2+\frac{1}{12}m\,a^2=\frac{1}{12}m\,(a^2+b^2)$

The moment of inertia about its corner is given by (Mass *moment of inertia)*

$I_{xx} = \frac{1}{3}mb^2 = \frac{1}{3}(ab)b^2 = \frac{1}{3}ab^3,$ $I_{yy} = \frac{1}{3}ba^3$ $I_o = I_{xx} + I_{yy} = \frac{1}{3}ab(a^2+b^2)$

$I_{x'x'} = \frac{1}{12}ab^3,$ $I_{y'y'} = \frac{1}{12}ba^3$ $I_{o'} = I_{x'x'} + I_{y'y'} = \frac{1}{12}ab(a^2+b^2)$

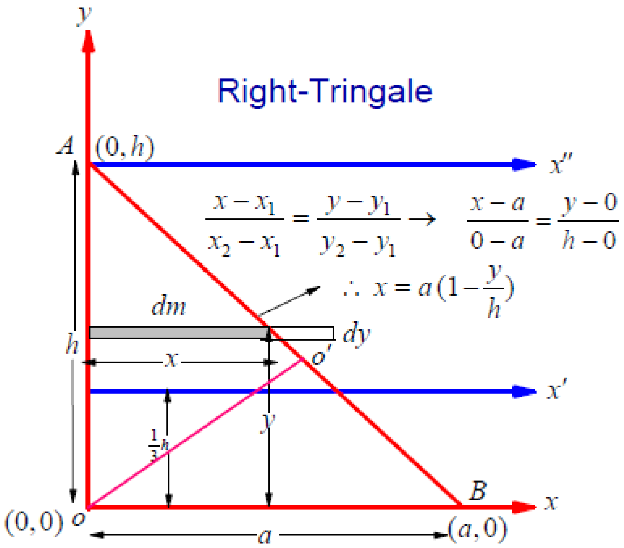
Uniform rectangular plate (a,b)	*Axis coincides with one of its sides*	*Axis passing through its centroid*	*Axis coincides to other side*
With respect toaxis I_{yy} −	$I_{yy} = \frac{1}{3}ma^2$	$I_{y'y'} = \frac{1}{12}ma^2$	$I_{y''y''} = \frac{1}{3}ma^2$
With respect toaxis I_{xx} −	$I_{xx} = \frac{1}{3}mb^2$	$I_{x'x'} = \frac{1}{12}mb^2$	$I_{x''x''} = \frac{1}{3}mb^2$
With respect to axis perpendicular to the plane oxy	$I_{zz} = \frac{1}{3}m(a^2+b^2)$	$I_{z'z'} = \frac{1}{12}m(a^2+b^2)$	$I_{z''z''} = \frac{1}{3}m(a^2+b^2)$

Example 3: Determine the mass moment of inertia for right Triangular Plate (Right-angled triangle)?

Solution

We consider a uniform strip with the length (x) and thickness (dy), such that it is parallel to x - axis, as shown in below Figure. Then

$$dm = \rho x\, dy \to m = \rho\int_0^h x\, dy = \rho\int_0^h a(1-\frac{y}{h})dy = a\rho\left[y-\frac{y^2}{2h}\right]_0^h == a\rho\left[h-\frac{h^2}{2h}\right] \to m = \frac{1}{2}ah\rho$$



Then moment of Inertia with respect to $x-$ axis:

$$dI_{xx}=y^2dm=\rho x y^2\,dy\quad\rightarrow I_{xx}=\rho\int_0^h x y^2\,dy\ \text{, but}\ \frac{x}{a}+\frac{y}{h}=1\ \rightarrow\ x=a(1-\frac{y}{h})$$

$$I_{xx}=\rho\int_0^h a(1-\frac{y}{h})y^2\,dy=\rho a\int_0^h(y^2-\frac{y^3}{h})\,dy=\rho a\left[\frac{y^3}{3}-\frac{y^4}{4h}\right]_0^h$$

$$I_{xx}=\rho a\left[\frac{h^3}{3}-\frac{h^4}{4h}\right]=\frac{1}{12}\rho a h^3(4-3)=\frac{1}{12}\rho a h^3\frac{m}{\frac{1}{2}ah\rho}=\frac{1}{6}mh^2$$

$$\therefore\ I_{xx}=\frac{1}{6}mh^2$$

Then moment of Inertia with respect to $x'-$ axis:

$$I_{xx}=I_{x'x'}+m\left(\frac{1}{3}h\right)^2\rightarrow\quad I_{x'x'}=\frac{1}{6}mh^2-\frac{1}{9}mh^2=\frac{1}{18}mh^2(3-2)=\frac{1}{18}mh^2\qquad I_{x'x'}=\frac{1}{18}mh^2$$

Then moment of Inertia with respect to $x''-$ axis:

$$I_{x''x''}=I_{x'x'}+m\left(\frac{2}{3}h\right)^2=\frac{1}{18}mh^2+\frac{4}{9}mh^2=\frac{1}{18}mh^2(1+8)=\frac{9}{18}mh^2\qquad I_{x''x''}=\frac{1}{2}mh^2$$

Also, $I_{yy}=\frac{1}{6}ma^2,\qquad I_{y'y'}=\frac{1}{18}ma^2,\qquad I_{y''y''}=\frac{1}{2}ma^2.$

$$I_{zz}=I_{xx}+I_{yy}=\frac{1}{6}ma^2+\frac{1}{6}mh^2=\frac{1}{6}m(a^2+h^2)\quad I_{z'z'}=I_{x'x'}+I_{y'y'}=\frac{1}{18}ma^2+\frac{1}{18}mh^2=\frac{1}{18}m(a^2+h^2)$$

Again, $I_{AB}=\frac{1}{6}m(oo')^2$

where $\frac{1}{2}(oo')\,AB$, $AB=\sqrt{(0-a)^2+(h-0)^2}=\sqrt{a^2+h^2}$

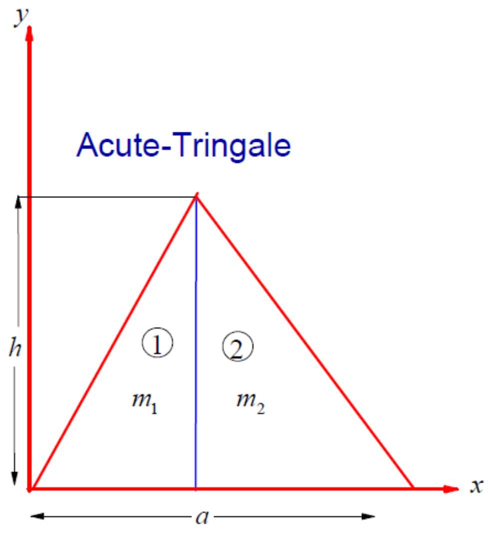
$$I_{AB}=\frac{1}{6}m(oo')^2=\frac{a^2h^2}{6(a^2+h^2)}m\text{, Also }\frac{1}{2}a\,h=\frac{1}{2}(oo')\,AB=\frac{1}{2}(oo')\sqrt{a^2+h^2}\rightarrow oo'=\frac{a\,h}{\sqrt{a^2+h^2}}$$

Right Triangular Plate of height h and bass a	About its corner	*About its center of mass*	*About its* vertex
About its base	$I_{xx}=\frac{1}{6}mh^2$	$I_{x'x'}=\frac{1}{18}mh^2$	$I_{x''x''}=\frac{1}{2}mh^2$
About its height	$I_{yy}=\frac{1}{6}ma^2$	$I_{y'y'}=\frac{1}{18}ma^2$	$I_{y''y''}=\frac{1}{2}ma^2$
About vertical axis	$I_{zz}=\frac{1}{6}m(a^2+h^2)$	$I_{z'z'}=\frac{1}{18}m(a^2+h^2)$	$I_{z''z''}=\frac{1}{6}m(3a^2+h^2),\ I_{z''z''}=\frac{1}{6}m(a^2+3h^2)$

Example 4: The Mass Moment of inertia of acute triangular plate?

Solution

We divide the *acute triangular plate to two right triangular plate as is shown in Figure*



The Moment of inertia of about $x-$ axis for the two right triangular plate is given as

$$(I_{xx})_1=\frac{1}{6}m_1h^2,\qquad (I_{xx})_2=\frac{1}{6}m_2h^2,$$

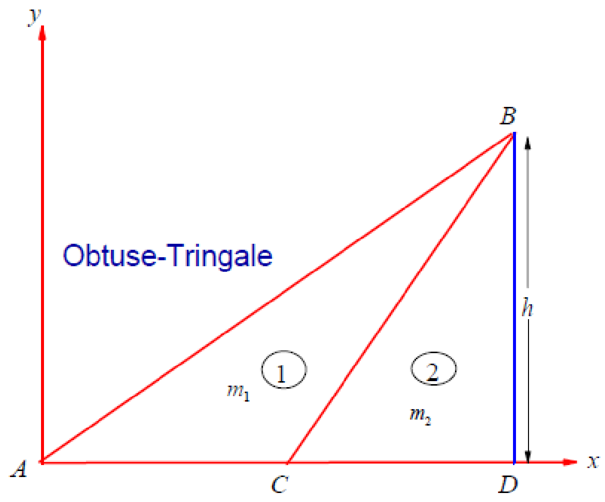
For the acute triangular plate

$$I_{xx} = (I_{xx})_1 + (I_{xx})_2 = \frac{1}{6}m_1 h^2 + \frac{1}{6}m_2 h^2 = \frac{1}{6}(m_1 + m_2)h^2 = \frac{1}{6}mh^2$$

Example 5: The Mass Moment of inertia of obtuse triangular plate?

Solution

We divide the *obtuse triangular plate to two right- triangular plate as is shown below Figure*



The Moment of inertia of about $x-$ axis for the two right triangular plate is given as

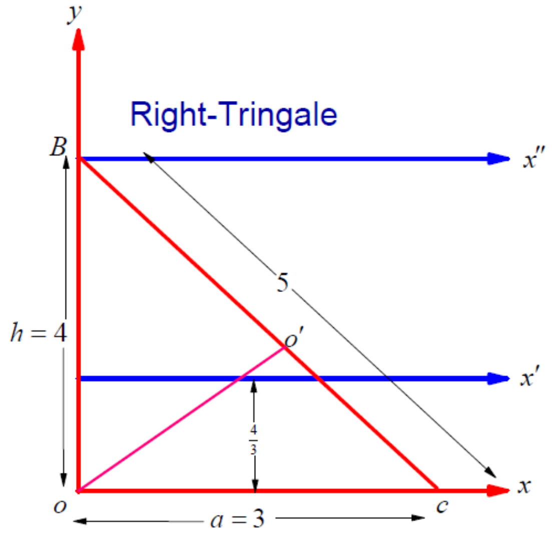
$$(I_{xx})_{ABD} = \frac{1}{6}(m_1 + m_2)h^2, \quad (I_{xx})_{CBD} = \frac{1}{6}m_2 h^2$$

For the acute triangular plate

$$(I_{xx})_{ABC} = (I_{xx})_{ABD} + (I_{xx})_{CBD} = \frac{1}{6}(m_1 + m_2)h^2 - \frac{1}{6}m_2 h^2 = \frac{1}{6}m_1 h^2$$

Example 6 : Find the Mass Moment of inertia of right- *triangular plate as is shown in figure about all different axes?*

Solution



From the Figure it is clear that $I_{xx} = \frac{1}{6} m h^2$, $I_{yy} = \frac{1}{6} m a^2$, $I_{BC} = \frac{a^2 h^2}{6(a^2+h^2)} m$

$$I_{xx} = \frac{1}{6} m h^2 = \frac{1}{6} m (4)^2 = \frac{16}{6} m = \frac{8}{3} m, \qquad I_{yy} = \frac{1}{6} m a^2 = \frac{1}{6} m (3)^2 = \frac{9}{6} m = \frac{3}{2} m$$

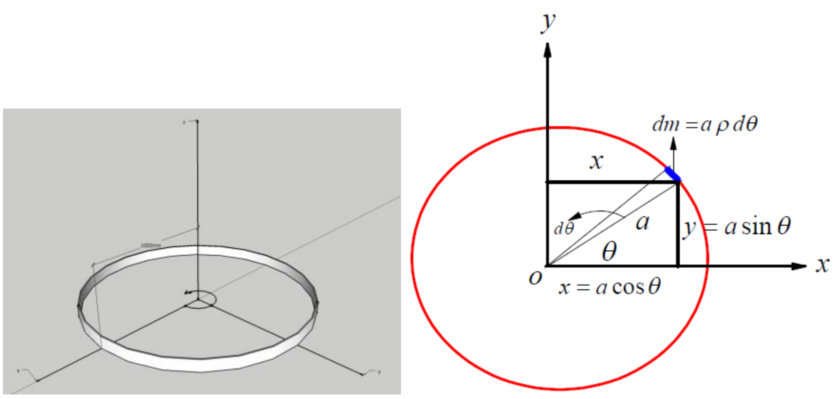
$$I_{BC} = \frac{a^2 h^2}{6(a^2+h^2)} m = \frac{(3)^2 (4)^2}{6((3)^2+(4)^2)} m = \frac{(9)(16)}{6(9+16)} m = \frac{(9)(16)}{6(25)} m = \frac{24}{25} m$$

Note that $3 < 4 < 5, \quad I_{xx} = \frac{8}{3} m > I_{yy} = \frac{3}{2} m > I_{BC} = \frac{24}{25} m$

Example 7: The Mass Moment of inertia of Circular Ring?

Solution

We select a small element has the mass dm at any point located at distance (x, y) from the origin point



The Moment of inertia about $z-$axis (The axis is passing through the center (z-axis) and is perpendicular to the Ring) is given as

$$dI_{zz} = a^2 dm \,........\quad I_{zz} = \int a^2\, dm = a^2 \int_0^m dm \rightarrow \quad I_{zz} = a^2\, m$$

From the Perpendicular axis theorem (Here, the distance between the tangent and the diameter is a) $I_{zz} = I_{xx} + I_{yy}$. So $I_{xx} + I_{yy} = ma^2$

But I_{xx} and I_{yy} are symmetric, so $I_{xx} = I_{yy}$, Then

$I_{xx} = I_{yy} = \frac{1}{2} ma^2$ (The moment of inertia of a ring about of its diameter or the axis passes through the diameter)

From the parallel axis theorem $I_{y'y'} = I_{yy} + ma^2 \rightarrow I_{y'y'} = \frac{1}{2} ma^2 + ma^2 \rightarrow I_{y'y'} = \frac{1}{2} ma^2$

$$I_{x'x'} = I_{yy} + ma^2 \rightarrow I_{x'x'} = \frac{1}{2} ma^2 + ma^2 \rightarrow I_{x'x'} = \frac{1}{2} ma^2$$

Moment of inertia about an axis is passing through the edge of Ring and perpendicular to its plane and parallel an axis is passing through the center (z-axis) and is perpendicular to the Ring

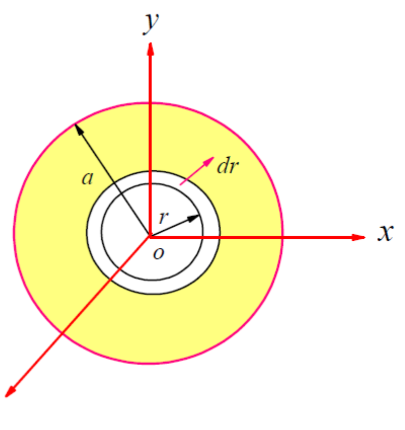
$$I_{z'z'} = I_{zz} + ma^2 \rightarrow I_{z'z'} = ma^2 + ma^2 \rightarrow I_{z'z'} = 2ma^2$$

Circular Ring	For Vertical axis	About axis in the plane of Circular Ring and passes in the its center *The moment of inertia of the ring about of its diameter*
Axis of rotation	$I_{zz} = ma^2$	$I_{xx} = I_{yy} = \frac{1}{2} ma^2$
Axis of rotation	$I_{z'z'} = 2ma^2$	$I_{x'x'} = I_{y'y'} = \frac{3}{2} ma^2$

Example 8: Find the Mass Moment of inertia of Circular area *?*

Solution

We divide the Circular area to the *small Circular Rings, we selected one of them has mass* (dm)*,thickness* (dr) *and raids* (r).



So, $dm = 2\pi r\rho\, dr \rightarrow m = 2\pi\rho\int_0^a r\, dr \rightarrow m = 2\pi\rho \left.\frac{r^2}{2}\right|_0^a = \pi a^2 \rho$

$$I_{zz} = \int r^2\, dm = \int r^2 (2\pi r\rho\, dr) = 2\pi\rho\int_0^a r^3\, dr = \left.\frac{2\pi\rho r^4}{4}\right|_0^a = \frac{\pi\rho a^4}{2} = \frac{\pi\rho a^4}{2}\frac{m}{m} = \frac{\pi\rho a^4}{2}\frac{m}{\pi a^2 \rho}$$

$$I_{zz} = \frac{1}{2} m a^2$$

From the Perpendicular axis theorem

$I_{zz} = I_{xx} + I_{yy}$. So $I_{xx} + I_{yy} = \frac{1}{2} m a^2$.

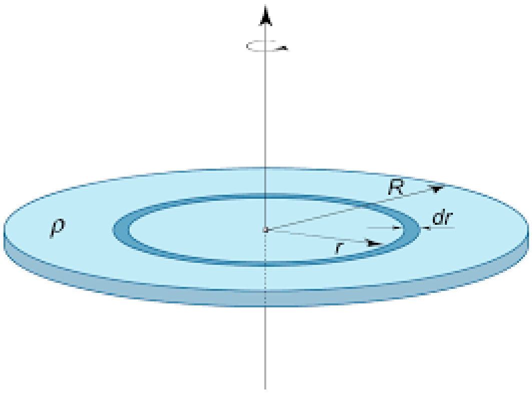
But I_{xx}, I_{yy} are symmetric, so $I_{xx} = I_{yy}$. Then $I_{xx} = I_{yy} = \frac{1}{4} m a^2$

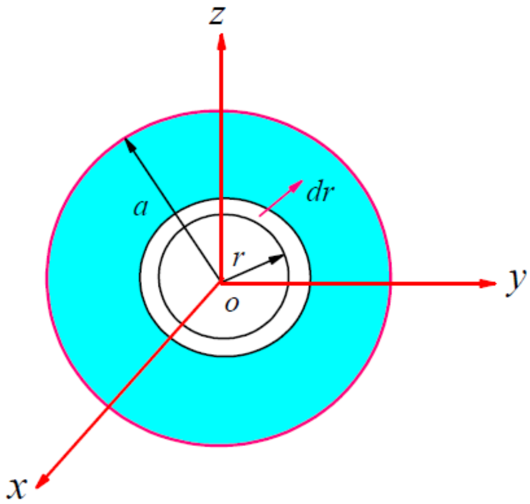
Circular area	For Vertical axis	About axis in the plane of Circular Ring and passes in the its center
Axis of rotation	$I_{zz} = \frac{1}{2} m a^2$	$I_{xx} = I_{yy} = \frac{1}{4} m a^2$
Axis of rotation	$I_{z'z'} = \frac{3}{2} m a^2$	$I_{x'x'} = I_{y'y'} = \frac{5}{4} m a^2$

Example 9: Find the Moment of inertia of Thin Disc?

Solution

We divide the solid Disc to the small Circular Rings, we selected one of them has mass (dm),thickness (dr),distraction thickness (Δz) and raids (r).





$$dm = 2\pi\, r\rho\,\Delta z\, dr \rightarrow\; m = 2\pi\,\rho\,\Delta z\int_0^a r\, dr \;\rightarrow m = 2\pi\,\rho\,\Delta z\left.\frac{r^2}{2}\right|_0^a = \pi\, a^2\rho\,\Delta z$$

So, the Moment of inertia of thin Disc is

$$I_{zz} = \int r^2\, dm = \int r^2 (2\pi\, r\,\rho\,\Delta z\, dr) = 2\pi\rho\,\Delta z\int_0^a r^3\, dr = 2\pi\rho\,\Delta z\left.\frac{r^4}{4}\right|_0^a = \pi\rho\,\Delta z\,\frac{a^4}{2}$$

$$I_{zz} = \frac{\pi\rho\, a^4}{2}\frac{m}{m} = \frac{\pi\rho\Delta z\, a^4}{2}\frac{m}{\pi\, a^2\rho\,\Delta z} \quad\rightarrow\quad I_{zz} = \frac{1}{2}ma^2$$

From the Parallel axis theorem $I_{z'z'} = I_{zz} + m\,a^2 \rightarrow \quad I_{z'z'} = \frac{3}{2}m\,a^2$

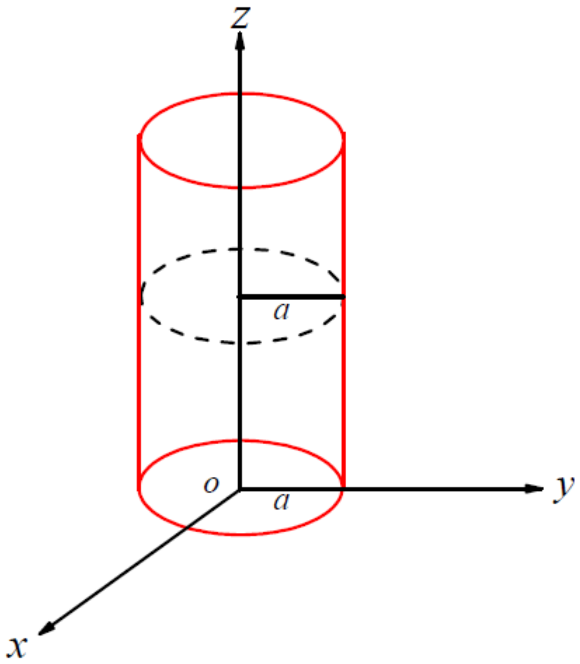
From the Perpendicular axis theorem $I_{zz} = I_{xx} + I_{yy}$. So $I_{xx} + I_{yy} = \frac{1}{2}m\,a^2$.

But I_{xx}, I_{yy} are symmetric, so $I_{xx} = I_{yy}$. Then $I_{xx} = I_{yy} = \frac{1}{4}m\,a^2$

Example: 10: Derive the *Mass* moment of inertia of Hollow Cylinder?

Solution

Take the hollow cylinder as the corresponding shape, divide it into an infinite number of regular circular rings and take one of these rings with the mass (dm) and the radius (a).



Then the moment of inertia of this ring is given as $dI_{zz} = a^2 dm$.

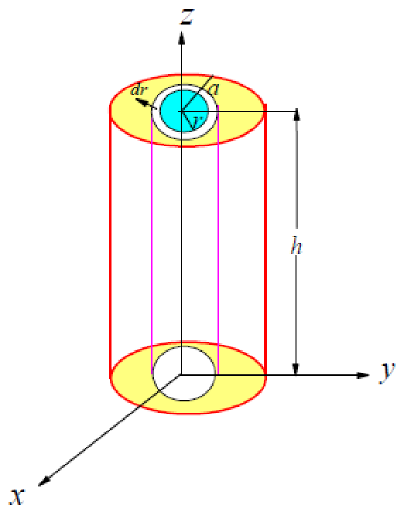
Then, the total moment of Hollow Cylinder

$$I_{zz} = \int_0^m a^2\, dm = m a^2 \quad \rightarrow \quad I_{zz} = m a^2$$

Example: 11: Derive the Mass moment of inertia of Solid Cylinder?

Solution

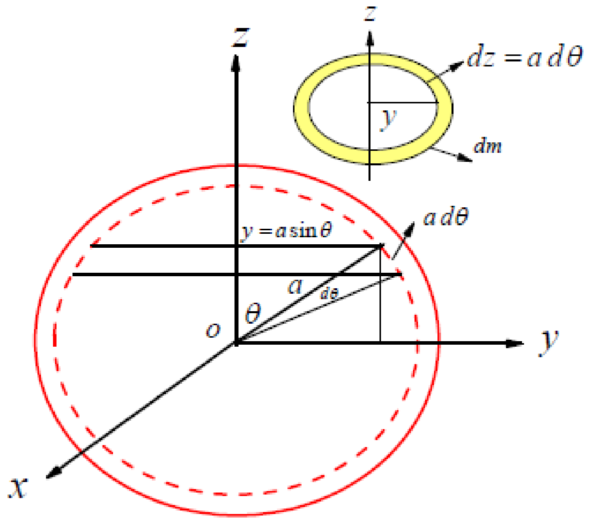
We divide the Solid Cylinder it into an infinite number of thin discs and take one of these discs with the mass (dm) and the radius (a).

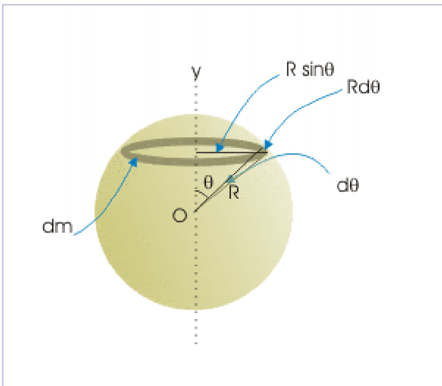


Then the moment of inertia of this disc is given as. $dI_{zz} = \frac{1}{2} a^2 \, dm$. Then the total moment of Hollow Cylinder $I_{zz} = \int_0^m \frac{1}{2} a^2 \, dm = \frac{1}{2} m a^2 \quad \rightarrow \quad I_{zz} = \frac{1}{2} m a^2$

Example: 11: Derive the Mass moment of inertia of Hollow Sphere?

Solution





We divide the Hollow Sphere into a number of small circular rings and we consider one of them with the mass (dm), the radius (y) and thickness (dz).

$$dm = 2\pi \, y \, \rho \, dz = 2\pi \, (a \sin\theta) \, \rho \, a \, d\theta \rightarrow m = 2\pi \, \rho \, a^2 \int_0^{\pi} \sin\theta \, d\theta \rightarrow m = -2\pi \, \rho \, a^2 \cos\theta \Big|_0^{\pi} =$$

$$= -2\pi \, \rho \, a^2 \left(\cos(\pi) - \cos(0)\right) = -2\pi \, \rho \, a^2 \left(-1-1\right) = 2\pi \, \rho \, a^2 \left(1+1\right) == 4\pi \, \rho \, a^2$$

The moment of inertia of this circular ring is given as $dI_{zz} = y^2 dm$.

Then the total moment of Hollow Sphere $I_{zz} = \int y^2\, dm$, then

$$I_{zz} = \int y^2\, dm = 2\pi\rho a^4 \int_0^{\pi} (\sin)^2 \sin\theta\, d\theta = 2\pi\rho a^4 \left[\int_0^{\pi} (1-\cos^2\theta)\sin\theta\, d\theta \right]$$
$$= 2\pi\rho a^4 \left[\int_0^{\pi} \sin\theta\, d\theta - \int_0^{\pi} (\cos\theta)^2\, d(-\sin\theta) \right]$$
$$= 2\pi\rho a^4 \left[-\cos\theta + \frac{1}{3}(\cos\theta)^3 \right]_0^{\pi} = 2\pi\rho a^4 \left[-\cos(\pi) + \frac{1}{3}(\cos(\pi))^3 - \left\{ -\cos(0) + \frac{1}{3}(\cos(0))^3 \right\} \right]$$

$$= 2\pi\rho a^4 \left[1 + \frac{1}{3} - \left\{ -1 + \frac{1}{3} \right\} \right] = 2\pi\rho a^4 \left[1 + \frac{1}{3} + 1 - \frac{1}{3} \right] = 2\pi\rho a^4 \left[2 - \frac{2}{3} \right] = \frac{8}{3}\pi\rho a^4$$

$$I_{zz} = \frac{8}{3}\pi\rho a^4 \frac{m}{4\pi a^2 \rho} = \frac{2}{3} m a^2 \qquad \textit{Then} \quad I_{zz} = \frac{2}{3} m a^2$$

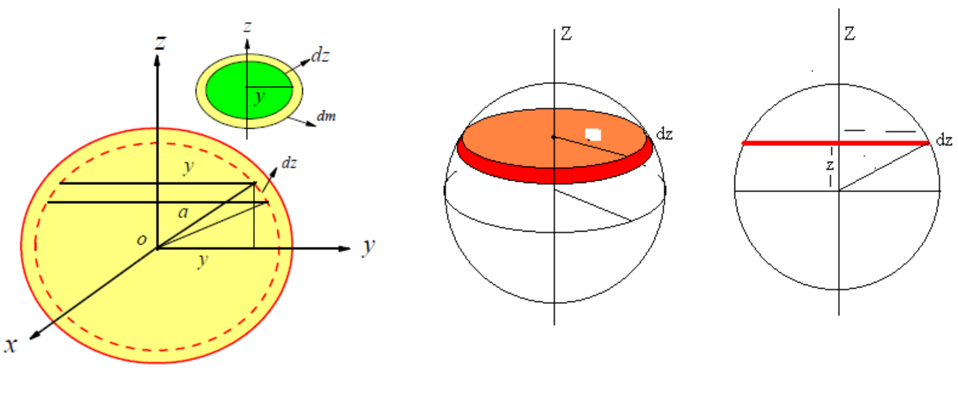
For the symmetric of axes $I_{xx} = I_{yy} = I_{zz} = \frac{2}{3} m a^2$

Also, we know $I_{xx} + I_{yy} + I_{zz} = 2I_o$, $2I_o = \frac{2}{3} m a^2 + \frac{2}{3} m a^2 + \frac{2}{3} m a^2 = \frac{6}{3} m a^2 = 2ma^2$

$I_o = ma^2$

Example: 12: Derive the Mass moment of inertia of Solid Sphere?

Solution



We divide the solid sphere into a number of hollow sphere and take one of these sphere with mass (dm), radius (r) and thickness (dr). Then the moment inertia of this sphere around oz

axis is $dI_{zz} = \frac{2}{3}(dm)r^2$, for whole sphere the moment inertia is given as $I_{zz} = \int \frac{2}{3}(dm)r^2$, where

$$dm = 4\pi r^2 \rho\, dr \rightarrow m = 4\pi \rho \int_0^a r^2\, dr = \frac{4}{3}\pi a^3 \rho.$$ Then

$$I_{zz} = \int \frac{2}{3}(dm)r^2 = \int_0^a \frac{2}{3}\left(4\pi r^2 \rho\, dr\right) r^2 = \frac{8}{3}\pi \rho \int_0^a r^4\, dr = \frac{8}{3}\pi \rho \left.\frac{r^4}{5}\right|_0^a = \frac{8}{15}\pi \rho a^5$$

$$I_{zz} = \frac{8}{15}\pi \rho a^5 \frac{m}{m} = \frac{8}{15}\pi \rho a^5 \frac{m}{\frac{4}{3}\rho a^3} = \frac{2}{5}ma^2$$ Then $$I_{zz} = \frac{2}{5}ma^2$$

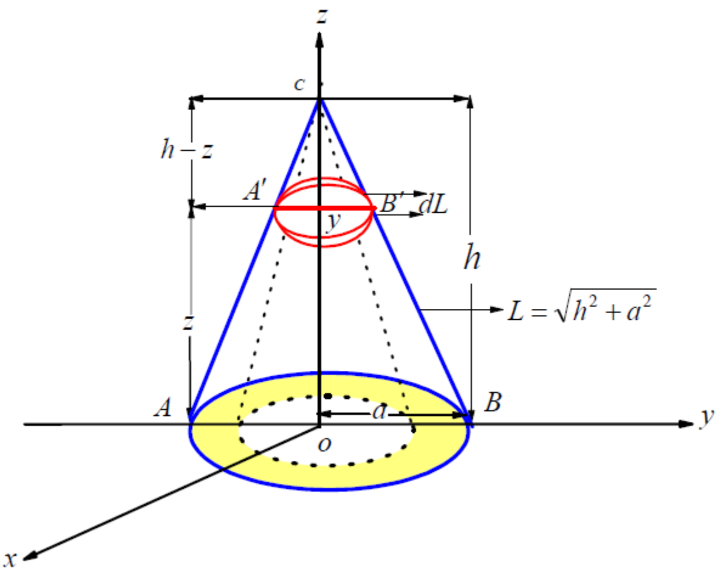
Where the axes are Symmetrical $I_{xx} = I_{yy} = I_{zz} = \frac{2}{5}ma^2$

Also $I_{xx} + I_{yy} + I_{zz} = 2I_o$, Then $2I_o = \frac{2}{5}ma^2 + \frac{2}{5}ma^2 + \frac{2}{5}ma^2 = \frac{6}{5}ma^2$ $I_o = \frac{3}{5}ma^2$

Example: 13: Find the Mass moment inertial for the Hollow Circular Cone ?

Solution

Divide the Hollow Circular Cone into a number of small circular rings and take one of these rings with mass (dm), radius (y) and thickness (dL), which is located higher (z) than the base of the cone with radius (a). Note that it is similar to triangles ABC and $A'B'C$, we have $\frac{h-z}{h} = \frac{y}{a} \rightarrow y = \frac{a}{h}(h-z) \rightarrow z = \frac{h}{a}(a-y)$



The moment of inertia of this circular ring is given as $dI_{zz} = y^2 dm$.

Then the total moment of Hollow Circular Cone $I_{zz} = \int y^2\, dm$

Note that , where $dm = 2\pi\, y\,\rho\, dL \rightarrow\ m = 2\pi\,\rho \int_0^h y\, dL$

$$dL = \sqrt{1+\left(\frac{dz}{dy}\right)^2}\ dy = \sqrt{1+\left(\frac{h}{a}\right)^2}\ dy = \frac{1}{a}\sqrt{a^2+h^2}\ dy = \frac{L}{a}\ dy.$$ Then

$$dm = 2\pi\,\rho \int_0 y\, dL == 2\pi\,\rho \int_0^a y\frac{L}{a}dy = \ 2\pi\,\rho\frac{L}{a}\frac{y^2}{2}\Big|_0^a = 2\pi\,\rho\frac{L}{a}\frac{a^2}{2} \rightarrow m = \pi\, a\, L\,\rho.$$ Then

$$I_{zz} = \int_0^a y^2\, dm = \int_0^a y^2\left(2\pi\, y\,\rho\ dL\right) = 2\pi\ \ \rho\int_0^a y^3\,\frac{L}{a}dy = 2\pi\ \ \rho\frac{L}{a}\frac{y^4}{4}\Big|_0^a = 2\pi\ \ \rho\frac{L}{a}\frac{a^4}{4} = \pi\, L\rho\ \frac{a^3}{2}$$

$$= \pi\, L\rho\ \frac{a^3}{2}\ \frac{m}{m} = \pi\, L\rho\ \frac{a^3}{2}\ \ \frac{m}{\pi\, a\, L\,\rho} = \frac{1}{2}m\, a^2 \qquad\qquad I_{zz} = \frac{1}{2}m\, a^2$$

Again, $dL = \sqrt{1+\left(\frac{dy}{dz}\right)^2}\ dz = \sqrt{1+\left(\frac{a}{h}\right)^2}\ dz = \frac{1}{h}\sqrt{h^2+a^2}\ dz = \frac{L}{h}\ dz$

$$dm = 2\pi\,\rho \int_0 y\, dL == 2\pi\,\rho \int_0^h \frac{a}{h}z\ \frac{L}{h}dz = \ 2\pi\,\rho\, a\frac{L}{h^2}\frac{z^2}{2}\Big|_0^h = 2\pi\,\rho\, a\frac{L}{h^2}\ \frac{h^2}{2} \rightarrow m = \pi\, a\, L\,\rho$$

$$I_{zz} = \int_0^a y^2\, dm = \int_0^a y^2\left(2\pi\, y\,\rho\ dL\right) = 2\pi\ \ \rho\int_0^h\left(\frac{a}{h}z\right)^3\frac{L}{h}dz = 2\pi\ \ \rho\frac{a^3L}{h^4}\int_0^h z^3\, dz == 2\pi\ \ \rho\frac{a^3L}{h^4}\ \frac{z^4}{4}\Big|_0^a$$

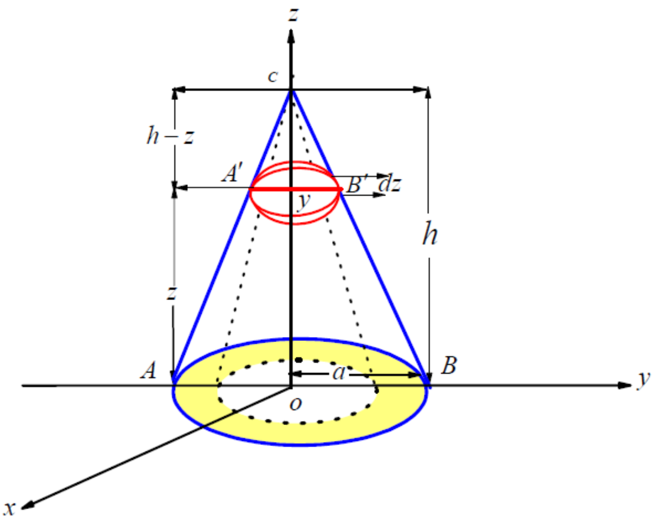
$$= 2\pi\ \ \rho\frac{a^3L}{h^4}\ \frac{h^4}{4} = \pi\, L\rho\ \frac{a^3}{2}\ \ = \pi\, L\rho\ \frac{a^3}{2}\ \frac{m}{m} = \pi\, L\rho\ \frac{a^3}{2}\ \ \frac{m}{\pi\, a\, L\,\rho} = \frac{1}{2}m\, a^2 \qquad I_{zz} = \frac{1}{2}m\, a^2$$

Example 14: Find the Mass moment inertial for the Solid Circular Cone?

Solution

We divide the Solid Circular Cone into a number of small Disks and take one of them with mass (dm), radius (y) and thickness (dz), which is located higher (z) than the base of the cone with radius (a). Note that it is similar to triangles ABC and $A'B'C$, we have

$$\frac{h-z}{h}=\frac{y}{a}\rightarrow y=\frac{a}{h}(a-z)\rightarrow z=\frac{h}{a}(a-y)$$



$$dm=\pi y^2 \rho\, dz\rightarrow\ m=\pi\rho\int_0^h\left(\frac{a}{h}(h-z)\right)^2 dz=\pi\rho\frac{a^2}{h^2}\int_0^h\left(h^2-2hz+z^2\right)dz$$

$$=\pi\rho\frac{a^2}{h^2}\left(h^2 z-2h\frac{z^2}{2}+\frac{z^3}{3}\right)\Bigg|_0^h=\pi\rho\frac{a^2}{h^2}\left(h^3-h^3+\frac{h^3}{3}\right)\rightarrow m=\frac{1}{3}\pi a^2 h\,\rho$$

The moment of inertia of this Disk is given as $dI_{zz}=y^2 dm$.

Then the total moment of Solid Circular Cone $I_{zz}=\int y^2\,dm$, that is given

$$I_{zz}=\frac{1}{2}\int_0^a y^2\,dm=\frac{1}{2}\int_0^h y^2\left(\pi y^2\rho\, dz\right)=\frac{1}{2}\pi\rho\int_0^h y^4\,dz=\frac{1}{2}\pi\rho\int_0^h\left(\frac{a}{h}(h-z)\right)^4 dz=\frac{1}{2}\pi\rho\left(\frac{a}{h}\right)^4\int_0^h (h-z)^4\,dz$$

$$=\frac{1}{2}\pi\rho\left(\frac{a}{h}\right)^4\frac{(h-z)^5}{-5}\Bigg|_0^h=\frac{1}{2}\pi\ \rho\frac{a^4}{h^4}\frac{h^5}{5}=\frac{1}{10}\pi\ \rho a^4 h==\frac{1}{10}\pi\ \rho a^4 h\,\frac{m}{m}==\frac{1}{10}\pi\ \rho a^4 h\ \frac{m}{\frac{1}{3}\pi a^2 h\rho}=\frac{3}{10}m a^2$$

$I_{zz}=\dfrac{3}{10}m a^2$, Also

$$I_o=\int_0^h z^2\,dm=\int_0^h z^2\left(\pi y^2\rho\, dz\right)=\pi\rho\int_0^h z^2 y^2\,dz=\pi\rho\int_0^h z^2\left(\frac{a}{h}(h-z)\right)^2 dz$$

$$=\pi\rho\left(\frac{a}{h}\right)^2\int_0^h\left(h^2 z^2-2h z^3+z^4\right)dz=\pi\rho\frac{a^2}{h^2}\left(h^2\frac{z^3}{3}-h\frac{z^4}{2}+\frac{z^5}{5}\right)\Bigg|_0^h=\pi\rho\frac{a^2}{h^2}h^5\left(\frac{10-15+6}{30}\right)$$

$$=\pi\ \rho\frac{a^2}{h^2}\frac{h^5}{30}=\frac{1}{30}\pi\ \rho a^2 h^3=\frac{1}{30}\pi\ \rho a^2 h^3\ \frac{m}{m}=\frac{1}{30}\pi\ \rho a^2 h^3\ \frac{m}{\frac{1}{3}\pi a^2 h\rho}=\frac{1}{10}m h^2.$$

Then $I_o=\dfrac{1}{10}m h^2$

II. Area Moment of Inertia

Area moment of inertia also known as second area moment or 2^{nd} moment of area is a property of a two-dimensional plane shape where it shows how its points are dispersed in an arbitrary axis in the cross-sectional plane. This property basically characterizes the deflection of the plane shape under some load.

Area moment of inertia is usually denoted by the letter I for an axis in a plane. The dimension unit of second area moment is Length to the power of four which is given as L^4. If we take the International System of Units, its unit of dimension is meter to the power of four or m^4. If we take the Imperial System of Units it can be inches to the fourth power, in^4.

We will come across this concept in the field of structural engineering often. Here the area moment of inertia is said to be the measure of the flexural stiffness of a beam. It is an important property that is used to measure the resistance offered by a beam to bending or in calculating a beam's deflection. Here we have to look at two cases.

First, a beam's resistance to bending can be easily described or defined by the planar second moment of area where the force lies perpendicular to the neutral axis.

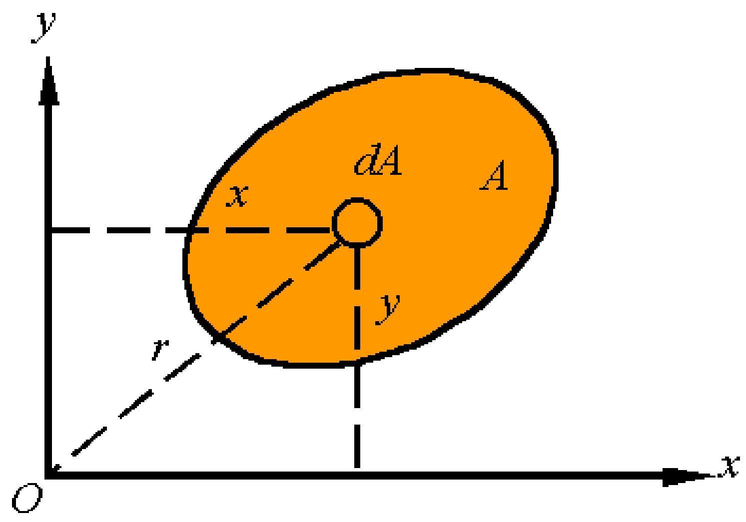
Secondly, the polar second moment of area can be used to determine the beam's resistance when the applied moment is parallel to its cross-section. It is basically the beams ability to resist torsion

Area Moment Of Inertia Formulas

The area moment of inertia for the area is given in below figure can be expressed mathematically as:

Referenced to x-axis is given by $I_{xx} = y^2\, dA,$

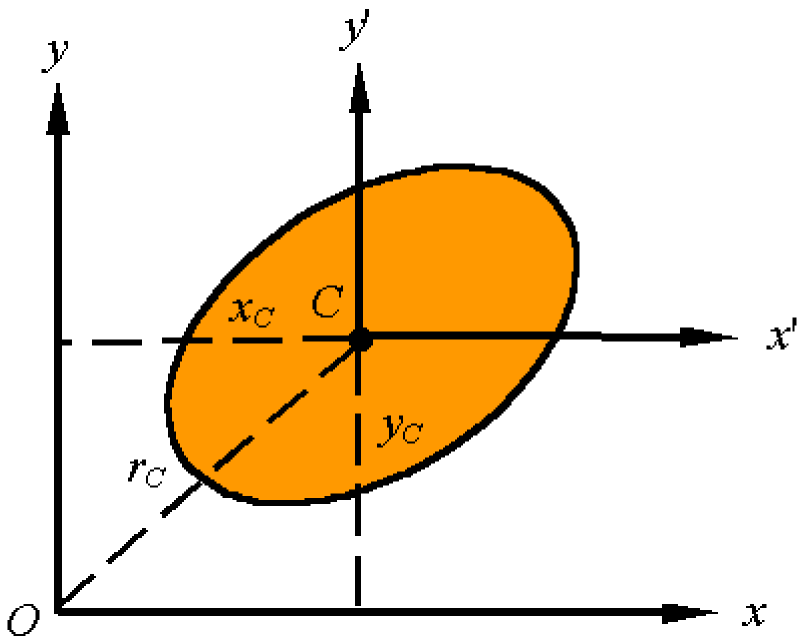
Referenced to y-axis is given by $I_{yy} = x^2\, dA,$



Referenced to o -point is given by $I_O = r^2 dA = (x^2 + y^2)\, dA = I_{xx} + I_{yy}$

The parallel axis theorem

The parallel axis theorem is a relation between the moment of inertia about an axis passing through the centroid and the moment of inertia about any parallel axis.



The parallel axis theorem states that

$$I_{xx} = I_{x'x'} + A\bar{y}^2, \quad I_{yy} = I_{y'y'} + A\bar{x}^2$$

A simple recap of the Basics:

- Moments of inertia are always positive.

- Minimum moments of inertia axes always pass through the center of mass.
- Moments of inertia are a measure of the mass distribution of a body about a set of axes. Think of a rotating ice skater. If the person stretches the arms out, she slows down and speeds up otherwise. Hence the smaller the inertia the more concentrated or closer the mass is about a particular axis.

- Area moments of inertia are for a particular section or a 2D surface.

- Products of inertia can be positive, negative or zero.

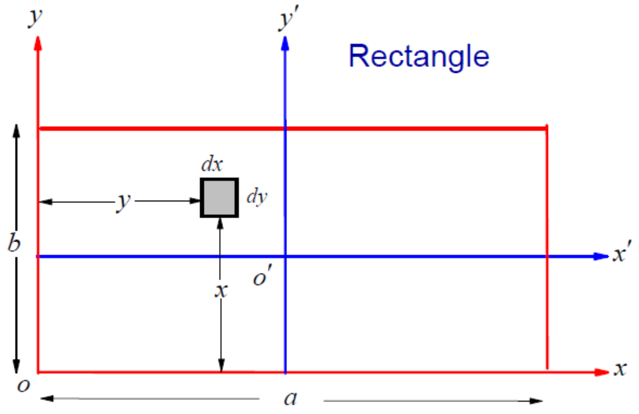
- Products of inertia are a measure of the symmetry of a body about a set of axes. They are zero about any axis normal to a plane of symmetry.

- For any given point on a section, for example the centroid or any other point, there exists a set of axes oriented in such a way that all products of inertia are zero.

Example 1: Find the Area moment of inertia of a rectangular section about a horizontal axis passing through base?

Solution

We consider a uniform element with the length (dx) and thickness (dy) as is shown in below Figure



The Area moment of inertia about its vertical corner is given by

$$dI_{yy}=x^2dA=x^2\,dx\,dy \quad\rightarrow\; I_{yy}=\int_0^b\int_0^a x^2\,dx\,dy=\left[\frac{x^3}{3}\right]_0^b[y]_0^b=\frac{1}{3}ba^3 \quad \therefore\; I_{yy}=\frac{1}{3}ma^2$$

From the parallel Axis Theorem $\quad I_{yy}=I_{y'y'}+m\left(\frac{1}{2}a\right)^2\rightarrow$

$$I_{y'y'}=\frac{1}{3}ba^3-\frac{1}{4}(ab)a^2=\left(\frac{4-3}{12}\right)ba^3=\frac{1}{12}ba^3 \qquad \therefore\; I_{y'y'}=\frac{1}{12}ba^3$$

Similarly, we can prove that: $I_{xx}=\frac{1}{3}ab^3,\qquad I_{x'x'}=\frac{1}{12}ab^3$

For axis is perpendicular ox,oy $\;I_{zz}=I_{xx}+I_{yy}=\frac{1}{3}ab^3+\frac{1}{3}ba^3=\frac{1}{3}ab(a^2+b^2)$

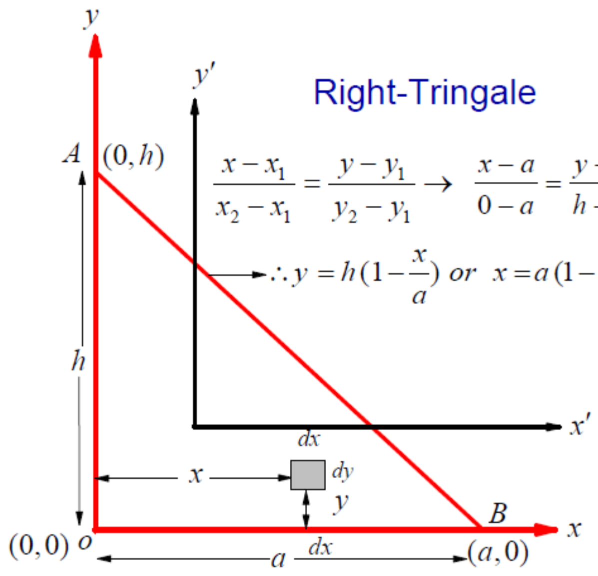
For axis is perpendicular ox',oy' : $\;I_{z'z'}=I_{x'x'}+I_{y'y'}=\frac{1}{12}ab^3+\frac{1}{12}ba^3=\frac{1}{12}ab(a^2+b^2)$

Uniform rectangular plate (a,b)	Axis coincides with one of its sides	Axis passing through its centroid	Axis coincides to other side
With respect toaxis $I_{yy}-$	$I_{yy}=\frac{1}{3}ba^3$	$I_{y'y'}=\frac{1}{12}ba^3$	$I_{y''y''}=\frac{1}{3}ba^3$
With respect toaxis $I_{xx}-$	$I_{xx}=\frac{1}{3}ab^3$	$I_{x'x'}=\frac{1}{12}ab^3$	$I_{x''x''}=\frac{1}{3}ab^3$
With respect to axis perpendicular to the plane oxy	$I_{zz}=\frac{1}{3}ab(a^2+b^2)$	$I_{z'z'}=\frac{1}{12}ab(a^2+b^2)$	$I_{z''z''}=\frac{1}{3}ab(a^2+b^2)$

Example 2: Find the Area moment of inertia of a triangular section about a horizontal axis passing through base?

Solution

We consider a uniform element with the length (dx) and thickness (dy) as is shown in below Figure



$$I_{yy}=\int\int x^2\,(dx\,dy) \rightarrow I_{xy}=\int_0^h\int_0^{a(1-\frac{y}{h})} x^2 dx\,dy=\frac{a^3}{3}\int_0^h (1-\frac{y}{h})^3\,dy$$

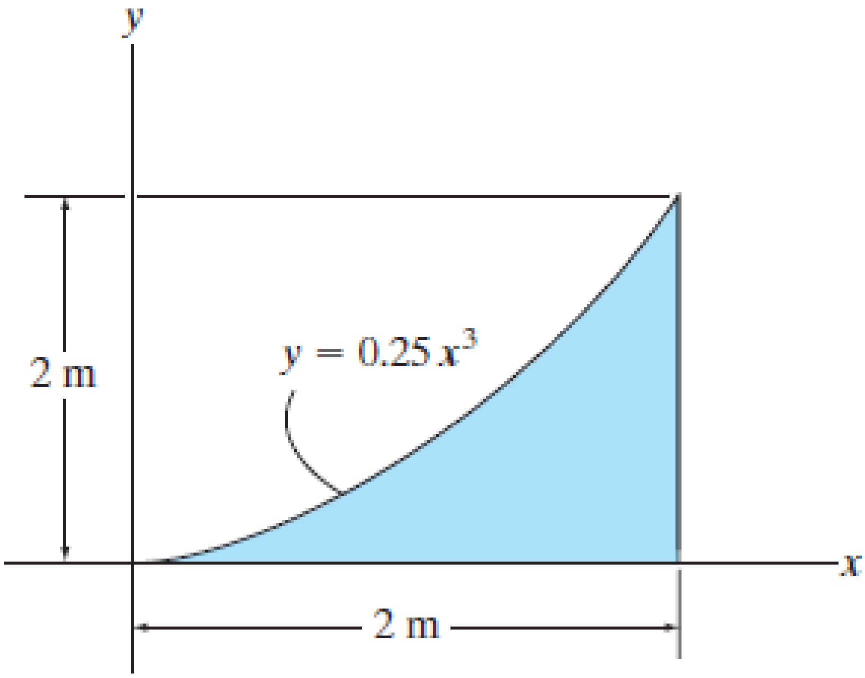
$$=\frac{a^3}{3h^3}\int_0^h (h-y)^3\,dy=\frac{a^3}{h^2}\int_0^h (h^3-3h^2y+3y^2h-y^3)\,dy$$

$$=\frac{a^3}{h^3}\left[h^3y-\frac{3y^2}{2}h^2+y^3h-\frac{y^4}{4}\right]_0^h=\frac{a^3}{3h^3}\left[h^4-\frac{3}{2}h^4+h^4-\frac{h^4}{4}\right]=\frac{a^3h^4}{12h^3}[8-6-1]=\frac{1}{12}a^3h$$

$$\therefore\ I_{yy}=\frac{1}{12}ha^3$$

Right Triangular Plate of height h and bass a	About its corner	About its center of mass	About its vertex
About its base	$I_{xx}=\frac{1}{12}ah^3$	$I_{x'x'}=\frac{1}{36}ah^3$	$I_{x''x''}=\frac{1}{4}ah^3$
About its height	$I_{yy}=\frac{1}{12}ha^3$	$I_{y'y'}=\frac{1}{36}ha^3$	$I_{y''y''}=\frac{1}{4}ha^3$
About vertical axis	$I_{zz}=\frac{ah}{12}(a^2+h^2)$	$I_{z'z'}=\frac{ah}{36}(a^2+h^2)$	$I_{z''z''}=\frac{ah}{12}(a^2+3h^2)$, $I_{z''z''}=\frac{ah}{12}(3a^2+h^2)$

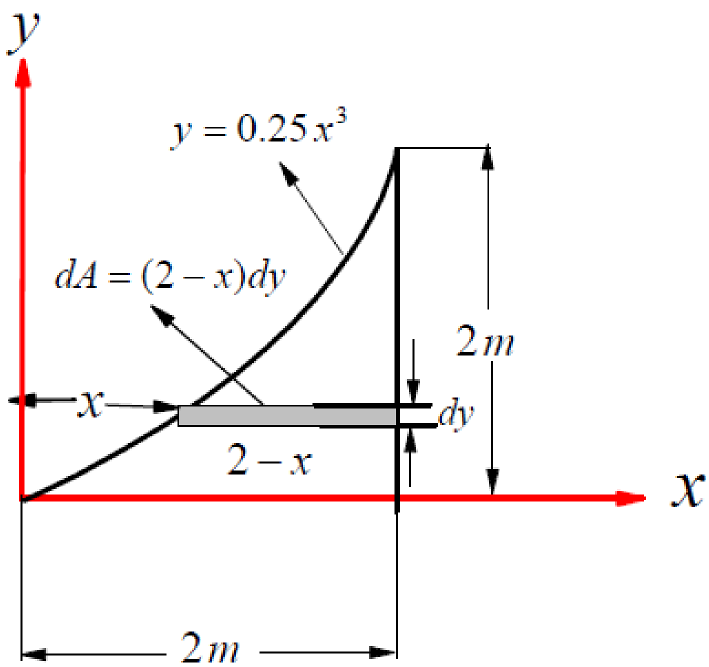
Example 3: Determine Area the moment of inertia of the shaded area with respect to $ox,\ oy-$axes?



Solution

The Area Moment of inertia with respect to x- axis

We consider a uniform strip line parallels to the $x-$axis with the length $(2-x)$ and thickness (dy) as is shown in below Figure. Then $dA=(2-x)\,dy$



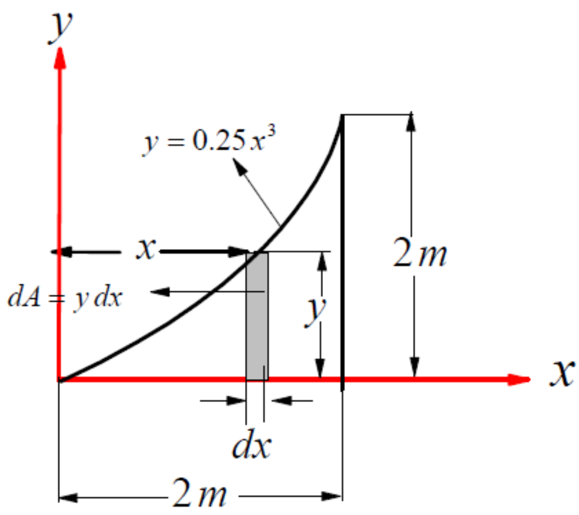
So, the area moment of inertia with respect to x -axis is given as

$$I_{xx} = \int y^2 dA = \int_0^2 y^2 \left(2 - x \right) dy = \int_0^2 y^2 \left(2 - \left(4y\right)^{\frac{1}{3}} \right) dy = \int_0^2 \left(2y^2 - \left(4\right)^{\frac{1}{3}} y^{\frac{7}{3}} \right) dy$$

$$I_{xx} = \left(\frac{2}{3} y^3 - \frac{3}{10} \left(4\right)^{\frac{1}{3}} y^{\frac{10}{3}} \right)_0^2 = \frac{2}{3}(2)^3 - \frac{3}{10}\left(4\right)^{\frac{1}{3}} (2)^{\frac{10}{3}} = \frac{16}{3} - \frac{3}{10}\left(4\right)^{\frac{1}{3}} (2)^{\frac{10}{3}} = \frac{8}{15} = 0.53334\, m^4$$

<u>The Area Moment of inertia with respect to y- axis</u>

While if we consider a uniform strip line parallels to the $y-$axis with the length (y) and thickness (dx) as is shown in below Figure. Then $dA = y\,dx$



So, the area moment of inertia with respect to y -axis is given as

$$I_{yy} = \int x^2 dA = \int_0^2 x^2 y\, dx = \int_0^2 x^2 \left(0.25\right) x^3\, dx = \int_0^2 \left(0.25\right) x^5\, dy = \frac{(2)^6}{24} = \frac{64}{24} = \frac{8}{3} = 2.67\, m^4$$

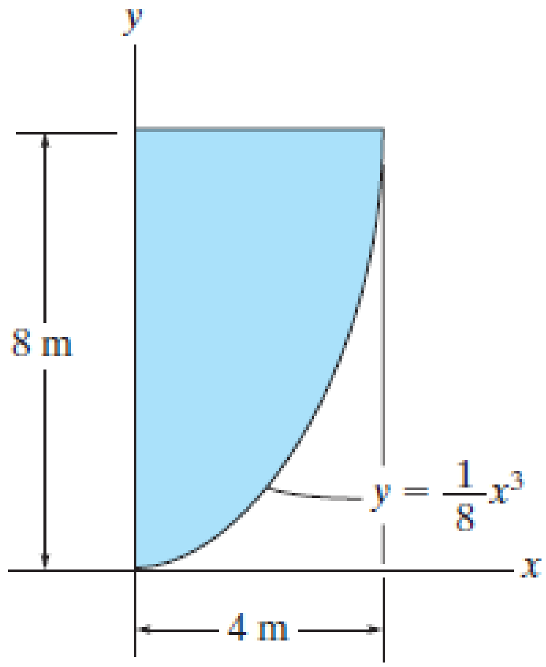
<u>Again, the Area Moment of inertia with respect to x- axis</u>

If we consider the previous Figure (second Figure) we can find the Area moment of inertia as

$$I_{xx} = \int \frac{1}{3} y^3 dx = \int_0^2 \frac{1}{3} y^3 dx = \frac{1}{3}\int_0^2 \left(0.25 x^3\right)^3 dx = \frac{1}{3}\left(0.25\right)^3 \int_0^2 x^9\, dx = \frac{1}{3}\left(0.25\right)^3 \frac{1}{10}\left(x^{10}\right)_0^2$$

$$I_{xx} = \frac{1}{3}\frac{15625}{1000000}\frac{1024}{10} = \frac{1}{3}\frac{16000000}{10000000} = \frac{1}{3}\frac{16}{10} = \frac{1}{3}\frac{8}{5} = \frac{8}{15} = 0.53334\, m^4$$

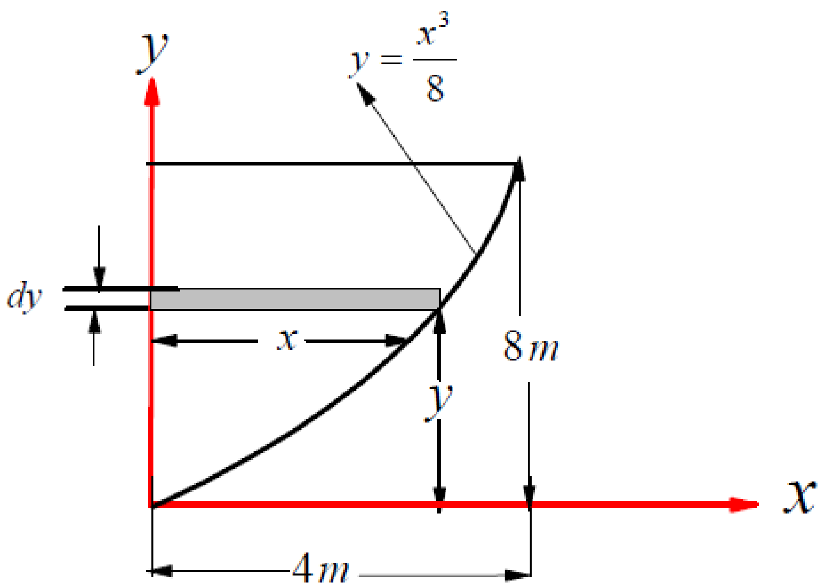
Example 4: Determine the Area moment of inertia of the shaded area with respect to $ox, oy-$axes?



Solution

The Area Moment of inertia with respect to x- axis

We consider a uniform strip line parallels to the $x-$axis with the length (x) and thickness (dy) as is shown in below Figure. Then $dA = x\,dy$



So, the area moment of inertia with respect to x -axis is given as

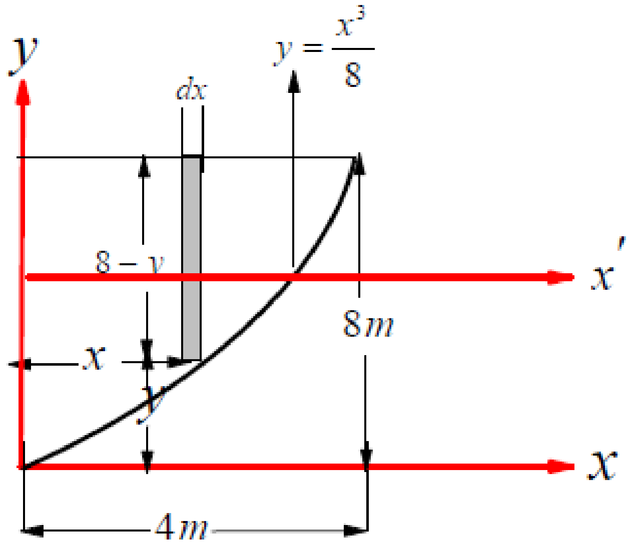
$$I_{xx}=\int y^2dA=\int_0^8 y^2x\,dy=\int_0^8 y^2(2)y^{\frac{1}{3}}\,dy=2\int_0^8 y^{\frac{7}{3}}\,dy=2\frac{3(8)^{\frac{10}{3}}}{10}=\frac{3}{5}(1024)=614.4\,m^4$$

From the above Figure, the area moment of inertia with respect to y -axis is given by

$$I_{yy}=\int \frac{1}{3}x^3dy=\int_0^8\frac{8}{3}y\,dy=\frac{8}{3}\frac{1}{2}\left(y^2\right)_0^8=\frac{8}{3}\frac{1}{2}(64)=\frac{256}{3}=85.3334m^4$$

The Area Moment of inertia with respect to y- axis

While if we consider a uniform strip line parallels to the $y-$axis with the length $(8-y)$ and thickness (dx) as is shown in below Figure. Then $dA=(8-y)\,dx$.



So, the area moment of inertia with respect to y -axis is given as

$$I_{yy}=\int x^2dA=\int_0^4 x^2(8-y)dx=\int_0^4 x^2\left(8-\frac{x^3}{8}\right)dx=\int_0^4\left(8x^2-\frac{x^5}{8}\right)dx=$$

$$I_{yy}=\left(\frac{8}{3}x^3-\frac{1}{48}y^6\right)_0^4=\frac{8}{3}(4)^3-\frac{1}{48}(4)^6=(4)^4\left(\frac{2}{3}-\frac{16}{48}\right)=(4)^4\left(\frac{2}{3}-\frac{1}{3}\right)=\frac{256}{3}=85.3334m^4$$

The Area Moment of inertia with respect to x- axis

If we consider the previous Figure (second Figure) we can find the Area moment of inertia with respect to $x-$ axis (from the parallel axis theorem) as $I_{xx}=I_{x'x'}+dm\,(\bar{y})^2$

Where $I_{x'x'} = \frac{1}{12}(8-y)^3 dx = \frac{1}{12}(512-192y+24y^2-y^3)\,dx$

$\bar{y} = \frac{1}{2}(8-y)+y = \frac{1}{2}(8+y) \rightarrow \bar{y}^2 = \left[\frac{1}{2}(8+y)\right]^2 = \frac{1}{4}(64+16y+y^2)$,

$dm = (8-y)\,dx$ Then $I_{xx} = I_{x'x'} + dm\,(\bar{y})^2$ becomes

$$I_{xx} = \frac{1}{12}(512-192y+24y^2-y^3)\,dx + (8-y)\,dx\left(\frac{1}{4}(64+16y+y^2)\right)$$

$$= \left\{\frac{1}{12}(512-192y+24y^2-y^3)+128+32y+2y^2-\frac{1}{4}(64y+16y^2+y^3)\right\}dx$$

$$= \left\{\frac{128}{3} - \underset{2+6+8=0}{16y} + \underset{3+7+9=0}{2y^2} - \frac{1}{12}y^3 + 128 + \underset{2+6+8=0}{32y} + \underset{3+7+9=0}{2y^2} - \underset{2+6+8=0}{16y} - \underset{3+7+9=0}{4y^2} - \frac{1}{4}y^3\right\}dx$$

$$= \left\{\frac{512}{3} - \frac{1}{3}y^3\right\}dx = \frac{1}{3}\left\{512 - y^3\right\}dx = \frac{1}{3}\left\{512-(\frac{x}{8})^3\right\}dx$$

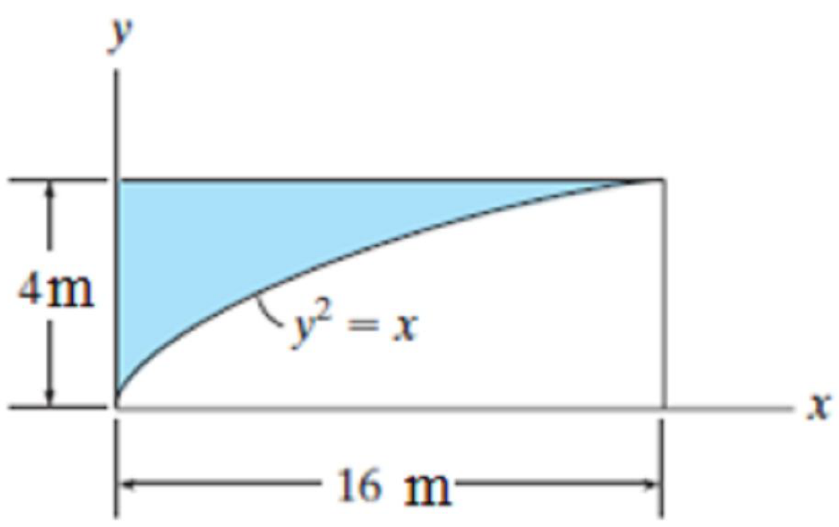
For all the Area, we have

$$I_{xx} = \int \frac{1}{3}\left\{512-(\frac{x}{8})^3\right\}dx = \frac{1}{3}\int_0^4\left\{512-(\frac{1}{8})^3x^3\right\}dx = \frac{1}{3}\left[512x - \frac{1}{4}(\frac{1}{8})^3x^4\right]_0^4$$

$$= \frac{1}{3}\left[512(4)-\frac{1}{4}(\frac{1}{8})^3(4)^4\right] = \frac{1}{3}\left[512(4)-(\frac{1}{8})^3(4)^3\right] = \frac{1}{3}\left[512(4)-\frac{(4)(4)(4)}{(8)(8)(8)}\right]$$

$$= \frac{1}{3}\left[512(4)-\frac{1}{8}\right] = \frac{2}{7}m^4$$

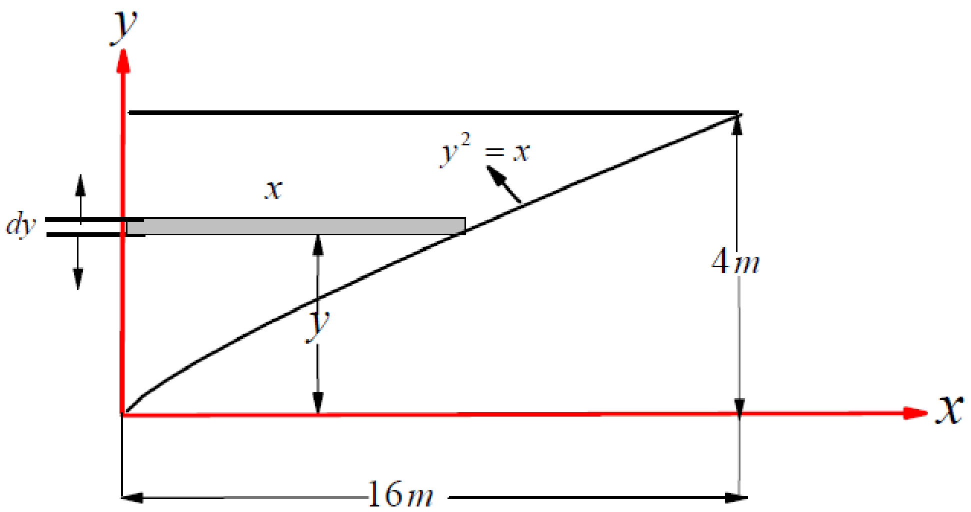
Example 5: Determine Area the moment of inertia of the shaded area with respect to ox, oy – axes?



Solution

The Area Moment of inertia with respect to x- axis

We consider a uniform strip line parallels to the $x-$axis with the length (x) and thickness (dy) as is shown in below Figure. Then $dA = x\,dy$

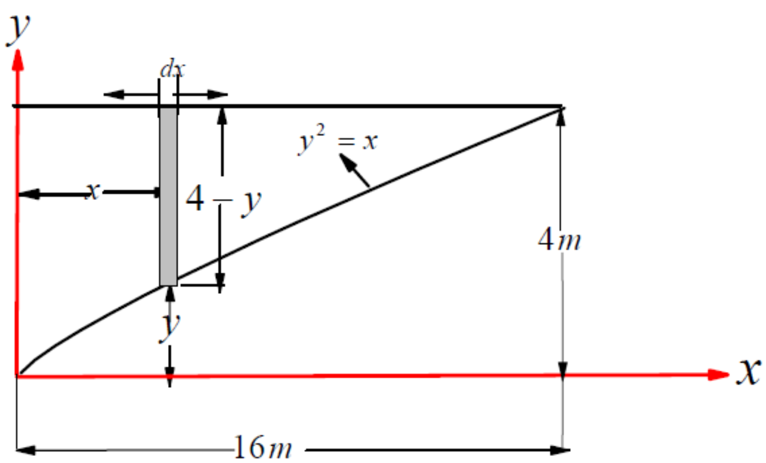


So, the area moment of inertia with respect to x-axis is given as

$$I_{xx} = \int y^2 dA = \int_0^4 y^2 x\,dy = \int_0^4 y^2 (y)^2\,dy = \int_0^4 y^4\,dy = \frac{(4)^5}{5} = \frac{1024}{5} = 204.8\,m^4$$

The Area Moment of inertia with respect to y- axis

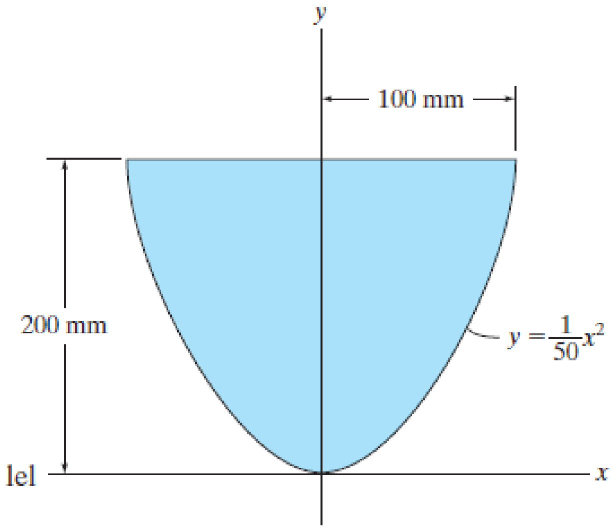
While if we consider a uniform strip line parallels to the $y-$axis with the length $(4-y)$ and thickness (dx) as is shown in below Figure. Then $dA = (4-y)\,dx$



$$I_{yy} = \int x^2 dA = \int_0^{16} x^2(4-y)dx = \int_0^{16} x^2\left(4-(x)^{\frac{1}{2}}\right)dx = \int_0^{16}\left(4x^2-(x)^{\frac{5}{2}}\right)dx = \frac{4(16)^3}{3} - \frac{2(16)^{\frac{7}{2}}}{7}$$

$$I_{yy} = 5461.333 - 4861.1428 = 780.2\,m^4$$

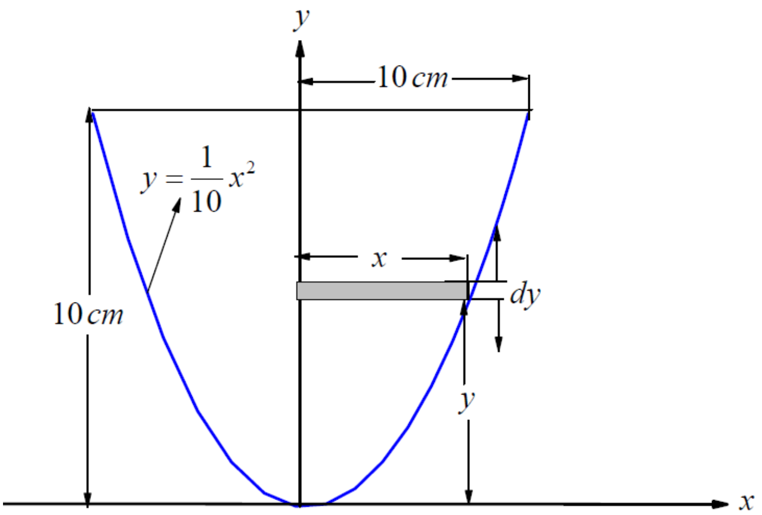
Example 6: Determine the Area moment of inertia of the shaded area with respect to $ox,\ oy-$ axes?



Solution

<u>The Area Moment of inertia with respect to x- axis</u>

We consider a uniform strip line parallels to the $x-$ axis with the length (x) and thickness (dy) as is shown in below Figure. Then $dA = x\,dy$



So, the area moment of inertia with respect to x-axis is given a

$$I_{xx}=2\int y^2 dA=2\int_0^{10} y^2 x\,dy=2\int_0^{10} y^2(10y)^{\frac{1}{2}}\,dy=2(10)^{\frac{1}{2}}\int_0^{10} y^{\frac{5}{2}}\,dy=2(10)^{\frac{1}{2}}\left(\frac{2}{7}y^{\frac{7}{2}}\right)_0^{10}$$

$$I_{xx}=2(10)^{\frac{1}{2}}\left(\frac{2}{7}(10)^{\frac{7}{2}}\right)=\frac{4}{7}10^4\,m^4$$

The Area Moment of inertia with respect to y- axis

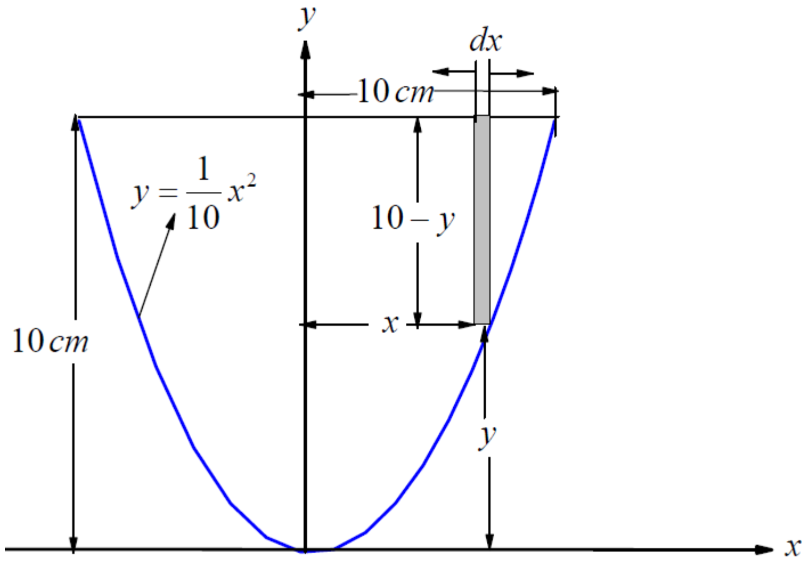
If we consider the previous Figure (first Figure) we can find the Area moment of inertia with respect to $y-$ axis as $I_{yy}=\int\frac{1}{3}x^3dy$

$$I_{yy}=2\int\frac{1}{3}x^3dy=2\int_0^{10}\frac{1}{3}(10y)^{\frac{3}{2}}\,dy=\frac{2(10)^{\frac{3}{2}}}{3}\int_0^{10}(y)^{\frac{3}{2}}\,dy=\frac{2(10)^{\frac{3}{2}}}{3}\frac{2(10)^{\frac{5}{2}}}{5}$$

$$I_{yy}=\frac{4}{15}(10)^{\frac{3}{2}+\frac{5}{2}}=\frac{4}{15}(10)^4=\frac{4}{15}10^4\,m^4$$

Again the Area Moment of inertia with respect to y- axis

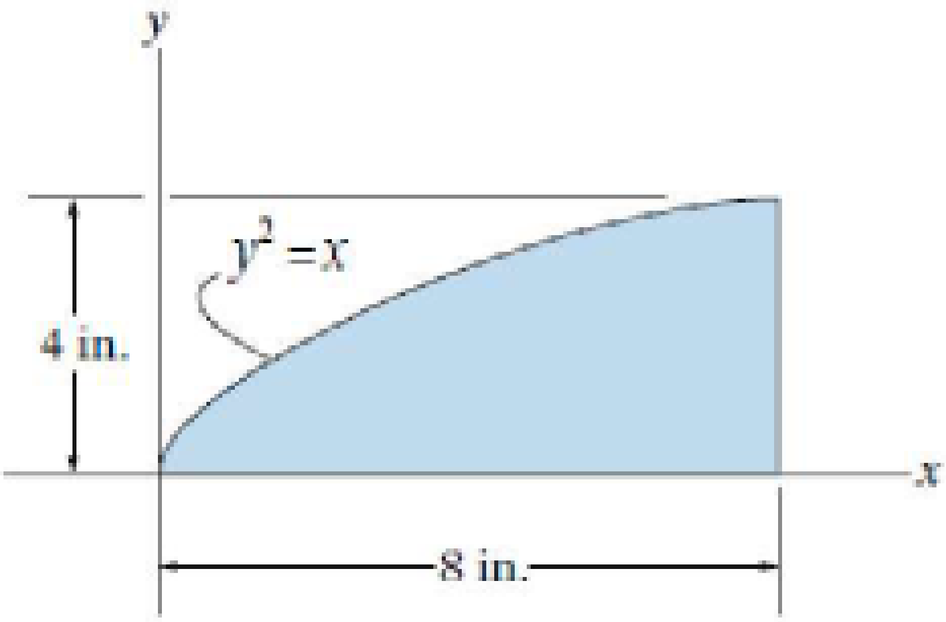
While if we consider a uniform strip line parallels to the $y-$axis with the length $(10-y)$ and thickness (dx) as is shown in below Figure. Then $dA=(10-y)\,dx$



$$I_{yy}=2\int x^2dA=2\int_0^{10}x^2(10-y)dx=2\int_0^{10}x^2\left(10-\frac{1}{10}x^2\right)dx=2\int_0^{10}\left(10x^2-\frac{1}{10}x^4\right)dx$$

$$=2\left(\frac{10}{3}x^3-\frac{1}{50}x^5\right)_0^{10}=2\left(\frac{10^4}{3}-\frac{10^4}{50}\right)=\frac{2}{150}(50-30)10^4=\frac{40}{150}10^4=\frac{4}{15}10^4$$

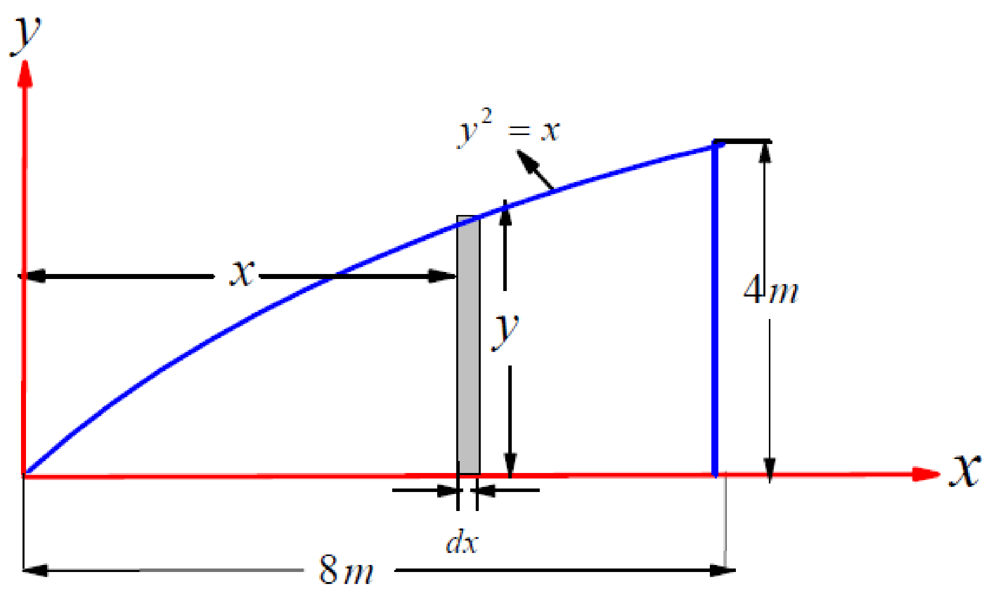
Example 7: Determine the Area moment of inertia of the shaded area with respect to $ox,\ oy-$axes?



Solution

The Area Moment of inertia with respect to x- axis

We consider a uniform strip line parallels to the $y-$axis with the length (y) and thickness (dy) as is shown in below Figure. Then $dA = y\,dx$



So, the area moment of inertia with respect to x -axis is given as

$$I_{xx} = \int \frac{1}{3} y^3 dx = \frac{1}{3}\int_0^4 y^3 (2y\,dy) = \frac{2}{3}\int_0^4 y^4 dy = \frac{2}{15}\left(y^5\right)_0^4 = \frac{2}{15}(4)^5 = \frac{2048}{15} m^4 = 136.533 m^4$$

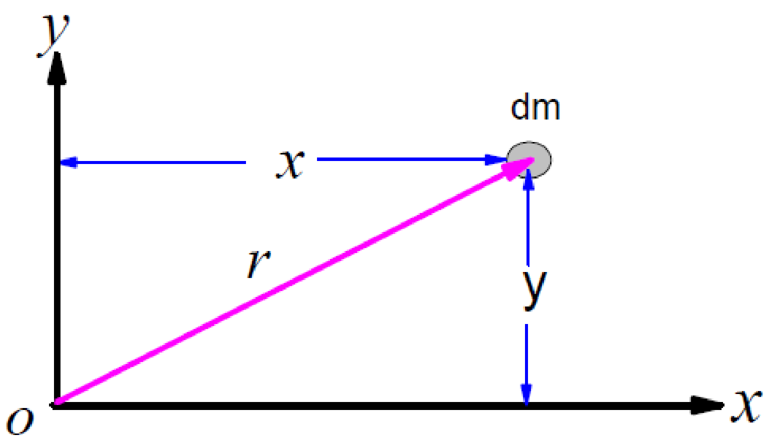
<u>The Area Moment of inertia with respect to y- axis</u>

$$I_{yy} = \int x^2 dA = \int_0^8 x^2 y\ dx = \int_0^8 x^2 (x)^{\frac{1}{2}}\ dx = \int_0^8 x^{\frac{5}{2}}\ dx = \frac{2}{7}\left[x^{\frac{7}{2}} \right]_0^8 = \frac{2048\sqrt{2}}{7} m^4$$

III. Products of Inertia of mass

Products of Inertia of mass

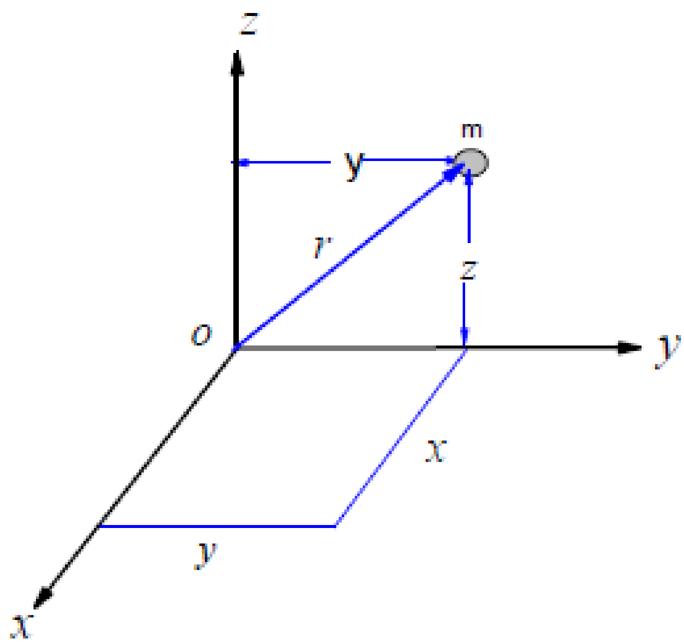
(1) If the body is located in a plane as shown below figure and has mass (dm). Then the product of inertia with respect to the axes ox, oy is given by



$$I_{xy} = x\,y\,dm \tag{1}$$

Note that $I_{xy} = x\,y\,dm = I_{yx} = y\,x\,dm$

(2) For the body in space



With respect to the axes ox, oy

$$I_{xy} = xy\,dm \qquad (2)$$

With respect to the axes ox, oz

$$I_{xz} = xz\,dm \qquad (3)$$

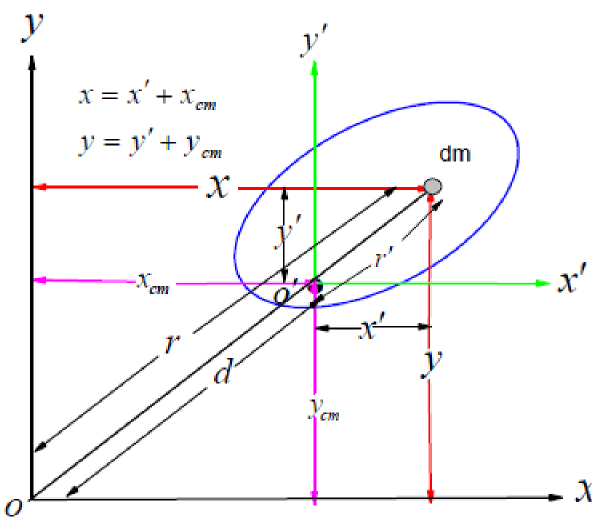
While, With respect to the axes oy, oz

$$I_{yz} = yz\,dm \qquad (4)$$

Product of inertia can be positive or negative value as oppose the moment of inertia. The calculation of the product of inertia isn't different much for the calculation of the moment of inertia. The units of the product of inertia are the same as for moment of inertia.

Parallel-axis theorem for products of inertia

For any rigid body has mass (m) and the center of mass (x_{cm}, y_{cm}) as shown below figure



Dividing the body into a number of small elements. Taking a small element whose mass (dm) and its coordinate with respect to the original axes is (x, y). With respect to axes parallel to the original axes and passing through the center of mass the element has the coordinate (x', y').

For the original axes (x, y), the inertial product of mass (dm) is given by

$$I_{xy} = xy\,dm \qquad (1)$$

For the total mass (m)

$$I_{xy} = \int x\, y\, dm \tag{2}$$

From the above Figure $x = x' + x_{cm}, \quad y = y' + y_{cm}$ and into Eq. (2), we have

$$I_{xy} = \int x\, y dm = \int \left\{ \left(x' + x_{cm} \right) \left(y' + y_{cm} \right) \right\} dm = \int \left\{ x'y' + x'y_{cm} + x_{cm} y' + x_{cm}\, y_{cm} \right\} dm$$

$$I_{xy} = \int x'y'dm + y_{cm} \int x'dm + x_{cm} \int y'dm + x_{cm}\, y_{cm} \int dm \tag{3}$$

But, it is well-known that

$$\int x'y'dm = I_{x'y'}, \quad x_{cm}\, y_{cm} \int dm = x_{cm}\, y_{cm}\, m,$$
$$\bar{x} = \frac{\int x'\, dm}{\int dm} \to \int x'\, dm = \bar{x} \int dm, \quad \bar{y} = \frac{\int y'\, dm}{\int dm} \to \int y'\, dm = \bar{y} \int dm \tag{4}$$

From Eq. (4) into Eq. (3), we have

$$I_{xy} = I_{x'y'} + y_{cm}\, \bar{x} \int dm + x_{cm}\, \bar{y} \int dm + x_{cm}\, y_{cm} m$$

$$I_{xy} = I_{x'y'} + y_{cm}\, \bar{x}\, m + x_{cm}\, \bar{y}\, m + x_{cm}\, y_{cm} m \tag{5}$$

But the coordinate $(\bar{x}, \bar{y})$ is the center of mass from the center of mass and it is equal to zero. Substituting in (5) we get

$$I_{xy} = I_{x'y'} + m\, x_{cm}\, y_{cm} \tag{6}$$

Where I_{xy} is the product of inertia with respect to the two axes $ox, \ oy$, while $I_{x'y'}$ is the product of inertia with respect to the two axes $o'x'$, $o'y'$ and x_{cm}, y_{cm} are the distance of the center of gravity from the two axes $ox, \ oy$, respectively.

Notes

(1)-The product of inertia is a product of different coordinates, so it can be positive or negative quantity

(2)- For the product of inertia is $I_{xy} = I_{yx}, \; I_{yz} = I_{zy}, \; I_{zx} = I_{xz}$

(3)- If the Products of Inertia are zero with respect to any two planes. It is said that the axis of intersection of these two axes is a principal axis.

(4) If the inertia product is neglected with respect to any two principal coordinates ox, oy, it is said that the tow axis ox, oy are principal axes

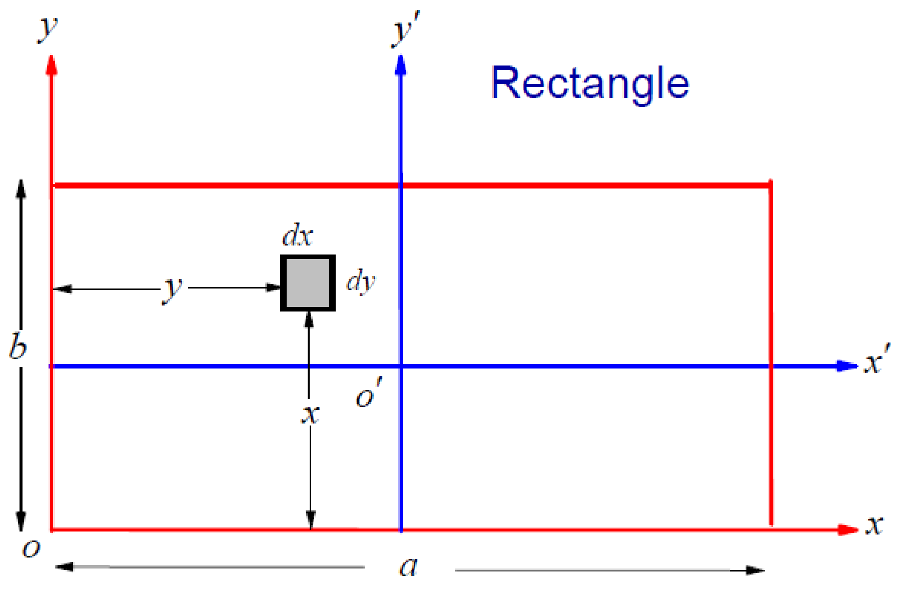
(5)- Any symmetry axis in a flat plate with any perpendicular axis , then these axes are called the principal axes

(6) - The product of inertia is finished for the two axes are perpendicular, if each other and one or both axes of symmetry.

Example 1: Find the Product of Inertia of a thin uniform rectangular plate?

Solution

We divide the plate to small uniform strip, we consider one of them with the length (dx) and thickness (dy) as in Figure, where the density is ρ .



$$dm = \rho\, dx dy \rightarrow m = \rho \int_0^b \int_0^a dx dy = \rho a \int_0^b dy = ab\rho \rightarrow \; m = a b \rho$$

With respect to $ox,\; oy$, we have $dI_{xy} = (dm) x y$.

For the total plate, we have

$$I_{xy} = \int \int x y \left(\rho\, dx dy\right) \rightarrow I_{xy} = \rho \int_0^b \int_0^a x y\, dx dy = \frac{a^2}{2}\frac{b^2}{2}\rho = \frac{a^2 b^2}{4}\rho\frac{m}{m} = \frac{a^2 b^2}{4}\rho\frac{m}{a b \rho} = \frac{1}{4} mab$$

$$I_{xy} = \frac{1}{4} m a b$$

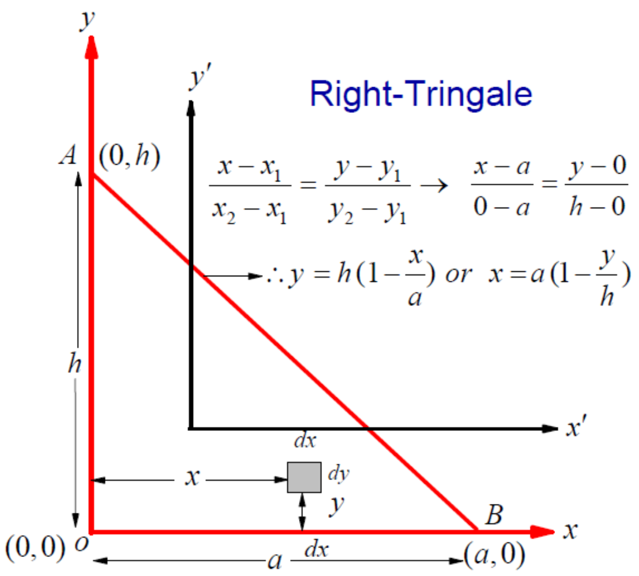
From the theory of parallel axes for the product of inertia, the product of inertia with respect to ox', oy' is given $I_{xy} = I_{x'y'} + m x_{cm} y_{cm} \rightarrow \frac{1}{4} mab = I_{x'y'} + m(\frac{1}{2}a)(\frac{1}{2}b) \rightarrow = I_{x'y'} = \frac{1}{4} mab - \frac{1}{4} m ab = 0$

$I_{x'y'} = 0$. So, the axes ox', oy' are symmetric axes.

Example 2: Find the Product of Inertia of a thin uniform triangular plate?

Solution

$$dm = \rho\, dx dy \rightarrow m = \rho \int_0^h \int_0^{a(1-\frac{y}{h})} dx dy = \rho \int_0^h a(1 - \frac{y}{h}) dy = a\rho \left[y - \frac{y^2}{2h} \right]_0^h = a\rho \left[h - \frac{h^2}{2h} \right] = \frac{1}{2} a h \rho$$



With respect to $ox,\ oy$, we have $dI_{xy} = (dm)\,x\,y$. For the total plate, we have

$$I_{xy} = \int\int x\,y\,(\rho\,dx\,dy) \rightarrow I_{xy} = \rho\int_0^h\int_0^{a(1-\frac{y}{h})} x\,y\,dx\,dy = \frac{a^2}{2}\rho\int_0^h (1-\frac{y}{h})^2\,y\,dy$$

$$= \frac{a^2}{2h^2}\rho\int_0^h (h-y)^2\,y\,dy = \frac{a^2}{2h^2}\rho\int_0^h (h^2-2yh+y^2)\,y\,dy = \frac{a^2}{2h^2}\rho\int_0^h (yh^2-2y^2h+y^3)\,dy$$

$$= \frac{a^2}{2h^2}\rho\left[\frac{y^2h^2}{2} - \frac{2y^3}{3}h + \frac{y^4}{4}\right]_0^h = \frac{a^2}{2h^2}\rho\left[\frac{h^4}{2} - \frac{2h^4}{3} + \frac{h^4}{4}\right] = \frac{a^2h^4}{24h^2}\rho[6-8+3] = \frac{a^2h^4}{24h^2}\rho$$

$$= \frac{a^2h^2}{24}\rho\frac{m}{m} = \frac{a^2h^2}{24}\rho\frac{m}{\frac{1}{2}\rho ah} = \frac{1}{12}mah$$

$$I_{xy} = \frac{1}{12}mah$$

From the theory of parallel axes for the product of inertia, the product of inertia with respect to ox', oy' is given

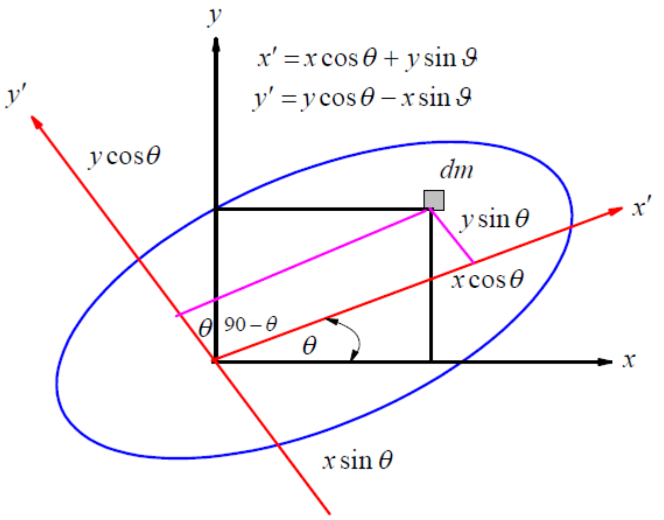
$$I_{xy} = I_{x'y'} + m\,x_{cm}\,y_{cm} \rightarrow \frac{1}{12}mah = I_{x'y'} + m(\frac{1}{3}a)(\frac{1}{3}h) \rightarrow\ = I_{x'y'} = \frac{1}{12}mah - \frac{1}{9}m\,ah = \frac{1}{72}(6-8)$$

$$I_{x'y'} = -\frac{1}{36}mah$$

$$I_{x''y''} = I_{x'y'} + m\,x_{cm}\,y_{cm} \rightarrow I_{x''y''} = -\frac{1}{36}mah + m(\frac{2}{3}a)(\frac{-1}{3}h) \rightarrow\ = I_{x'y'} = -\frac{1}{36}mah - \frac{2}{9}m\,ah = \frac{1}{36}(-1-8)m\,ah$$

$$I_{x''y''} = \frac{-1}{4}m\,ah$$

IV. Moments of Inertia about inclined axis



For ox' $dI_{x'x'} = (dm)\, y'^2$

$$I_{x'x'} = \int y'^2\, dm \qquad (1)$$

$$y' = y\cos\theta - x\sin\theta$$

$$I_{x'x'} = \int (y\cos\theta - x\sin\theta)^2\, dm = \int y^2\cos^2\theta\, dm + \int x^2\sin^2\theta\, dm - 2\int xy\cos\theta\sin\theta\, dm$$

$$I_{x'x'} = \cos^2\theta \int y^2\, dm + \sin^2\theta \int x^2\, dm - 2\cos\theta\sin\theta \int xy\, dm$$

$$I_{x'x'} = I_{xx}\cos^2\theta + I_{yy}\sin^2\theta - I_{xy}\ \sin 2\theta \qquad (2)$$

For oy'

$$I_{y'y'} = \int x'^2\, dm \qquad (3)$$

$$x' = x\cos\theta + y\sin\theta$$

$$I_{y'y'} = \int (x\cos\theta + y\sin\theta)^2\, dm = \int x^2\cos^2\theta\, dm + \int y^2\sin^2\theta\, dm + 2\int xy\cos\theta\sin\theta\, dm$$

$$I_{y'y'} = \cos^2\theta \int x^2\, dm + \sin^2\theta \int y^2\, dm + 2\cos\theta\sin\theta \int xy\, dm$$

$$I_{y'y'} = I_{yy}\cos^2\theta + I_{xx}\sin^2\theta + I_{xy}\sin 2\theta \qquad (4)$$

For $I_{x'y'}$

$$I_{x'y'} = \int x'y'\,dm \qquad (5)$$

$$x' = x\cos\theta + y\sin\theta, \quad y' = y\cos\theta - x\sin\theta$$

$$I_{x'y'} = \int (x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta)dm$$

$$I_{x'y'} = \int x\,y\cos^2\theta\,dm - \int x^2\sin\theta\cos\theta\,dm + \int y^2\cos\theta\,\sin\theta\,dm - \int x\,y\sin^2\theta\,dm$$

$$I_{x'y'} = \cos^2\theta\int x\,y\,dm - \sin\theta\cos\theta\int x^2\,dm + \sin\theta\cos\theta\int y^2 dm - \sin^2\theta\int x\,y\,dm$$

$$I_{x'y'} = \cos^2\theta\,I_{xy} - \cos\theta\sin\theta\,I_{yy} + \cos\theta\sin\theta\,I_{xx} - \sin^2\theta\,I_{xy}$$

$$I_{x'y'} = (\cos^2\theta - \sin^2\theta)I_{xy} + \sin\theta\,\cos\theta(I_{xx} - I_{yy})$$

$$I_{x'y'} = \frac{I_{xx} - I_{yy}}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

$$I_{x'y'} = \frac{I_{xx} - I_{yy}}{2}\sin 2\theta + I_{xy}\cos 2\theta \qquad (6)$$

From Eq. (6), the maximum angle happens at $I_{x'y'} = 0$

$$\tan 2\theta = \frac{2I_{xy}}{I_{yy} - I_{xx}} \qquad (7)$$

Eq. (2)……. $I_{x'x'} = I_{xx}\cos^2\theta + I_{yy}\sin^2\theta - \sin 2\theta\,I_{xy}$

$$I_{x'x'} = I_{xx}\frac{1+\cos 2\theta}{2} + I_{yy}\frac{1-\cos 2\theta}{2} - \sin 2\theta\,I_{xy}$$

$$I_{x'x'} = \frac{I_{xx} + I_{yy}}{2} + \frac{I_{xx} - I_{yy}}{2}\cos 2\theta - I_{xy}\sin 2\theta \qquad (8)$$

Eq. (4)……. $I_{y'y'} = I_{yy}\cos^2\theta + I_{xx}\sin^2\theta + I_{xy}\sin 2\theta$

$$I_{y'y'} = I_{yy}\frac{1+\cos 2\theta}{2} + I_{xx}\frac{1-\cos 2\theta}{2} + I_{xy}\sin 2\theta$$

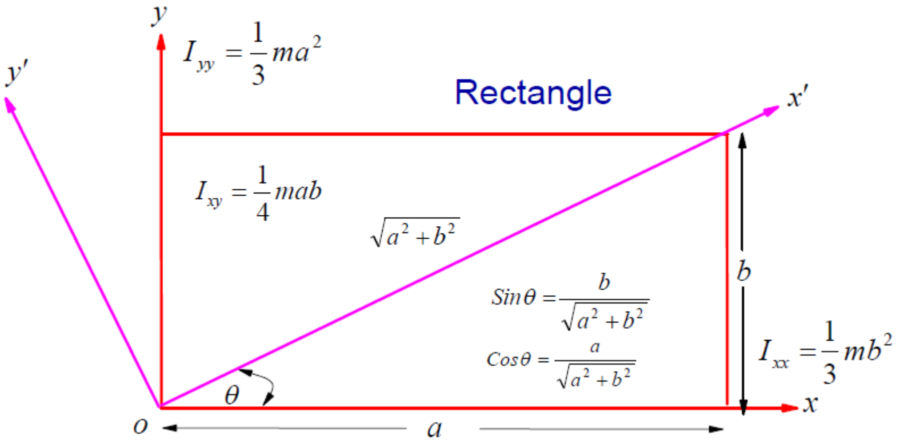
$$I_{y'y'} = \frac{I_{yy} + I_{xx}}{2} + \frac{I_{yy} - I_{xx}}{2}\cos 2\theta + I_{xy}\sin 2\theta \qquad (9)$$

Add 1+2 and 8+9, we have

$$I_{x'x'} + I_{y'y'} = I_{xx} + I_{yy} \qquad (10)$$

Example 3: Find the moment of inertia with respect to a diagonal of the rectangular plate?

Solution



It is well-known

$$I_{x'x'}=I_{xx}\cos^2\theta+I_{yy}\sin^2\theta-I_{xy}\sin 2\theta \tag{1}$$

$$I_{y'y'}=I_{yy}\cos^2\theta+I_{xx}\sin^2\theta+I_{xy}\sin 2\theta \tag{2}$$

$$I_{x'y'}=\frac{I_{xx}-I_{yy}}{2}\sin 2\theta+I_{xy}\cos 2\theta \tag{3}$$

Where

$$I_{xx}=\frac{1}{3}mb^2,\quad I_{yy}=\frac{1}{3}ma^2,\quad I_{xy}=\frac{1}{4}mab \tag{4}$$

Then From Eq. 1-4, we have

$$I_{x'x'}=I_{xx}\cos^2\theta+I_{yy}\sin^2\theta-I_{xy}\sin 2\theta \quad I_{x'x'}==\frac{1}{3}mb^2\cos^2\theta+\frac{1}{3}ma^2\sin^2\theta-\frac{1}{4}mab\,(2\sin\theta\cos\theta)$$

$$I_{x'x'}=\frac{1}{3}mb^2\left(\frac{a}{\sqrt{a^2+b^2}}\right)^2+\frac{1}{3}ma^2\left(\frac{b}{\sqrt{a^2+b^2}}\right)^2-\frac{2}{4}mab\left(\frac{a}{\sqrt{a^2+b^2}}\right)\left(\frac{b}{\sqrt{a^2+b^2}}\right)$$

$$I_{x'x'}=\frac{1}{3}m\frac{a^2b^2}{a^2+b^2}+\frac{1}{3}m\frac{a^2b^2}{a^2+b^2}-\frac{1}{2}m\frac{a^2b^2}{a^2+b^2}=m\frac{a^2b^2}{a^2+b^2}\frac{1}{6}(2+2-3)=\frac{1}{6}m\frac{a^2b^2}{a^2+b^2}$$

$$I_{x'x'}=\frac{1}{6}m\left(\frac{a^2b^2}{a^2+b^2}\right) \tag{5}$$

$$I_{y'y'}=I_{yy}\cos^2\theta+I_{xx}\sin^2\theta+I_{xy}\sin 2\theta \qquad I_{y'y'}==\frac{1}{3}ma^2\cos^2\theta+\frac{1}{3}mb^2\sin^2\theta+\frac{1}{4}mab\,(2\sin\theta\cos\theta)$$

$$I_{y'y'}=\frac{1}{3}ma^2\left(\frac{a}{\sqrt{a^2+b^2}}\right)^2+\frac{1}{3}mb^2\left(\frac{b}{\sqrt{a^2+b^2}}\right)^2+\frac{2}{4}mab\left(\frac{a}{\sqrt{a^2+b^2}}\right)\left(\frac{b}{\sqrt{a^2+b^2}}\right)$$

$$I_{y'y'}=\frac{1}{3}m\frac{a^4}{a^2+b^2}+\frac{1}{3}m\frac{b^4}{a^2+b^2}+\frac{1}{2}m\frac{a^2b^2}{a^2+b^2}=\frac{1}{3}m\frac{a^4+b^4}{a^2+b^2}+\frac{1}{2}m\frac{a^2b^2}{a^2+b^2}$$

$$I_{y'y'}=\frac{1}{6(a^2+b^2)}m\left(2a^4+3a^2b^2+2b^4\right) \qquad (6)$$

$$I_{x'y'}=\frac{I_{xx}-I_{yy}}{2}\sin 2\theta+I_{xy}\cos 2\theta$$

$$I_{x'y'}=\left(I_{xx}-I_{yy}\right)\sin\theta\cos\theta+I_{xy}\left(\cos^2\theta-\sin^2\theta\right)$$

$$I_{x'y'}=\left(\frac{1}{3}mb^2-\frac{1}{3}ma^2\right)\left(\frac{a}{\sqrt{a^2+b^2}}\right)\left(\frac{b}{\sqrt{a^2+b^2}}\right)+\frac{1}{4}mab\left\{\left(\frac{a}{\sqrt{a^2+b^2}}\right)^2-\left(\frac{b}{\sqrt{a^2+b^2}}\right)^2\right\}$$

$$I_{x'y'}=\frac{1}{3}m\left(b^2-a^2\right)\left(\frac{ab}{a^2+b^2}\right)+\frac{1}{4}mab\left\{\frac{a^2}{a^2+b^2}-\frac{b^2}{a^2+b^2}\right\}=\frac{1}{12}m\left(b^2-a^2\right)\left(\frac{ab}{a^2+b^2}\right)(4-3)$$

$$I_{x'y'}=\frac{1}{12}m\left(b^2-a^2\right)\left(\frac{ab}{a^2+b^2}\right) \qquad (7)$$

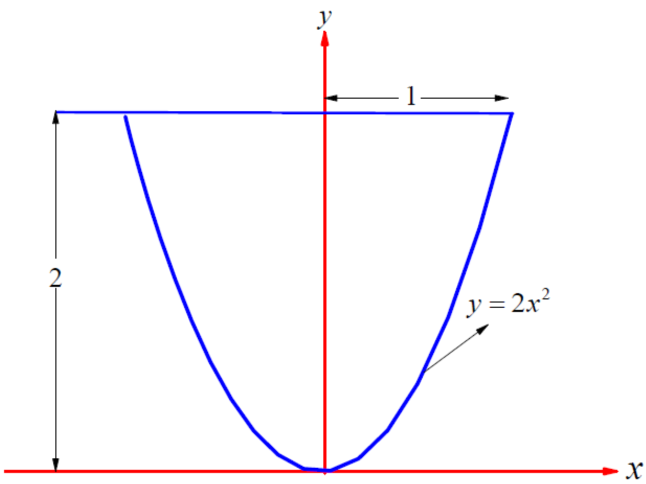
Note that at $\theta=45^O$, we have $a=b$, Then

$$I_{x'x'}=\frac{1}{6}m\left(\frac{a^2b^2}{a^2+b^2}\right)=\frac{1}{12}m,$$

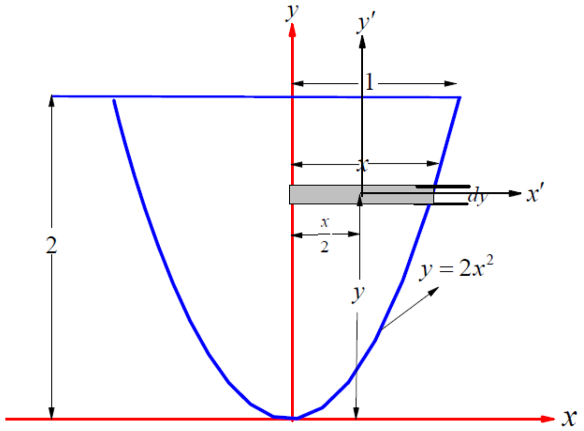
$$I_{y'y'}=\frac{1}{6(a^2+b^2)}m\left(2a^4+3a^2b^2+2b^4\right)=\frac{1}{6(2)}(7)m=\frac{7}{12}m, \qquad (8)$$

$$I_{x'y'}=\frac{1}{12}m\left(b^2-a^2\right)\left(\frac{ab}{a^2+b^2}\right)=0$$

Example 4: Determine the product of inertia I_{xy} of the right half of the parabolic area, bounded by the $y=2m$ and $x=0$?



Solution



$$dm = \rho\, x\, dy \rightarrow m = \rho \int_0^2 \left(\frac{y}{2}\right)^{\frac{1}{2}} dy = \frac{\rho}{\frac{3}{2}\sqrt{2}} [y]^{\frac{3}{2}} \Big|_0^2 = \frac{2\rho}{3\sqrt{2}} [2]^{\frac{3}{2}} = \frac{2\rho}{3\sqrt{2}} [8]^{\frac{1}{2}} = \frac{4\rho}{3\sqrt{2}} [2]^{\frac{1}{2}} \rightarrow \; m = \frac{4}{3}\rho$$

$$dI_{xy} = dI_{x'y'} + dm\, x_{cm}\, y_{cm}$$

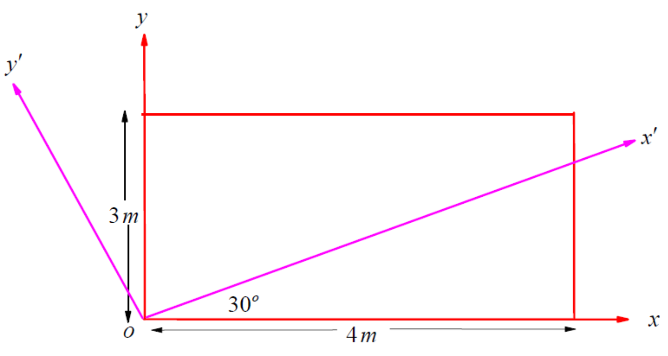
$$I_{xy} = \int dm\, x_{cm}\, y_{cm} = \int (\rho\, x\, dy)(\frac{1}{2}x)\, y = \frac{1}{2}\rho \int_0^2 x^2\, y\, dy = \frac{1}{2}\rho \int_0^2 \frac{y}{2}\, y\, dy = \frac{1}{4}\rho \int_0^2 y^2\, dy$$

$$I_{xy} = \frac{1}{2}m$$

$$= \frac{1}{4}\rho \frac{y^3}{3}\Big|_0^2 = \frac{1}{4}\rho \frac{8}{3} = \frac{8}{12}\rho = \frac{2}{3}\rho = \frac{2}{3}\rho\, \frac{m}{m} = \frac{2}{3}\rho\, \frac{m}{\frac{4}{3}\rho} = \frac{1}{2}m$$

Exercise

Find the moment of inertia about $o\,x'$ and $o\,y'$ axis also the product of inertia for rectangular plate as is shown Figure (3×4)?



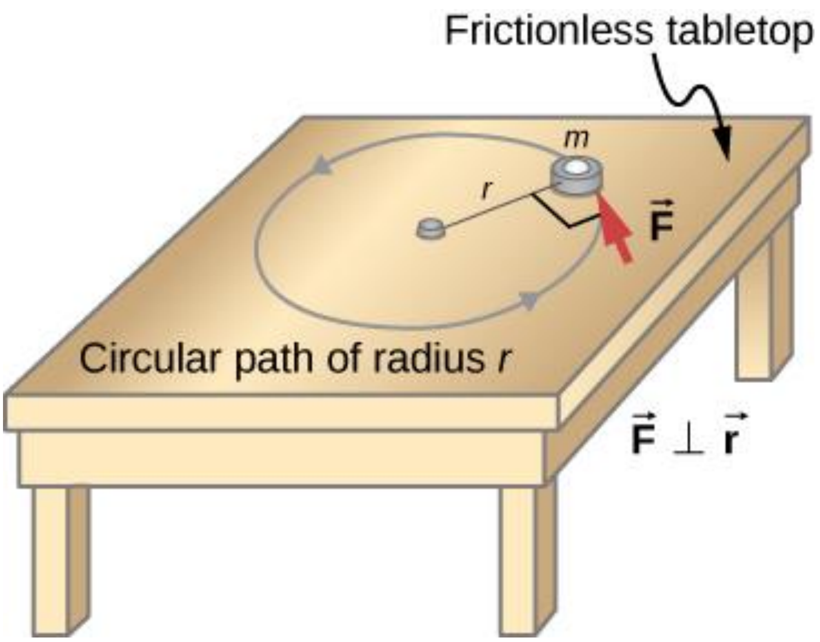
Chapter 3

Application

Newton's second law for rotation

We have thus far found many counterparts to the translational terms used throughout this text, most recently, torque, the rotational analog to force. This raises the question: Is there an analogous equation to Newton's second law $\sum \vec{F} = m\vec{a}$, which involves torque and rotational motion? To investigate this, we start with Newton's second law for a single particle rotating around an axis and executing circular motion. Let's exert a force $\vec{F}$ on a point mass m that is at a distance r from a pivot point (see below Figure). The particle is constrained to move in a circular path with fixed radius and the force is tangent to the circle. We apply Newton's second law to determine the magnitude of the acceleration $a = \frac{F}{m}$ in the direction of $\vec{F}$.

Recall that the magnitude of the tangential acceleration is proportional to the magnitude of the angular acceleration by $a = r\alpha$



Substituting this expression into Newton's second law, we obtain $F = m\,r\,\alpha$

Multiply both sides of this equation by r, we have $r\,F = m\,r^2\,\alpha$

Note that the left side of this equation is the torque about the axis of rotation, where r is the lever arm and F is the force, perpendicular to r. Recall that the moment of inertia for a point particle is $I = m\,r^2$. The torque applied perpendicularly to the point mass in above Figure is therefore $\tau = I\,\alpha$

The torque on the particle is equal to the moment of inertia about the rotation axis times the angular acceleration. We can generalize this equation to a rigid body rotating about a fixed axis.

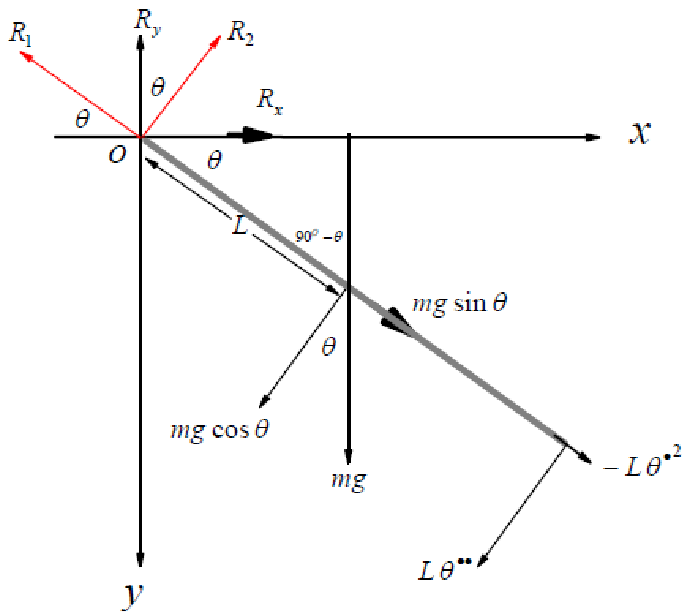
If more than one torque acts on a rigid body about a fixed axis, then the sum of the torques equals the moment of inertia times the angular acceleration:

$$\sum_i \tau_i = I\,\alpha$$

The term $I\,\alpha$ is a scalar quantity and can be positive or negative (counterclockwise or clockwise) depending upon the sign of the net torque. Remember the convention that counterclockwise angular acceleration is positive. Thus, if a rigid body is rotating clockwise and experiences a positive torque (counterclockwise), the angular acceleration is positive.

Example-1: A uniform rod of length $2L$ and mass M is pivoted (is hinged) at one end and the other one is free to rotate in the vertical plane. If the rod is beginning the rotation when it was horizontally. Prove that the horizontal reaction will be maximum when the Rod tilts on the horizontal at an angle ? $\frac{11}{8}mg$ and in this case the vertical reaction is given as $\frac{\pi}{4}$

Solution



The motion of center of Rod

$$m(-L\theta^{\bullet 2}) = mg\sin\theta - R_1 \;\rightarrow\; mL\theta^{\bullet 2} = R_1 - mg\sin\theta \qquad (1)$$

$$mL\theta^{\bullet\bullet} = mg\cos\theta - R_2 \qquad (2)$$

The rotation of motion (at then of Rod)

$$\frac{d}{dt}\left(I_O\,\theta^{\bullet}\right) = M_o \;\rightarrow\; I_O\,\theta^{\bullet\bullet} = M_o \qquad (3)$$

Eq. (3) maybe written as

$$\frac{1}{3}m(2L)^2\theta^{\bullet\bullet} = (mg\cos\theta)\;(L) \rightarrow \quad \theta^{\bullet\bullet} = \frac{3g}{4L}\cos\theta \qquad (4)$$

$$\theta^{\bullet}\frac{d\theta^{\bullet}}{d\theta} = \frac{3g}{4L}\cos\theta \;\rightarrow\; \int\theta^{\bullet}d\theta^{\bullet} = \frac{3g}{4L}\int\cos\theta\,d\theta \;\rightarrow\; \frac{\theta^{\bullet 2}}{2} = \frac{3g}{4L}\sin\theta + c_1$$

At the start point $\theta = 0$ and $\theta^{\bullet} = 0$, then $c_1 = 0$

$$\theta^{\bullet 2} = \frac{3g}{2L}\sin\theta \qquad (5)$$

From Eq, (5) into Eq. (1) $mL\left(\frac{3g}{2L}\sin\theta\right) = R_1 - mg\sin\theta \rightarrow R_1 = mL\left(\frac{3g}{2L}\sin\theta\right) + mg\sin\theta$

$$R_1 = \frac{5}{2}mg\,\sin\theta \qquad (6)$$

From Eq, (4) into Eq. (2)

$$mL\left(\frac{3g}{4L}\cos\theta\right)=mg\cos\theta-R_2 \quad\rightarrow\quad R_2=mg\cos\theta-\frac{3}{4}mg\cos\theta$$

$$R_2=\frac{1}{4}mg\cos\theta \qquad (7)$$

But

$$R_x=R_2\sin\theta-R_1\cos\theta \qquad (8)$$

$$R_y=R_1\sin\theta+R_2\cos\theta \qquad (9)$$

Then

$$R_x=(\frac{1}{4}mg\cos\theta)\sin\theta-(\frac{5}{2}mg\ \sin\theta)\ \cos\theta\rightarrow \quad R_x=-\frac{9}{4}mg\ \sin\theta\ \cos\theta$$

$$R_x=-\frac{9}{8}mg\ \sin 2\theta \qquad (10)$$

$$R_y=(\frac{5}{2}mg\ \sin\theta)\sin\theta+(\frac{1}{4}mg\cos\theta)\cos\theta \ \rightarrow\ R_y=\frac{5}{2}mg\ \sin^2\theta+\frac{1}{4}mg\cos^2\theta$$

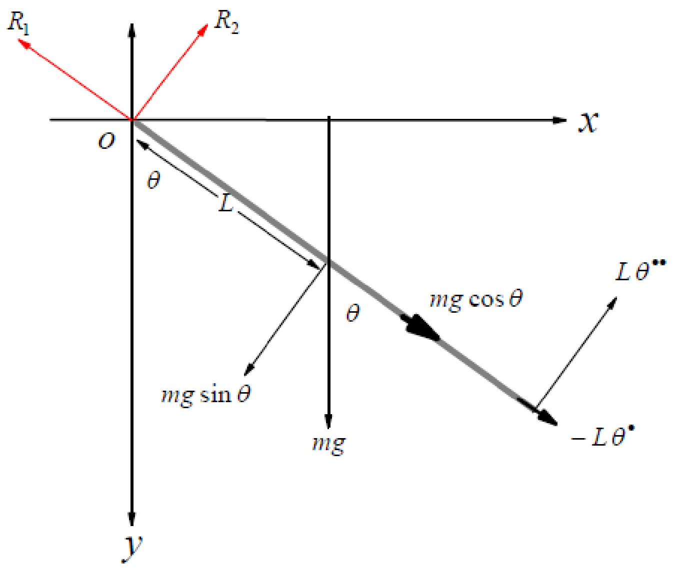
$$R_y=\left(\frac{5}{2}\sin^2\theta+\frac{1}{4}\cos^2\theta\right)mg \qquad (11)$$

From Eq. (9) $R_x=\frac{9}{8}mg\ \sin 2\theta$, R_x is maximum if $\sin 2\theta$ is maximum and $\sin 2\theta$ is maximum if $\sin 2\theta=1$, then $2\theta=\frac{\pi}{2}\rightarrow \quad \theta=\frac{\pi}{4}$

In this case $R_y(\theta=\frac{\pi}{4})=\left(\frac{5}{2}\left(\frac{1}{\sqrt{2}}\right)^2+\frac{1}{4}\left(\frac{1}{\sqrt{2}}\right)^2\right)mg \quad\rightarrow\quad R_y=\frac{11}{8}mg$

Example-2: A uniform rod of length L and mass M is pivoted (is hinged) at one end and the other one is free to rotate in the vertical plane. If the rod is beginning the rotation when it was vertically with angle velocity $\sqrt{\frac{3g}{L}}$. Find the reaction at the hinged point at $\theta=\frac{\pi}{3}$ and prove that the Rod move angle θ in time $t=2\sqrt{\frac{L}{3g}}\ln\left(\sec(\frac{\theta}{2})+\tan(\frac{\theta}{2})\right)$.

Solution



The motion of center of Rod

$$-mL\theta^{\bullet 2} = mg\cos\theta - R_1 \quad \rightarrow \quad mL\theta^{\bullet 2} = R_1 - mg\cos\theta \qquad (1)$$

$$mL\theta^{\bullet\bullet} = R_2 - mg\sin\theta \qquad (2)$$

The rotation of motion (At then of Rod)

$$\frac{d}{dt}\left(I_O\,\theta^{\bullet}\right) = M_o \quad \rightarrow \quad I_O\,\theta^{\bullet\bullet} = M_o \qquad (3)$$

Eq. (3) maybe written as

$$\frac{1}{3}m(2L)^2\theta^{\bullet\bullet} = (-mg\sin\theta)\ (L) \rightarrow \quad \theta^{\bullet\bullet} = -\frac{3g}{4L}\sin\theta \qquad (4)$$

$$\theta^{\bullet}\frac{d\theta^{\bullet}}{d\theta} = -\frac{3g}{4L}\sin\theta \quad \rightarrow \quad \int \theta^{\bullet} d\theta^{\bullet} = -\frac{3g}{4L}\int \sin\theta\ d\theta \quad \rightarrow \quad \frac{\theta^{\bullet 2}}{2} = \frac{3g}{4L}\cos\theta + c_1$$

At the start point $\theta = 0$ and $\theta^{\bullet} = \sqrt{\dfrac{3g}{L}}$, then $c_1 = \dfrac{3g}{2L} - \dfrac{3g}{4L} = \dfrac{3g}{4L}$

$$\frac{\dot{\theta}^{2}}{2}=\frac{3g}{4L}\cos\theta+\frac{3g}{4L} \quad \rightarrow \quad \dot{\theta}^{2}=\frac{3g}{2L}(1+\cos\theta) \qquad (5)$$

Note that

$$\cos(\theta)=\cos(\frac{\theta}{2}+\frac{\theta}{2})=\cos^2(\frac{\theta}{2})-\sin^2(\frac{\theta}{2})=\cos^2(\frac{\theta}{2})-(1-\cos^2(\frac{\theta}{2}))=2\cos^2(\frac{\theta}{2})-1$$

From Eq. (5), we have

$$\dot{\theta}=\frac{d\theta}{dt}=\sqrt{\frac{3g}{2L}\left(2\cos^2(\frac{\theta}{2})\right)} \rightarrow \frac{d\theta}{dt}=\sqrt{\frac{3g}{L}}\cos(\frac{\theta}{2}) \rightarrow \int\frac{1}{\cos(\frac{\theta}{2})}d\theta=\sqrt{\frac{3g}{L}}\int dt$$

$$2\int\sec(\frac{\theta}{2})\,d(\frac{\theta}{2})=\sqrt{\frac{3g}{L}}\int dt \rightarrow \quad =2\ln\left(\sec(\frac{\theta}{2})+\tan(\frac{\theta}{2})\right)=\sqrt{\frac{3g}{L}}\,t+c_2$$

At the start point $\theta=0$ and $t=0$

$$2\ln(\sec(0)+\tan(0))=c_2 \quad \rightarrow \quad c_2=2\ln(1+0)=0$$

$$t=2\sqrt{\frac{L}{3g}}\ln\left(\sec(\frac{\theta}{2})+\tan(\frac{\theta}{2})\right) \qquad (6)$$

$$R_1=\frac{1}{2}\left(3+5\cos\theta\right)mg \quad mL\left(\frac{3g}{2L}(1+\cos\theta)\right)=R_1-mg\cos\theta \rightarrow R_1=mL\left(\frac{3g}{2L}(1+\cos\theta)\right)+mg\cos\theta$$

(7)

$$mL\left(-\frac{3g}{4L}\sin\theta\right)=R_2-mg\sin\theta \quad \rightarrow \quad R_2=mg\sin\theta-\frac{3}{4}mg\sin\theta$$

$$R_2=\frac{1}{4}mg\sin\theta \qquad (8)$$

At $\theta=\frac{\pi}{3}$

$$R_1=\frac{1}{2}\left(3+5\cos(\frac{\pi}{3})\right)mg=\frac{1}{2}\left(3+\frac{5}{2}\right)mg=\frac{11}{4}mg \quad \rightarrow \quad R_1=\frac{11}{4}mg \qquad (9)$$

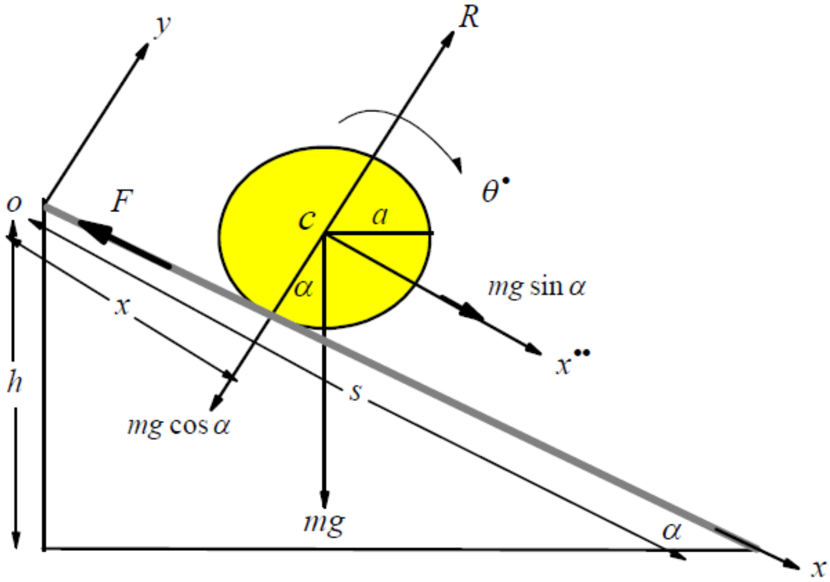
$$R_2=\frac{1}{4}mg\sin\theta=\frac{1}{4}mg\sin(\frac{\pi}{3})=\frac{1}{4}mg(\frac{\sqrt{3}}{2}) \quad \rightarrow \quad R_2=\frac{\sqrt{3}}{8}mg \qquad (10)$$

Example 3: A body rolls down an inclined plane without slipping. Describe the motion of the body?

Solution

First draw a free body diagram of the body, which down the plane:

We can write both of the Linear and rotation equations of motion



Linear equations of motion (Equations of motion of center of gravity)

$$m\ddot{x} = mg\sin\alpha - F \qquad (1)$$

$$m(0) = R - mg\cos\alpha \qquad (2)$$

Equation of Rotational Motion of a Rigid Body

$$\frac{d}{dt}\left(I_c\dot{\theta}\right) = M_c \;\rightarrow\; I_c\ddot{\theta} = M_c \qquad (3)$$

$$I_c\ddot{\theta} = (F)\,(a) \rightarrow \quad F = \frac{I_c}{a}\ddot{\theta} \qquad (4)$$

$$m\ddot{x} = mg\sin\alpha - \frac{I_c}{a}\ddot{\theta} \qquad (5)$$

Pure rolling

$x^{\bullet} = a\theta^{\bullet}$, then $x^{\bullet\bullet} = a\theta^{\bullet\bullet}$

$$m\,x^{\bullet\bullet} = mg\sin\alpha - \frac{I_c}{a^2}x^{\bullet\bullet} \;\rightarrow\; x^{\bullet\bullet} + \frac{I_c}{m\,a^2}x^{\bullet\bullet} = g\sin\alpha$$

$$x^{\bullet\bullet} = \frac{g\sin\alpha}{1+\dfrac{I_c}{m\,a^2}} \;\rightarrow\; a\theta^{\bullet\bullet} = x^{\bullet\bullet} = \frac{g\sin\alpha}{1+\dfrac{I_c}{m\,a^2}} \qquad (6)$$

$$v^2 = v_0 + 2x^{\bullet\bullet}x$$

$$v^2 = 0 + 2\left(\frac{g\sin\alpha}{1+\dfrac{I_c}{m\,a^2}}\right)s \;\rightarrow\; v^2 = 2\left(\frac{g\sin\alpha}{1+\dfrac{I_c}{m\,a^2}}\right)\frac{h}{\sin\alpha} \rightarrow\; v^2 = \frac{2g\,h}{1+\dfrac{I_c}{m\,a^2}}$$

$$v = \sqrt{\frac{2g\,h}{1+\dfrac{I_c}{m\,a^2}}} \qquad (7)$$

$$F = \frac{I_c}{a}\frac{1}{a}\left(\frac{g\sin\alpha}{1+\dfrac{I_c}{m\,a^2}}\right) \rightarrow F = \left(\frac{I_c}{m\,a^2 + I_c}\right)m\,g\sin\alpha \quad or \quad F = \frac{1}{\dfrac{m\,a^2}{I_c}+1}m\,g\sin\alpha \qquad (8)$$

$$F < \mu R$$

$$F < \mu R \;\rightarrow \mu > \frac{F}{R} \;\rightarrow\; \mu > \frac{\left(\dfrac{I_c}{m\,a^2 + I_c}\right)m\,g\sin\alpha}{mg\;\cos\alpha}$$

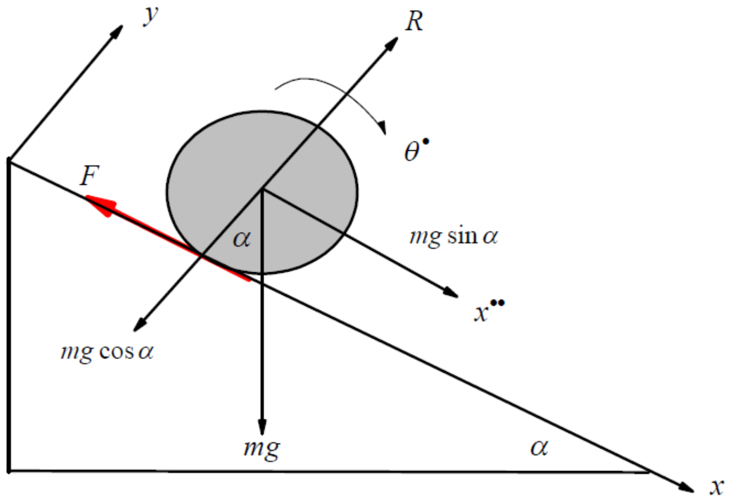
$$\mu > \left(\frac{I_c}{m\,a^2 + I_c}\right)\tan\alpha \qquad (9)$$

$$\mu = \left(\frac{I_c}{m\,a^2 + I_c}\right)\tan\alpha$$

Example 4: A Solid Cylinder of mass m and radius a rolls without slipping down an inclined plane whose incline angle with the horizontal is theta. Determine the acceleration of the cylinder's center of mass, and the minimum coefficient of friction that will allow the cylinder to roll without slipping on this incline?

Solution

First draw a free body diagram of the cylinder, which down the plane:
We can write both of the Linear and rotation equations of motion



Linear equations of motion (Equations of motion of center of gravity)

$$m\ddot{x} = mg\sin\alpha - F \qquad (1)$$

$$mg\cos\alpha = R \qquad (2)$$

Rotational Motion of a Cylinder

$$\frac{d}{dt}\left(I_c\dot{\theta}\right) = M_c \rightarrow I_c\ddot{\theta} = M_c \qquad (3)$$

that can be written as

$$\frac{1}{2}ma^2\ddot{\theta} = (F)\ (a) \rightarrow F = \frac{1}{2}ma\ddot{\theta} \qquad (4)$$

The necessary condition for rolling without slipping is the contact point have zero velocity (the condition for no sliding is). i. e. $\overset{\bullet}{x} = a\overset{\bullet}{\theta} \to \overset{\bullet\bullet}{x} = a\overset{\bullet\bullet}{\theta}$. Substitute in Eq. (4), we have

$$F = \frac{1}{2} m \overset{\bullet\bullet}{x} \qquad (5)$$

Again, Substituting from Eq. (5) into Eq. (1), we have

$$m\overset{\bullet\bullet}{x} = mg\sin\alpha - \frac{1}{2} m \overset{\bullet\bullet}{x} \;\to\; \overset{\bullet\bullet}{x} + \frac{1}{2}\overset{\bullet\bullet}{x} = g\sin\alpha \;\to\; \frac{3}{2}\overset{\bullet\bullet}{x} = g\sin\alpha$$

$$\overset{\bullet\bullet}{x} = \frac{2}{3} g\sin\alpha \qquad (6)$$

Substituting from Eq. (6) into Eq. (5), we have

$$F = \frac{1}{2} m \left(\frac{2}{3} g\sin\alpha\right) \;\to\; F = \frac{1}{3} m g\sin\alpha \qquad (7)$$

Again, the necessary condition for rolling without slipping is the static coefficient and is generally lower than the static coefficient of friction. i. e. $F < \mu R$

$$\mu > \frac{F}{R} \qquad (8)$$

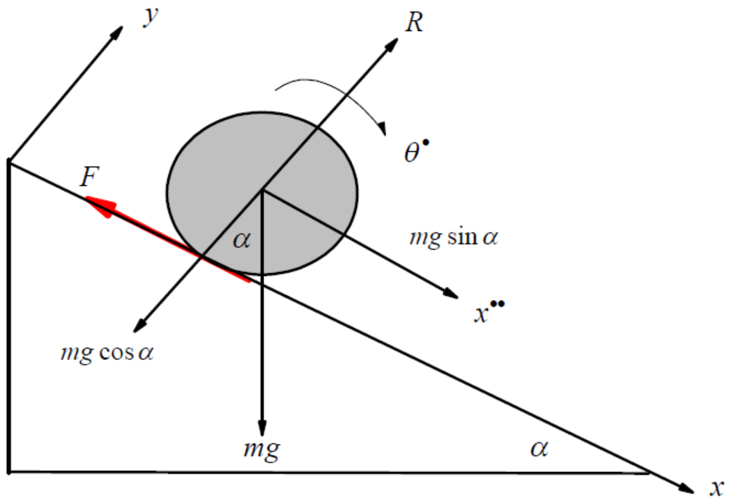
$$\mu > \frac{\frac{1}{3} m g\sin\alpha}{mg\cos\alpha} \;\to\; \mu > \frac{1}{3}\frac{\sin\alpha}{\cos\alpha} \;\to\; \mu > \frac{1}{3}\tan\alpha$$

Example 5: Calculate the minimum coefficient of friction necessary to keep a thin circular ring from sliding as it rolls down a plane inclined at an angle θ with respect to the horizontal plane.

Solution

First draw a free body diagram of the ring, which down the plane:

We can write both of the Linear and rotation equations of motion



Linear equations of motion (Equations of motion of center of gravity)

$$m x^{\bullet\bullet} = mg\,\sin\alpha - F \qquad (1)$$

$$mg\,\cos\alpha = R \qquad (2)$$

Rotational motion equations

$$\frac{d}{dt}\left(I_c\,\theta^{\bullet}\right) = M_c \;\rightarrow\; I_c\,\theta^{\bullet\bullet} = M_c \qquad (3)$$

that can be written as

$$m a^2 \theta^{\bullet\bullet} = (F)\ (a) \rightarrow \quad F = m a \theta^{\bullet\bullet} \qquad (4)$$

The necessary condition for rolling without slipping is the contact point have zero velocity (the condition for no sliding is). i. e. $x^{\bullet} = a\theta^{\bullet} \rightarrow x^{\bullet\bullet} = a\theta^{\bullet\bullet}$. Substitute in Eq. (4), we have

$$F = m\ x^{\bullet\bullet} \qquad (5)$$

Again, Substituting from Eq.. (5) into Eq. (1), we have

$$m x^{\bullet\bullet} = mg\sin\alpha - m\,x^{\bullet\bullet} \;\rightarrow\; x^{\bullet\bullet} + x^{\bullet\bullet} = g\sin\alpha \;\rightarrow\; 2\ x^{\bullet\bullet} = g\sin\alpha$$

$$x^{\bullet\bullet} = \frac{1}{2}\,g\sin\alpha \qquad (6)$$

Substituting from Eq.. (6) into Eq. (5), we have

$$F = m\left(\frac{1}{2} g\sin\alpha\right) \quad \rightarrow \quad F = \frac{1}{2} m g \sin\alpha \qquad (7)$$

Again, the necessary condition for rolling without slipping is the static coefficient and is generally lower than the static coefficient of friction. i. e. $F < \mu R$

$$\mu > \frac{F}{R} \qquad (8)$$

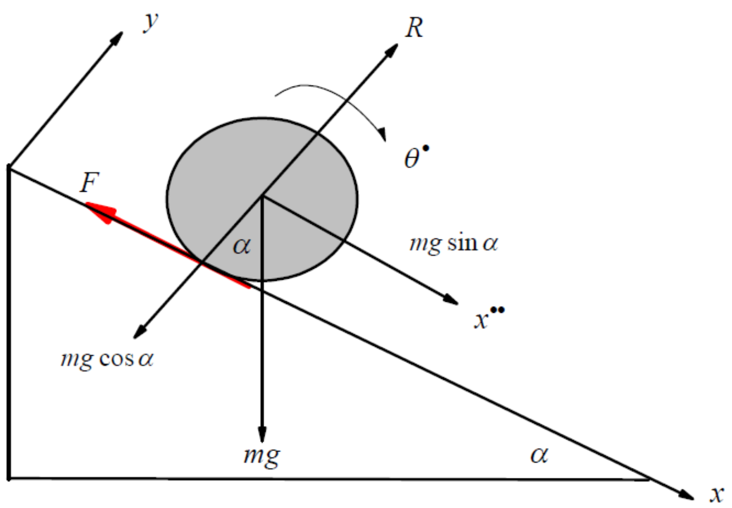
$$\mu > \frac{\frac{1}{2} m g \sin\alpha}{mg\cos\alpha} \quad \rightarrow \quad \mu > \frac{1}{2}\frac{\sin\alpha}{\cos\alpha} \quad \rightarrow \quad \mu > \frac{1}{2}\tan\alpha$$

Example 6: A uniform solid sphere of mass m and radius a rolls without slipping down an inclined plane whose incline angle with the horizontal is theta. Determine the acceleration of the ball's center of mass, and the minimum coefficient of friction that will allow the ball to roll without slipping on this incline?

Solution

First draw a free body diagram of the sphere, which down the plane:

We can write both of the Linear and rotation equations of motion



Linear equations of motion (Equations of motion of center of gravity)

$$m\ddot{x} = mg\sin\alpha - F \qquad (1)$$

$$mg\cos\alpha = R \qquad (2)$$

Rotational motion equations

$$\frac{d}{dt}\left(I_c\dot{\theta}\right) = M_c \rightarrow I_c\ddot{\theta} = M_c \qquad (3)$$

that can be written as

$$\frac{2}{5}ma^2\ddot{\theta} = (F)\ (a) \rightarrow F = \frac{2}{5}ma\ddot{\theta} \qquad (4)$$

The necessary condition for rolling without slipping is the contact point have zero velocity. i. e. $\dot{x} = a\dot{\theta} \rightarrow \ddot{x} = a\ddot{\theta}$. Substitute in Eq. (4), we have

$$F = \frac{2}{5}m\ddot{x} \qquad (5)$$

Again, Substituting from Eq. (5) into Eq. (1), we have

$$m\ddot{x} = mg\sin\alpha - \frac{2}{5}m\ddot{x} \rightarrow \ddot{x} + \frac{2}{5}\ddot{x} = g\sin\alpha \rightarrow \frac{7}{5}\ddot{x} = g\sin\alpha$$

$$\ddot{x} = \frac{5}{7}g\sin\alpha \qquad (6)$$

Substituting from Eq. (6) into Eq. (5), we have

$$F = \frac{2}{5}m\left(\frac{5}{7}g\sin\alpha\right) \rightarrow F = \frac{2}{7}mg\sin\alpha \qquad (7)$$

Again, the necessary condition for rolling without slipping is the static coefficient and is generally lower than the static coefficient of friction. i. e. $F < \mu R$

$$\mu > \frac{F}{R} \qquad (8)$$

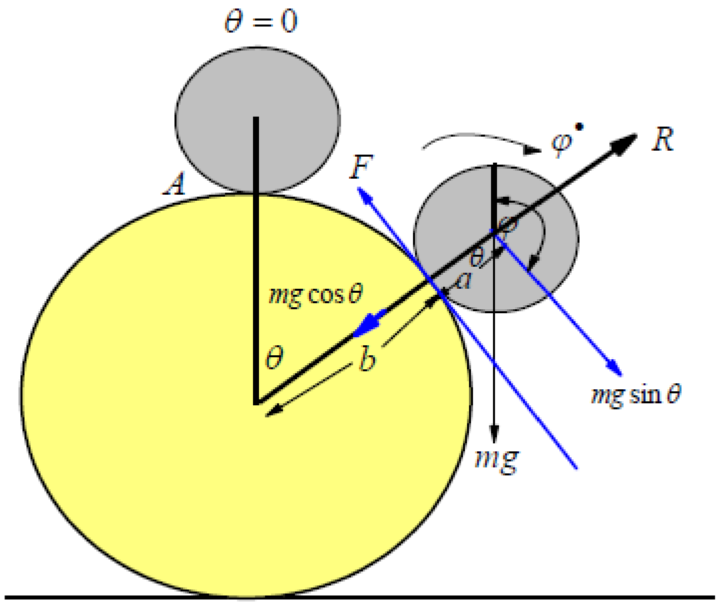
$$\mu > \frac{\frac{2}{7}mg\sin\alpha}{mg\cos\alpha} \rightarrow \mu > \frac{2}{7}\frac{\sin\alpha}{\cos\alpha} \rightarrow \mu > \frac{2}{7}\tan\alpha$$

Example 7: A uniform sphere of radius a initially at rest rolls without slipping down from the top of a rough sphere of radius b. Find the angular velocity of the ball at the instant it breaks off the sphere and show that the angle $\cos^{-1}\left(\frac{10}{17}\right)$ with the vertical?

Solution

First draw a free body diagram of the sphere, which down the plane:

We can write both of the Linear and rotation equations of motion



Note that

$\vec{v} = (v_r, v_\theta) = (\dot{r}, r\dot{\theta})$

$\vec{a} = (a_r, a_\theta) = (\ddot{r} - r\dot{\theta}^2, r\ddot{\theta} + 2\dot{r}\dot{\theta})$

<u>Equations of motion of Center of Gravity</u>

$m(a+b)\ddot{\theta} = mg\ \sin\theta - F$ (1)

$$-m(a+b)\dot{\theta}^{2} = mg\cos\theta - R \qquad (2)$$

Rotational motion equation

$$\frac{d}{dt}\left(I_c\,\dot{\varphi}\right) = M_c \;\rightarrow\; I_c\,\ddot{\varphi} = M_c \qquad (3)$$

That can be written as

$$\frac{2}{5}ma^2\,\ddot{\varphi} = (F)\;(a) \rightarrow \quad F = \frac{2}{5}ma\,\ddot{\varphi} \qquad (4)$$

The condition for pure rolling is $(a+b)\theta = a\varphi \rightarrow$, then $(a+b)\dot{\theta} = a\dot{\varphi}$

$$(a+b)\ddot{\theta} = a\ddot{\varphi} \quad \rightarrow \quad \ddot{\varphi} = \frac{a+b}{a}\ddot{\theta} \qquad (5)$$

Substituting from Eq. (5) into Eq. (4), we have

$$F = \frac{2}{5}m(a+b)\;\ddot{\theta} \qquad (6)$$

Again, substituting from Eq. (6) into Eq. (1), we have

$$m(a+b)\;\ddot{\theta} = mg\sin\theta - \frac{2}{5}m(a+b)\;\ddot{\theta} \quad \rightarrow \quad \frac{7}{5}m(a+b)\;\ddot{\theta} = mg\sin\theta$$

$$\ddot{\theta} = \frac{5}{7(a+b)}g\sin\theta \qquad (7)$$

$$\dot{\theta}\frac{d\dot{\theta}}{d\theta} = \frac{5}{7(a+b)}g\sin\theta \quad \rightarrow \int\dot{\theta}\,d\dot{\theta} = \frac{5}{7(a+b)}g\int\sin\theta\,d\theta$$

$$\frac{\dot{\theta}^2}{2} = -\frac{5}{7(a+b)}g\cos\theta + c_1 \qquad (8)$$

At the initial motion $\theta = 0$, then $\dot{\theta} = 0$

Then in Eq. (8), we have $c_1 = \frac{5}{7(a+b)}g$ and again in Eq. (8), we have

$$\frac{\dot{\theta}^2}{2} = -\frac{5}{7(a+b)} g \cos\theta + \frac{5}{7(a+b)} g = \frac{5}{7(a+b)} g\,(1-\cos\theta)$$

$$\dot{\theta}^2 = \frac{10\,g}{7(a+b)}(1-\cos\theta) \quad \rightarrow \quad \dot{\theta}^2\,(a+b) = \frac{10g}{7}(1-\cos\theta) \qquad (9)$$

Substituting from Eq. (9) into Eq. (2), we have

$$-m\left(\frac{10}{7} g\,(1-\cos\theta)\right) = mg\ \cos\theta - R$$

$$R = mg\ \cos\theta - \frac{10}{7} mg\,(1-\cos\theta) \quad \rightarrow \quad R = \frac{17}{7} mg\ \cos\theta - \frac{10}{7} mg \qquad (10)$$

When the ball instant breaks off the sphere

At the instant, that the ball breaks off the sphere, the reaction equals zero, so from Eq. (10), we have

$$\frac{17}{7} mg\ \cos\theta - \frac{10}{7} mg = 0 \quad \rightarrow \quad \frac{17}{7} mg\ \cos\theta = \frac{10}{7} mg \quad \rightarrow \quad 17\ \cos\theta = 10$$

$$\cos\theta = \frac{10}{17} \quad \rightarrow \quad \theta = \cos^{-1}\left(\frac{10}{17}\right) \quad \rightarrow \quad \theta = 53.968^{O} \qquad (11)$$

In this case the angle will be maximum ($\theta = \theta_{\max}$)

Where the velocity is given by $\vec{v} = (v_r,\ v_\theta) = (\dot{r},\ r\,\dot{\theta})$ حيث $\vec{v} = (0,\ r\,\dot{\theta})$

$$v = r\,\dot{\theta} = (a+b)\sqrt{\frac{10\,g}{7(a+b)}(1-\cos\theta)} = \sqrt{\frac{10\,g}{7}(a+b)(1-\cos\theta)}$$

At the moment ($\theta = \theta_{\max}$)

$$v = \sqrt{\frac{10\,g}{7}(a+b)(1-\frac{10}{17})} = \sqrt{\frac{10\,g}{7}(a+b)(\frac{17-10}{17})} = \sqrt{\frac{10\,g}{7}(a+b)(\frac{7}{17})}$$

$$v = \sqrt{\frac{10\,g}{17}(a+b)} \qquad (12)$$

APPLIED MATHEMATICS 4

PROF. MAHDY

Second year

SVU

Chapters

1. Damped and forced oscillations

2. Motion with variable mass

3. Motion in space

4. Motion relative to earth

SVU

Chapters

DAMPED AND FORCED OSCILLATIONS

INTRODUCTION

In order to investigate the damped and forced harmonic oscillations we need to take a look on differential equation of higher order or in particular of order two with constant coefficients. A linear differential equation is an equation in which the dependent variable and its derivatives appear only in the first degree. A linear differential equation of order n of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \dots a_{n-1} \frac{dy}{dx} + a_n y = Q(x) \qquad (1)$$

where $a_1, a_2, \dots a_{n-1}, a_n$ are constants and $Q(x)$ is any function of x is called a linear differential equation with constant coefficients.

For convenience, the operators $\frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3} \dots, \frac{d^n}{dx^n}$ are also denoted by $D, D^2, D^3, \dots, D^n$, respectively.

Thus the equation (1) can also be written as

$$D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \dots a_{n-1} Dy + a_n y = Q(x) \qquad (2)$$

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots a_{n-1} D + a_n) y = Q(x) \qquad (3)$$

If $y = f(x)$ is a solution of the homogeneous ODE

$$D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \dots a_{n-1} Dy + a_n y = 0$$

and $y = \phi(x)$ is any particular solution of the equation (2) not containing any arbitrary constant, then

$$y = f(x) + \phi(x)$$

is the general solution of ODE (2).

Thus the method of solving a linear equation is divided into two parts:

First, we find the general solution of the equation (3). It is called the complementary function (C.F.). It must contain as many arbitrary constants as is the order of the given differential equation.

Next, we find a solution of (2) which does not contain an arbitrary constant. This is called the particular integral (P.I.). If we add (C.F.) and (P.I.), we get the general solution of (2). Thus the general solution of ODE (2) is

$$y = \text{C.F.+P.I.}$$

♦ Determination of complementary solution (C.S.)

Consider a linear *n*th order differential equation with constant coefficients of the form $f(D) = 0$, i.e.,

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + ...a_{n-1} D + a_n) y = 0 \quad (4)$$

This is equivalent to

$$((D - m_1)(D - m_2)(D - m_3)...(D - m_n)) y = 0 \quad (5)$$

The solution of any one of the equations

$$(D - m_1) y = 0, (D - m_2) y = 0, ..., (D - m_n) y = 0 \quad (6)$$

is also a solution of (5) and we know that the general solution of

$$(D - m_1) y = 0 \text{ is } y = Ae^{m_1 x}$$

Hence we can assume that a solution of the equation (5) is of the form $y = e^{mx}$

Then, substituting e^{mx} for y in (1), so that

$$Dy = me^{mx}, D^2 y = m^2 e^{mx}, ...D^n y = m^n e^{mx} \text{ we get}$$

$$(m^n + a_1 m^{n-1} + a_2 m^{n-2} + ...a_{n-1} m + a_n)e^{mx} = 0$$

Or $m^n + a_1 m^{n-1} + a_2 m^{n-2} + ...a_{n-1} m + a_n = 0$ because $e^{mx} \neq 0$

Hence e^{mx} will be a solution of (4) if m has the value obtained from the equation

$$m^n + a_1 m^{n-1} + a_2 m^{n-2} + ...a_{n-1} m + a_n = 0 \qquad (7)$$

The equation (7) is called the auxiliary equation (A.E.) and is obtained by putting $D = m$ in $f(D) = 0$

It will give in general *n* roots, say $m_1, m_2, m_3, ..., m_n$

Now we will consider three cases of these roots.

♦ Case I: If all the roots of the Auxiliary Equation (A.E.) are distinct:

If the roots $m_1, m_2, m_3, ..., m_n$ are all distinct, then $e^{m_1 x}, e^{m_2 x}, ...e^{m_n x}$ are all distinct and linearly independent. So the general solution of (1) in this case is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}...... + c_n e^{m_n x} \qquad (8)$$

♦ Case II: Auxiliary equation having equal roots:

If two roots are equal say $m_1 = m_2$, then the solution (8) becomes

$$y = c_1 e^{m_1 x} + c_2 e^{m_1 x} + c_3 e^{m_3 x}...... + c_n e^{m_n x}$$

Or

$$y = (c_1 + c_2)e^{m_1 x} + c_3 e^{m_3 x}...... + c_n e^{m_n x}$$

Now ($c_1 + c_2$) can be replaced by single constant say *c*.

Therefore this solution has only (*n* - 1) arbitrary constants and so it is not the general solution.

To obtain the general solution, consider the differential equation $(D - m_1)^2 y = 0$ in which the two roots are equal.

This can be written as $(D - m_1)(D - m_1)y = 0 \qquad (9)$

Now putting $(D-m_1)y=v$ we get $(D-m_1)v=0$

or $dv/dx=m_1 v$ or $dv/v=m_1 dx$ (variables being separated).

Then by integrating,

$\log v=m_1 x+\log c_1$ or $\log(v/c_1)=m_1 x$ or $v=c_1 e^{m_1 x}$.

Thus putting it in (9) $(D-m_1)y=v=c_1 e^{m_1 x}$

or $Dy-m_1 y=c_1 e^{m_1 x}$ or $\dfrac{dy}{dx}-m_1 y=c_1 e^{m_1 x}$ $\left[D=\dfrac{d}{dx}\right]$

or $$e^{-m_1 x}\frac{dy}{dx}-m_1 e^{-m_1 x}y=c_1 \quad \Rightarrow \frac{d}{dx}\; e^{-m_1 x}y\; =c_1$$
$$\Rightarrow d\;\; e^{-m_1 x}y\; =c_1 dx$$

by integrating this equation we get

$$e^{-m_1 x}y=c_1 x+c_2 \qquad \Rightarrow y=(c_1 x+c_2)e^{m_1 x}$$

Hence the general solution of $f(D)y=0$ in this case is

$$y=(c_1+c_2 x)e^{m_1 x}+c_3 e^{m_3 x}......+c_n e^{m_n x}$$

Similarly if three roots of the auxiliary equation are equal say, $m_1=m_2=m_3$, the general solution of $f(D)y=0$ will be

$y=(c_1+c_2 x+c_3 x^2)e^{m_1 x}+c_4 e^{m_4 x}......+c_n e^{m_n x}$ and so on.

Case III: Auxiliary equation having complex roots:

Let the two roots of the auxiliary equation be complex, say $m_1=\alpha+i\beta$ and $m_2=\alpha-i\beta$, (where $i^2=-1$).

The solution corresponding to these two roots will be

$$\begin{aligned} y &= c_1 e^{(\alpha+i\beta)x}+c_2 e^{(\alpha-i\beta)x} \\ &= c_1 e^{\alpha x}e^{i\beta x}+c_2 e^{\alpha x}e^{-i\beta x} \\ &= c_1 e^{\alpha x}(\cos\beta x+i\sin\beta x)+c_2 e^{\alpha x}(\cos\beta x-i\sin\beta x) \\ &= (c_1+c_2)e^{\alpha x}\cos\beta x+i(c_1-c_2)e^{\alpha x}\sin\beta x \end{aligned}$$

or $\quad y = e^{\alpha x}(A_1 \cos \beta x + A_2 \sin \beta x)$ where $A_1 = c_1 + c_2, A_2 = i(c_1 - c_2)$

or $\quad y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$ by changing the constants.

If the imaginary roots are repeated, say $\alpha + i\beta$ and $\alpha - i\beta$ occur twice then the solution will be

$$y = e^{\alpha x} \ (c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x$$

and so on.

⬄**Note 1:** The expression $e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$ can also be written as

$$c_1 e^{\alpha x} \sin(\beta x + c_2) \qquad \text{or} \qquad c_1 e^{\alpha x} \cos(\beta x + c_2)$$

⬄**Note 2:** If a pair of the roots of the auxiliary equation is irrational *i. e.*, they are $\alpha \pm i\sqrt{\beta}$, where β positive, then the corresponding term in the C.F. is will be $\quad e^{\alpha x} \ c_1 \cosh \sqrt{\beta} x + c_2 \sinh \sqrt{\beta} x$

or $\quad c_1 e^{\alpha x} \sinh(\sqrt{\beta} x + c_2) \quad$ or $\quad c_1 e^{\alpha x} \cosh(\sqrt{\beta} x + c_2)$

If these irrational roots are repeated twice, then the corresponding portion of the solution will be

$$e^{\alpha x} \ (c_1 + c_2 x) \cosh \sqrt{\beta} x + (c_3 + c_4 x) \sinh \sqrt{\beta} x$$

■Illustrative Examples■

❒ **EXAMPLE:** Solve $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$

❒ **Solution:** The given differential equation is $(D^2 - 7D + 12)y = 0$.

Then the auxiliary equation is $m^2 - 7m + 12 = 0$

or $(m-3)(m-4) = 0 \quad \Rightarrow m = 3 \text{ and } 4$.

Hence the solution is $y = c_1e^{3x} + c_2e^{4x}$

❒ **EXAMPLE:** Solve $(D^3 + 6D^2 + 11D + 6)y = 0$

❒ **Solution:** The auxiliary equation is $m^3 + 6m^2 + 11m + 6 = 0$

or $\Rightarrow m = -1, -2, -3$.

Hence the solution is $y = c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x}$

❒ **EXAMPLE:** Solve $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$ given that when $t = 0$, $x = 0$ and $dx/dt = 0$

❒ **Solution:** The auxiliary equation is $m^2 - 3m + 2 = 0 \quad \Rightarrow m = 1, 2$.

Hence the solution is $x = c_1e^{t} + c_2e^{2t}$ (1)

where c_1 and c_2 are arbitrary constants.

Now $x = 0$ when $t = 0$ $\Rightarrow 0 = c_1 + c_2$ (*)

Also $dx/dt = c_1e^{t} + 2c_2e^{2t}$ and $dx/dt = 0$ when $t = 0 \Rightarrow 0 = c_1 + 2c_2$ (**)

Solving (*) and (**), we get, $c_1 = c_2 = 0$.

Now putting values of c_1 and c_2 in (1), we get the required solution as $x = 0$

❒ **EXAMPLE:** Solve $\frac{d^3y}{dx^2} - 8y = 0$.

❒ **Solution:** The auxiliary equation is given by $m^3 - 8 = 0$

or $(m-2)(m^2 + 2m + 4) = 0$ i.e., $m = 2$ and $m = -1 \pm i\sqrt{3}$.

Hence the solution is written as

$$y = e^{-x}(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x) + c_3 e^{2x}$$

❒ **EXAMPLE:** Solve $\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} - 2\frac{dy}{dx} - y = 0$.

❒ **Solution:** The auxiliary equation is given by

$$m^4 - 2m^3 - 2m - 1 = 0$$
$$\text{or } (m^4 - 1) - 2m(m^2 + 1) = 0$$
$$\text{or } (m^2 - 1)(m^2 + 1) - 2m(m^2 + 1) = 0$$
$$\text{or } (m^2 + 1)(m^2 - 2m - 1) = 0$$

That is the roots are $m = \pm i$ and $m = 1 \pm \sqrt{2}$

Hence the solution is written as

$$y = (c_1 \cos x + c_2 \sin x) + e^x(c_3 \cosh\sqrt{2}x + c_4 \sinh\sqrt{2}x)$$

PROBLEMS

Solve the following ordinary differentia equations

1- $\frac{d^2y}{dx^2} + (a+b)\frac{dy}{dx} + aby = 0$

2- $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 0$

3- $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 10y = 0$

4- $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$

5- $\frac{d^4y}{dx^4} + 2\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$

DAMPED AND FORCED OSCILLATIONS

Simple harmonic motion has constant amplitude and goes on forever. For many real oscillating systems, SHM is not a very good model: usually the amplitude of the oscillations gradually decreases, and the motion dies away.

When you find that a model is unsatisfactory, you need to look again at your assumptions. Real oscillating systems are almost always *damped*: that is they are affected to some degree by the resistive forces of friction and/or air resistance. They perform ***damped oscillations***.

In many systems the damping force is proportional to the speed of the object. This is often represented on a diagram by a device called a linear dashpot, as shown in Figure 1.

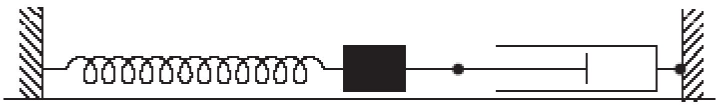


Figure 1

A dashpot exerts a force on the system which is proportional to the rate at which it is being extended or compressed, and which acts in the direction opposite to that of the motion. This is illustrated in Figure 2.

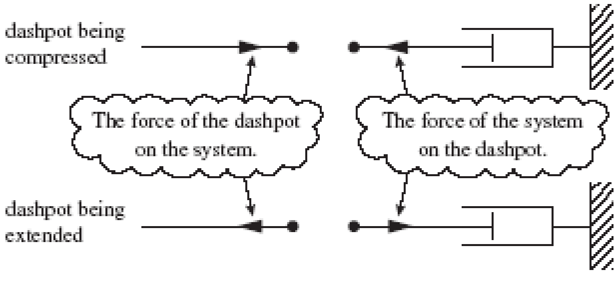


Figure 2

The force *R* that the dashpot exerts on the system at time *t* is given by

$$R = \beta \frac{dL}{dt}$$

where the constant of proportionality β is called the *dashpot constant* (or the *damping constant*). The amount of travel still left in the dashpot (see Figure 3)

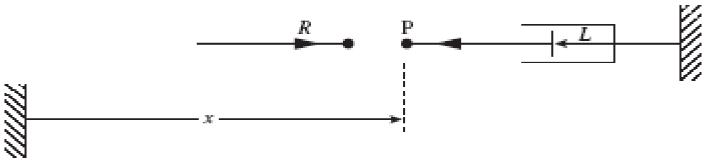


Figure 3

is denoted by *L*.

It is important to be clear in your mind about the direction of the force *R* and the signs involved. Look at the point P on the moving part of the dashpot.

⇔When P is moving from right to left, *L* is increasing; dL/dt is positive and the force is in the same direction as that marked for *R* in Figure 3. The dashpot is opposing the right to left motion.

⇔ When P is moving from left to right, *L* is decreasing, dL/dt is negative and the force is in the opposite direction to that marked for *R* in Figure 3. The dashpot is now opposing the left to right motion.

Thus the sign of the dashpot force looks after itself as the motion changes.

However you will not usually be interested in the quantity *L* so much as the distance of the point P from some fixed point of the system. This distance is shown as x in Figure 3. All the systems that you will meet in this book are set up so that as x increases, *L* decreases, and vice versa, so

$$\frac{dx}{dt} = -\frac{dL}{dt}$$

Consequently the force that the dashpot exerts on the system is given by

$$R = -\beta \frac{dx}{dt}$$

in the direction of increasing x.

The general equation for damped oscillations

The differential equation of motion in the spring-mass-dashpot system above is an example of the general differential equation of a linearly damped system:

$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0$$

Or by dividing by *m* $\quad \frac{d^2y}{dt^2} + \alpha\frac{dy}{dt} + \omega^2 y = 0$

where and α, ω are positive constants. For a spring-mass-dashpot system, $\alpha = \beta/m$ where β is the dashpot constant, and $\omega^2 = k/m$ where k is the stiffness of the spring. The quantity $(\omega/2\pi)$ is called the *natural frequency* of the system: it is the same whether the system is damped or undamped.

The solution of this ordinary differential equation can give several different types of motion,

depending on the relative sizes of the parameters and α, ω. The auxiliary equation is given by $\lambda^2 + \alpha\lambda + \omega^2 = 0$

This has two solutions

$$\lambda_1 = \frac{-\alpha + \sqrt{\alpha^2 - 4\omega^2}}{2} \text{ and } \lambda_2 = \frac{-\alpha - \sqrt{\alpha^2 - 4\omega^2}}{2}$$

The discriminant, $\alpha^2 - 4\omega^2$, determines the nature of the solution. There are three possibilities, as follows.

♦**Over-damping:** $\alpha^2 - 4\omega^2$, is positive, and the system does not oscillate see Figure 4.

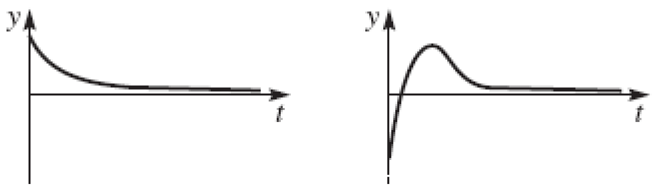


Figure 4

♦**Under-damping**: $\alpha^2 - 4\omega^2$, is negative, and oscillations occur oscillate see Figure 5.

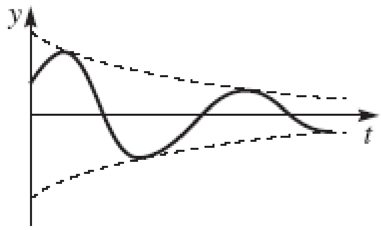


Figure 5

♦**Critical damping:** $\alpha^2 - 4\omega^2 = 0$, Figure 6

Critical damping is the borderline between overdamping and underdamping. It is not obvious in a physical situation when damping is critical, since the pattern of motion for critical damping can be very similar to that in the overdamped case.

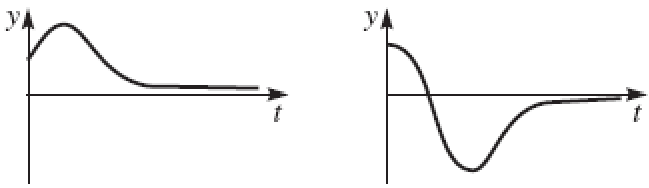


Figure 6

■Illustrative Examples■

❐ **EXAMPLE:** A simple oscillating system is being modelled as a damped spring-mass oscillator, in which an object of mass 2 kg is attached to fixed points by a spring of natural length 0.5 m, stiffness 20 Nm^{-1} and by a dashpot of constant 12 Nm^{-1} s. The spring-mass dashpot system lies on a smooth horizontal surface, as shown in the Figure 7.

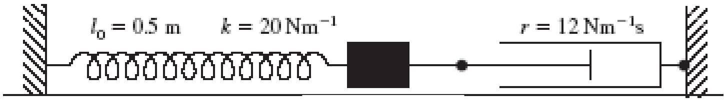


Figure 7

i) Formulate the differential equation of motion for this system.

The system is released from rest when the spring length is 0.6 m.

ii) Find the particular solution of the differential equation that models this situation.

❐ **SOLUTION**

i) Figure 8 shows the spring-mass-dashpot system at some general time *t* (seconds),

when the extension of the spring is x. The horizontal forces are the tension in the spring, *T*, and the damping force *R*.

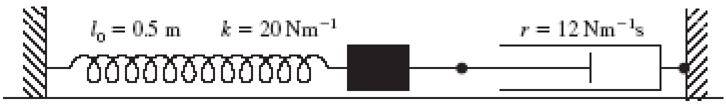


Figure 8

The tension in the spring is $T = kx = 20x$.

The dashpot force is $R = -\beta \frac{dx}{dt} = -12\frac{dx}{dt}$

Applying Newton's second law $F = ma$ at any instant gives

$$2\frac{d^2x}{dt^2} = -12\frac{dx}{dt} - 20x \qquad F = R - T$$

Dividing both sides by 2, and rearranging, you obtain the equation of motion of the spring-mass dashpot system:

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 10x = 0$$

ii) The auxiliary equation for the differential equation is

$$\lambda^2 + 6\lambda + 10 = 0 \qquad \Rightarrow \lambda = -3 \pm i$$

The general solution of the differential equation is $x = Ae^{-3t}\sin(t+\varepsilon)$

At the start of the motion, the length of the spring is 0.6 m and the object is at rest, so the initial conditions are $x = 0.1,\ dx/dt = 0$ as $t = 0$ and _

when $t = 0,\ x = 0.1 \quad \Rightarrow A\sin\varepsilon = 0.1$ (1)

when $t = 0,\ \frac{dx}{dt} = 0 \quad \Rightarrow -3A\sin\varepsilon + A\cos\varepsilon = 0 \qquad \Rightarrow \tan\varepsilon = 1/3$

From Eq. (1) we see that $\sin\varepsilon = 0.1/A$ which is positive, and so ε must be an angle in the first quadrant.

$$\Rightarrow \varepsilon = 0.322 \text{ (radians) and } A = 0.1 / \sin 0.322 = 0.316$$

The particular solution in this case is

$$x = 0.316e^{-3t}\sin(t + 0.322)$$

The initial amplitude of the motion is 0.316 m and the period

$$\frac{2\pi}{\omega} = \frac{2\pi}{1} = 2\pi \ (\omega = 1)$$

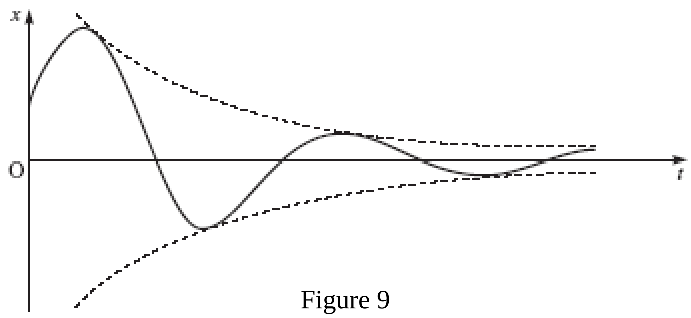


Figure 9

The amplitude decays exponentially. In this case, the oscillation decays very quickly. Figure 9 shows the graph of a typical damped oscillation. There are many real situations where the oscillations decrease gradually in amplitude like this. Oscillations of this type are often called *lightly damped* or *underdamped*.

❒ **EXAMPLE:** A particle of mass 20 kg executes a simple harmonic motion on X axis. Initially the particle position was at 4 m from the center and has a velocity 16 ms^{-1} and acceleration 80 ms^{-2} directed towards the center. A resistance force acted on the particle of magnitude value per unit of mass is $8v$ where v indicates the instant velocity. Find the position and velocity of the particle in terms of time and the periodic time and the frequency.

❒ **SOLUTION**

Since, for S.H.M. $F \propto x \Rightarrow F = Kx, \quad M\ddot{x} = Kx$

Due to the boundary conditions:

$$20(80) = 4K \quad \Rightarrow K = 400$$

when the resistance force acts then the equation of motion becomes

$$M\ddot{x} = -R - Kx \qquad \Rightarrow 20\ddot{x} = -8(20)\dot{x} - 400x \qquad (R = 8\dot{x} \times M)$$

$$\Rightarrow \ddot{x} + 8\dot{x} + 20x = 0$$

The auxiliary equation for the differential equation is

$$\lambda^2 + 8\lambda + 20 = 0 \qquad \Rightarrow \lambda = -4 \pm 2i$$

Therefore, the general solution of the differential equation is

$$x = e^{-4t}(A\sin 2t + B\cos 2t) \qquad \text{or} \qquad x = Ae^{-4t}\sin(2t + \varepsilon)$$

where A and B are arbitrary constants, and to obtain their values, at the start of the motion, i.e., initial conditions are $x = 4,\ dx/dt = 16$ as $t = 0$

when $t = 0, x = 4 \quad \Rightarrow B = 4$ when $t = 0, \dfrac{dx}{dt} = 16 \quad \Rightarrow B = 16$

The particular solution in this case is

$$x = 4e^{-4t}(4\sin 2t + \cos 2t) \qquad (1)$$

which gives the position of the particle at any instant and it illustrates that the motion of under-damping type with resistance factor e^{-4t}, it is clear that the amplitude of the motion reduces as Figure 9, to get the velocity in terms of time, differentiate Eq. (1)

$$\dot{x} = -16e^{-4t}(4\sin 2t + \cos 2t) + 4e^{-4t}(8\sin 2t - 2\cos 2t)$$
$$\Rightarrow \dot{x} = 16e^{-4t}(\cos 2t - 4.5\sin 2t+) \qquad (2)$$

Eq. (1) can be re-written as

$$x = 4\sqrt{17}e^{-4t}\sin(2t + \varepsilon) \qquad (\tan\varepsilon = 1/4)$$

That is the amplitude of the motion is $4\sqrt{17}e^{-4t}$ and the period

$$\tau = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi \ (\omega = 2)$$

The frequency is $\nu = 1/\tau = 1/\pi$

❒ **EXAMPLE:** The differential equation for a particle moves along straight line is $\ddot{x} + 5\dot{x} + 4x = 0$, initially the point at $x = 1, \dot{x} = 2$ as $t = 0$. Prove that the point reaches its maximum distance after time $(1/3)\ln 2$.

❒ **SOLUTION**

Since, $\ddot{x} + 5\dot{x} + 4x = 0$

The auxiliary equation for the differential equation is

$$\lambda^2 + 5\lambda + 4 = 0 \qquad \Rightarrow \lambda = -4, -1$$

Therefore, the general solution of the differential equation is

$$x = Ae^{-4t} + Be^{-t}$$

where A and B are arbitrary constants, and to obtain their values, at the start of the motion, i.e., initial conditions are $x = 1,\ \dot{x} = 2$ as $t = 0$

when $t = 0,\ x = 1 \quad \Rightarrow A + B = 1$ at $t = 0,\ \dot{x} = 2 \quad \Rightarrow 2 = -4A - B$

this gives $A = -1, B = 2$ The particular solution in this case is

$$x = 2e^{-t} - e^{-4t} \qquad \text{and} \qquad \dot{x} = 4e^{-4t} - 2e^{-t}$$

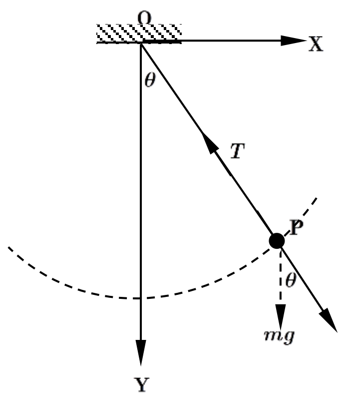
This is a over-damping motion, and the point reaches its maximum distance when $\dot{x} = 0$ that is

$$0 = 4e^{-4t} - 2e^{-t} \quad \Rightarrow 2e^{-4t} = e^{-t} \quad \text{or} \quad 2 = e^{3t} \quad \therefore t = \frac{1}{3}\ln 2$$

❒ **EXAMPLE:** A particle of mass *m* is attached to one end of a string, of length b, the other end of which is tied to a fixed point O, immersed in a viscous medium, where the resistance of the medium is proportional to the velocity with constant of proportionality equals $2m(g/b)^{1/2}$; g being the gravity acceleration. Initially the particle has been given a small angular displacement from rest with respect to vertical axis. Find the angular displacement and velocity at any instant.

❒ **SOLUTION**

Let the particle make an angle θ after time t as shown in the figure, then by using polar coordinate system the governing differential equation of motion is



$$mb\ddot{\theta} = -mg\sin\theta - Rb\dot{\theta} \qquad (R = 2m(g/b)^{1/2})$$

$$\Rightarrow \ddot{\theta} + \frac{g}{b}\sin\theta + \frac{2g}{b}\dot{\theta} = 0$$

But the angular displacement is small such that $\sin\theta \approx \theta$ and $\cos\theta = 1$

Hence the previous equation becomes

$$\ddot{\theta} + 2\sqrt{\frac{g}{b}}\dot{\theta} + \frac{g}{b}\theta = 0$$

The auxiliary equation for the differential equation is

$$\lambda^2 + 2\sqrt{\frac{g}{b}}\dot{\theta} + \frac{g}{b} = 0 \qquad \Rightarrow \lambda = \sqrt{\frac{g}{b}}, \sqrt{\frac{g}{b}} \quad \text{(repeated)}$$

Therefore, the general solution of the differential equation is

$$\theta = (A + Bt)e^{\sqrt{g/b}t}$$

where A and B are arbitrary constants, and this equation illustrates that the particle in critical damped harmonic motion.

■ Remember that in polar coordinate the velocity and acceleration components are given by $\underline{v} = (\dot{r}, r\dot{\theta}),\ \underline{a} = (\ddot{r} - r\dot{\theta}^2, r\ddot{\theta} + 2\dot{r}\dot{\theta})$)

and in particular case $r = b$ these components become

$\underline{v} = (0, b\dot{\theta}),\ \underline{a} = (-b\dot{\theta}^2, b\ddot{\theta})$

♦ Damped forced oscillations

Earlier in this chapter you saw the effect on simple harmonic motion of introducing a linear damping device called a dashpot. The oscillations, if they occurred, decayed to zero. If the damping constant, β, was large compared with the stiffness of the spring and the mass of the object, oscillations did not occur at all.

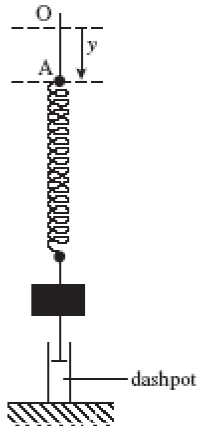


Figure 10

You have also seen the effect of forcing an undamped system. Most real systems do have an element of damping, so in this section we explore the effect of including a linear dashpot in the system. Figure 10 shows the spring-mass system which, as before, is forced to oscillate. A linear dashpot has been added below the object.

♦ Modeling forced vibrations: the undamped case

In order to understand the mathematics of forced oscillations, including resonance, we look at the simplest suitable case, that of an object hanging on a light, perfectly elastic spring, without damping. (The case in which both forcing and damping occur is considered later in this chapter).

In Figure 11 the top end A of the spring is forced to vibrate so that its displacement at time *t* is $y = A\sin\Omega t$. (This can be achieved experimentally, to a reasonable approximation, by attaching the supporting string over a pulley to a rotating cam.)

If the natural length of the spring is ℓ_0, the stiffness of the spring is k and the object has mass *m*, then in equilibrium, point A coincides with O, the object is at rest and the extension of the spring is $\gamma = mg/k$. At a general time *t* during the forced motion, the extension of the spring below the equilibrium position is

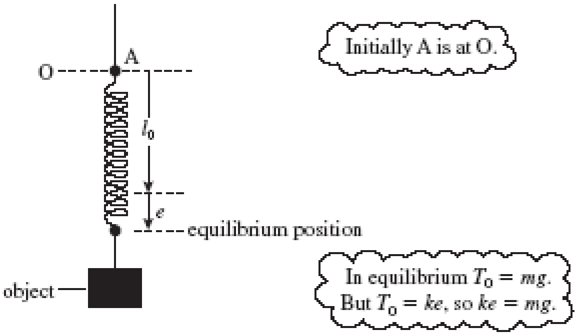


Figure 11

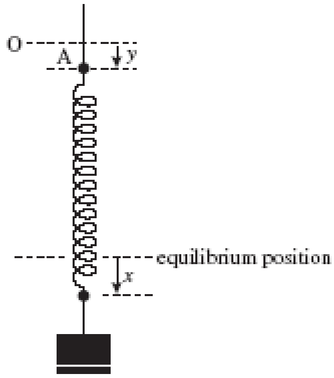


Figure 12

denoted by x (Figure 12).

There are two forces acting on the object, the force of gravity mg and the tension *T*. The acceleration of the object is $\ddot{x}$. Applying Newton's second law gives $m\ddot{x} = mg - T$

The extension of the spring is ($\gamma + x - y$), so the tension in the spring is

$T = k(\gamma + x - y)$.

The equation of motion is therefore

$$m\frac{d^2x}{dt^2} = mg - k(\gamma + x - y)$$

Expanding the right-hand side and recalling that $mg - k\gamma = 0$ (Figure 11), this becomes

$$m\frac{d^2x}{dt^2} + kx = ky$$

Dividing both sides by *m* and putting $\omega^2 = k/m$

$$\frac{d^2x}{dt^2} + \omega^2 x = \omega^2 y \quad \text{or} \quad \frac{d^2x}{dt^2} + \omega^2 x = \omega^2 f(t)$$

In the system we have described, $y = A\sin\Omega t$. This is *forced harmonic motion*, and $\Omega/2\pi$ is called the forcing frequency. The differential equation of motion may be written as $$\ddot{x} + \omega^2 x = A'\omega^2 \sin\Omega t \qquad (*)$$

The complementary function is given by $A\cos\omega t + B\sin\omega t$

For the particular integral, try $x = p\sin\Omega t + q\cos\Omega t$. This gives

$$\Rightarrow \frac{dx}{dt} = \Omega(p\cos\Omega t - q\sin\Omega t) \quad \text{and} \quad \frac{d^2x}{dt^2} = -\Omega^2(p\cos\Omega t + q\sin\Omega t)$$

Substituting these in the differential equation of motion gives

$-\Omega^2(p\cos\Omega t + q\sin\Omega t) + \omega(p\sin\Omega t + q\cos\Omega t) = A'\omega^2\sin\Omega t$ Equating coefficients:

$$\sin\Omega t: \quad -p\Omega^2 + p\omega^2 = A'\omega^2$$
$$\cos\Omega t: \quad -q\Omega^2 + q\omega^2 = 0$$

Normally you would use l and m in the trial function, but in this example p and q are used instead because l and m are representing length and mass.

Assuming $\Omega \neq \omega$, this gives $p = \dfrac{A'\omega^2}{\omega^2 - \Omega^2}$ and $q = 0$

The particular integral is therefore $\dfrac{A'\omega^2}{\omega^2 - \Omega^2}\sin\Omega t$

The general solution of the differential equation (*) is therefore

$$x = A\cos\omega t + B\sin\omega t + \frac{A'\omega^2}{\omega^2 - \Omega^2}\sin\Omega t, \qquad \Omega \neq \omega$$

Describing forced oscillations

You should have made the following deductions.

❒ The terms $A\sin\omega t + B\cos\omega t$ represent the natural or free oscillations of the system, as they would occur if the cam were not rotating. The natural frequency of the system is $\omega/2\pi$

The term $\dfrac{A'\omega^2}{\omega^2 - \Omega^2}\sin\Omega t$ represents the oscillations caused by rotating cam.

❒ As the value of Ω approaches that of ω, the quantity $\omega^2 - \Omega^2$ in the denominator tends to zero: the forced oscillations increase in amplitude. Consequently a small input amplitude A leads to a much larger output amplitude $A'\omega^2/(\omega^2 - \Omega^2)$. This effect is known as *resonance.*

❒ This solution is only valid in cases where Ω does not actually equal ω .

Over the last few pages we have set up and solved a differential equation to model a simple case of forced oscillations. This has given us a mathematical explanation for the phenomenon of resonance which we described earlier in this chapter.

<u>The case when</u> $\Omega = \omega$

Resonance occurs when the frequency of the driving function is the same as the natural frequency of the system. Looking back at the differential equation for the general spring-mass system (equation $\frac{d^2x}{dt^2} + \omega^2 x = A'\omega^2 \sin \Omega t$), this occurs when $\Omega = \omega$ and so the differential equation becomes

$$\frac{d^2x}{dt^2} + \omega^2 x = A'\omega^2 \sin \omega t$$

Since it is unaffected by the function on the right-hand side of the equation, the complementary function is still $A\sin\omega t + B\cos\omega t$. To obtain the particular integral, given the function on the right-hand side, you would normally try $x = A\sin\omega t + B\cos\omega t$ but this is included in the complementary function. So you multiply the usual trial function by the independent variable.

In this case try $x = t(A\sin\omega t + B\cos\omega t)$.

Differentiating this gives

$$\dot{x} = (A\sin\omega t + B\cos\omega t) + t(A\omega\cos\omega t - B\sin\omega t)$$

And $$\ddot{x} = 2\omega(A\cos\omega t - B\sin\omega t) - t\omega^2(A\sin\omega t + B\cos\omega t)$$

Substituting these into the differential equation gives

$$\begin{aligned} 2\omega(A\cos\omega t - B\sin\omega t) &- t\omega^2(A\sin\omega t + B\cos\omega t) \\ &+ t\omega^2(A\sin\omega t + B\cos\omega t) = A'\omega^2 \sin\omega t \end{aligned}$$

Equating coefficients:

$$\begin{aligned} \sin\omega t: &\quad -2B\omega - \omega^2 At + \omega^2 At = A'\omega^2 \quad \Rightarrow B = -\frac{1}{2}A'\omega \\ \cos\omega t: &\quad 2A\omega - \omega^2 Bt + \omega^2 Bt = 0 \quad \Rightarrow A = 0 \end{aligned}$$

So a particular integral is $-\frac{A'\omega t}{2}\cos\omega t$ and the general solution in the case when $\Omega = \omega$ is given by

This is called the forcing term.

$$x = A\sin\omega t + B\cos\omega t - \frac{A'\omega t}{2}\cos\omega t$$

Note that as *t* increases the forcing term dominates the solution. It represents an oscillation whose amplitude is proportional to *t* and so grows linearly with time. This is a mathematical description of resonance. It occurs when the forcing frequency is identical to the natural frequency of the system.

Drawing graphs of forced oscillations:

To illustrate the general results we have just established, we take a particular set of values for the variables involved and specify the initial conditions.

The general solution of

$$\frac{d^2x}{dt^2} + \omega^2 x = A'\omega^2 \sin\omega t \qquad (\Omega \neq \omega)$$

is given by $$x = A\sin\omega t + B\cos\omega t + \frac{A'\omega^2}{\omega^2 - \Omega^2}\sin\Omega t$$

Let us take the following values:

the stiffness of the spring: $k = 20\ \text{Nm}^{-1}$

the mass of the object: $m = 0.2$ kg

the amplitude of the forcing motion: A'= 0.02 m

acceleration due to gravity: $g = 10\ \text{ms}^{-2}$,

and we assume that initially the object is stationary at the equilibrium position so that when t = 0, $x = \dot{x} = 0$

Since $x = \frac{1}{100 - \Omega^2}(2\sin\Omega t - 0.2\Omega\sin 10t) \qquad (\Omega \neq \omega)$

Graphs in Figure 13 show the variation of x with t for various values of Ω

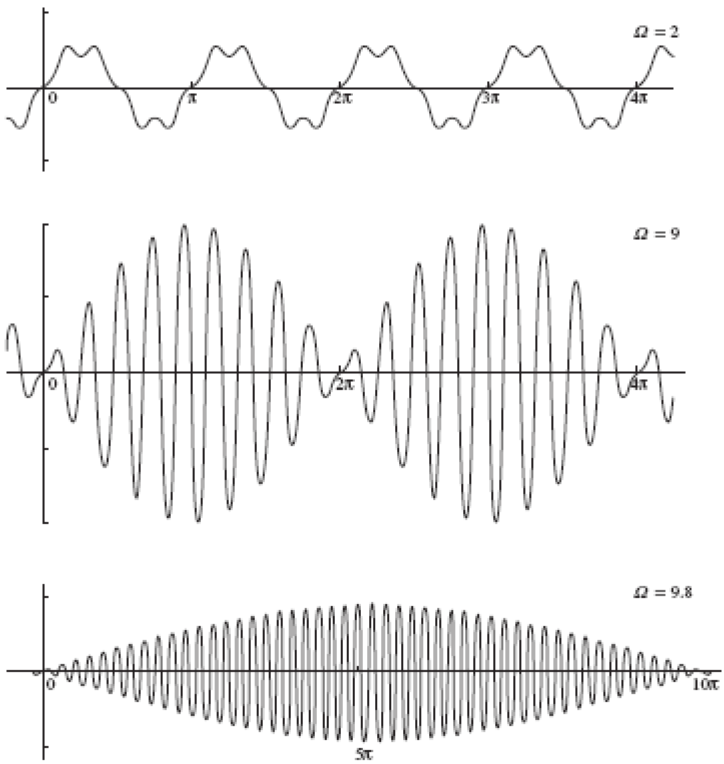


Figure 13

In the case of $(\Omega = \omega)$. The solution may be written as

$$x = 0.01(\sin 10t - 0.1t\cos 10t) \qquad (\Omega = \omega)$$

Figure 14 shows the graph of this solution.

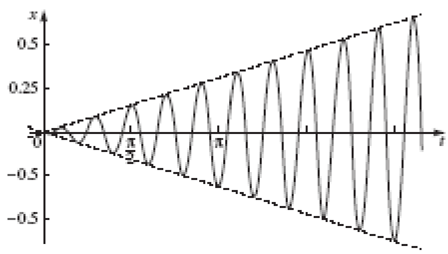


Figure 14

As before, the spring has stiffness 20 Nm^{-1}, the object has mass 0.2 kg and the amplitude of the forcing oscillation is 2 cm. The dashpot constant is 1 $Nm^{-1}s$ and g is taken to be 10 ms^{-2}. The object starts from rest in its equilibrium position.

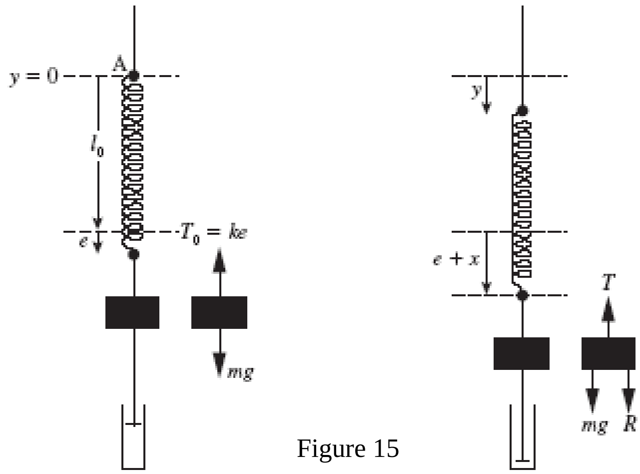


Figure 15

The first step is to formulate a differential equation to model this system. Figure 15 shows the system and the forces acting, first in equilibrium and then at some general time t during the motion.

In equilibrium, the extension of the spring is $\gamma = \frac{mg}{k} = \frac{2}{20} = 0.1$. At the general time t the object is displaced a distance x below the equilibrium level.

The acceleration of the object is $\frac{d^2x}{dt^2}$

The net force on the object in the direction of positive x (i.e. downwards) is

$$mg + R - T$$

Applying Newton's second law at any instant t gives

$$m\frac{d^2x}{dt^2} = mg + R - T$$

The extension of the spring is $\gamma + x - y$ so $T = 20(0.1 + x - y)$

Let the length of the dashpot at time t be L, and its length in equilibrium be L_0.

Then $L = L_0 - x$ and $\dfrac{dL}{dt} = -\dfrac{dx}{dt}$

The dashpot force R is given by $\beta\dfrac{dL}{dt}$, where β is the dashpot constant. So in this case,

$$R = 1\frac{dL}{dt} = -\frac{dx}{dt} \qquad (\beta = 1)$$

The equation of motion becomes

This is mg.

$$0.2\frac{d^2x}{dt^2} = 2 - \frac{dx}{dt} - 20(0.1 + x - y)$$

$$\therefore \frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 100x = 100y$$

If the displacement of the forcing point is $y = 0.02\sin\Omega t$, the differential equation modeling the system is

$$\therefore \frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 100x = 2\sin\Omega t$$

The next stage is to solve the differential equation, which is (as before) a nonhomogeneous linear equation with constant coefficients. Its auxiliary equation is

$$\lambda^2 + 5\lambda + 100 = 0 \qquad \Rightarrow \lambda = -2.5 \pm 9.68i$$

The complementary function is therefore $e^{-2.5t}(A\sin 9.68t + B\cos 9.68t)$.

For the particular integral, try $x = p\sin\Omega t + q\cos\Omega t$.

Differentiating this gives

$$\frac{dx}{dt} = \Omega(p\cos\Omega t - q\sin\Omega t), \qquad \frac{d^2x}{dt^2} = -\Omega^2(p\sin\Omega t + q\cos\Omega t)$$

Substituting these in the differential equation for the system, gives

$$-\Omega^2(p\sin\Omega t + q\cos\Omega t) + 5\Omega(p\cos\Omega t - q\sin\Omega t) + 100(p\sin\Omega t + q\cos\Omega t) = 2\sin\Omega t$$

Equating coefficients:

$$\sin\Omega t: \quad -p\Omega^2 - 5q\Omega + 100p = 2$$
$$\cos\Omega t: \quad -q\Omega^2 + 5p\Omega + 100q = 0$$

Solving these equations for *p* and *q* gives

$$p = \frac{2(100-\Omega^2)}{(100-\Omega^2)^2 + 25\Omega^2}, \quad \text{and} \quad q = -\frac{10\Omega}{(100-\Omega^2)^2 + 25\Omega^2},$$

The particular integral is therefore

$$\frac{2(100-\Omega^2)}{(100-\Omega^2)^2 + 25\Omega^2}\sin\Omega t - \frac{10\Omega}{(100-\Omega^2)^2 + 25\Omega^2}\cos\Omega t$$

The general solution is the sum of the complementary function and the particular integral:

$$x = e^{-2.5t}(A\sin 9.68t + B\cos 9.68t) + \frac{2(100-\Omega^2)}{(100-\Omega^2)^2 + 25\Omega^2}\sin\Omega t - \frac{10\Omega}{(100-\Omega^2)^2 + 25\Omega^2}\cos\Omega t$$

As *t* increases, the natural damped oscillations, given by the complementary function,

decay because of the $e^{-2.5t}$ term, leaving

$$\frac{2}{\sqrt{(100-\Omega^2)^2 + 25\Omega^2}}$$

This is called the *steady state* solution. It is the particular integral of the differential equation. It describes the oscillations that occur after the unforced oscillations have died away. Figure 16 shows graphs of this steady state solution for two values of Ω.

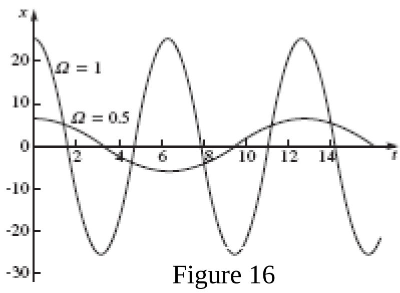


Figure 16

Remember that in the undamped case the value of Ω for resonance was calculated by setting the denominator to zero in the particular integral. Catastrophic resonance does not occur in the damped case, because the denominator of each part of the particular integral is always greater than zero. However, the amplitude of the forced vibrations does still depend on the value of Ω.

The amplitude of the steady state oscillations is the square root of the sum of the squares of the coefficients of $\cos\Omega t$ and $\sin\Omega t$ in the steady state solution, i.e.

$$\sqrt{\left(\frac{2(100-\Omega^2)}{(100-\Omega^2)^2+25\Omega^2}\right)^2+\left(\frac{10\Omega}{(100-\Omega^2)^2+25\Omega^2}\right)^2}$$

This can be simplified to

$$\frac{2}{\sqrt{(100-\Omega^2)^2+25\Omega^2}}$$

This result shows how the amplitude of the steady state solution depends on Ω. Figure 17 shows a graph of this amplitude against Ω

When Ω and ω are close in value to each other, then it follows that the forcing frequency $\Omega/2\pi$ and the natural frequency $\omega/2\pi$ are close in value. When this is the case, the steady state oscillations become large compared with the input amplitude. The motion of the system at these relatively large amplitudes is still called resonance, though the amplitude of the vibrations does not increase without limit as it does in the undamped case.

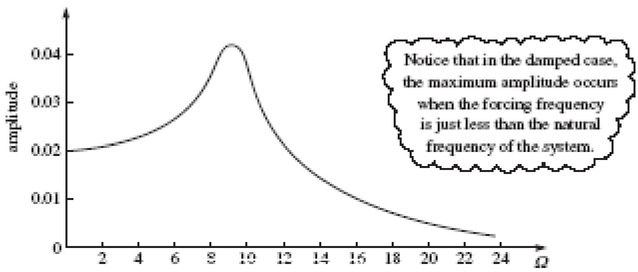


Figure 17

You have now seen the effect of linear damping on the system. What would be the effect of varying the damping constant, β? To predict this, look at the differential equation for the same damped spring-mass system (m = 0.2 and k = 20), but this time use a general damping constant β. The equation becomes

$$0.2\frac{d^2x}{dt^2}+\beta\frac{dx}{dt}+20x=0.4\sin\Omega t$$

We know that the complementary function decays and that the steady state oscillations are given by the particular integral. In this case it is

$$x = \frac{0.4(20 - 0.2\Omega^2)}{(20 - 0.2\Omega^2)^2 + \beta^2\Omega^2}\sin\Omega t - \frac{0.4\beta\Omega}{(20 - 0.2\Omega^2)^2 + \beta^2\Omega^2}\cos\Omega t$$

The amplitude of the steady state forced oscillation is

$$\frac{2}{\sqrt{(100 - \Omega^2)^2 + 25\beta^2\Omega^2}}$$

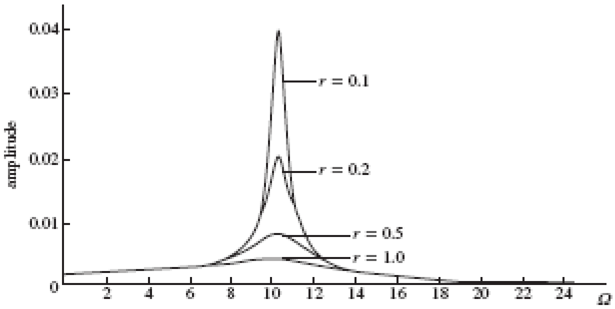


Figure 18

Figure 18 shows graphs of the steady state amplitude against Ω for different values of β. The graphs show that as β decreases (i.e. the amount of damping is reduced), the amplitude at the resonant frequency increases. In each case resonance occurs when Ω is very near in value to ω (in this case 10). In any real system there is always some damping but, as you can see, if the damping constant is small the resonance can still be damaging.

Activity: The previous example involved a particular case of damped forced motion in which the various parameters of the system were given particular values.

i) Show that the differential equation modelling the general case may be written. $\dfrac{d^2x}{dt^2} + \alpha\dfrac{dx}{dt} + \omega^2 x = A\omega^2 \sin\Omega t$

ii) Find

a) the general solution;

b) the particular solution corresponding to $x = \dot{x} = 0$ at t = 0

c) the amplitude of the steady state oscillations.

PROBLEMS

1 An electrical circuit consists of a 0.2 henry inductor, a 1 ohm resistor and a 0.8 farad capacitor in series. The charge q coulombs on the capacitor is modelled by the differential equation

$$0.2\frac{d^2q}{dt^2} + \frac{dq}{dt} + \frac{1}{0.8}q = 0.$$

Initially q is 2 and $\frac{dq}{dt}$ (the current in ampères) is 4.

i) Find an equation for the charge as a function of time.

ii) Sketch the graphs of charge and current against time. Describe how the charge and current change.

iii) What is the charge on the capacitor and the current in the circuit after a long period of time?

2 The temperature of a chemical undergoing a reaction is modelled by the differential equation

$$2\frac{d^2T}{dt^2} + \frac{dT}{dt} = 0$$

where T is the temperature in °C and t is the time in minutes.

For a particular experiment, the temperature is initially 50°C, and it is 45°C one minute later.

i) Find an expression for the temperature T at any time.

ii) What will the temperature be after two minutes?

iii) Sketch a graph of T against t.

iv) What is the steady state temperature?

3 The angular displacement from its equilibrium position of a swing door fitted with a damping device is modelled by the differential equation

$$\frac{d^2\theta}{dt^2} + 4\frac{d\theta}{dt} + 5\theta = 0.$$

The door starts from rest at an angle of $\frac{\pi}{4}$ from its equilibrium position.

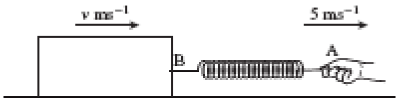
i) Find the general solution of the differential equation.

ii) Find the particular solution for the given initial conditions.

iii) Sketch a graph of the particular solution, and hence describe the motion of the door.

iv) What does your model predict as t becomes large?

4 The diagram shows a block of mass 10 kg being dragged across a horizontal surface by means of an elastic spring AB of unstretched length 0.5 m. The end A is being pulled with constant speed 5 ms^{-1}. The block is attached to the end B of the spring.



The block is subject to a resistance force given by $4v$ N, where v is the speed of the block in ms^{-1}. The only other horizontal force acting on the block is the tension in the spring given by $T = k(y - 0.5)$ N, where y is the length of the spring in metres and k is a constant (the stiffness of the spring).

i) By considering the horizontal forces acting on the block, write down the equation of motion of the block in terms of $\frac{dv}{dt}$, v, y and k. Use the fact that $\frac{dy}{dt} = 5 - v$ to eliminate v, and show that $10\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + ky = 0.5k + 20$.

ii) You are given that $k = 20$. Find the general solution of the differential equation.

At the beginning of the motion, the spring is unstretched and the block is at rest.

iii) Find an expression for y at time $t > 0$.

iv) What are the limiting values of v and y after a long period of time? Explain briefly how these values would be affected if a different spring were used with a lower value of k.

5 An electrical circuit consists of a 1 henry inductor and a 10^{-4} F capacitor in series with a sinusoidal power source.

The charge q, in coulombs (C), stored in the capacitor is given by the differential equation

$$\frac{d^2q}{dt^2} + 10\,000q = 1000\sin\Omega t$$

where t is the time in seconds. Initially the charge q and current $\frac{dq}{dt}$ in the circuit are both zero.

i) Find the particular solution for the charge q.

ii) State the value of Ω for which resonance occurs. Find the particular solution for $q(t)$ in this case. Calculate the time at which the charge first exceeds 10 000 C.

iii) Use a graphics calculator or computer to graph the solution in part ii).

6 A sphere of mass m and radius a is falling vertically through a liquid which produces a linear resistance force. The motion of the sphere is modelled by the differential equation

$$m\frac{d^2x}{dt^2} + ra\frac{dx}{dt} = mg$$

where x is the distance fallen in t seconds and r is a constant. The sphere is released from rest so that $x = 0$ and $\frac{dx}{dt} = 0$ when $t = 0$.

i) Find the solution for x and hence the velocity of the sphere as a function of time.

ii) Draw a graph of the velocity against time, and describe the motion of the sphere.

7 During the design of a car, part of the suspension system is tested by subjecting it to violent displacements. One such test is modelled by the differential equation

$$\ddot{x} + 2k\dot{x} + x = 1$$

where x is displacement, and initially $x = 0$ and $\dot{x} = 0$. The parameter k (> 0) is known as the damping coefficient and can be varied during the tests.

For optimum road holding a 'hard' suspension is desirable and it is believed that to achieve this the damping should be critical.

i) Find the value of k for critical damping.

ii) Determine x as a function of time t in this case.

For a more comfortable ride a 'soft' suspension is proposed in which $k = 0.6$.

iii) Determine x as a function of time t for the 'soft' suspension.

iv) Find the maximum displacement of the 'soft' suspension.

8 The current in an electrical circuit consisting of an inductor, resistor and capacitor in series with an alternating power source, is described by the equation

$$\frac{d^2I}{dt^2} + 25\frac{dI}{dt} + 100I = -170 \sin 20t$$

where I is the current in amperes and t is the time in seconds after the power source is switched on.

i) Find the general solution.

When $t = 0$, $\dfrac{dI}{dt} = I = 0$.

ii) Find the solution.

The exponentially decaying terms in the solution describe what is known as the transient current. The non-decaying terms describe the steady state current.

iii) Write down an expression for the steady state current for the solution in part ii). Why would this expression remain unchanged if the initial conditions were different?

iv) Express the steady state current in the form $R \sin (20t + \alpha)$, where R and α are to be determined. Verify that, after only 1 second, the magnitude of the transient current is close to 1% of the steady state amplitude, R.

MOTION WITH VARIABLE MASS

So far in this course we have dealt exclusively with the dynamics of particles and bodies whose masses remain constant during the motion. In certain applications we can't make this assumption. A rocket is propelled by ejecting burnt fuel which causes the mass of the rocket to decrease substantially as the rocket accelerates. A raindrop falling through a damp atmosphere coalesces with smaller droplets which increase its mass. In both of these illustrations the mass of the body may be thought of as varying with time: the term "variable mass" is slightly misleading since we don't intend to mean that mass is being created or being destroyed, but that it is being removed or added to the body.

Suppose that a body having variable mass $m(t)$ is moving with velocity $\underline{v}(t)$. At time $t+\delta t$ let its main mass be $m(t+\delta t)$ and its velocity be $\underline{v}(t+\delta t)$. The body has either gained or lost incrementally mass $-m(t+\delta t)+m(t)$ depending on the sign of this difference. For the sake of discussion let us suppose that an increment of mass has broken from the main body with absolute velocity $\underline{u}(t)$, Fig. 1. At time t this mass $m(t)-m(t+\delta t)$ (which will be positive in this case) experiences a sudden velocity change from $\underline{v}(t)$ to $\underline{u}(t)$. For the whole system the momentum at time $t+\delta t$ is

$$m(t+\delta t)\underline{v}(t+\delta t)+\{m(t)-m(t+\delta t)\}\underline{u}(t+\delta t)$$

whilst at time t it was $m(t)\underline{v}(t)$. We are only concerned with the translation of the body: any rotational effects on the body will not be considered here. The change of momentum is

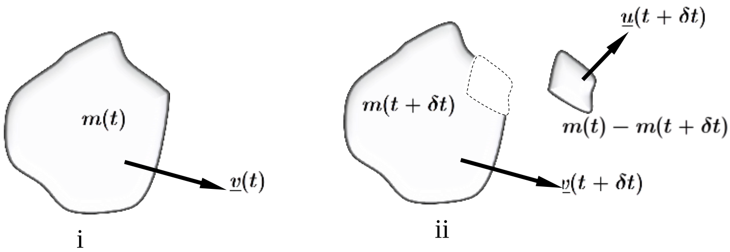


Fig. 1 Body of mass undergoing incremental mass change (i) shows the body immediately before the mass $m(t)-m(t+\delta t)$ is ejected with absolute velocity $u(t)$ (b) shows the situation at time $t+\delta t$ with the main mass now $m(t+\delta t)$

$$\begin{aligned} &m(t+\delta t)\underline{v}(t+\delta t)+\{m(t)-m(t+\delta t)\}\underline{u}(t+\delta t)-m(t)\underline{v}(t)\\ &=\{m(t+\delta t)-m(t)\}\underline{v}(t)+m(t)\{\underline{v}(t+\delta t)-\underline{v}(t)\}\\ &\qquad -\{m(t+\delta t)-m(t)\}\underline{u}(t+\delta t)\end{aligned}$$

where the previous line has been prepared for division by the time increment δt. We now divide the right-hand side above by δt so that

$$\frac{\{m(t+\delta t)-m(t)\}}{\delta t}\underline{v}(t)+\frac{\{\underline{v}(t+\delta t)-\underline{v}(t)\}}{\delta t}m(t)-\frac{\{m(t+\delta t)-m(t)\}}{\delta t}\underline{u}(t+\delta t)$$

From the definition of the derivative

$$\frac{dm}{dt}=\lim_{\delta t\to 0}\frac{m(t+\delta t)-m(t)}{\delta t}$$

etc. Consequently the rate of change of the linear momentum of the body becomes (taking limit as δt approach zero)

$$\frac{dm}{dt}\underline{v}(t)+m(t)\frac{d\underline{v}}{dt}-\frac{dm}{dt}\underline{u}(t)=(\underline{v}(t)-\underline{u}(t))\frac{dm}{dt}+m(t)\frac{d\underline{v}}{dt}$$

Suppose that the body is also subject to an external force $\underline{F}$. Then Newtonian's second law of motion is now interpreted in the form, force equals the rate of change of the linear momentum of the body which is the same as the previous hypothesis $\underline{F}=m(d\underline{v}\,/\,dt)$ if the mass of the body remains constant. Hence it follows that the equation of motion assumes the form

$$\underline{F}=(\underline{v}-\underline{u})\frac{dm}{dt}+m\frac{d\underline{v}}{dt} \qquad \text{....(1)}$$

where we have now dropped the time arguments of $m, \underline{v}$ and $\underline{u}$.

It may appear at first sight that Eq. (1) is inconsistent with the expression

$$\frac{d}{dt}(m\underline{v}) = \underline{v}\frac{dm}{dt} + m\frac{d\underline{v}}{dt}$$

which may be thought as the rate of change of momentum $m\underline{v}$. However, any disposal or accretion of mass which involves a velocity difference will have a continuous impulsive effect on the remaining mass. Hus the disposal of the increment $m(t) - m(t+\delta t)$ involves a velocity difference $\underline{u} - \underline{v}$. Hence over the time interval this means that the remaining mass experiences an impulse

$$(m(t) - m(t+\delta t))(\underline{v}(t) - \underline{u}(t))$$

The corresponding force $\underline{F}_1$, say, as a continuous function of time is given by

$$\underline{F}_1 = \lim_{\delta t \to 0} \frac{(m(t) - m(t+\delta t))}{\delta t}(\underline{v}(t) - \underline{u}(t))$$

Thus we could interpret Eq. (1) also as

$$\underline{F} + \underline{F}_1 = m\frac{d\underline{v}}{dt}$$

Notice that eq. (1) becomes

$$\underline{F} = m\frac{d\underline{v}}{dt}$$

If $\underline{u} = \underline{v}$. In this case mass is being lost or acquired but at zero relative velocity. If $\underline{u} = \mathbf{0}$, then

$$\underline{F} = m\frac{d\underline{v}}{dt} + \underline{v}\frac{dm}{dt} = \frac{d}{dt}(m\underline{v})$$

This corresponds, for example, to the case of the raindrop falling through a stationary cloud of droplets. Equation (1) is the fundamental relation for motion with variable mass. However, in order to be able to analyze a problem we shall still need to specify the rate of mass change and its velocity in addition to the external force.

■ Illustrative Examples ■

❒ EXAMPLE

A balloon of mass M contains a bag of sand of mass m_0, and the balloon is in equilibrium. The sand is released at a constant rate and is disposed of in a time t_0. Find the height of the balloon and its velocity when all the sand has been released. Assume that the balloon experiences a constant upthrust and neglect air resistance.

❒ SOLUTION

In equilibrium the up-thrust F must balance the weight of the balloon and sand

$$F = (M + m_0)g \qquad \text{....(1)}$$

Let m be the mass of the sand at time t where $0 \le t \le t_0$ then

$$M = m_0\left(1 - \frac{1}{t_0}\right) \qquad \text{.....(2)}$$

since the sand is released at a constant rate. The velocity of the sand relative to the balloon is zero on release with the result that $v = u$ in Eq. (1) . Let x be the subsequent displacement of the balloon. Its equation of motion becomes

$$(M + m_0)g - (M + m)g = (M + m) \qquad \text{....(3)}$$

where $v = \dot{x}$. On substituting for m from Eq. (2) into Eq. (3)

$$\frac{dv}{dt} = \frac{m_0 gt}{(M + m_0)t_0 - m_0 t}$$

$$= -g\frac{\pm(M + m_0)t_0 - m_0 t}{(M + m_0)t_0 - m_0 t}$$

$$= -g + \frac{(M + m_0)gt_0}{(M + m_0)t_0 - m_0 t}$$

This is a variable separable equation with solution

$$v = -gt - \frac{(M+m_0)gt_0}{m_0}\ln\left(1 - \frac{m_0 t}{(M+m_0)t_0}\right) \qquad \text{....(4)}$$

where the initial condition $v = 0$ when $t = 0$ has been used.

The differential equation for the displacement is

$$\frac{dx}{dt} = -gt - \frac{(M+m_0)gt_0}{m_0}\ln\left(1 - \frac{m_0 t}{(M+m_0)t_0}\right)$$

is gain of separable type with solution

$$x = -\int\left(gt + \frac{g}{k}\ln(1-kt)\right)dt + A, \qquad k = \frac{m_0}{(M+m_0)t_0}$$

$$= A - \frac{1}{2}gt^2 - \frac{gt}{k}\ln(1-kt) - g\int\frac{tdt}{1-kt} \quad \text{(integrating by parts)}$$

$$= A - \frac{1}{2}gt^2 - \frac{gt}{k}\ln(1-kt) - g\int\left(-1 + \frac{1}{1-kt}\right)dt$$

$$= A - \frac{1}{2}gt^2 - \frac{gt}{k}\ln(1-kt) + \frac{gt}{k} + \frac{g}{k^2}\ln(1-kt)$$

$$= A - \frac{1}{2}gt^2 + \frac{g}{k^2}(1-kt)\ln(1-kt) + \frac{gt}{k}$$

Taking the initial condition to be $x = 0$ when $t = 0$, we see that $A = 0$. Thus

$$x = \frac{gt}{k} - \frac{1}{2}gt^2 + \frac{g}{k^2}(1-kt)\ln(1-kt) \qquad \text{....(5)} \blacksquare$$

All equations and solutions hold only during the time interval $0 \le t \le t_0$
at time $t = t_0$ the balloon has reached a height

$$x_0 = \frac{gt_0^2}{2m_0^2}\left((2M+m_0)m_0 + 2M(M+m_0)\ln\left(\frac{M}{M+m_0}\right)\right)$$

and is moving with speed

$$v_0 = \frac{gt_0}{m_0}\left((M+m_0)\ln\left(\frac{M+m_0}{M}\right) - m_0\right)$$

❐ EXAMPLE

A spherical raindrop of initial radius a, falling freely, receives in each instant an increase of volume equal to λ times its surface at that instant; determine the velocity at the end of time t, and the distance fallen through in that time.

❐ SOLUTION

Let m be the mass and r the radius of the raindrop when it has fallen through a distance y in time t. Also let v be the velocity at that instant so that

$$\frac{dy}{dt} = v \qquad \text{....(1)}$$

Since raindrop is falling freely under gravity

$$P = mg \qquad \text{....(2)}$$

Now $m =$ mass of raindrop at time t equals

$$m = \frac{4}{3}\pi r^3 \rho \qquad \text{...(3)}$$

$$\Rightarrow \frac{dm}{dt} = 4\pi r^2 \rho \frac{dr}{dt}$$

But given, rate of increase of mass i.e., ($\Rightarrow dm / dt = \lambda \; 4\pi r^2 \; \rho$)

Therefore, equating two values of dm / dt, we have

$$\Rightarrow 4\pi r^2 \rho \frac{dr}{dt} = \lambda \rho (4\pi r^2) \qquad \text{or} \qquad \frac{dr}{dt} = \lambda$$

Integrating $\quad r = \lambda t + A \quad$ (A is a constant of integration)

Initially at $t = 0$, $r = a$ then $A = a$

Hence $\quad r = \lambda t + a \qquad$ ...(4)

$$m = \frac{4}{3}\pi \rho (\lambda t + a)^3 \qquad \text{....(5)} \quad \text{(Equations (3) and (4))}$$

Also the relative velocity $V = 0$, since the mass is picked up from rest. Hence Hence the equation of variable mass namely

$$\frac{d}{dt}(mv) = P + V\frac{dm}{dt} \quad \text{becomes}$$

$$\frac{d}{dt}\left(\left(\frac{4}{3}\pi\rho(\lambda t + a)^3\right)v\right) = \frac{4}{3}\pi\rho(\lambda t + a)^3 g \qquad \text{(using Eq. (2) and (5))}$$

or $\frac{d}{dt}\ (\lambda t + a)^3 v\ = (\lambda t + a)^3 g \qquad \Rightarrow d\ (\lambda t + a)^3 v\ = (\lambda t + a)^3 g\, dt$

Integrating,

$$(\lambda t + a)^3 v = \frac{g}{4\lambda}(\lambda t + a)^4 + A$$

Initially $v = 0,\ t = 0 \quad \therefore A = -(ga^4 / 4\lambda)$

$$(\lambda t + a)^3 v = \frac{g}{4\lambda}\ (\lambda t + a)^4 - a^4$$

or

$$v = \frac{g}{4\lambda}\left((\lambda t + a) - \frac{a^4}{(\lambda t + a)^3}\right) \qquad \text{....(6)} \blacksquare$$

This equation gives velocity at any time t

Now from Eq. (1) and (6) we get

$$\frac{dy}{dt} = \frac{g}{4\lambda}\left((\lambda t + a) - \frac{a^4}{(\lambda t + a)^3}\right)$$

Integrating;

$$y = \frac{g}{8\lambda^2}\left((\lambda t + a)^2 + \frac{a^4}{(\lambda t + a)^2}\right) + B$$

Initially $y = 0,\ t = 0 \quad \therefore B = -(ga^2 / 4\lambda^2)$. Hence

$$y = \frac{g}{8\lambda^2}\left((\lambda t + a)^2 + \frac{a^4}{(\lambda t + a)^2} - 2a^2\right)$$

$$= \frac{g}{8\lambda^2}\left((\lambda t + a) - \frac{a^2}{(\lambda t + a)}\right)^2$$

$$= \frac{g}{8\lambda^2}\left(\frac{(\lambda t + a)^2 - a^2}{(\lambda t + a)}\right)^2 = \frac{gt^2}{8}\left(\frac{(\lambda t + 2a)^2}{(\lambda t + a)^2}\right) \qquad \text{....(7)} \blacksquare$$

This gives required distance.

❒ EXAMPLE

A spherical raindrop of radius a cm., falls from rest through a vertical height h, receiving throughout the motion an accumulation of condensed vapour at the rate of μ grams per square cm. per second, no vertical force but gravity acting, show that when the raindrop reaches the ground its radius will be

$$\mu\sqrt{\frac{2h}{g}}\left(1+\sqrt{1+\frac{ga^2}{2h\mu^2}}\right)$$

❒ SOLUTION

Proceeding exactly as a previous example, the height h from Eq. (7) is given by

$$h=\frac{gt^2}{8}\left(\frac{(\mu t+2a)^2}{(\mu t+a)^2}\right)\quad \text{(replacing } \lambda \text{ by } \mu\text{)}$$

$$\Rightarrow 2\sqrt{\frac{2h}{g}}=t\,\frac{\mu t+2a}{\mu t+a} \qquad \text{....(1)}$$

Also radius r at any time t is

$$r=a+\mu t \quad \text{from Eq.(4) previous Ex.}$$
$$\Rightarrow t=(r-a)/\mu$$

Substituting values of t and $a+\mu t$ in Eq. (1), we have

$$\Rightarrow 2\sqrt{\frac{2h}{g}}=\frac{r-a}{\mu}\left(\frac{r+a}{r}\right)$$
$$\Rightarrow 2\mu r\sqrt{\frac{2h}{g}}=r^2-a^2$$
$$\Rightarrow r^2-\left(2\mu\sqrt{\frac{2h}{g}}\right)r-a^2=0$$

Solving this equation we have (neglect – sign before square root)

$$r=\frac{2\mu\sqrt{\frac{2h}{g}}+\sqrt{4\mu^2\frac{2h}{g}+4a^2}}{2}$$
$$=\mu\sqrt{\frac{2h}{g}}+\sqrt{\frac{2h\mu^2}{g}+a^2}$$

$$= \mu\sqrt{\frac{2h}{g}} + \mu\sqrt{\frac{2h}{g}}\left(\sqrt{1+\frac{ga^2}{2h\mu^2}}\right)$$
$$= \mu\sqrt{\frac{2h}{g}}\left(1+\sqrt{1+\frac{ga^2}{2h\mu^2}}\right)$$

❒ EXAMPLE

A trailer full of sand is pulled by a constant force F, Sand leaks out at the rate of λ units of mass per second. Prove that the velocity at the end of time is

$$-\frac{F}{\lambda}\ln\left(1-\frac{\lambda t}{M}\right)$$

and the distance moved during that time t is

$$\frac{FM}{\lambda^2}\left\{\left(1-\frac{\lambda t}{M}\right)\ln\left(1-\frac{\lambda t}{M}\right)+\frac{\lambda t}{M}\right\}$$

where M is the initial mass of the trailer and contents.

❒ SOLUTION

Let m be the mass and v the velocity of the trailer and its contents after any time t.

$$\Rightarrow m = M - \lambda t \qquad ...(1)$$

so that $$dm/dt = \lambda \qquad ....(2)$$

$$V = \text{ velocity of leaking sand } = v \qquad ...(3)$$

$$P = F$$

Hence equation of motion when mass varies namely

$$\frac{d}{dt}(mv) = P + V\frac{dm}{dt} \quad \text{gives on using Eqs. (1), (2) and (3)}$$

$$\frac{d}{dt}((M-\lambda t)v) = F + v(-\lambda)$$
$$\Rightarrow (M-\lambda t)\frac{dv}{dt} - \lambda v = F - \lambda v \quad \text{or} \quad \frac{dv}{dt} = \frac{F}{M-\lambda t}$$

Integrating after separation variables, we get

$$v = -\frac{F}{\lambda}\ln\ M - \lambda t\ + A$$

Initially $v = 0$, at $t = 0$ $\Rightarrow A = \frac{F}{\lambda}\ln M$

$$\Rightarrow v = -\frac{F}{\lambda}\ln\ M - \lambda t\ + \frac{F}{\lambda}\ln M$$

$$= -\frac{F}{\lambda}\ln\left(\frac{M - \lambda t}{M}\right) = -\frac{F}{\lambda}\ln\left(1 - \frac{\lambda t}{M}\right) \qquad \text{....(4)} \blacksquare$$

Now if x is the distance described by the trailer in time t, then

$$\frac{dx}{dt} = v \qquad \text{.....(5)}$$

From Eqs. (4) and (5), we have

$$\frac{dx}{dt} = -\frac{F}{\lambda}\ln\left(1 - \frac{\lambda t}{M}\right) \qquad \Rightarrow dx = -\frac{F}{\lambda}\ln\left(1 - \frac{\lambda t}{M}\right)dt$$

Integrating,

$$x = -\frac{F}{\lambda}\int \ln\left(1 - \frac{\lambda t}{M}\right)dt + B$$

or

$$x = -\frac{F}{\lambda}\left\{t\ln\left(1 - \frac{\lambda t}{M}\right) - \int t\frac{1}{1 - \frac{\lambda t}{M}}\left(-\frac{\lambda}{M}\right)dt\right\} + B \quad \text{Integrating by parts}$$

$$= -\frac{Ft}{\lambda}\ln\left(1 - \frac{\lambda t}{M}\right) - F\int \frac{t}{M - \lambda t}dt + B$$

$$= -\frac{Ft}{\lambda}\ln\left(1 - \frac{\lambda t}{M}\right) + \frac{F}{\lambda}\int \frac{M - \lambda t - M}{M - \lambda t}dt + B$$

$$= -\frac{Ft}{\lambda}\ln\left(1 - \frac{\lambda t}{M}\right) + \frac{F}{\lambda}t + \frac{FM}{\lambda^2}\ln\left(1 - \frac{\lambda t}{M}\right) + B$$

Initially $x = 0$, at $t = 0$ $\Rightarrow B = 0$

$$x = -\frac{Ft}{\lambda}\ln\left(1 - \frac{\lambda t}{M}\right) + \frac{F}{\lambda}t + \frac{FM}{\lambda^2}\ln\left(1 - \frac{\lambda t}{M}\right)$$

$$= \frac{FM}{\lambda^2}\left\{\left(1 - \frac{\lambda t}{M}\right)\ln\left(1 - \frac{\lambda t}{M}\right) + \frac{\lambda t}{M}\right\} \blacksquare$$

❒ EXAMPLE

A rocket whose total initial mass (fuel+ shell) is m_0 ejects fuel at a constant rate cm_0 and at a velocity V relative to the case. Deduce that the lowest rate of fuel consumption that will permit the rocket to rise at once is $c = g/V$. Assuming this design condition is met, Obtain the greatest speed and height reached by the rocket.

❒ SOLUTION

The equation of motion of variable mass is

$$\frac{d}{dt}(mv) = P + V\frac{dm}{dt} \qquad .....(1)$$

where m is mass at time t.

Here $P = -mg$ (since mass is moving in upward direction)

$$\frac{dm}{dt} = -cm_0 \quad \text{as mass is ejected} \qquad .....(2)$$

Integrating Eq. (2) $m = -cm_0 t + A$,

Initially $m = m_0, t = 0$ then $A = m_0$ $\qquad m = m_0(1 - ct) \quad .....(3)$

Substituting values of P and V from above in Eq. (1), we get

$$m\frac{dv}{dt} + v\frac{dm}{dt} = -mg + (v - V)\frac{dm}{dt}$$
$$m\frac{dv}{dt} = -mg - V\frac{dm}{dt}$$
$$\frac{dv}{dt} = -g + \frac{cVm_0}{m} \qquad \text{(from Eq.(2))} \quad ....(4)$$
$$\frac{dv}{dt} = -g + \frac{cV}{1 - ct} \qquad \text{(from Eq.(3))} \quad ....(5)$$

Integrating, we have

$$v = -gt - V\ln\ 1 - ct\ + B$$

PROBLEMS

1. A rocket of total mass M contains a proportion εM $(0 < \varepsilon < 1)$ as fuel. If the exhaust speed c is constant show that the final speed of the rocket is independent of the rate at which the fuel is burnt.

2. A rocket of mass M ejects fuel at a constant rate k with exhaust speed c. Show that the rocket will not rise initially from the Earth's surface unless $k > Mg/c$.

3. A rocket of total mass $M + m_0$ contains fuel of mass εM $(\varepsilon < 1)$. The payload is of mass m_0 and $(1 - \varepsilon)M$ is the mass of the rocket casing. Suppose it is technically possible to discard the casing continuously at a constant rate whilst the fuel is burning so that no casing remains when the fuel is burnt. If the fuel is burnt at the constant rate k show that the casing must be discarded at the rate $(1 - \varepsilon)k/\varepsilon$. Verify that, if $\varepsilon = 0.83$ and $m_0 = M/100$, the rocket's final speed will be approximately $3.8c$.

4. A liquid oxygen rocket has an exhaust speed of 2440 m/s. How far will a single-stage rocket burning liquid oxygen travel from the Earth if its fuel/total mass ratio is $\frac{2}{3}$ and the fuel is burnt in 150 s? Assume g to be constant.

5. A balloon of mass 400 kg has suspended from it a rope of mass 100 kg and length 100 m. The buoyancy force of the balloon is sufficient to support a mass of 450 kg. Initially it is falling at its terminal speed of 10 m/s due to air resistance which is proportional to the square of its speed. Show that if m is the total mass of the balloon and rope t s after the rope has first touched the ground, then the equation of motion can be written as

$$m\frac{d^2m}{dt^2} + g(m - 450) = 0$$

By writing $\dfrac{d^2m}{dt^2}$ as $\dfrac{dm}{dt}\dfrac{d}{dm}\left(\dfrac{dm}{dt}\right)$, solve the differential equation and find the speed of the balloon:

(i) when 50 m of rope lies on the ground,
(ii) when the balloon hits the ground.

Give a physical explanation for the speeds you obtain.

6. A rocket consists of a payload of mass m propelled by two stages of masses M_1 (first stage) and M_2 (second stage). Each stage has the same exhaust speed c and contains the same proportion $\varepsilon(<1)$ of fuel. Show that the final speed of the rocket is given by

$$v = -c\ln\left(1 - \frac{\varepsilon M_1}{M_1 + M_2 + m}\right) - c\ln\left(1 - \frac{\varepsilon M_2}{M_2 + m}\right).$$

If $\varepsilon = 0.83$ and $M_1 = 9M_2$, show that the maximum payload which can be given a final velocity of $2.5c$ is $0.019\,(M_1 + M_2)$.

A rocket of initial mass M of which εM $(0 < \varepsilon < 1)$ is fuel, burns the fuel at a constant rate k and ejects the exhaust gases with speed c. The rocket takes off from rest and rises vertically under (constant) gravity. If the air resistance is assumed to be $\alpha \times$ (speed of the rocket), find the speed of the rocket as a function of time whilst the fuel is burning.

A rocket is fired from an aircraft flying horizontally with speed V. The fuel is burnt at a constant rate k and ejected at a constant speed c. The attitude control of the rocket always maintains it in a horizontal position. If the total mass of the rocket is M find the path of the rocket during its powered flight. Assume that g is constant.

MOTION IN THREE DIMENSION

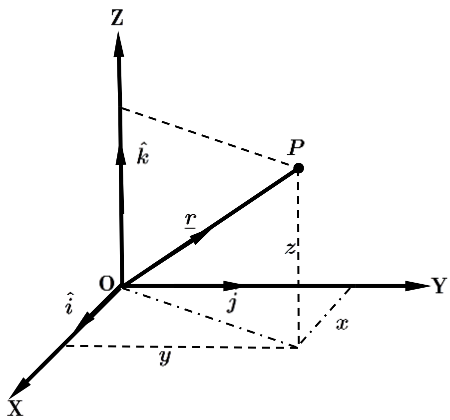
MOTION OF A PARTICLE IN THREE DIMENSIONS

The motion in two dimensions has been undertaken into account and we defined three types of coordinates, i.e., Cartesian coordinates (x, y), polar coordinates (r, θ) and intrinsic coordinates (S, ψ). This part deals with the motion in space or motion in three dimensions and we will consider three types of coordinate systems namely, Cartesian, cylindrical and spherical coordinates systems.

♦ Cartesian coordinate

Let $\mathbf{P}(x, y, z)$ be the Cartesian coordinates of a point P at time t, with respect to (w.r.t.) the fixed coordinate axes **OX,OY** and **OZ.** If $\underline{r} = \underline{\mathbf{OP}}$ is the position vector of w.r.t. the origin **O** and $\hat{i}, \hat{j}, \hat{k}$ are the unit vectors along the axes respectively, thus

$$\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$$



If $\underline{v}$ represents the velocity vector and $\underline{a}$ the acceleration vector of P, then the velocity is

$$\underline{v} = \frac{d\underline{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

That is the components of velocity of parallel to the coordinate axes are

$$\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \text{ respectively.}$$

These are positive in the direction of x, y and z increasing respectively.

The resultant velocity of P is given by

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

and the acceleration vector is

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k}$$

Thus the acceleration of P, parallel to the coordinate axes is

$$\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2} \text{ respectively}$$

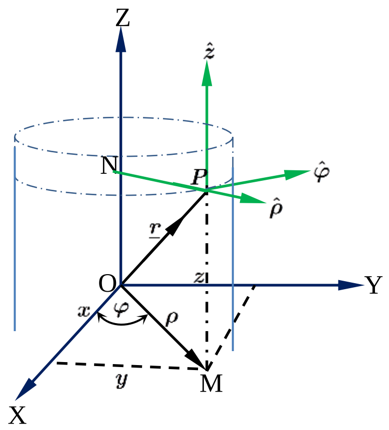
These are positive in the direction of x, y and z increasing respectively.

The resultant acceleration of P is given by

$$\sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2}$$

Now we will investigate the second type of coordinate system.

♦ Cylindrical coordinates:



Consider $\mathbf{P}(x, y, z)$ is a point whose projection on the xy–plane is $\mathbf{M}(x, y)$ has the cylindrical coordinates (ρ, φ, z) where $\rho = \mathbf{OM},\ \varphi = \angle\mathbf{XOM}$ and $z = \mathbf{PM}$. The limits of ρ, φ, z are $0 \leq \rho \leq \infty,\ 0 \leq \varphi \leq 2\pi$ and $-\infty \leq z \leq \infty$.

Due to the figure besides, we have

$$x = \rho\cos\varphi, \quad y = \rho\sin\varphi, \quad z = z \qquad (1)$$

With the help of Eq. (1), expressing ρ, φ and z in terms of x, y, z we get

$$\rho = \left(x^2 + y^2\right)^{1/2}, \quad \varphi = \tan^{-1}\left(y/x\right), \quad z = z \qquad (2)$$

The coordinate surfaces in the cylindrical coordinate system are

(i) $\rho = c_1$ i.e., $x^2 + y^2 = c_1^2$ i.e., right circular cylinders having the z- axis as a common axis.

(ii) $\varphi = c_2$ i.e., $y = x\tan c_2$ i.e., half planes through the z-axis

(iii) $z = c_3$ i.e., planes parallel to the xy–plane, as in the cartesian coordinate system.

Now, the point P is the point of intersection of these three coordinate surfaces. The coordinate curves for ρ, φ and z are respectively straight lines perpendicular to the z–axis, horizontal circles with centers on the z–axis and lines parallel to the z–axis.

The usual mutually perpendicular unit vectors $\hat{\rho}, \hat{\varphi}, \hat{z}$ in cylindrical coordinate system are as follows.

(i) The first unit vector $\hat{\rho}$ is normal to the cylindrical surface $\rho = \mathbf{const}$ in the direction of increasing radius ρ.

(ii) The second unit vector $\hat{\varphi}$ is tangential to the cylindrical surface perpendicular to the half plane $\varphi = \textbf{cosnt}$ and pointing in the direction of increasing azimuth angle φ.

(iii) The third unit vector $\hat{z}$ is the usual Cartesian unit vector $\hat{k}$ in the direction of increasing z.

Then we have

$$\hat{\rho} = \cos\varphi\,\hat{i} + \sin\varphi\,\hat{j}, \quad \hat{\varphi} = y = -\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j}, \quad \hat{z} = \hat{k} \qquad (3)$$

where $\hat{i}, \hat{j}, \hat{k}$ are the usual unit vectors in the cartesian coordinate system.

To obtain the components of velocity and acceleration of a moving point in terms of cylindrical coordinates let (x, y, z) be the coordinates of any point P moving in space at any instant t. If (ρ, φ, z) be its cylindrical coordinates, therefore

$$x = \rho\cos\varphi, \qquad y = \rho\sin\varphi, \qquad z = z \qquad (4)$$

Let $\underline{r}$ be the position vector of P, then

$$\underline{r} = x\hat{i} + y\hat{j} + z\hat{k} \qquad \text{Or} \qquad \underline{r} = \rho\cos\varphi\hat{i} + \rho\sin\varphi\hat{j} + z\hat{k} \qquad (5)$$

Now, the velocity vector $\underline{v}$ of P is the rate of change of displacement. Hence, using Eq. (5), we have

$$\begin{aligned}\underline{v} = \frac{d\underline{r}}{dt} &= \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \\ &= (\dot{\rho}\cos\varphi - \rho\sin\varphi\dot{\varphi})\hat{i} + (\dot{\rho}\sin\varphi + \rho\cos\varphi\dot{\varphi})\hat{j} + \dot{z}\hat{k}\end{aligned}$$

where dot denotes the derivatives w.r.t. time (t). Re-writing previous equation we get

$$\underline{v} = \dot{\rho}(\underbrace{\cos\varphi\hat{i} + \sin\varphi\hat{j}}_{\hat{\rho}}) + \rho\dot{\varphi}(\underbrace{-\sin\varphi\hat{i} + \cos\varphi\hat{j}}_{\hat{\varphi}}) + \dot{z}\underbrace{\hat{k}}_{\hat{z}}$$

Or $\underline{v} = \dot{\rho}\hat{\rho} + \rho\dot{\varphi}\hat{\varphi} + \dot{z}\hat{z}$ (6)

where $\hat{\rho}, \hat{\varphi}, \hat{z}$ are usual mutually perpendicular unit vectors in the cylindrical coordinate system given by

$$\hat{\rho} = \cos\varphi\hat{i} + \sin\varphi\hat{j}, \quad \hat{\varphi} = -\sin\varphi\hat{i} + \cos\varphi\hat{j}, \quad \hat{z} = \hat{k} \qquad (7)$$

If v_ρ, v_φ, v_z be the components of velocity of P in the directions $\hat{\rho}, \hat{\varphi}, \hat{z}$ respectively, then from Eq. (6) we have

$v_\rho = \dot{\rho}$ along NP, $v_\varphi = \rho\dot{\varphi}$ perpendicular to the plane ZOMP and $\dot{z}$ parallel to **OZ**

◈Note: $\hat{k}$ is constant unit vector $d\hat{k}/dt = 0$ so $d\hat{k}/dt = d\hat{z}/dt = 0$.

Again to get the components of acceleration at P, since the acceleration vector $\underline{a}$ of P is the rate of change of its velocity vector $\underline{v}$. Hence Using Eq. (6)

$$\begin{aligned} \underline{a} = \frac{d\underline{v}}{dt} &= \frac{d}{dt}(\dot{\rho}\hat{\rho}) + \frac{d}{dt}(\rho\dot{\varphi}\hat{\varphi}) + \frac{d}{dt}(\dot{z}\hat{z}) \\ &= \ddot{\rho}\hat{\rho} + \dot{\rho}\frac{d\hat{\rho}}{dt} + \dot{\rho}\hat{\varphi} + \rho\ddot{\varphi}\hat{\varphi} + \rho\dot{\varphi}\frac{d\hat{\varphi}}{dt} + \ddot{z}\hat{z} \end{aligned} \qquad (8)$$

where $\ddot{\rho} = d^2\rho/dt^2, \ \ \ddot{\varphi} = d^2\varphi/dt^2, \ddot{z} = d^2z/dt^2$

Differentiation relations (7) w.r.t. time, we get

$$\begin{aligned} \frac{d\hat{\rho}}{dt} &= -\dot{\varphi}\sin\varphi\hat{i} + \dot{\varphi}\cos\varphi\hat{j} = \dot{\varphi}(-\sin\varphi\hat{i} + \cos\varphi\hat{j}) = \dot{\varphi}\hat{\varphi}, \\ \frac{d\hat{\varphi}}{dt} &= -\dot{\varphi}\cos\varphi\hat{i} - \dot{\varphi}\sin\varphi\hat{j} = -\dot{\varphi}(\cos\varphi\hat{i} + \sin\varphi\hat{j}) = -\dot{\varphi}\hat{\rho} \end{aligned}$$

That is $d\hat{\rho}/dt = \dot{\varphi}\hat{\varphi}, \ \ d\hat{\varphi}/dt = -\dot{\varphi}\hat{\rho}$. Hence Eq. (8) becomes

$$\underline{a} = \ddot{\rho}\hat{\rho} + \dot{\rho}\dot{\varphi}\hat{\varphi} + \dot{\rho}\dot{\varphi}\hat{\varphi} + \rho\ddot{\varphi}\hat{\varphi} - \rho\dot{\varphi}^2\hat{\rho} + \ddot{z}\hat{z}$$

Or $$\underline{a} = \ddot{\rho} - \rho\dot{\varphi}^2 \ \ \hat{\rho} + (\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})\hat{\varphi} + \ddot{z}\hat{z} \qquad (9)$$

If a_ρ, a_φ, a_z be the components of acceleration of point P in the directions $\hat{\rho}, \hat{\varphi}, \hat{z}$ respectively, then by Eq. (9) we get

$$\begin{cases} a_\rho = \ddot{\rho} - \rho\dot{\varphi}^2 \qquad \text{(along NP)}, \\ a_\varphi = \rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi} = \dfrac{1}{\rho}\dfrac{d}{dt}\left(\rho^2\dfrac{d\varphi}{dt}\right) \qquad \text{(perpendicular to the plane ZOMP)} \\ a_z = \ddot{z} \ \ \text{parallel to OZ} \end{cases}$$

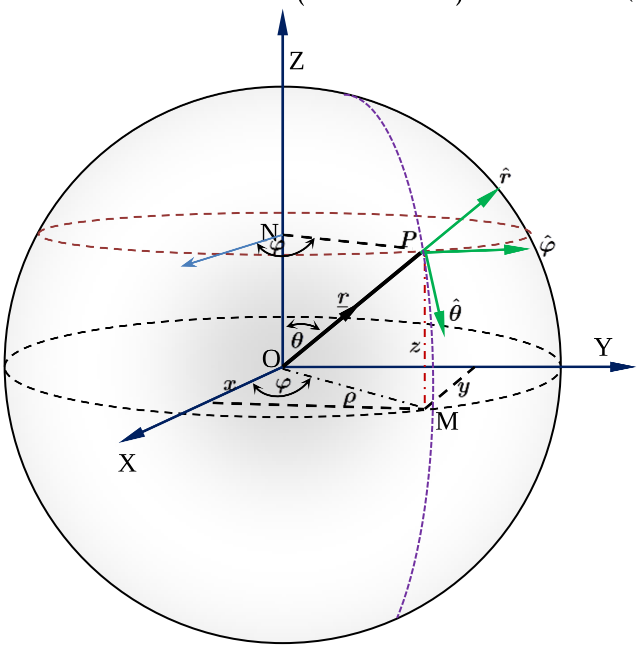
♦ Sphereical coordinates

Let $\mathbf{P}(x, y, z)$ be any point whose projection on the xy–plane is $\mathbf{M}(x, y)$ has the spherical coordinates (r, θ, φ) where $r = \mathbf{OP}$, $\theta = \angle\mathbf{ZOP}$ and $\varphi = \angle\mathbf{XOM}$. The limits on r, θ, φ are $0 \le r \le \infty$, $0 \le \theta \le 2\pi$ and $0 \le \varphi \le 2\pi$. Due to the figure besides, we have

$$x = r \sin\theta \cos\varphi, \quad y = r \sin\theta \sin\varphi, \quad z = r\cos\theta \qquad (10)$$

With the help of Eq. (10), expressing r, θ and φ in terms of x, y, z we get

$$\rho = \ x^2 + y^2 + z^2 \ ^{1/2}, \quad \theta = \tan^{-1}\left(\frac{(x^2 + y^2)^{1/2}}{z}\right), \quad \varphi = \tan^{-1}\left(\frac{y}{x}\right) \qquad (11)$$



The coordinate surfaces in the spherical coordinate system are

(i) $r = c_1$ i.e., $x^2 + y^2 + z^2 = c_1^2$ i.e., concentric spheres centered at the origin.

(ii) $\theta = c_2$ i.e., $x^2 + y^2 = z^2 \tan^2 c_2$ i.e., right circular cones with axis as z–axis and vertices at the origin.

(iii) $\varphi = c_3$ i.e., $y = x\tan c_3$ i.e., half planes through the z–axis.

Now, the point P is the point of intersection of these three coordinate surfaces. The coordinate curves for r, θ and φ are respectively straight lines passing through the origin, vertical circles with centre at the origin and the horizontal circles with centers on the z–axis.

The usual mutually perpendicular unit vectors $\hat{r}, \hat{\theta}, \hat{\varphi}$ in spherical coordinate system are in the direction of tangents to the r, θ and φ coordinate curves. These unit vectors are directed respectively in the direction of r–increasing, θ increasing and φ increasing respectively. Thus, we have

$$\begin{aligned}\hat{r} &= \sin\theta(\cos\varphi\,\hat{i} + \sin\varphi\,\hat{j}) + \cos\theta\hat{k},\\ \hat{\theta} &= \cos\theta(\cos\varphi\,\hat{i} + \sin\varphi\,\hat{j}) - \sin\theta\hat{k},\\ \hat{\varphi} &= -\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j},\end{aligned} \tag{12}$$

To find the components of velocity and acceleration of a moving point in terms of spherical coordinates let (x, y, z) be the coordinates of any point P moving in space at any instant t. If (r, θ, φ) be its spherical coordinates, then

$$x = r\sin\theta\cos\varphi, \qquad y = r\sin\theta\sin\varphi, \qquad z = r\cos\theta \tag{13}$$

Let $\underline{r}$ be the position vector of P, then

$$\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Or $\underline{r} = r\sin\theta\cos\varphi\hat{i} + r\sin\theta\sin\varphi\hat{j} + r\cos\theta\hat{k}$ (14)

Now, the velocity vector $\underline{v}$ of P is the rate of change of displacement. But the displacement $\underline{r} = r\hat{r}$ where $|\underline{r}| = r$, we have

$$\begin{aligned}\underline{v} = \frac{d\underline{r}}{dt} &= \frac{d}{dt}(r\hat{r})\\ &= \dot{r}\hat{r} + r\frac{d\hat{r}}{dt}\end{aligned} \tag{15}$$

From Eq. (14) $\hat{r} = \underline{r}/r = \sin\theta\cos\varphi\hat{i} + \sin\theta\sin\varphi\hat{j} + \cos\theta\hat{k}$

Differentiation both sides w.r.t time, we get

$$\frac{d\hat{r}}{dt} = (\dot{\theta}\cos\theta\cos\varphi - \dot{\varphi}\sin\theta\sin\varphi)\hat{i} + (\dot{\theta}\cos\theta\sin\varphi + \dot{\varphi}\sin\theta\cos\varphi)\hat{j} - \dot{\theta}\sin\theta\hat{k}$$

$$= \dot{\theta}(\underbrace{\cos\theta\cos\varphi\hat{i} + \cos\theta\sin\varphi\hat{j} - \sin\theta\hat{k}}_{\hat{\theta}})) + \dot{\varphi}\sin\theta(\underbrace{-\sin\varphi\hat{i} + \cos\varphi\hat{j}}_{\hat{\varphi}}) \qquad (16)$$

$$\Rightarrow \frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta} + \dot{\varphi}\sin\theta\hat{\varphi}$$

Using Eqns. (15) and (16) reduces to

$$\underline{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\dot{\varphi}\sin\theta\hat{\varphi} \qquad (17)$$

If v_r, v_θ, v_φ be the components of velocity of P in the directions $\hat{r}, \hat{\theta}, \hat{\varphi}$ respectively, then from Eq. (17) we have

$v_r = \dot{r}$ along OP in the direction of $\underline{r}$ increasing, $v_\theta = r\dot{\theta}$ perpendicular to OP in the plane ZOMP in the direction of θ increasing. and $v_\varphi = r\sin\theta\ \dot{\varphi}$ in the direction perpendicular to the plane ZOMP in the direction of φ increasing.

Again, the acceleration vector $\underline{a}$ of P is the rate of change of velocity $\underline{v}$ of P, Using Eq. (17), we get

$$\begin{aligned}\underline{a} = \frac{d\underline{v}}{dt} &= \frac{d}{dt}(\dot{r}\hat{r}) + \frac{d}{dt}(r\dot{\theta}\hat{\theta}) + \frac{d}{dt}(r\dot{\varphi}\sin\theta\hat{\varphi}) \\ &= \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt} \qquad (18) \\ &\quad + \dot{r}\dot{\varphi}\sin\theta\hat{\varphi} + r\ddot{\varphi}\sin\theta\hat{\varphi} + r\dot{\varphi}\dot{\theta}\cos\theta\hat{\varphi} + r\dot{\varphi}\sin\theta\frac{d\hat{\varphi}}{dt}\end{aligned}$$

Differentiation relations (12) w.r.t. time, we get

$$\begin{aligned} d\hat{r}/dt &= (\dot{\theta}\cos\theta\cos\varphi - \dot{\varphi}\sin\theta\sin\varphi)\hat{i} + (\dot{\theta}\cos\theta\sin\varphi + \dot{\varphi}\sin\theta\cos\varphi)\hat{j} - \dot{\theta}\sin\theta\hat{k}, \\ &= \dot{\theta}(\cos\theta\cos\varphi\hat{i} + \cos\theta\sin\varphi\hat{j} - \sin\theta\hat{k}) + \dot{\varphi}\sin\theta(-\sin\varphi\hat{i} + \cos\varphi\hat{j}),\end{aligned}$$

Thus

$$d\hat{r}/dt = \dot{\theta}\hat{\theta} + \dot{\varphi}\sin\theta\hat{\varphi}$$

Similarly $$d\hat{\theta}/dt = -\dot{\theta}\hat{r} + \dot{\varphi}\cos\theta\hat{\varphi},$$

and $$d\hat{\varphi}/dt = -\dot{\varphi}(\cos\varphi\,\hat{i} + \sin\varphi\,\hat{j}) = -\dot{\varphi}(\sin\theta\hat{r} + \cos\theta\hat{\theta})$$

If a_r, a_θ, a_φ be the components of acceleration of point P in the directions $\hat{r}, \hat{\theta}, \hat{\varphi}$ respectively, i.e., along OP, perpendicular to OP in the plane ZOMP in the direction of θ increasing and perpendicular to the plane ZOMP in the direction of φ increasing, then we have

$$\begin{cases} a_r = \ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2 \sin^2\theta, \\ a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\dot{\varphi}^2 = \dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dot{\theta}\right) - r\sin\theta\cos\theta\dot{\varphi}^2, \\ a_\varphi = \ddot{\varphi}r\sin\theta + 2\dot{r}\dot{\varphi}\sin\theta + 2r\cos\theta\dot{\theta}\dot{\varphi} = \dfrac{1}{r\sin\theta}\dfrac{d}{dt}\left(r^2\dot{\varphi}\sin^2\theta\right) \end{cases} \tag{20}$$

Now we will consider the following two particular cases

◈**Particular case I:** suppose the given particle moves in such a way that the position $P(r,\theta,\varphi)$ of the particle at any time is such that $r = \text{const.} = L$, say (that is the particle moves over or inner a sphere of radius L). In that case the coordinates of P become $P(L,\theta,\varphi)$ and clearly $\dot{r} = \ddot{r} = 0$ $(r = L)$. Hence the velocity Eq. (17) and acceleration Eq. (20) take the following form

$$\underline{v} = L\dot{\theta}\hat{\theta} + L\dot{\varphi}\sin\theta\hat{\varphi} \qquad \text{i.e.,} \quad (v_r = 0, v_\theta = L\dot{\theta}, v_\varphi = L\dot{\varphi}\sin\theta)$$

$$\begin{cases} a_r = -L(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta), \\ a_\theta = L\ddot{\theta} - L\sin\theta\cos\theta\dot{\varphi}^2, \\ a_\varphi = \ddot{\varphi}L\sin\theta + 2L\cos\theta\dot{\theta}\dot{\varphi} = \dfrac{L}{\sin\theta}\dfrac{d}{dt}\left(\dot{\varphi}\sin^2\theta\right) \end{cases}$$

◈**Particular case II:** suppose the given particle moves in such a way that the position $P(r,\theta,\varphi)$ of the particle at any time is such that $\theta = \text{const.} = \alpha$, say (that is the particle moves over or inner a cone of angle α). In that case the coordinates of P become $P(r,\alpha,\varphi)$ and clearly $\dot{\theta} = \ddot{\theta} = 0$ $(\theta = \alpha)$. Hence the velocity Eq. (17) and acceleration Eq. (20) take the following form

$$\underline{v} = \dot{r}\hat{r} + L\dot{\varphi}\sin\alpha\hat{\varphi} \qquad \text{i.e.,} \quad (v_r = \dot{r}, v_\theta = 0, v_\varphi = r\dot{\varphi}\sin\alpha)$$

and

$$\begin{cases} a_r = \ddot{r} - r\dot{\varphi}^2 \sin^2 \alpha, \\ a_\theta = -r \sin\alpha \cos\alpha \dot{\varphi}^2, \\ a_\varphi = \ddot{\varphi} r \sin\alpha + 2\dot{r}\dot{\varphi}\sin\alpha = \dfrac{\sin\alpha}{r} \dfrac{d}{dt}\left(r^2\dot{\varphi}\right) \end{cases}$$

Also in this case we can use the cylindrical coordinate $P(\rho, \varphi, z)$ where $\rho = z\tan\alpha$ such that the velocity and acceleration are given by

$$\underline{v} = \dot{z}\tan\alpha\hat{\rho} + z\dot{\varphi}\tan\alpha\hat{\varphi} + \dot{z}\hat{z}$$

$$\underline{a} = \left(\ddot{z}\tan\alpha - \dot{\varphi}^2 z\tan\alpha\right)\hat{\rho} + \left(\frac{\tan\alpha}{z}\frac{d}{dt}\left(z^2\dot{\varphi}\right)\right)\hat{\varphi} + \ddot{z}\hat{z}$$

Finally, the above results can be further simplified while solving problems related to a particular geometry of the problem (spheres, cones,..). Students are advised to remember these results for direct application in problems.

■Illustrative Examples■

❒ **EXAMPLE:** A particle moves over a spiral smooth wire of vertical axis and whose radius and angle are a, α with parametric equations $x = a\cos\theta$, $y = a\sin\theta$ and $z = a\theta\tan\alpha$. Initially the particle is projected with velocity V from position $\theta = 0$. Prove that the particle will stop after one revolution if $V^2 = 4\pi ag\tan\alpha$ and takes time $2\sqrt{a\pi/(g\sin\alpha\cos\alpha)}$.

❒ **SOLUTION:**

Due to the Figure shown, equations of motion in XYZ directions are

$$m\ddot{x} = -R\cos\theta, \qquad m\ddot{y} = -R\sin\alpha, \qquad m\ddot{z} = -mg$$

Multiply the first part by $\dot{x}$ and the second part by $\dot{y}$ and the third part by $\dot{z}$ then adding three parts, we have,

$$m(\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z}) = -(\dot{x}R\cos\theta + \dot{y}R\sin\alpha + mg\dot{z})$$

$$\Rightarrow \frac{1}{2}m\frac{d}{dt}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = -(\dot{x}R\cos\theta + \dot{y}R\sin\alpha + mg\dot{z}) \qquad (1)$$

But the parametric equations are $x = a\cos\theta$, $y = a\sin\theta$ and $z = a\theta\tan\alpha$

Therefore, $\dot{x} = -a\dot{\theta}\sin\theta$, $\dot{y} = a\dot{\theta}\cos\theta$ and $\dot{z} = a\dot{\theta}\tan\alpha$ (2)

Now Eq. (1) reduces to

$$\frac{1}{2}m\frac{d}{dt}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = -a\dot{\theta}(-R\sin\theta\cos\theta + R\cos\theta\sin\alpha) - mg\dot{z}$$

$$\therefore \frac{1}{2}m\frac{d}{dt}\underbrace{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}_{v^2} = -mg\frac{dz}{dt} \qquad \Rightarrow \frac{1}{2}mdv^2 = -mgdz$$

Integrating we get $\frac{1}{2}mv^2 + mgz = A$, (3), A being an arbitrary constant, this equation stated that the sum of kinetic energy and potential energy is constant. Apply this equation at initial position and when it stops after one revolution, i.e., $\theta = 2\pi$ **i.e.,** $z = 2\pi\tan\alpha$

$$\frac{1}{2}mV^2 + mg(0) = \frac{1}{2}m(0)^2 + mg(2a\pi\tan\alpha) \quad \Rightarrow V^2 = 4ag\pi\tan\alpha$$

Now, if we want to determine the arbitrary constant A, let $v = V$, $\theta = 0$ in Eq. (3), we get

$$\frac{1}{2}mV^2 + mg(0) = A \qquad \Rightarrow A = 2amg\pi\tan\alpha \quad (V^2 = 4ag\pi\tan\alpha)$$

hence Eq. (3) reduces to $\frac{1}{2}mv^2 + mgz = 2amg\pi\tan\alpha,$ (4)

Since,

$$\begin{aligned} v^2 &= \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \\ &= a^2\dot{\theta}^2(\sin^2\theta + a^2\cos^2\theta + \tan^2\theta) \\ &= a^2\dot{\theta}^2(1+\tan^2\theta) = a^2\dot{\theta}^2\sec^2\theta \qquad \therefore v = a\dot{\theta}\sec\alpha \end{aligned}$$

Hence, Eq. (4)

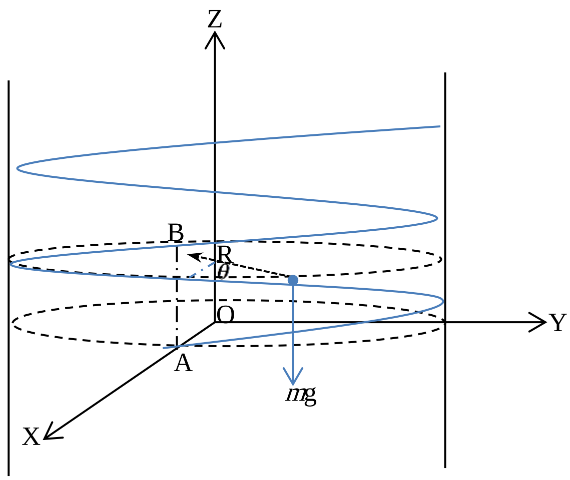
$$v^2 + 2gz = 4ag\pi\tan\alpha \quad \Rightarrow a^2\dot{\theta}^2\sec^2\alpha + 2ga\theta\tan\alpha = 4ag\pi\tan\alpha$$

$$a\left(\frac{d\theta}{dt}\right)^2 = (4g\pi\tan\alpha - 2g\theta\tan\alpha)\cos^2\alpha = 2g(2\pi-\theta)\sin\alpha\cos\alpha$$

$$\therefore \frac{d\theta}{dt} = \sqrt{\frac{2g(2\pi-\theta)\sin\alpha\cos\alpha}{a}} \qquad \Rightarrow \int_0^{2\pi}\frac{d\theta}{\sqrt{2\pi-\theta}} = \int_0^{\tau}\sqrt{\frac{2g\sin\alpha\cos\alpha}{a}}dt$$

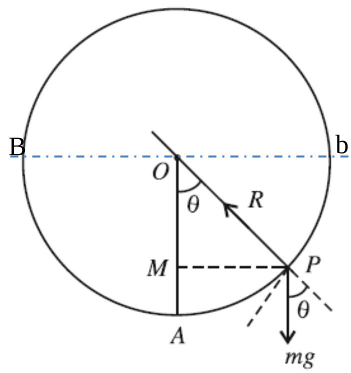
$$2\sqrt{2\pi-\theta}\Big|_{2\pi}^{0} = \sqrt{\frac{2g\sin\alpha\cos\alpha}{a}}\tau \qquad \Rightarrow \tau = 2\sqrt{\frac{\pi a}{g\sin\alpha\cos\alpha}}$$

which gives the time to stop after one revolution



❒ **EXAMPLE:** A heavy particle of mass *m* moves inside a smooth sphere of radius L; show that, if the velocity be that due to the level of the center, the reaction of the surface will vary as the depth below the center.

❒ **SOLUTION:**



Let $P(r,\theta,\varphi)$ its position at time *t*, such that OP makes an angle θ with the downward vertical through the center O of the sphere.

The velocity of the particle at P

$\underline{v} = L\dot{\theta}\hat{\theta} + L\dot{\varphi}\sin\theta\hat{\varphi}$ and

$v^2 = L^2(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) \quad or \quad v^2/L^2 = \dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta$

Let R be the reaction of the surface of the sphere at P, which will act along PO. At the point P $r = L$ (const.), therefore $\dot{r} = \ddot{r} = 0$, and since the equations of motion of the particle are given by

$m\underline{a} = \underline{F}$ or $ma_r = F_r, \quad ma_\theta = F_\theta, \quad ma_\varphi = F_\varphi$ then

$$-mL(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) = mg\cos\theta - R, \qquad (2)$$

$$mL(\ddot{\theta} - \sin\theta\cos\theta\dot{\varphi}^2) = -mg\sin\theta, \qquad (3)$$

$$\frac{mL}{\sin\theta}\frac{d}{dt}\,\dot{\varphi}\sin^2\theta = 0 \qquad (4)$$

Using (1) and (2) gives

$$\begin{aligned} R &= mL(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) + mg\cos\theta \\ &= mg\cos\theta + mv^2/L \end{aligned} \qquad (5)$$

Given that the velocity v at P is that due to the level Bb of the center, i.e., the velocity at P is that of a particle falling freely from Bb upto P. Hence using

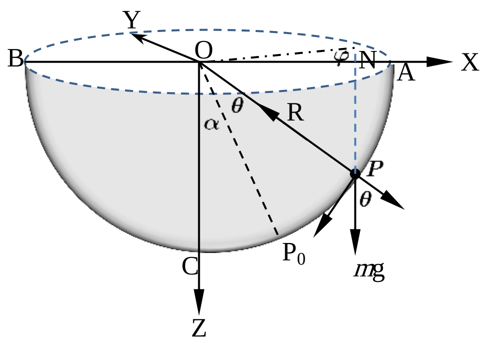
$v^2 = u^2 + 2gh$, we get $v^2 = 0 + 2g(\text{OM}) = 2gL\cos\theta$ then Eq. (5) reduces to

$R = mg\cos\theta + (m/L)(2gL\cos\theta) = 3mg\cos\theta = (3mg/a)(a\cos\theta)$

i.e., the reaction $R = (3mg/a)(\text{OM})$, showing that the reaction R at P will vary as the depth OM below the center $R \propto \text{OM}$.

❒ **EXAMPLE:** A particle is projected horizontally along the interior surface of a smooth hemisphere of radius b whose axis is vertical and its vertex is downwards: the point of projection being at an angular distance from the lowest point, show that the initial velocity so that the particle may just ascend to the rim of the hemisphere is $(2bg\sec\alpha)^{1/2}$.

❒ **SOLUTION:** Let ABC be a hemisphere whose axis OC is vertical and its vertex C is downwards. At any instant t, let P be the position of the given particle of mass m such that $\angle COP = \theta$. Let R be the reaction of the sphere along PO. Let spherical polar coordinates of P be (r,θ,φ) here $r = b$ so that $\dot{r} = \ddot{r} = 0$. Hence the usual components of acceleration of P in terms of spherical coordinates are given by



$$a_r = -b(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta),\quad a_\theta = b(\ddot{\theta} - \dot{\varphi}^2\sin\theta\cos\theta),\quad a_\varphi = \frac{b}{\sin\theta}\frac{d}{dt}\,\dot{\varphi}\sin^2\theta$$

Governing equations of motion are given by

$$ma_r = mg\cos\theta - R \quad \text{i.e.,} \quad -mb(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) = mg\cos\theta - R\,, \qquad (1)$$

$$ma_\theta = -mg\sin\theta \quad \text{i.e.,} \quad b(\ddot{\theta} - \dot{\varphi}^2\sin\theta\cos\theta) = -g\sin\theta, \qquad (2)$$

$$ma_\varphi = 0 \quad \text{i.e.,} \quad \frac{b}{\sin\theta}\frac{d}{dt}\,\dot{\varphi}\sin^2\theta = 0 \qquad (3)$$

Integrating Eq. (3), $\dot{r}^2\dot{\varphi} = A$, A being an arbitrary constant, **(4)**

Let initially the given particle be projected horizontally with velocity V along the interior surface the hemisphere from the point P_0 such that $\angle P_0OC = \alpha$. Let v_r, v_θ, v_φ be the components of velocity of the moving point particle in spherical coordinates. Then, by the condition of the problem, $v_r = v_\theta = 0$,

$v_\varphi = V$. But $v_\varphi = b\dot{\varphi}\sin\theta$. So initially at P_0, when $\theta = \alpha, v_\varphi = V$, let $\dot{\varphi}\big|_{t=0} = \dot{\varphi}_0$. Thus

$$V = b\dot{\varphi}_0 \sin\alpha \qquad \text{so that} \qquad \dot{\varphi}_0 = V/b\sin\alpha \qquad (5)$$

Now, since $\dot{\varphi}\big|_{t=0} = \dot{\varphi}_0$ when $\theta = \alpha$ hence Eq. (4) gives

$$A = \dot{\varphi}_0 \sin^2\alpha \qquad \text{or} \qquad A = \sin^2\alpha(V/b\sin\alpha) = (V/b)\sin\alpha, \text{ by (5)} \quad (6)$$

Eq. (4) gives $\dot{\varphi}\sin^2\theta = (V/b)\sin\alpha$ *or* $\dot{\varphi} = (V\sin\alpha/b\sin^2\theta)$ (7)

Using Eq. (7), Eq. (2) gives

$$b(\ddot{\theta} - (V^2\sin^2\alpha/b^2\sin^4\theta)\sin\theta\cos\theta) = -g\sin\theta$$

$$\Rightarrow \dot{\theta}\frac{d\dot{\theta}}{d\theta} = (V^2\sin^2\alpha/b^2)\sin^{-3}\theta\cos\theta - (g/b)\sin\theta \qquad \text{as} \quad \ddot{\theta} = \dot{\theta}\frac{d\dot{\theta}}{d\theta}$$

$$\Rightarrow 2\dot{\theta}d\dot{\theta} = (V^2\sin^2\alpha/b^2)\sin^{-3}\theta\cos\theta - (g/b)\sin\theta \;\; d\theta$$

Integrating, $\dot{\theta}^2 = -(V^2\sin^2\alpha/b^2\sin^2\theta) + (2g/b)\cos\theta + B$ (8)

where B is an arbitrary constant. Initially at P_0, when $\theta = \alpha, v_\theta = 0$, i.e., $b\dot{\theta} = 0$ or $\dot{\theta} = 0$, then Eq. (8) gives $B = (V^2/b^2) - (2g/b)\cos\alpha$ (9)

Subtracting Eq. (9) from (8), we have

$$\dot{\theta}^2 = (V^2/b^2)(1 - \sin^2\alpha \,/\, \sin^2\theta) + (2g/b)(\cos\theta - \cos\alpha)$$

By the problem, the particle just ascends to the rim A of the hemisphere. So at A, when $\theta = \pi/2, \dot{\theta} = 0$. Hence the above relation reduces to

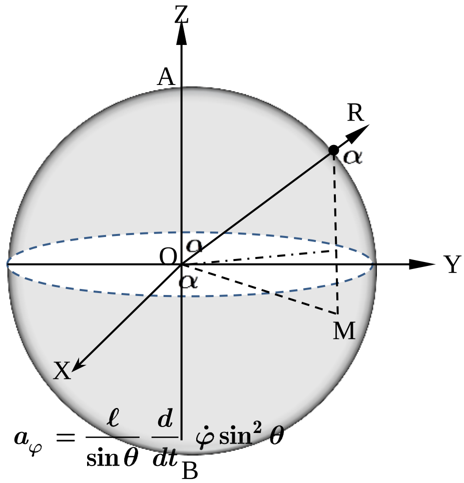
$$0 = (V^2/b^2)(1 - \sin^2\alpha) - (2g/b)\cos\alpha \quad \text{or} \quad V^2\cos^2\alpha/b^2 = (2g/b)\cos\alpha$$

Thus $\quad V^2 = 2gb\,\text{secs}\,\alpha \qquad$ or $\qquad V = (2gb\,\text{secs}\,\alpha)^{1/2}$

This we need to prove.

❒ **EXAMPLE:** A particle moves on a smooth sphere of radius ℓ under no forces except the pressure of the surface; Show that its path is given by the equation $\cot\theta = \cot\beta\cot\varphi$ where θ and φ are its angular coordinates, suppose $\varphi = 0$ when $\theta = \beta$.

❒ **SOLUTION:** Let AB be the vertical diameter of the given sphere. At any instant t, let P be the position of the moving particle of mass *m* such that $\angle AOP = \theta$. Let R be the pressure of the sphere. Let the spherical polar coordinates of P be (r,θ,φ) here $r = \ell$ so that $\dot{r} = \ddot{r} = 0$. Hence the usual components of acceleration of P in terms of spherical coordinates are given by



$$a_r = -\ell(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta),\quad a_\theta = \ell(\ddot{\theta} - \dot{\varphi}^2\sin\theta\cos\theta),\quad a_\varphi = \frac{\ell}{\sin\theta}\frac{d}{dt}\dot{\varphi}\sin^2\theta$$

Since the particle moves under no forces except the pressure R along OP, so there is no force on the particle in θ and φ directions. That is we have

$$a_\theta = 0 \quad \text{or} \quad \ddot{\theta} - \dot{\varphi}^2\sin\theta\cos\theta = 0 \qquad (1)$$

$$a_\varphi = 0 \quad \text{or} \quad \frac{\ell}{\sin\theta}\frac{d}{dt}\dot{\varphi}\sin^2\theta = 0 \qquad (2)$$

Integrating Eq. (2), $\dot{r}^2\dot{\varphi} = A$, A being an arbitrary constant,

$$\Rightarrow \frac{d\varphi}{dt} = A/\sin^2\theta \qquad (3)$$

Using Eq. (3), Eq. (1) gives $\ddot{\theta} - (A^2/\sin^4\theta)\sin\theta\cos\theta = 0$

or $\dot{\theta}\dfrac{d\dot{\theta}}{d\theta} = \dfrac{A^2\cos\theta}{\sin^3\theta} \qquad \Rightarrow 2\dot{\theta}d\dot{\theta} = 2A^2\cos\theta\sin^{-3}\theta d\theta$

Integrating, $\dot{\theta}^2 = -A^2/\sin^2\theta + B \qquad (4)$

where B is an arbitrary constant. Initially let $\theta = \beta, \dot{\theta} = 0$, then Eq. (4) gives

$$B = A^2/\sin^2\beta$$

Therefore Eq. (4) gives

$$\left(\frac{d\theta}{dt}\right)^2 = A^2\left(\frac{1}{\sin^2\beta} - \frac{1}{\sin^2\theta}\right) = A^2\left(\frac{\sin^2\theta - \sin^2\beta}{\sin^2\beta\sin^2\theta}\right)$$

$$\Rightarrow \frac{d\theta}{dt} = \left(\frac{A}{\sin\beta\sin\theta}\right)(\sin^2\theta - \sin^2\beta)^{1/2} \quad (5)$$

Dividing Eq. (3) by Eq. (5), we have

$$\frac{d\varphi}{d\theta} = \frac{A}{\sin^2\theta} \times \left(\frac{\sin\beta\sin\theta}{(\sin^2\theta - \sin^2\beta)^{1/2}}\right)$$

$$\Rightarrow \; d\varphi = \frac{\csc^2\theta d\theta}{(\csc^2\beta - \csc^2\theta)^{1/2}} = \frac{\csc^2\theta d\theta}{(\cot^2\beta - \cot^2\theta)^{1/2}}$$

Putting $\cot\theta = u$ so that $-\csc^2\theta d\theta = du$

$$d\varphi = -\frac{du}{(\cot^2\beta - u^2)^{1/2}}$$

Integrating, $\varphi = \cos^{-1}(u/\cot\beta) + C = \cos^{-1}(\cot\theta/\cot\beta) + C \quad (6)$

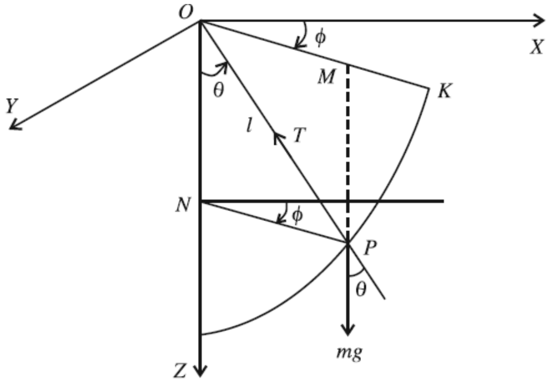
Initially, when $\theta = \beta$ let $\varphi = 0$ so Eq. (6) gives C = 0

Equation (6) becomes $\varphi = \cos^{-1}(\cot\theta/\cot\beta)$ or $\cot\theta = \cot\beta\cot\varphi$

which is required path of the particle.

❐ **EXAMPLE:** A particle is attached to one end of a string, of length b, the other end of which is tied to a fixed point O. When the string is inclined at an acute angle α to the downward-drawn vertical, the particle is projected horizontally and perpendicular to the string with a velocity V. Find the resulting motion of the particle. Also find the tension of the string at any instant.

❐ **SOLUTION**



Let P be the position of the particle at time *t*, such that the polar coordinates of P w.r.t. coordinate axes through O, with Z-axis along downward vertical are $P(l,\theta,\varphi)$. Here $r = l\ (\text{const.})$. If T is the tension in the string, then the equations of motion of the particle are

$$-ml(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) = mg\cos\theta - T, \qquad (1)$$
$$ml(\ddot{\theta} - \sin\theta\cos\theta\dot{\varphi}^2) = -mg\sin\theta, \qquad (2)$$
$$\frac{ml}{\sin\theta}\frac{d}{dt}\ \dot{\varphi}\sin^2\theta\ = 0 \qquad (3)$$

From Eq. (3) by integrating, we get

$$\dot{\varphi}\sin^2\theta = A\ \ (\text{const.}) \qquad (3)$$

From Eq. (4) $A = \sin^2\alpha\dfrac{V}{l\sin\alpha} = \dfrac{V\sin\alpha}{l}$

That is Eq. (4) reduces to $\sin^2\theta\dfrac{d\varphi}{dt} = \dfrac{V\sin\alpha}{l}$ (5)

Substituting the value of from (5) in (2)

$$\frac{d^2\theta}{dt^2} - \frac{V^2\sin^2\alpha}{l^2}\frac{\cos\theta}{\sin^3\theta} = -\frac{g}{l}\sin\theta$$

Multiplying both sides by $2d\theta/dt$ and then integrating, we get

$$\left(\frac{d\theta}{dt}\right)^2 + \frac{V^2\sin^2\alpha}{l^2\sin^2\theta} = \frac{2g}{l}\cos\theta + B \qquad (6)$$

But initially $\dot{\theta} = 0$ when $\theta = \alpha \quad \Rightarrow B = V^2/l^2 - (2g/l)\cos\alpha$

From Eq. (6) we have

$$\left(\frac{d\theta}{dt}\right)^2 + \frac{V^2 \sin^2\alpha}{l^2 \sin^2\theta} = \frac{V^2}{l^2} + \frac{2g}{l}(\cos\theta - \cos\alpha)$$

$$\Rightarrow \left(\frac{d\theta}{dt}\right)^2 = \frac{V^2}{l^2}\left(1 - \frac{\sin^2\alpha}{\sin^2\theta}\right) + \frac{2g}{l}(\cos\theta - \cos\alpha) \qquad (7)$$

$$= \frac{V^2}{l^2}\left(\frac{\sin^2\theta - \sin^2\alpha}{\sin^2\theta}\right) + \frac{2g}{l}(\cos\theta - \cos\alpha)$$

$$= \frac{V^2}{l^2}\left(\frac{\cos^2\alpha - \cos^2\theta}{\sin^2\theta}\right) + \frac{2g}{l}(\cos\theta - \cos\alpha)$$

$$= \frac{2g}{l}\frac{(\cos\theta - \cos\alpha)}{\sin^2\theta}\left(\frac{V^2}{2gl}(\cos\theta + \cos\alpha) - \sin^2\theta\right)$$

$$= \frac{2g}{l}\frac{(\cos\theta - \cos\alpha)}{\sin^2\theta}\left(2n^2(\cos\theta + \cos\alpha) - \sin^2\theta\right) \text{taking } V^2 = 4\lg n^2$$

If $d\theta/dt = 0$ then $2n^2(\cos\theta + \cos\alpha) - \sin^2\theta = 0$

Since $\cos\theta - \cos\alpha \neq 0$ as $\theta = \alpha$ is the initial position

$$\Rightarrow \cos^2\theta + 2n^2\cos\theta - (1 - 2n^2\cos\alpha) = 0$$

$$\Rightarrow \cos\theta = -n^2 \pm (1 - 2n^2\cos\alpha + n^4)^{1/2}$$

If $d\theta/dt = 0$ for $\theta = \beta$ then $\cos\beta = -n^2 + (1 - 2n^2\cos\alpha + n^4)^{1/2}$ (8)

neglecting - sign which is in admissible as θ is acute angle. Hence the motion of the particle is confined between $\theta = \alpha$ and $\theta = \beta$ given by (8)

The motion of the particle will remain above or below the starting point $\theta = \alpha$, according as $\beta >$ or $< \alpha$

i.e., according as $\cos\beta <$ or $> \cos\alpha$

i.e., according as $-n^2 \pm (1 - 2n^2\cos\alpha + n^4)^{1/2} <$ or $> \cos\alpha$

i.e., according as $1 - 2n^2\cos\alpha + n^4 <$ or $> (n^2 + \cos\alpha)^2$

i.e., according as $n^2 >$ or $< \sin^2\alpha/(4\cos\alpha)$

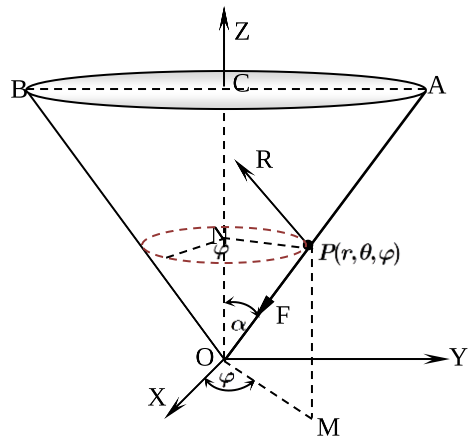
i.e., according as $V^2 >$ or $< gl\tan\alpha\sin\alpha$

To find the tension T at any instant, substituting the values of $d\theta/dt$, $d\varphi/dt$

from (5) and (7) in (1), we get $\quad T = m\ V^2/l + g(3\cos\theta - 2\cos\alpha)$

❒ **EXAMPLE:** A particle of mass m moves on the inner surface of a smooth cone of vertical angle 2α, being acted on by a force towards the vertex of the cone, and its direction of motion always cuts the generators at a constant angle β. Find the motion and the law of force.

❒ **SOLUTION:** Let a particle of mass m moves on the inner surface of the cone OAB whose axis is OC and $\angle AOC = \alpha$. Let F be the force acting on it towards the vertex and R the reaction of the cone acting along perpendicular PN to OA. At any time t let P be the position of the particle such that $OP = r$. Let spherical polar of P be (r, θ, φ) where $\theta = \alpha$ so that $\dot{\theta} = 0$. Hence the usual components of velocity v_r, v_θ, v_φ and components of acceleration a_r, a_θ, a_φ are



$$v_r = \dot{r}, \qquad v_\theta = 0, \qquad v_\varphi = r\dot{\varphi}\sin\alpha \qquad (1)$$

$$\begin{cases} a_r = \ddot{r} - r\dot{\varphi}^2 \sin^2\alpha, \\ a_\theta = -r\sin\alpha\cos\alpha\dot{\varphi}^2, \\ a_\varphi = \ddot{\varphi} r\sin\alpha + 2\dot{r}\dot{\varphi}\sin\alpha = \dfrac{\sin\alpha}{r}\dfrac{d}{dt}\ r^2\dot{\varphi} \end{cases} \qquad (2)$$

Hence the governing equations of motion are

$$ma_r = F \quad \text{or} \quad m(\ddot{r}\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\alpha) = -F, \qquad (3)$$

$$ma_\theta = -R \quad \text{or} \quad mr\sin\alpha\cos\alpha\dot{\varphi}^2 = R, \qquad (4)$$

$$ma_\varphi = 0 \quad or \quad \frac{m\sin\alpha}{r}\frac{d}{dt}\ r^2\dot{\varphi} = 0 \qquad (5)$$

Also, since the direction of motion always cuts OP at angle β, From Eq. (3) by integrating, we get $\tan\beta = v_\varphi / v_r = r\dot{\varphi}\sin\alpha/\dot{r}$ **(6)**

Integrating Eq. (5) we have, $\dot{r}^2\dot{\varphi} = A$, A being constant, **(7)**

Substituting the value of $\dot{\varphi}$ from Eq. (7) into Eq. (6), we have

$$\dot{r} = r\sin\alpha\cot\beta(A/r^2), \qquad \text{or} \qquad \dot{r} = (A/r)\sin\alpha\cot\beta \qquad (8)$$

Differentiation both sides of Eq. (8), we obtain

$$\ddot{r} = A(-\dot{r}/r^2)\sin\alpha\cot\beta \quad \text{or} \quad \ddot{r} = -A^2(1/r^3)\sin^2\alpha\cot^2\beta \quad (\text{by Eq.}(8)) \qquad (9)$$

Substituting the value of $\dot{\varphi}$ and $\ddot{r}$ from Eq. (7) into Eq. (9) in Eq. (3), we have

$$-F = -A^2(1/r^3)\sin^2\alpha\cot^2\beta - r\sin^2\alpha(A^2/r^4)$$

$$\Rightarrow \quad F' = (A^2/r^3)\sin^2\alpha(\cot^2\beta + 1)$$

$$\Rightarrow \quad F' = A^2\sin^2\alpha\csc^2\beta/r^3 \quad \Rightarrow F' = \mu/r^3 \qquad (F' = F/m) \qquad (10)$$

where $\mu = A^2\sin^2\alpha\csc^2\beta$ (11)

Equation (10) illustrates that force varies inversely as cube of the distance from the vertex O. This gives the required law of force.

Now, with the help of Eq. (1), the velocity v at any time t at P is given by

$$v^2 = v_r^2 + v_\theta^2 + v_\varphi^2 = \dot{r}^2 + (r\dot{\varphi}\sin\alpha)^2 \qquad \text{or} \qquad v^2 = \dot{r}^2 + (r\dot{\varphi}\sin\alpha)^2$$

$$v^2 = (A^2/r^2)\sin^2\alpha\cot^2\beta + r^2\sin^2\alpha(A/r^2)^2 \quad \text{using Eq.}(7)\text{ and }(8)$$

$$\Rightarrow v^2 = (A^2/r^2)\sin^2\alpha(\cot^2\beta + 1) = (A^2\sin^2\alpha/\sin^2\beta)(1/r^2) = \mu'^2/r^2$$

Thus $\Rightarrow v = \mu'/r$ i.e., the velocity varies inversely as OP.

Using Eq. (7), Eq. (4) reduces to

$$\begin{aligned} R/m = r\sin\alpha\cos\alpha(A/r^2)^2 &= (A^2/r^3)\sin\alpha\cos\alpha \qquad (12) \\ &= (A^2/r^3)\sin^2\alpha\csc^2\beta(\sin^2\beta\cos\alpha/\sin\alpha) \\ &= (F/m)(\sin^2\beta\cos\alpha/\sin\alpha) \quad \text{by Eq.}(10) \end{aligned}$$

Thus $R\sin\alpha = F\sin^2\beta\cos\alpha$ (13)

which gives relation between R and F. Re-writing Eq. (6),

$$\dot{r}/\dot{\varphi} = r\sin\alpha\cot\beta, \qquad \text{or} \qquad \frac{dr/dt}{d\varphi/dt} = r\sin\alpha\cot\beta$$

$$\text{or} \quad \frac{dr}{d\varphi} = r\sin\alpha\cot\beta \qquad \Rightarrow (1/r)dr = \sin\alpha\cot\beta\, d\varphi$$

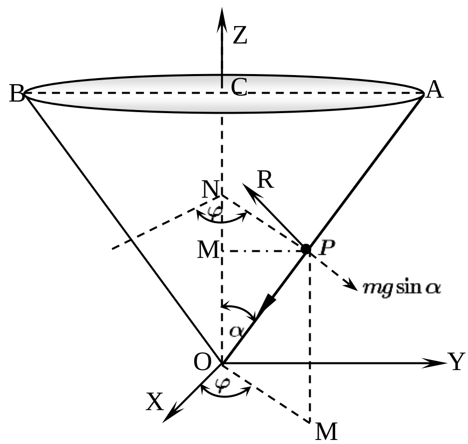
Integrating; $\log r = \varphi\sin\alpha\cot\beta + B$ (14); where, B is an arbitrary constant.

Let initially $r = r_0$ when $\varphi = 0$ (say), So Eq. (14) gives $B = \log r_0$

Then Eq. (14) reduces to $\log r - \log r_0 = \varphi\sin\alpha\cot\beta$ or $r = r_0 e^{\varphi\sin\alpha\cot\beta}$ with represents the path of the moving particle.

❐ **EXAMPLE:** A smooth conical surface is fixed with its axis vertically and vertex downward. A particle of mass *m* is in steady motion on its side in a horizontal circle and is slightly disturbed. Show that the time of a small oscillation about this state of steady motion is $2\pi(L/3g\cos\alpha)^{1/2}$, where α is the semi-vertical angle of the cone and L is the length of the generator of the circle of steady motion.

❐ **SOLUTION**: Let a particle of mass *m* moves on the inner surface of the cone OAB whose axis is OC and $\angle\mathbf{AOC} = \alpha$. Let R be the reaction of the cone acting along perpendicular PN to OA. At any time t let P be the position of the particle such that $\mathbf{OP} = r$. Let spherical polar of P be (r,θ,φ) where $\theta = \alpha$ so that $\dot{\theta} = 0$. Hence the usual components of velocity and acceleration in terms of spherical coordinates are given by



$$a_r = \ddot{r} - r\dot{\varphi}^2\sin^2\alpha, \quad a_\theta = -r\sin\alpha\cos\alpha\dot{\varphi}^2, \quad a_\varphi = \frac{\sin\alpha}{r}\frac{d}{dt}\left(r^2\dot{\varphi}\right)$$

Therefore, the equations of motion are given by

$$ma_r = -mg\cos\alpha \quad \text{i.e.,} \quad \ddot{r} - r\dot{\varphi}^2\sin^2\alpha = -g\cos\alpha, \qquad (1)$$

$$ma_\theta = mg\sin\alpha - R \quad \text{i.e.,} \quad -mr\sin\alpha\cos\alpha\dot{\varphi}^2 = mg\sin\alpha - R \qquad (2)$$

$$ma_\varphi = 0 \quad \text{i.e.,} \quad \frac{m\sin\alpha}{r}\frac{d}{dt}\left(r^2\dot{\varphi}\right) = 0 \qquad (3)$$

Integrating Eq. (3) $\quad r^2\dot{\varphi} = A$, A being a constant. **(4)**

Initially le the particle be at P_0 such that $OP_0 = L$. In this position the particle is in steady motion in a horizontal circle moving with angular velocity ω (say). Thus, initially, when $r = L, \dot{\varphi} = \omega$. Hence Eq. (4) gives $L^2\dot{\varphi} = A$.

Then Eq. (4) becomes $\quad r^2\dot{\varphi} = L^2\omega \quad \Rightarrow \dot{\varphi} = L^2\omega / r^2$ **(5)**

Substituting the value of $\dot{\varphi}$ from Eq. (5) in Eq. (1) we get,

$$\ddot{r} = \frac{L^4\omega^2 \sin\alpha\cos\alpha}{r^3} - g\cos\alpha \qquad (6)$$

But for steady motion, $\ddot{r} = 0$ when $r = L$. So Eq. (6) gives

$$0 = \frac{L^4\omega^2 \sin\alpha\cos\alpha}{l^3} - g\cos\alpha \quad \Rightarrow \quad g\cos\alpha = L\omega^2 \sin\alpha\cos\alpha$$

Putting $r = L + \tilde{r}$, where $r = L + \tilde{r}$ is small. Eq. (6) gives

$$\frac{d^2\tilde{r}}{dt^2} = \frac{L^4\omega^2 \sin\alpha\cos\alpha}{(L+\tilde{r})^3} - g\cos\alpha = L\omega^2 \sin\alpha\cos\alpha\left(1+\frac{\tilde{r}}{L}\right)^{-3} - g\cos\alpha$$

$$= L\omega^2 \sin\alpha\cos\alpha\left(1+\frac{3\tilde{r}}{L}\right) - L\omega^2 \sin\alpha\cos\alpha$$

Here we use Eq. (7) and the fact that $\tilde{r}$ is small.

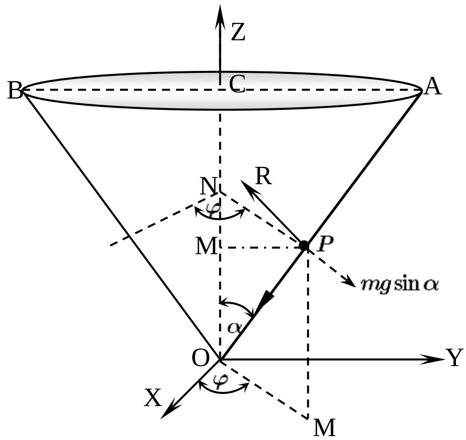
$\ddot{\tilde{r}} = -(3/L)g\tilde{r}\cos\alpha$ using Eq.(5) $\qquad (\ddot{\tilde{r}} = -\omega^2\tilde{r},\ \omega^2 = (3/L)g\tilde{r}\cos\alpha)$

which represents a S.H.M. of time period Components of velocity at Pare given

by $2\pi/\omega = 2\pi(L/3g\cos\alpha)^{1/2}$

❒ **EXAMPLE:** A hollow right circular cone is placed with its vertex downward and axis vertical, and a point of mass *m* on its interior surface at a height *h* above the vertex a particle is projected horizontally along the surface with a velocity $(2gh/(n^2+n))^{1/2}$. Show that the lowest point of its path will be at height above the vertex of the cone.

❒ **SOLUTION:** Let a particle of mass *m* moves on the inner surface of the cone OAB whose axis is OC and $\angle \mathbf{AOC} = \alpha$. Let R be the reaction of the cone acting along perpendicular PN to OA. At any time t let P be the position of the particle such that $\mathbf{OP} = r$. Let spherical polar of the point P



be (r,θ,φ) where $\theta = \alpha$ so that $\dot{\theta} = 0$. Hence the usual components of velocity and acceleration in terms of spherical coordinates are given by

$$a_r = \ddot{r} - r\dot{\varphi}^2 \sin^2\alpha, \quad a_\theta = -r\sin\alpha\cos\alpha\dot{\varphi}^2, \quad a_\varphi = \frac{\sin\alpha}{r}\frac{d}{dt}\; r^2\dot{\varphi}$$

Therefore, the equations of motion are given by

$$ma_r = -mg\cos\alpha \qquad \text{i.e.,} \qquad \ddot{r} - r\dot{\varphi}^2\sin^2\alpha = -g\cos\alpha, \qquad (1)$$

$$ma_\theta = mg\sin\alpha - R \qquad \text{i.e.,} \qquad -mr\sin\alpha\cos\alpha\dot{\varphi}^2 = mg\sin\alpha - R \qquad (2)$$

$$ma_\varphi = 0 \qquad \text{i.e.,} \qquad \frac{m\sin\alpha}{r}\frac{d}{dt}\; r^2\dot{\varphi} = 0 \qquad (3)$$

Integrating Eq. (3) $\qquad r^2\dot{\varphi} = A$, A being a constant. **(4)**

Components of velocity at Pare given by

$$v_r = \dot{r}, \qquad v_\theta = 0, \qquad v_\varphi = r\dot{\varphi}\sin\alpha \qquad (5)$$

Given that initially the particle is projected horizontally with velocity $(2gh/(n^2+n))^{1/2}$ from a point P_0 (say) such that OM_0 equals the height of P_0 above O which equals h

Hence initially, at P_0, we have $v_r = \dot{r} = 0,\ v_\theta = 0$ and $v_\varphi = (2gh/(n^2+n))^{1/2}$

Now, at P_0, $r = OP_0 = h\sec\alpha$. Also, let at P_0, $\dot{\varphi}\big|_{t=0} = \dot{\varphi}_0$. Hence Eq. (5) gives $(2gh/(n^2+n))^{1/2} = v_\varphi = h\dot{\varphi}_0 \sec\alpha\sin\alpha$

giving $\dot{\varphi}_0 = (\cos\alpha/h\sin\alpha)(2gh/(n^2+n))^{1/2}$ **(6)**

Since at P_0, $r = OP_0 = h\sec\alpha$ and $\dot{\varphi} = \dot{\varphi}_0$, Eq. (4) gives

$h^2\sec^2\alpha\dot{\varphi}_0 = A \quad or \quad A = (h/\sin\alpha\cos\alpha)(2gh/(n^2+n))^{1/2}$, by Eq. (6)

Therefore, Eq. (4) gives

$$\dot{\varphi} = A/r^2 = (h/r^2\sin\alpha\cos\alpha)(2gh/(n^2+n))^{1/2} \qquad (7)$$

Substituting the value of $\dot{\varphi}$ from Eq. (7) in Eq. (1), we get

$$\dot{r}\frac{d\dot{r}}{dr} - \frac{rh^2\sin^2\alpha}{r^4\sin^2\alpha\cos^2\alpha}\left(\frac{2gh}{n^2+n}\right) = -g\cos\alpha, \quad \text{as } (\ddot{r} = \dot{r}\frac{d\dot{r}}{dr})$$
$$\Rightarrow 2\dot{r}d\dot{r} = \left(\frac{4gh^3r^{-3}}{(n^2+n)\cos^2\alpha} - 2g\cos\alpha\right)dr$$

Integrating

$$\dot{r}^2 = \frac{2gh^3}{r^2(n^2+n)\cos^2\alpha} - 2gr\cos\alpha + B, \qquad (8)$$

where B is an arbitrary constant.

But initially at P_0, $r = OP_0 = h\sec\alpha,\ \dot{r} = 0$. So Eq. (8) gives

$$0 = -2gh/(n^2+n) - 2gh + B, \qquad \text{or} \qquad B = 2gh(n^2+n+1)/(n^2+n)$$

Then Eq. (8) re-written as

$$\dot{r}^2 = \frac{2gh^3}{r^2(n^2+n)\cos^2\alpha} - 2gr\cos\alpha + \frac{2gh(n^2+n+1)}{n^2+n}, \qquad (9)$$

Now, from Eq. (9) $\dot{r}$ is again zero when

$$0 = \frac{2gh^3}{z^2(n^2+n)} - 2gz + \frac{2gh(n^2+n+1)}{n^2+n}, \text{ where } z = r\cos\alpha$$

$$\Rightarrow (n^2+n)z^3-(n^2+n+1)hz^2+h^3=0$$
$$\Rightarrow (n^2+n)z^3-(n^2+n)hz^2-hz^2+h^3=0$$
$$\Rightarrow (n^2+n)z^2(z-h)-h(z-h)(z+h)=0$$
$$\Rightarrow (z-h)(n(n+1)z^2-hz-h^2)=0$$
$$\Rightarrow (z-h)(n(n+1)z^2-(n+1)hz+nhz-h^2)=0$$
$$\Rightarrow (z-h)((n+1)z(nz-h)+h(nz-h))=0$$
$$\Rightarrow (z-h)(nz-h)((n+1)z+h)=0$$

Thus, $z=r\cos\alpha=h,\ h/n$ or $-h/(n+1)$,

Now the negative value of r is inadmissible. Again, $r\cos\alpha=h$ corresponds to the initial position of P_0 of the particle. Hence the required lowest point of the particle path will be at height equals $r\cos\alpha=h/n$ above the vertex O of the cone.

○Note that examples of motion over a cone surface can be solved using cylindrical coordinate

Now we will re-solve this example using cylindrical coordinate:

The acceleration in terms of cylindrical coordinates are given by

$$a_\rho=\ddot{\rho}-\rho\dot{\varphi}^2,\quad a_\varphi=\frac{1}{\rho}\frac{d}{dt}\rho^2\dot{\varphi}\ ,\quad a_z=\ddot{z}$$

Note there is a relation between ρ and z i.e., $\rho=z\tan\alpha$, Therefore, $\dot{\rho}=\dot{z}\tan\alpha$ and $\ddot{\rho}=\ddot{z}\tan\alpha$ hence the acceleration components are re-written as

$$a_\rho=(\ddot{z}-z\dot{\varphi}^2)\tan\alpha,\quad a_\varphi=\frac{\tan\alpha}{z}\frac{d}{dt}z^2\dot{\varphi}\ ,\quad a_z=\ddot{z}$$

Therefore the equations of motion become

$$m(\ddot{z}-z\dot{\varphi}^2)\tan\alpha=-R\cos\alpha, \qquad (1)$$
$$m\frac{\tan\alpha}{z}\frac{d}{dt}z^2\dot{\varphi}=0, \qquad (2)$$
$$m\ddot{z}=R\sin\alpha-mg \qquad (3)$$

By deleting R between Eq. (1) and (3), multiply Eq. (1) by $\sin\alpha$ and Eq. (3) by $\cos\alpha$ then adding, we get

$$\ddot{z}(\sin\alpha\tan\alpha+\cos\alpha)-z\dot{\varphi}^2\tan\alpha\sin\alpha=-g\cos\alpha \qquad (4)$$

Integrating Eq. (2) $z^2\dot{\varphi}=A$, A being an arbitrary constant. **(5)**

Given that initially the particle is projected horizontally with velocity $(2gh/(n^2+n))^{1/2}$ from a point P_0 (say) such that OM_0 equals the height of P_0 above O which equals h

Hence initially, at P_0, we have

$$v_\rho=\dot{\rho}=0,\quad v_\varphi=\rho\dot{\varphi}=(2gh/(n^2+n))^{1/2},\quad v_z=\dot{z}=0 \qquad (6)$$

Now, at P_0, $\rho=h\tan\alpha$. Also, let at P_0, $\dot{\varphi}\big|_{t=0}=\dot{\varphi}_0$. Hence Eq. (6) gives $(2gh/(n^2+n))^{1/2}=v_\varphi=h\tan\alpha\dot{\varphi}_0$

giving $\dot{\varphi}_0=(\cos\alpha/h\sin\alpha)(2gh/(n^2+n))^{1/2}$ **(7)**

Since at P_0, $\rho=h\tan\alpha$ **i.e.,** $z=h$ and $\dot{\varphi}=\dot{\varphi}_0$, Eq. (5) gives

$h^2\dot{\varphi}_0=A$ **or** $A=(h/\tan\alpha)(2gh/(n^2+n))^{1/2}$, by Eq. (7)

Therefore, Eq. (5) gives

$$\dot{\varphi}=A/z^2=(h/z^2\tan\alpha)(2gh/(n^2+n))^{1/2} \qquad (8)$$

Substituting the value of $\dot{\varphi}$ from Eq. (8) in Eq. (4), we get

$$\ddot{z}(\sin\alpha\tan\alpha+\cos\alpha)-(h^2/z^4\tan^2\alpha)(2gh/(n^2+n))z\tan\alpha\sin\alpha=-g\cos\alpha$$

$$\Rightarrow \ddot{z}\sec\alpha-(h^2/z^3)(2gh/(n^2+n))\cos\alpha=-g\cos\alpha$$

$$\Rightarrow \ddot{z}-(h^2/z^3)(2gh/(n^2+n))\cos^2\alpha=-g\cos^2\alpha$$

$$\Rightarrow \dot{z}\frac{d\dot{z}}{dz}-(h^2/z^3)(2gh/(n^2+n))\cos^2\alpha=-g\cos^2\alpha$$

This equation is similar to the previous derived equation with ($z=r\cos\alpha$)

$$\ddot{r}-\frac{h^2}{r^3\cos^2\alpha}\left(\frac{2gh}{n^2+n}\right)=-g\cos\alpha, \qquad \textbf{Obtained previous}$$

Then one completes the solution as previous.

❒ **EXAMPLE:** A particle moves over the interior surface about the vertical axis. Initially at a height h above the vertex a particle is projected horizontally along the surface with a velocity V. Prove that the equation of motion of the particle and determine the maximum vertical distance when the particle projected $V^2 = 4gh$.

❒ **SOLUTION:**

The acceleration in terms of cylindrical coordinates are given by

$$a_\rho = \ddot{\rho} - \rho\dot{\varphi}^2, \quad a_\varphi = \frac{1}{\rho}\frac{d}{dt}\,\rho^2\dot{\varphi}\;, \quad a_z = \ddot{z}$$

Note there is a relation between ρ and z i.e., $\rho^2 = 4az$, Therefore, $\dot{\rho} = \dot{z}\tan\alpha$ and $\ddot{\rho} = \ddot{z}\tan\alpha$ hence the acceleration components are re-written as following

$$a_\rho = (\ddot{z} - z\dot{\varphi}^2)\tan\alpha, \quad a_\varphi = \frac{\tan\alpha}{z}\frac{d}{dt}\,z^2\dot{\varphi}\;, \quad a_z = \ddot{z}$$

Therefore the equations of motion become

$$m(\ddot{\rho} - \rho\dot{\varphi}^2) = -R\sin\psi, \qquad (1)$$

$$m\frac{1}{\rho}\frac{d}{dt}\,\rho^2\dot{\varphi} = 0, \qquad (2)$$

$$m\,\ddot{z} = R\cos\psi - mg \qquad (3)$$

By deleting R between Eq. (1) and (3), multiply Eq. (1) by $\sin\alpha$ and Eq. (3) by $\cos\alpha$ then adding, we get

$$\ddot{\rho}\cos\psi + \ddot{z}\sin\psi - \rho\dot{\varphi}^2\cos\psi = -g\sin\psi$$

$$\ddot{\rho} + \frac{\rho\ddot{z}}{2a} - \rho\dot{\varphi}^2 = -\frac{g\rho}{2a} \qquad (\tan\psi = \rho/2a)$$

Since $\rho^2 = 4az \quad \Rightarrow \ddot{\rho} = \dfrac{2a\ddot{z}}{\rho} - \dfrac{4a^2\dot{z}^2}{\rho^3}$

$$\Rightarrow \frac{2a\ddot{z}}{\rho} - \frac{4a^2\dot{z}^2}{\rho^3} + \frac{\rho\ddot{z}}{2a} - \rho\dot{\varphi}^2 = -\frac{g\rho}{2a} \qquad (\tan\psi = \rho/2a) \qquad (4)$$

Now, integrating Eq. (2) $\rho^2\dot{\varphi} = A$, A being an arbitrary constant. **(5)**

Given that initially the particle is projected horizontally with velocity V from a point P_0 (say) such that OM_0 equals the height of P_0 above O which equals h

Hence initially, at P_0, we have

$$v_\rho = \dot{\rho} = 0, \quad v_\varphi = \rho\dot{\varphi} = V, \quad v_z = \dot{z} = 0 \qquad (6)$$

Now, at P_0, $\rho^2 = 4az$. Also, let at P_0, $\dot{\varphi}\big|_{t=0} = \dot{\varphi}_0$. Hence Eq. (6) gives $V = v_\varphi = \rho\dot{\varphi}_0$ (7)

Eq. (5) gives $\rho_0 \; \rho\dot{\varphi}_0 = A$ **or** $A = 2V(ah)^{1/2}$, by Eq. (7)

Therefore, Eq. (5) re-written as

$$\rho^2\dot{\varphi} = 2V(ah)^{1/2} \quad \Rightarrow \rho\dot{\varphi}^2 = 4V^2ah/\rho^3 \qquad (8)$$

Substituting the value of $\dot{\varphi}$ from Eq. (8) in Eq. (4), we get

$$\frac{2a\ddot{z}}{\rho} - \frac{4a^2\dot{z}^2}{\rho^3} + \frac{\rho\ddot{z}}{2a} - \frac{4V^2ah}{\rho^3} = -\frac{g\rho}{2a}$$

Multiply by $2a/\rho$, we get

$$\ddot{z} + \frac{4a^2\ddot{z}}{\rho^2} - \frac{8a^3\dot{z}^2}{\rho^4} - \frac{8V^2a^2h}{\rho^4} = -g$$

Substituting $\rho^2 = 4az$, we have

$$\frac{a\ddot{z}}{z} - \frac{a\dot{z}^2}{2z^2} - \frac{V^2h}{2z^2} + \ddot{z} = -g \qquad (9)$$

Multiply by, we obtain

$$\frac{2a\dot{z}\ddot{z}}{z} - \frac{a\dot{z}^3}{z^2} - \frac{V^2h\dot{z}}{z^2} + 2\dot{z}\ddot{z} = -2\dot{z}g$$

$$\Rightarrow a\frac{d}{dt}\left(\frac{\dot{z}^2}{z}\right) + V^2h\frac{d}{dt}\left(\frac{1}{z}\right) + \frac{d}{dt}\dot{z}^2 = -2g\frac{dz}{dt}$$

$$\Rightarrow a\,d\left(\frac{\dot{z}^2}{z}\right) + V^2h\,d\left(\frac{1}{z}\right) + d\dot{z}^2 = -2gdz$$

then integrate

$$a\left(\frac{\dot{z}^2}{z}\right) + V^2h\left(\frac{1}{z}\right) + \dot{z}^2 = -2gz + B \qquad (10)$$

B being an arbitrary constant that can be determined using initial condition, i.e., $z = h,$ when $\dot{z} = 0 \Rightarrow B = V^2 + 2gh$, .

Eq. (10) reduces to

$$a\left(\frac{\dot{z}^2}{z}\right)+V^2h\left(\frac{1}{z}\right)+\dot{z}^2 \quad = V^2+2gh-2gz \qquad (11)$$

To obtain the two planes that the motion is between them we put in Eq. (11)

$$V^2h\left(\frac{1}{z}\right) \quad = V^2+2gh-2gz$$

$$\Rightarrow \quad 2gz^2 - z(V^2+2gh)+V^2h = 0$$

Solving this quadrant equation we have,

$$\begin{aligned} z &= \frac{V^2+2gh}{4g} \pm \frac{\sqrt{(V^2+2gh)^2-8gV^2h}}{4g} \\ &= \frac{V^2+2gh}{4g} \pm \frac{\sqrt{V^4+4gV^2h+4g^2h^2-8gV^2h}}{4g} \\ &= \frac{V^2+2gh}{4g} \pm \frac{\sqrt{(V^2-2gh)^2}}{4g} \\ &= \frac{V^2+2gh}{4g} \pm \frac{V^2-2gh}{4g} \end{aligned}$$

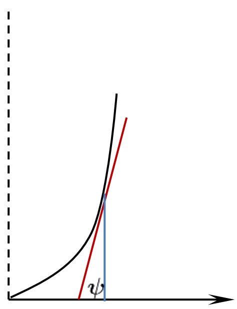
where, plus and minus signs gives the roots, i.e., $z = V^2/2g, h$

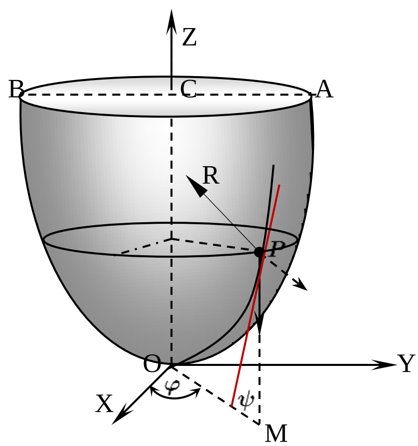
This means that the motion of the particle is subtended between the two planes

$z_1 = V^2/2g$ and $z_2 = h$ initial plane of projection

To calculate the maximum vertical distance when ' ue

in $z_1 = V^2/2g$ hence $z = 2h$.





PROBLEMS

1. A heavy particle is projected with velocity V from the end of a horizontal diameter of a sphere of radius a along the inner surface, the direction of projection making an angle β with the equator. If the particle never leaves the surface, prove that $3 \sin^2 \beta < 2 + (V^2/3ga)^2$.

2. A particle constrained to move on a smooth spherical surface is projected horizontally from a point at the level of the centre so that its angular velocity relative to the centre is ω. If $\omega^2 a$ be very great compared with g, show that its depth z below the level of the centre at time t is $(2g/\omega^2) \sin^2 (\omega t/2)$ approximately.

3. A particle describes a rhumb-line on a sphere in such a way that its longitude increases uniformly; show that the resultant acceleration varies as the cosine of the latitude and that its direction makes with the normal an angle equal to the latitude.

4. A particle moves on the inside of a smooth sphere, of radius a, under a force perpendicular to and acting from a given diameter, which equals $(\mu \sin \theta/\cos^4 \theta)$ when the particle is at an angular distance θ from that diameter; if, when the angular distance of the particle is γ, it is projected with velocity $(\mu a)^{1/2} \sec \gamma$ in a direction perpendicular to the plane through itself and the given diameter, show that its path is a small circle of the sphere, and find the reaction of the sphere.

5. A particle moves on the surface of a smooth sphere and is acted on by a force in the direction of the perpendicular from the particle on a diameter and equal to μ/(distance)^3. Show that it can be projected so that its path will cut the meridians at a constant angle.

6. A particle moves on the interior of a smooth sphere, of radius a, under a force producing an acceleration $\mu\omega^n$ along the perpendicular ω drawn to a fixed diameter. It is projected with velocity V along the great circle to which this diameter is perpendicular and is slightly disturbed from its path; show that the new path will cut the old one m times in a revolution, where $m^2 = 4(1 - \mu a^{n+1}/V^2)$.

A particle is attached to one end of a string, of length a, the other end of which is tied to a fixed point O. When the string is inclined at an acute angle α to the downward-drawn vertical the particle is projected horizontally and perpendicular to the string with a velocity V. Find the resulting motion.

A heavy particle is projected horizontally along the inner surface of a smooth spherical shell of radius $a/\sqrt{2}$ with velocity $(7ag/3)^{1/2}$ at a depth 2a/3 below the centre. Show that it will rise to a height a/3 above the centre, and that pressure on the sphere just vanishes at the highest point of the path.

A particle moves on the surface of a smooth sphere along a rhumb-line, being acted on by a force parallel to the axis of the rhumb-line. Show that the force varies inversely as the fourth power of the distance from the axis and directly as the distance from the meridian plane perpendicular to axis.

A thin straight hollow smooth tube is always inclined at an angle α to the upward drawn vertical, and revolves with uniform velocity ω about a vertical axis which intersects it. A heavy particle is projected from the stationary point of the tube with velocity $(g/\omega) \cot \alpha$; show that in time t it has described a distance $(g \cos \alpha/\omega^2 \sin^2 \alpha)(1 - e^{-\omega \sin \alpha \cdot t})$. Find also the reaction of the tube.

1. A smooth circular cone, of angle 2α, has its axis vertical and its vertex, which is pierced with a small hole, downwards. A mass M hangs at rest by a string which passes through the vertex, and a mass m attached to the upper end describes a horizontal circle on the inner surface of the cone. Find the time T of a complete revolution, and show that small oscillations about the steady motion take place in the time $T \operatorname{cosec} \alpha\{(M+m)/3m\}^{1/2}$.

2. A point moves with a constant velocity on a cone so that its direction of motion makes a constant angle with a plane perpendicular to the axis of the cone. Show that the resultant acceleration is perpendicular to the axis of the cone and varies inversely as the distance of the point from the axis.

3. At the vertex of a smooth cone of vertical cone 2α, fixed with its axis vertical and vertex downwards, is a centre of repulsive force $\mu/(\text{distance})^4$. A weightless particle is projected horizontally with velocity $(2\mu \sin^3 \alpha/c^3)^{1/2}$ from a point, distant c from the axis, along the inside of the surface. Show that it will describe a curve on the cone whose projection on a horizontal plane is $1-(c/r) = 3\tanh^2\{(\theta/2)\sin\alpha)\}$.

4. A particle moves on a smooth cone under the action of a force to the vertex varying as the square of the distance. If the cone be developed into a plane, show that the path becomes a conic section.

5. If a particle moves on the inner surface of a right circular cone under the action of a force from the vertex, the law of repulsion being $m\mu(a\cos^2\alpha/r^3 - 1/2r^3)$, where 2α is the vertical angle of the cone, and if it be projected from an apse at distance a with velocity $(\mu/a)^{1/2}\sin\alpha$, show that the path will be a parabola.

[show that the plane of the motion is parallel to a generator of the cone]

6. A particle is constrained to move on a smooth conical surface of vertical angle 2α and describes a plane curve under the action of an attraction to the vertex, the plane of the orbit cutting the axis of the cone at a distance a from the vertex. Show that the attractive force must vary as $(1/r^2) - (a/r^3)\cos\alpha$.

NON-INERTIAL REFERENCE FRAME

INTRODUCTION

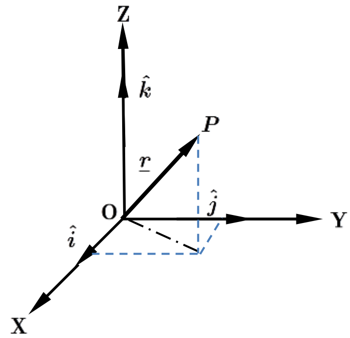
The advantage of choosing an inertial reference frame to describe dynamic motion was made evident in the previous studying. It is always possible to express the equations of motion for a system in an inertial frame. But there are types of problems for which these equations would be extremely complex, and it becomes easier to treat the motion of the system in a non-inertial frame of reference.

To illustrate, for example, the motion of a particle on or near the surface of the Earth, it is tempting to do so by choosing a coordinate system fixed with respect to Earth. We know, however, that Earth undergoes a complicated motion, compounded of many different rotations (and hence accelerations) with respect to an inertial reference frame identified with the fixed stars. Earth's coordinate system is, therefore, a non-inertial frame of reference; and, although the solutions to many problems can be obtained to the desired degree of accuracy by ignoring this distinction, many important effects result from the non-inertial nature of the Earth coordinate system. In fact, we have already studied non-inertial systems when we studied Sun-Earth orbits are observed on Earth's surface, which is a non-inertial system.

In analyzing the motion of rigid bodies in the following chapters, we also find is convenient to use non-inertial reference frames and therefore make use of much of the development presented here.

♦Time rate of change of vector in Fixed frame

Let $\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of point P in the Fixed system frame. In this case, the time rate of change of the vector $\underline{r}$ according to these fixed frames we get



$$\left(\frac{d\underline{r}}{dt}\right)_{\text{Fixed}} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

Since the time rate of change of the unit vectors $\hat{i}, \hat{j}, \hat{k}$ is zero because these unit vectors have fixed direction and constant length (unity) i.e.,

$$\frac{d\hat{i}}{dt} = \frac{d\hat{j}}{dt} = \frac{d\hat{k}}{dt} = 0 \qquad (1)$$

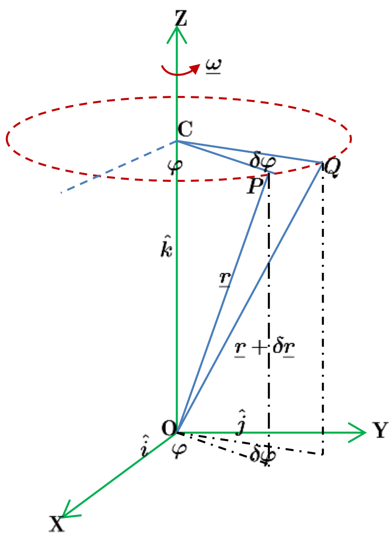
♦Time rate of change of constant vector

Let $\underline{r}$ be a vector of constant magnitude i.e., $|\underline{r}| = \text{constant}$ and it rotates with constant angular velocity ω around z-axis, i.e., $\underline{\omega} = \omega\hat{k}$, where the angle between the vector $\underline{r}$ and z-axis is constant and equals α Now

$$\underline{OQ} = \underline{r}(t+\delta t) = \underline{r}(t) + \delta\underline{r}(t)$$

we observe that $\underline{OQ} = \underline{OP} + \underline{PQ} = \underline{r}(t) + \underline{PQ}$ thus,

$$\underline{PQ} = \delta\underline{r}(t)$$



Then the time rate of change of vector $\underline{r}$ is

$$\frac{d\underline{r}}{dt} = \lim_{\delta t\to 0}\frac{\underline{r}(t+\delta t) - \underline{r}(t)}{\delta t}$$
$$= \lim_{\delta t\to 0}\frac{\delta\underline{r}(t)}{\delta t} = \lim_{\delta t\to 0}\frac{\underline{\underline{PQ}}}{\delta t}$$

Since the length $\underline{PQ}$ is small enough then it can be considered to be equal to the arc length from the circle whose its center is C and we get

$$PQ = PQ = CP\delta\varphi = r\sin\alpha\delta\varphi$$

when δt approaches to zero, therefore, $PQ \to 0$ and hence,

$$\begin{aligned}
\frac{d\underline{r}}{dt} &= \lim_{PQ\to 0}\frac{\underline{PQ}}{PQ}\times\lim_{\delta t\to 0}\frac{PQ}{\delta t} \\
&= \lim_{PQ\to 0}\frac{\underline{PQ}}{PQ}\times\lim_{\delta t\to 0}\frac{r\sin\alpha\delta\varphi}{\delta t} \\
&= r\sin\alpha\underbrace{\lim_{\delta t\to 0}\frac{\delta\varphi}{\delta t}}_{\dot{\varphi}}\times\lim_{PQ\to 0}\frac{\underline{PQ}}{PQ}
\end{aligned}$$

$$\therefore\ \frac{d\underline{r}}{dt} = r\dot{\varphi}\sin\alpha\lim_{PQ\to 0}\frac{\underline{PQ}}{PQ}$$

where $\dot{\varphi}$ represents the angular velocity of P rotation in the circle of center C which equals the angular velocity ω of rotation vector $\underline{r}$ then

$$\frac{d\underline{r}}{dt} = r\omega\sin\alpha\lim_{PQ\to 0}\frac{\underline{PQ}}{PQ}$$

But it is clear that $\underline{PQ}/PQ$ is a unit vector in direction of $\underline{PQ}$ and when PQ approaches zero ($PQ \to 0$) then $P \to Q$ that is the unit vector becomes in the direction of tangential to the circle whose its center is C at P that is

$$\frac{d\underline{r}}{dt} = r\omega\sin\alpha\ \hat{u} \qquad (1)$$

where $\hat{u}$ is a unit vector perpendicular to $\hat{k}$, in addition is perpendicular to the radius CP, that $\hat{u}$ is normal to the plane of vectors $\hat{k}$ and $\underline{CP}$, hence, $\hat{u}$ is normal to $\underline{OP} = \underline{r}$ and from the definition of vector product we have,

$$\underline{\omega}\wedge\underline{r} = r\omega\sin\alpha\ \hat{u} \qquad (2)$$

By comparing the two Equations (1) and (2) we have

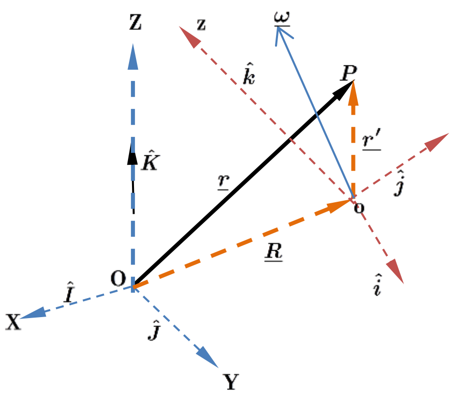
$$\frac{d\underline{r}}{dt} = \underline{\omega}\wedge\underline{r} \qquad (3)$$

The last equation (3) gives us the time rate of change of a vector with constant length and rotates with a constant angular velocity around an axis (here z-axis). So the time rate of change of unit vectors $\hat{i},\hat{j},\hat{k}$ (where $|\hat{i}|=|\hat{j}|=|\hat{k}|=1$)

$$\frac{d\hat{i}}{dt}=\underline{\omega}\wedge\hat{i},\quad \frac{d\hat{j}}{dt}=\underline{\omega}\wedge\hat{j},\quad \frac{d\hat{k}}{dt}=\underline{\omega}\wedge\hat{k} \qquad (*)$$

♦ Motion Referred to a Moving Coordinate System

Suppose that the position of a point P (Figure below) is determined with respect to n oxyz coordinate system, while at the same time this coordinate system moves with a translational velocity $\underline{\dot{R}}$ and an angular velocity $\underline{\omega}$ with respect to a "fixed" OXYZ coordinate system. This is the type of coordinate system which might become necessary, for example, in a long range ballistics problem for which the motion of the Earth would have to be taken into account. In such a problem the measurements would be made with respect to the Earth, and the motion of the Earth relative to some coordinate system fixed with respect to certain stars would be considered. We shall now derive a general expression for the acceleration of a point referred to a coordinate system which itself is moving.



In the analysis to follow, we shall always measure the vectors $\underline{R}$ and $\underline{r}$ in the fixed OXYZ system. The unit vectors $(\hat{i},\hat{j},\hat{k})$ always have the direction of the moving coordinate axes, while the unit vectors $(\hat{I}, \hat{J}, \hat{K})$ always have the direction of the fixed coordinate axes.

By the absolute displacement $\underline{r}$ of the point P is meant the displacement measured with respect to the fixed OXYZ system. By differentiating this absolute displacement we obtain the absolute velocity $\underline{\dot{r}}$. and the absolute acceleration $\underline{\ddot{r}}$.

$$\underline{r} = X\hat{I} + Y\hat{J} + Z\hat{K}$$
$$\underline{\dot{r}} = \dot{X}\hat{I} + \dot{Y}\hat{J} + \dot{Z}\hat{K}$$
$$\underline{\ddot{r}} = \ddot{X}\hat{I} + \ddot{Y}\hat{J} + \ddot{Z}\hat{K}$$

During these differentiations, the unit vectors it, $(\hat{I}, \hat{J}, \hat{K})$ are treated as constants due to Equation (1), since neither their magnitudes nor their directions change with time. If we wish to express the absolute motion in terms of motion measured in the moving oxyz system, we have,

$$\underline{r} = \underline{R} + \underline{r'} = \underline{R} + x\hat{i} + y\hat{j} + z\hat{k}$$

where the directions of the $(\hat{i}, \hat{j}, \hat{k})$ unit vectors are known with respect to the fixed system. However, the unit vectors are changing direction with time, since they rotate with the oxyz system. In taking the derivatives $\underline{\dot{r}}$ and $\underline{\ddot{r}}$, therefore, the time derivatives of these unit vectors must be included,

$$\underline{\dot{r}} = \underline{\dot{R}} + \underline{\dot{r}'} = \underline{\dot{R}} + \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k} + x\frac{d\hat{i}}{dt} + y\frac{d\hat{j}}{dt} + z\frac{d\hat{k}}{dt}$$

The derivatives of the unit vectors are given by Equations (2), so

$$\underline{\dot{r}} = \underline{\dot{R}} + \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k} + \underline{\omega} \wedge (x\hat{i} + y\hat{j} + z\hat{k})$$

The quantity $(\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k} = \underline{\dot{r}'}_r)$ represents the velocity of the point P, measured relative to the moving coordinate system, which we shall call the relative velocity $\underline{\dot{r}'}$. Using this notation, the expression for P becomes,

$$\underline{\dot{r}} = \underline{\dot{R}} + \underline{\dot{r}'}_r + \underline{\omega} \wedge \underline{r'}$$

The acceleration of *P* may be found by a second differentiation,

$$\ddot{\underline{r}} = \ddot{\underline{R}} + \ddot{\underline{r}}'$$
$$= \ddot{\underline{R}} + (\ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}) + 2\underline{\omega} \wedge (\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}) + \dot{\underline{\omega}} \wedge (x\hat{i} + y\hat{j} + z\hat{k})$$
$$+ \underline{\omega} \wedge (x\hat{i} + y\hat{j} + z\hat{k})$$

Writing($\ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k} = \ddot{\underline{r}}_r$), which we call the relative acceleration of the point P, the expression for $\ddot{\underline{r}}$ can be written as,

$$\ddot{\underline{r}} = \ddot{\underline{R}} + \underline{\omega} \wedge (\underline{\omega} \wedge \underline{r}') + \dot{\underline{\omega}} \wedge \underline{r}' + \ddot{\underline{r}}'_r + 2\underline{\omega} \wedge \dot{\underline{r}}'_r \qquad (3)$$

The first three terms in this expression for $\ddot{\underline{r}}$ represent the absolute acceleration of a point attached to the moving coordinate system, coincident with the point P at any given time. This may be seen by noting that for a point fixed in the moving system $\dot{\underline{r}}'_r = \ddot{\underline{r}}'_r = \mathbf{0}$. The fourth term $\ddot{\underline{r}}'_r$ represents the acceleration of P relative to the moving system. The last term $2\underline{\omega} \wedge \dot{\underline{r}}'_r$, is sometimes called the acceleration of Coriolis, after G. Coriolis (1792-1843)) a French engineer who first called attention to this term and $\underline{\omega} \wedge (\underline{\omega} \wedge \underline{r}')$ is known as centrifugal acceleration.

◘ When ($\underline{R} = \mathbf{0}$) i.e., P places at the origin of oxyz coordinates

$$\underline{v}_F = \left(\frac{d\underline{r}}{dt}\right)_F = \left(\frac{d\underline{r}}{dt}\right)_r + \underline{\omega} \wedge \underline{r} = \underbrace{\dot{\underline{r}}}_{\underline{v}_r} + \underline{\omega} \wedge \underline{r}$$

$$\underline{a}_F = \left(\frac{d\underline{v}_F}{dt}\right)_F = \left(\frac{d\underline{v}_F}{dt}\right)_r + \underline{\omega} \wedge \underline{v}_F$$
$$= \ddot{\underline{r}} + \underline{\omega} \wedge \dot{\underline{r}} + \underline{\omega} \wedge (\dot{\underline{r}} + \underline{\omega} \wedge \underline{r})$$
$$= \underbrace{\ddot{\underline{r}}}_{\underline{a}_r} + 2\underline{\omega} \wedge \underbrace{\dot{\underline{r}}}_{\underline{v}_r} + \underline{\omega} \wedge (\underline{\omega} \wedge \underline{r}) + \dot{\underline{\omega}} \wedge \underline{r}$$

■Illustrative Examples■

❒ **EXAMPLE:** A set of coordinate system axes **Oxyz** (non-inertial) rotates with angular velocity $\underline{\omega}$ with respect to a set of inertial axes **OXYZ** (Fixed axes) where $\underline{\omega}=2\hat{i}-3\hat{j}+5\hat{k}$ If $\underline{r}=\sin t\hat{i}-\cos t\hat{j}+e^{-t}\hat{k}$ then find

$\left(\frac{d\underline{r}}{dt}\right)_r$, $\left(\frac{d\underline{r}}{dt}\right)_F$, $\left(\frac{d^2\underline{r}}{dt^2}\right)_r$, $\left(\frac{d^2\underline{r}}{dt^2}\right)_F$

❒ **SOLUTION:**

$$\left(\frac{d\underline{r}}{dt}\right)_r=\dot{\underline{r}}=\cos t\hat{i}+\sin t\hat{j}-e^{-t}\hat{k},$$

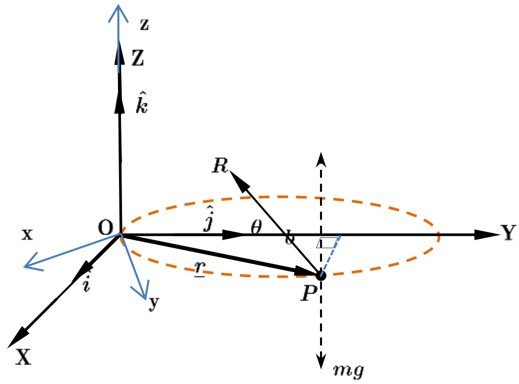
$$\left(\frac{d\underline{r}}{dt}\right)_F=\left(\frac{d\underline{r}}{dt}\right)_r+\underline{\omega}\wedge\underline{r}=\cos t\hat{i}+\sin t\hat{j}-e^{-t}\hat{k}+\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\2&-3&5\\\sin t&-\cos t&e^{-t}\end{vmatrix}$$
$$=(6\cos t-3e^{-t})\hat{i}+(6\sin t-2e^{-t})\hat{j}+(3\sin t-2\cos t-e^{-t})\hat{k},$$

$$\left(\frac{d^2\underline{r}}{dt^2}\right)_r=\ddot{\underline{r}}=-\sin t\hat{i}+\cos t\hat{j}+e^{-t}\hat{k},$$

$$\left(\frac{d\underline{r}}{dt}\right)_F=\left(\frac{d\underline{r}}{dt}\right)_r+\underline{\omega}\wedge\underline{r}=\cos t\hat{i}+\sin t\hat{j}-e^{-t}\hat{k}+\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\2&-3&5\\\sin t&-\cos t&e^{-t}\end{vmatrix}$$
$$=(6\cos t-3e^{-t})\hat{i}+(6\sin t-2e^{-t})\hat{j}+(3\sin t-2\cos t-e^{-t})\hat{k},$$

$$\left(\frac{d^2\underline{r}}{dt^2}\right)_F=\left(\frac{d}{dt}\left(\frac{d\underline{r}}{dt}\right)_F\right)_F=\left(\frac{d}{dt}\left(\frac{d\underline{r}}{dt}\right)_F\right)_r+\underline{\omega}\wedge\left(\frac{d\underline{r}}{dt}\right)_F=.......$$

❐ **EXAMPLE:** A smooth circular wire of radius b rotates with constant angular velocity ω in a horizontal plane about a perpendicular axis to its plane at a point O on its circumference. Prove that the equation of motion for a smooth small bead slides over the wire is $\ddot{\theta} + \omega^2 \sin\theta = 0$ where θ represents the angle between the diameters that passing through the bead and the point O.



❐ **SOLUTION:**

Let P be the position of the ring at instant t where $\underline{r} = b\sin\theta\hat{i} + b(1+\cos\theta)\hat{j}$ according to rotating frame, and the fixed and rotating frames are choosing as illustrated in the figure OXYZ represents the rotating frame, while Oxyz represents fixed frame. Now $\underline{\omega} = \omega\hat{k}$ and

$$\underline{r} = b\sin\theta\hat{i} + b(1+\cos\theta)\hat{j} \qquad \Rightarrow \dot{\underline{r}} = b\dot{\theta}\cos\theta\hat{i} - b\dot{\theta}\sin\theta\hat{j}$$

$$\Rightarrow \underline{v}_{\mathrm{F}} = \left(\frac{d\underline{r}}{dt}\right)_F = \dot{\underline{r}} + \underline{\omega} \wedge \underline{r}$$

$$= b\dot{\theta}\cos\theta\hat{i} - b\dot{\theta}\sin\theta\hat{j} + b\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \omega \\ \sin\theta & 1+\cos\theta & 0 \end{vmatrix}$$

$$= b(\dot{\theta}\cos\theta - \omega(1+\cos\theta))\hat{i} + b(\omega\sin\theta - \dot{\theta}\sin\theta)\hat{j}$$

and the acceleration becomes

$$\Rightarrow \underline{a}_{\mathrm{F}} = \left(\frac{d\underline{v}_{\mathrm{F}}}{dt}\right)_{\mathrm{F}} = \dot{\underline{v}}_{\mathrm{F}} + \underline{\omega} \wedge \underline{v}_{\mathrm{F}}$$

$$= b(\dot{\theta}\cos\theta - \omega(1+\cos\theta))\hat{i} + b(\omega\sin\theta - \dot{\theta}\sin\theta)\hat{j}$$

$$+ b\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \omega \\ \dot{\theta}\cos\theta - \omega(1+\cos\theta) & \omega\sin\theta - \dot{\theta}\sin\theta & 0 \end{vmatrix}$$

$$= \underbrace{b(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta + 2\omega\dot{\theta}\sin\theta - \omega^2\sin\theta)}_{a_x}\hat{i}$$

$$+ \underbrace{b(-\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta + 2\omega\dot{\theta}\cos\theta - \omega^2(1+\cos\theta))}_{a_y}\hat{j}$$

The forces acting on the ring within its motion are its weight mg downwards, reaction of the wire which we solve it into two components namely, R passing through the center of the wire and the other component N upwards direction where N = mg, the equations of motion in OX and OY directions are

$$ma_x = -R\sin\theta, \qquad ma_y = -R\cos\theta,$$

or $mb(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta + 2\omega\dot{\theta}\sin\theta - \omega^2\sin\theta) = -R\sin\theta$

and $mb(-\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta + 2\omega\dot{\theta}\cos\theta - \omega^2(1+\cos\theta)) = -R\cos\theta$

Dividing the last two equations we get

$$\frac{mb(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta + 2\omega\dot{\theta}\sin\theta - \omega^2\sin\theta)}{mb(-\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta + 2\omega\dot{\theta}\cos\theta - \omega^2(1+\cos\theta))} = \frac{-R\sin\theta}{-R\cos\theta}$$

or

$$\frac{(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta + 2\omega\dot{\theta}\sin\theta - \omega^2\sin\theta)}{(-\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta + 2\omega\dot{\theta}\cos\theta - \omega^2(1+\cos\theta))} = \frac{\sin\theta}{\cos\theta}$$

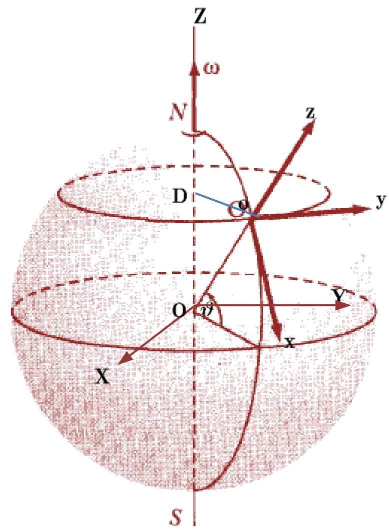
$$\Rightarrow \cos\theta(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta + 2\omega\dot{\theta}\sin\theta - \omega^2\sin\theta)$$
$$= \sin\theta(-\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta + 2\omega\dot{\theta}\cos\theta - \omega^2(1+\cos\theta))$$

$$\ddot{\theta}(\cos^2\theta + \sin^2\theta) + \omega^2\sin\theta = 0$$

or $$\ddot{\theta} + \omega^2\sin\theta = 0$$

which gives the equation of motion of the small bead.

♦Motion relative to the Earth



As known, the Earth spins about its polar axis once every about 24 hours or 8.64×10^4 s (not strictly true in a sidereal frame due to the orbit of the Earth about the Sun but sufficiently accurate for our purposes here), which means that each point of the Earth disrobe a circle about the axis. Consider a point P on the Earth's surface with latitude ϑ (latitude represents the angle between the radius to P and the equatorial plane; as seen in the Figure). The Earth angular speed is

$$\omega = \frac{2\pi}{8.64 \times 10^4} = 7.27 \times 10^{-5} \ \text{rad}\,\text{s}^{-1}$$

The acceleration of P due to the spin of the Earth will vary with the latitude of point P. If is the distance of P from the polar axis, the acceleration of P towards to the axis is $r\omega^2 = b\cos\vartheta \cdot \omega^2 = 3.36\cos\vartheta \ \text{cm}\,\text{s}^{-2}$

Gravitational pull. It is approximately 981 cm s^{-2} (its value of course not constant since the Earth is not perfectly spherical shape). Both accelerations are small compared with this typical acceleration due to gravity. For this reason we can take the Earth as an inertial frame for local phenomena on the Earth. It must be emphasized, however, that is not always safe to assume that small accelerations effective over long intervals of time can be ignored. A small sustained acceleration can produce significant impacts.

The motion of Earth with respect to an inertial reference frame is dominated by Earth’s rotation about its own axis. The effects of the other motion (e.g., the revolution about the Sun and the motion of the solar system with respect to the local galaxy) are small by comparison. If we place the fixed or inertial frame OXYZ at the center of Earth (at point O) and the moving reference frame oxyz on the surface of Earth at latitude ϑ as seen from the figure, we can describe the motion of a moving object close to the surface of Earth.

$$\underline{\omega} = -\omega\cos\vartheta\hat{i} + \omega\sin\vartheta\hat{k} \qquad \text{and} \qquad \underline{r}' = \boldsymbol{R} + \underline{r}$$

where $\underline{r}', \underline{r}$ are position vectors of the particle according to O and o respectively. The acceleration here differs from the obtained previous with the value of acceleration of o with respect to O which equals $\omega^2(oD)$ where D is the center of latitude ϑ of point o, that is differs by the value $\omega^2(R\cos\vartheta)$ which is a small value and so can be neglected. Hence the acceleration of the particle with respect to o is given by

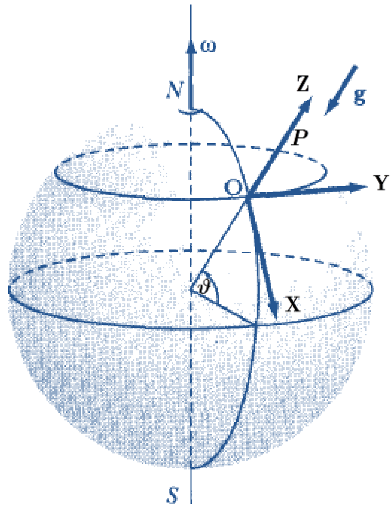
$$\underline{a}_{\mathrm{r}} = \underline{g} - 2\underline{\omega}\wedge\underline{v}_{\mathrm{r}} - \underline{\omega}\wedge(\underline{\omega}\wedge\underline{r}) - \dot{\underline{\omega}}\wedge\underline{r}$$

Here the angular velocity of the Earth is constant thus, $\dot{\underline{\omega}} = \mathbf{0}$ and also the term $\underline{\omega}\wedge(\underline{\omega}\wedge\underline{r})$ is small enough to be ignored and previous equation reduces to

$$\underline{a}_{\mathrm{r}} = \underline{g} - 2\underline{\omega}\wedge\underline{v}_{\mathrm{r}}, \qquad \underline{v}_{\mathrm{r}} \equiv (\dot{x}, \dot{y}, \dot{z})$$

This is the equation that we will use to discuss the motion of objects close to the surface of Earth.

❐ **APPLICATION 1:** Find the horizontal deflection from the plumb line caused by the Coriolis force acting on a particle falling freely in Earth's gravitational field from a height h above Earth's surface.



❐ **SOLUTION:**

The acceleration due to gravity is the effective one and is along the plumb line. We choose a Z-axis directed vertically outward (along –g) from the surface of Earth.

$$\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$\Rightarrow \underline{v}_r = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$$
$$\Rightarrow \underline{a}_r = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}$$

and the acceleration becomes

$$\Rightarrow \underline{a}_r = \underline{g} - 2\underline{\omega} \wedge \underline{v}_F, \qquad \underline{g} = -g\hat{k}$$

$$= -g\hat{k} - 2\omega \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\cos\vartheta & 0 & \sin\vartheta \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix}$$

The components of acceleration are

$$\ddot{x} = 2\omega\dot{y}\sin\vartheta \qquad (1)$$
$$\ddot{y} = -2\omega(\dot{x}\sin\vartheta + \dot{z}\cos\vartheta) \qquad (2)$$
$$\ddot{z} = -g + 2\omega\dot{y}\cos\vartheta \qquad (3)$$

Integrate Eq. (1) and (3), we get

$$\dot{x} = 2\omega y\sin\vartheta + c_1$$
$$\dot{z} = -gt + 2\omega y\cos\vartheta + c_2$$

From the boundary conditions at (t = 0), $\underline{v} = \underline{0}$, $\underline{r} = h\hat{k} \Rightarrow c_1 = c_2 = 0$

Therefore,

$$\dot{x} = 2\omega y\sin\vartheta \qquad (4)$$
$$\dot{z} = -gt + 2\omega y\cos\vartheta \qquad (5)$$

Substituting Eq. (4) and (5) in Eq. (2) we have

$$\begin{aligned}\ddot{y} &= -2\omega(2\omega y \sin^2\vartheta - gt\cos\vartheta + 2\omega y\cos^2\vartheta) \\ &= -2\omega(2\omega y - gt\cos\vartheta) \\ &= 2\omega gt\cos\vartheta - 4\omega^2 y \\ \Rightarrow \ddot{y} &= 2\omega gt\cos\vartheta \qquad (7)\end{aligned}$$

Neglecting the term $4\omega^2 y$ and integrate Eq. (7) we have $\dot{y} = \omega gt^2\cos\vartheta + c_3$

Again from the boundary conditions we obtain $c_3 = 0$

$$\therefore \dot{y} = \omega gt^2\cos\vartheta \qquad \Rightarrow y = \frac{1}{3}\omega gt^3\cos\vartheta + c_4$$

Since $c_4 = 0$ $\qquad \Rightarrow y = \frac{1}{3}\omega gt^3\cos\vartheta$

$$\dot{x} = 2\omega y\sin\vartheta = \frac{2}{3}\omega^2 gt^3\cos\vartheta\,\sin\vartheta$$
$$\dot{z} = -gt + \frac{2}{3}\omega^2 gt^3\cos^2\vartheta$$

We ignore the terms containing ω^2

$$\dot{x} = 0, \qquad \dot{z} = -gt$$

Integrating

$$x = c_5, \qquad z = -\frac{1}{2}gt^2 + c_6$$

Due to the boundary conditions we get $c_5 = 0, \quad c_6 = h$

$$x = 0, \qquad y = \frac{1}{3}\omega gt^3\cos\vartheta, \qquad z = h - \frac{1}{2}gt^2$$

These three equations give the position of the particle at any time and the particle reaches to the Earth i.e., (z = 0) when $T = (2h/g)^{1/2}$

Now when the particle arrives to Earth, its position is

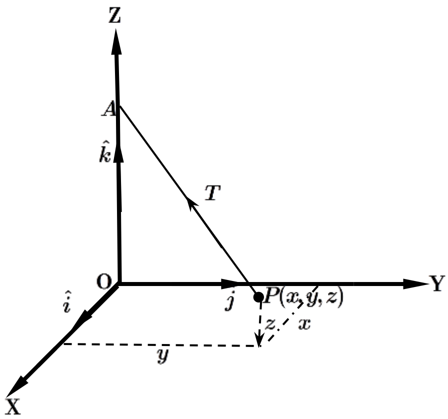
$$x = 0, \qquad y = \frac{1}{3}\omega g(2h/g)^{3/2}\cos\vartheta, \qquad z = 0$$

This means that the particle deflect to east of distance $\frac{1}{3}\omega g(2h/g)^{3/2}\cos\vartheta$

♦Foucault pendulum

The impact of Coriolis force on the motion of a pendulum produces a precession or rotation with time of the plane of oscillation. Describe the motion of this system, called a Foucault pendulum.

In order to illustrate this impact, let us select a set of coordinate axes with origin at the equilibrium point of the pendulum and z-axis along the local vertical. We are interested only in the rotation of the plane of oscillation- that is, we want to consider the motion of the pendulum bob in x-y plane (the horizontal plane). We therefore, limit the motion to oscillation of small amplitude, with the horizontal excursions small compared with the length of the pendulum. Under this assumpition, dz/dt is small compared with dx/dt and dy/dt and can be neglected. As clear



$$\underline{\omega} = -\omega\cos\vartheta\hat{i} + \omega\sin\vartheta\hat{k}$$

$$\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\underline{T} = -T\cos\alpha\,\hat{i} - T\cos\beta\,\hat{j} - T\cos\gamma\hat{k}$$

$$\cos\alpha = \frac{x}{L}, \qquad \cos\beta = \frac{y}{L}, \qquad \cos\gamma = \frac{L-z}{L}$$

Now the equation of motion are

$$m\underline{a} = \underline{T} + m\underline{g} - 2m\underline{\omega}\wedge\underline{v}$$

Since, $\underline{v} = \dot{x}\hat{i} + \dot{y}\,\hat{j} + \dot{z}\hat{k}, \qquad \underline{a} = \ddot{x}\hat{i} + \ddot{y}\,\hat{j} + \ddot{z}\hat{k}$

$$\underline{\omega}\wedge\underline{v} = \omega\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\cos\vartheta & 0 & \sin\vartheta \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix}$$
$$= \omega(-\dot{y}\sin\vartheta\,\hat{i} + (\dot{x}\sin\vartheta + \dot{z}\cos\vartheta)\hat{j} - \dot{y}\cos\vartheta\hat{k})$$

Thus, the equations of interest are

$$m\ddot{x} = -T\frac{x}{L} + 2m\omega\dot{y}\sin\vartheta,$$
$$m\ddot{y} = -T\frac{y}{L} - 2m\omega(\dot{x}\sin\vartheta + \dot{z}\cos\vartheta),$$
$$m\ddot{z} = -T\frac{(L-z)}{L} - mg + 2m\omega\dot{y}\cos\vartheta$$

For small oscillations, so we can consider that the particle moves in the horizontal plane XOY i.e., $\dot{z} = \ddot{z} = 0$ and hence, $T = mg - 2m\omega\dot{y}\cos\vartheta$, we have

$$m\ddot{x} = -m\frac{x}{L}(g - 2m\omega\dot{y}\cos\vartheta) + 2m\omega\dot{y}\sin\vartheta,$$
$$m\ddot{y} = -m\frac{y}{L}(g - 2m\omega\dot{y}\sin\vartheta) - 2m\omega\dot{x}\sin\vartheta,$$

Or
$$\ddot{x} = -\frac{g}{L}x + \frac{2\omega\cos\vartheta}{L}x\dot{y} + 2\omega\dot{y}\sin\vartheta,$$
$$\ddot{y} = -\frac{g}{L}y + \frac{2\omega\sin\vartheta}{L}y\dot{y} - 2\omega\dot{x}\sin\vartheta,$$

We note that the equation for $\ddot{x}$ contains a term in $\dot{y}$ and that the equation for $\ddot{y}$ contains a term in $\dot{x}$. Such equations are called coupled equations. Since the quantities x, y, ω are very small, so we can ignore the terms involve quantities $\omega x\dot{y}, \omega y\dot{y}$ then previous equations become

$$\ddot{x} = -\frac{g}{L}x + 2\omega\dot{y}\sin\vartheta,$$
$$\ddot{y} = -\frac{g}{L}y - 2\omega\dot{x}\sin\vartheta,$$

A solution for this pair of coupled equations can be affected by adding the first of the above equations to i times the second.

This equation is identical with the equation that describes damped oscillations, except that here the term corresponding to the damping factor is purely imaginary. The solution is given by ($Q = x + iy$, $k^2 = g/L$, $\tilde{\omega} = \omega\sin\vartheta$)

$$\ddot{Q} + 2i\tilde{\omega}\dot{Q} + k^2 Q = 0, \qquad (1)$$

The roots of auxiliary equation are

$$\lambda^2 + 2i\tilde{\omega}\lambda + k^2 = 0, \qquad \lambda = \frac{-2i\tilde{\omega} \pm \sqrt{-4\tilde{\omega}^2 - 4k^2}}{2} = i\ \ -\tilde{\omega} \pm \sqrt{\tilde{\omega}^2 + k^2}$$

The solution of differential equation (1) is

$$Q(t) = e^{-i\tilde{\omega}t}\quad Ae^{i\sqrt{\tilde{\omega}^2+k^2}t} + Be^{-i\sqrt{\tilde{\omega}^2+k^2}t}$$

If Earth were not rotating, that is $\tilde{\omega} = 0$, then the equation for $Q(t)$ would become

$$\ddot{\tilde{Q}} + k^2\tilde{Q} = 0, \qquad\qquad (\tilde{\omega} = 0)$$

from which it is seen that α corresponds to the oscillation frequency of the pendulum. This frequency is clearly much greater than the angular frequency of the Earth's rotation. Therefore, $k \gg \tilde{\omega}$, and the equation for $Q(t)$ becomes

$$Q(t) = e^{-i\tilde{\omega}t}\quad Ae^{ikt} + Be^{-ikt}$$

We can interpret this equation more easily if we note that the equation for $\tilde{Q}$ has the solution

$$\tilde{Q} = Q(t)\big|_{t=0} = X(t) + iY(t) = Ae^{ikt} + Be^{-ikt}$$

Thus

$$\begin{aligned} &Q(t) = \tilde{Q}(t)e^{-i\tilde{\omega}t} \\ &\Rightarrow x(t) + iy(t) = (X(t) + iY(t))(\cos\tilde{\omega}t - i\sin\tilde{\omega}t) \\ &\qquad\qquad\qquad = X(t)\cos\tilde{\omega}t + Y(t)\sin\tilde{\omega}t + i(Y(t)\cos\tilde{\omega}t - X(t)\sin\tilde{\omega}t) \end{aligned}$$

Equating real and imaginary parts,

$$x(t) = X(t)\cos\tilde{\omega}t + Y(t)\sin\tilde{\omega}t, \qquad y(t) = Y(t)\cos\tilde{\omega}t - X(t)\sin\tilde{\omega}t$$

or in matrix form $$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos\tilde{\omega}t & \sin\tilde{\omega}t \\ -\sin\tilde{\omega}t & \cos\tilde{\omega}t \end{pmatrix}\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} \qquad (\oplus)$$

from which (x, y) may be obtained from (X, Y) by the application of a rotation matrix of the familiar from

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Thus the angle of rotation $\theta = \tilde{\omega} t$, and the plane of oscillation of the pendulum therefore rotates with a frequency $\tilde{\omega} = \omega \sin \vartheta$ around Z-axis and the time of revolution is $2\pi/\tilde{\omega} = 2\pi/\omega \sin \vartheta$. The observation of this rotation gives a clear demonstration of the rotation of Earth.

PROBLEMS

If a particle is projected vertically upward to a height h above a point on Earth's surface at a northern latitude λ, show that it strikes the ground at a point $\frac{4}{3}\,\omega\cos\lambda\cdot\sqrt{8h^3/g}$ to the west. (Neglect air resistance, and consider only small vertical heights.)

If a projectile is fired due east from a point on the surface of Earth at a northern latitude λ with a velocity of magnitude V_0 and at an angle of inclination to the horizontal of α, show that the lateral deflection when the projectile strikes Earth is

$$d = \frac{4V_0^3}{g^2}\cdot\omega\sin\lambda\cdot\sin^2\alpha\,\cos\alpha$$

where ω is the rotation frequency of Earth.

Determine how much greater the gravitational field strength g is at the pole than at the equator. Assume a spherical Earth. If the actual measured difference is $\Delta g = 52\ \text{mm/s}^2$, explain the difference. How might you calculate this difference between the measured result and your calculation?

In the preceding problem, if the range of the projectile is R_0' for the case $\omega = 0$, show that the change of range due to the rotation of Earth is

$$\Delta R' = \sqrt{\frac{2R_0'^3}{g}}\cdot\omega\cos\lambda\left(\cot^{1/2}\alpha - \frac{1}{3}\tan^{3/2}\alpha\right)$$

Obtain an expression for the angular deviation of a particle projected from the North Pole in a path that lies close to Earth. Is the deviation significant for a missile that makes a 4,800-km flight in 10 minutes? What is the "miss distance" if the missile is aimed directly at the target? Is the miss distance greater for a 19,300-km flight at the same velocity?

Show that the small angular deviation ε of a plumb line from the true vertical (i.e., toward the center of Earth) at a point on Earth's surface at a latitude λ is

$$\varepsilon = \frac{R\omega^2\sin\lambda\cos\lambda}{g_0 - R\omega^2\cos^2\lambda}$$

where R is the radius of Earth. What is the value (in seconds of arc) of the maximum deviation? Note that the entire denominator in the answer is actually the effective g, and g_0 denotes the pure gravitational component.