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INTRODUCTION

The subject of Dynamics is generally divided into two branches: the first one, is called *Kinematics*, is concerned with the geometry of motion apart from all considerations of force, mass or energy; the second, is called *Kinetics*, is concerned with the effects of forces on the motion of bodies.

In order to describe the motion of a particle (or point) two things are needed,

(i) a frame of reference,
(ii) a time-keeper.

It is not possible to describe absolute motion, but only motion relative to surrounding objects; and a suitable frame of reference depends on the kind of motion that it is desired to describe. Thus if the motion is rectilinear the distance from a fixed point on the line is a sufficient description of the position of the moving point; and in more general cases systems of two or of three rectangular axes may be chosen as a frame of reference. For example, in the case of a body projected from the surface of the Earth a set of axes with the origin at the point of projection would be suitable for the description of motion relative to the Earth. But, for the description of the motion of the planets, it would be more convenient to take a frame of axes with an origin at the Sun's center (Polar co-ordinates).

■ Definitions

1. Mass: The mass of a body is the quantity of matter in the body. The unit of mass used in England is a pound and is defined to be the mass of a certain piece of platinum kept in the Exchequer Office.

2. A Particle (point): is a portion of matter which is indefinitely small in size, or which, for the purpose of our investigations, is so small that the distances between its different parts may be neglected.

3. A Body: may be regarded as an indefinitely large number of indefinitely small portions, or as a conglomeration of particles.

4. A Rigid Body is a body whose parts always preserve an invariable position with respect to one another

<u>KINEMATICS IN ONE DIMENSION</u>

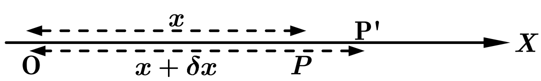
<u>RECTILINEAR MOTION</u>

When a point (or particle) moves along a straight line, its motion is said to be a rectilinear motion. Here in this chapter we shall discuss the motion of a point (or particle) along a straight line which may be either horizontal or vertical.

■ <u>Velocity and Acceleration</u>

Suppose a particle moves along a straight line **OX** where **O** represents a fixed point on the line. Let **P** be the position of the particle at time t, where $\mathbf{OP} = x$ and $\mathbf{P'}$ be the position of the particle at time $t + \delta t$, with $\mathbf{OP'} = x + \delta x$. Therefore $\dfrac{\delta x}{\delta t}$ represents the average rate of displacement or the average velocity during the interval δt. If this ratio be independent of the interval δt, i.e. if it has the same value for all intervals of time, then the velocity is constant or uniform, and equal distances will be traversed in equal times. Whether the ratio $\delta x / \delta t$ be constant or not, its limiting value as δt tends to zero is defined to be the measure of the *velocity* (also known as instantaneous velocity) of the moving point at time t. But this limiting value is the differential coefficient of x with regard to t, so that if we denote the velocity by v, we have

$$v = \lim_{\delta t \to 0} \frac{\delta x}{\delta t} = \frac{dx}{dt} = \dot{x}$$



Again, *Acceleration* is similarly defined as the rate of change of velocity. Thus, if $v, v + \delta v$ denote the velocities of the moving point at times $t, t + \delta t$, then

δv is the change of velocity in time δt and $\frac{\delta v}{\delta t}$ is the average rate of change of velocity during the interval δt. If this ratio is independent of the interval δt, then the acceleration is constant or uniform, or equal increments of velocity take place in equal intervals. Whether the ratio $\frac{\delta v}{\delta t}$ be constant or not, its limiting value as δt tends to zero is defined to be the measure of the acceleration of the moving point at time t. But this limiting value is the differential coefficient of v with regard to t, so that if we denote the acceleration by a, we have

$$\begin{aligned} a &= \lim_{\delta t \to 0} \frac{\delta v}{\delta t} = \frac{dv}{dt} \\ &= \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} = \ddot{x} \end{aligned}$$

■ Other Expression for Acceleration

Let $v = \frac{dx}{dt}$. We can write (using chain rule in Differentiation)

$$a = \ddot{x} = \frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{dv}{dt} = \frac{dv}{dx} \times \underbrace{\frac{dx}{dt}}_{v} = v\frac{dv}{dx}$$

Therefore, $\frac{d^2x}{dt^2}$, $\frac{dv}{dt}$ and $v\frac{dv}{dx}$ are three expressions for representing the acceleration and any one of them can be used to suit the convenience in working out the problems.

■ Remember

The law of acceleration in a particular problem may be given by expressing the acceleration as a function of the time t, or the distance x, or the velocity v. The problem of further investigating the motion can then be solved as follows:

➤If acceleration is given as a function of the time t say $\varphi(t)$ so

$$a = \varphi(t) \qquad \Rightarrow \frac{dv}{dt} = \varphi(t)$$
$$\Rightarrow dv = \varphi(t)dt$$
$$\Rightarrow v = \int \varphi(t)dt + c_1$$

And then $\quad \because \quad v = \int \varphi(t)dt + c_1 \qquad \Rightarrow \dfrac{dx}{dt} = \int \varphi(t)dt + c_1$

$$\Rightarrow dx = \quad \int \varphi(t)dt + c_1 \quad dt$$

$$\therefore x \;= \int \quad \int \varphi(t)dt + c_1 \quad dt + c_2$$

➤If acceleration is given as a function of the distance x say $f(x)$ so

$$a = f(x) \qquad \Rightarrow v\frac{dv}{dx} = f(x) \qquad \Rightarrow vdv = f(x)dx$$
$$\Rightarrow v^2 = 2\int f(x)dx + c_3$$

Further, $\quad \because v^2 = 2\int f(x)dx + c_3$

$$\Rightarrow \frac{dx}{dt} = \mp\sqrt{2\int f(x)dx + c_3}$$

$$\Rightarrow \mp\frac{dx}{\sqrt{2\int f(x)dx + c_3}} = dt$$
$$\Rightarrow t + c_4 = \mp\int \frac{dx}{\sqrt{2\int f(x)dx + c_3}}$$

➤Again, Acceleration is given as a function of velocity v say $\varphi(v)$

$$a = \varphi(v) \qquad \Rightarrow \frac{dv}{dt} = \varphi(v)$$
$$\Rightarrow \frac{dv}{\varphi(v)} = dt \qquad \text{by integrating}$$
$$\Rightarrow t = \int \frac{dv}{\varphi(v)} + c_5$$

or we may connect velocity with distance by writing

$$v\frac{dv}{dx} = \varphi(v) \quad \Rightarrow \frac{vdv}{\varphi(v)} = dx \qquad \therefore x = \int \frac{vdv}{\varphi(v)} + c_6$$

where, $c_1 - c_6$ are constants of integration

■ Illustrative Examples ■

Example

A particle moves along a straight line such that its displacement x from a fixed point on the line (origin) at time t is given by $x = t^3 - 9t^2 + 24t + 6$. Determine the instant when the acceleration becomes zero, the position of the particle at this instant and the velocity of the particle then.

Solution

Since, $x = t^3 - 9t^2 + 24t + 6$. Differentiating with respect to time (w.r.t),

the velocity $$v = \frac{dx}{dt} = 3t^2 - 18t + 24,$$

and the acceleration is $$a = \frac{dv}{dt} = 6t - 18$$

Now the acceleration vanishes i.e. $a = 0$ when $6t - 18 = 0 \quad \Rightarrow t = 3$

When $t = 3$, the position is given by $x = 3^3 - 9(3^2) + 24(3) + 6 = 24$ units.

Again when $t = 3$ the velocity is given by $v = 3(3^2) - 18(3) + 24 = -3$, this means that at $t = 3$ the velocity of the particle equals 3 units and in the opposite direction of x.

Example

If at time t the displacement x of a particle moving away from the origin is given by $x = A\cos t + B\sin t$, where A, B are constants. Find the velocity and acceleration of the particle at in terms of time.

Solution

Given that $x = A\cos t + B\sin t$

Differentiating with respect to time (w.r.t), we obtain the velocity of the particle

$$v = \frac{dx}{dt} = B\cos t - A\sin t$$

Differentiating again, one get the acceleration at any time,

$$\begin{aligned} a = \frac{dv}{dt} &= -A\cos t - B\sin t \\ &= -(\underbrace{A\cos t + B\sin t}_{x}) \\ &= -x \end{aligned}$$

Note that the acceleration proportional to the displacement.

Example

A point moves along a straight line where its distance x from a fixed point on the line is given by $x = A\cos(\mu t + \epsilon)$. Prove that its acceleration varies as the distance measured from the origin and is directed towards the origin.

Solution

Since we have $x = A\cos(\mu t + \epsilon)$

Differentiating w.r.t $x = A\cos(\mu t + \epsilon)$, we get

$$\frac{dx}{dt} = -\mu A\sin(\mu t + \epsilon)$$

Differentiation again $\frac{d^2x}{dt^2} = -\mu^2 \underbrace{A\cos(\mu t + \epsilon)}_{x} = -\mu^2 x$

That is the acceleration varies as the distance x from the origin. The negative sign "-" indicates that it is in the negative sense of the x-axis, i.e., towards the origin.

Example

A point moves along a straight line such that its distance x from a fixed point on it and the velocity v are related by $v^2 = \mu(b^2 - x^2)$. Prove that the acceleration varies as the distance from the origin and is directed towards the origin.

Solution

Since we have $v^2 = \mu(b^2 - x^2)$

Differentiating w.r.t x, we obtain

$$2v\frac{dv}{dx} = \mu(-2x) \qquad \therefore v\frac{dv}{dx} = a = -\mu x$$

Hence the acceleration varies as the distance x from the origin. The negative sign "-" indicates that it is in the direction of x decreasing, i.e., towards the origin.

Example

A particle moves along a straight line such that its distance x from a fixed point on it and the time at any time t are related by $x = 2(1 - e^{-t})$. Find the velocity in terms of distance and the acceleration in terms of velocity.

Solution

In order to obtain the velocity with differentiating the function of position x with respect to time, we get

$$x = 2\ 1 - e^{-t} \qquad \Rightarrow v = \frac{dx}{dt} = 2e^{-t} \qquad \textbf{Note} \quad \frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$$

$$\because x - 2 = -2e^{-t} \qquad \Rightarrow v = 2 - x$$

This equation illustrates the relation between velocity and distance.

Now to get the relation between acceleration and velocity

$$\because a = v\frac{dv}{dx} = v(-1) = -v \qquad \textbf{Note} \quad \frac{dv}{dx} = -1 \quad \therefore a = -v$$

Example

A mass moves along a straight line such that its acceleration at any time t is given by $6t + 2$. Initially the mass at rest placed at the origin point. Determine the velocity and distance as a function of time. Determine the position of the mass after 5 sec.

Solution

Since we have $a = 6t + 2,$ $\quad a = \frac{dv}{dt} \quad \Rightarrow \frac{dv}{dt} = 6t + 2$

Thus, by separation of variables we get

$$dv = \ 6t + 2 \ \ dt \qquad \Rightarrow \int dv = \int \ 6t + 2 \ \ dt$$

$$\therefore\ v = 3t^2 + 2t + c_1$$

From initial conditions at $t = 0$, $v = 1$ then $c_1 = 0$

Again, $\therefore\ v = 3t^2 + 2t$ this equation gives the relation between velocity and time. Since $v = \frac{dx}{dt}$ that is

$\frac{dx}{dt} = 3t^2 + 2t \qquad \Rightarrow dx = \ 3t^2 + 2t \ \ dt$ (Separation variables)

$\int dx = \int \ 3t^2 + 2t \ \ dt$ or $x = t^3 + t^2 + c_2$

From initial conditions at $t = 0$, $x = 0$ then $c_2 = 0$, i.e.

$x = t^3 + t^2$

this equation gives the relation between distance and time.

The position at $t = 5$ is $x\big|_{t=5} = 5^3 + 5^2 = 150$

Example

A point moves along a straight line according to $v = u + bx$, where u, b are constants. Find the velocity and acceleration in terms of time and the acceleration in terms of distance and also as a function of velocity.

Solution

Velocity and acceleration can be obtained by differentiation the function of position and then velocity with respect to time, therefore

$$v = u + bx \quad \Rightarrow a = \frac{dv}{dt} = b\frac{dx}{dt} = bv = b(u + bx) \qquad \Rightarrow a = b(u + bx)$$

This equation gives the acceleration as a function of velocity $a = bv$ and as a function of distance $a = b(u + bx)$

Again to get the velocity and acceleration as functions of time

$$\because v = u + bx \quad \Rightarrow \frac{dx}{dt} = b(u + bx) \qquad \Rightarrow \frac{dx}{u + bx} = bdt$$

Multiply the previous relation by b and then integrate

$$\int \frac{bdx}{u + bx} = \int b^2 dt \quad \Rightarrow \ln(u + bx) = b^2\, t + C$$

Where C is integration constant , the last relation can be rewritten as

$$\because \ln(u + bx) = b^2\, t + C \qquad \Rightarrow \ln v = b^2\, t + C \quad \text{Or}$$
$$\Rightarrow v = Ae^{b^2 t}, \quad A = e^C$$

This is the relation between velocity and time, also the acceleration given by

$$a = bv = bAe^{b^2 t}$$

Example

A Point moves along a straight line with retardation $a = -2v^2$. Find the position at any instance if the point starts from origin with initial velocity equals unity.

Solution

The motion under retardation where $a = -2v^2$ but we know $a = \dfrac{dv}{dt}$, so

$$\because a = -2v^2 \qquad \Rightarrow \frac{dv}{dt} = -2v^2$$

By separation of variables and integrate, we obtain

$$-\int \frac{dv}{v^2} = \int 2dt + c_1 \qquad \Rightarrow \frac{1}{v} = 2t + c_1$$

The integration constant c_1 can be evaluated as $v = 1$ when $t = 0$, hence $1 = 2(0) + c_1 \quad \therefore c_1 = 1$ then the velocity can be obtained by

$$\frac{1}{v} = 2t + 1 \quad \text{but} \quad v = \frac{dx}{dt} \quad \therefore \frac{dt}{dx} = 2t + 1 \quad \text{Or} \quad \frac{dt}{2t+1} = dx$$

Again by integrating we get

$$\frac{2dt}{2t+1} = 2dx \quad \Rightarrow \ln(2t+1) = 2x + c_2$$

From initial condition $x = 0$ when $t = 0$ then $c_2 = 0$ and the relation between distance and time becomes

$$x = \frac{1}{2}\ln(2t+1)$$

Example

A particle starts from rest at a distance h from the origin O with retardation $-4x^{-3}$. Prove that the particle reach to distance ℓ from O in time $\frac{h}{2}\sqrt{h^2 - \ell^2}$ and then find its velocity at this position.

Solution

Since we have been given the retardation as $a = -16x^{-3}$ and $a = v\frac{dv}{dx}$ therefore,

$$\therefore v\frac{dv}{dx} = -4x^{-3} \qquad \Rightarrow vdv = -4x^{-3}\ dx$$

By integrating, we obtain

$$\therefore \int vdv = -\int 4x^{-3}\ dx + c_1 \qquad \text{Or} \qquad \frac{1}{2}v^2 = \frac{2}{x^2} + c_1 \quad \text{Or} \quad v^2 = \frac{4}{x^2} + c$$

The integration constant c can be evaluated as $v = 0$ when $x = h$, hence $0 = \frac{4}{h^2} + c$ i.e. $c_1 = -\frac{4}{h^2}$ and then we get

$$v^2 = \frac{4}{x^2} - \frac{4}{h^2} = \frac{4(h^2 - x^2)}{x^2 h^2} \qquad \therefore v = \pm \frac{2}{h} \frac{\sqrt{h^2 - x^2}}{x}$$

We will consider the minus sign since the motion of the particle towards the origin –in decreasing x - and use $v = \dfrac{dx}{dt}$

$$\therefore \frac{dx}{dt} = -\frac{2}{h} \frac{\sqrt{h^2 - x^2}}{x} \qquad \Rightarrow -\frac{xdx}{\sqrt{h^2 - x^2}} = \frac{2}{h} dt \quad \textbf{Or}$$

$$\Rightarrow -\int \frac{xdx}{\sqrt{h^2 - x^2}} = \int \frac{2}{h} dt + c_2$$

$$\Rightarrow \sqrt{h^2 - x^2} = \frac{2}{h} t + c_2$$

To obtain the constant c_2 when $x = h$ as $t = 0$ and then $c_2 = 0$ so

$$\therefore \sqrt{h^2 - x^2} = \frac{2}{h} t \qquad \textbf{Or} \qquad t = \frac{h}{2} \sqrt{h^2 - x^2}$$

The spent time to reach to a distance ℓ from origin point is $t = \dfrac{h}{2} \sqrt{h^2 - \ell^2}$, to determine the velocity at this position, we put $x = \ell$ in velocity relation, that is

$$v\big|_{x=\ell} = \frac{2\sqrt{h^2 - \ell^2}}{h\ell}$$

Example

A Point moves along a straight line according to the relation $v = (1 + x^2)t$. Find the distance as a function of time if the point starts its motion from the origin.

Solution

Since $v = (1 + x^2)t$ thus

$$\frac{dx}{dt} = (1 + x^2)t \qquad \Rightarrow \frac{dx}{1 + x^2} = tdt \qquad \therefore \int \frac{dx}{1 + x^2} = \int tdt + c_1$$

$$\therefore \tan^{-1} x = \frac{1}{2}t^2 + c_1$$

From initial condition where the point starts its motion at origin

$$\therefore \tan^{-1} 0 = \frac{1}{2}0^2 + c_1 \quad \Rightarrow 0 = 0 + c_1 \therefore c_1 = 0 \qquad \therefore x = \tan\left(\frac{1}{2}t^2\right)$$

Note that

$$\int \frac{f'dx}{1+f^2} = \tan^{-1} f$$

PROBLEMS

❒ A particle moving in a straight line is subject to a resistance which produces the retardation kv^3, where v is the velocity and k is a constant. Show that v and t (the time) are given in terms of x (the distance) by the equations $v = \dfrac{u}{kux+1}$, $t = \dfrac{1}{2}kx^2 + \dfrac{x}{u}$, where u is the initial velocity.

❒ If the relation between x and t is of the form $t = bx^2 + kx$, find the velocity v as a function of x, and prove that the retardation of the particle is $2bv^3$.

❒ A particle is projected vertically upwards with speed u and moves in a vertical straight line under uniform gravity with no air resistance. Find the maximum height achieved by the particle and the time taken for it to return to its starting point.

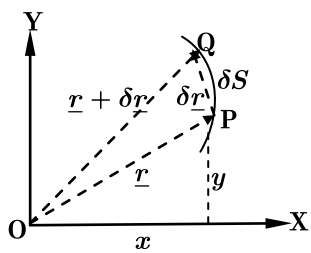
Kinematics in Two Dimensions

■ Velocity in Cartesian Coordinates

The velocity vector of a particle (or point) moving along a curve is the rate of change of its displacement with respect to time.

Let P and Q be the positions of a particle moving along a curve at times t and $t+\delta t$ respectively. With respect to $\mathbf{O}$ as the origin of vectors, let $\underline{\mathbf{O}P} = \underline{r}$ and $\underline{\mathbf{O}Q} = \underline{r} + \delta\underline{r}$ Then $\underline{\mathbf{P}Q} = \underline{\mathbf{O}Q} - \underline{\mathbf{OP}} = \delta\underline{r}$ represents the displacement of the particle in time δt and $\frac{\delta \underline{r}}{\delta t}$ indicates the average rate of displacement (or average velocity) during the interval δt. The limiting value of the average velocity $\frac{\delta \underline{r}}{\delta t}$ as δt tends to zero ($\delta t \to 0$) is the velocity. Therefore if the vector $\underline{v}$ represents the velocity of the particle at time t then

$$\underline{v} = \lim_{\delta t \to 0} \frac{\delta \underline{r}}{\delta t} = \frac{d\underline{r}}{dt} = \dot{\underline{r}}$$



Where $\underline{r}$ is the position vector of the particle.

Now, if $\underline{r} = x\,\hat{i} + y\,\hat{j}$

Then $\underline{v} = \frac{d\underline{r}}{dt} = \underbrace{\frac{dx}{dt}}_{v_x}\hat{i} + \underbrace{\frac{dy}{dt}}_{v_y}\hat{j} = \dot{x}\,\hat{i} + \dot{y}\,\hat{j}$

Note that $(\dot{x}, \dot{y})$ are called the components or resolved parts of the velocity $\underline{v}$ along the axes x and y respectively. The speed of the particle at P is given by

$$|\underline{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \frac{ds}{dt}$$

Also the angle θ which the direction of $\underline{v}$ makes with OX is

$$\tan\theta = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{dy}{dx}$$

■ Acceleration in Cartesian Coordinates

The acceleration vector of a particle moving along a curve is defined as the rate of change of its velocity vector.

if $\underline{v}$ and $\underline{v}+\delta\underline{v}$ are the velocities of a particle moving along a curve at times t and $t+\delta t$ respectively, then $\delta\underline{v}$ is the change in velocity vector in time δt and $\dfrac{\delta\underline{v}}{\delta t}$ is the average and then

$$\underline{a}=\lim_{\delta t\to 0}\frac{\delta\underline{v}}{\delta t}=\frac{d\underline{v}}{dt}=\frac{d}{dt}\left(\frac{d\underline{r}}{dt}\right)=\frac{d^2\underline{r}}{dt^2}$$

Substituting for $\underline{v}=\dfrac{dx}{dt}\hat{i}+\dfrac{dy}{dt}\hat{j}$ we have,

$$\underline{a}=\frac{d}{dt}\left(\frac{dx}{dt}\hat{i}+\frac{dy}{dt}\hat{j}\right)=\underbrace{\frac{d^2x}{dt^2}}_{a_x}\hat{i}+\underbrace{\frac{d^2y}{dt^2}}_{a_y}\hat{j}=\ddot{x}\,\hat{i}+\ddot{y}\,\hat{j}$$

Here, $(\ddot{x},\ddot{y})$ are called the components of the acceleration $\underline{a}$ along the axes x and y respectively. The magnitude of the acceleration is given by

$$|\underline{a}|=\sqrt{\left(\frac{d^2x}{dt^2}\right)^2+\left(\frac{d^2y}{dt^2}\right)^2}$$

Again, the angle φ which the direction of $\underline{a}$ makes with OX is

$$\tan\varphi=\frac{d^2y}{dt^2}\Big/\frac{d^2x}{dt^2}$$

■ Illustrative Examples ■

Example

A point moves along the curve $x = t^3 + 1,\ \ y = t^2$ where, t is the time. Determine the components of velocity and acceleration at $t = 1$

Solution

Let $\underline{r}$ be the position vector of the particle at time t, therefore

$$\underline{r} = x\,\hat{i} + y\,\hat{j} = (t^3 + 1)\,\hat{i} + t^2\,\hat{j}$$

Then the velocity vector is

$$\underline{v} = \frac{d\underline{r}}{dt} = 3t^2\,\hat{i} + 2t\,\hat{j} \quad \text{and} \qquad \underline{v}\Big|_{t=1} = 3(1)^2\,\hat{i} + 2(1)\,\hat{j} = 3\,\hat{i} + 2\,\hat{j}$$

Again the vector of acceleration is

$$\underline{a} = \frac{d\underline{v}}{dt} = 6t\,\hat{i} + 2\,\hat{j} \qquad \text{and} \qquad \underline{a}\Big|_{t=1} = 6(1)\,\hat{i} + 2\,\hat{j} = 6\,\hat{i} + 2\,\hat{j}$$

Example

The position of a moving point at time t is given by $x = 3\cos t,\ \ y = 2\sin t$

Find its path velocity and acceleration vectors.

Solution

Since the parametric equations are $x = 3\cos t,\ \ y = 2\sin t$ then

$$\left(\frac{x}{3}\right)^2 = \cos^2 t,\ \left(\frac{y}{2}\right)^2 = \sin^2 t \qquad \Rightarrow \left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \qquad \text{or}\ \ 4x^2 + 9y^2 = 36$$

This is a the path equation which represents an Ellipse

Velocity vector is $\underline{v} = -3\sin t\ \hat{i} + 2\cos t\ \hat{j}$

While the acceleration vector is

$$\underline{a} = -3\cos t\ \hat{i} - 2\sin t\ \hat{j} = -\underbrace{(3\cos t\ \hat{i} + 2\sin t\ \hat{j})}_{\underline{r}} = -\underline{r}$$

Example

A particle moves along the curve $y = 2x^2$ such that its horizontal component of velocity is constant and equals 2. Calculate the components of acceleration and velocity when $y = 8$.

Solution

Since the horizontal component of velocity equals 2, i.e. $\dot{x} = 2$, therefore by differentiating w.r.t t we get

$\ddot{x} = 0$ and $y = 2x^2 \quad \Rightarrow \dot{y} = 4x\dot{x} = 8x \qquad \therefore \ddot{y} = 8\dot{x} = 16$

That is the acceleration vector is given by

$\underline{a} = 16\,\hat{j}$

and the velocity components are $\dot{x} = 2$ and $\dot{y} = 8x$

Since as $y = 8$ gives $x = \pm 2$ thus, $\qquad \underline{v} = 2\,\hat{i} + 8(\pm 2)\,\hat{j}, \qquad \left|\underline{v}\right| = \sqrt{260}$

Example

A particle describes a plane curve such that its components of acceleration equal $(0, -\mu / y^2)$ with initial velocity $\sqrt{2\mu / b}$ parallel to $\mathbf{X}$-axis and the initial position $(0, b)$. Find the path equation.

Solution

Here we are given that

$$\frac{d^2x}{dt^2} = 0, \qquad \frac{d^2y}{dt^2} = -\frac{\mu}{y^2}$$

Note that $\dfrac{d^2y}{dt^2} = \dfrac{d}{dt}\left(\dfrac{dy}{dt}\right) = \dfrac{d}{dy}\left(\dfrac{dy}{dt}\right) \times \dfrac{dy}{dt} = \dot{y}\dfrac{d\dot{y}}{dy}$ chain rule

Then $\quad \dot{y}\dfrac{d\dot{y}}{dy} = -\dfrac{\mu}{y^2} \qquad \Rightarrow \dot{y}d\dot{y} = -\dfrac{\mu}{y^2}dy \qquad \Rightarrow \int \dot{y}d\dot{y} = -\int \dfrac{\mu}{y^2}dy$

$$\dot{y}^2 = \frac{2\mu}{y} + c_1 \qquad \left(\dot{y} = \frac{dy}{dt}\right)$$

Initially $\dfrac{dy}{dt}=0$ when $y=b$, thus $\quad c_1=-\dfrac{2\mu}{b}$

$\dot{y}^2=\dfrac{2\mu}{y}-\dfrac{2\mu}{b} \qquad \Rightarrow \dot{y}^2=\dfrac{2\mu}{y}-\dfrac{2\mu}{b}=2\mu\left(\dfrac{1}{y}-\dfrac{1}{b}\right)=\dfrac{2\mu}{b}\dfrac{b-y}{y}$

Hence

$$\frac{dy}{dt}=-\sqrt{\frac{2\mu}{b}}\sqrt{\frac{b-y}{y}} \qquad (1)$$

(Negative sign has been taken because the particle is moving in the direction of y decreasing)

Again from $\dfrac{d^2x}{dt^2}=0 \qquad \Rightarrow \dfrac{dx}{dt}=c_3$

Initially when $t=0$, $\dfrac{dx}{dt}=\sqrt{\dfrac{2\mu}{b}}$ thus $\quad c_3=\sqrt{\dfrac{2\mu}{b}}$

$$\therefore \frac{dx}{dt}=\sqrt{\frac{2\mu}{b}} \qquad (2)$$

By dividing the two equations (1) and (2) we get

$\dfrac{dy}{dx}=-\sqrt{\dfrac{b-y}{y}} \qquad \Rightarrow \sqrt{\dfrac{y}{b-y}}\,dy=-dx$, then by integrating

$$b\left(\sin^{-1}\sqrt{\frac{y}{b}}-\sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}}\right)=-x+c_2$$

Hint to get the integration $\displaystyle\int\sqrt{\frac{y}{b-y}}\,dy$ let us use the transformation

$$y=b\sin^2\theta \qquad \Rightarrow dy=2b\sin\theta\cos\theta\,d\theta$$

$$\therefore \int\sqrt{\frac{y}{b-y}}\,dy=\int\sqrt{\frac{b\sin^2\theta}{b-b\sin^2\theta}}\,2b\sin\theta\cos\theta\,d\theta$$

$$=\int\sqrt{\frac{b\sin^2\theta}{b\cos^2\theta}}\,2b\sin\theta\cos\theta\,d\theta$$

$$=\int\frac{\sin\theta}{\cancel{\cos\theta}}\,2b\sin\theta\cancel{\cos\theta}\,d\theta=2b\int\sin^2\theta\,d\theta$$

$$\because \sin^2\theta = \frac{1}{2}\ 1 - \cos 2\theta$$

$$\Rightarrow 2b\int \sin^2\theta\, d\theta = 2b\int \frac{1}{2}\ 1 - \cos 2\theta\ \ d\theta$$

$$= b\int\ 1 - \cos 2\theta\ \ d\theta = b\left(\theta - \frac{\sin 2\theta}{2}\right)$$

$$\therefore \int \sqrt{\frac{y}{b-y}}\, dy = b\left(\sin^{-1}\left(\sqrt{\frac{y}{b}}\right) - \sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}}\right)$$

The initial condition is $t = 0\ \ x = 0,\ \ y = b$ then from the equation

$$b\left(\sin^{-1}\sqrt{\frac{y}{b}} - \sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}}\right) = -x + c_2 \qquad \Rightarrow c_2 = b\frac{\pi}{2}$$

$$\therefore b\left(\sin^{-1}\left(\sqrt{\frac{y}{b}}\right) - \sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}}\right) = b\frac{\pi}{2} - x$$

$$\Rightarrow \sin^{-1}\sqrt{\frac{y}{b}} = \sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}} + \frac{\pi}{2} - \frac{x}{b}$$

$$\sqrt{\frac{y}{b}} = \sin\left(\sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}} + \frac{\pi}{2} - \sqrt{2\mu b}\, t\ \right)$$

$$= \cos\left(x - \sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}}\ \right)$$

$$y = b\cos^2\left(x - \sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}}\ \right)$$

PROBLEMS

❒ The position of a moving point at time t is given by $x = at^2,\ y = 2at$

Find its velocity and acceleration

❒ A particle moves with constant velocity parallel to the axis of $\mathbf{Y}$ and a velocity proportional to y parallel to the axis of $\mathbf{X}$. Prove that it will describe a parabola

❒ A particle is acted on by a force parallel to the axis of $\mathbf{Y}$ whose acceleration is λy and is initially projected with a velocity $a\sqrt{\lambda}$ parallel to the axis of $\mathbf{X}$ at a point where $y = a$. Prove that it will describe the catenary $y = a\cosh(x / a)$