Applied Mathematics 2

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First year

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SIMPLE HARMONIC MOTION

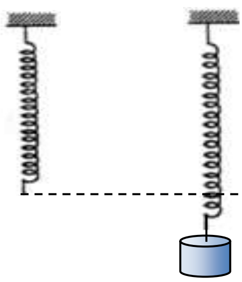
Oscillations are a particularly important part of mechanics and indeed of physics as a whole. This is because of their widespread occurrence and the practical importance of oscillation problems.

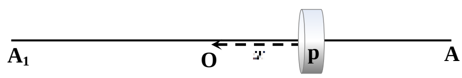
Most engineering materials are nearly elastic under working conditions. And, of course, all real things have mass. These ingredients, elasticity and mass, are what make vibration possible. Even structures which are fairly rigid will vibrate if encouraged to do so by the shaking of a rotating motor, the rough rolling of a truck, or the ground motion of an earthquake. The vibrations of a moving structure can also excite oscillations in flowing air which can in turn excite the structure further. This mutual excitement of fluids and solids is the cause of the vibrations in a clarinet reed, and may have been the source of the wild oscillations in the famous collapse of the Tacoma Narrows bridge. Mechanical vibrations are not only the source of most music but also of most annoying sounds. They are the main function of a vibrating massager, and the main defect of a squeaking hinge. Mechanical vibrations in pendulum or quartz crystals are used to measure time. Vibrations can cause a machine to go out of control, or a building to collapse. So, the study of vibrations, for better or for worse, is not surprisingly one of the most common applications of dynamics.

When an engineer attempts to understand the oscillatory motion of a machine or structure, she undertakes a vibration analysis. A vibration analysis is a study of the motions that are associated with vibrations. Study of motion is what dynamics is all about, so vibration analysis is just a part of dynamics. A vibration analysis could mean the making of a dynamical model of the structure one is studying, writing equations of motion using the momentum balance or energy equations and then looking at the solution of these equations. But, in practice, the motions associated with vibrations have features which are common to a wide class of structures and machines. For this reason, a special vocabulary and special methods of approach have been developed for vibration analysis. For example, one can usefully discuss resonance, normal modes, and frequency response, concepts which we will soon discuss, without ever writing down any equations of motion. We will first approach these concepts within the framework of the differential equations of motion and their solutions. But after the concepts have been learned, we can use them without necessarily referring directly to the governing differential equations.

■ **Definition**

A particle is said to execute Simple Harmonic Motion if it moves such that its acceleration is always directed towards a fixed point, and is proportional to the distance of the particle from the faced point.





The expressions for velocity and position of the particle at any instant are obtained as follows:

Suppose O be the fixed point in the line A_1OA and let P denote the particle after time t from moving with a velocity v in the positive direction from O to A. Let OP $= x$, then the acceleration is kx where k is a constant. Since the acceleration is in the direction opposite to that in which x increases, the equation of motion of the particle is given as

$$m\frac{d^2x}{dt^2} = -kx$$

Rearranging this equation, we get one of the most famous and useful differential equations of all time:

$$\ddot{x} + \frac{k}{m}x = 0$$

This equation appears in many contexts both in and out of dynamics. In non-mechanical contexts the variable x and the parameter combination k/m are replaced by other physical quantities. In an electrical circuit, for example, x might represent a voltage and the term corresponding to k/m might be 1/LC, where C is a capacitance and L an inductance. But even in dynamics the equation appears with other physical quantities besides k/m multiplying the x, and x itself could represent rotation, say, instead of displacement. In order to avoid being specific about the physical system being modeled, the harmonic oscillator equation is often written as

$$\ddot{x} + w^2x \ = 0$$

The constant in front of the x is called w^2 instead of just, say, w, for two reasons:

(i) This convention shows that w^2 is positive,

(ii) In the solution we need the square root of this coefficient, so it is convenient to have $\sqrt{w^2} = w$.

For the spring-block system, w^2 is k/m and in other problems w^2 is some other combination of physical quantities.

■ Solution of harmonic oscillator differential equation

$\because \dfrac{d^2x}{dt^2} = -w^2 x$ or $v\dfrac{dv}{dx} = -w^2 x \quad \Rightarrow vdv = -w^2 x dx$

Integrating previous equation, we have

$$\frac{1}{2}v^2 = c_1 - \frac{1}{2}w^2 x^2$$

where c_1 is an integral constant. As P is supposed to be moving in the direction OA and as the acceleration is given to be taking place in the opposite direction, the particle P must come to rest at some point in OA say at A, i.e., suppose $v = 0$ where $x = a$, so that

$$0 = c_1 - \frac{1}{2}w^2 a^2 \qquad \Rightarrow c_1 = \frac{1}{2}w^2 a^2$$

Therefore

$v^2 = w^2 a^2 - w^2 x^2 = w^2(a^2 - x^2)$ or $v = \pm w\sqrt{a^2 - x^2}$

This equation gives the value of the velocity v for any displacement x

As P is moving in the positive direction $v = w\sqrt{a^2 - x^2}$

$$\frac{dx}{dt} = w\sqrt{a^2 - x^2} \qquad \frac{dx}{\sqrt{a^2 - x^2}} = w\,dt$$

By integrating

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = wdt \qquad \Rightarrow \sin^{-1}\left(\frac{x}{a}\right) = wt + \epsilon \quad \text{Or}$$

$$x = a\sin(wt + \epsilon)$$

where ϵ is integration constant to be determined from the initial conditions. If t is measured from the instant when P is at O, i.e., if $x = 0$ when $t = 0$, then $\epsilon = 0$.

➢ Note 1 Velocity in terms of time t can be obtained by differentiating any of these equations involving x and t.

➢ Note 2 When the particle is on the left-hand side of O, the equation of motion is $\ddot{x} = -w^2 x$ acceleration in the direction of $P_1A = w^2\, OP_1$

$= w^2(-x) = -w^2x$ Hence the same equation that holds on the right-hand side of O, holds also on the left hand side.

The Equation $\pm w\sqrt{a^2 - x^2}$ gives the velocity of P in terms of its distance from O. Initially, when $x = 0$ at the point O, the velocity is maximum and equal to wa. As As the particle proceeds towards A, the acceleration being towards O, the velocity goes on decreasing as x increases. At A where $x = a$, it vanishes and the particle is, for an instant, at rest. Then owing to the acceleration towards O the particle moves in the negative direction with a velocity which increases numerically as x decreases and is the greatest at O where it is $-wa$. Due to this velocity, the particle proceeds further to the negative side of O, the velocity remaining negative and decreasing gradually in magnitude till the particle comes to rest at A_1 where $x = -a$. The acceleration being towards O, the particle then starts and moves towards O with a positive velocity which increases gradually till it is again maximum at O. The same motion is repeated again and again and the particle goes on oscillating indefinitely between A and A_1, the two positions of momentary rest.

The motion of the particle is oscillatory. All oscillatory motions are, however, not necessarily simple harmonic. In fact, simple harmonic motion is the simplest and most important case of oscillatory motion which occurs in nature and it is always dominated by the differential equation

$$\frac{d^2x}{dt^2} = -w^2x$$

The distance OA or OA_1 i.e., the distance of the center from one of the positions of rest is called the ***Amplitude***.

■ The Periodic time of Motion

The equation $x = a\sin wt$ gives the time form in terms of x, the distance of the particle measured from O. Since

$$x = a\sin wt = a\sin(wt + 2\pi) = a\sin\left(wt + \frac{2\pi}{w}\right)$$

And $\dfrac{dx}{dt} = aw\cos wt = aw\cos(wt + 2\pi) = aw\cos\left(wt + \dfrac{2\pi}{w}\right)$

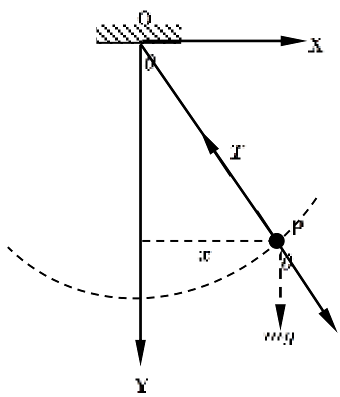
the particle has the same position, velocity and direction after time $t + \dfrac{2\pi}{w}$, $t + \dfrac{4\pi}{w}$ etc., as it had at the time t, i.e., the particle has a periodic motion, its periodic time τ being $\dfrac{2\pi}{w}$

■ The Frequency

The frequency of SHM is the number of complete oscillations in one second, so that if n denotes the frequency and τ the periodic time of the motion,

$$n\tau = 1 \quad \Rightarrow n = \frac{1}{\tau} = \frac{w}{2\pi}$$

■ Simple Pendulum



If a heavy particle is tied to one end of a light inextensible string with length and the other end of which is fixed, and oscillates in a vertical circle, we have what is called a Simple Pendulum. We now obtain the time of oscillation of such a pendulum when it is allowed to oscillate through a small angle only. Let O be the fixed point, A the lowest position of the particle, and P any position such that $\angle$**YOP**$=\theta$

The equations of motion in horizontal direction is (resolve the tension)

$$m\ddot{x} = -T\sin\theta \qquad \text{and} \qquad T\cos\theta = mg$$

Here we suppose that the motion of mass m in X direction only. Now, when the angle θ is small enough so the approximations $\cos\theta \cong 1$ and $\sin\theta \cong \theta$ can be applied and the equation of motion, $m\ddot{x} = -T\sin\theta$ becomes

$$m\ddot{x} = -mg\frac{x}{L} \qquad \text{or} \qquad \ddot{x} = -\frac{g}{L}x$$

which is similar to $\ddot{x} = -w^2 x$ with $w^2 = \dfrac{g}{L}$ Or $w = \sqrt{\dfrac{g}{L}}$

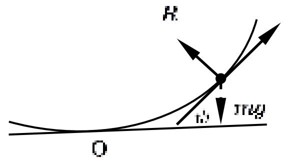
So, a simple pendulum moves like a SHM with periodic time of motion equals

$$2\pi\sqrt{\frac{L}{g}}$$

■ The Cycloid Pendulum

We have illustrated that the motion of a simple pendulum is simple harmonic motion only when the angle of swing is so small that $\sin\theta$ is very nearly equal to θ and the amplitude to the motion is so small that it may be treated as infinitesimal. If, however, the amplitude of motion is not small and the particle supposed to be constrained to move under gravity, along the arc of a smooth cycloid in a vertical plane, the equation of motion of the particle along the tangent to the curve is

$$m\frac{d^2S}{dt^2} = -mg\sin\psi \qquad (1)$$



where ψ is the angle which the tangent to the curve makes with the horizontal and S the length of its arc measured from the vertex, the cycloid being placed with its vertex downwards and axis vertical. We know by the Calculus that the intrinsic equation of the cycloid is

$$S = 4a\sin\psi \qquad (2)$$

Note S being measured from the vertex where $\psi = 0$, and a being the radius of the generating circle. From equations (1) and (2), we have

$$\frac{d^2S}{dt^2} = -\frac{g}{4a}S \qquad (3)$$

this formula shows that the motion represents simple harmonic no matter how great the amplitude. The time of a complete oscillation is given as $2\pi\sqrt{4a / g}$ which is constant for oscillations, small or large. Thus, if a particle is constrained to move along a smooth cycloid curve, its period of motion is absolutely independent of the amplitude. (This is an answer to the question which interested the mathematicians of the 18th century in what curve should that the bob of a pendulum swing in order that the period of oscillation may be absolutely independent of the amplitude?)

The oscillations on a cycloid are called isochronous because the period is the same for large or small oscillations. This important property of a cycloid finds its application in the formation of clocks. A cycloid pendulum may be constructed by causing the cord of the pendulum to wind and unwind itself on the evaluate of the path.

In order to find the pressure of the curve on the particle, we write its equation of motion in the direction of the normal at the point, namely

$$m\frac{v^2}{\rho} = R - mg\cos\psi$$

where ρ is the radius of curvature of the curve, R the normal pressure and v the velocity of the particle obtained from equation (3) by integration.

Note (i). The students acquainted with elements of differential equation will note that

$$S = A\cos wt + B\sin wt \text{ where } w = g / 4a$$

is the most general solution of differential equation (3).

Note a. Since $S = 4a\sin\psi = 4a\frac{dy}{ds} \Rightarrow S^2 = 8ay,$ y and S being measured from the vertex of the cycloid

■ Hooke's Law

The ' extension' of a stretched elastic string means the ratio of the increment in length to the unstretched length. Thus if ℓ, is the natural or unstretched length and the stretched length is ℓ' then the extension is $(\ell' - \ell)\,/\,\ell$.

Hooke's Law is that the *tension of the string is proportional to the extension.* If T denote the tension and we state the law in the form

$$T = \lambda\left(\frac{\ell' - \ell}{\ell}\right)$$

where λ is called the modulus of elasticity of the string.
The extension or compression of a spiral spring follows the same law, but in this' case the length is measured along the axis of the helix and not along the wire that forms the spring; and when the spring is extended or compressed the force exerted by the spring is a tension or a thrust in the direction of the axis. The formula above may be used for compression as well as extension provided we regard a negative tension as a thrust. For when the spring is compressed the length ℓ' is less than the natural length ℓ, so that the formula would give a negative tension, i.e. a thrust of magnitude $T = \lambda\left(\frac{\ell - \ell'}{\ell}\right)$.

■ Motion of a Particle Attached to an Elastic String

Elastic Strings. If an elastic string or wire or a spiral spring is fixed at one point and pulled within limits at the other, it is found to increase in length, the extension being proportional to the tension of the string.

If different wires of the same material are considered, the extension is directly proportional to the product of the tension and the natural length and inversely as the area A of the cross-section. Thus if *x* denotes the extension, l the natural length and T the tension (in absolute units), then,

$$x = \frac{T\ell}{\lambda A} \quad \text{Or} \quad T = \lambda\frac{Ax}{\ell}$$

where λ is a constant depending on the material of the wire. If we take A =unit area, we have $T = \lambda\frac{x}{\ell}$

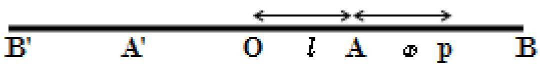
If ℓ is the natural length of an elastic string and l' the stretched length, then

$$T = \frac{\lambda}{\ell}(\ell' - \ell)$$

i.e., tension of an elastic string or a spring is proportional to the extension of the spring beyond its natural length. This is ***Hooke's law*** of elastic string and A is called the ***Modulus of Elasticity***.

When $x = \ell,\ T = \lambda$, so that λ for a string of unit cross-section is equal to the amount of force which would stretch it to twice its natural length.

Let one end of an elastic string be fixed to a point O on a smooth horizontal table and let OA=ℓ be its natural length.



If a particle of mass m is attached to the other end and if the particle is displaced along the line OA, a distance AB=b and P be position of the particle at any subsequent time so that AP=x, then the tension in the string is $T = \lambda\frac{x}{\ell}$; which acts in the direction PA and is directed towards A. The tension of the string being the only force which tends to move the particle, its equation of motion is

$$m\frac{d^2x}{dt^2} = -T = -\lambda\frac{x}{\ell} \quad \text{Or} \quad \frac{d^2x}{dt^2} = -\frac{\lambda}{\ell m}x \equiv -w^2x$$

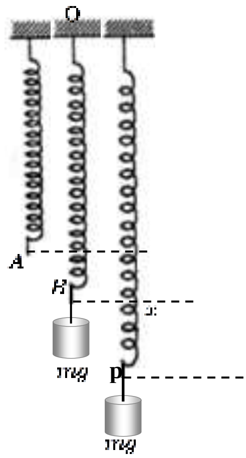
which shows that the motion about A is simple harmonic, the constant w equals $\sqrt{\frac{\lambda}{\ell m}}$. The periodic time of oscillation is $2\pi\sqrt{\frac{\ell m}{\lambda}}$

The particle will further move through to a point B' at an equal distance on the other side of and then back again and so on. The distance from A to A'

(OA= OA') and back to A is moved with the velocity which the particle acquires at A. The string being slack this velocity remains the same throughout this part. The periodic time obtained above refers to the time which the particle takes in moving from B to A, from A' to B' and then from B' to A' and from A to B. This is the only part where motion is simple harmonic.

■ Vertical Elastic string

Suppose that a particle of mass m is suspended from a fixed point by a string (or spring) OA of a natural length ℓ. Let OB be the length of the string when the mass hangs in equilibrium, then AB(=e), the extension of the string is given by



$$mg = \lambda\frac{AB}{\ell} = \lambda\frac{e}{\ell}$$

Now if the particle is displaced vertically from B it will oscillate in a vertical line about B and it will execute SHM which can be proved as follows:

Let P be the displaced position of the particle during its motion and let PB $= x$, then the tension, T, of the string in this position is given by

$T = \frac{\lambda}{\ell}(BA + x) = mg + \frac{\lambda}{\ell}x$ (from previous equation)

Then the resultant force acting on the particle in the direction BP

$$= mg - T = mg - (mg + \frac{\lambda}{\ell}x) = -\frac{\lambda}{\ell}x$$

Hence the equation of motion of the particle

$$m\frac{d^2x}{dt^2} = -\lambda\frac{x}{\ell} \quad \text{Or} \quad \frac{d^2x}{dt^2} = -\frac{\lambda}{\ell m}x \equiv -w^2x$$

which shows that the particle moves with simple harmonic motion having B, the position of equilibrium, as the center of oscillation. The period of motion is

$$2\pi\sqrt{\frac{\ell m}{\lambda}} = 2\pi\sqrt{\frac{e}{g}}$$

e being the extension of the string in the equilibrium position of the particle. By Equation $mg = \lambda \frac{AB}{\ell} = \lambda \frac{e}{\ell}$, e being proportional to $\frac{m}{\lambda}$, the A period depends on the weight which is hung on, and on the stiffness of the string or spring to which the particle is attached.

Note (i). At B, the ultimate position of equilibrium of the particle, the forces acting on it, viz., its weight and the tension of the string, balance. In all problems of this typo the position of this point must be obtained first.

Note (ii). The particle moves with Simple Harmonic motion only so long as the particle is below A, i.e., so long as the string remains stretched. If the particle rises above A (it will do so, for example when it is pulled down below, B, a distance greater than AB) the string will become slack and the part of the motion above A will be simply free vertical motion under gravity.

■ ILLUSTRATIVE EXAMPLES ■

‖ Example ▸

A point moves along a straight line such that its distance given by $x = 3\cos 2t + 4\sin 2t$. Prove that the motion of the point is simple harmonic motion and find its periodic time and amplitude.

‖ Solution ▸

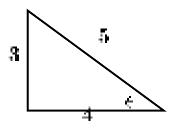
Since the position of the point is given by $x = 3\cos 2t + 4\sin 2t$ and by differentiating w.r.t t we get

$$\dot{x} = -6\sin 2t + 8\cos 2t,$$

again differentiating $\therefore \ddot{x} = -12\cos 2t - 16\sin 2t$

$$\text{Or } \therefore \ddot{x} = -4(\underbrace{3\cos 2t + 4\sin 2t}_{x}) = -2^2 x$$

This equation represents a simple harmonic motion with $\omega = 2$ since the acceleration varies with distance, where the periodic time is τ and given by $\tau = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{2} = \pi$ and the amplitude may be calculated as



$$3\cos 2t + 4\sin 2t = 5\left(\frac{3}{5}\cos 2t + \frac{4}{5}\sin 2t\right)$$
$$= 5\ \sin\epsilon\, cos2t + \cos\epsilon\sin 2t\ \ = 5\sin(2t + \epsilon)$$

that is the amplitude is $a = 5$

‖ Example ▸

A moving particle along a straight line where μ , $x = \mu - \mu\cos 2t$ is constant. Show that the motion of the point is simple harmonic motion and find its periodic time and amplitude.

‖ Solution ▸

Since the position of the point is given by $x = \mu - \mu\cos 2t$ and by differentiating twice w.r.t t we get

$$\frac{dx}{dt} = 2\mu\sin 2t \qquad \text{and} \qquad \frac{d^2x}{dt^2} = 4\underbrace{\mu\cos 2t}_{\mu - x} = 4(\mu - x) = -4(x - \mu)$$

This equation indicates a simple harmonic motion (SHM) with center $x = \mu$ and $\omega^2 = 4$. The periodic time is τ and given by $\tau = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{2} = \pi$.

(Hint: Let $y = x - \mu$ then the previous equation turn into $\ddot{y} = -2^2 y$, which represents a simple harmonic motion with center $y = 0$ ($x = \mu$)).

(Readers have to calculate the amplitude)

Example

A particle moves with SHM in a straight line. In the first second after starting from rest, it travels a distance a and in the next second it travels a distance b in the same direction. Prove that the amplitude of the motion is $2a^2 \,/\, (3a - b)$?

Solution

Measuring time t from the starting point and the distance x of the particle from the center of motion and denoting the amplitude by A, we have

$x = a\cos wt$

Now by the question when $t = 1, \quad x = A - a$

And when $t = 2, \quad x = A - a - b$

$$A - a = A\cos w \qquad \text{and} \qquad A - a - b = A\cos 2w = A\ \ 2\cos^2 w - 1$$

From these two equations we have

$$A - a - b = A\left(2\frac{(A-a)^2}{A^2} - 1\right) = \frac{1}{A}\left[A^2 - 4aA + 2a^2\right]$$

$$A^2 - aA - bA = A^2 - 4aA + 2a^2 \quad \text{Or} \quad (3a - b)A = 2a^2$$

$$\therefore A = \frac{2a^2}{3a - b}$$

Example

A point executing SHM has velocities u, u' and positions in two of its positions b, b' respectively. Show that the periodic time of motion is

$$2\pi\sqrt{\frac{b^2 - b'^2}{u'^2 - u^2}}$$

Solution

Let a be the amplitude of the simple harmonic motion then

$$v^2 = w^2(a^2 - x^2)$$

Therefore,

$$u^2 = w^2(a^2 - b^2) \qquad \text{and} \qquad u'^2 = w^2(a^2 - b'^2)$$

By subtracting

$$u'^2 - u^2 = w^2(b^2 - b'^2) \qquad \Rightarrow w^2 = \frac{u'^2 - u^2}{b^2 - b'^2} \qquad \text{Or} \qquad w = \sqrt{\frac{u'^2 - u^2}{b^2 - b'^2}}$$

Since $\tau = \frac{2\pi}{w}$ $\qquad \Rightarrow \tau = 2\pi\sqrt{\frac{b^2 - b'^2}{u'^2 - u^2}}$

Example

A body moving with SHM has an amplitude a and period T. Show that the velocity v at a distance x from the mean position is given by $v^2T^2 = 4\pi^2(a^2 - x^2)$

Solution

As we have $v^2 = w^2(a^2 - x^2)$ where a represents the amplitude

Also $\quad \because T = \frac{2\pi}{w} \qquad \therefore w = \frac{2\pi}{T}$

$$v^2 = \left(\frac{2\pi}{T}\right)^2(a^2 - x^2) \quad \Rightarrow v^2T^2 = 4\pi^2(a^2 - x^2)$$

Example

The speed v of a particle moving along the axis of x is given by the relation $v^2 = n^2(8bx - x^2 - 12b^2)$. Show that the motion is simple harmonic with its center at $x = 4b$ and amplitude $2b$. Find the time from $x = 5b$ to or $x = 6b$.

Solution

A particle is said to be its motion as simple harmonic motion if

$\ddot{x} = -w^2 x$

From the question we have $v^2 = n^2(8bx - x^2 - 12b^2)$ thus by differentiation

$$2v\frac{dv}{dx} = n^2(8b - 2x) \qquad \text{Or} \qquad \underbrace{v\frac{dv}{dx}}_{\ddot{x}} = -n^2(x - 4b)$$

So the particle moves as a SHM with center $x = 4b$

$$v = 0 \qquad \therefore\ 8bx - x^2 - 12b^2 = 0 \quad \Rightarrow (x - 6b)(x - 2b) = 0$$

Therefore $\qquad x = 6b$ and $x = 2b$

which gives the ended points of SHM and the amplitude is $2b$.

Example

At the ends of three successive seconds, the distances of a point moving with SHM, from its mean position, measured in the same direction are X_1, X_2, X_3. Find the periodic time of motion.

Solution

As known the general solution of simple harmonic motion is $x = a\sin(\omega t + \epsilon)$

Let the time to reach position X_1 is t and thus the time to reach position X_2 is $t + 1$ and $t + 2$ is the time to reach position X_3 and therefore,

$$\begin{aligned} X_1 &= a\sin(\omega t + \epsilon) \\ X_2 &= a\sin(\omega(t + 1) + \epsilon) \\ X_3 &= a\sin(\omega(t + 2) + \epsilon) \end{aligned}$$

From previous equations – by adding first and third equations- we have

$$X_1 + X_3 = a \quad \sin(\omega t + \epsilon) + \sin(\omega(t+2) + \epsilon)$$

$$= 2\underbrace{a\sin(\omega(t+1)+\epsilon)}_{X_2}\cos\omega$$

$$= 2X_2\cos\omega$$

Here we use the triangle relation

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\because X_1 + X_3 = 2X_2\cos\omega \qquad \Rightarrow \cos\omega = \frac{X_1 + X_3}{2X_2} \qquad \text{Or} \quad \omega = \cos^{-1}\left(\frac{X_1 + X_3}{2X_2}\right)$$

But the periodic time is given by $\tau = \frac{2\pi}{\omega}$ therefore,

$$\tau = \frac{2\pi}{\cos^{-1}\left(\frac{X_1 + X_3}{2X_2}\right)}$$

Where $-2X_2 \le X_1 + X_3 \le 2X_2$

Example

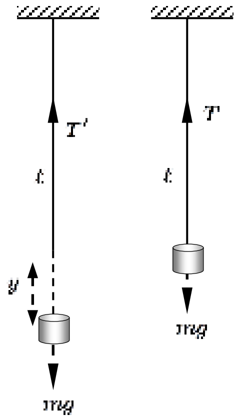
An elastic string supporting a heavy particle with mass m hangs in equilibrium. The particle is now pulled down below the equilibrium position through a small distance and let go then the particle done n complete oscillations per second. If ℓ represents the natural length of the string in the case of equilibrium. Find the natural length of string and evaluate the tension when the equals natural length.

Solution

Suppose that ℓ_0 represents natural length of string and T gives the tension in equilibrium after hangs mass m**,** in equilibrium case and from Hooke's law

$$mg = T = \frac{\lambda}{\ell_0}(\ell - \ell_0) \qquad (1)$$

After particle is pulled down below the equilibrium position a distance x then equation of motion becomes



$$m\ddot{x} = mg - T'$$

Where $T' = \dfrac{\lambda}{\ell_0}(\ell + x - \ell_0)$

$$\therefore\ m\ddot{x} = mg - \frac{\lambda}{\ell_0}(\ell + x - \ell_0)$$

$$= mg - \underbrace{\frac{\lambda}{\ell_0}(\ell - \ell_0)}_{mg} - \frac{\lambda}{\ell_0}x = -\frac{\lambda}{\ell_0}x$$

$$\ddot{x} = -\frac{\lambda}{\ell_0 m}x = -w^2 x$$

Which shows that the motion about a point of equilibrium is simple harmonic

motion, the constant w equals $\sqrt{\dfrac{\lambda}{m\ell_0}}$. The periodic time is $2\pi\sqrt{\dfrac{\ell_0 m}{\lambda}}$

Now since $n = \dfrac{w}{2\pi} \ \Rightarrow n = \dfrac{1}{2\pi}\sqrt{\dfrac{\lambda}{m\ell_0}} \ \Rightarrow 4\pi^2 n^2 m = \dfrac{\lambda}{\ell_0}$

Therefore, from Equation (1)

$$\cancel{m}g = \frac{\lambda}{\ell_0}(\ell - \ell_0) = 4\pi^2 n^2 \cancel{m}(\ell - \ell_0) \qquad \therefore\ \ell - \ell_0 = \frac{g}{4\pi^2 n^2}$$

Or $\ell_0 = \ell - \dfrac{g}{4\pi^2 n^2}$

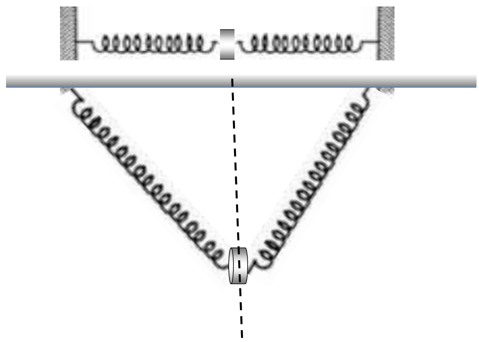
Which evaluate the natural length of string. To obtain the tension from Hooke's law

$$T = \frac{\lambda}{\ell_0}\ell_0 = 4\pi^2 n^2 m\ell_0 = 4\pi^2 n^2 m\left(\ell - \frac{g}{4\pi^2 n^2}\right) = m\ \ 4\pi^2 n^2 \ell - g$$

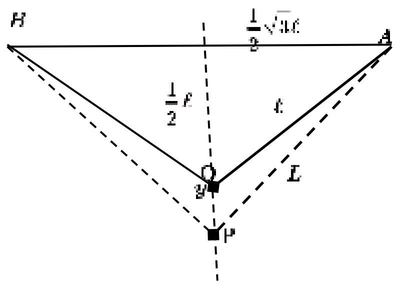
▌▌ Example ▶

A heavy particle is supported in equilibrium by two equal elastic strings with their other ends attached to two points in a horizontal plane and each inclined at an angle of 60° to the vertical. The modulus of elasticity is such that when the particle is suspended from any portion of the string its extension is equal to its natural length. The particle is displaced vertically a small distance and then released. Prove that the period of its small oscillations is $2\pi\sqrt{2\ell / 5g}$, where ℓ is the stretched length of either string in equilibrium.

Solution



Let m be the mass of the particle and λ the modulus of elasticity. Then by supposing the particle to be suspended from any portion of the string, since the extended length is double the natural length we find that $\lambda = mg$.



If ℓ_0 be the natural length of either string, we have, in the equilibrium position,

$$mg = 2\lambda \frac{\ell - \ell_0}{\ell_0}\cos 60^0 = \lambda \frac{\ell - \ell_0}{\ell_0}$$

but $\lambda = mg$, therefore $\ell_0 = \frac{1}{2}\ell$

Let y denote the vertical displacement and L the length of either string at time t. To find the period of small oscillations we want to obtain an equation of the form

$$\ddot{x} = -\omega^2 x$$

where ω is a constant. It will therefore be sufficient for our purpose to write down the equation of motion at time t and neglect all powers of x higher than the first.

We have

$$m\ddot{y} = mg - 2\lambda \frac{L - \ell_0}{\ell_0}\cos\angle\mathrm{OPA}$$

where **P** is the particle at time t, **O** is its equilibrium position and $\mathbf{PA{=}PB} = L$ are the strings.

Now

$$L^2 = \left(y + \frac{1}{2}\ell\right)^2 + \frac{3}{4}\ell^2 = \ell^2 + \ell y + y^2$$

Therefore $$L = \ell\left(1 + \frac{y}{\ell}\right)^{1/2} = \ell + \frac{1}{2}y$$

correct to the first power of x, and

$$\cos\angle\mathrm{OPA} = \frac{\frac{1}{2}\ell + y}{L} = \frac{\frac{1}{2}\ell + y}{\ell + \frac{1}{2}y}$$
$$= \frac{1}{2}\left(1 + \frac{2y}{\ell}\right)\left(1 - \frac{y}{2\ell}\right)$$
$$= \frac{1}{2}\left(1 + \frac{3y}{2\ell}\right)$$

to the first power of y . And hence

$$m\ddot{y} = mg - \frac{2\lambda\left(\ell + \frac{1}{2}y - \frac{1}{2}\ell\right)}{\frac{1}{2}\ell} \times \frac{1}{2}\left(1 + \frac{3y}{2\ell}\right)$$

Therefore,

$$\ddot{y} = g - g\left(1 + \frac{y}{\ell}\right)\left(1 + \frac{3y}{2\ell}\right) \qquad \text{Or} \qquad \ddot{y} = -\frac{5g}{2\ell}y$$

which represents a simple harmonic motion of period $2\pi\sqrt{2\ell / 5g}$.

PROBLEMS

❐ A point executing SHM has velocities u and v and accelerations a and b in two of its positions. Find the distance between the two positions and that the periodic time of motion

❐ If the displacement, velocity and acceleration at a particular instant of a particle describing SHM are respectively 3 in., 3 in./sec. and 3 in./sec^2, Find the greatest velocity of the particle and the period of motion.

❐ A point moving with SHM has a period of oscillation of л sec. and its greatest acceleration is 5 ft. /sec^2. Find the amplitude and the velocity when the particle is at a distance 1 ft. from the center of oscillation.

❐ A particle describing simple harmonic motion executes 100 complete Vibrations per minute and its speed at its mean position is 15 ft. per sec. What is the length of its path?

❒ A particle oscillates in a cycloid under gravity the amplitude of the motion being l and the periodic time being T. Show that its velocity at a time t measured from a position of rest is $w = \frac{2\pi l}{T} \sin \frac{2\pi t}{T}$

❒ A body is suspended from a fixed point by a light elastic string of natural length ℓ whose modulus of elasticity is equal to the weight of the body and makes vertical oscillations of amplitude a. Show that, if as the body rises through its equilibrium position it picks up another body of equal weight, the amplitude of the oscillation becomes $\left(\ell^2 + \frac{1}{2} a^2\right)^{1/4}$

IMPACT AND COLLISION OF ELASTIC BODIES

In this section we will integrate the equation of motion with respect to time and thereby obtain the principle of impulse and momentum. The resulting equation will be useful for solving problems involving force, velocity, and time. Using kinematics, the equation of motion for a particle of mass m can be written as

$$\sum F = ma = m\frac{dv}{dt}$$

where a and v are both measured from an inertial frame of reference. Rearranging the terms and integrating between the limits $v = v_1$ at $t = t_1$ and $v = v_2$, at $t = t_2$ we have

$$\sum \int_{t_1}^{t_2} F dt = m\int_{v_1}^{v_2} dv \qquad \Rightarrow \sum \int_{t_1}^{t_2} F dt = mv_2 - mv_1 \qquad \textbf{(*)}$$

This equation is referred to as the *principle of linear impulse and momentum*. From the derivation it can be seen that it is simply a time integration of the equation of motion. It provides a direct means of obtaining the particle's final velocity v_2 after a specified time period when the particle's initial velocity is known and the forces acting on the particle are either constant or can be expressed as functions of time. By comparison, if v_2 was determined using the equation of motion, a two-step process would be necessary; i.e., apply $\sum F = ma$ to obtain a, then integrate $a = \frac{dv}{dt}$ to obtain v_2

■ Linear Momentum

Each of the two vectors of the form $\boldsymbol{L} = m\boldsymbol{v}$ in Equation (*) is referred to as the particle's linear momentum. Since m is a positive scalar, the linear-momentum vector has the same direction as $\boldsymbol{v}$ and its magnitude mv has units of mass-velocity, e.g., kg.m/s, or slug. ft/s.

■ Linear Impulse

The integral $\boldsymbol{I} = \int \boldsymbol{F}dt$ in Equation (0) is referred to as the *linear impulse*. This term is a vector quantity which measures the effect of a force during the time the force acts. Since time is a positive scalar, the impulse acts in the same direction as the force, and its magnitude has units of force-time, e.g., N.s or lb·s. If the force is expressed as a function of time, the impulse can be determined by direct evaluation of the integral. In particular, if the force is constant in both magnitude and direction, the resulting impulse becomes

$$\boldsymbol{I} = \int_{t_1}^{t_2} \boldsymbol{F}_c dt = \boldsymbol{F}_c (t_2 - t_1)$$

■ Impact

This action occurs when two bodies collide with each other during a very short period of time, causing relatively large (impulsive) forces to be exerted between the bodies. The striking of a hammer on a nail, or a golf club on a ball, are common examples of impact loadings. In general, there are two types of impact. *Central impact* occurs when the direction of motion of the mass centers of the two colliding particles is along a line passing through the mass centers of the particles. This line is called the line of impact, which is perpendicular to the plane of contact. When the motion of one or both of the particles makes an angle with the line of impact, the impact is said to be *oblique impact.*

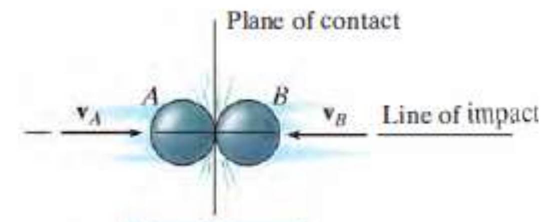
➢ **Elasticity** If we drop a ball of glass on to a marble floor, it rebounds almost to its original height but if the same ball were dropped on to a wooden floor, the distance through which it rebounds is much smaller. If further we allow an ivory ball and a wooden ball to drop from the same height upon a hard floor the heights through which they rebound are quite different. The velocities of these balls are the same when they reach the floor but since they rebound to different heights their velocities on leaving the floor are different.

Again, when a ball strikes against a floor or when two balls of any hard material collide, the balls are slightly compressed and when "they tend to recover their original shape, they rebound. The property of the bodies which causes these differences in velocities and which makes them rebound after collision is called ***Elasticity***. If a body does not tend to return to its original shape and does not rebound after collision, it is said to be ***Inelastic***.

In considering impact of elastic bodies, we suppose that they are smooth, so that the mutual action between them takes place only in the direction of their common normal at the point where they meet, there being no force in the direction perpendicular to their common normal.

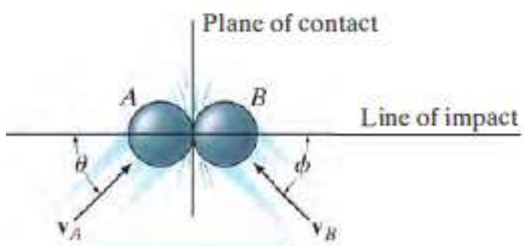
■ Definitions

When the, direction of each body is along the common normal at the point where they touch, the impact is said to be ***direct***.



Central impact

When the direction of motion of either or both, is not along the common normal at the point of contact the impact is said to be ***oblique.***



Oblique impact

■ Direct Impact of two Smooth Spheres

Suppose two smooth spheres of masses m and m' moving in the same straight line with velocities u and u', collide and stick together. The forces which act between them during the collision act equally but in opposite directions on the two spheres so that the total momentum of the spheres remain unaltered by the impact. If U be the common velocity of the spheres after the collision and if the velocities are all measured in the same direction, we have

$$(m+m')U = mu + m'u'$$

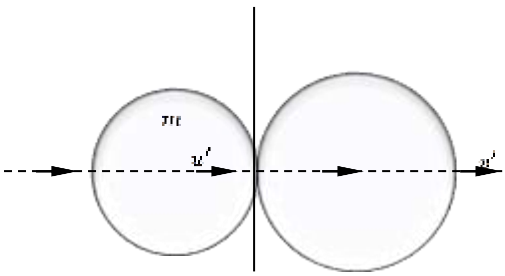
This equation is sufficient to determine the one unknown quantity U.

But we know, as a matter of ordinary experience, that when two bodies of any hard material impinge on each other, they separate almost immediately and a finite change of velocity is generated in each by their mutual action depending on the material of the bodies. Hence the spheres, if free to move, will have after impact, different velocities say v and v'.

The equation of momentum now becomes

$$mv + m'v' = mu + m'u' \qquad (1)$$

This single equation is not sufficient to determine the two unknown quantities v and v'.



Another relation between the velocities is supplied by ***Newton's Experimental Law*** which states that *when two bodies impinge directly, their relative velocity after impact is in a constant ratio to their relative velocity before impact, and is in the opposite direction.*

If bodies impinge obliquely, the same fact holds for their component velocities along the common nominal at the point of contact. The equation derived from this law for the above spheres is,

$$\frac{v-v'}{u-u'}=-e \qquad \text{Or} \qquad v-v'=-e(u-u') \qquad (2)$$

v, v', u and u' being all measured in the same direction.

The constant ratio, e is called ***the co-efficient of elasticity*** or ***restitution.*** It depends on the substances of which the bodies are made and is independent of the masses of the bodies and their velocities before impact. The value of e differs considerably for different bodies and varies from 0 to 1.

(i) When $e=0$, the bodies are said to be inelastic (Plastic impact). In this case we have from Equation (2) $v=v'$ i.e., if two inelastic spheres impinge they move with the same velocity after impact.

(ii) When $e=1$ the bodies are said to be perfectly elastic.

Both these are ideal eases never actually realized in nature.

In order to evaluate the velocities of the spheres after direct impact we solve Equations (1) and (2) and get

$$v=\frac{mu+m'u'-em'(u-u')}{m+m'} \quad \text{and} \quad v'=\frac{mu+m'u'+em(u-u')}{m+m'}$$

When $m=m'$ and $e=1$, we have $v=u'$ and $v'=u$ i.e., if two equal perfectly elastic spheres impinge directly they interchange their velocities after impact.

■ <u>Kinetic energy lost by direct impact</u>

In general, there is always a loss of kinetic energy whenever two bodies impinge. Since we have by algebra

$$(m+m')(mu^2+m'u'^2)=(mu+m'u')^2+mm'(u-u')^2$$

And $$(m+m')(mv^2+m'v'^2)=(mv+m'v')^2+mm'(v-v')^2$$

Subtracting these two equations and divide by $2(m+m')$ and using

$$mv + m'v' = mu + m'u' \text{ and } v - v' = -e(u - u')$$

Therefore, Loss in K.E. is

$$= \frac{1}{2}mu^2 + \frac{1}{2}m'u'^2 - \left(\frac{1}{2}mv^2 + \frac{1}{2}m'v'^2\right) = \frac{1}{2}\frac{mm'}{m+m'}(u-u')^2(1-e^2)$$

■ Oblique Impact of two smooth spheres

Suppose that at the moment of impact the direction of motion of the spheres is not along the line joining their centers. Let m, m' be the masses of the two spheres with centers A and B at the time of impact, u, u' the velocities just before impact, α, β the angles the directions of motion make with AB before impact, v, v' the velocities after impact, and θ, φ angles the directions of motion make with AB after impact.

Since the spheres are smooth, there is no impulse perpendicular to the line of centers and hence the resolved parts of velocities of the two spheres in the direction perpendicular to AB remain unaltered.

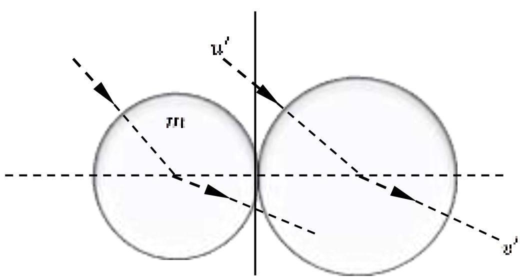
$$v\sin\theta = u\sin\alpha \quad \text{and} \quad v'\sin\varphi = u'\sin\beta \tag{3}$$

Since the impulsive forces acting during the collision on the two spheres along their line of centers are equal and opposite, the total momentum along AB remains unchanged.

$$mv\cos\theta + m'v'\cos\varphi = mu\cos\alpha + m'u'\cos\beta \tag{4}$$

By Newton's experimental law for relative velocities resolved along the common normal AB, we have

$$v\cos\theta - v'\cos\varphi = -e(u\cos\alpha - u'\cos\beta) \tag{5}$$



We deduce the following particular cases from the above equations:

(i) If $u' = 0$, from Equation (3) $\varphi = 0$, ($\because v' \neq 0$), i.e., if the sphere of mass m' were at rest, it will move along the line of centers after impact.

(ii) If $u' = 0$ and $m = em'$ from Equation (3) $\varphi = 0$ and then $\theta = 90^o$, so that if a sphere of mass m impinges obliquely on a sphere of mass m' at rest, the directions of motion of the spheres after impact will be at right angles if $m = em'$. This evidently holds true when the spheres are equal and perfectly elastic i.e. , when $u' = 0,\ e = 1$ and $m = m'$.

(iii) If $m = m'$ and $e = 1$ then, we have

$$v\cos\theta = u'\cos\beta \quad \text{and} \quad v'\cos\varphi = u\cos\alpha$$

i.e., if two equal-and perfectly elastic spheres impinge they interchange their velocities in the direction of their line of centers. Also in this case, by using Equation (3), we get: $\tan\theta\tan\varphi = \tan\alpha\tan\beta$

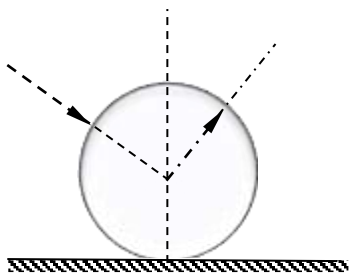
It follows that if two equal and perfectly elastic spheres impinge at right angles, their directions after impact will still be at right angles.

<u>The students advised to prove this particular case independently.</u>

Obtain the relation that describes the loss of kinetic energy in Oblique Impact

$$\therefore \text{Loss K.E.} = \frac{1}{2}\frac{mm'}{m+m'}(u\cos\alpha - u'\cos\beta)^2(1-e^2)$$

■ Impact against a Fixed Plane



Suppose a smooth sphere (or particle) of mass m, moving with a velocity u, strikes a smooth fixed plane in a direction making an angle α with the normal to the plane, and that it rebounds with velocity v making an angle θ with the normal. Then, since the plane is smooth, the component of the velocity along the plane must remain unaltered

$\therefore v \sin\theta = u \sin\alpha$ (6)

The plane being fixed its velocity is taken as zero. by Newton's experimental law for relative velocity along the common normal AN, we have

$\therefore v\cos\theta - 0 = -e(-u\cos\alpha - 0) \quad \Rightarrow \quad v\cos\theta = eu\cos\alpha$ (7)

Squaring and adding (6) and (7), we get

$$\therefore v^2 = u^2(\sin^2\alpha + e^2\cos^2\alpha)$$

Dividing (7) by (6) we have: $\cot\theta = e\cot\alpha$

These equations give the velocity and direction of motion of the sphere after impact. The following facts may be noted:

(1) If $\alpha = 0$ then by Equation (6), $\theta = 0$ and by Equation (7), $v = eu$ i.e., when the impact is direct, the direction of motion of the sphere is reversed after impact and its velocity is reduced in the ratio $e : 1$.

(2) If $e = 1$, therefore $\alpha = \theta$ and then $u = v$, i.e., when the plane is perfectly elastic, the angle of reflection is equal to the angle of incidence, and the velocity remains unchanged in magnitude.

(3) If $e = 0$, thus $\theta = 90^o$ and then $v = u\sin\alpha$, i.e., when the plane i perfectly inelastic, the sphere simply slides along the plane, its velocity parallel to the plane remaining unaltered.

(4) Loss of Kinetic energy ΔE

$$\begin{aligned} \Delta E &= \frac{1}{2}mu^2 - \frac{1}{2}mv^2 \\ &= \frac{1}{2}mu^2 - \frac{1}{2}mu^2(\sin^2\alpha + e^2\cos^2\alpha) \\ &= \frac{1}{2}mu^2(1-e^2)\cos^2\alpha \end{aligned}$$

■ ILLUSTRATIVE EXAMPLES ■

Example

A ball of mass $\mathbf{8\ Ib}$ moving with a velocity of $\mathbf{4\ ft\,sec^{-1}}$ is overtaken by a ball, of mass $\mathbf{12\ Ib}$ moving with a velocity of $\mathbf{9\ ft\,sec^{-1}}$, (i) in the same direction as the first, (ii) in the opposite direction. If $e = \mathbf{0.2}$ find the velocities of the balls after impact. Find also the loss of Kinetic energy in the first case.

Solution

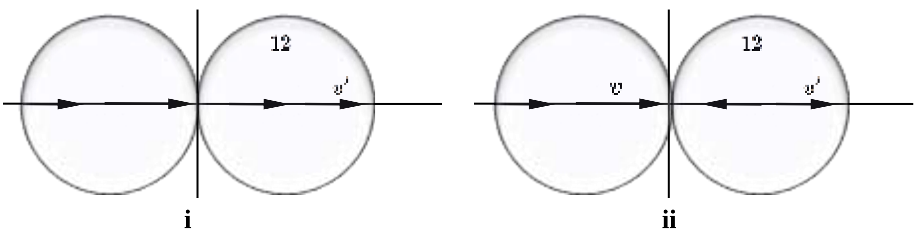
(i) Let the direction of motion of the first ball be taken as positive and let v, v' be the velocities after impact, then with consideration conservation of momentum.

$\mathbf{8v + 12v' = 8 \times 4 + 12 \times 9 = 140}$ and $v - v' = \mathbf{-0.2(4-9) = 1}$

which give $v = \mathbf{7.6}$ ft./s. and $v' = \mathbf{6.6}$ ft./s.

(ii) $\mathbf{8v + 12v' = 8 \times 4 - 12 \times 9 = -76}$ and $v - v' = -\mathbf{\frac{1}{5}(4-(-9)) = -2.6}$

which give $v = \mathbf{-5.36}$ ft./s. and $v' = \mathbf{-2.76}$ ft/s



In this case the first ball turns back after impact. It should be noted that the velocities are measured algebraically, that is, all velocities in one direction cert taken as positive while those in the opposite direction as negative.

Example

A ball A, moving with velocity u impinges directly on an equal ball B moving with velocity v in the opposite direction. If A be brought to rest by the impact, show that $\frac{u}{v} = \frac{1+e}{1-e}$ where e is the co-efficient of restitution.

Solution

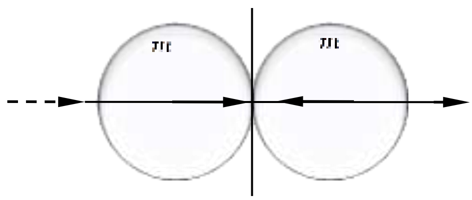
Let V be the velocity of B after impact and let m be the mass of each, then since A is reduced to rest after the impact, according to Conservation of momentum we obtain

$m \times 0 + m \times V = mu + m(-v)$ **Or** $V = u - v$ and

$0 - V = -e(u - (-v))$ **Or** $V = e(u + v)$

$\therefore u - v = V = e(u + v)$ **Or** $(1 - e)u = (1 + e)v$

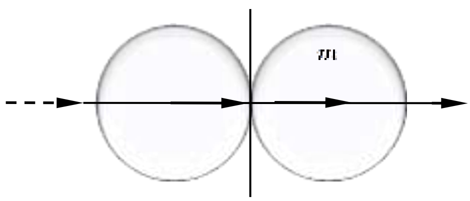
Hence $\frac{u}{v} = \frac{1+e}{1-e}$



Example

A ball with mass nm moving with velocity ua^{-1} impinges directly on another ball with mass m moving with velocity u in the same direction. If the ball with mass m be brought to rest by the impact, determine the co-efficient of restitution.

Solution



Let V be the velocity of the mass nm after impact (along the impact line since the balls impinge directly). According to the principle of the momentum along the impact line, we get

$$m(0) + nmV = mu + nm\left(\frac{u}{a}\right) \quad \Rightarrow nV = \left(1 + \frac{n}{a}\right)u \qquad \textbf{(1)}$$

From Newton's Experimental Law, we obtain

$$V - 0 = -e\left(\frac{u}{a} - u\right) \quad \Rightarrow V = eu\left(1 - \frac{1}{a}\right) \qquad \textbf{(2)}$$

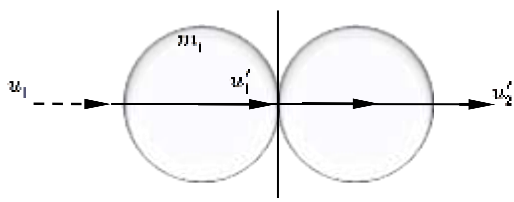
From these two equations (1) and (2)

$$ne\cancel{u}\left(1 - \frac{1}{a}\right) = \left(1 + \frac{n}{a}\right)\cancel{u} \quad \Rightarrow e = \frac{a+n}{n(a-1)}$$

Example

Let m_1, m_2 be the masses of two spheres impinge directly with velocities u_1, u_2 in the same direction. If e be the co-efficient of restitution. Prove that the loss of kinetic energy by impact is $\dfrac{(1-e^2)m_1m_2}{2(m_1+m_2)}(u_1-u_2)^2$

Solution



From the figure and according to the principle of the momentum along the impact line, we get Let u_1', u_2' be the velocities of the spheres after impact.

$$m_1u_1' + m_2u_2' = m_1u_1 + m_2u_2 \qquad \textbf{(1)}$$

By Newton's experimental law

$$u_1' - u_2' = -e(u_1 - u_2) \qquad \textbf{(2)}$$

Squaring equations (1) and (2) and multiply equation (2) by $m_1 m_2$ then adding we get

$$(m_1 u_1' + m_2 u_2')^2 + m_1 m_2 (u_1' - u_2')^2 = (m_1 u_1 + m_2 u_2)^2 + m_1 m_2 e^2 (u_1 - u_2)^2$$

By adding and subtracting the value $m_1 m_2 (u_1 - u_2)^2$ to the R.H.S. of previous equation

$$(m_1 + m_2)(m_1 u_1'^2 + m_2 u_2'^2) = (m_1 u_1 + m_2 u_2)^2 \pm m_1 m_2 (u_1 - u_2)^2 + m_1 m_2 e^2 (u_1 - u_2)^2$$

Or

$$(m_1 + m_2)(m_1 u_1'^2 + m_2 u_2'^2) = (m_1 + m_2)(m_1 u_1^2 + m_2 u_2^2) - m_1 m_2 (1 - e^2)(u_1 - u_2)^2$$

Dividing the last equation by $\frac{1}{2}(m_1 + m_2)$, we have

$$\frac{1}{2} m_1 u_1'^2 + \frac{1}{2} m_2 u_2'^2 = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 - \frac{m_1 m_2 (1 - e^2)(u_1 - u_2)^2}{2(m_1 + m_2)}$$

$$\Delta E = \left(\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2\right) - \left(\frac{1}{2} m_1 u_1'^2 + \frac{1}{2} m_2 u_2'^2\right) = \frac{m_1 m_2 (1 - e^2)(u_1 - u_2)^2}{2(m_1 + m_2)}$$

This relation illustrates that the total of kinetic energies of the two spheres after impact is less that the total of kinetic energies before impact by the value $\frac{m_1 m_2 (1 - e^2)(u_1 - u_2)^2}{2(m_1 + m_2)}$ and this values represents the loss of the kinetic energy by collision.

Example

A ball weighting one pound and moving with a velocity 8 ftsec^{-1}, impinges on a smooth fixed horizontal plane in a direction making 60^0 with the plane; find its velocity and direction of motion after impact, the co-efficient of restitution being 0.5. Find also the loss in Kinetic energy due to the impact.

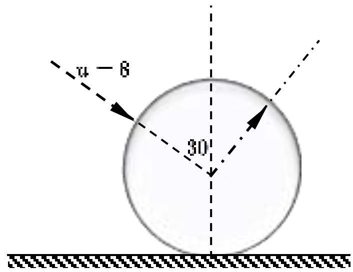
Solution

The direction of motion of the ball makes an angle of 30^0 with the normal to the plane. If after impact the ball moves in a direction making an angle θ with the normal with velocity v, then

$$v\cos\theta = eu\cos\alpha = \frac{1}{2}\times 8\times\cos 30 = 2\sqrt{3}$$

$$v\sin\theta = u\sin\alpha = 8\times\sin 30 = 4$$

$$\Rightarrow v^2 = 12+16 = 28 \quad \Rightarrow v = 5.29 \text{ ft sec}^{-1}$$



and

$$\cot\theta = \frac{\sqrt{3}}{2} \text{ i.e. } \theta = 49^o\, 6'$$

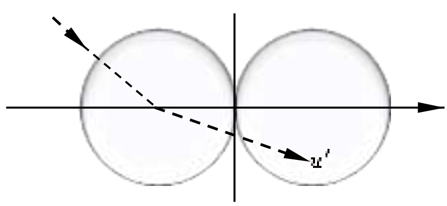
Loss of K.E. ΔE is

$$\Delta E = \frac{1}{2}mu^2 - \frac{1}{2}mv^2 = \frac{1}{2}m(8\times 8-28) = 18 \qquad (m=1)$$

Example

A smooth ball A , collides Obliquely with an equal smooth ball B. Just before impact B is stationary and A makes an angle of α with the line joining the centers of the spheres with velocity v in a direction making an angle of α at the instant of impact. If e is the co-efficient of restitution, find the resulting motion of the sphere A?

Solution



Since the momentum after impact along the line of centers = momentum before impact, we have, let be the velocity of the rest ball after collision

$$mu'\cos\theta + mV = mu\cos\alpha + 0 \qquad \therefore\ u'\cos\theta + V = u\cos\alpha \tag{1}$$

Again by Newton's experimental law

$$u'\cos\theta - V = -e(u\cos\alpha - 0) \tag{2}$$

By adding the equations (1) and (2)

$$\therefore\ 2u'\cos\theta = (1-e)u\cos\alpha \tag{3}$$

Now since the velocity of the sphere A perpendicular to AB remains the same, we have

$$u'\sin\theta = u\sin\alpha \tag{4}$$

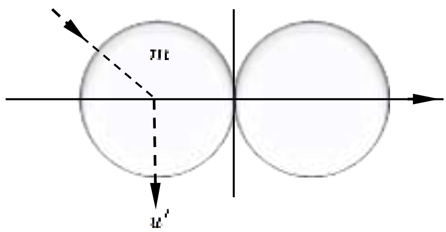
By dividing the equations (3) and (4) therefore

$$\frac{1}{2}\tan\theta = \frac{\tan\alpha}{1-e} \quad \Rightarrow \tan\theta = \frac{2\tan\alpha}{1-e} \quad \text{Or} \quad \theta = \tan^{-1}\left(\frac{2\tan\alpha}{1-e}\right)$$

Example

A sphere A, impinges obliquely on another sphere B at rest. If the direction of ball A after impact is perpendicular to the direction of ball B and the balls are perfectly elastic. Prove that the masses of the spheres are equivalent.

Solution



Let m' be the mass of the sphere B and hence its motion after impact will be along the line of impact and suppose its velocity will be V. Since the directions after impact make right angle, that is the velocity of the sphere A will be perpendicular to the line of impact. Let the sphere has a mass m and velocity u with an angle of α before the impact and has velocity u' after impact (perpendicular to the line of impact). According to the principle of constant of momentum along the line of impact we have

$mu'\cos 90 + m'V = mu\cos\alpha + 0 \quad \Rightarrow m'V = mu\cos\alpha$ **(1)**

According to Newton's experimental law

$u'\cos 90 - V = -e(u\cos\alpha - 0) \quad \Rightarrow V = u\cos\alpha \quad (e = 1)$ **(2)**

Substituting Equation (1) into Equation (2) we have $\therefore m = m'$

Example

A smooth sphere A moving with speed *u*, collides with an identical smooth sphere B which is moving in a perpendicular direction with the same speed *u*

The line of centers at the instant of impact is perpendicular to the direction of motion of sphere B. If the coefficient of restitution between the spheres is e. Prove that $\tan\varphi = \left(\frac{1+e}{2}\right)$, where φ is the angle through which sphere B is turned as a result of the impact.

Solution

Let V, u' be the velocities of the spheres after impact. From the figure and according to the principle of the momentum along the line of impact, we have,

$mV - mu'\cos\theta = mu\cos 90 + mu \quad \Rightarrow V - u'\cos\theta = u$ **(1)**

Again from Newton's experimental law

$V - (-u'\cos\theta) = -e(u - u\cos 90) \quad \Rightarrow V + u'\cos\theta = -eu$ **(2)**

Subtracting Equations (1) and (2)

$\therefore 2u'\cos\theta = (1+e)u$ **(3)**

Since the resolved parts of velocities of the two spheres in the direction perpendicular to the line of impact remain unaltered.

$u'\sin\theta = u$ **(4)**

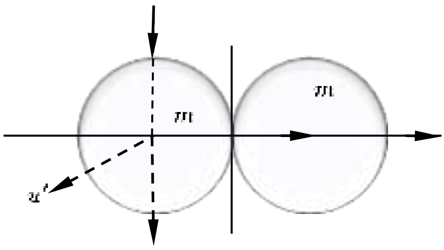
Now by dividing the equations (3) and (4)

$$\tan\theta = \frac{2}{1+e}$$

In order to determine the deviates of the velocity at an angle say φ where $\varphi = \frac{\pi}{2} - \theta$ therefore

$$\tan\varphi = \tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta = \frac{1+e}{2}$$

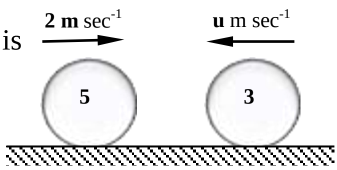
$$\tan\varphi = \frac{1+e}{2} \quad \text{Or} \quad \varphi = \tan^{-1}\left(\frac{1+e}{2}\right)$$



PROBLEMS

❒ A smooth sphere A of mass 5 kg is moving on a smooth horizontal surface with velocity (2i+3j) m s1. Another smooth sphere B of mass 3 kg and the same radius as A is moving on the same surface with velocity (4i-2j) m s1. The spheres collide when their line of centres is parallel to j. The coeffi cient of restitution between the spheres is 3/5. Find the velocities of both spheres after the impact.

❒ A smooth sphere P, of mass 5 kg, moving with a speed of 2 m/s collides directly with a smooth sphere Q, of mass 3 kg, moving in the opposite direction with a speed of u m/s on a smooth horizontal table. The coefficient of restitution for the collision is 0.5. As a result of the collision, sphere P is brought to rest.



(i) Find the value of u.

(ii) Find the speed of Q after the collision.

❒ An imperfectly elastic sphere whose elasticity is equal to tan 30 impinges upon a plane with a velocity such that the velocity after impact equals the velocity before impact $\times$ sin 45. Calculate the angles of incidence and reflection.

❒ If the masses of two balls be as 2:1 and their respective velocities before impact be as 1 : 2 in opposite directions. Evaluate the co-efficient of restitution, each ball moves back, after impact, with $\mathbf{5 / 6}$ of its original velocity.

sphere impinges directly on an equal sphere at rest; if the coefficient of restitution is e show that their velocities after the impact are as $\frac{1-e}{1+e}$.

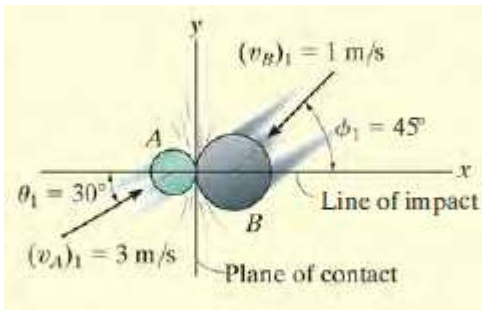
❒ Two bodies A and B whose elasticity is e, moving in opposite directions with velocities a and b, impinge directly upon each other ; determine their distance at time t after impact.

❒ Two equal balls moving with equal speeds impinge, their directions bring inclined at 30 and 60 to the line of centers at the time of impact; show that if $e=1$, the balls move in parallel directions after the impact, inclined at 45 to the line of centers

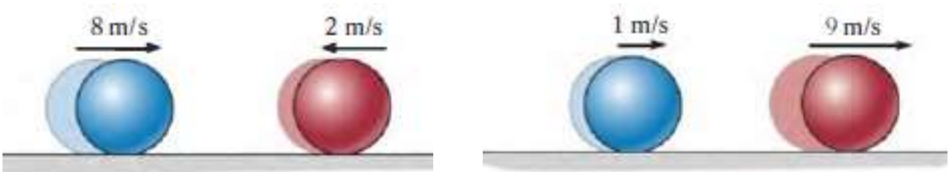
❒ body of moss M moving with a velocity v collides with another of mass m which rests on a table. Both are perfectly elastic and smooth and the body m is driven in a direction making an angle θ with the previous line of motion of the body M, show that its velocity is $\frac{2M}{M+m} v \cos\theta$

❒Two equal smooth spheres moving along parallel lines in opposite directions with velocities u and v. collide with the line of centers at an angle α with their direction of motion. If after impact their lines of motion are at right angles to one another, show that $\frac{(u-v)^2}{(u+v)^2} = \sin^2\alpha + e^2\cos^2\alpha$

❒ Two smooth disks A and E, having a mass of 1 kg and 2 kg, respectively, collide with the velocities shown in the Figure. If the coefficient of restitution for the disks is e = 0.75, determine the x and y components of the final velocity of each disk just after collision.



❒ Determine the coefficient of restitution e between ball A and ball B. The velocities of A and B before and after the collision are shown



Before collision After collision

ORBITAL MOTION

We have already illustrated the motion of a particle in a plane by writing down its equations of motion either in the directions of two fixed co-ordinate axes or in the direction of the tangent and normal to the path described by the particle. However a large number of dynamical problems, where a particle moves under a central force, are readily solved, as already pointed out, by writing the equations of motion in the direction of the radius vector and in a direction perpendicular to it. These equations are of the form (using polar coordinates)

$$F_r = m\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)$$

and

$$F_\theta = m\left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right) = m\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$$

where m is the mass of the particle and F_r and F_θ denote the sums of the components of the forces in the radial and transverse directions

Now, if a particle is moving only under the influence of a force having a line of action which is always directed toward a fixed point called the centre of force, the motion is called central-force motion. The path described by the particle is called a central orbit. This type of motion is commonly caused by electrostatic and gravitational forces. The position of the particle at any instant is defined by the polar co-ordinates r and θ referred to the centre of force $\mathbf{O}$ as the origin and any fixed line OX through O as the initial line.

♦ Definitions:

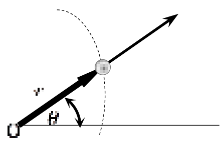
Central force. A force whose line of action always passes through a fixed point, is called a central force. The fixed point is known as the center of force

Central orbit. A central orbit is the path described by a particle moving under the action of a central force. The motion of a planet about the sun is an important example of a central orbit.

❒ **Theorem**. A central orbit is always a plane curve.

Proof.

Take the center of force $\mathbf{O}$ as the origin of vectors. Let P be the position of a particle moving in a central orbit at any time t and let $\underline{OP} = r$ .Then is the expression for the acceleration vector of the particle at the point P. Since the particle moves under the action of a central force with center at $\mathbf{O}$, therefore the only force acting on the particle at P is along the line OP or PO. So the acceleration vector of P is parallel to the vector $\underline{OP}$



$$\therefore \frac{d^2r}{dt^2} \text{ is parallel to } r$$

$$\Rightarrow \frac{d^2r}{dt^2} \wedge r = 0$$

$$\frac{d}{dt}\left(\frac{dr}{dt} \wedge r\right) = 0 \qquad \left(\frac{dr}{dt} \wedge \frac{dr}{dt} = 0\right)$$

Integration we have $\frac{dr}{dt} \wedge r = \text{Const vector} = h \text{ (say)}$ ....(1)

Taking dot product of both sides of Eq. (1) with the vector r, we get

$$r \bullet \left(\frac{dr}{dt} \wedge r\right) = r \bullet h$$

But the left hand member is a scalar triple product involving two equal vectors, and so it vanishes

$$r \bullet h = 0$$

Which shows that $\boldsymbol{r}$ is always perpendicular to a constant vector $\boldsymbol{h}$. Thus the radius vector $\boldsymbol{OP}$ is always perpendicular to a fixed direction and hence lies in a plane. Therefore the path of P is a plane curve

■ Differential Equation of Orbital Path

In order to find the differential equation of the path of a particle moving in a plane under a force which is directed to a fixed centre, we will consider the particle P shown in Fig. 1, which has a mass *m* and is acted upon only by the central force $\boldsymbol{F}$. The free-body diagram for the particle is shown in Fig. 2. Using polar coordinates (r,θ) the equations of motion are

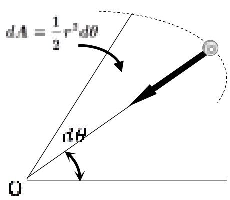


Fig. 1

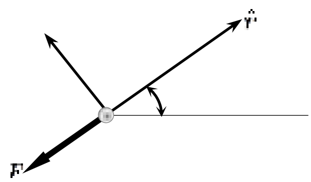


Fig. 2

$$F_r = ma_r, \qquad -F = m\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right) \qquad ....(1)$$

$$F_\theta = ma_\theta, \qquad 0 = m\left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right) \qquad ....(2)$$

The Equation (2) may be re-written in the form

$$\frac{1}{r}\left[\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\right] = 0$$

so that integrating yields

$$r^2\frac{d\theta}{dt} = h \qquad ....(3)$$

Here h represents the constant of integration.

To obtain the *path of motion,* $r = f(\theta)$, the independent variable t must be eliminated from Equations (1) and (2). Using the chain rule of calculus and Equation (3), the time derivatives of Equations (1) and (2) may be replaced by

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{h}{r^2}\frac{dr}{d\theta}$$

$$\begin{aligned}\frac{d^2r}{dt^2} &= \frac{d}{dt}\left(\frac{h}{r^2}\frac{dr}{d\theta}\right)\\ &= \frac{d}{d\theta}\left(\frac{h}{r^2}\frac{dr}{d\theta}\right)\frac{d\theta}{dt}\\ &= \frac{h}{r^2}\frac{d}{d\theta}\left(\frac{h}{r^2}\frac{dr}{d\theta}\right)\end{aligned}$$

Substituting a new dependent variable $r = u^{-1}$ into the Equation (2), we have

$$\frac{dr}{dt} = \frac{dr}{du}\frac{du}{d\theta}\frac{d\theta}{dt} = -h\frac{du}{d\theta}$$

$$\Rightarrow \frac{d^2r}{dt^2} = -h^2u^2\frac{d^2u}{d\theta^2} \qquad \text{....(4)}$$

As well as, the square of Equation (3) becomes

$$\left(\frac{d\theta}{dt}\right)^2 = h^2u^4 \qquad \text{...(5)}$$

Substituting these two Equations (4) and (5) into Equation (1) yields

$$-h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3 = -\frac{F}{m}$$

Or $$F = mh^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) \qquad \text{.....(6)}$$

This differential equation defines the path over which the particle travels when it is subjected to the central force $d\theta$. Equation (6) is important for the solution of two problems:

(i) Given the orbit, to determine the law of central force.

(ii) Given the law of central force, to determine the orbit.

■ Velocity Law

Since, $v_r = \frac{dr}{dt},\ v_\theta = r\frac{d\theta}{dt}$ then $v_r = -h\frac{du}{d\theta},\ v_\theta = h^2u^2$

Therefore, the velocity law describes as $v^2 = h^2\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right)$ ......(7)

which gives the velocity when the path is known.

■ Areal Velocity

When a particle moves along a plane curve, the rate of change of the area traced out by the radius vector joining the particle to a fixed point is called the areal velocity of the particle. Let the particle moves along the curve APQ and let it describes the arc PQ$= \delta s$ in time δt.

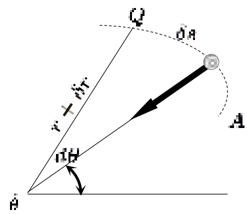


Fig. 3

Let (r,θ) be the co-ordinates of P and $(r+\delta r, \theta+\delta\theta)$ be those of Q, therefore the areal velocity $\dot{A}$ at P is given by

$$\begin{aligned}\dot{A} = \frac{dA}{dt} = \lim_{\delta t\to 0}\frac{\delta OPQ}{\delta t} &= \lim_{\delta t\to 0}\frac{1}{2}\frac{r(r+\delta r)\sin\delta\theta}{\delta t}\\ &= \lim_{\delta t\to 0}\frac{1}{2}\frac{r^2\sin\delta\theta}{\delta t} = \frac{1}{2}r^2\lim_{\delta t\to 0}\frac{\delta\theta}{\delta t}\\ &= \frac{1}{2}r^2\frac{d\theta}{dt} = \underbrace{\frac{1}{2}r^2\dot{\theta}}_{h} = \frac{h}{2}\end{aligned} \qquad (7)$$

From Fig. 3 notice that the shaded area described by the radius r, as r moves through an angle $\delta\theta$. In other words, the particle will sweep out equal segments of area per unit of time as it travels along the path.

■ Apse and Apsidal Distance

An apse is a point on central orbit at which the radius vector drawn from the center of a force is a maximum or minimum. The length of the radius vector at such a point is known as the *apsidal distance*. The analytical condition for a maximum or minimum value of the length of the radius vector is that $du/d\theta$

shall vanish and that the first differential co-efficient which does not vanish shall be of an even order.

Now if φ be the angle between the radius vector and the tangent to the curve, then by the Calculus,

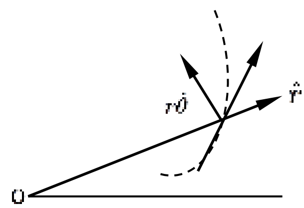
$$\tan\varphi = -u\frac{d\theta}{du} \qquad \text{Or} \qquad \frac{du}{d\theta} = -u\cot\varphi$$

so that when $$\frac{du}{d\theta} = 0 \Rightarrow \varphi = \frac{\pi}{2}$$

Hence the tangent at an apse is perpendicular to the radius vector. In the case of a planet moving round the sun in an ellipse, the ends of the major axis are the two apses, the one nearer to the sun is called, perihelion and the further one is called aphelion.

■ Conservation of Angular Momentum

The angular of momentum about the center of **O** represents by the moment of linear momentum about **O** – remember that $v \equiv (\dot{r}, r\dot{\theta})$ -



$$m\dot{r}(0) + mr\dot{\theta}(r) = m\underbrace{r^2\dot{\theta}}_{h} = mh = \text{constant}$$

That is the angular of momentum about **O** remains constant during the motion, this called the principle of Conservation of Momentum

■ Planetary Motion (Three Kepler's Laws)

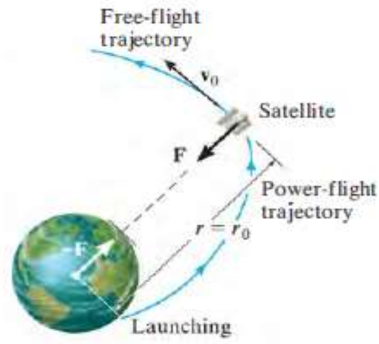
The laws according to which planets move round the sun are stated as follows:

(i) *The orbit of a planet round the sun is an ellipse, in one focus of which the center of the sun is situated.*

(ii) *The radius vector, drawn from the center of the sun to the planet describes equal areas in equal times.*

(iii) *The square of the periodic times of the various planets are proportional* *to the cubes of the semi-major axes of their orbits.*

These three laws were discovered by Kepler (1571-1630) and were deduced by him entirely from observations of the movements of the planets without any reference to the nature of the forces which control these movements.



For application, the force of gravitational attraction will be considered. Some common examples of central-force systems which depend on gravitation include the motion of the moon and artificial satellites about the earth, and the motion of the planets about the sun. As a typical problem in space mechanics, consider the trajectory of a space satellite or space vehicle launched into free-flight orbit with an initial velocity V_0, see the figure. It will be assumed that this velocity is initially parallel to the tangent at the surface of the earth, as shown in the figure. Just after the satellite is released into free flight, the only force acting on it is the gravitational force of the earth. (Gravitational attractions involving other bodies such as the moon or sun will be neglected, since for orbits close to the earth their effect is small in comparison with the earth's gravitation.) According to Newton's law of gravitation, force F will always act between the mass centers of the earth and the satellite, Figure 3. From Equation 1, this force of attraction has a magnitude of

$$F = G\frac{M_e m}{r^2}$$

where M_e and m represent the mass of the earth and the satellite, respectively, G is the gravitational constant, and r is the distance between the mass centers. To obtain the orbital path, we set $r = u^{-1}$ in the foregoing equation and substitute the result into Equation 6. We get

$$\frac{d^2u}{d\theta^2} + u = \frac{GM_e}{h^2}$$

This second-order differential equation has constant coefficients and is nonhomogeneous. The solution is the sum of the complementary and particular solutions given by

$$u = \frac{1}{r} = C\cos(\theta - \phi) + \frac{GM_e}{h^2}$$

This equation represents the free-flight trajectory of the satellite. It is the equation of a conic section expressed in terms of polar coordinates.

The type of path traveled by the satellite is determined from the value of the eccentricity of the conic section as

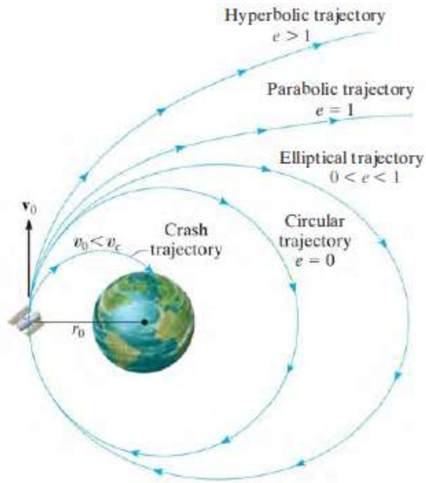
$$e = \frac{Ch^2}{GM_e}$$

$e = 0$ free-flight trajectory is a circle,

$e = 1$ free-flight trajectory is a parabola

$e < 1$ free-flight trajectory is an ellipse

$e > 1$ free-flight trajectory is a hyperbola



■ Illustrative Examples ■

Example

A particle describes the path $r = a\tan\theta$ under a force to the origin. Find its acceleration and velocity in terms of r.

Solution

Since, $r = a\tan\theta$ and let us consider $r = \dfrac{1}{u}$ then $au = \cot\theta$

By differentiation with respect to θ

$$a\frac{du}{d\theta} = -\csc^2\theta = -\;\;1 + \cot^2\theta\;\; = -\;\;1 + a^2u^2$$

Again

$$a\frac{d^2u}{d\theta^2} = 2\csc^2\theta\cot\theta = 2au\;\;1 + a^2u^2$$

$$\because F = mh^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$

$$\therefore F = mh^2u^2\;\;2u\;\;1 + a^2u^2\;\;+u\;\; = mh^2u^3\;\;2a^2u^2 + 3\;\; = \frac{h^2}{r^2}\left(3 + \frac{3a^2}{r^2}\right)$$

Also to get the velocity law

$$\because v^2 = h^2\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right) \quad \Rightarrow v^2 = h^2\left(\frac{\;\;1 + a^2u^2\;\;^2}{a^2} + u^2\right)$$

$$\therefore v^2 = \frac{h^2}{a^2r^4}\;\;a^4 + 3a^2r^2 + r^4$$

Example

Determine the law of force in the following orbits, the pole being the centre of attraction

(i) $r^2 = a^2\cos 2\theta$

(ii) $r = \dfrac{a}{\theta^2 + b}$

Solution

(i) Due to $r^2 = a^2\cos 2\theta$ and let us choose $r = u^{-1}$ therefore $\dfrac{1}{u^2} = a^2\cos 2\theta$

Now differentiate with respect to θ

$$-\cancel{2}\frac{1}{u^3}\frac{du}{d\theta} = -\cancel{2}a^2\sin 2\theta \quad \Rightarrow \frac{du}{d\theta} = a^2u^3\sin 2\theta$$

Again

$$\frac{d^2u}{d\theta^2} = 2a^2u^3\cos 2\theta + 3a^2u^2\frac{du}{d\theta}\sin 2\theta = 2u^3\underbrace{a^2\cos 2\theta}_{1/u^2} + 3a^2u^2\underbrace{\frac{du}{d\theta}}_{a^2u^3\sin 2\theta}\sin 2\theta$$

$$\therefore \frac{d^2u}{d\theta^2} = 2u + 3a^4u^5\sin^2 2\theta = 2u + 3a^4u^5\left(1-\cos^2 2\theta\right)$$
$$= 2u + 3a^4u^5 - 3a^4u^5\cos^2 2\theta$$
$$= 2u + 3a^4u^5 - 3u = 3a^4u^5 - u$$

$$\therefore \frac{d^2u}{d\theta^2} + u = 3a^4u^5$$

$$\because F = mh^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$

$$\therefore F = mh^2u^2\left(3a^4u^5\right) = 3ma^4h^2u^7 = \frac{3ma^4h^2}{r^7}$$

(ii) In the similar manner, we have $r = \dfrac{a}{\theta^2 + b}$ assume that $r = u^{-1}$ hence

$$\frac{1}{u} = \frac{a}{\theta^2 + b} \qquad \Rightarrow -\frac{1}{u^2}\frac{du}{d\theta} = -\frac{2a\theta}{\left(\theta^2 + b\right)^2}$$

$$\Rightarrow \frac{du}{d\theta} = \frac{2}{a}\cancel{u^2}\theta \times \frac{1}{\cancel{u^2}} = \frac{2}{a}\theta \qquad \therefore \frac{du}{d\theta} = \frac{2}{a}\theta$$

Now differentiate again with respect to variable θ

$$\therefore \frac{d^2u}{d\theta^2} = \frac{2}{a}$$

$$\therefore \frac{d^2u}{d\theta^2} + u = \frac{2}{a} + u$$

$$\because\ F = mh^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$
$$\therefore\ F = mh^2u^2\left(\frac{2}{a} + u\right) = \frac{mh^2}{r^3}\left(\frac{2}{a}r + 1\right)$$

Example

If a particle describes the cardioid $r = a(1 - \cos\theta)$ under a force to the pole, show that the force is proportional to the inverse fourth power of the distance. If P be the force at the apse ($\theta = \pi$) and V represents the velocity, prove that $3V^2 = 4aP$.

Solution

Since we have $r = a(1 - \cos\theta)$ and let us choose $r = u^{-1}$ therefore

$$\frac{1}{u} = a(1 - \cos\theta)$$

Now differentiate with respect to θ

$$-\frac{1}{u^2}\frac{du}{d\theta} = a\sin\theta \quad \Rightarrow \frac{du}{d\theta} = -au^2\sin\theta$$

Once again

$$\frac{d^2u}{d\theta^2} = -2au\frac{du}{d\theta}\sin\theta - au^2\cos\theta = 2a^2u^3\sin^2\theta - au^2\cos\theta$$

$$\therefore\ \frac{d^2u}{d\theta^2} = -au^2 \quad \cos\theta - 2au\sin^2\theta \quad = -au^2 \quad \cos\theta - 2au \quad 1 - \cos^2\theta$$

$$\begin{aligned}
\therefore\ \frac{d^2u}{d\theta^2} &= -au^2 \quad \cos\theta + 2au \quad 1 - \cos^2\theta \\
&= -au^2\left(\cos\theta - 2u\underbrace{a \quad 1 - \cos\theta}_{1/u}(1 + \cos\theta)\right) \\
&= -au^2 \quad \cos\theta - 2(1 + \cos\theta) \\
&= -au^2(-2 - \cos\theta) = -au^2(-3 + \underbrace{1 - \cos\theta}_{1/au}) \\
&= 3au^2 - u
\end{aligned}$$

$$\therefore \frac{d^2u}{d\theta^2} + u = 3au^2$$

$$\because F = mh^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$

$$\therefore F = mh^2u^2 \;\; 3au^2 \;\; = 3mah^2u^4 = \frac{3mah^2}{r^4}$$

At apse we have

$$\dot{r} = 0 \text{ Or } \frac{du}{d\theta} = 0 \qquad \Rightarrow -au^2 \sin\theta = 0 \qquad \therefore \sin\theta = 0 \qquad \Rightarrow \theta = \pi$$

$$\because h = r^2\dot{\theta} = \underbrace{(r\dot{\theta})}_{V} r = 2aV \quad \text{Note } r\Big|_{\theta=\pi} = a(1-\cos\pi) = 2a$$

But we derived the law of force

$$\therefore F = \frac{3mah^2}{r^4} \qquad \Rightarrow F\Big|_{\theta=\pi} = P = \frac{3ma(2aV)^2}{(2a)^4} \quad \therefore 3mV^2 = 4aP$$

Example

Show that the curve $r^n = a^n \cos n\theta$ can be described under a force to the pole varying inversely as $2n+3$ power of the distance

Solution

Since, $r^n = a^n \cos n\theta$ and let us take $r = u^{-1}$ thus $\frac{1}{u^n} = a^n \cos n\theta$

Now differentiate with respect to θ

$$-\cancel{n}\frac{1}{u^{n+1}}\frac{du}{d\theta} = -\cancel{n}a^n \sin n\theta \quad \Rightarrow \frac{du}{d\theta} = a^n u^{n+1} \sin n\theta$$

Once again

$$\frac{d^2u}{d\theta^2} = na^n u^{n+1} \cos n\theta + (n+1)a^n u^n \frac{du}{d\theta} \sin n\theta$$

$$= nu^{n+1} \underbrace{a^n \cos n\theta}_{1/u^n} + (n+1)a^n u^n \underbrace{\frac{du}{d\theta}}_{a^n u^{n+1} \sin n\theta} \sin n\theta$$

$$\therefore \frac{d^2u}{d\theta^2} = nu + (n+1)a^{2n}u^{2n+1}\sin^2 n\theta = nu + (n+1)a^{2n}u^{2n+1}(1-\cos^2 n\theta)$$
$$= nu + (n+1)a^{2n}u^{2n+1} - (n+1)u^{2n+1}\underbrace{a^{2n}\cos^2 n\theta}_{1/u^{2n}}$$

$$\therefore \frac{d^2u}{d\theta^2} = nu + (n+1)a^{2n}u^{2n+1} - (n+1)u = (n+1)a^{2n}u^{2n+1} - u$$

$$\therefore \frac{d^2u}{d\theta^2} + u = (n+1)a^{2n}u^{2n+1}$$

$$\because F = mh^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$
$$\therefore F = mh^2u^2 \ \ (n+1)a^{2n}u^{2n+1} \ = (n+1)ma^{2n}h^2u^{2n+3} = \frac{3ma^{2n}h^2}{r^{2n+3}}$$

Example

A particle moves under the action of a force to a fixed point varying inversely as the square of the distance r. Prove that the orbit is a conic section with one focus at the center of force.

Solution

Since, $F = \dfrac{\mu}{r^2} \equiv \mu u^2$ then

$$\because \frac{F}{mh^2u^2} = \frac{d^2u}{d\theta^2} + u \qquad \Rightarrow \frac{d^2u}{d\theta^2} + u = \frac{\mu u^2}{mh^2u^2} = \lambda \quad (\lambda = \frac{\mu}{mh^2})$$

$$\therefore \frac{d^2u}{d\theta^2} + u = \lambda$$

This is a differential equation which its general solution is

$$u = \frac{1}{r} = \mu(1 + \epsilon\cos(\theta - \alpha))$$

Where ϵ and α represent the constants of integration.

Example

A particle with mass $\mathbf{1\,gr}$ moves under an attractive force varies inversely as r^3 where the force equals $\mathbf{1\ Dyne}$ when $r = \mathbf{1\ cm}$. Find the path equation if

$\theta = 0$ when $r = 2$ cm and velocity $\frac{1}{2}$ cm sec^{-1} with direction makes an angle of $\frac{\pi}{4}$ with constant line.

Solution

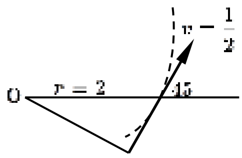
Since the attractive force varies inversely as cub of r, i.e. $F = \frac{\mu}{r^3}$ where μ is constant of proportional which can be evaluated from the condition $F = 1$ when $r = 1$ then $\mu = 1$, therefore $F = \frac{1}{r^3}$ and the path differential equation is

$$h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = u^3 \quad \Rightarrow h^2\left(\frac{d^2u}{d\theta^2} + u\right) = u \quad \text{Or} \quad \frac{d^2u}{d\theta^2} + \left(1 - \frac{1}{h^2}\right)u = 0 \quad (1)$$

The constant h can be obtained from the principle of conservation of angular momentum about the attractive point and then

$$\Rightarrow h = \frac{1}{2}(2\sin\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$$

Substituting in differential equation (1) we obtain



$$\frac{d^2u}{d\theta^2} - u = 0$$

$$\frac{du}{d\theta}\frac{d}{du}\left(\frac{du}{d\theta}\right) - u = 0 \quad \text{Or} \quad \left(\frac{du}{d\theta}\right)d\left(\frac{du}{d\theta}\right) - udu = 0$$

Then by integration

$$\left(\frac{du}{d\theta}\right)^2 = u^2 + c_1 \qquad (2)$$

Where c_1 is constant of integration and to determine c_1 we have to evaluate $\frac{du}{d\theta}$ as $r = 2$ which can be evaluated from velocity law

$$v^2 = h^2\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right) = \frac{1}{2}\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right)$$

Since $v = \frac{1}{2}$ when $u = \frac{1}{2}$ then

$$\frac{1}{4} = \frac{1}{2}\left(\left(\frac{du}{d\theta}\right)^2 + \frac{1}{4}\right) \qquad \Rightarrow \left(\frac{du}{d\theta}\right)^2\Bigg|_{r=2} = \frac{1}{4}$$

That is as $u = \frac{1}{2}$ we have $\left(\frac{du}{d\theta}\right)^2 = \frac{1}{4}$ therefore, the value of integration constant $c_1 = 0$ and from equation (2)

$$\left(\frac{du}{d\theta}\right)^2 = u^2 \quad \Rightarrow \frac{du}{d\theta} = -u \qquad \textbf{Or} \qquad \frac{du}{u} = -d\theta$$

Then by integration we have

$$\ln u = -\theta + c_2$$

Now from the initial condition $u = \frac{1}{2}$ as $\theta = 0$ we get $c_2 = \ln\frac{1}{2}$ **Or** $c_2 = -\ln 2$ and then

$$\ln u = -\theta - \ln 2 \quad \Rightarrow \ln r = \theta + \ln 2 \quad \textbf{Or} \quad r = 2e^{\theta}$$

Which gives the path equation.

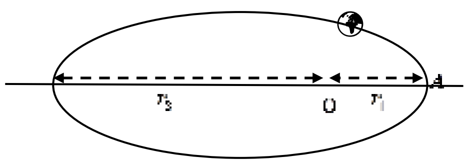
Example

If the ratio between the maximum value of angular velocity of a planet and the minimum value is γ^2. Prove that eccentricity of the planet trajectory is $\frac{\gamma - 1}{\gamma + 1}$.

Solution

According Kepler's law the planet moves around the sun in an ellipse path, in one focus of which the center of the sun is situated, where

$$r^2\dot{\theta} = h \quad \therefore \dot{\theta} = \frac{h}{r^2}$$



It's clear that the angular velocity $\dot{\theta}$ varies inversely as the square of distance of the sun r, therefore the greatest angular velocity occurs as r be smallest say $r = r_1$ where $r_1 = OA = a - ae$ and again the lowest angular velocity occurs as $r = r_2$ where $r_2 = OB = a + ae$

$$\Rightarrow \frac{\dot{\theta}_A}{\dot{\theta}_B} = \gamma^2 = \frac{r_2^2}{r_1^2}$$
$$= \frac{(1+e)^2}{(1-e)^2}$$

$$\Rightarrow \gamma = \frac{1+e}{1-e} \qquad \textbf{Or} \qquad e = \frac{\gamma - 1}{\gamma + 1}$$

Example

If a particle moves under the effect of a detractive central force to outside such that its path equation is $\theta = \theta(r)$. Prove that the force law is given by

$$-\frac{mh^2(2\theta' + r\theta'' + r^2\theta'^3)}{r^5\theta'^3}$$

where ' indicates differentiations with respect to r.

Solution

The law of detractive force is given by

$$F = -mh^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$

from the path equation θ is a function of r let as usual $r = \frac{1}{u}$ hence

$$\theta = \theta(r) \qquad \Rightarrow \frac{d\theta}{du} = \underbrace{\frac{d\theta}{dr}}_{\theta'}\frac{dr}{du} = \theta'\left(-\frac{1}{u^2}\right)$$

$$\Rightarrow \frac{du}{d\theta} = -\frac{u^2}{\theta'}$$

Once time differentiate we have

$$\begin{aligned}
\frac{d^2u}{d\theta^2} &= \frac{d}{du}\left(-\frac{u^2}{\theta'}\right)\frac{du}{d\theta} \\
&= -\frac{2u\theta' - u^2\left(-\frac{1}{u^2}\right)\theta''}{\theta'^2}\frac{du}{d\theta} \qquad \left(\theta'' = \frac{d^2\theta}{dr^2}\right) \\
&= -\frac{2u\theta' + \theta''}{\theta'^2}\left(-\frac{u^2}{\theta'}\right) \\
&= \frac{2u^3\theta' + u^2\theta''}{\theta'^3}
\end{aligned}$$

$$\begin{aligned}
\therefore F &= -mh^2u^2\left(\frac{2u^3\theta' + u^2\theta''}{\theta'^3} + u\right) \qquad \left(u = \frac{1}{r}\right) \\
&= -mh^2\left(\frac{2\theta' + r\theta'' + r^2\theta'^3}{r^5\theta'^3}\right)
\end{aligned}$$

Example

Prove that the areal velocity in Cartesian coordinate is $\frac{1}{2}(x\dot{y} - y\dot{x})$.

Solution

Since the relation between Cartesian (x, y) and Polar (r, θ) coordinates are

$$\tan\theta = \frac{y}{x}, \qquad r^2 = x^2 + y^2$$

And the areal velocity is given by $\dot{A} = \frac{1}{2}h = \frac{1}{2}r^2\dot{\theta}$

Then differentiating

$$\begin{aligned}
\tan\theta = \frac{y}{x} \qquad &\Rightarrow \dot{\theta}\sec^2\theta = \frac{\dot{y}x - \dot{x}y}{x^2} \\
&\Rightarrow \dot{\theta} = \frac{\dot{y}x - \dot{x}y}{x^2}\cos^2\theta
\end{aligned}$$

but $\cos^2\theta = \dfrac{x^2}{x^2 + y^2}$

$$\begin{aligned}
\dot{A} = \frac{1}{2}r^2\dot{\theta} &= \frac{1}{2}(x^2 + y^2)\frac{\dot{y}x - \dot{x}y}{x^2}\left(\frac{x^2}{x^2 + y^2}\right) \\
&= \frac{1}{2}\ \dot{y}x - \dot{x}y
\end{aligned}$$

PROBLEMS

❒ A particle is attracted to a point by a central force, and it is observed that the orbit of the particle is the spiral $r = e^{\theta}$. Determine the force that is acting.

❒ A particle moving under the influence of a central force, describes a circle through the center of the force. Prove that the force is attractive and inversely proportional to the fifth power of the distance

[Hint. Equation of the circle is $r = 2a\cos\theta$].

❒ If in a central orbit under the force ($\mu u^3(3 + 2a^2u^2)$), a particle be projected at a distance a with a velocity $\sqrt{5\mu} \;/\; a$ in a direction making $\tan^{-1}\dfrac{1}{2}$ with the radius, show that the equation to the path is $r = a\cot\left(\theta + \dfrac{\pi}{4}\right)$.

❒ Show that the curve $r = \dfrac{\ell}{1 + \epsilon\cos\theta}$ can be described under a force to the pole varying inversely as 2 power of the distance.