1 Introduction

We can think of a curve as a connected one-dimensional series of points. Sometimes all points in a curve lie in a plane. These curves are called *planar curves*, in contrast to *spatial curves* that are not contained in a plane. We study geometrical concepts for planar and spatial curves. Let us start with two different analytical approaches to describing curves.

2 Representation of a curve

2.1 Parametric representation

We could consider a curve as the trace left in the space by a point that is moving. Hence the coordinates of a generic point P of the curve C are functions

$$x = x(t),\ y = y(t),\ z = z(t),$$

called *coordinate functions*, where t is assuming all the values in an interval $I \subseteq \mathbb{R}$; that is, every value of the parameter is mapped to a point $P \in C$. The map

$$\begin{aligned} \vec{r}\colon I &\longrightarrow \mathbb{R}^3, \\ \vec{r}(t) &= (x(t), y(t), z(t))\,, \end{aligned}$$

is called a *parametric representation* or *parametrization* of C. In the following examples we see how a curve can be given by different parametric representations.

Examples

1. *Straight line.* A straight line containing a point of coordinates (a_1, a_2, a_3) with director vector (v_1, v_2, v_3) is given by the following parametric representation:

   $$\vec{r}(t) = (a_1 + tv_1, a_2 + tv_2, a_3 + tv_3)\,,$$

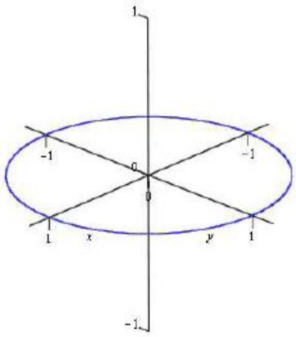
   where a_i, v_i are constants and at least one $v_i \neq 0$.

2. *Circle.* The circle is the set of all points in a plane that are at a given distance from a given point, the centre.

Let consider the circle C with radius r, center at the point of the coordinates $(a_1, a_2, 0)$ and contained in the plane $z = 0$. A parametric representation for C is given by

$$\vec{r}(t) = (a_1 + r\cos\alpha,\ a_2 + r\sin\alpha, 0)\,,\ \alpha \in [0, 2\pi),$$

where α is the angle formed by the point $P = \vec{r}(0)$, the origin of the coordinate O and the generic point $X = \vec{r}(t)$ of the circle.



Another parametric representation for C is:

$$\vec{r}(t) = (a_1 + r\cos 2\alpha,\ a_2 + r\sin 2\alpha, 0)\,,\ \alpha \in [0, \pi),$$

where we go over the circle at double speed.

Remark. As the functions in the previous parametrization of the circle are trigonometric functions the parametrization said to be *trigonometric.*

By solving the cartesian equation of the circle of radio r and center (a_1, a_2), that is, $(x(t) - a_1)^2 + (y(t) - a_2)^2 = r^2$, for x we obtain $y = a_2 + \sqrt{r^2 - (x - a_1)^2}$. Hence, we can consider the following parametrization of the circle,

$$\vec{r}(x) = \left(x,\ a_2 + \sqrt{r^2 - (x - a_1)^2}, 0\right)\,,\ x \in [a_1 - r, a_1 + r].$$

Remark. The previous parametrization is said to be *irrational* as its coordinates functions are irrational. The parametrization is said to be *rational* (resp. *polynomial*) if it depends on rational (resp. polynomial) functions.

Less well-known is the parameterization of the circle by rational functions. For simplicity, let consider the unit circle in $\mathbb{R}^2$ of center the origin of coordinates. The line through the point of coordinates $(0, 1)$ with slope m is given by $y = 1 + mx$. This line intersects the unit circle in one other point P and as we vary m we strike every point on the unit circle. The coordinates (x, y) of P satisfy both $x^2 + y^2 = 1$ and $y = 1 + mx$ so we have

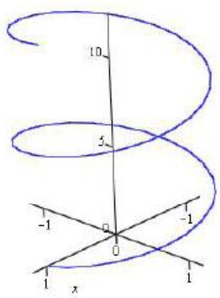
$$\begin{cases} x^2 + (1+mx)^2 = 1 \\ y = 1 + mx \end{cases} \Longrightarrow \begin{cases} x^2\left(1+m^2\right) + 2mx = 0 \\ y = 1 + mx \end{cases}$$
$$\Longrightarrow \begin{cases} x = \frac{-2m}{1+m^2} \text{ or } x = 0 \\ y = m(x+1) \end{cases}$$

Therefore $P = \left(\frac{-2m}{1+m^2}, \frac{1-m^2}{1+m^2}\right)$ or $P = (0, 1)$ and every point on the unit circle is of the form $\left(\frac{-2m}{1+m^2}, \frac{1-m^2}{1+m^2}\right)$ and

$$\vec{r}(m) = \left(\frac{-2m}{1+m^2}, \frac{1-m^2}{1+m^2}\right), \ m \in \mathbb{R},$$

is a rational parametrization of the circle.

3. *Circular helix.* Let consider the curve contained in a circular cylinder, for example the cylinder $x^2 + y^2 = r^2$ such that when x and y get again their initial values, z has increased $2\pi b$. Here's an illustration for the circular helix:



A parametric representation of the circular helix is given by

$$\vec{r}(t) = (r\cos t,\ r\sin t,\ bt)\,,\ t \in [0, 2k\pi),\ \text{with } k \in \mathbb{N}.$$

The parameter t is a measure of the angle that forms the x axis with the straight line that joins the point O with the projection of the generic point $P \in C$ with the plane xy.

2.1.1 Allowable change of parameter

Definition. We say that a function $t = t(s)$, $s \in J \subseteq \mathbb{R}$, is an *allowable change of parameter* if it verifies the following conditions:

1. $t = t(s)$ is a differentiable function of class 3,
2. $t\,'(s) \neq 0$, $\forall s \in J$.

Example The function $t(\theta) = \tan\frac{\theta}{2}$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, is an allowable change of parameter for any regular parametric representation $\vec{r}(t)$, $t \in I \subseteq \mathbb{R}$, of a curve C because the function $t(\theta) = \tan\frac{\theta}{2}$ is a function of class 3 in the interval $J = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and, moreover,

$$\frac{dt}{d\theta}(\theta) = \frac{1}{2\cos^2\frac{\theta}{2}} = \tfrac{1}{2}\left(1 + \tan^2\frac{\theta}{2}\right) \neq 0,\ \forall\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

2.1.2 Regular parametric representation.

Definition. The *tangent vector* of a curve C with parametrization $\vec{r}(t) = (x(t), y(t), z(t))$, $t \in I \subseteq \mathbb{R}$, at a point $P = \vec{r}(t_0)$ is the vector

$$\vec{r}\,'(t) = (x'(t), y'(t), z'(t))\,.$$

Definition. The map $\vec{r}(t) = (x(t), y(t), z(t))$, $t \in I \subseteq \mathbb{R}$ is a *regular parametric representation* of a curve C if the following conditions hold:

1. $\operatorname{Im}\vec{r} = C$,
2. $\vec{r}$ is a differentiable application of class C^3,
3. The tangent vector to the curve at any point $P \in C$ is never zero; that is,
$$\vec{r}\,'(t) = (x'(t), y'(t), z'(t)) \neq \vec{0} \text{ for every } t \in I.$$

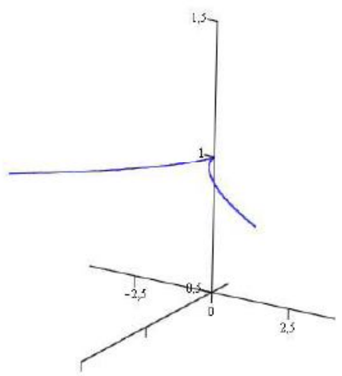
Definition. Let C be a curve with parametric representation $\vec{r}(t) = (x(t), y(t), z(t))$, $t \in I \subseteq \mathbb{R}$.

- A point $P = \vec{r}(t_0) \in C$ is said to be *singular* if $\vec{r}\,'(t_0) = \vec{0}$. If $\vec{r}\,'(t_0) \neq \vec{0}$ otherwise it is said to be *regular*.

- A point $P = \vec{r}(t_0) \in C$ is said to be a *double point* if there are $t_1, t_2 \in I$, $t_1 \neq t_2$, so that $\vec{r}(t_1) = \vec{r}(t_2) = P$.

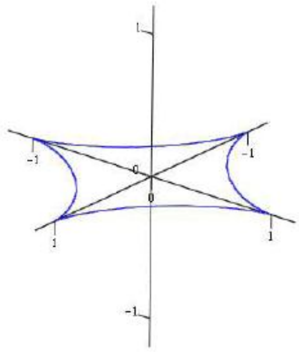
Examples

1. The polynomial parametrization $\vec{r}(t) = (t^2, t^3, 1)$ define a planar curve (contained in the plane $z = 1$). As $\vec{r}$ is a polyonimal parametrization it is differentiable of class C^∞.



We have: $\vec{r}\,'(t) = (2t, 3t^2, 0)$. Being $\vec{r}\,'(0) = \vec{0}$ the point $P = \vec{r}(0) = (0, 0, 1)$ is a singular point of the curve. The other point of the curves are regular points because $\vec{r}\,'(t) \neq \vec{0}$ if $t \neq 0$.

2. The parametrization $\vec{r}(t) = \left(\cos^3 t,\ \sin^3 t,\ 0\right)$, $t \in [0, 2\pi)$, define a planar curve (contained in the plane $z = 0$). As $\vec{r}$ is a trigonometric parametrization it is differentiable of class C^∞.



We have: $\vec{r}\ '(t) = \left(-3\cos^2 t \sin t,\ 3\sin^2 t \cos t,\ 0\right)$. Hence $\vec{r}\ '(t) = (0,0,0) \Leftrightarrow \sin t = 0$ or $\cos t = 0$; that is, if $t = 0, \pi/2, \pi, 3\pi/2$. Thus, the points

$$\begin{array}{ll} \vec{r}(0) = (1,0,0)\,, & \vec{r}(\pi/2) = (0,1,0)\,, \\ \vec{r}(\pi) = (-1,0,0)\,, & \vec{r}(3\pi/2) = (0,-1,0)\,, \end{array}$$

are singular points.

2.2 Implicit representation

A curve C can also be considered as the intersection of two surfaces. Hence, if $F(x,y,z) = 0$ and $G(x,y,z) = 0$ are the respective equations of two surfaces then the coordinates (x,y,z) of a generic point P of the curve C must satisfy both equations: $F(x,y,z) = 0$ and $G(x,y,z) = 0$. The equations $F(x,y,z) = 0$, $G(x,y,z) = 0$ are called *cartesian or implicit equations* of the curve.

Examples

1. *Straight line.* A straight line can be consider as the intersection of two planes. For example the straight line with parametrization $\vec{r}(t) = (a_1 + tv_1,\ a_2 + tv_2,\ a_3 + tv_3)$, $t \in \mathbb{R}$ can be seen as the intersection of the planes π_1, π_2 with equations:

$$\pi_1 \equiv \frac{x - a_1}{v_1} = \frac{y - a_2}{v_2}, \text{ and } \pi_2 \equiv \frac{x - a_1}{v_1} = \frac{z - a_3}{v_3}.$$

2. *Circle.*

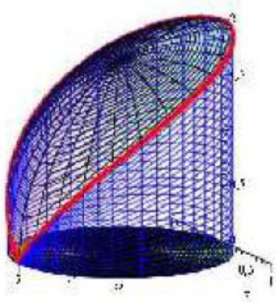
   A circle can be seen as the intersection of a sphere and a plane. For example, the unit cicle in the plane $z = 0$ ca be seen as the intersecction of the sphere of equation $x^2 + y^2 + z^2 = a^2$ and the plane $z = 0$.. It can also be seen as the intersection of the paraboloid of equation $x^2 + y^2 - z = a^2$ with the plane of equation $z = 0$ or as the intersection of the cylinder $x^2 + y^2 = a^2$ and the plane $z = 0$.

3. *Viviani's curve.* This curve is defined as the intersection of the hemisphere with a cylinder whose axis is parallel to a diameter of the sphere.

   For example, let us consider the hemisphere with center the origin of coordinates and radius 2 and the cylinder with axis the straight line $x = 0, y = 0$ and basis the circle with center $Z(1, 0, 0)$ and radius 1. Hence the Viviani's curve is the set of points satisfying

$$\begin{cases} x^2 + y^2 + z^2 = 4, \quad z \geq 0, \text{ (implicit equation of the hemisphere)}, \\ (x-1)^2 + y^2 = 1, \text{ (implicit equation of the cylinder)}. \end{cases}$$

   See the illustration bellow for a Viviani's curve:



   We are going to find a parametric representation of this curve. The circle $(x-1)^2 + y^2 = 1$ in $z = 0$ can be parameterized as follow:

$$x(t) = 1 + \cos t, \; y(t) = \sin t, \; z(t) = 0,$$

with $t \in [0, 2\pi)$. By substituting the values for x and y in the hemisphere equation we obtain:

$$\begin{aligned}
x^2+y^2+z^2=4 &\iff (1+\cos t)^2+(\sin t)^2+z^2=4 \\
&\iff 1+2\cos t+\cos^2 t+\sin^2 t+z^2=4 \\
&\iff z^2=2-2\cos t \\
&\iff z^2=2-2\left(\cos^2\tfrac{t}{2}-\sin^2\tfrac{t}{2}\right) \\
&\iff z^2=2\left(\cos^2\tfrac{t}{2}+\sin^2\tfrac{t}{2}\right)-2\cos^2\tfrac{t}{2}+2\sin^2\tfrac{t}{2} \\
&\iff z^2=4\sin^2\tfrac{t}{2} \\
&\iff z=2\sin\tfrac{t}{2} \text{ as } z\geq 0.
\end{aligned}$$

Note that for values of $t \in [0, 2\pi)$, $\sin \frac{t}{2}$ is always positive or null. Thus, one parametrization of the Viviani's curve is:

$$\vec{r}(t)=\left(1+\cos t,\ \sin t,\ 2\sin\tfrac{t}{2}\right), t \in [0, 2\pi).$$

3 The length of a curve

The first and simplest geometrical quantity associated with a curve is its length.

Definition. Let C be a curve with regular parametric representation $\vec{r}(t) = (x(t), y(t), z(t))$, $t \in I$. Then the *length* of C over the interval $I = (a, b)$ is given by

$$\text{length}[\vec{r}] = \int_a^b ||\vec{r}\,'(u)||du.$$

Remark. Let $\vec{\alpha}$ be a reparametrization of $\vec{r}$. Then

$$\text{length}[\vec{r}] = \text{length}[\vec{\alpha}].$$

That is the lenght of ac urve does not depend on the parametrization used to compute it.

Definition. Fix a number $a < c < b$. The *arc length function* s of a curve C with regular parametric representation $\vec{r}$: $(a, b) \to \mathbb{R}^3$ starting at c is defined by

$$s(t) = \int_c^t ||\vec{r}\,'(u)||du, \text{ where } t \in (a, b).$$

Remark. Note that

$$s\,'(t) = ||\vec{r}\,'(t)|| \neq 0, \ \forall t \in I,$$

as $\vec{r}(t)$ is a regular parametric representation. Therefore the arc length is an allowable change of parameter. As $s\,'(t) \neq 0$, $\forall t \in I$,the Inverse Function Theorem implies that $t \longmapsto s(t)$ has an inverse $s \longmapsto t(s)$ and that

$$t\,'(s) = \tfrac{1}{s\,'(t)} = \tfrac{1}{||\vec{r}\,'(t)||}.$$

Now define $\vec{\alpha}$ by $\vec{\alpha}(s) = \vec{r}(t(s))$. We have

$$\vec{\alpha}\,'(s) = \vec{r}\,'(t(s))t'(s) = \tfrac{\vec{r}\,'(t(s))}{||\vec{r}\,'(t(s))||},$$

and therefore

$$||\vec{\alpha}\,'(s)|| = \left\| \tfrac{\vec{r}\,'(t(s))}{||\vec{r}\,'(t(s))||} \right\| = \tfrac{||\vec{r}\,'(t(s))||}{||\vec{r}\,'(t(s))||} = 1.$$

Hence the unit-speed curves are said to be *parametrized by arc length* or they have a *natural parametric representation*.

Examples

1. Let C be the curve with parametric representation:

$$\vec{r}(t) = \left(\tfrac{\sqrt{2}}{2}\sin(t),\ \tfrac{\sqrt{2}}{2}t,\ \tfrac{\sqrt{2}}{2}\cos(t)\right),\ \forall\ t \in [0, 2\pi).$$

we have,

$$\vec{r}\,'(t) = \left(\tfrac{\sqrt{2}}{2}\cos(t),\ \tfrac{\sqrt{2}}{2},\ -\tfrac{\sqrt{2}}{2}\sin(t)\right),\ \forall\ t \in [0, 2\pi),$$

and

$$\begin{aligned} ||\vec{r}\,'(t)|| &= \sqrt{\left(\tfrac{\sqrt{2}}{2}\cos(t)\right)^2 + \left(\tfrac{\sqrt{2}}{2}\right)^2 + \left(-\tfrac{\sqrt{2}}{2}\sin(t)\right)^2} \\ &= \sqrt{\tfrac{1}{2}\cos^2(t) + \tfrac{1}{2} + \tfrac{1}{2}\sin^2(t)} \\ &= 1. \end{aligned}$$

Therefore t is the arc parameter and $\vec{r}$ is the natural or arc-length representation.

2. Let C be the curve with parametric representation:

$$\vec{r}(t) = (a\cos(t),\ a\sin(t),\ bt),\quad \forall t \in [0, +\infty),$$

with $a^2 + b^2 \neq 0$. We have:

$$\vec{r}\,'(t) = (-a\sin(t),\ a\cos(t),\ b),\quad \forall t \in [0, +\infty),$$

and

$$\begin{aligned} ||\vec{r}\,'(t)|| &= \sqrt{a^2\sin^2(t) + a^2\cos^2(t) + b^2} \\ &= \sqrt{a^2 + b^2}, \end{aligned}$$

thus, the arc length is given by the function:

$$s(t) = \int_0^t ||\vec{r}\,'(u)||du = \int_0^t \sqrt{a^2 + b^2}du = \sqrt{a^2 + b^2}t,$$

that is a linear application. We have,

$$s\,'(t) = \sqrt{a^2 + b^2} \neq 0,$$

that is an allowable change of parameter. We have:

$$t(s) = \frac{s}{\sqrt{a^2+b^2}},$$

and the natural parametric representation of C is the following:

$$\vec{r}(t(s)) = \left(a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right),\ a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right),\ b\frac{s}{\sqrt{a^2+b^2}}\right),$$

where the arc parameter s takes values in the interval $[0, +\infty)$.

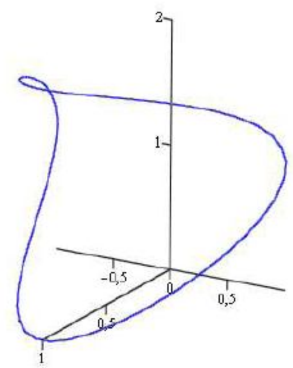
3. Let C be the curve intersection of the following surfaces:

$$\begin{aligned} S_1 &\equiv y^2 + (z-1)^2 = 1, \\ S_2 &\equiv x + y^2 = 1. \end{aligned}$$

A parametric representation of C is:

$$\vec{r}(t) = (1 - \sin^2(t), \sin(t), 1 + \cos(t)),\ \text{with } t \in [0, 2\pi).$$

In the picture below you can see the plot of the curve:



We have

$$\vec{r}\,'(t) = (-2\sin(t)\cos(t), \cos(t), -\sin(t)),\ \forall\ t \in [0, 2\pi),$$

and

$$
\begin{aligned}
||\vec{r}\,'(t)|| &= (-2\sin(t)\cos(t))^2 + \cos^2(t) + \sin^2(t) \\
&= 4\sin^2(t)\cos^2(t) + 1 \\
&= \sin^2(2t) + 1.
\end{aligned}
$$

As $||\vec{r}\,'(t)|| \neq 0$ for every t the parametrization $\vec{r}$ is regular but it is not the natural parametrization as $||\vec{r}\,'(t)|| \neq 1$, for some $t \in [0, 2\pi)$. The arc parameter function is given by the following expression:

$$s(t) = \int_0^t \sqrt{\sin^2(2u) + 1}du.$$

The inverse function of $s(t)$, which is needed to find the unit-speed parametrization, is too complicated to be of much use.

4. Let C be a curve with parametric representation:

$$\vec{r}(t) = \left((t+\pi)\cos(t),\ -(t+\pi)\sin(t),\ \tfrac{t^2}{2} + \pi t + \tfrac{\pi^2}{2}\right),\ t \in [-2\pi, 2\pi].$$

Hence

$$\vec{r}\,'(t) = (\cos(t) - (t+\pi)\sin(t),\ -\sin(t) - (t+\pi)\cos(t),\ 2t+\pi),$$

and $||\vec{r}\,'(t)||^2 = 1 + 5t^2 + 6\pi t + 2\pi^2$. Thus, $||\vec{r}\,'(t)|| = 0$ if and only if

$$
\begin{aligned}
1 + 5t^2 + 6\pi t + 2\pi^2 &= 0 \Longleftrightarrow t = \tfrac{-6\pi \pm \sqrt{36\pi^2 - 20(2\pi^2+1)}}{10} \notin \mathbb{R} \\
\text{as } 36\pi^2 - 20(2\pi^2+1) &= -4\pi^2 - 20 < 0.
\end{aligned}
$$

Hence the parametrization is regular but it is not the natural parametization as $||\vec{r}\,'(t)|| \neq 1$ for some $t \in [-2\pi, 2\pi]$. The arc parameter function is given by the following expression:

$$s(t) = \int_{-2\pi}^t ||\vec{r}\,'(u)||du = \int_{-2\pi}^t \sqrt{1 + 5u^2 + 6\pi u + 2\pi^2}du.$$

The inverse function of $s(t)$, which is needed to find the unit-speed parametrization, is an elliptic function that is too complicated to be of much use.

4 Local study of unit-speed curves

We first define the curvature and torsion of a *unit-speed* curve in $\mathbb{R}^3$, as the defienition are more straightforward in this case. The case of arbitrary-speed curves in $\mathbb{R}^3$ will be considered in the next section.

Therefore during this section $\vec{r}\colon I \to \mathbb{R}^3$ will be a natural (or unit-speed) parametrization of a curve $C \subset \mathbb{R}^3$.

4.1 Curvature of unit-speed curves

Definition. The function $\kappa\colon I \to \mathbb{R}$ defined as

$$\kappa(s) = ||\vec{r}\,''(s)||,$$

is called the *curvature* of $\vec{r}$.

Intuitively, curvature measures the failure of a curve to be a straight line. In fact, a straight line is characterized by the fact that its curvature vanishes at every value of the parameter s.

Geometrical interpretation of the curvature. The curvature measures the variation of the angle between the respective tangent lines of neighboring points of the curve. Let $\theta(s)$ the angle between the tangent line to C at the point $\vec{r}(s_0)$ and the the tangent line to C at the point $\vec{r}(s)$. We have:

$$\lim_{s\to s_0} \frac{\theta(s)}{|s-s_0|} = \lim_{s\to s_0} \frac{\theta(s)}{|s-s_0|} \frac{\sin\frac{\theta(s)}{2}}{\frac{\theta(s)}{2}} = \lim_{s\to s_0} \frac{2\sin\frac{\theta(s)}{2}}{|s-s_0|},$$

and taking into account the equality

$$||\vec{r}\,'(s) - \vec{r}\,'(s_0)|| = 2\sin\tfrac{\theta(s)}{2},$$

we have:

$$\lim_{s\to s_0} \frac{2\sin\frac{\theta(s)}{2}}{|s-s_0|} = \lim_{s\to s_0} \frac{||\vec{r}\,'(s)-\vec{r}\,'(s_0)||}{|s-s_0|} = ||\vec{r}\,''(s_0)||.$$

Definition. The vector field $\vec{t}(s) = \vec{r}\,'(s)$ is called the *unit tangent vector field* of the curve C with natural parametrization $\vec{r}$. Note that the unit tangent vector field satisfies $||\vec{t}(s)|| = 1$

Remark. Let $\vec{t}(s)$ be the unit tangent vector field of a curve C, then $0 = \vec{t}(s) \cdot \vec{t}\,'(s)$ as differentiating the equation $\vec{t}(s) \cdot \vec{t}(s) = 1$ we obtain $0 = (\vec{t}(s) \cdot \vec{t}(s))' = 2\vec{t}\,'(s) \cdot \vec{t}(s)$.

We now restrict ourselves to unit-speed curves whose curvature is strictly positive (that is, $\kappa(s) > 0$). Hence, the vector field $\vec{t}\,'(s) = \vec{r}\,''(s)$ never vanishes. Let us define the *principal normal vector field* and the *binormal vector field*.

Definition. The *principal normal vector field* is the unitary vector field $\vec{n}(s)$ is the unitary vector field in the direcction of the curvature vector field; that is,

$$\vec{n}(s) = \frac{\vec{t}\,'(s)}{\kappa(s)}. \tag{1}$$

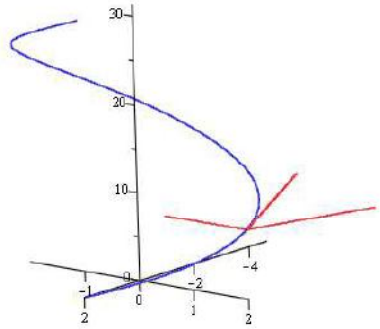
As $\vec{t}\,'(s) = \vec{r}\,''(s)$ and $\kappa(s) = ||\vec{r}\,''(s)||$. The *binormal vector field* $\vec{b}(s)$ is unitary vector field orthogonal to the tangent vector and to the principal normal vector; thas is,

$$\vec{b}(s) = \vec{t}(s) \wedge \vec{n}(s). \tag{2}$$

4.2 Frenet frame field

Definition. The triple $(\vec{t}(s), \vec{n}(s), \vec{b}(s))$ is called the *Frenet frame field* of the natural parametrization $\vec{r}(s)$ of a curve $C \in \mathbb{R}^3$. The Frenet frame field gives us an orthogonal *moving frame* along the curve.

The Frenet frame at a given point of a space curve is illustrated in the figure bellow



4.3 Torsion of unit-speed curves

In order to measure the separation of the curve from the plane containing the point $P = \vec{r}(s)$ and with characteristic vector $\vec{b}(s)$ we study the variation of the binormal vector field.

By differentiating the binormal vector field $\vec{b}(s) = \vec{t}(s) \wedge \vec{n}(s)$ we have:

$$\vec{b}'(s) = \vec{t}\,'(s) \wedge \vec{n}(s) + \vec{t}(s) \wedge \vec{n}'(s).$$

Let us compute $\vec{n}'(s)$. By differentiating the identity $\vec{n}(s) \cdot \vec{n}(s) = 1$, we obtain $\vec{n}(s) \cdot \vec{n}'(s) = 0$ and we can conclude that $\vec{n}'(s)$ is orthogonal to $\vec{n}(s)$. Therefore $\vec{n}'(s)$ is a linear combination of $\vec{t}(s)$ and $\vec{b}(s)$; that is, there exist functions $\mu(s)$, $\tau(s)$ such that

$$\vec{n}'(s) = \mu(s)\vec{t}(s) + \tau(s)\vec{b}(s). \tag{3}$$

By differentiating the identity $\vec{n}(s) \cdot \vec{t}(s) = 0$ we have:

$$\begin{aligned} 0 &= \left(\vec{n}(s) \cdot \vec{t}(s)\right)' \\ &= \vec{n}'(s) \cdot \vec{t}(s) + \vec{n}(s) \cdot \vec{t}\,'(s) \\ &= \Big(\mu(s)\vec{t}(s) + \tau(s)\vec{b}(s)\Big) \cdot \vec{t}(s) + \vec{n}(s) \cdot \kappa(s)\vec{n}(s) \\ &= \mu(s) + \kappa(s). \end{aligned}$$

Therefore $\mu(s) = -\kappa(s)$ and hence $\vec{n}'(s) = -\kappa(s)\vec{t}(s) + \tau(s)\vec{b}(s)$. By substituting $\vec{n}'(s)$ and $\vec{t}\,'(s) = \kappa(s)\vec{n}(s)$ into the expresion of $\vec{b}'(s)$ we have:

$$\begin{aligned} \vec{b}'(s) &= \vec{t}\,'(s) \wedge \vec{n}(s) + \vec{t}(s) \wedge \vec{n}'(s) \\ &= \kappa(s)\vec{n}(s) \wedge \vec{n}(s) + \vec{t}(s) \wedge \Big(\mu(s)\vec{t}(s) + \tau(s)\vec{b}(s)\Big) \\ &= \tau(s)\vec{t}(s) \wedge \vec{b}(s) \\ &= -\tau(s)\vec{n}(s). \end{aligned}$$

Finally, taking into account $\vec{n}(s) \cdot \vec{n}(s) = 1$ we obtain

$$\vec{b}'(s) \cdot \vec{n}(s) = -\tau(s).$$

Definition. The function defined as follows

$$\tau(s) = -\vec{b}'(s) \cdot \vec{n}(s).$$

is called *torsion or second curvature* of the curve $\vec{r}(s)$ the function $\tau(s)$ and it measures the variation of the binormal vector field.

4.4 Frenet-Serret formulas

Let $\vec{r}\colon I \to \mathbb{R}^3$ be a natural (or unit-speed) parametrization of a curve $C \subset \mathbb{R}^3$ with curvature $\kappa(s)$ and torsion $\tau(s)$. The vectors of the Frenet frame $\left(\vec{t}(s), \vec{n}(s), \vec{b}(s)\right)$ at any point $P = \vec{r}(s)$ of the curve, satisfy the following equations:

$$\begin{cases} \vec{t}\,'(s) = & & \kappa(s)\vec{n}(s), & \\ \vec{n}'(s) = & -\kappa(s)\vec{t}(s) & & +\tau(s)\vec{b}(s), \\ \vec{b}'(s) = & & -\tau(s)\vec{n}(s), & \end{cases}$$

called the *Frenet-Serret formulas.* The Frenet-Serret formulas are a system of ordinary differential equations for the vectors $\vec{t}(s)$, $\vec{n}(s)$ and $\vec{b}(s)$. Taking into account the theorem of existence and uniqueness of solutions of initial value problem for a system of ordinary differential equations we have the following result:
Theorem. Given two functions $\kappa, \tau\colon I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ of class C^1 with $\kappa(s) > 0$, $\forall s \in I$, there exists a unique curve $\vec{r}(s)$ up to transformations by the Euclidean group, such that $\kappa(s)$ is the curvature function of $\vec{r}(s)$ and $\tau(s)$ is its torsion function of $\vec{r}(s)$.

The above result tells us that the curvature and torsion functions; that is, $\kappa(s)$ and $\tau(s)$, determine the curve up to position at the space. For this reason, the equations $\kappa = \kappa(s)$ and $\tau = \tau(s)$ are called *intrinsic equations of the curve.*

4.4.1 Examples

1. Let C be a curve with parametric representation $\vec{r}(s)$, $s \in I$. Let check by using the Frenet-Serret formulas that the following conditions are equivalent:

   (a) The curve $\vec{r}(s)$ is a plane curve;

   (b) $\tau(s) = 0$ for every $s \in I$.

   Note that when both conditions hold, the binormal vector field $\vec{b}(s)$ is perpendicular tio the plane containing the curve $\vec{r}(s)$.

   The condition that a curve $\vec{r}(s)$ lie in a plane Π can be expressed analytically as

   $$(\vec{r}(s) - P) \cdot \vec{q} = 0$$

where P is a point in the plane and $\vec{q}$ is an unitary vector orthogonal to Π. By differentiating the above equation we have:

$$\vec{r}\,'(s) \cdot \vec{q} = 0 \text{ and } \vec{r}\,''(s) \cdot \vec{q} = 0.$$

Thus both $\vec{t}(s)$ and $\vec{n}(s)$ are orthogonal to $\vec{q}$. Since $\vec{b}(s)$ is also orthogonal to $\vec{t}(s)$ and $\vec{n}(s)$, it follows that

$$\vec{b}(s) = \vec{q}.$$

Therefore $\vec{b}'(s) = 0$ and from the third Frenet-Serret formula we deduce $\tau(s) = 0$.

Conversely, suppose $\tau(s) = 0$, for every $s \in I$. Then the third Frenet-Serret formula $\vec{b}'(s) = -\tau(s)\vec{n}(s)$ implies $\vec{b}'(s) = 0$ for every $s \in I$, therefore $\vec{b}(s) = \vec{b}$ is a constant vector field. Let consider $s_0 \in I$ and the function

$$f(s) = (\vec{r}(s) - \vec{r}(s_0)) \cdot \vec{b}.$$

As $f(s_0) = 0$ and $f'(s) = \vec{r}'(s) \cdot \vec{b} = 0$, we have $f \equiv 0$; that is, $(\vec{r}(s) - \vec{r}(s_0)) \cdot \vec{b} = 0$ and therefore the curve is contained in the plane orthogonal to $\vec{b}$ containing the point $\vec{r}(s_0)$.

2. Let C be a *circular helix* with parametrization

$$\vec{r}(t) = (a\cos(t),\ a\sin(t),\ bt), \quad t \in [0, +\infty),$$

where the radio $a > 0$ and b is the *incline* of the helix.

As

$$\vec{r}\,'(t) = (-a\sin(t),\ a\cos(t),\ b),$$

then $||\vec{r}\,'(t)|| = \sqrt{a^2 + b^2}$. The helix is one of the few curves for which a natural parametrization is easy to find. A natural parametrization of the circular helix is

$$\vec{\alpha}(s) = \left(a\cos\tfrac{s}{\sqrt{a^2+b^2}},\ a\sin\tfrac{s}{\sqrt{a^2+b^2}},\ b\tfrac{s}{\sqrt{a^2+b^2}}\right), \quad t \in [0, +\infty).$$

Let us compute the tangent, normal and binormal vector fields and the curvature and torsion functions of the helix. We have:

$$\begin{aligned} \vec{t}(s) &= \vec{\alpha}'(s) = \left(-\tfrac{a}{\sqrt{a^2+b^2}}\sin\tfrac{s}{\sqrt{a^2+b^2}},\ \tfrac{a}{\sqrt{a^2+b^2}}\cos\tfrac{s}{\sqrt{a^2+b^2}},\ \tfrac{b}{\sqrt{a^2+b^2}}\right) \\ &= \tfrac{1}{\sqrt{a^2+b^2}}\left(-a\sin\tfrac{s}{\sqrt{a^2+b^2}},\ a\cos\tfrac{s}{\sqrt{a^2+b^2}},\ b\right), \end{aligned}$$

and

$$\vec{\alpha}''(s) = \tfrac{1}{a^2+b^2}\left(-a\cos\tfrac{s}{\sqrt{a^2+b^2}},\ -a\sin\tfrac{s}{\sqrt{a^2+b^2}},\ 0\right).$$

As $a > 0$, we have

$$\kappa(s) = ||\vec{\alpha}''(s)|| = \tfrac{a}{a^2+b^2},$$

and

$$\begin{aligned}\vec{n}(s) &= \frac{\vec{\alpha}''(s)}{||\vec{\alpha}''(s)||} \\ &= \left(-\cos\tfrac{s}{\sqrt{a^2+b^2}},\ -\sin\tfrac{s}{\sqrt{a^2+b^2}},\ 0\right).\end{aligned}$$

By the formula $\vec{b}(s) = \vec{t}(s) \wedge \vec{n}(s)$, we obtain

$$\begin{aligned}\vec{b}(s) &= \begin{vmatrix} i & j & k \\ -\tfrac{a}{\sqrt{a^2+b^2}}\sin\tfrac{s}{\sqrt{a^2+b^2}} & \tfrac{a}{\sqrt{a^2+b^2}}\cos\tfrac{s}{\sqrt{a^2+b^2}} & \tfrac{b}{\sqrt{a^2+b^2}} \\ -\cos\tfrac{s}{\sqrt{a^2+b^2}} & -\sin\tfrac{s}{\sqrt{a^2+b^2}} & 0 \end{vmatrix} \\ &= \tfrac{1}{\sqrt{a^2+b^2}}\begin{vmatrix} i & j & k \\ -a\sin\tfrac{s}{\sqrt{a^2+b^2}} & a\cos\tfrac{s}{\sqrt{a^2+b^2}} & b \\ -\cos\tfrac{s}{\sqrt{a^2+b^2}} & -\sin\tfrac{s}{\sqrt{a^2+b^2}} & 0 \end{vmatrix} \\ &= \tfrac{1}{\sqrt{a^2+b^2}}\left(b\sin\tfrac{s}{\sqrt{a^2+b^2}},\ -b\cos\tfrac{s}{\sqrt{a^2+b^2}},\ a\right).\end{aligned}$$

Finally, as

$$\vec{b}\,'(s) = \tfrac{b}{a^2+b^2}\left(\cos\tfrac{s}{\sqrt{a^2+b^2}},\ \sin\tfrac{s}{\sqrt{a^2+b^2}},\ 0\right)$$

by comparing the expression for $\vec{b}\,'(s)$ with the expression for $\vec{n}(s)$ and by using the third Frenet-Serret formula: $\vec{b}'(s) = -\tau(s)\vec{n}(s)$ we deduce:

$$\tau(s) = \tfrac{b}{a^2+b^2}.$$

Remark. Both the curvature function and the torsion function of a helix are constant.

4.4.2 Computaion of the torsion function.

The torsion function of a curve with natural parametrization $\vec{r}(s)$ can be computed as follows:

$$\tau(s) = \frac{[\vec{r}\,'(s), \vec{r}\,''(s), \vec{r}\,'''(s)]}{||\vec{r}\,''(s)||^2}.$$

From the third Frenet-Serret formula we have: $\tau(s) = -\vec{b}'(s) \cdot \vec{n}(s)$.

By substituting

$$\begin{aligned} \vec{b}'(s) &= \left(\vec{t}(s) \wedge \vec{n}(s)\right)' \\ &= \vec{t}\,'(s) \wedge \vec{n}(s) + \vec{t}(s) \wedge \vec{n}'(s) \end{aligned}$$

into the above equations for the torsion and taking into account the expressions for $\vec{t}(s)$ and $\vec{n}(s)$ we obtain

$$\begin{aligned} \tau(s) &= -\vec{b}'(s) \cdot \vec{n}(s) \\ &= -\left(\vec{t}\,'(s) \wedge \vec{n}(s)\right) \cdot \vec{n}(s) - \left(\vec{t}(s) \wedge \vec{n}'(s)\right) \cdot \vec{n}(s) \\ &= -\left(\vec{t}(s) \wedge \vec{n}'(s)\right) \cdot \vec{n}(s) \\ &= -\left(\vec{r}\,'(s) \wedge \left(\tfrac{\vec{r}\,''(s)}{\kappa(s)}\right)'\right) \cdot \tfrac{\vec{r}\,''(s)}{\kappa(s)} \\ &= -\tfrac{1}{\kappa(s)^2}\left(\vec{r}\,'(s) \wedge \vec{r}\,'''(s)\right) \cdot \vec{r}\,''(s) \\ &= \tfrac{[\vec{r}\,'(s), \vec{r}\,''(s), \vec{r}\,'''(s)]}{||\vec{r}\,''(s)||^2}, \end{aligned}$$

where $[\vec{r}\,'(s), \vec{r}\,''(s), \vec{r}\,'''(s)]$ denotes the mixed product of the vectors fields $\vec{r}\,'(s)$, $\vec{r}\,''(s)$, $\vec{r}\,'''(s)$.

5 Local study of arbitrary-speed curves

For efficient computations of the curvature and torsion functions of an arbitrary-speed curve, we need formulas that avoid finding a natural parametrization explicitly.

Theorem. Let C be a curve with arbitrary regular parametric representation $\vec{r}\colon I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^3$ and nonzero curvature. Then

$$
\begin{aligned}
\vec{t}(t) &= \tfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|},\\
\vec{b}(t) &= \tfrac{\vec{r}\,'(t)\wedge\vec{r}\,''(t)}{\|\vec{r}\,'(t)\wedge\vec{r}\,''(t)\|},\\
\vec{n}(t) &= \vec{b}(t)\wedge\vec{t}(t),\\
\kappa(t) &= \tfrac{\|\vec{r}\,'(t)\wedge\vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3},\\
\tau(t) &= \tfrac{[\vec{r}\,'(t),\vec{r}\,''(t),\vec{r}\,'''(t)]}{\|\vec{r}\,'(t)\wedge\vec{r}\,''(t)\|^2}.
\end{aligned}
$$

Proof. Let $\vec{\alpha}$ be the natural parametrization of C. We have: $\vec{r}(t) = \vec{\alpha}(s(t))$ and therefore

$$
\begin{aligned}
\vec{r}\,'(t) &= \vec{\alpha}'(s(t))s'(t)\\
&= \vec{\alpha}'(s(t))||\vec{r}\,'(t)||\\
&= \vec{t}(t)||\vec{r}\,'(t)||,\\
\vec{r}\,''(t) &= \vec{\alpha}''(s(t))s'(t)^2 + \vec{\alpha}'(s(t))s''(t),\\
\vec{r}\,'(t)\wedge\vec{r}\,''(t) &= \vec{\alpha}'(s(t))s'(t)\wedge\big(\vec{\alpha}''(s(t))s'(t)^2 + \vec{\alpha}'(s(t))s''(t)\big)\\
&= s'(t)^3\vec{\alpha}'(s(t))\wedge\vec{\alpha}''(s(t))\\
&= s'(t)^3\kappa(s(t))\vec{t}(t)\wedge\vec{n}(t)\\
&= ||\vec{r}\,'(t)||^3\kappa(s(t))\vec{b}(t),\\
||\vec{r}\,'(t)\wedge\vec{r}\,''(t)|| &= ||\vec{r}\,'(t)||^3\kappa(s(t)),
\end{aligned}
$$

therefore we obtain the expressions for $\vec{t}(t)$, $\vec{b}(t)$ and $\kappa(s(t))$. Moreover, taking

$$\vec{r}\,'''(t) = \vec{\alpha}'''(s(t))s'(t)^3 + 3\vec{\alpha}''(s(t))s'(t)s''(t) + \vec{\alpha}'(s(t))s'''(t),$$

and the expression of the vector $\vec{r}\,'(t) \wedge \vec{r}\,''(t)$ into account, we have:

$$\begin{aligned}
& (\vec{r}\,'(t) \wedge \vec{r}\,''(t)) \cdot \vec{r}\,'''(t) \\
= & \left(s'(t)^3 \vec{\alpha}'(s(t)) \wedge \vec{\alpha}''(s(t))\right) \cdot \left(\vec{\alpha}'''(s(t))s'(t)^3 + 3\vec{\alpha}''(s(t))s'(t)s''(t) + \vec{\alpha}'(s(t))s'''(t)\right) \\
= & s'(t)^6 \left(\vec{\alpha}'(s(t)) \wedge \vec{\alpha}''(s(t))\right) \cdot \vec{\alpha}'''(s(t)) \\
= & ||\vec{r}\,'(t)||^6 \left[\vec{\alpha}'(s(t)), \vec{\alpha}''(s(t)), \vec{\alpha}'''(s(t))\right] \\
= & ||\vec{r}\,'(t)||^6 \tau(s(t)) \kappa(s(t))^2 \\
= & ||\vec{r}\,'(t)||^6 \tau(s(t)) \tfrac{||\vec{r}\,'(t) \wedge \vec{r}\,''(t)||^2}{||\vec{r}\,'(t)||^6} \\
= & \tau(s(t)) ||\vec{r}\,'(t) \wedge \vec{r}\,''(t)||^2,
\end{aligned}$$

and therefore

$$\tau(t) = \tfrac{[\vec{r}\,'(t), \vec{r}\,''(t), \vec{r}\,'''(t)]}{||\vec{r}\,'(t) \wedge \vec{r}\,''(t)||^2}.$$

5.0.3 Examples

1. Let C be a curve with parametric representation:

$$\vec{r}(t) = (t,\ -t^2,\ 1 + t^3), \quad t \in [0, +\infty).$$

   Let us find the elements of the Frenet frame, the curvature and the torsion at a generic point of the curve and at the point P of coordinates $(0, 0, 1)$. We have:

$$\vec{r}\,'(t) = (1,\ -2t,\ 3t^2), \quad t \in [0, +\infty).$$

   As $||\vec{r}\,'(t)|| = \sqrt{1 + 4t^2 + 9t^4} \neq 1$, the parameter t is not the arc parameter. Taking the derivative of the parametric representation we obatin

$$\begin{aligned}
\vec{r}\,''(t) &= (0,\ -2,\ 6t), \\
\vec{r}\,'''(t) &= (0,\ 0,\ 6) \\
\vec{r}\,'(t) \wedge \vec{r}\,''(t) &= (-6t^2,\ -6t,\ -2), \\
||\vec{r}\,'(t) \wedge \vec{r}\,''(t)|| &= \sqrt{36t^4 + 36t^2 + 4} \\
&= 2\sqrt{9t^4 + 9t^2 + 1}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\vec{t}(t) &= \tfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \tfrac{1}{\sqrt{1+4t^2+9t^4}}(1,\ -2t,\ 3t^2),\\
\vec{b}(t) &= \tfrac{\vec{r}\,'(t)\wedge\vec{r}\,''(t)}{\|\vec{r}\,'(t)\wedge\vec{r}\,''(t)\|} = \tfrac{1}{2\sqrt{1+9t^2+9t^4}}(-6t^2,\ -6t,\ -2),\\
\vec{n}(t) &= \vec{b}(t)\wedge\vec{t}(t) = \tfrac{1}{2\sqrt{1+9t^2+9t^4}}\tfrac{1}{\sqrt{1+4t^2+9t^4}}\begin{vmatrix} i & j & k\\ 1 & -2t & 3t^2\\ -6t^2 & -6t & -2\end{vmatrix}\\
&= \tfrac{1}{\sqrt{1+9t^2+9t^4}}\tfrac{1}{\sqrt{1+4t^2+9t^4}}(2t+9t^3,\ 1-9t^4,\ -3t-6t^3).
\end{aligned}$$

The curvature and the torsion are given by

$$\begin{aligned}
\kappa(t) &= \tfrac{\|\vec{r}\,'(t)\wedge\vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3} = \tfrac{2\sqrt{9t^4+9t^2+1}}{(1+4t^2+9t^4)^{3/2}},\\
\tau(t) &= \tfrac{[\vec{r}\,'(t),\vec{r}\,''(t),\vec{r}\,'''(t)]}{\|\vec{r}\,'(t)\wedge\vec{r}\,''(t)\|^2} = \tfrac{1}{36t^4+36t^2+4}\begin{vmatrix} 1 & -2t & 3t^2\\ 0 & -2 & 6t\\ 0 & 0 & 6\end{vmatrix}\\
&= \tfrac{-12}{36t^4+36t^2+4} = \tfrac{-3}{9t^4+9t^2+1}.
\end{aligned}$$

As $P = \vec{r}(0)$, the tangent vector, the binormal vector and the normal vector at P are:

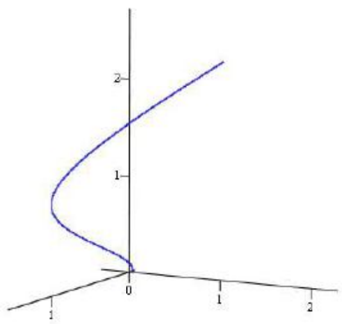
$$\begin{aligned}
\vec{t}(0) &= (1,0,0)\,,\\
\vec{b}(0) &= (0,0,-1)\,,\\
\vec{n}(0) &= \vec{b}(0)\wedge\vec{t}(0) = (0,-1,0)\,,
\end{aligned}$$

and $\kappa(0) = 2$, $\tau(0) = -3$.

2. Let C be a curve with parametric representation:

$$\vec{r}(t) = (e^t\cos(t),\ e^t\sin(t),\ e^t),\quad t\in[0,+\infty).$$

Let us find the elements of the Frenet frame, the curvature and the torsion at a generic point of the curve. In the figure bellow we have the plot of the curve:



By differentiating $\vec{r}(t)$ we obtain:

$$\begin{aligned}
\vec{r}\,'(t) &= \left(e^t\left(\cos(t)-\sin(t)\right),\ e^t\left(\sin(t)+\cos(t)\right),\ e^t\right),\\
\vec{r}\,''(t) &= \left(-2e^t\sin(t),\ 2e^t\cos(t),\ e^t\right),\\
\vec{r}\,'''(t) &= \left(-2e^t\left(\sin(t)+\cos(t)\right),\ 2e^t\left(\cos(t)-\sin(t)\right),\ e^t\right),\\
\vec{r}\,'(t)\wedge\vec{r}\,''(t) &= e^{2t}\left(\sin(t)-\cos(t),\ -\cos(t)-\sin(t),\ 2\right),\\
[\vec{r}\,'(t),\vec{r}\,''(t),\vec{r}\,'''(t)] &= 2e^{3t}.
\end{aligned}$$

Thus,

$$\kappa(t)=\frac{\|\vec{r}\,'(t)\wedge\vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}=\frac{e^{2t}\sqrt{6}}{e^{3t}(3)^{3/2}}=\frac{\sqrt{2}}{3e^t},$$

and

$$\tau(t)=\frac{[\vec{r}\,'(t),\vec{r}\,''(t),\vec{r}\,'''(t)]}{\|\vec{r}\,'(t)\wedge\vec{r}\,''(t)\|^2}=\frac{2e^{3t}}{6e^{4t}}=\frac{1}{3e^t}.$$

From

$$s=\int_0^t||\vec{r}\,'(u)||du=\int_0^t e^u\sqrt{3}du=\sqrt{3}\left(e^t-1\right),$$

we obtain

$$e^t=\tfrac{s}{\sqrt{3}}+1,$$

and therefore the intrinsic equations are

$$\kappa(s)=\tfrac{\sqrt{6}}{3\left(s+\sqrt{3}\right)},\quad \tau(s)=\tfrac{\sqrt{3}}{3\left(s+\sqrt{3}\right)}.$$

6 Lines and planes determined by the Frenet frame field

Review. The parametric equation of a line containing a point P of coordinates (a, b, c) and with director vector $\vec{v}$ of coordinates (v_1, v_2, v_3) is the following:

$$\vec{r}(\lambda) = \overrightarrow{OP} + \lambda\vec{v} \Longleftrightarrow \begin{cases} x(\lambda) = a + \lambda v_1 \\ y(\lambda) = b + \lambda v_2, \\ z(\lambda) = c + \lambda v_3, \end{cases} \quad \text{where } \lambda \in \mathbb{R}.$$

The cartesian equation of a plane Π containing a point P of coordinates (a, b, c) and with characteristic vector $\vec{v}$ of coordinates (v_1, v_2, v_3) is the following:

$$\begin{aligned} X \in \Pi &\Longleftrightarrow \overrightarrow{PX} \cdot \vec{v} = 0 \\ &\Longleftrightarrow (x - a, y - b, z - c) \cdot (v_1, v_2, v_3) = 0 \\ &\Longleftrightarrow (x - a)v_1 + (y - b)v_2 + (z - c)v_3 = 0. \end{aligned}$$

The parametric equation of a plane Π containing a point P of coordinates (a, b, c) and generated by the vector $\vec{u} = (u_1, u_2, u_3)$ and $\vec{w} = (w_1, w_2, w_3)$ is the following:

$$\vec{r}(s, t) = \overrightarrow{OP} + s\vec{u} + t\vec{w} \Longleftrightarrow \begin{cases} x(\lambda) = a + su_1 + tw_1, \\ y(\lambda) = b + su_2 + tw_2, \\ z(\lambda) = c + su_3 + tw_3, \end{cases} \quad \text{where } s, t \in \mathbb{R}.$$

Equivalently, a point X belongs to the plane Π if and only if $\overrightarrow{PX}$ is a linear combination of the vectors $\vec{u}$ and $\vec{w}$. Then if (x, y, z) are the coordinates of the point X they mush verify the following equation:

$$0 = \begin{vmatrix} x - a & y - b & z - c \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

6.1 Tangent line and normal plane

Definition The *tangent line* to a curve C at the point P is the straight line that has the same tangent vector at P as the curve C. We say that the tangent line has order of contact 1 with the curve C at the point P.

Thus, the tangent line to the curve C with tangent vector $\vec{t} = (v_1, v_2, v_3)$ at the point $P = (a, b, c)$ has the following parametrization: $\vec{r}(\lambda) = \overrightarrow{OP} + \lambda\vec{t}$, this is,

$$\begin{cases} x(\lambda) = a + \lambda v_1 \\ y(\lambda) = b + \lambda v_2, \\ z(\lambda) = c + \lambda v_3, \end{cases} \quad \text{with } \lambda \in \mathbb{R}.$$

Definition The *normal plane* to a curve C at a point P is the orthogonal plane to the tangent line that contains the point P.

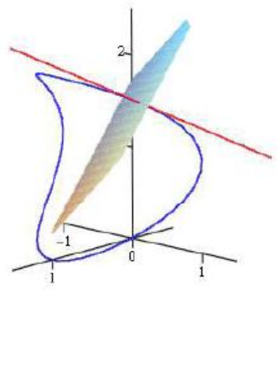
Let Π be the normal plane of the curve C with tangent vector $\vec{t} = (t_1, t_2, t_3)$ at the point $P = (a, b, c)$. A point X belongs to the normal plane if it verifies the following vectorial equation:

$$\overrightarrow{PX} \cdot \vec{t} = 0$$

That is, if (x, y, z) are the coordinates of the point X, the implicit equations of the plane is:

$$(x - a)t_1 + (y - b)t_2 + (z - c)t_3 = 0.$$

The following plot represents the tangent line and the normal plane of the curve with parametrization $\vec{r}(t) = (1 - \sin^2(t), \sin(t), 1 + \cos(t))$, $t \in [0, 2\pi)$, at the point $P = \vec{r}(\pi/4)$.



6.2 Principal normal line and osculating plane. Osculating circle.

Definition. The *principal normal line* to a curve C at a point P is the straight line that passes through the point P and whose direction vector is the normal vector to the curve at P. Thus the normal line to the curve C with normal vector $\vec{n} = (n_1, n_2, n_3)$ at the point $P = (a, b, c)$ has the following natural parametrization: $\vec{r}(\lambda) = \overrightarrow{OP} + \lambda\vec{n}$, this is,

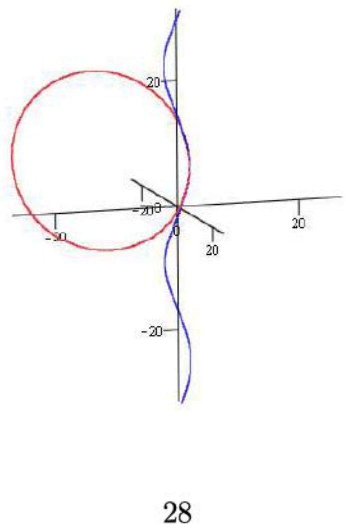
$$\begin{cases} x(\lambda) = a + \lambda n_1 \\ y(\lambda) = b + \lambda n_2, \quad \text{with } \lambda \in \mathbb{R}. \\ z(\lambda) = c + \lambda n_3, \end{cases}$$

Definition. The *osculating plane* to the curve C at a point P is the plane that contains the tangent and normal lines to the curve at the point P.

Let C be the curve with tangent vector $\vec{t} = (t_1, t_2, t_3)$ and normal vector $\vec{n} = (n_1, n_2, n_3)$ at the point $P = (a, b, c)$. A point X belongs to the osculating plane if and only if $\overrightarrow{PX}$ is a linear combination of the vectors $\vec{t}$ and $\vec{n}$. Then the coordinates (x, y, z) of X verify the following equation:

$$0 = \begin{vmatrix} x-a & y-b & z-c \\ t_1 & t_2 & t_3 \\ n_1 & n_2 & n_3 \end{vmatrix}.$$

Definition. We call *osculating circle* of the curve C at a point $P \in C$ the circle contained in the osculating plane of the curve C at P whose center, called *center of the curvature*, lies on the normal line and whose radius is $R(s_0) = 1/\kappa(s_0)$. See the picture bellow.



The osculating circle has *order of contact two* with the curve at a point $P \in C$; this means that it has the same tangent vector and the same curvature than the curve at the point P.

The center Z of the osculating circle at the point $P = \vec{\alpha}(s_0)$ satisfies the following equation:

$$\overrightarrow{OZ} = \overrightarrow{OP} + R(s_0)\vec{n}(s_0),$$

and the equations of the osculating circle is:

$$||\overrightarrow{ZX}|| = R(s_0).$$

6.2.1 Example.

The *evolute* of a curve is the locus of centers of curvature. Let us compute the evolute of the parabola contained in the plane $z = 0$, of equation $y = \frac{1}{2}x^2$. A parametrization of the parabola is $\vec{r}(t) = (t, \frac{1}{2}t^2, 0)$. By differentiating $\vec{r}(t)$ we obtain:

$$\vec{r}\,'(t) = (1, t, 0)\,, \quad \vec{r}\,''(t) = (0, 1, 0)\,, \quad \vec{r}\,'(t) \wedge \vec{r}\,''(t) = (0, 0, 1),$$

thus,

$$\vec{t}(t) = \tfrac{1}{\sqrt{1+t^2}}\,(1, t, 0)\,, \quad \vec{b}(t) = (0, 0, 1), \quad \vec{n}(t) = \tfrac{1}{\sqrt{1+t^2}}\,(-t, 1, 0)\,.$$

and the radius of curvature is given by the function:

$$R(t) = \tfrac{\|\vec{r}\,'(t)\|^3}{\|\vec{r}\,'(t) \wedge \vec{r}\,''(t)\|} = \left(1 + t^2\right)^{3/2}.$$

Thus, the locus of centers of curvature of the curve has the following parametrization:

$$\begin{aligned}
\vec{\beta}(t) &= \vec{r}(t) + R(t)\vec{n}(t) \\
&= \left(t, \tfrac{1}{2}t^2, 0\right) + \left(1 + t^2\right)^{3/2} \tfrac{1}{\sqrt{1+t^2}}\,(-t, 1, 0) \\
&= \left(t, \tfrac{1}{2}t^2, 0\right) + \left(1 + t^2\right)(-t, 1, 0) \\
&= \left(-t^3, 1 + \tfrac{3}{2}t^2, 0\right).
\end{aligned}$$

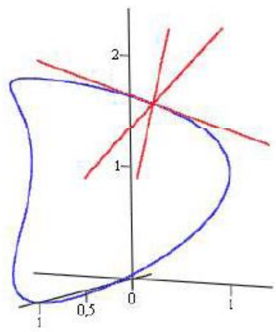
Note that the components of $\vec{\beta}(t)$ satisfy $y = 1 - \frac{3}{2}x^{2/3}$ and $z = 0$.

6.3 Binormal line and rectifying plane. Torsion.

Definition. The *binormal line* to the curve C at the point P is the line containing the point P and whose direction vector is the binormal vector to C at P. Thus the binormal line to the curve C with binormal vector $\vec{b} = (b_1, b_2, b_3)$ at the point $P = (a, b, c)$ has the following natural parametrization: $\vec{r}(\lambda) = \overrightarrow{OP} + \lambda\vec{b}$, this is,

$$\begin{cases} x(\lambda) = a + \lambda b_1 \\ y(\lambda) = b + \lambda b_2, \\ z(\lambda) = c + \lambda b_3, \end{cases} \quad \text{with } \lambda \in \mathbb{R}.$$

The following picture represents the tangent, normal and binormal lines of the curve with parametrization $\vec{r}(t) = (1 - \sin^2(t), \sin(t), 1 + \cos(t))$, $t \in [0, 2\pi)$, at the point $P = \vec{r}(\pi/4)$:



Definition. The *rectifying plane* of the curve C at the point $P = (a, b, c)$ is the plane containing the tangent and binormal lines to the curve at the point P. Thus the rectifying plane has the following vectorial equation:

$$\overrightarrow{PX} \cdot \vec{n} = 0 \iff (x - a)\, n_1 + (y - b)\, n_2 + (z - c)\, n_3 = 0.$$

Remark. If $\tau(s) = 0$, $\forall s$, then $\vec{b}'(s) = \vec{0}$, the binormal vector is constant and the curve C is contained in the osculating plane. Therefore, C is a plane curve.

Proposition. Let C be a curve of class C^3, then

$$C \text{ is plane if and only if the torsion is zero.}$$

6.4 Examples

1. Let C be a curve with parametrization $\vec{r}(t) = (a\cos(t),\ a\sin(t),\ b)$, $t \in [0, 2\pi)$ and $a \neq 0, 1$. Let us compute the tangent, normal and binormal lines and de tangent, osculating and rectifying planes at a generic point $X_0 \in C$ and at the point $P = (0, a, b)$.

   (a) By differentiating $\vec{r}(t)$ we obtain

$$\begin{aligned} \vec{r}\,'(t) &= (-a\sin(t),\ a\cos(t),\ 0), \\ \vec{r}\,''(t) &= (-a\cos(t),\ -a\sin(t),\ 0) \end{aligned}$$

   and therefore $||\vec{r}\,'(t)|| = a$ and

$$\begin{aligned} \vec{t}(t) &= \tfrac{\vec{r}\,'(t)}{||\vec{r}\,'(t)||} = (-\sin(t),\ \cos(t),\ 0), \\ \vec{r}\,'(t) \wedge \vec{r}\,''(t) &= \begin{vmatrix} i & j & k \\ -a\sin(t) & a\cos(t) & 0 \\ -a\cos(t) & -a\sin(t) & 0 \end{vmatrix} = (0, 0, a^2) \\ \vec{b}(t) &= \tfrac{\vec{r}\,'(t)\wedge\vec{r}\,''(t)}{||\vec{r}\,'(t)\wedge\vec{r}\,''(t)||} = (0, 0, 1), \\ \vec{n}(t) &= \vec{b}(t) \wedge \vec{t}(t) = \begin{vmatrix} i & j & k \\ -\sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos(t), \sin(t)). \end{aligned}$$

   Remark. Note that the Frenet frame $\left(\vec{t}(t), \vec{n}(t), \vec{b}(t)\right)$ is a positive-oriented frame as

$$\begin{vmatrix} -\sin(t) & -\sin(t) & 0 \\ \cos(t) & \sin(t) & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 > 0.$$

   Remark. Note that as $\vec{b}(t)$ is constant vector the curve is plane and it is contained at the osculating plane. A point X is at the osculating plane at a generic point $X_0 = \vec{r}(t_0) = (a\cos(t_0),\ a\sin(t_0),\ b)$ if and only if $\overrightarrow{X_0X} \cdot \vec{b}(t) = 0$. Hence the equation of the osculating plane at X_0 is:

$$(x - a\cos(t_0),\ y - a\sin(t_0),\ z - b) \cdot (0, 0, 1) = 0 \iff z = b.$$

At $P = (0, a, b) = \vec{r}(\frac{\pi}{2})$ we have:

$$\begin{aligned}
\vec{t}(\tfrac{\pi}{2}) &= (-\sin(\tfrac{\pi}{2}),\ \cos(\tfrac{\pi}{2}),\ 0) = (-1, 0, 0),\\
\vec{n}(\tfrac{\pi}{2}) &= (\cos(\tfrac{\pi}{2}), \sin(\tfrac{\pi}{2}), 0) = (0, 1, 0),\\
\vec{b}(\tfrac{\pi}{2}) &= (0, 0, 1).
\end{aligned}$$

(b) The tangent line at an arbitrary point $X_0 = \vec{r}(t_0)$ is the following:

$$\begin{aligned}
\vec{r}_t(\lambda) &= \vec{r}(t_0) + \lambda\vec{t}(t_0)\\
&= (a\cos(t_0),\ a\sin(t_0),\ b) + \lambda\,(-\sin(t_0),\ \cos(t_0),\ 0)\\
&= (a\cos(t_0) - \lambda\sin(t_0),\ a\sin(t_0) + \lambda\cos(t_0),\ b).
\end{aligned}$$

The principal normal line at an arbitrary point $X_0 = \vec{r}(t_0)$ is the following:

$$\begin{aligned}
\vec{r}_n(\lambda) &= \vec{r}(t_0) + \lambda\vec{n}(t_0)\\
&= (a\cos(t_0),\ a\sin(t_0),\ b) + \lambda\,(\cos(t_0),\ \sin(t_0),\ 0)\\
&= (a\cos(t_0) + \lambda\cos(t_0),\ a\sin(t_0) + \lambda\sin(t_0),\ b)\\
&= ((a+\lambda)\cos(t_0),\ (a+\lambda)\sin(t_0),\ b).
\end{aligned}$$

The binormal line at an arbitrary point $X_0 = \vec{r}(t_0)$ is the following:

$$\begin{aligned}
\vec{r}_b(\lambda) &= \vec{r}(t_0) + \lambda\vec{b}(t_0)\\
&= (a\cos(t_0),\ a\sin(t_0),\ b) + \lambda(0, 0, 1)\\
&= (a\cos(t_0) - \lambda\sin(t_0),\ a\sin(t_0) + \lambda\cos(t_0),\ b).
\end{aligned}$$

The tangent, normal and binormal lines at $P = \vec{r}(\frac{\pi}{2})$ are given by the following parametric equations:

$$\begin{aligned}
\vec{r}_t(\lambda) &= (a\cos(\tfrac{\pi}{2}) - \lambda\sin(\tfrac{\pi}{2}),\ a\sin(\tfrac{\pi}{2}) + \lambda\cos(\tfrac{\pi}{2}),\ b) = (-\lambda,\ a,\ b),\\
\vec{r}_n(\lambda) &= ((a+\lambda)\cos(\tfrac{\pi}{2}),\ (a+\lambda)\sin(\tfrac{\pi}{2}),\ b) = (0,\ a+\lambda,\ b),\\
\vec{r}_b(\lambda) &= (a\cos(\tfrac{\pi}{2}) - \lambda\sin(\tfrac{\pi}{2}),\ a\sin(\tfrac{\pi}{2}) + \lambda\cos(\tfrac{\pi}{2}),\ b) = (-\lambda,\ a,\ b).
\end{aligned}$$

(c) A point X is at the normal plane at $X_0 = \vec{r}(t_0) = (a\cos(t_0),\ a\sin(t_0),\ b)$ if and only if $\overrightarrow{X_0X} \cdot \vec{t}(t_0) = 0$. Hence, the equation of the normal plane is:

$$(x - a\cos(t_0),\ y - a\sin(t_0),\ z - b) \cdot (-\sin(t_0),\ \cos(t_0),\ 0) = 0$$

that is,

$$-(x - a\cos(t_0))\sin(t_0) + (y - a\sin(t_0))\cos(t_0) = 0.$$

A point X is at the rectifying plane at $X_0 = \vec{r}(t_0)$ if and only if $\overrightarrow{X_0X} \cdot \vec{n}(t_0) = 0$. Hence the equation of the rectifying plane is:

$$(x - a\cos(t_0),\ y - a\sin(t_0),\ z - b) \cdot (\cos(t_0),\ \sin(t_0),\ 0) = 0$$

that is,

$$(x - a\cos(t_0))\cos(t_0) + (y - a\sin(t_0))\sin(t_0) = 0.$$

And the equation of the osculating plane at X_0 is $z = b$.
The normal, rectifying and osculating planes at $P = \vec{r}(\frac{\pi}{2})$ are given by the following implicit equations:

$$\begin{aligned} \Pi_n &\equiv -(x - a) = 0, \\ \Pi_r &\equiv y = 0, \\ \Pi_o &\equiv z = b. \end{aligned}$$

2. *Viviani's curve.* Let C be the curve with parametrization:

$$\vec{r}(t) = \left(1 + \cos t,\ \sin t,\ 2\sin\tfrac{t}{2}\right), \quad t \in [0, 2\pi).$$

Let us compute the tangent, normal and binormal lines and de tangent, osculating and rectifying planes at the point $P = (0, 0, 2)$.

(a) By differentiating $\vec{r}(t)$ we obtain

$$\begin{aligned} \vec{r}\,'(t) &= (-\sin(t),\ \cos(t),\ \cos(\tfrac{t}{2})), \\ \vec{r}\,''(t) &= (-\cos(t),\ -\sin(t),\ -\tfrac{1}{2}\sin(\tfrac{t}{2})) \end{aligned}$$

and therefore $||\vec{r}\,'(t)|| = \sqrt{1 + \cos^2(\frac{t}{2})}$. We have: $P = \vec{r}(\pi)$, as $1 + \cos t = 0$, $\sin t = 0$ and $2\sin\frac{t}{2} = 2$ if and only if $t = \pi$. We have

$$\begin{aligned} \vec{r}\,'(\pi) &= (0,\ -1,\ 0), \\ \vec{r}\,''(\pi) &= (1,\ 0,\ -\tfrac{1}{2}), \end{aligned}$$

and therefore

$$\begin{aligned}
\vec{t}(\pi) &= \tfrac{\vec{r}\,'(\pi)}{||\vec{r}\,'(\pi)||} = (0,\ -1,\ 0),\\
\vec{r}\,'(\pi)\wedge\vec{r}\,''(\pi) &= \begin{vmatrix} i & j & k\\ 0 & -1 & 0\\ 1 & 0 & -\frac{1}{2}\end{vmatrix} = (\tfrac{1}{2},0,1),\\
\vec{b}(\pi) &= \tfrac{\vec{r}\,'(\pi)\wedge\vec{r}\,''(\pi)}{||\vec{r}\,'(\pi)\wedge\vec{r}\,''(\pi)||} = \tfrac{1}{\sqrt{\frac{1}{4}+1}}(\tfrac{1}{2},0,1) = \left(\tfrac{1}{\sqrt{5}},0,\tfrac{2}{\sqrt{5}}\right),\\
\vec{n}(\pi) &= \vec{b}(\pi)\wedge\vec{t}(\pi) = \begin{vmatrix} i & j & k\\ 0 & -1 & 0\\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}}\end{vmatrix} = \left(-\tfrac{2}{\sqrt{5}},0,\tfrac{1}{\sqrt{5}}\right).
\end{aligned}$$

The tangent line at $P=\vec{r}(\pi)$ is the following:

$$\begin{aligned}
\vec{r}_t(\lambda) &= \vec{r}(\pi)+\lambda\vec{t}(\pi)\\
&= (0,0,2)+\lambda(0,-1,0)\\
&= (0,-\lambda,2).
\end{aligned}$$

The principal normal line at $P=\vec{r}(\pi)$ is the following:

$$\begin{aligned}
\vec{r}_n(\lambda) &= \vec{r}(\pi)+\lambda\vec{n}(\pi)\\
&= (0,0,2)+\lambda\left(-\tfrac{2}{\sqrt{5}},0,\tfrac{1}{\sqrt{5}}\right)\\
&= \left(-\tfrac{2}{\sqrt{5}}\lambda,0,2+\tfrac{1}{\sqrt{5}}\lambda\right).
\end{aligned}$$

The binormal line at $P=\vec{r}(\pi)$ is the following:

$$\begin{aligned}
\vec{r}_b(\lambda) &= \vec{r}(\pi)+\lambda\vec{b}(\pi)\\
&= (0,0,2)+\lambda\left(\tfrac{1}{\sqrt{5}},0,\tfrac{2}{\sqrt{5}}\right)\\
&= \left(\lambda\tfrac{1}{\sqrt{5}},0,2+\lambda\tfrac{2}{\sqrt{5}}\right).
\end{aligned}$$

(b) A point X is at the normal plane at $P=\vec{r}(\pi)$ if and only if $\overrightarrow{PX}\cdot\vec{t}(\pi)=0$. Hence, the equation of the normal plane is:

$$(x,\ y,\ z-2)\cdot(0,-1,0)=0$$

that is, $\Pi_n\equiv y=0$.

A point X is at the rectifying plane at $P = \vec{r}(\pi)$ if and only if $\overrightarrow{PX} \cdot \vec{n}(\pi) = 0$. Hence the equation of the rectifying plane is:

$$(x,\ y,\ z-2) \cdot \left(-\tfrac{2}{\sqrt{5}}, 0, \tfrac{1}{\sqrt{5}}\right) = 0$$

that is,

$$\Pi_r \equiv -\tfrac{2}{\sqrt{5}}x + (z-2)\,\tfrac{1}{\sqrt{5}} = 0 \Longleftrightarrow -2x + z = 2.$$

A point X is at the osculating plane at $P = \vec{r}(\pi)$ if and only if $\overrightarrow{PX} \cdot \vec{b}(\pi) = 0$. Hence the equation of the osculating plane is:

$$(x,\ y,\ z-2) \cdot \left(\tfrac{1}{\sqrt{5}}, 0, \tfrac{2}{\sqrt{5}}\right) = 0$$

that is,

$$\Pi_o \equiv \tfrac{1}{\sqrt{5}}x + (z-2)\,\tfrac{2}{\sqrt{5}} = 0 \Longleftrightarrow x + 2z = 4.$$

Examples

3.1. Show that $\mathbf{x} = t\mathbf{e}_1 + (t^2+1)\mathbf{e}_2 + (t-1)^3\mathbf{e}_3$ is a regular parametric representation for all t.

$d\mathbf{x}/dt = \mathbf{e}_1 + 2t\mathbf{e}_2 + 3(t-1)^2\mathbf{e}_3$ is continuous and $|d\mathbf{x}/dt| = [1 + 4t^2 + 9(t-1)^4]^{1/2} \neq 0$ for all t. Hence $\mathbf{x}$ is regular for all t.

3.2. Show that the representation $x_1 = (1+\cos\theta)$, $x_2 = \sin\theta$, $x_3 = 2\sin(\theta/2)$, $-2\pi \leq \theta \leq 2\pi$, is regular

$dx_1/d\theta = -\sin\theta$, $dx_2/d\theta = \cos\theta$, $dx_3/d\theta = \cos(\theta/2)$
are continuous and

$$\left[\left(\frac{dx_1}{d\theta}\right)^2 + \left(\frac{dx_2}{d\theta}\right)^2 + \left(\frac{dx_3}{d\theta}\right)^2\right]^{1/2} = [1 + \cos^2(\theta/2)]^{1/2} \neq 0$$

Hence the representation is regular.

3.15. Compute the length of the arc $\mathbf{x} = 3(\cosh 2t)\mathbf{e}_1 + 3(\sinh 2t)\mathbf{e}_2 + 6t\mathbf{e}_3$, $0 \leq t \leq \pi$.

$$s = \int_0^\pi \left|\frac{d\mathbf{x}}{dt}\right| dt = \int_0^\pi |6\sinh 2t\mathbf{e}_1 + 6\cosh 2t\mathbf{e}_2 + 6\mathbf{e}_3|\, dt$$
$$= \int_0^\pi 6[\sinh^2 2t + \cosh^2 2t + 1]^{1/2}\, dt$$
$$= \int_0^\pi 6[2\cosh^2 2t]^{1/2}\, dt = \int_0^\pi 6\sqrt{2}\cosh 2t\, dt = 3\sqrt{2}\sinh 2\pi$$

3.17. Introduce arc length as a parameter along

$$\mathbf{x} = (e^t\cos t)\mathbf{e}_1 + (e^t\sin t)\mathbf{e}_2 + e^t\mathbf{e}_3, \quad -\infty < t < \infty$$

$$s = \int_0^t \left|\frac{d\mathbf{x}}{dt}\right| dt = \int_0^t |(e^t\cos t - e^t\sin t)\mathbf{e}_1 + (e^t\sin t + e^t\cos t)\mathbf{e}_2 + e^t\mathbf{e}_3|\,dt$$
$$= \int_0^t [e^{2t}(-2\cos t\sin t + 1) + e^{2t}(2\cos t\sin t + 1) + e^{2t}]^{1/2}\, dt = \sqrt{3}\int_0^t e^t\, dt = \sqrt{3}(e^t - 1)$$

Solving, $t = \log(s/\sqrt{3} + 1)$, $-\sqrt{3} < s < \infty$. Introducing arc length s as a parameter,

$$\mathbf{x} = (s/\sqrt{3} + 1)(\cos\log(s/\sqrt{3}+1)\mathbf{e}_1 + \sin\log(s/\sqrt{3}+1)\mathbf{e}_2 + \mathbf{e}_3)$$

3.25. Show that the representation

$$\mathbf{x} = t\mathbf{e}_1 + (t^2+2)\mathbf{e}_2 + (t^3+t)\mathbf{e}_3$$

is regular for all t

Example 4.1.

Along the helix $\mathbf{x} = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2 + bt\mathbf{e}_3$, $a, b \neq 0$, we have

$$\frac{d\mathbf{x}}{dt} = -a(\sin t)\mathbf{e}_1 + a(\cos t)\mathbf{e}_2 + b\mathbf{e}_3 \quad \text{and} \quad \left|\frac{d\mathbf{x}}{dt}\right| = (a^2+b^2)^{1/2}$$

Then

$$\mathbf{t} = \frac{d\mathbf{x}}{ds} = \frac{d\mathbf{x}}{dt}\frac{dt}{ds} = \frac{d\mathbf{x}}{dt}\Big/\frac{ds}{dt} = \frac{d\mathbf{x}}{dt}\Big/\left|\frac{d\mathbf{x}}{dt}\right| = (a^2+b^2)^{-1/2}(-a(\sin t)\mathbf{e}_1 + a(\cos t)\mathbf{e}_2 + b\mathbf{e}_3)$$

where we used the fact that $ds/dt = |d\mathbf{x}/dt|$ (Theorem 3.4). Observe that along the helix the unit tangent $\mathbf{t}$ makes a constant angle $\theta = \cos^{-1}(\mathbf{t}\cdot\mathbf{e}_3) = \cos^{-1} b(a^2+b^2)^{-1/2}$ with the x_3 axis.

Example 4.2.

The tangent line to the curve $\mathbf{x} = t\mathbf{e}_1 + t^2\mathbf{e}_2 + t^3\mathbf{e}_3$ at $t = 1$ is

$$\mathbf{y} = \mathbf{x}(1) + k\mathbf{x}'(1) \quad \text{or} \quad \mathbf{y} = (1+k)\mathbf{e}_1 + (1+2k)\mathbf{e}_2 + (1+3k)\mathbf{e}_3, \quad -\infty < k < \infty$$

The normal plane at $t = 1$ is

$$(\mathbf{y} - \mathbf{x}(1))\cdot\mathbf{x}'(1) = 0 \quad \text{or} \quad (y_1 - 1) + (y_2 - 1)2 + (y_3 - 1)3 = 0$$

or $y_1 + 2y_2 + 3y_3 = 6$.

Example 4.3.

Along the circle of radius a, $\mathbf{x} = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2$, $a > 0$, we have

$$\frac{d\mathbf{x}}{dt} = -a(\sin t)\mathbf{e}_1 + a(\cos t)\mathbf{e}_2, \quad \left|\frac{d\mathbf{x}}{dt}\right| = a$$

$$\mathbf{t} = \frac{d\mathbf{x}}{dt}\Big/\left|\frac{d\mathbf{x}}{dt}\right| = -(\sin t)\mathbf{e}_1 + (\cos t)\mathbf{e}_2$$

and $$\mathbf{k} = \dot{\mathbf{t}} = \frac{d\mathbf{t}}{ds} = \frac{d\mathbf{t}}{dt}\frac{dt}{ds} = \frac{d\mathbf{t}}{dt}\Big/\frac{ds}{dt} = \frac{d\mathbf{t}}{dt}\Big/\left|\frac{d\mathbf{x}}{dt}\right| = -\frac{1}{a}((\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2)$$

Note that $\mathbf{k}$ is directed towards the origin. The curvature is constant, equal to $|\kappa| = |\mathbf{k}| = 1/a$ and the radius of curvature is $\rho = 1/|\kappa| = a$. Hence as we expect, the radius of curvature of a circle is simply its radius.

Example 4.4.

Along the helix $\mathbf{x} = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2 + bt\mathbf{e}_3$, $a > 0$, $b \neq 0$, we have

$$\frac{d\mathbf{x}}{dt} = -a(\sin t)\mathbf{e}_1 + a(\cos t)\mathbf{e}_2 + b\mathbf{e}_3, \quad \left|\frac{d\mathbf{x}}{dt}\right| = (a^2+b^2)^{1/2}$$

$$\mathbf{t} = \frac{d\mathbf{x}}{dt}\Big/\left|\frac{d\mathbf{x}}{dt}\right| = (a^2+b^2)^{-1/2}(-a(\sin t)\mathbf{e}_1 + a(\cos t)\mathbf{e}_2 + b\mathbf{e}_3)$$

and

$$\begin{aligned}\mathbf{k} = \dot{\mathbf{t}} &= \frac{d\mathbf{t}}{dt}\Big/\left|\frac{d\mathbf{x}}{dt}\right| \\ &= (a^2+b^2)^{-1/2}(-a(\cos t)\mathbf{e}_1 \\ &\qquad - a(\sin t)\mathbf{e}_2)/(a^2+b^2)^{1/2} \\ &= -\frac{a}{a^2+b^2}((\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2)\end{aligned}$$

Example 4.5.

Along the third degree curve $\mathbf{x} = t\mathbf{e}_1 + \frac{1}{3}t^3\mathbf{e}_2$,

$$\frac{d\mathbf{x}}{dt} = \mathbf{e}_1 + t^2\mathbf{e}_2, \quad \left|\frac{d\mathbf{x}}{dt}\right| = (1+t^4)^{1/2}, \quad \mathbf{t} = \frac{d\mathbf{x}}{dt}\Big/\left|\frac{d\mathbf{x}}{dt}\right| = (1+t^4)^{-1/2}(\mathbf{e}_1 + t^2\mathbf{e}_2)$$

and
$$\mathbf{k} = \dot{\mathbf{t}} = \frac{d\mathbf{t}}{ds} = \frac{d\mathbf{t}}{dt}\Big/\left|\frac{d\mathbf{x}}{dt}\right| = -2t(1+t^4)^{-2}(t^2\mathbf{e}_1 - \mathbf{e}_2)$$

Example 4.7.

Consider the helix $\mathbf{x} = (\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2 + t\mathbf{e}_3$.

$$\mathbf{x}' = (-\sin t)\mathbf{e}_1 + (\cos t)\mathbf{e}_2 + \mathbf{e}_3, \quad |\mathbf{x}'| = \sqrt{2}$$

$$\mathbf{t} = \mathbf{x}'/|\mathbf{x}'| = (1/\sqrt{2})((-\sin t)\mathbf{e}_1 + (\cos t)\mathbf{e}_2 + \mathbf{e}_3)$$

and
$$\mathbf{k} = \dot{\mathbf{t}} = \mathbf{t}'/|\mathbf{x}'| = -(\tfrac{1}{2})((\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2)$$

and, since $\mathbf{k} \neq 0$ for all t,
$$\mathbf{n} = \mathbf{k}/|\mathbf{k}| = -((\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2)$$

The equation of the principal normal at $t = \pi/2$ is

$$\mathbf{y} = \mathbf{x}(\pi/2) + k\mathbf{n}(\pi/2) \quad \text{or} \quad \mathbf{y} = (1-k)\mathbf{e}_2 + \pi/2\mathbf{e}_3, \qquad -\infty < k < \infty$$

and the equation of the osculating plane at $t = \pi/2$ is

$$[(\mathbf{y} - \mathbf{x}(\pi/2))t(\pi/2)n(\pi/2)] = 0$$

or
$$\det\begin{pmatrix} y_1 & -1/\sqrt{2} & 0 \\ y_2 - 1 & 0 & -1 \\ y_3 - \pi/2 & 1/\sqrt{2} & 0 \end{pmatrix} = 0 \quad \text{or} \quad y_1 + y_3 = \pi/2$$

Example 4.8.

Referring to the helix in Example 4.4, we have

$$\mathbf{x} = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2 + bt\mathbf{e}_3, \quad a > 0, \quad b \neq 0$$

$$\mathbf{t} = (a^2+b^2)^{-1/2}(-a(\sin t)\mathbf{e}_1 + a(\cos t)\mathbf{e}_2 + b\mathbf{e}_3)$$

$$\mathbf{k} = -\frac{a}{a^2+b^2}((\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2), \quad \mathbf{n} = \frac{\mathbf{k}}{|\mathbf{k}|} = -((\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2)$$

$$\mathbf{b} = \mathbf{t} \times \mathbf{n} = \det\begin{pmatrix} \mathbf{e}_1 & -a(a^2+b^2)^{-1/2}\sin t & -\cos t \\ \mathbf{e}_2 & a(a^2+b^2)^{-1/2}\cos t & -\sin t \\ \mathbf{e}_3 & b(a^2+b^2)^{-1/2} & 0 \end{pmatrix}$$

$$= (a^2+b^2)^{-1/2}(b(\sin t)\mathbf{e}_1 - b(\cos t)\mathbf{e}_2 + a\mathbf{e}_3)$$

The equation of the binormal line at $t = t_0$ is

$$\mathbf{y} = \mathbf{x}(t_0) + k\mathbf{b}(t_0)$$

or
$$\mathbf{y} = (a\cos t_0 + kb(a^2+b^2)^{-1/2}\sin t_0)\mathbf{e}_1 + (a\sin t_0 - kb(a^2+b^2)^{-1/2}\cos t_0)\mathbf{e}_2$$
$$+ (bt_0 + ak(a^2+b^2)^{-1/2})\mathbf{e}_3, \qquad -\infty < k < \infty$$

Or, if we introduce the change in parameter $\theta = k(a^2+b^2)^{-1/2}$,

$$\mathbf{y} = (a\cos t_0 + \theta b\sin t_0)\mathbf{e}_1 + (a\sin t_0 - \theta b\cos t_0)\mathbf{e}_2 + (bt_0 + a\theta)\mathbf{e}_3, \qquad -\infty < \theta < \infty$$

The equation of the rectifying plane at $t = t_0$ is

$$(\mathbf{y} - \mathbf{x}(t_0)) \cdot \mathbf{n}(t_0) = 0$$

or
$$(y_1 - a\cos t_0)(-\cos t_0) + (y_2 - a\sin t_0)(-\sin t_0) = 0$$

or
$$y_1\cos t_0 + y_2\sin t_0 - a$$

Observe that rectifying planes are parallel to the x_3 axis.

Example 4.9.

We consider again the helix

$$\mathbf{x} = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2 + bt\mathbf{e}_3, \qquad a > 0,\ b \neq 0$$

Referring to Example 4.8, we have

$$\mathbf{b} = (a^2+b^2)^{-1/2}(b(\sin t)\mathbf{e}_1 - b(\cos t)\mathbf{e}_2 + a\mathbf{e}_3)$$

$$\dot{\mathbf{b}} = \frac{d\mathbf{b}}{ds} = \frac{d\mathbf{b}}{dt}\Big/\left|\frac{d\mathbf{x}}{dt}\right| = (a^2+b^2)^{-1}(b(\cos t)\mathbf{e}_1 + b(\sin t)\mathbf{e}_2)$$

The torsion is constant, equal to

$$\tau = -\dot{\mathbf{b}}\cdot\mathbf{n} = -(a^2+b^2)^{-1}(b(\cos t)\mathbf{e}_1 + b(\sin t)\mathbf{e}_2)\cdot((-\cos t)\mathbf{e}_1 - (\sin t)\mathbf{e}_2) = b/(a^2+b^2)$$

4.1. Find the equations of the tangent line and normal plane to the curve

$$\mathbf{x} = (1+t)\mathbf{e}_1 - t^2\mathbf{e}_2 + (1+t^3)\mathbf{e}_3$$

at $t = 1$.

$$\mathbf{x}' = \mathbf{e}_1 - 2t\mathbf{e}_2 + 3t^2\mathbf{e}_3, \quad \mathbf{x}(1) = 2\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3, \quad \mathbf{x}'(1) = \mathbf{e}_1 - 2\mathbf{e}_2 + 3\mathbf{e}_3.$$

The equation of the tangent line at $t = 1$ is

$$\mathbf{y} = \mathbf{x}(1) + k\mathbf{x}'(1) \quad \text{or} \quad \mathbf{y} = (2+k)\mathbf{e}_1 - (1+2k)\mathbf{e}_2 + (2+3k)\mathbf{e}_3$$

The equation of the normal plane is

$$(\mathbf{y}-\mathbf{x}(1))\cdot\mathbf{x}'(1) = 0 \quad \text{or} \quad (y_1-2) + (y_2+1)(-2) + (y_3-2)3 = 0 \quad \text{or} \quad y_1 - 2y_2 + 3y_3 = 10$$

4.2. Find the intersection of the x_1x_2 plane and the tangent lines to the helix

$$\mathbf{x} = (\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2 + t\mathbf{e}_3 \qquad (t > 0)$$

The tangent line at an arbitrary point $\mathbf{x}$ is

$$\mathbf{y} = \mathbf{x} + k\mathbf{x}' \quad \text{or} \quad \mathbf{y} = (\cos t - k\sin t)\mathbf{e}_1 + (\sin t + k\cos t)\mathbf{e}_2 + (t+k)\mathbf{e}_3$$

or, using $\mathbf{x}$ as the position vector,

$$x_1 = \cos t - k\sin t, \quad x_2 = \sin t + k\cos t, \quad x_3 = t + k$$

The equation of the x_1x_2 plane is $x_3 = 0$. Hence along the intersection, $t + k = 0$ or $k = -t$. Thus the intersection is the curve

$$x_1 = \cos t + t\sin t, \quad x_2 = \sin t - t\cos t, \quad x_3 = 0$$

4.3. Show that the tangent vectors along the curve $\mathbf{x} = at\mathbf{e}_1 + bt^2\mathbf{e}_2 + t^3\mathbf{e}_3$ where $2b^2 = 3a$, make a constant angle with the vector $\mathbf{a} = \mathbf{e}_1 + \mathbf{e}_3$.

$$\mathbf{x}' = a\mathbf{e}_1 + 2bt\mathbf{e}_2 + 3t^2\mathbf{e}_3$$

$$|\mathbf{x}'| = (a^2 + 4b^2t^2 + 9t^4)^{1/2} = (a^2 + 6at^2 + 9t^4)^{1/2} = a + 3t^2$$

where we have used $2b^2 = 3a$. Then the angle between the tangent $\mathbf{x}'$ and $\mathbf{a}$ is

$$\cos^{-1}\left\{\frac{(\mathbf{x}'\cdot\mathbf{a})}{|\mathbf{x}'|\,|\mathbf{a}|}\right\} = \cos^{-1}\left\{\frac{a+3t^2}{(a+3t^2)\sqrt{2}}\right\} = \cos^{-1}(1/\sqrt{2}) = \pi/4$$

4.5. Find the curvature vector $\mathbf{k}$ and curvature $|\kappa|$ on the curve

$$\mathbf{x} = t\mathbf{e}_1 + \tfrac{1}{2}t^2\mathbf{e}_2 + \tfrac{1}{3}t^3\mathbf{e}_3$$

at the point $t = 1$.

$$\mathbf{x}' = \mathbf{e}_1 + t\mathbf{e}_2 + t^2\mathbf{e}_3, \quad |\mathbf{x}'| = (1+t^2+t^4)^{1/2}$$

$$\mathbf{t} = \mathbf{x}'/|\mathbf{x}'| = (1+t^2+t^4)^{-1/2}(\mathbf{e}_1 + t\mathbf{e}_2 + t^2\mathbf{e}_3)$$

$$\begin{aligned}\mathbf{t}' &= (1+t^2+t^4)^{-1/2}(\mathbf{e}_2 + 2t\mathbf{e}_3) - (\mathbf{e}_1 + t\mathbf{e}_2 + t^2\mathbf{e}_3)(1+t^2+t^4)^{-3/2}(t+2t^3)\\ &= -(1+t^2+t^4)^{-3/2}[(2t^3+t)\mathbf{e}_1 + (t^4-1)\mathbf{e}_2 - (t^3+2t)\mathbf{e}_3]\end{aligned}$$

$$\mathbf{k} = \dot{\mathbf{t}} = \mathbf{t}'/|\mathbf{x}'| = -(1+t^2+t^4)^{-2}[(2t^3+t)\mathbf{e}_1 + (t^4-1)\mathbf{e}_2 - (t^3+2t)\mathbf{e}_3]$$

At $t = 1$ we have $\mathbf{k} = -\frac{1}{3}(\mathbf{e}_1 - \mathbf{e}_3)$ and $|\kappa| = |\mathbf{k}| = \frac{1}{3}\sqrt{2}$.

4.9. Let $\mathbf{x} = \mathbf{x}(s)$ be of class $\geqq 2$ and let $\Delta\theta$ denote the angle between the unit tangent $\mathbf{t}(s)$ at $\mathbf{x}(s)$ and $\mathbf{t}(s+\Delta s)$ at a neighboring point $\mathbf{x}(s+\Delta s)$, $\Delta s > 0$, as shown in Fig. 4-14(*a*). Show that the curvature

$$|\kappa| = \lim_{\Delta s \to 0} \frac{\Delta\theta}{\Delta s} = \frac{d\theta}{ds}$$

Namely, $|\kappa|$ is a measure of rate of change of direction of the tangent with respect to arc length.

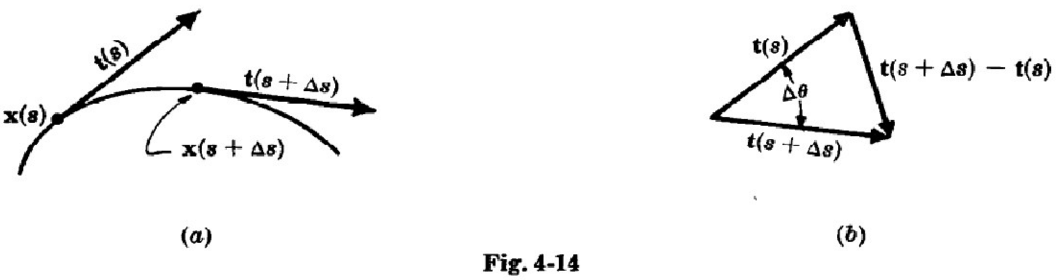


Fig. 4-14

Since t is a unit vector, $|\mathbf{t}(s+\Delta s) - \mathbf{t}(s)|$ is the base of an isosceles triangle with sides of length 1, as shown in Fig. 4-14(*b*). Hence

$$|\mathbf{t}(s+\Delta s) - \mathbf{t}(s)| = 2\sin\left(\tfrac{1}{2}\Delta\theta\right) = \Delta\theta + o(\Delta\theta)$$

where we have used the Taylor expansion for the sine function. Then

$$|\kappa| = |\dot{\mathbf{t}}| = \left|\lim_{\Delta s\to 0} \frac{\mathbf{t}(s+\Delta s) - \mathbf{t}(s)}{\Delta s}\right| = \lim_{\Delta s \to 0} \left|\frac{\mathbf{t}(s+\Delta s) - \mathbf{t}(s)}{\Delta s}\right|$$

$$= \lim_{\Delta s\to 0} \frac{\Delta\theta + o(\Delta\theta)}{\Delta s} = \lim_{\Delta s \to 0}\left[\frac{\Delta\theta}{\Delta s}\left(1 + \frac{o(\Delta\theta)}{\Delta\theta}\right)\right]$$

Since $\lim\limits_{\Delta s\to 0} \Delta\theta = 0$, then $\lim\limits_{\Delta s\to 0} \dfrac{o(\Delta\theta)}{\Delta\theta} = 0$ and $|\kappa| = \lim\limits_{\Delta s \to 0}\dfrac{\Delta\theta}{\Delta s} = \dfrac{d\theta}{ds}$.

4.11. Find a continuous unit principal normal and unit binormal along the curve

$$\mathbf{x} = (3t - t^3)\mathbf{e}_1 + 3t^2\mathbf{e}_2 + (3t + t^3)\mathbf{e}_3$$

$$\mathbf{x}' = (3 - 3t^2)\mathbf{e}_1 + 6t\mathbf{e}_2 + (3 + 3t^2)\mathbf{e}_3$$

$$|\mathbf{x}'| = 3[(1-t^2)^2 + (2t)^2 + (1+t^2)^2]^{1/2} = 3\sqrt{2}\,(1 + 2t^2 + t^4)^{1/2} = 3\sqrt{2}\,(1+t^2)$$

$$\mathbf{t} = \frac{x'}{|x'|} = \frac{1}{\sqrt{2}\,(1+t^2)}[(1-t^2)\mathbf{e}_1 + 2t\mathbf{e}_2 + (1+t^2)\mathbf{e}_3]$$

$$\mathbf{k} = \dot{\mathbf{t}} = \frac{\mathbf{t}'}{|\mathbf{x}'|} = \frac{-2t\mathbf{e}_1 + (1-t^2)\mathbf{e}_2}{3(1+t^2)^3}, \qquad |\mathbf{k}| = \frac{[(2t)^2 + (1-t^2)^2]^{1/2}}{3(1+t^2)^3} = \frac{1}{3(1+t^2)^2}$$

Since $\mathbf{k} \neq \mathbf{0}$ for all t, we can choose

$$\mathbf{n} = \frac{\mathbf{k}}{|\mathbf{k}|} = \frac{-2t}{1+t^2}\mathbf{e}_1 + \frac{1-t^2}{1+t^2}\mathbf{e}_2$$

and

$$\mathbf{b} = \mathbf{t}\times\mathbf{n} = \frac{1}{\sqrt{2}(1+t^2)^2}\det\begin{pmatrix} \mathbf{e}_1 & 1-t^2 & -2t \\ \mathbf{e}_2 & 2t & 1-t^2 \\ \mathbf{e}_3 & 1+t^2 & 0\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}(1+t^2)^2}[-(1+t^2)(1-t^2)\mathbf{e}_1 - 2t(1+t^2)\mathbf{e}_2 + (1+t^2)^2\mathbf{e}_3]$$

$$= \frac{1}{\sqrt{2}(1+t^2)}[(t^2-1)\mathbf{e}_1 - 2t\mathbf{e}_2 + (1+t^2)\mathbf{e}_3]$$

4.12. **Show that along a curve $\mathbf{x} = \mathbf{x}(t)$ the vector $\mathbf{x}''$ is parallel to the osculating plane and that its components with respect to t and n are $|\mathbf{x}'|'$ and $\kappa|\mathbf{x}'|^2$ respectively.**

Differentiating $\mathbf{x}' = \frac{d\mathbf{x}}{ds}\frac{ds}{dt} = \mathbf{t}s'$ with respect to t, we obtain

$$\mathbf{x}'' = \mathbf{t}s'' + \mathbf{t}'s' = \mathbf{t}s'' + \dot{\mathbf{t}}s'^2 = \mathbf{t}s'' + \mathbf{n}\kappa s'^2$$

where from equation (4.6), $\dot{\mathbf{t}} = \kappa\mathbf{n}$. It follows that $\mathbf{x}''$ is parallel to the osculating plane and that its components with respect to $\mathbf{t}$ and $\mathbf{n}$ are $|\mathbf{x}'|' = s''$ and $\kappa|\mathbf{x}'|^2 = \kappa s'^2$ respectively.

4.13. **(*a*) If $\mathbf{x}'$ and $\mathbf{x}''$ are linearly independent at a point $\mathbf{x}$ along $\mathbf{x} = \mathbf{x}(t)$, show that the osculating plane at $\mathbf{x}$ is $[(\mathbf{y}-\mathbf{x})\mathbf{x}'\mathbf{x}''] = 0$. (*b*) Use this formula to find the osculating plane to the curve $\mathbf{x} = t\mathbf{e}_1 + t^2\mathbf{e}_2 + t^3\mathbf{e}_3$ at $t = 1$.**

(*a*) We saw in the above problem that $\mathbf{x}''$ is parallel to the osculating plane and we know that $\mathbf{x}'$, being a multiple of $\mathbf{t}$, is parallel to the osculating plane. Since we are given that $\mathbf{x}'$ and $\mathbf{x}''$ are independent, it follows that $\mathbf{x}' \times \mathbf{x}''$ is a nonzero vector normal to the osculating plane at $\mathbf{x}$. Therefore the equation of the osculating plane at $\mathbf{x}$ is $[(\mathbf{y}-\mathbf{x})\mathbf{x}'\mathbf{x}''] = 0$.

(*b*) $\mathbf{x}' = \mathbf{e}_1 + 2t\mathbf{e}_2 + 3t^2\mathbf{e}_3$, $\mathbf{x}'' = 2\mathbf{e}_2 + 6t\mathbf{e}_3$

Thus the osculating plane at $t = 1$ is $[(\mathbf{y}-\mathbf{x}(1))\mathbf{x}'(1)\mathbf{x}''(1)] = 0$ or

$$\det\begin{pmatrix} y_1 - 1 & 1 & 0 \\ y_2 - 1 & 2 & 2 \\ y_3 - 1 & 3 & 6 \end{pmatrix} = 0$$

from which

$$6(y_1 - 1) - 6(y_2 - 1) + 2(y_3 - 1) = 0 \quad \text{or} \quad 3y_1 - 3y_2 + y_3 = 1$$

Chapter 3

In this chapter we are going to study the following concepts

1) Helices
2) Spherical Indicatrix
3) Involutes
4) Evolutes
5) Bertrand Curves

8. Helices. A curve traced on the surface of a cylinder, and cutting the generators at a constant angle, is called a *helix*. Thus the tangent to a helix is inclined at a constant angle to a fixed direction If then **t** is the unit tangent to the helix, and **a** a constant vector parallel to the generators of the cylinder, we have

$$\mathbf{t} \cdot \mathbf{a} = \text{const.}$$

and therefore, on differentiation with respect to s,

$$\kappa \mathbf{n} \cdot \mathbf{a} = 0.$$

Thus, since the curvature of the helix does not vanish, *the principal normal is everywhere perpendicular to the generators.* Hence the fixed direction of the generators is parallel to the plane of **t** and **b**; and since it makes a constant angle with **t**, it also makes a constant angle with **b**.

An important property of all helices is that the curvature and torsion are in a constant ratio.

To prove this we differentiate the relation $n.a = 0 \Rightarrow n'.a = 0$, where $' = \dfrac{d}{ds}$, obtaining

$$(\tau \mathbf{b} - \kappa \mathbf{t}) \cdot \mathbf{a} = 0.$$

Thus **a** is perpendicular to the vector $\tau\mathbf{b} - \kappa\mathbf{t}$. But **a** is parallel to the plane of **t** and **b**, and must therefore be parallel to the vector $\tau\mathbf{t} + \kappa\mathbf{b}$, which is inclined to **t** at an angle $\tan^{-1} \kappa/\tau$. But this angle is constant. Therefore the curvature and torsion are in a constant ratio.

Conversely we may prove that *a curve whose curvature and torsion are in a constant ratio is a helix.* Let $\tau = c\kappa$ where c is constant. Then since

$$\mathbf{t}' = \kappa\mathbf{n},$$

and
$$\mathbf{b}' = -\tau\mathbf{n} = -c\kappa\mathbf{n},$$

it follows that
$$\frac{d}{ds}(\mathbf{b} + c\mathbf{t}) = 0,$$

and therefore
$$\mathbf{b} + c\mathbf{t} = \mathbf{a},$$

where **a** is a constant vector. Forming the scalar product of each side with **t** we have

$$\mathbf{t}\cdot\mathbf{a} = c$$

Thus **t** is inclined at a constant angle to the fixed direction of **a**, and the curve is therefore a helix.

Finally we may show that the curvature and the torsion of a helix are in a constant ratio to the curvature κ_0 of the plane section of the cylinder perpendicular to the generators. Take the z-axis

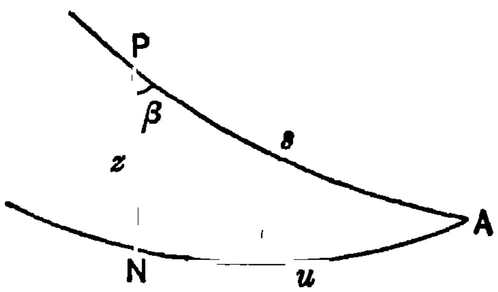


Fig 5.

parallel to the generators, and let s be measured from the intersection A of the curve with the xy plane. Let u be the arc-length of the normal section of the cylinder by the xy plane, measured from the same point A up to the generator through the current point (x, y, z). Then, if β is the constant angle at which the curve cuts the generators, we have

$$u = s \sin\beta,$$

and therefore
$$u' = \sin\beta.$$

The coordinates x, y are functions of u, while $z = s\cos\beta$. Hence for the current point on the helix we have

$$\mathbf{r} = (x,\ y,\ s\cos\beta),$$

so that
$$\mathbf{r}' = \left(\frac{dx}{du}\sin\beta,\ \frac{dy}{du}\sin\beta,\ \cos\beta\right),$$

and
$$\mathbf{r}'' = \left(\frac{d^2x}{du^2}\sin^2\beta,\ \frac{d^2y}{du^2}\sin^2\beta, 0\right).$$

Hence the curvature of the helix is given by

$$\kappa^2 = \mathbf{r}''^2 = \left\{\left(\frac{d^2x}{du^2}\right)^2 + \left(\frac{d^2y}{du^2}\right)^2\right\}\sin^4\beta = \kappa_0{}^2\sin^4\beta,$$

so that
$$\kappa = \kappa_0\sin^2\beta.$$

For the torsion, we have already proved that

$$\beta = \tan^{-1}\kappa/\tau,$$

so that
$$\tau = \kappa\cot\beta = \kappa_0\sin\beta\cos\beta.$$

From these results it is clear that *the only curve whose curvature and torsion are both constant is the circular helix.* For such a curve must be a helix, since the ratio of its curvature to its torsion is also constant. And since κ is constant it then follows that κ_0 is constant, so that the cylinder on which the helix is drawn is a circular cylinder.

Ex. Show that, for any curve,

$$[\mathbf{r}'',\ \mathbf{r}''',\ \mathbf{r}''''] = \kappa^5\frac{d}{ds}\left(\frac{\tau}{\kappa}\right).$$

This expression therefore vanishes for a helix and conversely, if it vanishes, the curve is a helix.

9. Spherical indicatrix. The locus of a point, whose position vector is equal to the unit tangent $\mathbf{t}$ of a given curve, is called the *spherical indicatrix of the tangent* to the curve Such a locus lies on the surface of a unit sphere, hence the name. Let the suffix unity be used to distinguish quantities belonging to this locus. Then

$$\mathbf{r}_1 = \mathbf{t},$$

and therefore
$$\mathbf{t}_1 = \frac{d\mathbf{r}_1}{ds_1} = \frac{d\mathbf{t}}{ds}\frac{ds}{ds_1} = \kappa\mathbf{n}\frac{ds}{ds_1},$$

showing that the tangent to the spherical indicatrix is parallel to the principal normal of the given curve. We may measure s_1 so that

$$\mathbf{t}_1 = \mathbf{n},$$

and therefore
$$\frac{ds_1}{ds} = \kappa.$$

For the curvature κ_1 of the indicatrix, on differentiating the relation $\mathbf{t}_1 = \mathbf{n}$, we find the formula

$$\kappa_1 \mathbf{n}_1 = \frac{d\mathbf{n}}{ds}\frac{ds}{ds_1} = \frac{1}{\kappa}(\tau\mathbf{b} - \kappa\mathbf{t}).$$

Squaring both sides we obtain the result

$$\kappa_1^2 = (\kappa^2 + \tau^2)/\kappa^2,$$

so that the curvature of the indicatrix is the ratio of the screw curvature to the circular curvature of the curve. The unit binormal of the indicatrix is

$$\mathbf{b}_1 = \mathbf{t}_1 \times \mathbf{n}_1 = \frac{\tau\mathbf{t} + \kappa\mathbf{b}}{\kappa\kappa_1}.$$

The torsion could be obtained by differentiating this equation; but the result follows more easily from the equation [cf. Examples I, (11)]

$$\kappa_1^2 \tau_1 \left(\frac{ds_1}{ds}\right)^6 = [\mathbf{r}_1', \mathbf{r}_1'', \mathbf{r}_1'''] = [\mathbf{t}', \mathbf{t}'', \mathbf{t}''']$$
$$= \kappa^3(\kappa\tau' - \kappa'\tau),$$

which reduces to
$$\tau_1 = \frac{(\kappa\tau' - \kappa'\tau)}{\kappa(\kappa^2 + \tau^2)}.$$

Similarly the *spherical indicatrix of the binormal* of the given curve is the locus of a point whose position vector is $\mathbf{b}$. Using the suffix unity to distinguish quantities belonging to this locus, we have

$$\mathbf{r}_1 = \mathbf{b},$$

and therefore
$$\mathbf{t}_1 = \frac{d\mathbf{b}}{ds}\frac{ds}{ds_1} = -\tau\mathbf{n}\frac{ds}{ds_1}.$$

We may measure s_1 so that

$$\mathbf{t}_1 = -\mathbf{n},$$

and therefore
$$\frac{ds_1}{ds} = \tau.$$

To find the curvature differentiate the equation $\mathbf{t}_1 = -\mathbf{n}$. Then

$$\kappa_1 \mathbf{n}_1 = \frac{d}{ds}(-\mathbf{n})\frac{ds}{ds_1} = \frac{1}{\tau}(\kappa\mathbf{t} - \tau\mathbf{b}),$$

giving the direction of the principal normal. On squaring this result we have

$$\kappa_1^2 = (\kappa^2 + \tau^2)/\tau^2.$$

Thus the curvature of the indicatrix is the ratio of the screw curvature to the torsion of the given curve. The unit binormal is

$$\mathbf{b}_1 = \mathbf{t}_1 \times \mathbf{n}_1 = \frac{\tau\mathbf{t} + \kappa\mathbf{b}}{\tau\kappa_1},$$

and the torsion, found as in the previous case, is equal to

$$\tau_1 = \frac{\tau\kappa' - \kappa\tau'}{\tau(\kappa^2 + \tau^2)}.$$

Ex. 1. Find the torsions of the spherical indicatrices from the formula

$$R^2 = \rho_1^2 + \sigma_1^2\rho_1'^2,$$

where $R = 1$ and $\rho_1 = 1/\kappa_1$ is known.

Ex. 2. Examine the spherical indicatrix of the principal normal of a given curve

10. Involutes. When the tangents to a curve C are normals to another curve C_1, the latter is called an *involute* of the former, and C is called an *evolute* of C_1. An involute may be generated

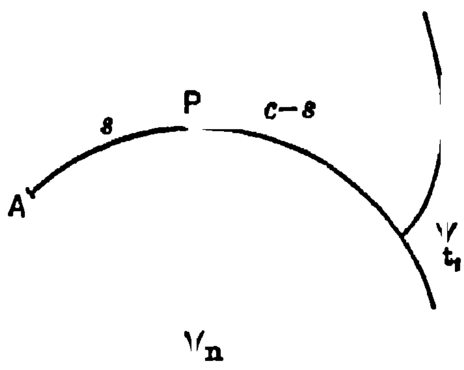


Fig. 6.

mechanically in the following manner Let one end of an inextensible string be fixed to a point of the curve C, and let the string be kept taut while it is wrapped round the curve on its convex side. Then any particle of the string describes an involute of C, since at each instant the free part of the string is a tangent to the curve C, while the direction of motion of the particle is at right angles to this tangent.

From the above definition it follows that the point $\mathbf{r}_1$ of the involute which lies on the tangent at the point $\mathbf{r}$ of the curve C is given by

$$\mathbf{r}_1 = \mathbf{r} + u\mathbf{t},$$

where u is to be determined. Let ds_1 be the arc-length of the involute corresponding to the element ds of the curve C. Then the unit tangent to C_1 is

$$\mathbf{t}_1 = \frac{d\mathbf{r}_1}{ds}\frac{ds}{ds_1} = \{(1 + u')\,\mathbf{t} + u\kappa\mathbf{n}\}\frac{ds}{ds_1}.$$

To satisfy the condition for an involute, this vector must be perpendicular to $\mathbf{t}$. Hence

$$1 + u' = 0,$$

so that

$$u = c - s,$$

where c is an arbitrary constant Thus the current point on the involute is

$$\mathbf{r}_1 = \mathbf{r} + (c - s)\,\mathbf{t},$$

and the unit tangent there is

$$\mathbf{t}_1 = (c - s)\,\kappa\,\frac{ds}{ds_1}\,\mathbf{n}.$$

Hence the tangent to the involute is parallel to the principal normal to the given curve. We may take the positive direction along the involute so that

$$\mathbf{t}_1 = \mathbf{n},$$

and therefore

$$\frac{ds_1}{ds} = (c - s)\,\kappa.$$

To find the curvature κ_1 of the involute we differentiate the relation $\mathbf{t}_1 = \mathbf{n}$, thus obtaining

$$\kappa_1\mathbf{n}_1 = \frac{\tau\mathbf{b} - \kappa\mathbf{t}}{\kappa(c - s)}.$$

Therefore, on squaring both sides, we have

$$\kappa_1^2 = \frac{\kappa^2 + \tau^2}{\kappa^2(c - s)^2}.$$

The unit principal normal to the involute is

$$\mathbf{n}_1 = \frac{\tau\mathbf{b} - \kappa\mathbf{t}}{\kappa\kappa_1(c - s)},$$

and the unit binormal

$$\mathbf{b}_1 = \mathbf{t}_1 \times \mathbf{n}_1 = \frac{\kappa\mathbf{b} + \tau\mathbf{t}}{\kappa\kappa_1(c - s)}.$$

Since the constant c is arbitrary, there is a single infinitude of involutes to a given curve; and the tangents at corresponding points of two different involutes are parallel and at a constant distance apart.

Ex. 1. Show that the torsion of an involute has the value

$$\frac{\kappa\tau' - \kappa'\tau}{\kappa(\kappa^2+\tau^2)(c-s)}.$$

Ex. 2. Prove that the involutes of a circular helix are plane curves, whose planes are normal to the axis of the cylinder, and that they are also involutes of the circular sections of the cylinder.

***11. Evolutes.** The converse problem to that just solved is the problem of finding the *evolutes* of a given curve C. Let $\mathbf{r}_1$ be the point on the evolute C_1 corresponding to the point $\mathbf{r}$ on C. Then, since the tangents to C_1 are normals to C, the point $\mathbf{r}_1$ lies in the normal plane to the given curve at $\mathbf{r}$. Hence

$$\mathbf{r}_1 = \mathbf{r} + u\mathbf{n} + v\mathbf{b},$$

where u, v are to be determined. The tangent to the evolute at $\mathbf{r}_1$ is parallel to $d\mathbf{r}_1/ds$, that is, to

$$(1-u\kappa)\mathbf{t} + (u' - v\tau)\mathbf{n} + (u\tau + v')\mathbf{b}$$

Hence, in order that it may be parallel to $u\mathbf{n} + v\mathbf{b}$ we must have

$$1 - u\kappa = 0,$$

and $$\frac{u' - v\tau}{u} = \frac{u\tau + v'}{v}.$$

The first of these gives $u = \frac{1}{\kappa} = \rho$, and from the second it follows that

$$\tau = \frac{v\rho' - \rho v'}{v^2 + \rho^2}.$$

Integrating with respect to s and writing $\psi = \int_0^s \tau\, ds$, we have

$$\psi + c = \tan^{-1}\left(-\frac{v}{\rho}\right),$$

so that $$v = -\rho\tan(\psi + c).$$

The point $\mathbf{r}_1$ on the evolute is therefore given by

$$\mathbf{r}_1 = \mathbf{r} + \rho\,\{\mathbf{n} - \tan(\psi + c)\,\mathbf{b}\}.$$

It therefore lies on the axis of the circle of curvature of the given curve, at a distance $-\rho\tan(\psi + c)$ from the centre of curvature.

The tangent to the evolute, being the line joining the points $\mathbf{r}$ and $\mathbf{r}_1$, is in the normal plane of the given curve C, and is inclined to the principal normal $\mathbf{n}$ at an angle $(\psi + c)$.

Let the suffix unity distinguish quantities referring to the evolute. Then on differentiating the last equation, remembering that $d\psi/ds = \tau$, we find

$$\mathbf{t}_1 \frac{ds_1}{ds} = \frac{d\mathbf{r}_1}{ds} = \{\rho' + \rho\tau \tan(\psi + c)\}\{\mathbf{n} - \tan(\psi + c)\,\mathbf{b}\}.$$

Thus the unit tangent to the evolute is

$$\mathbf{t}_1 = \cos(\psi + c)\,\mathbf{n} - \sin(\psi + c)\,\mathbf{b}$$

and therefore $$\frac{ds_1}{ds} = \frac{\kappa\tau \sin(\psi + c) - \kappa' \cos(\psi + c)}{\kappa^2 \cos^2(\psi + c)}.$$

The curvature of the evolute is obtained by differentiating the vector $\mathbf{t}_1$. Thus

$$\kappa_1 \mathbf{n}_1 \frac{ds_1}{ds} = \frac{d\mathbf{t}_1}{ds} = -\kappa \cos(\psi + c)\,\mathbf{t}.$$

The principal normal to the evolute is thus parallel to the tangent to the curve C. We may take

$$\mathbf{n}_1 = -\mathbf{t},$$

and therefore $$\kappa_1 = \kappa \cos(\psi + c)\frac{ds}{ds_1}$$

$$= \frac{\kappa^3 \cos^3(\psi + c)}{\kappa\tau \sin(\psi + c) - \kappa' \cos(\psi + c)}.$$

The unit binormal to the evolute is

$$\mathbf{b}_1 = \mathbf{t}_1 \times \mathbf{n}_1 = \cos(\psi + c)\,\mathbf{b} + \sin(\psi + c)\,\mathbf{n}.$$

The torsion is found by differentiating this. Thus

$$-\tau_1 \mathbf{n}_1 \frac{ds_1}{ds} = -\kappa \sin(\psi + c)\,\mathbf{t}$$

and therefore

$$\tau_1 = -\kappa \sin(\psi + c)\frac{ds}{ds_1}$$

$$= -\frac{\kappa^3 \sin(\psi + c)\cos^2(\psi + c)}{\kappa\tau \sin(\psi + c) - \kappa' \cos(\psi + c)}.$$

Thus the ratio of the torsion of the evolute to its curvature is $-\tan(\psi + c)$.

Since the constant c is arbitrary there is a single infinitude of evolutes The tangents to two different evolutes, corresponding to the values c_1 and c_2, drawn from the same point of the given curve, are inclined to each other at a constant angle $c_1 - c_2$.

Ex. 1. The locus of the centre of curvature is an evolute only when the curve is plane.

Ex. 2. A plane curve has only a single evolute in its own plane, the locus of the centre of curvature All other evolutes are helices traced on the right cylinder whose base is the plane evolute.

***12. Bertrand curves.** Saint-Venant proposed and Bertrand solved the problem of finding the curves whose principal normals are also the principal normals of another curve. A pair of curves, C and C_1, having their principal normals in common, are said to be conjugate or associate *Bertrand curves.* We may take their principal normals in the same sense, so that

$$\mathbf{n}_1 = \mathbf{n}.$$

The point $\mathbf{r}_1$ on C_1 corresponding to the point $\mathbf{r}$ on C is then given by

$$\mathbf{r}_1 = \mathbf{r} + a\mathbf{n} \quad \ldots\ldots \quad \ldots \ldots \quad \ldots\ldots \text{(i)},$$

where it is easily seen that *a is constant.* For the tangent to C_1 is parallel to $d\mathbf{r}_1/ds$, and therefore to

$$\mathbf{t} + a'\mathbf{n} + a(\tau\mathbf{b} - \kappa\mathbf{t})$$

This must be perpendicular to $\mathbf{n}$, so that a' is zero and therefore a constant. Further, if symbols with the suffix unity refer to the curve C_1, we have

$$\frac{d}{ds}(\mathbf{t}\cdot\mathbf{t}_1) = \kappa\mathbf{n}\cdot\mathbf{t}_1 + \mathbf{t}\cdot(\kappa_1\mathbf{n})\frac{ds_1}{ds} = 0,$$

showing that $$\mathbf{t}\cdot\mathbf{t}_1 = \text{const.}$$

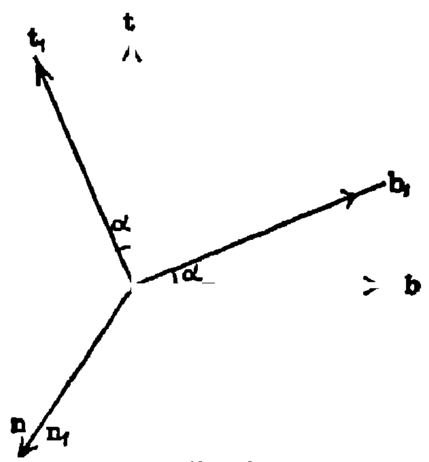


Fig. 7.

Thus *the tangents to the two curves are inclined at a constant angle* But the principal normals coincide, and therefore the binormals of the two curves are inclined at the same constant angle. Let α be the inclination of $\mathbf{b}_1$ to $\mathbf{b}$ measured from $\mathbf{b}$ toward $\mathbf{t}$. Then α is constant

On differentiating the above expression for $\mathbf{r}_1$ we have

$$\mathbf{t}_1 \frac{ds_1}{ds} = (1 - a\kappa)\,\mathbf{t} + a\tau\mathbf{b} \quad \text{......} \quad \text{...... ...(ii).}$$

Then, forming the scalar product of each side with $\mathbf{b}_1$, we obtain

$$0 = (1 - a\kappa)\sin\alpha + a\tau\cos\alpha.$$

Thus *there is a linear relation with constant coefficients between the curvature and torsion of C;*

$$\tau = \left(\kappa - \frac{1}{a}\right)\tan\alpha.$$

Moreover it is obvious from the diagram that

$$\mathbf{t}_1 = \mathbf{t}\cos\alpha - \mathbf{b}\sin\alpha$$

On comparing this with (ii) we see that

$$\left.\begin{aligned} \cos\alpha &= (1 - a\kappa)\frac{ds}{ds_1} \\ \sin\alpha &= -a\tau\frac{ds}{ds_1} \end{aligned}\right\} \quad \text{...} \quad \text{(iii).}$$

Now the relation between the curves C and C_1 is clearly a reciprocal one The point $\mathbf{r}$ is at a distance $-a$ along the normal at $\mathbf{r}_1$, and $\mathbf{t}$ is inclined at an angle $-\alpha$ to $\mathbf{t}_1$ Hence, corresponding to (iii), we have

$$\left.\begin{aligned} \cos\alpha &= (1 + a\kappa_1)\frac{ds_1}{ds} \\ \sin\alpha &= -a\tau_1\frac{ds_1}{ds} \end{aligned}\right\} \quad \text{..................(iv).}$$

On multiplying together corresponding formulae of (iii) and (iv) we obtain the relations

$$\left.\begin{aligned} \tau\tau_1 &= \frac{1}{a^2}\sin^2\alpha \\ (1 - a\kappa)(1 + a\kappa_1) &= \cos^2\alpha \end{aligned}\right\} \quad \text{...... ....(v).}$$

The first of these shows that *the torsions of the two curves have the same sign, and their product is constant.* This theorem is due to

Schell. The result contained in the second formula may be expressed as follows: *If P, P_1 are corresponding points on two conjugate Bertrand curves, and O, O_1 their centres of curvature, the cross ratio of the range (POP_1O_1) is constant and equal to* $\sec^2 \alpha$. This theorem is due to Mannheim.

Ex. 1. By differentiating the equation

$$(1-a\kappa)\sin\alpha + a\tau\cos\alpha = 0,$$

deduce the following results:

For a curve of constant curvature the conjugate is the locus of its centre of curvature.

A curve of constant torsion coincides with its conjugate.

Ex. 2. Show that a plane curve admits an infinity of conjugates, all *parallel* to the given curve

Prove also that the only other curve which has more than one conjugate is the circular helix, the conjugates being also circular helices on coaxial cylinders

EXAMPLES II

1. The principal normal to a curve is normal to the locus of the centre of curvature at points for which the value of κ is stationary

2. The normal plane to the locus of the centre of circular curvature of a curve C bisects the radius of spherical curvature at the corresponding point of C

3. The binormal at a point P of a given curve is the limiting position of the common perpendicular to the tangents at P and a consecutive point of the curve.

4. For a curve drawn on a sphere the centre of curvature at any point is the foot of the perpendicular from the centre of the sphere upon the osculating plane at the point

5. Prove that, in order that the principal normals of a curve be binormals of another, the relation

$$a(\kappa^2+\tau^2) = \kappa$$

must hold, where a is constant

6. If there is a one-to-one correspondence between the points of two curves, and the tangents at corresponding points are parallel, show that the principal normals are parallel, and therefore also the binormals. Prove also that

$$\frac{\kappa_1}{\kappa} = \frac{ds}{ds_1} = \frac{\tau_1}{\tau}.$$

Two curves so related are said to be deducible from each other by a *Combescure transformation.*

Chapter 2

Surfaces in $\mathbb{R}^3$

2.1 Definitions and Examples

First, we assume you know the definition of open sets and continuous maps from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Definition 2.1.1. *If $f : X \to Y$ is continuous and bijective, and if its inverse map $f^{-1} : Y \to X$ is also continuous, then f is called a homeomorphism and X and Y are said to be homeomorphic.*

Theorem 2.1.2 (Invariance of domain). *If $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is an injective continuous map, then $V = f(U)$ is open and f is a homeomorphism between U and V.*

Definition 2.1.3. *A subset $S \subset \mathbb{R}^3$ is a regular surface if for each $p \in S$, there exists a neighbourhood $W \subset \mathbb{R}^3$ and a map $\sigma : U \to W \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $W \cap S \subset \mathbb{R}^3$, such that*

- *σ is smooth*
- *σ is homeomorphism*
- *at all points $(u, v) \in U$, $\sigma_u \times \sigma_v \neq 0$.*

The mapping σ is called a *(regular) parametrization* or a *chart.* We will call its image a *coordinate patch.* A collection of charts such that every point of S is contained in a coordinate patch is called an *atlas.* The condition 3 above means σ_u and σ_v are linearly independent, or $d\sigma_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one to one.

For any point of a regular surface S, there might be more than one charts.

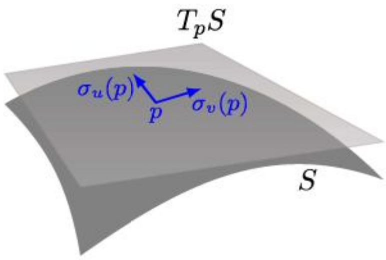
Proposition 2.1.4. *Let $\sigma : U \to S$, $\tilde{\sigma} : V \to S$ be two charts of S such that $p \in \sigma(U) \cap \tilde{\sigma}(V) = W$. Then the transition map $h = \sigma^{-1} \circ \tilde{\sigma} : \tilde{\sigma}^{-1}(W) \to \sigma^{-1}(W)$ is a diffeomorphism.*

Proof is omitted. It is another application of Inverse Function Theorem. The diffeomorphism h gives a *reparametrization.*

Definition 2.1.5. *A reparametrization of surface is a composition $\sigma \circ f : V \to \mathbb{R}^3$ where $f : V \to U$ is a diffeomorphism.*

Since the Jacobian df is invertible, let $f(x, y) = (u(x, y), v(x, y))$, $(\sigma \circ f)_x$ and $(\sigma \circ f)_y$ are linearly independent if and only if σ_u and σ_v are. So the following is well defined.

Definition 2.1.6. *The tangent plane T_pS of a surface S at the point p is the vector space spanned by $\sigma_u(p)$ and $\sigma_v(p)$.*



This space is independent of parametrization. One should think of the origin of the vector space as the point p.

Definition 2.1.7. *The unit vector*

$$\mathbf{N}_\sigma(u, v) = \frac{\sigma_u \times \sigma_v}{||\sigma_u \times \sigma_v||}$$

is the standard normal to the surface at point $p = \sigma(u, v)$.

Here are examples of parametrized surfaces. For the pictures of these, look at Hichin's notes.

Example:

1. A plane:

$$\sigma(u, v) = \mathbf{a} + u\mathbf{b} + v\mathbf{c}$$

   for constant vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{b} \times \mathbf{c} \neq 0$. The normal vector

$$\mathbf{N} = \frac{\mathbf{b} \times \mathbf{c}}{||\mathbf{b} \times \mathbf{c}||}$$

2. A cylinder:

$$\sigma(u, v) = (a \cos u, a \sin u, v), a > 0$$

$$\mathbf{N} = (\cos u, \sin u, 0)$$

3. A cone (without cone point):

$$\sigma(u,v) = (au\cos v, au\sin v, u)$$

4. A helicoid:

$$\sigma(u,v) = (au\cos v, au\sin v, v)$$

5. A sphere (minus a half circle connecting poles) in spherical coordinates: $U = (-\frac{\pi}{2}, \frac{\pi}{2}) \times (0, 2\pi)$.

$$\sigma(u,v) = (a\cos v\cos u, a\sin v\cos u, a\sin u)$$

$$\mathbf{N} = -\frac{1}{a}\sigma$$

6. A torus

$$\sigma(u,v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u),$$

$a > b$ are constants.

7. A surface of revolution

$$\sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$$

is obtained by rotating a plane curve (called profile curve) $\gamma(u) = (f(u), 0, g(u))$ around z-axis. We assume $f(u) > 0$ for all u. We have

$$\sigma_u = (f_u\cos v, f_u\sin v, g_u), \sigma_v = (-f\sin v, f\cos v, 0).$$

So

$$\sigma_u \times \sigma_v = (-f\dot{g}\cos v, -f\dot{g}\sin v, f\dot{f}), ||\sigma_u \times \sigma_v||^2 = f^2(\dot{f}^2 + \dot{g}^2) \neq 0$$

8. A generalized cylinder

$$\sigma(u,v) = \gamma(u) + v\mathbf{a}.$$

$$\sigma_u = \dot{\gamma}, \sigma_v = \mathbf{a}$$

σ is regular if γ is never tangent to the ruling $\mathbf{a}$.

But usually, a surface has more than one patches. That is the reason why we need more preparation of surfaces local theory than that of curves. For curves, only one patch is enough since the topology is simpler. The following example shows how a closed (i.e. compact without boundary) surface is different from a closed curve, where we can use a periodic one patch parametrization.

Example 2.1.8. *The unit sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3; x^2+y^2+z^2=1\}$ is a regular surface. We let $\sigma_1 : U \subset \mathbb{R}^2 \to \mathbb{R}^3$.*

$$\sigma_1(x,y) = (x, y, \sqrt{1-x^2-y^2}), (x,y) \in U = B_1(0)$$

is a parametrization. Please check the 3 conditions (general statement is Proposition 2.1.9).

Similarly $\sigma_2(x,y) = (x,y,-\sqrt{1-x^2-y^2}), (x,y) \in U = B_1(0)$ is also a parametrization. And $\sigma_1(U) \cup \sigma_2(U)$ covers S^2 minus equator $z=0$.

With 4 more parametrizations

$$\sigma_3(x,z) = (x, \sqrt{1-x^2-z^2}, z)$$

$$\sigma_4(x,z) = (x, -\sqrt{1-x^2-z^2}, z)$$

$$\sigma_5(y,z) = (\sqrt{1-y^2-z^2}, y, z)$$

$$\sigma_6(y,z) = (-\sqrt{1-y^2-z^2}, y, z)$$

they cover S^2. So S^2 is a regular surface.

To check each σ_i is a parametrization, one could prove the following more general result, whose proof is left as an exercise.

Proposition 2.1.9. *If $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ is a smooth function in an open set U of $\mathbb{R}^2$, then the graph of f, i.e. $\sigma(x,y) = (x,y,f(x,y))$ for $(x,y) \in U$ is a regular surface.*

2.1.1 Compact surfaces

A subset X of $\mathbb{R}^3$ is *compact* if it is closed and bounded (i.e. X is contained in some open ball).

Non-examples: A plane is not compact. The open disc $\{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 < 1, z = 0\}$ is not compact.

There are very few compact surfaces:

Example 2.1.10. *Any sphere is compact. Let us consider the unit sphere S^2.*

It is bounded because it is contained in the open ball $D_2(0)$.

To show S^2 is closed, i.e. the complement is open: if $||p|| \neq 1$, say $||p|| > 1$. Let $\epsilon = ||p|| - 1$, $D_\epsilon(p)$ does not intersect S^2. This is because if $q \in D_\epsilon(p)$, then $||q|| \geq ||p|| - ||p-q|| > ||p|| - \epsilon = 1$.

Other examples are torus $\Sigma_1 = T^2$, and surface of higher "genus" $\Sigma_{g\geq 2}$.

Theorem 2.1.11. *For any $g \geq 0$, Σ_g has an atlas such that it is a smooth surface. Moreover, every compact surface is diffeomorphic to one of Σ_g.*

2.1.2 Level sets

There is another family of regular surfaces: the level sets. Suppose that $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is smooth. For each $p \in U$, we have Jacobian

$$df_p = \nabla f(p) = (f_{x_1}, f_{x_2}, \cdots, f_{x_n})(p).$$

Definition 2.1.12. *We say $p \in U$ is a critical point if $df_p = 0$. Otherwise it is regular.*

The image $f(p)$ of a critical point is called a critical value. $t \in \mathbb{R}$ is a regular value if every point of the level set $f^{-1}(t)$ is regular.

The following shows the notions of regular surface and regular value coincide in some sense.

Theorem 2.1.13. *If $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ is a smooth function and $t \in f(U)$ is a regular value of f, then $f^{-1}(t)$ is a regular surface in $\mathbb{R}^3$.*

Proof. Let p be a point of $f^{-1}(t)$. Without loss, we assume $f_z(p) \neq 0$. Define $F : U \subset \mathbb{R}^3 \to \mathbb{R}^3$ by

$$F(x, y, z) = (x, y, f(x, y, z)).$$

Its Jacobian is

$$dF_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f_x & f_y & f_z \end{pmatrix}.$$

$det(dF_p) = f_z \neq 0$.

Thus by Inverse Function Theorem, we have a neighbourhood $V \subset \mathbb{R}^3$ of p and $W \subset \mathbb{R}^3$ of $F(p)$ such that $F : V \to W$ is invertible and $F^{-1} : W \to V$ is smooth, i.e. $F^{-1}(u, v, w) = (u, v, g(u, v, w))$ with $(u, v, w) \in W$ and g smooth. Especially $g(u, v, t) = h(u, v)$ is smooth, where h takes value from $W' = \{(u, v) | (u, v, t) \in W\} \subset \mathbb{R}^2$. Since $F(f^{-1}(t) \cap V) = \{(u, v, t)\} \cap W$, the graph of $h(u, v)$ is $F^{-1}(u, v, t) = f^{-1}(t) \cap V$. Hence $h : W' \to f^{-1}(t) \cap V$ is a parametrization containing p. Hence by Proposition 2.1.9 $f^{-1}(t)$ is a regular surface. □

Example 2.1.14.
- *$f(x, y, z) = x^2 + y^2 + z^2$. $\nabla f = (2x, 2y, 2z)$. Thus $f^{-1}(t)$ is an embedded surface for all $t > 0$. It is a sphere of radius t.*
- *$f(x, y, z) = x^2 + y^2 - z^2$. $\nabla f = (2x, 2y, -2z)$. $f^{-1}(0)$ is a cone which is singular at the origin. $f^{-1}(t)$ is a regular surface for $t \neq 0$. It is a hyperboloid — 1-sheeted for $t > 0$ and 2-sheeted for $t < 0$.*

2.2 The First Fundamental Form

Choose a parametrization $\sigma : U \to \mathbb{R}^3$ of S, such that $p \in \sigma(U)$ and $\sigma(u_0, v_0) = p$. A curve γ lies on S and passes through p when $t = t_0$ if $\gamma(t) = \sigma(u(t), v(t))$ with $u(t_0) = u_0$ and $v(t_0) = v_0$. By Inverse Function Theorem, both u and v are smooth.

Since $||\dot{\gamma}||^2 =< \dot{\gamma}, \dot{\gamma} >= E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2$, where

$$E = \sigma_u \cdot \sigma_u, F = \sigma_u \cdot \sigma_v, G = \sigma_v \cdot \sigma_v,$$

the arc length of such a curve from $t = a$ to $t = b$ is

$$\int_a^b ||\dot{\gamma}(t)||dt = \int_a^b \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}dt.$$

Definition 2.2.1. *The first fundamental form of a surface in $\mathbb{R}^3$ is the expression*

$$I = Edu^2 + 2Fdudv + Gdv^2.$$

This is just the quadratic form

$$Q(\mathbf{v}, \mathbf{v}) = \mathbf{v} \cdot \mathbf{v}$$

on the tangent plane written in terms of the basis σ_u and σ_v. (And we assume the formal computations $du(\sigma_u) = dv(\sigma_v) = 1, du(\sigma_v) = dv(\sigma_u) = 0$.) So it tells us how the surface S inherits the inner product of $\mathbb{R}^3$. It is represented in this basis by the symmetric matrix

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

It is clear that the first fundamental form only depends on S and p. Especially, it does not depend on the parametrization. A reparametrization $\tilde{\sigma} = \sigma \circ f$ will change it to the same form $\bar{E}dx^2 + 2\bar{F}dxdy + \bar{G}dy^2$ which is identical to the one calculated from coordinate change

$$du = u_x dx + u_y dy, dv = v_x dx + v_y dy,$$

where $f(x, y) = (u(x, y), v(x, y))$. It helps us to make measurement (e.g. Length of curves, angles, areas) on the surface directly, so we say a property of S is *intrinsic* if it can be expressed in terms of the first fundamental form.

Example:

1. Plane $\sigma(u, v) = \mathbf{a} + u\mathbf{b} + v\mathbf{c}$ with $\mathbf{b} \perp \mathbf{c}$ and

$$||\mathbf{b}|| = ||\mathbf{c}|| = 1.$$

$\sigma_u = \mathbf{b}, \sigma_v = \mathbf{c}$, so

$$E = ||\mathbf{b}||^2 = 1, F = \mathbf{b} \cdot \mathbf{c} = 0, G = ||\mathbf{c}||^2 = 1.$$

The first fundamental form is

$$I = du^2 + dv^2.$$

2. Surface of revolution

$$\sigma(u, v) = (f(u) \cos v, f(u) \sin v, g(u)).$$

We could assume the profile curve $\gamma(u) = (f(u), 0, g(u))$ is unit-speed, i.e. $f_u^2 + g_u^2 = 1$, and $f > 0$. We have

$$\sigma_u = (f_u \cos v, f_u \sin v, g_u), \sigma_v = (-f \sin v, f \cos v, 0).$$

So

$$E = f_u^2 + g_u^2 = 1, F = 0, G = f^2.$$

Hence

$$I = du^2 + f(u)^2 dv^2$$

The unit sphere S^2 is a special case where $u = \theta, v = \phi, f(\theta) = \cos\theta, g(\theta) = \sin\theta$. We have

$$I = d\theta^2 + \cos^2\theta d\phi^2$$

3. Generalized cylinder $\sigma(u, v) = \gamma(u) + v\mathbf{a}$. We assume γ is unit-speed, $\mathbf{a}$ is a unit vector, and $\dot{\gamma} \perp \mathbf{a}$. Since $\sigma_u = \dot{\gamma}, \sigma_v = \mathbf{a}$,

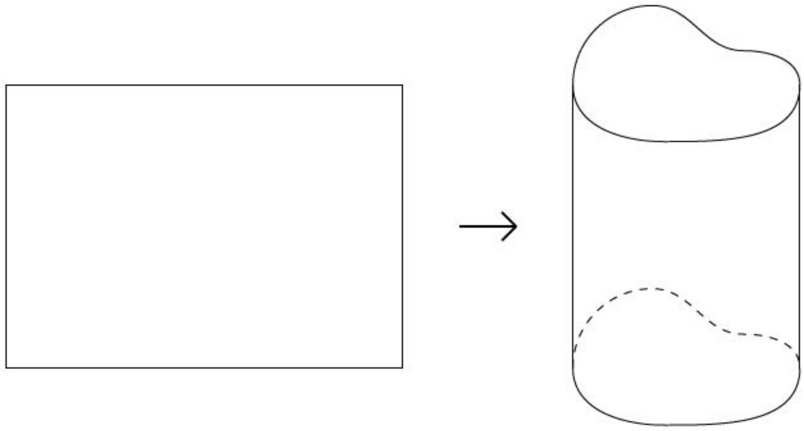
$$I = du^2 + dv^2.$$

Exercise: Calculate the first fundamental form for all other examples in previous section.

Observe that the first fundamental form of a generalized cylinder is the same as that of a plane! This is not a coincidence. The reason is the generalized cylinder is obtained from bending a piece of paper. Or it could be cut through one of its ruling to a flat paper. This is called a local isometry.

2.3 Length, Angle, Area: Isometric, Conformal, Equiareal

In this section, we explore several intrinsic properties.



2.3.1 Length: Isometry

Definition 2.3.1. *Two surfaces S_1 and S_2 are isometric if there is a diffeomorphism $f : S_1 \to S_2$ which maps curves in S_1 to curves in S_2 of the same length. The map f is called an isometry.*

The map from a plane to a cylinder is not an isometry since it is not a diffeomorphism. But indeed it has the second property. A smooth map like this is called a *local isometry*. This suggests us to look at this definition for a coordinate patch.

Theorem 2.3.2. *The coordinate patches U_1 and U_2 are isometric if and only if there exist parametrizations $\sigma_1 : V \to \mathbb{R}^3$ and $\sigma_2 : V \to \mathbb{R}^3$ with the same first fundamental form, and $\sigma_1(V) = U_1, \sigma_2(V) = U_2$.*

Proof. Suppose such parametrizations exist, then the identity map is an isometry since the first fundamental form determines the length of curves.

Conversely, assume U_1, U_2 are isometric. And let the charts be $\sigma_1 : V_1 \to \mathbb{R}^3$ and $\sigma_2 : V_2 \to \mathbb{R}^3$. So we could assume the diffeomorphism is realized by $f : V_1 \to V_2$. Then

$$\sigma_2 \circ f, \sigma_1 : V_1 \to \mathbb{R}^3$$

are parametrizations from the same open set $V = V_1$. So the fundamental forms are defined using same coordinate (u, v) as

$$E_1 du^2 + 2F_1 dudv + G_1 dv^2, E_2 du^2 + 2F_2 dudv + G_2 dv^2.$$

We have

$$\int_I \sqrt{E_1\dot{u}^2 + 2F_1\dot{u}\dot{v} + G_1\dot{v}^2}dt = \int_I \sqrt{E_2\dot{u}^2 + 2F_2\dot{u}\dot{v} + G_2\dot{v}^2}dt$$

for all curves and all intervals. Take derivative, we have

$$\sqrt{E_1\dot{u}^2 + 2F_1\dot{u}\dot{v} + G_1\dot{v}^2} = \sqrt{E_2\dot{u}^2 + 2F_2\dot{u}\dot{v} + G_2\dot{v}^2}$$

for all $u(t)$ and $v(t)$. Hence $E_1 = E_2, F_1 = F_2, G_1 = G_2$. □

2.3.2 Angle: conformal

One notices that the dot product inherited from $\mathbb{R}^3$ is also preserved under isometry (and *vice versa*), since it is determined by the first fundamental form:

$$\mathbf{v} \cdot \mathbf{w} = \frac{1}{2}(||\mathbf{v} + \mathbf{w}||^2 - ||\mathbf{v}||^2 - ||\mathbf{w}||^2).$$

Hence, the angle is also an intrinsic invariant. Let us define it.

Look at two curves α, β on the surface S intersecting at $t = 0$. The angle between them at $t = 0$ is given by

$$\cos\theta = \frac{\dot{\alpha} \cdot \dot{\beta}}{||\dot{\alpha}||||\dot{\beta}||}, 0 \leq \theta \leq \pi.$$

Everything is expressed in terms of the coefficients of the first fundamental form.

Definition 2.3.3. *Two surfaces S_1 and S_2 are conformal if there is a diffeomorphism f which preserves the angle for any pair of curves.*

Notice the invariance of the expression of $\cos\theta$ if we scale the first fundamental form by a positive function λ^2. Hence we have a similar characterization as for isometry.

Theorem 2.3.4. *The coordinate patches U_1 and U_2 are conformal if and only if there exist parametrizations $\sigma_1 : V \to \mathbb{R}^3$ and $\sigma_2 : V \to \mathbb{R}^3$ with $\sigma_1(V) = U_1, \sigma_2(V) = U_2$, and $E_2 = \lambda^2 E_1, F_2 = \lambda^2 F_1, G_2 = \lambda^2 G_1$ in V, where λ^2 is a nowhere zero differentiable function in V.*

We call them locally conformal. The most important property of conformal maps is the following.

Theorem 2.3.5. *Any two regular surfaces are locally conformal.*

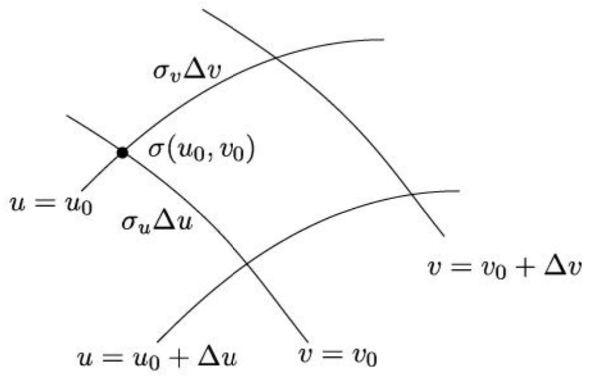
To prove the theorem, we need to choose a special parametrization. For a neighborhood of any point of a regular surface by *Isothermal parametrization*, in which the first fundamental form is $\lambda^2(u, v)(du^2 + dv^2)$.

2.3.3 Area: equiareal

Let us focus on a parametrized surface $\sigma : U \to \mathbb{R}^3$. There are two families of curves $u = const$ and $v = const$. Fix $(u_0, v_0) \in U$, we have the following picture.

The area of the "parallelogram" is

$$||\sigma_u \Delta u \times \sigma_v \Delta v|| = ||\sigma_u \times \sigma_v||\Delta u \Delta v.$$



Local "parallelogram"

Definition 2.3.6. *The area $A_\sigma(R)$ of the part $\sigma(R)$ of $\sigma : U \to \mathbb{R}^3$ for region $R \subset U$ is*

$$A_\sigma(R) = \int_R ||\sigma_u \times \sigma_v|| dudv = \int_R \sqrt{EG - F^2} dudv.$$

The second equality follows from

$$||\sigma_u \times \sigma_v||^2 = ||\sigma_u||^2 ||\sigma_v||^2 - (\sigma_u \cdot \sigma_v)^2 = EG - F^2.$$

As a corollary, we know the area of a surface patch is unchanged by reparametrization.

There is a characterization for equiareal map.

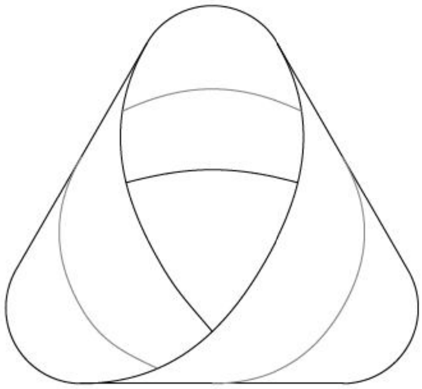
Theorem 2.3.7. *A diffeomrphism $f : U_1 \to U_2$ is equiareal, i.e. it takes any region in S_1 to a region of same area in S_2, if and only if for any surface patch σ on S_1, the first fundamental forms of charts σ and $f \circ \sigma$ satisfy*

$$E_1G_1 - F_1^2 = E_2G_2 - F_2^2.$$

We summarize that being isometric is a stronger condition than being conformal or equiareal.

2.4 The Second Fundamental Form

The first fundamental form describe the intrinsic geometry of a surface, namely independent of the choice of its sitting in $\mathbb{R}^3$. The second fundamental form describes how the surface is bent in $\mathbb{R}^3$.



Möbius band

2.4.1 Normals and orientability

A unit normal to surface S at p, up to sign, is a unit vector perpendicular to T_pS. Recall that we define standard unit normal for a parametrization $\sigma : U \to \mathbb{R}^3$ as

$$\mathbf{N}_\sigma = \frac{\sigma_u \times \sigma_v}{||\sigma_u \times \sigma_v||}.$$

However, we do not always have a smooth choice of the unit normal at any point of S. For instance, the Möbius band is such an example. Intuitively, if we walk along the middle circle of it, after one turn, the normal vector $\mathbf{N}$ will come back as $-\mathbf{N}$. In other words, we cannot make a consistent choice of a definite "side" on Möbius band. But apparently, $\mathbf{N}_\sigma$ is a smooth choice on one surface patch. Actually, the reason of this phenomenon is $\mathbf{N}_\sigma$ depends on the choice of patches.

Let $\tilde{\sigma} : \tilde{U} \to \mathbb{R}^3$ be another. Then

$$\tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}} = (\frac{\partial u}{\partial \tilde{u}}\frac{\partial v}{\partial \tilde{v}} - \frac{\partial u}{\partial \tilde{v}}\frac{\partial v}{\partial \tilde{u}})\sigma_u \times \sigma_v = \det J(\Phi)\sigma_u \times \sigma_v,$$

where $J(\Phi)$ is the Jacobian of the transition map $\Phi = \tilde{\sigma}^{-1} \circ \sigma$. So $\mathbf{N}_{\tilde{\sigma}} = \pm\mathbf{N}_\sigma$. The sign is that of $\det J(\Phi)$.

Definition 2.4.1. *A surface S is orientable if we have a smooth choice of unit normal at any point of S. Such a choice of unit normal vector field is called an orientation of S.*

A surface with a chosen orientation is called oriented.

Example 2.4.2. *Every compact surface in $\mathbb{R}^3$ is orientable. This is because every compact surface is diffeomorphic to one of Σ_g.*

The next follows from the above discussion.

Proposition 2.4.3. *A surface S is orientable if there exists an atlas $\mathcal{A}$ of S such that for transition map Φ between any two charts in $\mathcal{A}$, we have* $\det J(\Phi) > 0$.

After on, without particular mentioning, our surface will be orientable.

2.4.2 Gauss map and second fundamental form

Let $S \subset \mathbb{R}^3$ be a surface with an orientation $\mathbf{N}$, we have the *Gauss map*

$$\mathcal{G} : S \to S^2, p \mapsto \mathbf{N}_p,$$

where $\mathbf{N}_p$ is the unit normal of S at p. The rate at which $\mathbf{N}$ varies across S is measured by the derivative. It is denoted as $D_p\mathcal{G} : T_pS \to T_{\mathcal{G}(p)}S^2$. But as planes in $\mathbb{R}^3$, $T_{\mathcal{G}(p)}S^2$ and T_pS are parallel since both are perpendicular to $\mathbf{N}$. So we actually look at the *Weingarten map*

$$\mathcal{W}_{p,S} = -D_p\mathcal{G} : T_pS \to T_pS.$$

It is defined as the unique linear map determined by

$$\mathcal{W}(\sigma_u) = -\mathbf{N}_u, \mathcal{W}(\sigma_v) = -\mathbf{N}_v$$

for any parametrization σ.

Exercise: Prove $\mathcal{W}$ is independent of the choice of surface parametrization.

Parallel to the discussion of first fundamental form, we have

Definition 2.4.4. *The second fundamental form of an oriented surface is the expression*

$$II = Ldu^2 + 2Mdudv + Ndv^2$$

where $L = \sigma_{uu} \cdot \mathbf{N}, M = \sigma_{uv} \cdot \mathbf{N}, N = \sigma_{vv} \cdot \mathbf{N}$.

There is another expression. Note that $\sigma_u \cdot \mathbf{N} = 0$, we have

$$(\sigma_u \cdot \mathbf{N})_u = \sigma_{uu} \cdot \mathbf{N} + \sigma_u \cdot \mathbf{N}_u = 0$$

and similarly

$$\sigma_{vu} \cdot \mathbf{N} + \sigma_v \cdot \mathbf{N}_u = 0, \sigma_{uv} \cdot \mathbf{N} + \sigma_u \cdot \mathbf{N}_v = 0, \sigma_{vv} \cdot \mathbf{N} + \sigma_v \cdot \mathbf{N}_v = 0.$$

Hence we also have

$$L = -\sigma_u \cdot \mathbf{N}_u$$

$$M = -\sigma_u \cdot \mathbf{N}_v = -\sigma_v \cdot \mathbf{N}_u$$

$$N = -\sigma_v \cdot \mathbf{N}_v$$

Hence the second fundamental form is the symmetric bilinear form

$$II(\mathbf{w}) = \mathcal{W}_{p,S}(\mathbf{w}) \cdot \mathbf{w} =< \mathcal{W}_{p,S}(\mathbf{w}), \mathbf{w} > .$$

It is represented by

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

in terms of basis σ_u and σ_v.

There is a third interpretation. Recall that the curvature of a curve could be understand as $\mathbf{t}' \cdot \mathbf{n}$, or the second term of Taylor expansion of $\gamma(s)$. We could understand the second fundamental form in a similar way. We look at Taylor expression

$$\sigma(u+\Delta u, v+\Delta v)-\sigma(u,v) = \sigma_u \Delta u+\sigma_v \Delta v+\frac{1}{2}(\sigma_{uu}(\Delta u)^2+2\sigma_{uv}\Delta u \Delta v+\sigma_{vv}(\Delta v)^2)+R$$

where $\lim_{\Delta u, \Delta v \to 0} \frac{R}{(\Delta u)^2+(\Delta v)^2} = 0$. Since $\sigma_u \cdot \mathbf{N} = \sigma_v \cdot \mathbf{N} = 0$,

$$(\sigma(u+\Delta u, v+\Delta v) - \sigma(u,v)) \cdot \mathbf{N} = \frac{1}{2}(L(\Delta u)^2 + 2M\Delta u \Delta v + N(\Delta v)^2) + R'.$$

The fourth interpretation is more geometric: we take surface $\sigma(u,v)$ and push it inwards a distance t along its normal to get a family of surfaces

$$R(u,v,t) = \sigma(u,v) - t\mathbf{N}(u,v).$$

We calculate the first fundamental form $Edu^2 + 2Fdudv + Gdv^2$ of R which depends on t, then the derivative

$$\frac{1}{2}\frac{\partial}{\partial t}(Edu^2 + 2Fdudv + Gdv^2)|_{t=0} = Ldu^2 + 2Mdudv + Ndv^2$$

where $Ldu^2 + 2Mdudv + Ndv^2$ is the second fundamental form of σ. So it describes how the first fundamental form varies along the unit normal direction.

Example:

1. Plane $\sigma(u,v) = \mathbf{a} + u\mathbf{b} + v\mathbf{c}$ has $\sigma_{uu} = \sigma_{uv} = \sigma_{vv} = 0$. So the second fundamental form vanishes.

2. Surface of revolution

   $$\sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u)).$$

   We again assume $f_u^2 + g_u^2 = 1$ and $f > 0$. We have

   $$\sigma_u = (f_u \cos v, f_u \sin v, g_u), \sigma_v = (-f \sin v, f \cos v, 0).$$

So

$$\sigma_u \times \sigma_v = (-fg_u \cos v, -fg_u \sin v, ff_u), ||\sigma_u \times \sigma_v|| = f.$$

Hence

$$\begin{aligned} \mathbf{N} &= (-g_u \cos v, -g_u \sin v, f_u), \\ \sigma_{uu} &= (f_{uu} \cos v, f_{uu} \sin v, g_{uu}), \\ \sigma_{uv} &= (-f_u \sin v, f_u \cos v, 0), \\ \sigma_{vv} &= (-f \cos v, -f \sin v, 0). \end{aligned}$$

So the second fundamental form

$$II = (f_u g_{uu} - f_{uu} g_u) du^2 + f g_u dv^2.$$

There are two special cases:

(a) Unit sphere: $u = \theta$, $v = \phi$, $f(\theta) = \cos\theta$, $g(\theta) = \sin\theta$.

$$II = d\theta^2 + \cos^2\theta d\phi^2$$

the same as its first fundamental form.

(b) Unit cylinder: $f(u) = 1$, $g(u) = u$. So

$$II = dv^2.$$

This is different from that of a plane, although their first fundamental forms are the same.

These examples tells us second fundamental form is an extrinsic concept, although it is not independent of the first fundamental form.

Exercise: Prove the converse of Example 1: If the second fundamental form vanishes, it is part of a plane.

2.5 Curvatures

2.5.1 Definitions and first properties

The shape of a surface influences the curvature of curves on the surface.

Let $\gamma(t) = \sigma(u(t), v(t))$ be a unit-speed curve on an oriented surface S. Hence $\dot\gamma = \dot u \sigma_u + \dot v \sigma_v \in T_{\gamma(t)}S$, which means $\dot\gamma \perp \mathbf{N}$. So $\mathbf{N}$, $\dot\gamma$ and $\mathbf{N} \times \dot\gamma$ is a right handed orthonormal basis of $\mathbb{R}^3$. Since $\ddot\gamma \perp \dot\gamma$,

$$\ddot\gamma = \kappa_n \mathbf{N} + \kappa_g \mathbf{N} \times \dot\gamma \tag{2.1}$$

Here κ_n is called the *normal curvature* and κ_g is called the *geodesic curvature* of γ. Notice when σ is a plane and γ a plane curve, the geodesic curvature is just the signed curvature κ_s.

On a general (non-oriented) surface, only magnitudes of κ_n and κ_g are well defined.

Proposition 2.5.1. *1.* $\kappa_n = \ddot{\gamma} \cdot \mathbf{N}, \kappa_g = \ddot{\gamma} \cdot (\mathbf{N} \times \dot{\gamma})$.

2. $\kappa^2 = \kappa_n^2 + \kappa_g^2$.

3. $\kappa_n = \kappa \cos \psi, \kappa_g = \kappa \sin \psi$, *where* κ *is the curvature of* γ *and* ψ *is the angle between* $\mathbf{N}$ *and* $\mathbf{n}$ *of* γ.

Proof. The first is obtained by multiplying $\mathbf{N}$ and $\mathbf{N} \times \dot{\gamma}$ respectively to (2.1).

The second is by multiplying $\ddot{\gamma}$ to it.

For the last notice $\ddot{\gamma} = \kappa \mathbf{n}$. Comparing the coefficients of (2.1) and

$$\kappa \mathbf{n} = \kappa \cos \psi \mathbf{N} + \kappa \sin \psi \mathbf{N} \times \dot{\gamma}$$

gives us the equalities. □

Proposition 2.5.2. *If* γ *is a unit-speed curve on* S,

$$\kappa_n = II(\dot{\gamma}).$$

In other words, for $\gamma(t) = \sigma(u(t), v(t))$,

$$\kappa_n = L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2$$

Proof. Since $\mathbf{N} \cdot \dot{\gamma} = 0$, $\mathbf{N} \cdot \ddot{\gamma} = -\dot{\mathbf{N}} \cdot \dot{\gamma}$. So

$$\kappa_n = \mathbf{N} \cdot \ddot{\gamma} = -\dot{\mathbf{N}} \cdot \dot{\gamma} = < \mathcal{W}(\dot{\gamma}), \dot{\gamma} > = II(\dot{\gamma}).$$

□

So κ_n only depends on the point p and the tangent vector $\dot{\gamma}(p)$, but not the curve γ.

Theorem 2.5.3 (Meusnier's Theorem). *Let* $p \in S$, $\mathbf{v} \in T_pS$ *a unit vector. Let* Π_θ *be the plane containing* $\mathbf{v}$ *and making angle* $\theta \neq 0$ *with* T_pS. *Suppose* Π_θ *intersects* S *in a curve with curvature* κ_θ. *Then* $\kappa_\theta \sin \theta$ *is independent of* θ.

Proof. Let $\gamma_\theta = \Pi_\theta \cap S$, and parametrize it by arc length.

Then at p, $\dot{\gamma}_\theta = \pm \mathbf{v}$, so $\ddot{\gamma}_\theta \perp \mathbf{v}$ and $\parallel \Pi_\theta$ since γ_θ is a plane curve. Thus $\psi = \frac{\pi}{2} - \theta$ and $\kappa_\theta \sin \theta = \kappa_n$, independent of θ. □

The Weingarten map is a linear map. It could be viewed as a symmetric 2×2 matrix after fixing basis, say σ_u, σ_v, since the second fundamental form is a symmetric bilinear form. Its determinant and trace are two invariant associate with it, which is independent of the choice of basis.

Definition 2.5.4. *Let $\mathcal{W}_p$ be the Weingarten map at $p \in S$. Then the Gaussian curvature*

$$K = \det(\mathcal{W}_p),$$

and mean curvature

$$H = \frac{1}{2} trace(\mathcal{W}_p).$$

For a linear map/matrix, we also look at their eigenvalues and eigenvectors. For $\mathcal{W}$, the eigenvalues are real numbers since it is symmetric.

So at $p \in S$, there are κ_1, κ_2 and a basis $\{\mathbf{t}_1, \mathbf{t}_2\}$ of T_pS such that

$$\mathcal{W}(\mathbf{t}_1) = \kappa_1 \mathbf{t}_1, \mathcal{W}(\mathbf{t}_2) = \kappa_2 \mathbf{t}_2.$$

Moreover, if $\kappa_1 \neq \kappa_2$, then $< \mathbf{t}_1, \mathbf{t}_2 >= 0$. We call κ_1, κ_2 *principal curvatures*, and $\mathbf{t}_1, \mathbf{t}_2$ *principal vectors.* Points of the surface with $\kappa_1 = \kappa_2$ is called *umbilical points*, where $\mathcal{W}_p$ is $\kappa_1 \cdot I_{2\times 2}$ and every direction is a principal direction.

Hence, for any points, there is an orthonormal basis of T_pS consisting of principal vectors. We also know that

$$H = \frac{1}{2}(\kappa_1 + \kappa_2), K = \kappa_1 \cdot \kappa_2.$$

Theorem 2.5.5 (Euler's Theorem)**.** *Let γ be a curve on an oriented surface S, and let κ_1, κ_2 be the principal curvatures with principal vectors $\mathbf{t}_1, \mathbf{t}_2$. Then the normal curvature of γ is*

$$\kappa_n = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta,$$

where θ is the angle from $\mathbf{t}_1$ to $\dot{\gamma}$ in the orientation of T_pS (which is denoted as $\widehat{\mathbf{t}_1\dot{\gamma}}$).

Proof. We assume $\{\mathbf{t}_1, \mathbf{t}_2\}$ is an orthonormal basis and $\widehat{\mathbf{t}_1\mathbf{t}_2} = \frac{\pi}{2}$. So

$$\dot{\gamma} = \cos\theta \mathbf{t}_1 + \sin\theta \mathbf{t}_2.$$

Then

$$\kappa_n = II(\dot{\gamma}) = \cos^2\theta \cdot II(\mathbf{t}_1) + 2\sin\theta\cos\theta < \mathcal{W}(\mathbf{t}_1), \mathbf{t}_2 > + \sin^2\theta \cdot II(\mathbf{t}_2).$$

Here, recall $II(\mathbf{v}) =< \mathcal{W}(\mathbf{v}), \mathbf{v} >$.

Finally, the conclusion follows since

$$< \mathcal{W}(\mathbf{t}_i), \mathbf{t}_j >=< \kappa_i \mathbf{t}_i, \mathbf{t}_j >= \begin{cases} \kappa_i & i = j \\ 0 & i \neq j \end{cases}$$

□

We want to remark that Meusnier's Theorem and Euler's Theorem are most ancient results on the theory of surfaces.

Corollary 2.5.6. *The principal curvatures at a point of a surface are maximum and minimum of the normal curvature of all curves on the surface that pass through this point.*

Proof. If $\kappa_1 \geq \kappa_2$, then $\kappa_1 \geq \kappa_n \geq \kappa_2$. □

2.5.2 Calculation of Gaussian and mean curvatures

Now we want to calculate Gaussian curvature K and mean curvature H in terms of first and second fundamental forms. Let $\sigma(u,v)$ be a chart, and

$$I = Edu^2 + 2Fdudv + Gdv^2, II = Ldu^2 + 2Mdudv + Ndv^2.$$

We denote

$$\mathcal{F}_I = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \mathcal{F}_{II} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

Proposition 2.5.7. *Let σ be a parametrization. Then the matrix $\mathcal{W}_p$ with respect to the basis $\{\sigma_u, \sigma_v\}$ of T_pS is $\mathcal{F}_{II}\mathcal{F}_I^{-1} = (\mathcal{F}_I^{-1}\mathcal{F}_{II})^T$.*

Proof. We know that $\mathcal{W}(\sigma_u) = -\mathbf{N}_u, \mathcal{W}(\sigma_v) = -\mathbf{N}_v$. So the matrix of $\mathcal{W}$ is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

where

$$-\mathbf{N}_u = a\sigma_u + b\sigma_v, -\mathbf{N}_v = c\sigma_u + d\sigma_v.$$

Paring each with σ_u, σ_v, we have

$$L = aE + bF, M = aF + bG, M = cE + dF, N = cF + dG,$$

i.e.

$$\mathcal{F}_{II} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathcal{F}_I$$

□

Corollary 2.5.8.

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)}, K = \frac{LN - M^2}{EG - F^2}.$$

Proof.

$$K = \det(\mathcal{F}_I^{-1}\mathcal{F}_{II}) = \frac{\det \mathcal{F}_{II}}{\det \mathcal{F}_I} = \frac{LN - M^2}{EG - F^2}.$$

$$\begin{aligned}\mathcal{F}_I^{-1}\mathcal{F}_{II} =& \frac{1}{EG-F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} L & M \\ M & N \end{pmatrix} \\ =& \frac{1}{EG-F^2}\begin{pmatrix} LG - MF & MG - NF \\ ME - LF & NE - MF \end{pmatrix}\end{aligned}$$

So

$$H = \frac{1}{2}trace(\mathcal{F}_I^{-1}\mathcal{F}_{II}) = \frac{LG - 2MF + NE}{2(EG - F^2)}.$$

□

Example 2.5.9. *Surface of revolution*

$$\sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u)).$$

We again assume $f_u^2 + g_u^2 = 1$ *and* $f > 0$.

$$I = du^2 + f^2dv^2, II = (f_u g_{uu} - f_{uu}g_u)du^2 + fg_u dv^2.$$

Hence

$$K = \frac{(f_u g_{uu} - f_{uu}g_u)fg_u}{f^2}.$$

Taking derivative on $f_u^2 + g_u^2 = 1$, *we have*

$$f_u f_{uu} + g_u g_{uu} = 0.$$

So

$$(f_u g_{uu} - f_{uu}g_u)g_u = -f_{uu}(f_u^2 + g_u^2) = -f_{uu},$$

and

$$K = -\frac{f_{uu}f}{f^2} = -\frac{f_{uu}}{f}.$$

Especially, for a unit sphere $u = \theta, v = \phi, f(\theta) = \cos\theta, g(\theta) = \sin\theta$. *We thus have* $K = 1$.

Gauss uses another way to define K, roughly speaking it is the ratio of the area changed under Gaussian map $\mathcal{G}$, or

$$\lim_{R \to p} \frac{Area(\mathcal{G}(R))}{AreaR}.$$

Next theorem makes it precisely.

Theorem 2.5.10. *Let $\sigma : U \to \mathbb{R}^3$ be a parametrization, with $(u_0, v_0) \in U$. Let $R_\delta = \{(u,v) \in \mathbb{R}^2 | (u-u_0)^2 + (v-v_0)^2 \leq \delta^2\}$. Then*

$$\lim_{\delta \to 0} \frac{A_{\mathbf{N}}(R_\delta)}{A_\sigma(R_\delta)} = |K|,$$

where K is the Gaussian at $\sigma(u_0, v_0)$.

Proof. Recall that

$$\frac{A_{\mathbf{N}}(R_\delta)}{A_\sigma(R_\delta)} = \frac{\int_{R_\delta} ||\mathbf{N}_u \times \mathbf{N}_v|| dudv}{\int_{R_\delta} ||\sigma_u \times \sigma_v|| dudv}.$$

$$\begin{aligned} \mathbf{N}_u \times \mathbf{N}_v =& (a\sigma_u + b\sigma_v) \times (c\sigma_u + d\sigma_v) \\ =& (ad - bc)\sigma_u \times \sigma_v \\ =& \det(\mathcal{F}_I^{-1}\mathcal{F}_{II})\sigma_u \times \sigma_v \\ =& K\sigma_u \times \sigma_v \end{aligned}$$

So we could choose δ small, such that $|K(u,v) - K(u_0, v_0)| < \epsilon$ if $(u,v) \in R_\delta$. So

$$|K(u_0, v_0)| - \epsilon < \frac{A_{\mathbf{N}}(R_\delta)}{A_\sigma(R_\delta)} < |K(u_0, v_0)| + \epsilon.$$

This finishes the proof. □

2.5.3 Principal curvatures

Let us come back to principal curvatures. They are the roots κ of $\det(\mathcal{F}_I^{-1}\mathcal{F}_{II} - \kappa I) = 0$, which is

$$\det(\mathcal{F}_{II} - \kappa\mathcal{F}_I) = 0.$$

$\mathbf{t} = \xi\sigma_u + \eta\sigma_v$ is a principal vector if

$$(\mathcal{F}_{II} - \kappa\mathcal{F}_I)\begin{pmatrix}\xi \\ \eta\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}.$$

Example 2.5.11. *For unit sphere*

$$I = II = d\theta^2 + \cos^2\theta d\phi^2.$$

So principal curvatures are repeated roots $\kappa = 1$ and thus every tangent vector is principal, every point is umbilical.

Example 2.5.12. *For cylinder*

$$\sigma(u,v) = (\cos v, \sin v, u).$$

$$I = du^2 + dv^2, II = dv^2.$$

Principal curvatures are solutions of

$$\det\begin{pmatrix}0-\kappa & 0\\ 0 & 1-\kappa\end{pmatrix}=0.$$

So $\kappa = 0, 1$, *and no point is umbilical.*

The principal vector

$$\mathbf{t}_1 = \sigma_u = (0,0,1), \mathbf{t}_2 = \sigma_v = (-\sin v, \cos v, 0).$$

Proposition 2.5.13. *Let* S *be a connected surface of which every point is umbilical. Then* S *is an open subset of a plane or a sphere.*

Proof. For every tangent vector $\mathbf{t}$, $\mathcal{W}(\mathbf{t}) = \kappa\mathbf{t}$ where κ is the principal curvature. Since $\mathcal{W}(\sigma_u) = -\mathbf{N}_u, \mathcal{W}(\sigma_v) = -\mathbf{N}_v$, then

$$\mathbf{N}_u = -\kappa\sigma_u, \mathbf{N}_v = -\kappa\sigma_v.$$

Hence by taking derivatives,

$$\kappa_v\sigma_u = \kappa_u\sigma_v.$$

Since σ_u and σ_v are linearly independent, $\kappa_u = \kappa_v = 0$. Thus $\kappa \equiv C$.

If $\kappa = 0$, $\mathbf{N}$ is constant. Then $(\mathbf{N}\cdot\sigma)_u = (\mathbf{N}\cdot\sigma)_v = 0$, so $\mathbf{N}\cdot\sigma \equiv C$. Thus $\sigma(U)$ is an open subset of the plane $P\cdot\mathbf{N} \equiv C$.

If $\kappa \neq 0$, $\mathbf{N} = -\kappa\sigma + \mathbf{a}$. Hence

$$||\sigma - \frac{1}{\kappa}\mathbf{a}||^2 = ||-\frac{1}{\kappa}\mathbf{N}||^2 = \frac{1}{\kappa^2}.$$

So $\sigma(U)$ is an open subset of the sphere with centre $\kappa^{-1}\mathbf{a}$ and radius $|\kappa|^{-1}$.

To complete the proof, notice that each patch is contained in a plane or a sphere. But if the images of two patches intersect, they must clearly be part of the same plane or same sphere. So complete the proof. □

Principal curvature at $p \in S$ provides the information about shape. We choose the coordinates as following: p is the origin, T_pS is the xy-plane in $\mathbb{R}^3$, principal vectors $\mathbf{t}_1 = (1,0,0)$ and $\mathbf{t}_2 = (0,1,0)$ and $\mathbf{N} = (0,0,1)$. We could always choose such a coordinate up to an isometry, i.e. rotation and translation, of $\mathbb{R}^3$.

Let σ be a parametrization with $\sigma(0,0) = 0$ (point p). The tangent plane is $\{(x,y,0)\} = s\sigma_u(0,0) + t\sigma_v(0,0)$. Taylor expansion gives us

$$\sigma(s,t) = \sigma(0,0)+s\sigma_u(0,0)+t\sigma_v(0,0)+\frac{1}{2}(s^2\sigma_{uu}(0,0)+2st\sigma_{uv}(0,0)+t^2\sigma_{vv}(0,0))+\cdots$$

If x, y (hence s, t) are small, we have $\sigma(s,t) \approx (x,y,z)$ where

$$z \approx \frac{1}{2}(s^2\sigma_{uu}(0,0) + 2st\sigma_{uv}(0,0) + t^2\sigma_{vv}(0,0))\cdot\mathbf{N} = \frac{1}{2}(Ls^2 + 2Mst + Nt^2).$$

Since

$$\mathcal{W}(\mathbf{t}) = x\mathcal{W}(\mathbf{t}_1) + y\mathcal{W}(\mathbf{t}_2) = \kappa_1 x\mathbf{t}_1 + \kappa_2 y\mathbf{t}_2 = (\kappa_1 x, \kappa_2 y, 0)$$

for $\mathbf{t} = (x, y, 0)$, hence

$$Ls^2 + 2Mst + Nt^2 =< \mathcal{W}(\mathbf{t}), \mathbf{t} >= \kappa_1 x^2 + \kappa_2 y^2.$$

So near p, S is approximated by $z = \frac{1}{2}(\kappa_1 x^2 + \kappa_2 y^2)$.

There are 4 cases of local behaviour:

1. *Elliptic* if $K_p > 0$, so $z = \frac{1}{2}(\kappa_1 x^2 + \kappa_2 y^2)$ is an elliptic paraboloid.
2. *Hyperbolic* if $K_p < 0$, it is a hyperbolic paraboloid.
3. *Parabolic* if one of κ_1, κ_2 is zero, and the other is non-zero. It is a parabolic cylinder.
4. *Planar* if both $\kappa_1 = \kappa_2 = 0$ (or $\mathcal{W}_p \equiv 0$). We need higher derivatives to know the shape.

2.6 Gauss's Theorema Egregium

Since the definitions of curvatures involve the second fundamental form, they are usually not intrinsic. But actually Gaussian curvature K is an intrinsic invariant.

Theorem 2.6.1 (Gauss's Theorema Egregium)**.** *The Gaussian curvature K of a surface is invariant of the first fundamental form.*

In this section, we prove it by detailed calculations.

For regular surface S, and a chart $\sigma : U \to S$, $\sigma_u, \sigma_v, \mathbf{N}$ would be a basis. We express $\sigma_{uu}, \sigma_{uv}, \sigma_{vv}$ by

$$\sigma_{uu} = \Gamma_{11}^1\sigma_u + \Gamma_{11}^2\sigma_v + L_1 \cdot \mathbf{N} \tag{2.2}$$

$$\sigma_{uv} = \Gamma_{12}^1\sigma_u + \Gamma_{12}^2\sigma_v + L_2 \cdot \mathbf{N} \tag{2.3}$$

$$\sigma_{vv} = \Gamma_{22}^1\sigma_u + \Gamma_{22}^2\sigma_v + L_3 \cdot \mathbf{N} \tag{2.4}$$

Here Γ_{ij}^k are called *Christoffel symbols.*

First, by taking dot product with $\mathbf{N}$, $L_1 = L, L_2 = M, L_3 = N$.

Next, we claim Γ_{ij}^k only depends on the first fundamental form. More precisely,

$$\begin{aligned}
\Gamma_{11}^1 = \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)}, \Gamma_{11}^2 = \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)} \\
\Gamma_{12}^1 = \frac{GE_v - FG_u}{2(EG - F^2)}, \Gamma_{12}^2 = \frac{EG_u - FE_v}{2(EG - F^2)} \\
\Gamma_{22}^1 = \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)}, \Gamma_{22}^2 = \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}
\end{aligned} \tag{2.5}$$